The Role of Skills and Technology in the Determination of Optimal Assignments within the Firm

Glenn M. MacDonald

James R. Markusen

Citation of this paper:
RESEARCH REPORT 8211

THE ROLE OF SKILLS AND TECHNOLOGY IN THE DETERMINATION OF OPTIMAL ASSIGNMENTS WITHIN THE FIRM*

by

Glenn M. MacDonald
and
James R. Markusen

Abstract

The proposition that within the firm workers will be assigned to activities on the basis of comparative advantage is examined. It is shown that the optimal assignment depends upon the way in which individual skills interact with technology to produce final output. One implication of this interaction is that optimal assignments need not be consistent with input-based definitions of assignment by comparative advantage but are generally consistent with output-based measures. The reason is that the latter measures embody the interaction of skills with technology while the former do not.

May, 1982

*Comments provided by workshop participants at the University of Western Ontario, University of British Columbia, and Simon Fraser University are gratefully acknowledged.
1. INTRODUCTION

In Adam Smith's pin factory there are three reasons for the division of labor. First, if each activity performed within the firm requires a different type of human capital, specialization raises the fraction of time any particular kind of capital is put to use, and so raises the return to its accumulation. Specialization therefore encourages investment in skills. Second, when the fraction of time spent at each activity is raised, the returns to inventing machines which render performance of such tasks less arduous become greater. Finally, if stopping one activity and starting another is costly, there is obvious economy in reducing the number of such switches.

In Smith's world, all workers (at least at the outset) are equally proficient in the performance of each type of task. When this is not the case, division of labor arises to exploit such heterogeneity.

Specialization induced by investment in skills and new machinery is an intrinsically dynamic phenomenon. The static problem of how specialization arises when the work force is heterogeneous and/or when switching activities is costly is both of independent interest and logically prior to analysis of investment decisions. Accordingly, the focus herein is on the static problem.

The nonstochastic assignment of heterogeneous workers in the absence of switching costs has been analyzed by Rosen [1978]. For a particular type of technology (described below) a very strong result is obtained. Suppose there are just two activities ($\alpha$ and $\beta$) in which workers might engage, and
many types of workers. Then, in an optimal assignment, each worker participates in only one activity. Further, when worker types are ordered according to declining comparative advantage at activity $\alpha$ relative to activity $\beta$, there is a marginal worker type such that all workers with comparative advantage greater than that of this marginal worker type perform only activity $\alpha$, the rest performing $\beta$. Technology matters only for the determination of the marginal worker type.

That the pattern of comparative advantage is the primary determinant of optimal assignment accords well with intuition, but less well with experience. The "Peter Principle" is a standard example within the firm. Though typically stated in a hierarchical context, its essential ingredient is the following: If one worker is to be reassigned from $\alpha$ to $\beta$, the worker who is most able at $\alpha$ will be chosen. As a second example, consider administrative assignments in an academic environment. The scholars having the greatest opportunity cost in terms of research frequently seem to receive the bulk of the administrative load; the best athlete is made team captain; the most attractive student is elected class president; and so on.

For all these cases it is possible to contrive a pattern of skills which allows assignment by comparative advantage to hold. Alternatively, it is possible to introduce auxiliary hypotheses in each instance ("we do not know what is being maximized in an academic setting...”). Both approaches are neither scientific nor necessary.

The route followed herein is to examine the conditions under which assignment by comparative advantage in terms of worker skills is in fact optimal in a standard maximization problem. The theories upon which both intuition (at least that of the authors) and Rosen's result rest contain assumptions which effectively deny the technology used to produce output, an
important role in the determination of optimal assignment of workers to activities. Once these assumptions are relaxed a richer collection of assignment structures arise as optimal behavior; the situations which are counterintuitive from the standpoint of the structure of skills coincide quite neatly with the conditions necessary for such assignments to be optimal.

The investigation proceeds as follows. The structure of skills possessed by workers is set out first. Attention is then directed to the technology through which activities yield output. Section 4 specifies the definition of "assignment by comparative advantage" more carefully. Next, the optimal assignment of a homogeneous collection of workers with costs of switching activities is examined. The case of heterogeneous workers with and without switching costs is then pursued. The final two sections evaluate the extent to which the results are sensitive to one particular aspect of the technology employed in the earlier sections, and summarize the results.

2. SKILLS

Workers are endowed with the ability to perform two activities: $\alpha$ and $\beta$. Only one activity may be performed at a point in time.

There are many types of workers, where types are indexed by $s$, $s \in [0,1]$. Note that since the index $s$ is unidimensional, this specification of skills is of the "one-factor" type.\(^4\)

A worker of type $s$ can perform $t_i(s) > 0$ ($i=\alpha, \beta$) of activity $i$ in one period of work. Without loss of generality, the one factor setup implies worker types may be ordered so that $t_\alpha / t_\beta$ declines in $s$. Furthermore, the analysis to follow is simplified if it is assumed that $t_\alpha$ and $t_\beta$ are monotonic and differentiable functions of $s$. Given this, the ordering of worker types implies
\[
\frac{t'_{i}}{t'_{i}} > 0
\]

where \( t'_{i} = dt_{i}/ds \). If \( t'_{\alpha} \) and \( t'_{\beta} \) have the same sign, \( s \) is an unambiguous ability measure. For example if \( t'_{\alpha} \) and \( t'_{\beta} \) are negative, workers of type \( s = 0 \) are the most productive at both tasks. When \( t'_{\alpha} \) and \( t'_{\beta} \) are of opposite signs, no such ability index exists.\(^3\)

3. TECHNOLOGY

In the assignment problem, the nature of the interaction between skills and technology is very important. Therefore a careful examination of the technology is required. Suppose the firm has hired \( N(s) \) workers of type \( s \). Within any type, let workers be indexed by \( \omega \), \( 0 \leq \omega \leq N(s) \), and denote by \( \varphi(\omega, s)(\varphi > 0) \) the fraction of the working period that worker \( \omega \) of type \( s \) spends performing activity \( \alpha \).\(^5\)

To better understand the relatively special nature of both the technology used by Rosen and that used herein, it is useful to start with a very general technology and explicitly impose restrictions upon it. The general production relationship is described by the real-valued nonlinear functional

\[
q = F(t_{\alpha}, t_{\beta}, \varphi, N)
\]

where \( q \) represents the level of output.

The type of production relationship with which economists are usually familiar involves a much higher level representation of the skill-time-number of workers relationship. This representation is obtained by viewing \( N(s) \) as fixed, suppressing the (exogenous) \( t_{i}(s) \) functions, and choosing \( \varphi(\omega, s) \) to maximize output. This yields the maximum output attainable from the input function \( N(s) \):

\[
q = G(N).
\]
Here $G(\cdot)$ is a nonlinear functional defined on non-negative functions $N(s)$, $0 \leq s \leq 1$, and is analogous to the standard production function wherein there are finitely many inputs.

Returning to (2), the type of technology employed by Rosen is obtained by assuming that (i) the production functional depends not on $t_i(s)$ and $\varphi(\omega, s)$ separately, but rather on the amount of activity $i$ performed by person $\omega$ of type $s$, say $\tau_i[t_i(s), \varphi(\omega, s)]; (ii) \tau_i[\cdot]$ can be written as $\tau_{\alpha}(s) = t_{\alpha}(s)\varphi(\omega, s)$ and $\tau_{\beta}[\cdot] = t_{\beta}(s)[1 - \varphi(\omega, s)];$ and (iii) only the total quantity of activity $i$ performed is of any consequence. Under these assumptions (2) can be rewritten as

$$q = F(T_{\alpha}, T_{\beta})$$

(4)

where

$$T_{\alpha} = \int_0^1 t_{\alpha}(s) \int_0^s \varphi(\omega, s)d\omega ds$$

(5)

and

$$T_{\beta} = \int_0^1 t_{\beta}(s) \int_0^s [1 - \varphi(\omega, s)]d\omega ds.$$

(6)

When might (4) be an appropriate a priori specification? The crucial aspect of (4) is that a proportional increase in the amount of time a worker spends at an activity generates an equiproportionate increase in the quantity of that activity (condition (ii) above). The activities are perfectly replicable. In particular, there is no room for increasing or decreasing returns to scale in performance of activities. Another way to look at this is to view the $t_i$ as skills which, in conjunction with time ($\varphi$), yield productive activities ($\tau_i$) according to the production functions $\tau_{\alpha} = \varphi t_{\alpha}$ and $\tau_{\beta} = (1 - \varphi) t_{\beta}$. If $t_i$ is simply the repetition of activity $i$, and this can be accomplished without fatigue (energy is the fixed factor) or warmup, these production relations are acceptable. If either of these
conditions fail, the homogeneous specification will be inappropriate.

