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Generalized 4/2 Factor Model

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Abstract

We investigate portfolio optimization, risk management, and derivative pricing for a factor stochastic model that considers the 4/2 stochastic volatility on the common/systematic factor as well as on the intrinsic factor. This setting allows us to capture stochastic volatility and stochastic covariation among assets. The model is also a generalization of existing models in the literature as it includes the mean reverting property and spillover effect to capture wider types of financial assets. At a theoretical level we identify conditions for well-defined changes of measure. A quasi-closed form solution within a 4/2 structured model is obtained for a portfolio optimization problem. In the numerical section, a sensitivity analysis reveals a substantial impact on the implied volatility surface and risk measures level due to small changes in the 3/2 component b. In addition, commonality loading, spillover effect, and dependency among common factors are also influential with regards to implied volatility and risk measures.

Keywords – Stochastic covariance, 4/2 model, portfolio optimization, option pricing, implied volatility, risk measure

Lay Summary

A Factor model is a financial model that employs some correlated factors or characteristics to explain or calculate a financial variable. In mathematical finance, factor models are widely used to model the relationship between asset returns and underlying risk factors. The goal of the model is to measure the expected return and forecast/manage security risk. Our model decomposes the risk across the market into common factors and intrinsic factors in a clear manner. Specifically, the common factors are exogenous (observable or not) variables explaining the systematic risk in the market, while the intrinsic factors relate to companies' or assets' inherent risks. Moreover, each factor follows a 4/2 structured volatility process, which is a superposition of the well-known 1/2 Heston process and the 3/2 process brought up by Grasselli (2017). This setting allows us to capture stochastic volatility and stochastic covariation among assets. The model is also a generalization of existing models in the literature as it includes the mean reverting property and spillover effect to capture wider types of financial assets. In this thesis, we investigate portfolio optimization, risk management, derivative pricing, and sensitivity analysis on important parameters within this generalized 4/2 factor model.

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Nomenclature

X_t	asset price
Y_t	logarithm of asset X_t
$\mathbb{P}$	real-world probability measure
$\mathbb{Q}$	putable risk-neutral measure
$\mathbb{E}^{\mathbb{Q}}_{x,v,t}[.] = \mathbb{E}^{\mathbb{Q}}[._t = x, v_t = v]$	conditional expectation
n	number of assets
p	number of common factors
L	mean reverting level
c	driver of excess return from common factor
$\tilde{c}$	driver of excess return from intrinsic factor
F	a set of common factor
$\tilde{F}$	intrinsic factor
a	common factor loading
b	3/2 component within the 4/2 structure
Z, W, $\widetilde{W}$, B	standard Brownian motion
ρ, $\tilde{\rho}$	correlations between Brownian motions
v, $\tilde{v}$	CIR processes
α, $\tilde{\alpha}$	mean reverting speed of CIR process
θ, $\tilde{\theta}$	mean reverting level of CIR process
ξ, $\tilde{\xi}$	volatility of CIR process
β	spillover effect
V	common variance
$\widetilde{V}$	intrinsic variance
r	risk-free rate
$C(t, K)$	option price
K	strike price
T	time to maturity
σ	implied volatility
M_t	risk-free asset
P_t	wealth
π_t	proportion of wealth investing in risky asset
Θ	driver of correlation among assets

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1 Introduction

In this thesis, we introduce and study a multivariate 4/2 stochastic volatility factor model that combines, on each factor component, the well-known 1/2 Heston type process (Heston, 1993) and the 3/2 process (Platen, 1997) for the modeling of the factor's volatility. This combination was brought up by Grasselli (2017) to model stock prices with stochastic volatility and named 4/2 stochastic process due to its superposition of 1/2 and 3/2 processes. Our model describes a factor representation of a given stock assuming the risk of the stock can be explained by a set of common factors and an intrinsic factor. Our idea is to utilize the 4/2 models as underlying on a factor structure for asset prices, and we also allow for a mean-reverting structure on the assets. This helps in modelling other asset classes like multiple commodity behavior or multiple volatility indexes.

The remaining parts of the introduction are structured as follows: we firstly give a brief review of the most commonly studied factor models in literature and illustrate the connection to our factor model. The reason we formulate our model in a factor representation can be evidenced by the advantage of factor analysis. Then, we introduce our model framework based on factor analysis, together with stochastic volatility and an advanced 4/2 structure on each factor. Moreover, we present some flexibilities that can be incorporated to our model. Furthermore, a summary of the contributions of this thesis is discussed. Finally, the agenda of this work is presented.

In mathematical finance, factor models are quite popular in modelling the relationship between asset returns and underlying, market-driven risk factors. These models resolve the deficiencies of requiring large samples of historical data to produce precise analyses, in principle, one can get more accurate risk estimates assuming a small set of common factors to explain the risk across the market. For instance, *the capital asset pricing model* (CAPM) is a one-factor model where only a systematic risk (a market index) is used in explaining cross-section stock returns. Another popular example is *the Fama and French Three-Factor Model* (Fama and French, 1993), which was developed based on CAPM by considering two more common factors, the size risk (effect of market capitalization) and value risk (book-to-price ratio). These factor models assume that all the variation in financial assets' expected return can be decomposed and explained by a set of common risk factors (systematic) and a residual. All of the factors in CAPM and Fama and French model fit beautifully in our factor model as our common factors. The residual is further interpreted as intrinsic factor in our model, which captures asset's specific/inherent risk (non-systematic).

The factor structured models also lead to a highly structured covariance matrix among assets.

That is, the systemic/common risk factors are the same for each asset, and a higher marginal variance can either be caused by higher loading of one of the common risk factors or higher intrinsic risk. More importantly, factor models reduce the dimension of the parametric space, and keep the "curse of dimensionality" under control. For example, assume there are n assets, a standard constant covariance model would require $n(n+1)/2$ parameters, on the other hand in a factor model, if $p \ll n$, then one would require $n(p+1)$ parameters (linear rather than quadratic on dimensionality). The factor model can be very appealing by its simple explanation and fewer parameters requirement. The factors depend on the asset classes and the objectives of the modeler. There is a rich literature dedicates to selecting factors that can encompass the variation in the market, such as The Barra Risk Factor Analysis (Bender and Nielsen, 2012), which incorporates over 40 data metrics to identify and measure risk factors in terms of industry risk, investment theme risk, and company specific risk.

The formulation of our model decomposes the risk across the market into common factors and intrinsic factors in a clear manner. The independence of the common and the intrinsic factors, each with its own stochastic volatility process, enables a straightforward economics interpretation and also facilitates the statistical analysis. As mentioned before, in our model, the common factors are exogenous (observable or not) variables explaining the systematic risk in the market, while the intrinsic factors relate to companies' or assets' inherent risks. Note, most continuous-time models in the literature work directly with the covariance matrix, i.e. no insightful decomposition of it. For example, if you have two assets, you can take data and estimate the covariance of these two assets. Another alternative to our factor approach could be principal component analysis (PCA). PCA aims at finding components that explain most of the variance, and it splits the total variance into different components. Similarly to latent factors, the component/factor may not have economic interpretation. Therefore, our model has a more straightforward economics interpretation compared to estimating covariance directly or PCA. Moreover, compared to estimating covariance directly, when the number of risk factors is much smaller than that of financial assets, the risk of financial instruments can be efficiently computed through the chosen risk factors using a smaller parametric space. Thus, the setting of our model also facilitates statistical analysis.

Our model applies the setting of CAPM to get a decomposition of the covariance explained by common factors and variance explained by intrinsic factors, and each factor follows the advanced 4/2 stochastic volatility process. The benefits and necessity of considering a stochastic model can be shown by its capacity to capture established stylized facts. Nevertheless, both CAPM and Fama and French model do not entertain stochastic volatility on the factors, and they assume the factors are as simple as Gaussian processes. In this regard, our model fills an important gap in economic literature and mathematical literature by considering stochastic volatility on each factor as well. Therefore, our model can capture stochastic volatility, highly structured stochastic covariation among assets (Engle, 2002), multiple factors in the volatility (Heston et al., 2009), as well as leverage effect. To see this, note that in a factor decomposition, advanced processes for the individual common factors and the intrinsic factors lead, thanks to the commonality loadings, to advanced covariance and variance processes for the underlying assets.

Furthermore, in our model, since the volatility of each factor follows the 4/2 framework, the volatility of an asset follows a combination of independent $p+1$ (p common factors and one intrinsic factor) 4/2 models. The superiority of the 4/2 structure has been analyzed quite comprehensively in Grasselli (2017). For example, the 1/2 process and the 3/2 process predict differently when the instantaneous volatility increases: the 1/2 process forecasts the implied volatility flattening skews whereas the 3/2 model predicts steepening skews, both present in implied volatilities evolutions. Additionally, when there is a sudden increase in stock price, the 1/2 process requires high volatility-of-volatility parameter to comply with this change, and thus has a high risk in violating the Feller condition. On the other hand, the 3/2 model admits extreme paths with spikes in instantaneous volatility. Therefore, the author claims that the two processes complement each other and they can reproduce both "smile" and "skew" pattern of the implied volatility surface. Factors are usually assets/indexes themselves, therefore they should follow 4/2 structured processes.

Another flexibility of this model is that we can easily transform the mean reverting model into a non mean reverting one by setting the long term mean reverting level equal to risk-free interest rate and the mean reverting speed to zero. Thereby, assets, such as equity, can also be studied under this structure of factor decomposition and 4/2 structured instantaneous volatility. Another generalization provided in our model is allowing interdependence of the drift between assets, namely the spillover effect.

The main contributions of this work are: firstly, we identify a set of conditions that produce well-defined changes of measure and avoids local martingales for a multivariate factor 4/2 model; hence, it can be used for risk-neutral pricing purposes involving multiple assets. Note that we are dealing with incomplete market. We assume that we know the equivalent martingale measure we need to pick for option pricing among infinitely many. A proper Euler-based simulation setting is also implemented. Secondly, a quasi-closed form solution for a portfolio optimization problem in the context of Expected Utility Theory is obtained within a 4/2 structured model. We also identify situations where the solution can be presented in fully closed-form.

Thirdly, in a one factor model, we develop sensitivity analyses of implied volatility surface and important risk measures with respect to common factor loading a and the newly $3/2$ component b, this is done for both a mean reverting and a non mean reverting one common factor model. In terms of implied volatility surface, for a mean reverting model, we found an increase of up to 91.7% due to the presence of the $3/2$ component b and an increase of up to 55% due to common factor loading a. In terms of risk measures, there is an increase of 28.6% in value at risk and an increase of 36.6% in expected shortfall due to the presence of the $3/2$ component b, while there is a jump of 35.3% in value at risk resulting from considering stochastic correlation between assets. In general, it can be observed that the impact from the $3/2$ component and the common factor loading can be crucial with different underlying processes for common and intrinsic factors. Moreover, the sensitivity analysis for a non mean reverting model is consistent with that on a mean reverting model. Further, we compare the value of risk implied by our model with respect to important parameters under two forms of market price of risk, one is proportional to a $4/2$ structured volatility process while the other one is proportional to a $1/2$ structured volatility process. It turns out that the $1/2$ structured market price of risk persistently shows higher impacts on risk compared to the $4/2$ structured market price of risk.

Fourthly, inspired by the importance of the presence of the $3/2$ component and the necessity of considering multiple factors, we extend the implied volatility and risk measure sensitivity analyses to a two factor model. For a mean reverting model, we examine the impact of the spillover effects β, driver of correlation among assets Θ, and the $3/2$ component b. Specifically, the impact of the spillover effects on implied volatility surface varies with its sign. For instance, changing the spillover from the other asset from negative to positive can lead to a 50% increase in value at risk and a 36.3% increase in expected shortfall. Also, a very small value of the $3/2$ component b can have a jump of 36% on asset's expected return and an increase of 18.12% on its variance of return. Meanwhile, a stabilization effect can be evidenced by its decreasing variation in implied volatility surface from 11% to 3.9% if no spillover effect, and from 100% to 50% in the presence of spillover effect. Another observation is that the driver of correlation among assets Θ changes the leverage effect. Furthermore, for a non mean reverting model, a very small value of the $3/2$ component b can result in a 33.03% increase in asset's expected return and 41% increase in its return's variance. In addition, we study the parameters from the underlying CIR process with different driver of correlation among assets Θ to have a more comprehensive insight. The analysis shows that the impact of κ, θ, and ξ differs for different choices of correlation Θ. In general, the mean reverting level θ can change the trend of implied volatility surface, and volatility of volatility ξ infuses more variance as we would expect.

This work is organized as follows: in section 2, we specify the model, describe the simulation, and identify a set of conditions that produce well-defined changes of measure. In section 3, a portfolio optimization problem is considered within a 4/2 structured model. A one common factor and one intrinsic factor case is studied under a mean reverting and a non mean reverting model respectively in section 4. In particular, the impacts from commonality loading a, the 3/2 component b, and different market prices of risk are examined. In section 5, a two common factors case is investigated in terms of implied volatility surface and risk measures. In section 6, a more advanced model is defined and discussed. The conclusions are given in section 7.

2 Model Specification

In this section, we first define and introduce our generalized 4/2 structured factor model under measure $\mathbb{P}$ followed by a description of model simulation. Also, the possible change of measure under specified market price of risk is discussed. The necessary conditions on parameters are provided to ensuring well-defined changes of measure. Later on, we extract implied volatilities from an exponential OU process and a Gaussian process for comparison. For the convenience of the presentation and discussion of the model, a compilation of notation is provided in the Nomenclature section that prior to the content.

2.1 Model specification

Suppose $X_t = (X_1(t), ..., X_n(t))'$ is a vector of assets. Assume a probability space (Ω, $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\in[0,T]}$, $\mathbb{P}$) where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a right-continuous information filtration generated by the involving standard Brownian motions, then the dynamics for single asset $X_i(t)$ under $\mathbb{P}$-measure is defined as

$$\frac{dX_i(t)}{X_i(t)} = \left\{ L_i - \sum_{j=1}^{n} \beta_{ij} ln(X_j(t)) \right\} dt + \sum_{j=1}^{p} a_{ij} \frac{dF_j}{F_j} + \frac{d\tilde{F}_i}{\tilde{F}_i}, \tag{2.1}$$

where L_i is the long term average of asset $X_i(t)$, a_{ij} and β_{ij} are arbitrary constants.

Here the processes F_j, $j = 1, ..., p$ represent independent common factors and can be interpreted as the regressors on a Fama and French d-factor capital asset pricing model (Fama and French, 2015) or simply as the latent factor in a factor analysis. In this setting, the parameters a_{ij} for asset X_i, $j = 1, ..., p$, can be explained as either the "betas" in a regression or the factor loadings from a factor analysis. This provides a framework to interpret and estimate these parameters. The process $\tilde{F}_i$ represents the intrinsic factor that is related to asset X_i. Further, each common factor F_j or the intrinsic factor $\tilde{F}_i$ is assumed to follow a 4/2 stochastic volatility process,

$$\begin{aligned}
\frac{dF_j}{F_j} &= \left(c_j(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})^2 \right) dt + (\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}}) dW_j^{\mathbb{P}}(t), \\
\frac{d\tilde{F}_i}{\tilde{F}_i} &= \left(\tilde{c}_i(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2 \right) dt + (\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}}) d\widetilde{W}_i^{\mathbb{P}}(t), \\
dv_j(t) &= \alpha_j(\theta_j - v_j(t))dt + \xi_j \sqrt{v_j(t)} dB_j^{\mathbb{P}}(t), j = 1, ..., p \\
d\tilde{v}_i(t) &= \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t))dt + \tilde{\xi}_i \sqrt{\tilde{v}_i(t)} d\widetilde{B}_i^{\mathbb{P}}(t), i = 1, ..., n
\end{aligned} \tag{2.2}$$

with quadratic variation structure $\left\langle dB_j^{\mathbb{P}}(t), dW_j^{\mathbb{P}}(t) \right\rangle = \rho_j dt$, $\left\langle d\widetilde{B}_i^{\mathbb{P}}(t), d\widetilde{W}_i^{\mathbb{P}}(t) \right\rangle = \tilde{\rho}_i dt$, and zero otherwise. Additionally, c_j and $\tilde{c}_i$ are arbitrary constants. The terms $\left(c_j(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})^2\right)$ and $\left(\tilde{c}_i(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2\right)$ represent the premia/excess return associated with the Brownian risks from common factors and intrinsic factor, respectively.

In the language of factor decomposition, the variance of the underlying asset is categorized into two sources: the commonality or systematic variances, that captures co-variation of asset returns, are driven by $V_j(t) = (\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})^2$ for $j = 1, ..., p$ (in matrix form $\Lambda_{n\times p} = A\text{diag}\left(V^{\frac{1}{2}}(t)\right)$); while the remaining intrinsic variance, that explains the variance of asset itself, corresponds to $\widetilde{V}_i(t) = (\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2$ with $\Psi = \text{diag}\left(\tilde{V}(t)\right)$. In matrix form, a factors decomposition of the quadratic variation of asset prices can be expressed as

$$\Sigma(t)dt = (\Lambda\Lambda' + \Psi)\,dt = \left(A\text{diag}(V(t))A' + \text{diag}(\widetilde{V}(t))\right)dt$$

where $(a_{ij})_{n\times p}$ is the ij^{th} entry of a possibly orthogonal matrix A (assume $n = p$ if necessary), which determines the dependency or correlation structure among risky assets. In this vein, the constants c_j and $\tilde{c}_i$ drives the risk premium (excess return) of asset $X_i(t)$ associated with the volatility risk from the common factors and the volatility risk from the intrinsic factors respectively.

In addition, the randomness driving commonality volatility v_j and the asset-related randomness $\tilde{v}_i$ follow standard CIR processes, which means α_j, θ_j, and ξ_j are positive constants satisfying $\alpha_j\theta_j \geq \frac{\xi_j^2}{2}$ (Feller condition). Similarly, $\tilde{\alpha}_i$, $\tilde{\theta}_i$, and $\tilde{\xi}_i$ are positive constants satisfying $\tilde{\alpha}_i\tilde{\theta}_i \geq \frac{\tilde{\xi}_i^2}{2}$. Note that the Feller condition in CIR model guarantees that the process remains positive. Thus, it is a popular model for volatility due to its positiveness and mean-reverting properties.

Furthermore, β_{ij} is the modification of the mean reverting level of asset i due to changes in asset j. It captures the interdependence in the drift between assets $X_i(t)$ and $X_j(t)$. In other words, when $i \neq j$, the model allows asset j to influence the mean/trend of asset i. This is known as the spillover of asset X_j on asset X_i on the level of expected return. When $j = i$, it is the mean reverting speed of asset X_i. Additionally, if $\beta_{ij} = 0$, the process of asset price follows a generalized Geometric Brownian motion with stochastic drift and volatility.

Based on the quadratic variation relationship defined in this model, if we assume $B_j^{\mathbb{P}}$, $B_j^{\mathbb{P}}(t)^{\perp}$, $\tilde{B}_i^{\mathbb{P}}(t)$, $\tilde{B}_i^{\mathbb{P}}(t)^{\perp}$ are independent Brownian motions with $0 \leq \rho_j \leq 1$ and $0 \leq \tilde{\rho}_i \leq 1$, then it follows

$$\begin{aligned} dW_j^{\mathbb{P}}(t) &= \rho_j dB_j^{\mathbb{P}}(t) + \sqrt{1-\rho_j^2}dB_j^{\mathbb{P}}(t)^{\perp} \\ d\widetilde{W}_i^{\mathbb{P}}(t) &= \tilde{\rho}_i d\widetilde{B}_i^{\mathbb{P}}(t) + \sqrt{1-\tilde{\rho}_i^2}d\widetilde{B}_i^{\mathbb{P}}(t)^{\perp}. \end{aligned} \tag{2.3}$$

By observing the form of the dynamics followed by the asset price, it is easy to consider the logarithmic form. The dynamics of log price $Y_i(t) = ln(X_i(t))$ under $\mathbb{P}$-measure is then given by

$$
\begin{aligned}
dY_i(t) = &\left\{ L_i + (c_i - \frac{1}{2}) \sum_{j=1}^{n} a_{ij}^2 (\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})^2 - \sum_{j=1}^{n} \beta_{ij} Y_j(t) + (\tilde{c}_i - \frac{1}{2})(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2 \right\} dt \\
&+ \sum_{j=1}^{n} a_{ij} (\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})(\rho_j dB_j^{\mathbb{P}}(t) + \sqrt{1-\rho_j^2} dB_j^{\mathbb{P}}(t)^{\perp}) \\
&+ (\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})(\tilde{\rho}_i d\widetilde{B}_i^{\mathbb{P}}(t) + \sqrt{1-\tilde{\rho}_i^2} d\widetilde{B}_i^{\mathbb{P}}(t)^{\perp}), \\
dv_j(t) = &\ \alpha_j(\theta_j - v_j(t))dt + \xi_j \sqrt{v_j(t)} dB_j^{\mathbb{P}}(t), j = 1, ..., n \\
d\tilde{v}_i(t) = &\ \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t))dt + \tilde{\xi}_i \sqrt{\tilde{v}_i(t)} d\widetilde{B}_i^{\mathbb{P}}(t), i = 1, ..., n
\end{aligned}
$$

where $c_i = \sum_{j=1}^{n} c_j = \sum_{j=1}^{n} \left(\rho_j \lambda_j + \sqrt{1-\rho_j^2} \lambda_j^{\perp} \right)$, $\tilde{c}_i = \tilde{\rho}_i \tilde{\lambda}_i + \sqrt{1-\tilde{\rho}_i^2} \tilde{\lambda}_i^{\perp}$. Note that λ_j, $\lambda_j^{\perp}$, $\tilde{\lambda}_i$, and $\tilde{\lambda}_i^{\perp}$ are the rates/derives that featuring individual independent risk factors.

2.2 Model Simulation

In this subsection, we depict the Euler approximation method in a general way first. Then by applying it to simulate our model, a possible issue of negative values for CIR processes is discussed and a proper correction is further provided.

2.2.1 Review of Euler Method

For a general diffusion process (SDE),

$$dX(t) = a(X(t), t)dt + \sigma(X(t), t)dW(t),$$

where $a(X(t), t)$ is the drift term, $\sigma(X(t), t)$ is the diffusion term, and $W(t)$ is a Brownian motion. The Euler approximation of the general diffusion process is then given by

$$X(t_{k+1}) = X(t_k) + a(X(t_k), t_k)\Delta t + \sigma(X(t_k), t_k)\sqrt{\Delta t} Z,$$

where the discretize size $\Delta t = t_{k+1} - t_k = \frac{T}{N}$ for $k = 0, ..., N-1$, the discretize points are $0 = t_0 < t_1 < ... < t_N = T$ on time interval $[0, T]$, and the initial value is $X(t_0) = X(0) = X_0$. Since ΔW is a Brownian motion, such that $\Delta W \sim N(0, \Delta t)$. So we can replace ΔW as $\sqrt{\Delta t} Z$ where Z is a standard normal.

2.2.2 Euler Approximation

Now, let us apply Euler method to approximate our generalize 4/2 factor model. The discretization of $v_j(t)$, $\tilde{v}_i(t)$ and $Y_i(t)$ for simulations looks as follow: Firstly, discretize v_j and $\tilde{v}_i$ processes, given the initial value $v_j(t_0) = v_j(0)$ for all j, and $\tilde{v}_i(t_0) = \tilde{v}_i(0)$ for all i:

$$\begin{aligned}
v_j(t_k) &= v_j(t_{k-1}) + \alpha_j(\theta_j - v_j(t_{k-1}))\Delta t + \xi_j \sqrt{v_j(t_{k-1})}\Delta B_j(t_{k-1}), \\
\tilde{v}_i(t_k) &= \tilde{v}_i(t_{k-1}) + \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t_{k-1}))\Delta t + \tilde{\xi}_i \sqrt{\tilde{v}_i(t_{k-1})}\Delta \widetilde{B}_i(t_{k-1}).
\end{aligned}$$

Here, we need to notice that although the CIR process itself is guaranteed to be positive, the Euler discretization has a nonzero probability of becoming negative in the next time step. In practice, there are two corrections for this problem (Lord et al., 2010): *absorption* and *reflection*. Absorption means simply ignores the negative values, and take them as zero instead. Reflection means taking absolute values. However, absorption can not be applied with regard to our asset processes because it may drive volatility of the underlying asset to infinity due to the flipped CIR term $\frac{b_j}{\sqrt{v_j(t_k)}}$. Thus, we opt for the reflection method in the simulation scheme for $v_j(t_k)$ and $\tilde{v}_i(t_k)$, this means

$$\begin{aligned}
\hat{v}_j(t_k) &= v_j(t_{k-1}) + \alpha_j(\theta_j - v_j(t_{k-1}))\Delta t + \xi_j \sqrt{v_j(t_{k-1})}\Delta B_j(t_{k-1}), \\
v_j(t_k) &= |\hat{v}_j(t_k)| \\
\hat{\tilde{v}}_i(t_k) &= \tilde{v}_i(t_{k-1}) + \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t_{k-1}))\Delta t + \tilde{\xi}_i \sqrt{\tilde{v}_i(t_{k-1})}\Delta \widetilde{B}_i(t_{k-1}), \\
\tilde{v}_i(t_k) &= |\hat{\tilde{v}}_i(t_k)|.
\end{aligned}$$

Diop (2004) has proven that the Euler discretization with the reflection fix for the Heston has a weak convergence of order 1 in the time step. Specifically, for a scheme that converges weakly with order p, Duffie et al. (1995) has proven that for the optimal combination of the number of path N and discretization size Δt, the root mean square error (RMSE) for a European call under Heston has $\mathcal{O}(N^{-p/(2p+1)})$ convergence. To be more clear, RMSE is defined as $\sqrt{\text{bias}(\hat{C})^2 + \text{variance}(\hat{C})}$, where $\hat{C}$ is a Monte Carlo estimator of European option prices.

Next, given the initial value $Y_i(t_0) = Y_i(0)$ and the quadratic variation relationships in (2.3), the discretization of process $Y_i(t)$ follows

$$
\begin{aligned}
Y_i(t_{k+1}) = Y_i(t_k) + &\left[L_i + (c_i - \frac{1}{2})\sum_{j=1}^{n} a_{ij}^2(\sqrt{v_j(t_k)} + \frac{b_j}{\sqrt{v_j(t_k)}})^2 - \sum_{j=1}^{n}\beta_{ij}Y_j(t_k)\right. \\
&\left. + (\tilde{c}_i - \frac{1}{2})(\sqrt{\tilde{v}_i(t_k)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t_k)}})^2\right]\Delta t \\
&+ \sum_{j=1}^{n} a_{ij}(\sqrt{v_j(t_k)} + \frac{b_j}{\sqrt{v_j(t_k)}})(\rho_j \Delta B_j(t_k) + \sqrt{1-\rho_j^2}\Delta B_j(t_k)^{\perp}) \\
&+ (\sqrt{\tilde{v}_i(t_k)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t_k)}})(\tilde{\rho}_i \Delta \widetilde{B}_i(t_k) + \sqrt{1-\tilde{\rho}_i^2}\Delta \widetilde{B}_i(t_k)^{\perp}).
\end{aligned}
$$

Moreover, we can replace $\Delta B(t_k)$ as $\sqrt{\Delta t}Z_k$, where Z_k's are iid standard normal for $k = 0, ..., N-1$. Given iid standard normal vector $Z_k^{\perp}$, $\widetilde{Z}_k$, and $\widetilde{Z}_k^{\perp}$, similarly replacement for $\Delta B(t_k)^{\perp}$, $\Delta\widetilde{B}(t_k)$, $\Delta\widetilde{B}(t_k)^{\perp}$. Thus, for asset $i = 1, ..., n$, we will have:

$$
\begin{aligned}
v_j(t_k) &= v_j(t_{k-1}) + \alpha_j(\theta_j - v_j(t_{k-1}))\Delta t + \xi_j\sqrt{v_j(t_{k-1})}\sqrt{\Delta t}Z_{k-1}, j = 1, ..., n; \\
\tilde{v}_i(t_k) &= \tilde{v}_i(t_{k-1}) + \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t_{k-1}))\Delta t + \tilde{\xi}_i\sqrt{\tilde{v}_i(t_{k-1})}\sqrt{\Delta t}\widetilde{Z}_{k-1}; \\
Y_i(t_{k+1}) &= Y_i(t_k) + \left[L_i + (c_i - \frac{1}{2})\sum_{j=1}^{n} a_{ij}^2(\sqrt{v_j(t_k)} + \frac{b_j}{\sqrt{v_j(t_k)}})^2 - \sum_{j=1}^{n}\beta_{ij}Y_j(t_k)\right. \\
&\quad\left. + (\tilde{c}_i - \frac{1}{2})(\sqrt{\tilde{v}_i(t_k)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t_k)}})^2\right]\Delta t \\
&\quad + \sum_{j=1}^{n} a_{ij}(\sqrt{v_j(t_k)} + \frac{b_j}{\sqrt{v_j(t_k)}})(\rho_j\sqrt{\Delta t}Z_{k-1} + \sqrt{1-\rho_j^2}\sqrt{\Delta t}Z_{k-1}^{\perp}) \\
&\quad + (\sqrt{\tilde{v}_i(t_k)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t_k)}})(\tilde{\rho}_i\sqrt{\Delta t}\widetilde{Z}_{k-1} + \sqrt{1-\tilde{\rho}_i^2}\sqrt{\Delta t}\widetilde{Z}_{k-1}^{\perp}).
\end{aligned}
$$

2.3 Change of Measure

Our focus is on using asset prices under the 4/2 generalized factor model to price European call option and extract implied volatilities from it, so the risk-neutral process is required. Therefore, in this section, we study the possible changes of measure for our model by specifying the market price of risk. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{Q})$ denote the probability space under measure $\mathbb{Q}$, and $B_j^{\mathbb{Q}}$, $B_j^{\mathbb{Q}}(t)^{\perp}$, $\widetilde{B}_i^{\mathbb{Q}}$, $\widetilde{B}_i^{\mathbb{Q}}(t)^{\perp}$ are Brownian motions under $\mathbb{Q}$.

Excess return/risk premium is the return that in excess of the risk-free rate. In order to be comparable to Geometric Brownian motion, we should think of $L_i - \sum_{j=1}^{n}\beta_{ij}Y_j(t)$ as r if we are dealing with mean reverting model under risk-neutral world. Then, the excess return of common/intrinsic factor follow

$$\text{excess return of } F_j = c_j(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})^2$$
$$\text{excess return of } \tilde{F}_i = \tilde{c}_i(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2$$

where $c_j = \rho_j\lambda_j + \sqrt{1-\rho_j^2}\lambda_j^{\perp}$ and $\tilde{c}_i = \tilde{\rho}_i\tilde{\lambda}_i + \sqrt{1-\tilde{\rho}_i^2}\tilde{\lambda}_i^{\perp}$. In this representation, λ_j, $\lambda_j^{\perp}$, $\tilde{\lambda}_i$, and $\tilde{\lambda}_i^{\perp}$ are drivers of the change of measure for each individual independent risk factors/Brownian motions. Then, for each risk factor/Brownian that composes F_j or $\tilde{F}_i$, its market price of risk is the ratio of its excess return to its volatility, such as

$$MPR_{B_j^{\mathbb{P}}} = \frac{\rho_j\lambda_j(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})^2}{(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})}, \quad MPR_{B_j^{\mathbb{P}\perp}} = \frac{\sqrt{1-\rho_j^2}\lambda_j^{\perp}(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})^2}{(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})} \quad \text{for} \quad j = 1, ..., n$$
$$MPR_{\tilde{B}_i^{\mathbb{P}}} = \frac{\tilde{\rho}_i\tilde{\lambda}_i(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2}{(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})}, \quad MPR_{\tilde{B}_i^{\mathbb{P}\perp}} = \frac{\sqrt{1-\tilde{\rho}_i^2}\tilde{\lambda}_i^{\perp}(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2}{(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})} \quad \text{for} \quad i = 1, ..., n$$

That is, the Radon-Nikodym derivative for each risk factor/Brownian follows

$$\frac{d\mathbb{Q}_{B_j^{\mathbb{P}}}}{d\mathbb{P}_{B_j^{\mathbb{P}}}} = \exp\Bigg(-\frac{1}{2}\rho_j^2\lambda_j^2\int_0^t \Bigg((\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})^2 \Bigg) du + \rho_j\lambda_j \int_0^t \Bigg((\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}}) dB_j^{\mathbb{P}}(t) \Bigg) \Bigg)$$
$$\frac{d\mathbb{Q}_{\tilde{B}_i^{\mathbb{P}}}}{d\mathbb{P}_{\tilde{B}_i^{\mathbb{P}}}} = \exp\Bigg(-\frac{1}{2}\tilde{\rho}_i^2\tilde{\lambda}_i^2\int_0^t \Bigg((\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2 \Bigg) du + \tilde{\rho}_i\tilde{\lambda}_i \int_0^t \Bigg((\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}}) d\widetilde{B}_i^{\mathbb{P}}(t) \Bigg) \Bigg)$$

The RN derivative for Brownians $B_j^{\mathbb{P}}(t)^{\perp}$ and $\widetilde{B}_i^{\mathbb{P}}(t)^{\perp}$ for all $i,\ j = 1, ..., n$, are similar.

To be more specific, combining the quadratic variation relationship in equation 2.3, the above RN derivatives imply a change of measure where the excess return of the underlying asset is proportional to its variance, i.e.,

$$\begin{cases} dB_j^{\mathbb{Q}}(t) = \lambda_j \left(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}} \right) dt + dB_j^{\mathbb{P}}(t), dB_j^{\mathbb{Q}}(t)^{\perp} = \lambda_j^{\perp} \left(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}} \right) dt + dB_j^{\mathbb{P}}(t)^{\perp} \\ d\widetilde{B}_i^{\mathbb{Q}}(t) = \tilde{\lambda}_i \left(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}} \right) dt + d\widetilde{B}_i^{\mathbb{P}}(t), d\widetilde{B}_i^{\mathbb{Q}}(t)^{\perp} = \tilde{\lambda}_i^{\perp} \left(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}} \right) dt + d\widetilde{B}_i^{\mathbb{P}}(t)^{\perp} \end{cases}$$

Denote this form of market price of risk as MPR_1. As pointed out by Grasselli (2017), a risk-neutral measure may not exist in a 4/2 model, which is a feature inherited from having a 3/2 model (e.g. $\frac{1}{\sqrt{v(t)}}$) (see also Platen and Heath (2006) and Baldeaux et al. (2015)). This failure may cause the discounted asset price process to be a strict $\mathbb{Q}$-local martingale, and not a true $\mathbb{Q}$-martingale that equivalent to the historical measure $\mathbb{P}$. In the next propositions we entertain the above changes of measure for asset i with constants λ_j, $\lambda_j^{\perp}$, $\tilde{\lambda}_i$, and $\tilde{\lambda}_i^{\perp}$ then identifying the parametric conditions needed for the existence of a valid risk-neutral measure $\mathbb{Q}$ for this family of market price of risk.

