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A Convexity Theorem For Symplectomorphism Groups

Seyed Mehdi Mousavi
The University of Western Ontario

Supervisor
Professor Martin Pinsonnault
The University of Western Ontario

Graduate Program in Mathematics

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Seyed Mehdi Mousavi

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Chief Advisor:

____
Professor Martin Pinsonnault

Advisory Committee:

____
Professor Tatyana Barron

____
Professor Rasul Shafikov

Examining Board:

____
Professor Matthias Franz

____
Professor Niky Kamran

____
Professor Hristo Sendov

____
Professor Gordon Sinnamon

The thesis by

Seyed Mehdi Mousavi

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____
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Professor Terry Sicular

Abstract

In this thesis, the topological, algebraic and geometric properties of the subgroup of equivariant symplectomorphisms of a symplectic toric manifold are studied. It is shown that this subgroup is a maximal Abelian and path connected "Lie subgroup" of the full symplectomorphism group which is flat and totally geodesic with respect to the canonical weak metric induced by the canonical Kaehler metric. Moreover, an infinite-dimensional version of the Schur-Horn-Kostant convexity theorem is presented which is obtained after an appropriate completion of algebras.

Keywords: Infinite-dimensional Lie groups, Weak Riemannian metric, Maximal tori, Symplectomorphisms, Toric manifolds, Equimeasurable rearrangements, Schur-Horn-Kostant convexity.

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To my late grandparents
Mohammad-Reza
and
Asieh

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Introduction

The study of infinite-dimensional Lie groups was pioneered by V. I. Arnold. In a celebrated paper in 1966, Arnold employed the language of infinite-dimensional groups to study the Euler equation of a perfect fluid [7]. Arnold's ideas were recast in a rigorous language via Sobolev completions by Ebin and Marsden in 1970 [26]. Ebin and Marsden employed Arnold's approach to elaborate further on the existence and uniqueness of solution of some classical equations. Since then, Arnold's geometric framework has been used effectively to study other classical equations and the study of infinite-dimensional Lie groups has been an active area of research.

Despite all the efforts, the theory of infinite-dimensional Lie groups remains somehow unsatisfactory. For instance, there exist several competing definitions for smooth structures on these Lie groups and there is no bijection between infinite-dimensional Lie algebras and infinite-dimensional Lie groups. However, it is believed that among infinite-dimensional Lie groups, symplectomorphism groups are somewhat better behaved and form a class of objects between finite-dimensional Lie groups and general diffeomorphism groups. For instance, it is well known that, up to conjugacy, the maximal torus in a compact Lie group is unique. For symplectomorphism groups of 4-dimensional manifolds, we know that there are only finitely many conjugacy classes of maximal finite-dimensional tori in $Ham(M, \omega)$ [57]. Similarly, the number of inequivalent toric actions on any symplectic manifold is also finite [45]. In this thesis we focus on two aspects of this philosophy, namely, the existence of infinite-dimensional maximal tori with good Lie theoretic properties in symplectomorphism groups of toric manifolds, and a proof of an infinite-dimensional version of the Schur-Horn-Kostant convexity theorem.

The first chapter of the thesis is devoted to background material. We recall some of the main techniques and tools from the theory of infinite-dimensional Lie groups that we need.

The crux of this thesis are Chapter 2 and 3. In Chapter 2 we show that the group of all equivariant symplectomorphism, $\mathscr{D}^s_\omega(M,T)$, can play the role of a maximal torus in the group of symplectomorphisms of toric manifolds, $\mathscr{D}^s_\omega(M)$. More precisely, we show that $\mathscr{D}^s(M,T)$ is a closed, infinite-dimensional and path-connected submanifold of $\mathscr{D}^s_\omega(M)$ which is flat and totally geodesic with respect to a "canonical" weak Riemannian metric. Moreover, it is a maximal Abelian subgroup of $\mathscr{D}^s_\omega(M)$. This provides an extension of previous results by Bao and Ratiu [10] and El Hadrami [30].

Finally in the last chapter, following [12], we prove a Schur-Horn-Kostant convexity theorem for $\mathscr{D}^s_\omega(M,T) \hookrightarrow \mathscr{D}^s_\omega(M)$ after an appropriate completion of algebras. In doing so, we relax the hypothesis that was originally assumed in the work of Bloch, Flaschka and Ratiu [12]. The sharper version of the convexity theorem states that the image of the orbit of the semi-group $\overline{Ham(M,\omega)}$ (a certain completion of the group of Hamiltonian symplectomorphisms) through the "spectrum" λ of a function $f \in L^p(M,\nu_\omega)$ under an averaging map is weakly compact and convex. Moreover, its set of extreme point is the orbit of Weyl semi-group $\overline{\mathbf{W}}$ through λ. Many of the ideas and results of the Chapter 3 were known in special cases (the annulus $\mathcal{A} = [0,1] \times \mathbb{S}^1$ and $\mathbb{P}^1$) and plan for extending them to toric manifolds was laid down in [14], but this program was never pursued.

Chapter 1

Background Material

In this chapter we present background material. We begin with a review of Riemannian geometry and its applications to Lie groups. Then we briefly review aspects of differential geometry on spaces of mappings, and the classical Schur-Horn-Kostant convexity theorem.

1.1 Riemannian Geometry and Lie Groups

In this section we review some classical facts from Riemannian geometry on Lie groups. The material of this section can be found in standard references on Riemannian geometry like [66] and [22]. The material about the connection map can be found in complete detail in [66] and we follow notation of this book closely.

Let M be an n-dimensional manifold. We denote the space of all smooth (C^∞) vector fields on M by $\mathfrak{X}(M)$. If for each $p \in M$ we assign an inner product $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ such that depends smoothly on p, then we say (M, g) is a Riemannian manifold and g is called a smooth Riemannian metric on M.

Definition 1.1.1. *A linear connection* $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$, *is a bilinear map that satisfies the following conditions for every* $f \in \mathscr{C}^\infty(M, \mathbb{R})$ *and* $X, Y \in \mathfrak{X}(M)$*:*

- $\nabla_{fX}Y = f\nabla_X Y$.

- $\nabla_X fY = f\nabla_X Y + (Xf)Y$.

We recall the following theorem which is called the fundamental theorem of Riemannian geometry.

Theorem 1.1.2. *[Fundamental Theorem of Riemannian Geometry] Let (M, g) be a Riemannian manifold, then there is a unique linear connection ∇ on M that satisfies the following conditions:*

(1) *∇ is torsion free, that is*

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

(2) *∇ is compatible with g in the sense that*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

(3) $$g(\nabla_X Y, Z) = \frac{1}{2}\Big\{Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)\Big\}.$$

This linear connection is called the Levi-Civita connection of g. Observe that the third formula can be considered as the explicit definition of the connection.

Locally, any connection ∇ on TM is characterized by its Christoffel symbols Γ_{ij}^k which, on a chart $\big(U, \varphi = (x^1, \ldots, x^n)\big)$, are defined by the equation

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k.$$

One can easily see that

$$(\nabla_X Y)\Big|_U = \sum_{k=1}^{n} \Big(X(Y^k) + \Gamma_{ij}^k X^i Y^j\Big)\partial_k$$

where $X = \sum_{i=1}^n X^i \partial_i$ and $Y = \sum_{j=1}^n Y^j \partial_j$. The Christoffel symbols can be computed explicitly as follows

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell=1}^{n} g^{k\ell}\Big(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\Big).$$

Any connection on TM gives rise to a decomposition of TTM into vertical and horizontal subbundles. First, note that $T_pM = \pi_M^{-1}(p)$ is a submanifold of TM whose tangent space is $T_{(p,u)}T_pM = \mathsf{Ker}(T_{(p,u)}\pi_M)$, which is called the vertical subspace at (p,u), and which is denoted by $V_{(p,u)}$. With respect to the chart $\big(\overline{U}, \overline{\varphi} = (x^i; y^i; \xi^i; \eta^i)\big)$ of TTM obtained from a chart $\big(U, \varphi = (x^1, \ldots, x^n)\big)$ of M we have

$$V_{(p,u)} = \Big\{(x^i(p); y^i(p); 0; \eta^i) \;\Big|\; \eta^i \in \mathbb{R}\Big\}$$

and the isomorphism $\iota_{(p,u)} : T_{(p,u)}T_pM \longrightarrow T_pM$ is given by

$$\iota_{(p,u)}\Big(x^i(p); y^i(p); 0; \eta^i\Big) = (x^i(p); \eta^i).$$

Now we find a natural complement $H_{(p,u)}$ to $V_{(p,u)}$ in $T_{(p,u)}TM$ using the Levi-Civita connection ∇. Define the vector fields $E_1, \ldots, E_n$ on $\pi_M^{-1}(U)$ by

$$E_k = \partial_{x_k} - \sum_{i,j=1} y^i(\Gamma_{ki}^j \circ \pi_M)\partial_{y_j}.$$

The vectors $E_1(p,u), \ldots, E_n(p,u)$ form a basis for a subspace $H_{(p,u)}$ which is called the horizontal space. Similar to $V_{(p,u)}$ one can easily see that

$$H_{(p,u)} = \Big\{\Big(x^i(p); y^i(p); \xi^i; -\sum_{j,k=1}^{n}\big(\Gamma^i_{jk}(p)y^k(p)\xi^j\big)\Big) \;\Big|\; \xi^i \in \mathbb{R}\Big\}.$$

By construction $H_{(p,u)} \cap V_{(p,u)} = 0$ and $T_{(p,u)}TM = V_{(p,u)} \oplus H_{(p,u)}$. Hence any $\eta \in T_{(p,u)}TM$ can be written uniquely as $\eta = \eta_v + \eta_h$, where η_v is the vertical component and η_h is the horizontal component of η.

Definition 1.1.3. *The connection map (or connector), $K : TTM \longrightarrow TM$, of the Levi-Civita connection ∇ is the projection to the vertical component $V_{(p,u)} \simeq T_pM$, that is, $K(\eta) = \iota_{(p,u)}(\eta_v)$ where $\pi_{TM}(\eta) = (p, u)$.*

In local coordinates we can represent K as

$$K(x^i; y^i; \xi^i; \eta^i) = \big(x^i; \eta^i + \sum_{j,k=1}^{n} \Gamma^i_{jk} y^k \xi^j\big).$$

and a simple computation reveals the following relation between K and ∇

$$K(TX(u)) = \nabla_u X$$

where $X \in \mathfrak{X}(M)$ and $u \in TM$.

Recall that any compact connected Lie group G can be endowed with a biinvariant metric g [16, See Corollary 3.7]. The Levi-Civita connection ∇ compatible with (G, g) evaluated on left invariant vector fields $X, Y \in \mathfrak{X}(M)$ satisfies

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

which follows from the third formula in the Fundamental theorem of Riemannian Geometry 1.1.2. Now if we consider a maximal torus $T \hookrightarrow G$ then it follows that T is flat with respect to biinvariant metrics on G, i.e., the curvature tensor vanishes since $\mathfrak{t}$ is Abelian. Next, we show that T is totally geodesic, and in fact that any closed Lie subgroup is totally geodesic. This is a direct consequence of the fact that Lie-theoretic exponential coincides with the Riemannian exponential of biinvariant metrics. However, since this is not the case for the infinite-dimensional groups that we will later consider we prefer to give a direct argument. We need to show that for every $t \in T$ and $v \in \mathfrak{t}$, the geodesic passing through t at time $s = 0$ with speed v remains in T. First, note that $c(s) = \exp^G(sv)$, the integral curve through the identity of the left invariant vector field V generated by v, is a geodesic since

$$\begin{aligned} \dot{c}(s) &= V(c(s)) \\ \Longrightarrow \nabla_{\dot{c}}\dot{c} &= \nabla_V V \\ &= \frac{1}{2}[V,V] \\ &= 0 \end{aligned} \tag{1.1.1}$$

On the other hand, for any $h \in G$ the left translation L_h is an isometry so any geodesic passing through h at time $s = 0$ can be written as $c(s) = h\exp^G(su)$ for some $u \in \mathfrak{g}$. In particular, if $h \in T$ then $\dot{c}(0) \in T_hT$ if and only if $u \in \mathfrak{t}$ and hence $\exp^G(su) = \exp^T(su) \in T$ for every s. This implies that T is a totally geodesic submanifold of G with respect to biinvariant metrics.

Finally, we gather all the main properties of the maximal torus $T \hookrightarrow G$ in the following theorem.

Theorem 1.1.4. *Let G be a compact connected Lie group and $T \hookrightarrow G$ a maximal torus of G. Then T*

- *Topologically: is a path-connected, finite-dimensional, smooth and closed submanifold of G.*

- *Algebraically: is a maximal Abelian subgroup of G and its Weyl group, $W = {}^{N(T)}/_{T}$ is finite.*

- *Geometrically: is a totally geodesic and flat Riemannian submanifold of G with respect to biinvariant metrics on G.*

Our goal in chapter 2 will be to find an analog of a maximal torus in the group of symplectomorphisms of toric manifolds that have most of these properties.

1.2 Manifold Structures on Spaces of Maps

Let us begin with recalling theory of Sobolev spaces for a domain $\Omega \subset \mathbb{R}^n$ endowed with the Lebesgue measure. For more detailed discussion about Sobolev spaces one can look into [31, 47]. Ebin, Marsden and Fischer's paper [47] contains a full discussion about Sobolev spaces on manifolds and other relevant material about manifolds of maps. Given a function $u \in L^1(\Omega)$ and a multi-index α the distributional derivative of u, $D^\alpha u$, satisfies

$$\langle D^\alpha u, \rho \rangle = (-1)^{|\alpha|} \int_\Omega u D^\alpha \rho$$

for every smooth compactly supported function $\rho \in \mathscr{C}_c^\infty(\Omega)$. A distribution Ξ is called an L^p distribution if there is an $f \in L^p(\Omega)$ such that

$$\langle \Xi, \rho \rangle = \int_\Omega f \rho$$

for every $\rho \in \mathscr{C}_c^\infty(\Omega)$. For given $k, p \in \mathbb{N}$ we define

$$W_k^p(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \forall\, |\alpha| \leq k,\ D^\alpha u \in L^p(\Omega) \right\}.$$

Here the derivatives are taken in the distributional sense. The space W_k^p can be turned into a Banach space with the norm

$$\|u\|_{W_k^p} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p}.$$

In particular, for $p = 2$, $W_k^2(\Omega)$ is a Hilbert space and is denoted by $H^k(\Omega)$. We can also define W_k^p as the completion of the set of all smooth function $u \in \mathscr{C}^\infty\left(\overline{\Omega}\right)$ for which $\|u\|_{W_k^p} < \infty$, provided that $\overline{\Omega}$ is a compact manifold with boundary [4]. By the Sobolev Embedding Theorem there is a continuous linear inclusion $H^s(\Omega) \hookrightarrow C^k\left(\overline{\Omega}\right)$ for every $s > k + n/2$. These results can be easily extended to the space of vector valued Sobolev functions $H^s(\Omega, \mathbb{R}^m)$.

Now we turn our attention to manifolds. Suppose M and N are d and ℓ dimensional manifolds possibly with boundary. In addition, suppose M is compact. A map $f : M \longrightarrow N$ is called of class H^s, $s > d/2$, if for any $m \in M$ there is a chart (U, φ) of M around m and a chart (V, ψ) of N around $f(m)$ such that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1} \in H^s\left(\varphi(U), \mathbb{R}^\ell\right)$. We denote by $H^s(M, N)$ the set of all H^s maps $f : M \longrightarrow N$. We claim that this definition does not depend on the choice of the charts. This can be easily checked using the following local ω-Lemma.

Theorem 1.2.1. *[Local ω-Lemma, [47]] Let U be a an open, bounded subset of $\mathbb{R}^p$, and suppose $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is C^∞-smooth. Then for $s > p/2$, ω_h : $H^s(U, \mathbb{R}^n) \longrightarrow H^s(U, \mathbb{R}^m)$ defined by $\omega_h(f) = h \circ f$ is a C^∞-smooth map.*

We remark that it is necessary to assume that M is compact in order for the definition of H^s maps to be independent of the choice of Riemannian metrics on M [31]. It is also important to assume $s > d/2$ to ensure independence on the choice of the charts [47]. Now we explain how we can construct a chart for $H^s(M, N)$, provided that N is compact and without boundary. First we need to find a candidate for our modelling space, and as in the finite-dimensional setting the modelling space can be chosen to be the tangent space to $H^s(M, N)$ at any given point. So we look for a candidate for the tangent space to $H^s(M, N)$ at point $\eta \in H^s(M, N)$. For simplicity, take a smooth path $(t, m) \in (-\epsilon, \epsilon) \times M \longmapsto \eta_t(m) \in N$, such that at $t = 0$ it coincides with η. Hence $\frac{d}{dt}\Big|_{t=0} \eta_t(m) \in T_{\eta(m)}N$, this suggests that a tangent vector $X_\eta \in T_\eta H^s(M, N)$ is a map $X_\eta : M \longrightarrow TN$ such that $X_\eta(m) \in T_{\eta(m)}N$, so

$$\begin{aligned} T_\eta H^s(M, N) &:= H^s(\eta^* TN \to M) \\ &= \Big\{ X \in H^s(M, TN) \;\Big|\; \pi_N \circ X = \eta \Big\} \end{aligned}$$

where $H^s(\eta^* TN \to M)$ is the space of all H^s sections of the pull-back bundle $\eta^* TN \to M$. $H^s(\eta^* TN \to M)$ is a Hilbertable or Hilbertian space, that is, it can be turned into a Hilbert space but not in a canonical way [47, 24].

Now to construct a chart for $H^s(M, N)$, we choose a Riemannian metric on N and we consider its associated exponential map $\exp_p : T_pN \longrightarrow N$ which is defined as follows. Given $v_p \in T_pN$, there is a unique geodesic γ_{v_p} through p whose speed at p is v_p. The exponential map is defined by $\exp_p(v_p) = \gamma_{v_p}(1)$. It is known that the exponential map is a diffeomorphism from a neighbourhood of 0 in T_pN to a neighbourhood of p in N. However, by Hopf-Rinow theorem, as we have assumed N is compact and without boundary, N is geodesically complete, i.e., the exponential map is defined on the whole T_pN.

This map can be extended to a map $\exp : TN \longrightarrow N$ such that $\exp(v_p) = \exp_p(v_p)$. We use this map to define a map $\overline{\exp}_\eta : T_\eta H^s(M,N) \longrightarrow H^s(M,N)$ as follows

$$X \longmapsto \exp \circ X.$$

To check that this map takes values in $H^s(M,N)$ one can use the Local ω-Lemma 1.2.1. To show that the transition maps for the charts are smooth we need the following Global ω-Lemma [55, See Theorem 11.3].

Theorem 1.2.2. *[Global ω-Lemma, [55]] Let M be compact and E, F be two vector bundles over M and $h : E \longrightarrow F$ be a smooth fiber-preserving map. Then for $s > d/2$, the map $\omega_h : H^s(E \to M) \longrightarrow H^s(F \to M)$*

$$\omega_h(z) = h \circ z$$

is a smooth map and its Fréchet derivatives are given by $D^k\omega_h = \omega_{F^k h}$, where $F^k h$ denotes the k-th fiber derivative of h.

