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Tensions Between Mathematics and Science Disciplines: Creative Opportunities to Enrich Teaching Mathematics and Science

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Abstract

An application in mathematics is any context, within science or broader, which involves or requires some kind of quantitative thinking. For instance, arguments involving risk, chance, or uncertainty use probabilistic concepts. Every time we interpolate or extrapolate from a given set of data we employ functional relationships. In discussing dynamics of drug absorption, we use exponential or more complex mathematical models. Describing viruses infecting bacteria or studying interactions between species in an ecosystem requires that we use mathematics tools.

In this paper I study certain teaching and learning situations, named tensions, which arise when the students, practitioners, or instructors engage with applications in mathematics that require modifications to our cognitive models. Tensions can be identified in the ways we formulate the problem of our inquiry, in defining the objects we study, in the implicit and explicit assumptions we make, in the interpretations of results of experiments and mathematical calculations, in visual interpretations, and in other situations.

Using specific examples, I illustrate that these tensions could be viewed as living in a specific zone of proximal development. This concept provides a framework within which we contrast what a single discipline can achieve, compared to fresh new visions and insights generated when the diverse views of mathematics and other science disciplines are brought together.

Keywords: Modeling, applications in mathematics, definition, language, interpretation

Introduction

Mathematics modeling (i.e., working with applications[1]) involves identifying and formulating a problem, making observations, experimenting and collecting data, conceptualizing, identifying mathematical techniques and tools, choosing relevant assumptions, making and testing hypotheses, graphing and calculating, approximating, interpreting, comparing different solutions, and so on. Even a brief look at this list suggests that engaging with mathematics is close to the ways scientists engage with physics, biology, or any other discipline.

Published papers, conference presentations, and panels attest to the need and importance of teaching applications in science within mathematics courses (Baker 1975; Berlin & Lee, 2005; Davidson, Miller, & Metheny, 1995; Matthews, Adams, & Goss, 2009; Pang & Good, 2000; Stillman, 2012; Stillman, Kaiser, Blum, & Brown, 2013). Applications are now an integral part of curricula of many mathematics courses. And

[1] Often called "real-life applications" or "true applications"

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yet, there is a feeling that we are still not doing it right: applications discussed in high schools and university classrooms tend to be artificial; textbooks relegate application questions to the end of a section instead of giving them a more prominent place, such as using an application to motivate a new concept or algorithm. As well, there are numerous problems with method of teaching students to apply mathematical concepts. Gainsburg (2008) writes “research suggests that real-world connections (applications) are infrequent and cursory in mathematics classrooms” (p. 200). Many teachers view the true contexts of word problems as irrelevant distractions (Chapman, 2006). When teachers do connect mathematics to the real world, it is usually not to promote higher-order thinking but to teach procedures instead (Zeuli & Ben-Avie, 2003). Furthermore, anecdotal evidence suggests that teaching of applications in tertiary math courses might suffer from similar problems. Blum and Niss (1991) point at a potential source of problems: “Many mathematics teachers from school to university are afraid of not having enough time to deal with problem solving, modelling and applications in addition to the wealth of compulsory mathematics included in the curriculum.” (p. 53).

Greater effort should be made in the education system to focus on mathematical applications because these are many benefits of this process. When working with applications in mathematics, a learner or a teacher/instructor encounter certain situations which we name, for their dynamic nature, tensions. Tensions arise when cognitive models that have been created within the context of (pure) mathematics need to be modified to accommodate for the broader context of an application. They demand to be explored and understood, and the resulting enriched cognitive model allows for switching between (the pure math and the application) contexts.

Tension*s* can be identified in the ways we formulate the problem of our inquiry, in defining the objects we study, in the notation used, in the implicit and explicit assumptions we make, in the interpretations of results of experiments and mathematical calculations, or in visual interpretations, and so on. They are (at least metaphorically) related to the notion of a boundary, in particular, to the concepts of boundary objects and boundary crossing (Akkerman & Bakker, 2011).

