1971

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Citation of this paper:
RESEARCH REPORT 7123

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by

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September, 1971
ABSTRACT

This paper examines how Hicks's analysis of the elasticity of the market derived demand for a factor of production by a competitive industry is modified when the assumption of constant returns to scale is relaxed.

The effects of a change in the supply of one factor on the use of the other by the industry and on its own demand price are successively analyzed.

The implications of the analysis for Marshall's four rules are considered and a "fifth rule" about the effect of a change in the returns to scale on the elasticity of the derived input demand is formulated.
THE INDUSTRIAL DEMAND FOR FACTOR INPUTS
WITH NON-CONSTANT RETURNS TO SCALE

1. Introduction

A major contribution of Professor Hicks to the theory of derived input demand has been to show how Marshall's conclusions on the matter, which are formulated as "four rules for the things on which the elasticity of derived demand depends" (1932, p. 241), were modified when substitution between the factors is allowed for. He has shown in particular, that the effect of a change in the relative share of a factor on the elasticity of its own derived demand by a competitive industry in long-run equilibrium depended crucially on the relative magnitude of the elasticity of substitution between the factors and the elasticity of the market demand for output (1932 and 1961).

One important assumption made by Professor Hicks in his analysis, is that constant returns to scale prevail in the industry. The purpose of this paper is to examine what modifications result when this assumption is relaxed. The industry's production function will still be assumed to be homogeneous, but not necessarily of degree one.

This new approach, however, raises at least two problems: (1) Is it possible to find sufficient assumptions about the technology of the individual firms to justify the existence of a homogeneous production for a competitive industry? (2) Are non constant returns to scale compatible with perfect competition? The second question has been answered in the affirmative, and rather convincingly, by Chipman (1965, pp. 736-52) and (1970, pp. 347-52). No attempt will therefore be made here to expound on this point. The first question will be considered in the next section and a formal relation between the individual firm's and the industry's production functions will be established.
Having thereby justified our major departure from Hicks's assumptions, we shall then proceed to investigate in the rest of the paper the effect of a change in the supply of one factor on its own demand price and on the use of the other factor by the industry under these new conditions. A formula of the elasticity of the derived input demand which generalizes Hicks's to the case of non constant returns to scale will be given and the implications of the analysis for Marshall's four rules will be examined. In addition, a 'fifth rule', dealing with the effect of the returns to scale on the elasticity of the derived input demand will be proposed. The mathematical appendix provides a number of results on homogeneous production functions which are used in sections 3 and 4.
Throughout this paper, the following notation will be used:

\[ q = \text{output of the industry} \]
\[ a = \text{factor input whose supply shifts} \]
\[ b = \text{the other factor of production} \]
\[ f(a,b) = \text{production function of the industry} \]
\[ f_i = \text{marginal physical product of factor } i \text{ to the industry, } i = a, b \]
\[ f'_i = \text{marginal physical product of factor } i \text{ to each firm in the industry, } i = a, b \]
\[ MRP_i = \text{marginal revenue product of factor } i \text{ to the industry, } i = a, b \]
\[ P = \text{price per unit of output} \]
\[ P_a = \text{price per unit of input } a \]
\[ P_b = \text{price per unit of input } b \]
\[ k = \text{relative share of factor } a \]
\[ 1-k = \text{relative share of factor } b \]
\[ \rho = \text{degree of homogeneity of the industry's production function} \]
\[ * = \text{logarithmic differential of variables so designated--i.e.,} \]
\[ X^* = dX/X. \text{ In addition the following notation will be used for partial logarithmic derivative:} \]
\[ (X)^*_a = \frac{\delta \ln X}{\delta \ln a} \]
\[ \sigma = \text{elasticity of substitution between } a \text{ and } b \text{ in production--i.e.,} \]
\[ \sigma = (a/b)^* / (f'_e / f'_a)^* \]
\[ \eta = \text{elasticity of the demand for the industry's output--i.e.,} \]
\[ \eta = - q^* / p^* \]
\[ e = \text{elasticity of the supply of factor } b \text{ to the industry--i.e.,} \]
\[ e = b^* / p_b^* \]
\( \mu = \text{relative change of } b \text{ with respect to } a \text{ when the supply of factor } a \text{ shifts---i.e., } \mu \equiv \frac{b^*}{a^*} \)

\( \lambda = \text{elasticity of the derived demand for } a \text{---i.e., } \lambda \equiv \frac{a^*}{P_a^*} \).

2. **Homogeneity and the Industry's Production Function**

In his analysis of the long run market derived demand for a factor, Hicks assumed that the industry's production function was homogeneous of degree one. In this section, we shall try to show how it is possible to relax the assumption of linearity while still assuming homogeneity in such a way that non-constant returns to scale be compatible with competitive behavior. This will be achieved by introducing economies of scale external to each firm in the industry but internal to the industry in a manner similar to the technique used by Chipman (1965, 1970) and Aoki (1967). In what follows we shall first make some assumptions about the technology of the individual firm in the industry then show that these assumptions are sufficient for the industry's production function to be homogeneous of some degree \( \rho \) not necessarily equal to one.

The economies of scale used here will be what Chipman defines as "parametric economies of scale". Essentially, it will be assumed that each individual firm believes its level of production to depend only on its inputs while actually it depends also on the level of industrial output. The economies of scale arise from the fact that a change in the individual firm's level of production affects the industrial output, but that this effect is ignored by the individual entrepreneur who treats the industry's output as a 'parameter'. These economies are in fact partly internal since they benefit the expanding firm as well as the remaining firm in the industry, hence they should really be called "parametric economies" with the term 'external' dropped altogether. We shall however follow the example of Chipman himself who still adheres to "external"...,"out of deference to the Marshallian tradition, and because it describes the preponderant effect" (1970, p. 350).
Formally, we shall make the following assumptions:

(a) All the firms in the industry have the same production function given for the $i$th firm by:

$$q_i = cf^i(a_i, b_i)$$

(b) $f^i(a_i, b_i)$ is homogeneous of degree one and quasi concave with respect to $a$ and $b$.