We can similarly show that $C^k(M,N)$ is a smooth Banach manifold with tangent space at $\eta \in C^k(M,N)$ given by

$$T_\eta C^k(M,N) = C^k(\eta^* TN \to M).$$

Finally we remark that if $s > k + d/2$ the Sobolev Embedding Theorem holds, that is, there is a smooth inclusion

$$H^s(M,N) \hookrightarrow C^k(M,N).$$

Assume M is a oriented compact without boundary then $H^s(M,M)$ is an infinite-dimensional manifold which contains several natural submanifolds. By $C^k_{\mathscr{D}}(M)$ we denote the set of all orientation preserving diffeomorphisms of M of class C^k, and we write $\mathscr{D}^s(M) = H^s(M,M) \cap C^1_{\mathscr{D}}(M)$ for the set of all orientation preserving diffeomorphisms of M of Sobolev class s, where $s > d/2 + 1$. It is known that

$$\mathscr{D}^s(M) = \left\{ \eta \in H^s(M,M) \,\middle|\, \eta \text{ is a bijection and } \eta^{-1} \in H^s(M,M) \right\}$$

see [26]. Moreover, $\mathscr{D}^s(M)$ is a topological group which is open in $H^s(M,M)$. As $\mathscr{D}^s(M)$ is open its "Lie algebra" $T_e\mathscr{D}^s(M)$ is $\mathfrak{X}^s(M) := H^s(TM \to M)$ which is the set of all H^s vector fields on M. In the case one needs to consider manifolds with boundary, one needs to first construct the boundaryless double, $\widetilde{M}$, in order to define $\mathscr{D}^s(M) = H^s(M,\widetilde{M}) \cap C^1_{\mathscr{D}}(M)$. Then $\mathscr{D}^s(M)$ is a submanifold of $H^s(M,\widetilde{M})$ and $T_e\mathscr{D}^s(M) = \mathfrak{X}^s_{\|}(M)$, where $\mathfrak{X}^s_{\|}(M)$ is the set of all H^s vector fields that are parallel to the boundary [26].

If we endow M with a volume form ν, or with a symplectic form ω, then we can consider the subgroups of volume preserving diffeomorphisms $\mathscr{D}^s_\nu(M)$, and of symplectomorphisms $\mathscr{D}^s_\omega(M)$. It can be shown, using the Submersion Theorem, that $\mathscr{D}^s_\nu(M)$ and $\mathscr{D}^s_\omega(M)$ are submanifolds of $\mathscr{D}^s(M)$ whose tangent spaces at the identity are given by

$$T_e\mathscr{D}^s_\nu(M) = \mathfrak{X}^s_\nu(M), \quad T_e\mathscr{D}^s_\omega(M) = \mathfrak{X}^s_\omega(M)$$

respectively, where $\mathfrak{X}^s_\nu(M)$ is the set of all divergence free vector fields and $\mathfrak{X}^s_\omega(M)$ is the set of all symplectic vector fields [26]. Also, we point out that similar results hold when M has boundary. In that case, the tangent spaces at the identity consist of vector fields that are parallel to the boundary [26].

Finally we remark that $\mathscr{D}(M)$, the full group of C^∞ diffeomorphisms, can be endowed with the structure of a Fréchet Lie group. We will not consider Fréchet manifolds in our work. However, some of our results do hold in the Fréchet category. We refer to [43] for a complete treatment of differential calculus on Fréchet spaces.

1.3 $\mathscr{D}^s(M)$ as a Lie Group

The group $\mathscr{D}^s(M)$ is a differentiable Hilbert manifold so it is natural to ask if the composition and the inversion maps are smooth with respect to this differentiable structure. Let $\mu : \mathscr{D}^s(M) \times \mathscr{D}^s(M) \longrightarrow \mathscr{D}^s(M)$ be the composition map. To compute the tangent map $T_{(\eta,\xi)}\mu : T_\eta\mathscr{D}^s(M) \times T_\xi\mathscr{D}^s(M) \longrightarrow T_{\eta\circ\xi}\mathscr{D}^s(M)$ at the point (η, ξ), we choose smooth paths $\eta(t), \xi(t)$ in $\mathscr{D}^s(M)$ such that $\eta(0) = \eta, \xi(0) = \xi$ and $\dot{\eta}(0) = V_\eta \in T_\eta\mathscr{D}^s(M)$, $\dot{\xi}(0) = W_\xi \in T_\xi\mathscr{D}^s(M)$. Then

$$\begin{aligned} T_{(\eta,\xi)}\mu(V_\eta, W_\xi) &= \frac{d}{dt}\Big|_{t=0}\mu(\eta(t), \xi(t)) \\ &= \frac{d}{dt}\Big|_{t=0}[\eta(t) \circ \xi(t)] \\ &= V_\eta \circ \xi + T\eta \circ W_\xi. \end{aligned}$$

Since the term $T\eta$ appears in the right hand side of the above formula, the tangent map does not take values in $T_{\eta\circ\xi}\mathscr{D}^s(M)$, which is due to a loss of derivative in $T\eta$. Hence μ is not smooth. Similarly we can check that the inversion map is not smooth. However, we have the following theorem about the group operations on $\mathscr{D}^s(M)$.

Theorem 1.3.1. *[25, 27, 47, 26] Let M be a compact, connected, and orientable manifold possibly with boundary.*

(1) $(\mathscr{D}^s(M), \circ)$ *is a topological group.*

(2) *(Global α-Lemma) If $\eta \in \mathscr{D}^s(M)$, then the right translation*

$$R_\eta : \mathscr{D}^s(M) \longrightarrow \mathscr{D}^s(M), \quad R_\eta(\zeta) = \zeta \circ \eta$$

is smooth.

(3) *(ω-Lemma) If $\eta \in \mathscr{D}^{s+\ell}(M)$, then the left translation*

$$L_\eta : \mathscr{D}^s(M) \longrightarrow \mathscr{D}^s(M), \quad L_\eta(\zeta) = \eta \circ \zeta$$

is of class C^ℓ.

(4) *(Composition) More generally, the composition map*

$$\mu : \mathscr{D}^{s+\ell}(M) \times \mathscr{D}^s(M) \longrightarrow \mathscr{D}^s(M), \quad \mu(\eta, \zeta) = \eta \circ \zeta$$

is of class C^ℓ.

(5) *(Inversion) The inversion map*

$$\nu : \mathscr{D}^{s+\ell}(M) \longrightarrow \mathscr{D}^s(M), \quad \nu(\eta) = \eta^{-1}$$

is of class C^ℓ.

It follows from Theorem 1.3.1 that $\mathscr{D}^s(M)$ is not a Lie group in the usual sense. Nevertheless, it shares enough of the properties of Lie groups to make analogies that can be used effectively in the infinite-dimensional setting.

For example, since the right-multiplication is smooth we can identify each tangent space $T_\eta \mathscr{D}^s(M)$ with $T_e\mathscr{D}^s(M) = \mathfrak{X}^s_{\parallel}(M)$ and so $\mathfrak{X}^s_{\parallel}(M)$ can serve as the Lie algebra of $\mathscr{D}^s(M)$. Similarly, we can talk about right-invariant vector fields on $\mathscr{D}^s(M)$. If $u \in \mathfrak{X}^{s+\ell}_{\parallel}(M)$ then by the ω-Lemma, the map $u^R : \mathscr{D}^s(M) \to T\mathscr{D}^{s+\ell}$, $u^R(\eta) = u \circ \eta$, defines a right-invariant vector field of class C^ℓ on $\mathscr{D}^s(M)$. Conversely, if $\widetilde{u}$ is a right invariant vector field of class C^ℓ then $\widetilde{u}(e) \in \mathfrak{X}^{s+\ell}_{\parallel}(M)$, and we see that the space of right-invariant vector fields of class C^ℓ on $\mathscr{D}^s(M)$ is isomorphic to $\mathfrak{X}^{s+\ell}_{\parallel}(M)$ [26].

Let $\ell \geq 1$ then for any $u, v \in \mathfrak{g}^{s+\ell} := \mathfrak{X}^{s+\ell}_{\parallel}(M)$ we define their Lie bracket by

$$[u, v]_{\mathfrak{g}} := [u^R, v^R]_{\mathscr{D}^s}(e).$$

It follows that $[u^R, v^R]_{\mathscr{D}^s}(e) = [u, v]_M$ so the Lie bracket on the Lie algebra $\mathfrak{X}^s_{\parallel}(M)$ induced from the differential structure on $\mathscr{D}^s(M)$ is the same as the usual Lie bracket on $\mathfrak{X}^{s+l}_{\parallel}(M)$. Note that $[u, v] \notin \mathfrak{X}^{s+\ell}_{\parallel}(M)$ due to a loss of derivative, it is in $\mathfrak{X}^{s+\ell-1}_{\parallel}(M)$.

Finally, we remark that it is possible to define the Lie group exponential map for $\mathscr{D}^s(M)$ but it is not a local diffeomorphisms, contrary to finite-dimensional Lie groups [26, 47].

1.4 A Weak Riemannian Metric on $\mathscr{D}^s(M)$

The idea of existence of a right invariant metric and its physical applications is pioneered by Arnold [7] and has been studied in the rigorous language of analysis by Ebin and Marsden [26]. We briefly review the relevant materials from [26] and [47]. For a compact Lie group G there exists a biinvariant metric, and one can consider the exponential map of this metric. We have seen that the Lie group exponential map is the same as the exponential map of any biinvariant metric on G. In the case of $\mathscr{D}^s(M)$ there is a right invariant metric which is not left invariant and so the two exponential maps will generally not coincide. The existence of general biinvariant metrics on diffeomorphism groups has been studied by Smolentsev, see [71] for a comprehensive survey of this problem. However, if we are looking for "nice" biinvariant metrics on $\mathscr{D}^s(M)$, nice in the sense that it is biinvariant and it has a compatible Levi-Civita connection that admits a smooth geodesic spray, then it seems unlikely that such metrics exist if we do not impose specific condition on M. The main point is that the existence of such a metric implies that both the Lie theoretic exponential and the Riemannian exponential must coincide. Hence they have to have the same regularity. In general, the former is merely continuous and into, while the latter is smooth and onto [47].

Let $(M, \langle , \rangle)$ be a compact and orientable Riemannian n-manifold without boundary. Given a point $p \in M$, we denote by $\langle , \rangle_p$ the inner product on T_pM. Now we define a metric on $\mathscr{D}^s(M)$ as follows. For any $\eta \in \mathscr{D}^s(M)$ and any $u, v \in T_\eta \mathscr{D}^s(M)$ we define

$$(u, v)_\eta = \int_M \langle u(p), v(p) \rangle_{\eta(p)} \, d\nu. \tag{1.4.2}$$

Obviously, this is a symmetric bilinear form on each tangent space $T_\eta \mathscr{D}^s(M)$, but it is not right invariant. The topology induced by this inner product is the H^0 topology which is clearly weaker that the H^s topology. Hence the metric is called a *weak Riemannian metric*. This metric is smooth in the sense that it is a smooth section of the vector bundle of bounded bilinear maps, $B\left(T_\eta \mathscr{D}^s(M), T_\eta \mathscr{D}^s(M)\right)$. Indeed, the fact that $\eta \longmapsto (,)_\eta$ is a smooth section that belongs to $B\left(T_\eta \mathscr{D}^s(M), T_\eta \mathscr{D}^s(M)\right)$ simply follows from ω-Lemma 1.2.2.

It is interesting to see that there is a Levi-Civita connection compatible with this Riemannian metric. One may try to mimic the finite-dimensional argument to prove the existence of such a connection but as the Riemannian metric is weak this just shows the uniqueness of such a connection and no conclusion can be made about the existence. We now recall the existence of the Levi-Civita connection compatible with $(,)$ that is established in [26]. A more detailed discussion also can be found in [27].

Let ∇ be the Levi-Civita connection of $(M, \langle , \rangle)$. We know that there is a corresponding connection map $K : TTM \longrightarrow TM$. Recall that the relation between K and ∇ is given by

$$\nabla_u X = K(TX(u)).$$

We define a connection map $\overline{K} : TT\mathscr{D}^s(M) \longrightarrow T\mathscr{D}^s(M)$ as follows. Note that

$$\begin{aligned} T\mathscr{D}^s(M) &= \left\{ X \in H^s(M, TM) \,\middle|\, \pi_M \circ X \in \mathscr{D}^s(M) \right\} \\ TT\mathscr{D}^s(M) &= \left\{ Y \in H^s(M, TTM) \,\middle|\, \pi_{TM} \circ Y \in T\mathscr{D}^s(M) \right\} \end{aligned}$$

and set $\overline{K} : TT\mathscr{D}^s(M) \longrightarrow T\mathscr{D}^s(M)$ as $\overline{K}(Y) = K \circ Y$. Since K is smooth, by the ω-Lemma 1.2.2 $\overline{K}$ is also a smooth map. As in the finite-dimensional setting we define

$$\overline{\nabla}_X Y = \overline{K}(TY(X))$$

where X, Y are smooth vector fields on $\mathscr{D}^s(M)$. In [26] it is shown that the previous formula induces a genuine connection, namely

Theorem 1.4.1. *[26] $\overline{\nabla}$ is the Levi-Civita connection compatible with the weak Riemannian metric $(,)$ on $\mathscr{D}^s(M)$.*

It is very interesting to consider two smooth (C^∞) right invariant vector fields $\widetilde{u}$ and $\widetilde{v}$ on $\mathscr{D}^s(M)$, corresponding to $u, v \in \mathfrak{X}(M)$, and compute $\overline{\nabla}_{\widetilde{v}}\widetilde{u}$ at $\eta \in \mathscr{D}^s(M)$. This gives

$$\begin{aligned}
\left[\overline{\nabla}_{\widetilde{v}}\widetilde{u}\right](\eta) &= \left[\overline{K}(T\widetilde{u}(\widetilde{v}))\right](\eta) \\
&= \left[K \circ Tu \circ v\right](\eta) \\
&= \left[K(Tu(v))\right] \circ \eta \\
&= \left[\nabla_v u\right] \circ \eta
\end{aligned}$$

One should compare the above formula to the corresponding formula for finite-dimensional Lie groups.

We can restrict the weak Riemannian metric 1.4.2 to $\mathscr{D}^s_\nu(M)$, where ν is the volume of the metric on M. This metric is going to be right invariant and one should expect to have an induced compatible Levi-Civita connection on $\mathscr{D}^s_\nu(M)$. But this is not true in general as we only have a weak Riemannian structure. Nevertheless, the Hodge decomposition will make it possible to construct a Levi-Civita connection on $\mathscr{D}^s_\nu(M)$.

Let $H^s(\Omega^k(M))$ be the space of all k-forms on M of Sobolev class s and recall that we have the exterior derivative operator

$$d : H^{s+1}(\Omega^k(M)) \longrightarrow H^s(\Omega^{k+1}(M))$$

which is a first order differential operator, that is, it drops the differentiability degree by 1. Also, the Hodge star operator

$$* : H^s(\Omega^k(M)) \longrightarrow H^s(\Omega^{n-k}(M))$$

is defined by

$$\alpha_1 \wedge *\alpha_2 = \ll \alpha_1, \alpha_2 \gg \nu,$$

where $\alpha_1, \alpha_2 \in H^s(\Omega^k(M))$. Here $\nu = \sqrt{\det(g_{ij})}dx^1 \wedge \ldots \wedge dx^n$ is the Riemannian volume on $(M, \langle,\rangle)$ and if $\alpha_i = \sum_{m_\ell} \alpha_i^{m_1 \ldots m_k} dx^{m_1} \wedge \ldots \wedge d^{m_k}$ then $\ll \alpha_1, \alpha_2 \gg = \sum_{m_\ell n_\ell} g_{m_1 n_1} \ldots g_{m_k n_k} \alpha_1^{m_1 \ldots m_k} \alpha_2^{n_1 \ldots n_k}$ is the induced inner product by $\langle,\rangle$ on the space of k-forms.

The adjoint of d with respect to the inner product

$$\{\alpha, \beta\} = \int_M \alpha \wedge *\beta = \int_M \ll \alpha, \beta \gg \nu$$

on $H^s(\Omega^k(M))$ is $\delta : H^{s+1}(\Omega^k(M)) \longrightarrow H^s(\Omega^{k-1}(M))$ and is given by $\delta = (-1)^{n(k+1)+1} * d *$. Given any vector field $X \in \mathfrak{X}^{s+1}(M)$, its divergence $div(X)$ can be computed in terms of δ as

$$div(X) = -\delta(X^\flat)$$

where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the pairing induced by the metric on M and whose inverse is denoted by $\sharp$.

The Laplace operator $\Delta : H^s(\Omega^k(M)) \longrightarrow H^{s-2}(\Omega^k(M))$ is defined by $\Delta = d\delta + \delta d$. The classical Laplacian is essentially $-\Delta(f)$ for $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. More detailed discussion about Hodge star operator, δ and their properties can be found in [44].

A k-form α is called harmonic if $\Delta\alpha = 0$. It turns out that all harmonic forms are smooth [47].

Theorem 1.4.2. *[Hodge Theorem, [47]] Let M be a compact orientable n-manifold without boundary. Then any k-form $\alpha \in H^s(\Omega^k(M))$ can be written as*

$$\alpha = d\beta + h + \delta\gamma,$$

where $\beta \in H^{s+1}(\Omega^{k-1}(M))$, $\gamma \in H^{s+1}(\Omega^{k+1}(M))$, and h is a harmonic smooth k-from. Moreover, $d\beta$, $\delta\gamma$, and h are mutually H^0-orthogonal and hence, are determined uniquely.

Now let X be an H^s vector field on M, $s \geq 0$, and consider $X^\flat$. By the Hodge Theorem 1.4.2, $X^\flat$ can be decomposed as $X^\flat = d\alpha + h + \delta\beta$. Here as $k = 1$, α is a function of Sobolev class $(s+1)$ and obviously we have $\delta(h + \delta\beta) = 0$. Hence, if we put $p = \alpha$ and $Y = (h + \delta\beta)^\sharp$ then we have the following

Lemma 1.4.3. *[47] Let $X \in H^s(TM \to M)$, $s \geq 0$. Then there is an unique divergence free H^s vector field Y and a gradient vector field $\mathsf{grad}(p)$ such that*

$$X = Y + \mathsf{grad}(p).$$

Moreover, putting $P(X) = Y$, P is a bounded linear operator in H^0 topology.

Observe that the Hodge decomposition can be performed in the stronger H^s topology, in which case we obtain an H^s-continuous projection P.

