1970

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Citation of this paper:
RESEARCH REPORT 7011

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AN EXTENSION OF THE MUNDELL APPROACH

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May, 1970

Work on this paper was begun while I was a post-doctoral fellow at the University of Chicago under support from the Canada Council and the University of Chicago. Valuable comments on earlier drafts were received from Professor E. J. Kane, Peter Howitt, Ron Wirick and the Faculty Workshop at Western.
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I. Introduction

Recent contributions by R. A. Mundell [1963], [1965], [1969] have enriched traditional macro theory by emphasizing the monetary dynamics implicit in the IS-LM analysis. As important as the analytical implications of these articles is the geometrical apparatus Mundell devises to carry out the analyses. The purpose of this paper is to extend Mundell's geometry to relate more closely to the burgeoning literature on money and economic growth. A convenient property of the geometry is that it presents the money-growth relationship within the familiar IS-LM framework (or within a growing-economy analog of the IS-LM framework). In specifying the theoretical model that underlies the geometry we shall rely heavily on the work of Stein [1966], [1969]. For example, we shall carry over his distinction between short- and long-run equilibrium as well as his distinction between "Keynes-Wicksell" (hereafter referred to as K-W) and "neoclassical" growth models. Since the IS curve presupposes an investment function that can be different from the savings function, most of our analysis will be couched in terms of the K-W model. However, we shall also adapt our geometry to apply to the neoclassical monetary growth models.

The paper proceeds as follows. Part II presents the K-W model and discusses briefly the various equations. Next we geometrize the model and generate implications relating to the interaction of money, growth and traditional macro theory. Part IV investigates some of the analytics related to the model, e.g., stability. In V we present some implications of the geometry for the optimal growth rate of money and the payment of interest on money balances. The final section relates the geometry to a neoclassical growth model.

II. A Keynes-Wicksell Growth Model

The richness of the "Keynes-Wicksell" monetary growth model can best be appreciated if we consider an economy with three stores of value: money, bonds, and capital. A central feature of this type of model is that savings and investment decisions are assumed to be made independently of each other. [Stein, 1969, p. 162.]

The partially annotated model that follows embodies these features.
(1) \( y = f(k); \ f'(k) > 0; \ f''(k) < 0; \ y = \frac{Y}{L}; \ k = \frac{K}{L}. \)

...L.H. production function expressed in labor-intensive form.

(2) \( \frac{I}{L} = k\phi(f'(k)-r) + nk; \ \phi < 0. \)

...Investment function.

(3) \( \frac{S}{L} = \psi(k,m); \ \psi_k > 0; \ \psi_m < 0. \)

...Savings function.

(4) \( \frac{I}{L} = \frac{S}{L} \)

...Savings equals Investment

(5) \( m = \theta(k,i); \ \theta_k > 0; \ \theta_i < 0; \ m = \frac{M}{PL}. \)

...Money market equilibrium curve.

(6) \( \lambda = \frac{\dot{K}}{K} = \frac{1}{k} \frac{I}{L} = [\phi(f'(k)-r) + n]. \)

...Growth rate of capital stock.

(7) \( u - \pi = \lambda; \) where \( u = \frac{M}{M} \) and \( \pi = \frac{P}{P}. \)

...Required change in real balances to accommodate \( \lambda \) (also defines the actual rate of inflation).

(8) \( i = f'(k) + \hat{\pi} \)

(9) \( r = i - \pi \)

(10) \( \hat{\pi} = u - n \)

...Formulation of inflation expectations.

(11) \( \frac{k}{k} = \frac{\dot{K}}{K} - \frac{L}{L} = \lambda - n = 0 \)

(12) \( \frac{m}{m} = \frac{\dot{M}}{M} - \frac{P}{P} - \frac{I}{L} = u - \pi - n = 0 \)

Balanced-growth-equilibrium conditions.

(13) \( r = f'(k) \)

(14) \( \hat{\pi} = \pi \)

8 endogenous variables: \( y, k, m, \lambda, r, i, \pi, \hat{\pi}. \)

