On vector-valued automorphic forms on bounded symmetric domains

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Abstract

The objective of the present study is to investigate the behaviour of the inner products of vector-valued Poincaré series associated to submanifolds of a quotient of the unit ball $B^n_C$ in $C^n$ for large weight and how vector-valued automorphic forms could be constructed via Poincaré series.

The notion of automorphic forms was viewed by Jules Henri Poincaré as a generalization of both trigonometric and elliptic functions and he named them Fuchsian functions, after the mathematician Lazarus Fuchs. Under Poincaré’s definition, an automorphic function is one which is analytic in its domain and is invariant under a discrete infinite group of linear fractional transformations.

T.Foth [45],[46], [47] has worked on Poincaré series on bounded symmetric domains and discussed in particular, the case when the bounded symmetric domain is $B^n_C$, and associated a Poincaré series to specific submanifolds in ball quotients. Motivated by this study, last chapter of this dissertation is devoted to generalizing these results to vector-valued Poincaré series on irreducible bounded symmetric domains. We give estimates for the case when the domain is the complex unit ball. These illustrate the relation between properties of automorphic forms and geometric properties of submanifolds. In addition, we prove in chapter 2 that vector-valued Poincaré series on an irreducible bounded symmetric domain span the space of holomorphic vector-valued automorphic forms.
DEDICATED
To
My Parents and My Family
whose sacrifices made it possible
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Preface

It is an interesting general question: how is the geometry of a space related to analytic properties of functions on this space, (holomorphic or $C^\infty$ functions)?

Does the algebra of $C^\infty$ functions on a smooth manifold determine this manifold up to an isomorphism, for example?

One of the primary tasks in the theory of compact Riemann surfaces is studying all meromorphic functions on all Riemann surfaces. It looks difficult but the powerful tool in easing this problem is the Uniformization Theorem which allows the construction of functions on Riemann surfaces. Uniformization Theorem states that any simply connected Riemann surface is biholomorphic to either the Riemann sphere $\mathbb{CP}^1$, the complex number space $\mathbb{C}$, or the unit disc $D$. Therefore, the universal cover $\tilde{M}$ of any Riemann surface $M$ is biholomorphic to one of the above surfaces. Based on that, $M$ can be represented as a quotient of one of these surfaces by a discrete group $\Gamma$ of automorphic transformations. In this case, $\Gamma$-invariant meromorphic functions on the universal cover correspond to meromorphic functions on the original surface $M$. $\Gamma$-invariant meromorphic functions on $\tilde{M}$ can be constructed as ratios of holomorphic functions which are $\Gamma$-invariant. More generally, holomorphic functions on $\tilde{M}$ which satisfy

$$J(\gamma, z)f(\gamma z) = f(z), \quad \gamma \in \Gamma$$

are called automorphic forms, where the automorphy factors $\{J\}$ here are nowhere zero holomorphic functions satisfying a cocycle condition

$$J(\gamma_1 \gamma_2, z) = J(\gamma_1, \gamma_2 z)J(\gamma_2, z).$$

Formally, one can construct an automorphic form as a Poincaré series

$$\sum_{\gamma \in \Gamma} J(\gamma, z)f(\gamma z)$$

for some given holomorphic function $f$, and here convergence needs some care.

If $\Gamma$ is a group of $2 \times 2$ matrices acting on $D$ by fractional linear transformations, the determinant of Jacobi matrix of transformation $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
at \( z \) is given by
\[
J(\gamma, z) = (cz + d)^{-2}
\]
Then \( J^k(\gamma, z) \) for some integer (weight) \( k \geq 2 \) is an automorphy factor. Thus, Poincaré series can be established if the function \( f \) is bounded on \( D[104] \). Poincaré series of Bohr-Sommerfeld curves on the upper half plane \( H \approx D \) have been studied by D. Borthwick, T.Paul and A. Uribe [23]. As a corollary of the asymptotic expansion, D. Borthwick, T.Paul and A. Uribe [23] found that the Poincaré series associated to certain elements of \( \Gamma \) are non-vanishing for large weight \( k \).

The automorphic forms on \( \mathbb{C} \) are usually called theta functions. Eisenstein forms, the coefficients of Weierstrass \( \wp \)-function, give an example of the automorphic forms on \( \mathbb{C} \) [85]. Recall that the Weierstrass \( \wp \)-function is a meromorphic function on \( \mathbb{C} \) which is doubly periodic and every elliptic function is a rational function of \( \wp \) and its derivative \( \wp' \). Studying this automorphic function on \( \mathbb{C} \) leads to a significant fact that the complex tori are elliptic curves.

For a complex submanifold of a complex projective space, a particular method to construct automorphic forms will lead us to the Poincaré series. \( \{ J(\gamma, z) \} \) determine a holomorphic line bundle back on \( M \) such that the automorphic functions represent holomorphic sections pulled back to the universal cover. Therefore, automorphic forms are used to construct holomorphic sections of line bundles on \( M \), particularly of the canonical line bundle \( K_M \). Thus, sections of large powers of \( K_M \) can be produced by Poincaré series. J.Kollár [74] proved the existence of nonzero holomorphic sections of quadratic and higher powers of \( K_M \) by using Riemann-Roch theorem of Atiyah [7] and a vanishing theorem of Andreotti-Vesentini [5]. J.Kollár [74] as well studied the case when the universal cover is a ball or a bounded open subset of \( \mathbb{C}^n \).

In fact, an automorphic form is a generalization of a certain class of periodic functions. Generally, let \( \tilde{M} \) be a topological manifold which is locally compact with a properly discontinuous group action of a discrete group \( \Gamma \). Then an automorphic form \( f : \tilde{M} \rightarrow \mathbb{C} \) of weight \( k \) is a holomorphic function verifying the functional equation
\[
J^k(\gamma, z)f(\gamma z) = f(z), \quad \gamma \in \Gamma
\]
where $J$ is an automorphy factor. When $\tilde{M} = G/K$ where $G$ is locally compact Lie group and $K$ is closed subgroup, then this quotient is a Riemannian manifold. In addition, if the $G$-action is transitive and $\Gamma$ is a subgroup of $G$ which acts freely and properly discontinuously on $\tilde{M}$, then $M = \Gamma \backslash \tilde{M}$ is a manifold inheriting the structure of the ambient manifold $\tilde{M}$. This way would enable us to study the geometry aspects of a manifold since we could address its geometry compared with that of the known ambient manifold.

For compact manifolds, the well-known Whitney Embedding theorem states that every compact $n$-dimensional real manifold can be embedded into $\mathbb{R}^{2n}$. In contrast from the Liouville’s theorem, the analogous result does not hold for compact complex submanifolds of $\mathbb{C}^n$. This fact gives impetus to study submanifolds of other space such as the complex projective space $\mathbb{C}P^n$. The question arisen here is: Can every complex manifold be embedded in $\mathbb{C}P^n$?. Kodaira Embedding theorem (see [60] or [83]) and Tian’s work provide valuable information for compact complex manifolds and bring algebraic geometry to the study of complex projective manifolds.

When one attempts to construct automorphic functions associated to any compact Kaehler manifold $M$ with a positive hermitian holomorphic line bundle, one may realize that the Poincaré map is not always surjective. J.Kollár [74] and others have established results providing conditions on the Bergman kernel on the universal cover $\tilde{M}$ with the vector bundle $\tilde{E}$ such that the Poincaré map is surjective. Those assumptions on the Bergman kernel on $\tilde{M}$ are known to be satisfied when the universal cover is a bounded symmetric domain. Irreducible bounded symmetric domains in $\mathbb{C}^n$, $n > 1$, are classified into six types and the unit ball $B_\mathbb{C}^n$ is isomorphic to one of them. Recently, the study of automorphic function on bounded symmetric domains got impetus with J.Kollár work ([74]) on Poincaré series when the ambient space is a ball or a bounded open subset of $\mathbb{C}^n$ [23], [45], [46], [47]. T.Foth [45] has worked on Poincaré series on bounded symmetric domains and as a crucial finding, she showed that any $\mathbb{C}$-valued holomorphic automorphic form of sufficiently large weight on an irreducible bounded symmetric domain in $\mathbb{C}^n$, $n > 1$, is the Poincaré series of a polynomial in $z_1, z_2, ..., z_n$ and she gave an upper bound on the degree of this polynomial and gave an explicit basis in the space of holomorphic automorphic forms as well. In 2002, T.Foth [47] described a relative Poincaré series associated to Bohr-Sommerfeld tori in the quotient $\Gamma / B_\mathbb{C}^2$, where $\Gamma$ is a discrete subgroup of $SU(2, 1)$ and computed
the asymptotics of it. She extended her study on a submanifold of a quotient of the ball \( B_n^C \). Inspired by the work on the topic, new results on the study of automorphic forms associated to submanifolds of a ball quotient are presented in the last chapter of this thesis. Moreover, these automorphic forms are vector-valued.

The thesis comprises of three chapters. The first chapter is introductory and serves the purpose of fixing the notations and developing the basic concepts keeping in view the pre-requisites of the subsequent chapters and also to make the thesis self contained.

Section 1 reviews basic facts about one of fundamental families of function spaces in analysis, namely \( L^p \) spaces. They play an important role as model examples for the general theory of topological and normed vector spaces. The special importance of \( L^p \) spaces may come from the fact that they are considered as partial generalization of the fundamental \( L^2 \) space of square integrable functions which is of independent interest. This section contains the definition and properties relevant to measure theory, particularly, density, and completeness. A reference for the results incorporated in section 1 is [44].

Section 2 of the first chapter is devoted to the study of CR-submanifolds in the setting of almost Hermitian manifolds, Kaehler manifolds. Integrability conditions of the canonical distributions on a CR-submanifold of the underlying ambient manifolds are discussed and the geometry of the leaves of the distributions are studied. Consequently, the conditions are worked out under which a CR-submanifold reduces to a CR-product. The relevant results of B.Y.Chen [31], [32], [33], [34], [35], [36], Chen and Blair [19], N.Sato [97], V.A.Khan and K.A.Khan [69], [68] are incorporated in the section.

Sections 3 and 4 deal with hyperbolic geometry. Hyperbolic geometry is a rich area which is beautifully interconnected with many other branches of mathematics such as Riemannian geometry, complex analysis, symplectic geometry, algebraic geometry and group theory. Hyperbolic geometry is a non-Euclidean geometry, where the parallel postulate of Euclidean geometry is replaced by another. There are three classical models for complex hyperbolic space \( H^n_C \), namely the unit ball model in \( C^n \), the projective ball model in \( P^n_C \) and the Siegel domain model. In this study, we use the projective ball model, and the unit ball model. The unit ball in \( C^n \) which has a nat-
ural metric of constant negative holomorphic sectional curvature called the Bergman metric forms a model for complex hyperbolic n-space analogous to the ball model of real hyperbolic 2-space. This is analogous to but different from the real hyperbolic space. In the complex case, the sectional curvature is constant on complex lines, but it is not when we consider real 2-planes. An alternative description of $H^2_n$ is the Siegel domain. This is analogous to the the half space model of $H^2_n$. Another standard model for complex hyperbolic space is called the projective model that is given by projectivising the set of complex lines on which this form is negative. By taking a suitable form and making a choice of section we can recover the ball model and the Siegel domain model. All holomorphic isometries of complex hyperbolic space are given by the projectivisation of unitary matrices preserving the Hermitian form, $PU(n,1)$. They are therefore Möbius transformations. In section 3, we recall that for n-dimensional complex hyperbolic space, totally geodesic subspaces are are either embedded copies of $H^m_C$ or $H^m_R$ for $1 \leq m \leq n$ and that every totally real r-plane in $H^r_C$ is isometric to the real hyperbolic space $H^r_R$. Section 4 deals analogously with the unit ball model and the projective ball model in of the real hyperbolic space. We have included the results of [50], [29], [49], [109], [25], [30] in section 3 and results of [4], [49],[50], [99], [82], [89] in section 4.

Section 5 contains a brief introduction to characteristic classes associated with complex vector bundles which are known as Chern classes and some related results obtained from [60], [71], [83], [88], [105]. It gives basic definitions, characterization of positivity and negativity of vector bundles based on their first Chern classes, and we discuss how the first Chern class plays a role in embedding compact complex manifolds into projective spaces. These relevant results are obtained from [60], [71], [83], [88], [105].

The Bergman kernel is a reproducing kernel for the Hilbert space of all square integrable holomorphic functions on a domain $D$ in $\mathbb{C}^n$. Bergman kernel and its associated metric are invariant under the automorphism group of the domain since the $L^2$ inner product on this space is invariant under biholomorphisms of $D$. The Bergman kernel has become an important tool in the complex analysis (see [43], [77]). Its fundamental significance comes out from its reproducing properties, biholomorphic invariance, and relationship to the Bergman metric and its contribution in constructing Poincaré series and then automorphic forms on $D$, when $D$ is a bounded symmetric
domain. Therefore, it is important to obtain concrete information about the Bergman kernel. On the disc, the ball, and the polydisc, the kernel may be computed explicitly (see e.g. [95] or [77]). Analogously, it can be computed on the bounded symmetric domains (see [59]). Generally, it is quite difficult to obtain specific, concrete information about this kernel. In Section 6, we introduce appropriate definitions, theorems and facts related to Bergman kernel, the references are [28], [75], [76], [83], [86], [96], [105], [106], [114].

A Poincaré series is a generalization of the classical theta series that is associated to any discrete group of symmetries of a complex domain. In case $X = D/\Gamma$ is a compact quotient of a bounded domain $D$ in $\mathbb{C}^n$ and for $k$ sufficiently large, the space of holomorphic sections of $K_X^k$ is generated by the Poincaré series of bounded holomorphic functions on $D$, where $K_X$ is the canonical line bundle of $X$ (see [74], [85]). Therefore, the Poincaré map is surjective which implies very ampleness of $K_X^k$[111]. Hence, Poincaré series has a major contribution to constructing modular forms on $D$ in a certain way as shown in the other two chapters of this study. Section 8 is a short introduction to Poincaré series.

In chapters 2 and 3, we construct vector-valued Poincaré series on irreducible bounded symmetric domains, and give estimates in the case when the domain is the complex unit ball. Chapter 2 splits into two parts. First section of chapter 2 is devoted to a review of standard facts about $\mathbb{C}$-valued automorphic forms on the complex upper half plane whereas the second section is devoted to the study of $\mathbb{C}$-valued automorphic forms on irreducible bounded symmetric domains building on work by T.Barron (see [45], [46], [47]). We introduce the definition of vector-valued automorphic forms and prove that vector-valued Poincaré series span the space of holomorphic vector-valued automorphic forms in Theorem 2.2.2. In chapter 3, we extend the study of $\mathbb{C}$-valued automorphic forms associated to submanifolds of irreducible bounded symmetric domains to the setting of vector-valued automorphic forms. Asymptotics of the inner product of two Poincaré series of submanifolds of the complex unit ball and some examples are provided in this chapter. The main results in this chapter are Theorems 3.3.1, 3.3.2, 3.3.3. Examples 3.3.1- 3.3.11 contain explicit calculations for specific submanifolds. In particular, Examples 3.3.1, 3.3.2, 3.3.3, 3.3.10 illustrate statements of the main theorems, while Examples 3.3.5 and 3.3.11 provide insight in situations not covered by statements of the theorems. Some of the results and calculations
that appear in Chapter 2 and 3 contained in [3].

At the end of the thesis, references have been given which by no means are comprehensive but mention only the papers and books referred to in the main body of the thesis.
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Chapter 1
Preliminaries

1.1 $L^p$ spaces

Any automorphic form on a bounded symmetric domain $D$ in $\mathbb{C}^n$ can be formally constructed through Poincaré series and is related to the Bergman kernel which is a reproducing kernel for the Hilbert space of all square integrable holomorphic functions on $D$. Moreover, if the Bergman projection on $L^2$ can be extended to bounded linear maps on $L^1$ and $L^\infty$, this would provide one condition of surjectivity of Poincaré series map (see [74] Theorem 7.12). Here in this section we introduce the basic structural facts about the $L^p$ spaces. Essential material for a study of function spaces in [6], [21], [44], [70] will be provided in this section.

Let $(X, A, \mu)$ be a measure space, where $X$ denotes the underlying space, $A$ is the $\sigma$-algebra of measurable sets, and $\mu$ is the measure on $X$. If $1 \leq p < \infty$, the space $L^p(X, A, \mu)$, simply $L^p(X)$ or $L^p$, consists of all complex-valued measurable functions on $X$ that satisfy

$$\int_X |f(x)|^p d\mu(x) < \infty$$

(1.1)

The $L^p$ norm of $f \in L^p(X, A, \mu)$ is defined by

$$\|f\|_{L^p} = \left( \int_X |f(x)|^p d\mu(x) \right)^{1/p}$$

When $p = 1$ the space $L^1(X, A, \mu)$ consists of all integrable functions on $X$. 
We note here $\|f\|_{L^p} = 0$ does not imply that $f = 0$, but merely $f = 0$ almost everywhere for the measure $\nu$. Therefore, it introduces the equivalence relation, in which $f$ and $g$ are equivalent if $f = g$ a.e. Then,

**Definition 1.1.1.** ([44], [21]) Let $(X, A, \nu)$ be a measure space and $1 \leq p < \infty$. The *space* $L^p$ consists of all equivalence classes of complex-valued measurable functions on $X$ which satisfy (1.1).

**Example 1.1.1.** $X = \mathbb{R}^d$ is a common example of $L^p$ spaces with *Lebesgue measure* $\nu$ which is often used in practice. There, we have

$$\|f\|_{L^p} = \left( \int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}$$

**Example 1.1.2.** If $\mathbb{N}$ is equipped with counting measure, then $L^p(\mathbb{N})$ consists of all sequences $f = \{x_n \in \mathbb{R} : n \in \mathbb{N}\}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

We write this sequence space as $\ell^p(\mathbb{N})$, with norm

$$\|f\|_{\ell^p} = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

**Definition 1.1.2.** Let $f$ a measurable function on a measure space $(X, A, \nu)$. The *essential supremum* of $f$ on $X$ is

$$\text{ess sup}_X f = \inf \{ \ell \in \mathbb{R} : \nu \{x \in X : f(x) > \ell\} = 0 \}.$$ 

Equivalently,

$$\text{ess sup}_X f = \inf \{ \sup_X g : g = f \text{ pointwise a.e.} \}.$$ 

We say that $f$ is *essentially bounded* on $X$ if

$$\text{ess sup}_X |f| < \infty$$
Definition 1.1.3. ([44], [21]) Let \((X, A, \nu)\) be a measure space. The space \(L^\infty(X)\) consists of all equivalence classes of essentially bounded measurable functions \(f : X \to \mathbb{C}\) with norm
\[
\|f\|_{L^\infty} = \text{ess sup}_X |f|
\]

Therefore, we have \(|f(x)| \leq \|f\|_{L^\infty}\) for almost every \(x\). Indeed, if \(E = \{x : |f(x)| > \|f\|_{L^\infty}\}\), and \(E_n = \{x : |f(x)| > \|f\|_{L^\infty} + 1/n\}\), then we have \(\nu(E_n) = 0\), and \(E = \bigcup E_n\), hence \(\nu(E) = 0\).

1.1.1 Hölder and Minkowski inequalities

If the two exponents \(p\) and \(q\) satisfy \(1 \leq p, q \leq \infty\), and the relation
\[
\frac{1}{p} + \frac{1}{q} = 1
\]
holds, we say that \(p\) and \(q\) are conjugate or dual exponents. Here, we use the convention \(\frac{1}{\infty} = 0\). Note that \(p = 2\) is self-dual, also \(p = 1, \infty\) corresponds to \(q = \infty, 1\) respectively.

Theorem 1.1.1 (Hölder). Suppose \(1 \leq p \leq \infty\) and \(1 \leq q \leq \infty\) are conjugate exponents. If \(f \in L^p\) and \(g \in L^q\), then \(fg \in L^1\) and
\[
\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.
\]

Proof. The idea of the proof relies on the arithmetic-geometric mean inequality: if \(A, B \geq 0\), and \(0 \leq \alpha \leq 1\), then
\[
A^\alpha B^{1-\alpha} \leq \alpha A + (1 - \alpha)B.
\]

To obtain (1.2), assume \(B \neq 0\), and replacing \(A\) by \(AB\), then it suffices to prove that \(A^\alpha \leq \alpha A + (1 - \alpha)\). If we let \(f(x) = x^\alpha - \alpha x - (1 - \alpha)\), then \(f'(x) = \alpha(x^{\alpha-1} - 1)\). Thus \(f(x)\) increases when \(0 \leq x \leq 1\) and decreases when \(1 \leq x\), and we see that the continuous function \(f\) attains a maximum at \(x = 1\), where \(f(1) = 0\). Therefore \(f(A) \leq 0\), as desired.

Now we prove Hölder’s inequality as follows: If either \(\|f\|_{L^p} = 0\) or \(\|g\|_{L^q} = 0\), then \(fg = 0\) a.e. and the inequality is obviously verified. Therefore, we may assume that neither of these norms vanish, say \(\|f\|_{L^p} = \|g\|_{L^q} = 1\).
We now need to prove that $\|fg\|_{L^1} \leq 1$. If we set $A = |f(x)|^p, B = |g(x)|^q,$ and $\alpha = \frac{1}{p}$ so that $1 - \alpha = \frac{1}{q}$, then (1.2) gives

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q.$$ Introducing this inequality yields $\|fg\|_{L^1} \leq 1$, and then by replacing $f$ by $f^\alpha \|f\|_{L^p}$ and $g$ by $g^\beta \|g\|_{L^q}$, the proof of the H"older inequality is complete.

The following theorem gives us the triangle inequality for the $L^p$ norm.

**Theorem 1.1.2 (Minkowski).** If $1 \leq p \leq \infty$ and $f, g \in L^p$, then $f + g \in L^p$ and $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.

**Proof.** The case $p = 1$ is obtained by integrating $|f(x) + g(x)| \leq |f(x)| + |g(x)|$. When $p > 1$ and both $f$ and $g$ belong to $L^p$, we consider separately the cases $|f(x)| \leq |g(x)|$ and $|g(x)| \leq |f(x)|$, so we get

$$|f(x) + g(x)|^p \leq 2^p(|f(x)|^p + |g(x)|^p)$$

Therefore integrating this inequality, we find $f + g \in L^p$. Now consider

$$(f + g)^p = (f + g)(f + g)^{p-1}$$

Let $q$ be the conjugate exponent of $p$, then $(p - 1)q = p$, so it is easily seen that $(f + g)^{p-1}$ belongs to $L^q$. Now, we note that

$$|f(x) + g(x)|^p \leq (\|f\| + \|g\|)(f(x) + g(x))^{p-1}$$

Therefore, applying Hölder’s inequality to the two terms on the right-hand side of the above inequality gives

$$\|f + g\|_{L^p}^p \leq (\|f\|_{L^p} + \|g\|_{L^p})(f + g)^{p-1}_{L^q}.$$ (1.3)

However, using once again $(p - 1)q = p$, we get

$$\|(f + g)^{p-1}\|_{L^q} = \|f + g\|_{L^p}^p$$

By multiplying both sides of (1.3) by $\|(f + g)^{p/q}\|_{L^p}$, then using $p - p/q = 1$, and supposing that $\|f + g\|_{L^p} > 0$, we find

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p},$$

so the proof is done. \(\Box\)

Hence, the spaces $L^p$ are normed vector spaces.
1.1.2 Completeness of $L^p$

The triangle inequality makes $L^p$ into a metric space with distance $d(f, g) = \|f - g\|_{L^p}$. Moreover, $L^p$ is complete in the sense that every Cauchy sequence in the norm $\|\cdot\|_{L^p}$ converges to an element in $L^p$.

**Theorem 1.1.3. ([44], [70] Theorem 3.2.2) The space $L^p(X, A, \mu)$ is complete in the norm $\|\cdot\|_{L^p}$.**

**Proof.** First, suppose that $1 \leq p < \infty$. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $L^p$, and consider a subsequence $\{f_{n_i}\}_{i=1}^\infty$ of $\{f_n\}$ with the following property $\|f_{n_{i+1}} - f_{n_i}\|_{L^p} \leq 2^{-i}$ for all $i \geq 1$. We now consider the series whose convergence will be seen below

$$f(x) = f_{n_1}(x) + \sum_{i=1}^\infty (f_{n_{i+1}}(x) - f_{n_i}(x))$$

and

$$g(x) = |f_{n_1}(x)| + \sum_{i=1}^\infty |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

and the corresponding partial sums

$$S_K(f)(x) = f_{n_1}(x) + \sum_{i=1}^K (f_{n_{i+1}}(x) - f_{n_i}(x))$$

and

$$S_K(g)(x) = |f_{n_1}(x)| + \sum_{i=1}^K |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

The triangle inequality for $L^p$ implies

$$\|S_Kg\|_{L^p} \leq \|f_{n_1}\|_{L^p} + \sum_{i=1}^K \|f_{n_{i+1}} - f_{n_i}\|_{L^p} \leq \|f_{n_1}\|_{L^p} + \sum_{i=1}^K 2^{-i}$$

Letting $K \to \infty$, and applying the monotone convergence theorem proves that $\int g^p < \infty$, and therefore the series defining $g$, and hence the series defining $f$ converges almost everywhere, and $f \in L^p$. We now show that $f$
is the limit of the sequence \( \{f_n\} \). We observe that

\[
f(x) = f_{n_1}(x) + \lim_{K \to \infty} \sum_{i=1}^{K-1} (f_{n_{i+1}}(x) - f_{n_i}(x))
\]

\[
= \lim_{K \to \infty} [f_{n_1}(x) + (f_{n_2}(x) + f_{n_1}(x)) + \ldots + (f_{n_K}(x) + f_{n_{K-1}}(x))]
\]

\[
= f_{n_K}(x)
\]

\[
f_{n_K}(x) \to f(x) \quad a.e.
\]

Now we show that \( \|f_n - f\|_{L^p} \). Since \( f_n \) is Cauchy, for a given \( \varepsilon > 0 \) there exists \( N \) so that for all \( n, m > N \) such that \( \|f_n - f_m\|_{L^p} < \frac{\varepsilon}{2} \). For all \( K \),

\[
|f(x) - S_K(f)(x)|^p \leq 2^p \left( \max(\{|f(x)|, |S_K(f)(x)|\}) \right)^p
\]

\[
\leq 2^p |f(x)|^p + 2^p |S_K(f)(x)|^p
\]

\[
\leq 2^{p+1} |g(x)|^p.
\]

Then, we may apply the dominated convergence theorem to get \( \|f_{n_K} - f\|_{L^p} \to 0 \) as \( K \to \infty \). If \( n_K \) is chosen so that \( n_K > N \), and \( \|f_{n_K} - f\|_{L^p} < \frac{\varepsilon}{2} \), then the triangle inequality implies

\[
\|f_n - f\|_{L^p} \leq \|f_n - f_{n_K}\|_{L^p} + \|f_{n_K} - f\|_{L^p} < \varepsilon
\]

whenever \( n > N \). This concludes the proof of the theorem.

Second, suppose that \( p = \infty \). If \( \{f_n\} \) is Cauchy in \( L^\infty \), then for every \( m \in \mathbb{N} \) there exists an integer \( k \in \mathbb{N} \) such that we have

\[
|f_j(x) - f_n(x)| < \frac{1}{m}
\]

for all \( j, n \geq k \) and \( x \in N_{j,n,m}^c \), where \( N_{j,n,m} \) is a null set. Let

\[
N = \bigcup_{j,n,m \in \mathbb{N}} N_{j,n,m}.
\]

Then \( N \) is a null set, and for every \( x \in N^c \) the sequence \( \{f_n(x) : n \in \mathbb{N}\} \) is Cauchy. We define a measurable function \( f : X \to \mathbb{C} \), unique up to pointwise a.e. equivalence, by

\[
f(x) = \lim_{n \to \infty} f_n(x)
\]

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for $x \in N^c$. Letting $n \to \infty$ in (1.4), we find that for every $m \in \mathbb{N}$ there exists an integer $k \in \mathbb{N}$ such that

$$|f_j(x) - f(x)| \leq \frac{1}{m}$$

for $j \geq k$ and $x \in N^c$. It follows that $f$ is essentially bounded and $f_j \to f$ in $L^\infty$ as $j \to \infty$. This proves that $L^\infty$ is complete. \qed

**Corollary 1.1.1.** Suppose that $(X, A, \nu)$ is a measure space and $1 \leq p < \infty$. If $\{f_n\}$ is a sequence in $L^p(X)$ that converges in $L^p$ to $f$, then there is a subsequence $\{f_{n_j}\}$ that converges pointwise almost everywhere to $f$.

**Theorem 1.1.4.** Let $(X, A, \nu)$ be a measure space. Then

(i) $L^p(X)$ is a Banach space for $1 \leq p \leq \infty$,

(ii) $L^2(X)$ is a Hilbert space since its norm satisfies the parallelogram equality.

If the underlying space has finite measure, then relations between the various $L^p$ spaces is as follows

**Proposition 1.1.1.** ([44] Theorem 2.5, [21] corollary 13.3) If $X$ has finite positive measure, and $p_1 \leq p_2$, then $L^{p_2}(X) \subset L^{p_1}(X)$ and

$$\frac{1}{\nu(X)^{\frac{1}{p_1}}\|f\|_{L^{p_1}}} \leq \frac{1}{\nu(X)^{\frac{1}{p_2}}\|f\|_{L^{p_2}}}$$

We may assume that $p_2 > p_1$. Suppose $f \in L^{p_2}$, and set $F = |f|^{p_1}, G = 1, p = \frac{p_2}{p_1} > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, in Hölder’s inequality applied to $F$ and $G$. This yields

$$\|f\|_{L^{p_1}}^{p_1} \leq (\int_X |f|^{p_2})^{p_1/p_2} \cdot \nu(X)^{1-p_1/p_2}$$

In particular, we find that $\|f\|_{L^{p_1}} < \infty$.

Such inclusion does not hold when $X$ has infinite measure, for example, $f(x) = \frac{1}{x}$ belongs to $L^2([1, \infty))$ but clearly it does not belong to $L^1([1, \infty))$.

**Proposition 1.1.2.** ([44] Theorem 2.5) If $f \in L^{p_1}$ for some $0 < p_1 < \infty$ and every set of positive measure in $X$ has measure at least $m$, then $f \in L^{p_2}$ for all $p_1 < p_2 \leq \infty$, with $\|f\|_{L^{p_2}} \leq m^{\frac{1}{p_2}} - \frac{1}{p_1}\|f\|_{L^{p_1}}$. 

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The following proposition gives a relation between $L^\infty$ and $L^p$.

**Proposition 1.1.3.** ([21] Lemma 13.1) Suppose $f \in L^\infty$ is supported on a set of finite measure. Then $f \in L^p$ for all $p < \infty$, and

$$
\|f\|_{L^p} \to \|f\|_{L^\infty} \quad \text{as} \quad p \to \infty.
$$

**Proof.** Let $E$ be a measurable subset of $X$ with $\nu(E) < \infty$, and so that $f$ vanishes in the complement of $E$. If $\nu(E) = 0$, then $\|f\|_{L^\infty} = \|f\|_{L^p} = 0$ and there is nothing to prove. Otherwise

$$
\|f\|_{L^p} = \left(\int_E |f(x)|^p d\nu\right)^{1/p} \leq \left(\int_{E} \|f\|_{L^\infty}^p d\nu\right)^{1/p} \leq \|f\|_{L^\infty} \nu(E)^{1/p}.
$$

Since $\nu(E)^{1/p} \to 1$ as $p \to \infty$, we find that $\limsup_{p \to \infty} \|f\|_{L^p} \leq \|f\|_{L^\infty}$. On the other hand, given $\varepsilon > 0$, we have

$$
\nu(\{x : |f(x)| \geq \|f\|_{L^\infty} - \varepsilon\}) \geq \delta \quad \text{for some} \quad \delta > 0,
$$

hence

$$
\int_X |f|^p d\nu \geq \delta (\|f\|_{L^\infty} - \varepsilon)^p.
$$

Therefore $\liminf_{p \to \infty} \|f\|_{L^p} \geq \|f\|_{L^\infty} - \varepsilon$, and since $\varepsilon$ is arbitrary, we have $\liminf_{p \to \infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}$. Hence the $\liminf_{p \to \infty} \|f\|_{L^p}$ exists, and equals $\|f\|_{L^\infty}$.

1.1.3 Linear functionals on $L^p$

Given an exponent $1 \leq p \leq \infty$, and its dual exponent $q$. From Hölder’s inequality, we see that for any $g \in L^q$, the functional $\lambda_g : L^p \to \mathbb{C}$ defined by

$$
\lambda_g(f) := \int_X f \bar{g} d\nu
$$

is well defined on $L^p$, linear and continuous.

**Theorem 1.1.5.** (Riesz Representation Theorem for $L^p$) (see [44], [70], [70], [21] Theorem 13.26 & Theorem 13.28). Let $1 \leq p < \infty$, and assume $\nu$ is $\sigma$-finite. Let $\lambda : L^p \to \mathbb{C}$ be a continuous linear functional. Then there exists a unique $g \in L^q$ such that $\lambda = \lambda_g$. 

8
1.2 CR Submanifolds of Almost Hermitian Manifolds

Given an almost Hermitian manifold $\tilde{M}$, on a submanifold of an almost Hermitian manifold the action of the almost complex structure $J$ transforms a vector to a vector perpendicular to it and in particular, it gives rise to $J$-invariant and anti-invariant distributions. The study of invariant (or, holomorphic) submanifolds was initiated by E. Calabi and others in early 1950s (cf [26],[27]). Afterwards, it became an active and fruitful field in modern differential geometry. The study of anti-invariant (or, totally real) submanifolds was initiated in early 1970s (cf [32], [33]). Since then many differential geometers have contributed interesting results (cf[39], [42], [113] etc). In 1978, A. Bejancu ([14], [15]) introduced the notion of a CR-submanifold and generalized the above two classes of submanifolds. In fact, the class of CR submanifolds provides a single setting to study invariant and anti-invariant distributions on an almost Hermitian manifold. Due to our aim in the last chapter of this thesis is to associate Poincaré series on the complex unit ball $\mathbb{B}^n$ to specific examples of such submanifolds and study the relation between the geometry of these submanifolds and $L^1$-norm of their Poincaré series, we have studied the results of [14], [16],[17], [34], [35],[36], [68] in this section.

1.2.1 CR Submanifolds and CR-Product Submanifolds of Almost Hermitian Manifolds

On an almost complex manifold, there always exists a Riemannian metric $g$ compatible with the almost complex structure $J$ i.e., satisfying

$$g(JX, JY) = g(X, Y)$$

for any tangent vectors $X, Y$, by virtue of which $g$ is called a Hermitian metric. An almost complex (resp. a complex) manifold with a Hermitian metric is called an almost Hermitian (resp. a Hermitian) manifold. The fundamental 2-form $\omega$ of an almost Hermitian manifold $\tilde{M}$ with an almost complex structure $J$ and metric $g$ is defined by

$$\omega(X, Y) = g(X, JY)$$

Since $g$ is invariant by $J$, so is $\omega$ i.e.,

$$\omega(JX, JY) = \omega(X, Y)$$
for any $X,Y$. The almost complex structure $J$ is not, in general, parallel with respect to the Riemannian connection $\bar{\nabla}$ defined by the Hermitian metric $g$. In fact, we have the following formula:

$$4g((\bar{\nabla}XJ)Y, Z) = 6d\omega(X, JY, JZ) - 6d\omega(X, Y, Z) + g(N(Y, Z), JX)$$  (1.9)

for any vector fields $X,Y,Z$ on $\bar{M}$, where $N$ denotes the Nijenhuis tensor of $J$. In general, the Nijenhuis tensor field of two tensors $A$ and $B$ on a manifold $\bar{M}$ is a (1,2) tensor field defined by


Thus, the Nijenhuis tensor of a (1,1)-tensor field $A$ is given by


It is easy to verify that the Nijenhuis tensor of $J$ satisfies

$$N(JX, Y) = N(X, JY) = -JN(X, Y)$$  (1.11)

for all vector fields $X,Y$ on $\bar{M}$. The vanishing of the Nijenhuis tensor $N$ of $J$ is a necessary and sufficient condition for an almost complex manifold to be a complex manifold [112].

If we extend the Riemannian connection $\bar{\nabla}$ to be a derivation on the tensor algebra of $M$, then we have the following formula

$$(\bar{\nabla}XJ)Y = \bar{\nabla}XJY - J\bar{\nabla}XY$$  (1.12)

**Definition 1.2.1.** [112] A Hermitian metric on the almost complex manifold $\bar{M}$ is called Kaehler metric if the fundamental 2-form $\omega$ is closed. A complex manifold equipped with a Kaehler metric is said to be a Kaehler manifold.

Thus, by formula (1.9), an almost complex manifold $\bar{M}$ is Kaehler if and only if

$$(\bar{\nabla}XJ)Y = 0$$  (1.13)

for all $X, Y \in T\bar{M}$. In this case, the connection $\bar{\nabla}$ on $\bar{M}$ is said to be the Kaehlerian connection.
Example 1.2.1. A complex unit ball $\mathbb{B}_C^n$ is an example of a Kaehler manifold as we will see in Section 1.6.

On an almost Hermitian manifold $(\tilde{M}, J, g)$, (1.6) becomes

$$g(JX, X) = 0,$$

i.e., $JX \perp X$ for each vector field $X$ on $\tilde{M}$. Hence, for a submanifold $M$ of $\tilde{M}$ if $X \in T_p(M)$, $JX$ may or may not belong to $T_p(M)$. Thus, the action of the almost complex structure $J$ on the tangent vectors of the submanifold gives rise to its classification into invariant and anti-invariant submanifolds. These submanifolds are defined as:

Definition 1.2.2. A submanifold $M$ of an almost Hermitian manifold $\tilde{M}$, is said to be an invariant (or, holomorphic or, complex) submanifold if

$$JT_p(M) = T_p(M),$$

for all $p \in M$.

Definition 1.2.3. A submanifold $M$ of an almost Hermitian manifold $\tilde{M}$, is said to be a totally real (or, anti-holomorphic) submanifold if

$$JT_p(M) \subseteq T_p^\perp(M)$$

for all $p \in M$.

Definition 1.2.4. A distribution $D$ of dimension $r$ on a manifold $\tilde{M}$ is an assignment to each point $p$ of $\tilde{M}$, an $r$-dimensional subspace $D_p$ of the tangent bundle $T_p(\tilde{M})$.

It is called differentiable if every point $p$ has a neighbourhood $\Omega_p$ and $r$-differentiable vector fields on $\Omega_p$, say, $\{X_1, X_2, \ldots, X_r\}$ which form a basis of $D_q$ at every $q$ in $\Omega_p$. A vector field $X$ is said to belong to $D$ if $X_p \in D_p$ for all $p \in \tilde{M}$.

Definition 1.2.5. A distribution $D$ is called integrable (or, involutive) if $[X, Y] \in D$ whenever two vector fields $X, Y \in D$.

Definition 1.2.6. A connected submanifold $M$ of $\tilde{M}$ is called an integral manifold of the distribution $D$ if $f_*(T_p(M)) = D_p$ for all $p \in M$, where $f$ is the embedding of $M$ into $\tilde{M}$. If there is no other integral manifold of $D$ which contains $M$, $M$ is called a maximal integral manifold of $D$. 

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The classical *Theorem of Frobenius* can be formulated as follows.

**Theorem 1.2.1.** [72] Let $D$ be an integrable distribution on a manifold $\bar{M}$. Through every point $p \in \bar{M}$, there passes a unique maximal integral manifold $M(p)$ of $D$. Any integral manifold through $p$ is an open submanifold of $M(p)$.

**Definition 1.2.7.** A submanifold $M$ of $\bar{M}$ is called *auto-parallel* if for each $X \in T_p(M)$ and for each curve $\tau$ in $M$ starting from $p$, the parallel displacement of $X$ along $\tau$ with respect to the affine connection $\nabla$ of $M$ yields a vector field tangent to $M$.

