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A Treatise of PD-LGD Correlation Modelling

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Abstract

The provision in Paragraph 468 of Basel II Framework Document for calculating loss given default (LGD) requires that parameters used in Pillar I of Basel II capital estimations must be reflective of economic downturn conditions so that relevant risks are accounted for. This provision is based on the fact that the probability of default (PD) and LGD correlations are not captured in the proposed formula for estimating economic capital. To help quantify economic downturn LGD, the Basel Committee proposed establishing a functional relationship between long-run and downturn LGD.

To the best of our knowledge, the current proposed models that map out this relationship have the same underlying framework. This thesis presents a general factor PD-LGD correlation model within the conditional independence framework, where obligors' defaults are conditional on a common state of affairs in the economy. We highlight a mistake that is frequently made in specifying loss given default, which is, current studies ignore the difference between account-level potential loss and LGD. By correcting this mistake and deriving the correct distribution of potential loss and LGD, sensitivity analysis is conducted to ascertain the impact of the defective model on economic capital and parameter estimates. The relationship between the account and portfolio level correlations are explored. Finally, an empirical analysis is conducted to validate the proposed estimation scheme of parameters in the model.

Keywords: PD-LGD correlation, potential loss, realized loss, systematic and idiosyncratic risk factors, economic and regulatory capital.

Summary For Lay Audience

When Banks issue loans, they are required to set aside some funds to protect the credit issued in case of default. These funds are termed economic capital. It is imperative that the funds set aside adequately protect these positions even in stressful economic conditions. Knowing the right amount of money for this purpose is a concern. Financial Regulators require that the parameters used in estimating economic capital are reflective of bad economic conditions, however, the formula presented in their document does not reflect this. We have shown that existing proposed methods to address this are defective, which implies that the empirical findings based on these methods will be flawed as well. This flaw is fixed and exploratory work conducted.

To the one who holds the wind in his fists,

bounded the waters in a garment, established all the ends of the earth.

Your provisions are without bounds.

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Terms, Acronyms, and Symbols

- **Account-level Quantities** Quantities relating to a single representative borrower on a portfolio.
- **Default Rate** The frequency of default on an entire portfolio within a given planning horizon.
- **Exposure at Default EAD** Loan exposure to a borrower at the time of default within a given planning horizon.
- **Economic Capital EC** Amount of funds needed to remain solvent within a given planning horizon.
- **Loss Given Default LGD** The amount of (or percentage) loss on a loan's exposure in the event of default within a given planning horizon.
- **Probability of Default PD** The likelihood of a borrower's default within a given planning horizon.
- **Potential Loss** Possible future loss on the portfolio within a given planning horizon.
- **Realized Loss** Actualized loss on an exposure within a given planning horizon.
- **Portfolio-Level Quantities** Quantities relating to an entire portfolio.

- **EL** Expected Loss.
- **TTC** Through The Cycle.
- **UL** Unexpected Loss.
- **VaR** Value at Risk.
- ADR_p Asymptotic Portfolio-Level Default Rate.
- $ALGD_p$ Asymptotic Portfolio-Level Loss Given Default.
- ARL_p Asymptotic Portfolio-Level Realized Loss.
- DR_p Finite Portfolio-Level Default Rate.
- $\mathcal{D}$ Default Indicator.
- $\mathbb{L}$ Account-Level Loss Variable.
- $\mathbb{L}_p$ Portfolio-Level Loss Variable.
- LGD_A Expected Value of Account-Level Loss Given Default.
- $LGD_A^{(2)}$ Account-Level Joint Loss Given Default.
- LGD_p Finite Portfolio-Level Loss Given Default.
- M Maturity.
- $\mathcal{PL}$ Potential Loss.
- $\mathcal{RL}$ Realized Loss.
- $\mathcal{RL}_p$ Finite Portfolio-Level Realized Loss.
- α Dependence Parameter for Default Driver.
- β Dependence Parameter for Loss Driver.

- c Confidence Level.
- θ_I Correlation between Idiosyncratic Risk Drivers.
- θ_S Correlation between Systematic Risk Drivers.
- ρ_A Correlation between Default and Loss Drivers.
- ρ_p Correlation between Portfolio-Level Default Rate and Loss Given Default.

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Chapter 1

Introduction

1.1 Motivation

The cacophony associated with the 2007/2008 global financial crisis has triggered the revision of existing rules and regulations by international financial regulators — with the Basel Committee on Banking Supervision (BCBS) being the lead crusader [22, 90]. The changes in these rules and regulations are to bring a "sanctity" and "sanity" check to the financial community. The attempt to fix the grossly exposed deficiencies in the then existing financial rules and regulations by the crises resulted in putting forward the third Basel Accord (also known as Basel III). The guidelines presented in Basel III strengthen global capital and liquidity rules with the aim of promoting a banking sector that can withstand financial and economic stress. The underlying principle is to provide regulations that will help cushion the banking sector against shocks originating from stressful economic conditions [23]. The framework provided by Basel III does not supersede that provided by Basel II, thus the shortfalls inherent in Basel II are worth noting.

The focus of Basel II is the implementation of risk management principles rooted in three pillars: Minimum capital requirements, Supervisory review process, and Market discipline. The central theme of this thesis deals with minimum capital requirements calculated for credit risk — one of the three major risk components banks are faced with (the others are market and operational risk) [20]. Credit risk is the risk of one party under financial contractual obligation to renege.

The calculation of the capital requirement and accordingly economic capital involves the estimation of five risk parameters: The exposure at default (EAD) — the loan's exposure to borrowers at the time of default, probability of default (PD) — likelihood of default over a given period of time, loss given default (LGD) — the amount (or percentage) of money a bank expects to loose on a loan's exposure in the event of default, maturity (M) and correlations (ρ). The estimation of the parameters needed as inputs for estimating EAD, PD, and LGD, and their dependencies is a vital task. For instance, Credit Value at Risk is highly sensitive to the correlations between PDs and LGDs, therefore correlation modelling is a major factor in credit risk modelling [17–20, 23]. Basel II empowers banks to use different approaches to computing regulatory capital (or capital requirement). The internal ratings-based (IRB) approach, which is subject to supervisory review, allows banks to calculate their own risk parameters after meeting some minimum requirements or guidelines. The approaches for IRB takes two forms: The foundation approach, where banks can calculate their own probability of default (PD) parameter while other risk parameters are furnished by the bank's national supervisors. Advanced IRB permits banks to rely on their own risk assessments of their counterparties and associated exposures to arrive at capital requirements. In these approaches, the account-level capital requirement ($\mathcal{CR}$) is calculated as (see [20], paragraph 271-272)

$$\mathcal{CR} = EAD \cdot LGD \cdot \left[\Phi\left(\frac{\Phi^{-1}(PD) + \rho(PD) \cdot \Phi^{-1}(0.999)}{\sqrt{1-\rho(PD)^2}} \right) - PD \right] \cdot AF(PD, MAT), \tag{1.1}$$

where $AF(PD, MAT)$ is an adjustment factor dependent on PD and effective maturity MAT, $\rho(PD)$ is the correlation parameter expressed as a function of PD, Φ is the standard normal cumulative distribution and Φ^{-1} is its inverse. For a comprehensive note on the theory behind Eq. (1.1), see [32, 60]. Accurate estimation of capital requirement depends on accurate description of the loss distribution, which is very much dependent on the account-level correlation between PDs (default correlation between obliogrs), LGDs (LGD correlation between obliogrs), and PD and LGD.

The expression for capital requirement (Eq. (1.1)) does not account for PD-LGD correlations in that the Vasicek model used to derive this formula does not capture systematic correlation between PD and LGD [87]. To address this, the provision in paragraph 468 of [20] requires that the parameters for capital estimation are reflective of downturn economic conditions. This establishes the link between defaults and LGD over a complete economic cycle.

This thesis focuses on modelling of portfolio-level co-movements of PD and LGD, which is vital to credit risk assessment. We looked at, among others, the role of the correlation between the systematic and idiosyncratic risk drivers of portfolio defaults and losses on the correlation between PD and LGD within conditional independent framework, where obligors' defaults are conditional on common set of latent variables.

1.2 Literature

The severity of estimation errors of the risk parameters, EAD, PD and LGD and their dependencies on operations of banks, especially, their solvency in the advent of severe distress economic condition can not be overemphasized. The freedom accorded banks in the Basel II framework to quantifying these risk parameters renders the effect of model choice on the estimate of these parameters uncertain. The effect of (un)observable market-wide (systematic) risk factors on defaults and losses — either static or dynamic (see for example [30, 35, 41] and [77]) — has been a motivating factor for research on economic downturn effect on dependency of risk parameters.

Even though EAD estimates form a vital component of risk capital calculation, empirical studies and estimation schemes for EAD is scanty. Most of these papers focus on corporate loans (for example [11, 12, 14, 63] and [64]). These papers examined floating rate corporates bond performance in the US, factors that influence credit lines usage, among others. Other papers such as [2, 15, 79] and [85] focus on retail loans. In particular, proposed estimation schemes can be found in [58, 69] and [85].

Modelling default correlations among obligors on a bank's portfolio is vital to the accurate measurement of capital requirement. Generally, two modelling directions are followed in evaluating default correlations — correlated Brownian motion, where default correlations between obligors are directly modelled under assumption of multidimensional correlated Brownian motion for the firm's assets value falling below or at some exogenously prespecified level (examples,[73, 89]), and by obligors' exposure to systematic (common) risk factors (examples, CreditRisk$^+$ by Credit Suisse Financial Products and CreditPortfolioView by Mckinsey & Company). Overview of these credit models are respectively found in [28] and [36]. Pillar I of Basel II capital estimation is based on the latter direction, where the existence of a nonobservable contemporaneous risk factor responsible for PD correlations is assumed. Some empirical and theoretical work on PD estimations and correlations are respectively found in [33, 38, 45] and [37, 59, 67, 70, 72, 93]. For example, Koyluoglu and Hickman proposed a generalized model that is rooted in the underlying models of CreditRisk$^+$, CreditMetrics, and CreditPortfolioView [67].

LGD — one of the major components in the calculation of regulatory and economic capital naturally is of great interest to lenders and investors and has to be accurately estimated. For this reason, different modelling approaches for estimating or forecasting LGD have been proposed. We have those that are based on macroeconomic variables (examples, level of unemployment, interest rates), see for example, [25]. Others are based on characteristics of obligor [24, 39]. One of the well known works on predictive models for LGD can be found in the technical report of Moody's KMV LossCalc version 2.0 for dynamic prediction of LGD [57]. LossCalc is a statistical model that uses information on instrument, firm, industry, and economy as inputs to predict LGD. Empirical studies on LGD are also found in [6, 13, 62].

The notion that economic phenomena are incorporated into risk models for estimating PDs and LGDs has received much attention. This is backed by the argument that the current state of the business cycle is accounted for. This translates into accurate measurement of credit portfolio risk. The trade-off inherent in employing this approach is that regulatory capital obtained is higher during periods of economic downturn — pro-cyclical effects of bank capital requirements kick in, where regulatory and economic capital become higher and losses erode banks' capital and potentially hindering their credit supply. Some works on pro-cyclicality capital requirements are found in [56, 80, 81, 88]. In particular, Gordy and Howells provided evidence on pro-cyclicality of capital requirements and accordingly evaluated policy options in dampening pro-cyclicality [56]. Empirical and theoretical evidence of the link between PDs and LGDs (the equivalent of $1 - \textit{recovery rate}$) and its contribution to the pro-cyclicality effects is ubiquitous in the literature: see for example, the works in [1, 7, 8, 51, 61]. These papers show that there is a significant systematic variation of LGD and most importantly a positive correlation between these two quantities. Frye's model [51] is motivated by the work by Finger [46] and Gordy [54], where defaults are driven by a single systematic risk factor as opposed to correlated parameters. A detailed description of Frye's model is given in this chapter.

Summarizing, banks are required to estimate LGD parameters that capture downturn economic conditions. Choice of model to quantify downturn LGD has been challenging for industrial players. To help quantify downturn LGD, the Basel Committee proposed establishing a functional relationship between long-run and downturn LGD [16]. An alternative is banks providing graphs on downturn LGD based on their internal assessments of LGDs during adverse economic conditions (subject to supervisory standards).

1.3 Related Concepts

1.3.1 On Regulatory and Economic Capital

The importance of economic roles of banks were heightened in the wake of the 2007/2008 financial crises [3, 47, 76]. The smooth transmission of savings into productive economic activities is very vital. Banks are one part of this transmission mechanism. One of the vital economic roles of banks is the supply of credit: they lend money to sovereign entities, corporations and consumers [52, 65, 76]. Issuance of credit facilities comes with associated risk of loss of principal and interest. Some borrowers default on their debt obligations, causing financial loss to the banks. And thereby, potentially rendering them handicapped in fulfilling their own debt and contractual obligations. Regulatory and economic capital is therefore motivated in part by concern of banks protecting themselves [26] and over negative externalities that may arise from counterparties' default.

Since knowledge of the future is beyond the grasp of everyone, it is almost impossible to know in advance the severity of loss to a bank for a given period. However, a bank can forecast the level of possible credit loss the bank expects to incur (termed, expected loss (EL)). Expected loss can be viewed as an insurance for losses banks' anticipate from historical defaults, and therefore considered as part of the cost for conducting business [16]. Funds set aside to cover EL is not enough as the loss can exceed the anticipated level. Unexpected loss (UL) — the loss that exceeds expected loss — can arise because of credit risk or adverse interest rate shocks (see, example [4] and [40], of works integrating these two risks). Funds set aside to cover UL is economic capital (EC) if EL is covered by revenues [16]. When these funds reflect supervisory guidance and rules, then we have regulatory capital.

Determining the right level of capital against UL is a major task for banks in that there are trade-offs involved. Holding a reduced level of capital makes available economic resources that could be invested in profitable ventures, but more likelihood of inadequate funds to meet debt obligations in the occurrence of unexpected large losses. Conversely, holding a high level of capital freezes up funds that can be directed to profitable ventures, however, banks are better placed in a position to likely meet own-debt obligations.

Banks are unable to determine ahead of time, the exact number of defaults, actual losses, and the exact outstanding amount of loans in a given year. These factors introduce random effects into the modelling process of economic or regulatory capital. These random effects correspond to the risk parameters forming the foundation block of Basel II IRB [20]: PD, LGD, and EAD. The subsequent sections throw more light on the above mentioned factors as related to expected loss and the determination of economic capital.

1.3.2 Expected and Unexpected Losses

We define the expected loss by first observing that the loss variable of any obligor on any given portfolio is expressed as

$$\mathbb{L} = EAD \times LGD \times \mathcal{D}, \tag{1.2}$$

where $\mathcal{D}$ is the indicator variable associated with a default event in a given period of time. EAD, LGD and $\mathcal{D}$ are random variables and are measured with respect to a specified time period. LGD is a percentage value. Expected loss is therefore defined as follows:

Definition 1. *Given a loss variable $\mathbb{L}$ as described in Eq. (1.2), its expected value*

$$EL = \mathbb{E}[\mathbb{L}] \tag{1.3}$$

is the expected loss of the credit-risky asset of interest [28].

Proposition 1. *It follows from Definition 1 that the expression of Expected Loss of any credit-risky asset is given as*

$$EL = \mathbb{E}[EAD] \times \mathbb{E}[LGD] \times \mathbb{E}[\mathcal{D}], \tag{1.4}$$

provided EAD, LGD, and $\mathcal{D}$ are independent. Furthermore, if EAD and LGD assume constant values then

$$EL = EAD \times LGD \times \mathbb{E}[\mathcal{D}], \tag{1.5}$$

where $\mathbb{E}[\mathcal{D}]$ is the probability of an obligor defaulting in a certain time horizon.

Proof. Note that the expectation of any Bernoulli random variable is the probability of a success event. Therefore, the variable $\mathcal{D}$ being a Bernoulli variable has its expected value equal to the probability of obligor default. If EAD, LGD and $\mathcal{D}$ are independent, then the joint density factors into the product of the marginal densities and the result follows. □

Eq. (1.4) and (1.5) indicate that a high PD will trigger a high level of EL. The same holds for LGD and EAD, an upward movement in both will cause an upward movement of EL. To determine EL based on portfolio values, consider Figure 1.1. The figure shows the distribution of portfolio loss. The vertical and the horizontal axes show the likelihood of loss and values of loss respectively. The shaded area (green) under the right hand side of the curve represents the probability that a bank will be unable to meet its own-debt obligations given its profits and capital — assuming the bank has set aside capital to cover both expected and unexpected losses. One minus this probability is the confidence level at which (Credit) Value at Risk (VaR) is obtained. The VaR is the level of loss that corresponds to the confidence level. Thus, if the bank sets capital commensurable to the gap between EL and VaR, then this capital is what is termed economic capital if EL is covered by revenues. Economic capital is therefore the amount of risk capital that a bank estimates in order to remain solvent at a given confidence level within a given time horizon. And when capital is set to reflect supervisory guidance and rules, the resulting capital estimates is termed regulatory capital. That said, we give the general definition of economic capital by first observing the following definitions [28].

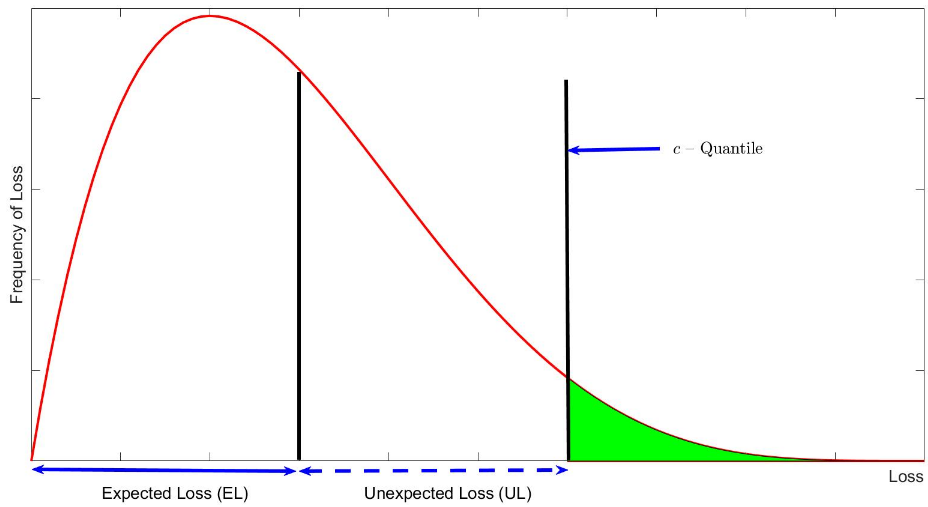


Figure 1.1: The Portfolio Loss distribution.

Definition 2. *A portfolio loss $\mathbb{L}_p$ is the sum of the collection of losses of individual obligors on the portfolio. Notationally,*

$$\mathbb{L}_p = \sum_{i=1}^{n} \mathbb{L}_i = \sum_{i=1}^{n} EAD_i \cdot LGD_i \cdot \mathcal{D}_i,$$

where n is the number of obligors on the portfolio. It follows that the expected portfolio loss is

$$\mathbb{E}\,\mathbb{L}_p = \sum_{i=1}^{n} \mathbb{E}[EAD_i \cdot LGD_i \cdot \mathcal{D}_i].$$

Definition 3. *The c-quantile of portfolio loss $\mathbb{L}_p$ is defined as*

$$Q(c) = \inf\{q > 0 | c \leq \mathsf{P}(\mathbb{L}_p \leq q)\}.$$

Definition 4. *The economic capital (EC) corresponding to a prescribed confidence level c is defined as the c-quantile of portfolio loss $\mathbb{L}_p$ minus the EL of the portfolio:*

$$EC_c = Q(c) - \mathbb{E}\,\mathbb{L}_P,$$

The c-quantile of portfolio loss $Q(c)$ is what is termed as the (Credit) Value at Risk (VaR) in the sequel provided above. For example, a confidence level of $c = 99.99\%$ implies that the economic capital will on the average sufficiently cover 9,999 out of 10,000 years under a one-year planning horizon.

1.4 Modelling dependencies using factor models

In statistical modelling there is the need to provide explanations for the variance of variables in terms of underlying factors and this need surfaces in credit risk. Factor models provide this platform. For instance, via factor models, the correlation between respective losses can be explained in terms of economic variables. Thus, explanations to large losses can be inferred. That is to say, once we can give a valid interpretation of the correlations between respective losses associated with individual obligors, we can as well give valid interpretations to the distribution of the total loss on the portfolio. Summarizing, factor models provide a way to express the correlation between defaults and losses among obligors, and correlation between default and loss by a representative obligor. The following subsections give a review of selected factor models on which illustrations in this thesis are based. These factor models, including any dependencies are specified at the acount-level while interest lies in the portfolio-level quantities, such as the portfolio level default rate or loss.

1.4.1 A Review of Selected Factor Models

Frye's Model

Frye's credit risk model incorporates factors that simultaneously affect default and the value of loan collateral [51]. By this specification, account-level LGD and default are dependent on a common risk factor representing depressing and buoyant years of the economy in addition to dependence on individual risk factors unique to these representative exposures. The model shows that the decrease (increase) in default rate and collateral value of loan is due to their dependency on common factor(s). In the spirit of the conditional approach suggested in [46] and [53], the proposed model may be described as follows: Let each obligor i have an exposure of \$1.00. The collateral value for each obligor is a random variable that follows a normal distribution determined by three parameters: its amount and volatilities – μ and σ respectively, and the sensitivity of systematic (common) risk factor q. The value of collateral is assumed to be determined within a one year period and that the default and loss drivers have a common systematic risk component and independent idiosyncratic risk components. We stick to the notation used in the original papers.

Frye's model is a single systematic risk factor model that assumes an exposure of \$1.00 per obligor i in a portfolio. A latent variable driving losses on the portfolio is defined as

$$C_i = q_i X + \sqrt{1 - q_i^2} Z_i$$

is fed into a function driving the underlying collateral value of the portfolio. The collateral value of an obligor i is presented as

$$\text{Collateral}_i = \mu(1 + \sigma C_i). \tag{1.6}$$

Since collateral is not expected to fall below 0, we assume $\mu > 0$ in our study. X and Z_i are independent standard normal random variables capturing economy wide and obligor-specific risk, respectively. $q_i \in [-1, 1]$, measures the degree of impact of X and Z_i on C_i. Eq. (1.6) implies that the value of collateral for obligor i is jointly determined by the systematic risk X, and an idiosyncratic risk factor Z_i, at the end of the planning horizon. The collateral is normally distributed with mean μ and standard deviation $\mu\sigma$. Furthermore, the default driver is expressed as

$$A_i = p_i X + \sqrt{1 - p_i^2} X_i,$$

where X_i is standard normally distributed and is independent of X and Z_i. $p_i \in [-1, 1]$ accounts for the sensitivity of A to X and X_i. A_i is the so called "financial condition" or "asset return" of obligor i that determines whether the representative obligor i defaults or not. The default event is modelled by observing

$$\mathcal{D}_i = \begin{cases} 1 \text{ if } A_i \leq \Phi^{-1}(PD_i) \\ 0 \text{ Otherwise} \end{cases},$$

where PD_i is the probability of default corresponding to obligor i. It is assumed that LGD associated with a representative obligor i is governed by

$$LGD_i = \max(0, 1 - \text{Collateral}_i).$$

By this specification, the portfolio-level loss is given as

$$\mathbb{L}_p = \sum_{i=1}^{n} LGD_i \mathcal{D}_i = \sum_{i=1}^{n} \max(0, 1 - \text{Collateral}_i) \cdot \mathcal{D}_i.$$

Frye analyzed the effect of collateral value on capital by using a conditional expected loss given the state of the economy (systematic risk factor). By this, the author demonstrated that the effect of collateral damage increases economic capital for all loans when model specification incorporates correlation between default probability and recoveries with a common factor compared to models with independent recoveries.

Pykhtin's Model

The specification of Frye's model allows for negative collateral values due to the normality assumption which leads to LGD_i larger than 1. Pykhtin's model [78] is a follow up to Frye's model in that it addresses this defect ($LGD > 1$) in Frye's model by assuming that the collateral corresponding to obligor i in the portfolio follows a log-normal distribution:

$$\text{Collateral}_i = \exp(\mu + \sigma R_i). \tag{1.7}$$

The loss driver R_i is defined as

$$R_i = \beta_i Y + \gamma_i \xi_i + \sqrt{1 - \beta_i^2 - \gamma_i^2}\eta_i, \tag{1.8}$$

where $Y, \xi_i, \eta_i, i = 1, 2, ...n$ are independent and identical distributed ($i.i.d$) $N(0,1)$ random variables. β_i and γ_i measure the sensitivity of R_i to Y, ξ_i and η_i. The default driver is given as

$$X_i = \alpha_i Y + \sqrt{1 - \alpha_i^2}\xi_i, \tag{1.9}$$

where α_i accounts for the degree of impact of Y and ξ_i on X_i. The expression for R_i implies that not only is the loss driver correlated with the default driver via a common systematic risk factor, Y, the loss driver is as well dependent on the idiosyncratic risk component ξ_i of default driver. In addition to Eq. (1.8) and (1.9), $\beta_i, \gamma_i > 0$ such that $\beta_i^2 + \gamma_i^2 \leq 1$ and $\gamma_i \leq \left(\frac{\alpha_i}{\sqrt{1-\alpha_i^2}}\right)\beta_i$.

By specifying a loss function as in Eq. (1.7) in the event of default, and making use of (1.7)-(1.9), Pykhtin derived the closed-form expression for the asymptotic portfolio-level expected loss and the contribution of each obligor to the portfolio loss. The effect of the correlation between default probability and recoveries on economic capital is illustrated by assuming that at the portfolio level, the individual with the largest exposure accounts for a negligible share of total portfolio exposure and thereby diversifying away idiosyncratic risk at the portfolio level.

Miu and Odzemir's (M & O Model)

Miu and Ozdemir [74] identified four different components of account-level PD and LGD correlations in their modelling approach. These components are outlined as follows:

1. Correlations between the systematic risk drivers of default and loss for a given obligor: This correlation arises from the impact systematic risk factors have on the value of asset(s) of counterparties. Asset value is linked to the severity of the effect of economic-wide risk factors which has potential effects on the likelihood of default by the counterparties and the recovery value.

2. Correlations between idiosyncratic risk drivers of default and loss for a given obligor: The presence of these correlations follow from the fact that the idiosyncratic risk factors affecting the value of the specific asset of a particular obligor will affect the value of the asset in question and thereby affecting the chances of default by the specific obligor, the recovery rate and the related LGD.

3. Correlations between the default risk drivers between different obligors: This component is captured as inputs in the risk weight formulas specified by the

Basel committee [20] and is also captured in most structural models.

4. Correlations between loss risk drivers between different borrowers: This component denotes the systematic risk factors that affect the LGDs of all borrowers, which may be independent of those affecting the likelihood of default by the firm (or one of the borrowers).

It is worth noting that this model produces correlation between $\mathcal{D}_i$ and LGD_j, since the model allows for correlation between the default driver of obligor i and loss driver of obligor j.

Based on the above components, they proposed time invariant models that are categorized into two — a model describing the (i) systematic and (ii) idiosyncratic credit risk and then conducted a comparative study on these models. The model on systematic credit risk is given as

$$\begin{cases} P_t = \beta_{PD} X_t + \sqrt{1-\beta_{PD}^2}\varepsilon_{PD,t}^i \\ L_t = \beta_{LGD} X_t + \sqrt{1-\beta_{LGD}^2}\varepsilon_{LGD,t}^i \end{cases},$$

where P_t and L_t are latent variables and are jointly normally distributed. These two variables account for the systematic risk factors relating to PD and LGD respectively. The pair $(\beta_{PD}, \beta_{LGD}) \in [-1,1]^2$ respectively determine the sensitivity of P_t and L_t to X_t and $\varepsilon_{PD,t}^i$ and $\varepsilon_{LGD,t}^i$ respectively. $X_t, \varepsilon_{PD,t}^i, \varepsilon_{LGD,t}^i$ are $i.i.d$ standard normal random variables. The model assumes homogeneous credit quality among individual obligors and further allows for correlation between the individual specific risk components of PD and LGD. By this, the equations governing the idiosyncratic credit risk for exposure i, are as follow:

$$\begin{cases} p_t^i = R_{PD}P_t + \sqrt{1-R_{PD}^2}e_{PD,t}^i \\ l_t^i = R_{LGD}L_t + \sqrt{1-R_{LGD}^2}e_{LGD,t}^i \end{cases},$$

where

$$\begin{cases} e_{PD,t}^i = \theta_{PD}x_t^i + \sqrt{1-\theta_{PD}^2}\varepsilon_{PD,t}^i \\ e_{LGD,t}^i = \theta_{LGD}x_t^i + \sqrt{1-\theta_{LGD}^2}\varepsilon_{LGD,t}^i \end{cases}. \tag{1.10}$$

$(R_{PD}, R_{LGD}) \in [-1,1]^2$ measures the sensitivity of p_t and i_t to P_t and L_t respectively and are assumed to be homogeneous across individual obligors. The pair (p_t, l_t) are correlated latent variables and the correlation is via Eq. (1.10). p_t^i and l_t^i are the individual PD and LGD risk drivers respectively. x_t^i denotes the specific risk driver for obligor i, and is assumed to be standard normally distributed. Also, $\varepsilon_{PD,t}^i$ and $\varepsilon_{LGD,t}^i$ are independent of x_t^i and are assumed to be mutually independent and normally distributed with zero means and unit standard deviations. Thus, $e_{PD,t}^i$ and $e_{LGD,t}^i$ are standard normally distributed. $\theta_{PD}, \theta_{LGD} \in (0,1)$. It is assumed that there is a one-to-one monotonic mapping between LGD_t^i and the l_t^i value defined as follows:

$$LGD_t^i = \mathcal{B}^{-1}(\Phi(l_t^i), a, b),$$

where $\mathcal{B}^{-1}(\cdot)$ denotes the beta inverse cumulative distribution function with shape parameters a and b. Default occurs when p_t^i is less than a particular constant threshold.

Witzany's Model

Witzany's model [92] is a two-factor model that captures retail portfolio probability of default and LGD and their dependency thereof. The proposed model is an extended version of the one-factor model proposed by Frye [51], Pykhtin [78], and Tasche [83]. It is assumed that the loss driver is dependent on two systematic factors – the systematic factor that drives probability of default and an additional systematic factor that captures, say fluctuations in economic conditions. The respective expressions governing default and loss drivers are as follows:

$$
\begin{aligned}
Y_{1,i} &= \sqrt{\rho_1} X_1 + \sqrt{1-\rho_1}\zeta_{1,i}, \\
Y_{2,i} &= \sqrt{\rho_2}\left(\omega X_1 + \sqrt{1-\omega^2} X_2\right) + \sqrt{1-\rho_2}\zeta_{2,i},
\end{aligned}
$$

where the pair (X_1, X_2) are the systematic risk factors and are independent standard normally distributed. The variables $\zeta_{1,i}$ and $\zeta_{2,i}$ account for the idiosyncratic risk components in the model. These variables ($\zeta_{1,i}$ and $\zeta_{2,i}$) are standard normally distributed and are independent of each other as well as X_1 and X_2. The pair $(\rho_1, \rho_2) \in [-1, 1]^2$ measures the sensitivity of $Y_{i,1}$ and $Y_{i,2}$ to the systematic risk factors and $\omega \in [-1, 1]$ parameterizes the link between PD and LGD systematic risk factors. The loss function given that default occurs takes the form

$$
LGD_i = F^{-1}(\Phi(Y_2)),
$$

where F^{-1} is the inverse of a cumulative distribution function. In this proposed modelling framework, F^{-1} is the cumulative inverse beta distribution function as in the case of Miu and Odzemir's modelling approach.

1.5 Contribution

The estimate of regulatory and economic capital needed to offset losses within a bank's portfolio of loans is dependent on the accurate representation of the portfolio loss distribution tail. As evident in the literature, losses and defaults are correlated: portfolio-level losses are dependent on the extent to which individual defaults are correlated with each other and the severity of the associated losses. Portfolio loss and default distributions are therefore linked. The link determines the portfolio loss tail distribution.

Default and loss dependency is driven by the degree of dependency on systematic risk factors. These factors represent sectoral (for examples of work along this line, see the technical documentation on CreditRisk$^+$ [34], also [5, 10] and [84]) or macroeconomic forces (see, for example, [42] and [51]) responsible for driving default and loss dynamics for all obligors. The effect of the correlation between the respective systematic risk drivers of default and loss on portfolio performance is worth investigating — the effect on the tail distribution of portfolio loss, and thus regulatory and economic capital.

This thesis contributes to the continual studies on portfolio credit risk modelling and focuses on the co-movement of probability of default (PD) and loss given default (LGD) (also termed, PD-LGD correlation) within the factor modelling approach. To the best of our knowledge, the current PD-LGD correlation models have the same underlying framework (compare, for example, [44, 51, 68, 74, 78, 82, 91] and [92]). We highlight a mistake that is frequently made in specifying LGD, which is, current studies ignore the difference between potential loss and loss given default at the account level. Generally the current literature defines loss given default at the account-level as

$$LGD = H(loss\ driver).$$

An examination of the above transformation of loss drivers to account-level LGDs reveals that occurrence of default is not incorporated in the transformation — Loss drivers are assumed independent of default drivers in determining LGDs. We argue that the correct transformation of loss drivers to account-level LGD should take into account the dependency of loss and default drivers. Thus, the transformation $H(\cdot)$ should transform loss drivers to potential losses $\mathcal{PL}$ so that we have

$$\mathcal{PL} = H(loss\ driver).$$

By proving the validity of our argument, we address the following research questions: How serious is the effect of this flaw on model-implied account-level loss distribution, parameter estimates and economic capital? What is the behaviour of LGD at the account and portfolio-level and the relationship between account-level default and loss and the default and loss relationship at the portfolio-level?

We examine the above research questions through the lens of Monte Carlo Simulation (MCS) and Analytical Approximation. We propose an estimation scheme — based on Method of Moments — for model parameters. Empirical analysis is conducted on model parameters. We conclude by proposing a future direction of the study by observing that the generalized PD-LGD model presented in this thesis assumes serially independent systematic and idiosyncratic risk factors. And that to adequately capture the portfolio risk, the model can be extended to incorporate serial dependence of the risk factors on the portfolio. The remaining part of the introduction outlines the thesis.

Chapter 2: Modelling Account-Level Quantities

This chapter presents a generalized credit risk model that nests existing factor credit risk models. The flaw inherent in the existing models is highlighted, which is to ignore the difference between account-level potential loss and loss given default. By fixing this defect and deriving the correct distribution of potential loss and LGD, sensitivity analysis is conducted to ascertain impact of defective model on risk metrics such as economic capital and parameter estimates.

Chapter 3: Comparing Account and Portfolio Level LGD

It is imperative that nobody uses credit risk models they do not understand at an intuitive level. The user should be able to establish a clear link between model parameters and statistical properties of model implied quantities. An intuitive understanding requires a clear link between the account and portfolio levels parameters, distributions, risk measures, etc. This chapter, therefore, investigates the relationship between the relative size of the correlation between the systematic risk factors to that of idiosyncratic risk factors and the correlation between portfolio-level default rate and LGD.

Chapter 4: Empirical Study on Model Parameters

We proposed a Method of Moments (MoM) based estimation scheme for model parameters. We established that the risk measure, Value at Risk, and the mean and standard deviation of the model implied portfolio-level LGD and the correlation between portfolio-level default rate and LGD are respectively not sensitive to the correlation between loss and default drivers. The marginal and dependency parameters in the model are estimated using both Monte Carlo simulation scheme and a data set on Charge-off and Delinquency Rates on loans and leases from the 100 largest banks — obtained from the Board of Governors from the Federal Reserve System (BGFRS) [29]. The Monte Carlo simulation scheme is to validate the proposed method. The BGFRS data implied estimates are used to estimate the model implied mean and variance of the portfolio-level LGD and the correlation between default rate and LGD. Comparison of these estimates indicates appreciable performance of our proposed model.

Chapter 5: Future Work

This chapter discusses a possible future work based on the findings presented in this document and results in the literature.

Chapter 2

Modelling Account-Level Quantities

This chapter presents generalized PD-LGD correlation model within conditional independent framework, where obligor's defaults are conditional on a common systematic risk factor. We highlight a mistake that is frequently made, which is to ignore the difference between potential loss and loss given default at the account level. A model that corrects this defect is proposed and the distribution associated with this "correct-model" is compared to that of the defective model. The parameter estimates from these respective distributions are also compared.

2.1 Nesting Models and Related Distributions

This section describes a generalized PD-LGD correlation model and its link to specific models in the literature. Relevant loss distributions needed in subsequent sections are derived.