\(^1\)There are only 3 independent equations in the model. Equations (2) and (3) can be eliminated by appropriate substitution into (4). Given (13) one can show a) that (8) and (9) imply (14); b) that (6) implies (11); c) only one of (10) and (7) is independent. These results make (12) redundant. Since \( f'(k) \) and \( k \) are uniquely related, one could replace the latter by the former in the list of endogenous variables. For expository (primarily diagrammatic) purposes the model was not abridged.
Because of the annotation the verbal description of the model can be fairly brief. Most variables (e.g., savings and investment) are standardized by labor, \( L \). Saving per worker is assumed to be positively related to capital intensity, \( k \), and negatively related to the level of real money balances per worker, \( m \), i.e., equation 3. Although we use \( k \) rather than \( y \) as an argument (for convenience) and, as mentioned above, express savings in per-worker terms this formulation is in general conformity with several savings functions employed in the literature, e.g., Stein [1969], Tobin [1965], Mundell [1965]. We follow Stein [1969, p. 163] in assuming that firms desire a ratio of capital to labor such that marginal product of capital, \( f'(k) \), is equal to the real rate of interest, \( r \). The latter rate is interpreted as the opportunity cost of capital so that whenever \( r < f'(k) \) firms will attempt to raise their capital-labor ratio. Equation 6, which defines the growth rate of capital stock, \( \lambda \), can be seen to embody this assumption since \( \lambda - n \) is the rate of change of the capital-labor ratio, \( k \). This equation is equivalent to Stein's equation 21 [1969, p. 163] if \( \phi = 1 \). Investment per worker is defined by equation (2) and is merely a transformation of equation (6). We treat equation (4) as the growth-model equivalent of the familiar IS curve. And equation (5) is the growth analog of the LM curve. The demand for real money balances per worker is positively related to \( k \) and inversely related to the opportunity cost of money (equals the money rate of interest) \( i \).

Equation (7) needs some explanation. The LM curve (equation 5) expresses the relation between \( m, k \) and the money rate of interest. But \( k \) can change and this will require a corresponding change in the supply of money balances, \( m \). Assuming that the demand for money is unit elastic with respect to \( k \), equation (7) gives the change in the supply of real balances \( \mu - \pi \) required to accommodate the change in \( k \), represented by \( \lambda \). In other words given \( \lambda \), the supply of real money balances must change either via a change in nominal balances, \( u \), or a change in the rate of inflation, \( \pi \). Note that (7) also defines the actual rate of inflation. Diagrammatic convenience dictates separating these two demand-supply equations for real balances, i.e., equations (5) and (7).

Corresponding to the three stores of value (capital, bonds, and money) in the model there are three rates of interest. The marginal productivity of capital is denoted by \( f'(k) \). The money rate of interest, \( i \), is defined as \( f'(k) \) plus the expected rate of inflation \( \hat{\pi} \) (equation (8)). It is assumed here that the speed of response in the bond market (which determines \( i \)) is infinite and

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2 Strictly speaking the growth rate of labor should be subtracted from both sides of the equation. Since it cancels, it is deleted.
that the money rate of interest is identical to the opportunity cost of money (i.e., no interest is paid on money). The real rate of interest, \( r \), equals the money rate minus the actual rate of inflation (equation (9)). Like Stein, we interpret \( r \) as the opportunity cost of capital and allow it to differ from \( f'(k) \).

Expectations concerning the rate of inflation are generated by equation (10), i.e., the expected rate of inflation equals the growth rate of nominal balances minus the growth rate of the labor force. This formulation, too, is adapted from Stein [1969, p. 159, equation (11b)]. In our model, then, \( \hat{\pi} > \pi \) whenever \( n < \lambda \) i.e., given \( n \), the greater the growth rate of capital stock the greater the expected level of inflation relative to the actual rate. Equivalently, whenever \( \lambda < n \) the real rate of interest is less than the marginal productivity of capital. The key assumption underlying the discrepancy between \( \pi \) and \( \hat{\pi} \) is that investors are assumed to be dead right in anticipating the level of inflation, i.e., they assume \( \pi = u - \lambda \), whereas the economy as a whole anticipates \( u - n \) to be the expected rate of inflation.

Capital deepens if \( \lambda > n \) and \( {\frac{m}{k}} \) increases if \( u - \lambda - n > 0 \). In balanced-growth equilibrium both \( \frac{k}{k} \) and \( \frac{m}{m} \) equal zero. Furthermore, balanced growth implies \( \pi = \hat{\pi} \) and \( r = f'(k) \). These four conditions are the last four equations of the model.

The rates of growth of nominal balances, \( u \), and the labor force, \( n \), are exogenous. Full employment is assumed at all times. Prior to treating the analytics of the model we present the diagrammatic exposition.