Thus, a distribution $D$ on a manifold $\bar{M}$ is auto-parallel if $\nabla_X Y \in D$ for each $X, Y \in D$. In general

**Definition 1.2.8.** if $D$ and $\bar{D}$ are two distributions on $\bar{M}$, we say that $D$ is $\bar{D}$-parallel if $\nabla_X Y \in D$ for all $X \in D$ and $Y \in D$. $D$ is said to be parallel if $\nabla_X Y \in D$ for all $X \in T\bar{M}$ and $Y \in D$.

If a distribution $D$ on $\bar{M}$ is auto-parallel, then it is clearly integrable and by Gauss formula $D$ is totally geodesic in $\bar{M}$. If $D$ is parallel, then the orthogonal complementary distribution $D^\perp$ is also parallel which implies that $D$ is parallel if and only if $D^\perp$ is parallel. In this case, $\bar{M}$ is locally the Riemannian product of the leaves of $D$ and $D^\perp$.

**Remark 1.2.1.** [17] In view of the above observation, throughout, the auto-parallelism of a distribution on a manifold $\bar{M}$ and the totally geodesicness of its leaves in $\bar{M}$ are treated equivalently and the two terms are used interchangeably.

Throughout this section, we study the CR-submanifold of an almost Hermitian manifold $\bar{M}$.

**Definition 1.2.9.** [34] A submanifold $M$ of an almost Hermitian manifold $(\bar{M}, g, J)$ is called a *CR-submanifold* if there exists a differentiable distribution $D : p \to D_p \subseteq T_pM$ on $M$ satisfying the following conditions

(a) $D$ is holomorphic, i.e., $JD_p = D_p$ , for each $x \in M$,

(b) The complementary orthogonal distribution $D^\perp : p \to D^\perp_p \subseteq T^\perp_pM$ is totally real , i.e, $JD^\perp_p \subseteq T^\perp_pM$ , for each $p \in M$.

**Example 1.2.2.** A *real hypersurface* of a complex manifold is a CR-submanifold.
Remark 1.2.2. On a CR-submanifold M, if \( \dim D^\perp_p = 0 \) (resp., \( \dim D_p = 0 \)), then M is a holomorphic (resp., totally real) submanifold. A CR-submanifold is called proper if it is neither holomorphic nor totally real.

If \( TM \) and \( T^\perp M \) denote the tangent and normal bundles respectively on a CR-submanifold M, then

\[
TM = D \oplus D^\perp
\]

and,

\[
T^\perp M = JD^\perp \oplus \mu
\]

where \( \mu \) is the orthogonal complementary distribution of \( JD^\perp \) in \( T^\perp M \).

Lemma 1.2.1. The orthogonal complementary distribution of \( JD^\perp \) in \( T^\perp M \) is invariant under \( J \).

Definition 1.2.10. A submanifold \( M \) of an almost Hermitian manifold is said to be a \textit{CR-product submanifold} if \( M \) is locally a Riemannian product of a holomorphic submanifold \( M_T \) and a totally real submanifold \( M_\perp \).

Thus, a CR-submanifold \( M \) of an almost Hermitian manifold is a CR-product if and only if both the distributions \( D \) and \( D^\perp \) on \( M \) are integrable and their leaves are totally geodesic in \( M \).

If a distribution \( D \) is parallel on a manifold, then it is easy to observe that its complementary distribution \( D^\perp \) is also parallel. In the other words, \( D \) is parallel if and only if \( D^\perp \) is parallel. To be more precise on a CR-submanifold, we have

Lemma 1.2.2. A CR-submanifold \( M \) of an almost Hermitian manifold is a CR-product submanifold if and only if

\[
\nabla_U X \in D
\]

(1.15)

or, equivalently,

\[
\nabla_U Z \in D^\perp
\]

(1.16)

for each \( U \in TM \), \( X \in D \) and \( Z \in D^\perp \).
1.2.2 CR-Submanifolds of Kaehler Manifolds

Throughout this part of the section we denote by $\bar{M}$ a Kaehler manifold, and by $M$ a CR-submanifold of $\bar{M}$. Let $\bar{\nabla}, \nabla$ be the Riemannian connection on $\bar{M}$ and $M$, respectively. Then

$$(\bar{\nabla}_U J)V = 0, \forall U, V \in TM$$ (1.17)

Lemma 1.2.3. Let $M$ be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then, we have

- $(i) g(\bar{\nabla}_U Z, X) = g(JA_{JZ}U, X)$
- $(ii) A_{JZ}W = A_{JW}Z$
- $(iii) A_{J\xi}X = -A_{\xi}JX$

for each $U \in TM$, $X \in D$, $Z, W \in D^\perp$ and $\xi \in \mu$, where $A$ is the shape operator (or Weingarten map or second fundamental tensor).

Proof. Using Weingarten formula,

$$g(JA_{JZ}U, X) = -g(A_{JZ}U, JX)$$
$$= g(\bar{\nabla}_U JZ, JX)$$
$$= g(J\bar{\nabla}_U Z, JX)$$
$$= g(\nabla U Z, X)$$

$$\Rightarrow g(JA_{JZ}U, X) = g(X, \nabla U Z)$$

This proves (i). $h$ and $A$ are the second fundamental forms and are related as

$$g(A_N U, W) = g(h(U, W), N)$$

for $U, W \in TM$ and $N \in T^\perp M$. Now, consider $g(A_{JZ}W, U)$, for $U \in TM$

$$g(A_{JZ}W, U) = g(h(U, W), JZ)$$
$$= g(\bar{\nabla}_U W, JZ)$$
$$= -g(J\bar{\nabla}_U W, Z) = -g(\bar{\nabla}_U JW, Z)$$
$$= g(A_{JW}U, Z) = g(A_{JW}Z, U)$$

The statement (ii) follows from the above equation due to by nondegeneracy of $g$. 

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Consider \( g(A_J \xi X, U) \), for \( U \in TM \), we have
\[
g(A_J \xi X, U) = g(h(X, U), J\xi) \\
= g(\bar{\nabla}_U X, J\xi) = -g(\bar{\nabla}_U JX, \xi) \\
= -g(h(U, JX), \xi) = -g(A_J JX, U)
\]
\( \Rightarrow A_J \xi X = -A_J JX \)

This proves (iii). \( \square \)

**Lemma 1.2.4.** The totally real distribution \( D^\perp \) on a CR submanifold of a Kaehler manifold is integrable.

*Proof.* Let \( M \) be a CR-submanifold of a Kaehler manifold \( \bar{M} \). Then by Weingarten formula, for any \( Z, W \in D^\perp \)
\[
\nabla_Z^\perp JW = \bar{\nabla}_Z JW + A_{JW}Z
\]
On using the previous lemma, the above equation gives
\[
\nabla_Z^\perp JW - \nabla_W^\perp JZ = \bar{\nabla}_Z JW - \bar{\nabla}_W JZ
\]
As \( \bar{M} \) is Kaehler, the right hand side of the above equation becomes \( J[Z, W] \in JD^\perp \), and the assertion is proved. \( \square \)

For the integrability of the holomorphic distribution, we have

**Lemma 1.2.5.** Let \( M \) be a CR-submanifold of a Kaehler manifold \( \bar{M} \). Then \( D \) is integrable if and only if
\[
g(h(X, JY), JZ) = g(h(JX, Y), JZ)
\]
for any \( X, Y \in D \) and \( Z \in D^\perp \).

*Proof.* For \( N \in T^\perp M \) and \( X, Y \in D \), we have
\[
g(\bar{\nabla}_X JY - \bar{\nabla}_Y JX, N) = g(h(X, JY) - h(JX, Y), N)
\]
As \( \bar{M} \) is Kaehler manifold by (1.17),
\[
g(J(\bar{\nabla}_X Y - \bar{\nabla}_Y X), N) = g(h(X, JY) - h(JX, Y), N)
\]
Taking account of the fact that $D$ is integrable if and only if the normal component of $J[X,Y]$ vanishes, it follows from the above equality that $D$ is integrable if and only if

$$h(X, JY) - h(JX, Y) = 0 \quad (1.18)$$

Now, consider for $\xi \in \mu$, $g(h(X, JY) - h(JX, Y), \xi)$

$$g(h(X, JY) - h(JX, Y), \xi) = g(A_{\xi} JY, X) - g(A_{\xi} JX, Y)$$

Using Lemma 1.2.4 (iii) we get

$$g(h(X, JY) - h(JX, Y), \xi) = g(A_{\xi} JX, Y) - g(A_{\xi} JY, X)$$

As the shape operator $A$ is symmetric endomorphism on $TM$, the right hand side is zero, i.e.,

$$g(h(X, JY) - h(JX, Y), \xi) = 0$$

Hence (1.18) holds if and only if

$$g(h(X, JY) - h(JX, Y), JZ) = 0$$

In other words, $D$ is integrable if and only if

$$g(h(X, JY), JZ) = g(h(JX, Y), JZ)$$

This completes the proof.

Lemma 1.2.6. Let $M$ be a CR-submanifold in a Kaehler manifold $\bar{M}$. Then

- (a) The leaves of $D^\perp$ are totally geodesic in $M$ if and only if

$$g(h(D, D^\perp), JD^\perp) = 0 \quad (1.19)$$

- (b) The leaves of the distribution $D$ are totally geodesic in $M$ if and only if

$$g(h(D, D), JD^\perp) = 0 \quad (1.20)$$

Proof. By definition, a leaf $M_\perp$ of $D^\perp$ is totally geodesic in $M$ if and only if

$$\nabla_Z W \in D^\perp, \forall Z, W \in D^\perp \ i.e.,$$

$$g(\nabla_Z W, JX) = 0, \forall X \in D$$

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or,
\[ g(W, \nabla_Z JW) = 0 \]
or,
\[ g(JW, \nabla_Z X) = 0 \]
or,
\[ g(JW, h(X, Z)) = 0, \]
for all \( Z, W \in D^\perp \) and \( X \in D \). This proves the statement (a). Similarly, a leaf \( M_T \) of \( D \) is totally geodesic in \( M \) if and only if \( \nabla_X Y \in D, \forall X, Y \in D \), i.e.,
\[ g(\nabla_X JY, Z) = 0 \]
or,
\[ g(JY, \nabla_X Z) = 0 \]
On using Lemma 1.2.4 (i), we get
\[ g(JY, JA_{JZ} X) = 0 \]
or,
\[ g(Y, A_{JZ} X) = 0 \]
or,
\[ g(h(X, Y), JZ) = 0 \]
for each \( X, Y \in D \) and \( Z \in D^\perp \). This proves (b), and completes the proof. \( \square \)

The condition of totally geodesicness in \( M \) of the leaves of \( D^\perp \) in the above lemma can also be written as
\[ g(A_{JD^\perp} D, D^\perp) = 0 \tag{1.21} \]
whereas the condition for the leaves of \( D \) to be totally geodesic in \( M \) can be stated as
\[ g(A_{JD^\perp} D, D) = 0 \tag{1.22} \]

It is easy to observe that the above condition ensures the integrability of \( D \) on \( M \). On combining the above two conditions, we obtain a characterization for a CR-submanifold of a Kaehler manifold to be a CR-product as:
Theorem 1.2.2. A CR-submanifold $M$ of a Kaehler manifold $\bar{M}$ is a CR-product if and only if $A_{JD^\perp}D = 0$.

Corollary 1.2.1. If the holomorphic distribution $D$ on a CR-submanifold $M$ of a Kaehler manifold $\bar{M}$ is integrable, then its leaves are totally geodesic in $M$ if and only if

$$h(X,Y) \in \mu$$

for each $X,Y \in D$.

From Theorem 1.2.2 and Codazzi equation, we get

Lemma 1.2.7. [34] Let $M$ be a CR-product of a Kaehler manifold $\bar{M}$. Then for any unit vectors $X \in D$ and $Z \in D^\perp$ we have

$$\bar{H}(X,Z) = 2||h(X,Z)||^2$$

where $\bar{H}(X,Z) = \bar{R}(X,JX,JZ,Z)$ is the holomorphic bisectional curvature of $X \wedge Z$.

Thus by Theorem 1.2.2 and the above lemma we obtain the following theorem

Theorem 1.2.3. [34] Let $M$ be a Kaehler manifold with negative holomorphic bisectional curvature. Then every CR-product in $\bar{M}$ is either a holomorphic submanifold or a totally real submanifold. Particularly, there is no proper CR-product in any complex hyperbolic space $H^n_C$.

1.3 Complex Hyperbolic Geometry

Hyperbolic geometry is a non-Euclidean geometry that satisfies all of Euclid’s postulates except the parallel postulate. In hyperbolic geometry, through a point not on a given line there are at least two lines parallel to the given line. This section is intended to give a quick introduction to hyperbolic geometry and description of hyperbolic space in several different ways. References for this material are [50], [29], [49], [109], [25], [30].
1.3.1 Hermitian form on $\mathbb{C}^{n,1}$

**Definition 1.3.1.** A *Hermitian form* on a complex vector space $V$ is a map $\langle ., . \rangle : V \times V \to \mathbb{C}$ which is bilinear over reals, linear in the first entry and conjugate linear on the second one. In the other words, for all $v, w, u \in V$ and $\alpha$ a complex scalar, we have

$$\langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle,$$
$$\langle \alpha v, u \rangle = \alpha \langle v, u \rangle,$$
$$\langle v, u \rangle = \overline{\langle u, v \rangle}.$$

It follows that

$$\langle v, v \rangle \in \mathbb{R},$$
$$\langle v, \alpha u \rangle = \overline{\alpha} \langle v, u \rangle.$$

To each $r \times r$ Hermitian matrix $H$, we can naturally associate a Hermitian form $\langle ., . \rangle : \mathbb{C}^r \times \mathbb{C}^r \to \mathbb{C}$ given by

$$\langle v, u \rangle = u^* Hv$$

where $^*$ denotes the complex conjugate transpose.

Let $\mathbb{C}^{n,1}$ be the complex vector space of complex dimension $(n+1)$ equipped with a non-degenerate, indefinite Hermitian form $\langle ., . \rangle$ of signature $(n,1)$, that is, $\langle ., . \rangle$ is given by a non singular $(n+1) \times (n+1)$ Hermitian matrix $H$ with $n$ positive eigenvalues and one negative eigenvalue.

There are two standard matrices which give different Hermitian forms on $\mathbb{C}^{n,1}$ which are called the *first* and *second Hermitian forms*. The *first Hermitian form* is defined as follows

$$\langle v, u \rangle = \sum_{i=1}^{n} v_i \bar{u}_i - v_{n+1} \bar{u}_{n+1}$$

and its associated Hermitian matrix $H$ is given by

$$H = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$
1.3.2 Models of complex hyperbolic space

(see [50]) If $z \in \mathbb{C}^{n,1}$ then we may define subsets $V_-, V_0$ and $V_+$ of $\mathbb{C}^{n,1}$ by:

\[
V_- = \{ z \in \mathbb{C}^{n,1} : \langle z, z \rangle < 0 \}
\]

\[
V_0 = \{ z \in \mathbb{C}^{n,1} : \langle z, z \rangle = 0 \}
\]

\[
V_+ = \{ z \in \mathbb{C}^{n,1} : \langle z, z \rangle > 0 \}
\]

since $\langle z, z \rangle \in \mathbb{R}$. We classify a vector $z \in \mathbb{C}^{n,1}$ to be negative, null or positive if $z$ belongs to $V_-, V_0$ or $V_+$, respectively.

- Define an equivalence relation on $\mathbb{C}^{n,1} - \{0\}$ by $z \sim w$ if and only if there is $\alpha \in \mathbb{C} - \{0\}$ such that $w = \alpha z$.

The standard projection map $\mathbb{P} : \mathbb{C}^{n,1} - \{0\} \to \mathbb{CP}^n$ is given by

\[
\mathbb{P}(z) = [z],
\]

where $[z]$ is the equivalence class of $z$. Since $\langle \alpha z, \alpha z \rangle = |\alpha|^2 \langle z, z \rangle$, $\alpha z$ is negative, null or positive if and only if $z$ is.

On the affine chart of $\mathbb{C}^{n,1}$ with $z_{n+1} \neq 0$, the projection map is defined by

\[
\mathbb{P} : \begin{bmatrix} z_1 \\ \vdots \\ z_{n+1} \end{bmatrix} \to \begin{bmatrix} z_1/z_{n+1} \\ \vdots \\ z_n/z_{n+1} \end{bmatrix}
\]

Complex hyperbolic space $H^n_\mathbb{C}$ is defined to be the subset of $\mathbb{CP}^n$ consisting of negative lines in $\mathbb{C}^{n,1}$. This is the projective model of complex hyperbolic space $H^n_\mathbb{C}$. That is, $H^n_\mathbb{C} = \mathbb{P}V_-$ and $\partial H^n_\mathbb{C} = \mathbb{P}V_0$.

- Taking the section such that $z_{n+1} = 1$ for the first Hermitian form,

\[
\begin{bmatrix} z_1 \\ \vdots \\ z_n \\ 1 \end{bmatrix} \in \mathbb{C}^{n,1}
\]

and

\[
\langle z, z \rangle = |z_1|^2 + \cdots + |z_n|^2 - 1 < 0.
\]
Then, $H^n_C$ is identified with the unit ball

$$B^n_C = \{ z \in \mathbb{C}^n : \langle z, z \rangle < 0 \}.$$ 

This forms the unit ball model of $H^n_C$.

Therefore,

$$H^n_C = \mathbb{P}V \approx B^n_C \approx SU(n, 1)/U(n),$$

where $SU(n, 1)$ is the group of matrices in $SL(n + 1, \mathbb{C})$ which preserves the first Hermitian form of type $(n, 1)$, and the isotropy group in $SU(n, 1)$ of $0 \in \mathbb{C}^n$ is $U(n) = \{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} \in SU(n, 1) : A^*A = I_n, |\alpha|^2 = 1, \alpha \det A = 1 \} \approx U(n)$. Also, denote the projective unitary group by $PU(n, 1) = U(n, 1)/U(1)$, where $U(n, 1)$ is the group of matrices in $GL(n + 1, \mathbb{C})$ which preserves the first Hermitian form of type $(n, 1)$, and $U(1)$ is canonically identified with $\{ e^{i\theta} | 0 \leq \theta \leq 2\pi \}$.

The group $SU(n, 1)$ acts on $B^n_C$ and its boundary by fractional linear transformations, so for

$$\gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU(n, 1)$$

where $A$ is $n \times n$ matrix, $b$ is $n \times 1$, $c$ is $1 \times n$ and $d$ is a complex number, and $z \in B^n_C$, we have

$$\gamma z = \frac{Az + b}{cz + d}$$

See from that

$$\langle \gamma z, \gamma w \rangle = \frac{\langle z, w \rangle}{(cz + d)(cw + d)}$$

and the Jacobian determinant of transformation $\gamma$ at $z$ is given by

$$J(\gamma, z) = (cz + d)^{-(n+1)}.$$ 

For the projective model, the $SU(n, 1)$-invariant Kaehler metric on $H^n_C$ is called Bergman metric and given by

$$ds^2 = -\frac{4}{\langle z, z \rangle^2} \det \begin{pmatrix} \langle z, z \rangle & \langle dz, z \rangle \\ \langle z, dz \rangle & \langle dz, dz \rangle \end{pmatrix}.$$ 

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Let $\zeta, \xi \in H^n_C$ be two points corresponding to vectors $z, w \in \mathbb{C}^{n+1}$, then the (hyperbolic) distance $\tau(., .)$ between them is defined by the formula

$$\cosh^2\left(\frac{\tau(z, w)}{2}\right) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}.$$  

For the ball model, we can find the distance between any two points by plugging their standard lift of those points into the aforementioned formula.

**Lemma 1.3.1.** *In the ball model of $H^n_C$, the volume form is given by*

$$dV = \frac{16}{(-\langle z, z \rangle)^{n+1}} dv$$

*where the volume element $dv$ is*

$$(1/2i)^n dz_j \wedge d\bar{z}_j = dx_j \wedge dy_j$$

*for $j = 1, \ldots, n$.*

### 1.3.3 Holomorphic and Totally Real Submanifolds of Complex Hyperbolic Space

A submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is called complex (resp. totally real) if each tangent space of $M$ is mapped into itself (resp. the normal space) by the almost complex structure $J$ of $M$ as addressed in section 1.2.

**Definition 1.3.2.** A linear real subspace $\tilde{\mathbb{R}}^{r+1}$ of $\mathbb{C}^{n+1}$ of real dimension $r+1$ which contains negative vectors is *totally real* with respect to the Hermitian form $H$ if $J(\tilde{\mathbb{R}}^{r+1})$ is $H$-orthogonal to $\tilde{\mathbb{R}}^{r+1}$. Then, a totally real subspace of $H^n_C$ is defined to be the intersection with $H^n_C$ of projectivisation of a totally real projective subspace $\tilde{\mathbb{R}}^{r+1}$ of $\mathbb{C}^{n+1}$. Such a plane is called an $\mathbb{R}^r$-plane, where $r \leq n$.

Note that every $\mathbb{R}^r$-plane is isometric to the real hyperbolic space $H^r_{\mathbb{R}}$ equipped with the projective model of hyperbolic geometry (see [30], Theorem 2.2.2).

In the same way we define a $\mathbb{C}^r$-plane:
Definition 1.3.3. Let $C^r$ be a complex projective subspace of $\mathbb{P}(\mathbb{C}^{n,1})$ of complex dimension $r$ that passes through the point $z \in H^2_C$. Then $C^r \cap H^2_C$ is obviously a complex holomorphic submanifold of $H^2_C$ and such a submanifold is called a $C^r$-plane.

Hence, any such subspace is the conjugate by an element of $PU(n,1)$ of the subspace $H^2_C$ of $H^2_C$.

Definition 1.3.4. [31] A submanifold $M$ of a Riemannian manifold $(\bar{M}, g)$ is called totally geodesic if any geodesic on the submanifold $M$ with its induced Riemannian metric $g$ is also a geodesic on the Riemannian manifold.

Therefore, a submanifold $M$ of $H^2_C$ is said to be totally geodesic if every geodesic in $H^2_C$ joining two points in $M$ is contained in $M$. Any two distinct points in $H^2_C$ determine a unique line in $\mathbb{P}(\mathbb{C}^{n,1})$, so there is a unique complex geodesic $L$ passing through them.

Theorem 1.3.1. (see [50], theorem 3.1.10) Let $M$ be a complex $m$-dimensional projective subspace of $\mathbb{P}(\mathbb{C}^{n,1})$ intersecting with $H^2_C$. Then $M \cap H^2_C$ is a totally geodesic holomorphic submanifold biholomorphically isometric to $H^m_C$.

If $x_1, x_2$ are in $H^2_R$, they span a unique complex geodesic $L$ which is invariant under the conjugation and the restriction of the conjugation on $\mathbb{C}^{n,1}$ to $L$ is an antiholomorphic involution. Hence, its fixed-point set $\nu = L \cap H^2_R$ of $L$ under the conjugation is an $\mathbb{R}$-linear subspace which is a geodesic in $L$ containing $x_1, x_2$.

Lemma 1.3.2. [50] The only totally geodesic subspaces of $H^2_C$ are either complex linear or totally real.

Here are some examples of complex linear and totally real submanifolds of $H^2_C$:

Example 1.3.1. If we let $\gamma = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in PU(3,1)$ act on $H^1_R$ (on its standard lift to $\mathbb{C}^{3,1}$), we get the submanifold $\{(ix,0) \mid x \in \mathbb{R}, \ |x|^2 < 1 \} = iH^1_R \approx H^1_R$. 

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Example 1.3.2. The submanifold $M_{\alpha_1,\alpha_2} = \{(r_1 e^{i\alpha_1}, r_2 e^{i\alpha_2}) \mid r_1, r_2 \in \mathbb{R} \text{ and } |r_1|^2 + |r_2|^2 < 1\} \approx \gamma H^2_R$, where $\gamma = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in PU(3,1)$.

Example 1.3.3. The submanifold $M_\alpha = \{(ir_1, r_2 e^{i\alpha}) \mid r_1, r_2 \in \mathbb{R} \text{ and } |r_1|^2 + |r_2|^2 < 1\} \approx \gamma H^2_R$, where $\gamma = \begin{pmatrix} i & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in PU(n,1)$.

Example 1.3.4. The small real ball $B^2_R(\frac{1}{\mu}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1|^2 + |x_2|^2 < \frac{1}{\mu^2}\}$, where $\mu$ is a large enough positive integer. It is clear that $B^2_R(\frac{1}{\mu}) \approx H^2_R$.

Hence, examples 1.3.1-1.3.4 are totally real submanifolds of $H^2_C$.

Example 1.3.5. The small complex ball $B^2_C(\frac{1}{\lambda}) = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < \frac{1}{\lambda^2}\}$, where $\lambda \in \mathbb{Z}^+$ and $\lambda > 1$. Here, $B^2_C(\frac{1}{\mu})$ is the complex linear submanifold of $H^2_C$.

Example 1.3.6. The submanifold $M_1 = \{(0, z_2) \mid z_2 \in B^1_C\}$.

Example 1.3.7. The submanifold $M_2 = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C}, z_1 = 1 \text{ and } |z_2|^2 < 1\}$.

$M_1$ and $M_2$ are complex lines in $\mathbb{C}^2$ and we can apply an element of $PU(n,1)$ to $M_2$ (or to any complex line in $H^2_C$) to get $M_1$. Moreover, the intersection of $M_1$ with the unit ball is just a disc $\{z_2 \mid z_2 \in \mathbb{C} \text{ and } |z_2|^2 < 1\}$ which is an embedded copy of $H^1_C$. The latter can be thought to be $H^2_R$ equipped with Poincaré ball model for real hyperbolic geometry.

Definition 1.3.5. A manifold $M$ is said to be symplectic if $M$ is a smooth manifold equipped with a closed nondegenerate differential 2-form $\omega$ called the symplectic form.

Definition 1.3.6. Lagrangian submanifold $N$ of a symplectic manifold $(M, \omega)$ is a submanifold where the restriction of the symplectic form $\omega$ to $N$ is vanishing, i.e. $\omega|_N = 0$, and $\dim N = 1/2 \cdot \dim M$.

Lemma 1.3.3. Any real submanifold isomorphic to $H^2_R$ in $H^2_C$ is Lagrangian.
1.4 Real Hyperbolic Geometry

Real hyperbolic geometry $H^n_R$ is similar to the complex one. It is useful to consider two models of $H^n_R$ that provided in [4], [49], [50], [99], [82], [89].

**Projective Model**

Real projective plane $\mathbb{PR}^n$ is defined analogously to the complex one $\mathbb{PC}^n$. Therefore, the collection of negative lines in $\mathbb{R}^{n,1}$ is defined to be the projective model of complex hyperbolic space $H^n_C$.

**Unit Ball Model**

By taking the set of all points such that $x_{n+1} = 1$ with real coordinates for the first Hermitian form, we then define the other standard form of $H^n_R$ which is the unit ball model $B^n_R$. In a special case when $n = 2$, the Cayley transform provides an isometry between the Poincaré disk model and the half-plane model.

Let $G = \text{Isom}H^n_R$, the group of isometries of $H^n_R$. A standard special orthogonal group $SO(n, 1)$ with indefinite signature $(n, 1)$, $SO(n, 1) = \{g \in GL(n+1, \mathbb{R}) : \langle gx, gy \rangle = \langle x, y \rangle, \det g = 1 \forall x, y \in \mathbb{R}^n\}$, acts transitively on the open unit ball in $\mathbb{R}^{n+1}$ by linear transformations as seen in case of complex balls. Hence, $G = SO(n, 1)$ and the isotropy group of the origin is

$$\overline{O(n)} = \left\{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} \in SO(n, 1) : A^T A = I_n, \alpha^2 = 1, \alpha \det A = 1 \right\},$$

which is a copy of orthogonal group $O(n) = \{g \in GL(n, \mathbb{R}) : \langle gx, gy \rangle = \langle x, y \rangle, \det g = 1 \forall x, y \in \mathbb{R}^n\}$, so

$$H^n_R \approx B^n_R \approx SO(n, 1)/\overline{O(n)}.$$

1.5 Complex Vector Bundles

The Chern classes are topological invariants associated with vector bundles on a smooth manifold. If the Chern classes of a pair of vector bundles do not agree, then the vector bundles are not isomorphic. In this section we study some relevant results obtained from [60], [71], [83], [88], [105]. We define the positivity and negativity of vector bundles based on the sign of their first
Chern classes. Kodaira Embedding Theorem tells us that a compact complex manifold with positive line bundle can be embedded into projective complex space, (hence, algebraic), which eases the work done in the last two chapters of this thesis.

Vector Bundles

**Definition 1.5.1** (Fiber bundles). Let $M$ be a smooth manifold. A manifold $E$ together with a smooth submersion $\pi : E \to M$ is called a *fiber bundle* of rank $m$ over $M$ if the following holds:

(i) There is a $m$-dimensional manifold $F$, called typical fibre of $E$, such that for any point $p \in M$ the fibre $E_p = \pi^{-1}(p)$ of $\pi$ over $p$ is a manifold isomorphic to $F$.

(ii) Any point $p \in M$ has a neighbourhood $U$, such that there is a diffeomorphism

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\psi_U} & U \times F \\
\downarrow \pi & & \downarrow \pi_p \\
U & \xrightarrow{id} & U
\end{array}
\]

and the diagram commutes, which means that every fibre $E_p$ is mapped to $\{p\} \times F$. $\psi_U$ is called a local trivialization of $E$ over $U$ and $U$ is a trivializing neighbourhood for $E$.

(iii) $\psi_{U|E_p} : E_p \to F$ is an isomorphism of manifolds.

Some more terminology: $M$ is called the base and $E$ the total space of this fiber bundle. $\pi : E \to M$ is said to be a real or complex fiber bundle corresponding to the typical fibre being a real or complex manifold.

A vector bundle of rank $r$ over $M$ is a fiber bundle as in the above definition for which the fibers $\pi^{-1}(p)$, $p \in M$ are $r$-dimensional real vector spaces, the manifolds $F$ in the local trivialization are vector spaces, and for each $p \in U$ the local trivialization $\psi_U$ restricts to a vector space isomorphism $\pi^{-1}(p) \to \mathbb{R}^r$.

A *subbundle* of $E$ is a vector bundle $F$ over $M$ such that $F_p$ is a vector subspace of $E_p$ for every $p \in M$.

A *section* of $E$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = Id_M$. The space of all sections of $E$ is denoted by $\Gamma(M, E)$.  

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Definition 1.5.2. Let $M$ be a smooth manifold. A rank $r$ complex vector bundle over $M$ is a smooth manifold $E$ together with a smooth submersion $\pi : E \to M$ such that

(i) Each fiber $E_p = \pi^{-1}(p)$ of $\pi$ over $p$ has a structure of $r$-dimensional complex vector space.

(ii) For any point $p \in M$ there is a neighbourhood $U$ and diffeomorphism

\[
\begin{array}{ccc}
\pi^{-1}(U) & \psi_U & U \times \mathbb{C}^r \\
\downarrow \pi & & \downarrow P_r \\
U & \xrightarrow{id} & U
\end{array}
\]

and the diagram commutes, which means that every fibre $E_p$ is mapped to $\{p\} \times \mathbb{C}^r$.

(iii) $\psi_U|_{E_p} : E_p \to \mathbb{C}^r$ is a vector space isomorphism onto $\{p\} \times \mathbb{C}^r$ for every $p \in U$.

A complex vector bundle of rank one is called a line bundle.

Definition 1.5.3. Let $\varphi : M \to N$ be a smooth map and $\pi : E \to N$ be a vector bundle, or indeed any fiber bundle, over $N$. the pullback of $E$ by $\varphi$ is

\[
\varphi^* E := \{(u, x) \in E \times M | \pi(u) = \varphi(x)\}.
\]

From the local triviality of $E$ we see that the map $\varphi^* E \to M$ given by $(u, x) \to x$ is a vector bundle, or fiber bundle, over $M$. The fibre $(\varphi^*E)_x$ is canonically identified with the fibre $E_{\varphi(x)}$ by the map $(u, x) \to u$.

Moreover, precomposition defines a pullback operation on sections of $E$ as: if $s$ is a section of $E$ over $N$, then the pullback section $\varphi^* s = s \circ \varphi$ is a section of $\varphi^* E$ over $M$.

Since $\varphi^* E$ is a subset of $E \times M$, its tangent space at some $(u, x)$ is a subset of $T_u E \times T_x M$. More precisely,

\[
T_{(u,x)}\varphi^* E = \{(V,X) \in T_u E \times T_x M | \pi_*(V) = \varphi_*(X)\}.
\]

Definition 1.5.4. Let $M$ be a complex manifold and let $\pi : E \to M$ be a complex vector bundle over $M$. $E$ is called a holomorphic vector bundle if there exists a trivialization with holomorphic transition functions. In other words, there is an open cover $\mathcal{U}$ of $M$ and for each $U \in \mathcal{U}$ a diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that
the following diagram commutes
\[
\begin{array}{ccc}
\pi^{(-1)}(U) & \xrightarrow{\psi_U} & U \times \mathbb{C}^r \\
\downarrow{\pi} & & \downarrow{p_r} \\
U & \xrightarrow{id} & U
\end{array}
\]

(ii) For every intersecting \(U\) and \(V\), we have
\[
\psi_U \circ \psi_V^{-1}(p, v) = (p, g_{UV}(p)v),
\]
where \(g_{UV} : U \cap V \to GL(r, \mathbb{C}) \subset \mathbb{C}^{r^2}\) are holomorphic functions.

**Definition 1.5.5.** A **holomorphic section of a holomorphic line bundle** \(\mathcal{L}\) is a holomorphic map \(\sigma : M \to \mathcal{L}\) such that \(\pi \circ \sigma\) is the identity map. The local trivialization \(\psi_U\), gives rise to a nonzero local section \(e_i\), which we call a local frame. All other local holomorphic sections over \(U_i\) can be written as
\[
f = f_i e_i
\]
where \(f_i\) is a holomorphic function on \(U_i\).

We write the space of global holomorphic sections as \(H^0(M, \mathcal{L})\) and \(H^0_{L^2}(M, \mathcal{L})\) is refered to the space of \(L^2\)-holomorphic sections on \(\mathcal{L}\).

**Example 1.5.1.** The holomorphic tangent bundle of a complex manifold \(M^n\) is holomorphic. More precisely, for a holomorphic atlas \((U, \phi_U)\) on \(M\), we define
\[
\psi_U : TM|_U \to U \times \mathbb{C}^n
\]
by \(\psi_U(X_p) = (p, (\phi_U)_*(X))\). Then transition functions \(g_{UV} = (\phi_U)_* \circ (\phi_V)^{-1}\) are holomorphic.

**Example 1.5.2.** The holomorphic cotangent bundle of a complex manifold \(M^n\) and more generally the bundle \(\wedge^{s,0}M\) are holomorphic. By using a holomorphic atlas of \(M\) and trivializing locally \(\wedge^{s,0}M\)
\[
dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_s} = \sum_{\beta_1, \ldots, \beta_q} \frac{\partial z_{\alpha_1}}{\partial w_{\beta_1}} \cdots \frac{\partial z_{\alpha_s}}{\partial w_{\beta_s}} dw_{\beta_1} \wedge \cdots \wedge dw_{\beta_s}
\]
shows that the transition functions are holomorphic. The holomorphic complex bundle \(K_M := \wedge^{n,0}M\) is called the **canonical bundle** of \(M\).
For every holomorphic bundle \( E \), we define the bundles
\[ \wedge^s t(E) := \wedge^s t M \otimes E \]
to be bundles of \( E \)-valued forms on \( M \) of type \((s, t)\). The space of sections of \( \wedge^s t E \) is denoted by \( \Omega^s t(E) \). We define the \( \bar{\partial} \)-operator as
\[ \Omega^s t(E) \xrightarrow{\bar{\partial}} \Omega^{s, t+1}(E) \]

**Definition 1.5.6.** A pseudo-holomorphic structure on a complex vector bundle \( E \) is a \( \bar{\partial} \)-operator satisfying the Leibniz rule
\[ \bar{\partial}(\omega \wedge \sigma) = (\bar{\partial} \omega) \wedge \sigma + (-1)^{s+t} \omega \wedge (\bar{\partial} \sigma), \quad \forall \omega \in \Omega^s t(M), \sigma \in \Omega^{l, m}(E). \]
If \( \bar{\partial}^2 = 0 \), then \( \bar{\partial} \) is called a holomorphic structure.

**Lemma 1.5.1.** A complex vector bundle \( E \) is holomorphic if and only if it has a holomorphic structure \( \bar{\partial} \).

**Definition 1.5.7.** Let \( E \rightarrow M \) be a complex bundle of rank \( r \) over a complex manifold \( M \). A Hermitian structure \( h \) on \( E \) is a smooth field of Hermitian products on the fibers of \( E \), more precisely, for every \( p \in M \), the map \( h : E_p \times E_p \rightarrow \mathbb{C} \) satisfies
\[ \begin{align*}
\bullet & \quad h(u, v) \text{ is } \mathbb{C}\text{-linear in } u \text{ for every } v \in E_p. \\
\bullet & \quad h(u, v) = \overline{h(v, u)}, \text{ for all } u, v \in E_p. \\
\bullet & \quad h(u, u) > 0, \text{ for all } u \neq 0 \\
\bullet & \quad h(u, v) \text{ is a smooth function on } M \text{ for every smooth sections } u \text{ and } v \text{ of } E.
\end{align*} \]

**Definition 1.5.8.** A complex vector bundle endowed with a Hermitian structure is called a Hermitian vector bundle

Let \( E \) be a complex vector bundle of rank \( r \) over some smooth manifold \( M \). Take a trivialization \((U_\alpha, \psi_\alpha)\) of \( E \) and a partition of unity \((f_\alpha)\) subordinate to \( \{U_\alpha\} \). For every \( p \in U_\alpha \), let \((h_\alpha)_p\) denote the pull back of the standard Hermitian metric on \( \mathbb{C}^r \) by the \( \mathbb{C}\)-linear map \( \psi_\alpha|_{E_p} \). Then \( h := \sum f_\alpha h_\alpha \) is a well defined Hermitian structure on \( E \). Then we conclude that
Lemma 1.5.2. Every complex vector bundle $E$ of rank $r$ admits Hermitian structures.

Remark 1.5.1. Let $M$ be an almost complex manifold with a Hermitian metric $g$ on $M$ (see Section 1.2.1). Then, $h(X,Y) := g(X,Y) - ig(JX,Y) = g(X,Y) - iω(X,Y)$ defines a Hermitian structure on the complex vector bundle $(TM,J)$. Conversely, any Hermitian structure $h$ on $TM$ as a complex vector bundle defines a Hermitian metric $g$ on $M$ by $g := Re(h)$.

The canonical bundle of $\mathbb{CP}^n$

We know that the complex projective space $\mathbb{CP}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$ and points of $\mathbb{CP}^n$ can be described by homogeneous coordinates, $z_0, \ldots, z_n$, where we identify $[z_0 : \cdots : z_n] = [λz_0 : \cdots : λz_n]$ for all $λ \in \mathbb{C} - \{0\}$.

To define the complex structure, we will use $n + 1$ charts. Let

$$U_i = \{[z_0 : \cdots : z_n]|z_i \neq 0\}$$

and

$$φ_i : U_i → \mathbb{C}^n$$

$$[z] = [z_0 : \cdots : z_n] → (\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i})$$

where the term $\frac{z_i}{z_i}$ is omitted.

Let us use coordinates $w_1, \ldots, w_n$ on $\mathbb{C}^n$, then

$$φ_0 : U_0 → \mathbb{C}^n$$

$$[1 : w_1 : \cdots : w_n] → (w_1, \cdots, w_n)$$

Therefore,

$$φ_i ◦ φ_0^{-1}(w_1, \ldots, w_n) = (\frac{1}{w_1}, \frac{w_2}{w_1}, \cdots, \frac{w_n}{w_1})$$

the transition functions are holomorphic.