Now we return to $(\mathscr{D}^s_\nu(M), (,))$ to define a Levi-Civita connection compatible with the right invariant metric $(,)$ obtained by restriction to $\mathscr{D}^s_\nu(M)$. Using 1.4.3, for any $\eta \in \mathscr{D}^s_\nu(M)$ we can define an H^0-orthogonal projection map $P_\eta : T_\eta \mathscr{D}^s(M) \longrightarrow T\mathscr{D}^s_\nu(M)$ by setting

$$P_\eta(u) = T_e R_\eta \circ P \circ T_\eta R_{\eta^{-1}}(u).$$

Indeed we have

$$T_\eta \mathscr{D}^s(M) = T_\eta \mathscr{D}^s_\nu(M) \oplus \left[\mathsf{grad}\big(H^{s+1}(M, \mathbb{R})\big)\right] \circ \eta$$

Theorem 1.4.4. *[26] The orthogonal projection* $\overline{P} : T\mathscr{D}^s(M)\Big|_{\mathscr{D}^s_\nu(M)} \longrightarrow T\mathscr{D}^s_\nu(M)$ *defined on each fiber by* P_η *is a smooth vector bundle morphism.*

As it is in the case of finite-dimensional Riemannian geometry $\widehat{\nabla} = \overline{P} \circ \overline{\nabla}$ is the Levi-Civita connection on $(\mathscr{D}^s_\nu(M), (,))$.

Theorem 1.4.5. *[26] Let* M *be a compact and orientable manifold without boundary. The* $\widehat{\nabla}$ *is the Levi-Civita connection compatible with the right invariant metric* $(,)$ *on* $\mathscr{D}^s_\nu(M)$.

Similar results hold for the case that M has smooth boundary [26]. More detailed discussion about the Hodge decomposition on manifolds with boundary can be found in [68].

Finally, we turn our attention to the symplectomorphism group $\mathscr{D}^s_\omega(M)$. We can restrict $(,)$ to $\mathscr{D}^s_\omega(M)$, as in the case of volume preserving diffeomorphisms , we are interested in the existence of a Levi-Civita connection on $(\mathscr{D}^s_\omega(M), (,))$. Again, the problem reduces to finding an H^0-orthogonal complement for the tangent space $T_e\mathscr{D}^s_\omega(M)$.

Given any compact symplectic manifold (M, ω) without boundary, there is a compatible almost complex structure $\mathbf{J}$ on TM such that $\langle -, - \rangle_{\mathbf{J}} := \omega(-, \mathbf{J}-)$ defines a Riemannian metric on M [46, See Proposition 4.1]. Given this fact and the Hodge decomposition 1.4.2, we can construct the H^0-orthogonal complement of the tangent space $T_e\mathscr{D}^s_\omega$ and hence we can construct a compatible Levi-Civita connection on $(\mathscr{D}^s_\omega(M), (,))$, where $(,)$ is obtained by integrating $\langle , \rangle_{\mathbf{J}}$. We will denote the corresponding projection and connection in this case by $\widehat{P}$ and $\widetilde{\nabla} := \widehat{P} \circ \overline{\nabla}$, respectively.

1.5 Classical Schur-Horn-Kostant Convexity

A well known result of Schur, proven in 1923, states that the diagonal $\mathbf{a} = (a_{11}, \ldots, a_{nn})$ of a hermitian matrix A with spectrum $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is contained in the convex hull of $S_n \cdot \boldsymbol{\lambda}$, where S_n acts on $\mathbb{R}^n$ by permutation of coordinates. In 1954, through a delicate argument, Horn showed that the converse of the Schur's theorem is true namely, each point in the convex hull can be obtained this way [36]. In 1973, Kostant generalized Schur-Horn theorem to all compact Lie groups where he reinterpreted his result as a property of the co-adjoint orbits [42]. In particular, Schur-Horn theorem is a property of (co)-adjoint orbits of the unitary group $\mathbf{U}(n)$. In 1982, Atiyah and Guillemin-Sternberg independently proved the convexity theorem for moment maps in symplectic geometry. This enabled them to further generalize Kostant's result. Since then, the Schur-Horn-Kostant theorem is understood as a special case of Atiyah-Guillemin-Sternberg convexity theorem [20, See Chapters 21 and 27] and [8, See Corollary IV.4.11]. We are interested in the classical proof of Schur-Horn convexity rather than the symplectic proof. As we will see later on, a similar argument can be used to construct an infinite-dimensional analog of Schur-Horn convexity theorem. A good exposition of Schur's and Horn's Theorems can be found in [11, See Theorem 6.2]. We will follow [11] and [49] and notations therein.

The closed convex hull of finitely many points in $\mathbb{R}^n$ is called a polytope. A vertex of a polytope Υ, is a point in Υ that cannot be written as a convex combination of two distinct points in Υ.

Definition 1.5.1. *Let $\kappa \in S_n$, be a permutation of n letters. The permutation matrix $X^\kappa = [x^\kappa_{ij}]$ is the $n \times n$ matrix defined by*

$$x^\kappa_{ij} = \begin{cases} 1 & \kappa(j) = i \\ 0 & \textit{otherwise.} \end{cases} \tag{1.5.3}$$

Definition 1.5.2. *An $n \times n$ matrix $A = [a_{ij}]$ is called doubly stochastic if*

(1) $\forall i, j : a_{ij} \geq 0$, *i.e., A is a non-negative matrix.*

(2) *The sum of entries in each row and in each each column is* 1*. That is*

$$\forall j : \sum_i a_{ij} = 1 \ (\iff A\mathbf{e} = \mathbf{e}) \quad , \quad \forall i : \sum_j a_{ij} = 1 \ (\iff \mathbf{e}^\intercal A = \mathbf{e}^\intercal).$$

where $\mathbf{e} = [1, \ldots, 1]^\intercal$.

All $n \times n$ doubly stochastic matrices form a polytope which is called the Birkhoff polytope of order n, and which is denoted by B_n. The following theorem characterizes all extreme points of B_n [11, See Theorem 5.2].

Theorem 1.5.3. *[Birkhoff-von Neumann Theorem, [11]] The vertices of the Birkhoff polytope B_n are exactly the $n \times n$ permutation matrices.*

Definition 1.5.4. *Let* $\mathbf{x} \in \mathbb{R}^n$*. Then the permutation polytope* $\Upsilon(\mathbf{x})$ *is defined by*

$$\Upsilon(\mathbf{x}) := \textbf{Conv}\left\{\kappa \cdot \mathbf{x} \;\middle|\; \kappa \in S_n\right\}.$$

Next we establish a relation between $\Upsilon(\mathbf{x})$ and doubly stochastic matrices. This is done through the theory of majorization [49]. For any $\mathbf{u} \in \mathbb{R}^n$ the decreasing rearrangement of $\mathbf{u}$, denoted by $\mathbf{u}^*$, is obtained by reordering the entries of $\mathbf{u}$ in a decreasing order.

Definition 1.5.5. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*, we say* $\mathbf{u}$ *majorizes* $\mathbf{v}$ *and we write* $\mathbf{v} \prec \mathbf{u}$ *if*

(1) *for all* $k \leq n$, $\sum_{i=1}^{k} v_i^* \leq \sum_{i=1}^{k} u_i^*$.

(2) $\sum_{i=1}^{n} v_i^* = \sum_{i=1}^{n} u_i^*$.

The following characterization of doubly stochastic matrices through majorization will enlighten the definition of doubly stochastic operators later [49, See Chapter 2 Theorem A.4.].

Theorem 1.5.6. *[49] An* $n \times n$ *matrix* P *is doubly stochastic if and only if* $P\mathbf{u} \prec \mathbf{u}$ *for all* $\mathbf{u} \in \mathbb{R}^n$*.*

Conversely, we can characterize majorization using doubly stochastic matrices [49, See Chapter 2 Theorem B.2.].

Theorem 1.5.7. *[49] Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*, then* $\mathbf{v} \prec \mathbf{u}$ *if and only if there is a doubly stochastic matrix* P *such that* $P\mathbf{u} = \mathbf{v}$*.*

From Birkhoff-von Neumann Theorem 1.5.3, and these two characterizations it follows that

$$\Upsilon(\mathbf{x}) = \left\{\mathbf{u} \in \mathbb{R}^n \;\middle|\; \mathbf{u} \prec \mathbf{x}\right\}. \tag{1.5.4}$$

Hence $\Upsilon(\mathbf{x})$ is the closed convex hull of $\{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^* = \mathbf{x}^*\}$. This fact will be interesting when we construct orbits in the infinite-dimensional analog of the Schur-Horn-Kostant convexity.

For proof of Horn's theorem we need the following definition of orthostochastic matrices.

Definition 1.5.8. *An $n \times n$ matrix $B = [b_{ij}]$ is called an orthostochastic matrix if there is an orthogonal matrix $Q = [q_{ij}]$ such that $a_{ij} = q_{ij}^2$.*

Finally, we mention the following theorem about characterizing majorization via orthostochastic matrices which will be used in the proof of Schur-Horn theorem [49, See Chapter 2 Theorem B. 6.].

Theorem 1.5.9. *[49] Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $\mathbf{v} \prec \mathbf{u}$ if and only if there is an orthostochastic matrix Q such that $Q\mathbf{u} = \mathbf{v}$.*

Now we are ready to state the classical Schur-Horn Theorem [11, See Theorem 6.2].

Theorem 1.5.10. *[Schur-Horn Theorem, [11]] Let n be a fixed positive integer, and $\lambda_1, \ldots, \lambda_n$ be real numbers. Assume $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$.*

(1) ***Schur's Theorem:*** *Let $A = [a_{ij}]$ be a complex Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the diagonal $\mathbf{a} = (a_{11}, \ldots, a_{nn})$ belongs to the permutation polytope $\Upsilon(\boldsymbol{\lambda})$.*

(2) ***Horn's Theorem:*** *Let $\mathbf{a} \in \Upsilon(\boldsymbol{\lambda})$, then there is an $n \times n$ real symmetric matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ and the diagonal $\mathbf{a}$.*

Proof. We briefly recall the classical proof of Schur's and Horn's Theorems. Let A be an $n \times n$ matrix with spectrum $\boldsymbol{\lambda}$, by spectral theorem from linear algebra we know that A is unitarily diagonalizable. That is, there is an unitary matrix U such that $U^*AU = \boldsymbol{\lambda}$. Hence we have

$$a_{mm} = \sum_{i=1}^{n} |u_{mi}|^2 \lambda_i.$$

But $U^*U = UU^* = \mathbb{I}_n$ and hence $[|u_{ij}|^2]$ is a doubly stochastic matrix. This concludes the Schur's Theorem. To prove Horn's Theorem note that by Theorem 1.5.9 we can reverse the proof of Schur's Theorem. □

We can write the Schur-Horn Theorem in the Kostant formulation which will become more handy later on. Let $\pi : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^n$ be the map that projects each matrix onto its diagonal, and let $\mathcal{H}_{\boldsymbol{\lambda}}$ be the set of all Hermitian matrices with the given spectrum $\boldsymbol{\lambda}$. The Schur-Horn Theorem 1.5.10 can be written in the Kostant formulation as

$$\pi(\mathcal{H}_{\boldsymbol{\lambda}}) = \Upsilon(\boldsymbol{\lambda}).$$

We can even do better. Let $\mathbf{U}(n)$ be the set of unitary $n \times n$ matrices and $\mathbf{T}^n = \underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n\ times}$ be a maximal torus inside $\mathbf{U}(n)$. The Lie algebra of $\mathbf{U}(n)$, $\mathfrak{u}(n)$, can be identified with the space of Hermitian matrices, $\mathcal{H}$. The adjoint of the inclusion $\mathfrak{t}^n = \mathbb{R}^n \hookrightarrow \mathfrak{u}(n) \simeq \mathcal{H}$, induced from the inclusion $\mathbf{T}^n \hookrightarrow \mathbf{U}(n)$, is $\pi \big|_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathbb{R}^n$ with respect to the canonical inner product on $\mathbb{R}^n$ and the following inner product on $\mathcal{H}$

$$\langle A, B \rangle_{\mathcal{H}} = \mathsf{tr}(AB^*).$$

For a given $A \in \mathcal{H}$ with spectrum $\boldsymbol{\lambda}$, the orbit $\mathscr{O}_A$ of the (co)-adjoint action through A is exactly $\mathcal{H}_{\boldsymbol{\lambda}^*}$. On the other hand, the Weyl group of $\mathbf{T}^n$ is S_n,

$\mathbf{W}_{\mathbf{T}^n} = S_n$. Hence we can rewrite the Kostant formulation of the Schur-Horn convexity as

$$\pi\left(\mathscr{O}_A\right) = \textsf{\textbf{Conv}}\left(\mathbf{W}_{\mathbf{T}^n} \cdot \boldsymbol{\lambda}\right).$$

We can obtain Schur-Horn-Kostant convexity theorem by applying the Atiyah-Guillemin-Sternberg convexity theorem. The action of $\mathbf{U}(n)$ on the coadjoint orbit $\mathcal{H}_{\boldsymbol{\lambda}}$ is Hamiltonian and its moment map is given by the inclusion $\iota : \mathcal{H}_{\boldsymbol{\lambda}} \hookrightarrow \mathfrak{u}(n) \simeq \mathcal{H}$. The restricted action of the maximal torus $\mathbf{T}^n \hookrightarrow \mathbf{U}(n)$ is also Hamiltonian and its moment map is given by $\pi : \mathcal{H}_{\boldsymbol{\lambda}} \longrightarrow \mathbb{R}^n$. By Atiyah-Guillemin-Sternberg convexity the image of π is the closed convex hull of the image of the fixed points of the $\mathbf{T}^n$-action. But the fixed pints of the $\mathbf{T}^n$-action are exactly the diagonal matrices in $\mathcal{H}_{\boldsymbol{\lambda}}$, hence $\pi(\mathcal{H}_{\boldsymbol{\lambda}}) = \textsf{\textbf{Conv}}(S_n \cdot \boldsymbol{\lambda})$. Observe that the symplectic approach to Schur-Horn theorem leads to a slightly weaker statement since it only guarantees the existence of unitary matrices (instead of symmetric) with a given spectrum and diagonal. However, the original statement can be recovered from a stronger version of the convexity theorem that holds in the presence of an anti-symplectic involution [23].

1.6 Symplectic Toric Manifolds

In this section we briefly review symplectic toric manifolds. A detailed discussion about symplectic toric manifolds can be found in [8, 21].

Let (M, ω) be a symplectic manifold and T be the n-torus. If T acts on M in a symplectic way such that there is a map $\mu : M \longrightarrow \mathbb{R}^n$ satisfies the following conditions

- It is invariant with respect to the T-action on M.

- For any $\mathbf{u} \in \mathbb{R}^n$, the component of μ along $\mathbf{u}$, $\mu_{\mathbf{u}} : M \longrightarrow \mathbb{R}$ defined by $\mu_{\mathbf{u}}(x) = \langle \mu(x), \mathbf{u} \rangle_{\mathbb{R}^n}$ is a Hamiltonian of the fundamental vector field U on M obtained from $\mathbf{u}$.

then we say that the action of T on M is Hamiltonian and μ is called the moment map. A $2n$-dimensional symplectic manifold endowed with an effective Hamiltonian action of an n-torus T is called a symplectic toric manifold.

Let (M, ω, T, μ) be a compact connected symplectic toric manifold. The celebrated Atiyah-Guillemin-Stenberg convexity theorem states that the image $\Delta = \mu(M)$ of the moment map is the convex hull of the image of the fixed points M^T. Conversely, Delzant's theorem asserts that any convex polytope satisfying some mild integrability properties is the moment image of a toric manifold which is unique up to equivariant symplectomorphisms.

Two important properties of toric manifolds are especially important in our work. First, the preimage of the interior of Δ can be endowed with a global coordinate system, the so called action-angle coordinates, that defines a diffeomorphism

$$\mu^{-1}(\mathring{\Delta}) \simeq \mathring{\Delta} \times T \tag{1.6.5}$$

Moreover, this diffeomorphism is symplectic, that is, the symplectic form can be written as

$$\omega = \sum_{i=1}^{n} dz_i \wedge d\theta_i$$

where z_i's are coordinates on Δ and θ_i's are coordinates on T. Here z_i's are action or moment coordinates and they are indeed the components of the moment map μ. We should point out that this is a special case of the well-known Arnold-Liouville Theorem [8, See Theorem III.3.3] that provides a local normal form near regular points of integrable systems. The second property is the Duistermaat-Hekman Theorem [20, See Theorem 30.3] which states that the pushforward of the symplectic measure under the moment map μ is the Lebesgue measure on Δ.

Chapter 2

A Maximal Torus in Symplectomorphism Groups

In this chapter, we investigate the properties of a certain maximal torus in the symplectomorphism groups of symplectic toric manifolds. We show that the set of all equivariant symplectomorphisms, $\mathscr{D}^s_\omega(M,T)$, has many of the properties of a finite-dimensional maximal torus mentioned in Theorem 1.1.4. Hence it can be considered as a good candidate for an analog of maximal torus in the infinite-dimensional setting.

2.1 Topological Properties of $\mathscr{D}^s_\omega(M,T)$

Let (M,ω,T,μ) be a $2n$-dimensional compact connected toric manifold without boundary and let $\mathscr{D}^s_\omega(M)$ be the set of all Sobolev symplectomorphisms of (M,ω), where $s > n+1$. We denote by $\mathscr{D}^s_\omega(M,T)$ the set of all equivariant symplectomorphisms of Sobolev class s. Ebin and Marsden showed that $\mathscr{D}^s_\omega(M)$ is a Hilbert submanifold of $\mathscr{D}^s(M)$ [26, See Theorem 4.2]. We will mimic their argument to show that $\mathscr{D}^s_\omega(M,T)$ is a submanifold of $\mathscr{D}^s_\omega(M)$. The main ideas of our proof are as follows:

(1) We will show that the set of all equivariant orientation preserving diffeomorphisms, $\mathscr{D}^s(M,T)$, is a submanifold of $\mathscr{D}^s(M)$.

(2) We will show that $\mathscr{D}^s_\omega(M,T)$ is a submanifold of $\mathscr{D}^s(M,T)$, and hence that it is a submanifold of $\mathscr{D}^s(M)$.

(3) Finally, the fact that $\mathscr{D}^s_\omega(M,T)$ is a submanifold of $\mathscr{D}^s_\omega(M)$ will follow as a simple application of the following 3-inclusion Lemma [30, See Lemma 3.2.5].

Lemma 2.1.1. *[3-inclusion Lemma, [30, 10]] Let A, B and C be such that $A \hookrightarrow C$ and $B \hookrightarrow C$ are submanifolds of C. If $A \subseteq B$ then $A \hookrightarrow B$ is a submanifold of B.*

We remark that the proof of the 3-inclusion Lemma given in [30] seems to be erroneous, but the proof can be corrected using the Immersion Theorem [2, See Theorem 3.5.6].