III. The Geometrics of the Keynes-Wicksell Model

Figure 1 geometrizes the model. The ordinate has a percent dimension so that it is capable of representing both rates of interest and rates of growth. Real balances per worker, \( m \), are measured along the horizontal axis. Even though \( k \) (and \( y \)) is an endogenous variable, it is a parameter in the diagram: both the IS and LM curves are drawn for a given level of \( k \). This means that the marginal productivity of capital, \( f'(k) \), is also fixed. The IS curve is upward sloping: for a given \( k \) and \( f'(k) \), an increase in \( m \) decreases \( \frac{s}{L} \) and, to generate an equivalent reduction in \( \frac{I}{L} \), must be offset by an increase in the real rate of interest, \( r \). If \( k \) were fixed at a higher level (and, therefore, \( f'(k) \) at a lower level) \( \frac{s}{L} \) would exceed \( \frac{I}{L} \) so that equilibrium in the goods market would require a greater level of \( m \). In other words an IS curve to the right of the one in the diagram is consistent with \( k \) being fixed at a higher level. The money-market-equilibrium (or LM) curve is downward sloping since an increase in \( m \) creates an excess supply of money which to be offset requires, for given \( k \), a
RATES OF INTEREST AND GROWTH

FIGURE 1

\[ m = \frac{M}{P_L} \]
fall in the opportunity cost of money, \( i \). Naturally for a higher level of \( k \) the
LM curve will shift upward and to the right: for each level of \( m \), demand for
real balances per worker will be larger and will require a higher money rate to
restore money market equilibrium. Note that the LM curve is constructed with
reference to the money rate and the IS curve with reference to the real rate,
both of which are measured by the same scale (call it the real-rate scale) on
the ordinate. The real rate is read from the IS curve and the money rate from
the LM curve.

Moving northeast along the IS curve, the levels of \( \frac{I}{L} \) and \( \frac{S}{L} \) while ob-
viously equal to each other are both falling—the former because \( r \) is higher and
the latter because \( m \) is higher. Therefore, for the given level of capital stock,
this implies that the growth rate of capital stock, \( \lambda \), is falling as one moves
upward along the IS curve. Equation (6) captures this relationship: given \( k \),
\( \lambda \) is related inversely to \( r \). We represent this in Figure 1 by the vertical dis-
tance (which will have percent dimension) between the IS curve and the GG curve.
This vertical distance, \( \lambda \), narrows for greater \( r \). In a related manner we con-
struct the NN curve with reference to the IS curve. The vertical distance be-
tween IS and NN is constant for all levels of \( m \) and \( i \) (since labor-force growth
is exogenous) and it is equal to \( n \).

**short-run equilibrium**

Assume that the growth rate of nominal balances, \( u \), equals zero. Equi-
librium cannot occur at A, the intersection of the LM and IS curves. This is so
because at A, \( \lambda = \text{AF} \) and \( \pi \) must fall at the rate AF in order that \( m \) increase to
accommodate the increased demand for money associated with \( \lambda \) (equation (7)). A
negative value for \( \pi \) requires that \( i \) be less than \( r \) which is not true at A.
Rather, short-run equilibrium will occur at B, the intersection of the LM curve
with GG. At B, \( \lambda = \text{CB} \) and prices are declining at this rate (\( \pi = \text{CB} \)). The money
rate of interest is DB and the real rate is DC, the difference being \( -\pi \), the
rate of inflation. This relationship corresponds to equation (9) of the model.
Real balances per worker equal OD.

By varying the rate of change of nominal money balances the monetary
authorities can alter the characteristics of short-run equilibrium. Consider a
value of \( u = \text{AF} \). This is illustrated in Figure 1 by drawing a line parallel to
LM but below it by the amount of the exogenous growth rate of \( M \) (designated by
Equilibrium will now be established at the point of intersection of the U'U' and GG curves, i.e., at F. Here, the growth rate of capital stock is fully accommodated by the nominal money expansion: \( \lambda = u = AF \). Hence \( \pi = 0 \) and the real and money rates should be equal. And they are, both equalling AQ. As a result, then, of increasing \( u \), the level of real balances per worker has fallen from OD to OQ. This result is familiar: an increase in the rate of inflation will increase the money rate of interest and will decrease the level of real balances. Indeed, the Mundell [1963] results are confirmed: an increase in \( u \) increases the money rate (from BD to AQ) and decreases the real rate (from CD to AQ). This is hardly surprising since Figure 1, for short-run equilibrium considerations, is merely an adaptation of the geometrics implicit in the Mundell articles [1963] and [1965].

The economic interpretation of the movement from B to F on the real side is that the decreased level of \( m \) increases the level of \( \frac{S}{L} \) while the lower real rate (lower because \( \pi \) is closer to \( \hat{\pi} \) at F than at B, i.e., FR vs. BH) accounts for the increase in \( \frac{1}{L} \) and, therefore, \( \lambda \).