Let $L$ be a line subbundle of the product vector bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ over $\mathbb{CP}^n$ in a natural manner. We use the Hermitian structure induced from the natural inner product in $\mathbb{C}^{n+1}$. In the open set $U_0$ of $\mathbb{CP}^n$ defined by $z_0 \neq 0$, we use the inhomogeneous coordinate system $w_i = \frac{z_i}{z_0}$. Then a local holomorphic frame field $s : U_0 → L$ given by

$$s(w_1, \ldots, w_n) = (1, w_1, \ldots, w_n) ∈ \mathbb{C}^{n+1}.$$
With respect to this frame field, the Hermitian structure $h$ of $\mathcal{L}$ is given by

$$h(s, s) = 1 + |w_1|^2 + |w_n|^2.$$  

Its curvature form $\Omega_{FS}$ is given by

$$\Omega_{FS} = -\partial \bar{\partial} \log(1 + |w_1|^2 + ... + |w_n|^2)$$

It follows that $c_1(\mathcal{L})$ is negative, where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$ as we will see later in this section. We note that $\mathbb{CP}^n$ has a natural Kaehler metric called the Fubini-Study metric defined as

$$\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log(|z_0|^2 + |z_1|^2 + ... + |z_n|^2)$$

At $(0, \cdots, 0, 1)$ this equals $(i \sum_j dz_j \wedge d\bar{z}_j)$. The corresponding Hermitian matrix is the identity, which is positive definite. Clearly, $\omega_{FS}$ is closed since it is locally exact. In local coordinates, say on $U_0$, it is given by

$$g_{ij} = \partial_i \bar{\partial}_j \log(|z_0|^2 + |z_1|^2 + ... + |z_n|^2).$$

On the complex projective space there is a distinguished holomorphic line bundle called the tautological line bundle.

**Definition 1.5.9.** The tautological line bundle is defined as the complex line bundle $\pi : \mathcal{L} \to \mathbb{CP}^n$ whose fibre $\mathcal{L}_z$ over some point $[z] \in \mathbb{CP}^n$ is the complex line $\{z\} \subset \mathbb{C}^{n+1}$.

We consider the canonical holomorphic charts $(U_\alpha, \phi_\alpha)$ on $\mathbb{CP}^n$ and the local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ of $\mathcal{L}$ defined by $\psi_\alpha([z], w) = ([z], w_\alpha)$, and then the transition functions:

$$\psi_\alpha \circ \psi_\beta^{-1}([z], \lambda) = ([z], g_{\alpha\beta}([z]) \lambda), \quad \text{where} \quad g_{\alpha\beta}([z]) = \frac{z_\alpha}{z_\beta}.$$  

This shows that the tautological bundle of $\mathbb{CP}^n$ is holomorphic.

We have another distinguished holomorphic line bundle of $\mathbb{CP}^n$ induced by the tautological line bundle.

**Definition 1.5.10.** The hyperplane line bundle of $\mathbb{CP}^n$ is the dual of the tautological line bundle of $\mathbb{CP}^n$. In other words, the fibre of the hyperplane line bundle over some point $[z] \in \mathbb{CP}^n$ is the set of $\mathbb{C}$-linear maps $\mathbb{C}z \to \mathbb{C}$.  

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Proposition 1.5.1. The canonical bundle of $\mathbb{CP}^n$ is isomorphic to the $(n + 1)$th power of the tautological bundle.

Let $V$ be a complex vector space, then $\mathbb{P}(V)$ denotes the projective space associated to $V$, i.e. $\mathbb{P}(V) := (V \setminus \{0\})/\mathbb{C}^*$. Note that, after choosing a basis of $V$ the complex manifold $\mathbb{P}(V)$ becomes isomorphic to $\mathbb{CP}^n$ with $n = \dim_{\mathbb{C}} V - 1$.

Similarly, let $E$ be a complex vector bundle of rank $r$ over $M$. At each point $x$ of $M$, let $\mathbb{P}(E_x)$ be the $(r - 1)$-dimensional projective space of lines through the origin in the fibre $E_x$. Let $\mathbb{P}(E)$ be the fibre bundle over $M$ whose fibre at $x$ is $\mathbb{P}(E_x)$. In other words,

$$\mathbb{P}(E) = (E - \{\text{zero section}\})/\mathbb{C}^*$$

Using the projection $p : \mathbb{P}(E) \to M$, we pull back the bundle $E$ to obtain the vector bundle $p^*E$ of rank $r$ over $\mathbb{P}(E)$. We define the tautological line bundle $\mathcal{L}(E)$ over $\mathbb{P}(E)$ as a subbundle of $p^*E$ as follows: The fibre $\mathcal{L}(E)_x$ at $x \in \mathbb{P}(E)$ is the complex line in $E_{p(x)}$ represented by $x$.

Let $\mathcal{L} \to M$ be a holomorphic line bundle over a complex manifold $M$ and $s_0, \ldots, s_n$ be sections of $\mathcal{L}$, then over an open subset $U$ of $M$, where at least one $s_j$ is nonzero, we obtain a holomorphic map

$$U \to \mathbb{CP}^n$$

$$z \mapsto [s_0(z) : \cdots : s_n(z)].$$

(1.23)

Definition 1.5.11. A line bundle $\mathcal{L}$ over $M$ is very ample if for suitable sections $s_0, \ldots, s_n$ of $\mathcal{L}$ the map (1.39) defines an embedding of $M$ into $\mathbb{CP}^n$. A line bundle $\mathcal{L}$ is ample if the tensor power $\mathcal{L}^k$ is very ample for suitable integer $k > 0$.

We denote by $\Omega^k_M := \Gamma(\wedge^k M \otimes \mathbb{C})$ the space of smooth complex-valued $k$-forms. Since the exterior derivative satisfies $d^2 = 0$, we have the following definition

Definition 1.5.12. A de Rham chain complex is a sequence of maps between vector spaces

$$0 \to \Omega^0_M \to \Omega^1_M \to \cdots \to \Omega^k_M \to 0.$$
Note that it terminates because there is no \((n+1)\)-form on an \(n\)-dimensional manifold.

**Definition 1.5.13.** The *de Rham cohomology group* is the cohomology class defined by

\[
H^k_{dR}(M, \mathbb{C}) := \frac{Z^k_M}{B^k_M}.
\]

Note that \(H^k_{dR}(M, \mathbb{C}) = 0\) if \(\mathbb{Z}^k_M = B^k_M\), i.e., the sequence is exact at \(\Omega^k_M\). Hence the notion of exact sequence and exact differential forms coincide. Similarly one can define over any field other than \(\mathbb{C}\), such as \(\mathbb{R}\) and \(\mathbb{Z}\).

**Chern Classes**

**Definition 1.5.14.** To every complex vector bundle \(E\) over a smooth manifold \(M\) one can associate a cohomology class \(c_1(E) \in H^2_{dR}(M, \mathbb{Z})\) called the *first Chern class* of \(E\) satisfying the following axioms:

(a) (Naturality) For every smooth map \(\phi : M \to N\) and complex vector bundle \(E\) over \(N\), one has \(\phi^*(c_1(E)) = c_1(\phi^*E)\), where the left term denotes the pull-back in cohomology and \(\phi^*E\) is the pull-back bundle.

(b) (Whitney sum formula) For every bundles \(E, F\) over \(M\) one has \(c_1(E \oplus F) = c_1(E) + c_1(F)\), where \(E \oplus F\) is the Whitney sum defined as the pull-back of the bundle \(E \times F \to M \times M\) by the diagonal inclusion of \(M \to M \times M\).

(c) (Normalization) The first Chern class of the tautological bundle of \(\mathbb{CP}^1\) is equal to \(-1\) in \(H^2_{dR}(\mathbb{CP}^1, \mathbb{Z}) \simeq \mathbb{Z}\), which means that the integral over \(\mathbb{CP}^1\) of any representative of this class equals \(-1\).

Let \(E \to M\) be a complex vector bundle. From the above definition, one can express the images in real cohomology of the Chern classes of \(E\) using the curvature of an arbitrary connection \(\nabla\) on \(E\).

**Definition 1.5.15.** Let \(M\) be a smooth manifold of dimension \(n\) and \(E \to M\) be a vector bundle over \(M\). A *connection* on \(E\) is a \(\mathbb{C}\)-linear differential operator \(\nabla : \Gamma(M, E) \to \Omega^1(E)\) satisfying the Leibniz rule

\[
\nabla(f \sigma) = df \otimes \sigma + f \nabla \sigma, \quad \forall f \in C^\infty(M), \sigma \in \Gamma(M, E),
\]
where $\Omega^1(E)$ denotes the space of $E$-valued 1-forms, or sections of $\wedge^1 M \otimes E$ and $C^r(M)$ is denoted the space of all functions with continuous partial derivatives up to order $r$ on $M$.

Note that we can extend any connection to the bundles of $E$-valued $s$-forms on $M$ by

$$\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^s \omega \wedge \nabla \sigma.$$  

**Definition 1.5.16.** The *curvature operator* of $\nabla$ is the $\operatorname{End}(E)$-valued 2-form $R$ defined by

$$R(\sigma) := \nabla(\nabla \sigma), \quad \forall \sigma \in \Gamma(M, E).$$

Let $\{\sigma_1, \ldots, \sigma_r\}$ are local sections of $E$ which form a basis of each fibre over some open set $U$, we define the *local connection forms* $\alpha_{ij} \in \Omega^1(U)$ relative to the choice of the basis by

$$\nabla \sigma_i = \alpha_{ij} \otimes \sigma_j.$$  

We also define the *local curvature 2-forms* $R_{ij}$ by

$$R(\sigma_i) = R_{ij} \otimes \sigma_j,$$

and compute

$$R_{ij} \otimes \sigma_j = R(\sigma_i) = \nabla(\alpha_{ij} \otimes \sigma_j) = (d\alpha_{ij}) \otimes \sigma_j - \alpha_{il} \wedge \alpha_{lj} \otimes \sigma_j, \quad (1.24)$$

showing that

$$R_{ij} = (d\alpha_{ij}) - \alpha_{il} \wedge \alpha_{lj}.$$  

Although the coefficients $R_{ij}$ of $R$ depend on the local basis of sections $\{\sigma_i\}$, its trace is a well-defined complex-valued 2-form on $M$, independent of the chosen basis, and can be expressed as $\operatorname{Tr}(R) = R_{ii}$ in the local basis $\{\sigma_i\}$. To compute it explicitly, we use the following summation trick:

$$\sum_{i,l}^r \alpha_{il} \wedge \alpha_{i} = \sum_{l,i}^r \alpha_{li} \wedge \alpha_{il} = - \sum_{i,l}^r \alpha_{il} \wedge \alpha_{li}$$

Thus, we get

$$\operatorname{Tr}(R) = d(\sum \alpha_{ii}), \quad (1.25)$$

This shows that $\operatorname{Tr}(R)$ is closed, being locally exact.
Lemma 1.5.3. The cohomology class $[\text{Tr}(R)] \in H^2_{dR}(M, \mathbb{C})$ of the closed 2-form $\text{Tr}(R)$ does not depend on $\nabla$.

Proof. If $\nabla$ and $\nabla'$ are connections on $E$, the Leibniz rule shows that their difference $\rho := \nabla' - \nabla$ is a zero-order operator, more precisely a smooth section of $\Omega^1(M) \otimes \text{End}(E)$. Thus $\text{Tr}(\rho)$ is a well-defined 1-form on $M$ and (1.25) implies

$$\text{Tr}(\tilde{R}) = \text{Tr}(R) + d(\text{Tr}(\rho)).$$

Let us choose an arbitrary Hermitian structure $h$ on $E$ and take $\nabla$ such that $h$ is $\nabla$-parallel and let $\{\sigma_i\}$ be a local basis adapted to $h$, then we have

$$0 = \nabla(\delta_{ij}) = \nabla(h(\sigma_i, \sigma_j)) = h(\nabla\sigma_i, \sigma_j) + h(\sigma_i, \nabla\sigma_j) = \alpha_{ij} + \bar{\alpha}_{ji}.$$ From (1.24) we get

$$\tilde{R}_{ij} = d\alpha_{ij} - \sum_{l=1}^r \bar{\alpha}_{il} \wedge \bar{\alpha}_{lj}$$

$$= -d\alpha_{ji} - \sum_{l=1}^r \alpha_{li} \wedge \alpha_{jl}$$

$$= -d\alpha_{ji} + \sum_{l=1}^r \alpha_{jl} \wedge \alpha_{li}$$

$$= -R_{ji}$$

This leads to that the trace of $R$ is a purely imaginary 2-form. Therefore, $[\text{Tr}(R)]$ is a purely imaginary class, in the sense that it has a representative which is a purely imaginary 2-form.

Theorem 1.5.1. Let $\nabla$ be a connection on a complex bundle $E$ over $M$. The real cohomology class

$$c_1(\nabla) := \left[ \frac{i}{2\pi} \text{Tr}(R) \right]$$

is equal to the image of $c_1(E)$ in $H^2_{dR}(M, \mathbb{R})$.

The proof of this theorem has been provided in [88].

Note that if $M$ is an almost complex manifold, we define the first Chern class of $M$ denoted by $c_1(M)$ to be the first Chern class of the tangent bundle $TM$, viewed as complex vector bundle:

$$c_1(M) := c_1(TM).$$
1.5.1 Positivity and Negativity of Vector Bundles

Proposition 1.5.2. (a) $c_1(E) = c_1(\wedge^k E)$, where $k$ denotes the rank of $E$.

(b) $c_1(E \otimes F) = rk(F)c_1(E) + rk(E)c_1(F)$.

(c) $c_1(E^*) = -c_1(E)$, where $E^*$ denotes the dual of $E$.

Let $\mathcal{L}$ be a holomorphic line bundle over a compact complex manifold $M$ of dimension $n$. Let $c_1(\mathcal{L}) \in H^2_{dR}(M, \mathbb{R})$ denote the (real) first Chern class of $\mathcal{L}$. We say that $c_1(L)$ is positive (resp. semi-positive, negative, semi-negative, of rank $\geq r$) and write $c_1(\mathcal{L}) > 0$ (resp. $c_1(\mathcal{L}) \geq 0$, $c_1(\mathcal{L}) < 0$, $c_1(\mathcal{L}) \leq 0$, rank $c_1(\mathcal{L}) \geq r$) if the cohomology class $c_1(\mathcal{L})$ can be represented by a closed real $(1,1)$-form

$$
\omega = \frac{i}{2\pi} \sum \omega_{ij} dz^i \wedge d\bar{z}^j
$$

such that at each point $z$ of $M$ the Hermitian matrix $(\omega_{ij}(z))$ is positive definite (resp. positive semi-definite, negative definite, negative semi-definite, of rank $\geq r$).

Definition 1.5.17. Let $\mathcal{L}$ be a holomorphic line bundle over a compact complex manifold $M$. Then $\mathcal{L}$ is **positive** (resp. **negative** if $c_1(\mathcal{L})$ is positive (resp. negative). Equivalently, a line bundle is positive if for a suitable a Hermitian metric $h$ the curvature form is a Kahler form.

Theorem 1.5.2. (**Kodaira embedding theorem**) Let $\mathcal{L}$ be a line bundle over a compact complex manifold $M$. Then $\mathcal{L}$ is ample if and only if the first Chern class $c_1(\mathcal{L})$ is positive.

One can readily prove the first direction. For the other direction, we need to show that for sufficiently large $k$, $\mathcal{L}^k$ has enough holomorphic sections to give rise to an embedding of the manifold. One approach to proceed is through Kodaira’s vanishing theorem for cohomology (see [52] p. 176) and another way is through studying the Bergman kernel for large powers of $\mathcal{L}$ as we will see in the next chapter.

Let $E$ be a holomorphic vector bundle of rank $r$ over a complex manifold $M$ of dimension $n$ and let $\mathbb{P}(E)$ as and

$$
\mathcal{L}(E) = \cup_{z \in M} \mathcal{L}(E_z)
$$
be the tautological line bundle over $\mathbb{P}(E)$. We define positivity of a vector bundle $E$ by positivity of the line bundle $\mathcal{L}(E)$. Thus, we say that a holomorphic vector bundle $E$ over $M$ is positive (resp. semi-positive of rank $\geq d$) if the first Chern class $c_1(\mathcal{L}(E))$ of the line bundle $\mathcal{L}(E)$ satisfies $c_1(\mathcal{L}(E)) > 0$ (resp. $c_1(\mathcal{L}(E)) \geq 0$ with rank $c_1(\mathcal{L}(E)) \geq d + r - 1$). We say that $E$ is negative (resp. semi-negative of rank $\geq d$) if its dual $E^\ast$ is positive (resp. semi-positive of rank $\geq d$).

### 1.5.2 Chern Class of Kaehler Manifolds

**Definition 1.5.18.** Let $E$ be a complex vector bundle over a manifold $M$ with a Hermitian structure $h$. Then a connection $\nabla$ on $E$ is a *Hermitian connection* if $h$ is parallel with respect to $\nabla$.

**Theorem 1.5.3.** For every Hermitian structure $h$ in a holomorphic bundle $E$ with holomorphic structure $\bar{\partial}$, there exists a unique Hermitian connection $\nabla$ called the Chern connection such that $\nabla^{0,1} = \bar{\partial}$, where $\nabla^{0,1} := \pi^{0,1} \circ \nabla$ and $\pi^{0,1} : \Lambda^1(E) \to \Lambda^{0,1}(E)$.

**Proposition 1.5.3.** On a Hermitian manifold $(M, h, J)$, the Chern connection coincides with the Levi-Civita connection if and only if $(M, h, J)$ is Kaehler.

Let $(M, \omega)$ be a compact Kaehler manifold. Since the Kaehler form $\omega$ is a closed real form, it defines a cohomology class $[\omega]$ in $H^2_{dR}(M, \mathbb{R})$.

Let $R$ be a curvature tensor of Levi-Civita connection $\nabla$ of a Kaehler manifold $M$. We denote by Ric its *Ricci tensor* $\operatorname{Ric}(X, Y) = \sum_i R(e_i, X, Y, e_i)$ where $\{e_i\}$ is a local orthonormal basis of $TM$.

**Definition 1.5.19.** The *Ricci form* $\rho$ of a Kaehler manifold is defined by $\rho(X, Y) := \operatorname{Ric}(JX, Y)$, for every $X, Y \in TM$.

The Ricci form is one of the most important objects on a Kaehler manifold.
Proposition 1.5.4. The Ricci form $\rho$ of a Kaehler manifold has the following properties:

(a) The Ricci form $\rho$ is closed;

(b) The cohomology class of $\rho$ is equal (up to some real multiple) to the Chern class of the canonical bundle of $M$;

(c) In local coordinates, $\rho$ can be expressed as $\rho = -i\partial \bar{\partial} \log \det(h_{i\bar{j}})$, where $\det(h_{i\bar{j}})$ denotes the determinant of the matrix $(h_{i\bar{j}})$ expressing the Hermitian metric.

1.6 Bergman Kernel and Poincaré Series on Bounded Symmetric Domains

The study of Bergman kernel originates from several complex variables. Bergman kernel for complex projective manifolds was studied by Tian, Zelditch, Catlin, Lu, among others. In this section, basic definitions and theorems related to Bergman kernel on bounded domains are reviewed in the first two parts. The third part of the section concerns the Bergman kernels on compact complex manifolds. Some results from [28], [75], [76], [83], [86], [96], [105], [106], [114] are incorporated in this section. Bergman kernel and Poincaré series on complex manifolds will be used to construct automorphic forms in the last two chapters.

1.6.1 Bergman Kernel on Domains in $\mathbb{C}^n$

Let $D$ be a bounded domain (a bounded connected open subset) in $\mathbb{C}^n$. The Bergman space is defined as a space $A^2(D)$ of holomorphic square integrable functions (or, $L^2$-functions), in other words,

$$A^2(D) := \{ f \text{ holomorphic on } D \mid \int_D |f(z)|^2dV(z) = \|f\|_{L^2} < \infty \}.$$  

Since $D$ is bounded domain, $A^2(D)$ is nontrivial (it contains all polynomials in $z_1, z_2, \cdots, z_n$). Since for any compact subset $K$ of $D$ and $z_0 \in K$ there is an $r(K) = r > 0$ such that a disc $B(z_0, r) \subset D$, so we get by applying the
mean value property of holomorphic function for \( z_0 \in K \),

\[
|f(z_0)| = \left| \frac{1}{\text{vol}(B(z_0, r))} \int_{B(z_0, r)} f(z) dV(z) \right| \\
\leq \frac{1}{\pi r^2} \int_{B(z_0, r)} |f(z)| dV(z)
\]

Since the radius \( r \) here is less than the distance \( d \) between \( z_0 \) and the boundary of \( D, \bar{D} \), and by using the Hölder inequality, we have

\[
|f(z_0)| \leq \frac{1}{\sqrt{\pi} r} \left( \int_{B(z_0, r)} |f(z)|^2 dV(z) \right)^{\frac{1}{2}} \\
\leq \frac{1}{\sqrt{\pi} r} \|f(z)\|_{L^2}.
\]

Hence,

\[
|f(z_0)| \leq C_K \|f(z)\|_{L^2}. \quad (1.26)
\]

By letting \( r \) tends to \( d \), we obtain

\[
|f(z_0)| \leq \frac{1}{\sqrt{\pi d}} \|f(z)\|_{L^2} \quad (1.27)
\]

Thus, any sequence \( \{f_i\} \) of holomorphic functions on \( D \), that converges in \( L^2(D) \) to a function \( h \), is uniformly convergent on compact subsets of \( D \) to a holomorphic function \( f \). From the fact that the convergence in \( L^2(D) \) implies pointwise convergence almost everywhere of a subsequence, we have \( h \in A^2(D) \) since \( h = f \) almost everywhere.

It follows that \( A^2(D) \) is a closed subspace of the Hilbert space \( L^2(D) \). Hence, \( A^2(D) \) is a Hilbert space with an induced inner product defined as

\[
\langle f, g \rangle := \int_D f(z) \overline{g(z)} dV(z)
\]

Next consider the point evaluation map

\[
\Psi_z : A^2(D) \rightarrow \mathbb{C} \\
f \mapsto f(z)
\]

The estimate (1.27) gives the following lemma
Lemma 1.6.1. For any fixed point \( z \in D \), the evaluation functional \( \Psi_z \) is a continuous linear functional on \( A^2(D) \).

Hence, Riesz Representation theorem (Theorem 1.1.5 & see [44]) implies that there exists a unique element \( K_z \), (or \( K(\cdot, z) \)), in \( A^2(D) \) such that

\[
\Psi_z(f) = \langle f, K_z \rangle \tag{1.28}
\]

for all \( f \in A^2(D) \).

Definition 1.6.1. The Bergman kernel function \( K_D : D \times D \to \mathbb{C} \) is uniquely defined by the following

- (a) \( K_D(., w) \in A^2(D), \forall w \in D \).
- (b) the reproducing property (1.28),

\[
f(z) = \langle f, K_D(\cdot, z) \rangle.
\]

For simplicity, we can write \( K_D(z, w) = K(z, w) \). Clearly for any \( w \in D \), by the estimates (1.26) and (1.27), the Bergman kernel satisfies

\[
\|K(., w)\|_{L^2} \leq C_K \tag{1.29}
\]

for every compact set \( K \), and

\[
\|K(., w)\|_{L^2} \leq \frac{1}{\sqrt{\pi d}} \tag{1.30}
\]

Lemma 1.6.2. The Bergman kernel function \( K \) satisfies a symmetry property

\[
K(z, w) = \overline{K(w, z)}
\]

for any \( z, w \in D \).

Proof. Let \( w \in D \). By the reproducing property of The Bergman kernel, we obtain

\[
K(z, w) = \langle K(., w), K(., z) \rangle = \overline{\langle K(., z), K(., w) \rangle} = \overline{K(w, z)}.
\]

\[
\square
\]
Since $A^2(D)$ is separable Hilbert space, the Bergman kernel function can be represented in terms of any orthonormal basis for $A^2(D)$.

**Theorem 1.6.1.** Let $\{e_i(z)\}_{i=1}^\infty$ be an orthonormal basis for $A^2(D)$. Then

$$K(z, w) = \sum_{i=1}^\infty e_i(z) \otimes \overline{e_i(w)}, \text{ for } (z, w) \in D \times D$$

(1.31)

where the series (1.31) converges uniformly on any compact subset of $D \times D$.

**Proof.** For any fixed $w \in D$, we have

$$K(z, w) = \sum_{i=1}^\infty \langle K(., w), e_i(.) \rangle e_i(z)$$

$$= \sum_{i=1}^\infty \langle e_i(., K(., w)) \rangle e_i(z)$$

$$= \sum_{i=1}^\infty \overline{e_i(w)} e_i(z)$$

This series converges in $L^2$-norm, and

$$\|K(., w)\|_{L^2}^2 = \sum_{i=1}^\infty |e_i(w)|^2$$

(1.32)

Since the convergence is dominated by $L^2$-convergence in $A^2(D)$, we obtain the pointwise convergence of (1.31). Hence, from the normal family argument (see Theorem 5.2.6 in [108]), it suffices to show that $\sum_{i=1}^m e_i(w)e_i(z)$, for any $m \in \mathbb{Z}$, is uniformly bounded on any compact set of $D \times D$. Let $K$ be a compact subset of $D$. For any $(z, w) \in K \times K$,

$$\left| \sum_{i=1}^m e_i(w)e_i(z) \right| \leq \sum_{i=1}^\infty |e_i(w)||e_i(z)|$$

$$\leq \left( \sum_{i=1}^\infty |e_i(w)|^2 \right)^{1/2} \left( \sum_{i=1}^\infty |e_i(z)|^2 \right)^{1/2}$$

Applying (1.29) and (1.32) in the last line shows that

$$\left| \sum_{i=1}^m e_i(w)e_i(z) \right| \leq C_K$$
for some constant $C_K > 0$ independent of choice of $m$. This completes the proof of the theorem.

Bergman kernel can be defined for arbitrary domains but it is very difficult to obtain explicit representations of it except for special cases such as for balls.

**Example 1.6.1.** Let $\Omega$ be the unit ball $B^n_C$. From Theorem 1.6.1, we should find an orthonormal basis of $A^2(B^n_C)$. Obviously, $\{z^\alpha\}$ is an orthogonal basis for $A^2(B^n_C)$, where the index $\alpha = (\alpha_1, \cdots, \alpha_n)$ runs over the multi-indices, since the unit ball is stable (invariant) under coordinate-wise rotations, i.e.,

$$ (z_1, z_2, \cdots, z_n) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2, \cdots, e^{i\theta} z_n) $$

then, the hermitian inner products of monomials

$$ \int_{B^n_C} z^\alpha \cdot \bar{z}^\beta dV(z) = \int_{B^n_C} (z_1^{\alpha_1} \cdots z_n^{\alpha_n}) \cdot (\bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}) dV(z) $$

must vanish unless $\alpha = \beta$. Using the fact that every $z \in \mathbb{C}^n$ can be written in polar coordinates as

$$ z = rw; \quad r = |z| \text{ and } w = \frac{z}{|z|} \in S_{2n-1} $$

Then, for any function $f$ over $\mathbb{C}^n$

$$ \int_{\mathbb{C}^n} f(z) dz = \int_{S_{2n-1}} \int_0^\infty f(rw) r^{2n-1} dr d\sigma_{2n-1}(w) \quad (1.33) $$

where $\sigma$ is the $2n - 1$ dimensional surface measure on the unit sphere $S_{2n-1}$.

Similarly, for any $z \in B^n_C$, put

$$ z = rw; \quad r = |z| \text{ and } w = \frac{z}{|z|} \in S_{2n-1} $$

and

$$ f(z) = |z|^{2\alpha} = |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} $$
Then

\[
\|z^n\|^2_{L^2} = \int_{B^n_c} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} dV(z)
\]

\[
= \int_{S_{2n-1}} \int_0^1 r^{2\alpha_1 + \cdots + 2\alpha_n} |w_1|^{2\alpha_1} \cdots |w_n|^{2\alpha_n} \cdot r^{2n-1} dr d\sigma_{S_{2n-1}}(w)
\]

\[
= \int_0^1 r^{2\alpha_1 + \cdots + 2\alpha_n + 2n-1} dr \cdot \int_{S_{2n-1}} |w|^{2\alpha} d\sigma_{S_{2n-1}}(w)
\]

\[
= \frac{1}{2\alpha_1 + \cdots + 2\alpha_n + 2n} \int_{S_{2n-1}} f(w) d\sigma_{S_{2n-1}}(w)
\]

Hence,

\[
\|z^n\|^2_{L^2} = \frac{1}{2\alpha_1 + \cdots + 2\alpha_n + 2n} \int_{S_{2n-1}} f(w) d\sigma_{S_{2n-1}}(w) \quad (1.34)
\]

Now, consider

\[
I = \int_{C^n} f(z) e^{-\pi |z|^2} dz = \Pi_1^n \int_{C} |z_j|^{2\alpha_j} e^{-\pi |z_j|^2} dz_j
\]

For each \(j\), use change of variable

\[
z_j = r_j w_j \quad (\text{or}, \ z_j = r_j e^{i\theta_j})
\]

we get

\[
I = \Pi_1^n \int_{S_1} \int_0^\infty r_j^{2\alpha_j} e^{-\pi r_j^2} \cdot r_j d\sigma_{S_1}(\theta)
\]

\[
= \Pi_1^n 2\pi \int_0^\infty r_j^{2\alpha_j} e^{-\pi r_j^2} \cdot r_j dr_j
\]

Using the substitution \(r_j^2 = s_j, 2r_j dr_j = ds_j\), and the fact that

\[
\int_0^\infty x^m e^{-nx^2} dx = \frac{1}{n^{m+1/2}} \Gamma\left(\frac{m+1}{n}\right), \quad (1.35)
\]

where \(m, n > 0\), we get

\[
I = \Pi_1^n \pi^{-\alpha_j} \Gamma(\alpha_j + 1) \quad (1.36)
\]

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Evaluating $I$ by using (1.33),

$$I = \int_{S_{2n-1}}^{\infty} \int_{0}^{\infty} f(rw) e^{-\pi r^2} \cdot r^{2n-1} drd\sigma_{2n-1}(w)$$

$$I = \int_{S_{2n-1}}^{\infty} r^{2\alpha_1 + \cdots + 2\alpha_n} f(w) e^{-\pi r^2} \cdot r^{2n-1} drd\sigma_{2n-1}(w)$$

$$= \int_{0}^{\infty} r^{2\alpha_1 + \cdots + 2\alpha_n + 2n-1} e^{-\pi r^2} dr \int_{S_{2n-1}} f(w) d\sigma_{2n-1}(w)$$

Use (1.35) for the first integral,

$$I = \frac{\Gamma(\alpha_1 + \cdots + \alpha_n + n)}{2\pi^{\alpha_1 + \cdots + \alpha_n + n}} \int_{S_{2n-1}} f(w) d\sigma_{2n-1}(w)$$

(1.37)

Therefore, from (1.36) and (1.37),

$$\int_{S_{2n-1}} f(w) d\sigma_{2n-1}(w) = \frac{2\pi^{\alpha_1 + \cdots + \alpha_n + n}}{\Gamma(\alpha_1 + \cdots + \alpha_n + n)} \cdot \Pi_{j=1}^{n} \pi^{-(\alpha_j + 1)} \Gamma(\alpha_j + 1)$$

Substituting in (1.34),

$$||z^\alpha||^2_{L^2(B^n)} = \frac{\pi^n}{\alpha_1 + \cdots + \alpha_n + n} \cdot \frac{\Gamma(\alpha_1 + 1) \cdots \Gamma(\alpha_n + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_n + n)}$$

$$= \frac{\alpha_1! \cdots \alpha_n!}{(\alpha_1 + \cdots + \alpha_n + n)!}$$

Now, the orthonormal basis of $A^2(B^n_C)$ are

$$\left\{ \sqrt{\frac{||\alpha||+n)!}{\pi^\alpha!} z^\alpha \right\}$$

where $|\alpha| = \sum_{j=1}^{n} \alpha_j$ and $\alpha! = \Pi_{j=1}^{n} \alpha_j!$. Subsequently, the explicit formula of Bergman kernel function on $B^n_C$ is given by

$$K(z, w) = \sum_{\alpha} \frac{(|\alpha| + n)!}{\pi^n(\alpha)!} z^\alpha \bar{w}^\alpha$$

$$= \frac{1}{\pi^n} \sum_{\alpha} \left( \frac{|\alpha|}{\alpha} \right) (|\alpha| + n) \cdots (|\alpha| + 1)(z\bar{w})^\alpha.$$
Since

\[(|\alpha| + n) \cdots (|\alpha| + 1) = \frac{\partial^n}{\partial t^n} \bigg|_{t=1} (t^{(|\alpha|+n)}) \]

we have,

\[\mathcal{K}(z, w) = \frac{1}{\pi^n} \sum_{\alpha} \left( \frac{|\alpha|}{\alpha} \right) \frac{\partial^n}{\partial t^n} \bigg|_{t=1} \left( t^{(|\alpha|+n)} \right) \cdots (|\alpha| + 1)(z \bar{w})^\alpha \]

\[= \frac{1}{\pi^n} \sum_{\alpha} \left\{ t^n \sum_{\alpha} \left( \frac{|\alpha|}{\alpha} \right) t^{(|\alpha|+n)}(z \bar{w})^\alpha \right\} \]

\[= \frac{1}{\pi^n} \sum_{\alpha} \left\{ t^n \sum_{N=0}^\infty \sum_{|\alpha|=N} \left( \begin{array}{c} N \\ \alpha \end{array} \right) (z \bar{w})^\alpha \right\} \]

\[= \frac{1}{\pi^n} \sum_{N=0}^\infty \left( t^{N+n} \right)(z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n)^N \]

\[= \frac{1}{\pi^n} \sum_{N=0}^\infty (N + n) \cdots (N + 1)(z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n)^N \]

\[= \frac{1}{\pi^n} \sum_{N=0}^\infty \left( t^{N+n} \right)(z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n)^N \]

\[= \frac{1}{\pi^n} \sum_{N=0}^\infty \frac{1}{(1 - t)^n} \]

\[= \frac{n!}{\pi^n (1 - z \bar{w})^{n+1}} \]

Therefore,

\[\mathcal{K}_{B^n}(z, w) = \frac{n!}{\pi^n (1 - z \bar{w})^{n+1}} \tag{1.38} \]

where \(z \bar{w} = \sum_{i=1}^n z_i \bar{w}_i\).

Since \(A^2(D)\) is closed subspace of the Hilbert space \(L^2(D)\), any \(f \in L^2(D)\) can be written as \(f = f_1 + f_2\), where \(f_1 \in A^2(D)\) and \(f_2 \in (A^2(D))^\perp\).

Moreover, one can define the orthogonal projection map

\[P : L^2(D) \rightarrow A^2(D)\]
such that
\[
Pf(z) = f_1(z) \\
= \langle f_1(\cdot), K(\cdot, z) \rangle \\
= \langle f_1(\cdot), K(\cdot, z) \rangle + \langle f_2(\cdot), K(\cdot, z) \rangle \\
= \langle f(\cdot), K(\cdot, z) \rangle
\]

Hence, the following theorem is obtained

**Theorem 1.6.2.** The Bergman projection \( P_D : L^2(D) \to A^2(D) \) is represented by
\[
P_Df(z) = \int_D K(z, w)f(w)dV(w),
\]
for any \( f \in L^2(D) \) and \( z \in D \)

**Theorem 1.6.3.** Let \( g : D_1 \to D_2 \) be a biholomorphic map between two domains \( D_1 \) and \( D_2 \) in \( \mathbb{C}^n \). Then
\[
K_{D_1}(z, w) = J_g(z)K_{D_2}(g(z), g(w))\overline{J_g(w)}
\]
for all \( z, w \in D_1 \), where \( J_g(z) \) is the determinant of the complex Jacobian of \( g \) at \( z \).

**Proof.** We have via the standard identification between \( \mathbb{C}^n \) and \( \mathbb{R}^{2n} \)
\[
(J_g)_R(z) = |J_g(z)|^2. \tag{1.39}
\]
let \( h \in A^2(D_1) \). Then, by change of variables, for \( z, w \in D_1 \)
\[
\int_{D_1} J_g(z)K_{D_2}(g(z), g(w))\overline{J_g(w)}h(w)dV(w)
\]
\[
= \int_{D_2} J_g(z)K_{D_2}(g(z), \zeta)\overline{J_g(g^{-1}(\zeta))h(g^{-1}(\zeta))(J_{g^{-1}})_R(\zeta)}dV(\zeta).
\]
using (1.39), we get
\[
J_g(z)\int_{D_2} K_{D_2}(g(z), \zeta)(J_g(g^{-1}(\zeta)))^{-1}h(g^{-1}(\zeta))dV(\zeta). \tag{1.40}
\]
By change of variables,
\[
(J_g(g^{-1}(\zeta)))^{-1}h(g^{-1}(\zeta)) \in A^2(D_2).
\]
Using the reproducing property of $K_{D_2}$ in (1.40), then we get
\[ J_g(z)(J_g(z))^{-1}h(g^{-1}(g(z))) = h(z) \]
Therefore, from the uniqueness of the kernel function, we have
\[ K_{D_1}(z, w) = J_g(z)K_{D_2}(g(z), g(w)) \frac{J_g(w)}{J_g(z)} \]
which completes the proof. \qed

Remark 1.6.1. The theorem above gives a transformation law of the Bergman kernel functions and describes the behavior of the Bergman kernel under biholomorphic mappings.

Theorem 1.6.4. Let $g : D_1 \rightarrow D_2$ be a biholomorphic map between two domains $D_1$ and $D_2$ in $\mathbb{C}^n$, and let $P_1, P_2$ be the Bergman projections on $D_1, D_2$ respectively. Then
\[ P_1(J_g \cdot (h \circ g)) = J_g \cdot (P_2(h) \circ g) \]
for all $h \in L^2(D_2)$, where $J_g(z)$ is the determinant of the complex Jacobian of $g$ at $z$.

Proof. Let $h \in L^2(D_2)$. Then $J_g \cdot (h \circ g) \in L^2(D_1)$ and from the previous theorem we get for any $z \in D_1$
\begin{align*}
P_1(J_g \cdot (h \circ g))(z) &= \int_{D_1} K_{D_1}(z, w)J_g(w)h((g(w))dV_{D_1}(w) \\
&= \int_{D_1} J_g(z)K_{D_2}(g(z), g(w))|J_g(w)|^2h((g(w))dV_{D_1}(w) \\
&= J_g(z)\int_{D_2} K_{D_2}(g(z), \zeta)h(\zeta)dV_{D_2}(\zeta) \\
&= J_g(z) \cdot (P_2(h) \circ g)(z). \quad \square
\end{align*}

To get a more developed transformation rule of Bergman kernel, we need to define the following

Definition 1.6.2. Let $D_1, D_2$ be domains. A continuous mapping $g : \Omega_1 \rightarrow D_2$ is said to be proper if the inverse image of any compact set is compact.
Proposition 1.6.1. Let $D_1, D_2$ be domains in $\mathbb{C}^n$ and let $g : D_1 \rightarrow D_2$ be proper and holomorphic. Then

(a) $g$ is an open-closed mapping;

(b) $g(D_1) = D_2$

(c) $g$ is non-degenerate, i.e., $J_g \neq 0$

(d) (Remmert’s Proper Mapping Theorem) for any analytic subset $A$ of $D_1$, the image $g(A)$ is analytic in $D_2$. In particular $g(E_g)$ is a proper analytic subset of $D_2$, where $E_g = \{ z \in D_1 : J_g(z) = 0 \}$.

