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Cognition Without Construction: Kant, Maimon, and the Transcendental Philosophy of Mathematics

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Abstract

In the *Critique of Pure Reason*, Immanuel Kant takes the ostensive constructions characteristic of Euclidean-style demonstrations to be the paradigm of both mathematical proofs and synthetic a priori cognition in general. However, the development of calculus included a number of techniques for representing infinite series of sums or differences, which could not be represented with the direct geometrical demonstrations of the past. Salomon Maimon's *Essay on Transcendental Philosophy* addresses precisely this disparity. Maimon, owing much to G. W. Leibniz, proposes that differentials of sensation achieve what Kantian constructions could not. More importantly, Maimon develops a kind of symbolic cognition that is not delimited by the constraint of the pure forms of intuition. The mind does not construct its objects, but constructs itself through inquiry into the real objects of thought.

Keywords: Kant, Maimon, Leibniz, Calculus, Construction, Cognition, Symbolic, Differential, Philosophy, Mathematics

Summary for Lay Audience

With the *Critique of Pure Reason* Immanuel Kant reorients metaphysics away from things considered independently of the mind, towards the invariant structures of cognition and how the mind must suppose these structures in every experience of things. While this might bring order to the flurry of affections that impose themselves on the mind, a consequence of the critical turn is that the non-empirical objects of mathematics must also find their ultimate source and validation in the experience of empirical objects. Kant develops his philosophy of mathematics according to the use, tradition, and rigor of constructions in geometry. By the early eighteenth century, however, calculus had challenged the role of perception and sensation in the mathematical sciences. And not only this, the prominence of symbolic notation over geometrical construction allowed mathematicians and philosophers to think real objects that could never be given in the domain of experience. This line of thought is inaugurated by G.W. Leibniz and further developed by Salomon Maimon in the eighteenth century. In the first chapter, I develop Kant's notion of 'construction' and its place within his philosophy of mathematics, and critical philosophy more generally. The second chapter explicates Kant's relationship with calculus, specifically with Isaac Newton's Method of Fluxions, and shows how the indefinite iteration of constructions cannot adequately represent the relevant properties of infinite series. Chapter 3 goes on to develop Maimon's response to Kant - the 'differentials of sensation' - together with Leibniz's analytic method of infinitesimals. The fourth and final chapter illustrates the use of differentials in cognition with an example from Leibniz's *De quadratura* and goes on to explicate some consequences for Kant's critical philosophy. It concludes by indicating a passage from representation to reality, where cognition determines the thinking subject as much as it determines the object of experience.

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Words cannot describe how thankful I am for the support of my parents – for how much they have given and how much they have cared.

Δ – I carry you with me, everywhere.

Lastly, Megan. With all the grace of a bird or a cat, you bring me out of myself.

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List of Abbreviations

(CPR) *Critique of Pure Reason* by Immanuel Kant,

(ETP) *Essay on Transcendental Philosophy* by Salomon Maimon,

(L) *Philosophical Papers and Letters* of Gottfried Wilhelm Leibniz, Edited by Leroy E. Loemker.

So we start in the middle with our cognition of things and finish in the middle again.

Salomon Maimon, *Essay on Transcendental Philosophy*

Introduction

Charles Sander Peirce one remarked that Kant "drew too hard a line between the operations of observation and of ratiocination. He allows himself to fall into the habit of thinking that the latter only begins after the former is complete…His doctrine of the ***schemata*** can only have been an afterthought, an addition to his system after it was substantially complete. For if the ***schemata*** had been considered early enough, they would have overgrown his whole work" (CP 1.35).[1] Kant intended the schema of a concept to bridge the gap between the world of things and our representations of those things. Our access to the world is mediated by our capacity to receive information through bodily sensations, and our ability to retain then compile this information according to a common set of relations. Our representations are not equal to the things they represent, since the existence of things does not depend on the mind that thinks them. On this model, concepts group together relations that are observed through experience. But what are the origin of these concepts? Are they already in the mind, or are they communicated to the mind through experience? To complicate matters, the rise and success of Newtonian physics indicated that there was a necessary relation between the world of motion and the mathematical models of motion. With empirical phenomena, it is possible to appeal to the senses as the bridge between mind and world. But the relation between pure mathematics and the world of experience subsists without as obvious of a support. In *Critique of Pure Reason*, Kant proposes an all-encompassing system of philosophy that accounts for the connections between the different faculties of the mind.

[1] This refers to *The Collected Papers of Charles Sander Peirce*, Volume 1, Paragraph 35.

Instead of searching after the ultimate nature of things (whether they have essential mathematical or conceptual properties), he turns inward and asks about the necessary conditions that must foreground our experience of things. This is the anticipatory character that Peirce criticizes. In addition, this places all activities of the mind under the jurisdiction of Kant's philosophical system. The domains of mathematics, natural science, and logic are delimited in advance by "a teacher in the ideal, who controls all of these and uses them as tools to advance the essential ends of human reason" (A839/B867). The ultimate goal of thinking is a single coherent science into which all cognitive pursuits are resolved: "One can regard the critique of pure reason as the true court of justice for all controversies of pure reason; for the critique is not involved in these disputes, which pertain immediately to objects, but is rather set the task of determining and judging what is lawful in reason in general in accordance with the principles of its primary institution" (A751/B779). The ideal legislator that presides over this court is the philosopher.[2] The relation of this system to the sensible world is confused if not lost.

Such a pursuit seems grandiose because it is. This thesis aims to show how developments in mathematical practice, roughly between the years 1660-1700, presented a fundamental challenge to this systematic way of philosophizing. The appraisal of Kant's philosophy in light of infinitesimal or differential calculus is most thoroughly developed in Salomon Maimon's *Essay on Transcendental Philosophy*. Maimon is critical of systematic philosophizing because he finds real value in reflecting on our cognitive capacities and delimiting the scope of our representations. He states, "In order to avoid all misunderstandings, I will make my opinion on this matter known to the world. I maintain that Kant's *Critique of Pure Reason* is, in its own way, as classical a work

[2] "From this point of view philosophy is the science of the relation of all cognition to the essential ends of human reason (*teleologia rationis humanae*), and the philosopher is not an artist of reason but the legislator of human reason" (A839/B867).

as Euclid's, and as incapable of refutation. In order to defend this claim I will take on all of his opponents” (ETP 338n.3). But this is not to remain content with the system as Kant formulates it. He continues, “But looked at from the other side I hold this system to be insufficient… The existence of ideas in the mind necessarily indicates some kind of use for them… As a result, our thinking essence can never be satisfied with sensible objects and its way of thinking them… it feels in itself an irresistible drive to extend these limits ever further and to discover a passage from the sensible to the intelligible world” (ETP 338-339n.3). One such method of passing from the sensible to the intelligible is through Euclidean-style constructions of geometrical objects. And here is where the schema of a concept is meant to bridge mind and world. Schemata turn the logical relations of concepts into the spatial and temporal relations of experience. Concepts, especially the concepts of mathematics, cannot relate to empirical objects “without immediately descending to conditions of sensibility, thus to the form of the appearances, to which, as their sole objects, they must consequently be limited, since, if one removes this condition, all significance, i.e., relation to the object, disappears, and one cannot grasp through an example what sort of thing is really intended by concepts of that sort” (CPR A240-241/B300). The paradigm of direct, ostensive construction proves the validity of mathematical relations by *showing* such relations in space and time. But, how is it possible to construct the summation of an infinite number of miniscule triangles to prove that it equals the area under a curve? This method of proof falters when concepts cannot be constructed in space and time, but are nonetheless real.

Mathematicians of the eighteenth century developed a new set of primarily algebraic techniques to solve this problem, and Maimon draws on these techniques to establish the passage from the sensible to the intelligible adequate to the demands of the objects of mathematics.[3] He replaces the tradition of constructing figures in space and time with a method of abstracting conceptual relations from the sensations and appearances given to consciousness. Maimon posits differentials or elements of sensation that are finite yet smaller than any assignable magnitude, and are "determinate units such that when they are added to themselves successively, an arbitrary finite magnitude then arises" (ETP 29n.2). The reciprocal operations of summation and differentiation give us a way of representing a finite magnitude composed of infinitely many, infinitely small differences. The construction is no longer the decisive element in proving the validity of mathematical claims, rather the rules according to which summation and differentiation operate are the ultimate criterion of reality. In other words, *where Kant would explain calculus in terms of consciousness, Maimon explains consciousness in terms of calculus.* The aim is not to inaugurate a 'return' to Maimon, since there is no 'Maimonian' paradigm to return to.[4] Instead, Maimon's work describes a way beyond Kant from both within his system and by bringing mathematics to bear on the totalizing ambitions of the *Critique*.

This thesis does not argue against Kant by appealing to the authority of mathematics. The claim is that restricting cognition to the domain of experience, and the forms of space and time, cannot account for the actual scope of cognition both in domain of philosophy and beyond. The finitude of consciousness is the point of departure for thinking by other means. Symbolic notation is not empty characteristic in need of an arithmetical or geometrical supplement, nor does it stand in for

[3] In addition to the central place of both the differentials of specific quantities, as well as the differential quotient or the derivative of a curve (expressed in Leibnizian terminology as *dy or dx*, and $\frac{dy}{dx}$ respectively), Maimon also makes reference to Newton's method of fluxions, Cavalieri's method of indivisibles, as well as the binomial theorem and the process of integration by means of infinite, recursive sums (See Maimon 33-35, 229). Also see (Duffy 2014, 241-242) for a discussion of Maimon's familiarity with mathematics across other works. A detailed discussion of Maimon's Leibnizian interpretation of the calculus, and Kant's Newtonian interpretation, follows below.

[4] Among the growing scholarship on Maimon, there has been a consistent relationship between Maimon and Gilles Deleuze, due to several mentions across various works. For a list of references, see Voss (2015, 77n.58).

the ratios and proportions that give mathematical concepts a real object. Marcelo Dascal explains that, for genuinely symbolic cognition, "instead of the 'thing' conferring its meaningfulness to the character, it is only the use of characters that makes 'things' accessible to our thoughts" (Dascal 1987, 68). Mathematical notation is an adequate symbolic language for representing the infinite *because a determinate object does not need to precede symbolization*. For algebra of Descartes' analytic geometry, it is only "once the ratios or proportions are constructed between the known and the unknown magnitudes according to the given conditions of the problem, [that] the algebraist can symbolize each such relationship by writing equations using the chosen notation" (Shabel 2003, 126). With the resources of infinite series expansions (such as the binomial theorem, Taylor series, or the Leibniz's use of the harmonic triangle), the proportions between terms cannot *all* be given, but the *rule* for determining the series can be grasped in a *finite* or *partial* summation. The difference between the thing and its concept is reduced to a degree of comprehension, since a finite mind cannot fully and adequately represent these objects. Even though the manipulation of a well-designed characteristic is a practice that takes place in space and time, it amplifies the content of cognition beyond direct or ostensive constructions, without presuming complete or direct intuition of things as they are in-themselves. This is the trajectory traced by Leibniz and Maimon, who agree that "the sign is an instrument for representing something as an object of intuition that is not [in fact] an object of intuition," which is exemplified by an interpretation of the calculus that attends not only to the spirit of discovery, but to its letter as well (ETP 278).[5] By freeing representation from the constraints of experience, the mind frees itself from the constraints of finitude.

[5] For a detailed explication of Leibniz's thought, see *La Sémiologie de Leibniz*. « Lorsque nous sassions distinctement chaque composante d'une idée, nous avons une connaissance « intuitive » de cette idée. Mais ce type de connaissance n'existe chez l'homme que pour les notions primitives et pour un petit nombre de notions composées peu complexes. La grande majorité de nos connaissances étant donc symbolique, dans la plupart de nos raisonnements nous

Chapter one is dedicated to explaining the role of geometrical construction in Kant's philosophy. Section 1.1 introduces some fundamental Kantian terminology in addition to laying out the structure of cognition and transcendental philosophy. 1.2 introduces the distinction between philosophical and mathematical cognition. This section highlights the connection between the terms *construction, imagination,* and *schema*. 1.3 shows how geometrical (ostensive) construction is the foundation of Kant's philosophy of mathematics, and how algebra has objective validity only through its use in geometry. 1.4 marks a turning point, where the centrality of construction is extended beyond mathematics alone. There is a bifurcation between sensations and their corresponding objects, which Maimon calls the 'universal antinomy of thought in general.' This antinomy is solved (not dissolved) by positing a vector through sensation and the understanding, which Maimon calls a differential of sensation. Chapter two describes Kant's relationship to calculus. 2.1 gives a brief account of Newton's Method of Fluxions, and shows how Kant was able to incorporate this kinematic conception of calculus into his philosophy. 2.2 explains how constructions are meant to prove the validity of a priori judgments. Kant distinguishes between ostensive and apagogic proof, where the former is the standard on which he built his system. However, calculus implicitly relies on apagogic modes of proof because it is impossible to construct infinite series or infinitely small magnitudes. Chapter 3 comes off the heels of the new symbolic paradigm that concludes chapter 2, and develops the notion of symbolic cognition as a response to the shortcomings of ostensive construction. 3.1 begins by reprising real definitions and their role in geometry. From this, it shows that a construction cannot ground synthetic judgments because the *reason* for the correspondence of certain properties cannot be given, only assumed.

n'effectuons que des manipulations de signes, sans jamais les remplacer par les idées correspondantes. Les signes sont donc, pour des raisons dérivées de la théorie de la connaissance, constitutifs de la pensée » (Dascal 1978, 209).