2.1.1 Nesting All the Models

Consider a portfolio of N loans and assume homogeneous parameters across individual obligors (or exposures), the general structure of each of these models is presented as

$$A_i = \alpha \cdot S_A + \sqrt{1-\alpha^2} \cdot I_{A,i}, \tag{2.1}$$

$$B_i = \beta \cdot S_B + \sqrt{1-\beta^2} \cdot I_{B,i}, \tag{2.2}$$

where i represents individual exposures. In the ensuing discussion, we drop i in favour of notational convenience. Associated with each exposure is the pair of latent variables A and B that governs default and loss scenarios. Here α and β are constants in the interval $[-1, 1]$, the pair (S_A, S_B) are the systematic risk drivers capturing prevailing economic conditions affecting A and B, respectively, and are common to all accounts. This pair (S_A, S_B) are assumed to be standard bivariate normally distributed[1] with correlation θ_S. The variables I_A and I_B account for the idiosyncratic risks on the portfolio through A and B, respectively, and are specific to each account and independent across accounts. They are bivariate standard normal with correlation θ_I. The pairs — (S_A, S_B) and (I_A, I_B) — are independent of one another.

These assumptions imply that (A, B) follows a standard bivariate normal distribution. The account-level correlation ρ_A is therefore obtained as

$$\begin{aligned}
\rho_A &= \mathbb{E}[AB] \\
&= \alpha\beta\,\mathbb{E}[S_A S_B] + \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\,\mathbb{E}[I_A I_B] \\
&= \alpha\beta\theta_S + \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I,
\end{aligned}$$

where $\theta_S = \mathbb{E}[S_A S_B]$ and $\theta_I = \mathbb{E}[I_{A,i} I_{B,i}]$. See Table A.1 for a comparison of the

[1]By this we mean bivariate normal with standard normal margins and arbitrary correlation.

parameters and correlation structure of the general model to that of the specific models presented in Chapter 1. It is tempting to think of ρ_A as dictating portfolio-level PD-LGD correlation but this is tricky — we will show that ρ_A does not tell us the whole story about the portfolio. At the account level then ρ_A does explain a lot — the question is how relevant the account-level relationship is, if one is ultimately interested in portfolio-level statistics such as economic capital? This question is addressed in Chapter 3 of this document. Table 2.1 presents the generalized correlation structure between pairs of account-level default and loss drivers.

Table 2.1: Generalized correlation structure between pairs of account-level default and loss drivers.

Correlations	Formulas
(A_i, B_i)	$\alpha\beta\theta_S + \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I$
(A_i, A_j)	α^2
(B_i, B_j)	β^2
(A_i, B_j)	$\alpha\beta\theta_S$

Remark 1. *A and B are independent if and only if $\alpha\beta\theta_S + \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I = 0$ which holds if and only if one of the following is true*

1. *There is no dependence between the systematic risk factors $(S_A,\ S_B)$ and no dependence between the idiosyncratic risk factors $(I_A,\ I_B)$ — that is $\theta_S = \theta_I = 0$.*

2. *The systematic risk factors $(S_A,\ S_B)$ are independent, and at least one of (α, β) has an absolute value of 1. Note that $\alpha(\beta)$ controls the relative importance of the systematic and idiosyncratic components on the default (loss) driver.*

3. *The idiosyncratic risk factors (I_A, I_B) are independent, and at least one of (α, β) is zero. That is $\theta_S \neq 0$, $\theta_I = 0$ and $\alpha = 0$ or $\beta = 0$ or $\alpha = \beta = 0$.*

4. *The pair (S_A, S_B) are dependent ($\theta_S \neq 0$), (I_A, I_B) are also dependent ($\theta_I \neq 0$), and $\alpha\beta\theta_S = -\sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I$.*

The above remark follows intuitively as: $\rho_A = 0$ if A and B are independent. Conversely, if $\rho_A = \alpha\beta\theta_S + \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I = 0$ — given that any of the statements 1-4 holds —(and the fact that (A, B) are jointly normal) it follows that A and B are independent.

Now, default occurs when A is less than a given threshold amount. In notational form, we have

$$\mathcal{D} = \mathbb{1}_{\{A \leq \Phi^{-1}(PD)\}}. \tag{2.3}$$

$\mathcal{D}$ and PD are default indicator and probability of default associated with a representative exposure over a given time horizon respectively. This document assumes a homogeneous PD across individual exposures. That is $PD_i = PD_j = PD$ for $i \neq j$. Φ^{-1} is the standard normal inverse cumulative distribution function. Associated with each exposure is the potential and realized loss and the loss in the event of default.

Potential loss is the possible future loss on the portfolio in a planning horizon, expressed notationally as

$$\mathcal{PL} = H(B),$$

where $H(\cdot)$ is a monotone function of B. It appears we are the first to use the terminology, potential loss. It seems useful to think of potential loss as being related to collateral. For instance, if collateral for an individual obligor $\mathcal{C}$ is non-negative then $\mathcal{PL} = \max(0, 1 - \mathcal{C})$. Realized loss on an exposure is the product of $\mathcal{D}$ and $\mathcal{PL}$. So we have

$$\mathcal{RL} = \mathcal{D} \cdot \mathcal{PL} = \mathbb{1}_{\{A \leq \Phi^{-1}(PD)\}} \cdot H(B).$$

Remark 2. *It is important to observe from the definition of $\mathcal{PL}$ and $\mathcal{RL}$ that at the end of a planning horizon, the potential and realized loss may be different for the same obligor — realized losses being zero for the non-defaulting obligors but potential losses not necessarily zero. So a representative account may have a different notional* *value of potential and realized loss. Notationally, If $\mathcal{D} = 1$ then $\mathcal{PL} = \mathcal{RL}$ and if $\mathcal{D} = 0$ then $\mathcal{PL}$ and $\mathcal{RL}$ could potentially be different.*

Note that by model specification (Eq. 2.3) we have written default as a decreasing (non-increasing) function of A. $\mathcal{PL}$ can be expressed as an increasing or a decreasing function of B. Suppose $H(\cdot)$ increases (decreases) with increasing (decreasing) B, then in order that default and potential loss have a positive relationship we require $\rho_A < 0$. This is because a negative correlation between A and B induces a positive relationship between default and potential loss — increasing values of A implies decreasing likelihood of default ($\mathcal{D} = 1$), B and $H(\cdot)$ and conversely, $\mathcal{D}$, B and $H(\cdot)$ increases as A decreases. From similar chain of reasoning, decreasing (increasing) $H(\cdot)$ with an increasing (decreasing) B implies a positive correlation between A and B — $\rho_A > 0$.

Summarizing, the imposition of a positive relationship between default and potential loss means that the inverse (or positive) assumption between $H(\cdot)$ and B requires a positive (or negative) correlation between A and B. That said, Algorithm 1 presents the procedure for calculating economic capital.

2.1.2 Linking Nested Models to Specific Models

To proceed with our discussion, it is worth noting that the specific models fit the general model by the restrictions imposed on θ_S, θ_I and $H(\cdot)$. Table A.1 in Appendix A.1 presents a comparison of the variables and parameters in the general model — Eq. (2.1) and (2.2)) — and the correlation structure thereof to that of the four specific models presented earlier in this document. The description of the function $H(\cdot)$ — the transformation of B – varies across models.

Algorithm 1 Estimating economic capital

1: Input parameters — (α, β): sensitivity parameter of systematic and idiosyncratic risk factors, PD: common probability of default, m: portfolio size, n: number of simulations, c: confidence level, θ_I: correlation between systematic risk factors (I_A, I_B), θ_S: correlation between idiosyncratic risk factors (S_A, S_B)

2: Generate n quantities of the pair (S_A, S_B). Denote the simulated values as $(s_{A,1}, s_{B,1}), (s_{A,2}, s_{B,2}), ..., (s_{A,n}, s_{B,n})$

3: For $j = 1$ to n

- a. Generate m quantities of the pair (I_A, I_B). Denote the simulated values as $(i_{A,1}, i_{B,1}), (i_{A,2}, i_{B,2}), (i_{A,3}, i_{B,3}), ..., (i_{A,m}, i_{B,m})$
    - i For $i = 1$ to m set
        - $A_i = \alpha s_j + \sqrt{1-\alpha^2} i_{A,i}$
        - $B_i = \beta s_j + \sqrt{1-\beta^2} i_{B,i}$
        - $\mathcal{D}_i = \begin{cases} 1 & \text{if } A_i \leq \Phi^{-1}(PD) \\ 0 & \text{Otherwise} \end{cases}$
        - $\mathcal{PL}_i = H(B_i)$
        - $\mathcal{RL}_i = \mathcal{D}_i \cdot \mathcal{PL}_i$
    - ii End
- b. Set $\mathbb{L}_{p,j} = \sum_{i=1}^{m} \mathcal{RL}_i$

4: End

5: Calculate the mean $\mu_{\mathbb{L}_p}$ and $c-$quantile $Q(c)$ of $\mathbb{L}_p$

- $\mu_{\mathbb{L}_p} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{L}_{p,j}$
- $Q(c) =$ empirical quantile of $\mathbb{L}_p$ at c

6: Calculate Economic capital EC_c

- $EC_c = Q(c) - \mu_{\mathbb{L}_p}$

Table 2.2 links the models presented in the introductory chapter of this document to the general model presented in Section 2.1.1. The definition of $H(\cdot)$ and the restrictions on the respective correlation parameters θ_S and θ_I are highlighted. $H(\cdot)$ in Miu and Odzemir's model is a strictly monotone function of B. $H(\cdot)$ in Frye and Pykhtin's models are expressed as a piecewise function of B, where both models have respectively, a linear and exponential components. The respective models has a general form as follows:

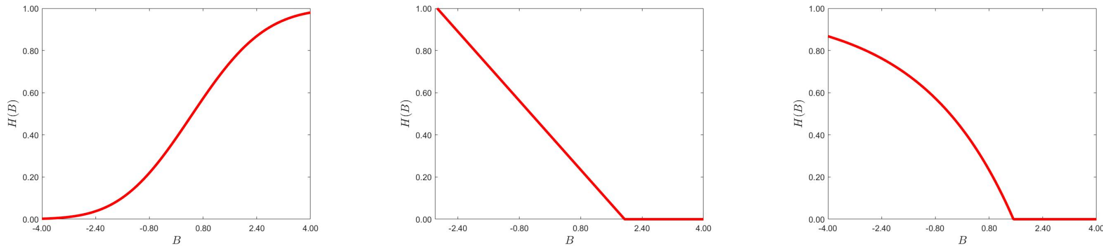
$$H(B) = \begin{cases} 0 & \text{if } B \geq \overline{H}^{-1}(0), \\ \overline{H}(B) & \text{if } B < \overline{H}^{-1}(0), \end{cases} \tag{2.4}$$

where $\overline{H}(\cdot)$ is a strictly monotone function of B. The function $H(\cdot)$ in Frye and Pykhtin's model decreases with respect to B and that of Miu and Odzemir and Witzany is an increasing function of B. Note that $\overline{H}(\cdot)$ is a strictly decreasing function within Frye's and Pykhtin's modelling framework but could be expressed as a strictly increasing function of B.

Figure 2.1 presents the graphs of the respective function $H(\cdot)$ of these models. The parameters used to plot the graphs are chosen such that the means and variances of $\mathcal{PL}$ are the same across models.

Table 2.2: Model specific $H(B)$, θ_S and θ_I. $\mathcal{B}^{-1}$ is the inverse of beta CDF.

Models	θ_S	θ_I	$H(B)$	$\overline{H}(B)$
Miu & Odzemir	arbitrary	arbitrary	$\mathcal{B}^{-1}(\Phi(B), \delta_1, \delta_2)$	NA
Witzany	arbitrary	0	$\mathcal{B}^{-1}(\Phi(B), \delta_1, \delta_2)$	NA
Frye	1	0	$\max(0, 1 - \mu(1 + \sigma B))$	$1 - \mu(1 + \sigma B)$
Pykhtin	1	arbitrary	$\max(0, 1 - \exp(\mu + \sigma B))$	$1 - \exp(\mu + \sigma B)$



(a) $H(\cdot)$ in Miu and Odzemir and Witzany's model: $\delta_1 = 2, \delta_2 = 3$.

(b) $H(\cdot)$ in Frye's model: $\mu = 0.6020$, $\sigma = 0.3310$

(c) $H(\cdot)$ in Pykhtin's model:$\mu = -0.5584$, $\sigma = 0.3660$

Figure 2.1: $H(\cdot)$ in specific models: Miu, Odzemir, Witzanny, Frye, and Pykhtin.

2.2 Conditional Distribution of B given $A \leq \Phi^{-1}(PD)$

Let $PD(b) = \mathsf{P}(A \leq \Phi^{-1}(PD)|B = b)$, the conditional probability of default, given the value of the loss driver. To derive the expression for $\mathsf{P}(A \leq \Phi^{-1}(PD)|B = b)$, consider the following properties of jointly normal random variables (X, Y) with corresponding means (μ_X, μ_Y), variances (σ_X^2, σ_Y^2) and correlation coefficient ρ_{XY} (see [27] for derivation of these properties):

- the conditional expectation of Y given $X = x$ satisfies the relation

$$\mathbb{E}[Y|X = x] = \mu_Y + \rho_{XY}\frac{\sigma_Y}{\sigma_X}(x - \mu_X),$$

- the conditional variance of Y given $X = x$ is governed by

$$\mathrm{Var}[Y|X = x] = \sigma_Y^2(1 - \rho_{XY}^2),$$

  and

- the conditional distribution of Y given $X = x$ therefore satisfies

$$Y|X = x \sim N(\mu_Y + \rho_{XY}\frac{\sigma_Y}{\sigma_X}(x - \mu_X), \sigma_Y^2(1 - \rho_{XY}^2)).$$

Since A and B are bivariate standard normal distributed, by the above properties we have

$$\begin{cases} \mathbb{E}[A|B=b] = \rho_A \cdot b, \\ \mathrm{Var}[A|B=b] = 1-\rho_A^2. \end{cases}$$

So we have

$$\begin{aligned} \mathsf{P}(A \leq \Phi^{-1}(PD)|B=b) &= \mathsf{P}\left(\frac{A-\rho_A \cdot b}{\sqrt{1-\rho_A^2}} \leq \frac{\Phi^{-1}(PD)-\rho_A \cdot b}{\sqrt{1-\rho_{A,B}^2}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(PD)-\rho_A \cdot b}{\sqrt{1-\rho_A^2}}\right). \end{aligned} \tag{2.5}$$

The conditional density of the loss driver B given that default occurred, which is the distribution modelling the probability of observing B when the default driver A falls below the default threshold $\Phi^{-1}(PD)$ is obtained as follows: Let $\widehat{\Phi}(b)$ be conditional cumulative distribution function of B given $A \leq \Phi^{-1}(PD)$ so that we have

$$\widehat{\Phi}(b) = \mathsf{P}(B \leq b | A \leq \Phi^{-1}(PD)) = \frac{\mathsf{P}(B \leq b, A \leq \Phi^{-1}(PD))}{\mathsf{P}(A \leq \Phi^{-1}(PD))}. \tag{2.6}$$

Consider

$$\mathsf{P}(B \leq b, A \leq \Phi^{-1}(PD)) = \int_{-\infty}^{b} \int_{-\infty}^{\Phi^{-1}(PD)} \phi(u,v) du dv,$$

where $\phi(u,v) = \phi(u|v)\phi(v)$ is the joint density of A and B, $\phi(v)$ is the marginal pdf of B and $\phi(u|v)$ is the conditional pdf of A given B. This gives

$$
\begin{aligned}
\mathsf{P}(B \le b, A \le \Phi^{-1}(PD)) &= \int_{-\infty}^{b} \int_{-\infty}^{\Phi^{-1}(PD)} \phi(u|v)\phi(v)dudv \\
&= \int_{-\infty}^{b} \phi(v) \left(\int_{-\infty}^{\Phi^{-1}(PD)} \phi(u|v)du \right) dv \\
&= \int_{-\infty}^{b} \phi(v)\mathsf{P}(A \le \Phi^{-1}(PD)|B = v)dv.
\end{aligned}
$$

Also, since A is standard normally distributed the denominator in Eq. (2.6) is

$$\mathsf{P}(A \le \Phi^{-1}(PD)) = \Phi(\Phi^{-1}(PD)) = PD.$$

Eq. (2.6) can therefore be rewritten as

$$\widehat{\Phi}(b) = \frac{\int_{-\infty}^{b} \phi(v)\mathsf{P}(A \le \Phi^{-1}(PD)|B = v)dv}{PD}.$$

By employing the fundamental theorem of calculus we get

$$\widehat{\phi}(b) = \frac{d}{db}\widehat{\Phi}(b) = \phi(b) \cdot \frac{\mathsf{P}(A \le \Phi^{-1}(PD)|B = b)}{PD},$$

thus from Eq. (2.5)

$$\widehat{\phi}(b) = \phi(b) \cdot \frac{\Phi\left(\frac{\Phi^{-1}(PD)-\rho_A \cdot b}{\sqrt{1-\rho_A^2}}\right)}{PD} = \phi(b) \cdot \frac{PD(b)}{PD}. \tag{2.7}$$

It is interesting to note that values of b which make default more likely have a greater likelihood under this conditional distribution, as might be expected.

Remark 3. *The above suggests that there exist a critical value b^* of b, below or above which the conditional distribution of B given default puts — relative to the unconditional distribution of B — more weight on the distribution of B. The direction* *of the weight is determined by the sign of ρ_A. So for example $\rho_A \leq 0$, there exist b^* above which $\widehat{\phi}(b) \geq \phi(b)$. See Figures 2.2 and 2.3.*

Having the derivation of the conditional distribution of B given $A \leq \Phi^{-1}(PD)$, observe the following theorem:

Theorem 1. *$\widehat{\Phi}(b) \geq \Phi(b)$ if $\rho_A \geq 0$ and conversely, $\widehat{\Phi}(b) \leq \Phi(b)$ if $\rho_A \leq 0$.*

We prove the above theorem by observing the theorem below (see [86], pages 8-12):

Theorem 2. *(Slepian Inequality). Let $X = (X_1, X_2, ..., X_n)$ be mean zero random normal vector with $n \times n$ covariance matrix Σ. Define $\mathbb{S} = (\rho_{i,j})$ and $\mathbb{T} = (\epsilon_{i,j})$ as two positive semidefinite correlation matrices. If $\rho_{i,j} \geq \epsilon_{i,j}$ for all $i, j = 1, 2, ...n$, then*

$$\mathsf{P}_{\Sigma=\mathbb{S}}(X_1 \leq \nu_1, X_2 \leq \nu_2, ..., X_n \leq \nu_n) \geq \mathsf{P}_{\Sigma=\mathbb{T}}(X_1 \leq \nu_1, X_2 \leq \nu_2, ..., X_n \leq \nu_n)$$

holds for all $\nu = (\nu_1, \nu_2, ..., \nu_n)$. Alternatively,

$$\mathsf{P}_{\Sigma=\mathbb{S}}(X_1 \geq \nu_1, X_2 \geq \nu_2, ..., X_n \geq \nu_n) \geq \mathsf{P}_{\Sigma=\mathbb{T}}(X_1 \geq \nu_1, X_2 \geq \nu_2, ..., X_n \geq \nu_n).$$

It follows from Theorem 2 that for bivariate normal random variables (X, Y) with correlation $\rho \leq (\geq)\ 0$ and zero mean vector (see corollary 2 of [86])

$$\mathsf{P}_{\mathbb{S}}(X \leq x, Y \leq y) \leq (\geq) \mathsf{P}(X \leq x) \cdot \mathsf{P}(Y \leq y).$$

In particular, for standard normal variates X and Y (see for example [75]), we have

$$\mathsf{P}(X \leq x, Y \leq y) \leq (\geq) \Phi(x) \cdot \Phi(y).$$

Theorem 2 implies that the quadrant probability $\mathsf{P}(\cdot)$ is a monotonically increasing function of $\rho_{i,j}$ and therefore this is maintained in the impact the co-movements of the components of the random vector X have on their joint probability. Now, we prove Theorem 1 as follows:

Proof. Recall that

$$\widehat{\Phi}(b) = \mathsf{P}(B \leq b | A \leq \Phi^{-1}(PD)) = \frac{\mathsf{P}(B \leq b, A \leq \Phi^{-1}(PD))}{\mathsf{P}(A \leq \Phi^{-1}(PD))}.$$

Case 1. Suppose $\rho_A \geq 0$. From Theorem 2, we have

$$\begin{aligned}
\widehat{\Phi}(b) &= \frac{\mathsf{P}(B \leq b, A \leq \Phi^{-1}(PD))}{\mathsf{P}(A \leq \Phi^{-1}(PD))} \\
&\geq \frac{\mathsf{P}(B \leq b) \cdot \mathsf{P}(A \leq \Phi^{-1}(PD))}{\mathsf{P}(A \leq \Phi^{-1}(PD))} \\
&= \mathsf{P}(B \leq b) = \Phi(b).
\end{aligned}$$

Case 2. Suppose $\rho_A \leq 0$. Theorem 2 implies

$$\begin{aligned}
\widehat{\Phi}(b) &= \frac{\mathsf{P}(B \leq b, A \leq \Phi^{-1}(PD))}{\mathsf{P}(A \leq \Phi^{-1}(PD))} \\
&\leq \frac{\mathsf{P}(B \leq b) \cdot \mathsf{P}(A \leq \Phi^{-1}(PD))}{\mathsf{P}(A \leq \Phi^{-1}(PD))} \\
&= \mathsf{P}(B \leq b) = \Phi(b).
\end{aligned}$$

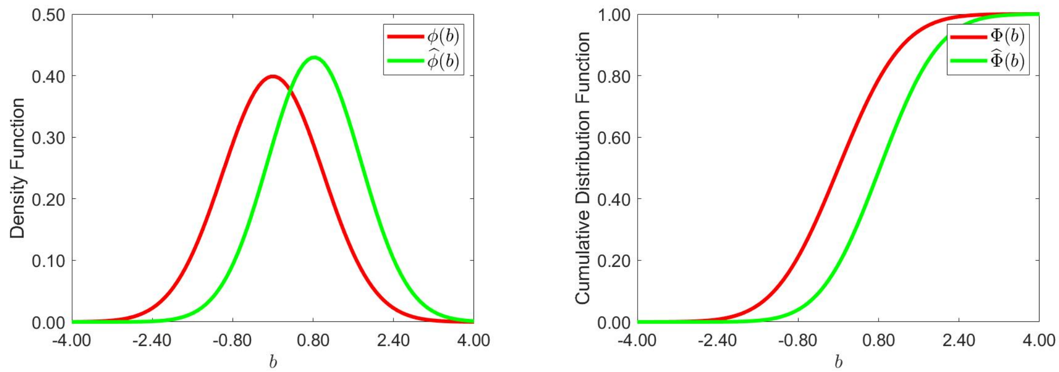
$\square$

2.2.1 Numerical Illustration of Theorem 1 ($\widehat{\Phi}(b)$ verses $\Phi(b)$)

We confirm Theorem 1 by comparing the unconditional and conditional distribution of the loss driver B given that $A \leq \Phi^{-1}(PD)$ under the respective correlation conditions ($\rho_A \geq 0$ and $\rho_A \leq 0$). The figures are produced by arbitrarily specifying a range of values of b from -4 to 4. Parameter values are specified as $PD = 0.05, \rho_A = -0.4$ and 0.4.

Figure 2.2 compares the density functions of the unconditional (red curve) and conditional (green) loss drivers B given default. The specification of $\widehat{\phi}(b)$ assumes a negative correlation ($\rho_A = -0.4$) between the loss driver B and the default driver A. The graphs show that $\phi(b)$ and $\widehat{\phi}(b)$ are different. This difference – the graph of $\widehat{\phi}(b)$ (green curve) overlapping to the right of the graph of $\phi(b)$ (red curve) – stems from the specification of the respective density functions. The inverse relationship between A and B forces the graph of $\widehat{\phi}(b)$ to overlap to the right of the graph of $\phi(b)$. As a result of this phenomenon the graph of the respective cumulative distribution functions of the unconditional and conditional loss drivers given default shows similar pattern where the graph of the cumulative distribution function of the conditional loss drivers given default (green curve) is to the right of the graph of the cumulative distribution function of the unconditional loss drivers as depicted in Figure 2.2b.

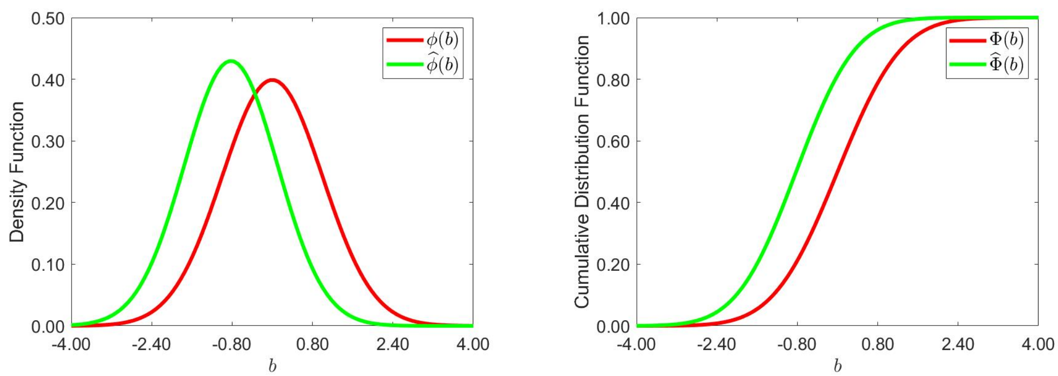
Figure 2.3 compares the conditional and unconditional density and cdf plots when default occurs for positive correlation between default and loss drivers ($\rho_A = 0.4$) — see Figures 2.3a-2.3b. The figure shows reverse of the patterns observed in the case for negative correlation between default and loss drivers — we observe that the conditional density curve of the loss driver given default overlaps to the left of the unconditional density curve of the loss driver. This pattern is reflected in the respective cdfs involving the conditional and unconditional loss drivers when default occurs, as expected from Theorem 1.



(a) Unconditional and conditional density function of B, given default occurred

(b) Unconditional and conditional cumulative distribution function of B, given default occurred

Figure 2.2: Distribution of unconditional and conditional loss driver, given default. $PD = 0.05, \rho_A = -0.4$.



(a) Unconditional and conditional density function of B, given default occurred

(b) Unconditional and conditional cumulative distribution function of B, given default occurred

Figure 2.3: Distribution of unconditional and conditional loss driver, given default. $PD = 0.05, \rho_A = 0.4$.

2.3 Account-Level Potential Loss and Loss Given Default (LGD) Distribution

It is important to relate model quantities to observable quantities. In order to think about observable quantities let us imagine we have a giant spreadsheet. In one column we have the default indicator telling us whether the account defaulted or not. In another column we have (i) blank cells in the non-default rows and (ii) numbers in the default rows.

The blank cells correspond to unobserved potential losses, the non-empty cells correspond to observed potential losses. So we only observe potential loss if its value is “switched on” by a default.

Remark 4. *The non-empty cells therefore contain observations drawn from the conditional distribution of $\mathcal{PL}$, given that default occurred ($\mathcal{D} = 1$). They do not contain observations from the marginal (unconditional) distribution of $\mathcal{PL}$. This means that we cannot use account-level loss given default data to estimate parameters of the unconditional distribution of $\mathcal{PL}$, because our data did not come from that distribution.*

Definition 5. *Account-level potential loss ($\mathcal{PL}$) distribution is denoted as $f_{\mathcal{PL}}$ and is defined as the unconditional distribution of $\mathcal{PL}$.*

Definition 6. *Account-level loss given default (LGD) distribution is denoted as f_{LGD} and is defined as the conditional distribution of $\mathcal{PL}$ given that $\mathcal{D} = 1$.*

The derivation of the distribution of the account-level $\mathcal{PL}$ and LGD under the general model and that of the specific models — Frye [51], Miu & Odzemir [74], Pykhtin [78] and Witzany [92] — are presented in the ensuing sections.

2.3.1 Account-Level $\mathcal{PL}$ Distribution under General Model

Define $F_{\mathcal{PL}} = \mathsf{P}(\mathcal{PL} \leq \ell)$. We derive $F_{\mathcal{PL}}$ and its density as follows: Suppose $f_{\mathcal{PL}}(\ell) = \frac{d}{d\ell}F_{\mathcal{PL}}(\ell)$ and assume $H(\cdot)$ is invertible and is an increasing function of B, then the marginal density of $\mathcal{PL}$ is $f_{\mathcal{PL}}$. We have the derived distribution as

$$\begin{aligned}\mathsf{P}(\mathcal{PL} \leq \ell) &= \mathsf{P}(H(B) \leq \ell) \\ &= \mathsf{P}(B \leq H^{-1}(\ell)) \\ &= \Phi(H^{-1}(\ell)).\end{aligned}$$

We therefore have

$$f_{\mathcal{PL}}(\ell) = \phi(H^{-1}(\ell)) \cdot \frac{d}{d\ell} H^{-1}(\ell), \tag{2.8}$$

where ϕ is the standard normal probability density function (pdf). If $H(\cdot)$ is a decreasing function of B, we have

$$\mathsf{P}(\mathcal{PL} \leq \ell) = 1 - \Phi(H^{-1}(\ell)),$$

which results in

$$f_{\mathcal{PL}}(\ell) = -\phi(H^{-1}(\ell)) \cdot \frac{d}{d\ell} H^{-1}(\ell).$$

For the case where $H(\cdot)$ is not strictly monotone — such as presented in Frye and Pykhtin's model — Eq. (2.4). The distribution takes a different form: The probability of $\mathcal{PL} = 0$ is given as

$$\mathsf{P}(\mathcal{PL} = 0) = \mathsf{P}(H(B) = 0) = \mathsf{P}(B \geq \overline{H}^{-1}(0)) = 1 - \Phi(\overline{H}^{-1}(0)) = \Phi(-\overline{H}^{-1}(0)).$$

And probability of $\mathcal{PL} \leq \ell$ given that $\mathcal{PL} > 0$ is derived as

$$
\begin{aligned}
F_{\mathcal{PL}}(\ell) = \mathsf{P}(\mathcal{PL} \leq \ell | \mathcal{PL} > 0) &= \frac{\mathsf{P}(\overline{H}(B) \leq \ell, \overline{H}(B) > 0)}{\mathsf{P}(\overline{H}(B) > 0)} \\
&= \frac{\mathsf{P}(\overline{H}^{-1}(\ell)) \leq B < \overline{H}^{-1}(0))}{\mathsf{P}(B < \overline{H}^{-1}(0))} \\
&= 1 - \frac{\Phi(\overline{H}^{-1}(\ell))}{\Phi(\overline{H}^{-1}(0))}.
\end{aligned}
$$

The resulting conditional pdf of $\mathcal{PL} \leq \ell$ given that $\mathcal{PL} > 0$ is

$$
f_{\mathcal{PL}}(\ell) = -\frac{\phi(\overline{H}^{-1}(\ell))}{\Phi(\overline{H}^{-1}(0))} \cdot \frac{d}{d\ell}\overline{H}^{-1}(\ell). \tag{2.9}
$$

Note that if $\overline{H}(B)$ is invertible and an increasing function of B, then

$$
f_{\mathcal{PL}}(\ell) = \frac{\phi(\overline{H}^{-1}(\ell))}{\Phi(-\overline{H}^{-1}(0))} \cdot \frac{d}{d\ell}\overline{H}^{-1}(\ell). \tag{2.10}
$$

2.3.2 Account-Level LGD Distribution under General Model

Since $\mathcal{PL}$ is driven by B, it implies that the underlying mechanism in obtaining loss given default (or observed $\mathcal{PL}$) is dependent on the specification of the conditionally realized B given that default occurred ($A \leq \Phi^{-1}(PD)$). By this, the distribution of loss given that default occurred is in part driven by the conditional distribution of B given $A \leq \Phi^{-1}(PD)$.

Suppose $H(\cdot)$ invertible and increases with respect to B. We obtain f_{LGD} as follows: Let $F_{LGD}(\ell) = \mathsf{P}(\mathcal{PL} \leq \ell | \mathcal{D} = 1)$ then we have

$$
\mathsf{P}(\mathcal{PL} \leq \ell | \mathcal{D} = 1) = \mathsf{P}(B \leq H^{-1}(\ell) | A \leq \Phi^{-1}(PD)) = \widehat{\Phi}(H^{-1}(\ell)).
$$

Similarly, let $f_{LGD}(\ell) = \frac{d}{d\ell} F_{LGD}(\ell)$ so that we have

$$f_{LGD}(\ell) = \frac{d}{d\ell} F_{LGD}(\ell) = \widehat{\phi}(H^{-1}(\ell)) \cdot \frac{d}{d\ell} H^{-1}(\ell). \tag{2.11}$$

By rearranging Eq. (2.8), we get

$$\frac{d}{d\ell} H^{-1}(\ell) = \frac{f_{\mathcal{PL}}(\ell)}{\phi(H^{-1}(\ell))}. \tag{2.12}$$

Using Eq. (2.11) and (2.12)

$$f_{LGD}(\ell) = f_{\mathcal{PL}}(\ell) \cdot \frac{\widehat{\phi}(H^{-1}(\ell))}{\phi(H^{-1}(\ell))}.$$

By making use of Eq. (2.7) and observing that $b = H^{-1}(\ell)$,

$$f_{LGD}(\ell) = f_{\mathcal{PL}}(\ell) \cdot \frac{PD(H^{-1}(\ell))}{PD}. \tag{2.13}$$

Observe $PD(H^{-1}(\ell)) = \mathsf{P}(A \leq \Phi^{-1}(PD) | B = H^{-1}(\ell))$. So for values of ℓ that make default more likely, $PD(H^{-1}(\ell)) > PD$. This leads to a critical value ℓ^* of ℓ, below or above which the account-level LGD distribution put less or more weight on the loss distribution compared to that of account-level $\mathcal{PL}$ distribution. This critical value is obtained as — see Eq. (2.7)

$$\ell^* = H\left(\frac{\Phi^{-1}(PD)\left(1 - \sqrt{1-\rho_A^2}\right)}{\rho_A} \right).$$

Now, $PD(H^{-1}(\ell)) > PD$ implies

$$\rho_A H^{-1}(\ell) < \Phi^{-1}(PD)\left(1 - \sqrt{1-\rho_A^2}\right).$$

Suppose $\rho_A > 0$,

$$H^{-1}(\ell) < \frac{\Phi^{-1}(PD)\left(1 - \sqrt{1 - \rho_A^2}\right)}{\rho_A}.$$

In this case, if $H(\cdot)$ is an increasing (decreasing) function of B, then $\ell < (>)\ell^*$. Suppose $\rho_A < 0$, then we have

$$H^{-1}(\ell) > \frac{\Phi^{-1}(PD)\left(1 - \sqrt{1 - \rho_A^2}\right)}{\rho_A},$$

thus $H(\cdot)$ an increasing (decreasing) function of B, implies $\ell > (<)\ell^*$. The above inequalities imply the following:

Cases where $H(\cdot)$ is expressed as strictly increasing function of B and positive correlation between default and $\mathcal{PL}$ is imposed — negative ρ_A — if $\ell > \ell^*$ then f_{LGD} will assign more weight to the account-level loss distribution than $f_{\mathcal{PL}}$ does. This also holds for cases where $H(\cdot)$ is expressed as a strictly decreasing function of B and positive correlation between default and $\mathcal{PL}$ is imposed — positive ρ_A. "Bad" values of ℓ correspond to high values of b.

Conversely, suppose negative correlation between default and $\mathcal{PL}$ is imposed, coupled with strictly increasing $H(\cdot)$ as B increases — positive ρ_A. In this case if $\ell < \ell^*$ then "bad" ℓ corresponds to small values of b and default is more likely — f_{LGD} in this case assigns more weight to the account-level loss distribution than that of $f_{\mathcal{PL}}$. This also holds when $H(\cdot)$ strictly decreases with an increasing B and correlation between default and $\mathcal{PL}$ assumes a negative value— negative ρ_A.

