

Supplementary Materials

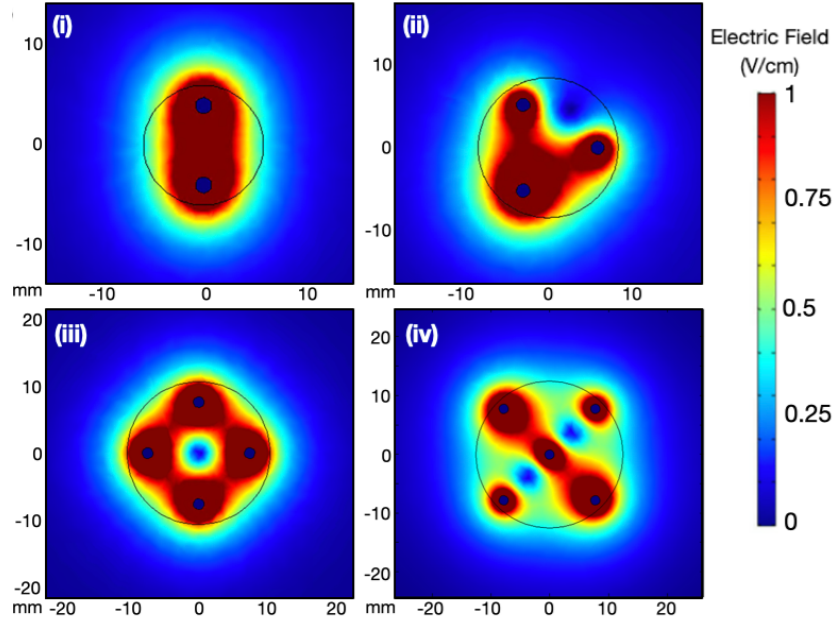


Fig. S-1. Average temporal electric field maps for optimized 2-5 electrodes configurations with no phase shifting.

Objective non-convexity test

To show that our objective function is not convex, we will present a counter example to the definition of convexity³⁶. For all points in the domain $(\vec{x}, \vec{y}) \in \mathbb{R}^n$ and all $\lambda \in [0,1]$:

$$F(\lambda\vec{x} + (1-\lambda)\vec{y}) \leq \lambda F(\vec{x}) + (1-\lambda)F(\vec{y}). \quad (9)$$

Using a random sample of parameters within the bounds of the problem, two parameter arrays can be tested to see if the inequality is satisfied. For continuous functions, it is sufficient to use $\lambda = 0.5$ to check if parameters satisfy the inequality³⁶.

We must first address the uncertainty in our objective function, which comes from the mesh grid discretization of the electric field. Small variations in the location of electrodes can result in uncertainty in the objective function. To overcome this uncertainty, random location variables (r, θ) are incremented every 0.5 mm for r and every $\pi/8$ degrees for θ . Our model also contains both phase shift and geometric symmetries that we must address in order to find a true counter example. Firstly, for any geometry and number of electrodes, with variable phase shifts relative to electrode 1 ($\varphi_1 = 0$), symmetry dictates that: $F(\varphi_2, \varphi_3, \dots, \varphi_n) = F(2\pi - \varphi_2, 2\pi - \varphi_3, \dots, 2\pi - \varphi_n)$. Next, since one electrode is held at a constant $\theta_1 = 0$, and we are dealing with a spherical tumor model, there is a symmetry along the x axis (perpendicular to the electrode insertion direction). Therefore,

$$F(\theta_2, \theta_3, \dots, \theta_n) = F(2\pi - \theta_2, 2\pi - \theta_3, \dots, 2\pi - \theta_n).$$

Both phase and location symmetries result in 4 symmetric regions, where $\theta_R = 2\pi - \theta$ and $\varphi_R = 2\pi - \varphi$:

- Region 1. θ, φ
- Region 2. θ_R, φ
- Region 3. θ, φ_R
- Region 4. θ_R, φ_R

All 4 of these configurations will result in the same objective value for any number of electrodes/contacts. If the pair of parameter points being tested by the inequality are in different symmetry regions, a false non-convex determination could be made. Since the region a parameter set belongs to is unknown, we will transform one parameter set to check the other 3 regions until both points are in the same region. With the restriction in place of removing symmetric solutions, if (9) is not satisfied by a pair of parameter points, then we have shown that the objective function is not convex.

An algorithm to test (9) was created, which starts by finding 50 random pairs of parameter points. The inequality is checked for the first pair of parameter sets. If (9) is not satisfied, all θ values in point 2 are changed to θ_R to check symmetric region 2. If the inequality is still not satisfied, all φ values in point 2 are changed to φ_R (with initial θ 's) to check symmetric region 3. If it is still not satisfied, both θ and φ are converted to θ_R and φ_R to check region 4. If the inequality is still not satisfied after all 4 symmetric regions have been checked, then the function is not convex.

For a 3-electrode single contact model with fixed location, the algorithm was tested for φ_2, φ_3 phase shift variables. It can be shown that the objective function for these parameters is convex. However, we will now show that the 3-electrode single contact model with variable location and phase is not convex. Running the algorithm for a 3 electrode, single contact model with variable location and variable phase, a counter example was found after 12 iterations. For the following variables ($\varphi_2, \varphi_3, r_1, r_2, r_3, \theta_2, \theta_3$), the objective function is not convex.

2.95, 3.36, 5.5 mm, 10.5 mm, 11 mm, 3.53, 0.39
5.65, 3.11, 9.5 mm, 8.5 mm, 11.5 mm, 5.50, 2.36

The objective was also tested for convexity for a 4 electrode, single contact model with fixed location. After 3 iterations a counter example was found for phase shift parameters ($\varphi_2, \varphi_3, \varphi_4$) of (4.08, 0.38, 3.37) radians and (3.89, 6.07, 1.98) radians. Lastly the objective was tested for the 3-electrode dual contact model, with fixed location. A counter example for the phase parameters ($\varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6$) was found after 7 iterations to be (4.05, 4.22, 0.08, 6.26, 2.39) radians and (4.20, 5.21, 2.10, 4.36, 5.54) radians respectively.

By finding counter examples to the definition of convexity, we are able to show that our objective function is not convex for full location and phase optimizations, phase (and location) optimizations of more than 3 electrodes, and phase (and location) optimizations of multi-contact electrode models.