(c) The term $c$ is treated as a constant by each firm but is in fact related to the industrial output $q$ as follows:

$$c = c_0 q^\theta$$

where $c_0$ is a positive number and $\theta$ is a scalar strictly less than one.

Since $f^i$ is quasi concave and linear homogeneous it is concave. Furthermore, being concave and linear homogeneous it is also super additive.\(^2\)

This implies that:

$$q = \sum q_i = \sum cf^i(a_i, b_i) \leq cf^i(\sum a_i, \sum b_i).$$

The equality holds if and only if,

$$\frac{\sum a_i}{\sum b_i} = \frac{a_i}{b_i}$$

for all $i$.

Since the firms behave as if their production function were homogeneous of degree one, and the industry is competitive, the proportion of factors used by each firm will depend uniquely on the relative prices of

---

\(^1\)The assumptions used here and the following analysis have been inspired by Aoki's treatment (1967, pp. 50-57).

\(^2\)See Fenchel (1953, p. 66).
the factors. In addition, all the firms have identical production functions and they are faced with the same cost conditions, hence, they will use the factors in the same proportion regardless of their scale of production. It follows therefore that condition (2) will always be satisfied in our model and that it makes no difference whether the increase in the industrial output results from the entry of new firms or from the expansion of the output of some firms already in the industry. Therefore, we have,

\[ q = cf_i(\Sigma a_i, \Sigma b_i) \]  

(3)

or substituting for \( c \) and solving for \( q \),

\[ q = \left[ c_0 f_i(\Sigma a_i, \Sigma b_i) \right]^{\frac{1}{1-\theta}} \]  

(4)

If we now let,

\[ \rho = \frac{1}{1-\theta}, \]

relation (4) can be written,

\[ q = \left[ c_0 f_i(\Sigma a_i, \Sigma b_i) \right]^\rho \]  

(5)

Since \( f_i \) is homogeneous of degree one, \( f \) will be homogeneous of degree \( \rho \). We can therefore distinguish between the industry's subjective production function (i.e., what each firm in the industry believes the industry's production function to be) given by relation (3) and the industry's objective production function given by relation (5).

Similarly, we can distinguish between the subjective marginal product of a factor (say \( a \)) to each individual firm, which is given by,

\[ \frac{\partial q_i}{\partial a_i} = c \frac{\partial f_i}{\partial a_i}, \]

from the objective marginal product of a to the industry,
\[
\frac{\partial q}{\partial a} = \rho [c_{o}f_{i}]^{\rho-1} c_{o} \frac{\partial f_{i}}{\partial a}.
\] (6)

Since,
\[
c = c_{o}q^{\theta}
\]
and,
\[
p = \frac{1}{1-\theta}
\]
it follows that
\[
c_{o} = cq^{\rho} = c(c_{o}f_{i})^{1-\rho}.
\]

If we substitute for \( c_{o} \) in relation (6) we can therefore write
\[
\frac{\partial q}{\partial a} = \rho c \frac{\partial f_{i}}{\partial a},
\]
or
\[
\frac{\partial q}{\partial a} = \rho \frac{\partial q_{i}}{\partial a_{i}}.
\] (7)

Hence, the objective marginal productivity of factor \( a \) to the industry is equal to the subjective marginal productivity of \( a \) to each individual firm multiplied by the degree of homogeneity of the industry's production function.

In what follows we shall denote respectively by \( f_{a} \) and \( f_{b} \), the objective marginal products of factors \( a \) and \( b \) to the industry and by \( f_{a}' \) and \( f_{b}' \), their subjective marginal products to each individual firm. In view of relation (7) we can therefore write:
\[
f_{a} = \rho f_{a}'
\]
and,
\[
f_{b} = \rho f_{b}'
\]

Since in a competitive equilibrium each factor is paid according to its marginal product to each firm and not to the industry, the total share
of the factors in total production will not depend on the degree of homogeneity
of the industry's production function but on the supply of entrepreneurship
and the level of normal profits.\textsuperscript{3} If one accepts Professor Hicks's contention
(1932, p. 234) that, in equilibrium, there is no function for the entrepreneur,
it can be assumed that the level of normal profits is equal to zero in the
long run and that factor shares exhaust total product. This is precisely what
is implied by our assumption that the subjective production function of each
individual firm is homogeneous of degree one. The relative shares of a and b
will be expressed respectively by,

\[ k = \frac{af_a'}{q} = \frac{af_a}{\delta q}, \]

and

\[ 1 - k = \frac{bf_b'}{q} = \frac{bf_b}{\delta q}. \]

This completes our justification of the assumptions made in this model.
We can now proceed to consider how Hicks's results on derived demand are modi-
fied by this new approach. We shall first investigate the effect of a change
in the supply of one factor on the use of the other in the next section, then
proceed in section 4 to derive a formula for the elasticity of the industry's
demand for a factor and examine the implications of the analysis for Marshall's
rules.

\textsuperscript{3} This point has been made by Mrs. Robinson (1934).
3. The Effect of a Change in the Supply of One Factor on the Use of the Other:

In this section we shall examine the effect of an increase in the supply of one factor, say a, on the quantity of the other factor, say b, used by a competitive industry in the case when the industry's production function is homogeneous of some degree \( \rho \) not necessarily equal to one. This will therefore generalize Hicks's results to cases of non constant returns to scale.

Let us call \( \mu \) the cross elasticity of the demand for b with respect to a. There are several reasons why we should be interested in \( \mu \). Not only is it useful per se to know the effect of a change in a on the demand for b but also, as was shown by Hicks (1961), this knowledge is quite illuminating to understand some of Marshall's rules. In addition, it is easy from an analytical point of view to deduce one from the other since there exists a relatively simple relation between the two.

Let us consider a competitive industry in long run equilibrium. The quantity supplied of the output is equal to the quantity demanded at the current market price and the factors are paid according to the value of their marginal products to each firm in the industry. We have therefore the relations,

\[
P_a = \frac{P f_a'}{f_a'}
\]

\[
P_b = \frac{P f_a}{f_a}
\]

\[
q = D(P)
\]

\[
q = f(a,b)
\]

where \( f_a' \) and \( f_b' \) are respectively the marginal products of a and b to each firm in the industry.