Proposition 1. *The change of measure described above is well defined for pricing purposes under the following conditions:*

$$\begin{aligned} \xi_j^2 &\leq 2\alpha_j\theta_j - 2\xi_j \max\left\{|b_j\lambda_j|, |b_j\lambda_j^{\perp}|, |b_j a_{1j}\rho_j|, |b_j a_{nj}\rho_j|\right\} & (2.4) \\ \tilde{\xi}_i^2 &\leq 2\tilde{\alpha}_i\tilde{\theta}_i - 2\tilde{\xi}_i \max\left\{\left|\tilde{b}_i\tilde{\lambda}_i\right|, \left|\tilde{\lambda}_i^{\perp}\tilde{b}_i\right|, \left|\tilde{b}_i\tilde{\rho}_i\right|\right\} & (2.5) \end{aligned}$$

$$\begin{aligned} \max\left\{|\lambda_j|, |\lambda_j^{\perp}|\right\} &< \frac{\alpha_j}{\xi_j} & (2.6) \\ \max\left\{\left|\tilde{\lambda}_i\right|, \left|\tilde{\lambda}_i^{\perp}\right|\right\} &< \frac{\tilde{\alpha}_i}{\tilde{\xi}_i} & (2.7) \end{aligned}$$

Moreover, if $\beta_{ij} = 0$ for $i, j = 1, .., n$, then the following must also be satisfied:

$$L_i = r,\ c_i = \sum_{j=1}^{n} \left(\rho_j\lambda_j + \sqrt{1-\rho_j^2}\lambda_j^{\perp}\right),\ \tilde{c}_i = \tilde{\rho}_i\tilde{\lambda}_i + \sqrt{1-\tilde{\rho}_i^2}\tilde{\lambda}_i^{\perp} \tag{2.8}$$

See proof in A1.

The mean-reverting 4/2 generalized factor model provides the flexibility in capturing the mean-reverting property of some financial assets, such as commodities and volatility indices. At the same time, by setting $\beta = 0$, and long term average of the underlying asset $L = r$, we can use the 4/2 factor model in a broad class of financial instruments as well. In this work, we will study both non-mean reverting and mean reverting 4/2 factor model to see how critical parameters a, b affect the implied volatility under one common factor model and two common factors model respectively.

Based on Girsanov's theorem, the dynamics for asset $X_i(t)$ under $\mathbb{Q}$-measure is defined as

$$\begin{aligned} \frac{dX_i(t)}{X_i(t)} &= \left\{L_i - \sum_{j=1}^{n} \beta_{ij} ln(X_j(t))\right\} dt \\ &\quad + \sum_{j=1}^{n} a_{ij}(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})dW_j^{\mathbb{Q}}(t) + (\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})d\widetilde{W}_i^{\mathbb{Q}}(t), \\ dv_j(t) &= (\alpha_j\theta_j - a_{ij}b_j\lambda_j\xi_j - (\alpha_j + a_{ij}\lambda_j\xi_j)v_j(t))\, dt + \xi_j\sqrt{v_j(t)}dB_j^{\mathbb{Q}}(t), j = 1, ..., n \\ d\tilde{v}_i(t) &= \left(\tilde{\alpha}_i\tilde{\theta}_i - \tilde{\lambda}_i\tilde{\xi}_i\tilde{b}_i - (\tilde{\alpha}_i + \tilde{\lambda}_i\tilde{\xi}_i)\tilde{v}_i(t)\right) dt + \tilde{\xi}_i\sqrt{\tilde{v}_i(t)}d\widetilde{B}_i^{\mathbb{Q}}(t), i = 1, ..., n \end{aligned}$$

Note that if it is non-mean reverting, the drift term is set to be the risk-free rate of return r.

However, the 4/2-liked structure of the instantaneous volatility is not trivial, and so is the structure of the excess return and the market price of risk. In addition, the 4/2-liked structure of market price of risk may lead to mathematical difficulties in other problems, i.e., finding an analytical solution in portfolio optimization. Therefore, we advocate another form of market price of risk that is proportional to the square root of $v(t)$ (namely MPR$_2$) instead of the instantaneous volatility of underlying asset, i.e., $\left(\sqrt{v(t)}+\frac{b}{\sqrt{v(t)}}\right)$, this is,

$$\begin{cases} dB_j^{\mathbb{Q}}(t)=\lambda_j\sqrt{v_j(t)}dt+dB_j^{\mathbb{P}}(t), dB_j^{\mathbb{Q}}(t)^{\perp}=\lambda_j^{\perp}\sqrt{v_j(t)}dt+dB_j^{\mathbb{P}}(t)^{\perp} \\ d\widetilde{B}_i^{\mathbb{Q}}(t)=\tilde{\lambda}_i\sqrt{\tilde{v}_i(t)}dt+d\widetilde{B}_i^{\mathbb{P}}(t), d\widetilde{B}_i^{\mathbb{Q}}(t)^{\perp}=\tilde{\lambda}_i^{\perp}\sqrt{\tilde{v}_i(t)}dt+d\widetilde{B}_i^{\mathbb{P}}(t)^{\perp} \end{cases}$$

Proposition 2. *The change of measure under the second form of market price of risk described above is well-defined for pricing purposes under the following conditions:*

$$\xi_j^2 \leq 2\alpha_j\theta_j-2\left|a_{ij}\rho_j b_j\right|\xi_j,\ i,j=1,...,n \tag{2.9}$$

$$\widetilde{\xi}_i^2 \leq 2\widetilde{\alpha}_i\widetilde{\theta}_i-2\left|\widetilde{\rho}_i\widetilde{b}_i\right|\widetilde{\xi}_i,\ i=1,...,n \tag{2.10}$$

$$\max\left\{\left|\lambda_j\right|,\left|\lambda_j^{\perp}\right|\right\} < \frac{\alpha_j}{\xi_j} \tag{2.11}$$

$$\max\left\{\left|\tilde{\lambda}_i\right|,\left|\tilde{\lambda}_i^{\perp}\right|\right\} < \frac{\tilde{\alpha}_i}{\widetilde{\xi}_i} \tag{2.12}$$

Moreover, if $\beta_{ij}=0$ for $i,j=1,..,n$, then the following must also be satisfied:

$$L_i=r,\ c_i=\sum_{j=1}^{n}\left(\rho_j\lambda_j+\sqrt{1-\rho_j^2}\lambda_j^{\perp}\right),\ \tilde{c}_i=\tilde{\rho}_i\tilde{\lambda}_i+\sqrt{1-\tilde{\rho}_i^2}\tilde{\lambda}_i^{\perp} \tag{2.13}$$

See proof in A1.

In line with this form of market price of risk, the dynamic for asset $X_i(t)$ under $\mathbb{P}$-measure evolves as

$$\begin{aligned} \frac{dX_i(t)}{X_i(t)} &= \left\{L_i+c_i\sum_{j=1}^{n}a_{ij}^2(v_j(t)+b_j)-\sum_{j=1}^{n}\beta_{ij}ln(X_j(t))+\tilde{c}_i(\tilde{v}_i(t)+\tilde{b}_i)\right\}dt \\ &\quad+\sum_{j=1}^{n}a_{ij}(\sqrt{v_j(t)}+\frac{b_j}{\sqrt{v_j(t)}})dW_j^{\mathbb{P}}(t)+(\sqrt{\tilde{v}_i(t)}+\frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})d\widetilde{W}_i^{\mathbb{P}}(t), \\ dv_j(t) &= \alpha_j(\theta_j-v_j(t))dt+\xi_j\sqrt{v_j(t)}dB_j^{\mathbb{P}}(t), j=1,...,n \\ d\tilde{v}_i(t) &= \tilde{\alpha}_i(\tilde{\theta}_i-\tilde{v}_i(t))dt+\tilde{\xi}_i\sqrt{\tilde{v}_i(t)}\widetilde{B}_i^{\mathbb{P}}(t), i=1,...,n \end{aligned} \tag{2.14}$$

with $\langle dB_j(t), dW_j(t)\rangle = \rho_j dt$, $\left\langle d\widetilde{B}_i^{\mathbb{P}}(t), d\widetilde{W}_i^{\mathbb{P}}(t)\right\rangle = \widetilde{\rho}_i dt$.

Under measure $\mathbb{Q}$, while the underlying asset follows the same type of process with both forms of market price of risk, the underlying CIR for of the common factor and the intrinsic factor exhibit different mean reverting levels,

$$\text{MPR}_1 \begin{cases} dv_j(t) &= \left(\alpha_j\theta_j - b_j\lambda_j\xi_j - (\alpha_j + \lambda_j\xi_j)v_j(t)\right)dt + \xi_j\sqrt{v_j(t)}dB_j^{\mathbb{Q}}(t), j = 1, ..., n \\ d\tilde{v}_i(t) &= \left(\tilde{\alpha}_i\tilde{\theta}_i - \tilde{\lambda}_i\tilde{\xi}_i\tilde{b}_i - (\tilde{\alpha}_i + \tilde{\lambda}_i\tilde{\xi}_i)\tilde{v}_i(t)\right)dt + \tilde{\xi}_i\sqrt{\tilde{v}_i(t)}d\widetilde{B}_i^{\mathbb{Q}}(t), i = 1, ..., n \end{cases} \tag{2.15}$$

$$\text{MPR}_2 \begin{cases} dv_j(t) &= \left(\alpha_j\theta_j - (\alpha_j + \lambda_j\xi_j)v_j(t)\right)dt + \xi_j\sqrt{v_j(t)}dB_j^{\mathbb{Q}}(t), j = 1, ..., n \\ d\tilde{v}_i(t) &= \left(\tilde{\alpha}_i\tilde{\theta}_i - (\tilde{\alpha}_i + \tilde{\lambda}_i\tilde{\xi}_i)\tilde{v}_i(t)\right)dt + \tilde{\xi}_i\sqrt{\tilde{v}_i(t)}d\widetilde{B}_i^{\mathbb{Q}}(t), i = 1, ..., n \end{cases} \tag{2.16}$$

Here are some documented properties about the parameters: negatively priced in volatility risk has been well-documented in Bakshi and Kapadia (2003), Chernov and Ghysels (2000), and Escobar et al. (2015), this means, λ_j and $\tilde{\lambda}_i < 0$. The underlying asset's risk premium driver c_i and $\tilde{c}_i$ should be positive, which is evidenced by Aït-Sahalia et al. (2007) and Escobar et al. (2015), whereas the correlation should be negative (ρ_j and $\tilde{\rho}_i < 0$) reasoned by the negative correlation between asset return and volatility (the "leverage effect") in Aït-Sahalia et al. (2007). Also, parameter a and 3/2 component b are positive.

2.4 Implied Volatility

Given the dynamic of the underlying asset, we can price a European call option on asset X_i for a specified strike price K and an expiry date T, such that

$$C(T, K) = e^{-rT}\mathbb{E}^{Q}[(X_i(T) - K)^+], \tag{2.17}$$

where $X_i(T)$ is approximated using Euler, and r is the risk-free interest rate.

Moreover, the implied volatility can be extracted by matching Black-Scholes option price formula with obtained simulated call prices and solve for volatility parameter, which is treated as a constant. In particular, if there is no mean reverting feature and $\beta = 0$, we take all sources of randomness as a constant, and denote it as σ with a Brownian motion $W^*(t)$, the dynamic of the risky asset becomes

$$dY_i(t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW^*(t) \tag{2.18}$$

It follows that

$$C(X_i(0), K) = X_i(0)N(d_1) - Ke^{-rT}N(d_2), \tag{2.19}$$

where,

$$d_1 = \frac{ln\frac{X_i(0)}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

That is, the simulated European call option prices are used as the left-hand side of BS formula (2.19) for some strike price K and the initial price of underlying asset X_i. On the other side, knowing the time to maturity and the risk-free interest rate, the implied volatility σ is the only unknown parameter to be solved for.

Note that when $\beta \neq 0$, the process has the mean-reverting property, and we need to treat the dynamics of $Y_i(t)$ as an O-U process instead,

$$dY_i(t) = (L_i - \frac{1}{2}\sigma^2 - \beta Y_i(t))dt + \sigma dW^*(t). \tag{2.20}$$

Further, we are dealing with matching call prices for $X_i(t)$ with a formula based on exponential O-U process. Fortunately, Detemple and Osakwe (2000) derived call option price formula for $X_i(t)$ with strike price K and maturity T such that

$$C(X_i(0), K) = e^{-rT}\left[X_i(0)^{\phi_T} exp\left\{\frac{\theta(\sigma)}{\beta}(1-\phi_T) + \frac{1}{2}a_T^2\right\}N(d + a_T) - KN(d)\right], \tag{2.21}$$

where,

$$\phi_T = e^{-\beta T}, \theta(\sigma) = L_i - \frac{1}{2}\sigma^2, a_T = \frac{\sigma}{\sqrt{2\beta}}(1-\phi_T^2)^{\frac{1}{2}},$$
$$d = \frac{1}{a_T}(\phi_T ln(X_i(0)) - ln(K) + \frac{\theta(\sigma)}{\beta}(1-\phi_T))$$

That is, the implied volatility σ for a mean reverting process can be obtained similarly by matching the simulated call option prices to the O-U formula (2.21) provided above.

3 Portfolio Optimization

In this section, we study the 4/2 model with respect to dynamic portfolio optimization within expected utility theory. We assume that interest rate is constant and the financial market is composed of one risk-free asset and one risky asset. The risky asset follows a 4/2 structured volatility and it can be traded continuously. The objective of the investor is to maximize the expected utility of terminal wealth in the finite horizon. Assume that risk preference of the investor is described by constant relative risk averse (CRRA) utility. The method of dynamic programming was used to obtain the Hamilton-Jacobi-Bellman (HJB) equation. By directly conjecturing the form of the solution to the HJB equation in the CRRA utility framework, a quasi-closed form solution is found under the 4/2 stochastic model. Besides, an analytical solution is available when we conjecture the solution to the HJB equation following an exponential affine form under a special setting. This chapter is a particular case of our generalized model by setting $\beta = 0$ and no intrinsic factor.

3.1 Problem setting

Let all the stochastic processes introduced below defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a right-continuous information filtration generated by the involving standard Brownian motions. The price process of the risk-free asset (Money Market) M_t evolves according to

$$dM_t = M_t r dt \ , M_0 = 1 \tag{3.1}$$

where the interest rate r is assumed to be constant.

The price process X_t of the risky asset follows

$$dX_t = X_t \left[\mu_t dt + (a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}) dW_t \right], \ X(0) = X_0 > 0 \tag{3.2}$$

$$dv_t = \kappa \left(\theta - v_t\right) dt + \sigma\sqrt{v_t} dZ_{1t}, \ v(0) = v_0 > 0 \tag{3.3}$$

where v_t is the variance driver, which follows a CIR with mean-reversion rate $\kappa > 0$, long-run mean $\theta > 0$ and volatility of volatility $\sigma > 0$. The Feller condition, i.e., $2\kappa\theta \geq \sigma^2$, is also imposed to keep the process v_t strictly positive. These two standard Brownian motions W_t and Z_{1t} are correlated with parameter $\rho \in [-1, 1]$, hence for convenience we will write $dW_t = \rho dZ_{1t} + \sqrt{1-\rho^2} dZ_{2t}$, where Z_{2t} is another standard BM and independent of Z_{1t}.

We assume the natural form of market price of risk, such that

$$\begin{cases} \lambda_1(v_t) = \bar{\lambda}_1\sqrt{v_t} \\ \lambda_2(v_t) = \bar{\lambda}_2(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}) \end{cases} \tag{3.4}$$

where $\lambda_1(v_t)$ is the market price of risk with respect to Z_{1t}, and $\bar{\lambda}_1$ is a constant. Similarly, $\lambda_2(v_t)$ is the market price respect to Z_{2t}, and $\bar{\lambda}_2$ is a constant. This choice of market price of risk is preferable because it makes the ratio of excess return relative to each risk factor proportional to its volatility. In other words, the excess return of the stock-driving risk factor is proportional to the variance of the underlying process (Heston, 1993) while the excess return of v_t is proportional to itself. Then, the stock dynamics can be rewritten as

$$dX_t = X_t\left[\left(r + [\rho\lambda_1(v_t) + \sqrt{1-\rho^2}\lambda_2(v_t)][a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}]\right)dt + (a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})(\rho dZ_{1t} + \sqrt{1-\rho^2}dZ_{2t})\right] \tag{3.5}$$

Substituting in the market price of risk, the dynamic of risky asset evolves as

$$\begin{aligned} dX_t &= X_t\left[\left(r + \bar{\lambda}_1\rho(av_t + b) + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\right)\right]dt \\ &+ X_t\left[(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})(\rho dZ_{1t} + \sqrt{1-\rho^2}dZ_{2t})\right] \end{aligned} \tag{3.6}$$

By investing a proportion π_t of wealth into risky asset and the remaining proportion $(1-\pi_t)$ of wealth into risk free asset, the wealth process for this investor in the historical measure evolves according to:

$$\begin{aligned} \frac{dP_t}{P_t} &= \pi_t\frac{dX_t}{X_t} + (1-\pi_t)\frac{dM_t}{M_t} \\ &= \left[r + \pi_t\left(\bar{\lambda}_1\rho(av_t + b) + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\right)\right]dt \\ &+ \pi_t\left[(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})(\rho dZ_{1t} + \sqrt{1-\rho^2}dZ_{2t})\right], P(0) = x_0 > 0. \end{aligned} \tag{3.7}$$

where x_0 is the initial wealth. Note that this is a self-financing portfolio. An investor who aims at maximizing utility from terminal wealth at time T with CRRA risk preference follows a power utility function, such as

$$u(x) = \frac{x^\gamma}{\gamma}, \tag{3.8}$$

where $\gamma < 1$ and $x \geq 0$. In addition, the power utility function has a well-defined limit when

$\gamma \to 0$. From L'Hôspital's rule, we have

$$\lim_{\gamma \to 0} \frac{x^\gamma}{\gamma} = \lim_{\gamma \to 0} \frac{x^\gamma ln x}{1} = ln x, \tag{3.9}$$

which is an important special case of logarithmic utility. That is, we can obtain the optimal strategies of a log-utility investor as the limit of the optimal strategies of the general CRRA investor as $\gamma \to 0$.

The goal/objective of the investor is to find an investment strategy that maximizes the terminal utility at time T. Mathematically, the objective function can be expressed as

$$J(x, v, t) = \sup_{\pi \in \mathcal{U}} \mathbb{E}_{x,v,t} \left[u(P_T) \right]$$

where $J(x, v, t)$ is the value function and $\mathcal{U}$ denotes the space of admissible strategies.

Definition 1. *An investment strategy π_t is said to be admissible if the following conditions are satisfied:*

1) π is progressively measurable.

2) For all $(x_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $t \in [0,\ T]$, the SDE 3.7 has a pathwise unique solution $\{P_t^\pi\}_{t \in [0,T]}$ under measure $\mathbb{Q}$ and

$$\mathbb{E}^{\mathbb{Q}}_{x_0, v_0, t_0} \left[u(P_t) \right] < \infty$$

where $\mathbb{E}^{\mathbb{Q}}_{x,v,t}[.] = \mathbb{E}^{\mathbb{Q}}[. \mid P_t = x, v_t = v]$ denote the conditional expectation.

3.2 Mathematical solution

According to the principles of dynamics programming, the Hamilton-Jacobi-Bellman (HJB) equation for such problem should satisfy:

$$\begin{aligned} 0 = sup_\pi \Bigg\{ & J_t + x \left(r + \pi \bar{\lambda}_1 \rho (a v_t + b) + \pi \bar{\lambda}_2 \sqrt{1 - \rho^2} (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 \right) J_x + \kappa (\theta - v) J_v \\ & + \frac{1}{2} x^2 \pi^2 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 J_{xx} + \frac{1}{2} \sigma^2 v J_{vv} + \pi x (a v_t + b) \sigma \rho J_{xv} \Bigg\} \end{aligned} \tag{3.10}$$

where J_x, J_v, J_{xx}, J_{vv}, and J_{xv} are first and second partial derivatives of function J. The function J should satisfy the boundary condition $J(x, v, T) = \frac{x^\gamma}{\gamma}$.

Proposition 3. *The solution to the HJB problem has the structure:*

$$J(x,v,t) = \frac{x^\gamma}{\gamma} h(t,v), \tag{3.11}$$

where h(t,v) satisfy the following equation, with terminal condition h(T,v)=1:

$$\begin{aligned} 0 &= h_t + \kappa(\theta - v)h_v + \frac{1}{2}\sigma^2 v h_{vv} + rh\gamma \\ &- \sigma\rho^2\bar{\lambda}_1 v\frac{\gamma h_v}{\gamma - 1} - \sigma\bar{\lambda}_2\rho\sqrt{1-\rho^2}(av+b)\frac{\gamma h_v}{\gamma-1} - \frac{1}{2}v\sigma^2\rho^2\frac{\gamma h_v^2}{(\gamma-1)h} \\ &- \frac{1}{2}\left[\bar{\lambda}_1\rho\sqrt{v} + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v} + \frac{b}{\sqrt{v}})\right]^2 \frac{\gamma h}{\gamma - 1} \end{aligned} \tag{3.12}$$

In this setting the optimal allocation in risky asset satisfies the equation:

$$\pi^* = \frac{\sqrt{v}\sigma\rho h_v}{(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})(1-\gamma)h} + \frac{\bar{\lambda}_1\rho\sqrt{v}}{(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})(1-\gamma)} + \frac{\bar{\lambda}_2\sqrt{1-\rho^2}}{(1-\gamma)} \tag{3.13}$$

Proof. Separating out the terms that involves π in equation (3.10) and denoting it as a function $g(\pi)$:

$$\begin{aligned} 0 = J_t + \kappa(\theta - v)J_v + \frac{1}{2}\sigma^2 v J_{vv} + sup_\pi \Bigg\{ x\left(r + \pi\bar{\lambda}_1\rho(av_t + b) + \pi\bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\right) J_x \\ + \frac{1}{2}x^2\pi^2(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 J_{xx} + \pi x(av_t + b)\sigma\rho J_{xv} \Bigg\} \end{aligned} \tag{3.14}$$

That is,

$$0 = J_t + \kappa(\theta - v)J_v + \frac{1}{2}\sigma^2 v J_{vv} + sup_\pi\Big\{g(\pi)\Big\} \tag{3.15}$$

The first order condition for finding the optimal investment strategy π^* is $g'(\pi) = 0$, where

$$\begin{aligned} g'(\pi) &= x\left(\bar{\lambda}_1\rho(av_t + b) + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\right) J_x \\ &+ x^2\pi(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 J_{xx} + x(av_t + b)\sigma\rho J_{xv} \end{aligned} \tag{3.16}$$

Solving for the candidate π^*:

$$\begin{aligned}\pi^* &= \frac{-x(av_t+b)\sigma\rho J_{xv} - x\left(\bar{\lambda}_1\rho(av_t+b)+\bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})^2\right)J_x}{x^2(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})^2 J_{xx}} \\ &= \frac{-\sqrt{v}\sigma\rho J_{xv} - \bar{\lambda}_1\rho\sqrt{v}J_x - \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})J_x}{x(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})J_{xx}}\end{aligned} \tag{3.17}$$

Substituting equation (3.17) back to the HJB Equation (3.10) and eliminating the "sup"

$$\begin{aligned}0 = J_t &+ \kappa(\theta-v)J_v + \frac{1}{2}\sigma^2 vJ_{vv} + x\left(r+\pi^*\bar{\lambda}_1\rho(av_t+b)+\pi^*\bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})^2\right)J_x \\ &+ \frac{1}{2}x^2(\pi^*)^2(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})^2 J_{xx} + \pi x(av_t+b)\sigma\rho J_{xv}\end{aligned} \tag{3.18}$$

That is,

$$\begin{aligned}0 &= J_t + \kappa(\theta-v)J_v + \frac{1}{2}\sigma^2 vJ_{vv} + rxJ_x \\ &+ x\frac{-\sqrt{v}\sigma\rho J_{xv} - \bar{\lambda}_1\rho\sqrt{v}J_x - \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})J_x}{x(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})J_{xx}}\left(\bar{\lambda}_1\rho(av_t+b)+\bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})^2\right)J_x \\ &+ \frac{1}{2}x^2\left[\frac{-\sqrt{v}\sigma\rho J_{xv} - \bar{\lambda}_1\rho\sqrt{v}J_x - \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})J_x}{x(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})J_{xx}}\right]^2(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})^2 J_{xx} \\ &+ \frac{-\sqrt{v}\sigma\rho J_{xv} - \bar{\lambda}_1\rho\sqrt{v}J_x - \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})J_x}{x(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})J_{xx}}x(av_t+b)\sigma\rho J_{xv}\end{aligned} \tag{3.19}$$

Further simplifications leads to:

$$\begin{aligned}0 = J_t &+ \kappa(\theta-v)J_v + \frac{1}{2}\sigma^2 vJ_{vv} + rxJ_x \\ &+ \frac{-\sqrt{v}\sigma\rho J_{xv} - \bar{\lambda}_1\rho\sqrt{v}J_x - \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v}+\frac{b}{\sqrt{v}})J_x}{J_{xx}}\left(\bar{\lambda}_1\rho\sqrt{v}+\bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v}+\frac{b}{\sqrt{v}})\right)J_x \\ &+ \frac{1}{2}\frac{\left[-\sqrt{v}\sigma\rho J_{xv} - (\bar{\lambda}_1\rho\sqrt{v}+\bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v}+\frac{b}{\sqrt{v}}))J_x\right]^2}{J_{xx}} \\ &+ \frac{-\sqrt{v}\sigma\rho J_{xv} - \bar{\lambda}_1\rho\sqrt{v}J_x - \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v}+\frac{b}{\sqrt{v}})J_x}{J_{xx}}\sqrt{v}\sigma\rho J_{xv}\end{aligned} \tag{3.20}$$

Canceling out and grouping terms,

$$\begin{aligned}0 &= J_t + \kappa(\theta - v)J_v + \frac{1}{2}\sigma^2 v J_{vv} + rxJ_x - \frac{\sqrt{v}\sigma\rho\left(\bar{\lambda}_1\rho\sqrt{v} + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v} + \frac{b}{\sqrt{v}})\right)}{J_{xx}}J_{xv}J_x \\ &- \frac{1}{2}\frac{v\sigma^2\rho^2}{J_{xx}}J_{xv}^2 - \frac{1}{2}\frac{\left[\bar{\lambda}_1\rho\sqrt{v} + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v} + \frac{b}{\sqrt{v}})\right]^2}{J_{xx}}J_x^2\end{aligned} \tag{3.21}$$

Assuming

$$J(x, v, t) = \frac{x^\gamma}{\gamma}h(t, v), \tag{3.22}$$

where $h(T, v) = 1$, $\forall v$. Thereby, it follows that

$$\begin{aligned}J_t = \frac{x^\gamma}{\gamma}h_t,\ J_v = \frac{x^\gamma}{\gamma}h_v,\ J_x = x^{\gamma-1}h \\ J_{vv} = \frac{x^\gamma}{\gamma}h_{vv},\ J_{xv} = x^{\gamma-1}h_v,\ J_{xx} = (\gamma-1)x^{\gamma-2}h\end{aligned}$$

Substituting the corresponding partial derivatives back into equation (3.21):

$$\begin{aligned}0 = &\frac{x^\gamma}{\gamma}h_t + \kappa(\theta - v)\frac{x^\gamma}{\gamma}h_v + \frac{1}{2}\sigma^2 v\frac{x^\gamma}{\gamma}h_{vv} + rx^\gamma h - \sigma\rho^2\bar{\lambda}_1 v\frac{x^\gamma h_v}{\gamma - 1} \\ &- \sigma\bar{\lambda}_2\rho\sqrt{1-\rho^2}(av + b)\frac{x^\gamma h_v}{\gamma - 1} - \frac{1}{2}v\sigma^2\rho^2\frac{x^\gamma h_v^2}{(\gamma-1)h} - \frac{1}{2}\left[\bar{\lambda}_1\rho\sqrt{v} + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v} + \frac{b}{\sqrt{v}})\right]^2\frac{x^\gamma h}{\gamma - 1}\end{aligned} \tag{3.23}$$

Multiplying term $\frac{\gamma}{x^\gamma}$ on both sides:

$$\begin{aligned}0 = &h_t + \kappa(\theta - v)h_v + \frac{1}{2}\sigma^2 v h_{vv} + rh\gamma \\ &- \sigma\rho^2\bar{\lambda}_1 v\frac{\gamma h_v}{\gamma - 1} - \sigma\bar{\lambda}_2\rho\sqrt{1-\rho^2}(av + b)\frac{\gamma h_v}{\gamma - 1} - \frac{1}{2}v\sigma^2\rho^2\frac{\gamma h_v^2}{(\gamma-1)h} \\ &- \frac{1}{2}\left[\bar{\lambda}_1\rho\sqrt{v} + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v} + \frac{b}{\sqrt{v}})\right]^2\frac{\gamma h}{\gamma - 1}\end{aligned} \tag{3.24}$$

□

We show next that this problem is not solvable under a conjecture of an exponential affine form of function h.

Corollary 0.1. *The solution in equation (3.10) does not follow an exponential affine structure of* *the form:*

$$J(x,v,t) = \frac{x^\gamma}{\gamma} exp\left\{ A(T-t) + B(T-t)v \right\} \tag{3.25}$$

where the functions $A(\tau)$ and $B(\tau)$ are only time dependent with time horizon $\tau(t) = T - t$. However, if $\rho = 1$, the problem is solvable with expressions for A, B as follows:

$$A(\tau(t)) = \gamma r \tau + \frac{2\theta\kappa}{k_2} \ln \left(\frac{2k_3 e^{\frac{k_1+k_3}{2}\tau}}{2k_3 + (k_1+k_3)\left(e^{k_3\tau} - 1\right)} \right), \tag{3.26}$$

$$B(\tau(t)) = \frac{k_0\left(e^{k_3\tau} - 1\right)}{2k_3 + (k_1+k_3)\left(e^{k_3\tau} - 1\right)}, \tag{3.27}$$

This leads to the explicit form of the optimal strategy,

$$\pi^* = \frac{-\sqrt{v}\sigma B}{(a\sqrt{v} + \frac{b}{\sqrt{v}})(\gamma - 1)} - \frac{\bar{\lambda}_1\sqrt{v}}{(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})(\gamma - 1)}, \tag{3.28}$$

where the auxiliary parameters k_0, k_1, k_2, k_3 satisfy $k_1^2 - k_0 k_2 > 0$, and are defined as follow:

$$k_0 := \frac{\gamma\bar{\lambda}_1^2}{1-\gamma} \tag{3.29}$$

$$k_1 := \left(\kappa - \bar{\lambda}_1 \frac{\gamma\sigma\rho}{1-\gamma}\right) \tag{3.30}$$

$$k_2 := \left(\sigma^2 + \frac{\gamma\sigma^2}{1-\gamma}\right) \tag{3.31}$$

$$k_3 := \sqrt{k_1^2 - k_0 k_2} \tag{3.32}$$

Proof. Assume that $h(t,v)$ is of exponentially affine form as well, such that

$$h(t,v) = \exp(A(\tau(t)) + B(\tau(t))v)\ , \tag{3.33}$$

with time horizon $\tau(t) = T - t$ and therefore boundary conditions

$$\begin{aligned} h(T,v) = 1 \forall v \Rightarrow A(0) = A(\tau(T)) = 0, \\ B(0) = B(\tau(T)) = 0. \end{aligned} \tag{3.34}$$

This leads to

$$h_t = (-A' - B'v)h,\ h_v = Bh,\ h_{vv} = B^2 h,\ \frac{h_v^2}{h} = \frac{B^2h^2}{h} = B^2 h = h_{vv}$$

Substituting into π^* (3.17), it can be expressed as

$$\pi^* = \frac{-\sqrt{v}\sigma\rho B}{(a\sqrt{v}+\frac{b}{\sqrt{v}})(\gamma-1)} - \frac{\bar{\lambda}_1\rho\sqrt{v}}{(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})(\gamma-1)} - \frac{\bar{\lambda}_2\sqrt{1-\rho^2}}{(\gamma-1)} \tag{3.35}$$

Substituting into equation (3.12), then

$$\begin{aligned}
0 &= (-A' - B'v)h + \kappa(\theta - v)Bh + \frac{1}{2}\sigma^2 vB^2h + rh\gamma \\
&- \sigma\rho^2\bar{\lambda}_1 v\frac{\gamma Bh}{\gamma-1} - \sigma\bar{\lambda}_2\rho\sqrt{1-\rho^2}(av+b)\frac{\gamma Bh}{\gamma-1} - \frac{1}{2}v\sigma^2\rho^2\frac{\gamma}{\gamma-1}B^2h \\
&- \frac{1}{2}\left[\bar{\lambda}_1\rho\sqrt{v} + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v}+\frac{b}{\sqrt{v}})\right]^2\frac{\gamma h}{\gamma-1}
\end{aligned}$$

Cancelling h:

$$\begin{aligned}
0 &= (-A' - B'v) + \kappa(\theta - v)B + \frac{1}{2}\sigma^2 vB^2 + r\gamma \\
&- \sigma\rho^2\bar{\lambda}_1 v\frac{\gamma B}{\gamma-1} - \sigma\bar{\lambda}_2\rho\sqrt{1-\rho^2}(av+b)\frac{\gamma B}{\gamma-1} - \frac{1}{2}v\sigma^2\rho^2\frac{\gamma}{\gamma-1}B^2 \\
&- \frac{1}{2}\left[\bar{\lambda}_1\rho\sqrt{v} + \bar{\lambda}_2\sqrt{1-\rho^2}(a\sqrt{v}+\frac{b}{\sqrt{v}})\right]^2\frac{\gamma}{\gamma-1}
\end{aligned} \tag{3.36}$$

Regrouping and separating out v:

$$\begin{aligned}
0 &= -A' + \kappa\theta B + r\gamma - \sigma\rho^2\bar{\lambda}_1 b\frac{\gamma B}{\gamma-1} - [ab\bar{\lambda}_2^2(1-\rho^2) + \bar{\lambda}_1\bar{\lambda}_2\rho\sqrt{1-\rho^2}b]\frac{\gamma}{\gamma-1} \\
&+ v\bigg[-B' - \kappa B + \frac{1}{2}\sigma^2B^2 - \sigma\rho^2\bar{\lambda}_1\frac{\gamma B}{\gamma-1} - \sigma\bar{\lambda}_2 a\rho\sqrt{1-\rho^2}\frac{\gamma B}{\gamma-1} - \frac{1}{2}\sigma^2\rho^2\frac{\gamma}{\gamma-1}B^2 \\
&- \frac{1}{2}(\bar{\lambda}_1^2\rho^2 + 2\bar{\lambda}_1\bar{\lambda}_2\rho\sqrt{1-\rho^2}a + a^2\bar{\lambda}_2^2(1-\rho^2))\frac{\gamma}{\gamma-1}\bigg] - \frac{1}{v}\left[\frac{1}{2}\bar{\lambda}_2^2(1-\rho^2)b^2\frac{\gamma}{\gamma-1}\right]
\end{aligned} \tag{3.37}$$

The term $1/v$ can not be eliminated and this is why the solution can not be as prescribed. However, if $\rho = 1$:

$$\begin{aligned}
0 &= -A' + \kappa\theta B + r\gamma - \sigma\bar{\lambda}_1 b\frac{\gamma B}{\gamma-1} \\
&+ v\left[-B' + \frac{1}{2}\left(\sigma^2 - \frac{\gamma\sigma^2}{\gamma-1}\right)B^2 - (\kappa + \sigma\bar{\lambda}_1\frac{\gamma}{\gamma-1})B - \frac{1}{2}\bar{\lambda}_1^2\frac{\gamma}{\gamma-1}\right]
\end{aligned} \tag{3.38}$$

We end up with a term that is linear in v, but both coefficients are linear differential equations.