There are many ways in which this specification might be relaxed. Roughly speaking, the approach taken herein involves introduction of diminishing marginal product at the individual level. Specifically, it is assumed that output is produced according to

\[ q = \int \int h[\varphi(\omega, s) t_\alpha(s), \xi(\omega, s) T_\beta] d\omega d\sigma \]  

(7)

with \( T_\beta \) as defined in (6),

\[ \xi(\omega, s) \geq 0, \]

and

\[ \int \int \xi(\omega, s) d\omega d\sigma = \text{constant}. \]

\( h[\cdot] \) is assumed to possess positive and diminishing marginal products, and both inputs are necessary for positive output. The reader can easily incorporate regions of increasing marginal product. The idea here is that final production takes place on an individual basis using the own input \( \varphi_\alpha \) and an allocation (\( \xi \)) of an input (\( T_\beta \)) which is produced under perfect replicability. \( q \) is simply aggregate individual output. For example, activity \( \beta \) could be the perfectly replicable production of parts, and individual output the finished product of assembly (activity \( \alpha \)) using the share \( \xi \) of these parts. In this case \( \xi(\cdot) \) integrates to unity; \( T_\beta \) is a private input. Alternatively, \( \xi \equiv 1 \) can be assumed (hence \( \int \int \xi = \int N \)), in which case \( T_\beta \) is a public input such as administrative activities which provide an environment for individual research (requiring activity \( \alpha \)).

In what follows \( \xi(\omega, s) \) will be assumed constant. This simplifies the algebra without materially affecting the results (see Section 7). Note that the value \( \xi \) takes on need not be specified. This highlights the point
that the answers to the questions under consideration do not depend in any important way on the degree to which some inputs used in the firm have public input characteristics.

One further point should be made concerning $h[.]$. When $h_{11} < 0$, it may be possible that the marginal product of time ($\varphi$) at activity $\alpha$, $t_{\alpha} h_{1} [.]$, is smaller for workers with larger $t_{\alpha}$. There is obviously no a priori reason to rule this out, but whether this occurs matters for what is to follow. It is easy to show that

$$\frac{\partial}{\partial t_{\alpha}} t_{\alpha} h_{1} \leq 0 \Rightarrow \eta \leq -1,$$

where

$$\eta = \frac{\varphi t_{\alpha}}{h_{1}} \cdot h_{11} < 0 \quad (8)$$

is the elasticity of the marginal product of activity $\alpha$.

4. DEFINITIONS OF ASSIGNMENT BY COMPARATIVE ADVANTAGE

One of the main questions addressed in the paper concerns the conditions under which optimal assignments, and assignments based solely on the structure of comparative advantage in terms of workers' skills, are in some sense qualitatively similar. If technology plays any role apart from the determination of a marginal worker type, then optimal assignments will necessarily be quantitatively distinct from those based solely on the structure of skills. This much is obvious and not very informative. More useful are notions that allow for some quantitative difference between assignments, and state that two assignments are qualitatively similar unless they differ in terms of some central attribute.
There are two obvious candidates. Recall that: (i) worker types are ordered so that comparative advantage at activity $\alpha$ relative to $\beta$ declines across workers types; and (ii) $\varphi(w,s)$ is the fraction of the working period worker $w$ of type $s$ spends at activity $\alpha$; and hence (iii) $\varphi(w,s)t'_{\alpha}(s)$ is the amount of task $\alpha$ preferred by that worker. One criterion for qualitative similarity of assignments is simply that $\varphi(w,s)$ be nonincreasing in $s$. When an optimal assignment satisfies this, workers having a stronger comparative advantage at activity $\alpha$ spend more time performing it. As time is an input to activity $\alpha$, this criterion is what will be referred to as the input-based criterion. Formally, where $\varphi$ is differentiable, an optimal assignment follows the pattern of comparative advantage if for all $w$ and $s$

$$\frac{\partial}{\partial s} \varphi(w,s) \leq 0. \tag{9}$$

A second criterion will be referred to as output-based: $\varphi(w,s)t'_{\alpha}(s)$ is nonincreasing in $s$. When an optimal assignment satisfies this, workers having a stronger comparative advantage at activity $\alpha$ perform more of that activity. Formally, an optimal assignment follows the pattern of comparative advantage if for all $w$ and $s$

$$\frac{1}{\varphi(w,s)} \frac{\partial \varphi}{\partial w} + \frac{t'_{\alpha}(s)}{t_{\alpha}(s)} \leq 0. \tag{10}$$

Since $\xi(w,s)$ is fixed (and hence so is the quantity of activity $\beta$ available to each worker), if (10) is satisfied those workers with a stronger comparative advantage at activity $\alpha$ produce output using a higher activity $\alpha$/activity $\beta$ ratio. When $\xi(\cdot)$ is allowed to vary, this latter criterion becomes, for all $w$ and $s$

$$\frac{1}{\varphi(w,s)} \frac{\partial \varphi}{\partial s} + \frac{t'_{\alpha}(s)}{t_{\alpha}(s)} - \frac{1}{\xi(w,s)} \frac{\partial \xi}{\partial s} \leq 0. \tag{11}$$
5. OPTIMAL ASSIGNMENT OF HOMOGENEOUS WORKERS
WHEN SWITCHING ACTIVITIES IS COSTLY

This section considers the problem of assigning a homogeneous
collection of workers when switching activities is costly. The principal
point is a fairly obvious one, namely that costs of switching activities
may yield a situation in which identical workers are assigned differently.
The structure of such assignments is completely characterized.

The specification of the cost of switching activities is a very
simple one. Whenever a worker switches activities, \( c ( \geq 0) \) units of output
are lost. More complicated structures can be accommodated at the cost
of some notational inconvenience.

Since all workers are identical, the index \( s \) can be suppressed. There
are then \( N \) workers to be assigned, and \( \omega \in [0,N] \) indexes workers. Output
(gross of switching costs) can then be written

\[
q = \int_0^N h[\phi(\omega)]t_\alpha T_\beta d\omega , \tag{12}
\]

where

\[
T_\beta = t_\beta \int_0^N [1 - \phi(\omega)] d\omega .
\]

Workers can be assigned to performing solely one activity or the other,
or both. The latter assignment involves a switch of activities and hence
causes the cost \( c \) to be incurred.

Since all workers are identical, without loss of generality the continuum
\([0,N]\) can be divided into three subsets: for \( \omega \in [0,\bar{\omega}] \), worker \( \omega \) performs
only activity \( \alpha \); for \( \omega \in [\bar{\omega},\bar{\omega}] \), some time is allocated to each activity; and
for \( \omega \in [\bar{\omega},N] \), the worker is entirely specialized to activity \( \beta \).

The only reason for treating workers differently is that switching costs
are saved. For workers who perform both tasks, this cost is sunk and thus
irrelevant. It follows that all workers engaging in both activities spend the same fraction of time at each: \( \phi(w) \equiv \bar{\phi} \) for \( w \in (w, \bar{w}) \). Combining this with the partitioning of the continuum implies (12) can be rewritten (netting out the switch cost)

\[
q = w h[t_{\alpha}, T_{\beta}] + (\bar{w}-w)[h[\bar{\phi} t_{\alpha}, T_{\beta}] - c] \tag{13}
\]

where

\[
T_{\beta} = t_{\beta} [(N-\bar{w}) + (1-\bar{\phi})(\bar{w}-w)] \tag{14}
\]

and \( h[0, T_{\beta}] \equiv 0 \) is assumed. The problem is reduced to picking \( w, \bar{w} \) and \( \bar{\phi} \) subject to the constraints \( 0 \leq w \leq \bar{w} \leq N \) and \( 0 < \bar{\phi} < 1 \).