Any proper holomorphic mapping between domains in $\mathbb{C}^n$ is of finite multiplicity. More precisely:

Theorem 1.6.5. Let $D_1, D_2$ be domains in $\mathbb{C}^n$ and $g : D_1 \rightarrow D_2$ be a holomorphic proper mapping. Then there is $m \in \mathbb{N}$ such that for $w \in D_2$, $g^{-1}(w) = \{ g_1^{-1}(w), \ldots, g_k^{-1}(w) \}$ and $k \leq m$. Moreover:

(a) $k = m$ for $w \in D_2 \setminus g(E_g)$;

(b) $k < m$ for $w \in g(E_g)$.

Such an $m$ is called the order of $g$.

Using similar arguments of Theorem 1.6.3 one may get a transformation rule for proper holomorphic mappings:

Theorem 1.6.6. Suppose $D_1, D_2$ be domains in $\mathbb{C}^n$ and $g : D_1 \rightarrow D_2$ be a proper holomorphic mapping of order $m$. Let $h_1, \ldots, h_m$ denote the local inverses to $g$ (defined locally on $D_2 \setminus g(E_g)$). Then the Bergman kernels transform in the following way:

$$\sum_{k=1}^{m} K_{D_1}(z, h_k(w)) J_{h_k}(w) = J_g(z) K_{D_2}(g(z), w)$$

for all $z \in D_1$ and $w \in D_2 \setminus g(E_g)$.
1.6.2 Bounded Symmetric Domains

A domain $D$ is a connected open subset of a finite-dimensional complex vector space $M$. Let $M$ be a complex Banach space and $D \subset M$ a bounded domain. Then $D$ is called symmetric if for every $z \in D$ there exists an $\gamma \in Aut(D)$ such that

- (i) $\gamma^2 = Id_D$,
- (ii) $z$ is an isolated fixed point of $\gamma$.

**Theorem 1.6.7.** (Cartan’s uniqueness theorem) Let $M, N$ be arbitrary complex Banach spaces and suppose that $D \subset M$ is a bounded domain, $\Lambda \subset N$ is an arbitrary domain and $f, g : D \rightarrow \Lambda$ are holomorphic mappings with $g$ biholomorphic. Then $f, g$ coincide if there exists at least one point $z \in D$ with $f(z) = g(z)$ and $df_z = dg_z$.

Condition (ii) in the definition of the bounded symmetric domains can be replaced by $\gamma(z) = z$ and $d\gamma_z = -Id_M$. By Cartan’s uniqueness theorem, the symmetry $\gamma$ about $z$ is uniquely determined by $z$ and will always be denoted by $\gamma_z$. The mapping $\Omega \rightarrow Aut(\Omega)$, $z \rightarrow \gamma_z$, is analytic and the subset $\{\gamma_z \gamma_w : z, w \in \Omega\}$ generates a connected subgroup of $Aut(\Omega)$ acting transitively on $\Omega$. Therefore every bounded symmetric domain is homogeneous.

**Example 1.6.2.** The unit ball is a bounded symmetric domain. The origin is the fixed point of the involution $\gamma_0(z) = -z$. Since $Aut(B^n_C)$ acts transitively on $B^n_C$, this shows that every other point $z \in B^n_C$ is also the fixed point of an involution, $\gamma_z = g_z \circ \gamma_0 \circ g_{-z}$.

**Example 1.6.3.** Let $\mathcal{H}$ be the complex upper half plane. Then $SL(2, \mathbb{R})$ acts on $\mathcal{H}$ by

$$z \rightarrow \frac{az + b}{cz + d}.$$ 

For any $z = x + iy \in \mathcal{H}$, we have $\gamma = \left( \begin{array}{cc} \sqrt{y} & z \\ \sqrt{y} & \sqrt{y} \end{array} \right) \in SL(2, \mathbb{R})$, such that $\gamma i = z$, hence $\mathcal{H}$ homogeneous. The isomorphism $z \rightarrow -1/z$ is a symmetry at $i \in \mathcal{H}$, and so $\mathcal{H}$ is symmetric.

Let $D$ be domain in $\mathbb{C}^n$ and $A^2(D)$ be a Hilbert space defined as before. Then, there is a unique Bergman kernel function $K : D \times D \rightarrow \mathbb{C}$ with
properties mentioned in Definition 1.6.1. Since $D$ is bounded, then every polynomial function on $D$ is $L^2$ and hence $K(z, z) > 0$ for every $z \in D$. Moreover, $\log(K(z, z))$ is smooth and the equations

$$g = \sum g_{ij}dz_id\bar{z}_j, \quad g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(K(z, z))$$

(1.41)

defines a hermitian metric (Bergman metric) on $D$, which is shown to have a negative curvature in [77], or [58]. Hence, we have the following theorem

**Theorem 1.6.8.** Every bounded domain in $\mathbb{C}^n$ has a canonical hermitian metric called Bergman metric which has negative curvature.

The Bergman metric is invariant under the action of $\text{Aut}(D)$. Hence a bounded symmetric domain $D$ is a hermitian symmetric domain.

**Example 1.6.4.** Let $D$ be the unit ball $B^n_{\mathbb{C}}$. Then from the explicit formula of the Bergman kernel on $B^n_{\mathbb{C}}$ obtained in Example 1.6.1 and the equations (1.41), the components of Bergman metric on $B^n_{\mathbb{C}}$ are given by

$$g_{ij}(z) = \frac{n!}{\pi^n} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \left( \frac{1}{(1 - z\bar{z})^{n+1}} \right)$$

Moreover, the Kaehler form, up to a positive constant factor, is

$$\omega = -i \frac{n!}{\pi^n} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \left( \frac{1}{(1 - z\bar{z})^{n+1}} \right).$$

(1.42)

**Definition 1.6.3.** A bounded symmetric domain is called irreducible if it is not biholomorphic to a Cartesian product of two other bounded symmetric domains.

Irreducible bounded symmetric domains were completely classified by E. Cartan. Let $D = G/K$ be an irreducible bounded symmetric domain in $\mathbb{C}^n$, $n > 1$, then $G$ is one of the following Lie groups: $SU(p, q)$, $SO^*(2p)$, $Sp(p, \mathbb{R})$, $SO_0(p, 2)$, $E_6^{(-14)}$, $E_7^{(-25)}$ and $K$ is a maximal compact subgroup of $G$ [1].

### 1.6.3 Bergman Kernel on Manifolds

Let $(M, \omega)$ be a compact Kaehler manifold with a holomorphic positive line bundle $\mathcal{L}$ and $h$ be the hermitian metric on $\mathcal{L}$ such that at a point $p \in M$,
with local coordinates \((z_1, \cdots, z_n)\) and frame \(\{e_L\}\) of \(\mathcal{L}\) in a neighborhood \(U_\alpha\), the Kaehler metric of the manifold and the hermitian metric of the line bundle satisfy the following relation locally
\[
\omega = \frac{i}{2\pi} g_{ij} dz^i \wedge d\bar{z}^j = -i \partial \bar{\partial} \log h_\alpha
\]
where \(h_\alpha\) and \(g_{ij}\) are the local representation of the hermitian metric \(h_\alpha = h(e_L, e_L)\) on \(\mathcal{L}\) and Riemannian metric \(g_{ij} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})\) on \(M\) respectively. Therefore, the hermitian metric \(h\) induces the \(L^2\)-inner product on \(\mathcal{L}\) as
\[
(s_1, s_2)_{L^2} = \int_M h(s_1, s_2) dV_M,
\]
where the volume form \(dV_M\) is given by \(\frac{\omega^n}{n!}\) and \(n\) is the dimension of \(M\). We also use \(|\cdot|_h\) to denote the pointwise norms of the metric \(h\). We refer to the space of holomorphic sections of \(\mathcal{L}\) by \(H^0(M, \mathcal{L})\), and by \(H^0_{L^p}(M, \mathcal{L})\) the space of \(L^p\)-holomorphic sections where \(p \geq 1\). Then by Hodge theorem ([83]), the space \(H^0(M, \mathcal{L})\) is finite-dimensional.

The orthogonal projection (or, the Bergman projection) on \(M\) with respect to \((\cdot, \cdot)_h\) is denoted by \(P\) and is given by
\[
P : L^2(M, \mathcal{L}) \mapsto H^0_{L^2}(M, \mathcal{L})
\]

**Definition 1.6.4.** Consider an orthonormal basis \(\{s_j\}\) of sections of \(\mathcal{L}\). The Bergman kernel associated to \(M, \mathcal{L}\) is defined as
\[
\mathcal{K}_{M,\mathcal{L}}(z, w) = \sum_{j=0}^N s_j(z) \otimes s_j(w),
\]
where \(\text{dim}(H^0(M, \mathcal{L})) = N + 1\) ([83]).

The restriction to the diagonal of the full Bergman kernel is a function
\[
\mathcal{K}_{M,\mathcal{L}} : M \longrightarrow \mathbb{R}
\]
\[
z \mapsto \sum_{j=0}^N |s_j(z)|^2_h \tag{1.43}
\]
It is easy to check that \(\mathcal{K}_{M,\mathcal{L}}\) is independent of the orthonormal basis chosen. The following lemma gives us an alternative definition of \(\mathcal{K}_{M,\mathcal{L}}(z, \bar{z})\) or, \(\mathcal{K}_{M,\mathcal{L}}\) for simplification.
Lemma 1.6.3. [105] For any \( z \in M \) we have
\[
\mathcal{K}_{M,L} = \sup \{|s(z)|_h^2 : \|s\|_{L^2} = 1\}.
\]

Proof. By considering any orthonormal basis containing \( s \), it is clear that \( \mathcal{K}_{M,L}(z,z) \geq |s(z)|_h^2 \) for any \( s \) such that \( \|s\|_{L^2} = 1 \). On the other hand, if \( E_z \subset H^0(M,\mathcal{L}) \) is the space of sections vanishing at \( z \) and if \( \mathcal{K}_{M,L}(z,z) > 0 \), then there must be a section which does not vanish at \( z \), and so \( E_z \) has codimension 1. Let \( s \) be in the orthogonal complement of \( E_z \), such that \( \|s\|_{L^2} = 1 \). Then it follows from the definition that \( \mathcal{K}_{M,L}(z,z) = |s(z)|_h^2 \) since every section orthogonal to \( s \) vanishes at \( z \).

Let \( \mathcal{L}^k \) be the \( k \)-th tensor power of \( \mathcal{L} \), extending the Hermitian metric \( h \) to \( h^k \) as well, and we get a corresponding Kaehler form \( k\omega \). We refer to the space of holomorphic sections of the \( k \)-th power of \( \mathcal{L} \) by \( H^0(M,\mathcal{L}^k) \), and by \( H^0_{L^p}(M,\mathcal{L}^k) \) the space of \( L^p \)-holomorphic sections where \( p \geq 1 \). Then by Hodge theorem ([83]), the space \( H^0(M,\mathcal{L}^k) \) is finite-dimensional. Repeating the above construction with this metric, we obtain a function \( \mathcal{K}_{M,L^k} \) on \( M \).

When manifolds are compact Kaehler, then it is imperative to look into the geometric aspects of the Bergman kernel. To this end, we have

Theorem 1.6.9. Given a compact Kaehler manifold \( M \) with a positive hermitian holomorphic line bundle \( \mathcal{L} \). By the Kodaira embedding theorem, we have a map \( \Phi : M \rightarrow \mathbb{CP}^{N_k} = \mathbb{P}(H^0(M,\mathcal{L}^k)^*) \) such that
\[
\Phi : z \mapsto [s_0(z) : \cdots : s_{N_k}(z)]
\]

Hence, the Bergman kernel can be used in the following way

Lemma 1.6.4. [105] Let \( M \) be a compact Kaehler manifold with a positive hermitian holomorphic line bundle \( \mathcal{L}^k \). Then
\[
\Phi^* \omega_{FS} = \omega + i\partial\bar{\partial} \log \mathcal{K}_{M,L^k}
\]

Proof. On the subset of \( M \) where \( s_0 \neq 0 \), we have
\[
\Phi^* \omega_{FS} = i\partial\bar{\partial} \log \left( 1 + \frac{|s_1|^2}{s_0} + \cdots + \frac{|s_{N_k}|^2}{s_0} \right)
\]
\[
= i\partial\bar{\partial} \log \left( 1 + \frac{|s_1|^2}{|s_0|^2_h} + \cdots + \frac{|s_{N_k}|^2}{|s_0|^2_h} \right)
\]
\[
= i\partial\bar{\partial} \log (\mathcal{K}_{M,L^k}) - i\partial\bar{\partial} \log (|s_0|^2_h)
\]
\[
= i\partial\bar{\partial} \log (\mathcal{K}_{M,L^k}) + \omega
\]

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The same argument works on the open sets where \( s_i \neq 0 \) for each \( i \) which cover \( M \).

**Lemma 1.6.5.** [105] For large \( k \), an orthonormal basis of \( H^0(M, \mathcal{L}^k) \) gives a map \( \Phi_k : M \rightarrow \mathbb{CP}^{N_k} \), where \( N_k + 1 = \dim(H^0(M, \mathcal{L}^k)) \), and

\[
\frac{1}{k} \Phi_k^* \omega_{FS} - \omega = O(k^{-2}), \text{ in } C^\infty.
\]

According to the above, any Kaehler metric can be approximated by algebraic metrics obtained as pullbacks of Fubini-Study metrics under projective embeddings.

Let \( \Gamma \) be a group of biholomorphisms acting transitively on \( M \), so that its action lifts to an action on \( \mathcal{L} \), and assume that \( \mathcal{L} \) is generated by \( L^2 \)-holomorphic sections, and \( h \) is a \( \Gamma \)-invariant hermitian metric on \( \mathcal{L} \). Then \( K_{M,\mathcal{L}} \) is nondegenerate \( \Gamma \)-invariant hermitian metric on the dual of \( \mathcal{L} \), \( \mathcal{L}^* \). Therefore, \( K_{M,\mathcal{L}} \) is a constant multiple of \( \Gamma \)-invariant metric \( h^* \). Moreover, \( \mathcal{L}^k \) is also generated by \( L^2 \)-holomorphic sections (see lemma 5.14 [74]), and \( K_{M,\mathcal{L}^k} \) is then a constant multiple of \( (h^*)^k \). Since for connected \( M \), \( K_{M,\mathcal{L}}(z,z) \) and \( K_{M,\mathcal{L}^k}(z,w) \) determine each other

\[
K_{M,\mathcal{L}^k}(z,w) = c(k)K_{M,\mathcal{L}}^k(z,w)
\]

where \( c(k) \) is a constant (see Example 7.7 [74]).

### 1.7 Poincaré Series

Throughout this section, we study Poincaré series on a bounded domain \( D \) and go through some definitions and theorems related to a group action. The references relevant to this section are [65], [85] and [111].

**Definition 1.7.1.** The action of a group \( \Gamma \) on a topological space \( X \) is said to be **properly discontinuous** if, for every \( x \in X \), there exists a neighborhood \( U_x \) such that

\[
\{ \gamma \in \Gamma \mid \gamma U_x \cap U_x \neq \emptyset \}
\]

is finite.

**Definition 1.7.2.** The action of a group \( \Gamma \) on a topological space \( X \) is said to be **free** if any \( \gamma \in \Gamma \) except the identity has no fixed point.
Theorem 1.7.1. Let $G$ be a locally compact group acting transitively on a topological space $X$ such that for a point $x_0 \in X$, the stabilizer $\Gamma_{x_0}$ of $x_0$ in $G$ is compact and

$$\varphi : G/\Gamma_{x_0} \to X$$

$$g\Gamma_{x_0} \to gx_0$$

is a homeomorphism. Then the following conditions on a subgroup $\Gamma$ of $G$ are equivalent:

(a) $\Gamma$ acts properly discontinuously on $X$;

(b) For any compact subsets $A$ and $B$ of $X$, $\{ \gamma \in \Gamma \mid |\gamma(A) \cap B \neq \emptyset \}$ is finite;

(c) $\Gamma$ is a discrete subgroup of $G$.

Proof. The equivalence of (a) and (b) is straightforward. We will show (b) is equivalent to (c). Given compact sets $A$ and $B$ of $X$, let $C = \pi^{-1}(A)$ be the lift of $A$ to $G$ and similarly $D = \pi^{-1}(B)$. Then $\gamma(A) \cap B \neq \emptyset$ implies $\gamma(C) \cap D \neq \emptyset$. That is, $\gamma \in \Gamma \cap (DC^{-1})$. Now, we want to prove that $\pi^{-1}(A)$, and $D$ is compact, and hence $DC^{-1}$ is compact. Take an open cover of $G = \cup U_i$ whose closures $\bar{U}_i$ are compact. Then $A \subset \cap \pi(U_i)$ where the union runs over only finitely many $i$. Thus $\pi^{-1}(A) \subset \cup U_i \Gamma_{x_0} \subset \cup \bar{U}_i \Gamma_{x_0}$. As the image of $\bar{U}_i \times \Gamma_{x_0}$ under the multiplication map, each $\bar{U}_i \Gamma_{x_0}$ is compact. Thus $\pi^{-1}(A)$ is a closed subset of a compact set, so it is compact. Therefore, if $\Gamma$ discrete, $\Gamma \cap DC^{-1}$ is finite, hence (c) implies (b). Finally, to prove (b) implies (c), let $U$ be a compact neighborhood of the identity $e$ in $G$. Let $x = \pi(e)$. Then

$$\Gamma \cap U \subset \{ g \in \Gamma \mid gx \in \pi(U) \}$$

Take $A = \{ x \}$ and $B = \pi(U)$. Then, by assumption (b), $\Gamma \cap U$ is a finite set, so $\Gamma$ is discrete.

Proposition 1.7.1. Let $\Gamma$ be a discrete subgroup of $G$, with all the hypotheses of the previous result. Then:

(a) For any $x$ in $X$, $\{ \gamma \in \Gamma \mid \gamma x = x \}$ is finite.

(b) For any $x$ in $X$, there is a neighborhood $U_x$ of $x$ such that if $\gamma \in \Gamma$ with $U_x \cap \gamma U_x \neq \emptyset$, then $\gamma x = x$.  

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(c) For any $x$ and $y$ in $X$, not in the same $\Gamma$-orbit, there exist neighborhoods $U_x$ and $U_y$ of $x$ and $y$ such that $\gamma U_x \cap U_y = \emptyset$ for every $\gamma \in \Gamma$.

Proof. For part (a), $\{ \gamma \in \Gamma \mid \gamma x = x \} = \pi^{-1}(x) \cap \Gamma$ where again $\pi$ is the map $g \mapsto gx$. By the previous proposition, inverse images of compact sets under $\pi$ are again compact, so the intersection is compact and discrete, hence finite. To prove (b), let $U$ be a compact neighborhood of $x$. By the previous theorem (b), there is a finite set $\{ \gamma_1, \ldots, \gamma_n \}$ in $\Gamma$ such that $U \cap \gamma_i U$. Reindexing if necessary, let $\gamma_1, \ldots, \gamma_n$ be the subset of $\gamma_i$’s which fix $x$. For each $i > s$, choose disjoint neighborhoods $U_i$ of $x$ and $V_i$ of $\gamma_i x$ and let

$$W = U \cap \left( \bigcap_{i>s} U_i \cap \gamma^{-1} V_i \right).$$

Then $W$ has the required property, since for $i > s$, $\gamma_i W \subset V_i$ but $V_i$ is disjoint from $U_i$ which contains $W$. To prove (c), we again use the previous theorem (b). Choose compact neighborhoods $A$ of $x$ and $B$ of $y$ and let $\gamma_1, \ldots, \gamma_n$ be the finite set in $\Gamma$ such that $\gamma_i A \cap B \neq \emptyset$. Since $x$ and $y$ are assumed to be inequivalent under $\Gamma$, we can find disjoint neighborhoods $U_i$ of $\gamma_i x$ and $V_i$ of $y$. Then

$$U = A \cap \gamma_1^{-1} U_1 \cap \cdots \cap \gamma_n^{-1} U_n, \quad V = B \cap V_1 \cap \cdots \cap V_n$$

gives the required pair of neighborhoods. \hfill \square

Corollary 1.7.1. Let $\Gamma$ be a discrete subgroup of $G$, with all the hypotheses of the previous theorem. Then the space $\Gamma \backslash X$ is Hausdorff.

Definition 1.7.3. Let $D$ be a bounded domain in $\mathbb{C}^n$. The set $G$ of biholomorphic mappings from $D$ onto itself has a natural group structure. Let $\Gamma$ be a subgroup of $G$ acting properly discontinuously on $D$, From Theorem 1.7.1, $\Gamma$ is a countable set. Denote by $J_\gamma$ the complex Jacobian of $\gamma$. Then for $k \geq 2$,

$$\theta_k(f)(z) = \theta(f)(z) := \sum_{\gamma \in \Gamma} f(\gamma(z)) J_\gamma^k(z) \quad (1.44)$$

is called the Poincaré series of a bounded holomorphic function $f$.

Lemma 1.7.1. The Poincaré series converges absolutely and uniformly on compact sets of $D$ for $k \geq 2$. 

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The proof of this lemma follows from Prop. 1, p. 44 [9] also it can be
derived from Poincaré lemma (section 37 in [103]).

The above lemma implies that \( \theta(f) \) is holomorphic on \( D \) and

\[
\theta(\gamma f)(z)J^k(z) = \theta(f)
\]

(i.e., \( \theta(f) \) is an automorphic form of weight \( k \) with respect to \( \Gamma \) which has
been addressed in the next chapter) since \( J \) satisfies the cocycle condition

\[
J_{\gamma \gamma'}(z) = J_{\gamma}(\gamma' z)J_{\gamma'}(z)
\]

for for each \( \gamma, \gamma' \in \Gamma \).

**Remark 1.7.1.** • From Proposition 1.7.1, if \( \Gamma \)-action is properly discontinuous and free then \( X := D/\Gamma \) has a canonical structure of a complex
manifold induced from that of \( D \).

• \( \theta(f) \) can be seen as a holomorphic section of \( K_X^k \) over \( X \), where \( K_X \)
is the canonical line bundle of \( X \) (see section 2.2). Hence, a Poincaré
map \( \theta \) can be defined as

\[
H^0_{L^1}(D, K_D^k) \rightarrow H^0(X, K_X^k).
\]

**Theorem 1.7.2.** For \( k \) large enough, there exist bounded holomorphic func-
tions \( f_0, \ldots, f_N \) and a point \( z_0 \) in \( D \) such that \( \theta(f_0)(z_0) \neq 0 \) and

\[
\det \left( \frac{\partial(\theta(f_i)/\theta(f_0))}{\partial z_j} \right)_{1 \leq i,j \leq N}(z_0) \neq 0. \tag{1.45}
\]

Therefore,

\[
z \rightarrow [\theta(f_0)(z) : \cdots : \theta(f_N)(z)] \in \mathbb{CP}^N
\]
is non-degenerate at \( z_0 \).

The theorem was proved by Siegel (see section 40 in [103]). Then if \( X \) is
compact without boundary,

\[
\mathcal{A} := \{ \theta(f) : f \text{ is a bounded holomorphic function on } D \}
\]
is a finite-dimensional subspace of \( H^0(X, K_X^k) \), by the Hodge theorem [83].
Let \( \sigma_0, \ldots, \sigma_N \) be a basis of \( \mathcal{A} \), then if \( z \rightarrow [\sigma_0(z), \ldots, \sigma_N(z)] \) defines an
embedding of \( X \) into \( \mathbb{CP}^N \), \( K_X^k \) is very ample.
The surjectivity of Poincaré map In [[74], Theorem 7.12], for $K \geq 2$ and compact quotients of a bounded symmetric domain $D$, $\theta$ is proved to be surjective as long as the Bergman projection $\mathcal{K}$ on $L^2$ extends to $L^1$ and $L^\infty$ and is reproducing on $L^\infty$. Two conditions of the surjectivity of the map have been reviewed by Kollár [74]. Condition 1 is that the Bergman projection $\tilde{P}_k$ for $(D, \tilde{h})$ extends to bounded linear maps on $L^1(D, K_k^D)$ and $L^\infty(D, K_k^D)$. For the first condition, In [[74], Proposition 7.13], it is sufficient that $\mathcal{K}(\cdot, w) \in L^1(D)$ with $\mathcal{K}(\cdot, w)_{L^1} \leq C$ for a uniform constant $C$ independent of $w$. Condition 2 is that $\tilde{P}_k$ is a reproducing on holomorphic sections. The surjectivity of the Poincaré map means $A = H^0(X, K_X^k)$ and implies ampleness of $K_X^k$. In [45] T.Foth defined the $\mathbb{C}$-valued Poincaré series in terms of Bergman kernel, as we will see in the next chapter, and the general definition of the vector-valued ones has been provided in the last chapter (see [100]).
Chapter 2

Automorphic Forms on Bounded Symmetric Domains

To study all holomorphic functions on all Riemannian manifolds, it is a lot more convenient to first embed a manifold into a known manifold and then study its geometry viz-a-viz that of the ambient manifold. In this chapter, we focus on a quotient space of an irreducible bounded symmetric domain $D$ by a discrete subgroup $\Gamma$ of $\text{Aut}(D)$, the group of all biholomorphic $\gamma : D \to D$, acting properly discontinuously on $D$. The aim of this chapter is to introduce a certain class of functions on $D$ known as automorphic forms. An automorphic form is a holomorphic (or, meromorphic) function that transforms nearly invariantly under the $\Gamma$-action. It is a generalization of a certain class of periodic functions. Theta function is an example of "automorphic form" on $\mathbb{C}^n$ that has been used in studying the problem of representing integers using binary quadratic forms which reflects the applications of the automorphic forms to number theory. The importance of the theory of automorphic forms is much wider, ranging through the fields of geometry, analysis and algebra as well (see [102], [40], [9], [22] and [100]). To construct an automorphic form for subgroups of $\Gamma$, a method of Poincaré series has been usually used, as in [85], which is similar to the standard way of constructing invariant functions. We will adapt the same technique and we will prove that Poincaré series span the space of automorphic forms in a certain way. Section 2.1 of this chapter is dedicated to give a base and a good understanding to the work done in the last chapter. The first section of this chapter gives the basic definitions, some theorems and how to construct the automorphic forms on the complex upper half plane $\mathcal{H}$. It introduces quotients of compactification of $\mathcal{H}$ as nat-
ural domains of automorphic forms. The second section generalizes the case in section 1 to quotient spaces of $D$ which are Riemannian manifolds with induced complex structure.

2.1 Automorphic Forms on the Upper Half Plane

In this section, we look over the geometry of the upper half plane $\mathcal{H}$. For a fixed choice of a discrete subgroup $\Gamma$, we study the quotient $\Gamma \backslash \mathcal{H}$ as a Riemann surface and then formulate a definition for automorphic forms for $\Gamma$. A Poincaré series is a widely used construction. If $\Gamma \backslash \mathcal{H}$ is compact and smooth, then holomorphic automorphic forms are Poincaré series of polynomials in $z$ [[57],[93]]. In this subsection we will discuss in more detail a classical construction of Poincaré series for the modular group $PSL(2, \mathbb{Z})$. We have included the results of [40], [56], [65], [85] and [87] in this section.

2.1.1 Linear fractional transformations

Let $GL(2, \mathbb{C})$ be the group of all $2 \times 2$ invertible matrices with complex entries. A linear fractional transformation on $\mathbb{C} \cup \{\infty\}$ is defined as follows. For any pair

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}), \quad z \in \mathbb{C} \cup \{\infty\},$$

we define

$$\gamma(z) = \frac{az + b}{cz + d} \quad \text{where} \quad \gamma(\infty) = \frac{a}{c}$$

From the theory of the Jordan canonical form, each matrix $\gamma$ (not scalar) is conjugate to one of the following forms:

$$\gamma = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \quad \text{or} \quad \gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \text{with} \quad a \neq b$$

according to whether it has repeated eigenvalues or distinct eigenvalues. Hence, each transformation is essentially one of:

$$z \mapsto z + a^{-1} \quad \text{or} \quad z \mapsto \lambda z, \lambda \neq 1.$$
Matrices $\gamma$ conjugate to the first of these cases are called **parabolic**. These matrices act by translation, and their lone fixed point is $\infty$. Those in the second class are divided into three groups. If $|\lambda| = 1$, they are called **elliptic**. If $\lambda$ is real and positive, they are called **hyperbolic**. All other matrices are known as **loxodromic**. These elements conjugate to the second class all have two fixed points. If we specialize to matrices with $\det(\gamma) = 1$, then the Jordan form of $\gamma$ is either

$$\gamma = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \text{ or } \gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \text{ with } a \neq \pm 1$$

The group of all matrices of $GL(2, \mathbb{C})$ with determinant 1 is denoted by $SL(2, \mathbb{C})$. Now, we may restrict our focus to transformations with real matrices, so if $\gamma \in GL(2, \mathbb{R})$, set

$$j(\gamma, z) = cz + d \text{ for } z \in \mathbb{C}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then one may check that

$$\det(\gamma)\text{Im}(z) = |j(\gamma, z)|^2\text{Im}(\gamma(z))$$

so that restricting $\gamma$ to $GL^+(2, \mathbb{R})$, invertible matrices with positive determinant, we send $\mathcal{H}$ to itself. Since scalar matrices induce the identity map, we may restrict our attention to $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$.

**Lemma 2.1.1.** The group $SL(2, \mathbb{R})$ acts transitively on $\mathcal{H}$. The isotropy group of the point $i$ is the subgroup $SO(2, \mathbb{R})$ of rotations.

Hence the isotropy group of any element $z \in \mathcal{H}$ is the set

$$\gamma SO(2, \mathbb{R})\gamma^{-1}, \text{ where } \gamma \in SL(2, \mathbb{R}) \text{ such that } \gamma(i) = z.$$ 

Therefore, an element of $SL(2, \mathbb{R})$ with at least one fixed point in $\mathcal{H}$ is either $\pm I$ or elliptic. The group $SL(2, \mathbb{R})$ also acts transitively on $\mathbb{R} \cup \{\infty\}$. Moreover, the isotropy subgroup of $\infty$ is

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \bigg| a \in \mathbb{R}^*, b \in \mathbb{R} \right\}$$

and the subset of all parabolic elements in this isotropy subgroup are those with $a = \pm 1$ and $b \neq 0$. Thus, any element $\gamma \neq \pm I$ in $SL(2, \mathbb{R})$ having at least one fixed point in $\mathbb{R} \cup \{\infty\}$ is either parabolic or hyperbolic.
Proposition 2.1.1. Let $\gamma \in SL(2, \mathbb{R})$ such that $\gamma \neq \pm I$. Then

(a) $\gamma$ is parabolic iff it has one fixed point on $\mathbb{R} \cup \{\infty\}$.

(b) $\gamma$ is elliptic iff $\gamma$ has one fixed point $z$ in $\mathcal{H}$ and the other fixed point is $\bar{z}$.

(c) $\gamma$ is hyperbolic iff $\gamma$ has two fixed points on $\mathbb{R} \cup \{\infty\}$.

Lemma 2.1.2. The Cayley transform

$$z \mapsto cz = \begin{pmatrix} 1 & -i \\ i & z + i \end{pmatrix} z = \frac{z - i}{z + i}.$$ 

maps isomorphically $\mathcal{H}$ onto the unit disk $D = \{z \in \mathbb{C} | |z| < 1\}$.

2.1.2 Fuchsian groups

As we are mainly interested in the quotient of $\mathcal{H}$ by a discrete subgroup of $SL(2, \mathbb{R})$, the most important examples of discrete subgroups are the full modular group $SL(2, \mathbb{Z})$ and its subgroups.

Definition 2.1.1. For a fixed discrete group $\Gamma$. The point $x \in \mathbb{R} \cup \{\infty\}$ such that $\gamma(x) = x$ for some parabolic element $\gamma \in \Gamma$ will be called a cusp of $\Gamma$. The point $z \in \mathcal{H}$ such that $\gamma(z) = z$ for an elliptic element $\gamma$ of $\Gamma$ will be called an elliptic point of $\Gamma$.

Proposition 2.1.2. If $z$ is an elliptic point of $\Gamma$ then $\{\gamma \in \Gamma | \gamma(z) = z\}$ is a finite cyclic group.

Proposition 2.1.3. The group $SL(2, \mathbb{Z})$ is generated by the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The following propositions determine the parabolic and elliptic elements of $SL(2, \mathbb{Z})$.

Proposition 2.1.4. The cusps of $SL(2, \mathbb{Z})$ are the points of $\mathbb{Q} \cup \{\infty\}$, and they all lie in a single $SL(2, \mathbb{Z})$-orbit.

Proposition 2.1.5. The elliptic points of $SL(2, \mathbb{Z})$ are all $SL(2, \mathbb{Z})$-equivalent to either $z = i$ or $z = e^{2\pi i/3} = (1 + i\sqrt{3})/2$. 

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Definition 2.1.2. ([85], [87]) Discrete subgroups of $SL(2, \mathbb{R})$ are known as Fuchsian groups.

Proposition 2.1.6. A Fuchsian group $\Gamma$ acts properly discontinuously on the upper half-plane $\mathcal{H}$.

Proof. We need to prove that for every compact subsets $U, V \subset \mathcal{H}$, the set $\{ \gamma \in \Gamma | \gamma U \cap V \neq \emptyset \}$ is finite. Because the group $G = SL(2, \mathbb{R})$ acts on $\mathcal{H}$ with compact stabilizer, the subset $\{ \gamma \in G | \gamma U \cap V \neq \emptyset \}$ is compact. Its intersection with the discrete subgroup $\Gamma$ is finite. \( \square \)

Corollary 2.1.1. For every Fuchsian group $\Gamma$, the quotient $\Gamma \backslash \mathcal{H}$ is a Hausdorff topological space.

Proof. First since the action of $SL(2, \mathbb{Z})$ on $\mathbb{C} \cup \{ \infty \}$, the quotient space is defined and we may regard $\Gamma \backslash \mathcal{H}$ as a subspace and then from Proposition 1.7.1 (c) the proof is complete. \( \square \)

Note that for a Fuchsian group $\Gamma$, the orbits for any point $z \in \mathcal{H}$ has no limit point in $\mathcal{H}$. However, it could have a limit point on the boundary $\mathbb{R} \cup \{ \infty \}$.

Definition 2.1.3. [92] A Fuchsian group $\Gamma$ is said to be of the first kind if every point of $\partial \mathcal{H} = \mathbb{R} \cup \{ \infty \}$ is a limit point of $\Gamma$.

Lemma 2.1.3. Assume that $\infty$ is a cusp of $\Gamma$. Then the stabilizer of $\infty$ is of the form

$$\Gamma_{\infty} = \left\{ \pm \left( \begin{array}{cc} 1 & h \\ 0 & 1 \end{array} \right)^m | m \in \mathbb{Z} \right\}, \quad \text{(for some } h > 0)$$

Let $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$. If $|ch| < 1$, then $c = 0$.

$\Gamma \backslash \mathcal{H}$ as a topological space. Let $\Gamma$ be a Fuchsian group of the first kind and endow $\Gamma \backslash \mathcal{H}$ with the quotient topology. Let $p : \mathcal{H} \longrightarrow \Gamma \backslash \mathcal{H}$ be the quotient map. There is a unique complex structure on $\Gamma \backslash \mathcal{H}$ such that a function $f$ on an open subset $U$ of $\Gamma \backslash \mathcal{H}$ is holomorphic if and only if $f \circ p$ is holomorphic on $p^{-1}(U)$. Thus $f \longmapsto f \circ p$ defines a one-to-one correspondence between holomorphic functions on $U \subset \Gamma \backslash \mathcal{H}$ and holomorphic functions on $p^{-1}(U)$ invariant under $\Gamma$, i.e., such that $g(\gamma z) = g(z)$ for all $\gamma \in \Gamma$. 

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For any group $\Gamma$ that has cusps, the quotient $\Gamma \backslash \mathcal{H}$ is not compact. But if $\Gamma$ is a Fuchsian group of the first kind, we may consider $X = \Gamma \backslash \mathcal{H}^*$, where $\mathcal{H}^*$ is the union of $\mathcal{H}$ with the set of cusps for $\Gamma$. For example, $SL(2, \mathbb{Z}) \backslash \mathcal{H}^*$ is compactified by adding a single point. Hence, the topology on $\mathcal{H}^*$ can be defined as follows:

- For $z \in \mathcal{H}$, the fundamental system of open neighborhoods for $z \in \mathcal{H}^*$ is just that for $z \in \mathcal{H}$.
- For $z$ as a cusp, the fundamental system of open neighborhoods at $z$ is the family

$$\{\gamma^{-1}U_l \mid l > 0\} \text{ where } U_l = \{z \in \mathcal{H} \mid \Im(z) > l\}, \text{ and } \gamma(z) = \infty.$$

We call a point in the quotient space $\Gamma \backslash \mathcal{H}^*$ an elliptic point (resp. a cusp) if its preimage in $\mathcal{H}^*$ under the canonical projection is an elliptic point (resp. a cusp).

$\Gamma \backslash \mathcal{H}$ as a Riemann surface. To define the complex structure on $\Gamma \backslash \mathcal{H}^*$, for any point $z \in \mathcal{H}^*$, there is an open neighborhood $U$ such that $p(z) \in U$.

$$\Gamma_z = \{\gamma \in \Gamma \mid \gamma(U) \cap U \neq \emptyset\},$$

where $\Gamma_z$ is a stabilizer of $z$. Then there is a natural injection of $\Gamma_z \backslash U \to \Gamma \backslash \mathcal{H}^*$, with $\Gamma_z \backslash U$ an open neighborhood of $p(z)$, the image of $z$ under the canonical projection to $\Gamma \backslash \mathcal{H}^*$. If $z$ is neither an elliptic point nor a cusp, then $\Gamma_z \subset \{\pm I\}$ so that the map $p : U \to \Gamma_z \backslash U$ is a homeomorphism. Then we may take the pair $(\Gamma_z \backslash U, p^{-1})$ as part of the complex structure.

If $z$ is an elliptic point of $\mathcal{H}^*$, then let $\Gamma_z = \Gamma_z / (\Gamma \cap \{\pm I\})$. Let $\alpha$ be a holomorphic isomorphism of $\mathcal{H}$ onto the unit disc $D$ with $\alpha(z) = 0$. Recall that $\Gamma_z$ is cyclic, say of order $n$. By Schwarz’ lemma, $\alpha \Gamma_z \alpha^{-1}$ consists of the transformations

$$D \to D : w \to e^{2\pi i k/n}w, \quad k \in [0, n - 1].$$

Then we can define the chart $\phi : \Gamma_z \backslash U \to \mathbb{C}$ by $\phi(p(z)) = \alpha^n(z)$. The resulting $\phi$ is a homeomorphism onto an open subset of $\mathbb{C}$.

The complex structure on $SL(2, \mathbb{Z}) \backslash \mathcal{H}^*$. We first define the complex structure on $SL(2, \mathbb{Z}) \backslash \mathcal{H}$. Write $p$ for the quotient map $\mathcal{H} \to SL(2, \mathbb{Z}) \backslash \mathcal{H}$. Let $z$ be a point of $\mathcal{H}$ mapping to $w \in SL(2, \mathbb{Z}) \backslash \mathcal{H}$.
If $z$ is not an elliptic point, we can choose a neighbourhood $U_z$ of $z$ such that $p$ is a homeomorphism $U_z \to p(U_z)$. We define $(p(U_z), p^{-1})$ to be a coordinate neighbourhood of $w$.