Let us define by \( \eta \) the elasticity of the demand for the output, by e the elasticity of the supply of factor b and by \( \sigma \) the elasticity of substitution between a and b:
\[ \eta = - \frac{q^*}{P^*} \]  
\[ e = \frac{b^*}{P_b^*} \]  
\[ \sigma = \frac{(a/b)^*}{(f_b^*/f_a^*)^*} \]  

A shift of \( S_a \), the supply curve of factor \( a \), will induce changes in the equilibrium values of \( q, b, a, P_a, b, P_b \) such that the relations (8) - (14) remain satisfied.

It was shown in the preceding section that \( f_a \) and \( f_b \), the marginal products of \( a \) and \( b \) to the industry are related in equilibrium to \( f_a^* \) and \( f_b^* \), the marginal products of \( a \) and \( b \) to each firm in the industry as follows:

\[ f_a = \rho f_a^* \]
\[ f_b = \rho f_b^* \]

where \( \rho \) is the degree of homogeneity of the industry's production function.

Substituting for \( f_b^* \) we can therefore write (9) as:

\[ P \cdot f_b = \rho P_b \]  

(15)

Since the industry is assumed to move from one position of equilibrium to another, the condition (15) must hold at the new position of equilibrium as well as at the old. Hence, the changes in \( a \) and \( b \) must be such that,

\[ P^* + f_b^* = P_b^* \]  

(16)

Differentiating totally with respect to \( a \) and \( b \) and gathering terms we have:

\[ \mu = \frac{(M_RP_a^*_b)^*}{P_b^*/P^* - (M_RP_b^*_b)^*} \]  

(17)

We see in relation (17) that the relative change of \( b \) with respect to a change in \( a \) is equal to the ratio of the change in the marginal revenue product of \( b \) resulting from a change in \( a \), \( b \) remaining constant, divided by the difference between the relative change in the price of \( b \) and the relative change in the marginal revenue product of \( b \) resulting from a change in \( b \), \( a \) remaining.
constant. We can therefore think of the effect of a change in the supply of a on b as taking place in two steps; first the firms increase their use of a, b remaining constant; they then adjust their use of b, a remaining constant, in order to restore the equality between the marginal revenue product of b and its price, disturbed by the change in a. Let us consider successively these two changes.

The exogenous increase in a, b remaining temporarily fixed, will induce an increase in output and in the ratio a/b. These changes, in turn, will affect P, the demand price of output and \( f_b \), the marginal product of b, hence their product, the marginal revenue product of b.

The relative change in P resulting from an increase in the supply of a, b remaining constant, can be expressed by:

\[
(P)^*_a = \frac{P^*_a}{q^*_a} (q)^*_a.
\]

Now, \( P^*/q^* = -\frac{1}{\eta} \)

by definition of \( \eta \). Furthermore,

\[
(q)^*_a = a \frac{f_a}{f}
\]

or,

\[
(q)^*_a = \rho k.
\]

It follows therefore that,

\[
(P)^*_a = -\frac{\rho k}{\eta}.
\]  \( (18) \)

We see in relation (18) that the demand price of output will tend to fall as a result of the increase in a. The magnitude of the fall will depend on the returns to scale \( \rho \), and on the elasticity of the demand for output, \( \eta \). The larger \( \rho \), the larger will be the increase in output, hence the fall in P, for any given \( \eta \). The smaller \( \eta \), the larger will be the fall in P for any change in output.
It is shown in the appendix that the relative change in the marginal physical product of b resulting from an increase in a, factor b being held fixed, is given by:

\[(f_b^*)^a = k\left(\frac{1}{\sigma} + (\rho - 1)\right)\]  \hspace{1cm} (19)

This result can be interpreted as follows: the increase in a has two effects on \(f_b\); a "substitution effect" resulting from the change in \(a/b\), and an "output effect" due to the movement to a higher isoquant. The smaller \(\sigma\), the elasticity of substitution, the larger will be the substitution effect; the larger \(\rho\), the larger will be the output effect, since \(f_b\) is homogeneous of degree \(\rho - 1\).

Combining relations (18) and (19) we can express the relative change in the marginal revenue product of b resulting from a change in a, b being held constant as follows:

\[(\text{MRP}_b^*)^a = k\left(\frac{1}{\sigma} + (\rho - 1) - \frac{\rho}{\eta}\right)\]  \hspace{1cm} (20)

Hence, an increase in a is more likely to induce an increase in the marginal revenue product of b if the demand for output is elastic and the elasticity of substitution is small. The role of the returns to scale is a little more complex; a large \(\rho\) tends to induce a large fall in \(P\) but also a large increase in \(f_b\). As can be seen in relation (20), whether the effect on \(P\) dominates the effect on \(f_b\) depends on the elasticity of the demand for output: If \(\eta\) is greater than one, large return to scale will have a more favourable effect on \(f_b\) than an unfavourable effect on \(P\). If \(\eta\) is less than one, however, the opposite will be true. When \(\rho = 1\) the effect of a on \(\text{MRP}_b\) depends only on the relative magnitude of \(\sigma\) and \(\eta\), which is Hicks's finding. Hence, in the "normal case" (\(\sigma < 1, \eta > 1\)) an increase in a is more likely to induce an
increase in the marginal revenue product of factor b, the larger the returns to scale. If \( \eta \) is less than one, however, the opposite will be true. Let us turn now to the compensating change in b.