Both of them have to be zero to satisfy the equation and boundary condition of h,

$$A' = \kappa\theta B + \gamma r \tag{3.39}$$

$$B' = \frac{1}{2}\underbrace{\left(\sigma^2 + \frac{\gamma\sigma^2}{1-\gamma}\right)}_{k_2} B^2 - \underbrace{\left(\kappa - \sigma\bar{\lambda}_1\frac{\gamma}{1-\gamma}\right)}_{k_1} B + \frac{1}{2}\underbrace{\frac{\gamma\bar{\lambda}_1^2}{1-\gamma}}_{k_0} \tag{3.40}$$

The equation (3.40) is a so called Riccati equation with auxiliary parameters $k_i, i \in \{0,1,2\}$, which can be solved. Let $A(\tau), B(\tau)$ be two time dependent functions satisfying the equations

$$A'(\tau) = \kappa\theta B(\tau) + \gamma r \tag{3.41}$$

$$B'(\tau) = \frac{1}{2}k_2 B(\tau)^2 - k_1 B(\tau) + \frac{1}{2}k_0 \tag{3.42}$$

and the boundary conditions $A(0) = 0, B(0) = 0$ with constants k_0, k_1, k_2 satisfying $k_1^2 - k_0k_2 > 0$. Define $k_3 := \sqrt{k_1^2 - k_0k_2}$. The right hand side of equation (3.42) has roots

$$B_{1,2} = \frac{k_1 \pm \sqrt{k_1^2 - k_0k_2}}{k_2} = \frac{k_1 \pm k_3}{k_2}$$

where $k_3 := \sqrt{k_1^2 - k_0k_2}$, which is well-defined due to the assumption made. Dissecting factors leads to

$$B'(\tau) = \frac{dB(\tau)}{d\tau} = \frac{k_2}{2}\left(B(\tau) - \frac{k_1+k_3}{k_2}\right)\left(B(\tau) - \frac{k_1-k_3}{k_2}\right). \tag{3.43}$$

Integrating yields

$$\begin{aligned}
\int_0^{B(\tau)} \frac{1}{\left(\beta(\tau) - \frac{k_1+k_3}{k_2}\right)\left(\beta(\tau) - \frac{k_1-k_3}{k_2}\right)} d\beta &= \int_0^{\tau} \frac{k_2}{2} dt \\
\frac{k_2}{2k_3}\left[\ln\left(\beta(\tau) - \frac{k_1+k_3}{k_2}\right) - \ln\left(\beta(\tau) - \frac{k_1-k_3}{k_2}\right)\right]_0^{B(\tau)} &= \frac{k_2}{2}\tau \\
\ln\frac{B(\tau) - \frac{k_1+k_3}{k_2}}{B(\tau) - \frac{k_1-k_3}{k_2}} - \ln\frac{k_1+k_3}{k_1-k_3} &= k_3\tau \\
\frac{B(\tau) - \frac{k_1+k_3}{k_2}}{B(\tau) - \frac{k_1-k_3}{k_2}} &= \frac{k_1+k_3}{k_1-k_3}e^{k_3\tau} \\
B(\tau)\left(1 - \frac{k_1+k_3}{k_1-k_3}e^{k_3t}\right) &= \frac{k_1+k_3}{k_2}\left(1 - e^{k_3\tau}\right) \ ,
\end{aligned}$$

where we implicitly assumed that $B(\tau) \neq \frac{k_1+k_3}{k_2}$ and $B(\tau) \neq \frac{k_1-k_3}{k_2}$. We finally obtain

$$\begin{aligned} B(\tau) =& \frac{(k_1+k_3)\left(1-e^{k_3\tau}\right)}{k_2\left(1-\frac{k_1+k_3}{k_1-k_3}e^{k_3\tau}\right)} = \frac{\left(k_1^2-k_3^2\right)\left(1-e^{k_3\tau}\right)}{k_2\left(k_1-k_3-(k_1+k_3)e^{k_3\tau}\right)} \\ =& \frac{k_0\left(1-e^{k_3\tau}\right)}{k_1-k_3-(k_1+k_3)e^{k_3\tau}} = k_0\frac{e^{k_3\tau}-1}{(k_1+k_3)e^{k_3\tau}-k_1+k_3} = \frac{k_0\left(e^{k_3\tau}-1\right)}{2k_3+(k_1+k_3)\left(e^{k_3\tau}-1\right)}. \end{aligned} \tag{3.44}$$

This leads to the following for $A(\tau)$,

$$\begin{aligned} A(\tau) =& \gamma r\tau + \int_0^\tau \theta\kappa B(t)dt \\ =& \gamma r\tau + \theta\kappa\int_0^\tau \frac{k_0\left(e^{k_3t}-1\right)}{2k_3+(k_1+k_3)\left(e^{k_3t}-1\right)}dt \end{aligned}$$

With $z(t) = 2k_3 + (k_1+k_3)\left(e^{k_3t}-1\right)$, i.e., $t = \frac{1}{k_3}\ln(\frac{z-2k_3+k_1+k_3}{k_1+k_3}) = \frac{1}{k_3}\ln(\frac{z+k_1-k_3}{k_1+k_2})$ and $dt = \frac{1}{k_3(z+k_1-k_3)}dz$, we obtain

$$\begin{aligned} A(\tau) =& \gamma r\tau + \theta\kappa\int_{z(0)}^{z(\tau)} \frac{k_0\left(\frac{z+k_1-k_3}{k_1+k3}-1\right)}{z}\frac{1}{k_3(z+k_1-k_3)}dz \\ =& \gamma r\tau + \theta\kappa\frac{k_0}{k_3}\int_{z(0)}^{z(\tau)} \frac{\left(\frac{z+k_1-k_3}{k_1+k3}-1\right)}{z(z+k_1-k_3)}dz \\ =& \gamma r\tau + \theta\kappa\frac{k_0}{k_3}\left(\int_{z(0)}^{z(\tau)}\frac{1}{z(k_1+k_3)}dz - \int_{z(0)}^{z(\tau)}\frac{1}{z(z+k_1-k_3)}dz\right) \end{aligned}$$

Note: $\frac{1}{z(z+a)} = \frac{C}{z} + \frac{D}{z+a} = \frac{C(z+a)+Dz}{z(z+a)} = \frac{Ca+(C+D)z}{z(z+a)} \Leftrightarrow Ca = 1,\ C+D = 0 \Leftrightarrow$ Solve for C,D to dissect factors. Here, $a = k_1 - k_3$, $C = \frac{1}{k_1-k_3}$, $D = -\frac{1}{k_1-k_3}$. It follows that

$$\begin{aligned} A(\tau) =& \gamma r\tau + \theta\kappa\frac{k_0}{k_3}\left(\int_{z(0)}^{z(\tau)}\frac{1}{z(k_1+k_3)}dz - \int_{z(0)}^{z(\tau)}\frac{1}{z(k_1-k_3)}dz + \int_{z(0)}^{z(\tau)}\frac{\frac{1}{k_1-k_3}}{z+k_1-k_3}dz\right) \\ =& \gamma r\tau + \theta\kappa\frac{k_0}{k_3}\left(\left[\frac{\ln z}{k_1+k_3}\right]_{z(0)}^{z(\tau)} + \left[\frac{\ln(\frac{z+k_1-k_3}{z})}{k_1-k_3}\right]_{z(0)}^{z(\tau)}\right) \\ =& \gamma r\tau + \theta\kappa\frac{k_0}{k_3}\left(\left[\frac{\ln z}{k_1+k_3}\right]_{z(0)}^{z(\tau)} - \left[\frac{\ln z}{k_1-k_3}\right]_{z(0)}^{z(\tau)} + \left[\frac{\ln(z+k_1-k_3)}{k_1-k_3}\right]_{z(0)}^{z(\tau)}\right) \end{aligned}$$

Note: $\frac{k_0}{k_3} = \frac{k_0 k_2}{k_3 k_2} = \frac{k_1^2 - k_3^2}{k_3 k_2} = \frac{(k_1+k_3)(k_1-k_3)}{k_3 k_2}$, where $k_3^2 = k_1^2 - k_0 k_2$

$$\begin{aligned}
A(\tau) =& \gamma r \tau + \frac{\theta\kappa}{k_3 k_2}\left((k_1 - k_3)\Big[\ln z\Big]_{z(0)}^{z(\tau)} + (k_1 + k_3)\Big[\ln(\frac{z + k_1 - k_3}{z})\Big]_{z(0)}^{z(\tau)} \right) \\
=& \gamma r \tau + \frac{\theta\kappa}{k_3 k_2}\left((k_1 - k_3)\Big[\ln z\Big]_{z(0)}^{z(\tau)} - \Big[(k_1 + k_3)\ln z\Big]_{z(0)}^{z(\tau)} + (k_1 + k_3)\Big[\ln(z + k_1 - k_3)\Big]_{z(0)}^{z(\tau)} \right) \\
=& \gamma r \tau + \frac{\theta\kappa}{k_3 k_2}\left(-2k_3\Big[\ln z\Big]_{z(0)}^{z(\tau)} + (k_1 + k_3)\Big[\ln(z + k_1 - k_3)\Big]_{z(0)}^{z(\tau)} \right)
\end{aligned}$$

Take $2k_3$ out, we get

$$A(\tau) = \gamma r \tau + \frac{2\theta\kappa}{k_2}\left(\left(\frac{k_1 + k_3}{2k_3}\right)\Big[\ln(z + k_1 - k_3)\Big]_{z(0)}^{z(\tau)} - \Big[\ln z\Big]_{z(0)}^{z(\tau)} \right)$$

Note: $z(t) + k_1 - k_3 = k_1 + k_3 + (k_1 + k_3)(e^{k_3 t} - 1) = (k_1 + k_3)(1 + e^{k_3 t} - 1) = (k_1 + k_3)e^{k_3 t}$ and $z(0) + k_1 - k_3 = 2k_3 - k_3 + k_1 = k_3 + k_1$, then

$$\begin{aligned}
A(\tau) =& \gamma r \tau + \frac{2\theta\kappa}{k_2}\left(\left(\frac{k_1 + k_3}{2k_3}\right)\ln\left(\frac{(k_1 + k_3)e^{k_3 t}}{k_1 + k_3}\right) - \Big[\ln z\Big]_{z(0)}^{z(\tau)} \right) \\
=& \gamma r \tau + \frac{2\theta\kappa}{k_2}\left(\left(\frac{k_1 + k_3}{2k_3}\right)\ln\left(e^{k_3 \tau}\right) - \ln\left(\frac{2k_3 + (k_1 + k_3)\left(e^{k_3 \tau} - 1\right)}{2k_3}\right) \right) \\
=& \gamma r \tau + \frac{2\theta\kappa}{k_2}\left(\ln\left(e^{\frac{k_1 + k_3}{2}\tau}\right) - \ln\left(\frac{2k_3 + (k_1 + k_3)\left(e^{k_3 \tau} - 1\right)}{2k_3}\right) \right) \\
=& \gamma r \tau + \frac{2\theta\kappa}{k_2}\ln\left(\frac{2k_3 e^{\frac{k_1 + k_3}{2}\tau}}{2k_3 + (k_1 + k_3)\left(e^{k_3 \tau} - 1\right)}\right).
\end{aligned}$$

□

We are currently exploring other market prices of risk and potential closed-form solutions to a portfolio optimization problem within a 4/2 framework. Once this is achieved, then we can explore the multidimensional case of factor models.

4 One Common Factor Model

We assume there are two assets, namely $X_1(t)$ and $X_2(t)$, with one common stochastic volatility component, and one intrinsic stochastic volatility factor each. The 4/2 generalized model is investigated under mean reverting and non mean reverting respectively. It is known that financial instruments, such as commodities, volatility index (VIX), foreign exchange rates and interest rates, etc., are characterized by a mean reverting property. In this regard, we use the novelty of 4/2 stochastic processes together with the stylized fact of mean-reverting to explore the impacts of common factor loading a and the 3/2 component b.

Moreover, under the consideration of two forms of market price of risk, i.e., proportional to the instantaneous volatility of the underlying asset and proportional to the square root of the underlying CIR process, we explore the impact on implied volatility due to variations on parameters of the model through the Absolute Relative Change (ARC) in implied volatility (IV), defined as

$$\mathrm{ARC} = |\frac{\mathrm{IV}_{\theta+\Delta\theta} - \mathrm{IV}_{\theta}}{\mathrm{IV}_{\theta}}|$$

where θ is a parameter of interest, this could be the mean reverting level of the common factor's underlying CIR process, and $\Delta\theta$ stands for the variation in θ per se. Further, we explore the impact of risk premiums/excess returns' driver c under different forms of market price of risk with respect to value at risk and expected shortfall. Also, how different forms of market price of risk react to zero correlation ($a_i = 0$) and stochastic correlation ($a_i \neq 0$) among assets are studied in terms of risk measures (i.e., VaR and ES). The impacts of presence and absence of the 3/2 component b on VaR and ES are measured under different forms of market price of risk as well.

In this chapter, we will study two scenarios, i.e., Scenario **A** and Scenario **B** corresponding to a mean reverting model and a non mean reverting model respectively. For each model, we will firstly study the impact of the 3/2 parameters b_1, $\tilde{b}_1$ and commonalities loading a_1 on implied volatility for asset X_1. To be more specific, the impacts from these parameters on implied volatility are visualized in a three-dimensional plot where the x-axis is strike prices, the y-axis is the targeted parameter, and the z-axis is the implied volatility while time to maturity is assumed to be 1 year. This sends a direct message about the influence of each one of the key parameters in the model. Then risk measures, value at risk (VaR) and expected shortfall (ES), are assessed with respect to the common factor loading a and the 3/2 component b for a portfolio investing on both underlying X_1 and X_2.

In the case of one common factor and one intrinsic factor each, we state a few theoretical features about the implied covariance of asset returns. Denote $V_1 = \left(\sqrt{v_1} + \frac{b_1}{\sqrt{v_1}}\right)^2$ as the common factor variance, and $\tilde{V}_i = \left(\sqrt{\tilde{v}_i} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i}}\right)^2$ for $i = 1, 2$ as the intrinsic factors variance. Then, for asset X_i and X_j, the instantaneous quadratic variation between assets is given by

$$\Sigma_{ij} dt = \left\langle \frac{dX_i}{X_i}, \frac{dX_j}{X_j} \right\rangle = \left(a_i a_j V_1(t)\right) dt \tag{4.1}$$

The instantaneous quadratic variation of asset X_i is given by

$$\Sigma_{ii} dt = \left\langle \frac{dX_i}{X_i}, \frac{dX_i}{X_i} \right\rangle = \left(a_i^2 V_1(t) + \tilde{V}_i(t)\right) dt \tag{4.2}$$

Applying Ito's lemma, we have

$$\begin{aligned} d\Sigma_{ii} &= d\Bigg(a_i^2 \big(\sqrt{v_1} + \frac{b_1}{\sqrt{v_1}}\big)^2 + \big(\sqrt{\tilde{v}_i} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i}}\big)^2 \Bigg) \\ &= \Bigg\{ a_i^2 \Big[\frac{v_1^2 - b_1^2}{v_1^2} \alpha_1 (\theta_1 - v_1) + \frac{b_1^2}{v_1^3} \xi_1^2 \Big] + \Big[\frac{\tilde{v}_i^2 - \tilde{b}_i^2}{\tilde{v}_i^2} \tilde{\alpha}_i (\tilde{\theta}_i - \tilde{v}_i) + \frac{\tilde{b}_i^2}{\tilde{v}_i^3} \tilde{\xi}_i^2 \Big] \Bigg\} dt \\ &+ a_i^2 \Big[\frac{v_1^2 - b_1^2}{v_1^2} \Big] \xi_1 dB_1 + \Big[\frac{\tilde{v}_i^2 - \tilde{b}_i^2}{\tilde{v}_i^2} \Big] \tilde{\xi}_i d\tilde{B}_i \end{aligned} \tag{4.3}$$

Therefore, the correlation process between the assets X_1 and X_2 then follows

$$\begin{aligned} \rho &= \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} = \textbf{Corr}\left(\frac{dX_1}{X_1}, \frac{dX_2}{X_2} \right) = \textbf{Corr}\left(dlnX_1, dlnX_2 \right) \\ &= \frac{a_1 a_2 V_1(t)}{\sqrt{\left(a_1^2 V_1(t) + \tilde{V}_1(t)\right)\left(a_2^2 V_1(t) + \tilde{V}_2(t)\right)}} \end{aligned} \tag{4.4}$$

Note that the correlation between assets is stochastic whenever $a_i \neq 0$; otherwise, the two assets are uncorrelated.

Further, the leverage effect that refers to negative correlation between volatility and returns (Black, 1976; Christie, 1982; Heston et al., 2009), is defined as

$$\begin{aligned} \text{leverage} &= \textbf{Corr}\left(dlnX_i, < dlnX_i > \right) = \textbf{Corr}\left(\frac{dX_i}{X_i}, d\Sigma_{ii} \right) \\ &= \frac{a_i^3 \left(\sqrt{v_1} + \frac{b_1}{\sqrt{v_1}}\right)\left(\frac{v_1^2 - b_1^2}{v_1^2}\right)\rho_1 + \left(\sqrt{\tilde{v}_i} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i}}\right)\left(\frac{\tilde{v}_i^2 - \tilde{b}_i^2}{\tilde{v}_i^2}\right)\tilde{\rho}_i}{\sqrt{\left(a_i^2\left(\sqrt{v_1} + \frac{b_1}{\sqrt{v_1}}\right)^2 + \left(\sqrt{\tilde{v}_i} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i}}\right)^2\right)\left(a_i^4\left(\frac{v_1^2 - b_1^2}{v_1^2}\right)^2 + \left(\frac{\tilde{v}_i^2 - \tilde{b}_i^2}{\tilde{v}_i^2}\right)^2\right)}} \end{aligned} \tag{4.5}$$

Initial values
$X_1(0) = 18$, $X_2(0) = 100$, $r = 0.06$, Schwartz (1997) $v_1(0) = \theta_1$, $\tilde{v}_1(0) = \tilde{\theta}_1$, $\tilde{v}_2(0) = \tilde{\theta}_2$ Commodity Drift, Schwartz (1997) $\beta_{11} = 0.301$, $\beta_{12} = 0$, $\beta_{21} = 0$, $\beta_{22} = 0.369$ $L_1 = 3.09\beta_{11} = 0.93$, $L_2 = 4.85\beta_{22} = 1.79$
Commodity St. Volatility, Heston (1993) and Schwartz (1997). Scenario **A** $\alpha_1 = \tilde{\alpha}_1 = \tilde{\alpha}_2 = 2$ $\theta_1 = 0.01, \tilde{\theta}_1 = 0.0753, \tilde{\theta}_2 = 0.0124$ $\xi_1 = \tilde{\xi}_1 = \tilde{\xi}_2 = 0.1$ $\rho_1 = \widetilde{\rho}_1 = \widetilde{\rho}_2 = -0.5$
Commodity St. Volatility, Heston et al. (2009) and Schwartz (1997). Scenario **B** $\alpha_1 = \tilde{\alpha}_1 = \tilde{\alpha}_2 = 0.2098$ $\theta_1 = 0.1633, \tilde{\theta}_1 = 0.0685, \tilde{\theta}_2 = 0.0689$ $\xi_1 = \tilde{\xi}_1 = \tilde{\xi}_2 = 0.1706$ $\rho_1 = \widetilde{\rho}_1 = \widetilde{\rho}_2 = -0.9$
New parameters $c_1 = c_2 = \tilde{c}_1 = \tilde{c}_2 = 0$ $a_1 = a_2 = 0.75$ $b_1 = \tilde{b}_1 = \tilde{b}_2 = 0.008$

Table 4.1: Baseline Parametric values

Table 4.1 gives a baseline parameter set for the one common factor, two dimensional 4/2 factor model to be used in the coming sections. The choice of parameters in scenario **A** come from combining the seminal work of Schwartz (1997) (Oil and Copper, Tables IV and V) and Heston (1993). Scenario **B** combines Schwartz (1997) (Oil and Copper, Tables IV and V) with Heston et al. (2009). In both cases we assume a simple market price of risk structure ($c_1 = c_2 = \tilde{c_1} = \tilde{c_2} = 0$). In addition, when we investigate the non mean reverting model, we assumed a constant risk-free interest rate of 0.06, which was approximately the average interest rate over the period considered for Oil and Copper in Schwartz (1997).

It is worth mentioning that the mean reverting level of intrinsic factor $\tilde{\theta}_i$, $i = 1, 2$ in the table are set to match the long term volatility estimated in Schwartz (1997) for each commodity, which are 0.334 for Oil (Table IV), and 0.233 for Copper (Table V):

$$\begin{aligned}
&\mathbb{E}\left[a_1^2\left(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}}\right)^2 + \left(\sqrt{\tilde{v}_1(t)} + \frac{\tilde{b}_1}{\sqrt{\tilde{v}_1(t)}}\right)^2\right] \\
= \;& a_1^2\left(\frac{2\alpha_1 b_1^2}{2\alpha_1\theta_1 - \xi_1^2} + 2b_1 + \theta_1\right) + \frac{2\tilde{\alpha}_1\tilde{b}_1^2}{2\tilde{\alpha}_1\tilde{\theta}_1 - \tilde{\xi}_1^2} + 2\tilde{b}_1 + \tilde{\theta}_1 = (0.334)^2
\end{aligned}$$

This explain the values of $\tilde{\theta}_i$ in the table.

4.1 Mean Reverting

The asset prices with the mean reverting property in the 4/2 generalized factor model follow the system of SDE for $i = 1, 2$, such that

$$
\begin{aligned}
dY_i(t) &= \left(L_i - \sum_{j=1}^{2} \beta_{ij} Y_j(t)\right) dt \\
&+ \left((c_i - \frac{1}{2}) a_i^2 (\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}})^2 + (\tilde{c}_i - \frac{1}{2})(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2\right) dt \\
&+ a_i \left(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}}\right) dW_1(t) + \left(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}}\right) d\widetilde{W}_i(t) \\
dv_1(t) &= \alpha_1(\theta_1 - v_1(t))dt + \xi_1 \sqrt{v_1(t)} dB_1(t) \\
d\tilde{v}_i(t) &= \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t))dt + \tilde{\xi}_i \sqrt{\tilde{v}_i(t)} d\widetilde{B}_i(t)
\end{aligned}
$$

with quadratic variation relationship $\langle dB_j(t), dW_j(t) \rangle = \rho_j dt$, $\left\langle d\widetilde{B}_i(t), d\widetilde{W}_i(t) \right\rangle = \widetilde{\rho}_i dt$ for $j = 1$; and $i = 1, 2$.

In order to ensure that the choice of parameters lead to reasonable assets behavior, we report the expected return, variance of return for each asset, as well as the correlation between two assets and the leverage effects in Table 4.2, Table 4.3 and Table 4.4, Table 4.5 under scenario **A** and scenario **B** for various choices of b and a respectively. Here we simulate $500,000$ paths with $dt = 0.1$ and consider the following scenarios for b: $b_1 = 0.008$, $\tilde{b}_1 = \tilde{b}_2 = 0$; $b_1 = 0$, $\tilde{b}_1 = \tilde{b}_2 = 0.008$; $b_1 = \tilde{b}_1 = \tilde{b}_2 = 0$ and $b_1 = \tilde{b}_1 = \tilde{b}_2 = 0.008$. That is, under scenario 1, only the common factor is assumed to follow 4/2 structure; under scenario 2, only the intrinsic factor is assumed to follow 4/2 structure; under scenario 3, the common factor and the intrinsic factor are both assumed to be Heston-like; and under scenario 4, both common factor and intrinsic factor follow a 4/2-liked instantaneous volatility structure.

Table 4.2: First four moments for scenarios on 3/2 component (b). Scenario **A**

	$b_1 = 0.008$, $\tilde{b}_i = 0$	$b_1 = 0$, $\tilde{b}_i = 0.008$	$b_1 = \tilde{b}_i = 0$	$b_1 = \tilde{b}_i = 0.008$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0495	0.0501	0.0509	0.0495
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0770	0.0778	0.0781	0.0767
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0686	0.0694	0.0626	0.0756
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0386	0.0402	0.0330	0.0481
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.3484	0.0921	0.1069	0.4888
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.4385	-0.4707	-0.4617	-0.3926
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.4195	-0.4650	-0.4608	-0.3492

Table 4.3: First four moments for scenarios on 3/2 component (b). Scenario **B**

	b_1 =0.008, $\tilde{b}_i$=0	$b_1 = 0$, $\tilde{b}_i$=0.008	b_1=$\tilde{b}_i$=0	$b_1 = \tilde{b}_i$=0.008
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0514	0.0505	0.0528	0.0497
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0773	0.0768	0.0787	0.0756
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0354	0.0660	0.0242	0.0988
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0354	0.0725	0.0244	0.0750
$\mathbf{Corr}(dlnX_1(T)\,, dlnX_2(T))$	0.7488	0.0098	0.4650	0.0145
$\mathbf{Corr}(dlnX_1(T)\,, < dlnX_1(T) >)$	-0.5843	-0.2882	-0.7517	-0.0386
$\mathbf{Corr}(dlnX_2(T)\,, < dlnX_2(T) >)$	-0.5823	-0.0575	-0.7441	-0.3260

Similarly, the key statistics under scenarios $a_1 = a_2 = 0$; $a_1 = 0.75$, $a_2 = 0$; $a_1 = 0$, $a_2 = 0.75$; and $a_1 = a_2 = 0.75$ are evaluated for commonalities loading a. To be more precise, under scenario 1, the common factor and thus the stochastic correlations between assets are ignored with $a_i = 0$; under scenario 2 and 3, one of the two asset ignores the common factor and only considers the intrinsic factor follows a 4/2 structure; and under scenario 4, both common factor and intrinsic factor follow a 4/2-liked instantaneous volatility structure for each asset.

Table 4.4: First four moments for scenarios on commonalities (a). Scenario **A**

	$a_1 = a_2 = 0$	$a_1 = 0.75$, $a_2 = 0$	$a_1 = 0$, $a_2 = 0.75$	$a_1 = a_2 = 0.75$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0505	0.0499	0.0502	0.0502
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0777	0.0780	0.0766	0.0763
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0674	0.0761	0.0675	0.0755
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0382	0.0381	0.0484	0.0478
$\mathbf{Corr}(dlnX_1(T)\,, dlnX_2(T))$	0.0011	-0.0001	-0.0010	0.3208
$\mathbf{Corr}(dlnX_1(T)\,, < dlnX_1(T) >)$	-0.4703	-0.2840	-0.4695	-0.4441
$\mathbf{Corr}(dlnX_2(T)\,, < dlnX_2(T) >)$	-0.4679	-0.4680	-0.3758	-0.4162

Table 4.5: First four moments for scenarios on commonalities (a). Scenario **B**

	$a_1 = a_2 = 0$	$a_1 = 0.75$, $a_2 = 0$	$a_1 = 0$, $a_2 = 0.75$	$a_1 = a_2 = 0.75$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0514	0.0496	0.0512	0.0500
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0768	0.0772	0.0756	0.0752
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0420	0.0733	0.0420	0.0857
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0507	0.0564	0.0746	0.0719
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0004	-0.0000	-0.0005	0.0039
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.1126	-0.0047	-0.3203	-0.2634
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.1600	-0.2544	-0.0164	-0.0111

4.1.1 Pricing Option

In this section, we price the European call option on the asset X_1 based on the mean reverting 4/2 structured one common factor and one intrinsic factor model. We explore the implied volatility surface with strike prices K: 15, 16.4, 17.8, 19.2, 20.6, 22 and expiry date T=1.0. By choosing these strike prices, we take into account in-the-money, at-the-money, and also out-of-the-money given the initial asset price is 18. Then, for each strike price and the expiry date, we can firstly get a simulated call option price by equation 2.17. Then, following the procedure of extracting the implied volatility, we match the exponential O-U option price formula in equation 2.21 with simulated call prices and solve for the volatility parameter σ in the dynamics of $Y(t)$ such that:

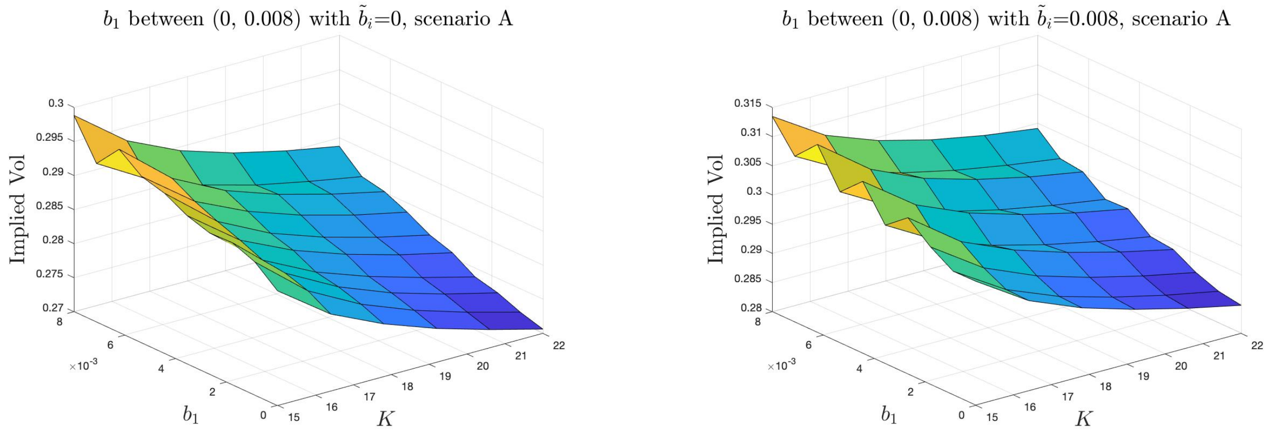
$$dY_1(t) = (L_1 - \frac{1}{2}\sigma^2 - \beta_{11}Y(t))dt + \sigma dW^*(t).$$

4.1.2 Sensitivity Analysis

In this subsection, we analyze the effects of the common factor loading a, the 3/2 parameter b from common factor and $\tilde{b}$ from intrinsic factor on implied volatility in scenario **A** and scenario **B** respectively. The purpose of investigating two scenarios of the parameter is that we can observe how the implied volatility reacts to the underlying CIR process driving common and intrinsic volatilities. In other words, whether the parameter a, b, and $\tilde{b}$ affect the implied volatility differently with respect to v or $\tilde{v}$.

In the study of parameter b, which represents the size of the 3/2 component on the common factor, the implied volatility surfaces are shown in terms of b_1 changing in the interval (0, 0.008) while $\tilde{b}_i$ equals to zero and 0.008. See Figure 4.1 for scenario **A** and Figure 4.2 for scenario **B**.

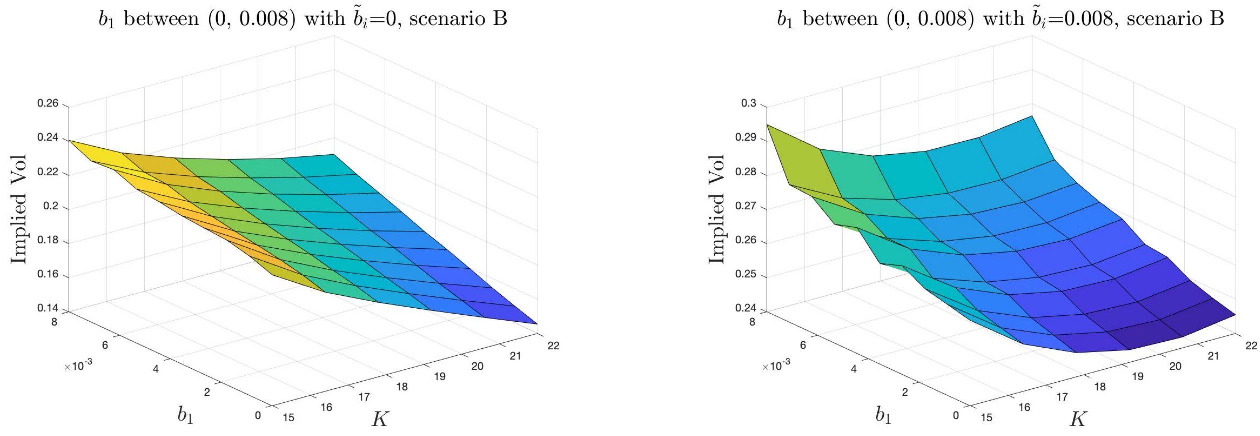
In scenario **A**, Figure 4.1 illustrates that even small changes in the common factor 3/2 component b_1 (from 0 to 0.008) can lead to a 10.37% and a 5.3% difference in implied volatility (from 0.27 to 0.298, and 0.285 to 0.3) respectively. In general, it can be observed that the overall shapes of implied volatility surfaces are similar whether the intrinsic factor contains the 3/2 component or not. The relative change declines by almost half due to the presence of the 3/2 component in the intrinsic factor. Besides, the joint effect of the common and intrinsic 3/2 components (b_1 and $\tilde{b}_1$) can be obtained by combining those two figures leading to a 15.9% change (from 0.27 to 0.313) in the presence of relatively small values of b's.



(a) $\tilde{b}_i$=0, b_1 between (0, 0.008) **(b)** $\tilde{b}_i$=0.008, b_1 between (0, 0.008)

Figure 4.1: Impact of b_1 (common factor, 3/2 component) on implied volatility. Scenario **A**

In scenario **B**, we observe that the impact of the intrinsic factor on volatility surface is more significant than that in Scenario **A**. There is about 31% (from 0.145 to 0.19) increase in implied volatility with the 3/2 component only in the common factor, while only half of the increase, 13.5% (from 0.245 to 0.278), can be observed in the presence of a 3/2 component in the intrinsic factor as well. In addition, a volatility "smile" can be seen in Figure 4.2b when the 3/2 component plays a role in both common factor and intrinsic factor. Similar to what we observe in scenario **A**, adding another ingredient into the intrinsic factor surely increases the level of the implied volatility surface. However, as a consequence of adding a 3/2 component in intrinsic factor, the stabilization effect in implied volatility is more transparent. This stabilization effect may result from the 3/2 component's feature of quickly mean-reverting when the process gets large. Moreover, the joint effect of the common and intrinsic 3/2 components in this case is 91.7% (0.145 to 0.278).

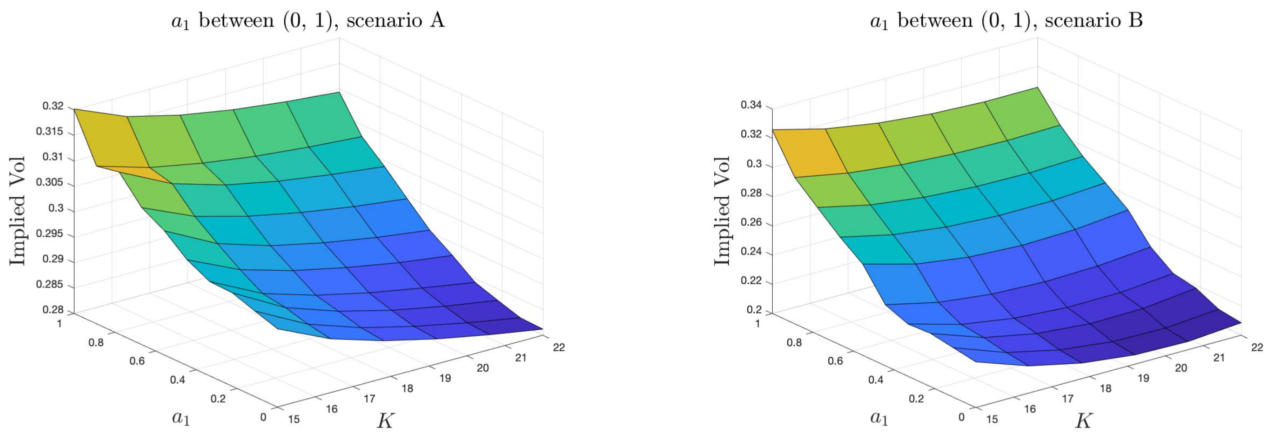


(a) $\tilde{b}_i$=0, b_1 between (0, 0.008) **(b)** $\tilde{b}_i$=0.008, b_1 between (0, 0.008)

Figure 4.2: Impact of b_1 (common factor, 3/2 component) on implied volatility. Scenario **B**

Combining scenario **A** and scenario **B**, on the one hand, the presence of 3/2 component b can improve the implied volatility level but also stabilize it. On the other hand, comparing these two scenarios, given different underlying CIR process for common and intrinsic factors, the impact of the 3/2 component can be crucial.