In this paper, I explore tensions and demonstrate that—instead of presenting an obstacle—they provide great opportunities to teach applications in somewhat novel and imaginative ways. As well, knowing the dynamics of tensions can provide teachers and instructors with a better understanding of relationships between mathematics and other sciences, which may prove helpful in interdisciplinary classrooms. Whenever we discuss a scientific phenomenon (say, interaction between competing species) using quantitative tools, tensions inevitably show up. I argue that, by contrasting the ways these tensions co-exist and operate within each science discipline, we can create rich teaching and learning situations that will deepen our understanding of both mathematics and science. I illustrate this point in several examples.

Framework

Teaching modelling in mathematics has two objectives: (a) to understand and solve a particular problem (short-term objective), and (b) to develop necessary skills to apply to a wide variety of problems which require quantitative reasoning (long-term objective). One of the key commitments of a genuine modelling process is to preserve authenticity. Commenting on modern approaches to modelling in mathematics, Galbraith (2013) proposes that “Authenticity be viewed in terms of four dimensions: content authenticity, process authenticity, situation authenticity, and product authenticity” (p. 33). It is this quest for authenticity that invites the appearance of *tensions* that I analyze in this paper.

Perhaps the most appropriate way to frame our research is provided by the notion of distributed expertise (Brown et al. 1993) and a related notion of the zone of proximal development (ZPD). Initially defined as “the ‘distance’ between what a learner can achieve alone, and with the assistance of a more advanced teacher or mentor” (Galbraith 2013, p. 36), the ZPD naturally generalizes to a relationship between a researcher and practitioner (i.e., a mathematician and a “user” of mathematics). Commenting on applying the concept of ZPD to groups of individuals, Galbraith writes that:

> “participants with partially overlapping ZPDs provide a changing mix of levels of expertise, so enabling many different productive partnerships [..] overlapping individual ZPDs can create a combined ZPD which promotes a higher vision of possibilities than either separately could provide. Here partnerships are located in the community itself, where the participants are professionally linked, typically as researchers and/or practitioners.”(p. 37)

This ecosystem of partially overlapping ZPDs, where the researchers and practitioners are most likely to interact, is inhabited by tensions: (a) between the notation and symbols in mathematics, and the way practitioners use them; (b) between the clarity and sharpness of a mathematical definition and the appearance of vagueness in definitions in life sciences; (c) between exactness of mathematics calculations and the necessity to approximate “real-life” quantities which are modelled; (d) between the mathematics’ irresponsibility toward reality and the interpretations of mathematical results in the context of an applied problem; and (e) between a rigidity of assumptions in a theorem and a fictional freedom of selecting quantities for a model.

Tensions

The limits on the size of this paper permit me to discuss only a few examples of tensions in the world of math and science (i.e., in the overlapping ZPDs of mathematics and its applications in other sciences). Although they succeed in giving a general idea, I decided to present additional scenarios in a separate publication.

Definition

A definition in mathematics is a statement that introduces something new—a new object, concept, or property of a mathematical object—based on previously established objects, concepts, and/or properties. Definitions are clear, concise and unambiguous; once established, they rarely change. The following statement, more than two thousand years old, defines a prime number:

> A prime number is a natural number that has exactly two distinct divisors: number 1 and itself.

Within this definition, clearly the context is natural numbers. Also, the requirement "two distinct divisors" removes 1 from the list of prime numbers[2]. The definition draws a clear line—there is no number that is both prime and non-prime[3].

On the other hand, scientists might operate with objects without clearly defining them. For example, the commonly used term "species" is taken for granted. Additionally, there is no generally accepted single definition of climate change. Sometimes, definitions are made purposely vague and misleading. For instance, the label "sodium free" on bottles of water does not imply that there is no sodium—but that the levels of sodium are below some low level[4]. It is impossible to find an agreement of what exactly the word organic, as in "organic produce," really means[5].

"Real-life" definitions are modified over time and space: the definition of ADHD (attention deficit hyperactivity disorder) changed within the last five years and so did the classification of overweight and obese in North America. In Japan, a person whose body mass index is higher than 25 is considered obese. In China, that threshold is 28, and in North America it is 30.