3.2 stresses the difference between actual and possible experience, and how the former involves a passive synthesis not effected by the understanding. Actual experience presupposes a rule that governs the sensible, but that the understanding has no insight into. The imagination has a role in both of these syntheses, in one case it passively determines some element, and in another it actively determines it. This element is the differential. 3.3 gives the Leibnizian background to Maimon's philosophy, and it also shows how the differential as infinitesimal quantity is not compatible with Kant's use of calculus (which relies on fluxions). Differentials are fictional entities that reduce difference to nothing and produce continuity between objects governed by the same rule. The differential quotient, or relation between two differentials, persists even when the individual terms are reduced to zero. Together with the law of continuity and Leibniz's new principle of equality, differentials become the vanishing difference between two things whose difference can be made smaller than any assignable magnitude. Chapter 4 is the final chapter which aims to draw some consequences from the role of differentials in cognition. 4.1 summarizes a demonstration from Leibniz's *De quadratura*, and reprises apagogic proof in relation to continuity and equality. This section gives a proper example of how differentials are able to extend beyond the bounds of intuition, and how they are fictional entities that cannot be circumscribed or inscribed in a construction. 4.2 returns to the philosophy of Maimon, and uses the above lessons to explain how differentials allow the understanding to abstract pure relations from actual appearances. Concepts can legitimately be exhibited by symbols, without recourse to the relations of space and time. 4.3 is both the final section and conclusion. The upshot of Maimon's philosophy is that cognition is not meant to organize all of nature according to concepts. The differential is a passage from the sensible to the intelligible, from the finite to the infinite. The idea of an infinite mind whose panoptic view adequately describes every object is a fiction that brings unity to cognition.

Nevertheless, it is a fiction that enables the mind to become more objective, more real; “a practical rule by which we go into ourselves, as it were, or better, by which we, as such, attain ever greater reality” (ETP 165).

Chapter One

1.1 Empirical Experience at the Center of Transcendental Philosophy

Kant does not give an explicit definition of the transcendental per se, but there is much to infer from its use as an adjective. Two instances are particularly clear. First, in the introduction to the 'Transcendental Logic,' Kant distinguishes general logic from transcendental logic. The former concerns only the relations that one can think between objects in general, including the objects of experience, and makes no distinctions based on the content or origin of its cognitions. As to the later, Kant writes: "Such a science, which would determine the origin, the domain, and the objective validity of such cognitions, would have to be called **transcendental logic**, since it has to do merely with the laws of the understanding and reason, but solely insofar as they are related to objects *a priori* and not, as in the case of general logic, to empirical as well as pure cognitions of reason without distinction" (CPR A57/B82). In this instance, the transcendental describes a kind of logic that is not only concerned only with the form of its cognitions (the laws of the understanding and reason), but also the relation between these laws and the ways that objects are necessarily represented in cognition. Transcendental logic is, in one respect, concerned with the *possibility* of the relation between form and matter, it does not look to empirical experience since the unity and significance of experience is the result of actually applying the rules of the understanding to the matter of sensations. This is the logic of truth insofar as "no cognition can contradict it without at the same time losing all content, i.e., all relation to any object, hence all truth" (CPR A62/B87). The other part of transcendental logic concerns its merely subjective use, and the principles of reason that mitigate illusions, since "it is very enticing and seductive to make use of these pure cognitions of the understanding and principles by themselves, and even beyond all bounds of experience, which however itself alone can give us the matter (objects) to which those pure concepts of the understanding can be applied" (CPR A63/B87). Kant warns that "the understanding falls into the danger of making a material use of the merely formal principles! of pure understanding through empty sophistries, and of judging without distinction about objects that are not given to us, which perhaps indeed could not be given to us in any way" (CPR A63/B88).

Second, in the introduction to the *Critique*, Kant begins to describe the kind of cognition necessary for a pure use of reason, which would be capable of addressing the relation between the particular claims of experience and the universal and necessary claims of reason. Such a discipline is metaphysics. If such a discipline were to bring itself to "certainty regarding either the knowledge or ignorance of objects, i.e., to come to a decision either about the objects of its questions or about the capacity and incapacity or reason for judging something about them, thus either reliably to extend our pure reason or else to set determinate and secure limits," it could rightly be called a science (CPR B21). It is doubtful whether this has ever been achieved, even today, but, the critical project is a first step in this pursuit, and as such, must take stock of the objects, capacities, and limits that have thus far composed the discipline. Kant writes, "I call all cognition transcendental that is occupied not so much with objects but rather with our mode of cognition of objects insofar as this is to be possible *a priori*" (CPR A12/B25). Transcendental cognition is the use of reason directed in assessing *how* and in *what ways* it is possible to have cognition of objects. As in the first instance, this involves analyzing the various faculties at work, and isolating those that do not depend on experience, or can be thought as logically proceeding their application to experience.

Though it may seem that the transcendental is a realm unaffected by actual, empirical experience, the opposite is the case. Kant writes, "There is no doubt whatever that all our cognition begins with experience; for how else should the cognitive faculty be awakened into exercise if not through objects that stimulate our senses and in part themselves produce representations, in part bring the activity of our understanding into motion…thus to work up the raw material of sensible impressions into a cognition of objects that is called experience?" (CPR B1).[6] By this Kant means that the proper use of the understanding is to make judgments about the appearances that present themselves to consciousness. Kant divides cognition into three (possibly four) faculties: Sensible intuition or Sensibility, the Understanding, and Reason.[7] All empirical (*a posteriori*) cognition concerns sensible intuition, which is the only receptive faculty of cognition. All intuitions are sensible since the qualitative component of experience is not the product of a spontaneous act of the understanding; rather, these qualities are representations of how objects 'affect' the senses. Though Kant writes "In whatever way and through whatever means a cognition may relate to objects, that through which it relates immediately to them, and at which all thought as a means is directed as an end, is **intuition**" this is not to say that sensible intuitions of an object are the things themselves (CPR A19/B33). All cognition deals with representations, whether empirical or pure. Since the matter of sensation must be given from without, any representation of empirical objects must depend on what is given in experience. For this reason, there are no definitions of empirical concepts, only explications of them since "I can never be certain that the distinct representation of a (still confused) given concept has been exhaustively developed unless I know that it is

[6] The next paragraph qualifies this statement: "But although all our cognition commences **with** experience, yet it does not on that account all arise **from** experience" (CPR B1). This is the difference between a posteriori and a priori cognitions, which will be explored below.
[7] This list will function to introduce the necessary groundwork, though the imagination will play a large role in what follows. There is some controversy about whether the imagination is a faculty in its own right, since it spans the divide between intuition and the understanding. Heidegger's influential interpretation in *Kant and the Problem of Metaphysics* places the imagination at the centre of cognition (Onof, Schulting 2015, 23-27). Deleuze also includes the imagination among the faculties of *re*presentation, and sensibility as a faculty of presentation (Deleuze 2013, 8-9). Recent commentaries have given more ground to the understanding as that which determines, in some way, the syntheses of the imagination. The imagination would be performing a discursive though 'non-' or 'pre-conceptual' synthesis that unifies intuition prior to judgments of the understanding. This is the thrust of Beatrice Longuenesse's and Michael Friedman's approach (Onof, Schulting 2015, 10, 16-19). Maimon's discussion of the imagination resembles the latter position, though he maintains that we can only represent empirical intuitions *as if* the imagination were governed by the understanding, if the work of the imagination is to be 'legitimate' and not mere play (ETP 134).

adequate to the object," and that it "can contain many obscure representations, which we pass by in our analysis though we always use them in application, the exhaustiveness of the analysis of my concept is always doubtful, and by many appropriate examples can only be made **probably** but never **apodictically** certain" (CPR A728/B756).

As sensible intuitions, all appearances have both matter and form *prior* to their determination and subsumption under a concept. The pure forms of any intuition are space and time, regardless of whether the matter is also given, that is, whether the intuition is pure or empirical. The representation of objects outside of the mind is space, outer sense; and the representation of objects inside the mind is time, inner sense.[8] Kant is explicit that the forms of intuition are not concepts, nor do they have their origin anywhere other than in intuition. In an especially clear footnote, he writes,

> Thus empirical intuition is not put together out of appearances and space (out of perception and empty intuition). The one is not to the other a correlate of its synthesis, but rather it is only bound up with it in one and the same empirical intuition, as matter and its form. If one would posit one of these two elements outside the other (space outside of all appearances), then from this there would arise all sorts of empty determinations of outer intuition, which, however, are not possible perceptions. (CPR A429/B457)

There is no such thing as an *absolute* space or *absolute* time that exists apart from the mind, since this would imply that space and time are forms of things as they are in themselves or apart from any representation. However, even if this were the case and there was a perfect isomorphism

[8] Notice that thoughts or mental representations in general are still *intuitions*. That is to say, the mind perceived empirically is not identical to the unity of apperception that receives these intuitions. As such, all judgments issuing from the understanding are spontaneous, i.e., not implicated in the empirical order of causes (CPR B155-156). This is significant, since Kant introduces a logical *order of reasons* that stands apart from the natural *order of causes*, and as such there can be no reduction of cognition to substance or matter of any kind (this terminology is taken from Brandom 2012). This opens up the possibility of genuinely *objective* judgments. That is to say, connections between concepts, though derived from experience are not grounded in any one experience, but range over a number of instances.

between the forms of intuition, and the relations amongst things themselves, there is no perception that could reveal this coincidence, since, by subjective necessity, all cognition of outer objects is necessarily represented in space. Space and time, as the forms of intuition that must precede any representation of sensation, are *pure* representations. By pure, or *a priori*, Kant names those features of cognition that the subject contributes of itself. This is the discovery that put mathematics at the forefront of scientific innovation, where the geometer must produce a figure "from what he himself thought into the object and presented (through construction) according to *a priori* concepts, and that in order to know something securely *a priori* he had to ascribe to the thing nothing except what followed necessarily from what he himself had put into it in accordance with its concept" (CPR Bxii). Not only does the subject give form to appearances, it also brings unity to the manifold of various qualities located in space and time. It does this by means of the *original unity of apperception*. In many ways, this is the key component of the entire critical system, and it deserves some elaboration.

As we have already mentioned, Kant takes the achievements of both mathematics and natural science as examples of *objective* truth. The subject matter of both sciences can be approached, not independently of all subjectivity, but *independently of any particular subject*. When Kant turns to metaphysics, he asks: What, in experience, must be true for all subjects with minds like my own, or, what is objective in experience? Before answering this, the term *experience* has a technical meaning for Kant. In the B edition of the 'Transcendental Deduction,' Kant makes a distinction between thinking and cognition. Where thinking only requires concepts, cognition requires two components, concepts and intuitions. For the sake of brevity, and because this will be treated later, Kant proposes twelve concepts that "spring pure and unmixed" from the understanding alone (CPR A67/B92). In contrast to the receptive nature of sensibility, which yields sensible intuitions, the understanding is both spontaneous and guided by principles that determine concepts. In sum, "All intuitions, as sensible, rest on affections, concepts therefore on functions. By a function, however, I understand the unity of the action of ordering different representations under a common one" (CPR A68/B93). Since intuitions are the only kind of representation immediately related to an object, concepts can only relate to objects by means of an intuition. Consequently, when Kant calls the understanding the faculty for judging, by judgment he means the representation (concept) of a representation (intuition) of an object by means of a rule (function).[9]

Kant isolates twelve distinct 'functions of judgment' from a fundamental unity of the understanding, that is, twelve ways that the understanding can impart unity to two or more representations.[10] From these are derived the twelve categories, also grouped in four, which, as concepts, act as the more general representations, and "are related to some representation of a still undetermined object" (CPR A69/B94).[11] The understanding can only bring unity to representations by means of concepts, and as such, the categories are not concepts *of* the functions of judgments, but *are* the functions of judgments in action, i.e., are the purely formal rules given content from intuition. Apart from the understanding, sensibility gives a diverse array of intuitions. Even though these intuitions have both form and matter,[12] "Only the spontaneity of our thought requires that

[9] "We can, however, trace all actions of the understanding back to judgments, so that the understanding in general can be represented as a faculty for judging" (CPR A69/B94).