Next, the impact of the weight a on the common factor is explored for scenarios **A** and **B** respectively. Note that when $a_i = 0$, the correlation between assets is zero. Figure 4.3a and Figure 4.3b display significant increases in implied volatility due to the commonality loading. Specifically, the change in implied volatility can increase up to 10.7% (from 0.28 to 0.31) in scenario **A** and up to 55% (from 0.2 to 0.31) in scenario **B**. It implies that, if we wrongly allocate the weight on the common factor or assumes wrong dependence between assets, it could result in huge differences in the implied volatility.



(a) a_1 between (0, 1). Scenario **A** **(b)** a_1 between (0, 1). Scenario **B**

Figure 4.3: Impact of commonality (a_1) on implied volatility

4.1.3 Risk Measures

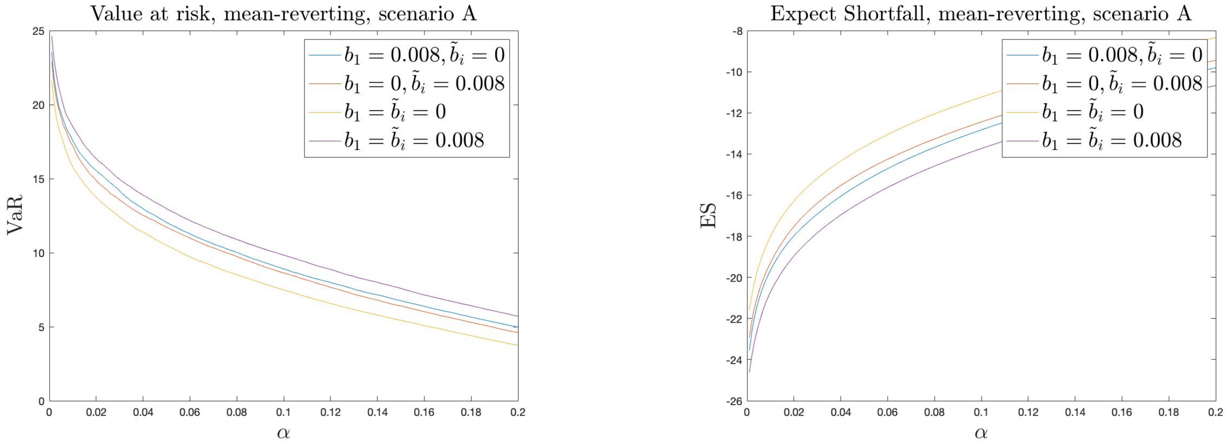
In this section, we look at the impact of b and a on important risk measures, in particular value at risk (VaR) and expected shortfall (ES). Value at risk is defined as a measure of loss in an investment. It is an assessment of the amount of capital needed to cover the potential loss. With respect to a portfolio, value at risk is the maximum loss with a certain confidence level within a certain time frame. For example, if a bank's 3-day 99% VaR is $1 million, they have about 1% chance that losses will exceed $1 million in 3 days. Further, ES is known as a conditional or average VaR, which is an expected return of the portfolio in the worst α% cases. In other words, VaR answers the question about "what is the value of our portfolio at risk if things go bad", while ES further gives the estimate of the expected loss if things did go bad. For clarity and the purpose of our calculations, these measures are defined as follows:

$$\alpha = P\left(X(T) \leq -VaR_{\alpha}\right) \tag{4.6}$$

$$ES = -\frac{1}{\alpha}\int_0^{\alpha} VaR_{\gamma} d\gamma \tag{4.7}$$

where, for simplicity, $X(T) = \omega_1(X_1(T) - X_1(0)) + \omega_2(X_2(T) - X_2(0))$ is the profit and loss portfolio with equal weights ($w_1 = w_2 = 1/2$). The rational is from DeMiguel et al. (2009) that, the simple and relatively low cost of implementing the 1/N naive-diversification rule can serve as a natural benchmark to assess the performance of more complex asset-allocation rules. Recall that the initial value of asset X_1 is \$18, and the initial value of asset X_2 is \$100. Hence, the initial budget for the equal weight portfolio is \$59. We will let α varies from 0.001 to 0.2 with a discretize size of 200.

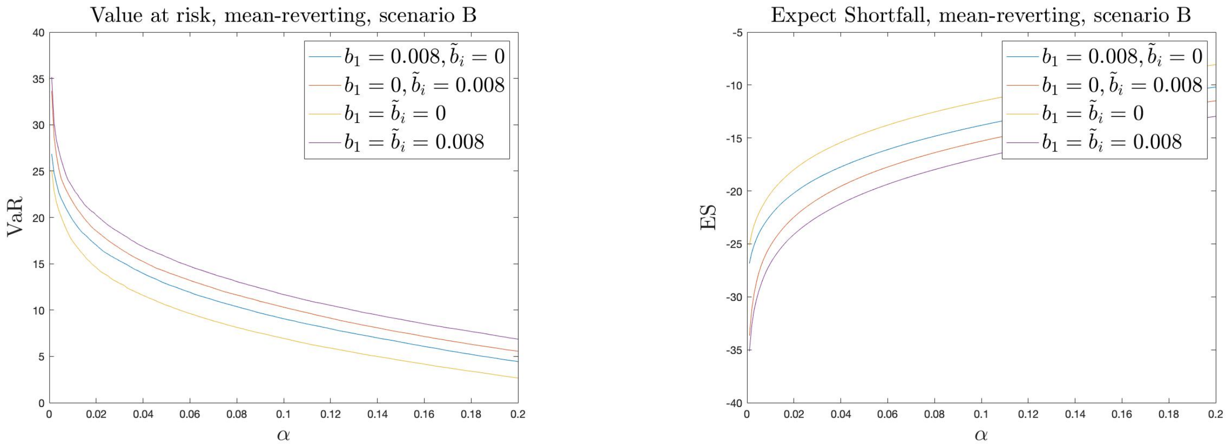
We first study the impact of b_1 and $\tilde{b}_i$ on VaR and ES for a fixed $\alpha = 0.01$. For scenario **A**, Figures 4.4a and 4.4b display a substantial increase in VaR, from \$16 (all b set to zero) to \$19.5 (all b set to 0.008), this is a 21% increase ($\alpha = 0.01$) due to the presence of b. That is, an investor would have to place 21% more capital aside in the presence of 3/2 components. Similarly ES increases from -\$21 in the presence of 3/2 components to -\$18.5 in the absence of it, this is a 13.5% increase in the average VaR.



(a) Value at Risk vs. α, various b **(b)** Expected Shortfall vs. α, various b

Figure 4.4: Impact of 3/2 components (b) on Risk measures. Scenario **A**

For scenario **B**, Figure 4.5a and Figure 4.5b also display a substantial increase in VaR, from \$17.5 (all b set to zero) to \$22.5 (all b set to 0.008), this is a 28.6% increase ($\alpha = 0.01$) due to the presence of b. In other words, 28.6% more capital is required in the presence of 3/2 components. Similarly ES increases from -\$27.5 with the 3/2 components to -\$20.5 in the absence of it, this is a 36.6% increase in the average VaR.

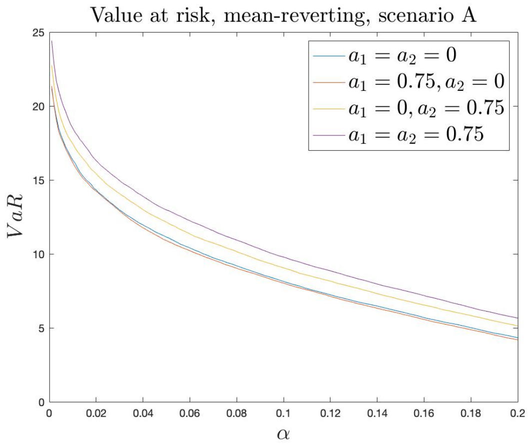


(a) Value at Risk vs. α, various b **(b)** Expected Shortfall vs. α, various b

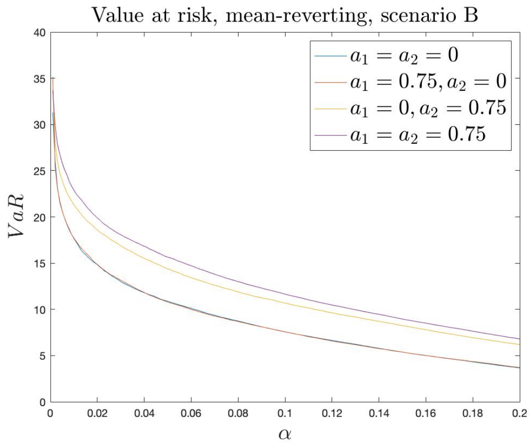
Figure 4.5: Impact of 3/2 components (b) on Risk measures. Scenario **B**

Combining scenario **A** and scenario **B**, the presence of a 3/2 component has a considerable impact on quantifying risk. Missing the 3/2 component or underestimating it would lead to wrongly calculating the value in the portfolio under risk, hence mistakenly evaluating the potential expected loss in worst-case scenarios.

A similar analysis is performed with respect to the commonality a, in the presence of stochastic volatility (in the common factor) versus in the absence of it. In other words, we are assessing the impact from a per se and the impact of the stochastic correlation produced by the 4/2 model. Figure 4.6a demonstrates an increase in VaR, from \$16 to \$18.5, this is a 15.6% increase ($\alpha = 0.01$) due to the stochastic correlation between assets. At the same time, Figure 4.6b indicates a jump from \$17 to \$23, which results in a 35.3% increase in the value at risk at $\alpha = 0.01$. It implies that ignoring the stochastic correlation between assets can make a huge difference in measuring risk, and the impact depends on the different underlying processes.



(a) Value at Risk vs. α, various a. Scenario **A**



(b) Value at Risk vs. α, various a. Scenario **B**

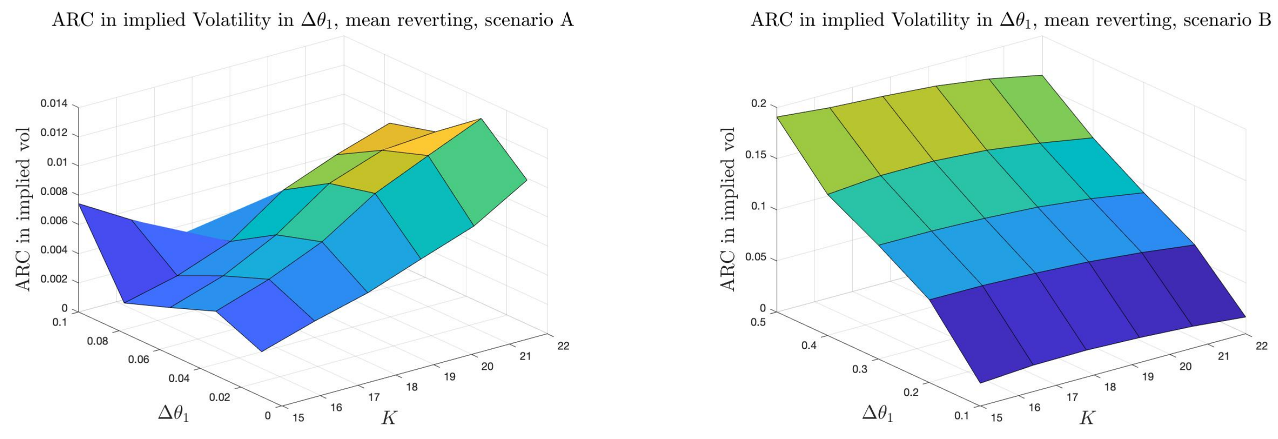
Figure 4.6: Impact of commonality (a) on Value at Risk.

4.1.4 Alternative Market Price of Risk

For a more direct comparison on how much effect the choice of the market price of risk makes on implied volatility, we investigate the absolute value of relative change (ARC) in implied volatility due to changes in the mean reverting level of the CIR process θ_j in the common factor.

Under scenario **A**, the mean reverting level of common factor volatility v_1 is 0.01, the ARC explored values of $\Delta\theta_1$ from 0.01 to 0.1 in mean reverting levels; on the other hand, the mean reverting level of v_1 is 0.1633 under scenario **B**, thus a range of 0.1 to 0.5 for variation $\Delta\theta_1$ is considered.

From Figure 4.7, the difference in the absolute relative changes in implied volatility under different underlying processes is substantial. For example, in scenario **A**, the change in implied volatility due to changes in the θ_1 is minor (a variation of 1.2% from 0.002 to 0.014), and it increases with the strike price. In contrast, in scenario **B**, the absolute relative change in implied volatility due to different mean reverting level is significant, and it is indifferent with respect to strike prices. Specifically, changes in θ_1 can result in an increase of 19% in the implied volatilities for all types of moneyness options.

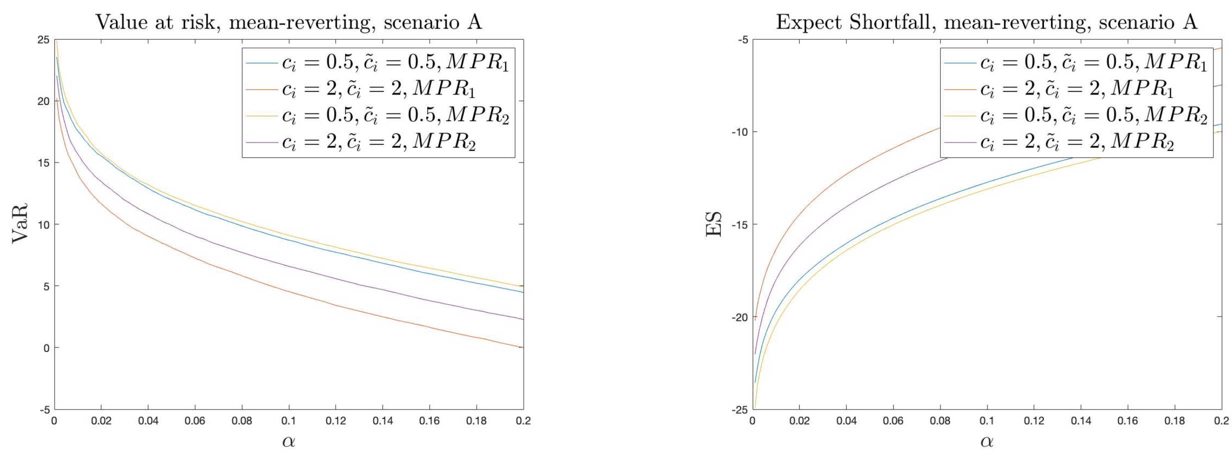


(a) ARC in mean reverting model. Scenario A

(b) ARC in mean reverting model. Scenario B

Figure 4.7: ARC in mean reverting model. Scenario **A** and **B**

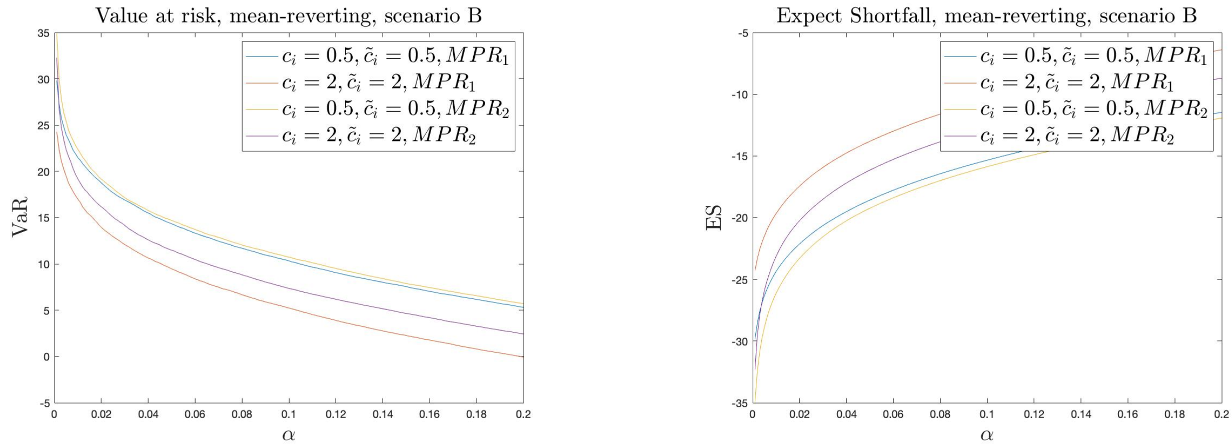
In terms of risk measures, by comparing the two scenarios in Figure 4.8 and Figure 4.9, it can be seen that the market price of risk MPR_2 always leads to a higher VaR and ES for the portfolio. In particular, if the driver of risk premium c_i or $\tilde{c}_i$ is relatively high, the change in risk measure is more significant. To be more precise, when $c_i = \tilde{c}_i = 2$, and $\alpha = 0.01$, there is a 15.4% (from \$13 to \$15) difference in scenario **A** with respect to VaR compared to $c_i = \tilde{c}_i = 0.5$, while a 17.6% (from \$17 to \$20) difference in scenario **B** can be observed. At the same time, there is a 9.4% (from \$-16 to \$-17.5) difference in scenario **A** with respect to ES caused by choice of market price of risk, while a 12.5% (from \$-20 to \$-22.5) difference in scenario **B** can be found. This is telling us that choice of market price of risk can make a significant difference in relevant risk measures, especially with a higher driver of excess return.



(a) Value at Risk vs. α, various c

(b) Expected Shortfall vs. α, various c

Figure 4.8: Impact of c on Risk measures, mean reverting model. Scenario **A**

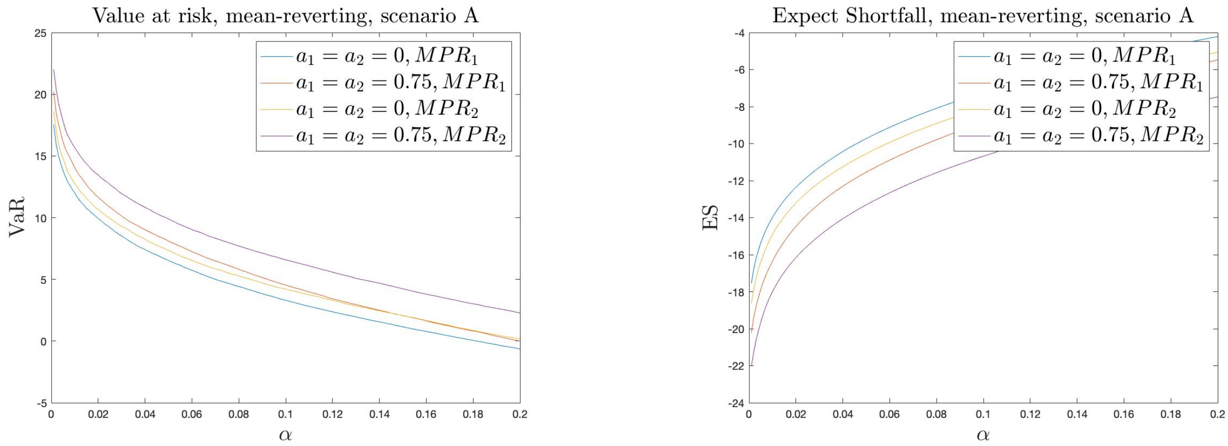


(a) Value at Risk vs. α, various c

(b) Expected Shortfall vs. α, various c

Figure 4.9: Impact of c on Risk measures, mean reverting model. Scenario **B**

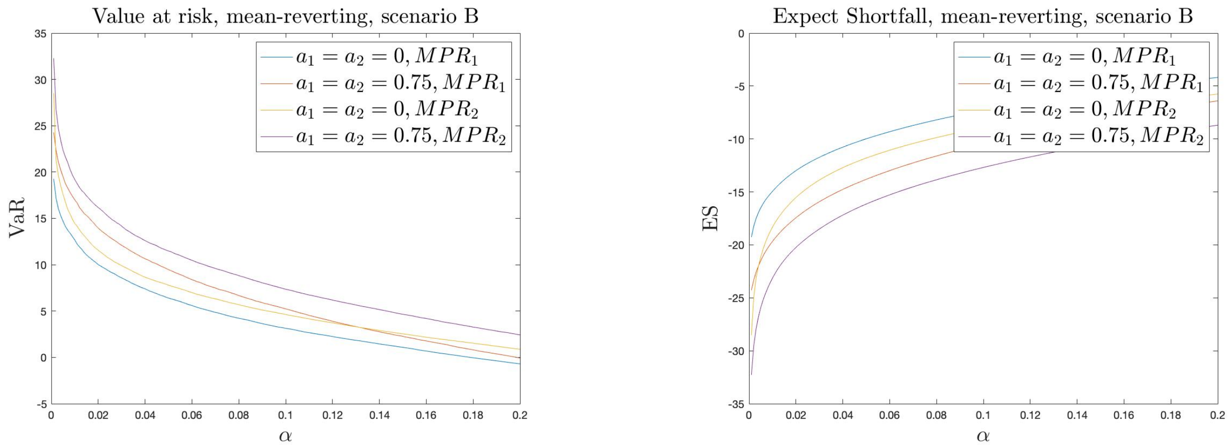
Figure 4.10 and Figure 4.11 present the impact of the common factor loading a with different forms of market price of risk on risk measures. Specifically, for each market price of risk, we consider no covariance ($a_i = 0$) and stochastic covariance ($a_i = 0.75$) among assets respectively. When there is no correlation between assets, it always demonstrates lower value at risk and ES than the stochastic covariance case with both forms of market price of risk. For example, with MPR_1, there is an increase of 7.7% (from \$13 to \$14) and 28% (from \$12.5 to \$16) in value at risk for scenario **A** and **B** respectively. Also, an increase of 14.3% (from -\$14 to -\$16) and 20% (from -\$15 to -\$18) in ES can be observed for scenario **A** and **B** respectively. On the other hand, with MPR_2, there is an increase of 18.5% (from \$13.5 to \$16) and 38.5% (from \$13 to \$18) in value at risk for scenario **A** and **B**. Likewise, there is an increase of 20% (from -\$15 to -\$18) and 28.6% (from -\$17.5 to -\$22.5) in ES for scenario **A** and **B** respectively. That is, different forms of market price of risk impact risk measures differently depending on the common factor loading and covariance between assets. Furthermore, by following market price of risk in the form of MPR_2, it exhibits a larger variation in risk measures if one ignored the stochastic correlation between assets.



(a) Value at Risk vs. α, various a

(b) Expected Shortfall vs. α, various a

Figure 4.10: Impact of a on Risk measures, mean reverting model. Scenario **A**

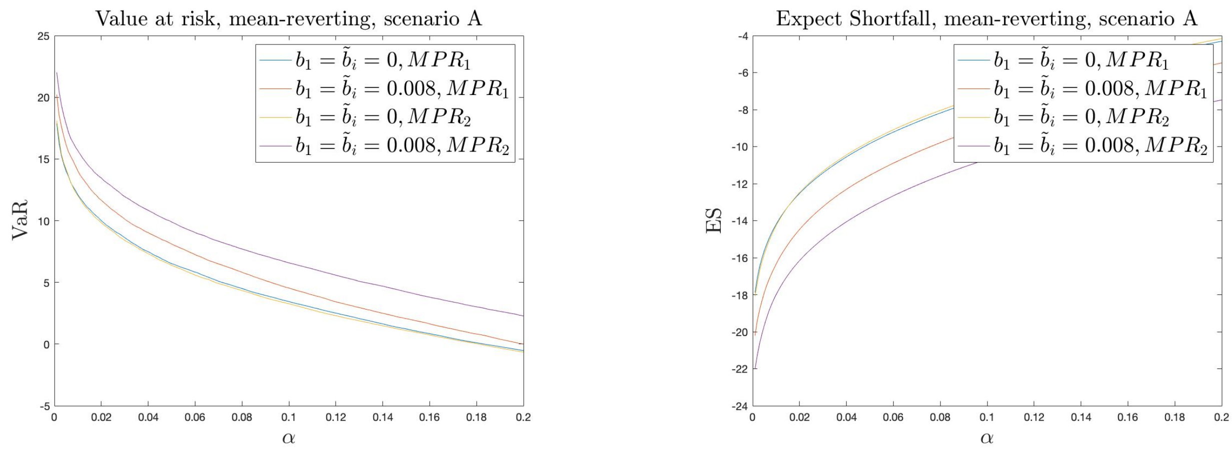


(a) Value at Risk vs. α, various a

(b) Expected Shortfall vs. α, various a

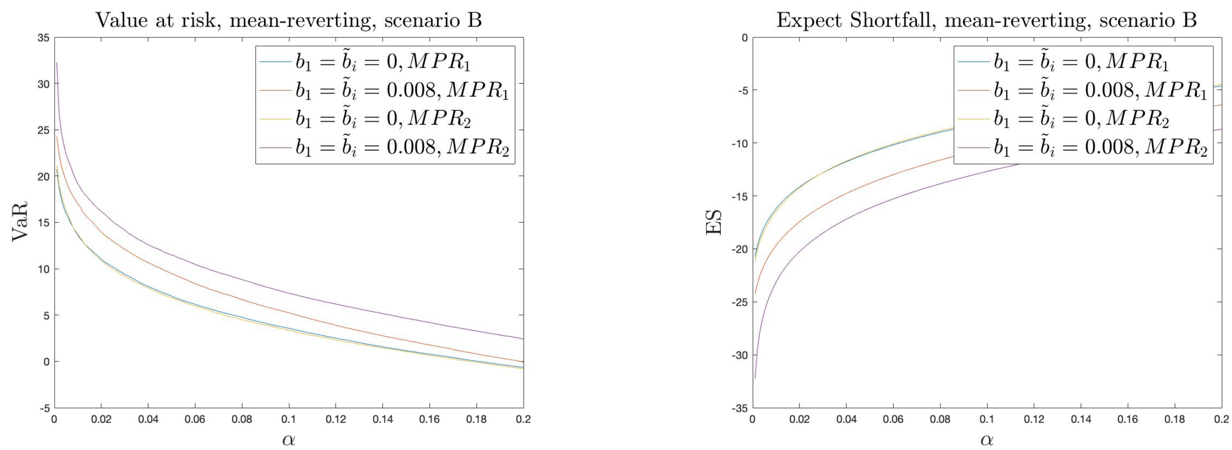
Figure 4.11: Impact of a on Risk measures, mean reverting model. Scenario **B**

Figure 4.12 and Figure 4.13 present impact of the $3/2$ component b with different forms of market price of risk in relevant risk measures. When there is no $3/2$ component, the risk measures behave similarly in both scenario **A** and **B** since MPR_1 and MPR_2 are identical. On the other hand, under the presence of the $3/2$ component in both common factor and intrinsic factor, there is a difference of 15.4% (from \$13 to \$15) in value at risk and a difference of 12.5% (from -\$16 to -\$18) in the expected shortfall in scenario **A**. Meanwhile, there is a difference of 12.5% (from \$16 to \$18) in value at risk and a difference of 18.4% (from -\$19 to -\$22.5) in the expected shortfall in scenario **B**. Thus, it can be evidenced that different forms of market price of risk gauge the risk differently, in particular, the MPR_2 exhibits more sensitive to risk than the other.



(a) Value at Risk vs. α, various b **(b)** Expected Shortfall vs. α, various b

Figure 4.12: Impact of b on Risk measures, mean reverting model. Scenario **A**



(a) Value at Risk vs. α, various b **(b)** Expected Shortfall vs. α, various b

Figure 4.13: Impact of b on Risk measures, mean reverting model. Scenario **B**

In summary, there is a difference arising from the choice of market price of risk in risk measures. The significance of the difference depends on the driver of risk premium c, the common factor loading a, and the 3/2 component b. In particular, between the two choices of market price of risk, the MPR_2 constantly shows its higher sensitivity than MPR_1 in quantifying risk.

4.2 Non Mean Reverting

We assume two assets, i.e., $X_1(t)$ and $X_2(t)$, with one common factor, and one intrinsic factor each. The asset prices without reverted mean thereby follow the system of SDE for $i = 1, 2$:

$$\begin{aligned} dY_i(t) &= \left(r + (c_i - \frac{1}{2})a_i^2(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}})^2 + (\tilde{c}_i - \frac{1}{2})(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2\right) dt \\ &\quad + a_i \left(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}}\right) dW_1(t) + \left(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}}\right) d\widetilde{W}_i(t) \\ dv_1(t) &= \alpha_1(\theta_1 - v_1(t))dt + \xi_1\sqrt{v_1(t)}dB_1(t) \\ d\tilde{v}_i(t) &= \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t))dt + \tilde{\xi}_i\sqrt{\tilde{v}_i(t)}d\widetilde{B}_i(t) \end{aligned}$$

with $\langle dB_j(t), dW_j(t)\rangle = \rho_j dt$, $\left\langle d\widetilde{B}_i(t), d\widetilde{W}_i(t)\right\rangle = \widetilde{\rho}_i dt$ for $j = 1$; $i = 1, 2$.

In this section, the baseline parameter set is assumed to be compatible with that of the mean reverting model. Under this assumption, it should be interpreted as modelling stocks that do not possess the mean reverting character. Therefore, here we study the impact from common factor loading a and the 3/2 component b or $\tilde{b}$ for assets with non mean reverting patterns.

4.2.1 Pricing Option

In this section, we price European call options on the asset X_1 via Monte Carlo simulations. We explore the implied volatility surface with strike prices K: 15, 16.4, 17.8, 19.2, 20.6, 22 and expiry date T=1 year. These strike prices take into account in-the-money, at-the-money, and also out-of-the-money given the initial asset price is 18. Then, for each strike price and expiry date, we can get a simulated call option price by equation 2.17. Then, following the procedure of extracting the implied volatility, we match the Black-Scholes option price formula in equation 2.19 with simulated call prices and solve for the implied volatility in the dynamics of $Y(t)$ such that:

$$dY_1(t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW^*(t).$$

4.2.2 Sensitivity Analysis

In this subsection, the sensitivity analysis on the implied volatility surface in terms of commonality loading a, and the 3/2 component parameter b or $\tilde{b}$ will be performed.

In order to ensure that the choice of parameters lead to reasonable assets behavior, we report the expected return, variance of return for each asset, as well as the correlation between two assets and the leverage effects in Tables 4.6, Table 4.7 and Table 4.8, Table 4.9 under scenario **A** and scenario **B** respectively.

Here we simulate 500, 000 paths with $dt = 0.1$ and consider the following scenarios for b: $b_1 = 0.008$, $\tilde{b}_1 = \tilde{b}_2 = 0$; $b_1 = 0$, $\tilde{b}_1 = \tilde{b}_2 = 0.008$; $b_1 = \tilde{b}_1 = \tilde{b}_2 = 0$ and $b_1 = \tilde{b}_1 = \tilde{b}_2 = 0.008$. Whenever b or $\tilde{b}_i$ is set to zero, its associated volatility factor would not have the 3/2 component and depend only on the squared CIR underlying process.

Table 4.6: First four moments for scenarios on 3/2 component (b). Scenario **A**

	b_1 =0.008, $\tilde{b}_i$=0	$b_1 = 0$, $\tilde{b}_i$=0.008	b_1=$\tilde{b}_i$=0	$b_1 = \tilde{b}_i$=0.008
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0608	0.0617	0.0618	0.0615
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0613	0.0625	0.0619	0.0620
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0870	0.0886	0.0783	0.0977
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0500	0.0528	0.0415	0.0643
$\mathbf{Corr}(dlnX_1(T)\,, dlnX_2(T))$	0.3593	0.0952	0.1134	0.5102
$\mathbf{Corr}(dlnX_1(T)\,, < dlnX_1(T) >)$	-0.4506	-0.4830	-0.4754	-0.3960
$\mathbf{Corr}(dlnX_2(T)\,, < dlnX_2(T) >)$	-0.4357	-0.4813	-0.4786	-0.3561

Table 4.7: First four moments for scenarios on 3/2 component (b). Scenario **B**

	b_1 =0.008, $\tilde{b}_i$=0	$b_1 = 0$, $\tilde{b}_i$=0.008	b_1=$\tilde{b}_i$=0	$b_1 = \tilde{b}_i$=0.008
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0613	0.0616	0.0620	0.0616
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0614	0.0625	0.0620	0.0621
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0455	0.0882	0.0301	0.1607
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0456	0.1045	0.0302	0.1163
$\mathbf{Corr}(dlnX_1(T)\,, dlnX_2(T))$	0.7641	0.0089	0.4686	0.0164
$\mathbf{Corr}(dlnX_1(T)\,, < dlnX_1(T) >)$	-0.5947	-0.2908	-0.7882	-0.0442
$\mathbf{Corr}(dlnX_2(T)\,, < dlnX_2(T) >)$	-0.5955	-0.0585	-0.7892	-0.3457

Similarly we will consider the following scenarios for a: $a_1 = a_2 = 0$; $a_1 = 0.75$, $a_2 = 0$; $a_1 = 0$, $a_2 = 0.75$ and $a_1 = a_2 = 0.75$. Table 4.8 and Table 4.9 present the key statistics for asset returns. Notice that when a_i is set to zero, it means that the ith asset ignores the stochastic common factor and the stochastic covariance among assets.

Table 4.8: First four moments for scenarios on commonalities (a). Scenario **A**

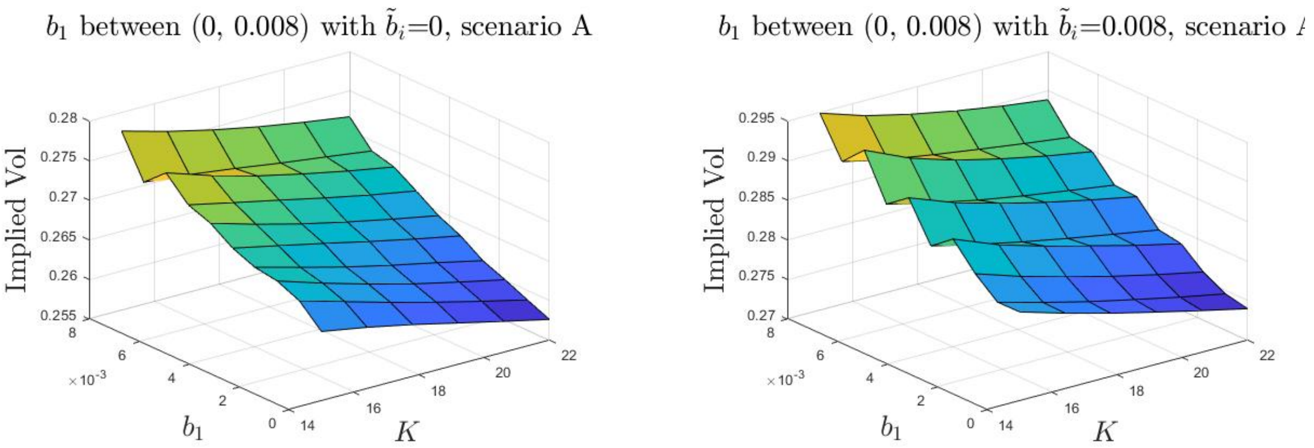
	$a_1 = a_2 = 0$	$a_1 = 0.75,\ a_2 = 0$	$a_1 = 0,\ a_2 = 0.75$	$a_1 = a_2 = 0.75$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0618	0.0619	0.0615	0.0622
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0621	0.0624	0.0619	0.0615
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0855	0.0986	0.0856	0.0976
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0496	0.0495	0.0648	0.0641
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0009	-0.0000	-0.0008	0.3317
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.4826	-0.2735	-0.4818	-0.4536
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.4834	-0.4835	-0.3921	-0.4319

Table 4.9: First four moments for scenarios on commonalities (a). Scenario **B**

	$a_1 = a_2 = 0$	$a_1 = 0.75,\ a_2 = 0$	$a_1 = 0,\ a_2 = 0.75$	$a_1 = a_2 = 0.75$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0617	0.0613	0.0615	0.0618
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0617	0.0620	0.0619	0.0615
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0559	0.0993	0.0558	0.1165
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.0694	0.0759	0.1027	0.0965
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0003	-0.0001	-0.0005	0.0041
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.1147	-0.0044	-0.3400	-0.2688
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.1834	-0.2696	-0.0166	-0.0128

In order to study the 3/2 component b on the common factor, the implied volatility surfaces are shown in terms of b_1 changing in the interval (0, 0.008) while $\tilde{b}_i$ either equals to zero or 0.008 in Figure 4.14 for scenario **A** and Figure 4.15 for scenario **B**.

