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RESEARCH PROGRAM:
IMPACT OF THE PUBLIC SECTOR ON LOCAL ECONOMIES

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1. Introduction

What effect does the stochastic nature of income have upon the structure of cities, particularly in regard to residential location?

For example, consider a residential city with identical inhabitants receiving the same stochastic income per capita. Suppose that the boundary of the city is determined by a rental condition for agricultural land. What happens if the variance of income is increased, while the population and mean income are fixed? Perhaps a more interesting and relevant comparison is to hold expected utility constant as the variance of per capita income is increased. Two ways of holding utility constant are considered. The first method is to raise mean income along with its variance. Population is held constant. A second method is to allow migration to take place to equate expected utility between different cities, when the mean income is held constant. The first of these comparisons then refers to cities which, for reasons of industrial structure, have a correlation between mean income and its variance. How does residential structure differ among such cities? The second comparison is between cities of equal mean income but where population adjusts so that utility is still constant among them. Again, what implications are there for residential structure?
2. **General Case**

Consider here as throughout the paper, a circular ring of residences surrounding a central CBD of radius I. The overall radius is R. All residents have identical utility functions,

\[ u = u(s, z, x) \]

where \( s \) is space, \( z \) is consumption, and \( x \) is the distance of the residence from the CBD. Transport costs are not considered explicitly here so that the dependence of utility on \( x \) is taken to account for the desire of individuals to locate close to the CBD. Since income is stochastic, the concavity of the above utility function is meaningful, and is assumed. The budget constraint is

\[ \tilde{y} = r(x)s + z \]

where \( \tilde{y} \) is random income, and \( r(x) \) is the rent paid at \( x \). The sequence of budget decisions is relevant. Assume it is as follows. Initially an individual is unaware of the income he will actually receive. Despite this, he must decide where to live (what value of \( x \)), and how much space, \( s \), to consume. After committing himself to these decisions a particular realization of \( \tilde{y} \) occurs. The amount of consumption that the individual obtains is then the residual of income over the amount already spent on space. This assumption basically generates the results of this paper.

The individual seeks to maximize expected utility

\[ E_u = E_u(s, \tilde{y} - r(x)s, x) \]

over choice of \( s \) and \( x \). The necessary conditions can be formulated as
\[ E(u_s(s, \tilde{y} - r(x)s, x)) = r(x)E u_z(s, \tilde{y} - r(x)s, x) \]

and

\[ E u(s, \tilde{y} - r(x)s, x) = u^* \]

where \( u^* \) is the common level of expected utility, independent of \( x \). The second condition determines the rental profile needed in order that expected utility be independent of \( x \). The level of expected utility, \( u^* \), is determined by conditions on total population and radius, \( R \). This will be done for the specific model of the next section.

Reconsider now the first necessary condition. Assuming that \( u_s(s, z, x) \) is a concave function of \( z \) and that \( u_z(s, z, x) \) is a convex function of \( z \), Jensen's Inequality implies

\[ u_s(s, E(\tilde{y}) - r(x)s, x) \geq E u_s(s, \tilde{y} - r(x)s, x) \]

\[ = r(x)E u_z(s, \tilde{y} - r(x)s, x) \geq r(x)u_z(s, E(\tilde{y}) - r(x)s, x) \]

Hence the amount of space purchased under uncertainty is less than the amount purchased when income is \( E(\tilde{y}) \) with certainty. This comparison is for a given \( x \) and \( r(x) \). In general equilibrium, one would expect uncertainty to also reduce \( r(x) \), but that \( s \) would still decline. This expectation is confirmed for the particular model of the next section.

Suppose that utility is now held constant in the above situation by allowing mean income to rise with its variance. Intuitively, this will reduce the decline in \( s(x) \) for a given \( x \) and \( r(x) \). Then in general equilibrium, \( r(x) \) will fall but by a lesser amount than in the uncompensated case. Again, \( s \) will decline but by a lesser amount. These conclusions are obtained for the particular model of the next section also.
What might happen if utility were held constant despite fixed mean income, by allowing \( r(x) \) to fall an appropriate amount? This would be the final general equilibrium outcome in a model where emigration from a city with increased uncertainty would equalize utility. Intuition might suggest that \( s \) would still decline. This is not always true, as is shown in the model. However, although \( s \) may rise at a particular \( x \), the rental profile falls far enough to make the city shrink so that the average space per person is unchanged, for the particular case studied.

3. A Particular Model - Logarithmic Utility and Small Variance of Income

The approach in this section is at once more specific and more approximate, in the interests of tractability. No doubt more general cases can be analyzed, however, the following is a first step.