If the marginal revenue product of b is larger than its price after the increase in a (b being held fixed temporarily), the firms in the industry will find it profitable to expand their use of b. Such an expansion will increase the industry's output, hence the demand price of output will fall. This fall will be larger the less elastic the demand for output and the larger the returns to scale. The increase in b will also affect its own marginal product. As in the case of a change in a, we can distinguish two effect; a substitution effect, resulting from the change in the ratio \( a/b \), and which is always negative, and an output effect which will be positive or negative depending on whether \( \rho \gtrless 1 \). The smaller \( \rho \) the larger will be the decline in \( f_b \) resulting from a change in b. If the returns to scale are large enough, however, the output effect may more than offset the substitution effect in which case the marginal product of b will increase with b. An increase in b will also affect its own price \( p_b \). This effect will depend on e, the elasticity of the supply of b. Hence the larger \( \eta, \rho, \sigma \) and e, the larger will be the increase in b necessary to restore equilibrium. In the extreme case when they are so large that the marginal revenue product of b increases faster than the price of b, as b increases, the industry will expand its output and its use of b indefinitely. Such a

\[4\text{We assume here a "Marshallian" adjustment process: the quantity supplied is increased when the "demand price" exceeds the "supply price". Such an adjustment process seems appropriate here since we are considering a long run situation (see Samuelson, Foundations, pp. 263-69, and Peter Newman, The Theory of Exchange, pp. 106-108.)}\]
possibility seems unlikely however, since one would expect that \( \eta \) and \( e \) will eventually decline when the market for the product becomes saturated (i.e., as substitution in consumption becomes more and more difficult) and when it becomes increasingly costly for the industry to bid \( b \) away from other industries.

If the exogenous change in \( a \) induces a reduction in the marginal revenue product of \( b \), the firms in the industry will want to reduce their use of \( b \). Such a decrease will cause a decline in \( P_b \) and an increase in \( P \) which will tend to bring the marginal revenue product of \( b \) in line with its price. The change in \( f_b \) can, however, either speed up or slow down this equilibrating reaction depending on whether \( f_b \) increases or decreases as the use of \( b \) declines. This in turn will depend on \( \sigma \) and \( \rho \). Formally, we have:

\[
(f_b)^* = (\rho - 1)(1-k) - \frac{k}{\sigma},
\]

\[
(P_b)^* = -\frac{\rho(1-k)}{\eta},
\]

and,

\[
\frac{P_b^*}{b^*} = \frac{5}{e}.
\]

Hence,

\[
\frac{P_b^*}{b^*} - (MRP_b)^* = \frac{1}{e} + \frac{\rho(1-k)}{\eta} + \frac{k}{\sigma} - (\rho - 1)(1-k).
\]

(21)

Making use of relations (20) and (21) in (17) we can write:

\[
\mu = \frac{ek[\eta - \sigma E]}{ek[\eta - \sigma E] + \sigma[\eta + \sigma E]},
\]

(22)

where,

\[
\mu = \frac{b^*}{\eta}
\]

and,

\[
E = \eta(1-p) + \rho.
\]

See the appendix for a derivation of \((f_b)^*_b\).
From this analysis we can therefore conclude that the direction of the change in \( b \) resulting from an exogenous increase in \( a \) depends only on the effect of this increase in \( a \) on the marginal revenue produced of \( b \). If \( \text{MRP}_b \) is increased, the use of \( b \) will be expanded; if \( \text{MRP}_b \) falls, the use of \( b \) will be reduced.

Whether \( \text{MRP}_b \) is reduced or increased by an increase in \( a \) will depend in turn on \( \eta \), \( \sigma \) and \( \rho \). Other things being equal, the larger \( \eta \) and the smaller \( \sigma \) the more likely it will be that an increase in \( a \) leads to an increase in \( \text{MRP}_b \) hence in the use of \( b \). Large returns to scale will have a favourable or unfavourable effect on the change in the marginal revenue product of \( b \) resulting from the increase in \( a \) depending on whether \( \eta \geq 1 \).

Whether the increase (decrease) in \( b \) will lead to a new position of equilibrium will depend on whether the marginal revenue product of \( b \) increases (decreases) faster than its price when \( b \) changes, a remaining fixed. An unstable behaviour can only arise in the case of increasing returns to scale and is more likely to occur when \( \eta \), \( \sigma \), \( \rho \), and \( e \) are large. Formally, the supply price of output, \( P_s \), must be such that the marginal revenue product of \( b \) is always equal to its price. Hence,

\[
P_s = \frac{P_b}{f_b}.
\]

It follows therefore that if we assume \( a \) to remain fixed, the elasticity of the supply curve of output is given by,

\[
\frac{q^*}{P_S} = \frac{\rho(1-k)}{\frac{1}{e} + \frac{k}{\sigma} - (\rho-1)(1-k)} \tag{23}
\]

Stability requires that the marginal revenue product of \( b \) increase (decrease) less than its price when \( b \) changes. This condition will be fulfilled, as we have seen, if the denominator of (23) is positive or if

\[
\eta \left[ \frac{1}{e} + \frac{k}{\sigma} - (\rho-1)(1-k) \right] + \rho(1-k) > 0 \tag{24}
\]
Comparing relation (23) with condition (24) we see that the system will always be stable when the slope of the supply curve is positive; i.e., when,

\[ \frac{1}{e} + \frac{k}{\sigma} - (\rho-1)(1-k) > 0. \]

If the supply curve is downward sloping, condition (24) will be fulfilled whenever,

\[ \eta < \frac{-\rho(1-k)}{\frac{1}{e} + \frac{k}{\sigma} - (\rho-1)(1-k)} \]

or, making use of (23), whenever,

\[ \eta < \frac{-q^*}{P^*_b} \]

which is the Marshallian condition of stability. It requires that the demand curve be less elastic than the supply curve at the position of equilibrium.