It is not reasonable to expect that real-life definitions[6] retain the clarity and sharpness of a mathematical definition. However, it is desirable that they be made as clear and as sharp as possible, to minimize misinterpretations and misuses. Horwitz and Wakefield (2012) point at an alarming finding:

> […] a systematic study of diagnostic practices […] found massive differences: New York psychiatrists diagnosed nearly 62 percent of their patients as schizophrenic, while in London only 34 received this diagnosis[7].

Another example of the issues that can arise from the lack of clear definitions comes from the British Medical Journal. Mistaking a vague term "size" for the length instead of the volume was behind a major error in the Journal's paper which attempted to dismiss the benefits of mammographic screening (Mittra, Baum, Thornton, & Houghton, 2000)[8].

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[2] This is important; without it, we would not have a unique factorization theorem, which is a cornerstone of number theory.

[3] composite

[4] The label on the bottle of Canadian Essence natural spring water states that "Products containing less that 5mg sodium per 100mL are permitted to state 'sodium-free.'"

[5] For a variety of definitions and conventions consult, for instance, the subsection "Legal definition" in Wikipedia entry "Organic food."

[6] Very often, other terms are used in lieu of "definition," such as: convention, rule, principle, standard, practice, and so on.

[7] This estimate is based on two major factors: attempts (mainly through Diagnostic and Statistical Manual of the American Psychiatric Association) to make psychiatric diagnoses more rigorous and predictable (which leads to simplistic, symptom-based diagnoses), and preferences and prejudices of each group (New York vs. London) of psychiatrists.

[8] The authors erroneously concluded that the lead time for mammographic detection of breast cancer is one doubling time ahead of the clinical breast examination detection. This would imply that there are no true benefits to mammography. The correct lead time is a significant, three doubling times interval—and thus mammography is most certainly beneficial.

To conclude, a student, taught about definitions in a (pure) mathematics context, needs to modify her cognitive model (and is thus entering a ZPD between a researcher and practitioner) in order to work with an application. The resulting, enriched cognitive model of the definition relaxes the rigidity of a mathematical definition to allow for a degree of vagueness and ambiguity.

Language and Interpretation

Math language is cold and precise, and it is mostly communicated through symbols and formulas. Narratives in other sciences are closer to the language used in everyday communication, with symbols and formulas replaced by abbreviations, verbal descriptions, metaphors, and analogies. The tensions between the two languages, when resolved, deepen our understanding and clarity. Consider the statement: The number of infected people will climb until it eventually plateaus at 1200. A mathematician would replace “climb” by “increase” and give a precise meaning (horizontal asymptote) to the visual metaphor of a plateau. The fact that the limit, as the time approaches infinity, is 1200 infected people does not mean much, as stated, to a practitioner. “Eventually” is a word that people “in the field” understand and can relate to: it means that some time, in the near (or not so near) future, the number of infected people will increase to around 1200.

The ability to translate mathematical facts into a language understandable by a layperson is an essential skill, which, unfortunately, is not given sufficient attention in various university programs. Unambiguous and clear communication—between professionals, and between a professional and a patient—is a crucial component of effective work in a health care setting. A survey of medical school students showed that while 90% correctly identified which of the two drugs offered works better (based on information about risk), only 61% were able to accurately interpret the given quantitative data[9] (Sheridan & Pignone, 2002). A large number of physicians and nurses have difficulties communicating information about risk and chance to their patients (for instance, in discussing dangers of a surgical procedure, or side-effects of a drug), which is a serious concern (Gigerenzer, 2003).

The graph (a) in Figure 1 shows the dependence of the heartbeat frequency h of a mammal on its body mass m, drawn from a given principle[10], but without taking context into consideration. With the meaning of m in mind, we redraw the diagram by identifying the domain of the function (interval from the mass of the lightest, to the mass of the heaviest mammal).

[9] The authors demonstrate that the ability to interpret data is strongly correlated to quantitative literacy (thus, this is not just a communication issue, but also a learning issue).