Theorem 2.1.2. *$\mathscr{D}^s(M,T)$ is a submanifold of $\mathscr{D}^s(M)$.*

Proof. Note that $\mathscr{D}^s(M)$ is open in $H^s(M,M)$ and charts for $H^s(M,M)$ are constructed using exponential map of a Riemannian metric on M, see section 1.2. As the manifolds structure of $H^s(M,M)$ is independent of the choice of the metric, we choose a metric on M which is invariant with respect to the T-action. This implies that exponential map and the T-action commute. Hence, for every $k \in T$ and $X \in \mathfrak{X}^s(M)$ which is in some open neighbourhood of the zero section we have

$$\begin{aligned} k_*(X) = X \quad &\Longleftrightarrow \quad T_p k(X(p)) = X(k \cdot p) \\ &\Longleftrightarrow \quad \exp(T_p k(X(p))) = \exp(X(k \cdot p)) \\ &\Longleftrightarrow \quad k \cdot \exp(X(p)) = \exp(X(k \cdot p)) \end{aligned}$$

It follows that $\overline{\exp}_e(X) \in \mathscr{D}^s(M,T)$ if and only if $X \in \mathfrak{X}^s(M,T)$ which is a closed subspace of $\mathfrak{X}^s(M)$. Hence we have a chart for $\mathscr{D}^s(M,T)$ around the identity. $\square$

Following the idea of [26] we define the subset of 2-forms

$$[\omega]_T^{s-1} = \omega + dH^s\left(\Omega_T^1(M)\right)$$

It follows from the invariant version of the Hodge Theorem 1.4.2 that $[\omega]_T^{s-1}$ is a closed affine subspace of $H^{s-1}\left(\Omega_T^2(M)\right)$. We define

$$\mathcal{G}^s = \left\{\eta \in \mathscr{D}^s(M,T) \;\middle|\; \eta^*(\omega) \in [\omega]_T^{s-1}\right\}$$

Note that the cohomology class of $\eta^*(\omega)$ depends on the homotopy class of η, i.e., the cohomology class of $\eta^*(\omega)$ depends only on the connected component of $\mathscr{D}^s(M,T)$ that contains η. Note also that if $\eta^*(\omega) = \omega + d\alpha$ for some $\alpha \in H^s\left(\Omega^1(M)\right)$ then by averaging this equation over T we can assume, without loss of generality, that α is T- invariant. Hence $\mathcal{G}^s$ is a collection of connected components of $\mathscr{D}^s(M,T)$ and hence it is open in $\mathscr{D}^s(M,T)$.

Theorem 2.1.3. *$\mathscr{D}^s_\omega(M,T)$ is a submanifold of $\mathscr{D}^s(M,T)$.*

Proof. We follow the proof of Theorem 4.2 of [26]. We define the map

$$\begin{aligned} \Phi_\omega : \mathcal{G}^s &\longrightarrow [\omega]_T^{s-1} \\ \eta &\longmapsto \eta^*(\omega). \end{aligned}$$

Note that Φ_ω is smooth [27] and that the preimage of ω is $\mathscr{D}^s_\omega(M,T)$. We will show that it is a submersion. The tangent map $T_e\Phi_\omega$ is given by

$$\begin{aligned} T_e\Phi_\omega : \mathfrak{X}^s(M,T) &\longrightarrow dH^s\left(\Omega^1_T(M)\right) \\ X &\longmapsto \mathscr{L}_X\omega = d\iota_X\omega. \end{aligned}$$

In order to show that Φ_ω is a submersion it is enough to show that $\flat_\omega$: $\mathfrak{X}^s(M,T) \longrightarrow H^s\left(\Omega^1_T(M)\right)$ defined by $\flat_\omega(X) := \iota_X\omega$ is onto. As ω is non-degenerate we need to show that $\flat_\omega(X) \in H^s\left(\Omega^1_T(M)\right)$ if and only if $X \in \mathfrak{X}^s(M,T)$. Let $k \in T$ and $X \in \mathfrak{X}^s(M,T)$ be arbitrary then we have

$$\begin{aligned} k^*\left(\flat_\omega(X)\right) &= \flat_\omega(X) \\ &\Updownarrow \\ \forall p \in M,\ \forall Y \in T_pM \quad &: \quad \omega\Big|_{k\cdot p}\left(X\Big|_{k\cdot p}, T_pk(Y)\right) = \omega\Big|_p\left(X\Big|_p, Y\right) \\ &\Updownarrow \\ \left(k^*\left(\omega\right)\right)\Big|_p\left(X\Big|_p, Y\right) &= \omega\Big|_p\left(X\Big|_p, Y\right) \end{aligned}$$

The last equality holds since $k^*(\omega) = \omega$. The converse of the statement can be proven similarly. □

Theorems 2.1.2 and 2.1.3 together imply that $\mathscr{D}^s_\omega(M,T)$ is a submanifold of $\mathscr{D}^s(M)$. It is also known that $\mathscr{D}^s_\omega(M)$ is a submanifold of $\mathscr{D}^s(M)$ and obviously $\mathscr{D}^s_\omega(M,T)$ is inside $\mathscr{D}^s_\omega(M)$. Hence a simple application of 3-inclusion Lemma 2.1.1 implies the following final result.

Theorem 2.1.4. *$\mathscr{D}^s_\omega(M,T)$ is a submanifold of $\mathscr{D}^s_\omega(M)$.*

We remark that $\mathscr{D}^s_\omega(M,T)$ is path-connected. The proof is given in [57] for smooth symplectomorphisms, since this proof also applies in the Sobolev setting we recall the main steps. Since the fibers of the moment map $\mu : M \longrightarrow \mathbb{R}^n$ are the orbits of the T-action any element in $\mathscr{D}^s_\omega(M,T)$ is of the form $\Psi_\phi(p) := \exp[\phi\circ\mu(p)]\cdot p$, where $\phi : \Delta \longrightarrow \mathbb{R}^n$ is H^s. Moreover, Ψ_ϕ defines a symplectomorphism if and only if ϕ has a symmetric Jacobian. The fact that ϕ is H^s follows from local normal form [8, See Proposition IV.4.21]. Since we can take the path $\gamma : [0,1] \longrightarrow \mathscr{D}^s_\omega(M)$ defined by $\gamma(t) := \Psi_{t\phi}$, $\mathscr{D}^s_\omega(M,T)$ is path-connected and sits inside $(\mathscr{D}^s_\omega(M))_e$, the connected component of $\mathscr{D}^s_\omega(M)$ containing the identity.

2.2 Algebraic Properties of $\mathscr{D}^s_\omega(M,T)$

For a compact Lie group, a maximal torus can be characterized as a path-connected maximal Abelian Lie subgroup. We have seen that $\mathscr{D}^s_\omega(M,T)$ is a path-connected "Lie subgroup" of $\mathscr{D}^s_\omega(M)$. Here we show that $\mathscr{D}^s_\omega(M,T)$ is a maximal Abelian subgroup of $\mathscr{D}^s_\omega(M)$.

Theorem 2.2.1. *$\mathscr{D}^s_\omega(M,T)$ is a maximal Abelian subgroup of $\mathscr{D}^s_\omega(M)$.*

Proof. Note that $\mathscr{D}^s_\omega(M,T)$ is indeed the centralizer of T considered as a subgroup of $\mathscr{D}^s_\omega(M)$. So we just need to show that it is Abelian and the maximality will follow automatically. On toric manifolds we have action-angle coordinates on the pre-image of the interior of the Delzant polytope $\Delta = \mu(M)$ [8, See Remark IV.4.19]. Note that $\mu^{-1}(\mathring{\Delta})$ is dense in M and that it is preserved by any equivariant symplectomorphism. So we just need to show that any two elements in $\mathscr{D}^s_\omega(M,T)$ commute over $\mu^{-1}(\mathring{\Delta})$. We write an equivariant symplectomorphism $\eta \in \mathscr{D}^s_\omega(M,T)$ restricted to $\mu^{-1}(\mathring{\Delta}) \simeq \mathring{\Delta} \times T$ as $(\mathbf{z},\boldsymbol{\theta}) \mapsto (H_1(\mathbf{z},\boldsymbol{\theta}),\ldots,H_n(\mathbf{z},\boldsymbol{\theta}),A_1(\mathbf{z},\boldsymbol{\theta})\ldots,A_n(\mathbf{z},\boldsymbol{\theta}))$. Here $\mathbf{z} = (z_1,\ldots,z_n)$ is the coordinate system on $\mathring{\Delta}$ and $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_n)$ is the coordinate system on T (moment-angle coordinates). Now we use two facts:

(1) η is equivariant which means that for any i we have

$$\begin{aligned} H_i(\mathbf{z}, \boldsymbol{\theta} + \boldsymbol{\alpha}) &= H_i(\mathbf{z}, \boldsymbol{\theta}) \\ A_i(\mathbf{z}, \boldsymbol{\theta} + \boldsymbol{\alpha}) &= A_i(\mathbf{z}, \boldsymbol{\theta}) + \alpha_i + k_i \end{aligned}$$

where $k_i \in \mathbb{Z}$ are constants. These imply that H_i's are only functions of $\mathbf{z}$ and $A_i(\mathbf{z}, \boldsymbol{\theta}) = \theta_i + \varphi_i(\mathbf{z})$.

(2) η is a symplectomorphism, which imposes conditions on the H_i's and the A_i's . The symplectic form can be written in the action-angle coordinates as $\omega = \sum_i dz_i \wedge d\theta_i$ so

$$\begin{aligned} \eta^*\omega = \omega \quad &\Longleftrightarrow \quad \sum_i dH_i \wedge dA_i = \sum_i dz_i \wedge d\theta_i \\ &\Longleftrightarrow \quad \sum_i \left((\sum_j \frac{\partial H_i}{\partial z_j} dz_j) \wedge (d\theta_i + \sum_l \frac{\partial \varphi_i}{\partial z_l} dz_l) \right) = \sum_i dz_i \wedge d\theta_i \\ &\Longleftrightarrow \quad \begin{cases} \frac{\partial H_i}{\partial z_i} = 1 & \\ \frac{\partial H_i}{\partial z_j} = 0, & i \neq j \\ \sum_i \left((\sum_j \frac{\partial H_i}{\partial z_j} dz_j) \wedge (\sum_l \frac{\partial \varphi_i}{\partial z_l} dz_l) \right) = 0 & \end{cases} \\ &\Longleftrightarrow \quad \begin{cases} H_i(z) = z_i + c_i \\ \frac{\partial \varphi_i}{\partial z_l} - \frac{\partial \varphi_l}{\partial z_i} = 0, \quad \forall i, l \end{cases} \end{aligned}$$

Since Δ is compact the constants c_i's are zero. The second condition means that $\varphi(z) = (\varphi_1(\mathbf{z}), \ldots, \varphi_n(\mathbf{z})) : \mathring{\Delta} \mapsto \mathbb{R}^n$ has a symmetric Jacobian which can be consider as the Hessian of some function $\phi : \mathring{\Delta} \mapsto \mathbb{R}$ for which $\varphi = \nabla \phi$.

We have shown that any equivariant symplectomorphism can be written on the open dense subset $\mu^{-1}(\mathring{\Delta})$ as $(\mathbf{z}, \boldsymbol{\theta}) \mapsto (\mathbf{z}, \boldsymbol{\theta} + \nabla\phi(\mathbf{z}))$ for some $\phi : \mathring{\Delta} \mapsto \mathbb{R}$.

Now choose two equivariant symplectomorphisms $\eta_1 : (\mathbf{z}, \boldsymbol{\theta}) \mapsto (\mathbf{z}, \boldsymbol{\theta} + \nabla\phi_1(\mathbf{z}))$ and $\eta_2 : (\mathbf{z}, \boldsymbol{\theta}) \mapsto (\mathbf{z}, \boldsymbol{\theta} + \nabla\phi_2(\mathbf{z}))$ then $\eta_2 \circ \eta_1 : (\mathbf{z}, \boldsymbol{\theta}) \mapsto (\mathbf{z}, \boldsymbol{\theta} + \nabla(\phi_1 + \phi_2)(\mathbf{z}))$ which is the same as $\eta_1 \circ \eta_2$. $\square$

2.3 Geometric Properties of $\mathscr{D}^s_\omega(M, T)$

For a compact Lie group endowed with a biinvariant metric we have seen that a maximal torus is flat and totally geodesic (see Theorem 1.1.4). Flatness followed from the fact that the curvature is tensorial and that the Levi-Civita connection on left invariant vector fields is given by $\nabla_X Y = \frac{1}{2}[X, Y]$. The fact that a maximal torus of a finite-dimensional Lie group is totally geodesic followed from a direct computation, see the discussion before the computation 1.1.1. There is another equivalent criterion to show that a submanifold is totally geodesic. Indeed, a submanifold is totally geodesic if and only if its second fundamental form vanishes [22, See Proposition 2.9] and [41].

We endow $\mathscr{D}^s_\omega(M)$ with the right invariant metric obtained by integrating the "canonical" toric metric on M coming from Delzant's construction. To prove that the infinite-dimensional torus $\mathscr{D}^s_\omega(M, T)$ is flat and totally geodesic with respect to this weak Riemannian structure we use three facts. First, we know that the Levi-Civita connection on $\mathscr{D}^s_\omega(M)$ evaluated on right invariant vector fields has a "very nice" form. Second, the curvature and the second fundamental form are tensorial. Third, for an arbitrary toric manifold Guillemin's result about the matrix representation of the canonical metric in symplectic coordinates allows to carry out computations.

As we mentioned above, Delzant's construction provides us with a "canonical" compatible almost complex structure $\mathbf{J}$ for which we have its representation in action-angle coordinates due to V. Guillemin [34]. We consider the metric $\langle -,-\rangle_{\mathbf{J}} := \omega(-,\mathbf{J}-)$ on M and we construct the corresponding weak Riemannian metric, $(,)$ on $\mathscr{D}^s_\omega(M)$. A good exposition of Guillemin's result can be found in [8], we will follow this reference and notations therein.

We begin by describing the Delzant polytope Δ by a set of inequalities

$$\langle \boldsymbol{x}, \boldsymbol{v}_r\rangle_{\mathbb{R}^n} \geq a_r, \quad r = 1, 2, \ldots, d$$

where $\boldsymbol{v}_r$ are the primitive elements in the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ that are normal to the r-th face of Δ and inward-pointing. Now, we define functions $\ell_r : \mathring{\Delta} \to \mathbb{R}$ by

$$\ell_r(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{v}_r\rangle_{\mathbb{R}^n} - a_r, \quad r = 1, 2, \ldots, d$$

Since $\boldsymbol{x} \in \mathring{\Delta}$ if and only if $\forall\ 1 \leq r \leq d\ :\ \ell_r(\boldsymbol{x}) > 0$. The function $h : \mathring{\Delta} \to \mathbb{R}$ defined by

$$h(\boldsymbol{x}) = \frac{1}{2}\sum_{r=1}^{d} \ell_r(\boldsymbol{x}) \ln \ell_r(\boldsymbol{x})$$

is smooth. With the above notations we have the following result about the canonical compatible almost complex structure $\mathbf{J}$.

Theorem 2.3.1. *[34, 8] The canonical compatible toric complex structure* $\mathbf{J}$ *on* (M, ω, T, μ) *in the action-angle coordinates* $(\boldsymbol{x}, \boldsymbol{\theta})$ *on* $\mu^{-1}(\mathring{\Delta}) \simeq \mathring{\Delta} \times T$ *is given by*

$$\mathbf{J} = \begin{bmatrix} 0 & \vdots & -H^{-1} \\ \cdots & \cdots & \cdots \\ H & \vdots & 0 \end{bmatrix}$$

where $H = \textsf{\textit{Hess}}_{\boldsymbol{x}}(h)$.

It follows that the matrix of the canonical Riemannian metric $\mathbb{G} := \langle -, - \rangle_{\mathbf{J}}$ in the action-angle coordinates is given by

$$\mathbb{G} = \begin{bmatrix} H & \vdots & 0 \\ \dots & \dots & \dots \\ 0 & \vdots & H^{-1} \end{bmatrix}$$

Next, we turn our attention to the geometric properties of $\mathscr{D}^s_\omega(M,T)$. First, we define a right invariant weak Riemannian metric, defined on the Lie algebra $\mathfrak{X}^s_\omega(M)$ of $\mathscr{D}^s_\omega(M)$ as follows

$$(X,Y) = \int_M \mathbb{G}(X,Y)\omega^n \tag{2.3.1}$$

We have pointed out in Section 1.4 that there is a Levi-Civita connection $\widetilde{\nabla}$ compatible with this weak metric on the symplectomorphism group $\mathscr{D}^s_\omega(M)$. To study the geometric properties of the inclusion of the torus $\mathscr{D}^s_\omega(M,T) \hookrightarrow \mathscr{D}^s_\omega(M)$ we need to show that the induced metric on $\mathscr{D}^s_\omega(M,T)$ admits a compatible Levi-Civita connection. The existence of such a Levi-Civita connection is not guaranteed in general, as we are working on a weak Riemannian manifold. Hence existence of an H^0-orthogonal complement for the Lie algebra $\mathfrak{X}^s_\omega(M,T)$ of $\mathscr{D}^s_\omega(M,T)$ is essential. We will find the H^0-orthogonal complement to the Lie algebra of $\mathscr{D}^s_\omega(M,T)$ which is the set $\mathfrak{X}^s_\omega(M,T)$ of T-invariant locally Hamiltonian vector fields.

Since the canonical Riemannian metric $\mathbb{G}$ is T-invariant, a simple computation shows that

$$\mathfrak{r}^s = \Big\{ X \in \mathfrak{X}^s_\omega(M) \;\Big|\; \int\limits_T k_*(X)dk = 0 \Big\}$$

is the H^0-orthogonal complement of $\mathfrak{X}^s_\omega(M,T)$ inside $\mathfrak{X}^s_\omega(M)$. One can write

$$\mathfrak{X}^s_\omega(M) = \mathfrak{X}^s_\omega(M,T) \oplus \mathfrak{r}^s$$

so for every $\eta \in \mathscr{D}^s_\omega(M)$ we have

$$T_\eta \mathscr{D}^s_\omega = \Big([\mathfrak{X}^s_\omega(M,T)] \circ \eta \Big) \oplus \Big([\mathfrak{r}^s] \circ \eta \Big)$$

This means that we have an H^0-orthogonal decomposition of the bundle $T\mathscr{D}^s_\omega(M)$ into closed subbundles

$$T\mathscr{D}^s_\omega = \widehat{\mathfrak{X}^s_\omega(M,T)} \oplus \widehat{\mathfrak{r}^s} \tag{2.3.2}$$

where the fibers of $\widehat{\mathfrak{X}^s_\omega(M,T)}$ and $\widehat{\mathfrak{r}^s}$ at any $\eta \in \mathscr{D}^s_\omega$ are $\mathfrak{X}^s_\omega(M,T) \circ \eta$ and $\mathfrak{r}^s \circ \eta$, respectively. So we have a projection

$$\begin{aligned} \widehat{P} : \mathfrak{X}^s_\omega(M) &\longrightarrow \mathfrak{X}^s_\omega(M,T) \\ X &\longmapsto \int\limits_T k_*(X)dk \end{aligned}$$

which induces a projection

$$\begin{aligned} \widehat{P}_\eta : T_\eta \mathscr{D}^s_\omega(M) &\longrightarrow T_\eta \mathscr{D}^s_\omega(M,T) \\ u &\longmapsto T_e R_\eta \circ \widehat{P} \circ T_\eta R_{\eta^{-1}}(u) \end{aligned}$$

for every $\eta \in \mathscr{D}^s_\omega(M,T)$. Finally, we can define a projection map, $\widetilde{P}$: $T\mathscr{D}^s_\omega(M)\Big|_{\mathscr{D}^s_\omega(M,T)} \longrightarrow T\mathscr{D}^s_\omega(M,T)$ which is defined on each fiber by $\widehat{P}_\eta$. The compatible Levi-Civita connection of the restriction of the metric 2.3.1 on $\mathscr{D}^s_\omega(M,T)$ is then given by $\widehat{\nabla} = \widetilde{P} \circ \widetilde{\nabla}$.