If we make use of the relations (17) - (22) the formal results on \( \mu \) can be summarized as follows:

1. An increase in the supply of \( a \) will induce a reduction of the use of \( b \) whenever \( (\text{MRP}_b)\) \( a \) is negative or whenever,

\[ \frac{\rho}{\eta} - (\rho-1) > \frac{1}{\sigma}. \]

If \( k \) is less than \( \frac{1}{2} \), \( \mu \) will always be larger than \(-1\). When \( k \) is larger than \( \frac{1}{2} \), \( \mu \) will be less than \(-1\) whenever,

\[ - (\text{MRP}_b)\) \( a > \frac{P^*_b}{b^*} - (\text{MRP}_b)\) \( b \]

or, whenever,

\[ (1-2k)[(\rho-1)\frac{\rho}{\eta}] > \frac{1}{e} + 2k. \]

2. An increase in the supply of \( a \) will induce an increase in the use of \( b \) whenever \( (\text{MRP}_b)\) \( a \) is positive, i.e., whenever,

\[ \frac{1}{\sigma} > \frac{\rho}{\eta} - (\rho-1). \]
In order to distinguish cases when $\mu$ is less than one from cases when $\mu$ is larger than one it is convenient to use a modified version of relation (17), expressed by:

$$
\mu = \frac{(\text{MRF}_b)_a^*}{(\text{MRF}_b)_a^* + [\text{P}_b^*/b^* - \text{MRF}]} 
$$

(25)

where,

$$
\text{MRF} = (\text{MRF}_b^d/b^*)_{a^*/b^*} = 1 = (\rho - 1) - \frac{\rho}{\eta}
$$

Relation (25) can easily be derived from relation (17) by noting that,

$$
(\text{MRF}_b^d)_a^* = \text{MRF} - (\text{MRF}_b^d)_b^*
$$

If $\text{P}_b^*/b^* > \text{MRF}$,

or if,

$$
\frac{1}{e} > (\rho - 1) - \frac{\rho}{\eta}
$$

$\mu$ will be less than one.

If $\text{MRF} > \text{P}_b^*/b^* > (\text{MRF}_b^d)_b^*$,

or if,

$$
(\rho - 1) - \frac{\rho}{\eta} > \frac{1}{e} > \frac{-k}{\sigma} + (1-k) [(\rho - 1) - \frac{\rho}{\eta}]
$$

$\mu$ will be larger than one.

When $\text{P}_b^*/b^* < (\text{MRF}_b^d)_b^*$ the system is unstable; an increase in the supply of $a$ induces an indefinite increase in the use of factor $b$.

A graphical illustration of these results can be given as follows: let us plot $\text{MRF}$ on the horizontal axis in Fig. 1 and $(\text{MRF}_a)_a^*$, $(\text{MRF}_b)_b^*$, $\text{MRF}_a$, $\text{MRF}_b$, and $1/e$ on the vertical axis. If we assume $k$, $\sigma$ and $e$ to be fixed, $(\text{MRF}_b)_a^*$ and $(\text{MRF}_b)_b^*$ are linear functions of $\text{MRF}$:

$$
(\text{MRF}_b)_a^* = \frac{k}{\sigma} + k \text{MRF}
$$

and,

$$
(\text{MRF}_b)_b^* = -\frac{k}{\sigma} + (1-k) \text{MRF}
$$
Fig. 1
Now,

\[(M_R^b)^* + (M_R^b)^*_a = \overline{M_R^b}\]

It follows therefore that \((M_R^b)^*\) is equal to zero when \((M_R^b)^*_a\) intersect the 45° line in the first quadrant of Fig. 1. Similarly, \((M_R^b)^*\) vanishes when \((M_R^b)^*_a\) intersect the 45° line in the third quadrant. The graphical representation of \(1/e\) is a straight line parallel to the horizontal axis.

The value of \(\mu\) in Fig. 1 will depend on the value of \(\overline{M_R^b}\), i.e., on the values of \(\eta\) and \(\rho\). If \(M_R^b\) is to the left of point A, \((M_R^b)^*_a\) will be negative hence \(\mu\) will be negative. Fig. 1 is drawn in such a way that \((M_R^b)^*_a\) and \((M_R^b)^*_b\) intersect in the first quadrant (i.e., \(k < \frac{1}{2}\)). It follows therefore that in quadrant 3, \((M_R^b)^*_a\) will always be larger in absolute value than \((M_R^b)^*_b\) hence that \(\mu\) will always be larger than -1. When \(\overline{M_R^b}\) is between A and B, \(\mu\) is positive and less than one. When \(\overline{M_R^b}\) is between B and C, \(\mu\) is positive and larger than one. When \(\overline{M_R^b}\) is to the right of C, the equilibrium is Marshall-unstable. We can therefore see in Fig. 1 that any change in \(\rho\) and \(\mu\) which tend to increase \(\overline{M_R^b}\) will also increase \(\mu\). Since

\[\overline{M_R^b} = (\rho - 1) - \frac{\rho}{\eta}\]

\(\overline{M_R^b}\) is non positive when \(\rho \leq 1\). This implies that the relative change in \(b\) is always less or equal to the relative change in \(a\) when non increasing returns to scale prevail in the industry.

Hence, one can state the following results:

1) Other things being equal, the more elastic the demand for output, the larger will be the increase in the use of one of the factors by a competitive industry in long run equilibrium if the supply of the other factor increases.

2) Other things being equal, and if the demand for output is elastic (inelastic), the larger (smaller) the returns to scale, the larger will be the increase in the use of one factor by a competitive industry if
the supply of the other factor increases.

The effect of changes in the other parameters \( e \) and \( \sigma \) can also be derived from Fig. 1 or from relation (22) and (25):

3) Other things being equal, the larger \( \sigma \), the smaller will be \( \mu \).

4) Other things being equal, the larger \( e \), the larger will be \( \mu \).

The effect of a change in \( k \) on \( \mu \) is slightly more complex: if we take the partial of \( \mu \) with respect to \( k \), we can write,

\[
\frac{\partial \mu}{\partial k} = \frac{\sigma e \left[ \eta - \sigma E \right] \left[ \eta + eE \right]}{\text{[square term]}}
\]

When \( E \) is positive the sign of \( \partial \lambda / \partial k \) will depend on whether \( \eta - \sigma E \geq 0 \) or on whether \( \mu \) is positive or negative.

When \( E \) is negative, the sign of \( \partial \lambda / \partial k \) depends on the sign of \( \eta + eE \) or on whether \( \mu \) is less or larger than one. Hence two cases must be considered:

5a) When \( E \) is positive (i.e., when \( \text{MMP} \) is negative) an increase in \( k \) will induce an increase (a reduction) in \( \mu \), other things being equal, when \( \mu \) is positive (negative).