In scenario **A**, Figure 4.14 shows that when the common factor 3/2 component b_1 varies from 0 to 0.008, there is a 5.8% difference in implied volatility (from 0.258 to 0.273) in the absence of 3/2 component in intrinsic factor, while a 5.47% difference in implied volatility in the presence of $\tilde{b}_i$ (from 0.274 to 0.289). The joint effect of the 3/2 component from common factor and intrinsic factor is 12% (from 0.258 to 0.289).

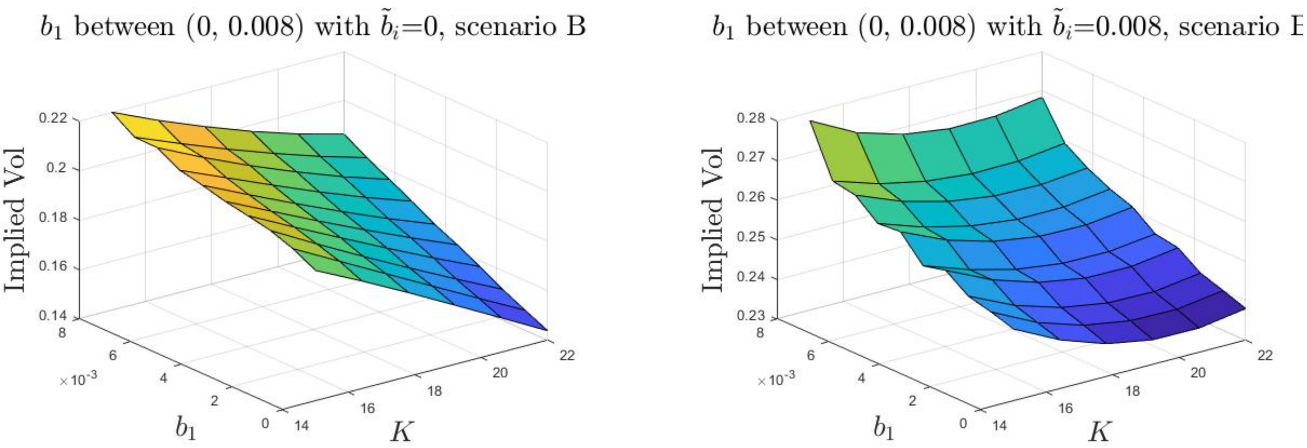


(a) $\tilde{b}_i$=0, b_1 between (0, 0.008) **(b)** $\tilde{b}_i$=0.008, b_1 between (0, 0.008)

Figure 4.14: Impact of b_1 (common factor, 3/2 component) on implied volatility. Scenario **A**

In scenario **B**, similar to what we find in the mean reverting model, it can be observed that the impact of intrinsic factor's 3/2 component on implied volatility surface is more significant than that of Scenario **A** (Figure 4.15 versus 4.14). Further, when there is no 3/2 component in the intrinsic factor, a 27.6% (from 0.145 to 0.185) increase in implied volatility can be seen. When there is a 3/2 component in the intrinsic factor, only an increase of 12.9% (from 0.239 to 0.27) is observed, nevertheless the presence of $\tilde{b}_i$ substantially shifts the implied volatility surface to a higher level (comparing Figure 4.15a to 4.15b). It is worth mentioning that the joint effect of the 3/2 component from the common factor and the intrinsic factor can be as large as 86.2% (from 0.145 to 0.27).

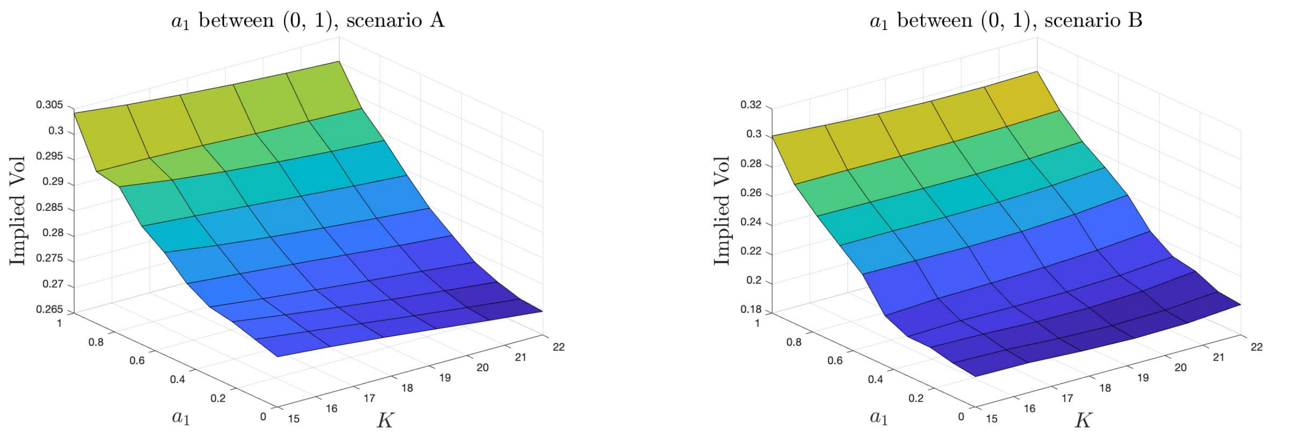
Moreover, by observing the impact of the 3/2 component under a non mean reverting model to a mean reverting model, the shape and the relative change on the implied volatility surfaces can be compared. In general, the mean reverting model leads to higher surfaces in all scenarios. That is, if one ever ignored the mean reverting property in modelling assets, it may lead to underestimating the associated implied volatility.



(a) $\tilde{b}_i$=0, b_1 between (0, 0.008) **(b)** $\tilde{b}_i$=0.008, b_1 between (0, 0.008)

Figure 4.15: Impact of b_1 (common factor, 3/2 component) on implied volatility. Scenario **B**

The study of the weight on the common factor a under a non mean reverting model is present in Figure 4.16a and Figure 4.16b. Specifically, the change in implied volatility can increase up to 11% (from 0.27 to 0.3) in scenario **A** and up to 50% (from 0.2 to 0.3) in scenario **B**. Hence, the common factor loading a per se and the stochastic covariation among assets have a vital influence on implied volatility.



(a) a_1 between (0, 1). Scenario **A** **(b)** a_1 between (0, 1). Scenario **B**

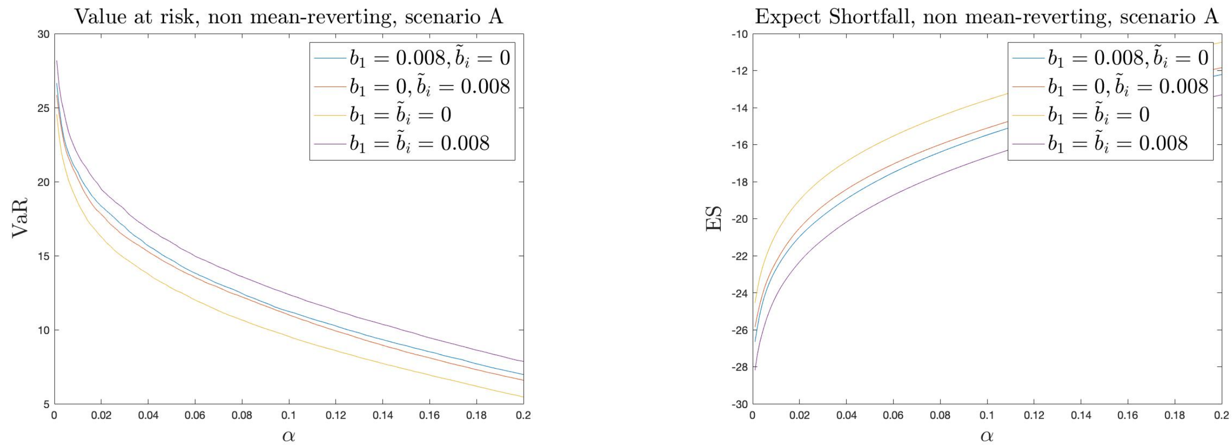
Figure 4.16: Impact of commonality (a_1) on implied volatility

In short, there are similarities in the sensitivity analysis for the mean reverting and non mean reverting model: firstly, the joint impacts from the 3/2 component b, $\tilde{b}$ and commonality loading a are non-negligible and substantial; secondly, these two types of the model indicate that a large impact comes from variation in the parameters in the underlying CIR process. Further, if we wrongly model mean-reverting assets as non mean reverting, there would be a large difference in implied volatility. More importantly, there is a profound difference in the interpretation between the mean reverting model and the non mean reverting model. Thus, it is critical to choose or develop a model that captured the stylized facts of a certain asset class.

4.2.3 Risk Measures

This section investigates the impact of b and a on risk measures of value at risk (VaR) and expected shortfall (ES) for the non mean reverting model. As per the mean reverting setting before, the risk measures are performed on the profit or loss of the portfolio with equal weights, such that $X(T) = \omega_1(X_1(T) - X_1(0)) + \omega_2(X_2(T) - X_2(0))$ where $X_1(0) = \$18$, $X_2(0) = \$100$ and the initial budget is \$59. In addition, the risk measures are produced by letting α vary from 0.001 to 0.2 with a discretization size of 200.

Firstly, we study the impact of b_1 and $\tilde{b}_i$ on VaR and ES for a fixed $\alpha = 0.01$. For scenario **A**, Figures 4.17a and 4.17b display a notable increase in VaR, from \$18 (all b set to zero) to \$22.5 (all b set to 0.008), this is a 25% increase ($\alpha = 0.01$) due to the presence of b. That is, an investor would have to place 25% more capital aside in the presence of 3/2 components. Similarly ES increases from -\$24 in the presence of 3/2 components to -\$21 in the absence of it, this is a 14.3% increase in the average/conditional VaR.



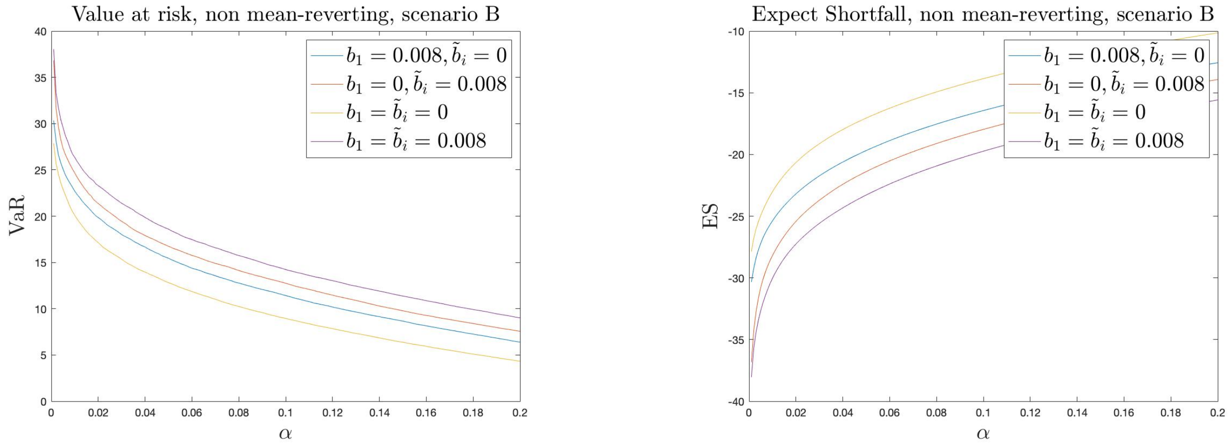
(a) Value at Risk vs. α, various b

(b) Expected Shortfall vs. α, various b

Figure 4.17: Impact of 3/2 components (b) on Risk measures. Scenario **A**

For scenario **B**, Figure 4.18a and Figure 4.18b demonstrate an increase in VaR, from \$17.5 (all b set to zero) to \$23 (all b set to 0.008), this is a 31.4% increase ($\alpha = 0.01$) due to the presence of b. Equivalently, 31.4% more capital is required in the presence of 3/2 components. Similarly ES increases from -\$30 with the 3/2 components to -\$22.5 in the absence of it, this is a 33% increase in the average VaR.

Combining these two scenarios, the 3/2 component b exhibits a greater impact in scenario **B** than in scenario **A**. That is, similar to mean reverting model, with different underlying processes for common and intrinsic factors, the influence from the 3/2 component b can be different and significant.

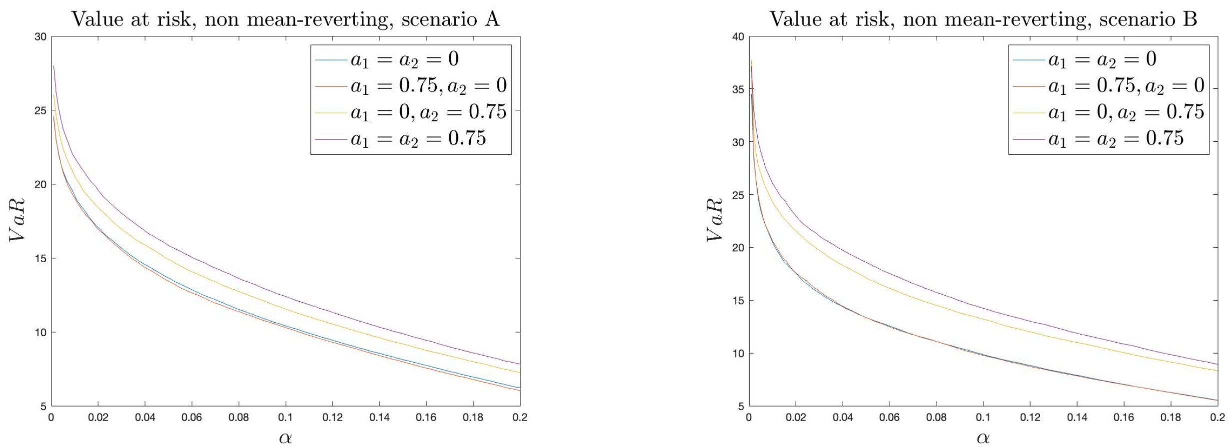


(a) Value at Risk vs. α, various b

(b) Expected Shortfall vs. α, various b

Figure 4.18: Impact of 3/2 components (b) on Risk measures. Scenario **B**

A similar analysis is performed with respect to the commonality loading a, which evaluates the impact of stochastic correlation among assets. Figure 4.19a demonstrates an increase in VaR, from 16 to 18, this is a 12.5% increase ($\alpha = 0.01$) due to elaborate the stochastic correlation (when $a_i \neq 0$). At the same time, 4.19b indicates a jump from 17 to 22.5, which results in a 32.3% increase in the value at risk at a confidence level of 99%. Thus, ignoring the stochastic correlation between assets can make a big difference in estimating risk under both the non mean reverting and the mean reverting model. Also, the significance of the impact depends on the underlying CIR process.



(a) Value at Risk vs. α, various a. Scenario **A**

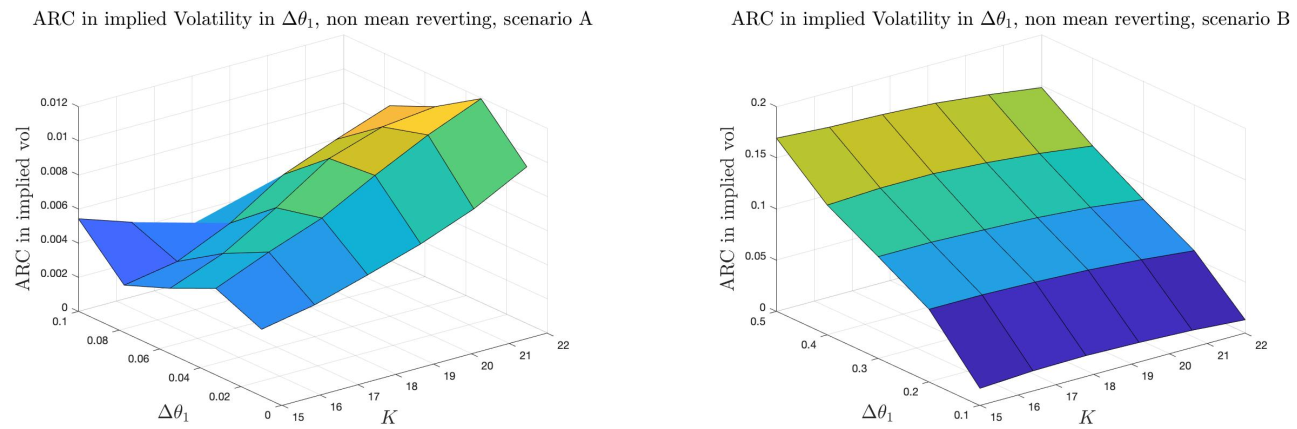
(b) Value at Risk vs. α, various a. Scenario **B**

Figure 4.19: Impact of commonality (a) on Value at Risk.

4.2.4 Alternative Market Price of Risk

Here we study the impact of different forms of market price of risk on implied volatility. We use the absolute value of relative changes (ARC) in implied volatility associated with the variation in the underlying process's parameters to measure such impact.

Under scenario **A**, the mean reverting level of volatility v_1 is 0.01, thus, the ARC in 4.20 explores values of $\Delta\theta_1$ from 0.01 to 0.1; similarly, the mean reverting level of v_1 is 0.1633 under scenario **B**, thus a range of 0.1 to 0.5 for the variation in the underlying process's mean reverting level is considered in Figure 4.20b.

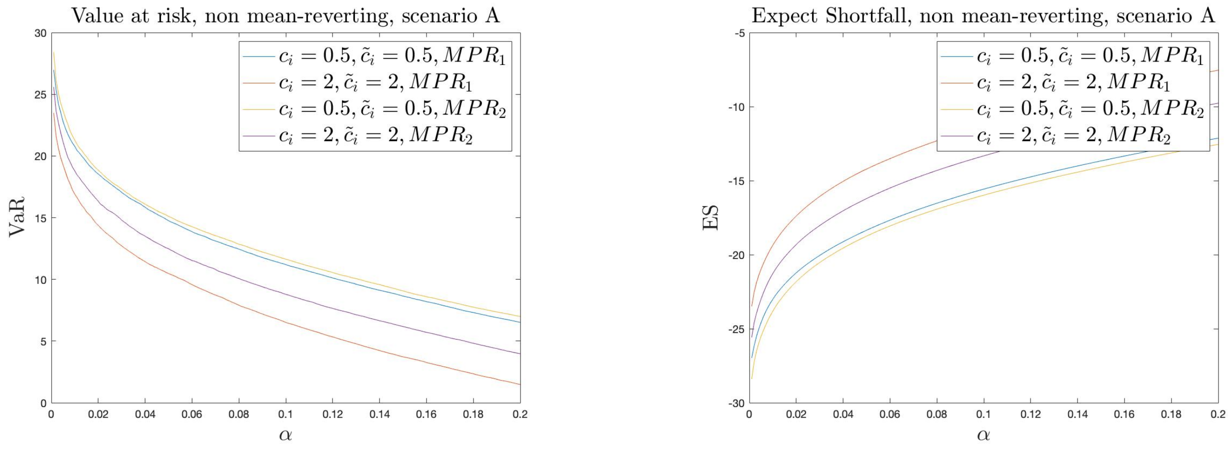


(a) ARC in non mean reverting model. Scenario A

(b) ARC in non mean reverting model. Scenario B

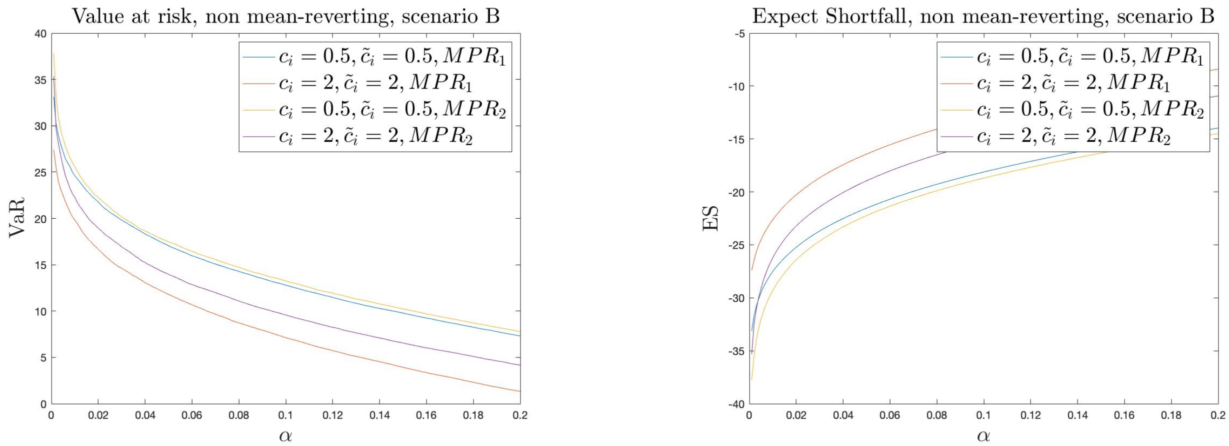
Figure 4.20: ARC in non mean reverting model. Scenario **A** and **B**

It can be seen from Figure 4.20 that the absolute relative changes in implied volatility exhibit very diverse shapes for different market prices of risk. For scenario **A**, the change in implied volatility due to the variation in θ_1 is as small as 1%, and it increases with the strike price as well. On the contrary, for scenario **B**, the absolute relative change in implied volatility due to θ_1 is substantial, up to 16%. However, it does not vary with the strike price.



(a) Value at Risk vs. α, various c **(b)** Expected Shortfall vs. α, various c

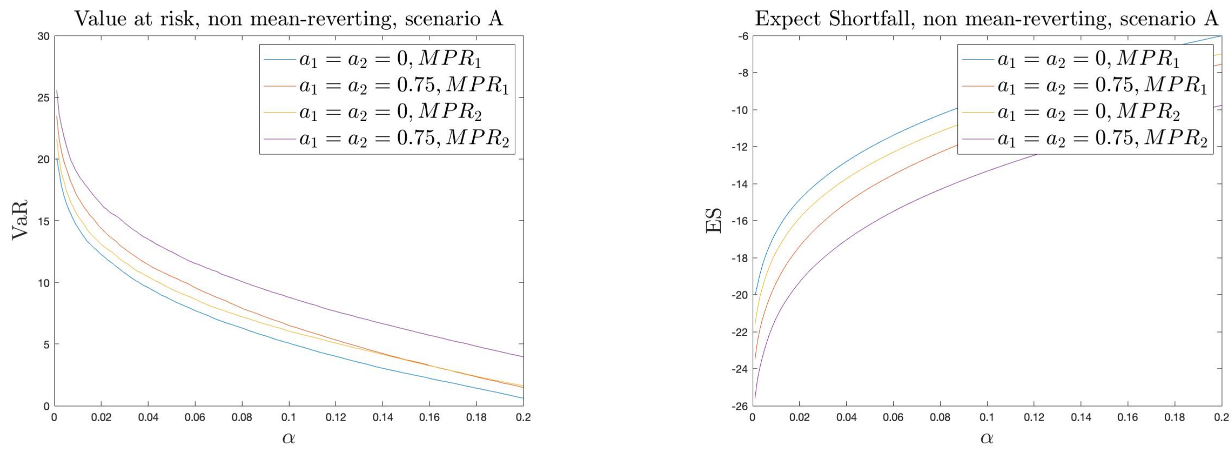
Figure 4.21: Impact of risk premium on Risk measures, non mean reverting model. Scenario **A**



(a) Value at Risk vs. α, various c **(b)** Expected Shortfall vs. α, various c

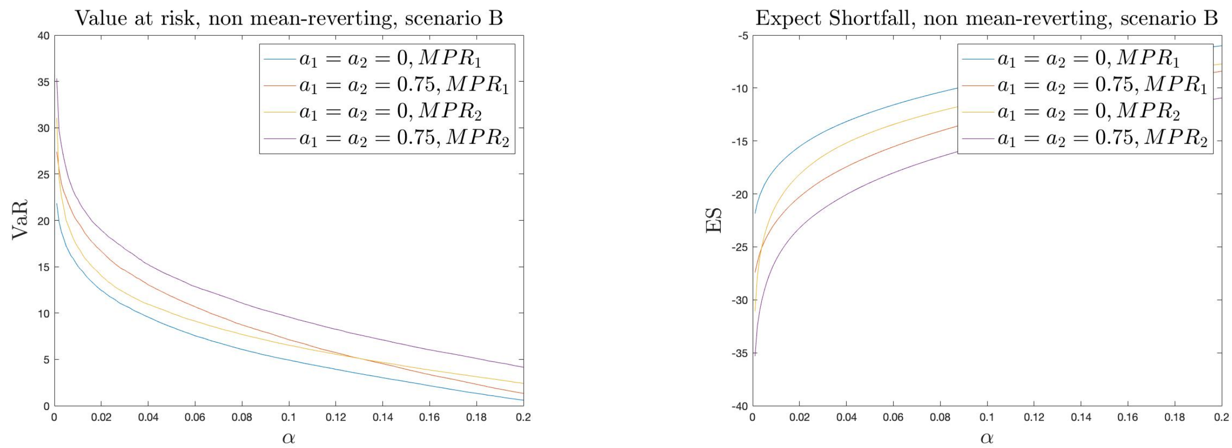
Figure 4.22: Impact of risk premium on Risk measures, non mean reverting model. Scenario **B**

In terms of risk measures, by looking at the two scenarios in Figure 4.21 and Figure 4.22, it is easy to find that the market price of risk MPR_2 always leads to larger VaR and ES for the portfolio. In particular, the higher the driver of risk premium c_i or $\tilde{c}_i$, the larger the difference between these two forms of market price of risk. For instance, when $c_i = \tilde{c}_i = 2$, and $\alpha = 0.01$, there is a 12.5% (from \$16 to \$18) difference in scenario **A** with respect to VaR caused by the choice of market price of risk, while a 18.4% (from \$19 to \$22.5) difference in scenario **B** can be observed. At the same time, there is a 10.5% (from \$-19 to \$-21) difference in scenario **A** with respect to ES caused by choice of market price of risk, while there is a 13.6% (from \$-22 to \$-25) difference in scenario **B**. Hence, the choice of market price of risk can make a difference in relevant risk measures, especially as the driver of risk premium is large.



(a) Value at Risk vs. α, various a **(b)** Expected Shortfall vs. α, various a

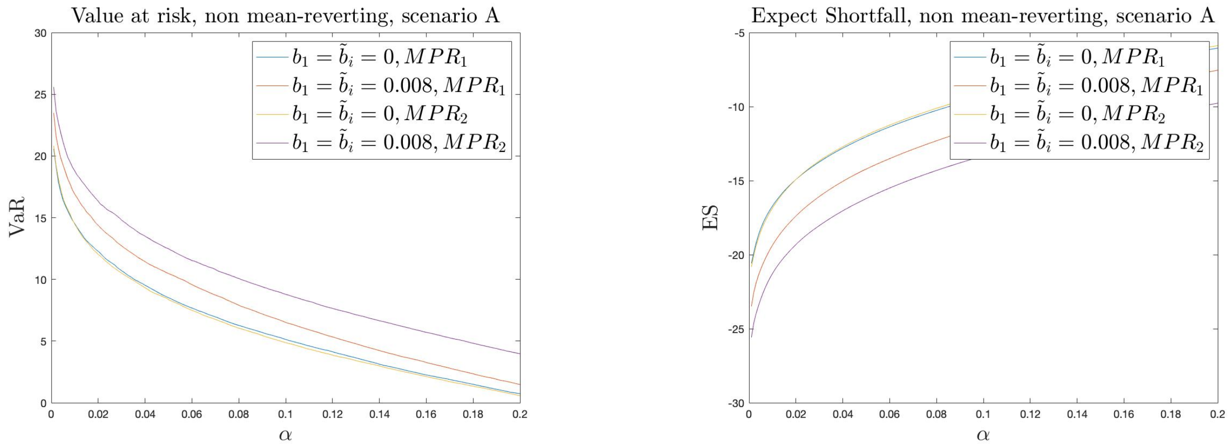
Figure 4.23: Impact of a on Risk measures, non mean reverting model. Scenario **A**



(a) Value at Risk vs. α, various a **(b)** Expected Shortfall vs. α, various a

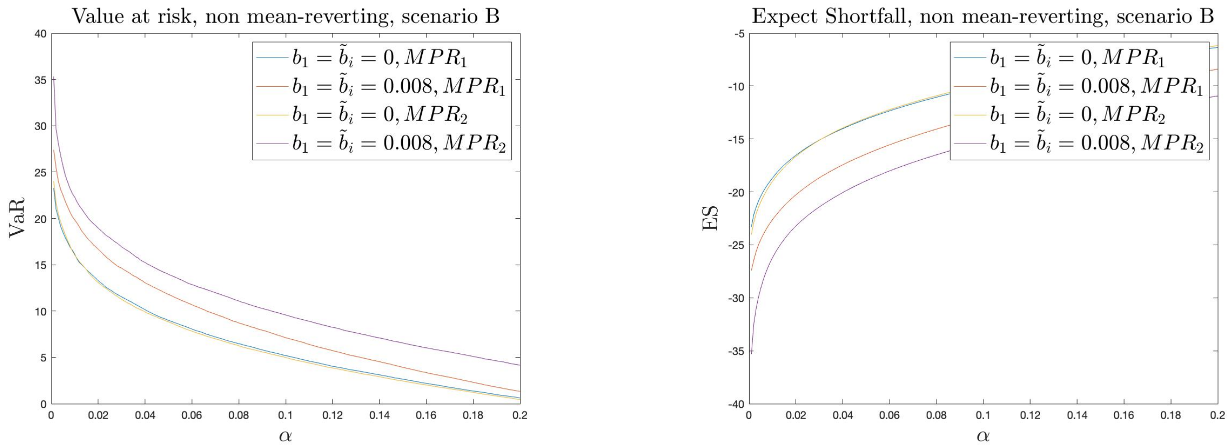
Figure 4.24: Impact of a on Risk measures, non mean reverting model. Scenario **B**

Figure 4.23 and Figure 4.24 present impact of the common factor loading a with different forms of market price of risk on risk measures. When there is no correlation between assets ($a_i = 0$), it reveals smaller VaR and ES than stochastic covariance case, i.e., $a_i = 0.75$, with both forms of market price of risk. For example, with MPR_1, there is an increase of 21.4% (from \$14 to \$17) and 26.7% (from \$15 to \$19) in value at risk for scenario **A** and **B** by taking stochastic covariance into account. Also, an increase of 15.6% (from -\$16 to -\$18.5) and 25.7% (from -\$17.5 to -\$22) in ES can be seen for scenario **A** and **B** respectively. On the other hand, with MPR_2, there is an increase of 20% (from \$15 to \$18) and 31.3% (from \$16 to \$21) in value at risk for scenario **A** and **B** after consideration of stochastic covariance. Likewise, there is an increase of 16.7% (from -\$18 to -\$21) and 23.8% (from -\$21 to -\$26) in ES for scenario **A** and **B** respectively. That is, MPR_2 shows higher impact on risk compared to MPR_1. In addition, both forms of market price of risk demonstrate that ignoring the stochastic covariance among assets will result in mistakenly underestimating the value at risk and the expected loss.



(a) Value at Risk vs. α, various b **(b)** Expected Shortfall vs. α, various b

Figure 4.25: Impact of b on Risk measures, non mean reverting model. Scenario **A**



(a) Value at Risk vs. α, various b **(b)** Expected Shortfall vs. α, various b

Figure 4.26: Impact of b on Risk measures, non mean reverting model. Scenario **B**

Figure 4.25 and Figure 4.26 present the impact of the 3/2 component b with different forms of market price of risk in VaR and ES. It is easy to see that when there is no 3/2 component, the risk measures behave similarly under different forms of market price of risk in scenario **A** and **B**. However, under the presence of the 3/2 component in risk factors, there is a difference of 8.6% (from \$17.5 to \$19) in value at risk and a difference of 15.8% (from -\$19 to -\$22) in the expected shortfall in scenario **A**. Meanwhile, there is a difference of 22.2% (from \$18 to \$22) in value at risk and a difference of 16% (from -\$22.5 to -\$26) in the expected shortfall in scenario **B**. Thus, it can be concluded that different forms of market price of risk combined with the 3/2 component cause larger variations in risk measures. Also, the MPR_2 exhibits more sensitive to risk than the other in the presence of the 3/2 component b.

5 Two Common Factors Model

From last chapter, we assume one common factor with loading a to be 0.75, which means 75% systematic variance can be captured by the single common factor. Equivalently, there still a 25% systematic variance cannot be interpreted by that common factor. It is intuitive and reasonable as the systematic variance in the financial market is almost impossible to be captured by only one variance factor. Also, according to Heston et al. (2009), a stochastic volatility model with two factors is likely to capture most variation in data and it is a better representation of the data than a single factor model. Therefore, it motivates us to consider a two common factor model in this chapter. Assume two assets are considered, i.e. $X_1(t)$ and $X_2(t)$, for $i = 1, 2$. We further denote $Y_1(t) = lnX_1(t)$, and $Y_2(t) = lnX_2(t)$. Assume there are two common variance drivers, i.e., $v_1(t)$ and $v_2(t)$ for $j = 1, 2$, and one intrinsic variance driver for each asset, i.e., $\tilde{v}_1(t)$ and $\tilde{v}_2(t)$.

Further, in this chapter, the model is generalized in incorporating the spillover effect. Specifically, we investigate the 4/2 generalized model with two common factors under a mean reverting model and a non mean reverting model respectively. For each model, we examine the impact of spillover effects and the dependency between common factors/assets in the presence and absence of the 3/2 component b with respect to implied volatility surface and important risk measures respectively. Note that we compare results within section, but not among sections.