If $z$ is equivalent to $i$, we may assume it to equal $i$. The map $z \mapsto \frac{z-i}{z+i}$ an isomorphism of an open disk $D(i)$ with centre 0 onto an open disk $D(0)$ with centre 0, and since the matrix $S$ fixes $i$ and has order 2, the action of $S$ on $D(i)$ is transformed into the automorphism $\sigma : z \mapsto -z$ of $D(0)$. Thus, $<S> \setminus D(i)$ is homeomorphic to $<\sigma> \setminus D(0)$, and we give $<S> \setminus D(i)$ the complex structure making this a bi-holomorphic isomorphism, i.e., $\frac{z-i}{z+i}$ is a holomorphic function defined in a neighbourhood of $i$, and $S$ maps $i$ to

$$\frac{-z^{-1} - i}{-z^{-1} + i} = -1 - iz = \frac{z - i}{z + i}$$

Hence, $z \mapsto (\frac{z-i}{z+i})^2$ is a holomorphic function defined in a neighbourhood of $i$ which is invariant under the action of $S$. It therefore defines a holomorphic function in a neighbourhood of $p(i)$, and we take this to the coordinate function near $p(i)$.

The case the point $z = e^{2\pi i/3}$ can be treated similarly. Apply a linear fractional transformation that maps $z$ to zero, and then take the cube of the map since it is fixed by $ST$, which has order 3.

The Riemann surface $SL(2, \mathbb{Z}) \setminus \mathcal{H}$ is not compact. The simplest way to compactify it is to add a point $\infty$ to $\mathcal{H}$. First we define the topology on $\mathcal{H}^*$ as follows: a fundamental system of neighbourhoods of a point in $\mathcal{H}$ is as before and a fundamental system of neighbourhoods for $\infty$ is formed of sets of the form $U = \{z \in \mathbb{C} | \text{Im}(z) > N\}$.

Consider the quotient space $SL(2, \mathbb{Z}) \setminus \mathcal{H}^*$. The function

$$q(z) = \begin{cases} e^{2\pi i z}, & z \neq \infty, \\ 0, & z = \infty \end{cases}$$

is a homeomorphism from $SL(2, \mathbb{Z}) \setminus \mathcal{H}^*$ onto the open disk $D(0)$ with centre 0. The function $q$ is invariant under the action of the stabilizer $<T>$ of $\infty$, and so defines a holomorphic function $q : <T> \setminus U \to D$, which can be taken to be the coordinate function near $p(\infty)$.

**The complex structure on $\Gamma \setminus \mathcal{H}^*$**. Let $\Gamma \subset SL(2, \mathbb{Z})$ of finite index. Then $\Gamma$ is a Fuchsian group of the first kind. The complement of $\Gamma \setminus \mathcal{H}$ in $\Gamma \setminus \mathcal{H}^*$ is the set of equivalence classes of cusps for $\Gamma$. First $\Gamma \setminus \mathcal{H}$ is given a
complex structure in exactly the same way as in the case \( \Gamma = SL(2, \mathbb{Z}) \). \( \Gamma \) must contain \( T^h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \) for some \( h \), and \( T^h \) is a parabolic element fixing \( \infty \), so the point \( \infty \) will always be a cusp. If \( h \) is the smallest power of \( T \) in \( \Gamma \), then the function

\[
q(z) = \begin{cases} 
  e^{2\pi i z/h}, & z \neq \infty, \\
  0, & z = \infty
\end{cases}
\]

is a homeomorphism from \( \Gamma \backslash \mathcal{H}^* \) onto the open disk \( V \) with centre 0 and radius \( e^{-2\pi c/h} \) for some \( c \). It therefore \( q = e^{2\pi i z/h} \) is a coordinate function near \( \infty \). For any other cusp \( \tau \neq \infty \), we know there is an element \( \gamma \in \Gamma(1) \) such that \( \tau = \gamma(\infty) \), and then \( z \rightarrow q(\gamma^{-1}(z)) \) is a coordinate function near \( \gamma(\infty) \).

**Theorem 2.1.1.** [87] Let \( \Gamma \) be a Fuchsian group. The quotient space \( \Gamma \backslash \mathcal{H}^* \) is Hausdorff.

**Corollary 2.1.2.** If \( \Gamma \backslash \mathcal{H}^* \) is compact, then the number of elliptic points and cusps of \( \Gamma \backslash \mathcal{H}^* \) is finite.

The following lemma gives an alternative definition of those groups.

**Lemma 2.1.4.** [30] A discrete subgroup \( \Gamma \) of \( SL(2, \mathbb{R}) \) is a Fuchsian group of first kind if \( \Gamma \backslash \mathcal{H}^* \) is compact.

**Theorem 2.1.2 (Siegel).** A discrete subgroup of \( SL(2, \mathbb{R}) \) is a Fuchsian group of first kind if and only if \( \Gamma \backslash \mathcal{H}^* \) has finite area.

**Definition 2.1.4.** A fundamental domain for \( \Gamma \) is a connected open subset \( \mathcal{F} \) of \( \mathcal{H} \) such that no two points of \( \mathcal{F} \) are equivalent under \( \Gamma \) and \( \mathcal{H} = \bigcup_{\gamma \in \Gamma} (\gamma \mathcal{F}) \) where \( \mathcal{F} \) denotes the closure of \( \mathcal{F} \).

We define the distance \( \tau(z, w) \) between any two points \( z, w \) in \( \mathcal{H} \) to be the infimum of all lengths of curves between \( z \) and \( w \) using the hyperbolic metric. Here we replace the usual Euclidean metric with

\[
d s = \frac{1}{y} \sqrt{dx^2 + dy^2}.
\]

One can show that the distance function \( \tau \) may be explicitly given by

\[
\tau(z, w) = \ln \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}.
\]
To any point $w$ in $\mathcal{H}$, we define the Dirichlet region for $\Gamma$ centered at $w$:

$$\mathcal{F}_w(\Gamma) = \{ z \in \mathcal{H} | \tau(z, w) \leq \tau(z, \gamma(w)) \quad \text{for all} \quad \gamma \in \Gamma \}.$$ 

**Definition 2.1.5.** If $w$ is not a fixed point of $\Gamma - \{I\}$, then $\mathcal{F}_w$ is called a Dirichlet fundamental domain for $\Gamma$.

**Proposition 2.1.7.** All fundamental domains $\mathcal{F}_w$ for $\Gamma$ of $\mathcal{H}$ have the same positive (but possibly infinite) volume

$$\int_{\mathcal{F}_w} d\mu$$

where $\mu$ is the Haar measure on $\mathcal{H}$ and $w$ is not a fixed point of $\Gamma - \{I\}$.

**Theorem 2.1.3.** Let $\Gamma$ be a Fuchsian group of the first kind. Then we have the following:

(a) Any Dirichlet region $\mathcal{F}_w$ which is a fundamental domain is a (hyperbolic) polygon with an even number of sides (where, if a side contains an elliptic point of order 2, we consider this as two sides).

(b) The sides of $\mathcal{F}_w$ can be arranged in pairs of equivalent sides. The elements $\gamma \in \Gamma$ which take one side to its pair generate $\Gamma$.

(c) Every fundamental domain has finite volume.

(d) $\Gamma$ is co-compact in $\mathcal{H}$ if and only if it contains no parabolic elements.

**Example 2.1.1.** The set of points

$$\mathcal{F} = \{ z \in \mathcal{H} | |z| > 1, -1/2 < Re(z) < 1/2 \}$$

is a fundamental domain for $SL(2, \mathbb{Z})$ that satisfies the conditions of the previous Theorem. It follows from the shape of this fundamental domain that $SL(2, \mathbb{Z}) \backslash \mathcal{H}^*$ is compact Riemann surface that is homeomorphic to the sphere.
2.1.3 Automorphic Forms for Fuchsian Groups

Definition 2.1.6. Let $\Gamma$ be a Fuchsian group of first kind. A modular function $f$ is a meromorphic function on the compact Riemann surface $\Gamma \backslash \mathcal{H}^*$. We often regard it as a meromorphic function on $\mathcal{H}^*$ invariant under $\Gamma$. Hence, from this point of view, a modular function $f$ for $\Gamma$ is a function on $\mathcal{H}$ satisfying the following conditions:

(i) $f$ is invariant under $\Gamma$, i.e.,

$$f(\gamma z) = f(z) \quad \text{for all } \gamma \in \Gamma;$$

(ii) $f$ meromorphic on $\mathcal{H}$;

(iii) $f$ meromorphic at the cusps.

For the cusp $\infty$, the last condition means the following: the stabilizer $\Gamma_{\infty}$ is generated by $T^h$ for some $h > 0$. As $f$ is invariant under $T^h$, $f(z + h) = f(z)$, the function $f$ can be expressed as a function $g(q)$ such that $q = e^{2\pi i z/h}$, defined on a punctured disk. Therefore, for $f$ to be meromorphic at $\infty$ means $g$ is meromorphic at $q = 0$.

For a cusp $\tau \neq \infty$, the last condition means the following: there is an element $\sigma \in \Gamma$ such that $\tau = \sigma(\infty)$ and then the function $z \mapsto f(\sigma z)$ is invariant under $\sigma \Gamma \sigma^{-1}$ which requires $f(\sigma z)$ to be meromorphic at $\infty$ in the above sense. The last condition only has to be checked for a set of representatives of the $\Gamma$-equivalence classes of cusps (which would be finite).

In the case of the full modular group $SL(2, \mathbb{Z})$, we have only one cusp. To be invariant under the full modular group means:

$$f\left(\frac{az + b}{cz + d}\right) = f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Since $T \in SL(2, \mathbb{Z})$, we have that $f(z + 1) = f(z)$. Any function satisfying this condition can be written in the form $f(z) = g(q)$, $q = e^{2\pi i z}$. As $z$ ranges over $\mathcal{H}$, $q(z)$ ranges over a punctured disk centred at 0 in $\mathbb{C}$. Therefore, the last condition means $g(q)$ is meromorphic on the whole disk, hence that $f$ has a Fourier expansion

$$f(z) = \sum_{n \geq -N_0} a_n q^n.$$
To construct a modular function, we have to construct a meromorphic function on $\mathcal{H}$ that is invariant under the action of $\Gamma$ and this is difficult. Therefore, it is easier to construct functions that transform in a certain way under the action of $\Gamma$ and the quotient of two such functions of same type will then be a modular function.

**Definition 2.1.7.** A modular form for a Fuchsian group of first kind $\Gamma$ and weight $k$ is a function $f$ on $\mathcal{H}$ such that

(i) $f(\gamma z) = (cz + d)^{2k} f(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H};$

(ii) $f$ holomorphic on $\mathcal{H}$

(iii) $f$ holomorphic at the cusps.

For the full modular group $SL(2, \mathbb{Z})$, note that (i) again implies that $f(z + 1) = f(z)$, so $f$ can be written as a function of $q = e^{2\pi i z}$, and condition (iii) then says that this function is holomorphic at 0, so that

$$f(z) = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i z}.$$

Note that some authors refer to a function satisfying only the first condition of the previous definition as being weakly modular of weight $k$, and a function satisfying all conditions with holomorphic replaced by meromorphic as being a meromorphic modular form of weight $k$ or classical automorphic forms. Consequently, an automorphic form of weight 0 is a modular function.

**Definition 2.1.8.** A modular form is a cusp form if it is zero at the cusps.

After conjugation, we can assume that the cusp is the point $\infty$.

**Lemma 2.1.5.** Suppose that $\infty$ is a cusp of a Fuchsian group $\Gamma$ of first kind with $\Gamma_\infty$, the stabilizer of $\infty$. Let $f$ be a modular form of weight $k$ and let $\sum a_n q_n$ with $q = e^{2\pi i z/h}$ denote its Taylor expansion near this cusp. Then the series $\sum a_n q_n$ converges absolutely and uniformly on every compact set in $\mathcal{H}$.

**Proof.** The function $z \to q = e^{2\pi i z/h}$ defines an isomorphism between $\Gamma_\infty \setminus \mathcal{H}$ and the punctured disc $D - \{0\}$, where $\Gamma_\infty$ is the stabilizer of $\infty$. By assumption modular form $f$ defines a holomorphic function on $D - \{0\}$ that extends holomorphically to $D$. Therefore, the Taylor series $\sum_{n=0}^{\infty} a_n q_n$ converges absolutely uniformly on every compact contained in $D$, (and $D$ is isomorphic to $\mathcal{H}$).
The forms of weight $k$ have interpretations in terms of $(dz)^k$-forms on $\Gamma \backslash \mathcal{H}^*$. A $k$-fold differential $\omega$ can be locally written in the form $\omega = f(z)(dz)^k$ on $\mathcal{H}$, where $f(z)$ is a meromorphic function. Now, we want to know under what conditions on $f$ the $k$-differential, $\omega$, is invariant under the action of $\Gamma$. Let $\gamma(z) = \frac{az + b}{cz + d}$, then
\[ \gamma^* \omega = \omega(\gamma(z)) = f(\gamma(z))(d(\gamma z))^k = f(\gamma z)(cz + d)^{-2k}(dz)^k \]
which gives the following result

**Remark 2.1.1.** A $k$-fold differential $\omega$ is invariant under the action of $\Gamma$ if and only if $f(z)$ is a meromorphic differential of weight $k$. We have one-to-one correspondences between the following sets:

- \{automorphic forms of weight $k$ on $\mathcal{H}$ for $\Gamma$\}
- \{meromorphic $k$-fold differentials on $\mathcal{H}^*$ invariant under the action of $\Gamma$\}
- \{meromorphic $k$-fold differentials on $\Gamma \backslash \mathcal{H}^*$\}.

**Definition 2.1.9.** An automorphy factor is a map
\[ j : \Gamma \times \mathcal{H} \rightarrow \mathbb{C}^* \]
such that

- (i) for each $\gamma \in \Gamma$, $\tau \mapsto j_\gamma(\tau)$ is a holomorphic function on $\mathcal{H}$;
- (ii) it satisfies the cocycle condition

\[ j_{\gamma \gamma'}(\tau) = j_\gamma(\gamma' \tau)j_\gamma(\tau) \]

for each $\gamma, \gamma' \in \Gamma$.

Note that if $j$ is an automorphy factor, so is $j^k$ for any integer $k$. One can generalize the definition of the automorphy factor into a map
\[ j : \Gamma \times D \rightarrow GL(m, \mathbb{C}) \]
with axioms (i) and (ii), where $D$ is an irreducible bounded symmetric domain in $\mathbb{C}^n$ and $m \geq 1$. 

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Example 2.1.2. Let $H$ be an open subset of $\mathbb{C}$ with a group $\Gamma'$ acting on it. Then, each $\gamma$ defines a map $H \rightarrow H$, and $(d\gamma)\tau$ is the map on the tangent space at $\tau$ defined by $\gamma$. As $H \subset \mathbb{C}$, the tangent spaces at $\tau$ and at $\gamma \tau$ are canonically isomorphic to $\mathbb{C}$, and so $(d\gamma)\tau$ can be regarded as a complex number. Hence, there is canonical automorphy factor $j_\gamma(\tau)$, namely,

$$
\Gamma \times H \rightarrow \mathbb{C}, \quad (\gamma, \tau) \mapsto (d\gamma)\tau.
$$

Suppose we have maps

$$
M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3
$$

of complex manifolds, then for any point $m_1 \in M_1$,

$$(d(\beta \circ \alpha))_{m_1} = (d\beta)_{\alpha(m_2)} \circ (d\alpha)_{m_1}.$$

Therefore, $j_\gamma(\tau) \overset{\text{def}}{=} (d\gamma)_\tau$ is an automorphy factor since

$$
j_{\gamma'\gamma}(\tau) = (d\gamma'\gamma')_\tau = (d\gamma)_{\gamma'\gamma}(d\gamma')_\tau = j_\gamma(\gamma'\tau)j_{\gamma'}(\tau).
$$

In case $\Gamma' = \Gamma(1)$ and $H = \mathcal{H}$, if $z \xrightarrow{\gamma} \frac{az+b}{cz+d}$, then

$$
d(\gamma z) = \frac{1}{(cz+d)^2}dz,
$$

and so $j_\gamma(\tau) = J_\gamma(\tau) = (cz+d)^{-2}$, and $J_\gamma(\tau)^k = (cz+d)^{-2k}$.

Now let $\Gamma'$ be a group acting freely and properly discontinuously on a Riemann surface $H$, and $X = \Gamma'\backslash H$. Write $p$ for the quotient map $H \rightarrow X$. Let $\pi : \mathcal{L} \rightarrow X$ be a line bundle on $X$; then

$$
p^*(\mathcal{L}) \overset{\text{def}}{=} \{(h, l) \subset H \times \mathcal{L} \mid p(h) = \pi(l)\}
$$

is a line bundle on $H$, and $\Gamma'$ acts on $p^*(\mathcal{L})$ through its action on $H$.

Theorem 2.1.4. There is a one-to-one correspondence between the set of pairs $(\mathcal{L}, i)$, where $\mathcal{L}$ is a line bundle on $\Gamma'\backslash H$ and $i$ is an isomorphism $H \times \mathbb{C} \approx p^*(\mathcal{L})$, and the set of automorphy factors.
Proof. Suppose we are given an isomorphism $i : H \times \mathbb{C} \longrightarrow p^*(\mathcal{L})$. Then the action of $\Gamma'$ on $p^*(\mathcal{L})$ can be transferred to an action of $\Gamma'$ on $H \times \mathbb{C}$ over $H$. For $\gamma \in \Gamma'$ and $(\tau, z) \in H \times \mathbb{C}$, write

$$\gamma(\tau, z) = (\gamma \tau, j_\gamma(\tau)z), \quad j_\gamma(\tau) \in \mathbb{C}^*.$$

Then

$$\gamma' \gamma (\tau, z) = \gamma' \gamma't(\tau, j_\gamma(\tau)z) = (\gamma' \gamma t, j_\gamma(\gamma' t), j_{\gamma'}(\tau)z).$$

Hence,

$$j_{\gamma' \gamma}(\tau) = j_\gamma(\gamma' \tau)j_{\gamma'}(\tau)$$

This is for the first direction $(\mathcal{L}, i) \longrightarrow j_\gamma(\tau)$. For the converse, use $i$ and $j$ to define an action of $\Gamma'$ on $H \times \mathbb{C}$, and define $\mathcal{L}$ to be $\Gamma' \backslash H \times \mathbb{C}$. 

Note that in case of the upper half plane, every line bundle on $\mathcal{H}$ is trivial, $p^*(\mathcal{L}) \approx \mathcal{H} \times \mathbb{C}$, and so the previous theorem gives us a classification of the line bundles on $\Gamma \backslash \mathcal{H}$.

Modular forms as sections of line bundles. Let $H$, $X$, $\mathcal{L}$, $i$ and $j$ be as in the above argument and let $\Gamma'((H, p^*(\mathcal{L}))$ be the space of sections of $p^*(\mathcal{L})$ over $H$. Then a holomorphic section $F : H \longrightarrow H \times \mathbb{C}$ can be written $F(\tau) = (\tau, f(\tau))$ with $f(\tau)$ a holomorphic map on $H$. Now, we define the group $\Gamma'(X, \mathcal{L})$ of sections of $\mathcal{L}$ over $X$ as

$$\Gamma'(X, \mathcal{L}) = \{ F \in \Gamma'(H, p^*(\mathcal{L})) | \text{F commutes with the action of } \Gamma' \}.$$

For $F$ to commute with the action of $\Gamma'$ means

$$F(\gamma \tau) = \gamma F(\tau)$$

i.e.,

$$(\gamma \tau, f(\gamma \tau)) = (\gamma \tau, j_\gamma(\tau)f(\tau)).$$

Hence

$$f(\gamma \tau) = j_\gamma(\tau)f(\tau).$$

Therefore, if $\mathcal{L}^k$ is the line bundle on $\Gamma \backslash \mathcal{H}$ corresponding to $j_\gamma(\tau)^{-k}$, where $j_\gamma(\tau)$ is the canonical automorphy factor in Example 2.1.2, then the above condition becomes

$$f(\gamma \tau) = (cz + d)^{2k} f(\tau),$$

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which guarantees the invariance under the action of $\Gamma$. The sections of $\mathcal{L}^k$ are then in natural one-to-one correspondence with the functions on $\mathcal{H}$ satisfying the first two conditions in the definition of the modular form for $\Gamma$ on $\mathcal{H}$. Since the line bundle $\mathcal{L}^k$ extends to a line bundle $\overline{\mathcal{L}}^k$ on the compactification $\Gamma \backslash \mathcal{H}^*$, the sections of $\overline{\mathcal{L}}^k$ are in natural one-to-one correspondence with the modular forms of weight $k$.

**Poincaré series for Fuchsian groups of the first kind**

Let $\Gamma$ be a Fuchsian group of the first kind. Note that the standard way of constructing invariant functions is as follows:

If $g$ is a function on $\mathcal{H}$, then

$$f(z) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma} g(\gamma z)$$

is invariant under $\Gamma$-action, provided the series converges absolutely.

Poincaré series follows a similar argument for constructing modular forms for $\Gamma$.

**Definition 2.1.10.** The Poincaré series of weight $k$ and character $n$ for $\Gamma$ is the series

$$\theta_{k,n}(z) = \theta_n(z) = \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} e^{2\pi i n \gamma(z)/h} J^k_\gamma(z),$$

where $\Gamma_{\infty}$ is the stabilizer of $\infty$ generated by translations $z \to z + h$ for some $h > 0$.

We need a set of representatives for $\Gamma_{\infty} \backslash \Gamma$. Note that

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + hc & b + hd \\ c & d \end{pmatrix}$$

Using this, it is easily to check that $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ and $\left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right)$ are in the same coset of $\Gamma_\infty$ if and only if $(c, d) = \pm (c', d')$ and $(a, b) = \pm (a', b') \mod h$. Thus a set of representatives for $\Gamma_{\infty} \backslash \Gamma$ can be obtained by taking one element of $\Gamma$ for each pair $(c, d)$, $c > 0$, which is the second row of a matrix in $\Gamma'$.

More generally, we may define the Poincaré series with respect to any cusp $s$ other than $\infty$ of $\Gamma$ by choosing $\sigma \in SL(2, \mathbb{R})$ such that $\sigma(s) = \infty$. Then $\Gamma_s = \sigma^{-1} \Gamma_{\infty} \sigma$. Then replacing $g(\gamma z)$ by $g(\sigma \gamma z)$ and $J_\gamma(z)$ by $J_{\sigma \gamma}(z)$, we obtain an invariant function by averaging over $\Gamma_s \backslash \Gamma'$.  

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Theorem 2.1.5. The Poincaré series $\theta_n$ for $k \geq 1$, $n \geq 0$, converges absolutely uniformly on compact subsets of $\mathcal{H}$. It converges absolutely uniformly on every fundamental domain $\mathcal{F}$ for $\Gamma$, and hence is a modular form of weight $k$ for $\Gamma$.

Proof. Now consider a term

$$(cz + d)^{-2k}e^{(2\pi i n\gamma(z)/h)}$$

in the Poincaré series as $k \to \infty$.

We have

$$|(cz + d)^{-2k}e^{(2\pi i n\gamma(z)/h)}| \leq |(cz + d)|^{-2k}$$

for all $z \in \mathcal{H}$ and $\gamma \in SL(2, \mathbb{Z})$, since $n \geq 0$ and $\gamma(z) \in \mathcal{H}$.

Since any pair $(c, d)$ occurs as a second row of a matrix in $\Gamma_\infty \backslash \Gamma$ at most $h$ times, the series $\sum_{(c,d) \neq (0,0)} |(cz + d)|^{-2k}$ converges absolutely and uniformly on compact sets for $k > 1$, see [54]. Hence, Poincaré series $\theta_n$ for $k > 1$, $n \geq 0$, converges absolutely uniformly on compact subsets of $\mathcal{H}$.

We select

$$U = \{x + iy_0 \mid |x| \leq 1/2\}$$

for some fixed $y_0 > 1$. The Lebesgue measure on $U$ is then $dx$. For $z \in U$, we find

$$|cz + d|^2 = (cx + d)^2 + c^2 y_0^2 \geq c^2 y_0^2.$$ 

If $c \neq 0$, since $c$ is an integer, the choice of $y_0 > 1$ leads to $c^2 y_0^2 > 1$, and hence

$$(cz + d)^{-2k}e^{(2\pi i n\gamma(z)/h)} \longrightarrow 0 \quad (2.1)$$

as $k \to \infty$, uniformly for $z \in U$ and $\gamma \in \Gamma$ with $c \neq 0$. This shows that the Poincaré series $\theta_n$ converges absolutely uniformly for $\text{Im} z \geq C$ and $r \leq \text{Re} z \leq t$. On the other hand, if $c = 0$, we have $\gamma \in \Gamma_\infty$. Therefore, we may assume $\gamma = Id$, and we then have

$$(cz + d)^{-2k}e^{(2\pi i n\gamma(z)/h)} = e^{(2\pi i nz/h)}$$

for all $k \geq 1$ and $z \in U$. Since Poincaré series $\theta_n$ converges for $n = 0$ and from the above equality we get

$$\infty > \sum_{\gamma \in \Gamma_\infty} 1 = m$$

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where \( m \) is the number of translations in \( \Gamma_{\infty} \backslash \Gamma' \). According to the fact that there is only one translation in \( \Gamma_{\infty} \backslash \Gamma' \), we have

\[
\lim_{\text{Im} z \to \infty} \theta_0(z) = 1
\]

Now, when \( n > 0 \),

\[
(cz + d)^{-2k} e^{(2\pi i n(z)/h)} = e^{(2\pi inz/h)} e^{(-2\pi iny/h)}
\]

so that

\[
\lim_{\text{Im} z \to \infty} \theta_n(z) = 0
\]

uniformly in \( U \) (or any finite strip \( r \leq \text{Re} z \leq t \)).

We shall study the behaviour of the series at any cusp \( s \) inequivalent to \( \infty \). Let \( \sigma(s) = \infty \) and \( \theta^*_n \) be the \( \sigma \)-transform of \( \theta_n \):

\[
\theta^*_n(z) = J_{\sigma^{-1}}^k(z) \theta_n(\sigma^{-1} z)
= J_{\sigma^{-1}}^k(z) \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma'} e^{(2\pi i n\gamma^{-1}(z)/h)} J_{\gamma^{-1}}^k(\sigma^{-1} z)
= \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma'} e^{(2\pi i n\gamma^{-1}(z)/h)} J_{\gamma^{-1}}^k(z)
= \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma'/\sigma^{-1}} e^{(2\pi i n\gamma(z)/h)} J_{\gamma}^k(z).
\]

The behaviour of \( \theta^*_n \) at \( \infty \) determines that of \( \theta_n \) at \( s \) because there are no translations in \( \Gamma_{\infty} \backslash \Gamma'/\sigma^{-1} \), otherwise \( s \) would be equivalent to \( \infty \). Note that \( \theta_n \) is holomorphic at \( \infty \) as a modular form by the Riemann’s theorem on removable singularities. Therefore, the Poincaré series \( \theta_n \) for \( k \geq 1, n \geq 0 \), converges absolutely uniformly on compact subsets of \( \mathcal{H} \) and the convergence on the fundamental domain follows from estimate (2.1) and the corresponding statement about \( \theta^*_n \).

From the argument above about the behaviour of the series at cusps, we immediately conclude the following.

**Corollary 2.1.3.** \( \theta_0(z) \) is zero at all cusps except \( \infty \) where \( \theta_0(\infty) = 1 \), i.e., \( \theta_0 \) is a modular form of weight \( k \) for \( \Gamma \). If \( n \geq 1 \), then \( \theta_n(z) \) is a cusp form.
The Hilbert space of cusp forms

We denote by $d\mu(z) = dx \wedge dy = \frac{i}{2}(dz \wedge d\bar{z})$ the standard plane area form and note that $d\mu(\gamma z) = |J_\gamma(z)|^2d\mu(z)$. The measure $\mu(U) = \iint_U y^{-2}d\mu(z)$ plays the same role as the usual measure $dxdy$ on $\mathbb{R}^2$. It is invariant under translation by elements of $PSL(2, \mathbb{R})$. This follows from the invariance of the differential $y^{-2}dxdy$.

Thus we can consider $\iint_{\mathcal{F}} y^{-2}d\mu(z)$ for any fundamental domain $\mathcal{F}$ of $\Gamma$, a Fuchsian group of the first kind, due to the invariance of the differential shows that this doesn’t depend on the choice of $\mathcal{F}$.

Lemma 2.1.6. Let $f$ and $g$ be modular forms of weight $k$ with respect to $\Gamma$. Then the differential

$$f(z)\overline{g(z)}y^{2k-2}dxdy$$

is invariant with respect to the action of $SL(2, \mathbb{R})$.

This is immediate from the transformation properties of $f$ and $g$ and our familiar identity

$$\text{Im}(\gamma z) = \text{Im}(z)|cz + d|^{-2}$$

together with the invariance of the differential $y^{-2}dxdy$ under the action of $SL(2, \mathbb{R})$.

Lemma 2.1.7. [85] Let $\mathcal{F}$ be a fundamental domain for $\Gamma$. Provided at least one of $f$ and $g$ is a cusp form,

$$\iint_{\mathcal{F}} f(z)\overline{g(z)}y^{2k-2}dxdy$$

converges.

Definition 2.1.11. Given two modular forms $f$ and $g$ of weight $k$ for $\Gamma$ such that at least one of $f$ and $g$ is a cusp form, we define the Petersson inner product of $f$ and $g$ by the integral

$$\langle f, g \rangle = \iint_{\mathcal{F}} f(z)\overline{g(z)}y^{2k-2}dxdy$$

In particular, for a cusp form $f$ we set

$$\|f\|^2 = \langle f, f \rangle = \iint_{\mathcal{F}} |f(z)|^2y^{2k-2}dxdy$$
By the preceding two lemmas, the Petersson inner product defines a positive-definite Hermitian form on the space of cusp forms $S_k(\Gamma)$ which endows $S_k(\Gamma)$ with the structure of a finite-dimensional Hilbert space.

**Theorem 2.1.6.** Let $f$ be a modular form of weight $k$ and $\theta_n(z)$, $n > 0$, the Poincaré series of weight $k$ for $\Gamma$. Then

$$\langle f, \theta_n \rangle = \frac{h^{2k}(2k-2)!}{(4\pi)^{2k-1}} n^{(1-2k)} a_n$$

where $h > 0$ such that $\Gamma_\infty$ is generated by translations $z \to z + h$, and $a_n$ is the $n$-th Fourier coefficient in the expansion of $f(z) = \sum a_n(z) e^{2\pi \imath nz/h}$.

**Proof.**

$$\langle f, \theta_n \rangle = \int_{\Gamma \setminus \mathbb{H}} y^{2k-2} f(z) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} e^{(2\pi \imath nz/h)J_\gamma^k(z)} d\mu$$

$$= \int_{\Gamma \setminus \mathbb{H}} \text{Im}z^{2k-2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} f(\gamma z) J_\gamma^k(z) e^{(2\pi \imath nz/h)J_\gamma^k(z)} d\mu$$

$$= \int_{\Gamma \setminus \mathbb{H}} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \text{Im}(\gamma z)^{2k-2} f(\gamma z) e^{(2\pi \imath nz/h)J_\gamma^k(z)} d\mu$$

$$= \int_0^h \int_0^h y^{2k-2} f(z) e^{(2\pi \imath nz/h)} dx dy$$

$$= \int_0^h \int_0^h y^{2k-2} \left( \sum_{m=0}^\infty a_m(z) e^{2\pi \imath nz/h} \right) e^{(2\pi \imath nz/h)} dx dy$$

The only one of these terms in the Fourier series to contribute is $m = n$, by orthogonality. Then, we get

$$\langle f, \theta_n \rangle = h a_n(z) \int_0^\infty y^{2k-2} e^{(-4\pi \imath ny/h)} dy = \frac{h^{2k} \Gamma(2k-1)}{(4\pi n)^{2k-1}} a_n.$$

**Corollary 2.1.4.** [85] The Poincaré series $\theta_n(z)$, $n \geq 1$, of weight $k$ spans the space $A(\Gamma, k)$ of modular forms of weight $k$ for $\Gamma$. 

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The space of all cusp forms $S_k(\Gamma)$ is a finite dimensional Hilbert space with the Petersson inner product. The set of Poincaré series $\theta_n(z), n \geq 1$, generate a linear subspace which is necessarily closed. A function orthogonal to this subspace must be zero since all of its Fourier coefficients vanish according to Theorem 2.1.6.

**Theorem 2.1.7.** [85] Every cusp form is a linear combination of Poincaré series $\theta_n(z), n \geq 1$.

In the one-dimensional case, compactifying $\Gamma \setminus \mathcal{H}$ presents no problem, and the Riemann-Roch theorem tells us the space of modular forms is of finite dimension. A set of modular forms that spans $S_k(\Gamma)$ can be expressed in terms of Poincaré series. In the higher dimensional case, Baily and Borel showed in [10] that the complex manifold $\Gamma \setminus D$, the quotient of a bounded symmetric domain $D$, can be embedded into projective space by using the Poincaré series (see Theorem 1.7.2), and that the closure of the image is a projective algebraic variety. Hence, $\Gamma \setminus D$ has a canonical structure of an algebraic variety.

### 2.2 Automorphic Forms on Bounded Symmetric Domains

#### 2.2.1 Preliminaries

Let $D = G/K \subset \mathbb{C}^n$ be an irreducible bounded symmetric domain, where $G$ is a real semisimple Lie group acting transitively on $D$ and $K$ a maximal compact subgroup of $G$. Let $z_1, \ldots, z_n$ be complex coordinates, $z_j = x_j + iy_j$ for $1 \leq j \leq n$ and denote the Euclidean volume form on $D$ by

$$dV_e = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$ 

Let $\mathcal{K}(.,.)$ be the Bergman kernel for $D$ as in Chapter 1, section 5 with the reproducing property (1.28):

$$f(z) = \int_D f(w)\mathcal{K}(z, w)dV_e,$$

$z \in D$, for all functions $f$ that are holomorphic on $D$ and such that $\int_D |f(z)|^2 dV_e(z) < \infty$. The volume form $dV(z) = \mathcal{K}(z, z)dV_e(z)$ is $G$-invariant.
Let \( k \in \mathbb{N} \) be a positive integer. It will be usually assumed that \( k \) is sufficiently large. The reproducing kernel for the Hilbert space of holomorphic functions on \( D \) satisfying \( \int_D |f(z)|^2 \mathcal{K}(z,z)^{-k}dV(z) < \infty \) is \( c(D,k)\mathcal{K}(z,w)^k \), where \( c(D,k) \) is a constant. The reproducing property is, for any such function \( f \):

\[
f(z) = c(D,k) \int_D f(w)\mathcal{K}(z,w)^k\mathcal{K}(w,w)^{-k}dV(w), \tag{2.2}
\]

\( z \in D \). The value of the constant \( c(D,k) \) is determined by (1.6) in [100]:

\[
c(D,k) \int_D \mathcal{K}(z,w)^k\mathcal{K}(w,z)^{k}\mathcal{K}(w,w)^{-k}dV(w) = K(z,z)^k \tag{2.3}
\]

for any \( z \in D \) (see [100]).

Let \( \Gamma \) be a discrete subgroup of \( G \) such that the quotient \( M = \Gamma \backslash D = \Gamma \backslash G/K \) is smooth and compact. Let \( \mathcal{F} \) be a Dirichlet fundamental domain for \( \Gamma \) (called a canonical fundamental domain in [101]) which is defined in analogous way to the one in Definition 2.1.5. Let \( m \) be a positive integer and \( \rho : \Gamma \to GL(m, \mathbb{C}) \) be a unitary representation of \( \Gamma \).

**Definition 2.2.1.** [9] A function \( f : D \to \mathbb{C} \) is called a \( \Gamma \)-automorphic form of weight \( k \) if \( f \) is holomorphic and

\[
f(\gamma z)J(\gamma, z)^k = f(z),
\]

for all \( \gamma \in \Gamma \) and \( z \in D \) and \( J(\gamma, z) \) denotes the determinant of the Jacobi matrix of \( \gamma \) at \( z \).

**Definition 2.2.2.** [100] Let \( m \) be a positive integer and let \( \rho : \Gamma \to GL(m, \mathbb{C}) \) be a unitary representation of \( \Gamma \). A vector-valued automorphic form of weight \( k \) for \( (\rho, \Gamma) \) is \( F = (F_j) \) where \( F_j : D \to \mathbb{C} \), \( j = 1, \ldots, m \), are holomorphic functions, and

\[
J(\gamma, z)^k F(\gamma z) = \rho(\gamma)F(z)
\]

for all \( \gamma \in \Gamma \) and \( z \in D \).

**Remark 2.2.1.** In a more general case when \( M \) is of finite volume and not compact the definitions should include an appropriate condition at cusps as in section 1. The condition "\( M \) is smooth" can be relaxed to allow \( \Gamma \) such as, for example, \( SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R}) \approx SU(1,1) \) or \( SU(2,1) \cap SL(3, \mathbb{Z}[i]) \).
Denote the space of holomorphic $\Gamma$-automorphic forms of weight $k$ on $D$ by $\mathcal{A}(\Gamma, k)$. Denote the space of holomorphic $(\rho, \Gamma)$-automorphic forms of weight $k$ on $D$ by $\mathcal{A}(\Gamma, m, k, \rho)$. Let $K_M$ be the canonical bundle on $M$ and $K_D$ be the canonical bundle on $D$.

We have been seen in the previous section modular forms are as sections of line bundles. Moreover, from the lifting of $\Gamma$-action on the line bundles,

$$
\begin{array}{ccc}
D & \xrightarrow{\rho} & M \\
\uparrow^\pi & & \uparrow^\pi \\
K_D = p^*(K_M) & \xrightarrow{\tilde{\rho}} & K_M
\end{array}
$$

the sections of $K_M^{\otimes k}$ are in natural one to one corresponding with the functions $f$ on $D$ satisfying

$$
f(\gamma z) = J^{-k}(\gamma, z)f(z)
$$

i.e.,

$$
\mathcal{A}(\Gamma, k) \cong H^0(M, K_M^{\otimes k})
$$

and similarly

$$
\mathcal{A}(\Gamma, m, k, \rho) \cong H^0(M, E_{\rho} \otimes K_M^{\otimes k}),
$$

where $E_{\rho}$ is the vector bundle on $M$ defined by $\rho$.

**Remark 2.2.2.** An irreducible bounded symmetric domain is a Stein manifold, and it is contractible. By the Oka principle, the holomorphic classification of vector bundles corresponds to the smooth classification, and the trivial bundle $D \times \mathbb{C}^m$ is the only rank $m$ holomorphic vector bundle on $D$ up to an isomorphism.

Define the inner product on the space $\mathcal{A}(\Gamma, m, k, \rho)$ as follows:

$$
(F, G) = \int_{\Gamma \backslash D} F(z)^T G(z) \mathcal{K}(z, z)^{-k} dV(z),
$$

for $F, G \in \mathcal{A}(\Gamma, m, k, \rho)$ and it is well-defined because $\rho$ is unitary and $F(z)^T G(z) \mathcal{K}(z, z)^{-k}$ is $\Gamma$-invariant. Similarly, define the inner product on the space $\mathcal{A}(\Gamma, k)$ by

$$
(f, g) = \int_{\Gamma \backslash D} f(z) g(z)^T \mathcal{K}(z, z)^{-k} dV(z),
$$

for $f, g \in \mathcal{A}(\Gamma, k)$. 

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2.2.2 Poincaré Series and a Spanning Result

Definition 2.2.3. [45] Choose \( p \in D \). The \( \mathbb{C} \)-valued Poincaré series is

\[
\theta_{k,p}(z) = \theta_p(z) = \sum_{\gamma \in \Gamma} \left( K(\gamma z, p)J(\gamma, z) \right)^k \in \mathcal{A}(\Gamma, k).
\]

Lemma 2.2.1. The series \( \sum_{\gamma \in \Gamma} \left( K(\gamma z, p)J(\gamma, z) \right)^k \) converges absolutely and uniformly on compact sets of \( D \) for \( k \geq 2 \).