5b) When \( E \) is negative (i.e., when \( \text{MMP} \) is positive) an increase in \( k \) will induce an increase (a decrease) in \( \mu \) whenever \( \mu \) is less (larger) than one.

This result can easily be seen in relation (23). If we divide both the numerator and the denominator by \( \left( \text{MMP}_b \right)^* \), we can write

\[
\mu = \frac{1}{\frac{1}{1/e - \text{MMP}} + \left( \text{MMP}_b \right)^*_{a}}
\]

An increase in \( k \) will increase \( \left( \text{MMP}_b \right)^* \) hence \( \mu \) will increase or decrease in (26) depending on whether \( 1/e - \text{MMP} \) is positive or negative.
4. The Elasticity of the Derived Demand and Marshall's Four Rules

In section 3 we examined how a change in the supply of one factor of production (a) affected the use of the other (b) by a competitive industry in long run equilibrium. In this section we shall use this result to determine the elasticity of the derived demand for factor a and see how Marshall's rules are affected when the assumption of constant returns to scale is relaxed.

Since the industry is assumed to move from one position of equilibrium to another when the supply of a changes, the demand price of a by the industry, \( P_a \), is equal to the marginal revenue product of a to each firm in the industry. Hence, the change in \( P_a \) must be equal to the change in the marginal revenue product of a to each firm in the industry. But,

\[
M{RP}_a^* = (M{RP}_a)_a^* a^* + (M{RP}_a)_b^* b^* \quad (27)
\]

Hence, the change in \( P_a \) must be such that:

\[
P_a^* = (M{RP}_a)_a^* a^* + (M{RP}_a)_b^* b^* \quad (28)
\]

Relation (28) expresses that the relative change in the price of a must be equal to the relative change in the marginal revenue product of a induced by the original change in a, factor b being held constant, plus the change in \( M{RP}_a \) induced by the equilibrating change in b, factor a being held constant. The change in \( P_a \) will therefore depend on how the exogenous change in a affects the use of b and on how the changes in the two factors affect the marginal revenue product of a.

In addition, since the industry is competitive, the condition,

\[
\frac{M{RP}_a}{M{RP}_b} = \frac{f_a}{f_b}
\]

must remain satisfied. It follows, therefore, that the relative change in the marginal revenue product of a and b must be such that,

\[
M{RP}_a^* - M{RP}_b^* = f_a^* - f_b^*
\]
But by definition of the elasticity of substitution,

\[ f_a^* - f_b^* = \frac{1}{\sigma} (b^* - a^*). \]

Hence, \( MRP_a^* \) and \( MRP_b^* \) must be such that,

\[ MRP_b^* - MRP_a^* = \frac{1}{\sigma} (a^* - b^*). \]  \hspace{1cm} (29)

Since the supply of factor \( b \) is a non-decreasing function of its price, the use of \( b \) can be increased only if its marginal revenue product increases. Conversely the use of \( b \) will decline whenever its marginal revenue product declines. It follows therefore that \( b^* \) and \( MRP_b^* \) will always be of the same sign in relation (29).

Because of the homogeneity property of the production function and the symmetry between \( a \) and \( b \), an increase in output along an expansion path will have the same relative effect on the marginal revenue products of the two factors; if we call, respectively, \( \overline{MRP}_a^* \) and \( \overline{MRP}_b^* \) the relative change in the marginal revenue products of \( a \) and \( b \) when the ratio \( a/b \) remains fixed, we have,

\[ \frac{\overline{MRP}_a^*}{\overline{MRP}_a^*} = \frac{\overline{MRP}_b^*}{\overline{MRP}_b^*} = (\rho - 1) - \frac{\rho}{\sigma}. \]  \hspace{1cm} (30)

Furthermore,

\[ \frac{\overline{MRP}_a^*}{\overline{MRP}_a^*} = (MRP_a^*)^* + (MRP_a^* b^*). \]  \hspace{1cm} (31)

and,

\[ \frac{\overline{MRP}_b^*}{\overline{MRP}_b^*} = (MRP_b^*)^* + (MRP_b^* b^*). \]  \hspace{1cm} (32)

If we now make use of relations (50) and (51) in relation to (46), we can write,

\[ \frac{\overline{MRP}_a^*}{\overline{MRP}_a^*} = MRP + (MRP_a^* b^* a^* \frac{b^*}{a^*} - 1) \]  \hspace{1cm} (33)

where,
\[
\frac{\text{MRP}_a}{\text{MRP}} = \frac{\text{MRP}_a^*}{a^*}.
\]

Relations (27) and (33) express differently the same change in \(\text{MRP}_a\). In relation (27) the change in the marginal revenue product of \(a\) is thought of as taking place first through an increase in \(a, b\) being held constant, then through an increase in \(b, a\) being held constant. In relation (33) there is first a change along the original expansion path, then a change in \(b, a\) being held constant, in order to bring \(\text{MRP}_a\) and the ratio \(a/b\) to their new equilibrium values.

If the change in the supply of \(a\) induces a reduction in \(b\), then the marginal revenue product of \(b\) must decline. It follows from relation (28) that the marginal revenue product of \(a\) must decline even more, hence that, in this case, the derived demand for \(a\) is always downward sloping. If the increase in the supply of \(a\) induces an increase in the use of \(b\) smaller than the increase in \(a\), then it follows from relation (33) that the marginal revenue product of \(a\) will always decline when \(\text{MRP}^*\) is negative. When \(\text{MRP}^*\) is positive, however, two subcases must be considered:

a) if \(\text{MRP}^* < (\text{MRP}_a^*)_b\),

there will exist a value of \(e, e_0\) such that

\[
\frac{\text{MRP}_a^*}{a^*} > 0,
\]

depending on whether

\[
e > e_0.
\]

b) if \(\text{MRP}^* > (\text{MRP}_a^*)_b\),

\[
\frac{\text{MRP}_a^*}{a^*} > 0 \text{ for all values of } e.
\]

When the change in \(b\) is larger than the change in \(a\) then the ratio \(a/b\) will decline; it follows therefore that \(\text{MRP}_a\) will increase with respect to \(\text{MRP}_b\), hence
both $P_a$ and $P_b$ will increase but $P_a$ will increase more than $P_b$.