**balanced-growth equilibrium**

But neither of these two short-run equilibrium positions represents balanced-growth equilibrium. Consider position B. The rate of growth of capital stock, \( \lambda \), equals CB which is less than \( n (=CH) \) so that capital intensity will fall over time. Similarly \( u-\pi-n < 0 \) at B so that \( m \) is also decreasing. A necessary condition for equilibrium is that \( \lambda = n \) and this occurs, for the given value of \( k \), only at E, the intersection of GG and NN. In order to attain this equilibrium the monetary authorities will have to establish a growth rate of nominal balances equal to SE, i.e., a growth rate of \( u'' \). The characteristics of balanced growth are comprehensively displayed by the geometrics of Figure 1.

Nominal balances are growing at rate SE which is greater than \( \lambda (=JE) \) so that the economy is inflating at the rate SJ. Since \( \pi = SJ \), \( i \) exceeds \( r \) by the same amount, i.e., \( i-r = SP-JP = SJ \). Balanced growth is satisfied since \( \lambda = n = JE \) (\( k = 0 \)) and \( u-n-\pi = SE-JE-SJ = 0 \) (\( \frac{\dot{m}}{m} = 0 \)). Income per worker is also constant and \( Y/Y \) is growing at the common rate \( JE (=n) \). Expected inflation equals actual inflation since \( \lambda = n \) and, therefore, the marginal productivity of capital equals

---

3 Since we have a fixed scale on the ordinate to represent both \( r \) and \( i \), an increase in \( M/M \) equal to \( u' \) in Figure 1 implies an increase in \( \pi \) by the same amount and hence an increase in \( i \) relative to \( r \) by the same amount. In terms of the fixed scale (the real rate) the LM curve shifts down by the distance \( u' \) since a lower rate on the ordinate (lower real rate) now represents the original money rate. The money rate is still read off the original LM curve, however. This technique is adopted from Mundell (1963).
the real rate, i.e., JP. While there are many possible real rates in short-run equilibrium, balanced-growth equilibrium requires \( r = f'(k) \) and this is possible only where \( u = SE \), given the value of \( k \) which underlays the diagram.

Figure 2 presents the opposite case. With \( u=0 \), the short-run equilibrium position A represents a warranted growth rate greater than the natural rate: \( \lambda \) is greater than \( n \). Balanced growth equilibrium, for the given value of \( k \), can occur only at E, the intersection of GG and NN. Not unlike the "impasse" described in the Tobin model [1965, pp. 675-77], an increase in \( m \) is required to absorb the excessive savings. In turn this requires a decrease in the opportunity cost of money which can be engineered via a decrease in nominal money growth by the rate \(-u\) in Figure 2. Balanced-growth equilibrium will be established at E, the common intersection of GG, NN, UU. Here \( r=NJ, i=FJ, \pi=-HF, \lambda=n=HE \) so that \( k/k=0 \), and \( u=n-n \) equals \(-EF-(-HF)=HE \) so that \( m/m \) also equals zero. Real balances per worker have increased from B to J, the increase providing the absorption of the required amount of S/L and the rise in \( r \) reducing I/L to the appropriate level.

At this juncture it is important to emphasize again that Figures 1 and 2 are constructed for a given value of \( k \) and, therefore, \( f'(k) \). The equilibrium points E in both diagrams are positions that make the given value of \( k \) equal to the balanced-growth equilibrium value. This raises an important issue. Suppose the monetary authorities do not move to establish a position of balanced-growth equilibrium. Will the system tend automatically, through a succession of short-run equilibrium positions, to balanced-growth equilibrium? And if so, how will this equilibrium differ from those described above? This gets us into the question of whether or not the model is stable, in part the subject of the next section. But we can indicate here just what stability implies in terms of the geometry. Consider position B in Figure 1 and assume throughout that the monetary authorities keep \( u = 0 \). Capital intensity will begin to fall since \( \lambda < n \) and \( m \) will also fall since \( u-n-n < 0 \) (again because \( \lambda < n \)). With lower \( k \) and, therefore, \( y \), the IS curve will shift up for every value of \( m \): savings will decrease and this will require a higher \( r \) in the goods market equilibrium. The LM curve will shift down for each \( m \) since the lower demand for money must be offset by a lower nominal rate. What happens to \( \lambda \) is crucial because if the fall in \( k \) results in

---

4Note that \( f'(k) \) equals JP for all levels of \( i \) and \( m \) in the diagram since \( k \) is given. This imposes constraints on the slopes of the curves. The real rate of interest will be above \( f'(k) \) by the amount that \( \pi \) is less than \( \hat{\pi} \). In turn, \( \pi-\hat{\pi}=n-\lambda \). Therefore, if we were to draw a horizontal line at the level of interest corresponding to \( f'(k) \), i.e., at height JP, the angle formed by the intersection of the IS curve and the horizontal through J must equal the angle formed by the intersection of the GG and NN curves, i.e., angle BEH in Figure 1.
FIGURE 2
the new \( \lambda \) being even smaller than the original one, \( k \) will keep on falling at an ever increasing rate. For equilibrium to be stable \( \lambda \) must increase. Since \( k \) is now lower, \( \lambda \) will increase on this account because \( k \) appears in the denominator of (4). What is required is that the real rate not rise sufficiently to reduce investment per worker to the point where it offsets the effect of \( k \), i.e., the effect of \( k \) on \( \lambda \) must dominate the effect of the change in the real rate on \( \lambda \).