[10] Heartbeat frequency is inversely proportional to the fourth root of the body mass.

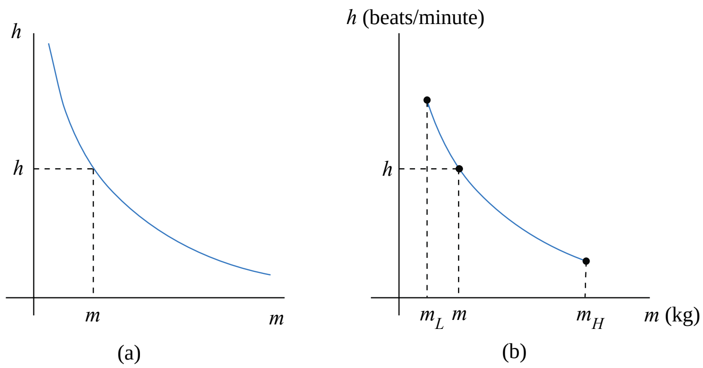

Figure 1. Graph of a function in mathematics (a) and its interpretation in biology (b).

Thus, an application (practitioner's viewpoint) brings something abstract, such as a function and its domain, into a more familiar and real space. For a mathematician, the graph (a) in Figure 2 represents a decreasing function, which approaches zero as *t* increases. The function (b) in Figure 2 has a relative minimum at t_0.

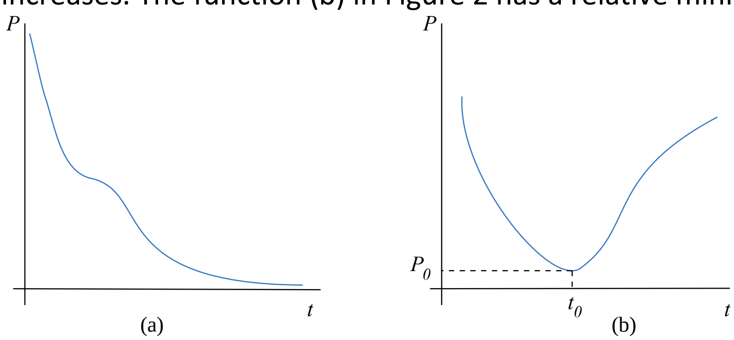

Figure 2. Modeling population dynamics.

For a population biologist, the two diagrams say something different, hence the tension. The population modeled by (a) in Figure 2 will go extinct after some time, and the tail of the graph has no meaning. The relative minimum value P_0 shown in graph (b) in Figure 2 could fall below the minimum that the population needs to survive; thus, the population will continue falling and go extinct, rather than increase as suggested by the graph.

Assumptions and Context

Working with assumptions brings one again into an overlapping ZPD of a (mathematics) researcher and practitioner (working on an application). Whereas mathematician makes an assumption, a practitioner hypothesizes, provides a best guess, works with a common belief, and so on.

Before we can model a predator-prey interaction between foxes and rabbits, we have to make important decisions. To reduce the complexity of the problem (thus increasing the chances that we can actually solve it), we *assume* that rabbits and foxes are the only inhabitants of the ecosystem we study. Next, we *assume* that there is plenty of food for rabbits, and that in the absence of foxes, rabbit population increases (modeled by some law, usually exponential or logistic). Next we *assume* that, on its own and without food, the population of foxes would dwindle, again according to some law. Finally, we *assume* that nothing else significantly affects the dynamics of the two populations. This example illustrates the necessity of making assumptions, as well as their importance in modelling. As in a previous case of the domain of a function, something abstract that we encounter in mathematics (assumptions which appear in every theorem), is brought, with the help of a scientific context, into the realm of a real, tangible object.

Ignoring assumptions is one of the defining characteristics of the surface approach to learning mathematics, and it is a serious problem (Galbraith & Stillman, 2001). Seino (2005) develops the “awareness of assumptions teaching principle,” to emphasize the role of making appropriate assumptions in mathematics. Quite often students show a correct use of an algorithm or a technique, but skip what they perceive as a “theoretical” part, which consists of checking whether or not the assumptions are satisfied. In grading assignments or test questions, anecdotal evidence suggests that instructors are guilty of the same crime: they assign full credit even when they realize that a student did not justify their work by checking assumptions.