Denoting optimal values with an asterisk, conditions implied by the optimality of \( w^*, \bar{w}^* \) and \( \bar{\phi}^* \) are

\[
\frac{\partial q}{\partial w} = h[t_{\alpha}, T_{\beta}] - [h[\bar{\phi} t_{\alpha}, T_{\beta}] - c] - t_{\beta} (1-\bar{\phi}) \Delta \begin{cases} 
\geq 0 & \text{if } w^* = \bar{w}^* \\
= 0 & \text{if } 0 < w^* < \bar{w}^* \\
\leq 0 & \text{if } w^* = 0,
\end{cases} \tag{15}
\]

\[
\frac{\partial q}{\partial \bar{w}} = h[\bar{\phi} t_{\alpha}, T_{\beta}] - c - t_{\beta} \bar{\phi} \Delta 
\begin{cases} 
\geq 0 & \text{if } \bar{w}^* = N \\
= 0 & \text{if } w^* < \bar{w}^* < N \\
\leq 0 & \text{if } w^* = \bar{w}^*,
\end{cases} \tag{16}
\]

and

\[
\frac{\partial q}{\partial \bar{\phi}} = (\bar{w}-w)[t_{\alpha}, h[\bar{\phi} t_{\alpha}, T_{\beta}] - t_{\beta} \Delta] = 0 \tag{17}
\]

where

\[
\Delta \equiv \frac{\partial q}{\partial T_{\beta}} = w h_2[t_{\alpha}, T_{\beta}] + (\bar{w}-w)h_2[\bar{\phi} t_{\alpha}, T_{\beta}] \tag{18}
\]

The interpretation of (15)-(17) is straightforward. Take (15). An increment to \( w \) involves one more worker performing activity \( \alpha \) full time, yielding \( h[t_{\alpha}, T_{\beta}] \) in output. Also there is one less worker performing each task part time. This saves the switch cost \( c \). These two effects comprise the return
to raising $\bar{w}$. The other side of the experiment is that the worker's output, $h[\bar{\varphi} t_{\alpha', T_2}]$ is lost, as is the contribution he makes to the output of all workers through performing activity $\beta$, $t_{\beta}(1-\bar{\varphi})\Delta$. These make up the costs of raising $\bar{w}$. The costs and returns to raising $\bar{w}$ can be treated similarly. $\bar{\varphi}$ is chosen to equate the marginal product of time at activity $\alpha$, $t_{\alpha}h_{\alpha}$, to the cost of foregone $T_2$, $t_{\beta}\Delta$.

The structure of the optimal assignment is easily obtained with a little manipulation of (15)-(16). (17) implies $t_{\alpha}h_{\alpha}[\bar{\varphi} t_{\alpha', T_2}] = t_{\beta}\Delta$.

Substitution for $t_{\beta}\Delta$ in $\frac{\partial q}{\partial w}$ and $\frac{\partial q}{\partial \bar{w}}$ yields

\[
\frac{\partial q}{\partial w} = h[t_{\alpha', T_2}] + c - \{h[\bar{\varphi} t_{\alpha', T_2}] + (1-\bar{\varphi})t_{\alpha}h_{\alpha}[\bar{\varphi} t_{\alpha', T_2}]\} \tag{19}
\]

and

\[
\frac{\partial q}{\partial \bar{w}} = \{h[\bar{\varphi} t_{\alpha', T_2}] - \bar{\varphi} t_{\alpha}h_{\alpha}[\bar{\varphi} t_{\alpha', T_2}]\} - c - h[0, T_2]. \tag{20}
\]

(Recall that $h[0, T_2] = 0$.) In (19), the term in braces is the cost of raising $w$. It is also a first order Taylor approximation to $h[t_{\alpha', T_2}]$ around the point $(\bar{\varphi} t_{\alpha', T_2})$. In (20), the term in braces is the cost of lowering $\bar{w}$ (i.e., making one more worker completely specialized to activity $\beta$). It is also a first order Taylor approximation to $0 = h[0, T_2]$ around the point $(\bar{\varphi} t_{\alpha', T_2})$.

The structure of the assignment can now be described. Since $h[\ ]$ is concave, the Taylor approximation always lies above it. Accordingly when $c = 0$, the costs of raising $w$ and reducing $\bar{w}$ always fell short of the returns. In other words, $w^* = 0$ and $\bar{w}^* = N$, in which case all workers are assigned identically and to part-time performance of each activity. As $c$ rises, the returns to raising $w$ and reducing $\bar{w}$ rise. However, since (for example) reassigning one worker from full-time performance of activity $\alpha$ to part-time involves a discrete loss of output ($h - h$ is not "small"), it
will not be optimal to do so unless \( c \) is sufficiently large. In other words, there is a critical switching cost \( c^* \) such that the optimal assignment is invariant to \( c \) so long as \( 0 \leq c < c^* \). When \( c \) exceeds \( c^* \), some workers will be assigned to full-time performance of one activity; either \( \bar{w}^* > 0 \) and \( \bar{w}^* = N \), or \( \bar{w}^* = 0 \) and \( \bar{w}^* < N \). (Figure 1 is drawn so that a cost slightly in excess of \( c^* \) yields some workers being assigned to activity \( \alpha \) full time.) For greater values of \( c \), some other workers will be assigned to the other activity full time; both \( \bar{w}^* > 0 \) and \( \bar{w}^* < N \). Finally, for large values of \( c \), no workers perform both tasks; \( \bar{w}^* = \bar{w}^* \).

In summary, the introduction of costs of switching activities can cause specialization among identical workers, provided those costs are sufficiently large. Small switching costs have no impact at all on the optimal assignment of homogeneous workers.

6. OPTIMAL ASSIGNMENT OF HETEROGENEOUS WORKERS

In this section, the optimal assignment of heterogeneous workers is considered for three situations: (i) no costs of switching activities; (ii) including costs of switching activities, but excluding the possibility of assigning identical workers to different activities; and (iii) including switching costs as in (ii) but allowing identical workers to be assigned differently.

A. Assignment in the Absence of Switching Costs

When workers are heterogeneous, output is given by

\[
q = \int_{0}^{1} \int_{0}^{N(s)} h(\varphi(w,s) \tau_\alpha(s), T_\beta) dw ds
\]
where $T_\beta$ is defined in (6).

Consider $\partial q/\partial \varphi(\omega, s)$:

$$
\frac{\partial q}{\partial \varphi(\omega, s)} = h[\varphi(\omega, s) t_\alpha(s), T_\beta] t_\alpha(s)
- t_\beta(s) \int_0^1 \int_0^1 h_2[\varphi(\omega, s) t_\alpha, T_\beta] d\omega ds.
$$

At the optimum, $T_\beta$ is necessarily constant across workers. It follows that if for some worker $\omega$ in group $s$, some $\varphi^*(\omega, s)$ is optimal, then that same value of $\varphi^*$ is optimal for all workers in that skill group. Thus the dependence of $\varphi(\omega, s)$ on $\omega$ may be suppressed, and the assignment problem written

$$
\max_{\varphi(s)} \int_0^1 h[\varphi(s) t_\alpha(s), T_\beta] N(s) ds,
$$

s.t. $0 \leq \varphi(s) \leq 1$

$$
T_\beta = \int_0^1 t_\beta(s) [1 - \varphi(s)] N(s) ds.
$$

Letting $\varphi^*(s)$ denote an optimal assignment, the necessary condition for a maximum is that for each worker type $s$:

$$
N(s) \left\{ t_\alpha(s) h_1[\varphi^*(s) t_\alpha, T_\beta] - t_\beta \int_0^1 h_2[\varphi^*(s) t_\alpha, T_\beta] N(s) ds \right\}
= \begin{cases} 
\geq 0 \text{ if } \varphi^*(s) = 1 \\
= 0 \text{ if } \varphi^*(s) \in (0, 1) \\
\leq 0 \text{ if } \varphi^*(s) = 0.
\end{cases}
\quad (21)
$$

The interpretation is straightforward. For each worker of type $s$, an increment to $\varphi$ raises output directly by $t_\alpha h_1[\cdot]$. However, some activity $\beta$ is lost in this reallocation, and this affects all workers; hence the second term $t_\beta h_2[\cdot]$. Now, note that in (21) (i) the integral factor in the second term does not vary across $s$; (ii) $N(s)$ may be cancelled; and (iii) $T_\beta$ does not vary with $s$. Accordingly (21) may be rewritten as
where $\mu$ is the constant "efficiency price" of one unit of activity $\beta$.\textsuperscript{12}

For simplicity attention is focused only on assignment functions $\varphi^*$ which are either nonincreasing or nondecreasing in $s$. The extension to the general case merely involves treating the results to follow as local.