Remark 5. *$f_{LGD} = f_{\mathcal{PL}}$ if and only if $\rho_A = 0$.*

Remark 6. *The assumption that positive correlation between $\mathcal{PL}$ and $\mathcal{D}$ implies that if $\rho_A \neq 0$, $f_{LGD} \neq f_{\mathcal{PL}}$ and $f_{LGD} > f_{\mathcal{PL}}$ for all values of $\ell > \ell^*$. Thus, LGD distribution attaches higher probability to large losses relative to that of $\mathcal{PL}$ distribution. This* *is illustrated in Figures 2.4- 2.6, where $\ell > \ell^*$ (value of ℓ at the intersection point of the graph of f_{LGD} (green curve) and $f_{\mathcal{PL}}$ (red curve)) depicts heavier tail distribution of LGD than that of $\mathcal{PL}$.*

Suppose $H(\cdot)$ is not a strictly monotone function of B and assumes the form in Eq. (2.4), then we have the probability of $\mathcal{PL} = 0$ given that $\mathcal{D} = 1$ as

$$F_{LGD}(0) = \mathsf{P}(\mathcal{PL} = 0|\mathcal{D} = 1) = \mathsf{P}(B \geq \overline{H}^{-1}(0)|\mathcal{D} = 1) = 1 - \widehat{\Phi}(\overline{H}^{-1}(0)),$$

and if $\overline{H}(\cdot)$ is an invertible decreasing function of B, the probability of $\mathcal{PL} \leq \ell$ given that $\mathcal{D} = 1$ and $\mathcal{PL} > 0$,

$$\begin{aligned} F_{LGD}(\ell) = \mathsf{P}(\mathcal{PL} \leq \ell|\mathcal{D} = 1, \mathcal{PL} > 0) &= \mathsf{P}(\overline{H}(B) \leq \ell|\mathcal{D} = 1, \overline{H}(B) > 0) \\ &= \frac{\mathsf{P}(\overline{H}^{-1}(\ell) \leq B < \overline{H}^{-1}(0)|A \leq \Phi^{-1}(PD))}{\mathsf{P}(B < \overline{H}^{-1}(0)|A \leq \Phi^{-1}(PD))} \\ &= 1 - \frac{\widehat{\Phi}(\overline{H}^{-1}(\ell))}{\widehat{\Phi}(\overline{H}^{-1}(0))}, \end{aligned}$$

which from Eq. (2.9) and (2.7) yields the conditional pdf of $\mathcal{PL} \leq \ell$ given that $\mathcal{D} = 1$ and $\mathcal{PL} > 0$ as

$$\begin{aligned} f_{LGD}(\ell) &= f_{\mathcal{PL}}(\ell) \cdot \frac{\widehat{\phi}(\overline{H}^{-1}(\ell))}{\phi(\overline{H}^{-1}(\ell))} \cdot \frac{\Phi(\overline{H}^{-1}(0))}{\widehat{\Phi}(\overline{H}^{-1}(0))} \\ &= f_{\mathcal{PL}}(\ell) \cdot \frac{PD(\overline{H}^{-1}(\ell))}{PD} \cdot \frac{\Phi(\overline{H}^{-1}(0))}{\widehat{\Phi}(\overline{H}^{-1}(0))}. \end{aligned} \tag{2.14}$$

If $\overline{H}(\cdot)$ is a strictly increasing function of B, then from Eq. (2.10) and (2.7) the conditional pdf of $\mathcal{PL} \leq \ell$ given that $\mathcal{D} = 1$ and $\mathcal{PL} > 0$ is derived as

$$f_{LGD} = f_{\mathcal{PL}}(\ell) \cdot \frac{PD(\overline{H}^{-1}(\ell))}{PD} \cdot \frac{\Phi(-\overline{H}^{-1}(0))}{\widehat{\Phi}(-\overline{H}^{-1}(0))}.$$

Finally, for $-\infty \leq \ell < 0$, the unconditional and conditional cdf of $\mathcal{PL}$, given default occurred are respectively zero. That is $\mathsf{P}(-\infty \leq \mathcal{PL} < 0) = \mathsf{P}(-\infty \leq \mathcal{PL} < 0 | A \leq \Phi^{-1}(PD)) = 0$.

2.3.3 Distribution of Account-Level $\mathcal{PL}$ and LGD under Specific Models

Miu and Odzemir's Model

Noting that $H^{-1}(\ell)$ under Miu and Odzemir's modelling framework is

$$H^{-1}(\ell) = \Phi^{-1}(\mathcal{B}(\ell), \delta_1, \delta_2),$$

where $\mathcal{B}$ denotes the beta cumulative distribution function with shape parameters δ_1 and δ_2, it follows that

$$\frac{d}{d\ell} H^{-1}(\ell) = \frac{\beta(\ell, \delta_1, \delta_2)}{\phi(\Phi^{-1}(\mathcal{B}(\ell, \delta_1, \delta_2)))},$$

where $\beta(\cdot)$ is the beta probability density function. From Eq. (2.8), the unconditional density of $\mathcal{PL}$ is the beta density function and is written as

$$f_{\mathcal{PL}}(\ell) = \frac{\Gamma(\delta_1 + \delta_2) \ell^{\delta_1 - 1} (1 - \ell)^{\delta_2 - 1}}{\Gamma(\delta_1)\Gamma(\delta_2)}. \tag{2.15}$$

Where $\Gamma(\cdot)$ is the gamma function. It follows that the expected value and variance of potential losses under this distribution are respectively expressed as

$$\begin{cases} \mathbb{E}[\mathcal{PL}] = \frac{\delta_1}{\delta_1+\delta_2}, \\ \mathrm{Var}[\mathcal{PL}] = \frac{\delta_1\delta_2}{(\delta_1+\delta_2)^2(\delta_1+\delta_2+1)}. \end{cases}$$

Making use of Eq. (2.13), the conditional density of $\mathcal{PL}$ given that default occurred is derived as

$$f_{LGD}(\ell) = f_{\mathcal{PL}}(\ell) \cdot \frac{PD(\Phi^{-1}(\mathcal{B}(\ell, \delta_1, \delta_2))}{PD}. \tag{2.16}$$

Figure 2.4 shows the density plots of potential loss and LGD for Miu and Ozdemir's model. The values for the correlation between default A and loss B drivers is assumed negative ($\rho_A = -0.4$). This is to impose a positive correlation between default and loss. $PD = 0.05, \delta_1 = 2, \delta_2 = 3$. The figure shows that the unconditional and conditional distribution of portfolio loss given that default occurred are different.

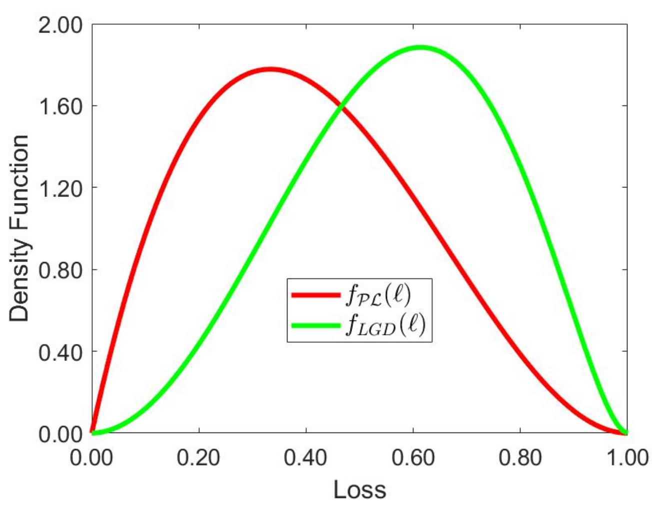


Figure 2.4: Miu and Ozdemir's model: Unconditional and conditional densities. $\rho_A = -0.4, \delta_1 = 2, \delta_2 = 3, PD = 0.05$.

Since the specification of potential loss within Witzany's modelling framework is the same as that of Miu and Odzemir's model, the densities are described by Eq. (2.15) and (2.16).

Frye's Model

From Table 2.2, the account-level $\mathcal{PL}$ distribution is derived as follows: Observe that

$$\overline{H}^{-1}(\ell) = \begin{cases} \frac{1-\ell-\mu}{\mu\sigma} & \text{if } \ell > 0, \\ \frac{1-\mu}{\mu\sigma} & \text{if } \ell = 0, \end{cases}$$

and

$$\frac{d}{d\ell}\overline{H}^{-1}(\ell) = -\frac{1}{\mu\sigma}.$$

For $\mathcal{PL} = 0$

$$\mathsf{P}(\mathcal{PL} = 0) = \mathsf{P}\left(B \geq \overline{H}^{-1}(0)\right) = \Phi\left(\frac{\mu - 1}{\mu\sigma}\right).$$

And from Eq. (2.9), the condtitional pdf of $\mathcal{PL} \leq \ell$ given that $\mathcal{PL} > 0$ is

$$f_{\mathcal{PL}}(\ell) = \frac{1}{\mu\sigma} \cdot \frac{\phi\left(\frac{1-\mu-\ell}{\mu\sigma}\right)}{\Phi\left(\frac{1-\mu}{\mu\sigma}\right)}.$$

The account-level LGD distribution is obtained as follows: The conditional probability of $\mathcal{PL} = 0$ given $\mathcal{D} = 1$ is

$$\mathsf{P}(\mathcal{PL} = 0|\mathcal{D} = 1) = 1 - \widehat{\Phi}\left(\frac{1-\mu}{\mu\sigma}\right),$$

and from Eq. (2.14) the conditional density of $\mathcal{PL}$ given $\mathcal{D} = 1$ for positive values of $\mathcal{PL}$ is

$$f_{LGD}(\ell) = \frac{1}{\mu\sigma \cdot PD} \cdot \frac{\phi\left(\frac{1-\ell-\mu}{\mu\sigma}\right)}{\widehat{\Phi}\left(\frac{1-\mu}{\mu\sigma}\right)} \cdot PD\left(\frac{1-\mu-\ell}{\mu\sigma}\right).$$

Having the expressions for the respective densities (conditional and unconditional) of $\mathcal{PL}$, we derive the first moment of $\mathcal{PL}$ by first observing that from Table 2.2, we have

$$\begin{aligned}
\mathbb{E}[\mathcal{PL}] &= \mathbb{E}[(1-(\mu+\mu\sigma B))\mathbb{1}_{\{\mu+\mu\sigma B\le 1\}}] \\
&= \mathbb{E}\left[(1-(\mu+\mu\sigma B))\mathbb{1}_{\left\{B\le\frac{1-\mu}{\mu\sigma}\right\}}\right] \\
&= \int_{-\infty}^{\frac{1-\mu}{\mu\sigma}} ((1-\mu)-\mu\sigma b)\phi(b)db \\
&= (1-\mu)\Phi\left(\frac{1-\mu}{\mu\sigma}\right)+\mu\sigma\phi\left(\frac{1-\mu}{\mu\sigma}\right).
\end{aligned} \tag{2.17}$$

and the second moment[2],

$$\begin{aligned}
\mathbb{E}[\mathcal{PL}^2] &= \mathbb{E}\left[\left(1-(\mu+\mu\sigma B)\right)^2\right)\mathbb{1}_{\{\mu+\mu\sigma B\le 1\}}] \\
&= \mathbb{E}\left[\left(1-(\mu+\mu\sigma B)\right)^2\right)\mathbb{1}_{\left\{B\le\frac{1-\mu}{\mu\sigma}\right\}}\right] \\
&= (1-2\mu+\mu^2)\Phi\left(\frac{1-\mu}{\mu\sigma}\right)+2\mu\sigma(1-\mu)\phi\left(\frac{1-\mu}{\mu\sigma}\right) \\
&\quad+(\mu\sigma)^2\left[\Phi\left(\frac{1-\mu}{\mu\sigma}\right)-\frac{1-\mu}{\mu\sigma}\phi\left(\frac{1-\mu}{\mu\sigma}\right)\right].
\end{aligned}$$

The variance is then obtained as

$$\mathrm{Var}(\mathcal{PL}) = \mathbb{E}[\mathcal{PL}^2]-\left(\mathbb{E}[\mathcal{PL}]\right)^2.$$

We use these quantities, $\mathrm{Var}(\mathcal{PL})$ and $\mathbb{E}[\mathcal{PL}]$, as the base for comparing the densities across the models presented in this section.

Figure 2.5 shows the graphs of unconditional (red curves) and conditional (green curves) density function of potential loss given that default occurred. The figure indicates that the distribution between the unconditional and conditional potential

[2] We used the relation $\frac{d\phi(b)}{db}=-b\phi(b)$ and $\frac{d^2\phi(b)}{db^2}=(b^2-1)\phi(b)$.

loss given default are different. Since $H(B)$ is a decreasing function of B, to impose a positive correlation between default and potential loss, we chose positive value of ρ_A — $\rho_A = 0.4$. For comparison purposes, the density curves are obtained by estimating the parameters μ and σ by assuming that the respective expected value and variance of the potential losses within the framework of Frye's model are equal to the respective expected value and variance of the potential loss obtained from Miu and Ozdemir's model — $\mu = 0.6020, \sigma = 0.3310$, and $PD = 0.05$.

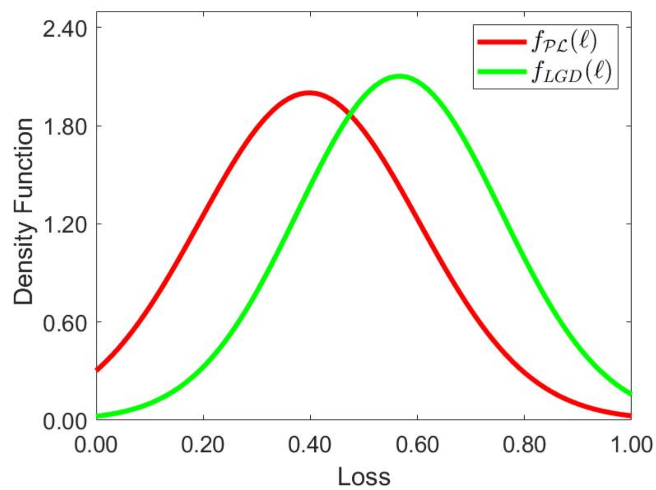


Figure 2.5: Frye's Model: Unconditional and conditional densities. $\mathsf{P}(\mathcal{PL} = 0) = 0.0259, \mathsf{P}(\mathcal{PL} = 0|\mathcal{D} = 1) = 0.0014$. $\mu = 0.6020, \sigma = 0.3310, \rho_A = 0.4, PD = 0.05$.

Pykhtin's Model

As in the case of Frye's model, Pykhtin' model takes the form of Eq. (2.4). We obtain the following from Table 2.2:

$$\overline{H}^{-1}(\ell) = \begin{cases} \frac{\ln(1-\ell)-\mu}{\sigma} & \text{if } \ell > 0, \\ -\frac{\mu}{\sigma} & \text{if } \ell = 0, \end{cases}$$

and

$$\frac{d}{d\ell}\overline{H}^{-1}(\ell) = -\frac{1}{(1-\ell)\sigma}.$$

The unconditional and conditional $\mathcal{PL}$ given $\mathcal{D} = 1$ are as follows: When $\mathcal{PL} = 0$,

$$\begin{cases} \mathsf{P}(\mathcal{PL} = 0) = \Phi\left(\frac{\mu}{\sigma}\right), \\ \mathsf{P}(\mathcal{PL} = 0 | \mathcal{D} = 1) = 1 - \widehat{\Phi}\left(-\frac{\mu}{\sigma}\right), \end{cases}$$

and for $\mathcal{PL} \leq \ell$, where $\ell > 0$

$$\begin{cases} f_{\mathcal{PL}}(\ell) = \frac{1}{(1-\ell)\cdot\sigma} \cdot \frac{\phi\left(\frac{\ln(1-\ell)-\mu}{\sigma}\right)}{\Phi\left(-\frac{\mu}{\sigma}\right)}, \\ f_{LGD}(\ell) = \frac{1}{(1-\ell)\sigma PD} \cdot \frac{\phi\left(\frac{\ln(1-\ell)-\mu}{\sigma}\right)}{\widehat{\Phi}\left(-\frac{\mu}{\sigma}\right)} \cdot PD\left(\frac{\ln(1-\ell)-\mu}{\sigma}\right). \end{cases}$$

Finally, $f_{\mathcal{PL}}(\ell) = f_{LGD}(\ell) = 0$ for $\ell < 0$.

The first and the second moments of potential loss under this setting are derived as

$$\begin{aligned} \mathbb{E}[\mathcal{PL}] &= \mathbb{E}[0 \cdot 1_{\{\overline{H}>1\}} + (1-\overline{H}) \cdot 1_{\{\overline{H}\leq 1\}}] \\ &= \mathbb{E}\, 1_{\{\overline{H}\leq 1\}} - \mathbb{E}\,\overline{H} \cdot 1_{\{\overline{H}\leq 1\}} \\ &= \mathbb{E}\, 1_{\{\exp(\mu+\sigma B)\leq 1\}} - \mathbb{E}\exp(\mu+\sigma B) \cdot 1_{\{\exp(\mu+\sigma B)\leq 1\}}, \end{aligned}$$

and making use of completing the square yields

$$E[\mathcal{PL}] = \Phi\left(-\frac{\mu}{\sigma}\right) - \exp\left(\frac{1}{2}(2\mu+\sigma^2)\right)\Phi\left(-\frac{\mu}{\sigma}, \sigma, 1\right)$$

and a similar approach gives

$$\mathbb{E}[\mathcal{PL}^2] = \Phi\left(-\frac{\mu}{\sigma}\right) - 2\exp\left(\frac{1}{2}(2\mu+\sigma^2)\right)\Phi\left(-\frac{\mu}{\sigma}, \sigma, 1\right) + \exp\left(2\left(\mu+\sigma^2\right)\right)\Phi\left(-\frac{\mu}{\sigma}, 2\sigma, 1\right)$$

respectively.

Figure 2.6 compares the graphs of the conditional and unconditional densities for the potential loss, given that default occurred. The graphs are obtained by first estimating the parameters μ and σ such that the first and second moments are equal to that obtained under Miu and Odzemir's setting. This is to guarantee a uniform comparison of the density curves across all the models. The graph shows similar pattern as in the case for Frye's model — distribution of the unconditional and conditional potential loss, given default are different.

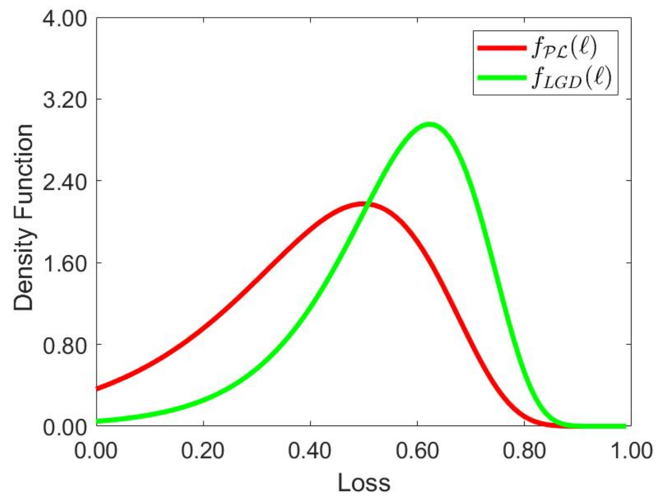


Figure 2.6: Pykhtin's Model: Unconditional and conditional densities. $\mathsf{P}(\mathcal{PL} = 0) = 0.0635, \mathsf{P}(\mathcal{PL} = 0|\mathcal{D} = 1) = 0.0056$. $\mu = -0.5584, \sigma = 0.3660, \rho_A = 0.4,$ and $PD = 0.05$.

Summary of Distribution Functions under Specific Models

Given the above background, Table 2.3 presents expressions for the pdfs linked to potential loss described in the above discussed four models.

Table 2.3: The expressions for the pdfs of potential loss linked to respective model under discussion

Models	Functions			
	$f_{\mathcal{PL}}(\ell)$		$f_{LGD}(\ell)$	
Miu & Odzemir's (Witzany's) Model	$\frac{\Gamma(\delta_1+\delta_2)\ell^{\delta_1-1}(1-\ell)^{\delta_2-1}}{\Gamma(\delta_1)\Gamma(\delta_2)}$		$\frac{\Gamma(\delta_1+\delta_2)\ell^{\delta_1-1}(1-\ell)^{\delta_2-1}}{\Gamma(\delta_1)\Gamma(\delta_2)} \cdot \frac{PD(\Phi^{-1}(\mathcal{B}(\ell,\delta_1,\delta_2))}{PD}$	
	Functions			
	$f_{\mathcal{PL}}(\ell)$		$f_{LGD}(\ell)$	
	$\mathcal{PL} \leq \ell$ given $\mathcal{PL} > 0$	$\mathcal{PL} = 0$	$\mathcal{PL} \leq \ell$ given $\mathcal{PL} > 0$	$\mathcal{PL} = 0$
Frye's Model	$\frac{1}{\mu\sigma} \cdot \frac{\phi\left(\frac{1-\ell-\mu}{\mu\sigma}\right)}{\Phi\left(\frac{1-\mu}{\mu\sigma}\right)}$	$\Phi\left(\frac{\mu-1}{\mu\sigma}\right)$	$\frac{1}{\mu\sigma PD} \frac{\phi\left(\frac{1-\ell-\mu}{\mu\sigma}\right)}{\widehat{\Phi}\left(\frac{1-\mu}{\mu\sigma}\right)} \cdot PD\left(\frac{1-\mu-\ell}{\mu\sigma}\right)$	$1 - \widehat{\Phi}\left(\frac{1-\mu}{\mu\sigma}\right)$
Pykhtin's Model	$\frac{1}{(1-\ell)\sigma PD} \frac{\phi\left(\frac{\ln(1-\ell)-\mu}{\sigma}\right)}{\Phi\left(\frac{-\mu}{\sigma}\right)}$	$\Phi\left(\frac{\mu}{\sigma}\right)$	$\frac{1}{(1-\ell)\sigma PD} \cdot \frac{\phi\left(\frac{\ln(1-\ell)-\mu}{\sigma}\right)}{\Phi\left(\frac{-\mu}{\sigma}\right)} \cdot PD\left(\frac{\ln(1-\ell)-\mu}{\sigma}\right)$	$1 - \widehat{\Phi}\left(\frac{-\mu}{\sigma}\right)$

2.4 Revealing Defect and Corrective Approach

The preceding sections present discussions on the respective distributions of account-level potential loss and loss given default, where these distributions are linked to specific models in the literature — (i) Frye, (ii) Miu and Ozdemir, (iii) Pykhtin and (iv) Witzany. We demonstrated the respective distributions are different.

Recall the account-level loss variable $\mathbb{L} = EAD \cdot LGD \cdot \mathcal{D}$. Empirical evidence suggests existence of correlation between the constituents of $\mathbb{L}$ [6, 7]. This implies that modelling and estimation schemes for these quantities should account for these correlations. The literature on credit risk has many models that capture this phenomenon — see for example, [44, 74, 91]. However, some proposed models and estimation schemes in the current literature implicitly assume independency of loss and default drivers in defining (or modelling) account-level LGD. By definition, LGD is defined as conditional loss given that default occurred, thus imposing dependency on loss and default drivers. The current literature assumes a definition of $\mathbb{L}$ that constitutes unconditional loss $\mathcal{PL}$ instead of LGD:

$$\mathbb{L} = EAD \cdot \mathcal{PL} \cdot \mathcal{D}, \tag{2.18}$$

where $\mathcal{PL}$ is a percentage value in this case. This leads to problems in calculating regulatory and economic capital: (1) If one assumes that the underlying assumptions in the derivation of the regulatory capital Eq. (1.1) holds, then estimates using Eq. (2.18) as input maybe misleading. (2) Addressing the dependency problem in Eq. (1.1) as a way of meeting the provision in paragraph 468 of [20], will be done via the triplet $(EAD, \mathcal{PL}, \mathcal{D})$ instead of $(EAD, LGD, \mathcal{D})$. (3) Since the current literature is concerned with parameter estimations that are ultimately used in finding an approximate distribution for account-level LGD, under the current framework, there is the problem of a mismatch of data and the targeted quantity to be estimated — LGD data is used to estimate potential loss distribution.

Our goal in this section is to explain the defect in the existing PD-LGD correlation models — which is the target distribution for the account-level loss given default is not what we think it is. We correct this defect by proposing a model that gives the correct distribution.

2.4.1 Explaining Defect

Several papers used the transformation — see for instance [44, 74, 83, 91, 92]

$$H(b) = \mathcal{B}^{-1}(\Phi(b), \delta_1, \delta_2), \tag{2.19}$$

where $\mathcal{B}$ is the cdf of beta distribution with scale parameters δ_1 and δ_2, which are the unknown parameters to be estimated. $\mathcal{B}^{-1}$ is the inverse cdf, δ_1 and δ_2 are to ensure that the model-implied and the observed LGD distributions match each other. The aforementioned papers used the observed LGD data to estimate the pair (δ_1, δ_2) by solving the equations

$$\begin{cases} \mu_{LGD} = \frac{\delta_1}{\delta_1+\delta 2}, \\ \sigma^2_{LGD} = \frac{\delta_1\delta_2}{(\delta_1+\delta_2+1)(\delta_1+\delta_2)^2}, \end{cases} \tag{2.20}$$

where μ_{LGD} and σ^2_{LGD} are the mean and variance of observed LGD.

If it is true that the account-level LGD distribution is beta, the proposed estimation scheme would have been the perfect way to estimate these parameters. Unfortunately, it is easy to see — from Eq. (2.8) and (2.13) —that the $\mathcal{PL}$ and LGD distributions are given by

$$\begin{cases} f_{\mathcal{PL}}(\ell) = \beta(\ell, \delta_1, \delta_2), \\ f_{LGD}(\ell) = \beta(\ell, \delta_1, \delta_2) \cdot \frac{PD(\Phi^{-1}(\mathcal{B}(\ell,\delta_1,\delta_2))}{PD}. \end{cases}$$

If $\rho_A \neq 0$, which is observed in general, then the observed account-level LGD does not follow a beta distribution and it is therefore erroneous to use LGD data as input in Eq. (2.20) to estimate δ_1 and δ_2 — a modelling error that has not been addressed in the literature.

The papers mentioned above do not make a distinction between account-level LGD and potential loss and we believe is the source of the modelling error. The ensuing section investigates the severity of this modelling error on parameter estimates and economic capital.

Before proceeding, we will give a general description of this error. Generally, parameter estimation involves the following: (1) Select a parametric family for the account-level LGD distribution $\{f_\psi : \psi \in \Psi\}$, where Ψ is the parameter space. (2) Set loss $H_\psi(b) = F_\psi^{-1}(\Phi(b))$, where F_ψ is the cdf of f_ψ and F_ψ^{-1} is its inverse. (3) Use LGD data to estimate parameter ψ. This approach would be correct if the LGD data came from f_ψ. Unfortunately, it is clear — from Eq. (2.8) and (2.13) — that the $\mathcal{PL}$ and LGD distributions take the form

$$\begin{cases} f_{\mathcal{PL}}(\ell) = f_\psi(\ell), \\ f_{LGD}(\ell) = f_\psi(\ell) \cdot \frac{PD(\Phi^{-1}(F_\psi(\ell))}{PD}. \end{cases}$$

Observe from the above equation that if $\rho_A \neq 0$ $f_{\mathcal{PL}}(\ell) \neq f_{LGD}(\ell)$, implying account-level LGD data does not come from $f_{\mathcal{PL}}(\ell)$ as it is implicitly assumed in the papers pointed out in this document. As such this approach is incorrect.

2.4.2 Effect of Defect on Parameter and Economic Capital Estimates

We explore the severity of this error using the correlation structure ($\theta_I = 0, \theta_S = 1$) in Frye's model, but used the transformation in Eq. (2.19) and assumed beta as the target distribution of the account-level loss given default. We employed method of moments, where we map the sample moments obtained for simulated account-level potential loss and loss given default to the respective population counterparts.

We estimate the parameters δ_1 and δ_2 by assuming 10000 economic scenarios. $\delta_1 = 0.3499, \delta_2 = 4.0354$ and $PD = 0.05$. Algorithm 2 outlines the estimation procedure. Note that the simulation procedure assumes a portfolio of one obligor (we are effectively working with account-level $\mathcal{PL}$ and LGD). The estimated values are used to estimate the corresponding values of EC and then the percentage differences are found. The percentage difference is calculated by subtracting the EC obtained from the estimate of δ_1 and δ_2 based on account-level LGD distribution from that obtained from the parameter estimate based on the account-level $\mathcal{PL}$ and then divided by the former.

Table 2.4 shows estimated values of δ_1 and δ_2 for varying values of ρ_A. The table compares the estimated values of the pair (δ_1, δ_2) under account-level $\mathcal{PL}$ and LGD with their respective true values — $\delta_1 = 0.3499$ and $\delta_2 = 4.0345$. The result shows clear discrepancy between the estimates. For example, $\rho_A = -0.4$ gives an estimate under the account-level LGD setting as 0.7737 and 3.309 for δ_1 and δ_2 respectively, which are different from the corresponding true values — $\delta_1 = 0.3499$ and $\delta_2 = 4.0345$. The deviation from the true values widens as ρ_A increases. The percentage difference in EC in the table indicates gross difference in the estimated EC based on the respective estimated parameter values δ_1 and δ_2 from the respective distributions of account-level LGD and $\mathcal{PL}$. This means estimated δ_1 and δ_2 from the $\mathcal{PL}$ distribution understates greatly EC. For instance, we have an understatement of EC of 54.26% and 41.07% at 90% and 99% confidence level (CL) respectively when $\rho_A = -0.2$.

Furthermore, using an approximate sampling distribution of the respective parameters, 95% confidence interval (CI) and the mean square error (MSE) are presented in Table 2.5 . The CI is obtained using the corresponding quantiles (2.5% and 97.5%) from the approximate distribution. We used a sample size of 1000. The CI and the MSE indicate that estimates using LGD data deviates grossly from the true values of the parameters.

Summarizing, we have demonstrated in this section that the observed LGD distribution obtained from the PD-LGD correlation models in the aforementioned papers is different from the targeted distribution. This modelling error greatly impacts EC estimates.

Algorithm 2 Estimating δ_1 and δ_2. We assumed a portfolio of one obligor

1: Input parameters — (α, β): sensitivity parameter of systematic and idiosyncratic risk factors, PD: common probability of default, n: number of simulated economic scenarios

2: Generate n quantities of the systematic risk driver, S_A. Denote the simulated values as $s_1, s_2, ..., s_n$.

3: Generate n quantities of the idiosyncratic risk drivers, I_A and I_B. Denote the simulated values for the pair I_A and I_B as $(i_{A,1}, i_{B,1}), (i_{A,2}, i_{B,2}), (i_{A,3}, i_{B,3}), ..., (i_{A,n}, i_{B,n})$.

4: For each of the observed systematic and idiosyncratic risk drivers i set

- $A_i = \alpha s_i + \sqrt{1-\alpha^2} i_{A,i}$
- $B_i = \beta s_i + \sqrt{1-\beta^2} i_{B,i}$
- $\mathcal{D}_i = \begin{cases} 1 & \text{if } A_i \leq \Phi^{-1}(PD) \\ 0 & \text{Otherwise} \end{cases}$
- $\mathcal{PL}_i = \mathcal{B}^{-1}(\Phi(B_i), \delta_1, \delta_2)$

5: Number of defaults $N_D = \sum_{i=1}^{n} \mathcal{D}_i$

6: Let $j_1, j_2, j_3, ..., j_{N_D}$ be defaulted exposures. For example, set

- $n = 10$
- $\begin{cases} D_i = 1 \text{ for } i = 1, 4, 7 \\ D_i = 0 \text{ for other values at } i \end{cases}$

So $N_D = 3, j_1 = 1, j_2 = 4, j_3 = 7$.

7: Define $LGD_k = \mathcal{PL}_{j_k}$

8: Calculate the means and variances of $\mathcal{PL}$, and LGD

- $\mu_{\mathcal{PL}} = \frac{1}{n} \sum_{i=1}^{n} \mathcal{PL}_i$
- $\mu_{LGD} = \frac{1}{N_D} \sum_{k=1}^{N_D} LGD_k$
- $\sigma^2_{\mathcal{PL}} = \frac{1}{n} \sum_{i=1}^{n} (\mathcal{PL}_i - \mu_{\mathcal{PL}})^2$
- $\sigma^2_{LGD} = \frac{1}{N_D} \sum_{i=1}^{N_D} (LGD_k - \mu_{LGD})^2$

9: Solve for δ_1 and δ_2 using

i
- $\mu_{\mathcal{PL}} = \frac{\delta_1}{\delta_1 + \delta_2}$
- $\sigma^2_{\mathcal{PL}} = \frac{\delta_1 \delta_2}{(\delta_1+\delta_2)^2(\delta_1+\delta_2+1)}$

ii
- $\mu_{LGD} = \frac{\delta_1}{\delta_1 + \delta_2}$
- $\sigma^2_{LGD} = \frac{\delta_1 \delta_2}{(\delta_1+\delta_2)^2(\delta_1+\delta_2+1)}$

Table 2.4: Estimated values of δ_1 and δ_2 under different values of ρ_A corresponding to different parameter value combination of α and β. The true values of δ_1 and δ_2 are 0.3499 and 4.0354 respectively. $PD = 0.05$. δ_1 and δ_2 are estimated by assuming a portfolio of one obligor ($N = 1$) and 10000 economic scenarios. EC is estimated using 1000 economic scenarios and number of obligors.

	Estimates using LGD data		Estimates using $\mathcal{PL}$ data		Percentage difference of EC	
Parameter values	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_1$	$\hat{\delta}_2$	90% CL	99% CL
$\rho_A = -0.2(\alpha = -0.6, \beta = 0.33333)$	0.5703	3.8056	0.3419	3.9430	54.26%	41.07%
$\rho_A = -0.4(\alpha = -0.520, \beta = 0.760)$	0.7737	3.3090	0.3434	4.0252	111.69%	52.77%
$\rho_A = -0.6(\alpha = -0.630, \beta = 0.950)$	1.2163	3.4421	0.3475	4.1248	177.93%	53.26%

Table 2.5: 95% Confidence Intervals of δ_1 and δ_2 under different values of ρ_A corresponding to different parameter value combination of α and β. The true values of δ_1 and δ_2 are 0.3499 and 4.0354 respectively. $PD = 0.05$. δ_1 and δ_2 are estimated by assuming a portfolio of one obligor ($N = 1$) and 10000 economic scenarios. 1000 sample size of estimates.

	CI using LGD data				CI using $\mathcal{PL}$ data			
Parameter values	$\hat{\delta}_1$	MSE	$\hat{\delta}_2$	MSE	$\hat{\delta}_1$	MSE	$\hat{\delta}_2$	MSE
$\rho_A = -0.2(\alpha = -0.6, \beta = 0.33333)$	[0.4338 0.5906]	0.0272	[2.9926 4.1602]	0.3499	[0.3464 0.3464]	0.0000	[4.0973 4.0973]	0.0038
$\rho_A = -0.4(\alpha = -0.520, \beta = 0.760)$	[0.6383 0.8572]	0.1618	[2.8226 3.8214]	0.5667	[0.3461 0.3461]	0.0000	[4.0395 4.0395]	0.0000
$\rho_A = -0.6(\alpha = -0.630, \beta = 0.950)$	[1.0713 1.3800]	0.7485	[3.0349 3.9911]	0.3517	[0.3396 0.3396]	0.0001	[3.9768 3.9768]	0.0034

2.4.3 Correcting Defect

In this section we show how to choose the transformation H so that the account-level LGD distribution comes from a target parametric family. In other words we have solved the problem that was identified in section 2.4.1. The objective is that given a target parametric family of distribution f_ψ for an account-level LGD data, we choose a transformation that defines $\mathcal{PL}$ such that the account-level distribution is f_ψ. In what follows, we let H_f denote a transformation under which $f_{LGD} = f$, where f is some pdf on [0, 1].

Theorem 3. *Suppose f is a pdf on $[0,1]$. There are two transformations, one increasing and the other decreasing, with the property that $f_{LGD} = f$. The increasing function is $H_f(b) = F^{-1}(\widehat{\Phi}(b))$ and the decreasing transformation is $H_f(b) = F^{-1}(1 - \widehat{\Phi}(b))$, where F is the cdf of f.*

Proof. Suppose H_f is an increasing function of B, then

$$\begin{aligned}\mathsf{P}(H_f(B) \leq \ell | A \leq \Phi^{-1}(PD)) &= \mathsf{P}(B \leq H_f^{-1}(\ell) | A \leq \Phi^{-1}(PD) \\ &= \widehat{\Phi}(H_f^{-1}(\ell)).\end{aligned}$$

Now for a choice of H_f such that

$$\mathsf{P}(H_f(B) \leq \ell | A \leq \Phi^{-1}(PD)) = F(\ell),$$

we should have

$$\widehat{\Phi}(H_f^{-1}(\ell)) = F(\ell),$$

This implies that

$$H_f^{-1}(\ell) = \widehat{\Phi}^{-1}(F(\ell)). \tag{2.21}$$

Now, let $b = H_f^{-1}(\ell)$. Noting that $\ell = H_f(b)$ and $b = \Phi^{-1}(F(H_f(b))$, Eq. (2.21) becomes

$$\widehat{\Phi}(b) = F(H_f(b)),$$

which implies

$$H_f(b) = F^{-1}(\widehat{\Phi}(b)).$$

Now, Suppose H_f is a decreasing function of B, then

$$\begin{aligned}\mathsf{P}(H_f(B) \leq \ell | A \leq \Phi^{-1}(PD)) &= \mathsf{P}(B \geq H_f^{-1}(\ell) | A \leq \Phi^{-1}(PD)) \\ &= 1 - \mathsf{P}(B \leq H_f^{-1}(\ell) | A \leq \Phi^{-1}(PD)) \\ &= 1 - \widehat{\Phi}(H_f^{-1}(\ell)).\end{aligned}$$

Now for a choice of H_f such that

$$\mathsf{P}(H_f(B) \leq \ell | A \leq \Phi^{-1}(PD)) = F(\ell),$$

we should have

$$1 - \widehat{\Phi}(H_f^{-1}(\ell)) = F(\ell).$$

Which is not different from

$$H_f^{-1}(\ell) = \widehat{\Phi}^{-1}(1 - F(\ell)). \tag{2.22}$$

Similar as above, let $b = H_f^{-1}(\ell)$. Noting that $\ell = H_f(b)$ and $b = \Phi^{-1}(1 - F(H_f(b)))$, Eq. (2.22) becomes

$$\widehat{\Phi}(b) = 1 - F(H_f(b)),$$

which means

$$H_f(b) = F^{-1}(1 - \widehat{\Phi}(b)).$$

□

2.5 Comparing Transformations

Let f be a pdf and F be its cdf. In this section we compare the transformations — $\widehat{H}(b)$ (right transformation) and $H(b)$ (wrong transformation). If f is the desired pdf for the account-level LGD, then $H(b)$ is the incorrect transformation that is currently used in some of the literature where as $\widehat{H}(b)$ is the correct transformation that has been identified in this thesis for the first time. Our first observation is as follows:

Proposition 2. *Suppose H is an increasing transformation. $H \geq \widehat{H}$ if $\rho_A \leq 0$. $H \leq \widehat{H}$ otherwise.*

Proposition 3. *Suppose H is a decreasing transformation. $H \geq \widehat{H}$ if $\rho_A \geq 0$. $H \leq \widehat{H}$ otherwise.*

Proof. The proof of Propositions 2 and 3 follows from Theorem 1, □

Propositions 2 and 3 imply that under the assumption of a positive correlation between account-level default and potential loss, H over transforms the loss driver. That is, simulations based on H overstate the account-level $\mathcal{PL}$. The converse — negative correlation between account-level default and potential loss — produces an understatement of account-level $\mathcal{PL}$ using H.