[10] Under four headings are: Quantity – universal, particular, singular; Quality – affirmative, negative, infinite; Relation – categorical, hypothetical, disjunctive; Modality – problematic, assertoric, apodictic (CPR A70/B95).

[11] The Categories are also grouped according to the functions of judgment: Quantity – unity, plurality, totality; Quality – reality, negation, limitation; Relation – inherence and subsistence, causality and dependence, community (reciprocity between agent and patient); Modality – possibility/impossibility, existence/non-existence, necessity/contingency (CPR A80/B106). Also following the functions of relation, causality and community relate concepts to concepts, where substance (and accident) relate a concept and an intuition.

[12] In the scholarship, there are two different ways of interpreting the form proper to sensations. Béatrice Longuenesse holds that the term 'formal intuition' describes the particular spaces and times of various sensations, which, when combined by the understanding, give rise to the pure forms of intuition considered as a singular abstract space and a singular abstract time (Longuenesse 2005, 72). Contrary to this, Henry E. Allison holds that space and time, as singular forms of intuition, are pre-conceptual unities (given all at once). All particular sensible intuitions are limitations of

this manifold first be gone through, taken up, and combined in a certain way in order for a cognition to be made out of this." He continues, "I call this action synthesis. By **synthesis** in the most general sense, however, I understand the action of putting different representations together with each other, and comprehending their manifoldness in one cognition" (CPR A77/B102-103). Judgments are able to unify representations because judgments subsume representations under a more *original* unity.[13] Kant explains that "the understanding is completely exhausted and its capacity entirely measured by these functions [of judgment]," which is a *transcendental unity* that is logically priori to category of unity, under which the categories unite the sensible manifold of intuition (CPR A79/B107).

Returning to the distinction between thinking and cognition, one can think whatever one likes since the concept itself is possible (if it is non-contradictory) without any sensible intuition. Concerning cognition, Kant writes, "the categories do not afford us cognition of things by means of intuition except through their possible application to **empirical intuition**, i.e., they serve only for the possibility of **empirical cognition**. This, however, is called **experience**. The categories consequently have no other use for the cognitions of things except insofar as these are taken as objects of possible experience" (CPR B147-148). Now we are closer to answering the question that spurred this discussion. The distinguishing feature of empirical cognition, or experience, is that sensation must be given, and givenness is only an affection of the *subject* by an object. However, *possible experience* does not require this subjective criterion insofar as it names the

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these pure forms, must be 'cut out' of them (See A32/B48). Formal intuitions are distinct regions of space treated as objects, as such, they are mediated by the concepts of the understanding (Allison 2004, 115). This issue will return when discussing mathematical constructions and the relations of Geometrical figures to objects. I lean more toward Longuenesse's position, since, even if there were something like a singular space for intuition, it could only be represented as a single space by the understanding, since there is no such *perceivable* object (See A429/B457). We will refer to the note on (CPR B160-161) below.

[13] Kant calls it the *original* unity of apperception since the unity under discussion proceeds the category of unity, and must ground the possibility of its use in experience (CPR B135).

formal or pure conditions requisite for the representation of a sensation, which must be true for any and all experience, or *experience as such*. Contrary to how it might seem, no particular experience of an object can be objective by itself. Instead, only insofar as the unity of the object stands as an emblem of the formal unity of consciousness can the experience of an object be abstracted from the particular, subjective features of experience. Kant expresses this differently in the A and B editions of the *Critique*, which nonetheless convey the same point. In the A edition, Kant frames it this way: all appearances, or empirical intuitions, are immediately related to an object that cannot itself be given in experience; since this object itself does not depend on any subjective faculty (only our sensation of it is subjective), it is a non-empirical component of experience that we can *think* but not *cognize*. Kant names this the 'transcendental object = *X*.' He writes:

> The pure concept of this transcendental object (which in all of our cognitions is really always one and the same = *X*) is that which in all of our empirical concepts in general can provide relation to an object, i.e., objective reality. Now this concept cannot contain any determinate intuition at all, and therefore concerns nothing but that unity which must be encountered in a manifold of cognition insofar as it stands in relation to an object. This relation, however, is nothing other than the necessary unity of consciousness, thus also of the synthesis of the manifold through a common function of the mind for combining it in one representation. (CPR A109).

This necessary, synthetic unity of consciousness is, as hinted above, the *transcendental unity of apperception*. This description reveals that the transcendental unity itself is *what is objective* in experience, and can be thought objectively as an empty object = *X*, or subjectively as the a priori unity to which all possible cognitions belong.[14] In the B edition, Kant introduces the term ‘I think’ as that which must accompany all representations, especially those given by intuition. Otherwise, one would find oneself in a situation where “something would be represented in me that could not be thought at all, which is as much as to say that the representation would either be impossible or else at least would be nothing for me” (CPR B132).[15]

It is important to stress that, even though appearances are given by means of the subject alone, by means of their form these appearances can be ordered objectively; that is, they can be represented according to concepts so that the circumstances of perception do not exhaust the possible ways one might experience an object. So, finally to answer the question, ‘What, in experience, must be true for all subjects with minds like my own, or, what is objective in experience?,’ one can say: The objective validity of experience is founded on empirical cognition, that is, the relation of concepts to intuitions, which are the only means for the understanding to represent objects. Since an object can never be given in itself, and cannot directly ground or unify representations of itself, the unified representation of an object is derived from the transcendental unity of apperception. This is not to say that there are no objects beyond consciousness but that the manifold of appearances, and the various qualities of sensation, can only be unified or *thought*

[14] Allison highlights that the transcendental object, and things-in-themselves, overlap since they are both the referents of sensation. Rather than commit Kant to radical idealism, wherein the mind perceives only itself in empirical objects, he takes Kant to be making a methodological point. The thing-in-itself, or object = *X*, is only ever *thought*, and can never be *perceived*. As such, it is a methodological exercise of the understanding; “it stipulates how an object must be considered, if it is to function in a transcendental account as “something corresponding to sensibility viewed as receptivity.” As such, the prohibition does not bring with it any ontological assumptions about the real nature of things or about a super-sensible realm” (Allison 2004, 70). This is not an ontological assumption, only a reflection on the transcendental limits of cognition.

[15] Perhaps the understanding cannot represent such a state to itself, but this does not make it absolutely impossible. In his *Monadology*, Leibniz distinguishes between *perception* and *apperception*, and maintains that one cannot discount “perceptions whose owners were not aware of them” §14. He also maintains that “we experience within ourselves a state in which we remember nothing and have no distinguishable perception, as when we fall into a swoon or are overcome by a deep and dreamless sleep” §20 (L 644).

as corresponding to objects by the understanding.[16] Any use of the understanding, even its use a priori, depends on its possible connection to appearances. In Kant's words:

> Understanding is, generally speaking, the faculty of cognitions. These consist in the determinate relation of given representations to an object. An object, however, is that in the concept of which the manifold of a given intuition is united. Now, however, all unification of representations requires unity of consciousness in the synthesis of them. Consequently the unity of consciousness is that which alone constitutes the relation of representations to an object, thus their objective validity, and consequently is that which makes them into cognitions and on which even the possibility of the understanding rests. (CPR B137).

It is by making objectively valid judgments that one either *clarifies* or *amplifies* the content of the judgment. Clarifying the concept is an analytic judgment because it clarifies the intension of a concept, its identity, or all of the varying marks or determinations that differentiate concepts from one another (CPR B11). Amplifying the concept is a synthetic judgment, since, as already mentioned, the act of synthesis involves connecting diverse representations or determinations, not by means of identity, but by means of the unity of experience. As we will see, space and time as the pure forms proper to sensibility, that is, sensible intuition, are what facilitate both synthetic empirical judgments and synthetic *pure* judgments. The former are judgments *of* experience, which means that two concepts that are not connected analytically (by identity) are thought together in the object. And, since empirical cognition (experience) is objective, the connection is

[16] In *Kant's Critical Philosophy*, Deleuze makes a similar point about the connection between the object = *X* as the correlate of the 'I think.' He writes, "the manifold would never be referred to an object if we did not have at our disposal objectivity as a form in general ('object in general', 'object = x'). Where does this form come from? The *object in general* is the correlate of the 'I think' or the unity of consciousness; it is the expression of the *cogito*, its formal objectivation" (Deleuze 2013, 15). The emphasis on the generality of this unity is meant to emphasize the role of the understanding from that of imagination, where the former "is not synthesis itself, it is the unity of synthesis and the expression of that unity" (ibid., 16). A positive characterization of the imagination follows below.

transcendentally necessary, though not logically necessary.[17] The latter are judgments of *possible experience*, where, "if we want to go beyond the given concept in an *a priori* judgment, we encounter that which is to be discovered *a priori* and synthetically connected with it, not in the concept but *in* the intuition that corresponds to it" (CPR B73). The synthetic connections are made by means of the pure forms of sensible intuition and concern how the pure concepts of the understanding (categories) must be adapted to space and time, so that they can have transcendental, in addition to logical, value.

1.2 Philosophical and Mathematical Cognition

The model of cognition centered on experience makes sense, since the aim of cognition is to sort the manifold sensations affecting the mind into a series of relations that obtain between objects, independent of the order of perception. This model is less obviously true of mathematical cognition, "For the object that it thinks it also exhibits *a priori* in intuition, and this can surely contain neither more nor less than the concept, since *through the explanation of the concept the object is originally given*, i.e., without the explanation being derived from anywhere else" [emphasis added] (CPR A730/B758). The primary difference between 1) philosophical and 2) mathematical cognition is this: 1) whether cognition a priori operates *in abstracto*, that is, whether concepts are treated only as universals without a determinate object; or, 2) whether cognition operates *in concreto*, i.e. whether concepts treat an individual, though pure, intuition that is not given in experience (CPR A735/B763). Though this distinction occurs more than five hundred

[17] In the *Prolegomena*, judgments of experience are contrasted with judgments of perception. Where the former are objective since they depend upon both the categories and the transcendental unity of apperception, the latter "require no pure concept of the understanding, but only the logical connection of perception in a thinking subject" (Kant 2001, 38). By logical Kant does not mean analytic, only that the connection is made formally and not transcendentally. Experience can yield objectively synthetic judgments only if the connections are required to cognize an object, not if the connections are established by affection or any other subjective ground.

pages into the *Critique*, mathematical cognition has been a staple throughout. In the preface to the B edition Kant writes:

> A new light broke upon the first person who demonstrated the isosceles triangle (whether he was called "Thales" or had some other name). For he found that what he had to do was not to trace what he saw in this figure, or even trace its mere concept, and read off, as it were, from the properties of the figure; but rather *that he had to produce the latter from what he himself thought into the object and presented (through construction) according to* a priori *concepts*, and that in order to know something securely *a priori* he had to ascribe to the thing *nothing except what followed necessarily from what he himself had put into it in accordance with its concept*. [emphasis added] (CPR Bxii).[18]

This last clause highlights the dynamics of mathematical cognition, namely, that the sole content of the object is derived from the rules of its composition, or, its concept thought in relation to intuition. And it is precisely the character of this relation to intuition that distinguishes the two types of cognition.

Kant's most clearly formulates this distinction in the first chapter of 'The discipline of pure reason.' He makes the distinction because, when one treats philosophy by the standards of mathematics, it becomes dogmatic, i.e. it does not ask the critical or transcendental questions about the use of reason. This is because mathematics supplies both the form *and matter* of its cognitions a priori, which is impossible for philosophy. Since this distinction will take us through to section 3, it is worth quoting in full:

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[18] This formulation returns with Maimon, who takes mathematical cognition as the standard of *all* cognition a priori. The objective validity of judgments depends on the understanding alone, since it is the *ground* of the connection. He writes: "All that the understanding can assume with certainty in the object is what it itself has put into it (in so far as it has itself produced the object itself in accordance with a self-prescribed rule), and not anything that has come into the object from elsewhere" (ETP 59-60).

> **Philosophical** cognition is **rational cognition** from **concepts**, mathematical cognition that from the **construction** of concepts. But to **construct** a concept means to exhibit *a priori* the intuition corresponding to it. For the construction of a concept, therefore, a **non-empirical** intuition is required, which consequently, as intuition, is an **individual** object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere ***imagination***, in pure intuition, or on paper, in empirical intuition, but in both cases completely *a priori*, without having had to borrow the pattern for it from any experience…Philosophical cognition thus considers the particular only in the universal, but mathematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless *a priori* and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept, to which this individual corresponds only as its ***schema***, must likewise be thought as universally determined. [emphasis added] (CPR A713-714/B741-742).

Three terms– construction, imagination, and schema – hold this quote together and name the key differences at play. The imagination and schema are introduced in close proximity to Kant's discussion of the categories because they are together the *reason* or *ground* that authorizes application of the categories to sensible intuitions. Constructions do not depend on sensibility in the same way since pure intuitions can be *spontaneously* produced by the imagination. Mathematical cognition, therefore, possesses a distinct relationship to the object s of experience.