Assume that utility is given by

\[
u(s,z,x) = \alpha \log s + \beta \log z - \gamma \log x\]

where \( \alpha, \beta, \gamma > 0 \). This is concave in \( s \) and \( z \). Also \( u_s(s,z,x) \) is concave, and \( u_z(s,z,x) \) is convex, considered as functions of \( z \).

Suppose that \( \tilde{y} \) is uniformly distributed on \([y - t, y + t]\) where \( y \) is mean income and \( t^2/3 \) is its variance. It is assumed that \( t \) is small.

The first necessary condition for the individual's maximization problem is then

\[
\frac{\alpha}{s} = \frac{\beta r(x)}{2t} \int_{y-t}^{y+t} \frac{dv}{v - r(x)s} = \frac{\beta r(x)}{2t} f(t)
\]

say. Now it is easily shown that
\[ f(0) = 0 \]

\[ f'(0) = \frac{2}{y - r(x)s} \]

\[ f''(0) = 0 \]

\[ f'''(0) = \frac{4}{(y - r(x)s)^3} \]

Hence, to second-order terms in \( t \),

\[ \frac{\alpha}{s} = \beta r(x) \left\{ \frac{1}{y - r(x)s} + \frac{t^2}{3(y - r(x)s)^3} \right\} \]

The appropriate solution to this equation when \( t = 0 \) is

\[ r(x)s = \frac{\alpha}{\alpha + \beta} y \]

Suppose then

\[ r(x)s = \frac{\alpha}{\alpha + \beta} y - \varepsilon \]

where \( \varepsilon > 0 \) is small since \( t \) is small. Then \( \varepsilon \) satisfies

\[ 3\alpha \left( \frac{\alpha}{\alpha + \beta} y + \varepsilon \right)^3 = 3\beta \left( \frac{\alpha}{\alpha + \beta} y - \varepsilon \right) \left( \frac{\alpha}{\alpha + \beta} y + \varepsilon \right)^2 \]

\[ + \beta \left( \frac{\alpha}{\alpha + \beta} y - \varepsilon \right) t^2 \]

Hence collecting terms of first-order in \( \varepsilon \) and second-order in \( t \), it can be shown that

\[ \varepsilon = \frac{\alpha t^2}{3\beta y} \]
Hence
\[ s = \left\{ \frac{\alpha}{\alpha + \beta} y - \frac{at^2}{3\beta y} \right\} / r(x) \]

Hence this determines \( s \) for a given value of \( r(x) \). The second necessary condition determines \( r(x) \) in terms of the common level of expected utility, \( u^* \). In fact,
\[ \alpha \log \left\{ \frac{\alpha}{\alpha + \beta} y - \frac{at^2}{3\beta y} \right\} + \frac{\beta}{2t} \int_{y-t}^{y+t} \log \left\{ v - \frac{\alpha}{\alpha + \beta} y + \frac{at^2}{\beta} \right\} \, dv 
- \gamma \log x - u^* = \alpha \log r(x) \]

Define
\[ F(t) = \int_{y-t}^{y+t} \log \left\{ v - \frac{\alpha}{\alpha + \beta} y + \frac{at^2}{\beta} \right\} \, dv \]

Then it can be shown that
\[ F(0) = 0 \]
\[ F'(0) = 2 \log \left\{ \frac{\beta}{\alpha + \beta} y \right\} \]
\[ F''(0) = 0 \]
\[ F'''(0) = \frac{2(\alpha^2 - \beta^2)}{\beta^2 y^2} \]

Hence
\[ F(t) = 2t \log \left\{ \frac{\beta}{\alpha + \beta} y \right\} + \frac{t^3}{3} \frac{\alpha^2 - \beta^2}{\beta^2 y^2} \]
Hence the second necessary condition can be reformulated as

\[ r(x) = e^{-u^*/\alpha} \left\{ \frac{\alpha}{\alpha + \beta} \right\}^{\beta/\alpha} x^{-\gamma/\alpha} \left\{ 1 - \frac{(\alpha + \beta)^2 t^2}{a \beta y^2} \right\} \]

This expression shows how \( r(x) \) decreases as \( t \) increases, for a given \( x \) and level of expected utility, \( u^* \).

There is a total population, \( N \), and a boundary rent condition of \( r^* \) at \( x = R \). Then

\[ N = 2\pi b \int_I^R \frac{x}{s(x)} \, dx \]

where \( b \) is the constant fraction of area devoted to residences. Also

\[ r(R) = r^* \]

where \( r^* \) is the value of agricultural land, say.