Under this restriction, there are indices $\underline{s}$ and $\bar{s}$ such that either

(a) $\varphi^*(s) = 1 \quad s \in [0, \underline{s}]$

$\varphi^*(s) \in (0, 1) \quad s \in (\underline{s}, \bar{s})$,

and $\varphi^*(s) = 0 \quad s \in [\bar{s}, 1]$,

or

(b) $\varphi^*(s) = 0 \quad s \in [0, \underline{s}]$

$\varphi^*(s) \in (0, 1) \quad s \in [\underline{s}, \bar{s}]$

and $\varphi^*(s) = 1 \quad s \in [\bar{s}, 1]$.

An assignment of type (a) occurs if, holding $\varphi(s)$ constant across $s$, $t_{\alpha} h_{\alpha} [\cdot] / t_{\beta}$ is falling in $s$. Thus in type (a) assignments, workers with a stronger comparative advantage in $\alpha$ spend a larger fraction of their time at $\alpha$. To see this, refer to (22). If $\varphi^*(s) = 1$ for some $s$, say $\tilde{s}$, then a declining $t_{\alpha} h_{\alpha} [\cdot] / t_{\beta}$ implies $\varphi^*(s) = 1$ for all $s < \tilde{s}$. As $t_{\alpha} h_{\alpha} [\cdot] / t_{\beta}$ falls across $s$, $s = \underline{s}$ is eventually reached and $\varphi^* < 1$ is optimal. Then holding $\varphi^*$ constant, an increase in $s$ further reduces $t_{\alpha} h_{\alpha} [\cdot] / t_{\beta}$. $h_{\beta} < 0$ implies equality in (22) can only be restored by lowering $\varphi^*$. Eventually $s = \bar{s}$ is reached and $\varphi^* = 0$ is optimal for all $s > \bar{s}$.\textsuperscript{12}
In a similar fashion an assignment of type (b) occurs if \( t_{\alpha_1} / t_\beta \) is rising in \( s \). Thus, in type (b) assignments, workers with a stronger comparative advantage in \( \alpha \) spend less time engaged in that activity.

The intuition is easy. A falling (rising) \( t_{\alpha_1} / t_\beta \) corresponds to the case where the marginal return to \( \varphi(s) \) declines (rises) across worker types relative to marginal cost.

Under what conditions does \( t_{\alpha_1} / t_\beta \) rise or fall? Using (8), differentiation of \( t_{\alpha_1} / t_\beta \) gives

\[
\frac{d}{ds} \left[ \frac{t_{\alpha_1}}{t_\beta} \right] \geq 0 = \frac{t'}{t_\beta} - \frac{t'}{t_\alpha} (1 + \eta) \geq 0. \tag{23}
\]

To examine (23), recall the restriction (equation (1)) \( t'/t_\beta - t'/t_\alpha > 0 \).

In conjunction with the issue of whether \(|\eta| \gtrless 1\), there are six cases, summarized in Table 1, wherein (*) denotes \( \frac{d}{ds} [t_{\alpha_1} / t_\beta] \).

| Table 1 |
|------------------|------------------|
| \( t'_\alpha > 0, \ t'_\beta > 0 \) | Case 1 (*) < 0 | Case 4 (*) < 0 |
| \( t'_\alpha < 0, \ t'_\beta > 0 \) | Case 2 (*) < 0 | Case 5 (*) \gtrsim 0 |
| \( t'_\alpha < 0, \ t'_\beta < 0 \) | Case 3 (*) \gtrsim 0 | Case 6 (*) > 0 |

Cases 1, 2 and 4 always yield a type (a) assignment; Case 6 invariably gives type (b), and either type (a) or type (b) may occur under Cases 3 and 5.

The explanation is easy enough. Looking across workers, \( t_\alpha / t_\beta \) always falls. Take Case 1 \((t'_\alpha > 0, \ t'_\beta > 0)\). If \( \varphi^* \) remained constant across workers,
since \( t_{\alpha} \) rises across workers, diminishing marginal product implies
\[ h_{1}[\varphi^*_{\alpha}, T_2] \] falls, reinforcing the decline in \( t_{\alpha}/t_{\beta} \). To raise \( t_{\alpha}h_{1}/t_{\beta} \)
back to equality with \( \mu \) requires a reduction in \( \varphi^* \) across workers: a
Type (a) assignment.  

In Case 2 (\( t'_{\alpha} < 0, t'_{\beta} > 0, |\eta| < 1 \), that \( t_{\alpha} \) falls across workers raises
\( h_{1} \), but since \( h_{1} \) is relatively unresponsive to changes in \( t_{\alpha} \) (holding \( \varphi^* \)
fixed) \( t_{\alpha}h_{1} \) falls across workers; a reduction in \( \varphi^* \) is again required to
restore equality in (22). Again this yields a Type (a) assignment.

In Case 5 (same as Case 2 except \( |\eta| > 1 \)), \( h_{1} \) is more responsive to
changes in \( t_{\alpha} \), and \( t_{\alpha}h_{1} \) rises across workers. Whether \( \varphi^* \) must rise or fall
across workers depends on how fast \( t_{\alpha}/t_{\beta} \) is falling. If the pattern of
comparative advantage changes quickly, Type (a) assignments emerge. More
moderate differences across individuals yield assignments of type (b)

For Case 3 (\( t'_{\alpha} < 0, t'_{\beta} < 0, |\eta| < 1 \), \( t_{\alpha}h_{1} \) again falls across workers.
But since \( t_{\beta} \) also falls, whether \( \varphi^* \) must increase or decline to restore
equality in (22) depends on the pattern of comparative advantage. But in
Case 6 (same as case 3 except \( |\eta| > 1 \), \( t_{\alpha}h_{1} \) rises across workers, making
a Type (b) assignment a necessity.

Overall, what is necessary for a Type (b) assignment to be optimal is
that workers with a comparative advantage at the activity which occurs
subject to diminishing returns \((\alpha)\) also have an absolute advantage at that
activity \((t'_{\alpha} > 0)\). This provides a framework within which to make sense out
of the somewhat counterintuitive examples cited in the Introduction. First,
each example contains an important element of individual production, hence
diminishing returns at the individual level is not unrealistic; and
second, each example involves the individuals who are most able at a given
activity spending a comparatively small fraction of their time performing it.
Now, how do the optimal assignments compare to those based upon the input- and output-based definitions of assignment by comparative advantage discussed in Section 4? Obviously, assignments of type (a) satisfy the input-based definition (\( \varphi' \leq 0 \)), while assignments of type (b) clearly do not. Also, for those workers who are not completely specialized, differentiation of (22) gives

\[
\frac{\varphi'}{\varphi} + \frac{t'_{\beta}}{t_{\alpha}} - \frac{t'_{\alpha}}{t_{\beta}} \geq \frac{1}{\eta} < 0. \tag{24}
\]

Recalling (10), the output-based definition of comparative advantage is always satisfied when \( \varphi \) can be varied to satisfy (22) with equality. Accordingly, optimal assignment yields a pattern of time allocation which follows the output-based definition of assignment by comparative advantage whenever such satisfaction is technologically possible.

B. Assignment with Switching Costs when all Workers of a Given Skill Type are Assigned in the Same Way

In this section the impact of switching costs on optimal assignment is examined under the restriction that all workers of a given type must be assigned in the same way.

One point is immediately obvious. If there are any workers who are not completely specialized in an optimal assignment (i.e., \( \varphi^\ast \in (0,1) \)), the switching costs are sunk as far as those workers are concerned. It follows that the results of the previous subsection apply directly to those worker types.

The other results are obtained from examination of the conditions implied when complete specialization of some workers is optimal. When there are costs of switching, output is given by
\[ q = \int_{0}^{1} h[\varphi(s) t_{\alpha}(s), T_{\beta}] N(s) ds - c \int_{\xi}^{1} N(s) ds, \]

where \( \omega \) is again suppressed because all workers within each group are assigned the same way by assumption, and

\[ \xi = [s | \varphi(s) \in (0,1)] \]

is the set of worker types which are not completely specialized. Denote the optimal time allocation by \( \varphi^*(s) \).

In Appendix 1 it is shown that a necessary condition for \( \varphi^*(s) = 1 \) to hold is

\[ t_{\alpha}(s) h_{1}[t_{\alpha}(s), T_{\beta}] - t_{\beta}(s) \int_{0}^{1} h_{2}[\varphi^*(s) t_{\alpha}(s), T_{\beta}] N(s) ds \]

\[ - \int_{0}^{1} \hat{H}(s) N(s) ds + c \geq 0, \tag{25} \]

where \( \hat{H}(s) \) is a strictly negative function of \( s \). The first three terms of (25) give the net loss of output which would be incurred if a worker of type \( s \) is reassigned from spending the fraction of the period \( \varphi \) at task \( \alpha \) to full time at task \( \alpha \). (25) states that the saving in switch costs must (at least) compensate for the lost output if \( \varphi^*(s) = 1 \) is to be optimal.