The imagination is given a distinct role in the 'Transcendental deduction of the pure concepts of the understanding,' and is discussed in the section entitled 'On the application of the categories to objects of the senses in general.' There is much to glean from its situation. For Kant, deduction is less associated with the logical or mathematical procedure of drawing necessary conclusions from a given set of premises, than it is with the legal procedure of providing proof for the question of what is lawful (*quid juris*).[19] The transcendental deduction is the procedure for providing proof by means of the laws or the rules of the understanding alone, of how pure concepts can apply to objects (CPR A85/B117). Thus far, we have described the categories in relation to appearances – the matter of sensation and the form of intuition are first given, and the understanding orders them as belonging to an object – but what has yet to be shown is *by what right* or *according to what reason* these heterogeneous representations can interact. Concerning sensibility, the rule governed action of the understanding is made legitimate by means of inner sense, or time.

The transcendental subject (the "I think") can only represent itself by means of intuition; that is, representation of its contents must occur in inner sense. All intuitions, then, must be capable of taking the form of time, since any perception through outer sense can be represented as a thought, or a content of the empirical mind. However, the situation becomes more complicated when considering contents of the mind that are *not* prompted by outer sense. In this case one must

[19] The full quote is as follows: "Jurists, when they speak of entitlements and claims, distinguish in a legal matter between the questions about what is lawful (*quid juris*) and that which concerns the fact (*quid facti*), and since they demand proof of both, they call the first, that which is to establish the entitlement or the legal claim, the **deduction"** (CPR A84/B116). Kant takes the question of fact as already settled, i.e., we know that this already occurs in mathematics and natural science. He begins the *Prolegomena* by writing in the preface, "there are enough of them [synthetic a priori judgments] which indeed are of undoubted certainty, and as our present method is analytical, we shall start from the fact that such synthetic but purely rational cognition actually exists; but we must inquire into the ground of this possibility and ask *how* such cognition is possible" (Kant 2001, 18). It might be contested that this assumption is valid only for the *Prolegomena*¸ since its method of presentation is 'analytical' as compared to the *Critique.* But, in the B introduction, Kant demonstrates that both mathematics and physics produce synthetic a priori judgments, and argues that, "since they are actually given, it can appropriately be asked how they are possible; for that they must be possible is proved through their actuality" (CPR B20). Since metaphysics is not yet a science, Kant infers that such a science is possible because the kernel of our 'natural predisposition' to abstract beyond experience is identical to that of mathematics and physics. The questions *quid facti* and *quid juris* will return with Maimon's objections to Kant.

turn to the imagination [*Einbildungskraft*], which "is the faculty for representing an object even **without its presence** in intuition" (CPR B151). The imagination at once belongs to sensibility – it is subjective because the intuitions do not originate from an object, but are *produced* – and to the understanding as an act of spontaneity. Kant remarks that, "the imagination is to this extent a faculty for determining the sensibility *a priori*, and its synthesis of intuitions, **in accordance with the categories**, must be the transcendental synthesis of the **imagination**, which is an effect of the understanding on sensibility and its first application (and at the same time the ground of all others) to objects of the intuition that is possible for us" (CPR B152).[20] This transcendental synthesis of the imagination is also called a *figurative* synthesis (*synthesis speciosa*), or the productive imagination, since any combination that is thought by the understanding is also exhibited in time. It is the synthesis that is responsible for the *duration* of thought, the time involved in passing from one representation to another. Kant provides these examples: "We cannot think of a line without **drawing** it in thought, we cannot think of a circle without **describing** it, we cannot represent the three dimensions of space at all without **placing** three lines perpendicular to each other at the same point…The understanding therefore does not **find** some sort of combination of the manifold already in inner sense, but **produces** it, by **affecting** inner sense" (CPR B154-155). The transcendental synthesis of the imagination does not describe the relationship of a concept to intuition in general,[21] but determines or structures intuition as the understanding thinks a concept. To reiterate, this is not the synthesis that brings already given representations together in one

[20] This synthesis does not belong to sensibility, since it is only a passive, determinable faculty, and it is not necessary to think the synthesis in terms of intuition either. "[The first synthesis of the understanding] is nothing other than the unity of the action of which it is conscious as such even without sensibility, but through which it is capable of itself determining sensibility internally with regard to the manifold that may be given to it in accordance with the form of its intuition" (CPR B153, B161).

[21] This is called the 'combination of the understanding,' or an intellectual combination. This can be thought as the general sort of combination that is true of the transcendental unity of apperception. This synthesis already assumes the connection between the rules of the understanding, and the determinate order in sensibility, which is not yet accounted for (CPR B151).

cognition. This synthesis produces order in inner sense, where the sole source of this order comes from the rules of the understanding. In this circumstance, Kant explains, “the understanding always determines the inner sense, in accordance with the combination that it thinks, to the inner intuition that corresponds to the manifold in the synthesis of the understanding” (CPR B156). Here, the ‘inner intuition’ stands for the implicit relation that, as properly transcendental, the categories posses to intuition in general. Sensibility can, at most, give the subjective associations between appearances, which it does not produce, but receives. In contrast, the understanding determines sensibility, not by producing a determinate intuition, but by adapting its pure rules to intuition. The product is not a determinate content, but *the determinateness of the combination itself*, in the form of a single, pure time.[22]

This leads to the second term, namely, schema. Different categories are thought by means of different rules, e.g. causality is thought by hypothetical judgment, and community (reciprocal cause-effect relations) is thought through disjunctions. These concepts can never be given empirical content, so they are cognized as the “representation of a general procedure of the imagination for providing a concept with its image;” this, Kant writes, “is what I call the schema for this concept” (CPR B180). Like the productive imagination, a schema straddles the gap between sensibility and understanding, but it is a general procedure, or a specific rule-set, for thinking a concept in intuition. As above, the transcendental status of these rules means they must be formulated in the language of intuition, and at least in pure time produced by the imagination.

[22] I am using ‘pure time’ in the way Kant uses ‘pure space’ in the ‘Analogies of Experience:’ “Space, prior to all things determining (filling or bounding) it, or which, rather, give an **empirical intuition** as to its form, is, under the name of absolute space, nothing other than the mere possibility of external appearances, insofar as they either exist in themselves or can be further added to given appearances.” Space ‘prior’ to determination which gives the form of empirical intuition (prior in a logical not temporal sense) is only the possibility of external appearances, just as pure time is the possibility of internal appearances. It is not itself an object since it is ‘not a possible perception,’ which supports the interpretation that ‘pure’ space and time can only be ‘represented as object’ in anticipation of empirical experience. The transcendental synthesis of the imagination orders inner sense by representing these appearances as parts of a pure time which is itself ordered by the categories.

Schemata are transcendental determinations of pure time. “Now a transcendental time-determination is homogeneous with the **category** (which constitutes its unity) insofar as it is **universal** and rests on a rule *a priori*. But it is on the other hand homogeneous with the **appearance** insofar as **time** is contained in every empirical representation of the manifold” (CPR A139/B178). All concepts have schemata, even empirical concepts. The schemata of empirical concepts, that is, those concepts that can be perceived or given by sensibility, are not any determinate intuition either, but only ‘a monogram of pure a priori imagination.’ This monogram or outline consists only of the marks or determinations that differentiate one concept from another. Given the example of a dog, one can “specify the shape of a four-footed animal in general, without being restricted to any single particular shape that experience offers me or any possible image that I can exhibit *in concreto*” (CPR B180). No individual or fully determinate representation can be given, insofar as a schema is only the specific rule set necessary for recognizing a concept in experience, or constructing it a priori.[23] Whence the schemata issue, Kant has nothing to say except “[the schematism] is an art hidden in the depths of the human soul, whose true operations we can divine…only with difficulty” (CPR A141/B181). As to their use, much more can be said.

In the *Jäsche Logic*, Kant discusses the possibility of a fully determinate concept: “The highest completed determination would yield a *thoroughly determinate* concept (*conceptus omnimode determinatus*), i.e. one to which no further determination might be added in thought” (Kant 1992, 596). In the note to this section he writes, **“**Since only individual things or individuals are thoroughly determinate, there can be thoroughly determinate cognitions only as *intuitions*, but

[23] In the A edition of the ‘Deduction of the pure concepts of the understanding,’ Kant divides cognition into three syntheses – apprehension, reproduction, and recognition – which do not have such neat divisions in the B edition. The first provides the unity of intuition, the second the particular rule, and the third the unity of rules. In the B edition, the unity of intuition is given by the unity of apperception, since all intuitions are related to the original unity through the unity of judgments (CPR B141). The third synthesis, recognition in the concept, is the objective unity of experience, the transcendental object, which is nothing but the transcendental unity thought immanent to objects. Recognition is used here to designate the empirical use of schemata, and construction to designate the a priori use of schemata.

not *as concepts*; in regard to the latter, logical determination can never be regarded as complete" (Ibid., 597). Returning to the distinction between philosophical and mathematical cognition, it is possible to see why philosophical cognition is 'rational cognition from concepts.' Apart from experience, the understanding can only give itself schemata, or the intuitive exhibition of its concepts. Even if the reproductive imagination supplies a distinct image in intuition, this "synthesis is subject solely to empirical laws, namely those of association," and cannot be said to correspond to an object at all (CPR B152). It is impossible to spontaneously produce an appearance, that is, to generate a sensible quality by means of specification, since, "even if we have a concept that we apply *immediately* to individuals, there can still be specific differences in regard to it, which we either do not note, or which we disregard" (Kant 1992, 595). The only means by which we represent individuals, or fully determinate qualities, is through experience.

This is not the case with *pure* sensible concepts, i.e., the concepts of mathematical objects. Recall that mathematical cognition "considers the universal in the particular, indeed even in the individual, yet nonetheless *a priori* and by means of reason…the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined" (CPR A714/B742). Where philosophical cognition considers objects that can never be adequately defined by their concepts, mathematical cognition considers objects whose determinations are adequately represented in their concept. This difference, as mentioned, is established by the fact that it is possible to *construct* the concept in pure intuition by means of the figurative synthesis and of the schemata alone. The act of synthesis is proper to the imagination, and the imagination only constructs a concept when the understanding gives unity to the act of imagination through schemata.[24] Michael Friedman argues that "Such constructive operations,

[24] Deleuze emphasizes that, for the imagination, schematizing is only one of its possible modes. He writes that the imagination "schematizes only in the speculative interest. When the understanding takes up the speculative interests,

have all the generality or universality of the corresponding concepts: they yield with appropriate inputs, *any and all* instances of these concepts" (Friedman 2012, 237). In other words, though no object can be given a priori, all of the determination sufficient to define the object can be exhibited in pure intuition without the object itself.

In *Kant and the Exact Sciences*, Friedman provides a detailed reading of Kant's philosophy of mathematics. He maintains that the objects of mathematics are empirical, but the schema is a kind of interface between the concept and the object.[25] He writes, "Their role is to provide something essential to the mathematical concepts themselves: namely, the possibility of a kind of rigorous representation of – more precisely, the possibility of a kind of rigorous reasoning with – these concepts that goes far beyond the resources of mere general logic as Kant understands it" (Friedman 1992, 125-126). Schemata are not the objects of mathematical cognition; those are reserved for the empirical cognition of figures. Kant is explicit about the necessity of experience, even for the concepts of mathematics: "all concepts and with them all principles, however *a priori* they may be, are nevertheless related to empirical intuitions, i.e., to *data* for possible experience. Without this they have no objective validity at all, but are rather a mere play, whether it be with

that is, when it becomes *determining*, then and only then is the imagination *determined* to schematize" (Deleuze 2013, 18). Deleuze leaves room for an *undetermined* or *free* use of the imagination that characterizes reflective judgment, as in the *Critique of the Power of Judgment*. Deleuze is influenced by Maimon when he argues that "in reflective judgment nothing is given from the standpoint of the active faculties; only a raw material presents itself, without really being 'represented'" (ibid., 60). In Maimonian terms, one reduces the given sensation of an object to its elements, and from there, one can let the construction of this object in intuition arise according to the rules that must have originally composed the elements. Beth Lord notes that Kant was composing the third *Critique* when he received Maimon's manuscript, and was probably engaged in solving the problem of the growth of scientific concepts induced by experiment and observation (Lord 2011, 128-129).