This exercise in curve shifting is not carried out here but the reader can do so and satisfy himself that, given stability, the new balanced-growth equilibrium (for \( u=0 \)) will be one with lower \( k \), \( y \), \( \pi \), and \( i \) and a higher \( r \) and \( f'(k) \) than at \( E \). The value for \( m \) will be lower than that at \( B \) but not necessarily lower than that at \( E \)—a lower \( y \) and higher \( i \) leaving the final impact on \( m \) uncertain. Note, however, that the monetary authorities can, by altering \( u \), bring the system to balanced-growth equilibrium: they can make any transitional short-run equilibrium value for \( k \) the balanced-growth value. For example, suppose the monetary authorities wish to deepen the capital intensity. In terms of Figure 1 they could set \( u > SE \) so that \( k \) would tend to increase over time. When \( k \) reached its appropriate value they would simply set \( u \) to attain balanced growth.

IV. Stability and Equilibrium

Stability of the system means that any accidental increase in \( k \) or \( m \) above its steady-state levels brings into play market forces to return it to its equilibrium levels. It can be shown that the model is stable if both \( \frac{d\lambda}{dm} \) and \( \frac{d\lambda}{dk} \) are negative, e.g., if the growth rate of capital stock falls as a result of an increase in \( m \) then stability obtains. Since the effect of a shift in \( k \) or \( m \) is to push the system out of equilibrium it is important to know whether desired investment or desired savings determines the actual amount of investment. Stein for example suggests that 'there is no presumption that either firms or consumers obtain their desired demands. Everyone is partially frustrated' [1969, p. 168] so that the growth of capital in his model is a linear combination of planned savings and planned investment. We shall proceed here on the arbitrary assumption that the behavior of savings in disequilibrium dominates and furthermore only treat the case of an increase in \( k \) above its equilibrium level. 5

5 In the mathematical section below this will also be the requirement of stability although the analysis is conducted in terms of the saving function.

6 On the investment side, note that if we differentiate equation (6) with respect to \( k \) we obtain \( \phi f''(k) - \phi \frac{dr}{dk} \), where \( \phi \) is some positive function. It is
not such a flagrant procedure since in the final analysis stability will, as in most money and growth models, have to be assumed rather than established.

Replacing \( \frac{I}{L} \) in equation (6) by \( \frac{S}{L} \) and in turn substituting equation (5) into the savings function yields:

\[
\lambda = \frac{1}{k} \psi \{k, \theta(k, f'(k) + \hat{n})\}
\]

where \( f'(k) + \hat{n} \) is substituted for \( i \). Differentiating (15) with respect to \( k \) and noting that \( \frac{dn}{dk} = 0 \) (since \( \hat{n} = u - n \)) yields the following stability condition

\[
\frac{\psi_k + \psi_m \theta_k + \psi_m \theta_i f''(k) - \psi_i}{k} < 0
\]

where \( \psi(\_\_\_\_) \) is the savings function, \( \psi_k \) and \( \psi_m \) are the partial derivatives of savings with respect to \( k \) and \( m \), and \( \theta_k \) and \( \theta_i \) are the partial derivatives of the demand for money function. The first three terms (which are \( > 0, < 0 \), and \( < 0 \) respectively) capture the impact of an increase in \( k \) on \( \frac{S}{L} \). Stability is assured if the sum of these three terms is \( < 0 \) or zero: Even if the increase in \( k \) leaves \( \frac{S}{L} \) unchanged, \( \lambda \) will fall because \( k \) is now greater and a given level of \( \frac{S}{L} \) now represents a lower \( \lambda \) (i.e., \( \lambda = \frac{1}{k} \frac{S}{L} \)). The fourth term in (16) represents this latter effect. Since the second and third terms of (16) are negative, a sufficient condition for stability is that \( \psi_k < \frac{\psi_i}{k} \), a condition we shall refer to below. Note that this discussion of stability parallels closely the Levhari-Patinkin discussion (1968, p. 731).