Proof. First, we observe that this statement is contained in the general framework of Chapters 5 and 7 of [74]. Now we will present an actual proof. This is a modification of the proof of Prop. 1, p. 44 [9], which will use that for a fixed \( p \in D \), \( \int_D |K(z, p)|^2 |dV_\varepsilon(z) < \infty \). Let \( A \) be a nonempty compact subset of \( D \). We’ll prove that the series \( \sum_{\gamma \in \Gamma} \left( K(\gamma z, p)J(\gamma, z) \right)^k \) converges absolutely and uniformly on \( A \), for \( k \geq 2 \). There are a compact subset \( B \) of \( D \) and \( \delta > 0 \) such that \( A \) is contained in the interior of \( B \), and for any \( a \in A \) there is a polydisc \( P_a \) of Euclidean volume \( \delta \), with center \( a \), such that \( P_a \subset B \). Let \( m_0 \) be the number of elements in \( \{ g \in \Gamma | gB \cap B \neq \emptyset \} \). First consider the case \( k = 2 \). For \( a \in A \)

\[
|K(\gamma a, p)^2 J(\gamma, a)| \leq \frac{1}{\delta} \int_{P_a} |K(\gamma z, p)^2 J(\gamma, z)|^2 |dV_\varepsilon(z) = \frac{1}{\delta} \int_{P_a} |K(w, p)^2 |dV_\varepsilon(w),
\]

where \( w = \gamma z \). We get:

\[
\sum_{\gamma \in \Gamma} |K(\gamma a, p)^2 J(\gamma, a)|^2 \leq \frac{1}{\delta} \sum_{\gamma \in \Gamma} \int_{P_a} |K(w, p)^2 |dV_\varepsilon(w) \leq \frac{m_0}{\delta} \int_D |K(w, p)^2 |dV_\varepsilon(w).
\]

The last inequality is justified by observing that if \( \gamma P_a \cap \gamma'/P_a \neq \emptyset \) for \( \gamma, \gamma' \in \Gamma \), then \( \gamma^{-1}\gamma' \in \{ g \in \Gamma | gB \cap B \neq \emptyset \} \), so each \( w \in D \) is in at most \( m_0 \) of the sets \( \gamma P_a \), \( \gamma \in \Gamma \). This settles the case \( k = 2 \). Therefore for \( a \in A \), \( |K(\gamma a, p)J(\gamma, a)| < 1 \) for all but at most finitely many \( \gamma \in \Gamma \). When \( |K(\gamma a, p)J(\gamma, a)| < 1 \), \( |K(\gamma a, p)J(\gamma, a)|^k \) is a decreasing function of \( k \geq 2 \). The desired statement follows. \( \square \)
Lemma 2.2.2. [45] For any \( f \in \mathcal{A}(\Gamma, k) \)

\[
(f, \theta_p) = f(p).
\]

Proof.

\[
(f, \theta_p) = c(D, k) \int f(z) \sum_{\gamma \in \Gamma} \mathcal{K}(\gamma z, p)^k \mathcal{J}(\gamma, z)^k \mathcal{K}(z, z)^{-k} dV(z)
\]

\[
= c(D, k) \sum_{\gamma \in \Gamma} \int f(w) \mathcal{K}(p, w)^k \mathcal{K}(w, w)^{-k} dV(w)
\]

\[
= c(D, k) \int_D f(w) \mathcal{K}(p, w)^k \mathcal{K}(w, w)^{-k} dV(w)
\]

\[
= f(p),
\]

where \( w = \gamma z \). We get the second step by using the property of \( f \) being automorphic form \( f \) and properties of Bergman kernel providing in Lemma 1.6.2 and Theorem 1.6.3.

From Theorem 1.6.6, this property reflects the fact that the Bergman kernel for \( K_{M}^{\otimes k} \) is the Poincaré series of the Bergman kernel for \( K_{D}^{\otimes k} \) (Theorem 2 84 or Theorem 1 81, see also Section 7 of [41]).

Theorem 2.2.1. [45] Let \( p_0, \ldots, p_N \) be points in \( D \) such that \( \pi(p_j), j = 0, \ldots, N \) are in general position on \( M \), i.e., \( \iota \circ \pi(p_0), \ldots, \iota \circ \pi(p_N) \) are not on the same hyperplane in \( \mathbb{CP}^N \). Then the Poincaré series \( \theta_{k,p_j}, j = 0, \ldots, N \) form a basis in \( \mathcal{A}(\Gamma, k) \), where \( N + 1 = \dim H^0(M, K_{M}^{\otimes k}) \).

Proof. The set \( \{ \theta_{k,p} \} \) consists of exactly \( N + 1 = \dim \mathcal{A}(\Gamma, k) \) elements. Assume that it is not a basis. Then there is \( f \in \mathcal{A}(\Gamma, k) \) which is not identically zero and not in the linear span of \( \{ \theta_{k,p_j} \} \). Then \( f \) is in the orthogonal complement of this subspace:

\[
(f, \theta_{k,p_j}) = 0
\]

for \( j = 0, \ldots, N \). Then by the preceding Lemma, \( f(p_0) = \cdots = f(p_N) = 0 \).

We know that there is a one-to-one correspondence between automorphic functions on \( D \) and sections of \( K_{M}^{\otimes k} \). Therefore, the corresponding section \( s \in H^0(M, K_{M}^{\otimes k}) \) vanishes at \( \pi(p_j), j = 0, \ldots, N \). From Theorem 1.6.9, where \( L = K_{M}^{\otimes k} \), and the fact that we can find an n-dimensional hyperplane
(affine space) in $\mathbb{C}^{n+1}$ passing through a set of $n+1$ points but not necessarily $n+2$ and since $s = \sum_{j=0}^{N} a_j s_j$; $a_j \in \mathbb{C}$, the locus in $\iota(M)$ where $s = 0$ is the intersection of $\iota(M)$ and a hyperplane in $\mathbb{CP}^N$. Since the points $\pi(p_j)$ are, by assumption, not on the same hyperplane, $s$ must be the zero section and hence $f$ is identically zero. Thus, we get a contradiction and the result follows.

In this section, we extend the spanning result to the setting of vector-valued functions.

**Definition 2.2.4.** [3] For an integrable holomorphic function $F : D \rightarrow \mathbb{C}^m$, the Poincaré series is defined as

$$\Theta_F(z) = \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) F(\gamma z) J(\gamma, z)^k,$$

where $J(\gamma, z)$ is the determinant of the complex Jacobian of the transformation $D \rightarrow D$ at $z$ defined by $\gamma \in \Gamma$.

If the series converges uniformly on compact sets in $D$, then $\Theta_F(z) \in A(\Gamma, m, k, \rho)$ because it is a holomorphic function by the Weierstrass theorem and for $g \in \Gamma$, $z \in D$

$$\Theta_F(gz) = \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) F(\gamma gz) J(\gamma, gz)^k = \sum_{\gamma \in \Gamma} \rho(g(\gamma g)^{-1}) F(\gamma gz) J(\gamma g, z)^k J(g, z)^k$$

$$= \rho(g) J(g, z)^{-k} \Theta_F(z).$$

Let us now generalize the construction from [45] by associating $m$ Poincaré series to a point $p \in D$:

$$\Theta_p^{(j)}(z) = c(D, k) \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) T_p(\gamma z) J(\gamma, z)^k, \quad j = 1, \ldots, m \quad (2.4)$$

where $T_p(z) = \left( \frac{(T_p)^{1}(z)}{(T_p)^{m}(z)} \right)$, $(T_p)^{j}(z) = \mathcal{K}(z, p)^k$ and $(T_p)^{i}(z) = 0$ for $i \neq j$ (i.e. $T_p(\gamma z)$ is the vector-function whose components, except for the $j$-th one, are zero, and $(T_p)^{j}(\gamma z) = \mathcal{K}(\gamma z, p)^k$).

We shall also use $\hat{\Theta}^{(j)}(z, p)$ to denote the function

$$\hat{\Theta}^{(j)} : D \times D \rightarrow \mathbb{C}^m$$

$$(z, p) \rightarrow \Theta_p^{(j)}(z).$$
Proposition 2.2.1. [3] Suppose \( k \) is sufficiently large.

(i) The series (2.4) converges absolutely and uniformly on compact sets.

(ii) For any \( H \in \mathcal{A}(\Gamma, m, k, \rho) \),

\[ (H, \Theta^{(j)}_p) = H_j(p). \]

Proof. Proof of (i): For \( 1 \leq i \leq m \)

\[
\left| \left( \frac{\rho^{-1}(\gamma^{-1})T_p(\gamma z)J(\gamma, z)^k}{\rho^{-1}(\gamma^{-1})T_p(\gamma z)J(\gamma, z)^k} \right)_i \right| = \left| \mathcal{K}(\gamma z, p)J(\gamma, z) \right|^k.
\]

The statement now follows from Lemma 2.2.1.

Proof of (ii): Denote \( w = \gamma z \) for \( \gamma \in \Gamma, z \in \mathcal{F} \). Note that \( H(z)^T = H(w)^T(\rho(\gamma)^{-1})^TJ(\gamma, z)^k \). We have:

\[
(H, \Theta^{(j)}_p) = c(D, k) \int_{\mathcal{F}} (H_1(z) \ldots H_m(z)) \sum_{\gamma \in \Gamma} \frac{\rho^{-1}(\gamma^{-1})T_p(\gamma z)J(\gamma, z)^k\mathcal{K}(z, z)^{-k}}{\rho^{-1}(\gamma^{-1})T_p(\gamma z)J(\gamma, z)^k} dV(z)
\]

\[
= c(D, k) \sum_{\gamma \in \Gamma} \int_{\gamma \mathcal{F}} H(w)^T\overline{T_p(w)J(\gamma, w)^k\mathcal{K}(w, w)^{-k}} dV(w)
\]

\[
= c(D, k) \sum_{\gamma \in \Gamma} \int_{\gamma \mathcal{F}} H_j(w)\overline{\mathcal{K}(w, p)^k\mathcal{K}(w, w)^{-k}} dV(w)
\]

\[
= c(D, k) \int_D H_j(w)\overline{\mathcal{K}(w, p)^k\mathcal{K}(w, w)^{-k}} dV(w)
\]

\[ = H_j(p). \]

\[ \square \]

Remark 2.2.3. Proposition 2.2.1 (ii), restated for the sections of the bundles corresponding to \( H \) and \( \Theta^{(j)}_p \), would mean that \( \Theta^{(j)}_p \) corresponds to the \( j \)-th row of the \( m \times m \) matrix representing the Bergman kernel of \( E_\rho \times K_M^{\otimes k} [80] \).

The assumptions in this Theorem are as in Subsection 2.2.1:

\( D \) is an irreducible bounded symmetric domain, \( \Gamma \) is a discrete subgroup of \( Aut(D) \) such that \( \Gamma \backslash D \) is smooth and compact, and \( \rho : \Gamma \to GL(m, \mathbb{C}) \) is a unitary representation of \( \Gamma \).

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Theorem 2.2.2. [3] For sufficiently large $k$, for sufficiently many points $p_1, \ldots, p_d$ in general position, the $\mathbb{C}$-linear span of $\{\Theta_{p_l}^{(j)}|1 \leq l \leq d, 1 \leq j \leq m\}$ is $\mathcal{A}(\Gamma, m, k, \rho)$.

Proof. The holomorphic vector bundle $W = E_\rho \otimes K_M^k$ is positive. Denote by $\mathbb{P}(W)$ the holomorphic vector bundle $W = E_\rho \otimes K_M^k$ is positive. Denote by $\pi : \mathbb{P}(W) \to M$ the projection, and by $L(W)$ the tautological line bundle over $\mathbb{P}(W)$ which is the subbundle of $\pi^*W$ with the fiber $L(W)_{\xi}$ at $\xi \in \mathbb{P}(W)$ being the complex line in $W_{\pi(\xi)}$ represented by $\xi$.

$$\begin{align*}
\mathbb{P}(W) &\to M \\
\uparrow &\downarrow \\
\pi^*W &\to W
\end{align*}$$

Also denote by $L(W^*)$ the tautological line bundle on $\mathbb{P}(W^*) = (W^* - \{\text{zero section}\})/\mathbb{C}^*$ and by $\hat{\pi} : \mathbb{P}(W^*) \to M$ the projection. $\mathbb{P}(W)$ can be thought of as a manifold by pairing $(x, [v])$; $[v] \in W_x$ and $[v] = \{Cv : v \in W_x\}$ and similarly for $\mathbb{P}(W^*)$. We note that $s$ produces a section $\hat{s}$ of $(L(W^*))^* \to \mathbb{P}(W^*)$. Specifically, $\hat{s} = h \circ s \circ \hat{\pi}$, where $h$ is the holomorphic surjection $\hat{\pi}^*W \to (L(W^*))^*$ given, fiberwise, by the quotient map $W_x \to W_x/\ker f$ over $(x, [f]) \in P(W^*)$, where $x \in M$, $f \in W_x^*, f \neq 0$. 

$$\begin{align*}
\mathbb{P}(W^*) &\to M \\
\uparrow &\downarrow \\
\hat{\pi}^*W &\to W \\
\downarrow &\downarrow \\
(L(W^*))^* &\to (L(W^*))^*
\end{align*}$$

Suppose $k$ is large enough, so that $(L(W^*))^* \to \mathbb{P}(W^*)$ is very ample. Let $d = \dim H^0(\mathbb{P}(W^*), (L(W^*))^*)$ and let $\hat{p}_1, \ldots, \hat{p}_d$ be points in $\mathbb{P}(W^*)$ in general position (i.e. such that their images under the projective embedding given by $(L(W^*))^*$ are not on the same hyperplane in $\mathbb{P}(H^0(\mathbb{P}(W^*), (L(W^*))^*))$). Define $p_j = \hat{\pi}(\hat{p}_j)$ for $j = 1, \ldots, d$. Now, to prove the statement of the theorem, suppose the opposite. Then there is $H \in \mathcal{A}(\Gamma, m, k, \rho)$ which is not identically zero and such that

$$(H, \Theta_{p_l}^{(j)}) = 0$$

for $1 \leq l \leq d, 1 \leq j \leq m$. By preceding Proposition (ii) $H(p_1) = \cdots = H(p_d) = 0$. Let $s$ be the section of $W$ corresponding to $H$. This section
vanishes at $p_1, \ldots, p_d$. Therefore $\tilde{s}(\tilde{p}_1) = \cdots = \tilde{s}(\tilde{p}_d) = 0$. Since $\tilde{p}_1, \ldots, \tilde{p}_d$
are in general position, we have the same argument of Theorem 2.2.1, so we
conclude that $\tilde{s} \equiv 0$. Assume that $s \neq 0$. Then there is a none zero $v \in W_x$
such that $s(x) = v \neq 0$. Since $\tilde{s}$ maps $x$ into zero,

$$h \circ s \circ \tilde{\pi}(x, [f]) = h \circ s(x) = h(v) = 0$$

Since $v$ is 1-form (i.e, it can be written in terms of coordinates) and $v \in \ker f$,
for any $f$ (because $h$ is determined by $f$), we get $v = 0$ which leads to
$s(x) = 0$. We get a contradiction which implies that $s = 0$ and $H = 0$. \qed

**Remark 2.2.4.** The results of this section hold in a more general setting:
when $D$ is a bounded domain in $\mathbb{C}^n$ which has a discrete subgroup of $\text{Aut}(D)$
such that $\Gamma \backslash D$ is smooth and compact.

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Chapter 3

Automorphic forms and submanifolds

In [45], [46] T. Barron (T. Foth) studied $\mathbb{C}$-valued automorphic forms for compact smooth $M$ as quotient of $\mathbb{B}_\mathbb{C}$ (in [46]), constructing explicitly for $p \in D$ the automorphic form $f_p$ with the property $(g, f_p) = g(p)$ for any other holomorphic automorphic form $g$. Such $f_p$ is constructed via Poincaré series and is related to the weighted Bergman kernel as seen in section 2.2 of Chapter 2. Choosing a $\Gamma$-invariant volume form on a submanifold, and integrating $f_p$, one can get automorphic forms associated to submanifolds of $D$. In this section we extend this framework to vector-valued holomorphic automorphic forms on irreducible bounded symmetric domains. Associating an automorphic form to a submanifold of a Kaehler manifold is an idea that is used in many contexts. In particular, relative Poincaré series can be associated to closed geodesics on a hyperbolic Riemann surface [66], [67]. In [78], [107] the submanifold is a closed geodesic or, more generally, a totally geodesic submanifold. To mention a somewhat different kind of such technique, there is a way to associate a section of a line bundle to a Bohr-Sommerfeld Lagrangian submanifold, which is used in semiclassical analysis and symplectic geometry. See, in particular, [12], [13], [23], [24], [38], [51], [63], [90]. This approach is also applicable to isotropic submanifolds [53]. In this section we take advantage of the fact that the Kähler manifold is $M = \Gamma \backslash D$, the holomorphic sections of the line bundle on $M$ can be viewed as holomorphic functions on $D$ (with the standard choice of the trivialization of the bundle pulled back to $D$), and we associate an automorphic form to a submanifold of a fundamental domain of $\Gamma$ in $D$, and not to a submanifold.
of $M$. The Bohr-Sommerfeld condition is not needed. A part of the content of this chapter is in [3].

Let $\Lambda$ be a $q$-dimensional submanifold of $D$ ($q \geq 1$) such that $\Lambda \subset \overline{B}(z_0, r_0) \subset D$, where $\overline{B}(z_0, r_0)$ is the closed ball centered at $z_0$ of radius $r_0$ with respect to the Euclidean metric, for some $z_0 \in D$, $r_0 > 0$. Let $\nu$ be a (real) $q$-form on $\Lambda$ such that $\int_\Lambda \nu > 0$. Set

$$
\Theta^{(j)}(z) = \int_\Lambda \hat{\Theta}^{(j)}(z, p) K(p, p)^{-\frac{q}{2}} \nu(p),
$$

(3.1)

where, $\hat{\Theta}^{(j)}(z, p) = \Theta^{(j)}(z)$ for $j = 1, \ldots, m$. We have: $\Theta^{(j)}_\Lambda \in \mathcal{A}(\Gamma, m, k, \rho)$ and

$$
(H, \Theta^{(j)}_\Lambda) = \int_\Lambda H_j(z) K(z, z)^{-\frac{q}{2}} \nu(z),
$$

(3.2)

for any $H \in \mathcal{A}(\Gamma, m, k, \rho)$.

### 3.1 Automorphic Forms Associated to Submanifolds of $\mathbb{B}^n_{\mathbb{C}}$

In what follows the domain $D$ will be $D = SU(n, 1)/S(U(n) \times U(1))$. The unit ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ is a bounded realization of $D$ (note that for $n = 1$, $D$ is the unit disc: $D = SU(1, 1)/U(1) \cong SL(2, \mathbb{R})/SO(2)$). Here $G = SU(n, 1) = \{ A \in SL(n + 1, \mathbb{C}) \mid A^t \sigma A = \sigma \}$, where $\sigma = \begin{pmatrix} 1_{n \times n} & 0 \\ 0 & -1 \end{pmatrix}$, and the action on $D$ is by fractional-linear transformations: for $\gamma = (a_{jk}) \in SU(n, 1)$, the corresponding automorphism $D \to D$ is

$$
z \mapsto \left( \frac{a_{11} z_1 + \ldots + a_{1n} z_n + a_{1,n+1} \bar{z}_1 + \ldots + a_{n+1,n} \bar{z}_n + a_{n+1,n+1} \bar{w}_1 + \ldots + a_{n,n+1} \bar{w}_n + a_{n+1,n+1}}{a_{n+1,1} z_1 + \ldots + a_{n+1,n} z_n + a_{n+1,n+1} \bar{z}_1 + \ldots + a_{n+1,n} \bar{z}_n + a_{n+1,n+1}} \right),
$$

where $z = (z_1, \ldots, z_n)$ in $D$. The complex Jacobian is

$$
J(\gamma, z) = (a_{n+1,1} z_1 + \ldots + a_{n+1,n} z_n + a_{n+1,n+1})^{-(n+1)}.
$$

We denote by $0$ the point $(0, \ldots, 0) \in \mathbb{B}^n_{\mathbb{C}}$. Define the Hermitian product by

$$
\langle z, w \rangle = z_1 \bar{w}_1 + \ldots + z_n \bar{w}_n - 1,
$$
for $z,w \in D$, and denote the distance between $z$ and $w$ with respect to the complex hyperbolic metric by $\tau(z,w)$. Note that
\[
\cosh^2 \frac{\tau(z,w)}{2} = \frac{\langle z,w \rangle \langle w,z \rangle}{\langle z,z \rangle \langle w,w \rangle}
\] (3.3)
(see e.g. [50] 3.1.7). In Example 1.6.1, equation (1.38) gives an explicit formula of $K$ for the ball (see e.g. [95] or [94])
\[
K(z,w) = \frac{n!}{\pi^n} (-\langle z,w \rangle)^{-(n+1)}.
\] (3.4)

**Lemma 3.1.1.** For $D = \mathbb{B}_C^n$, the constant $c(D,k)$ given in (2.3) is
\[
c(\mathbb{B}_C^n, k) = \left(\frac{(n+1)(k-1) + n}{n}\right).
\]

**Proof.** This follows from Theorem 2.2 in [115] with $\alpha = (n+1)(k-1)$ (the constant $c(D,k)$ comes out to be $c_n$ given by (2.2)[115]). This also can be verified in another way, by a direct calculation as follows:

For $D = \mathbb{B}_C^n$ with $z = 0$, we get:
\[
K(0,w) = K(w,0) = K(0,0) = \frac{n!}{\pi^n},
\]
and
\[
dV(w) = K(w,w)dV_e(w) = \frac{n!}{\pi^n} (-\langle w,w \rangle)^{-(n+1)}dV_e(w).
\]

Therefore from (2.3),
\[
c(\mathbb{B}_C^n, k) \frac{n!}{\pi^n} \int_{\mathbb{B}_C^n} (-\langle w,w \rangle)^{(n+1)(k-1)} \left(\frac{i}{2}\right)^n dw_1 \land d\bar{w}_1 \land ... \land dw_n \land d\bar{w}_n = 1.
\]

Put
\[
I = \int_{\mathbb{B}_C^n} (-\langle w,w \rangle)^{(n+1)(k-1)} \left(\frac{i}{2}\right)^n dw_1 \land d\bar{w}_1 \land ... \land dw_n \land d\bar{w}_n.
\]

To calculate this integral apply a change of variables $(w_1,\bar{w}_1) \to (R_1,\varphi_1)$, where $0 \leq R_1 \leq 1$, $0 \leq \varphi_1 < 2\pi$, $w_1 = R_1 e^{i\varphi_1} \sqrt{1 - |w_2|^2 - ... - |w_n|^2}$. We get:
\[
I = \int_{\mathbb{B}_C^n} (1 - |w_1|^2 - ... - |w_n|^2)^{(n+1)(k-1)} \left(\frac{i}{2}\right)^n dw_1 \land d\bar{w}_1 \land ... \land dw_n \land d\bar{w}_n
\]
\[
\left( \frac{i}{2} \right)^{n-1} \int_{\mathbb{R}^n} (1 - |w_2|^2 - \ldots - |w_n|^2)^{(n+1)(k-1)+1} (1 - R_1^{2i})^{(n+1)(k-1)} R_1 dR_1 \wedge d\varphi_1 \\
\wedge dw_2 \wedge d\bar{w}_2 \wedge \ldots \wedge dw_n \wedge d\bar{w}_n,
\]
then apply the change of variables \((w_2, \bar{w}_2) \rightarrow (R_2, \varphi_2)\), where \(0 \leq R_2 \leq 1\), \(0 \leq \varphi_2 < 2\pi\), \(w_2 = R_2 e^{i\varphi_2} \sqrt{1 - |w_3|^2 - \ldots - |w_n|^2}\) to transform the integral into

\[
\mathcal{I} = \left( \frac{i}{2} \right)^{n-2} \int_{\mathbb{R}^n} (1 - |w_3|^2 - \ldots - |w_n|^2)^{(n+1)(k-1)+2} (1 - R_2^2)^{(n+1)(k-1)} R_1 \\
(1 - R_2^2)^{(n+1)(k-1)+i} R_2 dR_1 \wedge d\varphi_1 \wedge dR_2 \wedge d\varphi_2 \wedge dw_3 \wedge d\bar{w}_3 \wedge \ldots \wedge dw_n \wedge d\bar{w}_n,
\]
and so on. At the end we will get:

\[
\mathcal{I} = (2\pi)^{n-1} \int_0^1 (1 - R_1^{2i})^{(n+1)(k-1)} R_1 dR_1 \int_0^1 (1 - R_2^2)^{(n+1)(k-1)+1} R_2 dR_2 \ldots \\
\int_0^1 (1 - R_{n-1}^{2i})^{(n+1)(k-1)+n-2} R_{n-1} dR_{n-1} \\
\int_{|w_n| \leq 1} (1 - |w_n|^2)^{(n+1)(k-1)+(n-1)i} \frac{i}{2} dw_n \wedge d\bar{w}_n,
\]
and with \(w_n = R_n e^{i\varphi_n}\), \(0 \leq R_n \leq 1\), \(0 \leq \varphi_n < 2\pi\), the last integral is

\[
2\pi \int_0^1 (1 - R_n^{2i})^{(n+1)(k-1)+n-1} R_n dR_n.
\]

Since

\[
2\pi \int_0^1 (1 - R_i^{2i})^{(n+1)(k-1)+i-1} R_i dR_i = \frac{1}{2[(n+1)(k-1) + i]},
\]
we get

\[
\mathcal{I} = \frac{(2\pi)^n}{2^n \{[(n+1)(k-1) + 1] + \ldots + [(n+1)(k-1) + n]\}}
\]
\[
= \frac{\pi^n [(n+1)(k-1)]!}{[(n+1)(k-1) + n]!}.
\]

An elementary calculation now yields the answer.
3.2 Asymptotics of Integrals

In proofs of the main theorems and in some examples in this chapter, the following technique for dealing with integrals with a large positive parameter will be useful.

**Laplace approximation for multivariable functions.** (see [62], [55]).

Let \( g : D \to \mathbb{R}, f : D \to \mathbb{C}, \) be smooth functions, where \( D \subset \mathbb{R}^d \) is a bounded domain, \( d \geq 1 \) and \( \lambda > 0 \) be a large positive parameter. As \( \lambda \to \infty \), consider an integral of the form

\[
I(\lambda) = \int_D f(y) e^{-\lambda g(y)} dy.
\]

If \( g(y) \) has a unique minimum at one interior nondegenerate critical point \( y_0 \in D \), then Laplace’s approximation is well-defined and is given by following asymptotic

\[
I(\lambda) \sim \left( \frac{2\pi}{\lambda} \right)^{d/2} |g''(y_0)|^{-1/2} f(y_0) e^{-\lambda g(y_0)}, \quad (\lambda \to \infty),
\]

and if \( g \) has its minimum stationary point on the boundary of \( D \) at \( a \), then

\[
I(\lambda) \sim \frac{1}{2} \left( \frac{2\pi}{\lambda} \right)^{d/2} |g''(a)|^{-1/2} f(a) e^{-\lambda g(a)}, \quad (\lambda \to \infty),
\]

where

\[ g''(y) = \frac{\partial^2 g(y)}{\partial y \partial y^T} \]

is the Hessian of \( g \).

As \( \lambda \to \infty \), the main contribution to the integral comes from a small neighborhood of \( y_0 \). In this neighborhood the terms of the third order in the Taylor expansion of \( g \) can be discarded. Also, the function \( f \) can be replaced by its value at \( y_0 \) since \( f \) is continuous at \( y_0 \). Then one can extend the region of integration to the whole \( \mathbb{R}^d \). By using the formula for the standard Gaussian integral, the leading asymptotics of the integral \( I(\lambda) \) are obtained as \( \lambda \to \infty \).

If \( g \) has its minimum on the boundary, a similar analysis to the foregoing yields

\[
I(\lambda) \sim \frac{1}{2} \left( \frac{2\pi}{\lambda} \right)^{d/2} |g''(a)|^{-1/2} f(a) e^{-\lambda g(a)}, \quad (\lambda \to \infty),
\]
We shall also need the following fact in proofs of the main theorems in this chapter.

**Stirling approximation.** \cite{55} \( N! \sim (\frac{N}{e})^N \sqrt{2\pi N} \) as \( N \to \infty \).

It follows from applying Laplace’s method to the integral \( \int_0^\infty x^{x+1} e^{-x(z-\ln z)} \, dz \).

On the other hand, by setting \( t = xz \), this integral equals \( \Gamma(x + 1) \).

**Remark 3.2.1.** As a consequence, we get: \( c(\mathbb{B}_c^n, k) \sim \frac{(n+1)(k-1)+n}{n!} \) as \( k \to \infty \).

The following theorem is a general form of Laplace theorem for multidimensional integrals.

**Theorem 3.2.1.** \cite{8} Let \( D \) be a bounded domain in \( \mathbb{R}^d \). Let \( f, g \) be smooth functions such that \( g : D \to \mathbb{R}, f : D \to \mathbb{C} \). Let \( y_0 \) be a nondegenerate critical point of the function \( g \) where it has the only maximum in \( D \). Consider the integral as \( \lambda \to \infty \)

\[
I(\lambda) = \int_D f(y) e^{\lambda g(y)} \, dy.
\]

Let \( 0 < \varepsilon < \pi/2 \). Then there is asymptotic expansion as \( \lambda \to \infty \) in the sector \( S_\varepsilon = \{ \lambda \in \mathbb{C} \mid |\arg(\lambda)| \leq \pi/2 - \varepsilon \} \)

\[
I(\lambda) \sim e^{\lambda g(y_0)} \lambda^{-d/2} \sum_{r=0}^\infty a_r \lambda^{-r}.
\]

The coefficients \( a_r \) are expressed in terms of the derivatives of the functions \( f \) and \( g \) evaluated at the point \( y_0 \).

The idea of the proof is the same as of Laplace approximation theorem but first we change the integration variable

\[
y = y_0 + \lambda^{-1/2} x,
\]

where \( x \) is the scaled fluctuation from the maximum point \( y_0 \). The interval of integration should be changed accordingly, so that the maximum point is now \( x = 0 \). Then, we expand both functions \( g \) and \( f \) in Taylor series at \( y_0 \). Therefore, as \( \lambda \to \infty \)

\[
I(\lambda) \sim \left( \frac{2\pi}{\lambda} \right)^{d/2} |g''(y_0)|^{-1/2} e^{\lambda g(y_0)} [f(y_0) + \frac{A}{\lambda} + \frac{B}{\lambda^2} + \cdots].
\]
3.3 Main Results

The setting in this section is the same as in Section 3.1, where $D = \mathbb{B}^n_\mathbb{C}$. For Theorems 3.3.1, 3.3.2 below, we will need the following additional assumptions (which will be somewhat modified for Theorem 3.3.3):

Let $\Gamma$ be a discrete subgroup of $G = SU(n, 1)$ such that the quotient $M = \Gamma \backslash \mathbb{B}^n_\mathbb{C}$ is smooth and compact and let $\pi : \mathbb{B}^n_\mathbb{C} \to M$ be the covering map and let $\mathcal{F}$ be a Dirichlet fundamental domain for $\Gamma$ [91]. Suppose $X$ and $Y$ are submanifolds of $\mathbb{B}^n_\mathbb{C}$ of dimensions $q_X > 0$ and $q_Y > 0$ respectively, such that $X = \pi^{-1}(X') \cap \mathcal{F}$, $X \cong X'$, and $Y = \pi^{-1}(Y') \cap \mathcal{F}$, $Y \cong Y'$, where $X'$ and $Y'$ are submanifolds of $M$. Let $\nu_X$ be a real $q_X$-form on $X$ such that $\int_X \nu_X > 0$ and let $\nu_Y$ is a real $q_Y$-form on $Y$ such that $\int_Y \nu_Y > 0$. Denote $\tilde{X} = \Gamma \cdot X$, $\tilde{Y} = \Gamma \cdot Y$. Define the $q_X$-form $\nu_{\tilde{X}}$ on $\tilde{X}$ by $\nu_{\tilde{X}} \big|_{\gamma^{-1}(X)} = \gamma^* \nu_X$ for each $\gamma \in \Gamma$. Define $\nu_{\tilde{Y}}$ the same way. Note that $\nu_{\tilde{X}}, \nu_{\tilde{Y}}$ are $\Gamma$-invariant.

Assume there is $\varepsilon > 0$ such that $\tau(z, w) \geq \varepsilon$ for all $z \in \tilde{X}, w \in \tilde{Y}$. Also assume $\int |K(z, w)|^{2} \nu_{\tilde{X}}(w) < \infty$ for any $z \in \mathcal{F}, \int |K(z, w)|^{2} \nu_{\tilde{Y}}(w) < \infty$ for any $z \in \mathcal{F}$ (the last condition is satisfied, for example, when $Y$ is a small ball and $\nu_{\tilde{Y}} = dV|_{\tilde{Y}}$, because $K(\cdot, w)$ is square-integrable on $D$). Denote $\delta_0 = \inf_{z \in X, w \in \partial \mathcal{F}} \tau(z, w)$. Assume $\delta_0 > 0$.

**Theorem 3.3.1.** For any $l \in \mathbb{N}$ there is a constant $C = C(l; n, X, Y, \Gamma, \nu_{\tilde{X}}, \nu_{\tilde{Y}})$ such that for $1 \leq r, j \leq m$, as $k \to \infty$

$$|(\Theta_{X}^{(r)}, \Theta_{Y}^{(j)})| \leq \frac{C}{k^l}.$$ 

**Proof.** Using (3.2), (3.1) and then (2.4), we get

$$|(\Theta_{X}^{(r)}, \Theta_{Y}^{(j)})| = \left| \int_{Y} (\Theta_{X}^{(r)}(z))_{j} K(z, z)^{-\frac{k}{2}} \nu_{Y}(z) \right|$$

$$= \left| \int_{Y} \int_{\tilde{X}} (\hat{\Theta}^{(r)}(z, \zeta)_{j} K(\zeta, \zeta)^{-\frac{k}{2}} \nu_{X}(\zeta) K(z, z)^{-\frac{k}{2}} \nu_{Y}(z) \right|$$

$$\leq c(\mathbb{B}^n_\mathbb{C}, k) \int_{\tilde{Y}} \int_{\tilde{X}} \sum_{\gamma \in \Gamma} |K(\gamma z, \zeta) J(\gamma, z)|^{k} K(\zeta, \zeta)^{-\frac{k}{2}} \nu_{X}(\zeta) K(z, z)^{-\frac{k}{2}} \nu_{Y}(z).$$
In the last inequality we used that $|\rho(\gamma^{-1})_{jr}| \leq 1$ since $\rho(\gamma^{-1})$ is a unitary matrix. Setting $\zeta = \gamma w$ and using the transformation law of Bergman kernel given in Theorem 1.6.3 and (3.4), we get:

$$|\langle \theta^{(r)}_X, \theta^{(j)}_Y \rangle| \leq c(\mathbb{B}^n_C, k) \int \int_{Y \in \Gamma_{\gamma^{-1}X}} |K(z, w)|^k K(w, w)^{-\frac{k}{2}+1} \nu_{\tilde{X}}(w) K(z, z)^{-\frac{k}{2}+1} \nu_Y(z)$$

$$= c(\mathbb{B}^n_C, k) \int \int_{Y \in \Gamma_{\gamma^{-1}X}} |K(z, w)|^k K(w, w)^{-\frac{k}{2}+1} \frac{\nu_{\tilde{X}}(w)}{K(w, w)} K(z, z)^{-\frac{k}{2}+1} \frac{\nu_Y(z)}{K(z, z)}$$

$$= c(\mathbb{B}^n_C, k) \int \int_{Y \in \Gamma_{\gamma^{-1}X}} \left( \frac{\langle z, w \rangle}{\langle w, w \rangle} \right)^{(n+1)(\frac{k}{2}-1)} |K(z, w)|^2 \frac{\nu_{\tilde{X}}(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)}$$

$$\int \int_{Y \in \Gamma_{\gamma^{-1}X}} \frac{\nu_{\tilde{X}}(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)} = \text{const}(\tilde{X}, Y, \nu_{\tilde{X}}, \nu_Y).$$

Since $\cosh \frac{\xi}{2} > 1$, and with Remark 3.2.1, the statement follows. \hfill \Box

**Theorem 3.3.2.** (i) For any $l \in \mathbb{N}$ there is a constant $C = C(l; n, X, \Gamma, \nu_X)$ such that for $r = 1, \ldots, m$, $j = 1, \ldots, m$, for $r \neq j$, as $k \to \infty$

$$|\langle \theta^{(r)}_X, \theta^{(j)}_X \rangle| \leq C \frac{k^l}{k^l}.$$

(ii) For $j = 1, \ldots, m$, and $1 \leq q_X \leq n$

$$\langle \theta^{(j)}_X, \theta^{(j)}_X \rangle \leq \text{const}(n, \Gamma, X, \nu_X) k^{n-\frac{nq_X}{2}}$$

as $k \to \infty$.

(iii) if $X \subset \{ z \in \mathbb{B}^n_C | y_1 = \ldots = y_n = 0 \}$, then for $j = 1, \ldots, m$

$$\langle \theta^{(j)}_X, \theta^{(j)}_X \rangle \sim \text{const}(n, \Gamma, X, \nu_X) k^{n-\frac{nq_X}{2}}$$

as $k \to \infty$.  

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Proof.

\[
(\Theta^{(r)}_X, \Theta^{(j)}_X) = \int_X \left( \Theta^{(r)}_X(z) \right)_j \mathcal{K}(z, z)^{-\frac{j}{2}} \nu_X(z)
\]

\[
= \int_X \int_X \left( \Theta^{(r)}(z, \zeta) \right)_j \mathcal{K}(\zeta, \zeta)^{-\frac{j}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{j}{2}} \nu_X(z)
\]

\[
= c(B^n_{C, k}) \int_X \int_X \sum_{\gamma \in \Gamma} (\rho(\gamma^{-1})_{jr})(\mathcal{K}(\gamma z, \zeta) J(\gamma, z))^k \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_X(z)
\]

\[
= I_1 + I_2,
\]

where \(I_1\) is the term with \(\gamma = \text{id}\) and \(I_2\) is the rest. \(I_1 = 0\) for \(r \neq j\) (because \(\rho(\gamma^{-1})_{jr} = 0\)), and for \(r = j\)

\[
I_1 = c(B^n_{C, k}) \int_X \mathcal{K}(z, \zeta)^k \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_X(z).
\]

Also,

\[
I_2 = c(B^n_{C, k}) \int_X \int_X \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} (\rho(\gamma^{-1})_{jr})(\mathcal{K}(\gamma z, \zeta) J(\gamma, z))^k \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_X(z).
\]

Using that \(\rho(\gamma^{-1})\) is a unitary matrix and setting \(\zeta = \gamma w\) we get:

\[
|I_2| \leq c(B^n_{C, k}) \int_X \int_X \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} |\mathcal{K}(\gamma z, \zeta) J(\gamma, z)|^k \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_X(z)
\]

\[
\leq c(B^n_{C, k}) \int_X \int_X \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} |\mathcal{K}(z, w)|^k \mathcal{K}(w, w)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_X(z)
\]

\[
= c(B^n_{C, k}) \int_X \int_X (\gamma^{-1})_{jr}(\mathcal{K}(z, w))^k \mathcal{K}(w, w)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_X(z)
\]

\[
= c(B^n_{C, k}) \int_X \int_X \frac{(z, w, \langle w, z \rangle \langle w, w \rangle)^{(n+1)(\frac{k}{2}-1)} |K(z, w)|^2 \nu_X(w) \nu_X(z)}{K(w, w) K(z, z)}
\]

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\[
= c(\mathbb{B}_C^n, k) \int_X \int_{\hat{X} \times X} \left( \cosh \frac{\tau(z, w)}{2} \right)^{-\alpha}(z, w) |K(z, w)|^2 \frac{\nu_X(w) \nu_X(z)}{K(w, w) K(z, z)}
\]
\[
\leq c(\mathbb{B}_C^n, k) \left( \cosh \frac{\delta_0}{2} \right)^{-(\alpha+1)(\beta-2)} \int_X \int_{\hat{X} \times X} |K(z, w)|^2 \frac{\nu_X(w) \nu_X(z)}{K(w, w) K(z, z)}.
\]

Since
\[
\int_X \int_{\hat{X} \times X} |K(z, w)|^2 \frac{\nu_X(w) \nu_X(z)}{K(w, w) K(z, z)} \leq \int_X \int_{\hat{X}} |K(z, w)|^2 \frac{\nu_X(w) \nu_X(z)}{K(w, w) K(z, z)} < \infty,
\]
and \(\cosh \frac{\delta_0}{2} > 1\), then using Remark 3.2.1, we see that \(I_2\) has the property:
for any \(l \in \mathbb{N}\) there is a constant \(C = C(l; n, X, \Gamma, \nu_X)\) such that
\[
|I_2| \leq c \frac{C}{k^l}
\]
as \(k \to \infty\). This completes the proof of (i).