[25] This differs from Lisa Shabel's reading of the schemata, where a priori constructions of the concept are equally the objects of mathematics, since both are sufficiently determined by a schema. For instance, "I claim that the schema of a pure sensible concept is that aspect of the pure intuition of a mathematical concept that distinguishes it from an empirical intuition of the same concept, namely, the procedure for constructing that concept according to a rule…The schema, or rule of construction, insures that *any* intuition of a triangle so constructed is adequate to represent the concept of triangle, and so represent it generally, since it will have been constructed on the same conditions and in accordance with the same rule as every other" (Shabel 2003, 112-113). This makes sense insofar as any empirical triangle, as an intuition, is never fully adequate to its concept and, as empirical intuition, contains more determinations than relevant. So, by attending only to the rules sufficient for construction it is easy to parse the relevant features of the intuition. However, the 'objective' character of mathematics becomes difficult to maintain.

representations of the imagination or of the understanding" (CPR A239/B298). Construction does not replace this demand; instead, construction resembles the figurative synthesis of the imagination in that intuitions arise alongside determinations of the concept. The difference is thus: where figurative synthesis gives pure concepts transcendental form, construction gives the pure forms of space and time a set of concepts. As Friedman writes: "pure intuition cannot be said to provide a model for Euclidean geometry at all; rather, it provides the one possibility for a rigorous and rational *idea* of space" (Freidman 1992, 94). For Kant, the two paradigms of pure mathematics are geometry and kinematics, which are the sciences of pure space and time, respectively. Since kinematics is less discussed in the *Critique*, and never by that name, the present discussion will focus on geometry, as well as arithmetic and algebra.[26]

1.3 Ostensive and Symbolic Construction

When discussing construction, Kant implicitly takes geometrical constructions, specifically the constructions utilized in Euclid's *Elements*, as the standard model. It is only in comparison to arithmetical and algebraic equations that Kant distinguishes how exactly the category of *quantity*, sometimes called magnitude, is applied differently in across these branches of mathematics. As mentioned above, there are four groups of categories – Quantity, Quality, Relation, Modality – the first two of which Kant calls mathematical, and the latter two dynamical. This division is founded on whether the category brings unity to intuitions, or if it brings unity to concepts. For instance, the category 'unity' is a universal judgment that ranges over intuitions and brings them under a common concept, where 'causality' is an hypothetical judgment, 'if-then,'

[26] Kant does discuss pure motion in the first *Critique*, but it is usually tied to drawing the lines of geometrical figure. There are also mention of 'flowing quantities,' which correspond to Newton's fluents and his kinematic interpretation of the calculus, see chapter 2. For an account of Kinematics as it is elaborated in Kant's *Metaphysical Foundations of Natural Science*, and other works, see Freidman (1992, 213-241).

that connects one concept to another concept by means of a rule. Currently, only the mathematical categorise are of interest; even then, it is the categories under quantity that will be the focus of discussion.

The principles for applying the categories of quantity and quality, the Axioms of Intuition and the Anticipations of Perception, are also called constitutive since "they justified applying mathematics to appearances, pertained to appearances with regard to their mere possibility, and taught how both their intuition and the real in their perception *could be generated* in accordance with the rules of mathematical synthesis, hence how in both cases numerical magnitudes and, with them, the determination of the appearances as magnitude could be used" [emphasis added] (CPR A178/B221). The constitutive function of these principles is what allows concepts to be constructed, since the 'particular' or 'individual' that functions as the 'object' of mathematical cognition can be given determinately, namely, as a "numerical magnitude." Without too much detail, in order to keep on topic, the concept of 'number' in the above quote has a very specific meaning. For Kant, the concept of number is the schema of 'pure magnitude' or all the categories that fall under quantity. It is through number that the pure rules that define the categories take form as "a representation that summarizes the successive addition of one (homogenous) unit to another" (CPR A142/B182). The relevant categories – unity, plurality, singularity – all compose the manifold of intuition, bring appearances together in different ways, and produce a synthetic unity [or unit, *Einheit*] *in* the manifold. These unities, or units, as parts make the representation of the whole possible because each part must be generated in order to be brought together. This is what defines extensive magnitudes (CPR A162/B203). The concepts of geometry, arithmetic, and algebra differ from the concepts of other a priori sciences, e.g., physics, since the objects of mathematics are not so much 'things' as they are determinate relations of space. In other words, constructions are composed of magnitudes, each of which is a determinate part of space, and the intuitive representation of number.

In an infamous footnote, Kant introduces the term ‘formal intuition’ to describe the act of taking the forms of intuition themselves as objects under the categories, rather than as the forms through which the categories first become operative in experience.[27] The controversy arises around Kant’s retroactive description of sensibility in light of the transcendental synthesis of the imagination. The footnote reads:

> Space, represented as **object** (as is really required in geometry), contains more than the mere form of intuition, namely the **comprehension** of the manifold given in accordance with the form of sensibility in an intuitive representation, so that the **form of intuition** merely gives the manifold, but the **formal intuition** gives unity of the representation. In the Aesthetic I ascribed this unity merely to sensibility, only in order to note that it precedes all concepts, though to be sure it presupposes a synthesis, which does not belong to the senses but through which all concepts of space and time first become possible. For since through it (as the understanding determines the sensibility) space or time are first **given** as intuitions, the unity of this a priori intuition belongs to space and time, and not to the concept of the understanding (§ 24). (CPR B160-161)

The last sentence is what causes trouble, since Kant maintained in the ‘Transcendental Aesthetic’ that the forms of intuition cannot be built up out of parts, because perception of these parts already presupposes that appearances have form, i.e., are sensations in space or time.[28] Here,

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[27] This footnote has been the subject of much discussion, since it implicates a number of terms active in both philosophical and mathematical cognition. For a thorough overview of the various interpretations, and an original reading, see ‘Space as Form of Intuition and as Formal Intuition: On the Note to B160 in Kant’s *Critique of Pure Reason*’ (Onof, Schulting 2015).

[28] See the third point in the ‘Metaphysical exposition of space:’ For, first, one can only represent a single space, and if one speaks of many spaces, one understands by that only parts of one and the same unique space. And these parts

he says that a synthesis is necessary to endow sensibility with unity, to produce objective, rule bound combination. This is given by the figurative synthesis which Kant mentions by citing (§ 24) where this is outlined. He concludes that the unity of intuition cannot be deduced from the transcendental unity of apperception, which belongs only to the understanding. By means of the figurative synthesis, and Kant is vague here, space and time are first *given* as the form sensible intuitions, as *spaces* and *times*. Béatrice Longuenesse takes this to mean that "When unified under the transcendental unity of apperception, before any concept, the form of intuition is again the form for a matter, the appearance or ''indeterminate object of empirical intuition''; this form, considered independently of any matter, is ''pure intuition'' or ''formal intuition'' (the space ''that is needed in geometry'')" (Longuenesse 2005, 72). There are two stages: 1) appearances considered with respect to sensibility alone have form since no matter is without form; this form is without unity in the transcendental sense because it is not *objective* but merely subjective; 2) the figurative synthesis determines sensibility by applying rules to appearances; once these rules are given intuitive form, the understanding can spontaneously represent these rules a priori as a cognition, without the need for a given appearance. Longuenesse also argues that, in the 'Table of Nothings' (CPR A292/B348) and elsewhere, Kant defines the pure forms of intuition as *ens imaginarium* – beings of the imagination – and writes, "To say that space and time are 'beings of imagination' is not to say that they are fictions (*Dichtungen*) of imagination.[29] It is to say, however,

cannot as it were precede the single all-encompassing space as its components (from which its composition would be possible), but rather are only thought in it. It is essentially single; the manifold in it, thus also the general concept of spaces in general, rests merely on limitations" (CPR A25/B39).

[29] Maimon characterizes the pure forms of space and time in both ways. "As intuition, space (and it is the same for time) is therefore an *ens imaginarium*: it arises because the imagination [*Einbildungskraft*] imagines as absolute what exists only in relation to something else: absolute place, absolute movement, and "the like are of this kind. Indeed the imagination even goes as far as determining these, its fictions [*Erdichtungen*], in manifold ways; the objects of mathematics arise in this way (the difference between the absolute and relative way of looking at things is merely subjective, and changes nothing in the object itself)" (ETP 19). For Maimon, space and time as formal intuitions are unified by concepts. Absolute space and time are thoroughly discursive because the understanding directs the imagination to hypostatize the relations that originally unified the manifold of sensibility.

that the imagination forms no imaginary representation without forming a representation of space and time" (ibid., 74). Pure intuition can only be imagined or thought in abstract, apart from any given intuition. But, the relation of pure space and time to any object can only have "empirical reality as the form of appearances" (ibid.).[30] That is to say, space and time can never themselves be empirically given, and have objective validity only as forms of appearance.

Returning to geometry, all construction a priori takes space as object. In doing so, any determination of pure space by concepts will necessarily take the *form* of a possible empirical appearance. Kant defines ostensive construction as construction of the *objects themselves* (CPR A717/B745). The philosopher begins from concepts, but the geometer begins with construction, "by means of which I put together in a pure intuition, just as in an empirical one, the manifold that belongs to the schema of a triangle in general and thus to its concept, through which general synthetic propositions must be constructed" (CPR A718/B746). Concepts are true of things in general, which can at most describe a priori the general conditions under which there can be experience. Constructions are true of individuals, where the *rules for construction* are true of the concept in general. The schema of an object is a way of reasoning with mathematical concepts that general logic cannot accomplish, and that sensation alone cannot either. Schemata ground the pure sensible concepts of mathematics and all of the synthetic a priori judgments of mathematics,

[30] Additional evidence comes from 'On Kästner's Treatises.' "Metaphysics must show how one can **have** the representation of space, geometry however teaches how one can **describe** a space, viz., exhibit one in the representation a priori (not by drawing). In the former, space is considered in the way it is **given**, before all determination of it in conformity with a certain concept of object. In the latter, one [i.e. a space] is **constructed** (*gemacht*). In the former it is **original** and only one (unitary) space, in the latter it is **derived** and hence there are (many) spaces, of which the geometer however, in accord with the metaphysician, must admit as a consequence of the foundational representation of space, that they can only be thought as parts of the unitary original space" (Kant 2014, 309). Just as the transcendental unity of apperception is original because logically prior to experience, so pure space and pure time are original. As such, they are not objects of experience, but only the possibility of appearances ordered objectively. Also, see ftn.22 above.

because intuition gives the understanding additional capacities that it did not have in itself, i.e. they allow the pure concepts to be governed by more than just the principle of contradiction.[31]

As an example, take Kant's distinction between *magnitudes* [*quanta*] and *mere magnitude* [*quantitatem*]. The former refers to the constructed geometrical figures as discussed above, namely, figures that take the form of possible appearances. The action of the understanding on pure space determines space according to the schema of a concept, which, because the categories are directed at intuition, thinks the possible relation to an object. The latter refers to the magnitude involved in counting or calculation, which are not figures but *operations*. Both Lisa Shabel and Michael Friedman give succinct definitions of each, which highlight different facets of the distinction. For Shabel, "Magnitudes in the first sense, as sized objects, are the constructible objects of geometry; these are conceived both quantitatively and qualitatively, that is, with respect to both their size and their shape, or figure. Magnitude in the second sense, as the size *of* an object, is to conceive magnitude in the first sense with respect to its quantitative aspect only; that is, to conceive of size without shape, quantity without quality" (Shabel 2003, 123). Two things are of note: firstly, the magnitude of geometrical figures involves the representation of quality in addition to quantity; secondly, that 'mere magnitude' is a more general characterization since figures necessarily have quantity, where mere magnitudes do not have quality. Friedman does not frame the distinction in terms of quantity and quality, but of construction and operation. For him, "*Quanta,* objects of intuition as magnitudes, are just the particular magnitudes there happen to be. These are given, in the first instance, by the axioms of Euclid's geometry, which postulate the construction…of all the relevant spatial magnitudes" (Friedman 1992, 114). As with Shabel, the

[31] Take, for instance, Kant's discussion of the concept of 'triangle:' "In fact it is not images of objects but schemata that ground our pure sensible concepts. No image of a triangle would ever be adequate to the concept of it. For it would not attain the generality of the concept, which makes this valid for all triangles, right or acute, etc., but would always be limited to one part of this sphere" (CPR A141/B180).

constructions are not empirical objects, but 'magnitude as object,' which is real or qualitative only in relation to sensibility. Things become more interesting with the second magnitude: "*Quantity*, the concept of a thing in general through the determination of magnitude, comprises the operations and concepts invoked by arithmetic and algebra for manipulating, and thereby calculating the specific magnitude of any magnitudes that happen to exist [*quanta*]" (ibid.). In addition to its generality, mere magnitude also comprises the operations that underly construction, namely, the calculation of numerical magnitudes that determine the specific ratios of the figure. The former is involved in ostensive construction, and the latter in symbolic construction.