Our model follows

$$dY_1(t) = \begin{pmatrix} L_1 + (c_1 - \frac{1}{2})\left[a_{11}^2\left(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}}\right)^2 + a_{12}^2\left(\sqrt{v_2(t)} + \frac{b_2}{\sqrt{v_2(t)}}\right)^2\right] - \beta_{11}Y_1(t) \\ \\ -\beta_{12}Y_2(t) + (\tilde{c}_1 - \frac{1}{2})\left(\sqrt{\tilde{v}_1(t)} + \frac{\tilde{b}_1}{\sqrt{\tilde{v}_1(t)}}\right)^2 \end{pmatrix} dt$$
$$+a_{11}\left(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}}\right)dW_1(t) + a_{12}\left(\sqrt{v_2(t)} + \frac{b_2}{\sqrt{v_2(t)}}\right)dW_2(t) + \left(\sqrt{\tilde{v}_1(t)} + \frac{\tilde{b}_1}{\sqrt{\tilde{v}_1(t)}}\right)d\widetilde{W}_1(t)$$

$$dY_2(t) = \begin{pmatrix} L_2 + (c_2 - \frac{1}{2})\left[a_{21}^2\left(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}}\right)^2 + a_{22}^2\left(\sqrt{v_2(t)} + \frac{b_2}{\sqrt{v_2(t)}}\right)^2\right] - \beta_{21}Y_1(t) \\ \\ -\beta_{22}Y_2(t) + (\tilde{c}_2 - \frac{1}{2})\left(\sqrt{\tilde{v}_2(t)} + \frac{\tilde{b}_2}{\sqrt{\tilde{v}_2(t)}}\right)^2 \end{pmatrix} dt$$
$$+a_{21}\left(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}}\right)dW_1(t) + a_{22}\left(\sqrt{v_2(t)} + \frac{b_2}{\sqrt{v_2(t)}}\right)dW_2(t) + \left(\sqrt{\tilde{v}_2(t)} + \frac{\tilde{b}_2}{\sqrt{\tilde{v}_2(t)}}\right)d\widetilde{W}_2(t)$$

with two common factor drivers v_1, v_2, and one intrinsic factor driver $\tilde{v}_1$, $\tilde{v}_2$ for each asset, that evolve according to:

$$
\begin{aligned}
&dv_1(t) = \alpha_1(\theta_1 - v_1(t))dt + \xi_1\sqrt{v_1(t)}dB_1(t), dv_2(t) = \alpha_2(\theta_2 - v_2(t))dt + \xi_2\sqrt{v_2(t)}dB_2(t);\\
&d\tilde{v}_1(t) = \tilde{\alpha}_1(\tilde{\theta}_1 - \tilde{v}_1(t))dt + \tilde{\xi}_1\sqrt{\tilde{v}_1(t)}d\widetilde{B}_1(t), d\tilde{v}_2(t) = \tilde{\alpha}_2(\tilde{\theta}_2 - \tilde{v}_2(t))dt + \tilde{\xi}_2\sqrt{\tilde{v}_2(t)}d\widetilde{B}_2(t);\\
&\langle dB_j(t), dW_j(t)\rangle = \rho_j dt, \left\langle d\widetilde{B}_i(t), d\widetilde{W}_i(t)\right\rangle = \tilde{\rho}_i dt, for\ j = 1,2; i = 1,2.
\end{aligned}
$$

For the purpose of demonstrating a few theoretical features of covariance of asset returns, we denote $V_j = \left(\sqrt{v_j} + \frac{b_j}{\sqrt{v_j}}\right)^2$ for $j = 1,2$ and $\tilde{V}_i = \left(\sqrt{\tilde{v}_i} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i}}\right)^2$ for $i = 1,2$. Then, for asset X_p and X_q, the instantaneous quadratic variation between assets is given by

$$
\Sigma_{pq}dt = \left\langle \frac{dX_p}{X_p}, \frac{dX_q}{X_q}\right\rangle = \left(a_{p1}a_{q1}V_1(t) + a_{p2}a_{q2}V_2(t)\right)dt \tag{5.1}
$$

The instantaneous quadratic variation of asset X_i is given by

$$
\Sigma_{pp}dt = \left\langle \frac{dX_p}{X_p}, \frac{dX_p}{X_p}\right\rangle = \left(a_{p1}^2V_1(t) + a_{p2}^2V_2(t) + \tilde{V}_p(t)\right)dt \tag{5.2}
$$

Applying Ito's lemma, we have

$$
\begin{aligned}
d\Sigma_{pp} &= d\left(a_{p1}^2\big(\sqrt{v_1} + \frac{b_1}{\sqrt{v_1}}\big)^2 + a_{p2}^2\big(\sqrt{v_2} + \frac{b_2}{\sqrt{v_2}}\big)^2 + \big(\sqrt{\tilde{v}_p} + \frac{\tilde{b}_p}{\sqrt{\tilde{v}_p}}\big)^2\right)\\
&= \left\{a_{p1}^2\Big[\frac{v_1^2 - b_1^2}{v_1^2}\alpha_1\big(\theta_1 - v_1\big) + \frac{b_1^2}{v_1^3}\xi_1^2\Big] + a_{p2}^2\Big[\frac{v_2^2 - b_2^2}{v_2^2}\alpha_2\big(\theta_2 - v_2\big) + \frac{b_2^2}{v_2^3}\xi_2^2\Big]\right.\\
&\left. + \Big[\frac{\tilde{v}_p^2 - \tilde{b}_p^2}{\tilde{v}_p^2}\tilde{\alpha}_p\big(\tilde{\theta}_p - \tilde{v}_p\big) + \frac{\tilde{b}_p^2}{\tilde{v}_p^3}\tilde{\xi}_p^2\Big]\right\}dt\\
&+ a_{p1}^2\Big[\frac{v_1^2 - b_1^2}{v_1^2}\Big]\xi_1 dB_1 + a_{p2}^2\Big[\frac{v_2^2 - b_2^2}{v_2^2}\Big]\xi_2 dB_2 + \Big[\frac{\tilde{v}_p^2 - \tilde{b}_p^2}{\tilde{v}_p^2}\Big]\tilde{\xi}_p d\tilde{B}_p
\end{aligned} \tag{5.3}
$$

Therefore, the correlation process between the assets X_1 and X_2 then follows

$$
\begin{aligned}
\rho &= \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\\
&= \frac{a_{11}a_{21}V_1(t) + a_{12}a_{22}V_2(t)}{\sqrt{\left(a_{11}^2V_1(t) + a_{12}^2V_2(t) + \tilde{V}_1(t)\right)\left(a_{21}^2V_1(t) + a_{22}^2V_2(t) + \tilde{V}_2(t)\right)}}
\end{aligned} \tag{5.4}
$$

Further, the leverage effect is defined as

$$\text{leverage} = \mathbf{Corr}\left(dlnX_p, < dlnX_p >\right) = \mathbf{Corr}\left(\frac{dX_p}{X_p}, d\Sigma_{pp}\right)$$
$$= \frac{a_{p1}^3\left(\sqrt{v_1}+\frac{b_1}{\sqrt{v_1}}\right)\left(\frac{v_1^2-b_1^2}{v_1^2}\right)\rho_1 + a_{p2}^3\left(\sqrt{v_2}+\frac{b_2}{\sqrt{v_2}}\right)\left(\frac{v_2^2-b_2^2}{v_2^2}\right)\rho_2 + \left(\sqrt{\tilde{v}_p}+\frac{\tilde{b}_p}{\sqrt{\tilde{v}_p}}\right)\left(\frac{\tilde{v}_p^2-\tilde{b}_p^2}{\tilde{v}_p^2}\right)\tilde{\rho}_p}{\sqrt{\left(a_{p1}^2\left(\sqrt{v_1}+\frac{b_1}{\sqrt{v_1}}\right)^2 + a_{p2}^2\left(\sqrt{v_2}+\frac{b_2}{\sqrt{v_2}}\right)^2 + \left(\sqrt{\tilde{v}_p}+\frac{\tilde{b}_p}{\sqrt{\tilde{v}_p}}\right)^2\right)\left(a_{p1}^4\left(\frac{v_1^2-b_1^2}{v_1^2}\right)^2 + a_{p2}^4\left(\frac{v_2^2-b_2^2}{v_2^2}\right)^2 + \left(\frac{\tilde{v}_p^2-\tilde{b}_p^2}{\tilde{v}_p^2}\right)^2\right)}} \tag{5.5}$$

To be in line with this stylized fact, in our model, the correlations ρ and $\tilde{\rho}$ should ensure negative correlations between asset return and its variance. In other words, if the asset is more risky (more volatile), then the price of that asset would be lower.

Moreover, in a factor decomposition perspective, the commonality loading matrix A is a 2×2 orthogonal matrix in this two common factor setting. We further assume the parametrization of A in the form of

$$A = \begin{bmatrix} cos(\Theta) & -sin(\Theta) \\ sin(\Theta) & cos(\Theta) \end{bmatrix}, \tag{5.6}$$

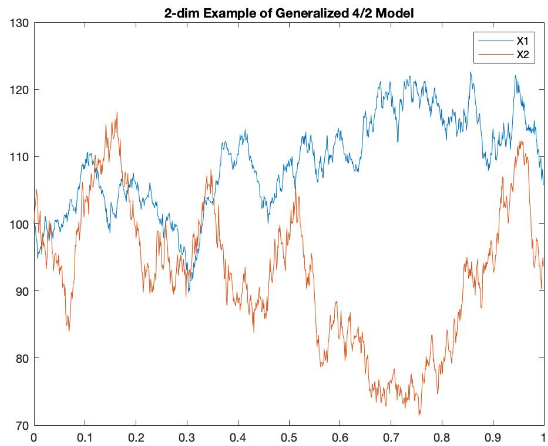
where Θ is the driver of correlation among assets, and it captures exactly the loadings.

Table 5.1 presents the baseline parameters employed in the following subsections. The parameter set is inspired by Graselli's numerical example (Grasselli, 2017). Then, by Euler approximation method, we can get a plot of the two assets X_1 and X_2 in Figure 5.1.

Table 5.1: Baseline Parameter Set for Two Common Factor Model

i	j	L	c	$\tilde{c}_i$	Θ	A
2	2	[0.02 0.02]	[0 0]	[0 0]	$\frac{\pi}{5}$	[0.8 -0.6;0.6 0.8]
b	β	Y	$v(0)$	$\tilde{v}(0)$	$\tilde{b}$	ρ
[0.008 0.005]	[0.12 -0.06; -0.1 0.06]	[log(100) log(100)]	[0.04 0.04]	[0.04 0.04]	[0.008 0.005]	[-0.7 -0.7]
α	θ	ξ	$\tilde{\alpha}$	$\tilde{\theta}$	$\tilde{\xi}$	$\tilde{\rho}$
[1.8 1.8]	[0.04 0.04]	[0.2 0.2]	[5 4]	[0.06 0.09]	[0.4 0.3]	[-0.7 -0.7]
T	n	dt				
1	50,000	0.01				

Figure 5.1: One Path of Asset Price under Two Common Factor Model



5.1 Pricing Options

In this section, we price a European call option based on our two common factors and one intrinsic factor 4/2 structured generalized factor model. Assume we take asset X_1 from our two-dimension example as the underlying asset of the European call option with risk-free interest rate 2%. Further, by letting $\beta_{12} \neq 0$, we price a European call option of asset X_1 by considering the impact of asset X_2 on X_1's long-term average return. In addition, we explore the implied volatility by plotting implied volatility surface, which is a three-dimensional plot with strike prices as X-axis, time to maturity as Y-axis, and corresponding implied volatility as Z-axis.

We assume the value of parameters follow the baseline parameter table 5.1, and the strike prices K are 90, 94, 98, 102, 106, 110 and expiry dates T are 0.2, 0.36, 0.52, 0.68, 0.84, 1.0 year. By choosing these strike prices, we take into account in-the-money, at-the-money, and also out-of-the-money given the initial asset price is 100. Then, for each strike price and each expiry date, we can get a simulated call price by Monte Carlo simulation as we described before.

5.2 Implied Volatility

Now, we can extract the implied volatility following the procedure of matching Black-Scholes option price formula with simulated call prices and then solving for the corresponding volatility, which is assumed to be constant. In the two common factor model, the method basically converts the three source of randomness explaining variance into a constant, denoted σ with a BM $W^*(t)$. To be more specific, we need to find σ such that we can match call prices on the model for X_1

with call prices for the model of Y presented next:

$$C(X_1(0), K) = e^{-rT}\left[X_1(0)^{\phi_T} exp\left\{\frac{\theta(\sigma)}{\beta_{11}}(1-\phi_T) + \frac{1}{2}a_T^2\right\}N(d+a_T) - KN(d)\right], \tag{5.7}$$

where,

$$\phi_T = e^{-\beta_{11}T}, \theta(\sigma) = L_1 - \frac{1}{2}\sigma^2, a_T = \frac{\sigma}{\sqrt{2\beta_{11}}}(1-\phi_T^2)^{\frac{1}{2}},$$
$$d = \frac{1}{a_T}(\phi_T ln(X_1(0)) - ln(K) + \frac{\theta(\sigma)}{\beta_{11}}(1-\phi_T))$$

Next, we explore whether the presence of b makes a difference by looking at the implied volatility surface for a mean reverting and a non mean reverting model respectively. Note that, except for the targeted parameter, the rest of the parameters follow the baseline setting. In addition, the expected return, the variance of return, as well as the correlation between two assets and the leverage effects are given in the table followed by each figure. There is no doubt that more reasonable key statistics can help us get a more reliable conclusion on the effects of the examined variable.

5.3 Mean Reverting

In this section, we investigate the presence and absence of the 3/2 component b on implied volatilities. For each case, the impacts of correlation Θ (thus matrix A), the spillover effect β_{ij}, and the 3/2 component b itself are explored.

5.3.1 Absence of b

In this subsection, we assume the structure of risk factors does not have the 3/2 component and follow a Heston-liked volatility process. Equivalently, when there is no presence of 3/2 parameter b

in a mean reverting model, the dynamics of our underlying asset $X_i(t)$ under measure $\mathbb{Q}$ becomes

$$
\begin{aligned}
d(lnX_1(t)) = dY_1(t) &= \left[L_1 - \frac{1}{2}\left(a_{11}^2 v_1(t) + a_{12}^2 v_2(t) + \tilde{v}_1(t)\right) - \beta_{11}Y_1(t) - \beta_{12}Y_2(t)\right]dt \\
&+ a_{11}\sqrt{v_1(t)}dW_1(t) + a_{12}\sqrt{v_2(t)}dW_2(t) + \sqrt{\tilde{v}_1(t)}d\widetilde{W}_1(t) \\
d(lnX_2(t)) = dY_2(t) &= \left[L_2 - \frac{1}{2}\left(a_{21}^2 v_1(t) + a_{22}^2 v_2(t) + \tilde{v}_2(t)\right) - \beta_{21}Y_1(t) - \beta_{22}Y_2(t)\right]dt \\
&+ a_{21}\sqrt{v_1(t)}dW_1(t) + a_{22}\sqrt{v_2(t)}dW_2(t) + \sqrt{\tilde{v}_2(t)}d\widetilde{W}_2(t) \\
dv_j(t) &= \alpha_j(\theta_j - v_j(t))dt + \xi_j\sqrt{v_j(t)}dB_j(t), j = 1, 2 \\
d\tilde{v}_i(t) &= \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t))dt + \tilde{\xi}_i\sqrt{\tilde{v}_i(t)}d\tilde{B}_i(t), i = 1, 2
\end{aligned}
$$

where $\langle dB_j(t), dW_j(t)\rangle = \rho_j dt, \left\langle d\widetilde{B}_i(t), d\widetilde{W}_i(t)\right\rangle = \tilde{\rho}_i dt$ for $j = 1, 2$; $i = 1, 2$.

Figure 5.2 displays the implied volatility surfaces with various β, and Figure 5.3 shows the implied volatility surface with various Θ's. The corresponding statistics of expected return, variance of return, the correlation between two assets, as well as the leverage effects are provided in the Table 5.2 and Table 5.3 respectively.

Figure 5.2: Two Common Factor Model, mean reverting, change β

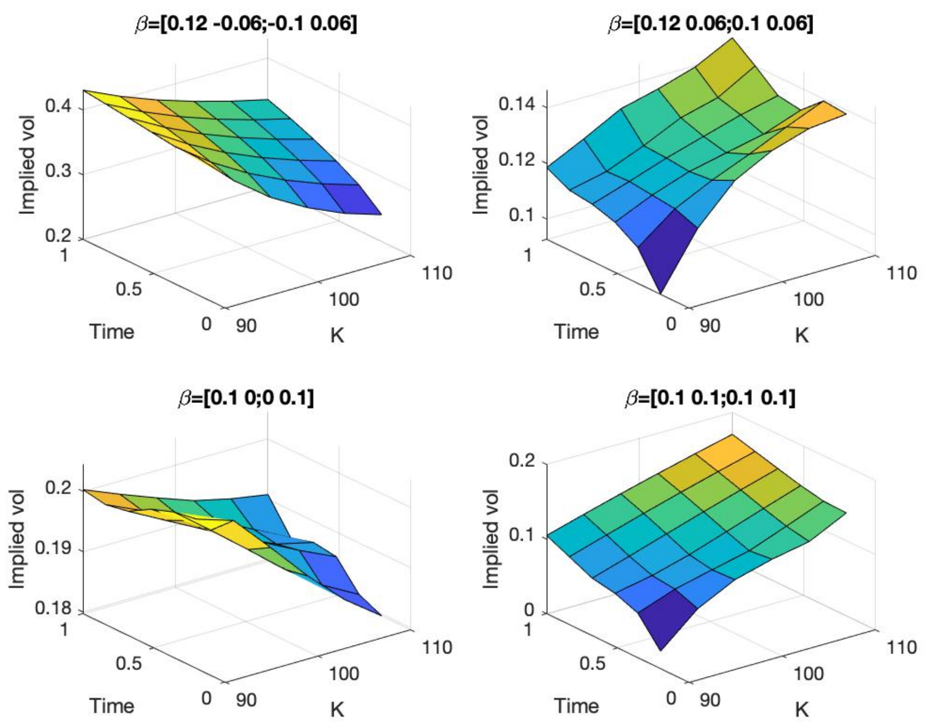


Table 5.2: Statistics for Two Common Factor Model, mean reverting, change β

β	[0.12 -0.06;-0.1 0.06]	[0.12 0.06;0.1 0.06]	[0.1 0;0 0.1]	[0.1 0.1;0.1 0.1]
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0579	0.0574	0.0581	0.0596
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.1615	0.1642	0.1563	0.1583
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.1356	0.1349	0.1353	0.1397
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.4767	0.4868	0.4576	0.4673
$\mathbf{Corr}(dlnX_1(T)\,, dlnX_2(T))$	0.0718	-0.0645	-0.0008	0.0944
$\mathbf{Corr}(dlnX_1(T)\,, < dlnX_1(T) >)$	-0.6300	-0.6221	-0.6209	-0.6169
$\mathbf{Corr}(dlnX_2(T)\,, < dlnX_2(T) >)$	-0.6151	-0.6161	-0.6138	-0.6136

In order to study the spillover effect β_{ij}, we consider the same mean reverting speed in cases 1 and 2 with negative and positive spillover effects respectively. In addition, we explore whether the existence of spillover effect has an impact on implied volatilities in cases 3 and 4 with the same mean reverting speed. In Figure 5.2, it can be seen that the spillover effect from the other asset displays a significant impact on the shape and the level of implied volatility surface. On the one hand, by comparing case 3 and case 4, the implied volatility surface changes from a decreasing shape (no spillover effect) into an increasing pattern by considering a positive spillover effect of asset X_2 on the current underlying asset X_1 of the option. However, with a spillover effect, the deviation in implied volatilities increased distinctly. To be more precise, the implied volatilities changes from 0.2 to 0.19 at maturity $T = 1$ without spillover effect, whereas it changes from 0.1 to 0.17 with spillover effect. On the other hand, by looking at case 1 and case 2, a negative influence from the other asset leads to downward implied volatilities while a positive influence from the other asset leads to increasing implied volatilities. However, negative spillover effects shows much higher implied volatilities (from 0.32 to 0.42 at maturity $T = 1$) than positive spillover effect (from 0.12 to 0.15 at maturity $T = 1$). Thereby, the impact of the spillover effect on implied volatilities is prominent and may depend on the positiveness of the spillover from the other asset.

Figure 5.3: Two Common Factor Model, mean reverting, change Θ

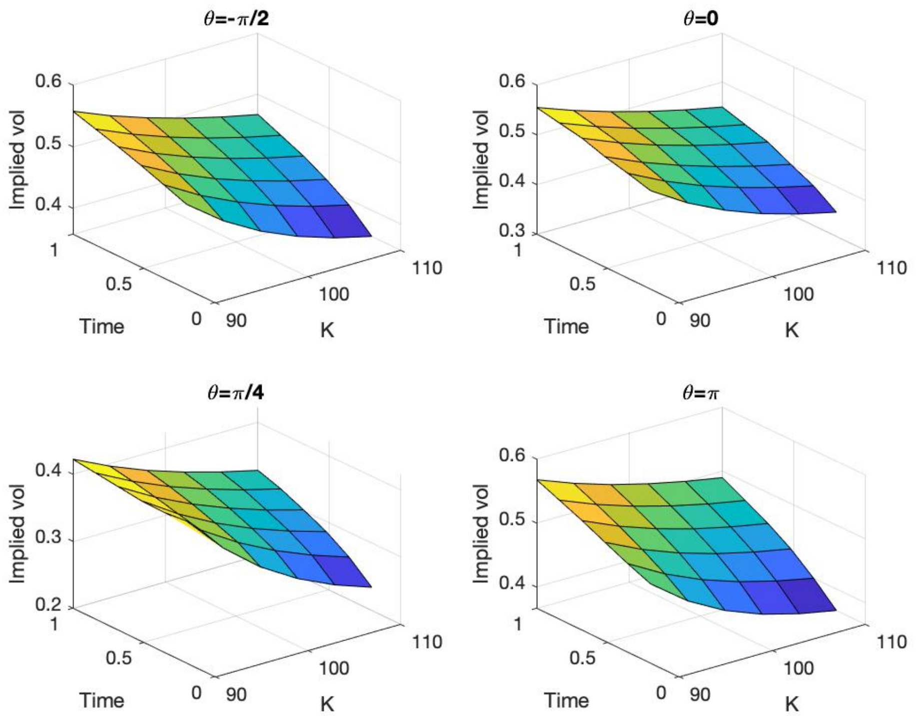


Table 5.3: Statistics for Two Common Factor Model, mean reverting, change Θ

Θ	$-\pi/2$	0	$\pi/4$	π
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0902	0.0928	0.0561	0.0911
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.1212	0.1231	0.1645	0.1209
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.2268	0.2349	0.1312	0.2305
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.3252	0.3344	0.4897	0.3296
$\mathbf{Corr}(dlnX_1(T)\,,dlnX_2(T))$	0.0741	0.0762	0.0811	0.0664
$\mathbf{Corr}(dlnX_1(T)\,,< dlnX_1(T) >)$	-0.4044	-0.6644	-0.5994	-0.1641
$\mathbf{Corr}(dlnX_2(T)\,,< dlnX_2(T) >)$	-0.4591	-0.6643	-0.5771	-0.2026

The impact from correlation Θ is explored for the cases $-\frac{\pi}{2}$, 0, $\frac{\pi}{4}$, and π respectively. From Figure 5.3, by varying the driver of correlation between assets, the level of the surface only slightly changes. In other words, the shape of the implied volatility surfaces does not change much in the correlation between assets but it changes the magnitudes of implied volatilities. Besides, as changes in Θ also variate the common factors' loading, as expected, the expected return and the variance of return can be different. Nevertheless, it can be noticed from $\Theta = 0$ and $\Theta = \pi$ that returns and variance of returns are very similar, the leverage effects change from -0.1641 to -0.6644 for asset X_1 and from -0.2026 to -0.6643 for asset X_2. Thus, the variation in the correlation between assets has a significant impact on the leverage effect.

5.3.2 Presence of b

In this subsection, the 3/2 component is considered on both common and intrinsic factors. That is, we allow all the explained factors to add a 3/2 component and thereby follow a 4/2 structured stochastic volatility process. Sensitivity analysis with regards to the 3/2 component b itself, the spillover effect β_{ij}, as well as the correlation between assets are implemented.

Figure 5.4, Figure 5.5, and Figure 5.6 display the implied volatility surface with changes in b, β, and Θ respectively. Moreover, Table 5.4, 5.5, and 5.6 report the associated statistics in expected return, variance of return, correlation, and leverage effect.

Figure 5.4: Two Common Factor Model, mean reverting, change b

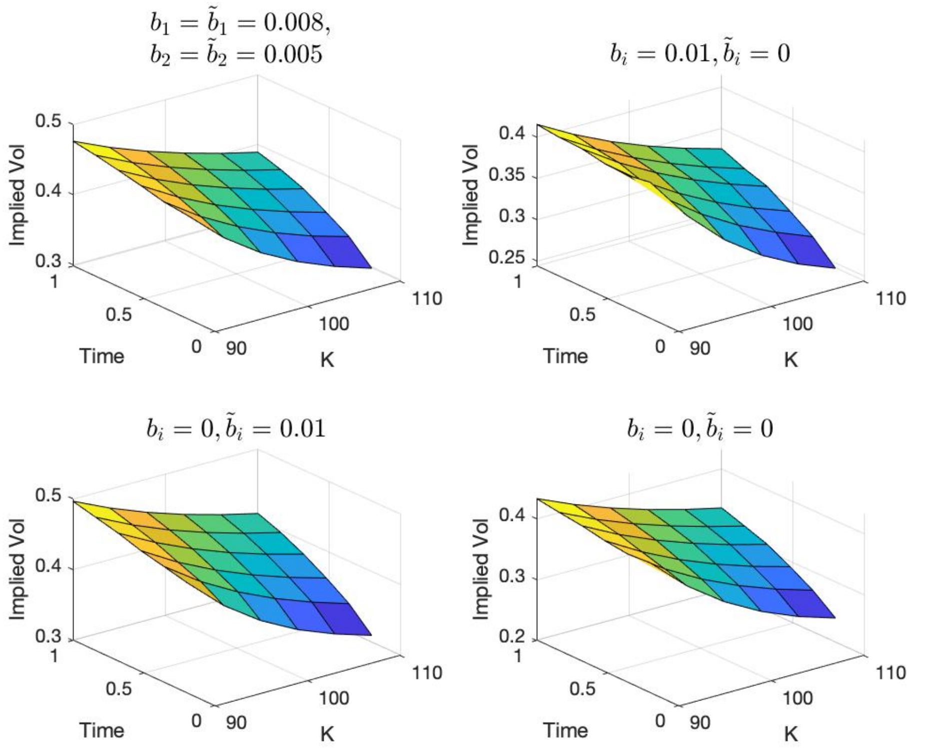


Table 5.4: Statistics for Two Common Factor Model, mean reverting, change b

	$b_1 = \tilde{b}_1 = 0.008$ $b_2 = \tilde{b}_2 = 0.005$	$b_i = 0.01$ $\tilde{b}_i = 0$	$b_i = 0$ $\tilde{b}_i = 0.01$	$b_i = 0$ $\tilde{b}_i = 0$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0740	0.0568	0.0791	0.0582
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.2057	0.2143	0.1877	0.1589
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.1789	0.1319	0.1951	0.1364
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.6646	0.7044	0.5870	0.4649
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0821	0.0814	0.0745	0.0788
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.6356	-0.6354	-0.6202	-0.6313
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.6069	-0.5839	-0.6273	-0.6156

In studying the impact from the 3/2 component b, we consider 4 cases: case 1 assumes both the common factor and the intrinsic factor follow 4/2 structure, case 2 assumes only common factors follow 4/2 structure, case 3 assumes only intrinsic factors follow 4/2 structure, and case 4 assumes all the factors to be Heston-like. Figure 5.4 reveals that the 3/2 component does not make an evident difference in shaping the implied volatility surfaces by comparing cases 1, 2, 3 to case 4. Nevertheless, by adding a 3/2 component into factor structure, both expected return and the variance from asset return can be influenced even with a very small b. For example, while $\tilde{b}_1$ changes from 0 to 0.01, the expected return of X_1 jumps up by 35.91% and its variance has an increase of 18.12%. In addition, by observing case 2 and case 3, the impact of adding the 3/2 component in the intrinsic factor is greater than adding it in common factors.

Figure 5.5: Two Common Factor Model, mean reverting with b, change β

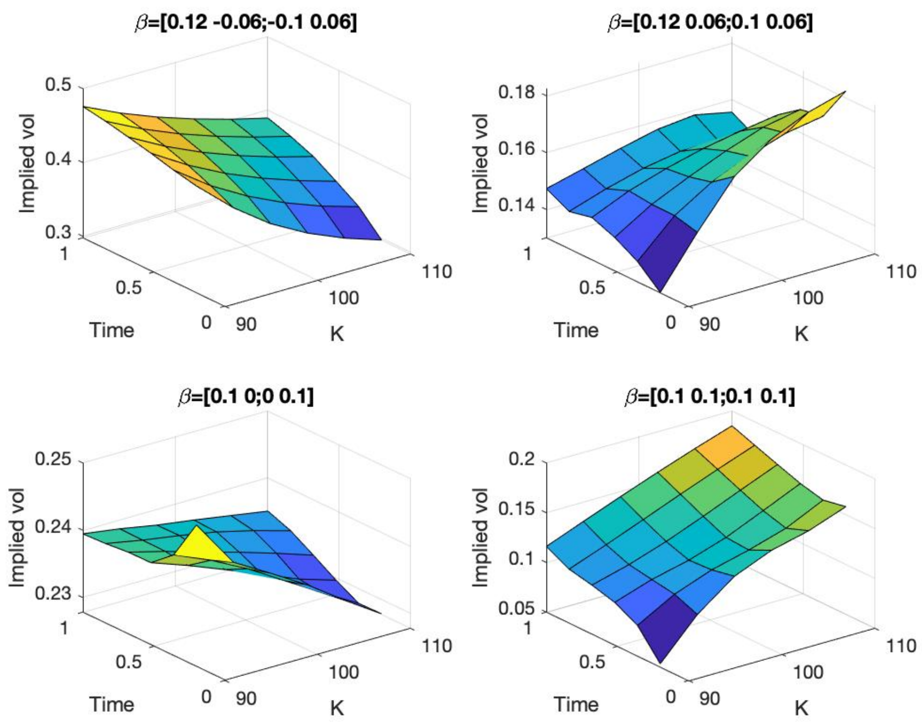


Table 5.5: Statistics for Two Common Factor Model, mean reverting, with b, change β

β	[0.12 -0.06;-0.1 0.06]	[0.12 0.06;0.1 0.06]	[0.1 0;0 0.1]	[0.1 0.1;0.1 0.1]
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0740	0.0746	0.0766	0.0781
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.2057	0.2049	0.1977	0.1932
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.1789	0.1804	0.1881	0.1909
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.6646	0.6606	0.6314	0.6062
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0821	-0.0671	0.0039	-0.0959
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.6356	-0.6173	-0.6296	-0.6186
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.6069	-0.6004	-0.6029	-0.6000

The variation in β shown in Figure 5.5 has a similar impact on implied volatility surface as that of

Figure 5.2 where there is no presence of 3/2 component b. That is, the positiveness of the spillover effect among assets has an impact on the trend of the implied volatility surfaces. Moreover, in case 3, when there is no spillover effect and all the other parameters stay the same, the 3/2 component brings an increase of 31.8% and 26.5% (from 0.0581 to 0.0766 for asset X_1 and from 0.1563 to 0.1977 for X_2) in the expected return of asset X_1 and X_2; while an 39% (from 0.1353 to 0.1881 for X_1) and 38% (from 0.4576 to 0.6314 for X_2) increase in the variance of assets' return respectively. Also, the 3/2 component b enlarges implied volatilities but a stabilization effect can be observed. For instance, when there is no spillover effect (case 3), a 11% (from 0.18 to 0.2) variation of implied volatility surface in the absence of b while a 3.9% (from 0.23 to 0.239) deviation of implied volatility surface with the 3/2 component b. When there is spillover effect (case 2), a 100% (from 0.01 to 0.14) variation of implied volatility surface without b while a 50% (from 0.12 to 0.18) deviation of implied volatility surface with b. That is, the 3/2 component b shows its capability in stabilizing the implied volatility surface.

Figure 5.6: Two Common Factor Model, mean reverting with b, change Θ

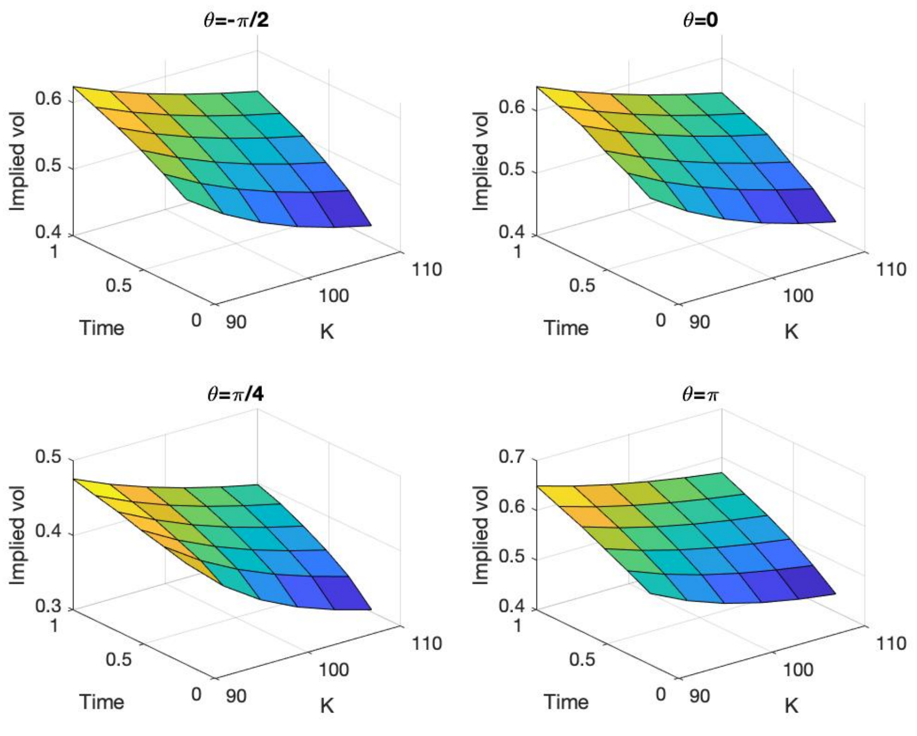


Table 5.6: Statistics for Two Common Factor Model, mean reverting, with b, change Θ

Θ	$-\pi/2$	0	$\pi/4$	π
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.1161	0.1268	0.0704	0.1247
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.1502	0.1448	0.2085	0.1429
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.3093	0.3436	0.1682	0.3358
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.4270	0.4102	0.6798	0.4070
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0713	0.0758	0.0762	0.0653
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.2664	-0.6225	-0.4740	-0.0799
$\mathbf{Corr}(lndX_2(T), < dlnX_2(T) >)$	-0.3677	-0.6421	-0.5643	-0.0505

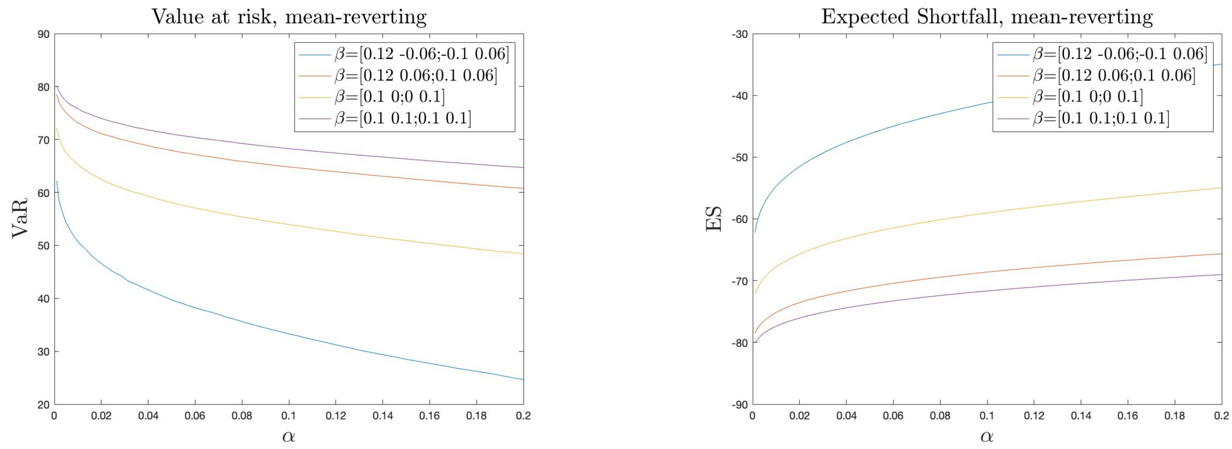
Figure 5.6 illustrates that the change in the correlation of assets does not have a strong effect in the overall shape of implied volatility surfaces. This observation is similar to what we obtained from Figure 5.3. Similarly, adding the 3/2 component, creates little variation in the implied volatility surface, although the expected return and the variance of asset return are clearly increased. Moreover, it is easy to notice that Θ can impact the leverage effects from -0.0505 (X_2 from case 4) to -0.6421 (X_2 from case 2) while not causing much change in the expected return or variance of asset's return.