Assuming stability we now turn to investigate the effect on the balanced-growth equilibrium values of the model stemming from an increase in \( u \), the rate of change of nominal money balances. For this purpose, appropriate substitution reduces the model to three equations in three unknowns, \( m, k, \) and \( \pi \):

\[
(17) \quad nk = \psi(k, m)
\]

\[
(18) \quad m = \theta(k, f'(k) + \pi)
\]

\[
(19) \quad u = \pi + n.
\]

Differentiating (17) - (19) with respect to \( u \) yields \( \frac{dm}{du} = 1 \) which enables the system to be reduced to a 2x2 case which possesses a positive determinant if the system is stable. It then can be shown that \( \frac{dk}{du} > 0 \) but that \( \frac{dm}{du} \) is indeterminate, this indeterminacy arising because the increase in \( i \) (tending to decrease \( m \))

Clear that \( f''(k) \) is negative. But it is also clear that \( \frac{dr}{dk} \) will equal \( f''(k) \) if \( \frac{dn}{dk} \) and \( \frac{dr}{dk} \) are zero, since \( r = i - \pi \) and \( i = f'(k) + \hat{n} \). Assuming that \( \hat{n} \) and \( \pi \) do not change, stability requires some lag in the speed of adjustment of the money rate to changes in the marginal productivity of capital. This is an eminently reasonable assumption but it is not embodied in our model as it is presently formulated.
is offset by the increase in $k$ (tending to increase $m$). However, if the sufficient condition for stability holds, i.e., $\psi_k < [\psi(k, m)]/k$, then it follows that $\frac{dm}{du} < 0$. Assuming the sufficient condition does obtain the complete effects of an increase in $u$ are: an increase in $k$, $\pi$, $\pi^*$, $\gamma$, $i$ and a decrease in $m$, $f'(k)$, and $r$ with $\lambda$ remaining unchanged. Differentiating (17) - (19) with respect to $n$ yields $\frac{dn}{du} = -1; \frac{dn}{dk} < 0; \frac{df'(k)}{dn} > 0; \frac{dr}{dn} > 0; \frac{d\lambda}{dn} = 1; \frac{dy}{dn} < 0$; $\frac{dm}{dn} = ?$ but probably $< 0$ and again an indeterminacy for $\frac{dm}{dn}$.

Most of these results can be confirmed by carrying out the appropriate shifts in $n$ and $u$ with the geometric apparatus. For example, from balanced-growth equilibrium position $E$ in Figure 1, let the value of $u$ fall from $SE$ to zero so that the short-run-equilibrium is at $B$ with the same level of $k$ and a larger $m$. But $k$ and $m$ will now begin falling since both $\lambda - n$ and $u - \pi - n$ are less than zero. Therefore $\frac{dk}{du}$ is positive. The value for $m$ will fall from its short-run level of $OD$, but it is not possible (unless the sufficient condition for stability is assumed) to know whether in the new equilibrium it will be greater or smaller than $OP$, its former equilibrium level. This is the indeterminacy referred to earlier.

V. Optimal Growth of Money

Thus far there has been no mention of whether or not the marginal productivity of capital equals the common growth rate, $n$, in balanced growth. If it does, then the conditions for maximum consumption per head are satisfied—the so-called "golden rule of accumulation". Figure 3 depicts such a state. The intersection of the LM, GG, and NN curves at $A$ satisfy balanced growth since $\mu = k = 0$ as the reader can establish. The real rate equals $f'(k)$ which equals $BA$. In Figure 3 this also equals the natural rate of growth, $n$. With $u = 0$ the rate of deflation equals $BA$ so that the money rate of interest equals zero. This configuration corresponds to a situation where the growth of money is optimal. In Marty's words (1961, p. 57)

Footnotes:
7 Intuitively the sufficient condition increases the value of the determinant, thus making $dk/du$ smaller. With a smaller increase in $k$, the real rate (equal in balanced-growth equilibrium to $f'(k)$ will fall less. Since $dn/du$ is unity, the smaller the fall in $r$ the greater will be the money rate. And the greater the money rate, the smaller $m$.

8 In Figure 3, that portion of the LM curve below the horizontal axis is cross-hatched to indicate that the money rate of interest cannot get below zero.
"There is in logic an optimal rate of growth of the money supply which maximizes social utility through time. From society's point of view, real balances are produced at zero marginal costs. In order to maximize social utility, people should be induced to hold the satiety level of real balances at each instant of time. This can be done by maintaining a rate of price decrease equal to the natural rate of interest, thereby reducing the money rate to zero."

It is important to realize that allowing the rate of deflation to equal the real rate (or \( f'(k) \)) will result in a zero money rate of interest only when the level of \( k \) is such that \( f'(k) = n \). Clearly this latter condition is not satisfied in Figure 1, for example, where \( f'(k) > n \) so that \( k \) is below the golden-rule level. Nor is it necessarily true that for the particular level of \( k \) such that \( f'(k) = n \) it is always possible to set \( i = 0 \). Consider Figure 3 and assume that the LM curve does not go through A but rather through B, parallel but distance BA above the shown LM curve. To generate balanced-growth equilibrium for this level of \( k \) the authorities have to set \( u = BA \). But this implies zero price inflation and a money rate equal to the real rate and not to zero.