In mathematics textbooks, as well as in lectures, students see a large number of (abstract) exercises and problems whose solutions are integers or simple fractions, such as 1/2 or 2/5. The cumulative effect of exposure to such situations skews students’ thinking to the point where they believe that their answer is not correct because it does not look “nice[11].” Working with models and applications in sciences removes this bias, and modifies students’ expectations of the kinds of answers they are supposed to get. (For instance, all coefficients in the differential equations for the predator-prey model are decimal numbers.)

Usual conventions (x is an independent variable, y is a dependent variable) are of no help when we consider modeling situations. Based on the context, we need to decide which quantity is a dependent variable (and use appropriate symbol), and which quantity (or quantities) it depends on. The remaining quantities are viewed as parameters, and often visualized as a family of curves[12]. Moreover, the choice of a coordinate system is no longer limited to the default xy-coordinate system: sometimes, we replace linear scales on one or both axes with logarithmic scales (for instance Richter scale is logarithmic). Again, something that does not usually leave the realm of abstract (such as the concept of a variable) is made palpable in an application. Mindless drawing of x and y coordinate axes is replaced by an informed decision about the type of a coordinate system that is the most appropriate for the model we work with.

Theorem

Based on assumptions and supported by a rock solid proof, a (mathematical) theorem establishes something new (fact, formula, property, or connection between

[11] Anecdotal evidence coming from author’s interactions with students working on applications questions. Students often view numbers 23, 3/4 and 0.5 as “nice”; however, 23/18 and 0.00102 are not considered “nice.”

[12] Families of curves are rarely fully discussed in mathematics textbooks.

objects). It is optimal in the sense that if even one of its assumptions is removed, the claim might no longer be true. Theorem is the only acceptable type of evidence in mathematics[13].

No other science has theorems[14]. Usually it is not even possible to list all assumptions that are behind a phenomenon in biology or geology. Certain facts (such as Schrödinger's equation in physics) are assumed to be true, without theoretical proof. Goldbach's conjecture[15] has been shown to be true for 400 thousand billion numbers. To rephrase, an experiment has been performed 400 thousand billion times, and it gave the same outcome every time. In any discipline this would be fairly significant, and quite likely lead to a new theory. However, a mathematician cannot accept this experiment as evidence that the conjecture is true for *all* positive numbers.

In science, we sometimes form theories based on outcomes from a single experiment (cosmology[16]). Although we cannot expect math-type proofs in geology or chemistry, we can borrow a healthy dose of math skepticism. The same questions that eventually lead us to consider a formal proof ("Why is this true?" or "How does this follow from that?" or "Is there a way I can verify this in a different way?"), we can, and must, ask whenever we study science.

Conclusion

I gave examples and discussed a few tensions (i.e., situations which arise when a cognitive model developed within a narrow scope of mathematics needs to be modified to accommodate for the concepts and notions from applications which are investigated using mathematics). Living in the ZPD, tensions enable us to look at a science discipline through the lens of mathematics, and vice versa. Tensions provoke questions and demand communication between science disciplines. They open doors through which mathematics enters into realms of other sciences, ultimately for a more active presence in textbooks and lectures in those disciplines.

I believe that, by experiencing tensions and seeing how they co-exist in various disciplines, students might enhance their understanding of all sciences. It is my hope that these comments might be helpful in enriching integrated science curricula.

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[13] In the sense that every claim needs to be supported by acceptable evidence (i.e., by a rigorous proof).

[14] No science (other than mathematics) is based on axioms. Examining textbooks in biology, chemistry, environmental science, and so on, we have not identified any theorems (in the rigid sense of a theorem in mathematics).

[15] Every even number greater than 2 is the sum of exactly two prime numbers.

[16] Since we do not know of any other universe but ours.

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