Since, as previously derived,

\[ r(x)s = \left\{ \frac{\alpha}{\alpha + \beta} y \right\} \left\{ 1 - \frac{(\alpha + \beta)^2 t^2}{3 \beta y^2} \right\}, \]

\[ s = e^{-u^*/\alpha} \left\{ \frac{\beta}{\alpha + \beta} \right\}^{-\beta/\alpha} x^{-\gamma/\alpha} \left\{ 1 - \frac{(a^2 - \beta^2) t^2}{6 a \beta y^2} \right\} \]

[This expression shows that it is possible that \( s \) increases with increasing uncertainty for a given level of expected utility, \( u^*, y \) and \( x \).]

Now

\[ N = e^{-u^*/\alpha} \left\{ \frac{\beta}{\alpha + \beta} \right\}^{\beta/\alpha} \left\{ 1 + \frac{(a^2 - \beta^2) t^2}{6 a \beta y^2} \right\} \int_0^R x^{1-\gamma/\alpha} \, dx \]
where the radius of the CBD, \( I \), is taken to be small, and \( \gamma < \alpha \) is also assumed.

Consider now the regime where \( N \) is fixed, as is \( y \). Then

\[
\frac{dN}{dt^2} = 0 = -\frac{1}{\alpha} \frac{d\mu^*}{dt^2} + \frac{\alpha^2 - \beta^2}{6\alpha y^2} + \frac{2\alpha - \gamma}{\alpha} \frac{1}{R} \frac{dR}{R dt^2}
\]

From substituting \( x = R \) in the expression for \( r(x) \), and differentiating with respect to \( t^2 \),

\[
0 = -\frac{1}{\alpha} \frac{d\mu^*}{dt^2} - \frac{\gamma}{\alpha} \frac{1}{R} \frac{dR}{dt^2} - \frac{(\alpha + \beta)^2}{6\alpha y^2}
\]

From the last two equations,

\[
\frac{1}{R} \frac{dR}{dt^2} = -\frac{\alpha + \beta}{6\alpha y^2} < 0
\]

Hence the city shrinks as uncertainty increases. Also

\[
\frac{d\mu^*}{dt^2} = -\frac{\alpha + \beta}{6\alpha y^2} \{ (\alpha - \gamma) + \beta \} < 0
\]

As is hardly surprising, expected utility falls. What happens to \( s(x) \)?

\[
\frac{ds(x)}{dt^2} = \frac{1}{\alpha} \frac{d\mu^*}{dt^2} - \frac{\alpha^2 - \beta^2}{6\alpha y^2} = -\left( \frac{2\alpha - \gamma}{\alpha} \right) \frac{\alpha + \beta}{6\beta y^2} < 0
\]

Hence, space per person falls at each radial distance \( x \). Finally, note that average density \( \rho \), say, satisfies
\[
\frac{dp}{d(t^2)} = \frac{2}{R} \frac{dR}{d(t^2)} - \frac{1}{N} \frac{dN}{d(t^2)} = \frac{\alpha + \beta}{3\beta y^2} < 0
\]

in this case.

Consider now regimes such that expected utility, \( u^* \), is constant. Firstly, suppose that mean income, \( y \), increases as its variance increases, but that total population, \( N \), is constant.

Then from the previously derived equation for \( N \)

\[
0 = \frac{\beta}{\alpha y} \frac{dy}{d(t^2)} + \frac{\alpha^2 - \beta^2}{6\alpha \beta y^2} + \frac{2\alpha - \gamma}{\alpha} \frac{1}{R} \frac{dR}{d(t^2)}
\]

From the boundary rent condition \( r(R) = r^* \),

\[
0 = \frac{(1 + \frac{\beta}{\alpha})}{y} \frac{dy}{d(t^2)} - \frac{\gamma}{\alpha} \frac{1}{R} \frac{dR}{d(t^2)} - \frac{(\alpha + \beta)^2}{6\alpha \beta y^2}
\]

Hence, from the last two equations,

\[
\frac{1}{R} \frac{dR}{d(t^2)} = \frac{-(\alpha + \beta)^2}{6\beta y^2 \{(2\alpha - \gamma) + 2\beta\}} < 0
\]

Thus the radius of the city falls, but not by as much as in the case where utility was allowed to fall. Also,

\[
\frac{1}{y} \frac{dy}{d(t^2)} = \frac{\alpha + \beta - \gamma}{2\alpha + 2\beta - \gamma} \frac{\alpha + \beta}{3\beta y^2} > 0
\]

which shows how fast mean income must rise to keep utility constant.

For \( s(x) \),
\[
\frac{1}{s(x)} \frac{ds(x)}{d(t^2)} = - \left( \frac{2\alpha - \gamma}{\alpha} \right) \frac{(\alpha + \beta)^2}{6\beta y^2 (2\alpha - \gamma + 2\beta)} < 0
\]

which shows that \(s(x)\) falls at each \(x\), but not by as much as in the previous case.