For (ii) and (iii), we also need to deal with \(I_1\). First use Fubini’s theorem to switch to the integral over \(X \times X\) with respect to the product measure, then choose and fix a sufficiently small \(\delta > 0\), and split \(I_1\) into two parts: \(I_1^{(1)}\), where the integration is over the part of \(X \times X\) where \(\tau(z, \zeta) \leq \delta\) and \(I_1^{(2)}\), where the integration is over the part of \(X \times X\) where \(\tau(z, \zeta) > \delta\). We have:
\[
I_1^{(2)} = c(\mathbb{B}_C^n, k) \int \int_{X \times X \tau(z, \zeta) > \delta} \frac{(\langle z, \hat{z} \rangle \langle \zeta, \hat{\zeta} \rangle)^{\alpha+1\beta}}{(-\langle z, \zeta \rangle)^{\alpha+1\beta}} \nu_X(z) \nu_X(\zeta),
\]
\[
|I_1^{(2)}| \leq c(\mathbb{B}_C^n, k) \int \int_{X \times X \tau(z, \zeta) > \delta} \frac{(\langle z, \hat{z} \rangle \langle \zeta, \hat{\zeta} \rangle)^{\alpha+1\beta}}{(\langle z, \zeta \rangle)^{\alpha+1\beta}} \nu_X(z) \nu_X(\zeta)
\]
\[
= c(\mathbb{B}_C^n, k) \int \int_{X \times X \tau(z, \zeta) > \delta} \left( \cosh \frac{\tau(z, \zeta)}{2} \right)^{-\alpha(k-1)} \nu_X(z) \nu_X(\zeta)
\]
\[
\leq c(\mathbb{B}_C^n, k) \frac{1}{(\cosh \frac{\delta}{2})^{\alpha(k-1)}} \int \int_{X \times X \tau(z, \zeta) > \delta} \nu_X(z) \nu_X(\zeta),
\]
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Therefore by Remark 3.2.1 and since $\cosh \frac{\delta}{2} > 1$ we see that $I^{(2)}_1$ has the property: for any $l \in \mathbb{N}$ there is a constant $\overline{C} = C(l; n, X, \delta, \nu_X)$ such that
\[
|I^{(2)}_1| \leq \frac{\overline{C}}{k^l},
\]
as $k \to \infty$.

It remains to investigate the term
\[
I^{(1)}_1 = c(B^n_C, k) \iint_{X \times X} \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle^{(n+1)k}}{\langle z, \zeta \rangle \langle \zeta, z \rangle} \nu_X(z) \nu_X(\zeta).
\]
In (ii),
\[
|I^{(1)}_1| \leq c(B^n_C, k) \iint_{X \times X} \left( \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle}{\langle z, \zeta \rangle \langle \zeta, z \rangle} \right)^{\frac{(n+1)k}{2}} \nu_X(z) \nu_X(\zeta).
\]
If $X$ is as in (iii), then for $z, \zeta \in X$ $\langle z, \zeta \rangle = \langle \zeta, z \rangle$ and
\[
|I^{(1)}_1| = c(B^n_C, k) \iint_{X \times X} \left( \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle}{\langle z, \zeta \rangle \langle \zeta, z \rangle} \right)^{\frac{(n+1)k}{2}} \nu_X(z) \nu_X(\zeta).
\]
We have:
\[
c(B^n_C, k) \iint_{X \times X} \left( \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle}{\langle z, \zeta \rangle \langle \zeta, z \rangle} \right)^{\frac{(n+1)k}{2}} \nu_X(z) \nu_X(\zeta)
\]
\[
= c(B^n_C, k) \iint_{X \times X} \left( \cosh \frac{\tau(z, \zeta)}{2} \right)^{-(n+1)k} \nu_X(z) \nu_X(\zeta)
\]
\[
= c(B^n_C, k) \iint_{X \times X} e^{-(n+1)k \ln \cosh \frac{\tau(z, \zeta)}{2}} \nu_X(z) \nu_X(\zeta)
\]
\[
= c(B^n_C, k) \iint_{X \times X} e^{-(n+1)k \ln \cosh \frac{\tau(z, \zeta)}{2}} \nu_X(z) \nu_X(\zeta).
\]

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Let $A_\zeta \in SU(n,1)$, $\zeta \in X$, be a continuous family of automorphisms $\mathbb{B}_C^n \to \mathbb{B}_C^n$ such that $A_\zeta \zeta = 0$. Note that $L = \bigcup_{\zeta \in X} A_\zeta(X)$ is bounded. Let $\{U_j\}$ be a finite cover of $L$ by open subsets of $\mathbb{B}^n$ with smooth boundary, $t_1^{(j)}, \ldots, t_{q_X}^{(j)}$ be local coordinates on $U_j \cap L$, and let $\psi^{(j)}(t)$ be a partition of unity subordinate to the cover $\{U_j\}$.

For a fixed $A_\zeta \in X$ let's deal with the integral

$$\int_{\{z \in X | \tau(z, \zeta) \leq \delta\}} e^{-(n+1)k \ln \cosh \frac{\tau(z, \zeta)}{2}} \nu_X(z)$$

$$= \int_{\{w \in A_\zeta(X) | \tau(w,0) \leq \delta\}} e^{-(n+1)k \ln \cosh \frac{\tau(w,0)}{2}} (A_\zeta^{-1})^* \nu_X(w),$$

where $w = A_\zeta z \left( \text{note: } \tau(z, w) = \tau(A_\zeta z, A_\zeta \zeta) \right)$. We have:

$$(A_\zeta^{-1})^* \nu_X \bigg|_{U_j} = f^{(j)}(t) dt_1^{(j)} \wedge \ldots \wedge dt_{q_X}^{(j)},$$

and the integral becomes

$$\sum_j \int_{\{w \in A_\zeta(X) | \tau(w,0) \leq \delta\} \cap U_j} e^{-(n+1)k \ln \cosh \frac{\tau(w,0)}{2}} \psi^{(j)}(t) f^{(j)}(t) dt_1^{(j)} \wedge \ldots \wedge dt_{q_X}^{(j)}.$$  

We will apply the multivariable Laplace method to the integral

$$\int_{\{w \in A_\zeta(X) | \tau(w,0) \leq \delta\} \cap U_j} e^{-(n+1)k \ln \left( \cosh \frac{\tau(w,0)}{2} \right)^2} \psi^{(j)}(t) f^{(j)}(t) dt_1^{(j)} \ldots dt_{q_X}^{(j)} \quad (3.5)$$

if $w = 0$ is in $U_j$ or on the boundary of $U_j$. The appropriate statement is, respectively, Theorem 3 p. 495 or (5.15) p. 498 in [110]. If $w = 0$ is not in $U_j$, then it follows that the contribution from the $j$-th integral is rapidly decreasing as $k \to \infty$, by an argument similar to the one that has already been used earlier.

To deal with (3.5), we need to show that the Hessian matrix $H_\zeta$ of the function $\ln \left( \cosh \frac{\tau(w,0)}{2} \right)^2 = -\ln(-\langle w, w \rangle)$ at $w = 0$ is positive definite. We have: for $l = 1, \ldots, q_X$, $p = 1, \ldots, q_X$

$$\frac{\partial}{\partial t_p} \left( -\ln(-\langle w, w \rangle) \right) = \frac{1}{\langle w, w \rangle} \sum_{r=1}^n \left( \frac{\partial \bar{w}_r}{\partial t_p} + \bar{w}_r \frac{\partial w_r}{\partial t_p} \right).$$

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\[
\frac{\partial^2}{\partial t_i \partial t_p} (- \ln(-\langle w, w \rangle)) = \frac{-1}{(\langle w, w \rangle)^2} \sum_{r=1}^{n} \left( \frac{\partial \bar{w}_r}{\partial t_r} + \bar{w}_r \frac{\partial w_r}{\partial t_r} \right) + \frac{1}{(-\langle w, w \rangle)} \sum_{r=1}^{n} \left( \frac{\partial w_r}{\partial t_i} \frac{\partial w_r}{\partial t_p} + \bar{w}_r \frac{\partial^2 w_r}{\partial t_i \partial t_p} + w_r \frac{\partial^2 w_r}{\partial t_i \partial t_p} \right).
\]

Then,
\[
\frac{\partial^2}{\partial t_i \partial t_p} (- \ln(-\langle w, w \rangle)) \bigg|_{w=0} = \sum_{r=1}^{n} \left( \frac{\partial w_r}{\partial t_i} \frac{\partial \bar{w}_r}{\partial t_p} + \bar{w}_r \frac{\partial^2 w_r}{\partial t_i \partial t_p} \right).
\]

Therefore \( H_\zeta = B_\zeta B_\zeta^T + \bar{B}_\zeta B_\zeta^T \), where \( B_\zeta \) is the \( q \times n \) matrix \( \begin{bmatrix} \frac{\partial w_1}{\partial t_1} & \cdots & \frac{\partial w_n}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_1}{\partial t_n} & \cdots & \frac{\partial w_n}{\partial t_n} \end{bmatrix} \).

It is clearly that \( H_\zeta \) is symmetric. The matrices \( H_\zeta, B_\zeta B_\zeta^T, \bar{B}_\zeta B_\zeta^T \) are positive semidefinite because for a vector \( v \in \mathbb{C}^n \)
\[
(B_\zeta \bar{B}_\zeta^T v)^T \bar{v} = (\bar{B}_\zeta^T v)^T B_\zeta v
\]
and
\[
(\bar{B}_\zeta B_\zeta^T v)^T \bar{v} = (B_\zeta^T v)^T \bar{B}_\zeta v.
\]

It remains to show that \( H_\zeta v = 0 \) implies \( v = 0 \). If \( H_\zeta v = 0 \) then \( B_\zeta^T v = 0 \).

Since the rank of a matrix \( B_\zeta \), denoted by \( rk(B_\zeta) \), is \( q \times \) equal to \( rk(B_\zeta^T) \) and by using the rank-nullity theorem
\[
rk(B_\zeta^T) + nullity(B_\zeta^T) = q
\]
we get
\[
\dim \ker B_\zeta^T = 0.
\]
Subsequently \( v = 0 \) which implies that \( H_\zeta \) is positive definite.

If \( w = 0 \) is in \( U_j \) then the integral (3.5) is asymptotic to
\[
\left( \frac{4\pi}{(n+1)k} \right)^{\frac{2q}{2}} \psi^{(j)}(t) f^{(j)}(t) \bigg|_{w=0} (\det H_\zeta)^{-\frac{1}{2}},
\]
and if \( w = 0 \) is on the boundary of \( U_j \) then the integral (3.5) is asymptotic to
\[
\frac{1}{2} \left( \frac{4\pi}{(n+1)k} \right)^{\frac{2q}{2}} \psi^{(j)}(t) f^{(j)}(t) \bigg|_{w=0} (\det H_\zeta)^{-\frac{1}{2}}.
\]
We conclude:

$$I_1 = c(\mathbb{B}^n_C, k) \int_{X} \int_{\{z \in X \mid \tau(z, \zeta) \leq \delta\}} e^{-(n+1)k \ln \cosh \frac{\tau(z, \zeta)}{2}} \nu_X(z) \nu_X(\zeta)$$

$$= c(\mathbb{B}^n_C, k) \sum_{j} \int_{X} \int_{\{w \in A_\zeta(X) \mid \tau(w, 0) \leq \delta\} \cap U_j} e^{-(n+1)k \ln \left( \cosh \frac{\tau(w, 0)}{2} \right)} \psi^{(j)}(t) f^{(j)}(t) dt_1 \ldots dt_{q_X} \nu_X(\zeta).$$

Therefore,

$$I_1 \sim c(\mathbb{B}^n_C, k) C k^{-\frac{2n}{m}},$$

and the statements (ii), (iii) now follow from Remark 3.2.1.

**Remark 3.3.1.** Recall that if \((a_k), (b_k)\) are two sequences of complex numbers, then notation \(a_k \sim b_k\) as \(k \to \infty\) means \(\lim_{k \to \infty} \frac{a_k}{b_k} = 1\).

**Remark 3.3.2.** The remainder in Theorem 3.3.2 is determined by \(I_2, I_1^{(2)}\), the error term in the Laplace approximation and the error in the Stirling formula.

**Remark 3.3.3.** In the proofs of Theorems 3.3.1, 3.3.2, it was essential that the domain is \(\mathbb{B}^n_C\).

Note that by using Theorem 3.2.1 for a totally real submanifold of \(\mathbb{B}^n_C\) with \(y_1, \ldots, y_n = 0\), we have

$$I_1^{(1)} \sim c(\mathbb{B}^n_C, k) C \sum_{r=0}^{\infty} k^{-\frac{2n}{m} - r}.$$  

In the first six examples, for a specific \(X\), we shall work out the term

$$I_1^{(1)} = c(\mathbb{B}^n_C, k) \int_{X} \int_{X \times X} \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle^{(n+1)k}}{\langle -z, \zeta \rangle^{(n+1)k}} \nu_X(z) \nu_X(\zeta),$$

that gives the leading order asymptotics in Theorem 3.3.2 (iii).
Example 3.3.1. Let $X_{\varphi} \subset \mathbb{B}_C^n$ be a (1-dimensional) line segment $z_1 = te^{i\varphi}$, $-\alpha < t < \alpha$, where $\alpha \in (0, 1)$ and $\varphi \in [0, \frac{\pi}{2}]$ are fixed, $z_j = 0$ for $j > 1$, and let $\nu_X = dt$. If $n = 1$ then $X$ is a Lagrangian submanifold of $\mathbb{B}^1$. For arbitrary $n$ such $X$ is totally real. We have:

$$ I_1^{(1)} = c(\mathbb{B}_C^n, k) \int_{-\alpha}^{\alpha} \int_{\{t : \tau(z, \zeta) \leq \delta\}} \left( \frac{(1 - t^2)(1 - T^2)}{(1 - tT)^2} \right)^{\frac{(n+1)k}{2}} dt \, dT. $$

Here $\zeta = Te^{i\varphi}$, and we note that $\{t : \tau(z, \zeta) \leq \delta\}$ is an interval. For a fixed $T$, denote $f(t) = \left( \frac{(1 - t^2)(1 - T^2)}{(1 - tT)^2} \right)$. We have:

$$ \frac{df}{dt} = \frac{2(1 - T^2)(T - t)}{(1 - tT)^3}, \quad \frac{d^2f}{dt^2} = \frac{2(1 - T^2)(3T^2 - 2tT - 1)}{(1 - tT)^4}, $$

and then

$$ f(T) = 1, \quad \frac{df}{dt} \bigg|_{t=T} = 0, \quad \frac{d^2f}{dt^2} \bigg|_{t=T} = -\frac{2}{(1 - T^2)^2} < 0, $$

$g(t) = \ln(f(t))$ has a maximum at $t = T$. Applying the 1-dimensional Laplace approximation formula ((1.5) [110] or (5.1.21) [20]) we get:

$$ \int_{\{t : \tau(z, \zeta) \leq \delta\}} e^{\frac{(n+1)k}{2}g(t)} dt \sim \left( \frac{-4\pi}{(n + 1)kg''(T)} \right)^{\frac{1}{2}} = \sqrt{\frac{2\pi}{(n + 1)k}}(1 - T^2), $$

hence

$$ I_1^{(1)} \sim c(\mathbb{B}_C^n, k) \sqrt{\frac{2\pi}{(n + 1)k}} \int_{-\alpha}^{\alpha} (1 - T^2) dT $$

$$ = 2c(\mathbb{B}_C^n, k) \sqrt{\frac{2\pi}{(n + 1)k}}(\alpha - \frac{\alpha^3}{3}) $$

$$ \sim c(n)(\alpha - \frac{\alpha^3}{3})k^{n - \frac{1}{2}}, $$

where $c(n) = 2\sqrt{\frac{2\pi(n+1)^n}{n!}}$. 

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Example 3.3.2. Let $X \subset \mathbb{B}_C^n (n \geq 2)$ be the circle of radius $0 < \alpha < 1$ in the $x_1 x_2$-plane centered at $(x_1, x_2) = (0, 0)$: $z_1 = x_1 = \alpha \cos \theta$, $z_2 = x_2 = \alpha \sin \theta$, $y_1 = y_2 = 0$, $z_j = 0$ for $j > 2$. Let $\nu_X = d\theta$. For arbitrary $n$ such $X$ is totally real. We have:

$$I_1^{(1)} = c(\mathbb{B}_C^n, k) \int_{\{z: \pi(z, \epsilon) \leq \delta\}} \left( \frac{1 - \alpha^2}{1 - \alpha^2 \cos(\theta - \varphi)} \right)^{(n+1)k} d\theta \ d\varphi.$$  

Here $\zeta_1 = \alpha \cos \varphi$, $\zeta_2 = \alpha \sin \varphi$, $Im(\zeta_1) = Im(\zeta_2) = 0$, $\zeta_j = 0$ for $j > 2$. For a fixed $\varphi$ denote $f(\theta) = \frac{1}{1 - \alpha^2 \cos(\theta - \varphi)}$. We have:

$$\frac{df}{d\theta} = \frac{\alpha^2(1 - \alpha^2) \sin(\varphi - \theta)}{(1 - \alpha^2 \cos(\theta - \varphi))^2},$$

$$\frac{d^2 f}{d\theta^2} = \alpha^2(1 - \alpha^2) \frac{2\alpha^2 \sin^2(\varphi - \theta) - \cos(\theta - \varphi)(1 - \alpha^2 \cos(\theta - \varphi))}{(1 - \alpha^2 \cos(\theta - \varphi))^3},$$

Then,

$$f(\varphi) = 1, \quad \frac{df}{d\theta} \bigg|_{\theta = \varphi} = 0, \quad \frac{d^2 f}{d\theta^2} \bigg|_{\theta = \varphi} = -\frac{\alpha^2}{(1 - \alpha^2)} < 0,$$

Therefore, $f$ has a maximum at $\theta = \varphi$. Applying the 1-dimensional Laplace approximation formula ((1.5) [110] or (5.1.21) [20]) we get:

$$\int_{\{z: \pi(z, \epsilon) \leq \delta\}} f(\theta)^{(n+1)k} d\theta \sim \left( \frac{-2\pi}{(n + 1)k f''(\varphi)} \right)^{\frac{1}{2}} = \sqrt{\frac{(1 - \alpha^2)^2 \pi}{\alpha^2(n + 1)k}},$$

hence

$$I_1^{(1)} \sim c(\mathbb{B}_C^n, k) \sqrt{\frac{(1 - \alpha^2)^2 \pi}{\alpha^2(n + 1)k}} \int_0^{2\pi} d\varphi = 2\pi c(\mathbb{B}_C^n, k) \sqrt{\frac{(1 - \alpha^2)^2 \pi}{\alpha^2(n + 1)k}}$$

$$\sim k^{n-\frac{1}{2}} c(n) \sqrt{\frac{1 - \alpha^2}{\alpha^2}},$$

where $c(n) = \frac{(n+1)^{n-\frac{1}{2}}}{n!} 2\pi \sqrt{2\pi}$. 

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Example 3.3.3. Let $X \subset \mathbb{B}_c^n (n \geq 2)$ be the disc $x_1^2 + x_2^2 < \alpha$, where $\alpha \in (0, 1)$ is fixed, $y_1 = y_2 = 0$, $z_j = 0$ for $j > 2$, and let $\nu_X = dx_1 \wedge dx_2$. For arbitrary $n$ such $X$ is totally real. If $n = 2$ then $X$ is a Lagrangian submanifold of $\mathbb{B}_c^2$.

\[ I_1^{(1)} = c(\mathbb{B}_c^n, k) \int_{X \{ \{x \in X \mid R(z, \zeta) \leq \delta \}} \left( \frac{(1 - x_1^2 - x_2^2)(1 - u_1^2 - u_2^2)}{(1 - x_1 u_1 - x_2 u_2)^2} \right)^{\frac{(n+1)k}{2}} dx_1 dx_2 du_1 du_2 \]

\[ = c(\mathbb{B}_c^n, k) \int_{X \{ \{x \in X \mid R(z, \zeta) \leq \delta \}} e^{\frac{(n+1)k}{2} \ln \frac{(1 - x_1^2 - x_2^2)(1 - u_1^2 - u_2^2)}{(1 - x_1 u_1 - x_2 u_2)^2}} dx_1 dx_2 du_1 du_2, \]

where $u_1 = Re(\zeta_1)$, $u_2 = Re(\zeta_2)$. For fixed $u_1$, $u_2$, let

\[ f(x_1, x_2) = -\ln \frac{(1 - x_1^2 - x_2^2)(1 - u_1^2 - u_2^2)}{(1 - x_1 u_1 - x_2 u_2)^2}. \]

We have: $f(u_1, u_2) = 0,$

\[ \frac{\partial f}{\partial x_j} = 2\left( \frac{x_j}{1 - x_1^2 - x_2^2} - \frac{u_j}{1 - x_1 u_1 - x_2 u_2} \right), \quad j = 1, 2 \]

\[ \left. \frac{\partial f}{\partial x_1} \right|_{(u_1, u_2)} = 0 \]

\[ \frac{\partial^2 f}{\partial x_j^2} = \frac{2}{(1 - x_1^2 - x_2^2)^2} - \frac{u_j^2}{(1 - x_1 u_1 - x_2 u_2)^2}, \quad j = 1, 2 \]

\[ \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{2}{(1 - x_1^2 - x_2^2)^2} - \frac{u_1 u_2}{(1 - x_1 u_1 - x_2 u_2)^2} \]

\[ H(u_1, u_2) = \left( \frac{\partial^2 f}{\partial x_1^2} \frac{\partial^2 f}{\partial x_2^2} - \left( \frac{\partial^2 f}{\partial x_1 \partial x_2} \right)^2 \right)_{(u_1, u_2)} = \frac{4}{(1 - u_1^2 - u_2^2)^3} > 0 \]

Using Laplace approximation in $\mathbb{R}^2$ ([110] p. 495 or Theorem 2 [61]) we get: for a fixed $\zeta$

\[ e^{-\frac{(n+1)k}{2} f(x_1, x_2)} dx_1 dx_2 \sim \frac{4\pi}{(n+1)k \sqrt{H(u_1, u_2)}} \]

\[ = \frac{2\pi}{(n+1)k} (1 - u_1^2 - u_2^2)^2. \]
Hence,

\[
I_1^{(1)} \sim c(\mathbb{B}_C^n, k) \frac{2\pi}{(n + 1)k} \int_X (1 - u_1^2 - u_2^2)^{3/2} \, du_1 \, du_2
\]

\[
= c(\mathbb{B}_C^n, k) \frac{2\pi}{(n + 1)k} \int_0^\alpha \int_0^\alpha (1 - r^2)^{3/2} r \, dr \, d\theta
\]

\[
= c(\mathbb{B}_C^n, k) \frac{2\pi}{(n + 1)k} \frac{2\pi}{5} (1 - (1 - \alpha)^{5/2})
\]

Consequently,

\[
I_1^{(1)} \sim k^{n-1/2} \frac{4\pi^2}{5} \frac{(n + 1)^{n-1}}{n!} (1 - (1 - \alpha)^{5/2}).
\]

**Example 3.3.4.** In this example, we will combine Example 3.3.1 and Example 3.3.3 and then we will see if we could compare the asymptotics of \((\Theta_X^{(j)}, \Theta_X^{(j)})\) and \((\Theta_Y^{(j)}, \Theta_Y^{(j)})\) to the asymptotics of \((\Theta_{X \times Y}^{(j)}, \Theta_{X \times Y}^{(j)})\) for two line segments \(X, Y\) in \(\mathbb{B}_C^n\) where \(n \geq 2\).

Set \(X = \{|x_1| < \alpha, y_1 = 0, z_j = 0 \text{ for } j > 1\}\) and \(Y = \{|x_2| < \alpha, x_1 = y_1 = y_2 = 0, z_j = 0 \text{ for } j > 2\}\), where \(\alpha \in (0, \frac{1}{\sqrt{2}})\) is a constant. By the Example 3.3.1 each of \((\Theta_X^{(j)}, \Theta_X^{(j)}), (\Theta_Y^{(j)}, \Theta_Y^{(j)})\) is asymptotic to \(2\sqrt{2\pi} \frac{(n + 1)^{n-1}}{n!} (\alpha - \frac{\alpha^3}{3}) k^{n-1/2}\). The calculation similar to the one in Example 3.3.3 gives that \((\Theta_{X \times Y}^{(j)}, \Theta_{X \times Y}^{(j)})\) is asymptotic to

\[
k^{n-1} \frac{2\pi}{2} \frac{(n + 1)^{n-1}}{n!} \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} (1 - u_1^2 - u_2^2)^{3/2} \, du_1 \, du_2.
\]

Switching to polar coordinates, we get:

\[
\int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} (1 - u_1^2 - u_2^2)^{3/2} \, du_1 \, du_2 = 8 \int_0^\alpha \int_0^\alpha (1 - r^2)^{3/2} r \, dr \, d\theta
\]

\[
= 8 \frac{\pi/4}{5} \int_0^\alpha \left(1 - (1 - \frac{\alpha^2}{\cos^2 \theta})^{5/2}\right) d\theta.
\]

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Here we have noticed that $X$ and $Y$ are line segments (totally real) of real dimension 1 and the asymptotic of $(\Theta_X^{(j)}, \Theta_X^{(j)})$ is $C^{(n+1)n-\frac{1}{2}}_{m!} k^{n-\frac{1}{2}}$ for some constant $C$, and similarly for $(\Theta_Y^{(j)}, \Theta_Y^{(j)})$ while $X \times Y$ is a totally real of real dimension 2 and hence the asymptotic of $(\Theta_{X \times Y}^{(j)}, \Theta_{X \times Y}^{(j)})$ is $C'^{(n+1)n-1}_{m!} k^{n-1}$ for some constant $C'$.

**Example 3.3.5.** Let $X \subset \mathbb{B}_C^n (n \geq 1)$ be the disc $x_1^2 + y_1^2 < \alpha$, where $\alpha \in (0, 1)$ is fixed, $z_j = 0$ for $j > 2$, and let $\nu_X = dx_1 \wedge dy_1$. For arbitrary $n$ such $X$ is a complex submanifold of $\mathbb{B}_C^n$.

$$|I_1^{(1)}| \leq c(\mathbb{B}_C^n, k) \int_X \int_{\{z \in X | r(z, \zeta) \leq \delta\}} \left( \frac{(1 - x_1^2 - y_1^2)(1 - u_1^2 - v_1^2)}{(1 - x_1 u_1 - y_1 v_1)^2 + (x_1 v_1 - y_1 u_1)^2} \right)^{(n+1)k} dx_1 dy_1 du_1 dv_1$$

$$= c(\mathbb{B}_C^n, k) \int_X \int_{\{z \in X | r(z, \zeta) \leq \delta\}} e^{\frac{(n+1)k}{2} \ln \left( \frac{(1 - x_1^2 - y_1^2)(1 - u_1^2 - v_1^2)}{(1 - x_1 u_1 - y_1 v_1)^2 + (x_1 v_1 - y_1 u_1)^2} \right)} dx_1 dy_1 du_1 dv_1,$$

where $u_1 = Re(\zeta_1)$, $v_1 = Im(\zeta_1)$. For fixed $u_1$, $v_1$, let

$$f(x_1, y_1) = -\ln \left( \frac{(1 - x_1^2 - y_1^2)(1 - u_1^2 - v_1^2)}{(1 - x_1 u_1 - y_1 v_1)^2 + (x_1 v_1 - y_1 u_1)^2} \right).$$

We have: $f(u_1, v_1) = 0$,

$$\left. \frac{\partial f}{\partial x_1} \right|_{(u_1, v_1)} = 2\left( \frac{x_1}{1 - x_1^2 - y_1^2} - \frac{u_1 - x_1 u_1^2 - x_1 v_1^2}{(1 - x_1 u_1 - y_1 v_1)^2 + (x_1 v_1 - y_1 u_1)^2} \right)_{(u_1, v_1)} = 0,$$

$$\left. \frac{\partial f}{\partial y_1} \right|_{(u_1, v_1)} = 2\left( \frac{y_1}{1 - x_1^2 - y_1^2} - \frac{v_1 - y_1 u_1^2 - y_1 v_1^2}{(1 - x_1 u_1 - y_1 v_1)^2 + (x_1 v_1 - y_1 u_1)^2} \right)_{(u_1, v_1)} = 0,$$

$$\left. \frac{\partial^2 f}{\partial x_1^2} \right|_{(u_1, v_1)} = \left. \frac{\partial^2 f}{\partial y_1^2} \right|_{(u_1, v_1)} = \frac{2}{(1 - u_1^2 - v_1^2)^2} > 0, \left. \frac{\partial^2 f}{\partial x_1 \partial y_1} \right|_{(u_1, v_1)} = 0,$$

$$H(u_1, u_2) = \left( \left. \frac{\partial^2 f}{\partial x_1^2} \right|_{y_1} - \left( \left. \frac{\partial^2 f}{\partial x_1 \partial y_1} \right|_{y_1} \right)^2 \right)_{(u_1, u_2)} = \frac{4}{(1 - u_1^2 - v_1^2)^4} > 0.$$
Using Laplace approximation in $\mathbb{R}^2$ ([110] p. 495 or Theorem 2 [61]) we get:

$$\int_{\{z : r(z, \zeta) \leq \delta\}} e^{-\frac{(n+1)k}{2}(x_1^2+y_1^2)} dx_1 dy_1 \sim \frac{4\pi}{(n+1)k \sqrt{H(u_1, u_2)}} = \frac{2\pi}{(n+1)k} (1-u_1^2-v_1^2)^{\frac{n-1}{2}}.$$

Hence,

$$|I_1^{(1)}| \leq c(\mathbb{B}^n, k) \frac{2\pi}{(n+1)k} \int_X (1-u_1^2-v_1^2)^{\frac{n-1}{2}} du_1 dv_1$$

$$= c(\mathbb{B}^n, k) \frac{2\pi}{(n+1)k} \int_0^{\alpha} \int_0 (1-r^2)^{\frac{n-1}{2}} rdrd\theta$$

$$= c(\mathbb{B}^n, k) \frac{2\pi}{(n+1)k} \frac{\pi}{3} (1-(1-\alpha)^3).$$

Consequently,

$$|I_1^{(1)}| \leq k^{n-1} \frac{2\pi^2 (n+1)^{n-1}}{3n!} (1-(1-\alpha)^3).$$

**Example 3.3.6.** Let $X_t$ and $Y_s \subset \mathbb{B}^n_\mathbb{C}$ be 1-dimensional (real) line segments defined in Example 3.3.1 with $\varphi = 0, \theta = \frac{\pi}{2}$ respectively and $\alpha \in (0, \frac{1}{\sqrt{2}})$ where Theorem 3.3.2 (iii) can be applied as shown above. Then $X_t \times Y_s$ is subset of a complex line in $\mathbb{C}^n$ where Theorem 3.3.2 (ii) can be applied. Any complex line can be thought of as a disc.

Set $X_t = \{x_1 = t : |x_1| < \alpha, y_1 = 0, z_j = 0 \text{ for } j > 1\}$ and $Y_s = \{y_1 = s : |y_1| < \alpha, x_1 = 0, z_j = 0 \text{ for } j > 2\}$, where $\alpha \in (0, \frac{1}{\sqrt{2}})$ is a constant. By the Example 3.3.1 each of $(\Theta_{X_t}^{(j)}, \Theta_{Y_s}^{(j)}), (\Theta_{X_t}^{(j)}, \Theta_{Y_s}^{(j)})$ is asymptotic to $2\sqrt{2\pi} (n+1)^{\frac{n-1}{2}} (1-\alpha) \frac{n}{3} \frac{1}{\sqrt{2}} k^{n-\frac{1}{2}}$. The calculation similar to the one in Example 3.3.5 gives that

$$\Theta_{X_t \times Y_s}^{(j)} \leq k^{n-1} \frac{2\pi^2 (n+1)^{n-1}}{n!} \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} (1-u_1^2-v_1^2)^{\frac{n-1}{2}} du_1 dv_1.$$

Switching to polar coordinates, we get:
\[
\int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} (1 - u_1^2 - v_1^2) du_1 dv_1 = 8 \int_{0}^{\pi/4} \int_{0}^{\pi/4} (1 - r^2)^2 r dr d\theta \\
= \frac{4}{3} \int_{0}^{\pi/4} \left( 1 - (1 - \frac{\alpha^2}{\cos^2 \theta})^3 \right) d\theta \\
= \frac{4}{3} \int_{0}^{\pi/4} \left( \frac{3}{\cos^2 \theta} - \frac{3}{\cos^4 \theta} + \frac{6}{\cos^6 \theta} \right) d\theta \\
= \frac{4}{3} \left\{ 3\alpha^2 \left[ \tan \theta \right]_0^\frac{\pi}{4} - 3\alpha^4 \left[ \tan \theta + \frac{1}{3} \tan^3 \theta \right]_0^\frac{\pi}{4} + \alpha^6 \left[ \tan \theta + \frac{1}{5} \tan^3 \theta + \frac{2}{3} \tan^3 \theta \right]_0^\frac{\pi}{4} \right\} \\
= \frac{4}{3} \left\{ 3\alpha^2 - 4\alpha^4 + \frac{28}{45} \alpha^6 \right\}.
\]

**Example 3.3.7.** Now, we check the asymptotic of \((\Theta_X^{(j)}, \Theta_X^{(j)})\), where \(X\) in \(B_1 \subset \mathbb{R}^2\) is \(\{z = (x, y) : x^2 + y^2 < \alpha^2\}\), for a small \(\alpha\). Hence, as in the beginning of the proof of Theorem 3.3.2:

\[(\Theta_X^{(r)}, \Theta_X^{(j)}) = I_1 + I_2,\]

where \(I_1\) is the term with \(\gamma = \text{id}\) and \(I_2\) is the rest, and

\[|I_2| \leq \frac{C}{k^d},\]

as \(k \to \infty\). Now consider,

\[
I_1 = \left(\frac{i}{2}\right)^2 c(B_1, k) \int_X \int_X \left( \frac{(1 - |z|^2)(1 - |\zeta|^2)}{(1 - z\zeta)^2} \right)^k \frac{dz d\bar{z}}{(1 - |z|^2)^2 (1 - |\zeta|^2)^2} d\zeta d\bar{\zeta}.
\]
Using polar coordinates with $z = re^{i\theta}$,

$$I_1 = \frac{i}{2} c(B^1_C, k) \int_{\mathbb{X}} (1 - |\zeta|^2)^k \int_0^\alpha \int_0^{2\pi} \frac{(1 - r^2)}{(1 - re^{i\theta}\zeta)^2} \frac{r d\theta dr}{(1 - r^2)^2 (1 - |\zeta|^2)^2} \frac{d\zeta d\bar{\zeta}}{\bar{\zeta}}.$$

Using the change of coordinates: $w = e^{i\theta}$, we get

$$I_1 = \frac{i}{2} c(B^1_C, k) \int_{\mathbb{X}} (1 - |\zeta|^2)^k \int_0^\alpha r(1 - r^2)^{k-2} \int_0^{2\pi} \frac{1}{(1 - re^{i\theta}\zeta)^2k} d\theta dr \frac{d\zeta d\bar{\zeta}}{\bar{\zeta}}.$$

Since $1 - z\bar{\zeta} \neq 0$, then zero is a simple pole of the function

$$\frac{1}{w(1 - rw\zeta)^{2k}},$$

therefore,

$$\int_{|w|=1} \frac{-i}{(1 - rw\zeta)^{2k}} \frac{dw}{w} = 2i\pi \text{Res}_{w=0} \left( \frac{-i}{w(1 - rw\zeta)^{2k}} \right) = 2\pi.$$

Hence,

$$I_1 = \frac{i}{2} (2\pi)c(B^1_C, k) \int_{\mathbb{X}} (1 - |\zeta|^2)^k \frac{d\zeta d\bar{\zeta}}{\bar{\zeta}} \int_0^\alpha r(1 - r^2)^{k-2} dr$$

$$= \frac{i}{2 k - 1} c(B^1_C, k) (1 - r^2)^{k-1} \int_0^\alpha (1 - |\zeta|^2)^k \frac{d\zeta d\bar{\zeta}}{\bar{\zeta}}$$

$$= \frac{i}{2 k - 1} c(B^1_C, k) \left( 1 - (1 - \alpha^2)^{k-1} \right) \int_0^\alpha (1 - |\zeta|^2)^k \frac{d\zeta d\bar{\zeta}}{\bar{\zeta}}$$

$$= \frac{\pi}{k - 1} c(B^1_C, k) \left( 1 - (1 - \alpha^2)^{k-1} \right) \int_0^{2\pi} \int_0 R(1 - R^2)^{k-1} \frac{d\varphi dR}{(1 - R^2)^2}.$$
\begin{align*}
I_1 &= -\frac{\pi^2}{(k-1)^2} c(B^n_C, k) \left( 1 - (1 - \alpha^2)^{k-1} \right) (1 - R^2)^{k-1} |^0_0 \\
&= \frac{\pi^2}{(k-1)^2} c(B^n_C, k) \left( 1 - (1 - \alpha^2)^{k-1} \right)^2 \\
&\sim \frac{\pi^2 (2k - 1)}{(k-1)^2} \left( 1 - (1 - \alpha^2)^{k-1} \right)^2 \\
&= O \left( \frac{1}{k} \right).
\end{align*}

From this example, we have noticed that in case of complex submanifolds of the unit ball, \((\Theta^{(j)}_X, \Theta^{(j)}_X)\) may behave differently than in case of totally real submanifolds with \(y_1, y_2, \ldots, y_n\) zeros.

**Example 3.3.8.** Let \(X \subset B^n_C (n \geq 1)\) be the circle of radius \(0 < \alpha < 1\) in the \(x_1 y_1\)-plane centered at \((0,0): z_1 = \alpha e^{i\theta}, z_j = 0\) for \(j > 2\). Let \(\nu_X = d\theta\), then similar to Example 3.3.7 we have:

\[
I_1 = c(B^n_C, k) \int_X \int_X \left( \frac{1 - |z|^2 (1 - |\zeta|^2)}{(1 - z\zeta)^2} \right)^{n+1-k} d\theta d\varphi
\]

\[
= c(B^n_C, k) \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1 - \alpha^2}{1 - \alpha^2 e^{-i\varphi}} \right)^{(n+1)k} d\theta d\varphi.
\]

Let \(w = e^{i\theta}\), then \(dw = iwd\theta\) and the integral becomes

\[
I_1 = c(B^n_C, k) \int_0^{2\pi} \int_{|w|=1} \left( \frac{1 - \alpha^2}{1 - \alpha^2 we^{-i\varphi}} \right)^{(n+1)k} -\frac{iwd\varphi}{w}.
\]

\[
= c(B^n_C, k) (1 - \alpha^2)^{(n+1)k} \int_0^{2\pi} 2i\pi \left( \frac{1}{w (1 - \alpha^2 we^{-i\varphi})^{(n+1)k}} \right) d\varphi
\]

\[
= (2\pi)^2 c(B^n_C, k) (1 - \alpha^2)^{(n+1)k}
\]

\[
\sim (2\pi)^2 (1 - \alpha^2)^{(n+1)k} \frac{((n + 1)k - 1)^n}{n!}.
\]

If \(n = 1\), then

\[
I_1 \sim (2\pi)^2 (1 - \alpha^2)^{2k} (2k - 1).
\]
The following shows an example of CR-submanifold $X$ which is proper (not totally real or complex), and we are going to check the asymptotic of $(\Theta_X^{(j)}, \Theta_X^{(j)})$.