The possibility of an upward sloping demand curve for $a$ raises the problem of stability. Whether the shift of the supply of $a$ leads to another equilibrium position will depend on the relative slope of the supply curve and the demand curve for $a$. If we assume a Marshallian adjustment process, stability will require that the supply of $a$ be steeper than the demand for $a$. If the condition is not fulfilled, the use of $a$ will increase indefinitely as well as the use of $b$ even if $b$'s market is stable. Hence instability in any of the factor markets is sufficient to ensure that the whole model will be unstable.

As was pointed out before, however, it seems unlikely that instability will actually lead to indefinite expansion of the industry's use of $a$ and $b$ and of its output since as output increases, substitution in consumption will become increasingly difficult (i.e., the elasticity of the demand for output will decline) and it will become increasingly difficult for the industry to bid resources away from other industries (i.e., the elasticity of the supply of the factors will decline).

If we make use of the results of the appendix we can formally obtain a formula for $\lambda$, the elasticity of the derived demand for $a$, comparable to Hicks's. It is shown in the appendix that,

$$ (f_b)_a^* = k\left[\frac{1}{\sigma} + (\rho - 1)\right]. $$

Because of the symmetry between $a$ and $b$ we can therefore write that,

$$ (f_a)_b^* = (1-k)\left[\frac{1}{\sigma} + (\rho - 1)\right]. $$

We recall from section 3 that,

$$ (P)_a^* = \frac{\partial k}{\eta}. $$

It follows again by symmetry that,
\[ (P)_{b}^{*} = - \frac{\rho(1-k)}{\eta} . \]

Hence,
\[ (MRP)_{b}^{*} = \frac{1-k}{\sigma} + (1-k)[(\rho-1) - \frac{\sigma}{\eta}] \quad (34) \]

Similarly, we could show that,
\[ (MRP)_{a}^{*} = - \frac{(1-k)}{\sigma} + k[(\rho-1) - \frac{\sigma}{\eta}] \quad (35) \]

If we now make use of (34) and (35) in (28) and substitute for \( \mu \) we can derive the following formula for \( \lambda \):
\[ \lambda = \frac{ek(\eta - \sigma E) + \sigma(\eta + eE)}{(\eta + eE) - k(\eta + \sigma E)} \quad (36) \]

where,
\[ \lambda = - \frac{a}{P_a}^{*} \]

and,
\[ E \equiv \eta(1-\rho) + \rho \quad (37) \]

One can easily check that (36) reduces to Hicks's formula when \( \rho \) (and therefore \( E \)) is set equal to one.

If we now take the partial derivatives of \( \lambda \) with respect to \( \eta \), \( \sigma \), \( e \) and \( k \), we obtain the following results:
\[ \frac{\partial \lambda}{\partial \sigma} = \frac{(1-k)(Ee + \eta)^2}{[\text{square term}]} \quad (38) \]
\[ \frac{\partial \lambda}{\partial \eta} = \frac{\rho \ k[e + e]^2}{[\text{square term}]} \quad (39) \]
\[ \frac{\partial \lambda}{\partial e} = \frac{k(1-k)(\eta + \sigma E)^2}{[\text{square term}]} \quad (40) \]
\[ \frac{\partial \lambda}{\partial k} = \frac{\eta + eE)(\eta - eE)(e + \sigma)}{[\text{square term}]} \quad (41) \]
Relations (38)-(40) confirm, respectively, Marshall's first, second and fourth laws. Marshall's third law, however, is slightly modified since in relation (41) the sign of $\partial \lambda / \partial k$ depends not only on the sign of $(\eta - \sigma E)$ but also on the sign of $(\eta + eE)$. If $E$ is positive $(\eta + eE)$ is positive; hence the sign of $\partial \lambda / \partial k$ is the same as the sign of $(\eta - \sigma E)$. The third law then reduces to something similar to Hicks's result: the effect of a change in $k$ on $\lambda$ depends on whether a reduction in $a$ induces a reduction or an increase in the use of $b$; if $b$ increases, $a$ must share with $b$ the price increase extracted from consumers; the smaller $(1-k)$ the less costly it is for factor $a$ to share. On the other hand if the reduction in $a$ induces a reduction in the use of $b$ then $a$ obtains its increased price both at the expense of the consumer and of factor $b$. It follows therefore that the larger $(1-k)$ the easier it is for $a$ to do so. 6 When $E$ is negative, however, a reduction in $a$ always induces a reduction of $b$ and $(\eta - \sigma E)$ is always positive. The sign of $\partial \lambda / \partial k$ therefore depends only on the sign of $(\eta + eE)$.

This latter expression in turn will be positive or negative depending on whether
\[ \frac{1}{e} \geq (\rho - 1) - \frac{\rho}{\eta}. \]

It was shown in section 3 that it was precisely the same condition which determined whether $\mu$ was less or larger than one. It follows therefore that when $E$ is negative, the sign of $\partial \lambda / \partial k$ depends on whether $\mu \leq 1$.

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6 It will be recalled that, 
\[ \eta - \sigma E \geq 0 \]
is the generalization of Hicks's condition 
\[ \mu - \sigma \geq 0 \]
for $\mu$ to be positive or negative.
This result can be explained as follows: when E is negative a downward movement along an expansion path will always tend to reduce the marginal revenue product of the factor because the returns to scale effect (ρ - 1) is stronger than the output effect ρ/η. It follows therefore that in this case MRP is positive in relation (33). This implies that a decline in a will always lead to a decline of MRP<sub>a</sub> and therefore of P<sub>a</sub> if the ratio a/b remains constant. If in addition to the movement along the original expansion path there is also an increase (a decline) in a/b (i.e., if b*/a* is larger (less) than one) this will reinforce (reduce) the decline in the marginal revenue product of a. Now, as can be seen from relation (35), an increase in k reduces the effect of a given relative change of b on the marginal revenue product of factor a (i.e., an increase in k reduces (MRP<sub>a</sub>*<sub>b</sub>). Furthermore, as was shown in section 4, an increase in k will bring about an increase or a reduction of μ depending on whether μ ≥ 1. It follows therefore that a larger k will tend to make

\[(MRP^*_a)_{b}^{*} [μ-1]\]

smaller in absolute value (in relation (33)). Hence an increase in k will induce an increase in MRP<sub>a</sub>*/a* when μ < 1 and a reduction in MRP<sub>a</sub>*/a* when μ > 1.