Before studying the geometric properties of $\left(\mathscr{D}^s_\omega(M,T), (,), \widehat{\nabla}\right)$, we recall the second fundamental form which is a bilinear symmetric map defined as follows

$$\begin{aligned} II_\eta : T_\eta\mathscr{D}^s_\omega(M,T) \times T_\eta\mathscr{D}^s_\omega(M,T) &\longrightarrow [T_\eta\mathscr{D}^s_\omega(M,T)]^\perp \\ (X,Y) &\longmapsto \left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widehat{\nabla}_{\widehat{X}}\widehat{Y}\right)(\eta) \end{aligned}$$

where on the right hand side of the definition the vector fields $\widetilde{X}, \widetilde{Y}$ are extensions of X, Y to $\mathscr{D}^s_\omega(M)$ while $\widehat{X}, \widehat{Y}$ are extensions of X, Y to $\mathscr{D}^s_\omega(M,T)$. It is known that the second fundamental form does not depend on the extensions of X, Y, i.e., it is tensorial.

It is known that the odd cohomology groups of toric manifolds are zero [21, See Theorem 3.3.1]. Hence $\mathfrak{X}^s_\omega(M)$ is the space of globally Hamiltonian vector fields and $\mathfrak{X}^s_\omega(M,T)$ is the space of T-invariant globally Hamiltonian vector fields.

We need the following technical lemma in order to prove that $\mathscr{D}^s_\omega(M,T)$ is flat and totally geodesic. The proof given in [10] for the case of $\mathcal{A} = [0,1]\times\mathbb{S}^1$ contains the main idea of the proof for other cases, i.e., one needs to look at the Christoffel symbols of the Levi-Civita connection ∇ on $(M, \mathbb{G})$ and evaluates $\nabla_{X_f}X_g$ for $f, g : [0,1] \longrightarrow \mathbb{R}$, which turns out to be zero. For the case of complex projective spaces $\mathbb{P}^i$, $i = 1, 2$, this is not true anymore as it has been observed by El Hadrami [30]. In the next step, one can naturally look into H^0-orthogonal decomposition of $\nabla_{X_f}X_g$ where $f, g : \mathring{\Delta} \longrightarrow \mathbb{R}$. El Hadrami noticed that $\nabla_{X_f} X_g$ is orthogonal to all $X_h \in \mathfrak{X}^s_\omega(M)$ with respect to the weak Riemannian metric and hence $\widetilde{\nabla}_{X_f^R} X_g^R = 0$ [30, See Lemmas 3.4.1 and 5.4.1]. We extend his idea to all toric manifolds using Guillemin's result 2.3.1.

Lemma 2.3.2. *For any $X_f \in \mathfrak{X}^s_\omega(M)$ and $X_{h_1}, X_{h_2} \in \mathfrak{X}^s_\omega(M,T)$ we have*

$$\left(X_f, \nabla_{X_{h_2}} X_{h_1}\right) = 0 \tag{2.3.3}$$

where ∇ is the the Levi-Civita connection compatible with the canonical Riemannian metric $\mathbb{G}$ on M.

Proof.

$$\begin{aligned}\left(X_f, \nabla_{X_{h_2}} X_{h_1}\right) = 0 \quad &\Longleftrightarrow \quad \int_M \mathbb{G}\left(X_f, \nabla_{X_{h_2}} X_{h_1}\right) = 0 \\ &\Longleftrightarrow \quad \int_{\mu^{-1}(\mathring{\Delta})} \mathbb{G}\left(X_f, \nabla_{X_{h_2}} X_{h_1}\right) = 0\end{aligned}$$

We note that on $\mu^{-1}(\mathring{\Delta}) \simeq \mathring{\Delta} \times T$ we have the action-angle coordinates $(\boldsymbol{x}, \boldsymbol{\theta})$ for which we have descriptions of ω and $\mathbb{G}$, and elements of $\mathfrak{X}^s_\omega(M,T)$ can be described as

$$\begin{aligned}X_h \in \mathfrak{X}^s_\omega(M,T) \quad &\Longleftrightarrow \quad h \text{ is an invariant function} \\ &\Longleftrightarrow \quad h \text{ is a function of } \boldsymbol{x} \\ &\Longrightarrow \quad X_h\Big|_{\mu^{-1}(\mathring{\Delta})} = -\sum_{j=1}^n \partial_{x_j} h \; \partial_{\theta_j}\end{aligned}$$

Let $\Gamma^\alpha_{\beta\gamma}$ be the Christoffel symbols of ∇ then

$$\nabla_{X_{h_2}} X_{h_1} = \sum_{i,j,k=1}^n \partial_{x_k} h_2 \; \partial_{x_j} h_1 \left(\Gamma^i_{j+n,k+n} \, \partial_{x_i} + \Gamma^{i+n}_{j+n,k+n} \, \partial_{\theta_i}\right) \tag{2.3.4}$$

Next, we compute Christoffel symbols in equation 2.3.4 using the matrix of $\mathbb{G}$ in Theorem 2.3.1

$$\begin{aligned}
\Gamma^{i}_{j+n,k+n} &= \frac{1}{2}\sum_{\ell=1}^{n} g^{i\ell}\left(\partial_{\theta_j}\, g_{k+n,\ell} + \partial_{\theta_k}\, g_{j+n,\ell} - \partial_{x_\ell}\, g_{j+n,k+n}\right) \\
&+ \frac{1}{2}\sum_{\ell=1}^{n} g^{i,\ell+n}\left(\partial_{\theta_j}\, g_{k+n,\ell+n} + \partial_{\theta_k}\, g_{j+n,\ell+n} - \partial_{\theta_\ell}\, g_{j+n,k+n}\right) \\
&= -\frac{1}{2}\sum_{\ell=1}^{n} g^{i\ell}\partial_{x_\ell}\, g_{j+n,k+n}
\end{aligned}$$

$$\begin{aligned}
\Gamma^{i+n}_{j+n,k+n} &= \frac{1}{2}\sum_{\ell=1}^{n} g^{i+n,\ell}\left(\partial_{\theta_j}\, g_{k+n,\ell} + \partial_{\theta_k}\, g_{j+n,\ell} - \partial_{x_\ell}\, g_{j+n,k+n}\right) \\
&+ \frac{1}{2}\sum_{\ell=1}^{n} g^{i+n,\ell+n}\left(\partial_{\theta_j}\, g_{k+n,\ell+n} + \partial_{\theta_k}\, g_{j+n,\ell+n} - \partial_{\theta_\ell}\, g_{j+n,k+n}\right) \\
&= -\frac{1}{2}\sum_{\ell=1}^{n} g^{i+n,\ell}\partial_{x_\ell}\, g_{j+n,k+n} \\
&= 0
\end{aligned}$$

Substituting the values of Christoffel symbols into equation 2.3.4 we get

$$\nabla_{X_{h_2}} X_{h_1} = \sum_{i,j,k,\ell=1}^{n} g^{i\ell}\partial_{x_k}h_2\ \partial_{x_j}h_1\ \partial_{x_\ell}g_{j+n,k+n}\partial_{x_i}.$$

Hence we have

$$\begin{aligned}
\left(X_f, \nabla_{X_{h_2}} X_{h_1}\right) &= \int_{\mathring{\Delta}}\int_{T} \mathbb{G}(X_f, \nabla_{X_{h_2}} X_{h_1}) \\
&= \sum_{i,j,k,\ell=1}^{n} \int_{\mathring{\Delta}} g^{i\ell}\partial_{x_k}h_2\ \partial_{x_j}h_1\ \partial_{x_\ell}g_{j+n,k+n} \int_{T} \mathbb{G}\left(X_f, \partial_{x_i}\right) \\
&= \sum_{j,k,\ell=1}^{n} \int_{\mathring{\Delta}} \partial_{x_k}h_2\ \partial_{x_j}h_1\ \partial_{x_\ell}g_{j+n,k+n} \int_{T} \partial_{\theta_\ell} f \\
&= 0
\end{aligned}$$

$\square$

Finally we prove the following theorem regarding the geometric properties of $\mathscr{D}^s_\omega(M,T)$.

Theorem 2.3.3. *$\mathscr{D}^s_\omega(M,T)$ is a flat and totally geodesic submanifold of $\mathscr{D}^s_\omega(M)$ with respect to the weak right invariant metric*

$$(X,Y) = \int_M \mathbb{G}(X,Y)\omega^n.$$

Proof. We will show that the curvature and the second fundamental form vanish using the fact that these two are tensorial. Let $X, Y \in \mathfrak{X}^{s+2}_\omega(M,T)$ and consider their right invariant extensions X^r, Y^r to $\mathscr{D}^s_\omega(M,T)$ and also their right invariant extensions X^R, Y^R to $\mathscr{D}^s_\omega(M)$. Note that for every $\eta \in \mathscr{D}^s_\omega(M,T)$ we have

$$\widetilde{\nabla}_{X^R} Y^R(\eta) = \widehat{\nabla}_{X^r} Y^r(\eta) + II(X \circ \eta, Y \circ \eta) \tag{2.3.5}$$

On the other hand $\widetilde{\nabla}_{X^R} Y^R(\eta) = \widehat{P}\left(\nabla_X Y\right)\right)(\eta)$, where

$$\widehat{P} : T_e\mathscr{D}^s(M)\Big|_{\mathscr{D}^s_\omega(M)} \longrightarrow T_e\mathscr{D}^s_\omega(M) \tag{2.3.6}$$

is the orthogonal projection. From Lemma 2.3.3 follows that $\widehat{P}(\nabla_X Y) = 0$, along with the orthogonal decomposition 2.3.5 this implies that II and $\widehat{\nabla}$ are zero when evaluated on right invariant vector fields. It follows readily that the second fundamental form is zero and that the curvature evaluated on right invariant vector fields is also zero. By tensoriality of the curvature, this in turn implies that curvature is zero. $\square$

2.4 Further Discussions and Comments

In this section we put all the properties of $\mathscr{D}^s_\omega(M,T)$ together to show how strong the analogy with the finite-dimensional setting is. We also discuss other smooth structures that one may be interested in, e.g., Fréchet and ILH Lie group structures. Finally we will compare our "maximal torus" $\mathscr{D}^s_\omega(M,T)$ to the one considered by Bao and Ratiu for the case of the finite cylinder and by El Hadrami in the case of $\mathbb{P}^i$, $i=1,2$.

Theorem 2.4.1. *Let (M,ω,T,μ) be a $2n$-dimensional compact connected toric manifold without boundary. Then for $s>n+1$, $\mathscr{D}^s_\omega(M,T)$*

- *Topologically: is a real, infinite-dimensional, smooth, path-connected and H^s-closed submanifold of $\mathscr{D}^s_\omega(M)$.*
- *Algebraically: is a maximal Abelian subgroup of $\mathscr{D}^s_\omega(M)$ and its Lie algebra is given by $\mathfrak{X}^s_\omega(M,T)$, T-invariant symplectic vector fields. Moreover its normal bundle is given by $\mathfrak{r}^s = \Big\{X \in \mathfrak{X}^s_\omega(M) \;\Big|\; \int_T k_*(X)dk = 0\Big\}$.*
- *Geometrically: is a flat and totally geodesic Riemannian submanifold of $\mathscr{D}^s_\omega(M)$ with respect to the canonical weak right invariant metric*

$$\big(X,Y\big) = \int_M \mathbb{G}(X,Y)\omega^n$$

when $s>n+2$.

We can endow $\mathscr{D}_\omega(M)$, the group of smooth symplectomorphisms, with a smooth regular Fréchet manifold structure [43, See Theorem 43.12]. We can consider $\mathscr{D}_\omega(M,T)$, the group of smooth equivariant symplectomorphisms, inside $\mathscr{D}_\omega(M)$. The proof of Theorem 43.12 in [43] which is based on the

Weinstein idea of Lagrangian submanifolds [74] can be modified such that we obtain an T-equivariant chart for $\mathscr{D}_\omega(M)$ for which $\mathscr{D}^s_\omega(M,T)$ is modelled on the space of invariant closed 1-forms on M. Algebraic and geometric properties can be proven along the same lines as the Sobolev completion case.

There is an intermediate setting between Sobolev completions and Fréchet structures introduced by Omori which is called ILH Lie group structure [52]. We recall the definition of an ILH Lie group [52, See Theorem 1.2.1].

Definition 2.4.2. *[52] A topological group G is called an ILH Lie group modelled on $\left\{\mathbb{E},\ \mathbb{E}^r \;\middle|\; r \geq r_0\right\}$ if there is a collection $\left\{G^s \;\middle|\; s \geq r_0\right\}$ of topological groups which satisfies the followings*

1. *G^s is a smooth Hilbert manifold modelled on a Hilbert space $\mathbb{E}^s$.*
2. *There is a smooth inclusion $G^{s+1} \hookrightarrow G^s$ and G^{s+1} is a dense subgroup of G^s.*
3. *$G = \bigcap_s G^s$ with the inverse limit topology.*
4. *The group multiplication $G \times G \longrightarrow G$ extends to a C^k map $G^{s+k} \times G^s \twoheadrightarrow G^s$.*
5. *The inversion map $G \longrightarrow G$ extends to a C^k map $G^{s+k} \rightarrowtail G^s$.*
6. *The right multiplication $R_\eta : G^s \longrightarrow G^s$ is smooth for all $\eta \in G^s$.*
7. *Let $\mathfrak{g}^s$ be the tangent space of G^s at identity and TG^s the tangent bundle of G^s. $dR : \mathfrak{g}^{s+k} \times G^s \longrightarrow TG^s$ defined by $dR(u, \eta) = T_\eta R(X)$ is a C^k map.*

It immediately follows from this definition and our construction that

Corollary 2.4.3. *$\mathscr{D}_\omega(M,T)$ is an ILH Lie subgroup of $\mathscr{D}_\omega(M)$.*

The existence of an infinite-dimensional analog of maximal torus has been studied by Bao and Ratiu for the group of volume preserving diffeomorphisms of the annulus $\mathcal{A} = [0,1] \times \mathbb{S}^1$ [10]. They showed that the set of all "pure twist" maps

$$\mathcal{T}^s = \left\{ \eta_\phi \;\middle|\; \eta_\phi(z,\theta) = (z, \theta + \phi(z)),\ \phi : [0,1] \to \mathbb{R} \text{ is } H^s \right\}$$

has all the properties of a finite-dimensional maximal torus stated in the Theorem 1.1.4, including that its Weyl group is finite. The maximal torus $\mathcal{T}^s$ can be viewed as the set of all symplectomorphisms that preserve the level sets of the moment map $\mathsf{pr}_1 : [0,1] \times \mathbb{S}^1 \longrightarrow [0,1]$, the projection onto the first component. This interpretation of $\mathcal{T}^s$ allowed El Hadrami to generalize this definition to all toric manifolds and to conjectured that the subgroup

$$\mathfrak{T}^s = \left\{ \eta \in \mathscr{D}^s_\omega(M) \;\middle|\; \mu \circ \eta = \mu \right\}$$

has many of the properties of the finite-dimensional maximal torus and hence can be viewed as an analog of maximal torus in the symplectomorphism group of a toric manifold [30]. He studied the set of twist maps $\mathfrak{T}^s$ for special cases of complex projective spaces $M = \mathbb{P}^i$, $i = 1, 2$. A glance through El hadrami's thesis reveals that the substantial difference between his argument and the one given by Bao and Ratiu is in the proof that $\mathfrak{T}^s$ is a submanifold of $\mathscr{D}^s_\omega(M)$. In an straightforward argument Bao and Ratiu showed that the subgroup of twist maps $\mathcal{T}^s$ form a submanifold of $\mathscr{D}^s(\mathcal{A})$. Indeed, they used the fact that the level sets of the moment map pr_1 are geodesics of the metric induced from $\mathbb{R}^3$ to show that the map $\Psi : H^s([0,1], \mathbb{R}) \to \mathscr{D}^s(\mathcal{A})$ defined by $\Psi(\phi) = \eta_\phi$ is an embedding [10]. Since the level sets of the moment map are no longer the geodesics of the canonical toric metric in the case of $\mathbb{P}^i$, El Hadrami produced a delicate argument employing the Submersion Theorem and transversality. However a gap in his argument has been discovered later [57]†.

We compare the equivariant symplectomorphisms and symplectomorphisms that preserve the Lagrangian foliation of M by Lagrangian tori, i.e., we compare $\mathscr{D}^s_\omega(M,T)$ and $\mathfrak{T}^s$. One can readily see that for the annulus $\mathcal{A} = [0,1] \times \mathbb{S}^1$, the two subgroups $\mathfrak{T}^s$ and $\mathcal{T}^s$ coincide. Surprisingly, the set of all equivariant symplectomorphisms $\mathscr{D}^s_\omega(\mathcal{A},\mathbb{S}^1)$ also coincides with $\mathcal{T}^s$. Hence for general toric manifolds we still have two a priori different subgroups, namely, $\mathscr{D}^s_\omega(M,T)$ and $\mathfrak{T}^s$. Naturally, one may ask about the connection between $\mathscr{D}^s_\omega(M,T)$ and $\mathfrak{T}^s$. The following lemma provides an answer to this question.

Lemma 2.4.4. *For all $s > n+1$, $\mathfrak{T}^s = \mathscr{D}^s{}_\omega(M,T)$.*

Proof. It is easy to see that

$$\eta \in \mathfrak{T}^{s+1} \Longleftrightarrow \eta_*(X_{\mu_i}) = X_{\mu_i}$$

where μ_i's are components of the moment map μ and X_{μ_i}'s are the corresponding Hamiltonian vector fields. Assume $Fl_{\mu_i}(t,p)$ is the flow of the vector field X_{μ_i} then

$$\eta_*(X_{\mu_i}) = X_{\mu_i} \Longleftrightarrow Fl_{\mu_i}(t,\eta(p)) = \eta(Fl_{\mu_i}(t,p))$$

As the components of μ form an integrable system we have $[X_{\mu_i}, X_{\mu_j}] = 0$, equivalently $Fl_{\mu_i}(t, Fl_{\mu_j}(s,p)) = Fl_{\mu_j}(t, Fl_{\mu_i}(s,p))$. Hence we can define an

†. Private communications with Tudor S. Ratiu.

$\mathbb{R}^n$-action which reduces to a torus action and the torus action defined this way is exactly the T-action on M. Hence

$$\eta_*(X_{\mu_i}) = X_{\mu_i} \Longleftrightarrow \eta \in \mathscr{D}^s_\omega(M, T).$$

$\square$

The main advantage of our definition of the maximal torus compared to El Hadrami's definition is that the proofs of the topological properties of the maximal torus become a lot simpler. Our approach also allows the study of the maximal torus $\mathscr{D}_\omega(M, T)$ as a Fréchet submanifold of $\mathscr{D}_\omega(M)$ and enables us to construct a chart for this submanifold. Finally, an important fact which is missing in El Hadrami's work, is the existence of the normal bundle $\mathfrak{r}^s$ which is crucial to define the compatible Levi-Civita connection, since our manifolds are only weakly Riemannian.