2.5.1 Comparing Distribution of Loss under $H(b)$ and $\widehat{H}(b)$

Without lost of generality, we assume an increasing transformation of B in this section. The value of ρ_A is negative. We set $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$ (correct transformation) and $H(b) = F^{-1}(\Phi(b))$ (wrong transformation).

The cdf of account-level potential loss $\mathcal{PL}$ under the transformation $H(b) = F^{-1}(\Phi(b))$ is given by

$$\begin{aligned}
\mathsf{P}(\mathcal{PL} \leq \ell) &= \mathsf{P}(H(B) \leq \ell) \\
&= \mathsf{P}(\Phi(B) \leq F(\ell)) \\
&= \mathsf{P}(B \leq \Phi^{-1}(F(\ell))) \\
&= \Phi[\Phi^{-1}(F(\ell))] \\
&= F(\ell).
\end{aligned}$$

The cdf of account-level LGD is

$$\begin{aligned}
\mathsf{P}(\mathcal{PL} \leq \ell | \mathcal{D} = 1) &= \mathsf{P}(H(B) \leq \ell | \mathcal{D} = 1) \\
&= \mathsf{P}(B \leq \Phi^{-1}(F(\ell)) | \mathcal{D} = 1) \\
&= \widehat{\Phi}[\Phi^{-1}\,(F(\ell))].
\end{aligned}$$

The cdf of account-level potential loss using the transformation $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$ is derived as

$$\begin{aligned}
\mathsf{P}(\mathcal{PL} \leq \ell) &= \mathsf{P}(\widehat{\Phi}(B) \leq F(\ell)) \\
&= \mathsf{P}(B \leq \widehat{\Phi}^{-1}(F(\ell))) \\
&= \Phi[\widehat{\Phi}^{-1}(F(\ell))]
\end{aligned}$$

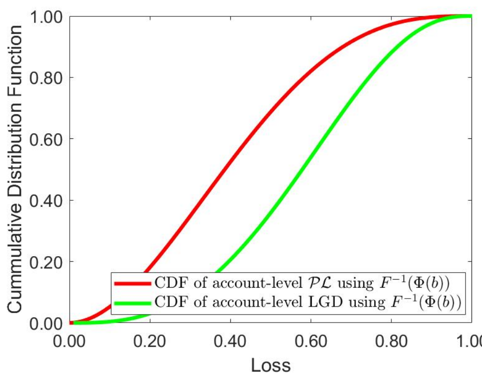
and the cdf of account-level LGD is

$$\mathsf{P}(\mathcal{PL} \leq \ell | \mathcal{D} = 1) = F(\ell).$$

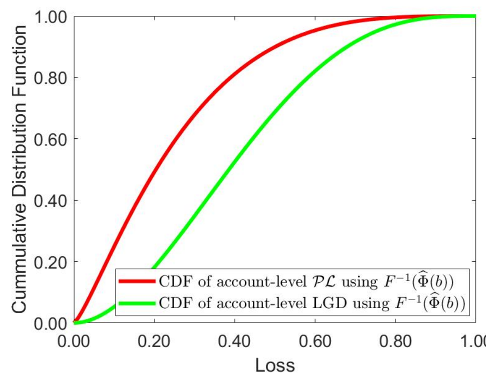
Table 2.6 presents a summary of the various distributions of potential loss under the different transformations presented above.

Table 2.6: Cumulative distribution function (cdf) of potential loss under different-transformations

	Transformation	
	$H(b) = F^{-1}(\Phi(b))$	$\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$
cdf of account-level $\mathcal{PL}$	$F(\ell)$	$\Phi[\widehat{\Phi}^{-1}(F(\ell))]$
cdf of account-level LGD	$\widehat{\Phi}[\Phi^{-1}(F(\ell))]$	$F(\ell)$



(a) Cumulative distribution function of account-level $\mathcal{PL}$ and LGD under transformation, $H(b) = F^{-1}(\Phi(b))$.

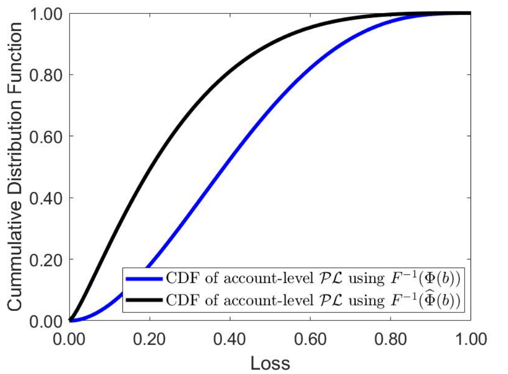


(b) Cumulative distribution function of account-level $\mathcal{PL}$ and LGD under transformation, $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$.

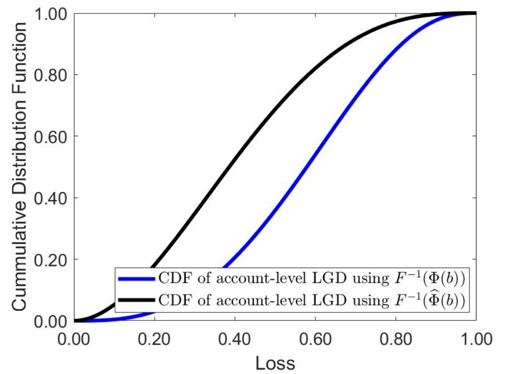
Figure 2.7: Cumulative distribution functions of account-level $\mathcal{PL}$ and LGD under different transformations. F is cdf of a beta random variable with scale parameters $\delta_1 = 2$ and $\delta_2 = 3$, $PD = 0.05, \rho_A = -0.4$.

Figures 2.7a and 2.7b compare the cdf of account-level $\mathcal{PL}$ (red curve) and LGD (green curve) under the respective transformations. Figure 2.7a compares the the distribution of the account-level $\mathcal{PL}$ and LGD under the transformation $H(b) = F^{-1}(\Phi(b))$. The figures show that the cdf of account-level $\mathcal{PL}$ and LGD are different.

Figure 2.8a compares the cdf of account-level $\mathcal{PL}$ under the respective transformations — $H(b)$ (blue curve) and $\widehat{H}(b)$ (black curve) and Figure 2.8b compares the account-level LGD under the respective transformations — $H(b)$ (blue curve)



(a) Cumulative distribution function of $\mathcal{PL}$ under transformations, $H(b) = F^{-1}(\Phi(b))$ and $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$.



(b) Cumulative distribution of LGD, $H(b) = F^{-1}(\Phi(b))$ and $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$.

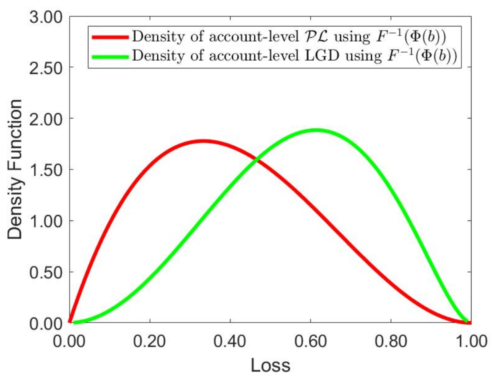
Figure 2.8: Cumulative distributions of account-level $\mathcal{PL}$ and LGD. $H(b) = F^{-1}(\Phi(b))$ and $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$. F is cdf of a beta random variable with scale parameters $\delta_1 = 2$ and $\delta_2 = 3$, $PD = 0.05, \rho_A = -0.4$.

and $\widehat{H}(b)$ (black curve). The figures show a disparity in the cdf under the respective transformations.

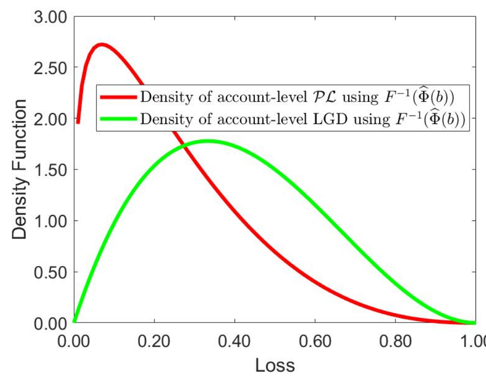
Table 2.7 summarizes the density functions of account-level $\mathcal{PL}$ and LGD under the respective transformations. A comparison of the graphs of these density functions under these transformations are presented in Figures 2.9 and 2.10. Figures 2.9a and 2.9b are the graphs of the density functions of the account-level $\mathcal{PL}$ and LGD under the transformations $H(b) = F^{-1}(\Phi(b))$ and $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$ respectively. The figures show disparity in the respective density functions under each transformation, thus corroborating the story presented by Figure 2.7 — the account-level $\mathcal{PL}$ and LGD using the two transformations defer. Figures 2.10a and 2.10b highlight the difference in the distributions of the account-level $\mathcal{PL}$ and LGD under the two transformations by comparing the respective density curves.

Table 2.7: Density function of potential loss under different transformations

	Transformation	
	$H(b) = F^{-1}(\Phi(b))$	$\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$
Density of account-level $\mathcal{PL}$	$f(\ell)$	$\frac{\phi(\widehat{\Phi}^{-1}(F(\ell)))f(\ell)}{\widehat{\phi}(\widehat{\Phi}^{-1}(F(\ell)))}$
Density of account-level LGD	$\frac{\widehat{\phi}(\Phi^{-1}(F(\ell)))f(\ell)}{\phi(\Phi^{-1}(F(\ell)))}$	$f(\ell)$

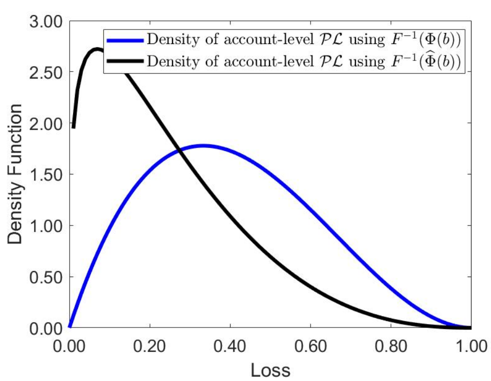


(a) Probability density functions of account-level $\mathcal{PL}$ and LGD under transformation, $H(b) = F^{-1}(\Phi(b))$.

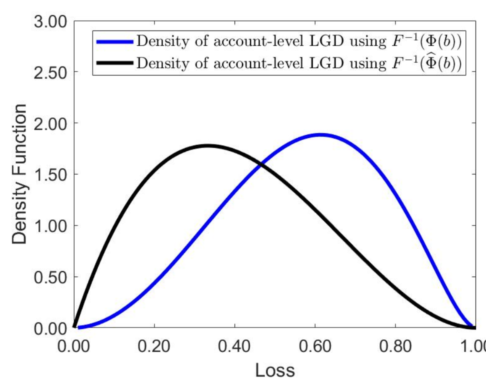


(b) Probability density functions account-level $\mathcal{PL}$ and LGD under transformation. $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$.

Figure 2.9: Density functions of account-level $\mathcal{PL}$ and LGD under different transformations. F is cdf of a beta random variable with scale parameters $\delta_1 = 2$ and $\delta_2 = 3$, $PD = 0.05, \rho_A = -0.4$.



(a) Unconditional density functions of $\mathcal{PL}$ under transformations, $H(b) = F^{-1}(\Phi(b))$ and $\widehat{H}(b) = F^{-1}(\Phi(b))$.



(b) Conditional density functions of $\mathcal{PL}$ given default under transformations, $H(b) = F^{-1}(\Phi(b))$ and $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$.

Figure 2.10: Density functions under transformations, $H(b) = F^{-1}(\Phi(b))$ and $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$. F is cdf of a beta random variable with scale parameters $\delta_1 = 2$ and $\delta_2 = 3$, $PD = 0.05, \rho_A = -0.4$.

2.5.2 Comparing Economic Capital (EC) under $H(b)$ and $\widehat{H}(b)$

Again, without loss of generality, we assumed an increasing transformation on the loss driver B and thereby impose a positive correlation between account-level $\mathcal{PL}$ and default. By this, the range of values of ρ_A are set on the interval $[-1, 0]$.

Table 2.8 further highlights the assertion that definition of potential loss using the transformation $H(b) = F^{-1}(\Phi(b))$ over transforms loss drivers leading to overstatement of EC. We considered confidence levels of 90% and 99% in estimating percentage EC — the fraction of estimated EC of portfolio value. See Algorithm 1 for estimation procedure. We assumed portfolio size of 1000 and 1000 economic scenarios. Under both confidence levels, we have estimated percentage EC corresponding to incorrect transformation ($H(b)$) bigger than estimated percentage EC corresponding to correct transformation ($\widehat{H}(b)$). Also, the gap between percentage EC under the respective transformations $H(b)$ and $\widehat{H}(b)$ increases with increasing correlation between default and loss drivers, A and B.

Table 2.8: Economic capital under the two transformations. The pair (S_A, S_B) assumes a perfect correlation between each other ($\theta_S = 1$) and the pair (I_A, I_B) are independent of each other ($\theta_I = 0$). PD is 0.05. F^{-1} is the inverse beta cumulative distribution function with scale parameters $\delta_1 = 0.3499, \delta_2 = 4.0354$, and mean and standard deviation, 0.0798 and 0.1166 respectively. Each exposure is 1.00. We have portfolio size of 1000 (1000 borrowers) and 1000 simulated pairs of (S_A, S_B) (economic scenarios). The EC is the difference between the mean realized losses and the respective confidence level, 90% and 99%. The EC is expressed as a percentage of portfolio value.

	α	β	Transformation	
			$H(b) = F^{-1}(\Phi(b))$	$\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b))$
Economic capital at 90% confidence level	-0.31	0.63	1.08%	0.68%
Economic capital at 99% confidence level			4.68%	3.50%
Economic capital at 90% confidence level	-0.59	0.67	1.54%	0.55%
Economic capital at 99% confidence level			11.29%	6.02%
Economic capital at 90% confidence level	-0.71	0.84	2.01%	0.23%
Economic capital at 99% confidence level			20.03%	8.24%

2.6 Conclusion

The correct specification of the distribution of account-level loss given default is a function of how accurately potential loss is defined. An erroneous specification of potential loss will inevitably lead to distribution of loss that may not capture appreciably the underlining risk on the portfolio. This chapter presents a generalized credit risk model that nests existing credit risk models. We highlight that existing models ignore the difference between account-level potential loss and loss given default. This leads to a distribution of loss given default that does not capture expected portfolio loss dynamics — the distributions presented in the literature is that of potential loss. We fixed this defect by proposing a function that transforms loss drivers to potential loss that reflects the difference in these concepts — potential loss and loss given default. By deriving the correct distributions under the respective defective and correct transformations we established that estimates of risk measures, for example, economic capital are grossly different under the defective and correct models. This is a concern to banks as they have to put in reserves funds that appreciably reflects risk on the portfolio. We further highlighted how sensitive parameter estimates are to the defect in the existing models.

Chapter 3

Comparing Account and Portfolio Level LGD

Chapter 2 presents a discussion on the distinction between account-level $\mathcal{PL}$ and LGD. This chapter looks at the behavior of LGD at the account and portfolio-level and the relationship between account-level default and loss relationship and default and loss relationship at the portfolio-level. In this thesis, average realized loss and portfolio level realized loss are used interchangeably.

3.1 Finite Portfolio-level Default Rate and Loss Given Default

Consider a portfolio of N exposures with a common exposure \$$X$ each. The portfolio-level default rate DR_p is defined as the proportion of defaults on the portfolio over a given time horizon. It is defined formally as

$$DR_p = \frac{1}{N} \sum_{i=1}^{N} \mathcal{D}_i.$$

The dollar value of the loss exposure i is $\$\mathcal{RL}_i = \$X \cdot \mathcal{D}_i \cdot \mathcal{PL}_i$. So the loss exposure expressed as a percentage of the dollar value of an exposure i is $\mathcal{RL}_i = \mathcal{D}_i \cdot \mathcal{PL}_i$. The dollar value of total realized loss on the entire portfolio (we may refer to this as portfolio-level realized loss) is the summation of the individual realized losses within a stipulated time period, and is expressed as

$$\$\mathcal{RL}_p = \sum_{i=1}^{N} \$\mathcal{RL}_i = \sum_{i=1}^{N} \$X \cdot \mathcal{D}_i \cdot \mathcal{PL}_i. \tag{3.1}$$

The total loss expressed as a percentage of the total notional value of the portfolio-level realized loss is

$$\mathcal{RL}_p = \frac{\sum_{i=1}^{N} \$X \cdot \mathcal{D}_i \cdot \mathcal{PL}_i}{\sum_{i=1}^{N} \$X} = \frac{\sum_{i=1}^{N} \mathcal{D}_i \cdot \mathcal{PL}_i}{N},$$

which is the same as the average percentage realized loss of individual exposures. Because we are multiplying by $\mathcal{D}_i$, the sum in Eq. (3.1) really only extends over defaulted exposures. The dollar value of portfolio-level realized loss and loss given default $\$LGD_p$ are therefore equal. We define LGD_p as the total loss expressed as a percentage of the total value of defaulted exposures (that is, the ratio of total loss on the portfolio to the total notional value of defaulted exposures):

$$LGD_p = \frac{\sum_{i=1}^{N} \$X \cdot \mathcal{D}_i \cdot \mathcal{PL}_i}{\sum_{i=1}^{N} \$X \cdot \mathcal{D}_i} = \frac{\sum_{i=1}^{N} \mathcal{D}_i \cdot \mathcal{PL}_i}{\sum_{i=1}^{N} \mathcal{D}_i}. \tag{3.2}$$

Eq. (3.2) implies that the percentage quote of portfolio-level LGD is the same as the average loss sustained on the defaulted exposures. For later use, we have LGD_p

rewritten as

$$LGD_p = \frac{\frac{1}{N}\sum_{i=1}^{N} \mathcal{D}_i \cdot \mathcal{PL}_i}{\frac{1}{N}\sum_{i=1}^{N} \mathcal{D}_i}. \tag{3.3}$$

So by extension, the percentage quote of portfolio-level LGD can be written in terms of the average percentage realized loss on the portfolio and the portfolio-level default rate (this would not be true if exposures are of different sizes).

The rest of this chapter (i) compares the LGD distribution at the portfolio-level (that is probability distribution of LGD_p) to the distribution of LGD at the account-level (that is conditional distribution of $\mathcal{PL}_i$ given $\mathcal{D}_i = 1$, which we study in great detail in the previous chapter) and (ii) investigates the extent to which portfolio-level relationships (specifically, the correlation between LGD_p and DR_p) are determined by account-level relationships (specifically, correlation between $\mathcal{D}_i$ and $\mathcal{PL}_i$).

3.2 Joint Distribution of DR_p and LGD_p

The exact distribution of $DR_p, \mathcal{RL}_p$ and LGD_p are extremely complicated for finite N. As originally noted by Vasicek, however, the limiting distribution of DR_p as $N \to \infty$ is surprisingly simple [87]. In this section, we show that the same is true for $\mathcal{RL}_p$ and LGD_p. And we derive the limiting joint distribution of DR_p and LGD_p.

3.2.1 Asymptotic Representation of $DR_p, \mathcal{RL}_p$ and LGD_p.

Given $S_A = s_A$ the default drivers $A_1, A_2, A_3, ...$ are independent and identical distributed (i.i.d.) normal with mean αs_A and standard deviation $1 - \alpha^2$. This means default indicators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, ...$ are i.i.d. Bernoulli variables with success probability

$$\mathsf{P}(\mathcal{D}_i = 1 | S_A = s_A) = \mathsf{P}(A_i \leq \Phi^{-1}(PD) | S_A = s_A) = \Phi\left(\frac{\Phi^{-1}(PD) - \alpha s_A}{\sqrt{1-\alpha^2}}\right), \quad (3.4)$$

where we note that Eq. (3.4) is the conditional default probability given $S_A = s_A$ of an individual exposure. By the law of large numbers, we know that for a sequence of i.i.d. Bernoulli random variables, the proportion of successes converges to the theoretical probability of success (the common probability of success) as $N \to \infty$. Thus given $S_A = s_A$, we have that

$$\lim_{N\to\infty} DR_p = ADR_p(s_A), \quad (3.5)$$

where

$$ADR_p(s_A) = \mathsf{P}(\mathcal{D}_i = 1 | S_A = s_A)$$

is the individual conditional default probability. In light of Eq. (3.5), we call $ADR_p(S_A)$ the asymptotic default rate (this will be the default rate of an infinitely large portfolio).

Now, given the realized values of the pair (S_A, S_B), say (s_A, s_B), the pairs $(A_1, B_1), (A_2, B_2), (A_3, B_3), ...$ are i.i.d. bivariate normal with mean vector $\mu_{A,B}$ and covariance matrix $\Sigma_{A,B}$ (see appendix B.1 for derivation),

$$\mu_{A,B} = \begin{bmatrix} \alpha s_A \\ \beta s_B \end{bmatrix}$$

and

$$\Sigma_{A,B} = \begin{bmatrix} 1-\alpha^2 & \sqrt{(1-\alpha^2)(1-\beta^2)}\theta_I \\ \sqrt{(1-\alpha^2)(1-\beta^2)}\theta_I & 1-\beta^2 \end{bmatrix}.$$

Note that the mean vector depends on the realized values of the pair (S_A, S_B) and $\Sigma_{A,B}$ does not. This means the sequence $\mathcal{D}_1 \cdot \mathcal{PL}_1, \mathcal{D}_2 \cdot \mathcal{PL}_2, \mathcal{D}_3 \cdot \mathcal{PL}_3, ...$ will be i.i.d. as well. Again, by the law of large numbers, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{D}_i \cdot \mathcal{PL}_i = \mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i | S_A = s_A, S_B = s_B]. \tag{3.6}$$

We define $ARL_p(s_A, s_B)$ as

$$ARL_p(s_A, s_B) = \mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i | S_A = s_A, S_B = s_B].$$

In light of Eq. (3.6), we call $ARL_p(S_A, S_B)$ the asymptotic portfolio-level realized loss.

Since the limit of the ratio of two functions is the ratio of the limit of the functions, provided that the limit in the denominator function is not zero, from Eq. (3.3),(3.5) and (3.6) we have

$$\lim_{N\to\infty} LGD_p = \frac{\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{D}_i \cdot \mathcal{PL}_i}{\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{D}_i} = \frac{ARL_p(S_A, S_B)}{ADR_p(S_A)},$$

where we note that $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{D}_i > 0$. We call the ratio of ARL_p and ADR_p the asymptotic portfolio-level LGD ($ALGD_p$).

Finding expression for $ARL_p(S_A, S_B)$ and $ALGD_p$

The conditional expectation defining ARL_p is of the form $\mathbb{E}[f(A) \cdot g(B) | S_A = s_A, S_B = s_B]$. By employing the tower property of conditional expectation, $\mathbb{E}[f(A) \cdot g(B) | S_A =$

$s_A, S_B = s_A]$ can take the form

$$
\begin{aligned}
&\mathbb{E}[\mathbb{E}[(f(A) \cdot g(B))|B = b, S_A = s_A, S_B = s_B]|S_A = s_A, S_B = s_B] \\
&\qquad = \mathbb{E}[g(B) \cdot \mathbb{E}[f(A)|B = b, S_A = s_A, S_B = s_B]|S_A = s_A, S_B = s_B] \\
&\qquad = \mathbb{E}[H(B)\, \mathbb{E}[\mathbb{1}_{\{A \le \Phi^{-1}(PD)\}}|B = b, S_A = s_A, S_B = s_B]|S_A = s_A, S_B = s_B] \\
&\qquad = \mathbb{E}[H(B)\mathsf{P}(A \le \Phi^{-1}(PD)|B = b, S_A = s_A, S_B = s_B)|S_A = s_A, S_B = s_B].
\end{aligned}
$$

Now (see appendix B.2)

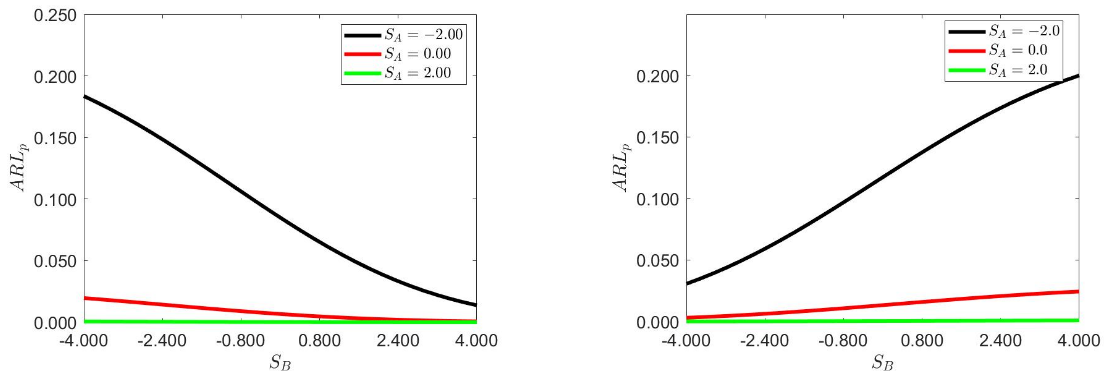
$$
\begin{aligned}
&\mathsf{P}(A \le \Phi^{-1}(PD)|B = b, S_A = s_A, S_B = s_B) \\
&\qquad = \Phi\left(\frac{\Phi^{-1}(PD) - \alpha s_A - \theta_I \sqrt{\frac{1-\alpha^2}{1-\beta^2}}(b - \beta s_B)}{\sqrt{(1-\alpha^2)(1-\theta_I^2)}}\right). \qquad (3.7)
\end{aligned}
$$

So we have

$$
\begin{aligned}
&ARL_p(s_A, s_B) \\
&= \int_{\mathbb{R}} H(b) \cdot \Phi\left(\frac{\Phi^{-1}(PD) - \alpha s_A - \theta_I \sqrt{\frac{1-\alpha^2}{1-\beta^2}}(b - \beta s_B)}{\sqrt{(1-\alpha^2)(1-\theta_I^2)}}\right) \cdot \phi(b; \beta s_B, 1-\beta^2)db, \qquad (3.8)
\end{aligned}
$$

where $\phi(\cdot)$ is the normal pdf with mean βs_B and variance $1 - \beta^2$. Note that the conditional pdf of the variate B given the realized values of S_A and S_B is normal with mean βs_B and variance $1 - \beta^2$. Figure 3.1 shows the graph of ARL_p against S_B for given values of S_A. It shows that for a given value of S_A, ARL_p is a monotone function of S_B. The nature of monotonicity (increasing or decreasing) is determined by the transformation $H(b)$. If H is a decreasing (increasing) function of b, ARL_p decreases (increases) with respective to S_B for a given value of S_A. See Figures 3.1a and 3.1b, and observe that even though the graph of ARL_p when $S_A = 2$ (green curve) appears horizontal, note that it is still increasing with S_B — the values of

ARL_p are very small.



(a) $H(b) = F^{-1}(1 - \widehat{\Phi}(b))$ (b) $H(b) = F^{-1}(\widehat{\Phi}(b))$

Figure 3.1: Asymptotic portfolio-level realized loss verses systematic risk factor S_B using Eq. (3.8). F is cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3, PD = 0.05, \rho_A = 0.4, \alpha = \beta = \sqrt{0.25}, \theta_S = 0.8, \theta_I = 0.2667$.

Remark 7. *Note that if $\theta_I = 0$ then ARL_p factors into the product of a function of s_A and a function of s_B since the drivers (A, B) are then independent given $S_A = s_A$ and $S_B = s_B$. Indeed if $\theta_I = 0$ then*

$$\begin{aligned} ARL_p(s_A, s_B) &= \Phi\left(\frac{\Phi^{-1}(PD) - \alpha s_A}{\sqrt{1-\alpha^2}}\right) \cdot \int_{\mathbb{R}} H(b) \cdot \phi(b; \beta s_B, 1-\beta^2) db, \\ &= ADR_p(s_A) \cdot h(s_B). \end{aligned}$$

In this case, the $ALGD_p$ is given as

$$ALGD_p(s_B) = h(s_B) = \int_{\mathbb{R}} H(b) \cdot \phi(b; \beta s_B, 1-\beta^2) db. \tag{3.9}$$

which is an decreasing (increasing) function of s_B when H is a decreasing (increasing) function of b — by reason of Figure 3.1.

Appendix D.1 discusses the behaviour of $ALGD_p$ with respect to θ_I. We answer the question, at what interval of θ_I will $ALGD_p$ be increasing (or decreasing).

3.2.2 Probability Density of ADR_p

Note that the asymptotic default rate is a random variable and is a decreasing function of S_A. From Eq. (3.4), the cdf of $ADR_p(S_A)$ is therefore obtained (for $0 < dr < 1$) as

$$\mathsf{P}(ADR_p(S_A) \leq dr) = \mathsf{P}(S_A \geq ADR_p^{-1}(dr)) = \Phi\left(\frac{\sqrt{1-\alpha^2}}{\alpha}\Phi^{-1}(dr) - \frac{\Phi^{-1}(PD)}{\alpha}\right),$$

where we use the fact that S_A is standard normal and

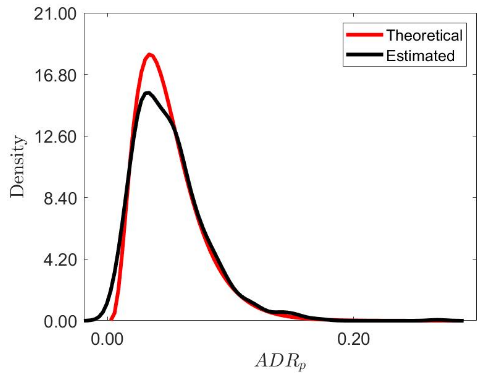
$$ADR_p^{-1}(dr) = \frac{\Phi^{-1}(PD) - \sqrt{1-\alpha^2}\Phi^{-1}(dr)}{\alpha}.$$

The resulting pdf is

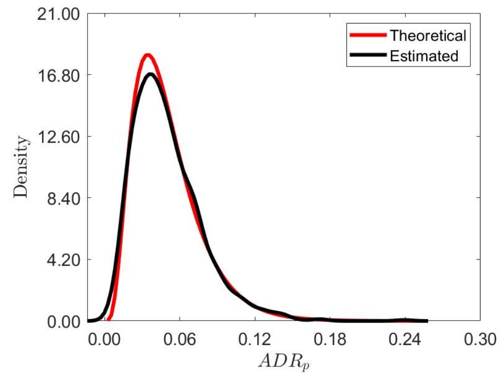
$$f_{ADR_p}(dr) = \frac{\sqrt{1-\alpha^2}}{\alpha \cdot \phi(\Phi^{-1}(dr))}\phi\left(\frac{\sqrt{1-\alpha^2}}{\alpha}\Phi^{-1}(dr) - \frac{\Phi^{-1}(PD)}{\alpha}\right), \tag{3.10}$$

a two parameter family with parameters PD and α. Figure 3.2 compares the estimated density of simulated values of finite portfolio-level default rate (black curve) and the density of asymptotic default rate ADR_p (red curve). Observe that as the portfolio size increases from 500 to 5000, we see very good approximation of the estimated density with the theoretical density. This visual agreement is confirmed by the Kolmogorov-Smirnov test, where using the kstest function in MATLAB we observed that for a portfolio size of 500, the p-value is 3%, and a portfolio size of 5000, the p-value is 67%. The null hypothesis is that the two data sets come from the same distribution.

The theoretical mean and variance are derived by first recalling that $ADR_p = \mathbb{E}[\mathcal{D}_i|S_A]$ $(= \mathsf{P}(\mathcal{D}_i = 1|S_A))$.



(a) $N = 500$



(b) $N = 5000$

Figure 3.2: Estimated density of simulated values of finite portfolio-level and density of asymptotic portfolio-level default rate using Eq. (3.10). $\alpha = 0.25, PD = 0.05$. Number of simulated systematic risk factors $m = 1000$. The ksdensity function in Matlab is used for density estimation.

The expected value of ADR_p is obtained as

$$\begin{aligned}\mu_{ADR_p} &= \mathbb{E}[ADR_p] \\ &= \mathbb{E}[\mathbb{E}[\mathcal{D}_i|S_A]] \\ &= \mathbb{E}[\mathcal{D}_i] = \mathsf{P}(\mathcal{D}_i) = PD, \end{aligned} \tag{3.11}$$

where we used the tower property. Using the fact that exposures are homogeneous

$$\begin{aligned}ADR_p^2 &= \mathbb{E}[\mathcal{D}_i|S_A] \cdot \mathbb{E}[\mathcal{D}_i|S_A] \\ &= \mathbb{E}[\mathcal{D}_i|S_A] \cdot \mathbb{E}[\mathcal{D}_j|S_A] \end{aligned} \tag{3.12}$$

Thus applying the property of conditional independence to Eq. (3.12), yields

$$ADR_p^2 = \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j|S_A]. \tag{3.13}$$

The variance of ADR_p is therefore

$$\begin{aligned}
\sigma^2_{ADR_p} &= \mathbb{E}[ADR_p^2] - PD^2 \\
&= \mathbb{E}[\mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j | S_A]] - PD^2 \qquad (3.14) \\
&= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j] - PD^2 \qquad (3.15) \\
&= PD^{(2)} - PD^2, \qquad (3.16)
\end{aligned}$$

where in moving from line (3.14) to (3.15) we use the tower property and

$$PD^{(2)} = \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j] = \mathsf{P}(\mathcal{D}_i = 1, \mathcal{D}_j = 1) = \mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)) \quad (3.17)$$

is the probability of two obligors defaulting simultaneously. $PD^{(2)}$ can be computed using bivariate normal pdf. Note that

$$\begin{aligned}
PD^{(2)} - PD^2 &= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j] - \mathbb{E}[\mathcal{D}_i] \cdot \mathbb{E}[\mathcal{D}_j] \\
&= \mathrm{Cov}(\mathcal{D}_i, \mathcal{D}_j),
\end{aligned}$$

$\mathrm{Cov}(\mathcal{D}_i, \mathcal{D}_j)$ is the covariance of default indicators.