Kant describes symbolic construction as the process of abstracting magnitude from the construction, where the mathematician "chooses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, extraction of roots, etc., and, after it has also designated the general concept of quantities in accordance with their different relations, it then exhibits all the procedures through which magnitude is generated and altered in accordance with certain rules in intuition" (CPR A717/B745). He says that this is characteristic of algebra, but seemingly counter to Friedman, not arithmetic. Shabel and Friedman disagree on the status of algebra, specifically, whether it is a kind of generalized arithmetic or a general representation of magnitude used for solving arithmetical or geometrical problems. Friedman's argument is mainly supported by Kant's letter to August Wilhelm Rehberg [1790], and his *Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality* [1766], in addition to Kant's discussion of arithmetic and calculation in the *Critique*. For instance, Kant's well-worn example of '7 + 5 = 12' is accompanied by his description of the process of counting.

In the 'Axioms of Intuition' Kant says that there are only singular formulas in arithmetic, not general axioms as in geometry, because "it is only the synthesis of that which is homogenous (of units) that is at issue here, the synthesis can take place only in a single way, even though the subsequent **use** of these numbers is general" (CPR A164/B205). Unlike constructions, arithmetic attends to the *synthesis* of units [*Einheiten*] which are pure extensive magnitudes ordered by the schema of quantity (number). Though individual portions of space are represented, these spaces are mere forms of intuition, and can (and indeed must) be filled with different matter. While discussing number, Kant explains that "the concept of magnitude seeks its standing and sense in number, but seeks this in turn in the fingers, in the beads of an abacus, or in strokes and points that are placed before the eyes" (CPR A240/B299). However, number as schema of quantity is a determination of intuition by the understanding. And as above, the transcendental synthesis of the imagination must precede any a priori cognition, wherein pure space is the 'matter' of the representation, and render the rules of the understanding intuitive so that they can act as form to the matter of an a priori cognition. The accumulation of synthetic unities can only be given in time, and the resultant numbers are only the 'mere quantity' produced by this synthesis in time. According to Friedman, "only the unboundedness of temporal succession can guarantee the infinity of the number series, and so on. For Kant, then, "The successive iteration made possible by the pure intuition of time, in other words, is a necessary condition for our possession of the *concept* of magnitude (*quantity*) itself: without such iteration we would be quite unable even *to think* the magnitude of any given thing" (Friedman 1992, 122). There can be no largest number because there can be no 'final' time. As a pure form of intuition, time is thought as the singular and infinite form through which all inner representations must be expressed.

One issue immediately arises: how are we to represent $\sqrt{2}$, or other irrational magnitudes if the resources of arithmetic only give determinate units? Friedman answers that,

> In particular, the "arithmetic of numbers" we are limited to cases where "the ratio of the magnitude to unity is determinate," whereas in "general arithmetic [algebra]" we can also consider "indeterminate magnitudes." And this means, I suggest, simply that the arithmetic of numbers is concerned only with rational magnitudes, whereas general arithmetic or algebra is also concerned with irrational or incommensurable magnitudes…by 'determinate ratio to unity' Kant here means rational ratio to unity, for even two incommensurable magnitudes certainly have a (definite) ratio – this, in fact, is the whole point of the theory of ratios. (ibid,.109-110, 111)[32]

Where an arithmetical representation is consigned to an indefinite series of decimals ($\sqrt{2} = 1.14121$ ...), algebra can adequately represent this value since it 'chooses a certain notation' for this magnitude and can "exhibit all the procedures through which magnitude is generated and altered in accordance with certain rules in intuition" (CPR A717/B745). That is, if a is the known positive magnitude, and x is the value of its root, then the ratio $1:x = x:a$, since the square root of a, Kant writes to Rehberg, is equivalent to "the mean geometrical proportional between 1 and a" (quoted in ibid., 110). That is, the ratio of 1 to x is that of x to a. Though this cannot be calculated, it can be *constructed*, namely, as a right-triangle. Geometrical construction does not need to give the numerical value because it can give an intuition of the object itself, and represent the ratio as a property of the thing.[33]

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[32] For a thorough account of Kant's relation to the Eudoxian theory of proportions, see (Sutherland 2004, 160-168); for the relationship between algebra, magnitude and proportion, see (Sutherland 2006, 554-558).

[33] Friedman makes this point explicit: "To be really possible [not just logically] in this context, I suggest, is to be constructible by a procedure of successive iteration in a finite number of steps…This much, in other words, is guaranteed by the mere concept of *quantity*, no matter what *quanta* there are. The possibility of a *quantum* with irrational magnitude, however, cannot be known in this way; for here an appeal to mere *quantity* or "number-intuition" is necessarily non-terminating. In this case, therefore, we must appeal to geometry in order to construct such a magnitude in a finite number of steps" (Friedman 1992, 118).

Lisa Shabel takes issue with lumping algebra and arithmetic together. She makes the historical point that, at the time, algebra was a *general method* for solving both arithmetical and geometrical problems. She spends a large part of *Mathematics in Kant's Critical Philosophy* devoted to the textbooks and mathematical treatises of the eighteenth century, in order to understand the function of algebra in *mathematical practice*, in addition to philosophical practice. She builds up to Christian Wolff's *Elementa Matheseos Universae*, the final version of which appeared in 1742 (Shabel 2003, 141n.1). Wolff and Leibniz are common targets in the *Critique* because they philosophize dogmatically, that is, without consideration of the bounds of human knowledge and experience.[34] Similarly, Kant takes Wolff's mathematical work and subjects it to critical scrutiny. In the 'De Algebra' section of the *Elementa*, Wolff employs algebraic equations to solve both numerical (arithmetical) and geometrical problems, where "the unknown quantity is isolated and expressed in terms of known quantities only" (ibid., 77). For arithmetical problems, the solution consists in substituting the numerical values of known quantities which, in turn, give the unknown numerical quantity. This is not the case for geometrical problems, since, counter to arithmetic, the *objects* of study cannot be introduced into the equation. Shabel explains, "the symbolic expression of a geometric magnitude is insufficient as a final solution of a geometric problem; just as the geometric schemes or figures were constructed to facilitate recognition of the relationships between the symbolically expressed magnitudes, so must a geometric figure be constructed to satisfactorily *show* the referent of the symbol for the sought-for unknown" (ibid., 78). This parallels the above account, especially with regard to geometrical (ostensive) constructions. The solution to a problem consists in the demonstration or *exhibition* [*Darstellung*]

[34] For instance, see the section 'On the amphiboly of the concepts of reflection,' which is primarily addressed at Leibniz, but Kant addresses both as if they shared a common rationalist position in the 'Remark' and elsewhere (CPR A273/B330).

of the concept, which is the form that any empirical instance must take. In addition to this, Shabel argues that symbolic construction is not a construction *with* symbols, but a construction *through* symbols. The symbols are not the objects, but *express* the objects and the procedures that are really constructed in intuition.[35] This is meant to include both geometrical figures and arithmetical equations, since both necessarily require relation to an object, i.e., an empirical form and the possibility of experience. At the highpoint of her argument, Shabel reintroduces arithmetic as an equally constructive discipline and bringing Kant in line with the prevailing *use* of mathematics:

> When Kant says at A717/B745 that algebra "achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves)" he does not mean to draw a strict distinction between symbolic construction, on the one hand, and ostensive/geometrical construction on the other. Insofar as algebra is a method applied to the solution of mathematical problems, the algebraic expression symbolizes the construction of arithmetic and geometric concepts in the form of figures. Thus, "symbolic constructions" are not *kinds* of constructions, that is, constructions of or out of symbols or characters. Rather, they are that which symbolize ostensive, or geometrical constructions. (ibid., 129)

Algebra does not construct any objects because its function is to symbolize the relations that compose these objects. Shabel calls this function 'heuristic,' which is true for Kant's critical philosophy, since all cognitions must relate back to experience as the condition of objectivity, no matter how abstract they may be. However, intuitions alone, even pure intuitions, prove nothing.

[35] Here, Kant is close to Leibniz's understanding of reasoning by signs. As Marcelo Dascal explains, "Les caractères, dit Leibniz, sont des « choses » au moyen desquelles les *relations* qui relient d'autres 'choses' entre elles sont *exprimées*" (Dascal 1978, 214). But for Leibniz, and for Maimon as well, these relations are conceptual rather than intuitive. As such, the intuitive ground of symbolic construction is adequately replaced by a characteristic that expresses the relations that constitute the object.

The Kantian construction of concepts is a process that objectively orders the forms of appearances without producing appearances. In some fundamental way, the sensible intuitions of experience and the pure intuitions of construction never meet one another. The epitome of synthetic a priori knowledge is the determinate combination of pure non-empirical concepts in the form of a pure, abstract, and imaginary intuition. Here, the supposedly essential relation to empirical appearances is tenuous. Perhaps, it is less that human sensible intuition grounds synthetic a priori judgments, and more that constructions turn intuition into an analogue of the understanding. After all, it is by means of *spontaneity* and *not passivity* that synthetic judgments amplify the content of concepts.

1.4 The Universal Antinomy of Thought in General

Now that the terms are set, the problem can be adequately posed. Construction is a specific, synthetic, act of cognition which is valid only for concepts that are also *real definitions,* that is, such definitions "exhibit originally the exhaustive concept of a thing within its boundaries" (CPR A727/B755). Kant elaborates in a note, that "**Exhaustiveness** signifies the clarity and sufficiency of marks; **boundaries**, the precision, that is, that there are no more of these than are required for the exhaustive concept; **original**, however, that this boundary-determination is not derived from anywhere else and thus in need of a proof, which would make the supposed definition incapable of standing at the head of all judgments about an object" (ibid.). No part or component of a constructible concept is beholden to empirical experience; the concept originates with the subject, and contains only the elements that the subject has decided it contains. All constructible concepts, that is to say all mathematical concepts, possess real definitions because they are arbitrary (CPR A729/B757). The only mention of 'real definitions' in the *Critique* is in relation to the concepts of mathematics. In the A edition, Kant includes this footnote in the '*Phenomena* and *Noumena*' section: “A real definition would therefore be that which does not merely make distinct a concept but at the same time its **objective reality**. Mathematical definitions, which exhibit the object in accordance with the concept **in intuition**, are of the latter sort [real definitions]” (CPR A242n).[36] Neither empirical concepts nor a priori concepts can have real definitions, since the truth of these concepts involves their agreement with an object, which is only *given* in appearance, so it is not possible to specify the marks of such a concept in advance (CPR A63/B87).[37] It is the fact that these objects are not *arbitrary* that lends them objective purport, that in some sense, the actual synthesis does not come from the subject, but from the object. Kant writes that, “There are only two possible cases in which synthetic representation and its objects can come together, necessarily relate to each other, and, as it were, meet each other: Either if *the object alone makes the representation possible* [*allein möglich macht*], or if the representation alone makes the object possible” [emphasis added] (CPR A92/B124). Obviously, the object does not induce a synthesis of its own accord, since empirical intuitions are still *re*presentation, which require a synthesis on the part of the subject. Instead, appearances are a posteriori syntheses that are produced by the *imagination*, and are given to the understanding as the matter for judgments, i.e. appearance as both form and matter of intuition.

After stating that constructions must be thought as forms of empirical appearances, Kant identifies the *formative synthesis* of the imagination as the element common to both construction

[36] In the B edition, he substitutes this note for a remark that any real definition must “immediately descen[d] to conditions of sensibility, thus to the form of the appearances, to which, as their sole objects, they must consequently be limited” (CPR A241/B300, modified).
[37] This has already been shown, but Kant insists on this point even in his late *Jäsche Logic*: “*Definitions of things* [*Sach-Erklärungen*] or *real definitions*, on the other hand [opposed to nominal], are ones that suffice for cognition of the object according to its inner determinations, since they present the possibility of the object from inner marks.” He continues, “**Note 2:** Objects of experience allow only nominal explanations…There are real definitions in mathematics, for the definition of an arbitrary concept is always *real*. **Note 3:** A definition is *genetic* if it yields a concept through which the object can be exhibited *a priori in concreto*; all mathematical definitions are of this sort” (Kant 1992, 634).

and apprehension – “this very same formative [*bildende*] synthesis by means of which we construct a figure in imagination is entirely identical with that which we exercise in the apprehension of an appearance *in order to make a concept of experience of it* - it is this alone that connects with this concept the representation of the possibility of such a thing” [emphasis added] (CPR A224/B271). Kant is not saying that the synthesis of construction is the same as the synthesis by means of which appearances are generated in sensibility. He is saying that it is identical with the synthesis of the understanding that orders appearances according to the categories and produces a determinate concept acquired a posteriori, from experience. ‘Analogies of Experience’ parses the difference between the subjective order of apprehension and the objective order of apperception. Concerning the former, Kant admits, “I would therefore not say that in appearance two states follow one another, but rather only that one apprehension follows the other, which is something merely **subjective**, and determines no object, and thus cannot count as the cognition of any object (not even in the appearance)” (CPR A195/B240). As we have seen, this is because the notion ‘object’ is applied to experience by means of the transcendental unity of apperception, or is the objective correlate of the unity of consciousness. This is true even of representations in inner sense; that is to say, *we cannot represent apprehension to ourselves except as ordered by the understanding*. For this reason, Kant subordinates apprehension to apperception, which is valid only for when the understanding *determines* the imagination to ‘reproduce’ what has been given from elsewhere.[38] There is no other way of representing alteration than as succession, one appearance proceeding or succeeding another:

[38] This language is borrowed from Deleuze’s treatment of the imagination in *Kant’s Critical Philosophy*. In the section ‘Role of Imagination,’ he writes “The schematism is an original act of the imagination: only the imagination schematizes. But it schematizes only when the understanding presides, or has the legislative power. It schematizes only in the speculative interest. When the understanding takes up the speculative interest, that is, when it becomes *determining*, then and only then is the imagination *determined* to schematize” (Deleuze 2013, 18). This sets up his later discussion of the *Critique of the Power of Judgment*, wherein the imagination is characterized as *free* and the understanding as *indeterminate* in judgments of reflection (ibid., 49). Also, see note 39 on the empirical imagination.