Finally, for average density \(\rho\)

\[
\frac{1}{\rho} \frac{d\rho}{d(t^2)} = - \frac{(\alpha + \beta)^2}{3\beta y^2 (2\alpha - \gamma + 2\beta)} < 0
\]

so that average density falls, but by less than in the previous case.

Finally, consider a regime such that utility is constant despite fixed \(y\), by allowing \(N\) to vary.

Then from the boundary rent condition,

\[
\frac{1}{R} \frac{dR}{d(t^2)} = - \frac{(\alpha + \beta)^2}{6\beta y^2} < 0
\]

which shows the city shrinks by a greater amount than in the first case, where utility was allowed to fall.

Now, from the expression for \(N\),

\[
\frac{dN}{d(t^2)} = \frac{- \alpha + \beta \{(\alpha - \gamma) + \beta\}}{3\beta y^2} < 0
\]

so that population falls, of course. Also,

\[
\frac{ds(x)}{d(t^2)} = - \frac{\alpha^2 - \beta^2}{s(x) \cdot 6\alpha \beta y^2} > 0 \quad \alpha < \beta
\]

\[
\frac{1}{s(x)} \frac{ds(x)}{d(t^2)} = \frac{\alpha^2 - \beta^2}{.6\alpha \beta y^2} < 0 \quad \alpha > \beta
\]
This demonstrates that space per person at a particular distance x will actually rise if \( \alpha < \beta \). In any case, even if \( s(x) \) falls, it falls by the least amount of any of the three cases.

What about average density, \( \rho \)?

\[
\frac{1}{\rho} \frac{d\rho}{d(t^2)} = -\frac{(\alpha + \beta)^2}{3\beta y^2} + \frac{\alpha + \beta}{3\beta y^2} \{\alpha - \gamma + \beta\} = -\frac{\alpha + \beta}{3\beta y^2} < 0
\]

Hence average density falls precisely as much as in the case where utility was allowed to fall.

4. **Conclusions**

Uncertainty in income lowers the rental profile and thereby shrinks a city. The average density of residences rises. These conclusions held for each of the three regimes studied, for the particular model used.

Firstly, when the population and per capita mean income are fixed, expected utility falls and space per capita falls at each distance x.

Secondly, if expected utility is held constant by allowing mean income to rise, this modifies the result for the first case as expected. Space per capita falls by a lesser amount, as does the rental profile and the radius of the city.

Lastly, however, when expected utility is held constant by allowing the population of the city to fall, it is possible that space per capita would rise at each particular x. The rental profile falls more rapidly than in the first case and, in fact, the radius of the city shrinks faster so that average density falls by the same amount.

The second and third cases might refer to comparisons of different
cities within the same country, where expected utility was equalized by migration. The results are depicted in Figure I.

To a certain point, the spatial nature of the problems analyzed in this paper is incidental. An individual must commit himself to a level of expenditure on one good prior to knowing his income and hence how much is available for a second good. For a given mean income, uncertainty reduces the amount of the first good he will purchase. It seems that this would be true even if an increase in his mean income compensates him for the increased uncertainty. This is an assertion that should be verified in the general case. However, the paper shows that when the price of the first good falls far enough that expected utility is constant, the individual may buy more of the first good when uncertainty increases. The general conditions for this should also be investigated, as should other possible applications.

The spatial nature of the problem is relevant when the model is closed. If the increased uncertainty is uncompensated, or if it is compensated by increasing mean income, the general equilibrium results are again consistent with the intuitive notion that space has become a less attractive commodity. However, if the compensation for increased uncertainty is by means of emigration, the general equilibrium results are more complex. It might be desirable to obtain greater generality in this spatial context also.

There does not seem to be much literature that bears directly on the question of uncertainty and urban structure. The urban model used here is quite standard—see Mills [1], for example. A standard treatment of uncertainty is Rothschild and Stiglitz [2].
Notes:

(1) $s^*(x)$ and $R^*$ refer to the original space per capita and radius.

(2) $s_1(x)$ and $R_1$ refer to the case where mean income, $y$, and population, $N$, are constant, but variance of income increases.

(3) $s_2(x)$ and $R_2$ refer to the case where $N$ is constant, and utility is held constant by increasing $y$ along with the variance of income.

(4) $s_3(x)$ and $R_3$ refer to the case where utility is held constant by allowing $N$ to decrease as the variance of income increases. Mean income, $y$, is held constant. The two possibilities are illustrated. Space per capita can actually increase as $s_3'(x)$ and $R_3'$ represent. Possibly space per capita does decline as $s_3''(x)$ and $R_3''$ represent. In either case average density should be the same as in (2).
REFERENCES.


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