Remark 8. *The variance of portfolio-level default rate is the same as the covariance between defaults.*

3.2.3 Mean and Variance of ARL_p

Using the tower property, the mean of ARL_p is derived as

$$\mu_{ARL_p} = \mathbb{E}[ARL_p] = \mathbb{E}[\mathbb{E}[\mathcal{RL}_i | S_A, S_B]] = \mathbb{E}[\mathcal{RL}_i].$$

The fact that exposures are homogeneous allows us to write

$$\begin{aligned} ARL_p^2 &= \mathbb{E}[\mathcal{RL}_i|S_A, S_B] \cdot \mathbb{E}[\mathcal{RL}_i|S_A, S_B] \\ &= \mathbb{E}[\mathcal{RL}_i|S_A, S_B] \cdot \mathbb{E}[\mathcal{RL}_j|S_A, S_B], \end{aligned}$$

so that

$$ARL_p^2 = \mathbb{E}[\mathcal{RL}_i \cdot \mathcal{RL}_j|S_A, S_B],$$

where we use the property of conditional independence. Hence, the expected value of ARL_p^2 is

$$\mathbb{E}[ARL_p^2] = \mathbb{E}[\mathcal{RL}_i \cdot \mathcal{RL}_j], \tag{3.18}$$

where we applied the tower property. Again, using the property of homogeneity of exposures

$$\mu_{ARL_p}^2 = \mathbb{E}[\mathcal{RL}_i] \cdot \mathbb{E}[\mathcal{RL}_j].$$

We can therefore express the variance of ARL_p as

$$\begin{aligned} \sigma_{ARL_p}^2 &= \mathbb{E}[\mathcal{RL}_i \cdot \mathcal{RL}_j] - \mathbb{E}[\mathcal{RL}_i] \cdot \mathbb{E}[\mathcal{RL}_j] \\ &= \mathrm{Cov}(\mathcal{RL}_i, \mathcal{RL}_j). \end{aligned}$$

$\mathrm{Cov}(\mathcal{RL}_i, \mathcal{RL}_j)$ is the covariance between account-level realized losses ($\mathcal{RL}_i$ and $\mathcal{RL}_j$)

Remark 9. *The variance of portfolio-level realized loss is equal to the covariance between the account-level realized loss on exposures.*

Alternatively, recall that $\mathcal{RL}_i = \mathcal{D}_i \cdot \mathcal{PL}_i$, hence we have the expected value of ARL_p as

$$\mu_{ARL_p} = \mathbb{E}[ARL_p] = \mathbb{E}[\mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i | S_A, S_B]] = \mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i].$$

Applying the tower property and conditioning on $\mathcal{D}_i$ gives

$$\begin{aligned} \mu_{ARL_p} &= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i] \\ &= \mathbb{E}[\mathcal{PL}_i | \mathcal{D}_i = 1] \cdot \mathsf{P}[\mathcal{D}_i = 1] = LGD_A \cdot PD, \end{aligned} \tag{3.19}$$

where

$$LGD_A = \mathbb{E}[\mathcal{PL}_i | \mathcal{D}_i = 1].$$

The variance is obtained as

$$\sigma^2_{ARL_p} = \mathbb{E}[ARL_p^2] - \mu^2_{ARL_p},$$

where

$$\begin{aligned} \mathbb{E}[ARL_p^2] &= \mathbb{E}[\mathcal{RL}_i \cdot \mathcal{RL}_j] \\ &= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_j] && (3.20) \\ &= \mathbb{E}[\mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_j | S_A, S_B]] && (3.21) \\ &= \mathbb{E}[(\mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i | S_A, S_B])^2] && (3.22) \end{aligned}$$

We use the tower property and the fact that exposures are homogeneous in moving from line (3.20) to (3.21) and from line (3.20) to (3.22) respectively. We use the

Jensen's inequality to get[1]

$$\begin{aligned}
\mathbb{E}[(\mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i|S_A, S_B])^2] &\leq \mathbb{E}[\mathbb{E}[(\mathcal{D}_i \cdot \mathcal{PL}_i)^2|S_A, S_B]] \\
&= \mathbb{E}[(\mathcal{D}_i \cdot \mathcal{PL}_i)^2] \\
&= \mathbb{E}[\mathcal{PL}_i^2|\mathcal{D}_i^2 = 1] \cdot \mathsf{P}(\mathcal{D}_i^2 = 1) \\
&= \mathbb{E}[\mathcal{PL}_i^2|\mathcal{D}_i = 1] \cdot PD.
\end{aligned}$$

Noting that the variance of account-level LGD is

$$\sigma_{LGD}^2 = \mathbb{E}[\mathcal{PL}_i^2|\mathcal{D}_i = 1] - (\mathbb{E}[\mathcal{PL}_i|\mathcal{D}_i = 1])^2,$$

we have

$$\mathbb{E}[\mathcal{PL}_i^2|\mathcal{D}_i = 1] = \sigma_{LGD}^2 + LGD_A^2.$$

Thus

$$\begin{aligned}
\sigma_{ARL_p}^2 &\leq PD[\sigma_{LGD}^2 + LGD_A^2] - PD^2 LGD_A^2 \\
&= PD[\sigma_{LGD}^2 + LGD_A^2(1 - PD)]. \qquad (3.23)
\end{aligned}$$

Remark 10. *Since* $0 \leq PD \leq 1$, *observe from Eq. (3.23) that extreme values of* PD, *say* $PD = 0$ *(no defaults at the account-level),* $\sigma_{ARL_p}^2 = \sigma_{LGD}^2 = 0$ *and* $PD = 1$ *(all account defaults),* $\sigma_{ARL_p}^2 \leq \sigma_{LGD}^2$.

[1]Observe that the expression in the outer expectation symbol $\mathbb{E}$ in line (3.22) takes the form $f(x) = x^2$. So we have $f((\mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i|S_A, S_B])) = (\mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i|S_A, S_B])^2 \leq \mathbb{E}[(\mathcal{D}_i \cdot \mathcal{PL}_i)^2|S_A, S_B] = \mathbb{E}[f(\mathcal{D}_i \cdot \mathcal{PL}_i)|S_A, S_B]$

3.2.4 Covariance of ADR_p and ARL_p

Again, we apply the homogeneity property of exposures to arrive at

$$\begin{aligned} ADR_p \cdot ARL_p &= \mathbb{E}[\mathcal{D}_i|S_A] \cdot \mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i|S_A, S_B] \\ &= \mathbb{E}[\mathcal{D}_j|S_A] \cdot \mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i|S_A, S_B], \end{aligned}$$

and applying the conditional independence property gives

$$ADR_p \cdot ARL_p = \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_i|S_A, S_B], \tag{3.24}$$

Applying the tower property to Eq. (3.24), we have

$$\begin{aligned} \mathbb{E}[ADR_p \cdot ARL_p] &= \mathbb{E}[\mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_i|S_A, S_B]] \\ &= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_i], \end{aligned} \tag{3.25}$$

$\mathbb{E}[ADR_p \cdot ARL_p]$ can be expressed as

$$\begin{aligned} \mathbb{E}[ADR_p \cdot ARL_p] &= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_i] \\ &= E[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_i|\mathcal{D}_i \cdot \mathcal{D}_j = 1] \cdot \mathsf{P}(\mathcal{D}_i \cdot \mathcal{D}_j = 1) \\ &= E[\mathcal{PL}_i|\mathcal{D}_i = 1, \mathcal{D}_j = 1] \cdot \mathsf{P}(\mathcal{D}_i = 1, \mathcal{D}_j = 1), \end{aligned}$$

thus the covariance of ADR_p and $ALGD_p$ is

$$\begin{aligned} \sigma_{ADR_p, ARL_p} &= \mathbb{E}[ADR_p \cdot ARL_p] - \mathbb{E}[ADR_p] \cdot \mathbb{E}[ARL_p] \\ &= LGD_A^{(2)} \cdot PD^{(2)} - PD^2 \cdot LGD_A, \end{aligned} \tag{3.26}$$

where

$$LGD_A^{(2)} = \mathbb{E}[\mathcal{PL}_i|\mathcal{D}_i = 1, \mathcal{D}_j = 1]$$

is the account-level LGD given joint defaults of two obligors — we call this quantity "account-level joint LGD". Section 3.3.2 presents discussion on computing $LGD_A^{(2)}$.

3.2.5 Deriving Joint Density of ADR_p and $ALGD_p$

Let $dr = ADR_p(s_A)$ and $\ell = \frac{ARL_p(s_A,s_B)}{ADR_p(s_A)}$. Since ADR_p is monotone and numerical evidence (Figure 3.1) suggests that ARL_p is monotone in s_B for a fixed value of s_A, it follows that (dr, ℓ) is a one-to-one transformation of (s_A, s_B). We let $s_A = g_1(dr)$ and $s_B = g_2(dr, \ell)$ be the components inverse transformation. Then $g_1(dr)$ is given by

$$s_A = g_1(dr) = \frac{\Phi^{-1}(PD) - \sqrt{1-\alpha^2}\Phi^{-1}(dr)}{\alpha}, \tag{3.27}$$

but generally, $g_2(dr, \ell)$ must be computed numerically. This can be done as follows: Set

$$g_{S_A,S_B}(s_A, s_B) = \ell - \frac{ARL_p(s_A, s_B)}{ADR_p(s_A)}.$$

Since s_A can be computed from Eq. (3.27), for given values of dr and ℓ, the function g_{S_A,S_B} reduces to a function of only s_B. We can then retrieve the value of s_B (or $g_2(dr, \ell)$) numerically by finding the zero of the following:

$$g_{S_B}(s_B) = \ell - \frac{ARL_p(s_A, s_B)}{ADR_p(s_A)}.$$

That is the value of s_B (or $g_2(dr, \ell)$) is the zero of $g_{S_B}(s_B)$.

Suppose $g_1(\cdot)$ and $g_2(\cdot)$ are partially differentiable on dr and ℓ so that we get the Jacobian as

$$J = \begin{vmatrix} \frac{\partial s_A}{\partial dr} & \frac{\partial s_A}{\partial \ell} \\ \frac{\partial s_B}{\partial dr} & \frac{\partial s_B}{\partial \ell} \end{vmatrix},$$

where $\frac{\partial s_A}{\partial dr} = \frac{\partial g_1(dr)}{\partial dr}$, $\frac{\partial s_A}{\partial \ell} = \frac{\partial g_1(DR_p)}{\partial \ell} = 0$, $\frac{\partial s_B}{\partial dr} = \frac{\partial g_2(dr,\ell)}{\partial dr}$, and $\frac{\partial s_B}{\partial \ell} = \frac{\partial g_2(dr,\ell)}{\partial \ell}$. We have

$$J = \frac{\partial g_1(dr)}{\partial dr} \cdot \frac{\partial g_2(dr,\ell)}{\partial \ell}.$$

It is straightforward to compute $\frac{\partial g_1(dr)}{\partial dr}$ using Eq. (3.27), however, computing $\frac{\partial g_2(dr,\ell)}{\partial \ell}$ is not that simple. We do the computation numerically. From derivative by first principle:

$$\frac{\partial g_2(dr,\ell)}{\partial \ell} = \lim_{\epsilon \to 0} \frac{g_2(dr, \ell+\epsilon) - g_2(dr,\ell)}{\epsilon},$$

where ϵ is a small change in ℓ and $g_2(dr, \ell+\epsilon)$ is the zero of the expression

$$g_{S_B,\epsilon}(s_B) = (\ell + \epsilon) - \frac{ARL_p(s_A, s_B)}{ADR_p(s_A)}.$$

By this, we can approximate $\frac{\partial g_2(dr,\ell)}{\partial \ell}$ as

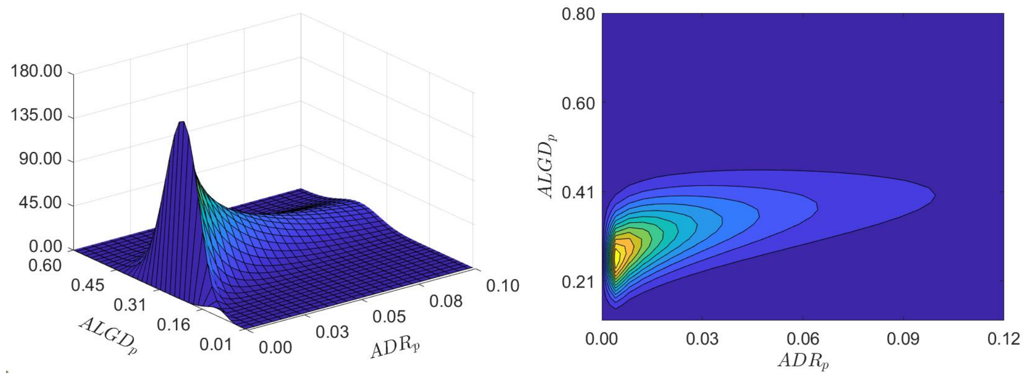
$$\frac{\partial g_2(dr,\ell)}{\partial \ell} \approx \frac{g_2(dr, \ell+\epsilon) - g_2(dr,\ell)}{\epsilon},$$

assuming very small value of ϵ. Having the computed value of J, the joint density of ADR_p and $ALGD_p$ is obtained using

$$f_{ADR_p, ALGD_p}(dr, \ell) = f_{S_A,S_B}(g_1(dr), g_2(dr,\ell))|J|, \tag{3.28}$$

where $f_{S_A,S_B}(g_1(dr), g_2(dr,\ell))$ is a bivariate normal density function of the pair (S_A, S_B). Figure 3.3 exhibits the numerically obtained joint density and associated contour plot. The contour, which is the height of the surface plot, are obtained by slicing the joint density (Eq. (3.28) surface of the pair $(ALGD_p, ADR_p)$ with planes parallel to the $ALGD_p - ADR_p$ plane and intersect at the corresponding values for the expression for the joint density of $ALGD_p$ and ADR_p evaluated at some given values for $ALGD_p$ and ADR_p.

Remark 11. *The closed form of the joint density of ADR_p and $ALGD_p$ presented in Eq. (3.28) can be numerically intensive to work with.*



(a) Surface plot of joint density of portfolio-level LGD and default rate.

(b) Contour plot of joint density of portfolio-level LGD and default rate.

Figure 3.3: Surface and contour plots (12 levels) of asymptotic portfolio-level LGD and default rate using potential loss defined as $H(b) = F^{-1}(1-\widehat{\Phi}(b))$. F is cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3, PD = 0.05, \rho_A = 0.4, \alpha = \beta = \sqrt{0.25}, \theta_S = 0.8, \theta_I = 0.2667$.

3.2.6 Marginal Density of $ALGD_p$

The general marginal density of $ALGD_p$ is

$$f_{ALGD_p}(\ell) = \int_{\mathbb{R}} f_{S_A,S_B}(g_1(dr), g_2(dr, \ell))|J|ddr, \tag{3.29}$$

where we used Eq. (3.28), and θ_I takes on arbitrary values.

Now for simplicity, we compare the finite and asymptotic portfolio-level LGD by considering $\theta_I = 0$ (that is independent idiosyncratic risk factors). With no idiosyncratic correlation, asymptotic portfolio-level LGD takes the form in Eq. (3.9). Suppose h is an increasing function of S_B, then the cdf of $ALGD_p$ is given by

$$\begin{aligned} F_p(\ell) &= \mathsf{P}(h(S_B) \leq \ell) \\ &= \mathsf{P}(S_B \leq h^{-1}(\ell)) \\ &= \Phi(h^{-1}(\ell)). \end{aligned}$$

Since S_B is normally distributed with 0 mean and standard deviation of 1. It follows that the density of $ALGD_p$ is derived as

$$\begin{aligned} f_p(\ell) &= \frac{d}{d\ell}\Phi(h^{-1}(\ell)) \\ &= \frac{\phi(h^{-1}(\ell))}{h'(h^{-1}(\ell))}. \end{aligned}$$

On the other hand if h is a decreasing function of S_B then the cdf of the $ALGD_p$ is derived as

$$\begin{aligned} F_p(\ell) &= \mathsf{P}(h(S_B) \leq \ell) \\ &= \mathsf{P}(S_B \geq h^{-1}(\ell)) \\ &= 1 - \mathsf{P}(S_B \leq h^{-1}(\ell)) \\ &= 1 - \Phi(h^{-1}(\ell)). \end{aligned}$$

The resulting density of $ALGD_p$ is therefore

$$\begin{aligned} f_p(\ell) &= -\frac{d}{d\ell}\Phi(h^{-1}(\ell)) \\ &= -\frac{\phi(h^{-1}(\ell))}{h'(h^{-1}(\ell))}. \end{aligned}$$

$h'(\cdot)$ is the derivative of h with respect to s and is given as (assuming existence of derivative under the integral)

$$h'(s) = \frac{\beta}{1-\beta^2} \cdot \int_{\mathbb{R}} H(b) \cdot (b - \beta s) \cdot \phi(b; \beta s, 1-\beta^2) db.$$

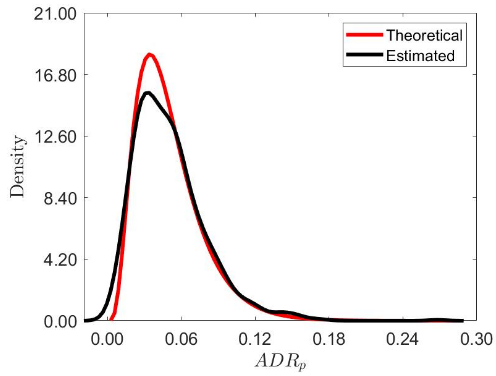
We compute $s_B = h^{-1}(\ell)$ by first setting

$$g_{S_B}(s_B) = \ell - h(s_B)$$

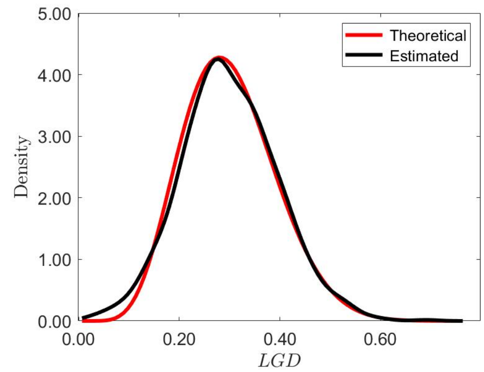
and then find the zero of $g_{S_B}(s_B)$ for a given value of ℓ. Note that this is numerically intensive procedure.

Figure 3.4 compares the estimated density of simulated values of finite portfolio-level LGD (LGD_p) (black curve) and the density of the asymptotic portfolio-level LGD ($ALGD_p$) (red curve) for independent idiosyncratic risk factors ($\theta_I = 0$). It illustrates that the estimated density of the simulated values of LGD_p agrees with the derived density function of $ALGD_p$ — as we increase the portfolio size from 500 to 5000, the estimated density approximates the theoretical density very well. The visual agreement is confirmed by the Kolmogorov-Smirnov test, where using the kstest function in MATLAB we observed that for a portfolio-size of 500, the p-value is 0.0009%, and a portfolio size of 5000, the p-value is 72%. The null hypothesis is that the two data sets come from the same distribution.

Figure 3.5 illustrates the densities of the account-level LGD (black curve) and the portfolio-level LGD (red curve). The transformation was chosen such that the account-level LGD distribution is beta with mean 40% and standard deviation 20%. Table 3.1 shows the statistical measures of the account and portfolio level LGD. The figure and table exhibit difference in the respective distributions. For instance, the skewness of both quantities shows a right skewed distribution for the account-level



(a) $N = 500$



(b) $N = 5000$

Figure 3.4: Estimated density of simulated values of finite portfolio-level LGD and the density of asymptotic portfolio-level LGD using potential loss defined as $H(b) = \mathcal{B}^{-1}(1 - \widehat{\Phi}(b))$. $\mathcal{B}$ is cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3$, and $\mathcal{B}^{-1}$ is the inverse. $PD = 0.05, \rho_A = 0.25, \alpha = \beta = \sqrt{0.25}, \theta_S = 1, \theta_I = 0$. Number of simulated systematic risk factors $m = 1000$. The ksdensity function Matlab is used for density estimation

LGD (skewness = 0.2857) and the portfolio-level LGD (skewness = 0.3715) and much variation in the observations of the account-level LGD (standard deviation = 0.2000) as compared to that of the portfolio-level LGD (standard deviation = 0.0921) — this could be attributed to diversification effect.

Remark 12. *This demonstrates that LGD can behave differently at the account and portfolio level. In particular, the means and standard deviations of these quantities are different. We explain the source of this difference in Section 3.3.*

Table 3.1: Statistical measure of asymptotic account and portfolio level LGD. We use the transformation $H(b) = \mathcal{B}^{-1}((1-\widehat{\Phi}(b)), \delta_1, \delta_2)$. $\mathcal{B}^{-1}$ is the inverse of the cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3, PD = 0.05, \rho_A = 0.25, \alpha = \beta = \sqrt{0.25}$. $\theta_S = 1, \theta_I = 0$.

Statistical Measure	Account-Level LGD	Portfolio-Level LGD
Mean	0.4000	0.3007
Median	0.3846	0.2944
Kurtosis	2.3571	2.9781
Skewness	0.2857	0.3715
Standard deviation	0.2000	0.0921

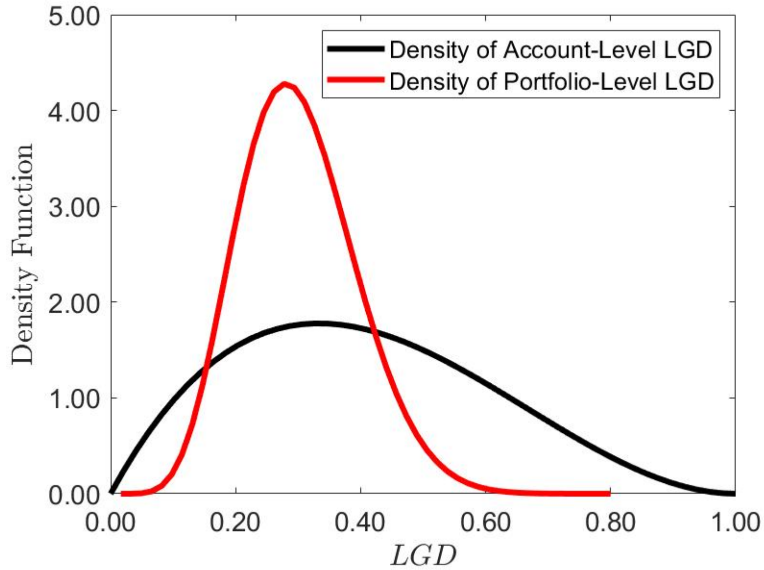


Figure 3.5: Densities of account-level LGD and portfolio-level LGD. Potential loss is defined as $H(b) = \mathcal{B}^{-1}((1-\widehat{\Phi}(b)), \delta_1, \delta_2)$. $\mathcal{B}^{-1}$ is the inverse of the cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3, PD = 0.05, \rho_A = 0.25, \alpha = \beta = \sqrt{0.25}, \theta_S = 1, \theta_I = 0$.

3.3 Account and portfolio level relationships

The correlation between default and loss drivers ρ_A summarizes the relationship between loss and default at the account level. It is natural to investigate whether it also determines the relationship between default and loss at the portfolio level. Specifically, we are looking at the extent to which the account level relationship between default and loss determines portfolio level relationship between default and loss.

Recall that the account-level correlation ρ_A is given as

$$\rho_A = \alpha\beta\theta_S + \sqrt{1-\alpha}\sqrt{1-\beta}\theta_I. \tag{3.30}$$

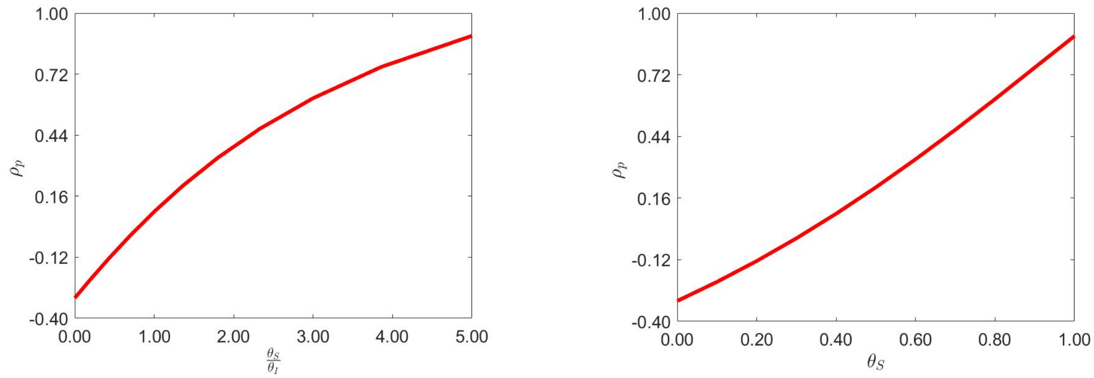
It is clear that for a given value of ρ_A, there are different possible combinations of θ_S and θ_I that are consistent with the values of ρ_A. If we vary (θ_S, θ_I) such that ρ_A is fixed, we are effectively varying the *source* of the account-level relation while its *degree* or *magnitude* is fixed. In order to determine whether portfolio-level relationship between default and loss is dictated by the *degree* or *source* of the account-level relationship we perform the following experiment. First we fixed α, β and ρ_A. Fixing ρ_A fixes the degree of account-level correlation. Next we created a grid of values in the unit interval $[0, 1]$ for θ_S. For each value of θ_S in the grid, we set

$$\theta_I = \frac{\rho_A - \alpha\beta\theta_S}{\sqrt{1-\alpha^2}\sqrt{1-\beta}}$$

and estimate the correlation between ADR_p and $ALGD_p$, ρ_p, as described in Algorithm 3, which is a simulation based estimation scheme. We chose this approach even though we could have used the joint density of ADR_p and $ALGD_p$ (Eq. (3.28)) because it is computationally easy.

Figure 3.6 illustrates the result of this experiment. We use the transformation of B such that account-level LGD distribution is beta with mean and standard deviation, 40% and 20% respectively. The figure illustrates a positive relationship between ρ_p and respectively $\frac{\theta_S}{\theta_I}$ and θ_S. Thus, even though the degree of account-level relationship has no bearing on portfolio-level relationship, the individual components impact portfolio-level relationship. In other words, the portfolio-level relationship is not determined by the degree of account-level relationship. It is determined by its source.

Of particular note, ρ_p has a mixture of negative and positive values for some parameter values of $\frac{\theta_S}{\theta_I}$. Indicating that if the correlation at the account-level is due primarily to correlation between systematic (relative to idiosyncratic) risk factors, the portfolio-level correlation is strong and positive (see Figure 3.6a). For example, if θ_S contributes about 4 times relative to θ_I to account-level relationship, the portfolio-level correlation is about 0.72. In this case, LGD estimated under economic downturn



(a) (b)

Figure 3.6: Relationship between the correlation between portfolio-level default rate and loss given default and correlation between risk factors using potential loss defined as $H(b) = \mathcal{B}^{-1}((1-\widehat{\Phi}(b)), \delta_1, \delta_2)$. $\mathcal{B}^{-1}$ is the inverse of the cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3, PD = 0.05, \rho_A = 0.4, \alpha = \beta = \sqrt{0.25}$.

condition will be greater than that estimated under through the cycle (TTC) condition. This is because downturn LGD is in sync with expected behavior of business cycles and thus changes with these cycles whereas TTC LGD is not cyclical and may be the representation of the cycle-average LGD over several periods, hence rendering LGD estimates relatively constant over the business cycle [74]. On the other hand, if the account-level correlation is primarily attributed to idiosyncratic (relative to systematic) risk factors, the portfolio-level correlation is strong and negative. In this case, we have an inverse relationship between portfolio-level default rate and LGD, thus suggesting lower estimates of LGD during downturn than estimates under TTC. A contradictory to what is expected in the Basel II [20].

Remark 13. *The above suggests that ensuring that the account-level relationship is positive does not necessarily guarantee that downturn LGD behaves as the Basel requirement says it should. We are fairly certain that this important discovery has not appeared in the literature.*

Figure 3.7 supports the observation in Figure 3.6. It shows the contour plots of the pair $(ADR_p, ALGD_p)$ for different parameter values of θ_S and θ_I. The figure

Algorithm 3 Calculating correlation between ADR_p and $ALGD_p$

1: Input parameters — (α, β): sensitivity of systematic and idiosyncratic risk factors, PD: common probability of default, n: number of simulations, θ_I: correlation between systematic risk factors (I_A, I_B), θ_S: correlation between idiosyncratic risk factors (S_A, S_B), ρ_A: correlation between default and loss drivers

2: For each θ_S

i set $\theta_I = \frac{\rho_A - \alpha\beta\theta_S}{\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}$

ii generate n observations of the pair (S_A, S_B)

iii using the observations of the pair (S_A, S_B) as inputs, set

- $ADR_p, = \Phi\left(\frac{\Phi^{-1}(PD) - \alpha s_A}{\sqrt{1-\alpha^2}}\right)$
- $ARL_p = \int_{\mathbb{R}} H(b) \cdot \Phi\left(\frac{\Phi^{-1}(PD) - \alpha s_A - \theta_I \sqrt{\frac{1-\alpha^2}{1-\beta^2}}(b - \beta s_B)}{\sqrt{(1-\alpha^2)(1-\theta_I^2)}}\right) \cdot \phi(b; \beta s_B, 1-\beta^2) db$
- $ALGD_p = \frac{ARL_p}{ADR_p}$

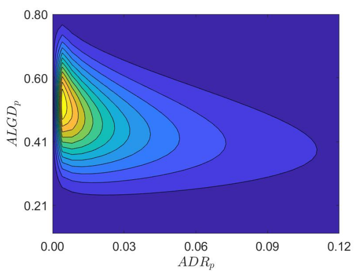
3: End for each θ_S

4: Calculate the correlation between ADR_p and $ALGD_p$

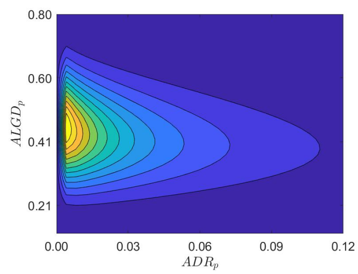
illustrates increasing concentration of the contour lines around the respective means of ADR_p and $ALGD_p$, for positive values of ρ_p ($\theta_S > 0.3402$, Figures 3.7c-3.7e) and negative values of ρ_p ($\theta_S < 0.3402$, Figures 3.7a and 3.7b). This shows the link between θ_S and the joint density via ρ_p. An important issue relating to the portfolio-level loss distribution, hence risk parameter estimations.

Figure 3.8 shows the density plot of $ALGD_p$ for varying parameter value combinations of the pair (θ_S, θ_I) given fixed value of ρ_A, α, and β. The figure indicates that the shape of the density of $ALGD_p$ is dependent on the values of θ_S and θ_I. In particular, the density plots show that as θ_S increases from 0.0 to 0.8, the likelihood of a portfolio realizing low values of losses increases. Figure 3.8b clearly exposes this observation, where the mean of $ALGD_p$ relates inversely to $\frac{\theta_S}{\theta_I}$. This echoes the story from Figures 3.6 and 3.7. For instance, $\frac{\theta_S}{\theta_I} = 0.00$ indicates that the source of the account-level relationship is primarily due to the the correlation between the idiosyncratic components. The portfolio-level correlation ρ_p in this case is strong and negative.

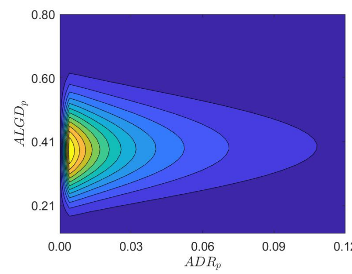
Remark 14. *The above observed phenomenon highlights the importance of dependency structure in a credit risk model in the accurate estimation of risk parameters — PD, LGD, EC. The natural question then is, what is the source of this phenomenon? The ensuing section explains the origin of this phenomenon.*



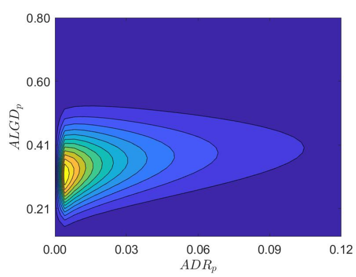
(a) $\theta_S = 0.00, \theta_I = 0.53$



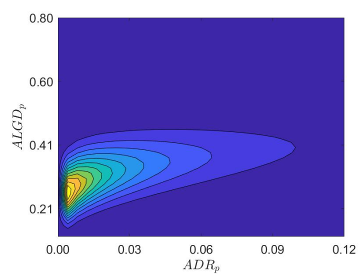
(b) $\theta_S = 0.20, \theta_I = 0.47$



(c) $\theta_S = 0.40, \theta_I = 0.40$



(d) $\theta_S = 0.6, \theta_I = 0.33$



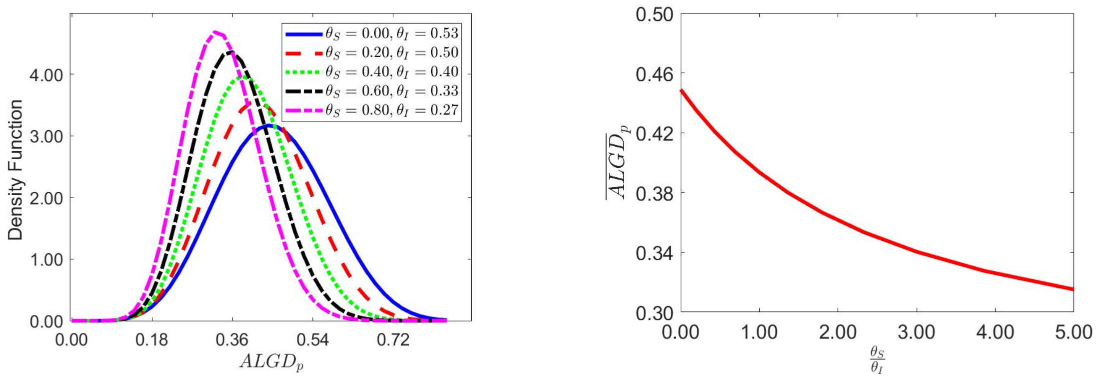
(e) $\theta_S = 0.8, \theta_I = 0.27$

Figure 3.7: Contour plots of the joint density of asymptotic portfolio-level LGD and default rate for different value combinations of correlation between systematic risk factors and idiosyncratic risk factors using potential loss defined as $H(b) = \mathcal{B}^{-1}((1 - \widehat{\Phi}(b)), \delta_1, \delta_2)$. $\mathcal{B}^{-1}$ is the inverse of the cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3$, $\rho_A = 0.4, \alpha = \sqrt{0.25}, \beta = \sqrt{0.25}, \delta_1 = 2, \delta_2 = 3, PD = 0.05$

3.3.1 Source of Account-Level Correlation

By definition, the correlation between asymptotic portfolio-level default rate and LGD is

$$\rho_p = \frac{\text{Cov}(ADR_p, ALGD_p)}{\sigma_{ADR_p} \cdot \sigma_{ALGD_p}},$$



(a) Density function of $ALGD_p$ for different value combinations of the pair (θ_S, θ_I)

(b) Mean of $ALGD_p$ verses θ_S

Figure 3.8: Density of asymptotic portfolio-level LGD $ALGD_p$ for different parameter value combinations of the correlation between the systematic θ_S and idiosyncratic θ_I risk factors and the plot of the mean of $ALGD_p$ against $\frac{\theta_S}{\theta_I}$ using potential loss defined as $H(b) = \mathcal{B}^{-1}((1-\widehat{\Phi}(b)), \delta_1, \delta_2)$. $\mathcal{B}^{-1}$ is the inverse of the cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3, PD = 0.05, \rho_A = 0.4, \alpha = \beta = \sqrt{0.25}$.

where σ_{ALGD_p} is the standard deviation of $ALGD_p$. Noting that $ALGD_p = \frac{ARL_p}{ADR_p}$ and $\mathbb{E}[ARL_p] = PD \cdot LGD_A$,

$$\begin{aligned}\mathrm{Cov}(ADR_p, ALGD_p) &= \mathbb{E}[ADR_p \cdot ALGD_p] - \mathbb{E}[ADR_p] \cdot \mathbb{E}[ALGD_p] \\ &= \mathbb{E}[ARL_p] - \mathbb{E}[ADR_p] \cdot \mathbb{E}[ALGD_p] \\ &= PD(LGD_A - \mathbb{E}[ALGD_p])\end{aligned}$$

is the covariance of ADR_p and $ALGD_p$. The sign of ρ_p is governed by the relative movement of ADR_p and $ALGD_p$, which is captured by $\mathrm{Cov}(ADR_p, ALGD_p)$ and is determined by the relative sizes of LGD_A and $\mathbb{E}[ALGD_p]$. $\mathrm{Cov}(ADR_p, ALGD_p)$ is positive if $LGD_A > \mathbb{E}[ALGD_p]$ and negative if $LGD_A < \mathbb{E}[ALGD_p]$. Hence, comprehending the sign of ρ_p is equivalent to comprehending the sign of $\mathrm{Cov}(ADR_p, ALGD_p)$. In fact, Figure 3.8b shows the dependency of $\mathbb{E}[ALGD_p]$ on θ_S. Noting that LGD_A is independent of θ_S, our discussion focuses on the influence of the relative size of θ_S on $\mathbb{E}[ALGD_p]$.