> In our case I must therefore derive the **subjective sequence** of apprehension from the **objective sequence** of appearances, for otherwise the former would be entirely undetermined and no appearance would be distinguished from any other. The former alone proves nothing about the connection of the manifold in the object, because it is entirely arbitrary. This connection must therefore consist in the order of the manifold of appearance in accordance with which the apprehension of one thing (that which happens) follows that of the other (which precedes) **in accordance with a rule**. (CPR A193/B238)

The subjective order is *derived* from the objective order only with relation to an object, or the intellectual representation of the thing itself, because, properly speaking, there are no objects (in a technical sense) outside of the mind, only 'things.' The only way to represent intuitions not generated by the understanding is to *reconstruct* them in inner sense. Though these objects are not mathematical, the objects of experience are knowable only by a similar process of exhibiting them in the intuition of a single pure time, and by means of the *same* formative synthesis. But as we have mentioned, the imagination can never give a thoroughly determined individual when it constructs a concept, only a schema or an image. The schema is the set of rules for constructing a concept in *pure* intuition, and the image is an arbitrary intuition produced by the subjective means of association and other psychological factors. As an intuition, the image includes an infinite number of determinations *not* prescribed by the understanding, which it can parse and represent again straying further from the empirical object. There is a double imperative: 1) the objective validity of a representation requires that the understanding alone construct it, since only it can relate representations to an object; 2) objectively valid representations must have matter, that is, a relation to a thing that gives determinate appearances to the understanding. These two demands cannot be satisfied in the same representation since construction cannot represent a given appearance, and a given appearance cannot be constructed. This is what Maimon calls *the universal antinomy of thought in general*.

In the 'Short Overview' of the *Essay on Transcendental Philosophy,* Maimon presents his solution to the problem, which does not free the mind from this double imperative, but gives the understanding the *idea* of its possible resolution. Maimon states, "For me the solution rests on this: that the understanding can and must be considered in two opposed ways. 1) As an absolute understanding (unlimited by sensibility and its laws). 2) As our understanding, in accordance with its limitation. So the understanding must think its objects according to two opposed laws" (ETP 227). The resolution of this problem rests on the possibility that these two modalities of the understanding might approach one another as a function of scientific inquiry into the nature of things. The aim is that the understanding will accumulate more determinations proper to its concepts, and so be able to construct if not a thoroughly determined individual, then one whose limitations are so minute that they would be near unassignable. Likewise, though the rules of the empirical imagination[39] are not transparent to the understanding, if there were something in an intuition by means of which one could reconstruct the occult rules of the imagination, it would be possible to gradually parse these rules so that they could be represented by the understanding.[40] These two processes are indefinite, because the finite mind would, at their point coincidence, become infinite or divine, capable of producing real objects by means of spontaneous intellection.

[39] This is a term that is employed by Kant in the 'Schematism' section of the *Critique*. The 'empirical faculty of the productive imagination' is meant to distinguish the production of images by subjective associations from the production of schemata by means of objective rules. That is, between the reproductive synthesis and the transcendental synthesis (CPR B181). Empirical imagination simply means the imagination as not determined to schematize.

[40] For example, when discussing the 'straight line' example employed by Kant to show how intuition connects the predicates 'straight' and 'shortest,' Maimon writes, "But the fact that we already have cognition of this proposition by means of intuition alone prior to its proof rests only on the following: we perceive its distinguishing mark or image in intuition (although it can only be made clear, not distinct) and so we already have a presentiment of the truth in advance (a presentiment that, I believe, must play no insignificant role in the power of invention [*Erfindungskraft*])." (ETP 70).

As Maimon writes, “The sufficient ground for a thing is the complete concept of the way it arises, and although we can approach ever more closely to it, we cannot reach it, because to explain the way that something arises we must presuppose something else that has already arisen (in accordance with the famous axiom: *ex nihilo nihil fit*) [nothing comes from nothing]” (ETP 392n.47). But it must nevertheless be possible to *think* this progression. In other words, it must be possible to think the complete *formal* and *material* determination of a representation, without actually producing a completely determined individual. The role of the empirical imagination is to supplement this deficit, and offer up sensations that were the product of some synthesis, which abide some implicit rule and involve some element by means of which the understanding can grasp this rule. And for this, one must posit something in a given sensible intuition that could be determined by both sensibility and the understanding, but which is nothing in itself.[41] This is what Maimon calls an *element*, or *differential of sensation*.

Sensibility gives qualities to the understanding which, in turn, represents them as units [*Einheiten*] of a magnitude.[42] In order to think the given quality itself, one would have to abstract from the quantitative mechanisms of representation, which would leave no formal component of intuition since the understanding necessarily represents appearances under the categories of quantity and quality, or, as magnitudes. Maimon begins the second chapter of the *Essay* with this example:

[41] This last criterion is essential to any critical philosophy, since nothing is immediately posited beyond consciousness. Such an idea is *regulative*, that is, it is “really only a schema for which no object is given, not even hypothetically, but which serves only to represent other objects to us, in accordance with their systematic unity, by means of the relation to this idea” (CPR A670/B698).
[42] This includes both extensive (space) and intensive (time) magnitudes. Kant employs the latter to explain the relation of qualities to the understanding, under the schema degree. In the next section we will see that “in all quality (the real of appearance) we can cognize *a priori* nothing more than their intensive **quantity**, namely that they have a degree, and everything else is left to experience” (CPR A176/B218).

> Considered in itself as a quality, every sensible representation must be abstracted from all quantity whether extensive or intensive. For example, the representation of the colour red must be thought without any finite extension, although not as a mathematical but rather as a physical point, or as the differential of an extension. It must further be thought without any finite degree of quality, but still as the differential of a finite degree. (ETP 27-28)

Maimon employs the term differential as it is used and understood by G.W. Leibniz. In a letter to Varignon [1702], he describes differentials as magnitudes that "are not at all fixed or determined but can be taken to be as small as we wish in our geometrical reasoning and so have the effect of the infinitely small in the rigorous sense" (L 543). Maimon uses this term because the calculus is concerned with a similar problem, namely, how to represent or generate qualitative behaviour of a curve by quantitative means alone.[43] The 'differential of intuition' is taken as the least magnitude of space or time that is not, as Kant and Maimon put it, intuition = 0.[44] Maimon explains, "Sensation is a modification of the cognitive faculty that is actualized within that faculty only passively (without spontaneity); but this is only an idea that we can approach by means of ever diminishing consciousness, but can never reach because the complete absence of consciousness = 0 and so cannot be a modification of the cognitive faculty" (ETP 168). There can be no experience of infinitesimal magnitudes since they are less than any assignable unit, and cannot be represented as a part of space or time. As such, they are called limit-concepts [*Gränzbegriffe*], concepts that represent the threshold of representation.

[43] There two primary methods developed to solve this problem – the method of fluxions, and the method of infinitesimals – which are attributed to Newton and Leibniz, respectively. The differences in their approaches, and the way Maimon adapts them to his project, will be thoroughly discussed in the next section.

[44] Kant uses this term in the 'Anticipations of perception,' when discussing the concepts of reality and negation. He has already established the possibility of a representation with only the pure a priori form of intuition (i.e., a schema), and introduces the possibility of "a synthesis of the generation of the magnitude of a sensation from its beginning, the pure intuition = 0, to any arbitrary magnitude" (CPR A165/B208). The schema of this kind of synthesis is *degree*, but this will be discussed in the next section.

Kant would dismiss the differential as an 'empty concept without object,' or an *ens rationis* [being of reason], because it is not self-contradictory, yet could never be perceived (CPR A292/B348). Maimon, however, does not introduce it for use in *determining* experience, but as a tool for *learning from* or *reflecting on* experience. The differential in itself is = 0; it is only an unassignable magnitude predicated of a given intuition. Maimon explains that, "With respect to intuition = 0, the differential of any such object in itself is $dx = 0$, $dy = 0$ etc.; however, their relations are not = 0, but can rather be given determinately in the intuitions arising from them" (ETP 32). The differentials, and the rules by means of which they are combined, are a way of thinking how the empirical imagination must unify the manifold, because *there is no other way the understanding can represent this process as it is in-itself*. Maimon takes issue with Kant's notion of experience because Kant does not see that representations of the understanding are essentially different from those of sensibility *only* because the former are produced according to *explicit* rules, and the latter according to *implicit* rules. This is an enormous difference since the understanding must provide a *reason* or *ground* for every determination of a concept. The understanding can never thoroughly determine a concept, or supply a sufficient ground, since this would amount to spontaneously producing an intuition of things themselves. Jan Bransen explains this in terms of finitude:

> The rules of our understanding do not enable us anything more than to know *possible* objects, since these rules are, characteristically, *incomplete*. Our minds can only grasp some small series of determinations: 'straightness' being of 'lines'; 'yellow' being of 'colour', and 'colour' being of 'surface', but we never arrive from 'surface' up to the mere 'determinable' in order to grasp the complete series of determinations that make up the *complete* concept, and, consequently, the *concrete* object of 'gold'…we need *intuitions*

merely to be *aware of the fact* – the experiential reality – *that* it happens to be the case that gold is yellow. (Bransen 1991, 89-90)

Kantian construction is synthetic only to the degree that it exhibits schemata in a priori intuition, where the construction can amplify concepts only to the degree that the intuitive supplement to the concept is represented in terms of *rules* and *determinations*, and not what is given. Similarly, experience does not ground appearances in sensible intuition, so much as it reproduces the given appearance in terms of pure, objective space and time – the product of the understanding's action on the imagination. To paraphrase Bransen on this point, the understanding only constructs *possible* objects, where sensibility only gives *actual* perceptions (ibid., 90).[45] The differential sits in between determinations of both the concept and the appearance of a fact or quality. As a limit-concept, the differential dons two masks: it accompanies intuition as the possibility of finding an adequate rule for the object, and it accompanies the concept as the undetermined yet determinable object towards which the understanding tends. In other words, it is a method of hypothesizing about what rules *might be required* for further determining or specifying the concept of an object. This method does not construct intuitions by means of concepts, but constructs concepts by means of intuitions, so to speak.[46] Nevertheless, the

[45] As mentioned in 1.1, possible experience means conformity with the structure of cognition a priori, or, "always only as formal and objective conditions of an experience in general" (CPR A223/B271). Actuality is the relation of a representation to a perception, which does not add anything to the definition of a concept, which is entirely formal. As such, no existence can be inferred from a concept alone. Actuality cannot represent direct connection with the thing itself, it nonetheless requires "its connection with some actual perception in accordance with the analogies of experience, which exhibit all real connection in an experience in general" (CPR A225/B272). The 'reality' of any representation is determined by its relation to an object, or, the possibility of empirical experience. This is not to say that existence can be inferred from concepts alone, but it does mean that the category of 'actuality' represents only the form of sensation in general (which gives only the *degree of reality*, not the reality itself). Maimon offers a new definition of actuality, where: "actuality is certainly that within which I perceive a synthesis; however, this synthesis does not proceed in accordance with the laws of. the understanding (the determinable and the determination), but merely in accordance with the laws of the imagination" (ETP 102).

[46] This sounds very much like a judgment of reflection as developed in *Critique of the Power of Judgment*. As mentioned in note 24, Kant and Maimon were concerned with the same problem, namely, how is it possible to learn from nature something that was not imposed onto nature by the mind. One must be able to incorporate new discoveries in to the already established framework, even if that requires changing some fundamental presuppositions. In any case,

differentials of sensation are not commensurable with geometrical constructions. They are not in experience, but their relations must be thought as giving rise to the objects of experience. One must look to the algebraic paradigm that problematized and superseded the geometrical standard of rigour in the late seventeenth century to find the tools for such a non-intuitive, symbolic kind of cognition.

experiment always yields order. Kant accounts for the continual production of order with the regulative idea of God or the system of nature. Maimon accounts for this with the regulative idea of an intuitive intellect, that humans approach ever more closely.