The point we are trying to make is that it may not be possible to set \( i = 0 \), and if it is possible the level of \( k \) which the authorities have to settle for may not be that value corresponding to the golden-rule conditions. With only one instrument, i.e., \( u \), it is not possible to set independent targets for both the money rate of interest and the level of \( k \).  

The payment of interest on money balances

Heretofore, money was assumed to bear a zero rate of interest so that the opportunity cost of holding money equalled the money rate \( f'(k) + \pi \). Now allow money balances to bear interest, say at rate \( v \). The money rate is still \( f'(k) + \pi \) but the opportunity cost of holding money is \( i - v \). The money rate entered the demand-for-money equation because it equalled the opportunity cost of money. With \( v > 0 \), the appropriate argument in equation (5) is \( i - v \) and not \( i \). This modification is readily introduced into our diagrammatic analysis. Consider Figure 2 and assume that \( v = EF \) and that the growth rate of nominal balances is zero \( (u = 0) \). The curve labelled LM no longer represents the combination of \( i \) and \( m \) that satisfy money market equilibrium. Rather it is the locus of points of \( m \) and the opportunity cost of money that equate the demand.

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9 In a recent paper, Marty recognizes this trade-off and comments: "If the authorities can use fiscal policy and engage in open market operations in equities, they can set both the optimal level of the money stock and the real rate" [1968, p. 863]. Our model has just one "instrument".
and supply for money. (This assumes that the public will hold each \( m \) at the same opportunity cost as before.) The curve labelled UU in Figure 2 gives the nominal rate that coincides with each \( m \). Therefore curve UU is drawn with respect to the money rate and the curve labelled LM is drawn with respect to the opportunity cost of money.

Will the introduction of \( v > 0 \) permit the authorities to pursue a policy of optimal money growth (i.e., \( f'(k) + \pi - v = 0 \)) for any value of \( k \)? Under our interpretation of the impact of an increase in \( v \) (i.e., \( \frac{dm}{dv} = -1 \)), the answer is no. Referring again to Figure 2 and setting \( v = EF \) and \( u = 0 \) we observe that this brings the system to balanced-growth equilibrium. The characteristics of this equilibrium differ from those described earlier for Figure 2. Specifically, the equilibrium nominal rate is now \( EJ \) (before it was \( FJ \)), the rate of deflation is now \( HE \) (compared to \( HF \) previously) and \( u = 0 \) (as against \( -EF \) before). What has not been altered is the equilibrium opportunity cost of money: an increase in \( v \) is offset by a corresponding decline in the rate of price deflation. What the payment of interest on money balances does permit is greater flexibility in setting the nominal interest rate and, therefore, the rate of inflation for each level of \( k \). But it does not alter the opportunity cost of money that is associated with each balanced-growth level of \( k \). In Figure 3, for example, payment of interest on money balances at a rate \( BA \) would still allow optimal money growth (i.e., opportunity cost of holding money balance is zero) but the nominal and real interest rates would now be equal, i.e., the value of \( \pi \) has increased from \( -BA \) to zero.

Implicit in the above discussion is the assumption that payment of interest on nominal money balances is not effected via an equivalent increase in the money stock. Stein [1969, p. 135] correctly points out that if the money supply "were a liability of the government which grows solely because a nominal rate of interest \( [v] \) is paid to the holders of money...monetary neutrality would result." Such a policy would imply \( u = v \) which in turn implies that the monetary authorities cannot shift the LM curve--the impact of a policy where \( u = v \) results in exactly offsetting shifts, leaving the curve unchanged and indeed unchangeable under this type of policy.

VI. Geometrics of the Neoclassical Model

The geometry of the Keynes-Wicksell model has been constructed with \( m \) rather than \( y \) or \( k \) on the horizontal axis. The advantage of having \( k \) on the horizontal axis is that the IS and LM curves (or the growth-model analogs of
these curves) have the familiar slopes. But rather than present the geometry of the Keynes-Wicksell model for the case where \( m \) is a parameter of the diagram we opt to adapt our geometrics to the neoclassical model and construct the neoclassical diagram with \( k \) on the horizontal axis. In the neoclassical model all savings is invested. Replacing equations (2), (3) and (6) of the K-W model is the following equation:

\[ \frac{1}{L} = \frac{S}{L} (k, m) \]

and the equation for \( \lambda \) is altered accordingly:

\[ \lambda = \frac{K}{K} = \frac{1}{k} \frac{S}{L} (k, m) \]