Proceeding in the same fashion for a worker of type \( s \) for which \( \varphi^*(s) = 0 \) yields the necessary condition

\[ t_{\alpha}(s^*) h_{1}[0, T_{\beta}] - t_{\beta}(s^*) \int_{0}^{1} h_{2}[\varphi^*(s) t_{\alpha}(s), T_{\beta}] N(s) ds \]

\[ + \int_{0}^{1} \hat{G}(s) N(s) ds = c \leq 0, \tag{26} \]
where $\hat{G}(s)$ is a strictly negative function of $s$. The interpretation here is as follows. If $\varphi^*(s) = 0$ is optimal, the gain in output associated with reassignment of workers of type $s$ from full time at task $\beta$ to the fraction $(1 - \varphi^*)$ of the period at activity $\beta$ cannot outweigh the added switch costs.

Having obtained (25) and (26), a comparison of optimal assignment with and without switch costs is straightforward. Again, attention is confined to type (a) and (b) assignments. Let $\varphi^0(s)$ be an optimal assignment for $c = 0$. $\varphi^0(s)$ solves (22) = 0 for $s \in (s, \bar{s})$. If (25) is evaluated for $\varphi^*(s) = \varphi^0(s)$, when $s = \bar{s}$ the result is

$$- \int_0^1 \hat{H}(s)ds + c > 0.$$

When $c = 0$ the firm is just indifferent between assigning workers of type $s$ to full time at activity $\alpha$ and almost full time. For $c > 0$, the former is strictly preferred. It is immediate that all workers who are completely specialized to activity $\alpha$ when $c = 0$, are also completely specialized when $c > 0$. Performing the same experiment for $s = \bar{s}$, (26) gives

$$\int_0^1 G(s)N(s)ds - c < 0,$$

in which case assigning workers of type $\bar{s}$ to full time performance of activity $\beta$ is strictly preferred to almost full time.

Still confining attention to type (a) assignments, unless $c$ is very large there will be some worker type $\underline{s} \geq \underline{s}$ such that (25) holds, and some $\bar{s} < \bar{s}$ such that (26) holds. For $s \in (\underline{s}, \bar{s})$, $\varphi^* \in (0,1)$ is optimal, where $\varphi^*(s)$ solves

$$t_{\alpha}(s)h_1[\varphi^*(s)t_{\alpha}(s), T_\beta] - t_{\beta}(s)\int_0^1 h_2[\varphi^*(s)t_{\alpha}(s), T_\beta]N(s)ds = 0 \quad (27)$$
Note that for \( s \) slightly in excess of \( \underline{s} \), (25) = 0 implies that a choice of \( \varphi^* \approx 1 \) will not satisfy (27). Indeed, (27) < 0 occurs, in which case it is necessary to choose \( \varphi^* \) a discrete amount below unity to satisfy (27). An analogous argument holds for \( s \) just below \( \bar{s} \).

The intuition is easy. Since the switch cost is discrete, the decision to bear this cost must involve a discrete increase in individual output over the completely specialized level. With a finite marginal product, this requires a discrete change in time allocated to activity \( \alpha \).

Assignments of type (b) can be treated in a fashion similar to the above. The results of this subsection are summarized in Figure 1.

C. Unconstrained Assignment with Switching Costs

In Section 5 optimal assignment of a homogeneous collection of workers was analyzed. It was shown that the introduction of costs of switching activities implies identical workers will not in general be assigned identically; switching costs induce specialization within identical groups of workers. In the previous subsection it was shown that the introduction of switch costs also raises the degree of specialization across worker types when workers are heterogeneous. The question to be addressed at this point is whether the analysis of the previous subsection (where identical workers were constrained to being assigned identically) is general, or does the existence of switch costs imply that identical workers will be assigned differently, as in Section 5? Do switch costs induce greater specialization within groups as well as across them?
Figure 1

Type (a)

Type (b)
The basic structure utilized for homogeneous workers is easily extended to the heterogeneous case. For workers of type \( s \), those for whom \( w \in [0, w(s)] \) perform only activity \( \alpha \); those with indices \( w \in [\tilde{w}(s), N(s)] \) perform activity \( \beta \) only. The rest, for whom \( w \in (\tilde{w}(s), \bar{w}(s)) \) spend \( \bar{\varphi}(s) \) at activity and \( 1 - \bar{\varphi}(s) \) performing activity \( \beta \).

Given this structure, output net of switch costs is

\[
q = \int_{0}^{1} [\bar{w}(s)h[t_{\alpha}(s), T_{\beta}] + [\tilde{w}(s) - w(s)]h[\bar{\varphi}(s)t_{\alpha}(s), T_{\beta}]]ds
\]

\[ - c\int_{0}^{1} [\tilde{w}(s) - w(s)]ds ,
\]

where

\[
T_{\beta} = \int_{0}^{1} \left\{ [N(s) - \tilde{w}(s)] + [1 - \bar{\varphi}(s)][\tilde{w}(s) - w(s)] \right\} ds ,
\]

and \( \tilde{w}(s) \geq w(s) \) for all \( s \).

For notational convenience, define \( \Delta \).

\[
\Delta \equiv \int_{0}^{1} [\bar{w}(s)h_{2}[t_{\alpha}(s), T_{\beta}] + [\tilde{w}(s) - w(s)]h_{2}[\bar{\varphi}(s)t_{\alpha}, T_{\beta}]]ds .
\]

It then follows that

\[
\frac{\partial q}{\partial w(s)} = h[t_{\alpha}(s), T_{\beta}] - h[\bar{\varphi}(s)t_{\alpha}(s), T_{\beta}] - [1 - \bar{\varphi}(s)]t_{\beta}(s)\Delta + c \quad (28)
\]

\[
\frac{\partial q}{\partial \bar{\varphi}(s)} = h[\bar{\varphi}(s)t_{\alpha}(s), T_{\beta}] - \bar{\varphi}(s)t_{\beta}(s)\Delta - c , \quad (29)
\]

and

\[
\frac{\partial q}{\partial \tilde{w}(s)} = [\tilde{w}(s) - w(s)]\left\{ h_{1}[\bar{\varphi}(s)t_{\alpha}(s), T_{\beta}]t_{\alpha}(s) - t_{\beta}(s)\Delta \right\} . \quad (30)
\]

One result is immediate. Suppose \( c = 0 \) and (30) can be satisfied by some
\( \bar{\varphi}(s) \in (0,1) \). Then an argument similar to that in Section 5 shows that \\
\( \partial q / \partial \bar{w}(s) < 0 \) and \( \partial q / \partial \bar{\varphi}(s) > 0 \), so that all workers of type \( s \) are assigned \\
in the same way.

Now suppose \( c = 0 \) and (30) cannot be satisfied by \( \bar{\varphi}(s) \in (0,1) \). That is; \\
either \( \bar{\varphi} = 1 \) or \( \bar{\varphi} = 0 \) for given \( s \). This is obviously equivalent to \( \bar{w}(s) = \bar{\omega}(s) = \bar{\bar{\omega}}(s) \).

Consider the choice of \( \bar{\omega}(s) \):

\[
\frac{\partial q}{\partial \bar{\omega}(s)} = h[T_{\alpha}(s), T_{\beta}] - t_{\beta}(s) \Delta.
\]

At the optimum, both \( T_{\beta} \) and \( \Delta \) do not vary across workers. Thus in general, \\
\( \partial q / \partial \bar{\omega}(s) = 0 \) will not hold for any \( \bar{\omega}(s) \in (0, N(s)) \). Therefore, provided \\
c = 0, \( \bar{\omega}(s) = 0 \) or \( \bar{\omega}(s) = N(s) \) will be optimal.\(^1\)

In sum then, except for a fortuitous satisfaction of (31), \( c = 0 \) \\
implies that identical workers will be assigned in precisely the same \\
fashion. Thus, barring coincidence, the analysis of Section 6A is general.

Returning to the case wherein \( c > 0 \), will identical workers ever \\
receive non-identical assignments?