Following Smolentsev [70] it is very tempting to define our maximal torus as the subgroup

$$\mathscr{D}_\omega(M, \mathbf{X}_\mu) = \left\{\eta \in \mathscr{D}_\omega(M) \;\middle|\; \forall\ 1 \leq i \leq n:\ \eta_*(X_{\mu_i}) = X_{\mu_i}\right\}$$

and try to use the Frobenius Theorem for ILH Lie groups [52] to prove that this subgroup is a genuine submanifold. However this approach leads to difficulties that have not been resolved yet.

We remark that the equation of the geodesics of any compatible, not necessarily canonical, weak Riemannian metric on $\mathscr{D}^s_\omega(M)$ has been studied recently by Ebin [28]. In this recent paper, Ebin has showed the global existence of the geodesics of these metrics. That is, the geodesics of any compatible weak Riemannian metric on $\mathscr{D}^s_\omega(M)$ is defined for all time and hence $\mathscr{D}^s_\omega(M)$ endowed with compatible weak metrics is geodesically complete.

Chapter 3

A Schur-Horn-Kostant Convexity

The existence of the maximal torus $\mathscr{D}^s_\omega(M,T)$ in the symplectomorphism group $\mathscr{D}^s_\omega(M)$ motivates a natural question. Is there any analog of Schur-Horn convexity theorem that holds for $\mathscr{D}^s_\omega(M,T) \hookrightarrow \mathscr{D}^s_\omega(M)$? This question has been studied for the annulus $\mathcal{A} = [0,1] \times \mathbb{S}^1$ by Bloch, Flaschka and Ratiu [12]. They showed that after an appropriate completion of the central extensions of the corresponding Lie algebras an analog of Schur-Horn convexity theorem holds. They expressed their results in a language analogous to Kostant's formulation of the classical Schur-Horn theorem. A similar result has been obtained by Bloch, El Hadrami, Flaschka and Ratiu for $M = \mathbb{P}^1$ [14].

In this chapter we prove a convexity theorem for the symplectomorphisms group of an arbitrary compact toric manifold. We will follow the method presented in [12, 14] and we will show that after an appropriate completion of the Lie algebras an analog of Schur-Horn-Kostant convexity Theorem holds. This infinite-dimensional analog of Schur-Horn-Kostant convexity theorem is interesting in many ways. First, although it is a pure result in functional analysis, it has a strong tie to the symplectic structure of M through the strong completion of the group of Hamiltonian symplectomorphisms. Second, the approach employed towards constructing the infinite-dimensional version of Schur-Horn convexity theorem is very similar to the original proof of the classical Schur-Horn convexity theorem presented in Section 1.5. Finally, it is one of the few convexity theorems that hold for infinite-dimensional algebras.

3.1 Doubly Stochastic Operators

We begin with a review of classical results about rearrangements, majorization and doubly stochastic operators on the space of functions. These are the key notions needed in the infinite-dimensional formulation of Schur-Horn convexity theorem. Our main reference for the material of this section is [19]. We should mention that the definition of doubly stochastic operators and their main properties have been studied rigorously for the first time by J. Ryff [63, 64, 65]. We restrict our attention to finite measure spaces which are easier to work with in rearrangement theory and which are sufficient for our purposes.

Definition 3.1.1. *[19] Let (X, Λ, ν) be a finite measure space, the distribution function d_f of $f \in \mathscr{M}(X, \nu)$ is defined by*

$$d_f(t) = \nu\left(\{x \in X : f(x) > t\}\right), \tag{3.1.1}$$

where $t \in \mathbb{R}$.

The distribution function d_f is decreasing and right-continuous. Now we consider the right inverse δ_f of d_f on $[0, \nu(X)]$.

Definition 3.1.2. *[19, See Definition 5.1] Let $f \in \mathscr{M}(X, \nu)$. The decreasing rearrangement of f is defined by*

$$\delta_f(t) = \inf\left\{x \in \mathbb{R} \;\middle|\; d_f(x) \le t\right\}, \tag{3.1.2}$$

where $t \in [0, \nu(X)]$.

Observe that the rearrangement of δ_f is δ_f itself. Here we have assumed that $\inf \emptyset = +\infty$ and $\inf \mathbb{R} = -\infty$. The rearrangement δ_f is a decreasing and right-continuous function [19]. Next, we define the notion of equimeasurability.

Definition 3.1.3. *[19, See Section I.3] Let* (X_i, Λ_i, ν_i)*,* $i = 1, 2$*, be two finite measure spaces such that* $\nu_1(X_1) = \nu_2(X_2)$*. Two functions* $f_i \in \mathscr{M}(X_i, \nu_i)$ *are called equimeasurable if* $d_{f_1} = d_{f_2}$ *and we write* $f_1 \sim f_2$*.*

It is known that $f_1 \sim f_2$ if and only if $\delta_{f_1} = \delta_{f_2}$ [19, See Proposition 5.3]. In particular, $\delta_f \sim f$ and hence δ_f is called the equimeasurable rearrangement of f by some authors. There is an interesting theorem stating that δ_f is unique in the sense that it is the only decreasing right-continuous function that is equimeasurable to f.

Theorem 3.1.4. *[19, See Theorem 5.2] Let* $f \in \mathscr{M}(X, \nu)$ *then* δ_f *is the only decreasing right-continuous Lebesgue measurable function on* $[0, \nu(X)]$ *such that* $\delta_f \sim f$*.*

There is another interesting connection between f and δ_f that can be explained using measure preserving transformations. This result was first proven by Ryff [64].

Definition 3.1.5. *Let* (X, Λ, ν) *be a measure space.* $A \in \Lambda$ *is called an atom if*

(1) $\nu(A) > 0$*.*

(2) *If* $B \in \Lambda$ *and* $B \subset A$ *then either* $\nu(B) = 0$ *or* $\nu(B) = \nu(A)$*.*

A measure space is called non-atomic if it does not have any atom.

Theorem 3.1.6. *[Ryff Theorem, [19]] Let* (X, Λ, ν) *be a non-atomic finite measure space. Given* $f \in \mathscr{M}(X, \nu)$*, there is a measure preserving transformation* $\sigma : X \to [0, \nu(X)]$ *such that* $f = \delta_f \circ \sigma$*,* ν*-almost everywhere.*

We remark that the measure preserving map σ in Theorem 3.1.6 is playing the role of "permutations" but it is not a measure preserving bijection in general. Also it is not always possible to find a measure preserving

$\gamma : [0, \nu(X)] \to X$ such that $f \circ \gamma = \delta_f$, more detail is given in the Remark right after Theorem 5.12 in [19].

The moments of a function $f \in L^\infty(X, \nu)$ defined by

$$\mathrm{I}_p = \int\limits_X f^p d\nu \ , \quad p \in \mathbb{Z}^+. \tag{3.1.3}$$

We are interested to see up to what extent the moments defined in the formula 3.1.3 determine the equimeasurability class of f and hence δ_f. Concerning this matter we will prove the following lemma.

Lemma 3.1.7. *Let (X_i, Λ_i, ν_i), $i = 1, 2$, be two finite measure spaces such that $\nu_1(X_1) = \nu_2(X_2)$ and let $f_i \in \mathscr{M}(X_i, \nu_i)$ be two essentially bounded functions then*

$$f_1 \sim f_2 \iff \int\limits_{X_1} f_1^p d\nu_1 = \int\limits_{X_2} f_2^p d\nu_2 \ , \quad \forall p \in \mathbb{Z}^+. \tag{3.1.4}$$

Proof. It is known that two essentially bounded positive functions are equimeasurable if and only if they have the same moments [5]. Take two essentially bounded functions f_1 and f_2 with the same moments, not necessarily positive. Choose $R > 0$ such that $-R \leq f_i \leq R$, almost everywhere. It can be easily verified that

$$f_1 \sim f_2 \iff \frac{f_1 + R}{R} \sim \frac{f_2 + R}{R}$$

In the other direction, note that $f_1^p \sim f_2^p$ for all $p \in \mathbb{Z}^+$ since the function $x \mapsto x^p$ is continuous [19, See Proposition 3.3]. Hence

$$\int\limits_{X_1} f_1^p d\nu_1 = \int\limits_{X_2} f_2^p d\nu_2.$$

□

The equimeasurable rearrangements can be used to define a pre-order relation which is indeed similar to the pre-order $\prec$ that was defined for n-tuples in 1.5.4.

Definition 3.1.8. *[19, See Definition 8.1] Let (X_i, Λ_i, ν_i), $i = 1, 2$, be two finite measure spaces such that $\nu_1(X_1) = \nu_2(X_2) = r$. We say $f_2 \in L^1(X_2, \nu_2)$ majorizes $f_1 \in L^1(X_1, \nu_1)$ and we write $f_1 \prec f_2$ if*

(1) $\forall t \in [0, r] : \int_0^t \delta_{f_1} \leq \int_0^t \delta_{f_2}$.

(2) $\int_0^r \delta_{f_1} = \int_0^r \delta_{f_2}$.

Recall that an $n \times n$ matrix A is doubly stochastic if and only if $\forall\ \mathbf{u} \in \mathbb{R}^n : A\mathbf{u} \prec \mathbf{u}$. A doubly stochastic operator is defined in a similar way.

Definition 3.1.9. *[19, See Definition 18.1] Given the conditions of definition 3.1.8, a linear operator $T : L^1(X_2, \nu_2) \longrightarrow L^1(X_1, \nu_1)$ is called doubly stochastic if for all $f \in L^1(X_2, \nu_2)$, $Tf \prec f$.*

The following characterization of doubly stochastic operators will be handy and useful.

Theorem 3.1.10. *[19, See Theorem 18.4] A linear mapping T that maps simple functions of (X_2, Λ_2, ν_2) into $L^1(X_1, \nu_1)$ has a unique extension to a doubly stochastic operator $L^1(X_2, \nu_2) \longrightarrow L^1(X_1, \nu_1)$ if and only if for all $A \in \Lambda_2$ we have*

(1) $0 \leq T\chi_A \leq \chi_{X_1}$.

(2) $\int_{X_1} T\chi_A = \nu_2(A)$.

By Theorem 3.1.10 to prove that a linear operator is doubly stochastic it is necessary and sufficient to check that conditions (1) and (2) hold.

Recall that for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \prec \mathbf{u}$ if and only if there is an $n \times n$ doubly stochastic matrix A such that $A\mathbf{u} = \mathbf{v}$. Similar result holds for majorization of functions.

Theorem 3.1.11. *[19, See Theorem 18.9] Let* (X_i, Λ_i, ν_i), $i = 1, 2$, *be two finite measure spaces such that* $\nu_1(X_1) = \nu_2(X_2)$ *and* $f_i \in L^1(X_i, \nu_i)$ *two integrable functions. Then* $f_1 \prec f_2$ *if and only if there is a doubly stochastic operator* $T : L^1(X_2, \nu_2) \longrightarrow L^1(X_1, \nu_1)$ *such that* $Tf_2 = f_1$.

For an interesting example of doubly stochastic operator, pick a finite measure space (X, Λ, ν) and a measure preserving transformation $\sigma : X \longrightarrow [0, \nu(X)]$. Then $T_\sigma : L^1[0, \nu(X)] \longrightarrow L^1(X, \nu)$ defined by $T_\sigma(f) = f \circ \sigma$ is a doubly stochastic operator.

Recall from the equation 1.5.4 that the permutation polytope is $\Upsilon(\mathbf{u}) = \left\{\mathbf{v} \;\middle|\; \mathbf{v} \prec \mathbf{u}\right\}$ and that its extreme points are $\left\{\mathbf{v} \;\middle|\; \mathbf{v}^* = \mathbf{u}^*\right\}$. Similarly, we can consider $\Omega_f = \left\{g \;\middle|\; g \prec f\right\}$ and $\Delta_f = \left\{g \;\middle|\; g \sim f\right\}$. Next, we review a few facts regarding these two sets. We will see that Ω_f is convex and weakly compact and that its set of extreme points is Δ_f.

Definition 3.1.12. *[19] Let* (X_i, Λ_i, ν_i), $i = 1, 2$, *be two measure spaces such that* $\nu_1(X_1) = \nu_2(X_2)$. *For a given* $f \in L^1(X_1, \nu_1)$ *we define*

(1) $\Omega_f = \left\{g \in L^1(X_2, \nu_2) \;\middle|\; g \prec f\right\}$.

(2) $\Delta_f = \left\{g \in L^1(X_2, \nu_2) \;\middle|\; g \sim f\right\}$.

Ω_f can be viewed equivalently as the orbit of action of doubly stochastic operators. That is

$$\Omega_f = \left\{ Tf \;\middle|\; T : L^1(X_1, \nu_1) \longrightarrow L^1(X_2, \nu_2) \text{ is doubly stochastic } \right\}.$$

If the measure spaces are non-atomic Ryff Theorem 3.1.6 gives a nice interpretation of Δ_f. That is

$$\Delta_f = \left\{ \delta_f \circ \sigma \;\middle|\; \sigma : X_2 \longrightarrow [0, \nu_2(X_2)] \text{ is measure preserving} \right\}. \tag{3.1.5}$$

We recall the following theorem regarding Ω_f and Δ_f.

Theorem 3.1.13. *[19, See Theorems 17.4; 20.3; 22.1; 22.12] Given any $f \in L^p(X_1, \nu_1)$, the subset $\Omega_f \subset L^p(X_2, \nu_2)$ is convex and weakly compact. Moreover, if (X_2, Λ_2, ν_2) is non-atomic then Ω_f is the weak closed convex hull of Δ_f and its set of extreme points is Δ_f.*

3.2 The Spectral Theorem

Let (M, ω, T, μ) be a $2n$-dimensional compact connected toric manifold without boundary with Delzant polytope Δ. The symplectic form induces a measure on the Borel algebra and we can pass to a completion of this measure in the sense that all the measure zero sets are going to be measurable [62, See Chapter 11 Section 1 Theorem 4]. Let (M, Γ, ν_ω) be the smallest completion of this symplectic measure and, without loss of generality, assume that the symplectic volume of M is 1.

Recall that we have the maximal torus $\mathscr{D}^s_\omega(M, T)$ inside $\mathscr{D}^s_\omega(M)$ and we have an inclusion of the corresponding Lie algebras $\mathfrak{X}^s_\omega(M, T) \hookrightarrow \mathfrak{X}^s_\omega(M)$. If we pass to the corresponding central extensions of these Lie algebras we get $H^{s+1}_T(M, \mathbb{R}) \hookrightarrow H^{s+1}(M, \mathbb{R})$. Similar to the approach presented in [12] we can pass to a completion of these function spaces. Namely, we can consider $L^2(\Delta)$ and $L^2(M, \nu_\omega)$.

Theorem 3.2.1. *[Spectral Theorem] Let $f \in L^2(M, \nu_\omega)$ then there is a unique, decreasing right-continuous function $\lambda \in L^2[0,1]$ such that $\lambda \sim f$.*

Proof. Take $\lambda = \delta_f$ and the result follows from Theorem 3.1.4. $\square$

We remark that if $f \in L^\infty(M, \nu_\omega)$ then λ will be determined uniquely by the moments of f by Lemma 3.1.7 and hence the Spectral theorem in [12] follows.

3.3 The Diagonalization and Orbit Theorems

Recall that in the classical Schur-Horn convexity theorem the orbit of the (co)-adjoint action through $A \in \mathcal{H}$ is the set of all Hermitian matrices that have the same spectrum as A. Hence our natural candidate for the orbit $\mathscr{O}_f$ passing through $f \in L^2(M, \nu_\omega)$ should be

$$g \in \mathscr{O}_f \Longleftrightarrow \delta_g = \delta_f$$

From the Definition 3.1.12 we can readily see that $\mathscr{O}_f = \Delta_f$. By analogy to the finite-dimensional setting we would like to have an action description for the orbit. Since (M, Γ, ν_ω) is non-atomic we can use the characterization of Δ_f given in 3.1.5 to get

$$\mathscr{O}_f = \left\{ \delta_f \circ \sigma \;\middle|\; \sigma : M \longrightarrow [0,1] \text{ is measure preserving} \right\}$$

Obviously, the set of all measure preserving transformations $\sigma : M \longrightarrow [0,1]$ does not have any algebraic structure. In order to express $\mathscr{O}_f$ as an orbit we need the following classical result from measure theory that guarantees the existence of a measure preserving bijection $q : M \longrightarrow [0, 1]$ [62, See Chapter 15 Section 5 Theorem 16].

Theorem 3.3.1. *[62] Let ν be a non-atomic probability Borel measure on an uncountable complete separable metric space X. Then $(X, \mathscr{B}(X), \nu)$ is isomorphic to the Lebesgue Borel measure on $[0, 1]$.*

Theorem 3.3.2. *[Diagonalization Theorem, [12]] Let $f \in L^2(M, \nu_\omega)$ and let λ be as in Spectral Theorem. Then there is a measure preserving transformation $\psi : (M, \nu_\omega) \longrightarrow (M, \nu_\omega)$ such that $f = (\lambda \circ q) \circ \psi$.*

Proof. The proof is exactly the same as the proof in [14]. By the Ryff Theorem 3.1.6 there is a measure preserving transformation $\phi : M \longrightarrow [0, 1]$ such that $f = \lambda \circ \phi$. Hence $f = (\lambda \circ q) \circ (q^{-1} \circ \phi)$. Take $\psi = q^{-1} \circ \phi$. □

The Diagonalization Theorem 3.3.2 enables us to write $\mathscr{O}_f$ as the orbit of the semi-group of measure preserving transformation, $\overline{Meas(M, \nu_\omega)}$, through $\lambda \circ q$. Note that here we are *not* defining $\overline{Meas(M, \nu_\omega)}$ as the completion of another space although such an interpretation is indeed possible, see the discussion in Section 3.4.

Theorem 3.3.3. *[Orbit Theorem] Let $f \in L^2(M, \nu_\omega)$. Then there is a unique, decreasing right-continuous function $\lambda \in L^2[0, 1]$ such that $f \in \mathscr{O}_{\lambda \circ q}$, where $\mathscr{O}_{\lambda \circ q}$ is the orbit of the action of the semi-group $\overline{Meas(M, \nu_\omega)}$ through $\lambda \circ q$. The orbit $\mathscr{O}_{\lambda \circ q}$ consists of all $g \in L^2(M, \nu_\omega)$ such that $\delta_g = \lambda$ or equivalently $g \sim f$.*

Proof. Take λ as in the Spectral Theorem 3.2.1. the result follows from the Ryff Theorem 3.1.6. □

We remark that if $f \in L^\infty(M, \nu_\omega)$ then from Theorem 3.1.4 it follows that the orbit $\mathscr{O}_{\lambda\circ q}$ consists of all $g \in L^\infty(M,\nu_\omega)$ such that

$$\int\limits_M g^p d\nu_\omega = \int\limits_M f^p d\nu_\omega, \quad \forall p \in \mathbb{Z}^+.$$

Hence the version of the orbit theorem in [12] follows.

3.4 The Completion of $Ham(M,\omega)$

We will see later on that the Schur-Horn-Kostant convexity theorem is really a result in pure functional analysis. However, we will also see that it has strong ties to the symplectic geometry of M. At first glance it seems that the only connection to symplectic geometry is through the symplectic volume which, after completion of the Poisson algebras, is lost. In particular, the convexity theorem no longer appears as a consequence of the existence of the maximal torus $\mathscr{D}^s_\omega(M,T)$. In this section we present another connection to the underlying symplectic geometry of our manifold. We show that the completion of $Ham(M,\omega)$ in the strong operator topology† is exactly $\overline{Meas(M,\nu_\omega)}$. This is a generalization of Proposition 3.1 in [12].