Remark 15. *To explain the origin of the phenomenon observed in Figure 3.6 (mixture of negative and positive sign for ρ_p), we need to establish the influence the relative sizes of θ_S and θ_I have on $Cov(ADR_p, ALGD_p)$ via their effect on $\mathbb{E}[ALGD_p]$. We do this by first finding the expression for $\mathbb{E}[ALGD_p]$.*

We know the expressions for $E[ARL_p]$ and $\mathbb{E}[ADR_p]$ (see Eq. (3.11) and (3.19)), so we are left with obtaining the expression for $\mathbb{E}[ALGD_p]$. Using quadratic Taylor Series Approximation, the approximation of $\mathbb{E}[ALGD_p]$ is (see Appendix B.3 for detailed discussion)

$$\mathbb{E}[ALGD_p] \approx \frac{\mu_{ARL_p}}{\mu_{ADR_p}} + \frac{\mu_{ARL_p}}{\mu^3_{ADR_p}}\sigma^2_{ADR_p} - \frac{1}{\mu^2_{ADR_p}}\sigma_{ADR_p,ARL_p}.$$

From Eq. (3.11) and (3.19)

$$\frac{\mu_{ARL_p}}{\mu_{ADR_p}} = LGD_A, \tag{3.31}$$

and from Eq. (3.11), (3.16), and (3.19)

$$\begin{aligned}\frac{\mu_{ARL_p}}{\mu^3_{ADR_p}}\sigma^2_{ADR_p} &= \frac{LGD_A}{PD^2}\left(PD^{(2)} - PD^2\right)\\ &= \frac{LGD_A}{PD^2} \cdot PD^{(2)} - LGD_A,\end{aligned} \tag{3.32}$$

and from Eq. (3.26) and (3.11)

$$\begin{aligned}\frac{1}{\mu^2_{ADR_p}}\sigma_{ADR_p,ARL_p} &= \frac{1}{PD^2}\left(LGD_A^{(2)} \cdot PD^{(2)} - PD^2 \cdot LGD_A\right)\\ &= \frac{1}{PD^2}LGD_A^{(2)} \cdot PD^{(2)} - LGD_A.\end{aligned} \tag{3.33}$$

Combining Eq. (3.31), (3.32) and (3.33) gives

$$\mathbb{E}[ALGD_p] \approx LGD_A - \frac{PD^{(2)}}{PD^2} \cdot \Delta LGD,$$

where

$$\Delta LGD = LGD_A^{(2)} - LGD_A. \tag{3.34}$$

Thus,

$$\text{Cov}(ADR_p, ALGD_p) \approx \frac{PD^{(2)}}{PD}\Delta LGD.$$

From Eq. (3.34), $\text{Cov}(ADR_p, ALGD_p)$ changes sign at $LGD_A^{(2)} = LGD_A$. Observe that account-level joint LGD, $LGD_A^{(2)}$, is governed by the parameters $\delta_1, \delta_2, \alpha, \beta, \theta_I$ and θ_S, so the value of ΔLGD is determined by values of these parameters. In particular θ_S or θ_I — the respective correlation between the systematic and idiosyncratic risk factors — highlights its value. Figure 3.9 compares the exact and approximated values of $\mathbb{E}[ALGD_p]$ using varying values of θ_S. The figure indicates a close match between the two quantities.

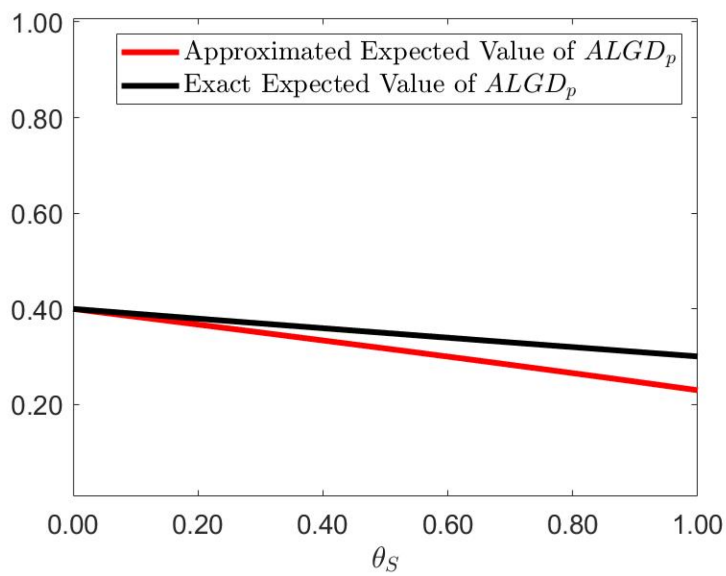


Figure 3.9: Exact and Approximated Values of $\mathbb{E}[ALGD_p]$. $\delta_1 = 2$, δ_3, $PD = 0.05$, $\rho_A = 0.25$, $\alpha = \beta = \sqrt{0.25}$, $\theta_I = 0$.

3.3.2 On $LGD_A^{(2)}$ and LGD_A

We know that $H(B)$ is a monotone function. This suggests that its mean will be a monotone function of the mean of B_i given that exposure i and j defaulted. Since we can use the result from [71] to get that conditional mean, we use this to explain the impact of the relative size of θ_S on $Cov(ADR_p, ALGD_p)$ through $\mathbb{E}[ALGD_p]$.

Now, from Eq. (11) in [71], we obtain (see Appendix B.4) $\mu_{\Phi^{-1}(PD)} = \mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD)]$ as

$$\mu_{\Phi^{-1}(PD)} = -\rho_A \cdot \frac{\phi(\Phi^{-1}(PD))}{PD}.$$

$\mu_{\Phi^{-1}(PD)}$ is the conditional expected value of loss driver given that default occurred. Also, $\mu^{(2)}_{\Phi^{-1}(PD)} = \mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)]$ is derived as (see Appendix B.4)

$$\mu^{(2)}_{\Phi^{-1}(PD)} = -(\rho_A + \alpha\beta\theta_S) \cdot \frac{\phi(\Phi^{-1}(PD))}{PD^{(2)}} \cdot \zeta,$$

where

$$\zeta = \Phi\left(\frac{\Phi^{-1}(PD)(1-\alpha^2)}{\sqrt{1-\alpha^4}}\right)$$

is the conditional probability of $A_i \leq \Phi^{-1}(PD)$ given $A_j = \Phi^{-1}(PD)$. $\mu^{(2)}_{\Phi^{-1}(PD)}$ is the conditional expectation of loss driver given simultaneous occurrence of defaults of two obligors. Note that $\mu^{(2)}_{\Phi^{-1}(PD)}$ is a monotone function of θ_S.

Theorem 4. $\mu^{(2)}_{\Phi^{-1}(PD)} = \mu_{\Phi^{-1}(PD)}$ *if and only if* $\frac{\theta_S}{\theta_I} = \frac{\theta_S^*}{\theta_I^*} = \frac{\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}{\alpha\beta}\left(\frac{PD^{(2)}-\zeta PD}{2\zeta PD - PD^{(2)}}\right)$.

Proof. Suppose $\mu^{(2)}_{\Phi^{-1}(PD)} = \mu_{\Phi^{-1}(PD)}$, then

$$\theta_S = \theta_S^* = \frac{\rho_A}{\alpha\beta}\left(\frac{PD^{(2)}}{\zeta PD} - 1\right).$$

Inserting θ_S^* in Eq. (3.30) and solving for θ_I gives

$$\theta_I = \theta_I^* = \frac{\rho_A}{\sqrt{1-\alpha^2}\sqrt{1-\beta^2}} \cdot (2\zeta PD - PD^{(2)}).$$

The result follows from the ratio of θ_S^* to θ_I^* $(\frac{\theta^*}{\theta_I^*})$.

Conversely, suppose $\frac{\theta_S}{\theta_I} = \frac{\theta_S^*}{\theta_I^*}$. This is equivalent to $\theta_S = \theta_S^*$. Inputting $\theta_S = \theta_S^*$ in the expression for $\mu^{(2)}_{\Phi^{-1}(PD)}$ gives the result. $\square$

Corollary 4.1. $\mathbb{E}[B_i|S_A, S_B, \mathcal{D}_i = 1, \mathcal{D}_j = 1]$ *is a monotone function of* θ_S*.*

Proof. Since

$$\mathbb{E}[B_i|\mathcal{D}_i = 1, \mathcal{D}_j = 1] = \mathbb{E}[E[B_i|S_A, S_B, \mathcal{D}_i = 1, \mathcal{D}_j = 1]|\mathcal{D}_i, \mathcal{D}_j = 1],$$

the result follows from Theorem 4. $\square$

Remark 16. $LGD_A^{(2)} = LGD_A$ *if and only if* $\frac{\theta_S}{\theta_I} = \frac{\theta_S^*}{\theta_I^*}$*. In other words there exist a threshold ratio of* θ_S *to* θ_I *for which* $LGD_A^{(2)} = LGD_A$*.*

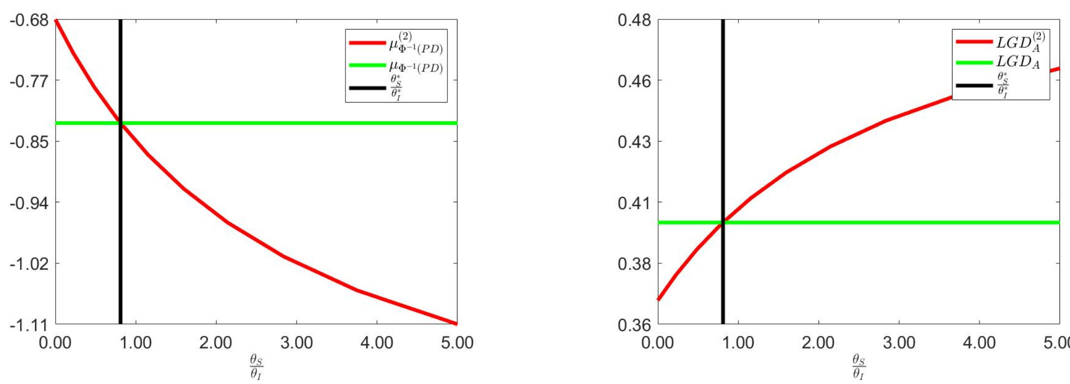
Proof. Note that

$$\begin{aligned} LGD_A^{(2)} &= \mathbb{E}[\mathcal{PL}_i|\mathcal{D}_i = 1, \mathcal{D}_j = 1] \\ &= \mathbb{E}[\mathbb{E}[\mathcal{PL}_i|S_A, S_B, \mathcal{D}_i = 1, \mathcal{D}_j = 1]|\mathcal{D}_i = 1, \mathcal{D}_j = 1] \\ &= \mathbb{E}[\mathbb{E}[H(B_i)|S_A, S_B, \mathcal{D}_i = 1, \mathcal{D}_j = 1]|\mathcal{D}_i = 1, \mathcal{D}_j = 1]. \end{aligned}$$

So $LGD_A^{(2)}$ is a monotone function of θ_S if and only if $\mathbb{E}[H(B_i)|S_A, S_B, \mathcal{D}_i = 1, \mathcal{D}_j = 1]$ is a monotone function of θ_S. $\square$

Figures 3.10 and 3.11 illustrate Remark 16. These figures are obtained by using fixed parameter values of ρ_A, α and β while the value of θ_S varies. Observe in Figure 3.10 that the value (black line) of $\frac{\theta_S}{\theta_I}$ for which the graphs (see Figure 3.10a) of $\mu^{(2)}_{\Phi^{-1}(PD)}$ (red line) and $\mu_{\Phi^{-1}(PD)}$ (green line) intersect equals the value (black line) at the intersection of the graphs (see Figure 3.10b) of $LGD_A^{(2)}$ (red line) and LGD_A (green line). This observation leads to the following remark.

Remark 17. *Because $Cov(ADR_p, ALGD_p) \geq 0$ if and only if $\frac{\theta_S}{\theta_I} \geq \frac{\theta_S^*}{\theta_I^*}$, and conversely, $Cov(ADR_p, ALGD_p) \leq 0$ if and only if $\frac{\theta_S}{\theta_I} \leq \frac{\theta_S^*}{\theta_I^*}$, we conclude that the sign of ρ_p is determined by $\frac{\theta_S}{\theta_I}$. It takes positive values for values of $\frac{\theta_S}{\theta_I} > \frac{\theta_S^*}{\theta_I^*}$ and negative values when $\frac{\theta_S}{\theta_I} < \frac{\theta_S^*}{\theta_I^*}$.*



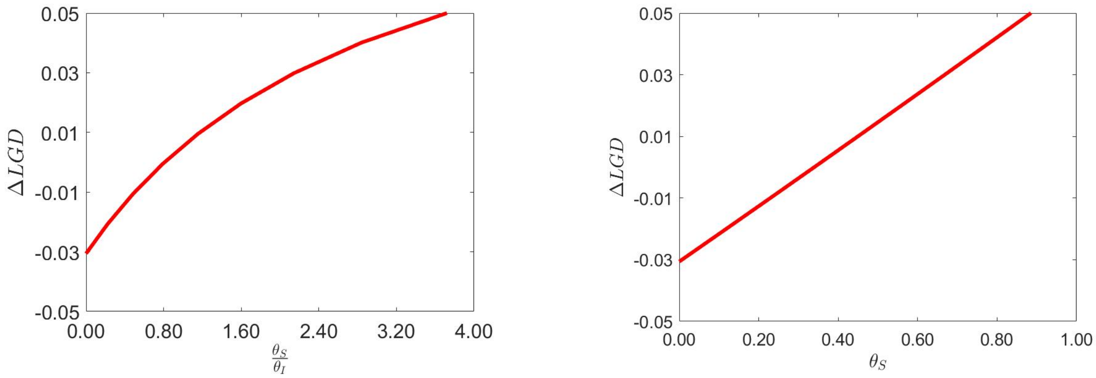
(a) $\mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD)]$ and $\mathbb{E}[B_i|A_i < \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)]$

(b) LGD_A and $LGD_A^{(2)}$

Figure 3.10: Conditional expectation of loss driver and account-level and potential loss. Potential loss is defined as $H(b) = \mathcal{B}^{-1}(1 - \widehat{\Phi}(b))$. $\mathcal{B}$ is cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3$, and $\mathcal{B}^{-1}$ is the inverse. $PD = 0.05, \rho_A = 0.4, \alpha = \beta = \sqrt{0.25}$.

3.4 Conclusion

PD-LGD correlations affect immensely the estimates of portfolio risk measures. Understanding the source of these correlations is of paramount importance as they can



(a) (b)

Figure 3.11: Difference in expected value of account-level LGD account-level joint LGD using potential loss defined as $H(b) = \mathcal{B}^{-1}((1 - \widehat{\Phi}(b)), \delta_1, \delta_2)$. $\mathcal{B}^{-1}$ is the inverse of the cumulative beta distribution with scale parameters $\delta_1 = 2$ and $\delta_2 = 3, PD = 0.05, \rho_A = 0.4, \alpha = \beta = \sqrt{0.25}$.

introduce additional variability into the losses on exposures in the event of defaults and the distribution of the overall losses on the portfolio.

In this chapter, we specify the expressions for asymptotic portfolio default rate and loss given default and their joint and marginal distribution. We use these expressions to explain the link between the account and portfolio level relationships.

We show that portfolio-level correlation is positively related to the correlation between the systematic risk factors. And that at some threshold the correlation between portfolio level default rate and loss given default transitions from positive to negative. Suggesting that with a fixed value of account-level relationship — correlation between default and loss drivers — portfolio-level correlations can vary.

Furthermore, we demonstrated that under certain conditions the expected value of the account and portfolio level loss given default can be equal, less or greater than each other. This threshold is determined by the relative size of the correlation between systematic risk factors to idiosyncratic risk factors. This phenomenon impacts estimates of downturn LGD via the portfolio-level relationship — the correlation between portfolio-level default rate and LGD. Default and LGD at the portfolio level can relate inversely or positively. This contradicts the expectation of Basel II, where it is believed that estimates of downturn LGD will be greater than TTC LGD estimates. As this determines the relative sizes of the estimates of LGD under each regime (downturn or TTC).

The above findings imply that parameter value choices in credit risk models determine largely the relative sizes of downturn and TTC LGD estimates, which means economic capital could be underestimated or overestimated.

Chapter 4

Empirical Study on Model Parameters

This chapter presents a methodology for estimating the parameters in the model. The methodology is based on the Method of Moments (MoM). Monte Carlo simulation is used to validate the proposed methodology. An empirical study on delinquency and charge-off rates from the Federal Financial Institutions Examination Council (FFIEC) Consolidated Reports of Condition and Income on the 100 largest banks in the United States is conducted, where the proposed method is applied.

4.1 Summary of Model of Interest

Recall from Section 2.1.1, the expressions governing default A and loss B drivers for a representative account i:

$$\begin{cases} A_i & = \alpha \cdot S_A + \sqrt{1-\alpha^2} \cdot I_{A,i} \\ B_i & = \beta \cdot S_B + \sqrt{1-\beta^2} \cdot I_{B,i} \end{cases},$$

where $-1 \leq \alpha, \beta \leq 1$, the pairs (S_A, S_B) and $(I_{A,i}, I_{B,i})$ are respectively standard bivariate normal systematic and idiosyncratic risk factors with respective correlation parameters θ_S and θ_I. The drivers (A_i, B_i) are bivariate standard normal with correlation parameter $\rho_A = \alpha\beta\theta_S + \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I$. These drivers are sensitive to the systematic and idiosyncratic risk factors through α and β. The default scenario is defined as

$$\mathcal{D} = \mathbb{1}_{\{A_i \leq \Phi^{-1}(PD)\}}.$$

We are interested in estimating the parameters in the above equations and LGD via the expression for potential loss. To carry out the estimation procedure, we assumed transformation of loss drivers to potential loss using the relation

$$\mathcal{PL} = \mathcal{B}^{-1}(1 - \widehat{\Phi}(b), \delta_1, \delta_2),$$

where $\mathcal{B}^{-1}$ is the inverse beta cdf. In all we have the following parameters to estimate: $\alpha, \beta, \theta_I, \theta_S, \delta_1, \delta_2, PD$, and ρ_A.

4.2 Data description

The exploratory study presented in this chapter is based on the dataset on delinquency (default (DR_p)) and charge-off (realized loss ($\mathcal{RL}_p$)) rates compiled from the quarterly Federal Financial Institutions Examination Council (FFIEC) Consolidated Reports of Condition and Income on 100 largest banks in the United States [29]. Charge-off rate is calculated as the difference in a bank's gross charge-offs and recoveries during a quarter written as a proportion of that quarter's average level of unpaid loans. These are seasonally adjusted. Charge-offs are values of "bad loans" that are written against loss reserves. Loans are considered delinquent, when they are past due at least thirty days. These loans could accrue interest or not. The delinquency rates are obtained as the ratio of the defaulted amount of loans (in monetary terms) to the total amount of outstanding loans and are seasonally adjusted. We calculate the LGD for each period as $\frac{\mathcal{RL}_p}{DR_p}$.

The loans of these banks are categorized under seven groups: Real Estate, Consumer, Commercial and Industrial (C & I), Leases, Agricultural, and All Loans and Leases (referred as Total Loans and Leases in the report). The Real Estate category has four subcategories — Residential, Commercial, Farmland and All. Also the Consumer category has three subcategories — Credit Cards, Other than Credit Cards and All. So in all, the study involves 11 time series.

The dataset on realized loss covers 138 quarters over a period of 1985 to 2019 for Commercial and Industrial, Leases, and Agricultural loans. That of default rates cover 130 quarters — from a period of 1987 to 2019. The Real estate dataset on defaults and realized losses runs from 1991 to 2019, a total of 114 quarters for its subcategories. Under consumer loans, we have 138 and 114 quarters respectively for charge-offs and delinquency rates for both categories, Credit Cards and Other than

Credit Cards. The category for All Consumers has 138 and 130 quarters for charge-offs and delinquency rates respectively. The total of these loans categories across each quarter has 138 quarters for both charge-offs and delinquency rates. We did not consider quarters with missing default rate in our computation. Table 4.1 and 4.2 present the respective summary statistics of delinquency and charge-off rates and LGDs of these loan categories.

Table 4.1: Summary statistics of delinquency (default) and charge-off rates (realized loss) of loan categories. Values are presented in percentages.

Loan Type	Delinquency/Default rate			Charge-off rate/Realized loss		
	Mean	Median	standard deviation	Mean	Median	standard deviation
Real Estate:						
Residential	4.50	2.83	3.44	0.50	0.17	0.73
Commercial	4.28	2.12	4.48	0.68	0.12	1.06
Farmland	3.80	3.41	1.81	0.24	0.14	0.39
All	4.75	3.66	2.99	0.59	0.22	0.71
Consumer:						
Credit Cards	4.00	4.17	1.19	4.66	4.30	1.66
Other	2.91	2.94	0.63	1.28	1.10	0.60
All	3.46	3.63	0.82	2.68	2.42	1.03
C & I	2.59	1.76	1.76	0.76	0.53	0.62
Leases	1.38	0.13	1.38	0.40	0.31	0.31
Agricultural	4.15	3.06	3.04	0.40	0.25	0.81
All Loans and Leases	3.73	2.86	1.89	1.04	0.80	0.66

4.3 Estimation Scheme and Results

The estimation scheme presented in this thesis is based on the Method of Moments (MoM). The MoM matches population moments of random variables to its sample counterparts, where the population moments are expressed as a function of the parameters of interest and these parameters are then recovered. The values obtained are the estimates of these parameters. The formal description is as follows: Let $\mathbb{B}_1, \mathbb{B}_2, ..., \mathbb{B}_T$ be a sequence of independent random sample from a population with

Table 4.2: LGD statistics for loan categories and the correlation between delinquency rates and LGD. Values are presented in percentages.

Loan Type	Mean	Median	standard deviation	ρ_p
Real Estate:				
Residential	8.50	6.54	7.15	44.10
Commercial	9.99	6.44	9.87	68.10
Farmland	5.30	41.65	68.63	32.36
All	9.40	7.84	7.34	66.11
Consumer:				
Credit Cards	119.68	114.84	28.85	-38.52
Other	44.52	40.34	17.64	-10.76
All	79.41	81.07	24.28	-31.29
C & I	29.50	28.76	14.05	-3.51
Leases	26.61	25.03	14.32	50.17
Agricultural	9.61	7.98	10.69	0.58
All Loans and Leases	28.52	28.03	8.73	-14.71

distribution function $f(b|\Theta)$, where Θ is a q-dimensional vector of parameters. Define the respective q^{th} sample and population moments as

$$\begin{cases} \overline{\mathbb{B}^q} = \frac{1}{T}\sum_{i=1}^{n} b_i^q, \\ \mathbb{E}[\mathbb{B}^q] = \mu_q(\Theta). \end{cases}$$

The MoM estimator $\widehat{\Theta}$ of Θ is then found by solving the system of equations for Θ that makes the population moments equal the sample moments

$$\begin{cases} \overline{\mathbb{B}^1} & = \mu_1(\Theta), \\ \overline{\mathbb{B}^2} & = \mu_2(\Theta), \\ & \vdots \\ \overline{\mathbb{B}^q} & = \mu_q(\Theta). \end{cases}$$

Even though MoM estimators often yields estimates that can be improved upon (for example, estimates can be out of parameter space), its choice is attractive in that its application is simple and typically yields appreciable results — see for example, [49, 55] for application of MoM to parameter estimation on portfolio credit risk models. The model discussed in this document presents the challenge of solving a system of five equations with seven unknowns. That is we have the problem of estimating seven parameters using five moments. Mapping those parameters to the five moments leaves us with two undetermined parameters. We have observed that necessary risk measures such as Value-at-Risk (VaR), the mean and standard deviation of the portfolio-level LGD, and portfolio-level correlation ρ_p do not depend on these two parameters.

Since we demonstrated in Chapter 3 that as the portfolio size increases, the distributions of the finite portfolio quantities approximates accurately that of their asymptotic quantities, we therefore based the estimation scheme on the asymptotic quantities — see Figures 3.2 and 3.4. Against this background, we want to choose values of parameters such that the empirical moments for finite portfolio equals that of the theoretical moments of asymptotic portfolio. So for instance, for large portfolio size, $\mathcal{RL}_p \approx ARL_p$, thus $\overline{\mathcal{RL}_p} \approx \overline{ARL_p}$. We are therefore interested in finding values of parameters such that $\overline{\mathcal{RL}_p} \approx \overline{ARL_p} = \mathbb{E}[ARL_p]$. This estimation approach is consistent with the dataset used for our analysis. Table 4.3 reports a summary of the model implied quantities and the parameter on which they depend. The ensuing subsections present the proposed estimation scheme.

Table 4.3: Model implied quantities and their parameter dependencies

Model implied quantities	Parameters
$\mathbb{E}[ADR_p]$	PD
$\mathbb{E}[ADR_p^2]$	α, PD
$\mathbb{E}[ARL_p]$	δ_1, δ_2, PD
$\mathbb{E}[ADR_p \cdot ARL_p]$	$\alpha, \beta\theta_S, \delta_1, \delta_2, \rho_A, PD$
$\mathbb{E}[ARL_p^2]$	$\alpha, \beta, \beta\theta_S, \delta_1, \delta_2, \rho_A, PD$

4.3.1 Estimating PD

Our estimation scheme starts with the estimation of default parameters. From Eq. (3.11), we note that $E[ADR_p] = E[D_i = 1] = PD$. So we can estimate PD by using the average of the delinquency rates:

$$\widehat{PD} = \overline{ADR_p} = \frac{1}{T}\sum_{i=1}^{T} ADR_{p,i}.$$

4.3.2 Estimating α

The estimation is based on the expression for asymptotic conditional default rate given S_A. From Eq. (3.13) and (3.17),

$$\mathbb{E}[ADR_p^2] = \mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)), \tag{4.1}$$

which is computed using the bivariate normal cdf. The mean vector and covariance matrix are expressed as

$$\mu_{A_i,A_j} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \Sigma_{A_i,A_j} = \begin{bmatrix} 1 & \alpha^2 \\ \alpha^2 & 1 \end{bmatrix}$$

respectively. Knowing the estimated value of PD implies we have one variable — α — to estimate from Eq. (4.1). This, we do by setting the sample second moment of ADR_p to its population counterpart and solve for α:

$$\overline{ADR_p^2} = \frac{1}{T}\sum_{i=1}^{T} ADR_{p,i}^2 = \mathbb{E}[ADR_p^2].$$

Figure 4.1 illustrates the graph of the empirical and theoretical second moments of ADR_p — respectively $\overline{ADR_p^2}$ and $\mathbb{E}[ADR_p^2]$ — as a function of α. We observe that α impacts $\mathbb{E}[ADR_p^2]$. This implies that it can affect other risk measures in the model as well. We also see that $\mathbb{E}[ADR_p^2]$ is monotone in α and it intersects the graph of $\overline{ADR_p^2}$ at a unique point.

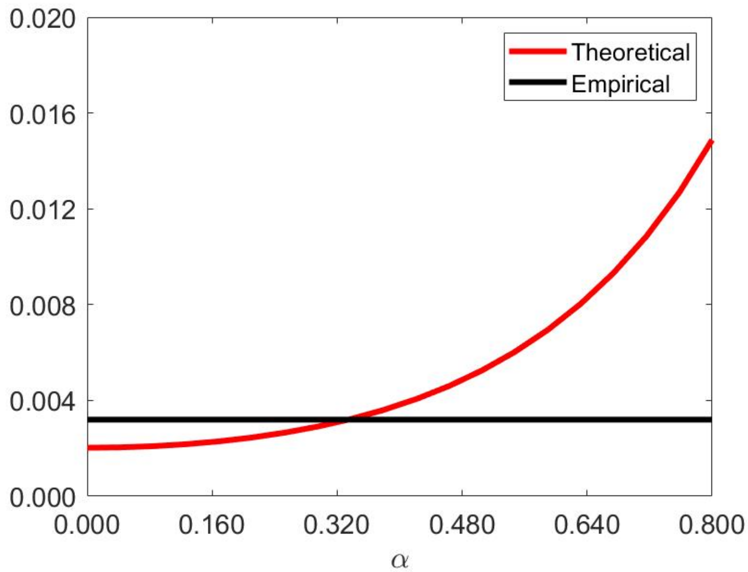


Figure 4.1: Empirical and theoretical second moments of ADR_p as a function of α using Eq. (4.1). The empirical moment is estimated using the data on seasonally adjusted delinquency rates on residential loans of 100 largest banks in the United States from the period 1985 to 2019 [29]. $\widehat{PD} = 0.0449$.

4.3.3 Estimating δ_1 and δ_2

Having the estimates for the default parameters, we estimate δ_1 and δ_2 as follows: From Eq. (3.19) , we know that

$$\mathbb{E}[ARL_p] = \mathbb{E}[\mathcal{PL}_i|\mathcal{D}_i = 1] \cdot PD.$$

Assuming the distribution of account-level LGD is modelled by beta distribution with scale parameters δ_1 and δ_2, the transformation of $\mathcal{PL}_i$ ensures that account-level LGD (LGD_i) has a beta distribution (see Table 2.7). Thus,

$$\mathbb{E}[LGD_i] = \mathbb{E}[\mathcal{PL}_i|\mathcal{D}_i = 1] = \frac{\delta_1}{\delta_1 + \delta_2},$$

which implies that

$$\mathbb{E}[ARL_p] = \frac{\delta_1}{\delta_1 + \delta_2} \cdot PD.$$

In practice, the distribution of account-level LGD is U-shaped (see for example the description of account-level LGD data in [31] and [66]). For this reason, we pick values of δ_1 and δ_2 between 0 and 1. Using the sample first moment of ARL_p as the proxy for its population counterpart, $\mathbb{E}[ARL_p]$, we can estimate the values of δ_1 and δ_2. Observe that given the respective values of PD and $\mathbb{E}[ARL_p]$, we can express the parameters δ_1 and δ_2 as a linear function of the other:

$$\delta_1 = \frac{\mathbb{E}[ARL_p]}{PD - \mathbb{E}[ARL_p]} \cdot \delta_2.$$

In our analysis, we assume the value of δ_2 as 0.5. The estimate for δ_1 is therefore, dependent on the first sample moments of ADR_p and ARL_p respectively. That is

$$\widehat{\delta_1} = \frac{\frac{1}{T}\sum_{i=1}^{T} ARL_{p,i}}{\frac{1}{T}\sum_{i=1}^{T} ADR_{p,i} - \frac{1}{T}\sum_{i=1}^{T} ARL_{p,i}} \cdot 0.5. \tag{4.2}$$

4.3.4 Identifying $\beta\theta_S$ and ρ_A

We now have the estimated values of α, PD, δ_1 and δ_2. We are left to estimate θ_S, θ_I and β. From Eq. (3.25), the expected value of the product of the asymptotic portfolio-level default rate and realized loss is

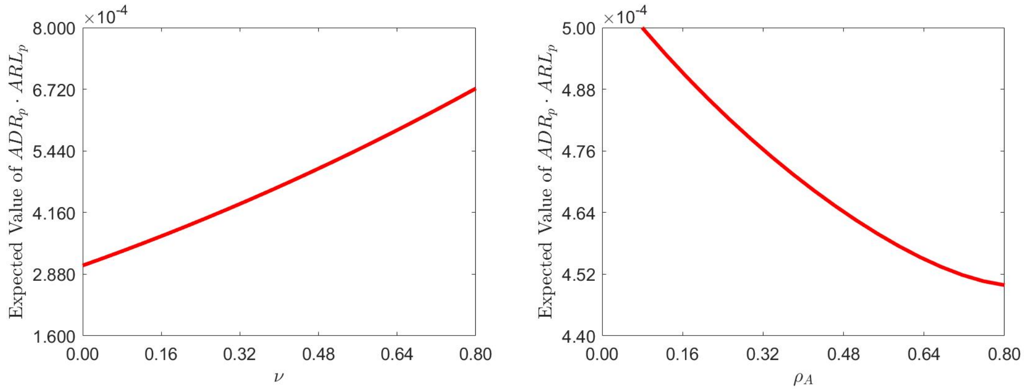
$$\begin{aligned}
\mathbb{E}[ADR_p \cdot ARL_p] &= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_i] \\
&= \mathbb{E}[\mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_i | B_i]] \\
&= \mathbb{E}[\mathcal{PL}_i \cdot \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j | B_i]] \\
&= \mathbb{E}[H(B_i) \cdot \mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD) | B_i)] \\
&= \int_{-\infty}^{\infty} H(b) \cdot \mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD) | B_i = b) \cdot \phi(b) \cdot db,
\end{aligned}$$

where $\phi(b)$ is the marginal density of B and is the standard normal pdf. The conditional distribution of the pair (A_i, A_j) given $B_i = b$, $\mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD) | B_i = b)$ is given as (see Appendix C.1)

$$A_i, A_j | B_i = b \sim N\left(\begin{bmatrix} \rho_A \cdot b \\ \alpha\beta\theta_S \cdot b \end{bmatrix}, \begin{bmatrix} 1 - \rho_A^2 & \alpha^2 - \alpha\beta\theta_S\rho_A \\ \alpha^2 - \alpha\beta\theta_S\rho_A & 1 - (\alpha\beta\theta_S)^2 \end{bmatrix} \right). \tag{4.3}$$

$H(\cdot)$ is a function of δ_1 and δ_2 and $\mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD) | B = b)$ is a function of ρ_A, α, and the product $\beta\theta_S$. Since we know the estimates of α, PD, δ_1 and δ_2 from the preceding sections, we set $\nu = \beta\theta_S$ in Eq. (4.3) and observe that $\mathbb{E}[ADR_p \cdot ARL_p]$ is a function of the pair (ρ_A, ν).

Figure 4.2a illustrates that, for a fixed value of ρ_A, $\mathbb{E}[ADR_p \cdot ARL_p]$ is a monotone function of ν, while Figure 4.2b illustrates that, for a fixed value of ν, $\mathbb{E}[ADR_p \cdot ARL_p]$ is a monotone function of ρ_A. One implication of this behaviour is that, for a fixed value of ρ_A, there is a unique value of ν for which the model-implied covariance is equal to the empirical covariance (that is $\mathbb{E}[ADR_p \cdot ARL_p] = \overline{ADR_p \cdot ARL_p}$) and in what follows we let $\nu(\rho_A)$ denote this value of ν.. In other words, for a given value of ρ_A, $\nu(\rho_A)$ is that value of ν that ensures the following



(a) $\mathbb{E}[ADR_p \cdot ARL_p]$ against ν. $\rho_A = 0.4$ (b) $\mathbb{E}[ADR_p \cdot ARL_p]$ against ρ_A. $\nu = 0.4$

Figure 4.2: $\mathbb{E}[ADR_p \cdot ARL_p]$ as a function of ν and ρ_A respectively. The Data used is seasonally adjusted charge-off (realized losses) and delinquency (default) rates on residential loans of 100 largest banks in the United States from the period 1985 to 2019 [29]. $\widehat{\delta}_1 = 0.0611, \widehat{\delta}_2 = 0.5, \widehat{\alpha} = 0.3344$, and $\widehat{PD} = 0.0450$.

equation holds

$$\overline{ADR_p \cdot ARL_p} = \frac{1}{T}\sum_{i=1}^{T} ADR_{p,i} \cdot ARL_{p,i} = \mathbb{E}[ADR_p \cdot ARL_p]. \tag{4.4}$$

Figure 4.3 shows the graph of $\nu(\rho_A)$. It is monotone in ρ_A. Any point on the curve will ensure that the model implied covariance equals the empirical covariance.

4.3.5 Identifying β

From Eq. (3.18), we know that

$$\mathbb{E}[ARL_p^2] = E[\mathcal{RL}_i \cdot \mathcal{RL}_j].$$

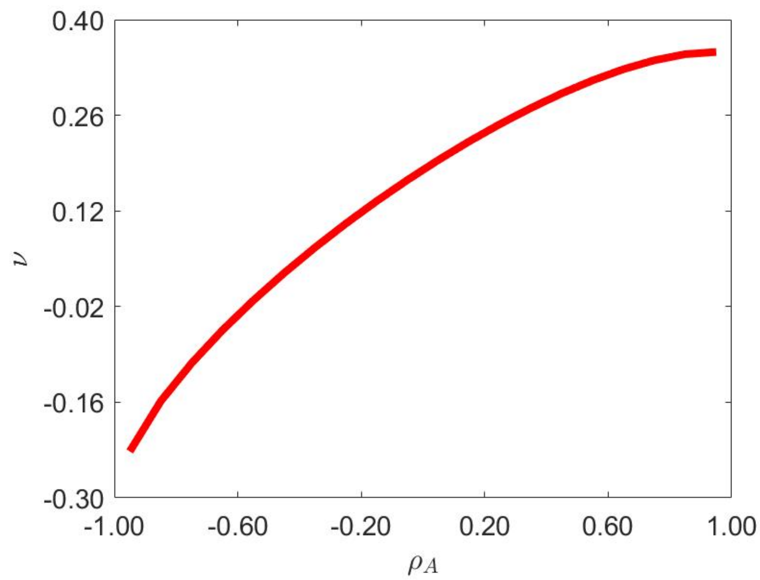


Figure 4.3: ν against ρ_A. The Data used is seasonally adjusted charge-off (realized losses) and delinquency rates on residential loans of 100 largest banks in the United States from the period 1985 to 2019 [29]. $\widehat{\delta_1} = 0.0611, \widehat{\delta_2} = 0.5$, and $\widehat{PD} = 0.0450$.