It is easy to see that identical assignments for identical workers \\
will be the rule rather than the exception. Examine (28)-(30). At the \\
optimum, neither \( \Delta \) nor \( T_{\beta} \) varies across worker types. This means that \\
at the optimum, (28)-(30) are not functions of \( \bar{w}(s) \) and \( \bar{\omega}(s) \). Thus (for \\
example) if it pays to raise \( \bar{w}(s) \) above zero, it generally pays to raise \\
it all the way to \( N(s) \).

In Appendix 2, this argument is spelled out precisely. It is 

shown that except for a particular configuration of parameter values, 

identical assignments for identical workers is optimal. The analysis of
Section 6B is therefore the general case. Specialization within types of workers is optimal only if the possibility of specialization across worker types is not available (as in Section 5).

7. EXTENSION TO THE CASE WHERE $\xi(s)$ IS NOT CONSTANT ACROSS WORKERS

The arguments of the previous sections can be extended to the technology (7), wherein the allocation of $T_\beta$ to each worker is a choice variable.

For example, consider the assignment of heterogeneous workers without switch costs. The firm's problem is (with $T_\beta$ written in full)

$$\max_{\varphi(s) \in [0, 1], \xi(s) \geq 0} \int_0^1 h[\varphi(s) t_\alpha(s), \xi(s) ] \int_0^{t_\beta(\nu)} [1 - \varphi(\nu)] N(\nu) d\nu ] N(s) ds.$$ 

With minor manipulation, and assuming interior solutions for both $\varphi(s)$ and $\xi(s)$, the necessary conditions can be written: for all $s$

$$h_1[t_\alpha(s) \varphi^*(s), \xi^*(s) T_\beta] \frac{t_\alpha(s)}{\xi^*(s) T_\beta(s)} = \mu_0$$

(32)

and

$$h_2[t_\alpha(s) \varphi^*(s), \xi^*(s) T_\beta] = \mu_1$$

(33)

where $\mu_0$ and $\mu_1$ are independent of $s$. Total differentiation of (32) and (33), plus manipulation analogous to that in Section 6A, gives

$$\frac{\varphi'}{\varphi} = \frac{t_\beta'}{t_\alpha'} \frac{1 + G}{G}$$

(34)

where $G$ is negative provided $h_{12}$ is not "too negative". (32) can be interpreted in the same fashion as (22).
Furthermore,

$$\frac{\xi'}{\xi} = - \frac{h_{21}}{h_{22}} \frac{\varphi^*}{s^* \beta} \cdot \frac{\varphi'}{t^*} \cdot \left[ \frac{\varphi}{\varphi} + \frac{t^*}{t^*} \right].$$

(35)

Substituting for $\varphi'/\varphi$ from (34), and using (1), gives

$$\text{sign}\left[ \frac{\xi'}{\xi} \right] = - \text{sign}[h_{12}].$$

(36)

The interpretation of (36) is easy. By the same arguments as above, $\varphi t$ falls as $s$ rises. According to (33), $\xi^*$ must change so as to hold $h_2(\cdot)$ constant, if $h_{12} > 0$, the decline in $\varphi^*_\alpha$ reduces $h_2$. Since $h_{22} < 0$, $\xi^*$ must fall to raise $h_2$ back to equality with $\mu_1$. Thus in the leading case ($h_{12} > 0$), workers who have a comparative advantage at performing the activity $\alpha$ receive more of the input $T_\beta$. This holds irrespective of whether the assignment to activity $\beta$ follows the pattern of comparative advantage.

Recall that when $\xi(s)$ is variable, the activity $\alpha$/activity $\beta$ ratio interpretation of the output-based nation of assignment by comparative advantage must be altered (see equation (11)). Using (35), (11) holds if and only if (since $\varphi'/\varphi + t^*/t^*_\alpha < 0$ from (34))

$$1 + \frac{h_{21}}{h_{22}} \frac{\varphi^*_\alpha}{T_\beta} > 0.$$  

(37)

Let

$$\eta^c = \frac{\varphi^*_\alpha}{h_2} h_{21}$$

be the cross elasticity of the marginal product of activity $\beta$, and

$$\eta_2 = \frac{s^*_\beta}{h_2} h_{22} < 0$$
be the elasticity of the marginal product of activity $\beta$. Multiplying and dividing by $h_2$ in the second term, (37) is equivalent to the plausible restriction

$$\eta^c < -\eta_2.$$  

The cases wherein there are costs of switching can be treated in a similar fashion.

8. SUMMARY

This essay has examined the optimal assignment of workers to activities, allowing for heterogeneity in the skills of the workforce and costs of switching activities. Earlier work was extended by assuming a production structure sufficiently general to allow technology to play a nontrivial role in determination of the optimal assignments. The principal gain obtained by doing so is that a richer class of assignments are possible under the present assumptions, and a clear characterization of the circumstances under which the various types of assignments are predicted becomes available. In particular, a class of commonly observed assignments, apparently suboptimal when viewed from the standpoint of workers' skills alone, constitute optimal behavior under the kind of conditions wherein they are typically observed.

The key to understanding certain paradoxical assignments lies in the fact that the optimal assignment is based on the interaction of skills with technology. Definitions of assignments by comparative advantage which are based on outputs embody this interaction while definitions based on inputs do not. It is not surprising therefore that optimal assignments are
consistent with the output definitions but not necessarily with the input definitions.

Herein, an input definition of assignment by comparative advantage is that workers with a stronger comparative advantage in task \( \alpha \) should spend more time at task \( \alpha \) (less time at task \( \beta \)). An output definition is that workers with a stronger comparative advantage in task \( \alpha \) should produce more of the output of task \( \alpha \). The latter definition is consistent with the optimal assignment while the former need not be.

An example of this is provided by the case of an academic department. Results have suggested that an individual with a stronger comparative advantage in research should be observed to produce more research. The theory does not predict that this individual should be observed to spend a higher fraction of his working time at research. It may thus be optimal after all that the best researchers are asked to take on extra administrative responsibilities.
FOOTNOTES

1 Rosen also makes some headway on the investment decision and how it is affected by changes in the extent of the market in his [1981] and [1982].

2 Note that our focus is on assignment within the firm. Related material may be found in Markusen [1979].

3 There is no doubt that the efficient allocation of workers to activities involves assignment by comparative advantage in some sense. Indeed, producing given output at least cost, and assignment by comparative advantage, may be viewed as the same. But this renders the latter concept metaphysical. Accordingly the phrase "assignment by comparative advantage" is reserved for the situation wherein the assignment follows the same pattern as that of comparative advantage in terms of workers' skills. See Section 4.

4 It is possible to allow \( t_\alpha \) and \( t_\beta \) to vary independently. Each worker is then described by his \((t_\alpha, t_\beta)\) pair. However, the comparative advantage issue then revolves around the manner in which assignments vary across workers as \( t_\alpha / t_\beta \) declines. The analysis is more cumbersome than the one pursued herein, but the conclusions are very similar.

5 \( N(s) \) is taken to be exogenous. Optimal choice of \( N(s) \) can be handled in a fashion similar to that in MacDonald [1981]. Should \( N(s) = 0 \) for some \( s \), this merely introduces some discontinuities in the analysis to follow. The extension to this case is notationally inconvenient, but differs from the analysis herein only in trivial aspects.
It is obvious that this approach lessens the tendency towards specialization. The issue is whether the backwards-seeming assignments of the Peter-Principle variety can arise as optimal behavior when there is diminishing marginal product.

As mentioned, there are other ways to relax (4). The only crucial aspect of the particular one used herein is that it will turn out that the efficiency price of a unit of activity $\beta$ does not vary across workers. The results may therefore be extended to other situations where this holds. For example, if production uses activity $\alpha$ subject to diminishing returns, and activity $\beta$ has a constant per unit opportunity cost elsewhere (as would be the case if home technology is linear homogeneous), then analogous results follow. The focus on within-firm allocation is purely for purposes of explaining what seem to be pervasive anomalous observations in that context.

Subscripts on the function $h[\ ]$ denote partial derivatives.

For $\tilde{\omega}^* = \tilde{\bar{\omega}}^*$, the correct conceptual experiment involves reductions in $\omega^*$ and increments to $\tilde{\bar{\omega}}^*$. Both experiments yield (17)-(19). Also, as $\varphi = 0$ corresponds to $\omega \in [0, \tilde{\bar{\omega}}]$, $0 < \varphi < 1$ can be assumed; $\varphi = 1$ is treated similarly. Hence (17) involves no inequalities. If $0 < \varphi < 1$ is never optimal, $\omega^* = \tilde{\omega}^*$ is implied.