Any $\eta \in Ham(M,\omega)$ induces an unitary operator

$$\begin{aligned} S_\eta : L^2(M,\nu_\omega) &\longrightarrow L^2(M,\nu_\omega) \\ f &\longmapsto f\circ\eta \end{aligned} \tag{3.4.6}$$

Theorem 3.4.1. *[Completion of $Ham(M,\omega)$] In the strong operator topology the completion of* $\left\{S_\eta \;\middle|\; \eta \in Ham(M,\omega)\right\}$ *is* $\left\{S_\xi \;\middle|\; \xi \in \overline{Meas(M,\nu_\omega)}\right\}$ *or*

†. On a Hilbert space H, a sequence P_n of operators converges strongly to P if $P_n(x) \longrightarrow P(x)$ for all $x \in H$.

equivalently, $\overline{Meas(M,\nu_\omega)}$ *is the strong closure of* $Ham(M,\omega)$*.*

We prove Theorem 3.4.1 in the following discussion. In doing so, we need some results that we are going to recall as we proceed. We will denote by $Meas(M,\nu_\omega)$ the set of all *invertible* measure preserving transformations of (M,ν_ω). It is known that the closure of $Meas(M,\nu_\omega)$ in the strong operator topology is $\overline{Meas(M,\nu_\omega)}$ [15]. Any $\zeta \in Meas(M,\nu_\omega)$ defines a unitary operator S_ζ as in 3.4.6. On the group of unitary operators, the strong operator topology and the weak operator topology coincide. Next, we recall some results from topology that allow us to compare the weak operator topology with convergence in measure. Our exposition closely follows [39].

Definition 3.4.2. *A topological space* (X,τ) *is called completely regular if for any* $x \in X$ *and any closed set* F*,* $x \notin F$*, there is a continuous function* $f : X \longrightarrow [0,1]$ *such that* $f(x) = 0$ *and* $f(F) = 1$*, i.e.,* x *and* F *can be separated by continuous functions.*

Typical examples of completely regular spaces are metric spaces.

Definition 3.4.3. *Let* X *be a set and* $D(X) = \left\{(x,x) \;\middle|\; x \in X\right\}$ *be the diagonal in* $X \times X$*. A diagonal uniformity on* X *is a collection* $\mathscr{U}$ *of subsets of* $X \times X$ *satisfying*

- $A \in \mathscr{U} \Longrightarrow D(X) \subset A$.
- $A, B \in \mathscr{U} \Longrightarrow A \cap B \in \mathscr{U}$.
- $A \in \mathscr{U} \Longrightarrow \exists\ E \in \mathscr{U} : E \circ E \subset A$.
- $A \in \mathscr{U} \Longrightarrow A^{-1} \in \mathscr{U}$.
- $A \subset B \subset X \times X$ *and* $A \in \mathscr{U} \Longrightarrow B \in \mathscr{U}$.

We remark that elements of each uniformity can be viewed as relations on X and hence their compositions and inversion make sense. For any metric space (X, d) we can define an uniformity $\mathscr{U}$ as follows

$$A \in \mathscr{U} \iff \exists\, r > 0 : \Big\{(x, y) \;\Big|\; d(x, y) < r\Big\} \subset A. \tag{3.4.7}$$

Given an uniformity $\mathscr{U}$ on X one can define a topology on X which is called the uniform topology induced by $\mathscr{U}$. For any $A \in \mathscr{U}$ define

$$A[x] = \Big\{y \in X \;\Big|\; (x, y) \in A\Big\}$$

A subset U of X is called open in the uniform topology if for any $x \in U$ there is an $A \in \mathscr{U}$ such that $A[x] \subset U$.

Definition 3.4.4. *A topological space (X, τ) is called uniformizable if there is an uniformity on X such that the uniform topology on X coincides with τ.*

It is known that a topological space is uniformizable if and only if it is completely regular [39, See Chapter 6 Corollary 17].

Definition 3.4.5. *Let (X, τ) be a topological space and $\mathscr{B}(X)$ the σ-algebra of Borel sets and ν a measure on $\mathscr{B}(X)$. The measure ν is called tight if for every $B \in \mathscr{B}(X)$ of finite measure we have*

$$\nu(B) = \sup \Big\{\nu(K) \;\Big|\; K \subset B,\; K \textit{ is compact} \Big\}.$$

It is easy to see that the completion of a tight measure is tight. For the tightness we recall Ulam's tightness theorem [56, See Theorem 3.2].

Theorem 3.4.6. *[Ulam's Tightness Theorem, [56]] If X is a separable, complete metric spaces then every probability measure on $\mathscr{B}(X)$ is tight.*

It follows that (M, ν_ω) is a tight measure space since it is a completion of a tight measure on the Borel algebra. We choose a uniformization $\mathscr{U}_\omega$ for (M, ω) and keep it fixed.

Definition 3.4.7. *[67] A sequence of measure preserving transformation $\{\sigma_n\}$ of (M, ν_ω) are said to be $\mathscr{U}_\omega$-convergent in measure to a measure preserving transformation σ of (M, ν_ω) if*

$$\lim_{n \to \infty} \nu_\omega^* \left(\left\{ x \in \sigma^{-1}(B) \,\middle|\, (\sigma_n(x), \sigma(x)) \notin V \right\} \right) = 0 \tag{3.4.8}$$

for any $V \in \mathscr{U}_\omega$ and any measurable set B. Here ν_ω^* is the outer measure corresponding to ν_ω.

In our case, since M is a toric manifold endowed with the canonical Riemannian metric 2.3.1 the symplectic volume coincides with the Riemannian volume. In particular, we can choose our uniformity $\mathscr{U}_\omega$ to be the one given in the formula 3.4.7. Thus to apply 3.4.8 it suffices to consider the sets V of the form $\left\{ (x, y) \,\middle|\, d(x, y) < r \right\}$. Hence the condition 3.4.8 becomes

$$\lim_{n \to \infty} \nu_\omega^* \left(\left\{ x \in M \,\middle|\, d(\sigma_n(x), \sigma(x)) \geq r \right\} \right) = 0. \tag{3.4.9}$$

Now we are ready to state a result that compares weak convergence to convergence in measure. We assume that the topological space (X, τ) is completely regular and Hausdorff, $\mathscr{U}$ is an uniformity compatible with τ and our measure is the completion of a tight measure on the Borel algebra.

Theorem 3.4.8. *[67] Let σ_n be a sequence of measure preserving transformations of (X, ν) and σ be a measure preserving transformation of (X, ν). If ν is locally finite then the following are equivalent:*

- $\sigma_n \to \sigma$ *weakly*†.

- σ_n $\mathscr{U}$*-converges in measure to* $\sigma.x$

In particular, for a sequence σ_n of invertible measure preserving transformations of (M, ν_ω) and an invertible measure preserving transformation σ of (M, ν_ω), $\sigma_n \to \sigma$ weakly if and only if 3.4.9 holds for any real number $r > 0$. Hence to approximate any $\eta \in Meas(M, \nu_\omega)$ by Hamiltonian symplectomorphisms in the strong operator topology we can approximate η in measure by Hamiltonian symplectomorphisms and this has been already done in [54].

Theorem 3.4.9. *[54] Let* $\eta \in Meas(M, \nu_\omega)$ *and every* $\epsilon > 0$ *there is a Hamiltonian symplectomorphism* $h : M \longrightarrow M$ *that* ϵ*-approximates* η *in measure, namely*

$$\nu_\omega^* \left(\left\{ x \in M \;\middle|\; d(\eta(x), h(x)) \geq \epsilon \right\} \right) < \epsilon.$$

3.5 The Projection π

In order to state our version of Schur-Horn-Kostant convexity we need the projection π that maps the orbit $\mathscr{O}_{\lambda \circ q}$ onto a convex set. We will see that π can be interpreted simply as an averaging process.

By Arnold-Liouville theorem [8, See Theorem III.3] we can disintegrate the measure ν_ω, i.e., for almost every $z \in \Delta$ we can find a unique probability measure ν_z on each T-orbit $\mu^{-1}(z)$ such that for every integrable function

†. On a Hilbert space H, a sequence P_n of operators converges weakly to P if $F(P_n(x)) \longrightarrow F(P(x))$ for all $x \in H$ and all $F \in H^*$.

$f : M \longrightarrow \mathbb{R}$ we have

$$\int_M f d\nu_\omega = \int_\Delta \int_{\mu^{-1}(z)} f d\nu_z dm$$

Equivalently, we write the measure ν_ω as the "product" of the measures ν_z, which is just the canonical measure on T, and the push-forward measure m which is just Lebesgue measure on Δ, see section 1.6.

Definition 3.5.1. *We define the projection* $\pi : L^2(M, \nu_\omega) \longrightarrow L^2(\Delta, m)$ *by*

$$[\pi(f)](z) = \int_{\mu^{-1}(z)} f d\nu_z$$

We call $\pi(f)$ *the zeroth Fourier coefficient of* f*.*

The projection π can be viewed as an averaging process that turns any element of $L^2(M, \nu_\omega)$ into a member of $L^2(\Delta)$. We remark that for the case of the annulus the projection π is exactly the zeroth Fourier coefficient [12].

3.6 The Weyl semi-group $\overline{\mathbf{W}}$

Recall that the classical Schur-Horn-Kostant convexity theorem for the unitary matrices involves the Weyl group of $\mathbf{T}^n \hookrightarrow \mathbf{U}(n)$, which is S_n. For the case of the annulus $\mathcal{A} = [0,1] \times \mathbb{S}^1$ it has been shown that the maximal torus $\mathscr{D}^s_\omega(M,T)$ has a finite Weyl group which is $\mathbb{Z}_2$ [10]. Because this Weyl group is finite it cannot be completed directly. Another closely related notion of Weyl group has been introduced in [57] in connection with the classification of maximal tori in symplectomorphism groups. Given a finite-dimensional torus

$T \hookrightarrow Ham(M, \omega)$, we can define its Weyl group to be

$$W = {}^{N_T} \Big/ {}_{C_T}$$

where N_T and C_T are the normalizer and centralizer of T in $\mathscr{D}_\omega(M)$. Note that in the case of a Hamiltonian toric action the centralizer C_T is equal to $\mathscr{D}_\omega(M, T)$. Hence the second definition of the Weyl group differs from the usual definition just in the numerator. More precisely, in the usual definition of the Weyl group we have $N\left(\mathscr{D}_\omega(M, T)\right)$ while for the second one we have N_T which sits inside $N\left(\mathscr{D}_\omega(M, T)\right)$. For the case of the annulus both notions coincide and we believe that they are the same if one considers an arbitrary symplectic toric manifold. Observe that, using the second definition of the Weyl group, we always obtain a finite group and hence it is too small to construct a convexity theorem [57].

Recall that any element of $\mathscr{D}^s_\omega(M, T)$ is of the form

$$\Psi_\phi(p) = \exp(\phi(\mu(p))) \cdot \ p$$

where $\phi : \Delta \longrightarrow \mathbb{R}^n$ is H^s with symmetric Jacobian. If we relax the condition of ϕ being H^s and allow ϕ to be measurable then we can complete $\mathscr{D}^s_\omega(M, T)$ to

$$\left\{ \ \Psi_\phi : \Delta \times T \longrightarrow \Delta \times T \ \middle| \ \Psi_\phi(\mathbf{z}, \boldsymbol{\theta}) = (\mathbf{z}, \boldsymbol{\theta} + \phi(\mathbf{z})) \text{ and } \phi \text{ is measurable } \right\}.$$

We compute the normalizer of this completion of $\mathscr{D}^s_\omega(M, T)$ in $\overline{Meas(M, \nu_\omega)}$ following the proof of Proposition 2.4 in [12]. We need to find all

$$(u(\mathbf{z}, \boldsymbol{\theta}), v(\mathbf{z}, \boldsymbol{\theta})) \in \overline{Meas(M, \nu_\omega)}$$

such that for every Ψ_α in the completion there is a Ψ_β in the completion such that

$$\begin{aligned} u(\mathbf{z}, \boldsymbol{\theta} + \alpha(\mathbf{z})) &= u(\mathbf{z}, \boldsymbol{\theta}) \\ v(\mathbf{z}, \boldsymbol{\theta} + \alpha(\mathbf{z})) &= v(\mathbf{z}, \boldsymbol{\theta}) + \beta(u(\mathbf{z}, \boldsymbol{\theta})) \end{aligned}$$

Thus u is a function of $\mathbf{z}$ alone and we have

$$v(\mathbf{z}, \boldsymbol{\theta} + \alpha(\mathbf{z})) - v(\mathbf{z}, \boldsymbol{\theta}) = \beta(u(\mathbf{z}))$$

From the above equation it follows that $v(\mathbf{z}, \boldsymbol{\theta}) = a(\mathbf{z})\boldsymbol{\theta} + b(\mathbf{z})$ where $b : \Delta \longrightarrow \mathbb{R}^n$ and $a : \Delta \longrightarrow \mathbb{R}$. Hence, the above formula can be recast as

$$a(\mathbf{z})\alpha(\mathbf{z}) = \beta(u(\mathbf{z})). \tag{3.6.10}$$

Equation 3.6.10 can be solved in β for every α if and only if $u : \Delta \longrightarrow \Delta$ is injective. Putting every thing together, we have shown that any element of the normalizer of the completion is of the form

$$\big(u(\mathbf{z}), a(\mathbf{z})\boldsymbol{\theta} + b(\mathbf{z})\big)$$

where $u : \Delta \longrightarrow \Delta$ is injective. Now we use the fact that this element is measure preserving and hence its components u and v must be measure preserving. This means that $u : \Delta \longrightarrow \Delta$ is measure preserving and $a : \Delta \longrightarrow \mathbb{R}$ is $+1$ or -1 almost everywhere. Thus our normalizer consists of elements of the form

$$\big(u(\mathbf{z}), \boldsymbol{\theta} + b(\mathbf{z})\big), \quad \big(u(\mathbf{z}), -\boldsymbol{\theta} + b(\mathbf{z})\big)$$

where $u : \Delta \longrightarrow \Delta$ is an injective measure preserving transformation. It readily follows that the Weyl group of the completion consists of the elements of the form

$$\big(u(\mathbf{z}), \boldsymbol{\theta}\big), \quad \big(u(\mathbf{z}), -\boldsymbol{\theta}\big)$$

where $u : \Delta \longrightarrow \Delta$ is an invertible measure preserving transformation. We consider the semi-group of measure preserving transformations which is the strong closure of the Weyl group of the completion of $\mathscr{D}^s_\omega(M, T)$.

Definition 3.6.1. *The Weyl semi-group* $\overline{\mathbf{W}}$ *is the set of all measure preserving transformations* $\sigma : (\Delta, m) \longrightarrow (\Delta, m)$.

3.7 Schur's Theorem

We are now ready to state our version of the Schur's theorem. Our proof follows [12] with some minor modifications.

Theorem 3.7.1. *[Schur's Theorem, [12]] Let* $f \in L^2(M, \nu_\omega)$*, let* $\pi(f)$ *be the "zeroth Fourier coefficient" of* f

$$\pi(f)(z) = \int\limits_{\mu^{-1}(z)} f d\nu_z$$

and let λ *be as in the spectral theorem 3.2.1. Then* $\pi(f)$ *belongs to the weak closed convex hull of the orbit of the Weyl semi-group* $\overline{\mathbf{W}}$ *through* $\lambda \circ r$ *where* $r : \Delta \longrightarrow [0, 1]$ *is a measure preserving bijection.*

Proof. By the Diagonalization Theorem 3.3.2 there is a measure preserving map $\sigma : M \longrightarrow M$ such that $f = (\lambda \circ q) \circ \sigma$. We define a doubly stochastic operator $P : L^2[0,1] \longrightarrow L^2(\Delta)$ by

$$(P\varkappa)(z) = \int\limits_{\mu^{-1}(z)} \big((\varkappa \circ q) \circ \sigma\big) d\nu_z$$

Obviously, $P\lambda = \pi(f)$ and in view of Theorem 3.1.11 P being a doubly stochastic operator implies that $\pi(f) \prec \lambda$. By Theorem 3.1.13 this means that $\pi(f)$ belongs to the weak closed convex hull of $\overline{\mathbf{W}}$ through $\lambda \circ r$. To show that P is indeed a doubly stochastic operator we use Theorem 3.1.10. Let $A \subseteq [0,1]$ be a measurable set and χ_A its characteristic function then we have

$$\big(\chi_A \circ q\big) \circ \sigma = \chi_{\big((q\circ\sigma)^{-1}(A)\big)}$$

So

$$\big(P\chi_A\big)(z) = \int\limits_{\mu^{-1}(z)} \chi_{\big((q\circ\sigma)^{-1}(A)\big)} \leq 1$$

and

$$\begin{aligned}
\int\limits_{\Delta} \big(P\chi_A\big)(z)dz &= \int\limits_{\Delta} \int\limits_{\mu^{-1}(z)} \big(\chi_A \circ q\big) \circ \sigma d\nu_z dz \\
&= \int\limits_{M} \big(\chi_A \circ q\big) \circ \sigma d\nu_\omega \\
&= \int\limits_{M} \big(\chi_A \circ q\big) d\nu_\omega \quad \text{since } \sigma \text{ is measure preserving} \\
&= \int\limits_{0}^{1} \chi_A dm \quad \text{since } q \text{ is measure preserving} \\
&= m(A).
\end{aligned}$$

□

Roughly speaking, Schur's Theorem 3.7.1 says that

$$\pi(\mathscr{O}_{\lambda\circ q}) \subseteq \mathsf{Conv}\Big(\overline{\mathbf{W}}\cdot(\lambda\circ r)\Big). \tag{3.7.11}$$

3.8 Horn's Theorem

In this section we prove the converse to Schur's Theorem 3.7.1 which will provide us with an analog of Horn's theorem in the infinite-dimensional setting. Our first goal is to prove a new representation for doubly stochastic operators that will lead to a sharper version of Horn's theorem compared to the one proven in [12].

Proposition 3.8.1. *Let $P : L^1[0,1] \longrightarrow L^1[0,1]$ be a doubly stochastic operator then there is a measure preserving transformation $\tau : [0,1]\times[0,1] \longrightarrow [0,1]$ such that*

$$\big(Pf\big)(x) = \int_0^1 f\circ\tau(x,y)dy.$$

Proof. First, we recall main points of the proof of Horn's Theorem in [12] in which they have produced a "kernel" τ.

(1) Let $\gamma(x,y) = P\chi_{[0,y]}(x)$, $\gamma(x,y)$ is increasing in y almost everywhere. That is, there is a measure zero set $E\subset[0,1]$ such that for every $x\notin E$, $\gamma(x,y)$ is increasing in y.