In summary, the spillover effect impacts the shape of the implied volatility and it can revert the trend of the surface. The 3/2 parameter b has a great impact on the implied volatility surface as a stabilizer. Further, the driver of correlation between assets Θ does not show much influence on the implied volatility surface, but it has a strong effect on leverage effects.

5.3.3 Risk Measures

This section investigates how parameter β, Θ and 3/2 component b make a difference on important risk measures, i.e., the value at risk (VaR) and the expected shortfall (ES). The mathematically definition of VaR and ES are given in equation 4.6 and equation 4.7. Similarly, the risk measures are evaluated over the profit or loss of an equal weights portfolio $X(T) = \omega_1(X_1(T) - X_1(0)) + \omega_2(X_2(T) - X_2(0))$, where $X_1(0) = X_2(0) = \$100$, and the initial budget of this portfolio is \$100. Besides, the risk measures are examined under the probability of gain/loss, i.e., α, varying from 0.001 to 0.2.

We firstly look at the influence from β on risk measures in Figure 5.7a and Figure 5.7b. As we know from implied volatility surfaces, β's variation can bring a remarkable change in the shape of it, this great influence can also be noticed in VaR and ES. Specifically, the existence of the spillover effect has a distinct reflection on risk measures. For example, when $\alpha = 0.01$ and the mean reverting speed is 0.1, there is a 16.9% (from \$65 to \$76) increase in VaR by having a positive effect from the other asset on the drift of current asset. That is, 16.9% more capital is required for potential losses. Also, if the undesirable scenario happens, there is a 14.7% more loss in the capital on average. Furthermore, keeping the mean reverting speed the same, a positive impact from the other asset may incur in larger VaR than that in the case of negative spillover. For instance, an increase of 50% (From \$50 to \$75) in VaR from negative spillover effect ($\beta_{12} = -0.06$) to positive spillover ($\beta_{12} = 0.06$), and a 36.3% (from -\$55 to -\$75) increase in ES can be witnessed.

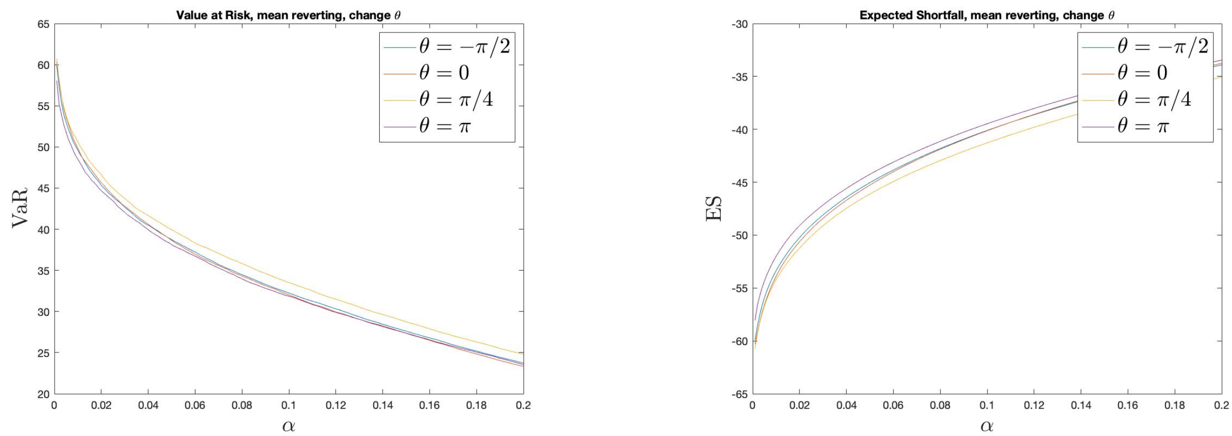


(a) Value at Risk vs. α, mean reverting, various β

(b) Expected Shortfall vs. α, mean reverting, various β

Figure 5.7: Impact of β on Risk measures

Secondly, the impact from the driver of assets' correlation Θ on risk measures are shown in Figure 5.8a and Figure 5.8b. For instance, the case of $\Theta = \pi/4$ results in larger VaR and ES compare to other cases. In other words, ignoring correlations among common factors or wrongly assigning it will possibly underestimate the risk exposures and mistakenly assess the expected losses.

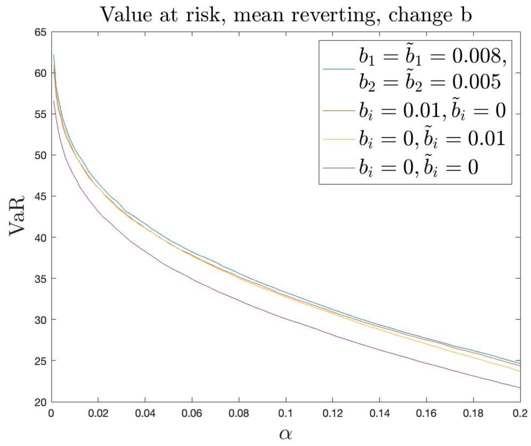


(a) Value at Risk vs. α, mean reverting, various Θ

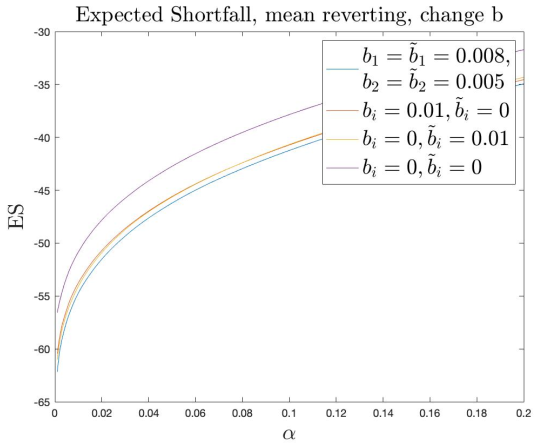
(b) Expected Shortfall vs. α, with b, various Θ

Figure 5.8: Impact of the Θ on Risk measures

Next, a similar analysis is performed with respect to the 3/2 component b in Figure 5.9a and Figure 5.9b. It is clear to realize that in the presence of the 3/2 component b, VaR and ES indicate higher risk compared to no b component. Specifically, Figure 5.9a demonstrates an 13% increase in VaR (from \$46 to \$52) at $\alpha = 0.01$ and Figure 5.9b indicates a 7.8% (from -\$51 to -\$55) increase in the ES. Therefore, the presence of the 3/2 component b has a certain impact in measuring risk.



(a) Value at Risk vs. α, mean reverting, various b



(b) Expected shortfall vs. α, mean reverting, various b

Figure 5.9: Impact of 3/2 component b on Risk measures

5.4 Non Mean Reverting

In this section, a non mean reverting model with two common factors and one intrinsic factor for each asset is studied. Specifically, the impacts from the presence of 3/2 component b, and the driver of correlation Θ among assets have been considered respectively. In addition, inspired by the one common factor example that different underlying processes (i.e., scenario **A** and scenario **B**) can exhibit different effects on implied volatility, a sensitivity analysis on parameters from underlying CIR process is included.

5.4.1 Absence of b

When we are under a non mean reverting model without presence of the $3/2$ component b, the dynamic of underlying asset $X_i(t)$ under measure $\mathbb{Q}$ becomes

$$\begin{aligned}
d(lnX_i(t)) = dY_i(t) &= \left[r - \frac{1}{2}\left(a_{i1}^2 v_1(t) + a_{i2}^2 v_2(t) + \tilde{v}_i(t)\right)\right]dt \\
&\quad + a_{i1}\sqrt{v_1(t)}dW_1(t) + a_{i2}\sqrt{v_2(t)}dW_2(t) + \sqrt{\tilde{v}_i(t)}d\widetilde{W}_i(t) \\
dv_j(t) &= \alpha_j(\theta_j - v_j(t))dt + \xi_j\sqrt{v_j(t)}dB_j(t), j = 1, 2 \\
d\tilde{v}_i(t) &= \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t))dt + \tilde{\xi}_i\sqrt{\tilde{v}_i(t)}d\widetilde{B}_i(t), i = 1, 2
\end{aligned}$$

where $\langle dB_j(t), dW_j(t)\rangle = \rho_j dt$, $\left\langle d\widetilde{B}_i(t), d\widetilde{W}_i(t)\right\rangle = \tilde{\rho}_i dt$ for $j = 1, 2$; $i = 1, 2$.

The analysis of implied volatility surfaces with respect to changes in α, θ, ξ subject to different correlation Θ is presented in Figure 5.10 and Figure 5.11. In addition, Figure 5.12 shows how Θ has an impact on the implied volatility surface. Similarly, the corresponding statistics for each figure are provided.

Figure 5.10: Two Common Factor Model Case, non mean reverting, $\Theta = \pi/5$, change α, θ, ξ

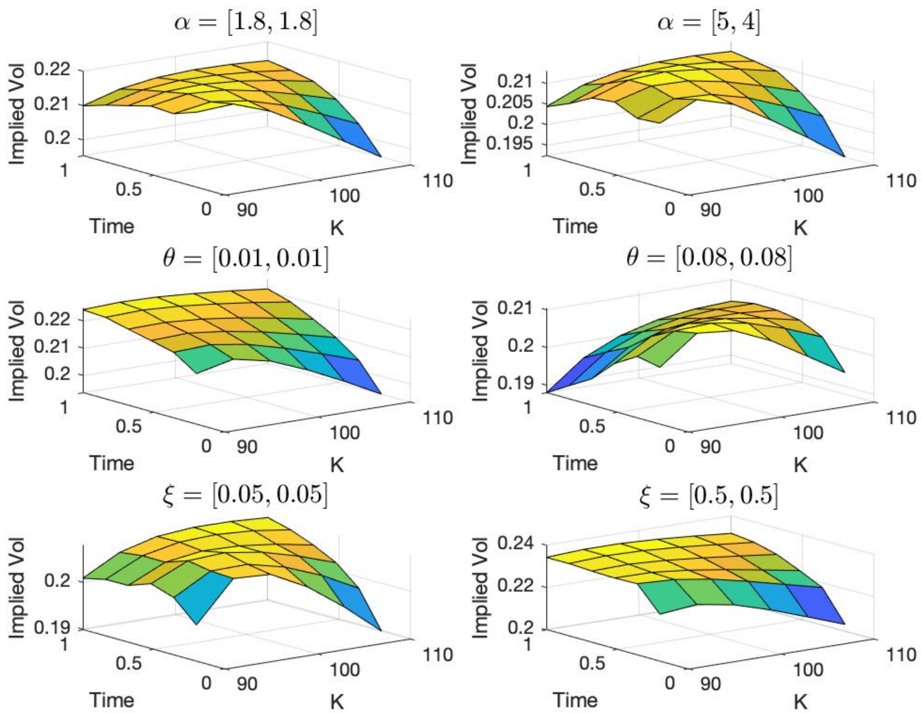


Table 5.7: Statistics for Two Common Factor Model, non mean reverting, change α, θ, ξ

	$\alpha = [1.8, 1.8]$	$\alpha = [5, 4]$	$\theta = [0.01, 0.01]$	$\theta = [0.08, 0.08]$	$\xi = [0.05, 0.05]$	$\xi = [0.5, 0.5]$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0651	0.0637	0.0651	0.0644	0.0623	0.0743
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.1720	0.1735	0.1357	0.2262	0.1738	0.1719
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.1561	0.1516	0.1557	0.1539	0.1476	0.1818
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.5283	0.5284	0.3804	0.7830	0.5247	0.5364
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0038	0.0016	0.0070	0.0064	0.0044	0.0038
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.6304	-0.6705	-0.6438	-0.6193	-0.6857	-0.5055
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.6114	-0.5941	-0.6430	-0.6085	-0.5351	-0.6004

Figure 5.10 is obtained by employing the baseline correlation $\Theta = \frac{\pi}{5}$. The impacts from the mean reverting speed of the underlying process α are explored by comparing the baseline with a higher reverting speed, i.e., $\alpha = [5, 4]$. The shape of the implied volatility does not change too much, but there is a slight decrease in the level of the implied volatility surface. Further, a small increase in the mean reverting level of the underlying process θ (from 0.01 to 0.08) is capable of turning the trend of the implied volatility from increasing in time to maturity to decreasing in it. Moreover, as we would expect, higher volatility of volatility ξ increases the implied volatility and makes it "less frown".

Figure 5.11: Two Common Factor Model, non mean reverting, $\Theta = -\pi$, change α, θ, ξ

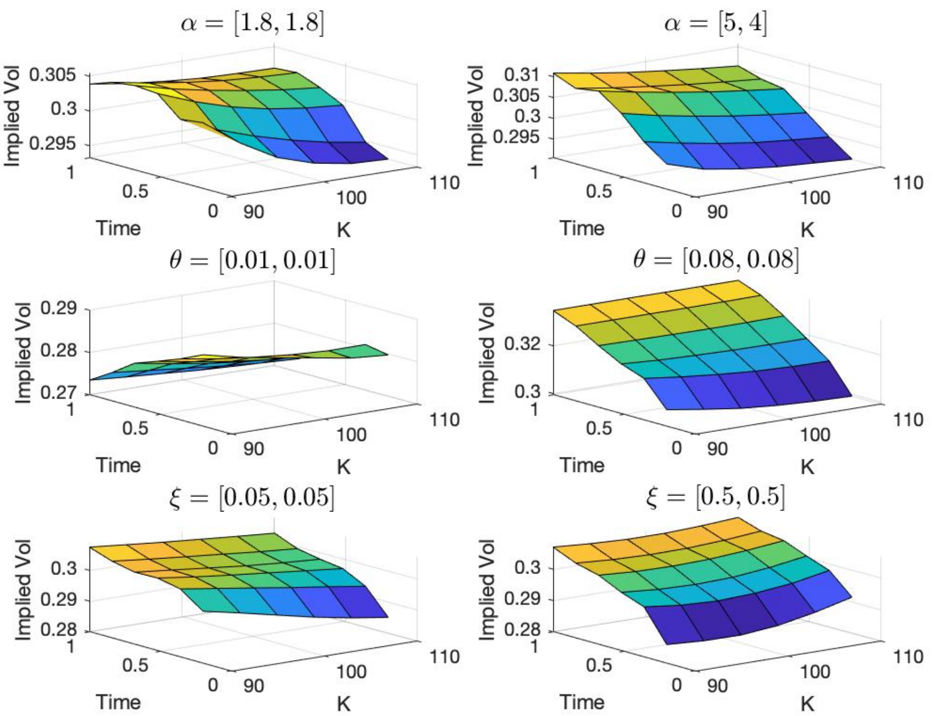


Table 5.8: Statistics for Two Common Factor Model, non mean reverting, change α, θ, ξ

	$\alpha = [1.8, 1.8]$	$\alpha = [5, 4]$	$\theta = [0.01, 0.01]$	$\theta = [0.08, 0.08]$	$\xi = [0.05, 0.05]$	$\xi = [0.5, 0.5]$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.1013	0.1039	0.0854	0.1232	0.1039	0.1027
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.1299	0.1282	0.1078	0.1497	0.1292	0.1278
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.2629	0.2688	0.2158	0.3356	0.2710	0.2763
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.3576	0.3536	0.2800	0.4352	0.3525	0.3619
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	-0.0080	-0.0057	-0.0004	-0.0018	-0.0014	0.0063
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.1660	-0.3184	-0.2908	-0.0463	-0.4374	-0.0976
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.2121	-0.3337	-0.3198	-0.0939	-0.4732	-0.0530

In Figure 5.11, we assume the correlation $\Theta = -\pi$ and similar variations on κ, θ, and ξ as that of in Figure 5.10, the changes due to mean reverting speed α does not show too much influence on the implied volatility surface. Additionally, changes in the mean reverting level θ not only have a significant impact on the level of the implied volatility surface, but also change the trend of the surface from downward sloping to an increasing sloping with regards to time to maturity. Besides, the increase in volatility of volatility changes a slightly downward implied volatility surface into a "smile" one in terms of strike price K.

By comparing the impact of κ, θ, and ξ under a different choice of correlation Θ, we observe that different driver of correlation Θ among assets produce very different shapes of implied volatility surface. In particular, the lesser changes come from the mean reverting speed α, while the mean reverting level θ is a "game-changer" in turning around the trend of the implied volatility surface. As for volatility of volatility ξ, it clearly infuses more variance in the asset and thus in implied volatility as well.

Figure 5.12: Two Common Factor Model, non mean reverting, change Θ

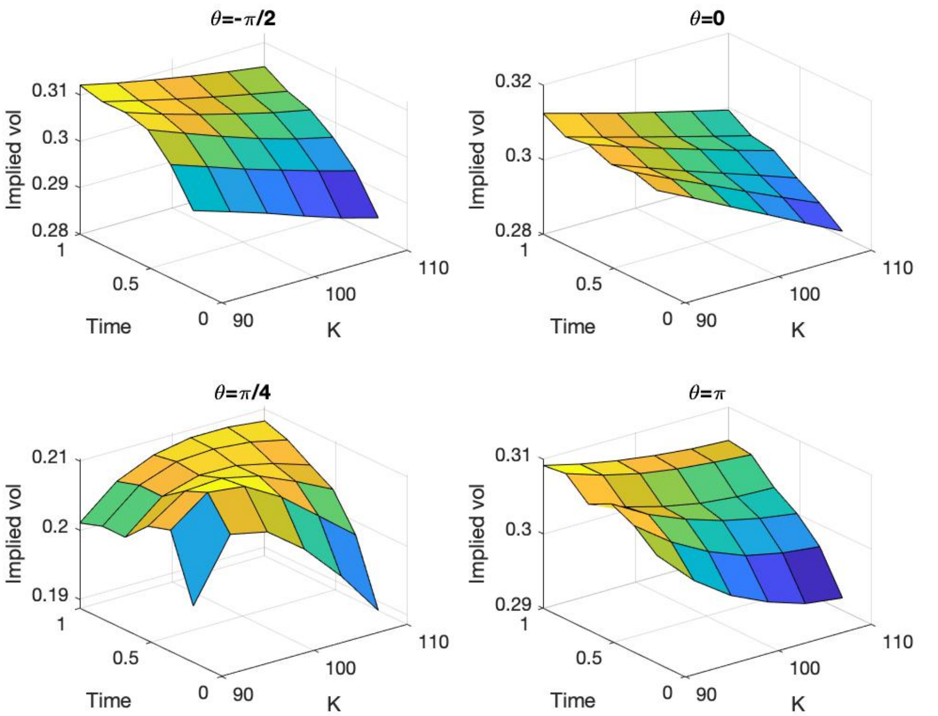


Table 5.9: Statistics for Two Common Factor Model, non mean reverting, change Θ

Θ	-$\pi/2$	0	$\pi/4$	π
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.1037	0.1059	0.0637	0.1012
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.1291	0.1303	0.1754	0.1281
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.2701	0.2811	0.1513	0.2630
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.3524	0.3592	0.5336	0.3562
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	-0.0056	-0.0029	0.0093	0.0079
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.4218	-0.6727	-0.6007	-0.1575
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.4382	-0.6654	-0.5851	-0.2137

The analysis of the impacts from changing the driver of correlation among assets Θ is displayed in Figure 5.12. To be more specific, we consider correlations of $-\frac{\pi}{2}$, 0, $\frac{\pi}{4}$, and π respectively. It can be observed that as Θ varies, the implied volatility surface appears differently in shapes and levels. In particular, there is a volatility "frown" in case 3 as $\Theta = \frac{\pi}{4}$, while a volatility "skew" can be observed in other cases. Also, when $\Theta = \frac{\pi}{4}$, its implied volatility level deviates between 0.19 and 0.21, while all the others have a deviation of 0.28 to 0.31. By looking at Table 5.9, it can be noticed that changes in Θ are powerful in controlling the leverage effects. For instance, when $\Theta = 0$, the leverage effect of asset X_1 and X_2 can be -0.6727 and -0.6654; when $\Theta = \pi$, they can end up with -0.1575 and -0.2137 respectively.

5.4.2 Presence of b

Here, we consider both the common factor and the intrinsic factor follow 4/2 structure. The dynamics of our underlying asset $X_i(t)$ is

$$
\begin{aligned}
d(lnX_i(t)) = dY_i(t) = \Bigg[r - \frac{1}{2}\Bigg(& a_{i1}^2(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}})^2 + a_{i2}^2(\sqrt{v_2(t)} + \frac{b_2}{\sqrt{v_2(t)}})^2 \\
& + (\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})^2\Bigg)\Bigg]dt + a_{i1}(\sqrt{v_1(t)} + \frac{b_1}{\sqrt{v_1(t)}})dW_1(t) \\
& + a_{i2}(\sqrt{v_2(t)} + \frac{b_2}{\sqrt{v_2(t)}})dW_2(t) + (\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}})d\widetilde{W}_i(t)
\end{aligned}
$$

Figure 5.13: Two Common Factor Model, non mean reverting, change b

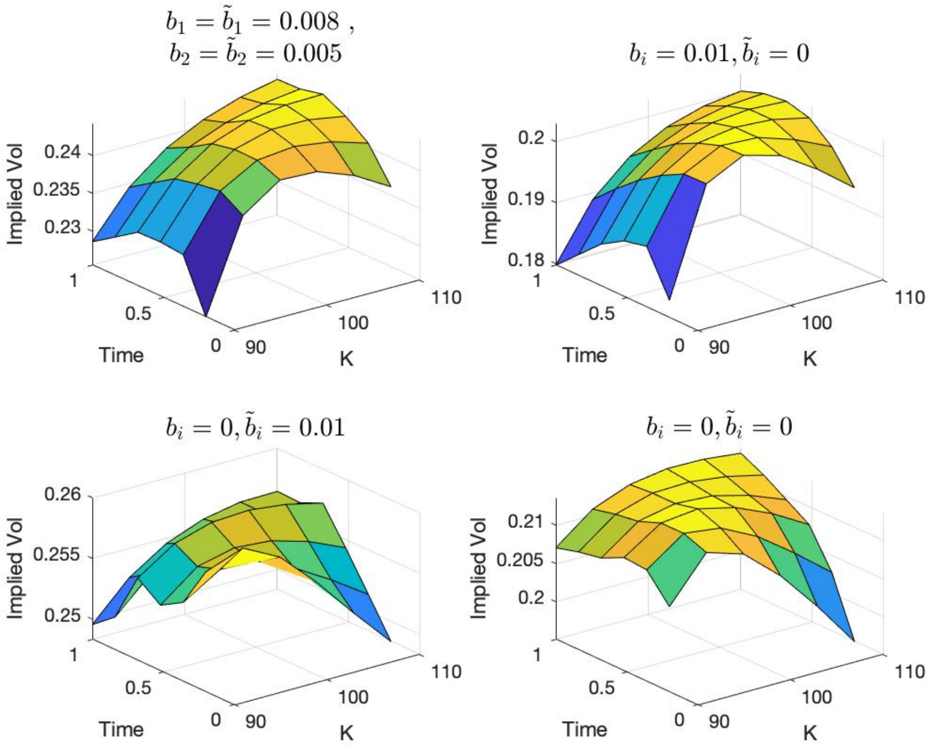


Table 5.10: Statistics for Two Common Factor Model, non mean reverting, change b

	$b_1 = \tilde{b}_1 = 0.008$ $b_2 = \tilde{b}_2 = 0.005$	$b_i = 0.01$ $\tilde{b}_i = 0$	$b_i = 0$ $\tilde{b}_i = 0.01$	$b_i = 0$ $\tilde{b}_i = 0$
$\mathbb{E}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.0866	0.0649	0.0873	0.0651
$\mathbb{E}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.2189	0.2301	0.1961	0.1730
$\mathbb{V}[\frac{X_1(T)-X_1(0)}{X_1(0)}]$	0.2167	0.1552	0.2194	0.1554
$\mathbb{V}[\frac{X_2(T)-X_2(0)}{X_2(0)}]$	0.7405	0.7880	0.6182	0.5252
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0055	0.0022	0.0086	0.0006
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.6352	-0.6391	-0.6172	-0.6333
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.6045	-0.5895	-0.6332	-0.6224

The 3/2 component b is studied in Figure 5.13. To be more specific, we consider four cases, the presence of 3/2 component in both common factor and intrinsic factor, presence of 3/2 component only in common factor, presence of 3/2 component only in intrinsic factor, as well as no presence of 3/2 component in the factor structure respectively. By comparing cases 1 and 4 in Figure 5.13, the presence of the 3/2 component results in a 33.03% and a 26.53% increase in the expected return of asset X_1 and X_2 respectively. This is accompanied by an increase of 39.45% and 41% in the variance of assets' return with a very small value of b or $\tilde{b}$. On the other hand, by comparing cases 2 and 3 in Figure 5.13, the impact from adding a 3/2 component to the intrinsic factor is greater than that of adding it to the common factor. It can be evidenced from the statistic table, i.e., Table 5.10, that assigning the same weight on b or $\tilde{b}$ makes a difference of -0.3% (from 0.0651 to 0.0649) and 34.1% (from 0.0651 to 0.0873) respectively compared to the absence of 3/2 component, in the expected returns of X_1. Also, a change of -0.13% (from 0.1544 to 0.1552) and 41.18% (from 0.1554 to 0.2194) in the variance of asset X_1's return can be observed.

Figure 5.14: Two Common Factor Model, non mean reverting, change Θ

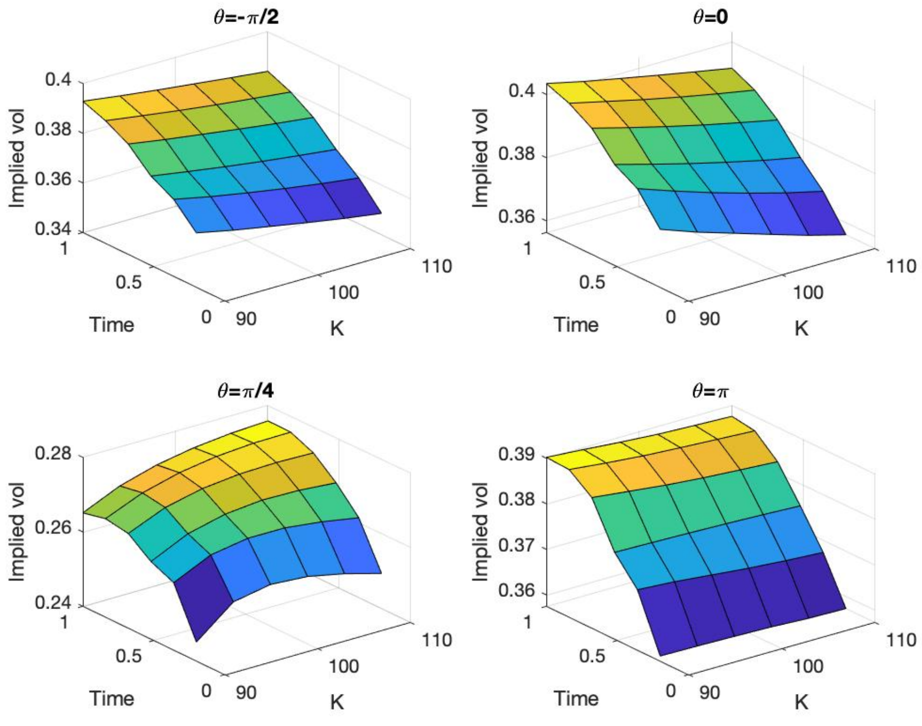


Table 5.11: Statistics for Two Common Factor Model, non mean reverting, change Θ

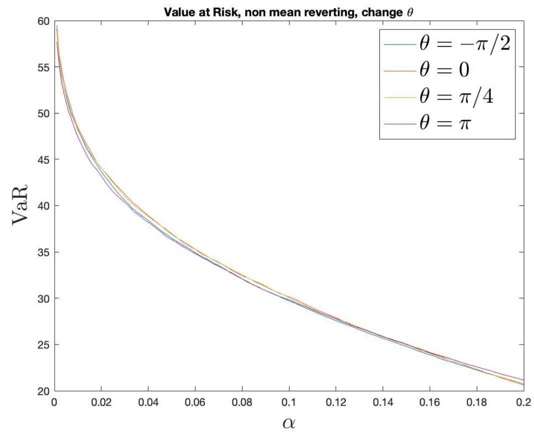
Θ	-$\pi/2$	0	$\pi/4$	π
$\mathbb{E}[\frac{X_1(t_{i+1})-X_1(t_i)}{X_1(t_i)}]$	0.1645	0.1750	0.1088	0.1722
$\mathbb{E}[\frac{X_2(t_{i+1})-X_2(t_i)}{X_2(t_i)}]$	0.1353	0.1302	0.1961	0.1249
$\mathbb{V}[\frac{X_1(t_{i+1})-X_1(t_i)}{X_1(t_i)}]$	0.4864	0.5304	0.2865	0.5268
$\mathbb{V}[\frac{X_2(t_{i+1})-X_2(t_i)}{X_2(t_i)}]$	0.3763	0.3588	0.6230	0.3395
$\mathbf{Corr}(dlnX_1(T), dlnX_2(T))$	0.0031	0.0035	0.0067	0.0073
$\mathbf{Corr}(dlnX_1(T), < dlnX_1(T) >)$	-0.4668	-0.6748	-0.6259	-0.2401
$\mathbf{Corr}(dlnX_2(T), < dlnX_2(T) >)$	-0.3716	-0.6708	-0.5549	-0.0971

The investigation of the influence of assets' correlation driver Θ on implied volatility surface is shown in Figure 5.14. Similar to what we observe from the sensitivity analysis on Θ in the absence of b, changes in Θ have a significant impact on leverage effects among all the other parameters. For example, when $\Theta = 0$, the leverage effect of asset X_1 and X_2 can be -0.6748 and -0.6708; when $\Theta = \pi$, they can end up with -0.2401 and -0.0971 respectively. On the other hand, adding the 3/2 component b to risk factors increases the implied volatilities, while the shapes and the levels of the surface are more persistent with variations in correlation Θ.

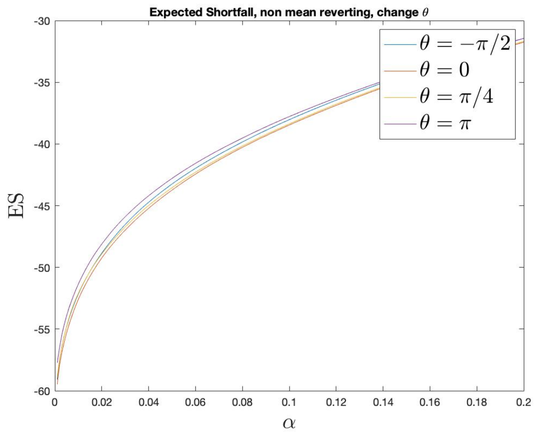
In conclusion, parameters from the underlying CIR process can have an important influence on implied volatility, especially the mean reverting level θ, which can change the trend of the implied volatility surface. Further, the 3/2 component b plays a role in affecting implied volatility, especially if included in the intrinsic factor. Moreover, the correlation Θ among assets have the advantage of charging the leverage effect.

5.4.3 Risk Measures

In this section, important risk measures are evaluated as functions of the correlation among assets Θ and the 3/2 component b respectively in the two common factors and one intrinsic factor 4/2 structured non mean reverting model. In the same fashion, we focus on the value at risk (VaR) and the expected shortfall (ES) as we mathematically defined in equation 4.6 and equation 4.7.



(a) Value at Risk vs. α, non mean reverting, various Θ



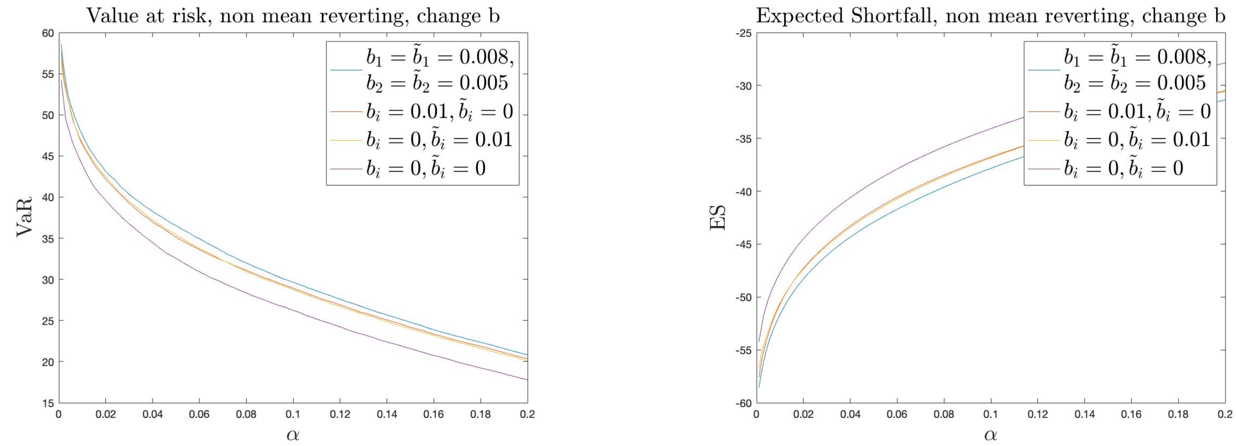
(b) Expected Shortfall vs. α, non mean reverting various Θ

Figure 5.15: Impact of θ on Risk measures

In line with what we found out before, correlation among assets Θ dominates the leverage effects, and it has a little effect on asset prices or their expected returns. By looking at the correlation between assets X_1 and X_2,

$$\begin{aligned}
\rho &= \frac{a_{11}a_{21}V_1(t) + a_{12}a_{22}V_2(t)}{\sqrt{\left(a_{11}^2 V_1(t) + a_{12}^2 V_2(t) + \tilde{V}_1(t)\right)\left(a_{21}^2 V_1(t) + a_{22}^2 V_2(t) + \tilde{V}_2(t)\right)}} \\
&= \frac{cos(\Theta)sin(\Theta)V_1(t) - sin(\Theta)cos(\Theta)V_2(t)}{\sqrt{\left(cos(\Theta)^2 V_1(t) + sin(\Theta)^2 V_2(t) + \tilde{V}_1(t)\right)\left(sin(\Theta)^2 V_1(t) + cos(\Theta)^2 V_2(t) + \tilde{V}_2(t)\right)}} \\
&= \frac{cos(\Theta)sin(\Theta)\left(V_1(t) - V_2(t)\right)}{\sqrt{\left(cos(\Theta)^2 V_1(t) + sin(\Theta)^2 V_2(t) + \tilde{V}_1(t)\right)\left(sin(\Theta)^2 V_1(t) + cos(\Theta)^2 V_2(t) + \tilde{V}_2(t)\right)}},
\end{aligned} \tag{5.8}$$

which is very small. Thus, both VaR and ES only show a slight difference for various Θ as shown in Figure 5.15. However, by looking at the expected loss, the case of ignoring correlation among assets, i.e., $\Theta = 0$, always hurts more compared to all others.