Note that the proposition that a "small" switching cost implies a "few" workers being assigned full time is clearly false.
Though the algebra is too complex to merit presentation, there is an interesting parallel with the results on assignment of heterogeneous workers (below).

Consider a reduction in \( t_\alpha / t_\beta \). There are three ways this can occur, of which consideration of one is sufficient.

Suppose \( t_\alpha \) falls and \( t_\beta \) rises. Referring back to (17), this implies \( t_\alpha h_1 [\varphi t_\alpha, T_\beta] \) must rise and hence that \( \varphi t_\alpha \) must fall. (Recall definition 2 of assignment by comparative advantage.) Consequently the level of \( c \) that will induce some workers to be assigned full time to task \( \alpha (\beta) \) rises (falls). Thus a reduction in \( t_\alpha / t_\beta \) induced by greater \( t_\beta \) and smaller \( t_\alpha \) shifts the "completely specialized" part of the assignment towards task \( \beta \). Whether \( \varphi \) also falls depends on the elasticity (\( \gamma \)) of the marginal product \( h_1 \).

Recalling (8), if \( |\gamma| > 1 \) the decline in \( t_\alpha \) raises \( t_\alpha h_1 \) for given \( \varphi \), so nothing is implied about \( \varnothing \). For \( |\gamma| < 1 \), \( \varphi \) must fall to raise \( t_\alpha h_1 \). (Recall definition 1 of assignment by comparative advantage.) These conclusions bear a strong family resemblance to those presented below for heterogeneous workers (Cases 2 and 5 of Table 1).

Indeed,

\[
\mu = \int_0^1 h_2 [\varphi^*(s) t_\alpha(s), T_2] N(s) ds.
\]

The case wherein \( \varphi^* = 0 \) or 1 proceeds analogously. Consequently, attention is directed towards the instances wherein (22) holds with equality.

For those workers for whom \( \varphi^* \) cannot be varied, as occurs when \( \varphi = 1 \) for example, \( \varphi t_\alpha \) may rise across workers; \( t_\alpha > 0 \) yields such a case. Hence the statement is that it is optimal for \( \varphi t_\alpha \) to fall across workers whenever it is possible for that to occur.
\[ \frac{\partial q}{\partial \bar{\omega}} = 0 \] is not impossible a priori. Indeed, if

\[ \frac{d}{ds} [h - t_\beta \Delta] = 0, \]

\[ \frac{\partial q}{\partial \bar{\omega}} = 0 \] could hold for a subinterval of the continuum of worker types.

Using \( \Delta = h/t_\beta \), this is easily shown to occur if and only if

\[ \zeta \frac{t'_\alpha}{t_\alpha} - \frac{t'_\beta}{t_\beta} = 0 \]

on an interval, where \( \zeta \in (0,1) \) is the output elasticity of activity \( \beta \) in \( h \).

For \( h[\cdot] \) concave, (*) can hold only if \( t'_\beta < 0 \) and \( t'_\alpha < 0 \).
REFERENCES


Appendix 1: Derivation of (25)

Assume \( \varphi^*(s^*) = 1 \) for some \( s^* \), and then define a comparison assignment

\[
\tilde{\varphi}(s) = \begin{cases} 
\varphi^*(s) & s \neq s^* \\
\varphi & s = s^*,
\end{cases}
\]

where \( \varphi \in (0,1) \); that is, \( \tilde{\varphi}(s) \) is identical to \( \varphi^*(s) \) except that workers of type \( s^* \) are specialized under \( \varphi^*(s) \) but not under \( \tilde{\varphi}(s) \). Let \( q^*, \tilde{q}, T^*_\beta, \tilde{T}_\beta, T^* \), and \( \tilde{T} \) denote output, total activity \( \beta \), and the set of workers for whom the switch cost is incurred under \( \varphi^* \) and \( \tilde{\varphi} \), respectively. Optimality of \( \varphi^* \) implies that for any \( \varphi \in (0,1) \)

\[
q^* - \tilde{q} > 0, \tag{A-1}
\]

where

\[
q^* = \int_0^1 h[\varphi^*(s)t_{\alpha}(s), T^*_\beta]N(s)ds - c \int_0^1 N(s)ds
\]

and

\[
\tilde{q} = \int_0^1 h[\varphi^*(s)t_{\alpha}(s), \tilde{T}_\beta]N(s)ds - c \int_{\tilde{\varphi}} \tilde{T} N(s)ds.
\]

It follows that

\[
q^* - \tilde{q} = \int_0^1 \{h[\varphi^*(s)t_{\alpha}(s), T^*_\beta] - h[\tilde{\varphi}(s)t_{\alpha}(s), \tilde{T}_\beta]\}N(s)ds + cN(s). \tag{A-2}
\]

Expand \( h[\tilde{\varphi}(s)t_{\alpha}(s), T^*_\beta] \) in a second order Taylor series around \( \varphi^*(s)t_{\alpha}(s) \) and \( T^*_\beta \):

\[
h[\tilde{\varphi}(s)t_{\alpha}(s), T^*_\beta] = h[\varphi^*(s)t_{\alpha}(s), T^*_\beta]
+ h_1[\varphi^*(s)t_{\alpha}(s), T^*_\beta](\tilde{\varphi} - \varphi^*(s))
+ h_2[\varphi^*(s)t_{\alpha}(s), T^*_\beta](\tilde{T}_\beta - T^*_\beta) + H(s) \tag{A-3}
\]
where, assuming \( h[\cdot] \) concave, \( H(s) \) is a negative semi-definite quadratic form. Now, from the definition of \( \tilde{\varphi} \)

\[
\begin{aligned}
\tilde{\varphi}(s) - \varphi^*(s) &= 0 \quad s \neq s^* \\
&= \tilde{\varphi} - 1 \quad s = s^*,
\end{aligned}
\tag{A-4}
\]

and hence

\[
\begin{aligned}
T_{\tilde{\varphi}} - T^*_B &= \int_0^1 \{ [1 - \tilde{\varphi}(s)] - [1 - \varphi^*(s)] \} t_{\tilde{\varphi}}(s)N(s)ds \\
&= (1 - \tilde{\varphi}) t_{\tilde{\varphi}}(s^*)N(s^*) \tag{A-5}.
\end{aligned}
\]

Substitution of (A-4) and (A-5) into (A-3), and the result into (A-2) yields

\[
q^* - \tilde{q} = (\tilde{\varphi} - 1) N(s^*) \left[ t_{\tilde{\varphi}}(s^*) h_1 \left[ t_{\tilde{\varphi}}(s^*), T_B \right] - t_B(s^*) \int_0^1 \left[ \varphi^*(s) t_{\tilde{\varphi}}(s), T_B^* \right] N(s) ds \\
- \int_0^1 \hat{H}(s) N(s) ds \right] + cN(s^*). \tag{A-6}
\]

where \( \hat{H}(s) \equiv H(s)/[(1 - \tilde{\varphi}) N(s^*)] < 0 \).

Now (A-1) must hold for any \( \tilde{\varphi} \), and \( \tilde{\varphi} = 1 \) raises the value of (A-6) under the assumed optimality of \( \varphi^* = 1 \). For \( \tilde{\varphi} = 1 \), (25) is implied.
Appendix 2: Analysis of (28)-(30)

At the optimum, and for each \( s \), (28)-(30) (equated to zero) constitute three equations in one unknown, \( \varphi(s) \). All three equations could happen to equal zero simultaneously without violating any constraints. It is therefore possible for identical workers to be treated differently for some particular worker type.

Is it possible that some nontrivial subset of workers can be treated this way? Suppose \( \underline{w}(s) \), \( \overline{w}(s) \) and \( \varphi(s) \) were all interior for some \( s \). If they are to remain interior across a sub-interval of the continuum of worker types, it must be that the \( \overline{\varphi}'/\overline{\varphi} \) expression generated by each equation is the same on that interval. If not, at least one equation cannot equal zero on the interval.