The second major difference between the K-W and neoclassical models is that in the latter there are only two stores of value - real money balances and real capital. Correspondingly, there are only two rates of interest, the marginal productivity of capital \( f'(k) \) and the opportunity cost of money. There is no nominal rate of interest in the familiar sense of the term, i.e., rate of return in bonds. Furthermore, there is no real rate of interest in the sense we employed this rate above—as the opportunity cost of capital. The reader may wish to equate \( r \) with \( f'(k) \) and \( i \) with the opportunity cost of money but these are identities rather than equilibrium conditions. There is no reason for expectations to go awry in the neoclassical model so that both expected and actual inflation equal \( u - \lambda \) (equation 7 of the K-W model). The complete neoclassical model, then, consists of equations (1), (20), (5), (21), (7), (8), (11), (12). Equations (13) and (14) become identities.

Figure 4 diagrams the model in terms of rates of interest and the capital-labor ratio. The level of real money balances, \( m \), is a parameter in this diagram in the same manner as \( k \) was in the previous diagrams. The \( LM \) curve is positively sloped for the usual reasons. The curve labelled IS is not really an investment=savings curve. (Savings always equals investment in the neoclassical model since savings is investment.) Rather IS is a marginal productivity of capital schedule or \( f'(k) \) schedule. But it is still possible to use this \( f'(k) \) schedule as a reference point in drawing the curves depicting \( \lambda \) and \( n \). Savings per worker is independent of any rate of interest but, given \( m \), it will be greater is the level of \( k \). But what about the rate of growth of capital stock, \( \frac{S}{K} \)? Is it positively or negatively related to \( k \)? Stability in the neoclassical model (and indeed in the stability analysis we presented above) implies \( \frac{d\lambda}{dk} < 0 \). Therefore the GG curve is drawn beneath the IS curve with the vertical
FIGURE 4
distance, \( \lambda \), narrowing for greater \( k \). The NN curve is drawn parallel to IS and below it by the rate \( n \). Note, therefore, that we have assumed stability in the construction of the diagram.

Balanced-growth equilibrium requires a growth rate of nominal balances equal to \( \lambda \) so that real balances per worker adjust to \( \lambda \), which must equal \( n \) for balanced growth, i.e., steady-state equilibrium occurs where the GG, NN and UU curves intersect in Figure 4. As drawn, \( f'(k) \) equals \( KJ \), the opportunity cost of money balances equals \( HJ \), and the rate of inflation, \( \pi \), equals \( HK \). Both \( k \) and \( m \) will remain constant, the former because \( \lambda = n = KE \) and the latter because \( u - \pi - n = HJ - HK = KE = 0 \). From balanced-growth equilibrium consider a decrease in \( u \) to zero such that a short run equilibrium is established at \( B \), the intersection of GG and LM. As in the Mundell [1965] analysis a decrease in inflation results in a fall in the money rate (in \( f'(k) \)) and also a fall in the desired level of \( k \). But at \( B \), \( k \) will tend to increase since \( n < \lambda \) and \( m \) will begin rising since \( u - n - \pi = 0 - CP - (\cdot CB) > 0 \). As a result of the increase in \( m \) the LM curve will shift to the right, i.e., a greater level of \( m \) requires a corresponding increase in demand for money which for any \( i \) means an increase in \( k \). The curve labelled IS is the \( f'(k) \) curve and, therefore, independent of \( m \). Likewise the NN curve remains fixed. But the GG curve does not. For an increased \( m \), \( S_L \) will fall and \( \lambda \) will be less for each value of \( k \): the GG curve shifts upward. Clearly then, the intersection of GG and NN will occur at a level of \( k \) which is below the previous balanced-growth equilibrium \( OJ \). And with the LM curve shifting to the right a position of common intersection of GG, NN and LM is assured. Since we have assumed stability the neoclassical model generates unambiguous results: \( \frac{dm}{du} < 0 \) and \( \frac{dk}{du} > 0 \). Monetary policy is not neutral. The reader is invited to conduct other experiments with the geometry. Indeed, most of the discussion of the Keynes-Wicksell model can be adopted easily to the neoclassical framework.\(^\text{10}\)

\(^{10}\) The geometry of the Keynes-Wicksell model constructed in terms of \( k \) on the horizontal axis differs somewhat from the neoclassical example. For example, the IS curve is an actual IS curve and is drawn with respect to the real rate. The marginal productivity of capital differs from the IS curve and crosses it only where \( \lambda = n \). The analysis and geometry of this case are available on request from the author.
REFERENCES