**Example 3.3.9.** Let $X = \{z = (x_1, y_1, x_2, 0) : x_1^2 + y_1^2 + x_2^2 < \alpha^2\}$ be a CR-submanifold of $\mathbb{B}^2$, where $\alpha \in (0, \frac{1}{\sqrt{2}})$ is fixed. As in the proof of Theorem 3.3.2:

$$(\Theta_X^{(j)}, \Theta_X^{(j)}) = I_1 + I_2,$$

where $I_1$ is the term with $\gamma = \text{id}$ and $I_2$ is the rest, and

$$|I_2| \leq \frac{C}{k^l},$$

as $k \to \infty$. For $I_1$, by using the change of coordinates:

$$x_1 = \rho \sin \phi \cos \theta, \quad y_1 = \rho \sin \phi \sin \theta, \quad x_2 = \rho \cos \phi,$$

where $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$, we get:

$$I_1 = c(\mathbb{B}_x^2, k) \int \int_{\mathbb{X}} \frac{(z, z) (\zeta, \zeta)}{(-\langle z, \zeta \rangle)^{3k}} \nu_X(z) \nu_X(\zeta)$$

$$= c(\mathbb{B}_x^2, k) \int \int_{\mathbb{X}} \rho^2 (1 - \rho^2)^{\frac{3k}{2}} (1 - u_1^2 - v_1^2 - u_2^2)^{\frac{3k}{2}} \sin \phi$$

$$\int_0^\alpha \int_0^{2\pi} \int_0^\alpha \left( 1 - \rho \sin \phi e^{i\theta} (u_1 - iv_1) - \rho \cos \phi u_2 \right) d\theta d\rho d\phi du_1 dv_1 du_2.$$

For fixed $u_1, v_1, u_2$, consider the integral

$$II = \int \int_{\mathbb{X}} \rho^2 (1 - \rho^2)^{\frac{3k}{2}} (1 - \rho \sin \phi e^{i\theta} (u_1 - iv_1) - \rho \cos \phi u_2)^2 \sin \phi d\theta d\rho d\phi,$$

and then use $w = e^{i\theta}$, we get.
\[ II = \int_{0}^{\pi} \sin \phi \int_{0}^{\alpha} \rho^2 (1 - \rho^2)^{\frac{3k}{2}} \int_{0}^{2\pi} \frac{d\theta d\rho d\phi}{(1 - \rho \sin \phi \rho e^{i\theta}(u_1 - iv_1) - \rho \cos \phi u_2)^{3k}} \]

\[ = -i \int_{0}^{\pi} \sin \phi \int_{0}^{\alpha} \rho^2 (1 - \rho^2)^{\frac{3k}{2}} \int_{|w|=1} \frac{dw d\rho d\phi}{w(1 - w \rho \sin \phi (u_1 - iv_1) - \rho \cos \phi u_2)^{3k}} \]

\[ = \int_{0}^{\pi} \sin \phi \int_{0}^{\alpha} \rho^2 (1 - \rho^2)^{\frac{3k}{2}} \frac{d\rho d\phi}{w(1 - w \rho \sin \phi (u_1 - iv_1) - \rho \cos \phi u_2)^{3k}} \]

\[ 2i\pi \text{Res}_{u=0} \frac{d\rho d\phi}{w(1 - w \rho \sin \phi (u_1 - iv_1) - \rho \cos \phi u_2)^{3k}} \]

Then, we get

\[ II = 2\pi \int_{0}^{\pi} \sin \phi \int_{0}^{\alpha} \rho^2 (1 - \rho^2)^{\frac{3k}{2}} \frac{d\rho d\phi}{(1 - \rho \cos \phi u_2)} \]

\[ = 2\pi \int_{0}^{\alpha} \rho^2 (1 - \rho^2)^{\frac{3k}{2}} \int_{0}^{\pi} \frac{\sin \phi}{3k} d\phi d\rho \]

\[ = \frac{2\pi}{(-3k + 1)u_2} \int_{0}^{\alpha} \rho(1 - \rho^2)^{\frac{3k}{2}} (1 + \rho u_2)^{-3k+1} - (1 - \rho u_2)^{-3k+1} d\rho \]

Now,

\[ \int_{0}^{\alpha} \rho(1 - \rho^2)^{\frac{3k}{2}} (1 + \rho u_2)^{-3k+1} d\rho = \int_{0}^{\alpha} \rho(1 - \rho^2)^{\frac{3k}{2}} (1 + \rho u_2)^{-3k} d\rho \]

\[ = \int_{0}^{\alpha} \rho(1 + \rho u_2) \left( \frac{1 - \rho^2}{(1 + \rho u_2)^2} \right)^{\frac{3k}{2}} d\rho. \]
Now to apply Laplace Method, we have
\[
\int_0^\alpha \rho(1 + \rho u_2) \left( \frac{1 - \rho^2}{(1 + \rho u_2)^2} \right)^{\frac{3k}{2}} d\rho, \quad (3.6)
\]
and
\[
f(\rho) = \rho(1 + \rho u_2) \quad \& \quad g(\rho) = \ln \frac{1 - \rho^2}{(1 + \rho u_2)^2}.
\]
Since the function \( g \) assumes its maximum at a critical point \( \rho = -u_2 \) if \( u_2 \leq 0 \) (in the domain),
\[
g'(\rho) = \frac{-2u_2 - 2\rho}{(1 - \rho^2)(1 + \rho u_2)} \quad \& \quad g''(\rho) \bigg|_{\rho=u_2} = \frac{-2}{(1 - u_2^2)^2} < 0,
\]
we get as \( k \to \infty \)
\[
\int_0^\alpha \rho(1 - \rho u_2) \left( \frac{1 - \rho^2}{(1 - \rho u_2)^2} \right)^{\frac{3k}{2}} d\rho \sim \sqrt{\frac{2\pi}{3k}} (-u_2)(1 - u_2^2)^{-\frac{3k}{2} + 2}.
\]
By appplying the same argument of integral (3.6) to the integral
\[
\int_0^\alpha \rho(1 - \rho u_2) \left( \frac{1 - \rho^2}{(1 - \rho u_2)^2} \right)^{\frac{3k}{2}} d\rho,
\]
we find that the function
\[
g(\rho) = \ln \frac{1 - \rho^2}{(1 - \rho u_2)^2}
\]
assumes its maximum at a critical point \( \rho = u_2 \) if \( u_2 \geq 0 \) (in the domain). Hence, as \( k \to \infty \)
\[
\int_0^\alpha \rho(1 - \rho u_2) \left( \frac{1 - \rho^2}{(1 - \rho u_2)^2} \right)^{\frac{3k}{2}} d\rho \sim \sqrt{\frac{2\pi}{3k}} (u_2)(1 - u_2^2)^{-\frac{3k}{2} + 2}.
\]
Whether \( u_2 \geq 0 \) or \( u_2 \leq 0 \), we get (according to the argument in the end of Example 3.1.11)
\[
II \sim \frac{2\pi}{3k - 1} \sqrt{\frac{2\pi}{3k}} (1 - u_2^2)^{-\frac{3k}{2} + 2}.
\]
Therefore,

\[
I_1 \sim \frac{2\pi}{3k - 1} \sqrt{\frac{2\pi}{3k}} c(\mathbb{B}_C^2, k) \int_X (1 - u_2^2)^{-\frac{3k}{2} + 2}(1 - u_1^2 - v_1^2 - u_2^2)^\frac{3k}{2} du_1 dv_1 du_2
\]

\[
= \frac{2\pi}{3k - 1} \sqrt{\frac{2\pi}{3k}} c(\mathbb{B}_C^2, k) \int_0^{2\pi} d\theta \int_0^\pi \sin \phi \int_0^\alpha \frac{\rho^2 (1 - \rho^2)^\frac{3k}{2}}{(1 - \rho^2 \cos^2 \phi)^\frac{3k}{2}} d\rho d\phi
\]

\[
= \frac{4\pi^2}{3k - 1} \sqrt{\frac{2\pi}{3k}} c(\mathbb{B}_C^2, k) \int_0^\pi \sin \phi \int_0^\alpha \frac{\rho^2 (1 - \rho^2)^\frac{3k}{2}}{(1 - \rho^2 \cos^2 \phi)^\frac{3k}{2}} d\rho d\phi.
\]

We conclude that \( I_1 \) is asymptotic to zero, since the integral

\[
\int_0^\alpha \frac{\rho^2 (1 - \rho^2)^\frac{3k}{2}}{(1 - \rho^2 \cos^2 \phi)^\frac{3k}{2}} d\rho
\]

goes to zero as \( k \to \infty \). In detail, set

\[
f(\rho) = \rho^2 (1 - \rho^2 \cos^2 \phi)^2 \quad \text{&} \quad g(\rho) = \ln \frac{1 - \rho^2}{(1 - \rho^2 \cos^2 \phi)^2}.
\]

The function \( g \) assumes its maximum at a critical point \( \rho = 0 \),

\[
g'(\rho) = \frac{2\rho (\cos^2 \phi - 1)}{(1 - \rho^2)(1 - \rho^2 \cos^2 \phi)} \quad \text{&} \quad g'(\rho) \bigg|_{\rho = \rho_2} = 2(\cos^2 \phi - 1) < 0,
\]

Hence, as \( k \to \infty \)

\[
\int_0^\alpha \frac{\rho^2 (1 - \rho^2)^\frac{3k}{2}}{(1 - \rho^2 \cos^2 \phi)^\frac{3k}{2}} d\rho \sim 0.
\]

Now, suppose \( X \) and \( Y \) are submanifolds of \( \mathbb{B}_C^2 \) of dimensions \( q_X > 0 \) and \( q_Y > 0 \) respectively, such that \( Y \subset X \) and \( X = \pi^{-1}(X') \cap F, X \cong X' \), and \( Y = \pi^{-1}(Y') \cap F, Y \cong Y' \), where \( X' \) and \( Y' \) are submanifolds of \( M \). Let \( \nu_X \) be a real \( q_X \)-form on \( X \) such that \( \int_X \nu_X > 0 \) and let \( \nu_Y \) is a real \( q_Y \)-form on
such that \( \int_Y \nu_Y > 0 \). Denote \( \tilde{X} = \Gamma X, \tilde{Y} = \Gamma Y \). Define the \( q_X \)-form \( \nu_{\tilde{X}} \) on \( \tilde{X} \) by \( \nu_{\tilde{X}} \bigg|_{\gamma^{-1}(X)} = \gamma^* \nu_X \) for each \( \gamma \in \Gamma \). Define \( \nu_Y \) the same way. Note that \( \nu_{\tilde{X}}, \nu_{\tilde{Y}} \) are \( \Gamma \)-invariant.

Let \( \sigma = \inf_{z \in Y, w \in \partial X} \tau(z, w) \), and assume \( \sigma > 0 \). Also assume \( \int_X |K(z, w)|^2 \nu_X(w) < \infty \) for any \( z \in F \). Let \( \delta_0 = \inf_{z \in X, w \in \partial F} \tau(z, w) \). Assume \( \delta_0 > 0 \). In this case the following theorem is a generalization of Theorem 3.3.2:

**Theorem 3.3.3.** (i) For any \( l \in \mathbb{N} \) there is a constant \( C = C(l; n, X, Y, \Gamma, \nu_X, \nu_Y) \) such that for \( r = 1, \ldots, m \), \( j = 1, \ldots, m \), for \( r \neq j \), as \( k \to \infty \)

\[
|\left( \Theta_X^{(r)}, \Theta_Y^{(j)} \right)| \leq \frac{C}{k^l}.
\]

(ii) For \( j = 1, \ldots, m \), if \( q_X \leq n \)

\[
\left( \Theta_X^{(j)}, \Theta_Y^{(j)} \right) \leq \text{const}(n, \Gamma, X, \nu_X, Y, \nu_Y) k^{n - \frac{2n}{k}},
\]

as \( k \to \infty \).

(iii) if \( X \subset \{ z \in B^n \mid y_1 = \ldots = y_n = 0 \} \), then for \( j = 1, \ldots, m \)

\[
\left( \Theta_X^{(j)}, \Theta_Y^{(j)} \right) \sim \text{const}(n, \Gamma, X, \nu_X, Y, \nu_Y) k^{n - \frac{2n}{k}},
\]

as \( k \to \infty \).

**Proof.** The idea of the proof is similar to the proof of Theorem 3.3.2 and the only difference will be illustrated in \( I_1^{(1)} \).

\[
\left( \Theta_X^{(r)}, \Theta_Y^{(j)} \right) = \int_Y \left( \Theta_X^{(r)}(z) \right) J K(z, z)^{-\frac{b}{2}} \nu_Y(z)
\]

\[
= \int_Y \int_X \left( \tilde{\Theta}(z, \zeta) \right) J K(\zeta, \zeta)^{-\frac{b}{2}} \nu_X(\zeta) K(z, z)^{-\frac{b}{2}} \nu_Y(z)
\]

\[
= c(B^n, k) \int_Y \int_X \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) J_r(\mathcal{K}(\gamma z, \zeta) J(\gamma, z))^k \mathcal{K}(\zeta, \zeta)^{-\frac{b}{2}} \nu_X(\zeta)
\]

\[
K(z, z)^{-\frac{b}{2}} \nu_Y(z)
\]

\[
= I_1 + I_2,
\]

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where $I_1$ is the term with $\gamma = \text{id}$ and $I_2$ is the rest. $I_1 = 0$ for $r \neq j$ since $\rho(\gamma^{-1})_{jr} = 0$, and for $r = j$

$$I_1 = c(B_\mathbb{C}^n, k) \int_\mathfrak{X} \int_\mathfrak{Y} \mathcal{K}(z, \zeta)^k \mathcal{K}(\zeta, \zeta)^{-\frac{3}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z).$$

Also

$$I_2 = c(B_\mathbb{C}^n, k) \int_\mathfrak{X} \int_\mathfrak{Y} \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} \rho(\gamma^{-1})_{jr} (\mathcal{K}(\gamma z, \zeta) J(\gamma, z)) \mathcal{K}(\zeta, \zeta)^{-\frac{3}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z).$$

Using that $\rho(\gamma^{-1})$ is a unitary matrix and setting $\zeta = \gamma w$ we get:

$$|I_2| \leq c(B_\mathbb{C}^n, k) \int_\mathfrak{X} \int_\mathfrak{Y} \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} |\mathcal{K}(\gamma z, \zeta) J(\gamma, z)\mathcal{K}(\zeta, \zeta)^{-\frac{3}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z)|$$

$$\leq c(B_\mathbb{C}^n, k) \int_\mathfrak{X} \int_\mathfrak{Y} \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} \int_\mathfrak{Y} \mathcal{K}(z, w)^k \mathcal{K}(w, w)^{-\frac{3}{2}} \nu_X(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z)$$

$$= c(B_\mathbb{C}^n, k) \int_\mathfrak{X} \int_\mathfrak{Y} |\mathcal{K}(z, w)|^k \mathcal{K}(w, w)^{-\frac{3}{2}} \nu_X(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z)$$

$$= c(B_\mathbb{C}^n, k) \int_\mathfrak{X} \int_\mathfrak{Y} \left( \frac{\langle z, z \rangle}{\langle z, w \rangle} \right)^{(n+1)(\frac{3}{2})-1} |\mathcal{K}(z, w)|^2 \frac{\nu_X(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)}$$

$$= c(B_\mathbb{C}^n, k) \int_\mathfrak{X} \int_\mathfrak{Y} \left( \cosh \frac{\tau(z, w)}{2} \right)^{-(n+1)(k-2)} |\mathcal{K}(z, w)|^2 \frac{\nu_X(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)}$$

$$\leq c(B_\mathbb{C}^n, k) \left( \frac{\delta_0}{2} \right)^{-(n+1)(k-2)} \int_\mathfrak{X} \int_\mathfrak{Y} |\mathcal{K}(z, w)|^2 \frac{\nu_X(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)}.$$
and
\[ \cosh \frac{\delta_0}{2} > 1, \]
then using Remark 2.3.1, we see that \( I_2 \) has the property: for any \( l \in \mathbb{N} \) there is a constant \( C = C(l; n, \tilde{X}, Y, \Gamma, \nu_{\tilde{X}}, \nu_Y) \) such that
\[ |I_2| \leq \frac{C}{k^l} \]
as \( k \to \infty \). This completes the proof of (i).

For (ii) and (iii) we also need to deal with \( I_1 \). First use Fubini’s theorem to switch to the integral over \( Y \times X \) with respect to the product measure, then choose and fix a sufficiently small \( \delta > 0 \) such that \( \delta < \sigma \), and split \( I_1 \) into two parts: \( I_1^{(1)} \), where the integration is over the part of \( Y \times X \) where \( \tau(z, \zeta) \leq \delta \) and \( I_1^{(2)} \), where the integration is over the part of \( Y \times X \) where \( \tau(z, \zeta) > \delta \). We have:

\[
I_1^{(2)} = c(\mathbb{B}_C^n, k) \iint_{Y \times X} \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle^{(n+1)k/2}}{(-\langle z, \zeta \rangle)^{(n+1)k}} \nu_X(\zeta)\nu_Y(z),
\]
\[
|I_1^{(2)}| \leq c(\mathbb{B}_C^n, k) \iint_{Y \times X} \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle^{(n+1)k/2}}{\langle z, \zeta \rangle^{(n+1)k}} \nu_X(\zeta)\nu_Y(z)
\]
\[
= c(\mathbb{B}_C^n, k) \iint_{Y \times X} \left( \cosh \frac{\tau(z, \zeta)}{2} \right)^{-(n+1)k} \nu_X(\zeta)\nu_Y(z)
\]
\[
\leq c(\mathbb{B}_C^n, k) \frac{1}{(\cosh \frac{\delta}{2})^{(n+1)k}} \iint_{Y \times X} \nu_X(\zeta)\nu_Y(z),
\]
Therefore by Remark 3.2.1 and since \( \cosh \frac{\delta}{2} > 1 \) we see that \( I_1^{(2)} \) has the property: for any \( l \in \mathbb{N} \) there is a constant \( C = C(l; n, X, Y, \delta, \nu_X, \nu_Y) \) such that
\[ |I_1^{(2)}| \leq \frac{C}{k^l} \]
as \( k \to \infty \).
It remains to investigate the term

\[ I^{(1)}_1 = c(B^n_C, k) \iint_{Y \times X \atop \tau(z, \zeta) \leq \delta} \frac{(\langle z, \zeta \rangle \langle \zeta, z \rangle)^{(n+1)k/2}}{(-\langle z, \zeta \rangle)^{(n+1)k}} \nu_X(\zeta) \nu_Y(z). \]

In (ii),

\[ |I^{(1)}_1| \leq c(B^n_C, k) \iint_{Y \times X \atop \tau(z, \zeta) \leq \delta} \left( \frac{(\langle z, \zeta \rangle \langle \zeta, z \rangle)}{\langle z, \zeta \rangle \langle \zeta, z \rangle} \right)^{(n+1)k/2} \nu_X(\zeta) \nu_Y(z). \]

If \( X \) is as in (iii), then for \( z, \zeta \in X \), \( \langle z, \zeta \rangle = \langle \zeta, z \rangle \) and

\[ |I^{(1)}_1| = c(B^n_C, k) \iint_{Y \times X \atop \tau(z, \zeta) \leq \delta} \left( \frac{(\langle z, \zeta \rangle \langle \zeta, z \rangle)}{\langle z, \zeta \rangle \langle \zeta, z \rangle} \right)^{(n+1)k/2} \nu_X(\zeta) \nu_Y(z). \]

We have:

\[
c(B^n_C, k) \iint_{Y \times X \atop \tau(z, \zeta) \leq \delta} \left( \frac{(\langle z, \zeta \rangle \langle \zeta, z \rangle)}{\langle z, \zeta \rangle \langle \zeta, z \rangle} \right)^{(n+1)k/2} \nu_X(\zeta) \nu_Y(z)
\]

\[ = c(B^n_C, k) \iint_{Y \times X \atop \tau(z, \zeta) \leq \delta} \left( \cosh \frac{\tau(z, \zeta)}{2} \right)^{-(n+1)k} \nu_X(\zeta) \nu_Y(z)
\]

\[ = c(B^n_C, k) \int_{Y} \int_{\{z \in X | \tau(z, \zeta) \leq \delta\}} e^{-(n+1)k \ln \cosh \frac{\tau(z, \zeta)}{2}} \nu_X(\zeta) \nu_Y(z). \]

By the same argument as in the proof of Theorem 3.3.2 (iii), we get the desired result. \( \square \)

**Example 3.3.10.** Let \( X = [\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \) and \( Y = [0, \frac{1}{2}] \) are two line segments in \( x_1 \)-axis. We have

\[
\{(y, x) \in Y \times X | \tau(x, y) \leq \delta, \delta > 0\} = \{(y, x) \in Y \times X : |x-y| \leq \delta\}
\]

\[ = \{(y, x) | -\delta + y \leq x \leq \delta + y\} \]

Therefore, \( x \) is in \( \delta \)-neighborhood of diagonal of \( Y \times Y \) if \( \{x \in X | \tau(x, y) \leq \delta\} \) for any \( y \in Y \), where \( \delta \) is a fixed sufficiently small real number.
\[
I^{(1)}_1 = c(B^n_C, k) \iint_{Y \times X \cap (\varepsilon, \varepsilon) \leq \delta} \left( \frac{\langle x, x \rangle \langle y, y \rangle}{\langle y, x \rangle} \right)^{(n+1)k/2} \nu_X(x) \nu_Y(y)
\]

\[
= c(B^n_C, k) \int_{Y} \int_{\{x \in X | (x, y) \leq \delta\}} \left( \frac{\langle x, x \rangle (y, y)}{\langle y, y \rangle (y, x)} \right)^{(n+1)k/2} \nu_Y(x) \nu_Y(y)
\]

\[
= c(B^n_C, k) \int_{Y} \int_{\{x \in X | (x, y) \leq \delta\}} \left( \frac{(1 - x^2)(1 - y^2)}{(1 - xy)^2} \right)^{(n+1)k/2} dxdy
\]

\[
= c(B^n_C, k) \int_{0}^{1} (1 - y^2)^{(n+1)k/2} \left( \int_{\{x \in X | (x, y) \leq \delta\}} \left( \frac{(1 - x^2)}{(1 - xy)^2} \right)^{(n+1)k/2} dx \right) dy
\]

Now,

\[
\int_{\{x \in X | (x, y) \leq \delta\}} \left( \frac{(1 - x^2)}{(1 - xy)^2} \right)^{(n+1)k/2} dx = \int_{\{x \in X | (x, y) \leq \delta\}} e^{-\frac{(n+1)k}{2} \left( -\ln \left( \frac{1 - x^2}{1 - xy} \right) \right)} dx
\]

Set

\[ g(x) = -\ln \left( \frac{1 - x^2}{1 - xy} \right), \]

Then

\[
\frac{\partial}{\partial x} g(x)|_{x=y} = \frac{2(x - y)}{(1 - x^2)(1 - xy)}|_{x=y} = 0, \quad \frac{\partial^2}{\partial x^2} g(x)|_{x=y} = \frac{2}{(1 - y^2)^2} > 0,
\]

Therefore, from Laplace Method we get

\[
\int_{\{x \in X | (x, y) \leq \delta\}} \left( \frac{(1 - x^2)}{(1 - xy)^2} \right)^{(n+1)k/2} dx \sim \sqrt{\frac{2\pi}{(n+1)k}} (1 - y^2)^{-\frac{(n+1)k}{2}k+1}
\]

Moreover,
\[ I_1^{(1)} \sim c(\mathbb{B}_n^2, k) \left( \frac{2\pi}{(n+1)k} \right)^{\frac{1}{2}} \int_0^{\frac{1}{2}} (1 - y^2) dy \]
\[ = c(\mathbb{B}_n^2, k) \left( \frac{2\pi}{(n+1)k} \right)^{\frac{11}{24}} \]
\[ = const(n, Y, \nu_Y) k^n^{-\frac{1}{2}}. \]

**Example 3.3.11.** Let \( X = [\frac{-1}{\sqrt{2}}, \frac{1}{2}] \) and \( Y = [0, \frac{1}{\sqrt{2}}] \) are two line segments in \( x_1 \)-axis. We have
\[
\{(y,x) \in Y \times X | \tau(x,y) \leq \delta, \delta > 0\} = \{(y,x) \in Y \times X : |x - y| \leq \delta\}
\[ = \{(y,x) | - \delta + y \leq x \leq \delta + y\} \]

Therefore, \( x \) is in \( \delta \)-neighborhood of diagonal of \((Y \cap X) \times (Y \cap X)\) for any \( y \in (Y \cap X) \) and \( \delta \) is a fixed sufficiently small real number. Let \( \delta_0 = \inf_{z \in X, w \in \partial F} \tau(z, w) \) and \( \delta_1 = \inf_{z \in Y, z \in \partial F} \tau(z, w) \). Assume \( \delta_0 > 0 \) and \( \delta_1 > 0 \).

\[
(\Theta^{(r)}_X, \Theta^{(j)}_Y) = \int_Y (\Theta^{(r)}_X(z)) J_K(z, z)^{-\frac{k}{2}} \nu_Y(z)
\[ = \int_Y \int_X \left( \hat{\Theta}^{(r)}(z, \zeta) J_K(z, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) K(z, z)^{-\frac{k}{2}} \nu_Y(z) \right)
\[ = c(\mathbb{B}_n^2, k) \int_Y \int_X \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) J_P(K(\gamma z, \zeta) J(\gamma, z))^k K(z, z)^{-\frac{k}{2}} \nu_X(\zeta)
\[ K(z, z)^{-\frac{k}{2}} \nu_Y(z). \]

We split \( X \) into two intervals, \( X_1 = [\frac{-1}{\sqrt{2}}, 0] \) and \( X_2 = [0, \frac{1}{2}] \) and \( Y = [0, \frac{1}{\sqrt{2}}] \cup [\frac{1}{2}, \frac{1}{\sqrt{2}}] \), so we get
\[
(\Theta^{(r)}_X, \Theta^{(j)}_Y) = I_1 + I_2 + I_3 + I_4
\]
where

\[ I_1 = c(B^n, k) \frac{1}{2} \int \int_{\gamma \in \Gamma} \rho(\gamma^{-1})_j r(\mathcal{K}(\gamma z, \zeta)J(\gamma, z))^{k} \mathcal{K}(\zeta, \zeta)^{-\frac{1}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z), \]

\[ I_2 = c(B^n, k) \frac{1}{2} \int \int_{\gamma \in \Gamma} \rho(\gamma^{-1})_j r(\mathcal{K}(\gamma z, \zeta)J(\gamma, z))^{k} \mathcal{K}(\zeta, \zeta)^{-\frac{1}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z), \]

\[ I_3 = c(B^n, k) \frac{1}{2} \int \int_{\gamma \in \Gamma} \rho(\gamma^{-1})_j r(\mathcal{K}(\gamma z, \zeta)J(\gamma, z))^{k} \mathcal{K}(\zeta, \zeta)^{-\frac{1}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z), \]

and

\[ I_4 = c(B^n, k) \frac{1}{2} \int \int_{\gamma \in \Gamma} \rho(\gamma^{-1})_j r(\mathcal{K}(\gamma z, \zeta)J(\gamma, z))^{k} \mathcal{K}(\zeta, \zeta)^{-\frac{1}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z). \]

\( I_2 \) has been addressed in Theorem 3.3.2 (iii) (same computation as in Example 3.3.10), so we have

\[ I_2 \sim c(B^n, k) C \sqrt{\frac{2\pi}{(n + 1)k}} \left( \frac{11}{24} \right). \]

One can apply Theorem 3.3.1 on \( I_3 \), so we have

\[ |I_3| \leq C \frac{k^d}{k^d}. \]

\( I_1 \) and \( I_4 \) follow the same argument, so it is enough to study one of them. Let us consider \( I_1 \). We split it into two parts: \( (I_1)_1 \) is the term where \( \gamma = \text{id} \) and \( (I_1)_2 \) is the rest. \((I_1)_1 = 0 \) for \( r \neq j \) since \( \rho(\gamma^{-1})_{jr} = 0 \), and for \( r = j \)

\[ (I_1)_1 = c(B^n, k) \int \int \mathcal{K}(z, \zeta)^{k} \mathcal{K}(\zeta, \zeta)^{-\frac{1}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z). \]
Also,

\[(\mathcal{I}_1)_2 = c(B_{\mathcal{C}}^n, k) \int_Y \int_X \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} \rho(\gamma^{-1}) \rho(\mathcal{K}(\gamma z, \zeta)J(\gamma, z)) \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \]

\[\mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z).\]

Using that \(\rho(\gamma^{-1})\) is a unitary matrix and setting \(\zeta = \gamma w\), we get:

\[|\mathcal{I}_1| \leq c(B_{\mathcal{C}}^n, k) \int_Y \int_X \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} |\mathcal{K}(\gamma z, \zeta)J(\gamma, z)|^{\frac{k}{2}} \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z)\]

\[\leq c(B_{\mathcal{C}}^n, k) \int_Y \int_X \int_{\gamma^{-1}X} |\mathcal{K}(z, w)|^{\frac{k}{2}} \mathcal{K}(w, w)^{-\frac{k}{2}} \nu_X(w) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z)\]

\[= c(B_{\mathcal{C}}^n, k) \int_Y \int_{\tilde{X} - X} |\mathcal{K}(z, w)|^{\frac{k}{2}} \mathcal{K}(w, w)^{-\frac{k}{2}} \nu_X(w) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z)\]

Since \(\delta_0 > 0\) and \(\delta_1 > 0\), there is \(\varepsilon > 0\) is such that \(\tau(z, w) \geq \varepsilon\) for all \(z \in \tilde{X} - X, w \in Y\).

\[|\mathcal{I}_1| \leq c(B_{\mathcal{C}}^n, k) \int_Y \int_{\tilde{X} - X} \left(\frac{\langle z, z \rangle \langle w, w \rangle}{\langle z, w \rangle \langle w, z \rangle}\right)^{(n+1)\left(\frac{k}{2} - 1\right)} |\mathcal{K}(z, w)|^2 \frac{\nu_X(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)}\]

\[= c(B_{\mathcal{C}}^n, k) \int_Y \int_{\tilde{X} - X} \left(\frac{\cosh \tau(z, w)}{2}\right)^{-(n+1)(k-2)} |\mathcal{K}(z, w)|^2 \frac{\nu_X(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)}\]

\[\leq c(B_{\mathcal{C}}^n, k) \left(\cosh \frac{\varepsilon}{2}\right)^{-(n+1)(k-2)} \int_Y \int_{\tilde{X} - X} |\mathcal{K}(z, w)|^2 \frac{\nu_X(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)}\]

Since

\[\int_Y \int_{\tilde{X} - X} |\mathcal{K}(z, w)|^2 \frac{\nu_X(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)} \leq \int_{\tilde{X}} \int_{\tilde{X}} |\mathcal{K}(z, w)|^2 \frac{\nu_X(w)}{K(w, w)} \frac{\nu_Y(z)}{K(z, z)} < \infty,\]

and

\[\cosh \frac{\varepsilon}{2} > 1,\]

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then using Remark 2.3.1, we see that $I_1^2$ has the property: for any $l \in \mathbb{N}$ there is a constant $C = C(l; n, X, Y, \Gamma, \nu_X, \nu_Y)$ such that

$$|(I_1)_2| \leq \frac{C}{k^l};$$

as $k \to \infty$.

If $r = j$ we also need to deal with $(I_1)_1$. First use Fubini’s theorem to switch to the integral over $Y \times X$ with respect to the product measure, then choose and fix a sufficiently small $\delta > 0$, and split $I_1$ into two parts: $(I_1)_1^{(1)}$, where the integration is over the part of $Y \times X$ where $\tau(z, \zeta) \leq \delta$ and $(I_1)_1^{(2)}$, where the integration is over the part of $Y \times X$ where $\tau(z, \zeta) > \delta$. We have:

$$(I_1)_1^{(2)} = c(B^n_C, k) \int_{Y \times X \atop \tau(z, \zeta) > \delta} \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle \langle \zeta, \zeta \rangle \langle z, z \rangle}{(n+1)k} \nu_X(\zeta) \nu_Y(z),$$

$$|(I_1)_1^{(2)}| \leq c(B^n_C, k) \int_{Y \times X \atop \tau(z, \zeta) > \delta} \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle \langle z, z \rangle}{(n+1)k} \nu_X(\zeta) \nu_Y(z)$$

$$= c(B^n_C, k) \int_{Y \times X \atop \tau(z, \zeta) > \delta} \left( \frac{\cosh \frac{\tau(z, \zeta)}{2}}{\cosh \frac{\delta}{2}} \right)^{(n+1)k} \nu_X(\zeta) \nu_Y(z)$$

$$\leq c(B^n_C, k) \frac{1}{(\cosh \frac{\delta}{2})^{(n+1)k}} \int_{Y \times X \atop \tau(z, \zeta) > \delta} \nu_X(\zeta) \nu_Y(z),$$

Therefore by Remark 3.2.1 and since $\cosh \frac{\delta}{2} > 1$, we see that $(I_1)_1^{(2)}$ has the property: for any $l \in \mathbb{N}$ there is a constant $C = C(l; n, X, Y, \delta, \nu_X, \nu_Y)$ such that

$$|I_1^{(2)}| \leq \frac{C}{k^l},$$

as $k \to \infty$.

It remains to investigate the term.
\[(I_1)^{(1)} = c(B_n, k) \int_0^1 \frac{1}{2} \int_{\{x \in [\frac{1}{2}, 0] : \tau(x,y) \leq \delta \}} \left( \frac{\langle x, y \rangle \langle y, y \rangle}{\langle x, y \rangle \langle y, x \rangle} \right)^{(n+1)k/2} \, dx \, dy \]

\[= c(B_n, k) \int_0^1 \left( 1 - y^2 \right)^{(n+1)k/2} \int_{\{x \in [\frac{1}{2}, 0] : \tau(x,y) \leq \delta \}} \left( \frac{1 - x^2}{(1 - xy)^2} \right)^{(n+1)k/2} \, dx \, dy \]

Now,

\[\int_{\{x \in [\frac{1}{2}, 0] : \tau(x,y) \leq \delta \}} \left( \frac{1 - x^2}{(1 - xy)^2} \right)^{(n+1)k/2} \, dx = \int_{\{x \in [\frac{1}{2}, 0] : \tau(x,y) \leq \delta \}} e^{-\frac{(n+1)k}{2} \left( -\ln \frac{1 - x^2}{(1 - xy)^2} \right)} \, dx \]

Set

\[g(x) = -\ln \frac{1 - x^2}{(1 - xy)^2}, \]

Then

\[\frac{\partial}{\partial x} g(x) \big|_{x=y} = \frac{2(x - y)}{(1 - x^2)(1 - xy)} \big|_{x=y} = 0, \quad \frac{\partial^2}{\partial x^2} g(x) \big|_{x=y} = \frac{2}{(1 - y^2)^2} > 0, \]

The function \(g(x)\) has no critical point \(y_0\) in the domain. Therefore, \((I_1)^{(1)}\) follows the argument: Consider the integral

\[I(\lambda) = \int_a^b f(y) e^{-\lambda g(y)} \, dy, \]

as \(\lambda \to \infty\), where \(f, g\) are smooth functions on \([a, b]\) such that \(g\) has no critical point in the interval. To tackle this problem we can choose a larger interval that contains a minimum stationary point of the function \(g\), so we have

\[\int_c^a f(y) e^{-\lambda g(y)} \, dy, \]

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and
\[ \int_{c}^{b} f(y) e^{-\lambda g(y)} dy, \]

Applying Laplace method of each integral,
\[ \int_{c}^{a} f(y) e^{-\lambda g(y)} dy \sim \left( \frac{2\pi}{\lambda} \right)^{1/2} |g''(y_0)|^{-1/2} f(y_0) e^{-\lambda g(y_0)} + e^{-\lambda g(y_0)} \lambda^{-1/2} \sum_{r=1}^{\infty} a_r \lambda^{-r}, \]

where the coefficients \( a_r \) are expressed in terms of the derivatives of the functions \( f \) and \( g \) evaluated at the point \( y_0 \).
\[ \int_{c}^{b} f(y) e^{-\lambda g(y)} dy \sim \left( \frac{2\pi}{\lambda} \right)^{1/2} |g''(y_0)|^{-1/2} f(y_0) e^{-\lambda g(y_0)} + e^{-\lambda g(y_0)} \lambda^{-1/2} \sum_{r=1}^{\infty} a_r \lambda^{-r}. \]

The coefficients \( a_r \) are expressed in terms of the derivatives of the functions \( f \) and \( g \) evaluated at the point \( y_0 \). Hence,
\[ I(\lambda) \sim e^{-\lambda g(y_0)} \lambda^{-1/2} \sum_{r=1}^{\infty} a_r \lambda^{-r}. \]

As a consequence,
\[ (\mathcal{I}_1)^{(1)} \sim \left\{ k^{\frac{1}{2}} \left( \frac{g^{(4)}(y_0)}{8(g''(y_0))^2} - \frac{5}{24} (g^{(3)})^2 \right) \right\} c(\mathbb{B}_C^n, k) \int_{0}^{\frac{\pi}{2}} dy. \]
\[ = \frac{1}{2} O(k^{\frac{1}{2} - 1}) c(\mathbb{B}_C^n, k), \]

which is dominated by \( k^{\frac{1}{2}} Cc(\mathbb{B}_C^n, k) \). Hence,
\[ (\Theta_X^{(r)}, \Theta_Y^{(j)}) \sim (const) c(\mathbb{B}_C^n, k) \sqrt{\frac{2\pi}{(n + 1)k}} \left( \frac{11}{24} \right). \]
3.4 Conclusion

There is extensive literature on vector-valued automorphic forms, going back to the foundational works by Borel, Harish-Chandra and Selberg. These functions have exciting applications in arithmetic geometry and frequently appear in the framework of string theory, as well as in other areas of mathematical physics.

Most of the more recent results are for vector-valued modular forms on the upper half plane. Insights related to vector-valued Poincaré series on bounded symmetric domains are contained, for example in [18] (where D. Bell points out that his methods allow to prove that the Poincaré series of polynomials in $z_1, \ldots, z_n$ span the space of vector-valued holomorphic automorphic forms on finite products of classical domains). The theorem obtained in Chapter 2 of this thesis provides a different kind of spanning result. The spanning set is constructed with the use of the Bergman kernel for the domain.

The results in Chapter 3 address, in a general sense, some version of the very first question asked in the thesis. To a submanifold of the ball we associate a sequence of functions on the ball and then ask how the Hilbert space norms of these functions behave in the limit. Many versions of the idea of associating of a section of a vector bundle to a submanifold have appeared in literature (at a very fundamental level this is analogous to a duality statement). The asymptotics that we get compare well with the existing results, including [23]. We get that, as the weight goes to infinity, the vector-valued Poincaré series associated to two submanifolds which are at a finite distance apart, are ”asymptotically orthogonal” in the Hilbert space of automorphic forms. We also get that for a totally real submanifold which is contained in a certain linear subspace of the ball, the norm of the Poincaré series grows, with the exponent in the leading term determined by the dimension of the submanifold. The calculations in examples show that this is not the case for complex submanifolds or CR (not totally real) submanifolds. Therefore, the asymptotic behavior of the automorphic forms reflects the properties of the submanifolds.
Bibliography


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