To these "four laws" we can now add a "fifth law" to take into account the effect of a change in the returns to scale on the elasticity of the derived demand for a. If we take the partial derivative of λ with respect to ρ, we can write:

\[\frac{∂λ}{∂ρ} = \frac{ηk(ε-σ)^2(η-1)}{[square\ term]}\] (42)

We can therefore state the fifth rule such as follows:

"Other things being equal, an increase in the returns to scale will make the derived demand for a factor more (less) elastic if the demand for output is elastic (inelastic)".
This law reflects the fact already noted in section 4 that large returns to scale have two opposite effects on the changes in the marginal revenue product of the factors: the larger $\rho$ the larger the output effect of an increase in $a$ hence, for any given $\eta$, the larger will be the decline of the price of output; the larger $\rho$ the larger, too, will be the increase in the marginal physical product of a resulting from a given change in $a$. Whether the returns to scale effect or the output effect dominates depends on the elasticity of the demand for output: the larger $\eta$ the smaller will be the decline in $P$ resulting from a given change in output. Hence the returns to scale effect is more likely to dominate the output effect when $\eta$ is large than when $\eta$ is small.

5. **Summary and Conclusion**

It has been shown in this paper that many of Hicks's results on derived demand are significantly modified when non constant returns to scale are allowed for in the model. These changes can be summarized as follows:

(a) Whether an increase in the supply of a factor (say $a$) induces an increase or a reduction of the use of the other (say $b$) by a competitive industry depends now not only on the ease of substitution between the factors and the elasticity of the demand for output but also on the returns to scale.

(b) The relative change in $b$ is no longer necessarily smaller than the relative change in $a$. In fact, if certain conditions hold, a given relative increase in $a$ can lead to an indefinite expansion of the use of $b$ by the industry.

(c) The derived demand for a factor is no longer necessarily downward sloping; (it can indeed become upward sloping) if the demand for output is elastic and the returns to scale are large enough. This may lead to an unstable behavior if the derived demand is steeper than the supply for the factor.

(d) Marshall's first, second and fourth laws remain valid but the third
law is modified: the effect of a change in the relative share of a factor on the elasticity of the derived demand depends not only on whether the use of b increases or declines when the supply of a increases, but also on whether the increase in b is larger or smaller than the increase in a.

In addition a "fifth rule" has been proposed to take into account the effect of a change in the returns to scale on the elasticity of the derived demand. This rule has been stated as follows: "Other things being equal, an increase in the returns to scale will make the derived demand for a factor more (less) elastic if the demand for output is elastic (inelastic)".
APPENDIX

SOME PROPERTIES OF HOMOGENEOUS PRODUCTION FUNCTIONS

Throughout this appendix we shall consider a production function homogeneous of degree \( \rho \), given by,

\[
q = f(a, b)
\]

such that the elasticity of substitution between the two factors of production \( a \) and \( b \) is always positive. For convenience, most of the results will be derived for factor \( b \). Since \( a \) and \( b \) play the same role in \( I \), however, similar results can be derived for \( a \) by symmetry. The notation will be the same as given at the end of section one.

**Proposition 1:** The elasticity of substitution between \( a \) and \( b \) is given by

\[
\sigma = \frac{f_a f_b}{f_a f_b - (\rho - 1) f_a f_b}
\]

**Proof:** From Euler's theorem the partial derivatives of \( f \) with respect to \( a \) and \( b \) will be homogeneous of degree \( \rho - 1 \). Their ratio will therefore be homogeneous of degree zero.

Let,

\[
g(a, b) = \frac{f_b}{f_a}
\]

Since \( g \) is homogeneous of degree 0, we can write:

\[
\frac{\partial g}{\partial a} + \frac{\partial g}{\partial b} = 0
\]

If we differentiate now \( g \) totally with respect to \( a \) and \( b \), we can write:

\[
dg = \frac{\partial g}{\partial a} da + \frac{\partial g}{\partial b} db
\]

If we make use now of the homogeneity property of \( g \) we can write,

\[
dg = \frac{\partial g}{\partial a} d\ln a - \frac{\partial g}{\partial b} d\ln b
\]

From which we can obtain,
\[ \sigma = \frac{\frac{f_a f_b}{f_{ab}}}{\frac{1}{f_{ab}} - (\rho - 1) \frac{f_a f_b}{f_{ab}}} \]

where,
\[ \sigma = \frac{d\ln (a/b)}{d\ln g} \]

is the elasticity of substitution between \( a \) and \( b \). Relation II reduces to Allen's result
\[ \sigma = \frac{f_a f_b}{f_{ab} f} \]

when \( \rho \) is equal to one. \(^7\)

Relation II can also be expressed as:
\[ f_{ab} = \frac{[1 + \sigma (\rho - 1)]}{\rho \sigma} \]

Where \( f_{ab} \) stands for the second order cross partial,
\[ f_{ab} = \frac{\partial^2 f}{\partial a \partial b} \]

From III we can easily derive,
\[ \frac{d\ln f_b}{d\ln a} = k[\frac{1}{\sigma} + (\rho - 1)] \]

where,
\[ k = \frac{a f_a}{\rho f} \]

Making use of the homogeneity property we can also write,
\[ f_{bb} = -f_b \frac{[k + \sigma (\rho - 1)(k - 1)]}{b \sigma} \]

and
\[ \frac{d\ln f_b}{d\ln b} = -\frac{k}{\sigma} + (1 - k)(\rho - 1) \]

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\(^7\) See Allen (1938), p. 343.
REFERENCES