So we have

$$\begin{aligned}
\mathbb{E}[ARL_p^2] &= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_i \cdot \mathcal{PL}_j] \\
&= \mathbb{E}[\mathcal{PL}_i \cdot \mathcal{PL}_j \cdot \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j | B_i, B_j]] \\
&= \mathbb{E}[H(B_i) \cdot H(B_j) \cdot \mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD) | B_i, B_j)] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(b_i) \cdot H(b_j) \cdot \\
&\quad \mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD) | B_i = b_i, B_j = b_j) \phi(b_i, b_j) \cdot db_i \cdot db_j,
\end{aligned}$$

where $\phi(b_i, b_j)$ is the bivariate normal pdf of the pair (B_i, B_j) with the respective mean vector and covariance matrix

$$\mu_{B_i,B_j} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \Sigma_{B_i,B_j} = \begin{bmatrix} 1 & \beta^2 \\ \beta^2 & 1 \end{bmatrix}.$$

The conditional distribution of the pair (A_i, A_j) given the pair $(B_i = b_i, B_j = b_j)$, $\mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)|B_i = b_i, B_j = b_j)$ has mean vector

$$\mu_{A_i,A_j|B_i=b_i,B_j=b_j} = \frac{1}{1-\beta^4}\begin{bmatrix} \rho_A(b_i - \beta^2 \cdot b_j) + \alpha\nu(b_j - \beta^2 \cdot b_i) \\ \alpha\nu(b_i - \beta^2 \cdot b_j) + \rho_A(b_j - \beta^2 \cdot b_i)] \end{bmatrix}$$

and covariance matrix

$$\Sigma_{A_i,A_j|B_i=b_i,B_j=b_j} = \frac{1}{1-\beta^4}\begin{bmatrix} \Gamma^*_{1,1} & \Gamma^*_{1,2} \\ \Gamma^*_{2,1} & \Gamma^*_{2,2} \end{bmatrix},$$

where

$$\begin{cases} \Gamma^*_{1,1} = \Gamma^*_{2,2} = (1-\beta^4) - \rho_A(\rho_A - \beta^2\alpha\nu) + \alpha\nu(\alpha\nu - \beta^2\rho_A), \\ \Gamma^*_{1,2} = \Gamma^*_{2,1} = \alpha^2(1-\beta^4) - \rho_A(\alpha\nu - \beta^2\rho_A) + \alpha\nu(\rho_A - \beta^2\alpha\nu). \end{cases}$$

Again, observe that both $H(b_i)$ and $H(b_j)$ are functions of δ_1 and δ_2. And $\mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)|B_i = b, B_j = b)$ is a function of $\alpha, \beta, \rho_A, PD$ and the product $\nu = \beta\theta_S$. Thus, $\mathbb{E}[ARL_p^2]$ is dependent on the parameters $\alpha, \beta, \delta_1, \delta_2, \rho_A, PD$ and ν.

Figure 4.4 illustrates that for given values of all the other parameters, $\mathbb{E}[ARL_p^2]$ is a monotone function of β. If it were possible to determine all the values of the other parameters it will be possible to determine β uniquely. But it is not possible to determine both ρ_A and ν, so it is not possible to determine β. That being said, if we fix the value of ρ_A, it is possible to determine the value of β uniquely as follows:

1. Fix the value of ρ_A.
2. Compute $\nu = \nu(\rho_A)$, the unique value of ν that ensures Eq. (4.4) is satisfied.

3. Given the values of ρ_A and $\nu(\rho_A)$, let $\beta(\rho_A)$ be that value of β that ensures that

$$\overline{ARL_p^2} = \frac{1}{T}\sum_{i=1}^{T} ARL_{p,i}^2 = \mathbb{E}[ARL_p^2]$$

is satisfied.

4. Note that for a given value of ρ_A, the value of θ_S is uniquely determined as $\theta_S(\rho_A) = \frac{\nu(\rho_A)}{\beta(\rho_A)}$, and $\theta_I(\rho_A) = \frac{\rho_A - \alpha\beta(\rho_A)\theta_S(\rho_A)}{\sqrt{1-\alpha^2)}\sqrt{1-\beta^2}}$.

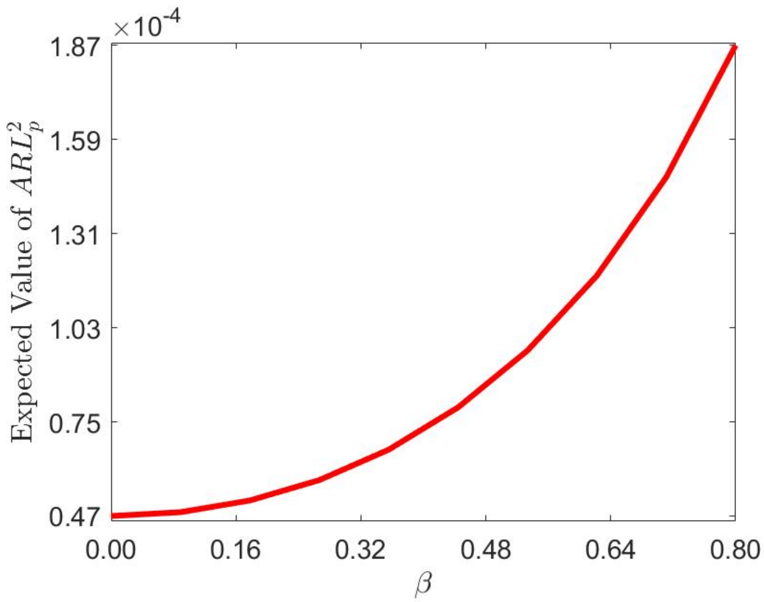
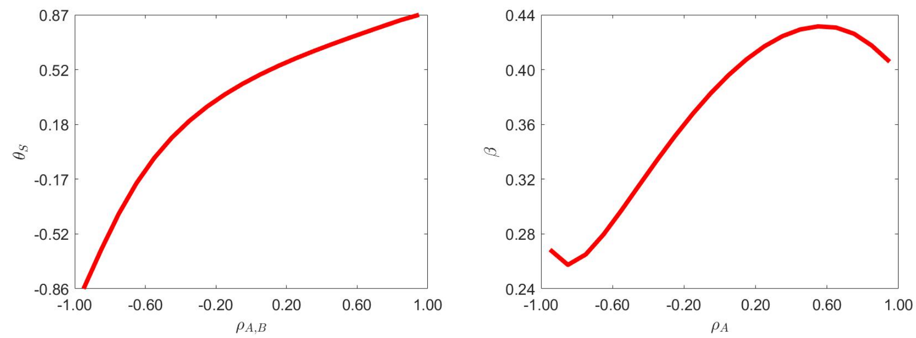


Figure 4.4: $\mathbb{E}[ARL_p^2]$ against β. The Data used is seasonally adjusted charge-off (realized losses) and delinquency rates on residential loans of 100 largest banks in the United States from the period 1985 to 2019 [29]. The value of $\widehat{\nu}$ (0.2921) corresponds to that value of ρ_A (0.4500) that satisfies Eq. (4.4). $\widehat{\delta}_1 = 0.0611, \widehat{\delta}_2 = 0.50$, and $\widehat{PD} = 0.0450$.

Figure 4.5 illustrates the relationship between ρ_A and respectively θ_S (see Figure 4.5a) and β (see Figure 4.5b). Again, notice that any point on the curve ensures that the model implied variance of realized loss is equal to the empirical variance. The same can be said of Figure 4.5b.



(a) θ_S against ρ_A

(b) β against ρ_A

Figure 4.5: Relationship between ρ_A and respectively θ_S and ν. The Data used is seasonally adjusted charge-off (realized losses) and delinquency (default) rates on residential loans of 100 largest banks from the period 1985 to 2019 in the United States [29]. $\widehat{\delta}_1 = 0.0611, \widehat{\delta}_2 = 0.5$, and $\widehat{PD} = 0.0450$.

4.4 Impact of Free Parameter on Model Outputs

The previous section allows us to uniquely determine all but one parameter. Without loss of generality, we have been using ρ_A as free parameter. But we could easily have allowed it to be ν, β or θ_S. In this section we consider the influence of the free parameter on the model output such as Value at Risk (VaR), the mean and standard deviation of portfolio-level LGD , μ_{ALGD_p} and σ_{ALGD_p} respectively, and the correlation between portfolio-level default rate and LGD, ρ_p.

To do so, for a given time series, we first estimate PD, δ_1 and δ_2 as described in Sections 4.3.1,4.3.2 and 4.3.3. Next, for each values of ρ_A in a large grid, we compute $\nu(\rho_A)$ and $\beta(\rho_A)$ as described in Sections 4.3.4 and 4.3.5 and thereby estimate $\theta_S(\rho_A)$ and $\theta_I(\rho_A)$. Finally, we simulate a large number of realizations of S_A and S_B and compute the corresponding values of ARL_p and ADR_p. And we used the estimated values to calculate the quantities of interest.

Figures 4.6a and 4.6b exhibit the sensitivity of respectively, 99.0% and 99.9% realized loss Value at Risk (VaR) to ρ_A. The figures show that the account-level correlation ρ_A does not have much impact on the VaR. Impact on the mean and standard deviation of portfolio-level LGD and the correlation between default rate and LGD are shown in Figures 4.7a-4.7c. ρ_A has negligible effect on VaR of realized loss, and the mean and standard deviation of portfolio-level LGD. ρ_A's impact on the correlation between portfolio-level default rate and LGD is not noticeable. The figures are produced from the time series of real estate data on single family residential mortgages described in section 4.2. Defaults parameters PD and α are set as $PD = 0.0449$ and $\alpha = 0.3344$ respectively while δ_1 and δ_2 are given as 0.0611 and 0.5.

This result suggests that regardless of the values chosen for ρ_A, the estimated risk measures will yield similar outcome. Thus the account-level correlation ρ_A does not impact portfolio-level quantities and for this reason, we can choose its value for convenience.

Furthermore, higher moments are not sensitive to changes in the model parameters. This is illustrated in Figure 4.8, where we observed that ρ_A does not impact $\mathbb{E}[ADR_p^3]$, $\mathbb{E}[ARL_p^3]$, $\mathbb{E}[ARL_p^2 \cdot ADR_p]$ and $\mathbb{E}[ARL_p \cdot ADR_p^2]$. The expression for $\mathbb{E}[ADR_p^3]$ is obtained as

$$
\begin{aligned}
\mathbb{E}[ADR_p^3] &= \mathbb{E}[\mathbb{E}[\mathcal{D}_i|S_A]^3] && (4.5)\\
&= \mathbb{E}[\mathbb{E}[\mathcal{D}_i|S_A] \cdot [\mathbb{E}[\mathcal{D}_j|S_A] \cdot [\mathbb{E}[\mathcal{D}_k|S_A]] && (4.6)\\
&= \mathbb{E}[\mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{D}_k|S_A]] && (4.7)\\
&= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{D}_k] && (4.8)\\
&= PD^{(3)},
\end{aligned}
$$

where we use the homogeneity, conditional independence and the tower properties in moving from lines (4.5) to (4.6), (4.6) to (4.7), and (4.7) to (4.8).

$$PD^{(3)} = \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{D}_k] = \mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD), A_k \leq \Phi^{-1}(PD))$$

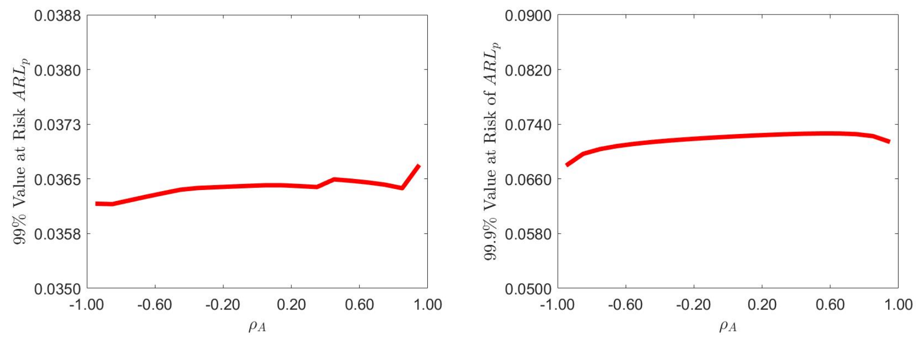
is the probability of simultaneous default by three obligors. $PD^{(3)}$ is calculated using three dimensional multivariate normal pdf with the mean vector and covariance matrix

$$\mu_{A_i,A_j,A_k} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \Sigma_{A_i,A_j,A_k} = \begin{bmatrix} 1 & \alpha^2 & \alpha^2 \\ \alpha^2 & 1 & \alpha^2 \\ \alpha^2 & \alpha^2 & 1 \end{bmatrix}$$

respectively. Similar as above, we derive the expression for $E[ARL_p^3]$ as

$$\begin{aligned} E[ARL_p^3] &= \mathbb{E}[\mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_j \cdot \mathcal{D}_k \cdot \mathcal{PL}_k | S_A, S_B]] \\ &= \mathbb{E}[\mathcal{D}_i \cdot \mathcal{PL}_i \cdot \mathcal{D}_j \cdot \mathcal{PL}_j \cdot \mathcal{D}_k \cdot \mathcal{PL}_k] \\ &= \mathbb{E}[\mathcal{PL}_i \cdot \mathcal{PL}_j \cdot \mathcal{PL}_k \cdot \mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{D}_k | B_i, B_j, B_k]]. \end{aligned}$$

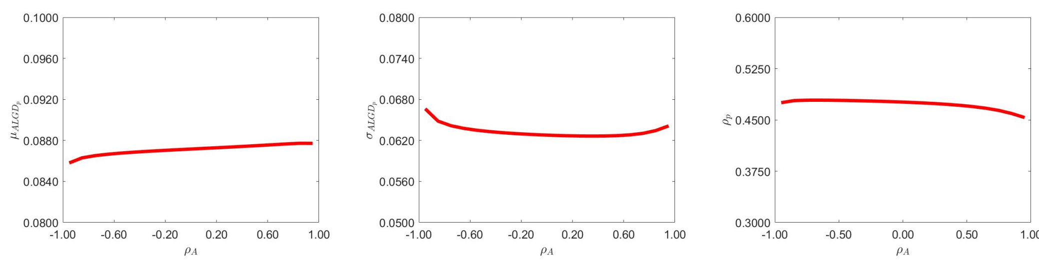
We compute $\mathbb{E}[\mathcal{D}_i \cdot \mathcal{D}_j \cdot \mathcal{D}_k | B_i, B_j, B_k]$ by first deriving its expression by following the work in Appendix C.



(a) 99.0% quantile of portfolio-level realized loss (99.0% VaR) against ρ_A

(b) 99.9% quantile of portfolio-level realized loss (99.9% VaR) against ρ_A

Figure 4.6: Impact of ρ_A on the risk measure, Value at Risk for 99.0% and 99.9% quantile of portfolio-level realized loss respectively. $\widehat{\delta_1} = 0.0611, \widehat{\delta_2} = 0.5$, and $\widehat{PD} = 0.0449$. These estimates are based on the residential loan explained in Section 4.2. Number of economic scenarios $m = 10000$. We used the asymptotic portfolio-level realized loss in obtaining the graphs.



(a) Mean of portfolio-level LGD against ρ_A

(b) Standard deviation of portfolio-level LGD against ρ_A

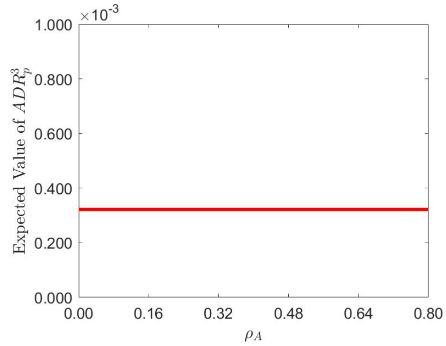
(c) Graph of correlation between portfolio-level default rate and LGD against ρ_A

Figure 4.7: Impact of ρ_A on the mean and standard deviation of portfolio-level LGD and the correlation between portfolio-level default rate and LGD. $\widehat{\delta_1} = 0.0611, \widehat{\delta_2} = 0.5$, and $\widehat{PD} = 0.0449$. These estimates are based on the residential loan explained in Section 4.2. Number of economic scenarios $m = 10000$. We used the asymptotic portfolio-level realized loss in obtaining the graphs.

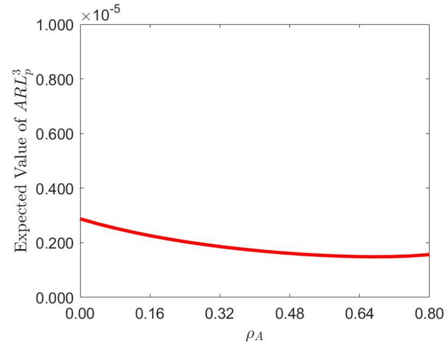
4.5 Estimation Result in the Reduced Model

4.5.1 Estimating Model Parameters

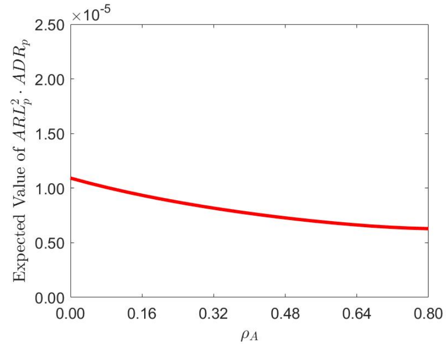
Based on the above discussion, we reduce the number of parameters in our model by assuming that the idiosyncratic risk factors are independent (that is $\theta_I = 0$) in the



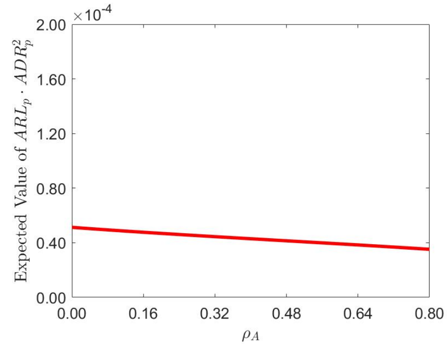
(a) $\mathbb{E}[ADR_p^3]$ against ρ_A.



(b) $\mathbb{E}[ARL_p^3]$ against ρ_A.



(c) $\mathbb{E}[ARL_p^2 \cdot ADR_p]$ against ρ_A.



(d) $\mathbb{E}[ARL_p \cdot ADR_p^2]$ against ρ_A.

Figure 4.8: Impact of ρ_A on $\mathbb{E}[ADR_p^3]$, $\mathbb{E}[ARL_p^3]$, $\mathbb{E}[ARL_p^2 \cdot ADR_p]$ and $\mathbb{E}[ARL_p \cdot ADR_p^2]$. The Data used is seasonally adjusted charge-off (realized losses) and delinquency (default) rates on residential loans of 100 largest banks in the United States from the period 1985 to 2019 [29]. $\widehat{PD} = 0.0450$, $\widehat{\alpha} = 0.3344, \widehat{\beta} = 0.3934, \widehat{\theta}_S = 0.5447, \widehat{\delta}_1 = 0.0611, \widehat{\delta}_2 = 0.5$.

remaining of our study. With $\theta_I = 0$, we have $\rho_A = \alpha\beta\theta_S$, which is determined by estimating α and $\nu = \beta\theta_S$ as described above. With this simplification $ADR_p = f(S_A)$ and $ARL_p = f(S_B)$ — this makes computing easier.

Table 4.4 shows the estimated values of parameters via Monte Carlo simulation scheme. We assumed particular values of model parameters and obtained the corresponding portfolio quantities $\overline{ADR_p}, \overline{ADR_p^2}, \overline{ARL_p}, \overline{ADR_p \cdot ARL_p}$ and $\overline{ARL_p^2}$. We then use these quantities to estimate the model parameters. This procedure is simplified as follows:

1. Choose parameter values of $\alpha, PD, \delta_1, \delta_2, \beta$ and ν.

2. Determine the model implied quantities — $\overline{ADR_p}, \overline{ADR_p^2}, \overline{ARL_p}, \overline{ADR_p \cdot ARL_p}$ and $\overline{ARL_p^2}$ — using the parameter values in step 1.

3. Using the quantities in step 2 and the method discussed in the previous Sections (4.3.1,4.3.2,4.3.4 and 4.3.5), determine the estimates of model parameters.

We see from Table 4.4 the estimates of these parameters match closely to their original values. For instance, the value of $\widehat{\alpha}$ (0.2508) is close to α (0.25). This indicate the accuracy of our proposed estimation scheme.

Table 4.5 shows parameter estimates from the data on the charge-off and delinquency rates from the different loan categories described in Section 1. The estimates are expressed in terms of the marginal (PD, δ_1, and δ_2) and dependence (α^2, β^2 and θ_S) parameters in the model. Some of the loan categories — subportfolios under consumer loans (Credit Cards and All) have the estimated values of δ_1 outside the assumed range (δ_1 sandwiched between 0 and 1) under this estimation scheme. This is due to the fact that the portfolio-level average default rate is less than the average realized loss — see Table 4.1 and Eq, (4.2). Consequently, no estimate can be provided for the dependence parameters β^2 and θ_S.

Table 4.6 compares the model implied quantities of portfolio-level LGD to that obtained from the time series data across the different loan categories described in Section 4.2. We do this by first using the estimated values of the parameters presented in Table 4.5 as inputs to compute simulated portfolio-level LGD and default rate of size, say N. We then calculate the respective means and the variances of the LGDs and the correlations between the default rate and LGD of both the simulated and the time series data. The Table shows that these quantities are pretty close with each other.

Table 4.4: Monte Carlo: Parameter estimates under the assumption of independent idiosyncratic risk factors. Number of economic-wide scenarios is 2000, number of obligors is 20000

PD	α	δ_1	β	θ_S
0.05	0.25	0.075	0.25	0.6
$\widehat{PD}$	$\widehat{\alpha}$	$\widehat{\delta}_1$	$\widehat{\beta}$	$\widehat{\theta}_S$
0.0501	0.2508	0.0740	0.2493	0.5708

Table 4.5: Parameter estimates assuming independent idiosyncratic risk factors

Loan Type	Marginal Parameters			Dependence Parameters			
	PD	δ_1	δ_2	α^2	β^2	θ_S	ρ_A
Real Estate:							
Residential	0.045	0.0611	0.5	0.1118	0.1548	0.5447	0.0717
Commercial	0.043	0.0949	0.5	0.1847	0.0910	0.6517	0.0845
Farmlands	0.0380	0.0339	0.5	0.0444	0.2726	0.3890	0.0428
All	0.0474	0.0710	0.5	0.0812	0.1132	0.6851	0.0657
Consumer:							
Credit Cards	0.040	-3.5518	0.5	0.0186	NaN	NaN	NaN
Other	0.0290	0.3947	0.5	0.0087	0.3106	-0.1131	-0.0059
All	0.0346	1.7345	0.5	0.0111	NaN	NaN	NaN
C & I	0.0259	0.2059	0.5	0.0738	0.2280	-0.1835	-0.0238
Leases	0.0138	0.2070	0.5	0.0202	0.1810	0.5343	0.0323
Agricultural	0.0415	0.0534	0.5	0.1004	0.5652	-0.0190	-0.0045
All Loans and Leases	0.0374	0.1932	0.5	0.0493	0.0945	0.0059	4.0263e-04

4.5.2 Comparing Economic Capital Under Correct and Wrong Transformations

This section compares EC estimates from the correct and wrong transformations of loss drivers to potential losses. Recall these transformations are respectively given as $\widehat{H}(b) = F^{-1}(\widehat{\Phi}(b)$ and $H(b) = F^{-1}(\Phi(b))$. We first compared the parameter estimates from these transformations. Table 4.7 presents these comparison. Observe that the estimates of β across the loan categories are approximately equal. How-

Table 4.6: Comparing model-implied quantities to observed data using estimates from Table 4.5

Loan Type	Mean of LGD		Variance of LGD		Correlation between DR and LGD	
	Simulated	Observed	Simulated	Observed	Simulated	Observed
Real Estate:						
Residential	0.0865	0.0850	0.0040	0.0051	0.5076	0.4410
Commercial	0.1244	0.0999	0.0036	0.0097	0.5979	0.6810
Farmland	0.0546	0.0530	0.0043	0.0047	0.3436	0.3236
All	0.1018	0.0940	0.0035	0.0054	0.6625	0.6611
Consumer:						
Credit Cards	NaN	1.1968	NaN	0.0833	NaN	-0.3852
Other	0.4501	0.4452	0.0362	0.03113	-0.1377	-0.1076
All	NaN	0.7941	NaN	0.0589	NaN	-0.3129
C & I	0.3124	0.2950	0.0236	0.0197	-0.1897	-0.0351
Leases	0.2711	0.2661	0.0166	0.0205	0.5209	0.5017
Agricultural	0.1004	0.0961	0.0225	0.0114	-0.0153	0.0058
All Loans and Leases	0.2806	0.2852	0.0088	0.0076	-0.0142	-0.1471

ever, that of θ_S and ρ_A are grossly different. For example, estimates of θ_S under the correct transformation is about twice of that obtained under the wrong transformation. These differences in the estimates are reflective in the EC estimates under both transformations. We have EC understated under wrong transformation for values of $\rho_A > 0$. And conversely, EC estimates are overstated under the wrong transformation when $\rho_A < 0$. Even though the difference might seem negligible, these estimates are reflective of the magnitude of the estimated account-level correlation ρ_A.

Table 4.7: Economic Capital Comparison under correct and wrong transformations.

Loan Type	Dependence Parameters						Economic Capital	
	β^2		θ_S		ρ_A		99%	
	Correct	Wrong	Correct	Wrong	Correct	Wrong	Correct	Wrong
Real Estate:								
Residential	0.1548	0.1494	0.5447	0.2491	0.0717	0.0322	3.17%	2.77%
Commercial	0.0910	0.0832	0.6517	0.2597	0.0845	0.0354	4.45%	4.02%
Farmlands	0.2726	02708	0.390	0.1866	0.0428	0.0205	1.64%	1.47%
All	0.1132	0.1083	0.6851	0.3240	0.0657	0.0304	2.83%	2.36%
Consumer:								
Credit Cards	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
Other	0.3106	0.3105	-0.1131	-0.0561	-0.0059	-0.0029	1.59%	2.91%
All	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
C & I	0.2280	0.2274	-0.1810	-0.0860	-0.0238	-0.0111	2.27%	2.44%
Leases	0.1810	0.1798	0.5343	0.2630	0.0323	0.0159	1.11%	0.95%
Agricultural	0.5652	0.5651	-0.0190	-0.0087	-0.0045	-0.0021	3.54%	3.56%
All Loans and Leases	0.0945	0.0945	0.0059	0.0028	4.0263e-04	1.9229e-04	2.22%	2.21%

4.5.3 Imputing Systematic Risk Factors

The estimated values in Table 4.5 can be used as inputs in the expressions for ADR_p and ARL_p to obtain the estimates for the systematic risk factors for each period. Thus, for a time series of ADR_p and ARL_p where i denotes the time index on ADR_p, s_A, s_B and ARL_p, we have

$$ADR_{p,i} = \Phi\left(\frac{\Phi^{-1}(\widehat{PD}) - \widehat{\alpha} s_{A,i}}{\sqrt{1-\widehat{\alpha}^2}}\right) \tag{4.9}$$

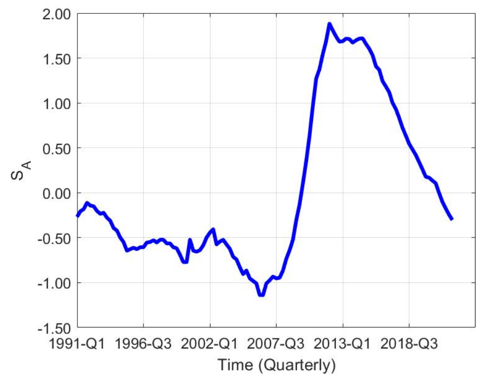
and

$$ARL_{p,i} = ADR_{p,i} \int_{\mathbb{R}} H(b) \cdot \phi(b; \widehat{\beta} s_{B,i}, 1-\widehat{\beta}^2) \cdot db. \tag{4.10}$$

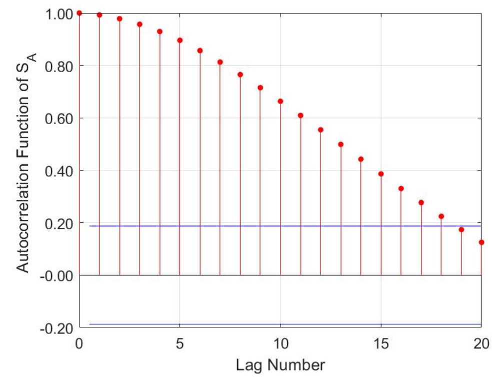
We solve for s_A and s_B in Eq. (4.9) and (4.10). Figures 4.9a–4.9d show the time series and auto-correlation plots of the recovered values of s_A and s_B. Observe that these risk factors are serially correlated. s_B is more volatile than s_A and is reflective in the time series data for LGD and default rate, where we notice that LGD is more volatile than default rate. Figure 4.10 confirms this observation.

4.6 Case Against Maximum Likelihood Estimation

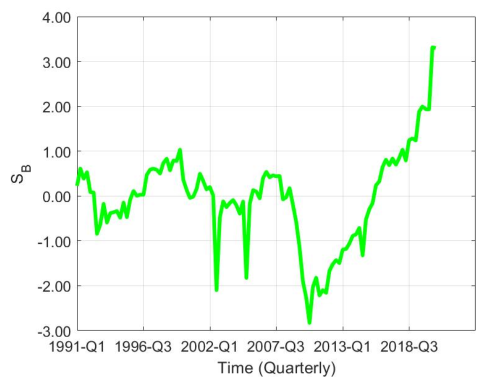
Arguably, Maximum Likelihood Estimation (MLE) can be used to estimate model parameters. However, this approach involved enormous computational effort and thus not practical under this setting. We demonstrate in this section why we did not employ the MLE scheme. We start by first assuming the pair $(ADR_p, ALGD_p)$ are serially independent time series, and then specify the respective likelihood functions



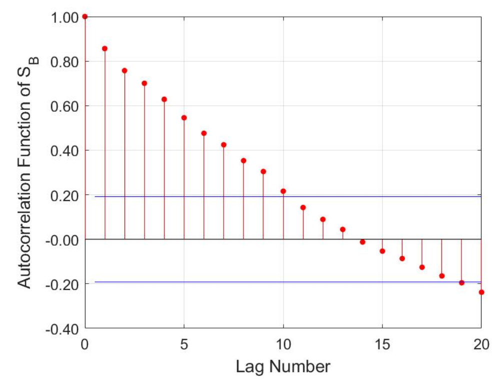
(a) Time series plot of systematic risk driver of defaults S_A



(b) Auto-correlation Function of systematic risk driver of defaults S_A



(c) Time series plot of systematic risk driver of losses S_B



(d) Auto-correlation Function of systematic risk driver of losses S_B

Figure 4.9: Time series and auto-correlation plots of systematic risk drivers. The Data used is seasonally adjusted charge-off (realized losses) and delinquency (default) rates on residential loans of 100 largest banks from the period 1991 to 2019 in the United States [29]. Parameter values used: $PD = 0.045, \delta_1 = 0.0611, \delta_2 = 0.5, \alpha = 0.3344, \beta = 0.3934$.

involving ADR_p and $ALGD_p$ as

$$\mathcal{L}_{ADR_p}(\alpha, PD) = \prod_{i=1}^{T} f_{ADR_p}(ADR_{p,i}) \tag{4.11}$$

and

$$\mathcal{L}_{ALGD_p}(\Theta) = \prod_{i=1}^{T} f_{ALGD_p}(ALGD_{p,i}), \tag{4.12}$$

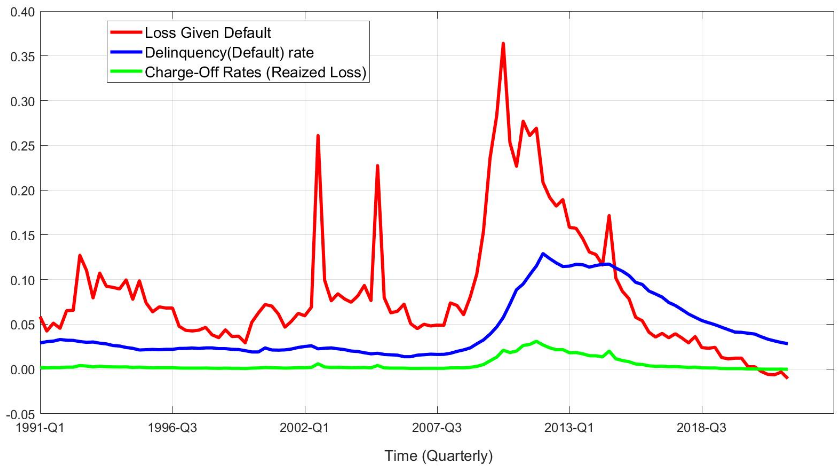


Figure 4.10: Time series plots of Residential Real Estate Loans. The Data used is seasonally adjusted charge-off (realized losses) and delinquency (default) rates on residential loans of 100 largest banks from the period 1991 to 2019 in the United States [29].

where $\Theta = (\alpha, \beta, \theta_S, \theta_I, \delta_1, \delta_2, PD)$, and the expressions for $f_{ADR_p}(\cdot)$ and $f_{ALGD_p}(\cdot)$ are presented in Eq. (3.10) and (3.29). It is easy (in terms of computational effort) to find parameter values for which $\mathcal{L}_{ADR_p}(\cdot)$ is optimized. However, that of $\mathcal{L}_{ALGD_p}(\cdot)$ is not. The singular evaluation of the likelihood function $\mathcal{L}_{ALGD_p}(\cdot)$ at $ALGD_{p,1} = 0.058419$ and $\Theta = (0.3344, 0.3934, 0.5447, 0.0611, 0.5, 0.045)$ is 5.2190. The computation time is 11.46 seconds. Thus using data points of 114 will give a computation time of approximately 22 minutes. Maximizing $\mathcal{L}_{ALGD_p}(\cdot)$ using the mle-function in MATLAB did not provide convergence for 200 iterations with total computation time of approximately 3 hours. This suggests that using the MLE approach for parameter estimation is not practical. Table 4.8 reports the computer system specifications for this experiment.

Table 4.8: Computer System Specifications

Device name	DESKTOP-1TNM4KN
Processor	Intel(R) Core(TM) m3-6Y30 CPU @0.90GHz 1.51 GHz
Installed	RAM 8.00 GB (7.87 GB usable)

4.7 Discussion and Conclusion

This chapter presents an estimation scheme based on method of moments (MoM) for parameters in the model discussed. Alternatively, Maximum Likelihood Estimation (MLE) could be used. The application of the MLE method involve specifying the densities of default rate and realized loss or LGD (or recovery rate) — see for example [44, 50]. We, however, make a case that the MLE approach under our modeling setting requires huge computational cost in terms of computing time. We make a case of over-specification of model parameters by demonstrating that the correlation between default and loss drivers on the risk measure, VaR for realized loss and that the degree of sensitivity of the mean and standard deviation of the portfolio-level LGD to this correlation is extremely weak. Therefore, the account-level correlation in and of itself has no influence on portfolio-level quantities. Against this background, we assumed an independent relationship between the idiosyncratic risk factors and obtained unique estimates for the model parameters.

The time series data applied to the estimation scheme have negative values in some periods (or quarters) for charge-off rates (realized losses) and thereby yielding negative values for LGD. The LGD calculated for some periods (or quarters) are more than 1. These values are outside the model assumed values for these quantities. However, since the estimation scheme is based on the sample moments of the realized loss and default rates this setback in the time series is accounted for. However, this yielded unstable results in the recovering of the systematic risk factors.

Owing to the theoretical property of conditional independence of default rate on the state of the economy (systematic risk factors) the estimation scheme is built on the assumption that default and loss drivers are serially independent. It is, therefore, worth noting that the proposed estimation scheme has the underlying assumption of serially independent default rates and realized loss (or LGD) time series. In practice, the observed default and realized loss time series exhibit auto-correlation (see for example Figure 4.10). This is traceable to the dynamic of the systematic risk factors, where we see that their serially correlated.

Against this background, the proposed estimation scheme in this thesis may not adequately capture the risk on the portfolio in that the presence of auto-correlation may systematically lead to low estimates [48]. However, under reasonable assumption, the proposed method will be a good estimation choice as demonstrated by the results from Monte Carlo procedure.

Chapter 5

Future Work

There is much evidence gathered on the sensitivity of LGD to systematic risk factors [66]. These evidence suggest that portfolio risk parameters estimates are sensitive to the systematic risk factors and therefore estimation of portfolio risk measures are reflective of the state of the economy. Indeed, the provision in Paragraph 468 of Basel II Framework Document for calculating loss given default (LGD) requires that parameters used in Pillar I of Basel II capital estimations must be reflective of economic downturn conditions so that relevant risks are accounted for and thus proposed establishing a functional relationship between long-run and downturn LGD [21].

This thesis has presented a generalized model on PD-LGD correlation. Underlying the modelling framework is the application of the two rating types commonly known as point-in-time (PIT) and TTC [81]. The latter ratings system evaluates customers by focusing on the permanent component of the default risk and the usage of a prudent migration policy while the former, evaluates customers subjected to the prevailing economic condition and thus incorporating cyclical and permanent effects in modelling default and LGD [9, 81].