(2) For every $x\notin E$ let $\tau : [0,1]\times[0,1] \longrightarrow [0,1]$ be

$$\tau(x,z) = \inf\{y \mid \gamma(x,y)\geq z\}.$$

Then τ is measurable and has the following important property

$$\tau(x,z) \leq a \Longleftrightarrow \gamma(x,a) \geq z. \tag{3.8.12}$$

(3) The property 3.8.12 implies that for any $A = (a,b]$ and any $x \notin E$ we have

$$\begin{aligned}
\int_0^1 \chi_A(\tau(x,z))dz &= \int_0^1 \chi_{\tau^{-1}(A)}(x,z)dz \\
&= \int_{\gamma(x,a)}^{\gamma(x,b)} dz \\
&= \gamma(x,b) - \gamma(x,a) \\
&= P\chi_A(x).
\end{aligned}$$

In particular, for every step function (and hence for every measurable function f that can be written as the limit of an increasing sequence of step functions) we have

$$(Pf)(x) = \int_0^1 f\big(\tau(x,z)\big)dz.$$

What has not been noticed† in [12] is that τ is a measure preserving transformation that plays the role of a kernel for P. Riesz approach to Lebesgue integral [72, See Chapter 10] guarantees that any $g \in L^1[0,1]$ can be decomposed as a difference of two integrable functions $g = h - k$ each of which

†. We should point out that El Hadrami states Proposition 3.8.1 in his thesis [30, See Theorem 6.2.4] but in his proof he refers to a corrected new version of [12] which we have not been able to find.

can be written as the limit of an increasing sequence of step functions almost everywhere, so

$$\begin{aligned}
\big(Pg\big)(x) &= \big(Ph\big)(x) - \big(Pk\big)(x) \\
&= \int_0^1 h\big(\tau(x,z)\big)dz - \int_0^1 k\big(\tau(x,z)\big)dz \\
&= \int_0^1 g\big(\tau(x,z)\big)dz
\end{aligned}$$

for all $x \notin E$. To show that τ is measure preserving, first recall that since P is doubly stochastic we have $\int_0^1 (P\chi_B)(x)dx = m(B)$ for every measurable set $B \subset [0,1]$. Hence

$$\begin{aligned}
m(B) &= \int_0^1 \int_0^1 \chi_B\big(\tau(x,z)\big)dzdx \\
&= \int_0^1 \int_0^1 \chi_{\tau^{-1}(B)} dzdx \\
&= m(\tau^{-1}(B)).
\end{aligned}$$

□

We remark that τ in Proposition 3.8.1 is not unique. To see this, take a non-trivial measure preserving transformation $\alpha : [0,1] \longrightarrow [0,1]$ then define $\widehat{\tau}(x,z) := \tau(x,\alpha(z))$. It is easy to see that both $\widehat{\tau}$ and τ define the same doubly stochastic operator.

Other similar parametrizations of doubly stochastic operators have been obtained previously [73]. Still there is an interesting question that is motivated by Proposition 3.8.1. For what choices of "kernel" τ is the corresponding doubly stochastic operator an extreme point of the convex set of doubly stochastic operators? It is known that for any measure preserving transformation

$\sigma : [0,1] \longrightarrow [0,1]$

$$\begin{aligned} T_\sigma : L^1[0,1] &\longrightarrow L^1[0,1] \\ f &\longmapsto f \circ \sigma \end{aligned}$$

is an extreme point [69], but this construction does not account for all extreme points. One can readily see that T_σ can be obtained by taking $\tau(x,z) := \sigma(x)$. On the other hand, if we let $\mathsf{pr}_1 : [0,1] \times [0,1] \longrightarrow [0,1]$ be the projection onto the first component $\mathsf{pr}_1(x,y) = x$ then pr_1 induces an operator $T_{\mathsf{pr}_1} : L^1[0,1] \longrightarrow L^1([0,1] \times [0,1])$ which is defined by $T_{\mathsf{pr}_1}(f) = f \circ \mathsf{pr}_1$. It is easy to see that $T^*_{\mathsf{pr}_1} : L^1([0,1] \times [0,1]) \longrightarrow L^1[0,1]$ is given by $T^*_{\mathsf{pr}_1}(g) = \int\limits_0^1 g(x,y)dy$. Hence $P(f) = T^*_{\mathsf{pr}_1} \circ T_\tau(f)$, where $T_\tau : L^1[0,1] \longrightarrow L^1([0,1] \times [0,1])$ is defined by $T_\tau(f) = f \circ \tau$. We can mimic the argument of Theorem 2 in [69] to show that if τ is an invertible measure preserving transformation then $P = T^*_{\mathsf{pr}_1} \circ T_\tau$ is an extreme doubly stochastic operator. In fact, since we have assumed that $\tau : [0,1] \times [0,1] \longrightarrow [0,1]$ is an invertible measure preserving transformation then $T_\tau : L^1[0,1] \longrightarrow L^1([0,1] \times [0,1])$ is an unitary operator which means that $T^*_\tau = T_{\tau^{-1}}$. Hence $P = \left(T^*_\tau \circ T_{\mathsf{pr}_1}\right)^* = \left(T_{\tau^{-1}} \circ T_{\mathsf{pr}_1}\right)^* = T^*_{\mathsf{pr}_1 \circ \tau^{-1}}$. So P is the adjoint of the operator $T_{\mathsf{pr}_1 \circ \tau^{-1}}$ which is extreme since $\mathsf{pr}_1 \circ \tau^{-1} : [0,1] \longrightarrow [0,1]$ is measure preserving. It is easy to see that the adjoint of an extreme doubly stochastic operator is extreme again and hence P is extreme. This method produces a new class of extreme doubly stochastic operators that are different from the previous ones that were obtained by taking $\tau(x,z) := \sigma(x)$. Also, it follows from our argument that any doubly stochastic operator can be decomposed as $T^*_\theta \circ T_\sigma$, where $\theta, \sigma : [0,1] \longrightarrow [0,1]$ are measure preserving, which is interesting on its own (one can compare with [64]). Moreover θ

can be chosen to be a fixed measure preserving transformation, i.e., it can be made "canonical". A detailed discussion about the theory of doubly stochastic operators and its history can be found in [40].

We now define an analog of Horn's orthostochastic matrices in our infinite-dimensional setting. This analog has nothing to do with Horn's theorem and is presented just for the sake of completeness. We need some results from measure theory that we are going to recall as we proceed. First, we provide an equivalent definition for non-atomic measure space when the measure is defined on a separable metric space X. Recall the following lemma [62, See 15.5 Lemma 14].

Lemma 3.8.2. *[62] Let ν be a Borel measure on a separable metric space X, and A an atom on ν. Then there is a point $x \in A$ such that $\nu(A \setminus \{x\}) = 0$.*

Lemma 3.8.2 means that all of the measure of an atom will be on a singleton $\{x\}$ inside that atom. Hence for a separable metric space X endowed with a Borel measure ν, ν is non-atomic if and only if $\forall\, x \in X : \ \nu(\{x\}) = 0$. Now we recall the following definition of doubly stochastic measures [73].

Definition 3.8.3. *A Borel measure ν on the unit square $[0,1] \times [0,1]$ is called doubly stochastic measure if*

$$\forall\ A, B \in \mathscr{B}(X)\ :\ \nu(A \times [0,1]) = m(A), \quad \nu([0,1] \times B) = m(B)$$

That is ν has the Lebesgue measure m as its marginal measures.

The criteria for a measure to be non-atomic can be used to show that all doubly stochastic measures on $[0,1] \times [0,1]$ are non-atomic. Obviously, any doubly stochastic measure is a probability measure so Theorem 3.3.1 will imply that (X, ν) is isomorphic to the Lebesgue Borel measure on $[0,1] \times [0,1]$.

Also, it is known that all the doubly stochastic measures can be realized by a pair of measure preserving transformations of $\alpha, \beta : [0,1] \longrightarrow [0,1]$, [73]. Namely,

(1) If $\alpha, \beta : [0,1] \longrightarrow [0,1]$ are measure preserving transformations then the map $(\alpha, \beta) : [0,1] \longrightarrow [0,1] \times [0,1]$ defined by $(\alpha, \beta)(x) = (\alpha(x), \beta(x))$ is a doubly stochastic measure.

(2) Every doubly stochastic measure ν on $[0,1] \times [0,1]$ can be realized as a pair (α, β) as in (1).

Let $\tau : ([0,1] \times [0,1], m) \longrightarrow [0,1]$ be a Borel measure preserving transformation then by choosing a Borel isomorphism $u : [0,1] \longrightarrow [0,1] \times [0,1]$ we can construct a measure preserving transformation $\tau \circ u : [0,1] \longrightarrow [0,1]$. Hence for any Borel measure preserving transformation $\sigma : [0,1] \longrightarrow [0,1]$, $(\tau \circ u, \sigma)$ will realize a doubly stochastic measure $\nu_{\tau,\sigma}$ as in (1) above. The map $\digamma_{\tau,\sigma} : \left([0,1] \times [0,1], \nu_{\tau,\sigma}\right) \longrightarrow ([0,1] \times [0,1], m)$ defined by $\digamma_{\tau,\sigma}(x,y) = \left(\tau(x,y), \sigma \circ u^{-1}(x,y)\right)$ is a measure preserving transformation. It follows easily that $\sigma \circ u^{-1} : [0,1] \longrightarrow ([0,1] \times [0,1], m)$, the second component of $\digamma_{\tau,\sigma}$, is measure preserving. Thus we have proven the following theorem.

Theorem 3.8.4. *Any Borel measure preserving map $\tau : ([0,1] \times [0,1], m) \longrightarrow [0,1]$ can be regarded as the first component of a Borel measure preserving map $\digamma_\nu : ([0,1] \times [0,1], \nu) \longrightarrow ([0,1] \times [0,1], m)$, where ν is a doubly stochastic measure. Conversely, any component of a Borel measure preserving map $\digamma_\nu : ([0,1] \times [0,1], \nu) \longrightarrow ([0,1] \times [0,1], m)$ is measure preserving.*

As any measure preserving map $\tau : ([0,1] \times [0,1], m) \longrightarrow [0,1]$ defines a doubly stochastic operator and conversely any doubly stochastic operator can be characterized by such a kernel, we can consider the measure preserving transformations between doubly stochastic measures and the canonical doubly stochastic measure as an analog of Horn's orthostochastic matrices. As we have pointed out, all doubly stochastic measures are isomorphic to the canonical one so the measure preserving automorphisms of the unit square endowed with the Lebesgue measure is a good analog of Horn's orthostochastic matrices. To support our argument more, we should point out that there is an one-to-one correspondence between doubly stochastic measures and doubly stochastic operators [40]. This answers a question asked by Bloch, Flaschka and Ratiu, see the remark at the end of page 527 in [12].

We now return to the proof of Horn's Theorem.

Theorem 3.8.5. *[Horn's theorem] Let $\lambda \in L^2[0,1]$ be decreasing, right-continuous function and let F belongs to the weak closed convex hull of $\overline{\mathbf{W}} \cdot (\lambda \circ r)$. Then there is a $f \in L^2(M, \nu_\omega)$ such that*

- $\pi(f)(z) = F(z)$.
- *f belongs to the orbit $\mathscr{O}_{\lambda \circ q}$.*

Proof. Let F be in the closed convex hull of the Weyl semi-group $\overline{\mathbf{W}}$ through $\lambda \circ r$ then there is a doubly stochastic operator $P : L^2[0,1] \longrightarrow L^2(\Delta)$ such that $P\lambda = F$. By Proposition 3.8.1 there is a measure preserving transformation $\beta : M \longrightarrow [0,1]$ such that for every $\varkappa \in L^2[0,1]$

$$(P\varkappa)(z) = \int\limits_{\mu^{-1}(z)} (\varkappa \circ \beta) d\nu_z$$

for almost every $z \in \Delta$. In particular

$$F(z) = \int\limits_{\mu^{-1}(z)} (\lambda \circ \beta) d\nu_z$$

Take $f = \lambda \circ \beta = (\lambda \circ q) \circ (q^{-1} \circ \beta)$. As $q^{-1} \circ \beta : M \longrightarrow M$ is measure preserving, f belongs to the orbit $\mathscr{O}_{\lambda \circ q}$. □

Roughly speaking, Horn's Theorem 3.8.5 says that

$$\pi(\mathscr{O}_{\lambda \circ q}) \supseteq \textsf{Conv}\Big(\overline{\mathbf{W}} \cdot (\lambda \circ r)\Big). \tag{3.8.13}$$

This provides the reveres inclusion to the Schur's Theorem inclusion 3.7.11.

We remark that in our Horn's Theorem 3.8.5 if one assumes that λ is essentially bounded then f is also essentially bounded and it has the same moments as λ, which implies the version of the Horn's theorem found in [12].

3.9 The Schur-Horn-Kostant Theorem

We put Schur's and Horn's Theorems together to get the Kostant formulation of the convexity theorem. Our construction allows us to replace L^2 functions by L^p functions everywhere and still obtain the Schur-Horn theorem in this new setting. We conclude by constructing a dictionary that illustrates the analogy between the finite-dimensional setting and the infinite-dimensional setting.

Theorem 3.9.1. *[Schur-Horn-Kostant Convexity] Let $\lambda \in L^2[0,1]$ be a decreasing, right-continuous. Let $\mathscr{O}_{\lambda \circ q}$ be the orbit of $\overline{Meas(M, \mu_\omega)}$ through $\lambda \circ q$. Then $\pi(\mathscr{O}_{\lambda \circ q}) \subset L^2(\Delta)$ is weakly compact, convex set. Its set of extreme points is $\overline{\mathbf{W}} \cdot (\lambda \circ r)$, the orbit of the Weyl semi-group through $\lambda \circ r$.*

Roughly speaking, the Schur-Horn-Kostant Theorem 3.9.1 says that

$$\pi(\mathscr{O}_{\lambda\circ q}) = \mathbf{Conv}\Big(\overline{\mathbf{W}}\cdot(\lambda\circ r)\Big)$$

or

$$\pi\Big(\overline{Meas(M,\mu_\omega)}\cdot(\lambda\circ q)\Big) = \mathbf{Conv}\Big(\overline{\mathbf{W}}\cdot(\lambda\circ r)\Big)$$

or

$$\pi\Big(\overline{Ham(M,\mu_\omega)}\cdot(\lambda\circ q)\Big) = \mathbf{Conv}\Big(\overline{\mathbf{W}}\cdot(\lambda\circ r)\Big)$$

Table 3.1: A Dictionary

$\mathbf{U}(n)$	$Ham(M,\omega)$
Finite-dimensional Lie group structure	Sobolev, $\cdots$
The maximal torus $\mathbf{T}^n$	$\mathscr{D}^{s+1}_\omega(M,T),\ \cdots$
$\mathfrak{g}^*$	$\overline{\{f\in C^\infty(M) \mid \int_M f=0\}} = \{f\in L^2(M,\mu_\omega) \mid \int_M f=0\}$
$\mathfrak{t}^*$	$\overline{\{f\in C^\infty_{\mathsf{T}}(M) \mid \int_M f=0\}} = \{f\in L^2(\Delta) \mid \int_\Delta f=0\}$
The coadjoint action	$\overline{Ham(M,\omega)} = \overline{Meas(M,\mu_\omega)}$
The coadjoint orbit $\mathscr{O}_A$	$\mathscr{O}_{\lambda\circ q}$
$\boldsymbol{\lambda}$: The Spectrum	$\lambda=\delta_f$, The Rearrangement of f
$\mathscr{O}_A=\Big\{B \;\Big|\; \boldsymbol{\lambda}(B)=\boldsymbol{\lambda}(A)\Big\}$	$\mathscr{O}_{\lambda\circ q}=\Big\{f\in L^2(M,\mu_\omega) \;\Big|\; \delta_f=\lambda\Big\}$
The moment map π	$\pi(f)=$ zeroth Fourier coefficient of f
The Weyl group $\mathbf{W}_{\mathbf{T}^n}$	The Weyl semi-group $\overline{\mathbf{W}}$
Closed convex hull	Weakly compact and convex
Schur-Horn-Kostant Convexity	Schur-Horn-Kostant Convexity

Schur-Horn-Kostant Convexity Theorems

3.10 Final Remarks

In this thesis, we have obtained some topological, geometrical and analytical results on the maximal Abelian subgroup $\mathscr{D}_\omega(M,T)$. We strongly believe that the analogy with the finite-dimensional case can be pushed much further. We would like to conclude by mentioning other possible directions of research that seems to be promising.

(1) Bundle structure: We know that topologically $\mathscr{D}_\omega(M,T)$ retracts to T. The recent result of Ebin about the geodesic completeness of $\mathscr{D}_\omega(M)$ suggests that we may be able to realize this retraction along geodesics. This would give us a very clear geometric understanding of the torus $\mathscr{D}_\omega(M,T)$.

(2) Geometry of the Quotient: An other project is to investigate the quotient $\mathscr{D}^s_\omega(M) / \mathscr{D}^s_\omega(M,T)$. For a compact connected Lie group G, the quotient of G by its maximal torus T has quite interesting properties from the geometric and the topological points of view. For, example G/T is a Kähler manifold which is called a *flag* manifold [6] and it can be viewed as an adjoint orbit. It would be interesting to see if there is any infinite-dimensional analog of the geometric quantization procedure in this context.†

(3) Hofer Geometry: It is shown in [57] that $\mathscr{D}_\omega(M,T)$ lies in the connected component of the identity of $\mathscr{D}_\omega(M)$, i.e., $\mathscr{D}_\omega(M,T)$ is a subgroup of $Ham(M,\omega)$. It is also known that $Ham(M,\omega)$ carries a Finsler structure which is called the *Hofer* metric [32, 59]. We know that $\mathscr{D}_\omega(M,T)$ is

†. The fact that the quotient should be Kähler was pointed out to us by Tudor S. Ratiu.

a closed submanifold of $Ham(M, \omega)$ by our result, but geometric properties of $\mathscr{D}_\omega(M, T)$ can be investigated further with respect to the Finsler structure on $Ham(M, \omega)$. Another project in this regard can be to study the diameter of $\mathscr{D}_\omega(M, T)$ with respect to the Hofer metric [58]. In this setting we can take a different approach and study the graphs of elements of $\mathscr{D}_\omega(M, T)$ as Lagrangian submanifolds with respect to the Hofer metric on Lagrangian submanifolds [53].

(4) Extensions to contact manifolds: Beyond symplectic manifolds, the next natural objects to study are toric contact manifolds. Indeed, these manifolds can be characterized in terms of the moment map image in a way similar to symplectic toric manifolds.

(5) Integrable Systems: It seems that some of our results may be extended to integrable systems, at least after imposing some conditions on the singular fibers [51].

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Vita

Name:	Seyed Mehdi Mousavi
Post-secondary Education and Degrees:	The University of Western Ontario London, Ontario, Canada 2008-2012, Ph.D.
	Iran University of Science and Technology Tehran, Iran 2004-2006, M.Sc.
	Iran University of Science and Technology Tehran, Iran 2000-2004, B.Sc.
Honors and Awards	Queen Elizabeth II Graduate Scholarship in Science and Technology 2011-2012
	Santoni Ontario Graduate Scholarship 2011-2012
	Western Graduate Thesis Research Award 2011
	Western Graduate Scholarship 2008-2011
Related Work Experience	Graduate Teaching Assistant The University of Western Ontario, 2008-2011
	Instructor Linear Algebra 1600B The University of Western Ontario, Winter 2012