(a) Value at Risk vs. α, non mean reverting, various b

(b) Expected Shortfall vs. α, non mean reverting, various b

Figure 5.16: Impact of the of 3/2 components b on Risk measures

The impact from the 3/2 component b on risk measure VaR and ES is explored in Figure 5.16. It is easy to notice that even for very small values of b, the presence of 3/2 component can cause a 6.8% (from \$48 to \$44) increase in VaR. That is a 6.8% more capital need to be prepared for possible losses. Further, a 10.6% (from -\$52 to -\$47) increase in average value at risk can also be evidenced due to the presence of b.

6 Advanced Model

In this section, we assume a more advanced model by employing different underlying processes on the 1/2 and the 3/2 components underlying the 4/2 structured common factors and intrinsic factors. That is, the 4/2 variance follows $(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{\sigma_j(t)}})$ instead of $(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}})$ in our proposed more advanced model, where $v_j(t)$ and $\sigma_j(t)$ are two different CIR processes with potential correlations. This separation of the 1/2 and 3/2 components was mentioned in Grasselli (2017) without further study. Note that, the model in Grasselli (2017) is embedded by assuming a perfect correlation between the 1/2 component and the 3/2 component. Here both of the components are driving from CIR underlying process. The advanced model is a generalization not only by separating the underlying CIR processes of the 1/2 and the 3/2 component but also due to incorporating the mean reverting property, the spillover effect and multi-factor structure.

This section lays out as follows: we firstly define and specify the model. Secondly, a two-dimensional example is presented. Then, the implied volatility surfaces are explored aiming at reproducing various shapes.

6.1 Define the Model

The process of asset price change in the more advanced model under $\mathbb{P}$ is given by

$$
\begin{aligned}
\frac{dX_i(t)}{X_i(t)} &= \begin{pmatrix} L_i + c_i \sum_{j=1}^{n} a_{ij}^2 \left(v_j(t) + \frac{b_j^2}{\sigma_j(t)} + 2\rho_{jj} \frac{b_j \sqrt{v_j(t)}}{\sqrt{\sigma_j(t)}} \right) \\ + \tilde{c}_i \left(\tilde{v}_i(t) + \frac{\tilde{b}_i^2}{\tilde{\sigma}_i(t)} + 2\tilde{\rho}_{ii} \frac{\tilde{b}_i \sqrt{\tilde{v}_i}}{\sqrt{\tilde{\sigma}_i}} \right) - \sum_{j=1}^{n} \beta_{ij}(t) Y_i(t) \end{pmatrix} dt \\
&\quad + \sum_{j=1}^{n} a_{ij} \left(\sqrt{v_j(t)} dW_j(t) + \frac{b_j}{\sqrt{\sigma_j(t)}} dZ_j(t) \right) + \left(\sqrt{\tilde{v}_i(t)} d\tilde{W}_i(t) + \frac{\tilde{b}_i}{\sqrt{\tilde{\sigma}_i(t)}} d\tilde{Z}_i(t) \right) \\
dv_j &= \alpha_j(\theta_j - v_j(t))dt + \xi_j \sqrt{v_j(t)} dB_j(t) \\
d\sigma_j &= \alpha_{js}(\theta_{js} - \sigma_j(t))dt + \xi_{js} \sqrt{\sigma_j(t)} dB_{js}(t) \\
d\tilde{v}_i &= \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t))dt + \tilde{\xi}_i \sqrt{\tilde{v}_i(t)} d\widetilde{B}_i(t) \\
d\tilde{\sigma}_i &= \tilde{\alpha}_{is}(\tilde{\theta}_{is} - \tilde{\sigma}_i(t))dt + \tilde{\xi}_{is} \sqrt{\tilde{\sigma}_i(t)} d\widetilde{B}_{is}(t) \\
\langle dB_j(t), dW_j(t) \rangle &= \rho_j dt, \ \left\langle d\widetilde{B}_i(t), d\widetilde{W}_i(t) \right\rangle = \tilde{\rho}_i dt \\
\langle dB_{js}(t), dZ_j(t) \rangle &= \rho_{js} dt, \ \left\langle d\widetilde{B}_{is}(t), d\widetilde{Z}_i(t) \right\rangle = \tilde{\rho}_{is} dt \\
\langle dZ_j(t), dW_j(t) \rangle &= \rho_{jj} dt, \ \left\langle d\widetilde{Z}_i(t), d\widetilde{W}_i(t) \right\rangle = \tilde{\rho}_{ii} dt
\end{aligned}
$$

where c_i and $\tilde{c}_i$ are the drivers in the risk premiums of the commonality factors and the intrinsic factors respectively. L_i is the mean reverting level of risky asset X_i. β determines the mean reverting speed of asset X_i if $j = i$; otherwise, β_{ij} reflects asset X_j's impact on current asset X_i's expected return, namely the spillover effects among assets.

Further, it can be easily observed that the correlation between the $1/2$ component and the $3/2$ component is governed by ρ_{jj} in the common factors and $\tilde{\rho}_{ii}$ in the intrinsic factors. That is, instead of prescribing $\sqrt{v}(t)$ (or $\sqrt{\tilde{v}(t)}$) perfectly correlated with $\frac{1}{\sqrt{\sigma(t)}}$ (or $\frac{1}{\sqrt{\tilde{\sigma}(t)}}$), they can have the flexibility in their dependency, as well as their mean reverting speed, mean reverting level and volatility of volatility in terms of their own CIR process. Besides, the Feller condition is assumed to be satisfied by each CIR process in the model.

In the language of factor decomposition, the variance of the underlying asset is explained by two sources: the commonality or systematic variances that captures co-variation of asset returns are given by $V_j(t) = \left(v_j(t) + \frac{b_j^2}{\sigma_j(t)} + 2\rho_{jj}\frac{b_j\sqrt{v_j(t)}}{\sqrt{\sigma_j(t)}}\right)$ for $j = 1, ..., p$, while the remaining intrinsic variance that explains the variance of asset itself corresponds to $\widetilde{V}_i(t) = \left(\tilde{v}_i(t) + \frac{\tilde{b}_i^2}{\tilde{\sigma}_i(t)} + 2\tilde{\rho}_{ii}\frac{\tilde{b}_i\sqrt{\tilde{v}_i}}{\sqrt{\tilde{\sigma}_i}}\right)$ for $i = 1, ..., n$. In addition, $(a_{ij})_{n\times p}$ is the ij^{th} entry of an orthogonal matrix A (assume $n = p$ if necessary), and it determines the dependency or correlation structure among risky assets.

Applying Ito's lemma, the stochastic process of $lnX_i(t){=}Y_i(t)$ is in the form of

$$\begin{aligned} dY_i(t) &= \left(\begin{array}{c} L_i + (c_i - \frac{1}{2})\sum_{j=1}^n a_{ij}^2\left(v_j(t) + \frac{b_j^2}{\sigma_j(t)} + 2\rho_{jj}\frac{b_j\sqrt{v_j(t)}}{\sqrt{\sigma_j(t)}}\right) \\ +(\tilde{c}_i - \frac{1}{2})\left(\tilde{v}_i(t) + \frac{\tilde{b}_i^2}{\tilde{\sigma}_i(t)} + 2\tilde{\rho}_{ii}\frac{\tilde{b}_i\sqrt{\tilde{v}_i}}{\sqrt{\tilde{\sigma}_i}}\right) - \sum_{j=1}^n \beta_{ij}(t)Y_i(t) \end{array}\right)dt \\ &\quad + \sum_{j=1}^n a_{ij}\left(\sqrt{v_j(t)}dW_j(t) + \frac{b_j}{\sqrt{\sigma_j(t)}}dZ_j(t)\right) + \left(\sqrt{\tilde{v}_i(t)}d\tilde{W}_i(t) + \frac{\tilde{b}_i}{\sqrt{\tilde{\sigma}_i(t)}}d\tilde{Z}_i(t)\right) \end{aligned}$$

In the spirit that c_i and $\tilde{c}_i$ are drivers of risk premiums of the commonality factors and the intrinsic factors respectively, the dynamic of underlying assets under risk-neutral measure $\mathbb{Q}$ can be tracked by eliminating the excess returns, such that

$$\begin{aligned} \frac{dX_i(t)}{X_i(t)} &= \left(L_i - \sum_{j=1}^n \beta_{ij}(t)Y_i(t)\right)dt \\ &\quad + \sum_{j=1}^n a_{ij}\left(\sqrt{v_j(t)}dW_j(t) + \frac{b_j}{\sqrt{\sigma_j(t)}}dZ_j(t)\right) + \left(\sqrt{\tilde{v}_i(t)}d\tilde{W}_i(t) + \frac{\tilde{b}_i}{\sqrt{\tilde{\sigma}_i(t)}}d\tilde{Z}_i(t)\right) \end{aligned}$$

Moreover, if there is no mean reverting property, i.e., $\beta_{ij} = 0$, the process of underlying asset under measure $\mathbb{Q}$ evolves as:

$$\frac{dX_i(t)}{X_i(t)} = rdt + \sum_{j=1}^{n} a_{ij}\left(\sqrt{v_j(t)}dW_j(t) + \frac{b_j}{\sqrt{\sigma_j(t)}}dZ_j(t)\right) + \left(\sqrt{\tilde{v}_i(t)}d\tilde{W}_i(t) + \frac{\tilde{b}_i}{\sqrt{\tilde{\sigma}_i(t)}}d\tilde{Z}_i(t)\right)$$

6.2 Two-dimension Example

In this subsection, the example of a two dimensional case with two common factors and one intrinsic factor each under a non mean reverting setting is presented. Firstly, the dynamic of asset price X_i for $i = 1, 2$ is given by

$$\begin{aligned} dY_i(t) &= \left(\begin{array}{c} L_i + (c_i - \frac{1}{2})\left[a_{i1}^2\left(v_1(t) + \frac{b_1^2}{\sigma_1(t)} + 2\rho_{11}\frac{b_1\sqrt{v_1(t)}}{\sqrt{\sigma_1(t)}}\right) + a_{i2}^2\left(v_2(t) + \frac{b_2^2}{\sigma_2(t)} + 2\rho_{22}\frac{b_2\sqrt{v_2(t)}}{\sqrt{\sigma_2(t)}}\right)\right] \\ +(\tilde{c}_i - \frac{1}{2})\left(\tilde{v}_i(t) + \frac{\tilde{b}_i^2}{\tilde{\sigma}_i(t)} + 2\tilde{\rho}_{ii}\frac{\tilde{b}_i\sqrt{\tilde{v}_i}}{\sqrt{\tilde{\sigma}_i}}\right) - \sum_{j=1}^{2}\beta_{ij}(t)Y_i(t) \end{array}\right) dt \\ &\quad + a_{i1}\left(\sqrt{v_1(t)}dW_1(t) + \frac{b_1}{\sqrt{\sigma_1(t)}}dZ_1(t)\right) + a_{i2}\left(\sqrt{v_2(t)}dW_2(t) + \frac{b_2}{\sqrt{\sigma_2(t)}}dZ_2(t)\right) \\ &\quad + \left(\sqrt{\tilde{v}_i(t)}d\tilde{W}_i(t) + \frac{\tilde{b}_i}{\sqrt{\tilde{\sigma}_i(t)}}d\tilde{Z}_i(t)\right) \end{aligned}$$

with

$$\begin{aligned} &dv_j = \alpha_j(\theta_j - v_j(t))dt + \xi_j\sqrt{v_j(t)}dB_j(t),\ d\sigma_i = \alpha_{js}(\theta_{js} - \sigma_j(t))dt + \xi_{js}\sqrt{\sigma_j(t)}dB_{js}(t), j = 1, 2 \\ &d\tilde{v}_i = \tilde{\alpha}_i(\tilde{\theta}_i - \tilde{v}_i(t))dt + \tilde{\xi}_i\sqrt{\tilde{v}_i(t)}d\widetilde{B}_i(t),\ d\tilde{\sigma}_i = \tilde{\alpha}_{is}(\tilde{\theta}_{is} - \tilde{\sigma}_i(t))dt + \tilde{\xi}_{is}\sqrt{\tilde{\sigma}_i(t)}d\widetilde{B}_{is}(t), i = 1, 2 \\ &\langle dB_j(t), dW_j(t)\rangle = \rho_j dt,\ \left\langle d\widetilde{B}_i(t), d\widetilde{W}_i(t)\right\rangle = \tilde{\rho}_i dt, i, j = 1, 2 \\ &\langle dB_{js}(t), dZ_j(t)\rangle = \rho_{js} dt,\ \left\langle d\widetilde{B}_{is}(t), d\widetilde{Z}_i(t)\right\rangle = \tilde{\rho}_{is} dt \\ &\langle dZ_j(t), dW_j(t)\rangle = \rho_{jj} dt,\ \left\langle d\widetilde{Z}_i(t), d\widetilde{W}_i(t)\right\rangle = \tilde{\rho}_{ii} dt \end{aligned}$$

Note the correlations must be such that the full correlation matrix is definite positive. Here we are dealing with a 16-dimensional vector: $(W_1, Z_1, B_1, B_{1s}, W_2, Z_2, B_2, B_{2s}, \widetilde{W}_1, \widetilde{Z}_1, \widetilde{B}_1, \widetilde{B}_{1s}, \widetilde{W}_2, \widetilde{Z}_2, \widetilde{B}_2, \widetilde{B}_{2s})$, but the correlation matrix can be easily written in block form (4 blocks) as shown below. This matrix is symmetric, so we only write the upper triangle, and those entries without specifying in the upper triangle are zeros.

$$
\begin{array}{cccccccccccccccc}
W_1 & Z_1 & B_1 & B_{1s} & W_2 & Z_2 & B_2 & B_{2s} & \widetilde{W}_1 & \widetilde{Z}_1 & \widetilde{B}_1 & \widetilde{B}_{1s} & \widetilde{W}_2 & \widetilde{Z}_2 & \widetilde{B}_2 & \widetilde{B}_{2s}
\end{array}
$$

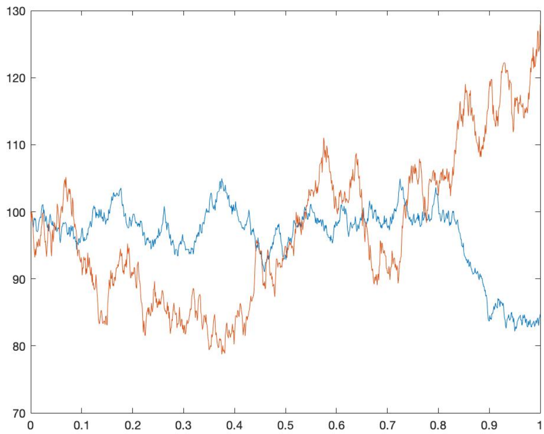
$$
\begin{pmatrix}
1 & \rho_{11} & \rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & \rho_{1s} & & & & & & & & & & & & \\
 & & 1 & 0 & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
 & & & & 1 & \rho_{22} & \rho_2 & 0 & & & & & & & & \\
 & & & & & 1 & 0 & \rho_{2s} & & & & & & & & \\
 & & & & & & 1 & 0 & & & & & & & & \\
 & & & & & & & 1 & & & & & & & & \\
 & & & & & & & & 1 & \widetilde{\rho}_{11} & \widetilde{\rho}_1 & 0 & & & & \\
 & & & & & & & & & 1 & 0 & \widetilde{\rho}_{1s} & & & & \\
 & & & & & & & & & & 1 & 0 & & & & \\
 & & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & & 1 & \widetilde{\rho}_{22} & \widetilde{\rho}_2 & 0 \\
 & & & & & & & & & & & & & 1 & 0 & \widetilde{\rho}_{2s} \\
 & & & & & & & & & & & & & & 1 & 0 \\
 & & & & & & & & & & & & & & & 1
\end{pmatrix}
$$

Also, one simulated path of assets X_1 and X_2 under the advanced model is shown in Figure 6.1 using Euler approximation. The employed parameters in producing Figure 6.1 are given in Table 6.1.

Table 6.1: Baseline Parameter Set for Advanced Model

i	j	L	c	$\tilde{c}$	A	v_0
2	2	[0.02 0.02]	[0 0]	[0 0.5]	[1 0;0 1]	[0.04 0.04]
$\tilde{v}_0$	b	β	Y(0)	σ_0	$\tilde{\sigma}_0$	$\tilde{b}$
[0.04 0.04]	[0.008 0.005]	[0.12 -0.06;-0.1 0.1]	[log(100) log(100)]	[0.04 0.04]	[0.04 0.04]	[0 0]
ρ_j	ρ_{js}	ρ_{jj}	$\tilde{\rho}_i$	$\tilde{\rho}_{is}$	$\tilde{\rho}_{ii}$	α
$[-0.7\ \ -0.8]$	[-0.7 -0.9]	[0.5 1]	[-0.9 -1]	[-0.9 -1]	[-0.1 -1]	[1.8 16]
θ	ξ	α_s	θ_s	ξ_s	$\tilde{\alpha}$	$\tilde{\theta}$
[0.04 0.04]	[0.2 0.15]	[5 15]	[0.4 0.04]	[0.4 0.02]	[0 0]	[0 0]
$\tilde{\xi}$	$\tilde{\alpha}_s$	$\tilde{\theta}_s$	$\tilde{\xi}_s$	dt	n	
[0 0]	[0 0]	[0 0]	[0 0]	0.001	100,000	

Figure 6.1: One Path of Asset Price under Advanced Model



6.3 Pricing Option and Implied Volatility

In this section, we price a European call option on an asset X with one common factor and no intrinsic factor under a non mean reverting model, i.e.,

$$\begin{aligned} dY(t) &= \left[r + (c_1 - \frac{1}{2}) a_{11}^2 \left(v_1(t) + \frac{b_1^2}{\sigma_1(t)} + 2\rho_{11} \frac{b_1 \sqrt{v_1(t)}}{\sqrt{\sigma_1(t)}} \right) \right] dt \\ &\quad + a_{11} \left(\sqrt{v_1(t)} dW_1(t) + \frac{b_1}{\sqrt{\sigma_1(t)}} dZ_1(t) \right) \\ dv_1 &= \alpha_1(\theta_1 - v_1(t))dt + \xi_1 \sqrt{v_1(t)} dB_1(t) \\ d\sigma_1 &= \alpha_{1s}(\theta_{1s} - \sigma_1(t))dt + \xi_{1s} \sqrt{\sigma_1(t)} dB_{1s}(t) \end{aligned}$$

where r is the risk free rate, $\langle dZ_1(t), dW_1(t) \rangle = \rho_{11} dt$, and $B_1 \perp B_{1s}$.

The intention of considering this special case is to see whether this simple setting can make use of the flexibility provided in our model in reproducing the volatility "smile" or "skew". Note that we set the parameter a_{11} a little different compared with Graselli's model where a_{11} has an impact on both $\sqrt{v_1(t)}$ and $\frac{b_1}{\sqrt{\sigma_1(t)}}$. Therefore, we will not consider $a_{11} = 0$, which will make the process not random at all.

We explore the implied volatility surface with strike prices K: 90, 94, 98, 102, 106, 110 and expiry dates T: 0.2, 0.36, 0.52, 0.68, 0.84 and 1 year. By choosing these strike prices, we take into account in-the-money, at-the-money, and out-of-the-money as well given the initial asset price is \$100. Then, for each strike price and the expiry date, we can firstly get a simulated call option price by equation 2.17 with $X(t)$ under risk-neutral measure $\mathbb{Q}$,

$$\frac{dX(t)}{X(t)} = rdt + a_{11}\sqrt{v_1(t)}dW_1(t) + \frac{a_{11}b_1}{\sqrt{\sigma_1(t)}}dZ_1(t)$$

Analogously, following the procedure of extracting the implied volatility, we match the Black-Scholes option price formula in equation 2.19 with simulated call prices and solve for the implied volatility in the dynamics of $Y(t)$ such that:

$$dY(t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW^*(t),$$

where the risk free interest rate $r = 0.02$. To be more detailed, we take all the randomness as a constant, such that

$$\sigma dW^*(t) = a_{11}\left(\sqrt{v_1(t)}dW_1(t) + \frac{b_1}{\sqrt{\sigma_1(t)}}dZ_1(t)\right).$$

It follows that

$$\sigma^2 dt = a_{11}^2\left(v_1(t) + \frac{b_1^2}{\sigma_1(t)} + 2\rho_{11}\frac{b_1\sqrt{v_1(t)}}{\sqrt{\sigma_1(t)}}\right)dt$$

Next figures explore the implied volatility surface under a parameterization of a_{11} and b_1 inspired by Grasselli (2017). That is, parameter a_{11} is predetermined and parameter b_1 is found by holding constant value of $a_{11}\left(\sqrt{v_1(0)} + \frac{b_1}{\sqrt{\sigma_1(0)}}\right)$ to $\sqrt{v_1(0)} = \sqrt{\sigma_1(0)} = \sqrt{0.04}$. Moreover, we assume the CIR process for the 3/2 component, i.e., σ_1, follows high volatility of volatility and high mean reverting speed relative to the CIR process for the 1/2 component, i.e., v_1. On the one hand, the advance for allowing different CIR processes in the constitution of 4/2 structured risk factor can be showed; on the other hand, the quick reverting as the process gets high in the 3/2 process can be utilized with the high volatility of volatility. Further, in order to make a comparison, the implied volatility surface under a one factor generalized 4/2 model that does not possess the advanced feature is presented as well. The following Table 6.2 shows the parameter set used to simulate the implied volatility surfaces in Figure 6.2.

Table 6.2: Parameter Set

j	r	Y	a$_{11}$	b	v_0	σ_0
1	0.02	log(100)	-0.4, -0.1, 0.5, 1	...	0.04	0.04
ρ_1	ρ_{1s}	ρ_{11}	α	θ	ξ	α_s
-0.7	-0.7	1	1.8	0.04	0.2	18
θ_s	ξ_s	dt	n			
0.04	2	0.01	50,000			

(a) Advance model with various a **(b)** Original model with various a

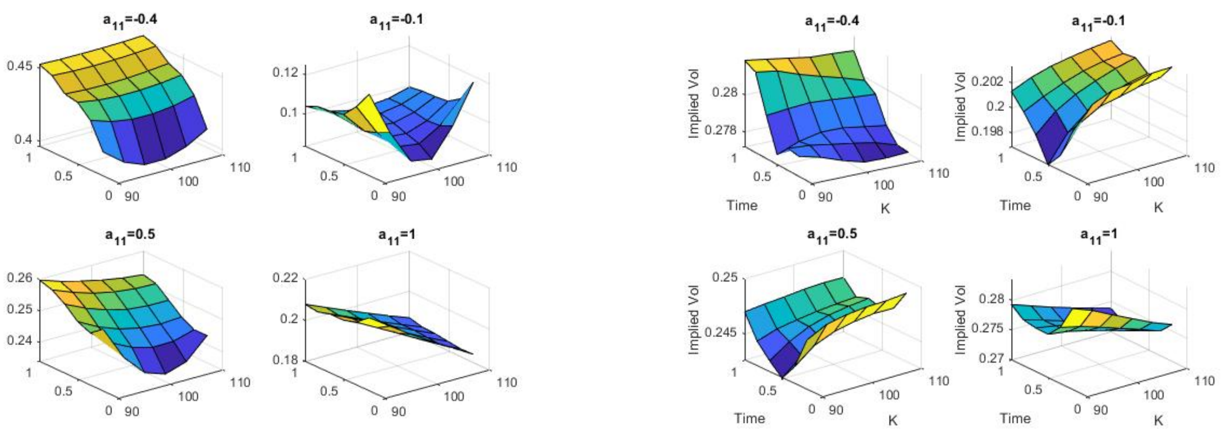


Figure 6.2: Implied volatility surface from changing a_{11}

Figure 6.2 shows that with the 1/2 component and the 3/2 component following different underlying CIR processes, the volatility "smile" can be obtained in the advanced model when $a_{11} = -0.1$ and $a_{11} = 0.5$. That is, the advanced model can deliver richer variation in the implied volatility surface. Also, a higher volatility of volatility and mean reverting speed of the 3/2 process can change the implied volatility surface in both shapes and variations significantly. For example, as $a_{11} = -0.4$, the implied volatilities lie between (0.4, 0.45), which is clearly higher than in the original model (case 1 in Figure 6.2b); as $a_{11} = -0.1$. they are within (0.02, 0.12), which is apparently below the original model (case 2 in Figure 6.2b). It implies that the effect of allowing a different CIR process for the 3/2 component is distinct and one should assign the parameters of the underlying process properly in order to capture the implied volatility.

7 Conclusion

This work proposed and investigated a generalized 4/2 factor model by considering systematic risk factors from the market and an independent risk factor from assets themselves in portfolio optimization, risk management, and option pricing. We generalized the model by allowing for mean reverting and spillover effects to make it compatible with a wider class of assets. We provided conditions on parameters that ensure well-defined changes of measure. In the context of EUT, a quasi closed-form solution of the optimal investment strategy was obtained, while analytical solution were available for special cases. The impact of the 3/2 component b and commonality loading a were explored with respect to implied volatility surface and two important risk measures in a setting of one common factor case. Besides, two forms of the market price of risk were compared in the context of implied volatility surfaces and risk measures. Further, a two common factor case was studied with respect to the correlation among assets Θ, the spillover effect β, as well as the 3/2 component b. Lastly, a more advanced model was defined and shown to capture richer volatility surfaces.

In the sensitivity analysis of the 4/2 factor model, we found that even small values in the 3/2 component b can lead to over 90% change in implied volatility and over 1/3 changes in risk measures. We also realized that ignoring the commonality loading that captures the dependency or covariance among assets would result in underestimating the value in the portfolio under risk. Another observation was that given different underlying CIR process for common and intrinsic factor, the impact of common factor loading a, the 3/2 component b, and the choice of market price of risk can be different and significant. The comparison of two types of market price of risk demonstrated that the one proportional to a 1/2 structure persistently shows higher impact on risk compared to a MPR proportional to a 4/2 structure. By examining both, mean reverting model and non mean reverting model, these findings are consistent. Furthermore, in the two common factor example, we captured the correlation among assets through Θ and realized its influence in also controlling the leverage effect. In addition, the spillover effect β showed its ability in changing the pattern of implied volatility surfaces.

Finally, we recognize many limitations in this study. Future research can help answer many open questions as for instance: univariate and multivariate 4/2 structures that allow for closed-form solutions in the context of EUT; estimation and calibration of our factor models to real data to confirm the importance of individual parameters and their impact on derivative pricing, risk measures and portfolio decisions; study the further the viability of decoupling 1/2 and 3/2 components within the 4/2 model.

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Appendix

A1 Change of measure conditions

Proof. Proof of Proposition 1.

The first step is to ensure the change of measure is well-defined and for this we use Novikov's condition, i.e. generically

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \lambda^2\left(\sqrt{\nu(t)}+\frac{b}{\sqrt{\nu(t)}}\right)^2 ds\right)\right] = e^{\lambda^2 bT}\mathbb{E}\left[\exp\left(\frac{\lambda^2}{2}\int_0^T \nu(s)ds + \frac{\lambda^2 b^2}{2}\int_0^T \frac{1}{\nu(s)}ds\right)\right] < \infty.$$

From Grasselli, in order for this expectation to exist, we need two conditions:

$$-\frac{\lambda^2}{2} > -\frac{\alpha^2}{2\xi^2} \implies |\lambda| < \frac{\alpha}{\xi} \tag{.1}$$

and

$$-\frac{\lambda^2 b^2}{2} \geq -\frac{(2\alpha\theta-\xi^2)^2}{8\xi^2} \implies |\lambda| \leq \frac{2\alpha\theta-\xi^2}{2|b|\xi} \implies \xi^2 \leq 2\alpha\theta - 2|\lambda||b|\xi \tag{.2}$$

The latter condition (.2) implies, in particular, that our volatility processes satisfy Feller's condition under $\mathcal{P}$ and $\mathcal{Q}$, in other words, it ensures all our CIR processes stay away from zero under both measures.

Applying equation .2 to our setting leads to $(i, j = 1, .., n)$:

$$\xi_j^2 \leq 2\alpha_j\theta_j - 2\xi_j \max\left\{|\lambda_j b_j|, \left|\lambda_j^{\perp} b_j\right|\right\} \tag{.3}$$

$$\widetilde{\xi}_i^2 \leq 2\widetilde{\alpha}_i\widetilde{\theta}_i - 2\widetilde{\xi}_i \max\left\{\left|\widetilde{\lambda}_i\widetilde{b}_i\right|, \left|\widetilde{\lambda}_i^{\perp}\widetilde{b}_i\right|\right\} \tag{.4}$$

Now we apply equation .1 producing two extra set of conditions $(i, j = 1, .., n)$:

$$\max\left\{|\lambda_j|, \left|\lambda_j^{\perp}\right|\right\} < \frac{\alpha_j}{\xi_j} \tag{.5}$$

$$\max\left\{\left|\widetilde{\lambda}_i\right|, \left|\widetilde{\lambda}_i^{\perp}\right|\right\} < \frac{\widetilde{\alpha}_i}{\widetilde{\xi}_i} \tag{.6}$$

The second step applies to the case $\beta_{ij} = 0$ for $i, j = 1, .., n$ and it is to ensure the drift of the asset price equal the short rate:

$$L_i = r,\, c_i = \sum_{j=1}^{n} \left(\rho_j \lambda_j + \sqrt{1-\rho_j^2}\lambda_j^{\perp} \right),\, \widetilde{c}_i = \widetilde{\rho}_i \widetilde{\lambda}_i + \sqrt{1-\widetilde{\rho}_i^2}\widetilde{\lambda}_i^{\perp}$$

For the most general case ($\beta_{ij} \neq 0$ for some i or j), the second step which should be adapted to any particular prescribed drift structure under the $\mathcal{Q}$-measure.

The third step is to ensure the drift-less asset price process is a true $\mathcal{Q}$-martingale and not just a local $\mathcal{Q}$-martingale:

$$\frac{dX_i(t)}{X_i(t)} = (.)\, dt + \sum_{j=1}^{n} a_{ij} \left(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}} \right) dW_j^Q(t) + \left(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}} \right) d\widetilde{W}_i^Q(t)$$

Here we test the martingale property using the Feller nonexplosion test for volatilities, hence considering the following $n^2 + n$ changes of Brownian motion for the volatility processes and checking the processes do not reach zero under the various measures:

$$dB_{ij}^Q(t) = a_{ij}\rho_j \left(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}} \right) dt + dB_j^P(t),\, d\tilde{B}_i^Q(t) = \widetilde{\rho}_i \left(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}} \right) dt + d\tilde{B}_i^P(t)$$

This leads to the following conditions:

$$\xi_j^2 \leq 2\alpha_j\theta_j - 2\left|a_{ij}\rho_j b_j\right| \xi_j,\, i,j = 1,...,n \tag{.7}$$

$$\widetilde{\xi}_i^2 \leq 2\widetilde{\alpha}_i\widetilde{\theta}_i - 2\left|\widetilde{\rho}_i\widetilde{b}_i\right| \widetilde{\xi}_i,\, i = 1,...,n \tag{.8}$$

We can combine the first and third steps .3, .7, .4 and .8 into the final conditions. □

Proof. Proof of Proposition 2.

The first step is to ensure the change of measure is well-defined and for this we use Novikov's condition, i.e. generically

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \lambda^2 \left(\sqrt{\nu(t)}\right)^2 ds\right)\right] = \mathbb{E}\left[\exp\left(\frac{\lambda^2}{2}\int_0^T \nu(s)ds\right)\right] < \infty.$$

From Grasselli, in order for this expectation to exist, we need next condition:

$$-\frac{\lambda^2}{2} > -\frac{\alpha^2}{2\xi^2} \implies |\lambda| < \frac{\alpha}{\xi} \tag{.9}$$

Now we apply equation .9 producing two extra set of conditions ($i, j = 1, .., n$):

$$\max\left\{|\lambda_j|, \left|\lambda_j^{\perp}\right|\right\} < \frac{\alpha_j}{\xi_j} \tag{.10}$$

$$\max\left\{\left|\widetilde{\lambda}_i\right|, \left|\widetilde{\lambda}_i^{\perp}\right|\right\} < \frac{\widetilde{\alpha}_i}{\widetilde{\xi}_i} \tag{.11}$$

Then, to ensure our volatility processes satisfy Feller condition under $\mathcal{P}$ and $\mathcal{Q}$. It ensures our CIR processes stay away from zero under both measures:

$$\xi_j^2 \leq 2\alpha_j\theta_j \tag{.12}$$

$$\widetilde{\xi}_i^2 \leq 2\widetilde{\alpha}_i\widetilde{\theta}_i \tag{.13}$$

The second step applies to the case $\beta_{ij} = 0$ for $i, j = 1, .., n$ and it is to ensure the drift of the asset price equal the short rate:

$$L_i = r,\, c_i = \sum_{j=1}^{n}\left(\rho_j\lambda_j + \sqrt{1-\rho_j^2}\lambda_j^{\perp}\right),\, \widetilde{c}_i = \widetilde{\rho}_i\widetilde{\lambda}_i + \sqrt{1-\widetilde{\rho}_i^2}\widetilde{\lambda}_i^{\perp}$$

For the most general case ($\beta_{ij} \neq 0$ for some i or j), the second step which should be adapted to any particular prescribed drift structure under the $\mathcal{Q}$-measure.

The third step is to ensure the drift-less asset price process is a true $\mathcal{Q}$-martingale and not just a local $\mathcal{Q}$-martingale:

$$\frac{dX_i(t)}{X_i(t)} = (.)\, dt + \sum_{j=1}^{n} a_{ij}\left(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}}\right) dW_j^Q(t) + \left(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}}\right) d\widetilde{W}_i^Q(t)$$

Here we test the martingale property using the Feller nonexplosion test for volatilities, hence considering the following $n^2 + n$ changes of Brownian motion for the volatility processes and checking the processes do not reach zero under the various measures:

$$dB_{ij}^Q(t) = a_{ij}\rho_j\left(\sqrt{v_j(t)} + \frac{b_j}{\sqrt{v_j(t)}}\right) dt + dB_j^P(t),\, d\tilde{B}_i^Q(t) = \widetilde{\rho}_i\left(\sqrt{\tilde{v}_i(t)} + \frac{\tilde{b}_i}{\sqrt{\tilde{v}_i(t)}}\right) dt + d\tilde{B}_i^P(t)$$

This leads to the following conditions:

$$\xi_j^2 \leq 2\alpha_j\theta_j - 2\left|a_{ij}\rho_j b_j\right|\xi_j,\; i, j = 1, ..., n \tag{.14}$$

$$\widetilde{\xi}_i^2 \leq 2\widetilde{\alpha}_i\widetilde{\theta}_i - 2\left|\widetilde{\rho}_i\widetilde{b}_i\right|\widetilde{\xi}_i,\; i = 1, ..., n \tag{.15}$$

We can combine the first and third steps .12, .14, .13 and .15 into the final conditions. □