Differentiation of (28)-(30), and substitution from (30) only, yields

\[
\frac{\varphi'}{\varphi} = -\frac{1}{2}\frac{t^{'}}{t_{\beta}} + \frac{t^{'}}{t_{\alpha}} \quad \text{(A-7)}
\]

from (28): \[ \frac{\varphi'}{\varphi} = \frac{t^{'}}{t_{\alpha}} - \frac{t^{'}}{t_{\beta}} \quad \text{(A-8)} \]

and

\[
\frac{\varphi'}{\varphi} = \frac{1}{\eta}\frac{t^{'}}{t_{\beta}} - \frac{t^{'}}{t_{\alpha}} (1 + \eta) \quad \text{(A-9)}
\]

(A-7) and (A-8) are obviously inconsistent. Therefore there can be no connected subset of the continuum for which both \( \underline{w}(s) \) and \( \overline{w}(s) \) are interior. Equivalently, at most two of (28)-(30) can equal zero, and if two equal zero, (30) must be one of them.
This scheme generates sixteen cases, which are listed in Table A-1. Examination of the table yields the conclusion that in only two cases (numbers 1 and 8) it is possible for non-identical treatment of identical workers to extend across an interval.

**Case 1:** \( \frac{\partial q}{\partial w(s)} = 0, \frac{\partial q}{\partial \phi(s)} = 0, \frac{\partial q}{\partial \omega(s)} > 0. \) Equating (A-7) and (A-8) shows that (28) = (30) = 0 on an interval if and only if \( \bar{\gamma} = -2. \)

**Case 8:** \( \frac{\partial q}{\partial w(s)} < 0, \frac{\partial q}{\partial \phi(s)} = 0, \frac{\partial q}{\partial \omega(s)} = 0. \) Equating (A-8) and (A-9) on an interval if and only if

\[
\begin{pmatrix}
\frac{t_B'}{t_B} - \frac{t_C'}{t_C} \\
\frac{t_B'}{t_B} - \frac{\bar{\phi} - 1}{\bar{\phi}} - \frac{t_C'}{t_C}
\end{pmatrix}
\]

The conditions necessary for cases 1 and 8 naturally cannot be ruled out on a priori grounds. Thus it is possible, though in a well-defined sense extremely unlikely, that identical workers will not be assigned identically. However, as this requires an accidental coincidence of parameter values, this situation may be ignored. The conclusion is that the analysis in Section 6B is effectively general.
Table A-1

1. \( \frac{\partial q}{\partial \bar{q}} = 0, \frac{\partial q}{\partial \bar{w}} = 0, \frac{\partial q}{\partial w} > 0 \) \Rightarrow \( \bar{\phi}(s) = 1 \ \forall \ \omega \in [0, \omega(s)] \)
   \( \bar{\phi}(s) \in (0, 1) \ \forall \ \omega \in [\omega(s), N(s)]. \)

2. \( \frac{\partial q}{\partial \bar{q}} = 0, \frac{\partial q}{\partial \bar{w}} = 0, \frac{\partial q}{\partial w} < 0 \) \Rightarrow \( \bar{\omega} = \bar{\tilde{\omega}} = \bar{\omega}. \ (31) \Rightarrow \bar{\phi} = 0 \ \text{or} \ \bar{\phi} = 1 \ \forall \ \omega. \)

3. \( \frac{\partial q}{\partial \bar{q}} = 0, \frac{\partial q}{\partial \bar{w}} < 0, \frac{\partial q}{\partial w} < 0 \) \Rightarrow \( \bar{\omega} = \bar{\tilde{\omega}} = N(s) \Rightarrow \bar{\phi} = 1 \ \forall \ \omega. \)

4. \( \frac{\partial q}{\partial \bar{q}} = 0, \frac{\partial q}{\partial \bar{w}} < 0, \frac{\partial q}{\partial w} > 0 \) \Rightarrow \( \bar{\omega} = 0, \bar{\tilde{\omega}} = 1. \)

5. \( \frac{\partial q}{\partial \bar{q}} = 0, \frac{\partial q}{\partial \bar{w}} > 0, \frac{\partial q}{\partial w} > 0 \) \Rightarrow \( \bar{\omega} = \bar{\tilde{\omega}} = N(s) \Rightarrow \bar{\phi} = 1 \ \forall \ \omega. \)

6. \( \frac{\partial q}{\partial \bar{q}} = 0, \frac{\partial q}{\partial \bar{w}} > 0, \frac{\partial q}{\partial w} < 0 \) \Rightarrow \( \bar{\omega} = \bar{\tilde{\omega}} = \bar{\omega}. \ (31) \Rightarrow \bar{\phi} = 0 \ \text{or} \ \bar{\phi} = 1 \ \forall \ \omega. \)

7. \( \frac{\partial q}{\partial \bar{q}} = 0, \frac{\partial q}{\partial \bar{w}} < 0, \frac{\partial q}{\partial w} = 0 \) \Rightarrow \( \bar{\omega} = \bar{\tilde{\omega}} = \bar{\omega}. \ (31) \Rightarrow \bar{\phi} = 0 \ \text{or} \ \bar{\phi} = 1 \ \forall \omega. \)

8. \( \frac{\partial q}{\partial \bar{q}} = 0, \frac{\partial q}{\partial \bar{w}} < 0, \frac{\partial q}{\partial w} = 0 \) \Rightarrow \( \bar{\phi} \in (0, 1) \ \forall \ \omega \in [0, \bar{\omega}] \)
   \( \bar{\phi} = 0 \ \forall \ \omega \in (\bar{\omega}, N(s)). \)
Table A-1 (cont'd.)

9. \ \frac{\partial \phi}{\partial \rho} > 0, \ \frac{\partial \phi}{\partial \omega} = 0, \ \frac{\partial \phi}{\partial \varphi} > 0 \ \Rightarrow \ \bar{\phi} = 1 \ \forall \omega.

10. \ \frac{\partial \phi}{\partial \rho} > 0, \ \frac{\partial \phi}{\partial \omega} = 0, \ \frac{\partial \phi}{\partial \varphi} < 0 \ \Rightarrow \ \bar{\omega} = \bar{\omega} = \bar{\omega}. \ \text{(31)} \Rightarrow \ \bar{\phi} = 1 \ \text{or} \ \bar{\phi} = 0 \ \forall \omega.

11. \ \frac{\partial \phi}{\partial \rho} < 0, \ \frac{\partial \phi}{\partial \omega} = 0, \ \frac{\partial \phi}{\partial \varphi} < 0 \ \Rightarrow \ \bar{\omega} = \bar{\omega} = \bar{\omega}. \ \text{(31)} \Rightarrow \ \bar{\phi} = 1 \ \text{or} \ \bar{\phi} = 0 \ \forall \omega.

12. \ \frac{\partial \phi}{\partial \rho} < 0, \ \frac{\partial \phi}{\partial \omega} = 0, \ \frac{\partial \phi}{\partial \varphi} > 0 \ \Rightarrow \ \text{equivalent to} \ \bar{\omega} = \bar{\omega} = \bar{\omega}.
\ \text{(31)} \Rightarrow \ \bar{\phi} = 0 \ \text{or} \ \bar{\phi} = 1 \ \forall \omega.

13. \ \frac{\partial \phi}{\partial \rho} < 0, \ \frac{\partial \phi}{\partial \omega} > 0, \ \frac{\partial \phi}{\partial \varphi} = 0 \ \Rightarrow \ \bar{\omega} = \bar{\omega} = \bar{\omega}. \ \text{(31)} \Rightarrow \ \bar{\phi} = 0 \ \text{or} \ \bar{\phi} = 1 \ \forall \omega.

14. \ \frac{\partial \phi}{\partial \rho} < 0, \ \frac{\partial \phi}{\partial \omega} < 0, \ \frac{\partial \phi}{\partial \varphi} = 0 \ \Rightarrow \ \bar{\phi} = 0 \ \forall \omega.

15. \ \frac{\partial \phi}{\partial \rho} > 0, \ \frac{\partial \phi}{\partial \omega} > 0, \ \frac{\partial \phi}{\partial \varphi} = 0 \ \Rightarrow \ \bar{\omega} = \bar{\omega} = \bar{\omega}. \ \text{(31)} \Rightarrow \bar{\phi} = 0 \ \text{or} \ \bar{\phi} = 1 \ \forall \omega.

16. \ \frac{\partial \phi}{\partial \rho} > 0, \ \frac{\partial \phi}{\partial \omega} < 0, \ \frac{\partial \phi}{\partial \varphi} = 0 \ \Rightarrow \ \text{equivalent to} \ \bar{\omega} = \bar{\omega}.
\ \text{(31)} \Rightarrow \ \bar{\phi} = 0 \ \text{or} \ \bar{\phi} = 1 \ \forall \omega.