The proposed parameter estimation scheme is inherently based on the assumption that the default and loss drivers are serially independent and that all individual obligors have the same PIT rating, which accounts for all current state of the economy — serially independent drivers means default and LGD time series are also serially independent. However, real world data on default and LGD time series are auto-correlated. Implying that, inherent in these data is the time dependency of portfolio risk drivers. It is argued by Frei et al. [48] that two reasons account for this: (1) due to the objective of stabilizing credit risk measures across economic cycles, in practice, ratings is done using the TTC instead of the PIT and (2) since changes in obligor's credit quality does not instantly impact ratings, ratings become dependent on the economic situation which leads to auto-correlated default and LGD time series. This may lead to unsatisfactory results under our proposed estimation scheme.

Against this background, one can improve on or reduced the biases of parameter estimates in the proposed method by adopting the method proposed in [48], where we may construct MoM estimators with correction terms incorporating the auto-correlation and finite length of default and LGD time series. In particular, the model takes the form:

$$\begin{aligned} A_t^i &= \alpha S_{A,t} + \sqrt{1-\alpha^2} I_{A,t}^i, \\ B_t^i &= \beta S_{B,t} + \sqrt{1-\beta^2} I_{B,t}^i, \end{aligned} \tag{5.1}$$

where the pair $(S_{A,t}, S_{B,t})$ are governed by a time series process. The process may be modelled as an Auto-Regressive (AR), Auto-Regressive Moving Average (ARMA), or Auto- Regressive Integrated Moving Average (ARIMA). We recommend the latter in that it provides the flexibility of the possibility of better capturing the dynamics in the real data and on the portfolio. Further more, the idiosyncratic component can also be modelled as a time series process.

Impact of account-level parameters on portfolio-level quantities may be one of the applications of the extended model (Eq. 5.1). The model may as well be used as a predictive model in forecasting future defaults and losses, therefore, it falls in place to test the predictability of the revised model. The model may be used to investigate how regulatory and economic capital evolve with time. Summarizing, the proposed extended model has the advantage of providing deep insight into the dynamics of both account and portfolio level quantities and risk measures.

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Appendix A

Chapter 2 Related Issues

A.1 Correlation Structure of the General Model and Specific Models

Table A.1: Variables, Parameters and Correlation Structure of Models

	General Model	Frye's Model	M & O Model	Pykhtin's Model	Witzany's Model
Parameters					
	α	p	R_{PD}	α	$\sqrt{\rho_1}$
	β	q	R_{LGD}	β	$\sqrt{\rho_2}$
	–	–	β_{PD}	γ	ω
	–	–	β_{LGD}	–	–
	–	–	θ_{PD}	–	–
	–	–	θ_{LGD}	–	–
PD drivers					
	A_i	A_i	p_i	X_i	$Y_{1,i}$
	S_A	X	$P = \beta_{PD}X + \varepsilon_{PD,i}$	Y	X_1
	$I_{A,i}$	X_i	$e_{PD,i} = \theta_{PD} \cdot x_i + \varepsilon_{PD,i}$	ξ_i	$\zeta_{1,i}$
Loss drivers					
	B_i	C_i	l_i	R_i	$Y_{2,i}$
	S_B	X	$L = \beta_{LGD}X + \varepsilon_{LGD,i}$	Y	$\omega X_1 + \sqrt{1-\omega^2}X_2$
	$I_{B,i}$	Z_i	$e_{LGD,i} = \theta_{LGD} \cdot x_i + \varepsilon_{LGD,i}$	$\frac{\gamma}{\sqrt{1-\beta^2}}\xi_i + \frac{\sqrt{1-\beta^2-\gamma^2}}{\sqrt{1-\beta^2}}\eta_i$	$\zeta_{2,i}$
Correlations					
Systematic	θ_S	1	$\beta_{PD}\beta_{LGD}$	1	ω
Idiosyncratic	θ_I	0	$\theta_{PD}\theta_{LGD}$	$\frac{\gamma}{\sqrt{1-\beta^2}}$	0
PD and Loss	$\alpha\beta\theta_S + \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I$	pq	$\beta_{PD}\beta_{LGD}R_{PD}R_{LGD} + \theta_{PD}\theta_{LGD}\sqrt{1-R_{PD}^2}\sqrt{1-R_{LGD}^2}$	$\alpha\beta + \sqrt{1-\alpha^2}\gamma$	$\sqrt{\rho_1\rho_2}\omega$

A.2 Linking Specific Models to Nested Models

A.2.1 Frye's Model

The model description in the introductory chapter guarantees θ_S and θ_I to be respectively one and zero. Defaults are governed by Eq. (2.1) and for brevity, losses are defined by collateral $\mathcal{C}$

$$\mathcal{C} = \mu(1 + \sigma B),$$

where B is given as in Eq. (2.2) with exception that $S_B = S_A$. Therefore $\mathcal{C}$ is normally distributed with mean μ and standard deviation $\mu\sigma$. The default condition is defined as in Eq. (2.3). Potential loss for a representative obligor is defined as

$$\mathcal{PL} = \max(0, 1 - \mathcal{C}), \tag{A.1}$$

implying that realized loss is given as

$$\mathcal{RL} = \max(0, 1 - \mathcal{C}) \cdot \mathcal{D}. \tag{A.2}$$

Table A.2 shows the correlation structure of the above discussed model. From the table, the correlation between default and loss drivers is the product of the parameters α and β. That is $\rho_A = \alpha\beta$, since the systematic and idiosyncratic risk components of Eq. (2.1) and (2.2) are respectively, perfectly correlated and independent of each other. ρ_A accounts for the correlation between $\mathcal{PL}$ and $\mathcal{D}$. Also, default drivers of individual obligors are correlated. A similar story can be said of the loss drivers.

Table A.2: Correlation structure of Frye's model

Correlations	Formulas
(A_i, B_i)	$\alpha\beta$
(A_i, A_j)	α^2
(B_i, B_j)	β^2
(A_i, B_j)	$\alpha\beta$

A.2.2 Miu and Odzemir's Model

The model takes the form of Eq. (2.1) and (2.2), where θ_S (correlation between S_A and S_B) is the product $\beta_{PD}\beta_{LGD}$ in their paper [74]. The parameters α and β are the same as the parameters R_{PD} and R_{LGD} in the paper [74] respectively. Also, θ_I is the product $\theta_{PD}\theta_{LGD}$ — correlation between I_A and I_B — in Miu and Ozdemir's paper [74]. The correlation structure is the same as those shown in Table 2.1.

The potential loss $\mathcal{PL}$ is defined as

$$\mathcal{PL} = \mathcal{B}^{-1}(\Phi(B), \delta_1, \delta_2), \tag{A.3}$$

where $\mathcal{B}^{-1}$ denotes the inverse of the beta cumulative distribution function with shape parameters δ_1 and δ_2.

A.2.3 Pykhtin's Model

Pykhtin's one factor credit risk model specifies a perfect correlation between the systematic risk factors ($\theta_S = 1$) and an arbitrary degree of correlation between idiosyncratic risk factors ($\theta_I \in [-1, 1]$). The default drivers (or the asset returns) are governed by Eq. (2.1). and the collateral value $\mathcal{C}$ takes the form

$$\mathcal{C} = \exp(\mu + \sigma B). \tag{A.4}$$

B in Eq. (A.4) drives the standardized returns on collateral. For each obligor i in this framework,

$$\begin{aligned} B_i &= \beta S_A + \gamma I_{A,i} + \sqrt{1-\beta^2-\gamma^2}\eta_i \\ &= \beta S_A + \sqrt{1-\beta^2}\left(\frac{\gamma}{\sqrt{1-\beta^2}} I_{A,i} + \sqrt{\frac{1-\beta^2-\gamma^2}{1-\beta^2}}\eta_i\right). \end{aligned}$$

By setting

$$I_{B,i} = \frac{\gamma}{\sqrt{1-\beta^2}} I_{A,i} + \sqrt{\frac{1-\beta^2-\gamma^2}{1-\beta^2}}\eta_i,$$

$$B_i = \beta S_A + \sqrt{1-\beta^2} I_{B,i},$$

which is in the form of Eq. (2.2). The variable S_A is the common systematic risk factor driving defaults and losses on the portfolio. In Pykhtin's paper, $S_A = Y, I_{A,i} = \varepsilon$ (see Table A.1 for a complete overview of the parameters and variables in the model or revisit the introductory chapter of this document). The potential and realized loss associated with an obligor is expressed as in the case of Frye's model — Eq. (A.1) and Eq. (A.2). Table A.3 presents the correlation structure for this model.

Table A.3: Correlation structure of Pykhtin's model

Correlations	Formulas
(A_i, B_i)	$\alpha\beta + \sqrt{1-\beta^2}\sqrt{(1-\alpha^2)}\theta_I$
(A_i, A_j)	α^2
(B_i, B_j)	β^2
(A_i, B_j)	$\alpha\beta$

A.2.4 Witzany's Model

In this modeling setting, α and β are the parameters $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$ in the Witzany's paper [92]. Table A.4 shows the correlation structure of the above discussed model. Given the model specification, the correlation between S_A and S_B, θ_S, is an arbitrary value and the correlation between the idiosyncratic risk factors, I_A and I_B is 0. The potential loss $\mathcal{PL}$ are determined by Eq. (A.3).

Table A.4: Correlation structure of Witzany's model

Correlations	Formulas
(A_i, B_i)	$\alpha\beta\theta_S$
(A_i, A_j)	α^2
(B_i, B_j)	β^2
(A_i, B_j)	$\alpha\beta\theta_S$

Appendix B

Derivations in Chapter 3

B.1 Conditional Mean and Covariance of A and B

Let

$$\mathcal{A} = \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} \alpha S_A + \sqrt{1-\alpha^2} I_{A,i} \\ \beta S_B + \sqrt{1-\beta^2} I_{B,i} \end{bmatrix}, \mathcal{I} = \begin{bmatrix} I_{A,i} \\ I_{B,i} \end{bmatrix} \text{ and } \mathcal{S} = \begin{bmatrix} S_A \\ S_B \end{bmatrix}.$$

Since by definition $\mathbb{E}[I_{A,i}] = \mathbb{E}[I_{B,i}] = 0$,

$$\mathbb{E}[\mathcal{A}|\mathcal{S} = \mathbb{S}] = \begin{bmatrix} \alpha s_A \\ \beta s_B \end{bmatrix},$$

where $\mathbb{S} = \begin{bmatrix} s_A \\ s_B \end{bmatrix}$. The corresponding covariance matrix of $\mathcal{A}$ given $\mathcal{S} = \mathbb{S}$ is derived as follows:

$$\begin{aligned}
\Sigma &= \mathbb{E}\left[(\mathcal{A} - \mathbb{E}[\mathcal{A}])(\mathcal{A} - \mathbb{E}[\mathcal{A}])^T | \mathcal{S} = \mathbb{S}\right] \\
&= \mathbb{E}\left[\begin{bmatrix} \sqrt{1-\alpha^2} I_{A,i} \\ \sqrt{1-\beta^2} I_{B,i} \end{bmatrix} \begin{bmatrix} \sqrt{1-\alpha^2} I_{A,i} \sqrt{1-\beta^2} I_{A,i} \end{bmatrix}\right] \\
&= \mathbb{E}\begin{bmatrix} \sqrt{1-\alpha^2} I_{A,i}^2 & \sqrt{1-\alpha^2}\sqrt{1-\beta^2} I_{A,i} I_{B,i} \\ \sqrt{1-\alpha^2}\sqrt{1-\beta^2} I_{A,i} I_{B,i} & \sqrt{1-\beta^2} I_{B,i}^2 \end{bmatrix} \\
&= \begin{bmatrix} 1-\alpha^2 & \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I \\ \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I & 1-\beta^2 \end{bmatrix}.
\end{aligned}$$

B.2 $\mathsf{P}(A \leq \Phi^{-1}(PD) | B = b, S_A = s_A, S_B = s_B)$

From the properties of joint normal random variables, we have

- the conditional expectation of A given that $B = b, S_A = s_A$ and $S_B = s_B$ is derived as

$$E[A|B = b, S_A = s_A, S_B = s_B] = \alpha s_A + \theta_I \frac{\sqrt{1-\alpha^2}}{\sqrt{1-\beta^2}}(b - \beta s_B),$$

- the conditional variance of A given $B = b$ is obtained as

$$\sigma^2_{A|B=b,S_A=s_A,S_B=s_B} = (1-\alpha^2)(1-\theta_I^2),$$

and

- the conditional distribution of A given $B = b$ is therefore obtained as

$$A|B = b, S_A = s_A, S_B = s_B \sim N\left(\alpha s_A + \theta_I \frac{\sqrt{1-\alpha^2}}{\sqrt{1-\beta^2}}(b - \beta s_B), (1-\alpha^2)(1-\theta_I^2)\right).$$

The above implies that

$$\begin{aligned}
\mathsf{P}(A \leq \Phi^{-1}(PD)|B = b, S_A = s_A, S_B = s_B) &= \mathsf{P}\left(Z \leq \frac{\Phi^{-1}(PD) - \alpha s_A - \theta_I \frac{\sqrt{1-\alpha^2}}{\sqrt{1-\beta^2}}(b - \beta s_B)}{\sqrt{(1-\alpha^2)(1-\theta_I^2)}}\right) \\
&= \Phi\left(\frac{\Phi^{-1}(PD) - \alpha s_A - \theta_I \frac{\sqrt{1-\alpha^2}}{\sqrt{1-\beta^2}}(b - \beta s_B)}{\sqrt{(1-\alpha^2)(1-\theta_I^2)}}\right).
\end{aligned}$$

Note that ρ_A given $S_A = s_A$ and $S_B = s_B$ is derived as

$$\begin{aligned}
\rho_{A|S_A=s_A,S_B=s_B} &= \frac{Cov(A, B|S_A = s_A, S_B = s_B)}{\sigma_{A|S_A=s_A,S_B=s_B} \cdot \sigma_{B|S_A=s_A,S_B=s_B}} \\
&= \frac{\sqrt{1-\alpha^2}\sqrt{1-\beta^2}\theta_I}{\sqrt{1-\alpha^2}\sqrt{1-\beta^2}} \\
&= \theta_I.
\end{aligned}$$

B.3 Taylor Series Approximation

For the purpose of our study, define a bivariate function $g : \Lambda \subset \mathbb{R}^2 \to \mathbb{R}$, where Λ is the induced sample space of two dimensional random variable $\mathcal{K} = (X, Y)$ defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, consisting of a sample space Ω, a $\sigma-$ Algebra $\mathcal{F}$ and a probability measure $\mathcal{P}$. The elements in $\mathcal{F}$ are events that can be measured. For instance, portfolio-level default rate DR_p, portfolio-level realized loss $\mathcal{RL}_p$ and loss given default LGD_p are claimed as measurable events. In mathematical notational form, we write Λ as

$$\Lambda = \{(X, Y) : X \in \Lambda_1, Y \in \Lambda_2\},$$

where Λ_1 and Λ_2 are subsets of $\mathbb{R}$, the set of real numbers. Let the mean vector $\mathcal{M}_{\mathcal{K}}$ and covariance matrix $\Sigma_{\mathcal{K}}$ of $\mathcal{K}$ be given as

$$\mathcal{M}_{\mathcal{K}} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \text{ and } \Sigma_{\mathcal{K}} = \begin{bmatrix} \sigma_X^2 & \sigma_{X,Y} \\ \sigma_{X,Y} & \sigma_Y^2 \end{bmatrix}.$$

It follows that the quadratic Taylor Series Approximation of $g(\cdot)$ around the respective means of X and Y is

$$\begin{aligned} g(X,Y) \approx{} & g(\mu_X, \mu_Y) + g_X(\mu_X, \mu_Y)(X - \mu_X) + g_Y(\mu_X, \mu_Y)(Y - \mu_Y) \\ & + \frac{1}{2}\Big[g_{XX}(\mu_X, \mu_Y)(X - \mu_X)^2 + 2g_{XY}(\mu_X, \mu_Y)(X - \mu_X)(Y - \mu_Y) \\ & + g_{YY}(\mu_X, \mu_Y)(Y - \mu_Y)^2\Big], \end{aligned} \tag{B.1}$$

where $g_X = \frac{\partial g}{\partial X}, g_{XX} = \frac{\partial^2 g}{\partial X^2}, g_Y = \frac{\partial g}{\partial Y}$ and $g_{YY} = \frac{\partial^2 g}{\partial Y^2}$.

From Eq. (B.1), the approximated mean for $g(X,Y)$ is obtained as

$$\begin{aligned} \mathbb{E}[g(X,Y)] \approx{} & g(\mu_X, \mu_Y) + g_X(\mu_X, \mu_Y)\,\mathbb{E}(X - \mu_X) + g_Y(\mu_X, \mu_Y)\,\mathbb{E}(Y - \mu_Y) \\ & + \frac{1}{2}\Big[g_{XX}(\mu_X, \mu_Y)\,\mathbb{E}(X - \mu_X)^2 + 2g_{XY}(\mu_X, \mu_Y)\,\mathbb{E}(X - \mu_X)(Y - \mu_Y) \\ & + g_{YY}(\mu_X, \mu_Y)\,\mathbb{E}(Y - \mu_Y)^2\Big] \\ ={} & g(\mu_X, \mu_Y) + \frac{1}{2}\Big[g_{XX}(\mu_X, \mu_Y)\sigma_X^2 + 2g_{XY}(\mu_X, \mu_Y)\sigma_{X,Y} + g_{YY}(\mu_X, \mu_Y)\sigma_Y^2\Big]. \end{aligned} \tag{B.2}$$

Since $ALGD_p = \frac{ARL_p(S_A, S_B)}{ADR_p(S_A)}$, let $g(X,Y) = \frac{Y}{X}$ so that $X = ADR_p$ and $Y =$

ARL_p. Thus, $\mu_X = \mu_{ADR_p}, \mu_Y = \mu_{ARL_p}, \sigma_X = \sigma_{ADR_p}, \sigma_Y = \sigma_{ARL_p}$ and $\sigma_{X,Y} = \sigma_{ADR_p,ARL_p}$. From Eq. (B.2), we obtain the expected value of $ALGD_p$ as

$$\mathbb{E}[ALGD_p] \approx \frac{\mu_{ARL_p}}{\mu_{ADR_p}} + \frac{\mu_{ARL_p}}{\mu^3_{ADR_p}}\sigma^2_{ADR_p} - \frac{1}{\mu^2_{ADR_p}}\sigma_{ADR_p,ARL_p}.$$

B.4 $\mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD)]$ and $\mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)]$

The moments of t-dimensional multivariate normal distribution $B^* = (B_1^*, B_2^*, ..., B_t^*)' \sim N(\underset{\sim}{0}, \Sigma)$ with double truncation in all variables $x_j \leq B^* \leq y_j$ can be obtained using

$$\mathbb{E}[B_i^*|x_1 \leq B_1^* \leq y_1, x_2 \leq B_2^* \leq y_2, ..., x_t \leq B_t^* \leq y_t] = \sum_{j=1}^{t} \sigma_{i,j}(f_j(x_j) - f_j(y_j)), \quad \text{(B.3)}$$

where $\sigma_{i,j}$ is the standard deviation of B_i variable when $i = j$, covariance of B_i^* and B_j^* when $i \neq j$. $f_j(\cdot)$ is the $j-th$ marginal truncated normal density [71]:

$$f_j(x_j) = \mathsf{P}(B_j^* = x_j|x_1 \leq B_1^* \leq y_1, x_2 \leq B_2^* \leq y_2, ..., x_t \leq B_t^* \leq y_t).$$

Deriving $\mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD)]$

We have a bivariate standard normal in this case. From Eq. (B.3)

$$\begin{aligned}\mathbb{E}[B_i^*|x_1 \leq B_1^* \leq y_1, x_2 \leq B_2^* \leq y_2] &= \sum_{j=1}^{2} \sigma_{i,j}(f_j(x_j) - f_j(y_j)) \\ &= \sigma_{i,1}(f_1(x_1) - f_1(y_1)) + \sigma_{i,2}(f_2(x_2) - f_2(y_2)).\end{aligned}$$

So

$$\begin{aligned}\mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD)] &= \mathbb{E}[B_i|-\infty \leq B_i \leq \infty, A_i \leq \Phi^{-1}(PD)] \\ &= \sigma_{B_i}(f_{B_i}(-\infty) - f_{B_i}(\infty)) + \sigma_{B_i,A_i}(f_{A_i}(-\infty) - f_{A_i}(\Phi^{-1}(PD))),\end{aligned}$$

where

$$\begin{aligned}f_{B_i}(b) &= \mathsf{P}(B_i = b|-\infty \leq B_i \leq -\infty, -\infty \leq A_i \leq \Phi^{-1}(PD)) \\ &= \frac{\mathsf{P}(-\infty \leq B_i \leq \infty, -\infty \leq A_i \leq \Phi^{-1}(PD)|B = b) \cdot \mathsf{P}(B_i = b)}{\mathsf{P}(A_i \leq \Phi^{-1}(PD))} \\ &= \frac{\mathsf{P}(-\infty \leq A_i \leq \Phi^{-1}(PD)|B = b) \cdot \mathsf{P}(B_i = b)}{PD} \\ &= \Phi\left(\frac{\Phi^{-1}(PD) - \rho_A \cdot b}{\sqrt{1-\rho_A^2}}\right) \cdot \frac{\phi(b)}{PD}\end{aligned}$$

and

$$\begin{aligned}f_{A_i}(a_i) &= \mathsf{P}(A_i = a_i|-\infty \leq B_i \leq \infty, -\infty \leq A_i \leq \Phi^{-1}(PD)) \\ &= \frac{\mathsf{P}(-\infty \leq B_i \leq \infty|A_i = a_i) \cdot \mathsf{P}(A_i = a_i)}{\mathsf{P}(A_i \leq \Phi^{-1}(PD))} \\ &= \frac{\phi(a_i)}{PD}.\end{aligned}$$

Thus

$$\mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD)] = -\rho_A \cdot \frac{\phi(\Phi^{-1}(PD))}{PD}.$$

Deriving $\mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)]$

Here, we have a 3-dimensional normal distribution. Similar as above, we have

$$\begin{aligned}
&\mathbb{E}[B_i|A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)] \\
&\quad = \mathbb{E}[B_i|-\infty \leq B_i \leq \infty, -\infty \leq A_i \leq \Phi^{-1}(PD), -\infty \leq A_j \leq \Phi^{-1}(PD)] \\
&\quad = \rho_{B_i}(f_{B_i}(-\infty) - f_{B_i}(\infty)) + \rho_A(f_{A_i}(-\infty) - f_{A_i}(\Phi^{-1}(PD)) \\
&\qquad + \alpha\beta\theta_S(f_{A_j}(-\infty) - f_{A_j}(\Phi^{-1}(PD)). \qquad \text{(B.4)}
\end{aligned}$$

The truncated marginal densities are obtained as

$$\begin{aligned}
f_{B_i}(b) &= \mathsf{P}(B_i = b|-\infty \leq B_i \leq \infty, A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)) \\
&= \frac{\mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)|B_i = b) \cdot \mathsf{P}(B_i = b)}{\mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD))} \\
&= \mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)|B_i = b) \cdot \frac{\phi(b)}{PD^{(2)}}, \qquad \text{(B.5)}
\end{aligned}$$

$$\begin{aligned}
f_{A_i}(a_i) &= \mathsf{P}(A_i = a_i|-\infty \leq B_i \leq \infty, A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)) \\
&= \frac{\mathsf{P}(A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)|A_i = a_i) \cdot \mathsf{P}(A_i = a_i)}{PD^{(2)}} \\
&= \frac{\mathsf{P}(A_j \leq \Phi^{-1}(PD)|A_i = a_i) \cdot \mathsf{P}(A_i = a_i)}{PD^{(2)}} \\
&= \Phi\left(\frac{\Phi^{-1}(PD) - \alpha^2 \cdot a_i}{\sqrt{1-\alpha^4}}\right) \cdot \frac{\phi(a_i)}{PD^{(2)}},
\end{aligned}$$

and

$$f_{A_j}(a_j) = \Phi\left(\frac{\Phi^{-1}(PD) - \alpha^2 \cdot a_j}{\sqrt{1-\alpha^4}}\right) \cdot \frac{\phi(a_j)}{PD^{(2)}}. \tag{B.6}$$

Inputting these densities (Eq. (B.5)-(B.6)) in Eq. (B.4) gives

$$\mathbb{E}[B_i | A_i \leq \Phi^{-1}(PD), A_j \leq \Phi^{-1}(PD)] = -(\rho_A + \alpha\beta\theta_S) \cdot \frac{\phi(\Phi^{-1}(PD))}{PD^{(2)}} \cdot \zeta.$$

where

$$\zeta = \Phi\left(\frac{\Phi^{-1}(PD)(1-\alpha^2)}{\sqrt{1-\alpha^4}}\right).$$

Appendix C

Derivations in Chapter 4

C.1 $\mathbb{E}[A_i, A_j | B_i = b]$ and $Cov(A_i, A_j | B_i = b)$

The joint distribution of A_i, A_j and B_i is multivariate normal with mean vector and covariance matrix is

$$\mu_{A_i,A_j,B_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \Sigma_{A_i,A_j,B_i} = \begin{bmatrix} 1 & \alpha^2 & \rho_A \\ \alpha^2 & 1 & \alpha\beta\theta_S \\ \rho_A & \alpha\beta\theta_S & 1 \end{bmatrix},$$

respectively. The conditional expectation $\mu_{A_i,A_j|B_i=b}$ and covariance $\Sigma_{A_i,A_j|B_i=b}$ of A_i and A_j given $B_i = b$ are derived by respectively partitioning the vector μ_{A_i,A_j,B_i} and the matrix Σ_{A_i,A_j,B_i} into the form

$$\mu_{A_i,A_j,B_i} = \begin{bmatrix} \mu^{[1]}_{A_i,A_j,B_i} \\ \mu^{[2]}_{A_i,A_j,B_i} \end{bmatrix} \text{ and } \Sigma_{A_i,A_j,B_i} = \begin{bmatrix} \Sigma^{[11]}_{A_i,A_j,B_i} & \Sigma^{[12]}_{A_i,A_j,B_i} \\ \Sigma^{[21]}_{A_i,A_j,B_i} & \Sigma^{[22]}_{A_i,A_j,B_i} \end{bmatrix},$$

where

$$\mu^{[1]}_{A_i,A_j,B_i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mu^{[2]}_{A_i,A_j,B_i} = 0,$$

$$\Sigma^{[11]}_{A_i,A_j,B_i} = \begin{bmatrix} 1 & \alpha^2 \\ \alpha^2 & 1 \end{bmatrix}, \quad \Sigma^{[12]}_{A_i,A_j,B_i} = \begin{bmatrix} \rho_A \\ \alpha\beta\theta_S \end{bmatrix}$$

$$\Sigma^{[21]}_{A_i,A_j,B_i} = \begin{bmatrix} \rho_A & \alpha\beta\theta_S \end{bmatrix}, \text{ and } \Sigma^{[22]}_{A_i,A_j,B_i} = 1.$$

So we have the [43]

$$\begin{aligned} \mu_{A_i,A_j|B_i=b} &= \mu^{[1]}_{A_i,A_j,B_i} + \Sigma^{[12]}_{A_i,A_j,B_i} \left[\Sigma^{[22]}_{A_i,A_j,B_i}\right]^{-1} (b - \mu^{[2]}_{A_i,A_j,B_i}) \\ &= \begin{bmatrix} \rho_A \cdot b \\ \alpha\beta\theta_S \cdot b \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \Sigma_{A_i,A_j|B_i=b} &= \Sigma^{[11]}_{A_i,A_j,B_i} - \Sigma^{[12]}_{A_i,A_j,B_i} \left[\Sigma^{[22]}_{A_i,A_j,B_i}\right]^{-1} \Sigma^{[21]}_{A_i,A_j,B_i} \\ &= \begin{bmatrix} 1-\rho_A^2 & \alpha^2 - \alpha\beta\theta_S\rho_A \\ \alpha^2 - \alpha\beta\theta_S\rho_A & 1-(\alpha\beta\theta_S)^2 \end{bmatrix}. \end{aligned}$$

Summarizing, conditional distribution of the pair (A_i, A_j) given $B_i = b$ is given as

$$A_i, A_j|B_i = b \sim N\left(\begin{bmatrix} \rho_A \cdot b \\ \alpha\beta\theta_S \cdot b \end{bmatrix}, \begin{bmatrix} 1-\rho_A^2 & \alpha^2 - \alpha\beta\theta_S\rho_A \\ \alpha^2 - \alpha\beta\theta_S\rho_A & 1-(\alpha\beta\theta_S)^2 \end{bmatrix}\right).$$

C.2 $\mathbb{E}[A_i, A_j | B_i = b_i, B_j = b_j]$ and $Cov(A_i, A_j | B_i = b_i, B_j = b_j)$

The mean vector and covariance matrix of the joint normal distribution of the pair (A_i, A_j, B_i, B_j) are respectively given as

$$\mu_{A_i,A_j,B_i,B_j} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \Sigma_{A_i,A_j,B_i,B_j} = \begin{bmatrix} 1 & \alpha^2 & \rho_A & \alpha\nu \\ \alpha^2 & 1 & \alpha\nu & \rho_A \\ \rho_A & \alpha\nu & 1 & \beta^2 \\ \alpha\nu & \rho_A & \beta^2 & 1 \end{bmatrix},$$

We obtain the conditional expectation $\mu_{A_i,A_j|B_i,B_j}$ and covariance matrix $\Sigma_{A_i,A_j|B_i,B_j}$ by employing the approach in Appendix C.1. We have

$$\begin{aligned} \mu_{A_i,A_j|B_i,B_j} &= \begin{bmatrix} \rho & \alpha\nu \\ \alpha\nu & \rho_A \end{bmatrix} \begin{bmatrix} 1 & \beta^2 \\ \beta^2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_i \\ b_j \end{bmatrix} \\ &= \frac{1}{1-\beta^4} \begin{bmatrix} \rho_A(b_i - \beta^2 \cdot b_j) + \alpha\nu(b_j - \beta^2 \cdot b_i) \\ \alpha(b_i - \beta^2 \cdot b_j) + \rho_A(b_j - \beta^2 \cdot b_i) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \Sigma_{A_i,A_j|B_i=b_i,B_j=b_j} &= \begin{bmatrix} 1 & \alpha^2 \\ \alpha^2 & 1 \end{bmatrix} - \begin{bmatrix} \rho_A & \alpha\nu \\ \alpha\nu & \rho_A \end{bmatrix} \begin{bmatrix} 1 & \beta^2 \\ \beta^2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho_A & \alpha\nu \\ \alpha\nu & \rho_A \end{bmatrix} \\ &= \frac{1}{1-\beta^4} \begin{bmatrix} \Gamma^*_{1,1} & \Gamma^*_{1,2} \\ \Gamma^*_{2,1} & \Gamma^*_{2,2} \end{bmatrix}, \end{aligned}$$

where

$$
\begin{cases}
\Gamma^*_{1,1} = \Gamma^*_{2,2} = (1-\beta^4) - \rho_A(\rho_A - \beta^2\alpha\nu) + \alpha\nu(\alpha\nu - \beta^2\rho_A), \\
\Gamma^*_{1,2} = \Gamma^*_{2,1} = \alpha^2(1-\beta^4) - \rho_A(\alpha\nu - \beta^2\rho_A) + \alpha\nu(\rho_A - \beta^2\alpha\nu).
\end{cases}
$$

Appendix D

Some Observations

D.1 $ALGD_p$ as a function of θ_I

The goal here is to determine the interval (conditions) under which asymptotic portfolio-level LGD increases (or decreases) as a function of θ_I. Recall that $ALGD_p = \frac{ARL_p(s_A,s_B)}{ADR_p(S_A)}$. Since $ADR_p(\cdot)$ is not a function of θ_I, by proving that $ARL_p(s_A, s_B)$ is an increasing (or decreasing) function of θ_I on a particular interval implies $ALGD_p$ is also an increasing (or decreasing) function of θ_I on that interval.

Now, let redefine ARL_p as a function of θ_I so that

$$ARL_p(\theta_I) = \int_{\mathbb{R}} G \cdot \Phi\left(Q(\theta_I)\right) db,$$

where

$$\begin{cases} G = H(b) \cdot \phi(b, \beta s_B, 1-\beta^2), \\ Q(\theta_I) = \frac{i_A - i_B\theta_I}{\sqrt{1-\theta_I^2}}. \end{cases}$$

And

$$\begin{cases} i_A = \frac{\Phi^{-1}(PD)-\alpha s_A}{\sqrt{1-\alpha^2}}, \\ i_B = \frac{b-\beta s_B}{\sqrt{1-\beta^2}}. \end{cases}$$

Observe that i_A is the critical value that drives the conditional default probability via s_A.

To demonstrate that $ARL_p(\theta_I)$ is an increasing (or decreasing) function on a particular interval we need to show that $ARL'_p(\theta_I) \geq 0$ on that interval. The first derivative of $ARL_p(\theta_I)$ is

$$ARL'_p(\theta_I) = \int_{\mathbb{R}} G \cdot Q'(\theta_I)\phi\left(Q(\theta_I)\right) db, \tag{D.1}$$

$Q'(\theta_I)$ in Eq. (D.1) is the first derivative of Q with respect to θ_I and is given as

$$\begin{aligned} Q'(\theta_I) &= \frac{\theta_I(i_A - i_B\theta_I)}{[1-\theta_I^2]^{\frac{3}{2}}} - \frac{i_A}{[1-\theta_I^2]^{\frac{1}{2}}} \\ &= \frac{\theta_I i_A - i_B}{[1-\theta_I^2]^{\frac{3}{2}}}. \end{aligned}$$

By definition $H(b)$ and $\phi(\cdot)$ are positive, thus $G \geq 0$. So for $ARL'_p(\theta_I) \geq 0$, $Q'(\theta_I) \geq 0$.

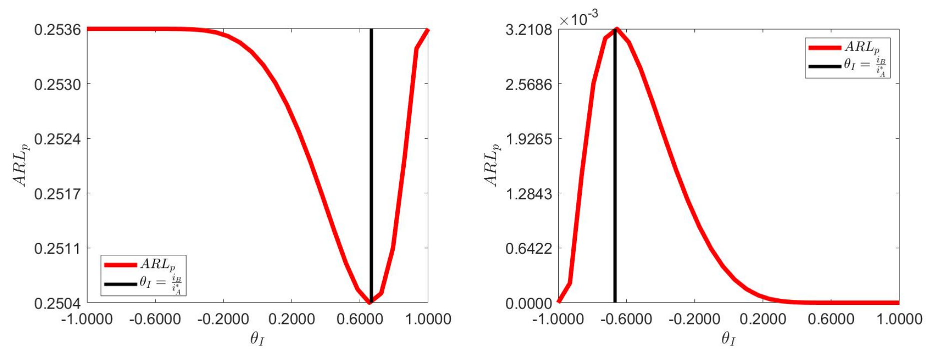
Proposition 4. *ARL_p decreases and then increases as a function of θ_I if and only if $i_A > 0$ and $|i_B| \leq i_A$, whereas ARL_p increases then decreases as a function of θ_I if and only if $i_A < 0$ and $|i_B| \leq |i_A|$. In both cases the point where the behaviour changes is $\theta_I = \frac{i_B}{i_A}$.*

Proof. Suppose ARL_p increases as a function of θ_I. This implies $Q'(\theta_I) \geq 0$. This means $\theta_I i_A - i_B \geq 0$. Thus, $i_A > 0$ for this inequality to hold for all i_B. Also, since $-1 \leq \theta_I \leq 1$, it follows that $|i_B| \leq i_A$. The same chain of reasoning holds for decreasing ARL_p.

Suppose $i_A > 0$ and $|i_B| \leq i_A$. Since $[1-\theta_I^2]^{\frac{3}{2}}$, it follows that for $Q'(\theta_I) \geq 0$, $\theta_I i_A - i_B \geq 0$. This implies $\theta_I \geq \frac{i_B}{i_A}$. Conversely, ARL_p decreases for $\theta_I \leq \frac{i_B}{i_A}$.

The proof of the second part of Proposition 4 follows from the above. $\square$

Figure D.1 illustrates numerically, Proposition 4. We see that at $\frac{i_B}{i_A}$ (black vertical line), the graph of ARL_p transitions from a decreasing to increasing function of θ_I for values of $i_A = 3$ and $i_B = 2$ (Figure D.1a), and from an increasing to decreasing function of θ_I for $i_A = -3$ and $i_B = 2$ (Figure D.1b).



(a) $i_A = 3, i_B = 2$ (b) $i_A = -3, i_B = 2$

Figure D.1: ARL_p as a function of θ_I. $\delta_1 = 2, \delta_2 = 3, PD = 0.05, \rho_A = 0.4$.

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