Chapter Two

2.1 Time, Motion, and the Method of Fluxions

Kant was very familiar with the scientific work of Isaac Newton, and took it as the model of natural science in the *Critique*. Though much has been written about Newtonian natural science and the possibility of its critical revision, this section will focus on Kant's use of the calculus in the first *Critique* alone.[47] Though it is never explicitly discussed, as in the case of geometry, analysis plays an important role in how the concept of magnitude is explicated, especially since all magnitude is *continuous* magnitude. As we have seen, Kant circumscribes the possibility of mathematics within the domain of transcendental idealism because he takes geometry to be the standard by which to judge mathematical science. He was able to use Newton's calculus to explicate part of his system because Newton's method was both geometrical and kinematic. As such, the motion of points, lines, etc. must assume time as an underlying component of continuous motion. This method was employed to avoid the question of actual infinitesimal magnitudes when solving problems with instantaneous velocities. For Kant, this was meant to elucidate the relation between perception and the understanding, namely, how can reality be a category of experience when *the real* is what the understanding can only receive from experience? His answer is "if it were supposed that there is something which can be cognized a priori in every sensation, as *sensation in general* (without a particular one being given), then this would deserve to be called an anticipation in an unusual sense, since it seems strange to anticipate experience precisely in what concerns its matter, which one can draw out of it" [emphasis added] (CPR A167/B209). One

[47] For a detailed account of how Kant revised Newtonian natural science see chapters 3,4, and the second half of *Kant and the Exact Sciences.*

is capable of anticipating sensations because each one must arise in experience, not all at once, but continuously.

For the present purposes, it suffices to present a simplified history of Newton's analytic and synthetic methods of calculus.[48] In keeping with Newton's mentor Isaac Barrow and 'the geometrical rigour of the ancients,' both methods were equally concerned with the motion of objects and the geometrical construction of such objects (Giorello 1992, 139).[49] Newton gives some justification for this in his *Introduction to the Quadrature of Curves* [1704], when he writes "I don't here consider Mathematical Quantities as composed of Parts extreamly small, but as generated by a continual motion…These Geneses are founded upon Nature, and are every Day seen in the motion of Bodies. And after this manner the Ancients by carrying moveable right Lines along immoveable ones in a Normal Position or Situation, have taught us the Geneses of Rectangles." (Newton 2010, 250). It is not so important to trace a genealogy of Newton's position, as it is to remark that Kant shares several of these convictions. Specifically, proceeding by finite quantities that one can exhibit and comprehend in experience, and insisting that the foundation of mathematical science remain the construction of geometrical figures in time. This later method is known as the 'Method of Fluxions' which was first developed in *De methodis serierum et*

[48] Niccolò Guicciardini identifies two moments in the development of Newton's method. Prior to his analysis of motion (1664), he described curves in terms of infinitely long, recursive, algebraic equations which could be obtained by means of the binomial theorem (where the coefficient of each term is determined by the preceding term). Around 1665, Newton identified the reciprocal nature of tangent and area problems, where the two equations were related by an infinite series whose terms could be transformed such that the equation describing a curve could yield an equation describing a line. A year later, he introduced the notion of fluent, or continually flowing quantity, as a way of reconciling the progressively more miniscule differences that can be interjected between points that approach one another without presupposing infinitely small quantities. This period is characterized as the 'analytic' method of fluxions because he employs various algebraic techniques to obtain his results. The second (1670 and on) method still includes fluents, though they are defined in terms of geometric ratios and proportions, such that there is a determinate mathematical object present at each stage of problem. For a detailed account, see (Gucciardini 1999, 17-22, 32-37).

[49] For instance, Barrow argued for the use of construction when defining geometrical objects with an almost Kantian flare: "For [constructions] not only explain the Nature of the Magnitude defined, but, at the same time, shew its possible Existence, and evidently discover the Method of its Construction: They not only describe what it is, but prove by Experiment, that it is capable of being such; and do put it beyond doubt how it becomes such" (quoted in Mancosu 1996, 98).

fluxionum (1670-71), and was given its determinate, synthetic form by the time Newton had published the *Principia* in 1687. He explains that, "Quantities, encreasing in equal times, and generated by this encreasing, are greater or less, according as their Velocity by which they encrease, and are generated, is greater or less; I endeavoured after a Method of determining the Quantities from the Velocities of their Motions or Increments, by which they are generated" (ibid.). He called the quantities the fluents, and the increments of velocity the fluxions. A fluent is not just a curve, but one that is defined by the rate of change of one term relative to another. And fluxions are infinitely small changes in velocity at a point, whose *ratio* gives the rate of change at that point. In both cases, the values are determined as functions of time. As David Bressoud writes (employing Leibnizian notation),

> To Newton, *fluxion* was a rate of change over time. Throughout his mathematical work, quantities that varied did so as a function of time. This is important. When Newton sought to find the slope of the tangent line to the curve $y = x^3$ at a point, he *always* treated it as implicit differentiation, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3x^2\frac{dx}{dt}}{\frac{dx}{dt}} = 3x^2$. This may seem cumbersome, but it conveys an important point that is often lost on calculus students: The derivative is telling us much more than the slope of a tangent line. It encodes the relationship of the rates of change of the two variables [*x* and *y*]. (Bressoud 2019, 87)

This 'encoded' relationship between the tangent to a curve and the area under this curve is explained in wholly kinematic terms. The insight of the method of fluxions is to introduce the fluent as a way of orienting the calculus around the motion of a point, so that the instantaneous fluctuations of a quantity do not exist solely as abstractions, but as parts of a real object. As Guicciardini writes: "For [Newton] reference to our intuition of continuous 'flow' provided a means to define the referents of the calculus: fluents, fluxions and moments" (Guicciardini 1999, 28). Imagine a point that traces a curvilinear path over a number of instants. This curve is a fluent insofar as it wholly defined by the continuously varying position of the point. At a given instant, the position of the point will be associated with a value of x and of y. Fluxions are the *nascent* or *vanishing* parts that compose a *moment*, in which x and y continue to vary. A moment can be divided such that, at its beginning, the values of x and y differ by an infinitesimal amount when compared with the end of the moment. So, the point can stray from its initial position and nascent quantities emerge, or it can tend to a final position where the difference vanishes. The moment gives a rate of change at an instant because these fluxions are small enough to be negligible, yet they facilitate the construction of miniscule triangle that is proportionate to another fixed triangle.

In his essay, 'The 'fine structure' of mathematical revolution,' Giulio Giorello gives an example of how fluxions, moments, and fluents are related (Figure 1, from Giorello 1992, 142), where the curve AN is the fluent, considered at a point B: "Newton (who had begun indicating his fluxions or 'instantaneous speeds' with dotted letters) believed that the shifting of the point from B to N took place in a 'minute particle of time', written as o. Indicating the instantaneous speeds

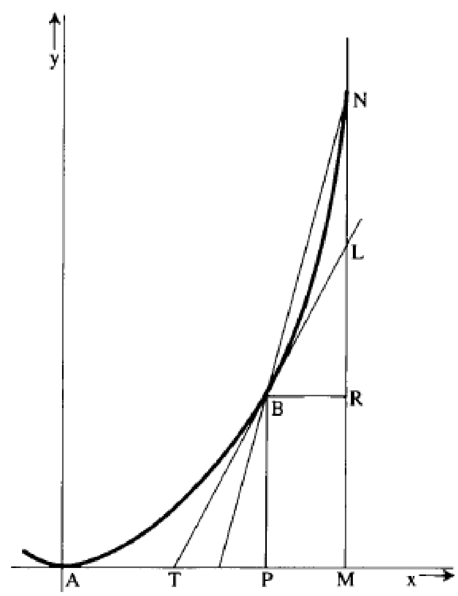

Figure 1

(that is, the two fluxions of x and y, respectively) by $\dot{x}$ and $\dot{y}$, and supposing that BR = $\dot{x}o$ and LR = $\dot{y}o$, the proportion TP:BP = BR:LR, obtained from the similarity of the triangles BTP and LBR, could be rewritten thus: subtangent[TL]:y = $\dot{x}o$: $\dot{y}o$." (ibid., 143). The fluxions are eliminated after they are used to construct the triangle LBR because they are nothing when one evaluates the curve only at B.[50]

In earlier works (*To Resolve Problems by Motion* [1666], and drafts from the same period), Newton did not emphasize or ensure the constant reference to a geometrical figure or object, and argued, according to Richard T.W. Arthur, that "to find the ratio of the velocities precisely at the beginning of the moment *o…, o* must be shrunk to zero, so that the extra terms in the expression of this ratio still depending on the quantity *o* will therefore also vanish, with the resulting expression yielding the "first ratio" of these velocities" (Arthur 2008, 13). The continuity between shrinking the duration of a moment to zero, and the kinematic character of the Method of Fluxions, is implicit in the way Newton handles these discrete moments. This earlier method is explicitly kinematic, since "it is implicit in the kinematic representation that the velocities *p* and *q* are the velocities at the very beginning of the moment *o*, so that the term for *po*:*qo* calculated by Newton's algorithm, which will still generally contain terms in *o*, will be closer to *p*:*q* the closer *o* is to 0" (ibid.).[51] It differs from the synthetic method as described above because there is no clear picture of what happens to the mathematical *objects* when *x* and *y* are substituted with their fluxions (see note 50). A synthetic method was necessitated by this blind, and thoroughly algebraic, manipulation of equations. Though fluents, fluxions, and moments are *manipulated* algebraically, this is still a properly kinematic interpretation of varying quantities, where the results of calculation

[50] Here the analytic and synthetic Method of Fluxions are blurred. In the analytic method, Newton works by substitution, replacing x with $x + \dot{x}o$ and y with $y + \dot{y}o$ in any equation and removing all terms with o as a factor (since it is negligible). The later synthetic method employs like triangles, the ratios of fluxions, and other geometrical operations. For examples, see (Guicciardini 1999, 22, 33-34). However, both methods are kinematic in nature, and describe the generation of fluents by means of variation in time, and fluxions as the minute elements of said variation.

[51] Here, the algorithm refers to the procedure of substitution as described in note 50. This is a generic procedure that operates on equations, mere symbols that can be manipulated without consideration of any object or thing that might be designated. Even if an ultimate geometrical interpretation is vital for Newton, the algorithmic manipulation of symbols is wholly effective *without* such a consideration.

can be geometrically represented. Again, imagine that point B is moving toward point N (Figure 1), the distance PM is the distance point B *would have* travelled as the moment comes to a close. From the synthetic point of view, Arthur insists, “there is no way to represent an instantaneous velocity geometrically save by showing the line segment…that a body would cover if it continued with that velocity for a time *o*. From this point of view, the moment *o* is more nearly a device enabling instantaneous velocities to be geometrically represented” (ibid.). Yet, from the analytic point of view, *o* is a symbol that facilitates a number of substitutions without necessarily referring to any real entity. All this is to say that, even at his most analytical, Newton presupposes time as the ‘third thing’ necessary to ground the operations of integration and differentiation.[52] Where, according to Newton, “These Geneses are founded upon Nature, and are every Day seen in the motion of Bodies,” for Kant this must be submitted to critical scrutiny, and be grounded in the form of inner sense.

In the ‘Anticipations of perception,’ Kant mentions ‘flowing quantities’ in his discussion of both extensive and intensive magnitudes. This term, ‘*fließende Größen,*’ was the standard translation of ‘fluent’ into German, and was familiar to Kant through the textbooks of Abraham

[52] Paolo Mancosu neatly summarizes the relation when he writes, “the central idea of the calculus is that the processes of differentiation and integration are, *grosso modo*, inverses of each other; or, more geometrically, that the determination of tangents to a given curve and the computation of the area between the axis and the curve are inverse problems” (Mancosu 1996, 154). H.J.M. Bos explains that, technically, the correct terms are differentiation and summation (not integration), since these operations 1) range over variables irrespective of their dependency; they are not assigned a specific variable as with a function $f(x)$. 2) Differentials and summations remain undetermined until the progression of variables is determined, or these operations are independent of the progression of variables. 3) Derivation and integration do not necessarily occur in geometrical contexts, but in such cases the derivative will reduce the dimension of the function from e.g. line to point, and the integral will increase it e.g. line to area, where differentiation and summation preserve the dimension but alter the ‘order’ of infinity i.e. a finite ratio obtains only between dx and dy, but not dx and ddx because dx is infinitely small compared to ddx (Bos 1974, 34-35). This distinction is important to the extent that continuity plays a fundamental role in the progression of variables, and that the relations between differentials, either when summed or explicated, are independent of the variables whose differences are actually manipulated; that the relations between terms tell something that is inaccessible at the level of terms themselves.

Kästner.[53] All magnitudes that can be applied to appearances are called *continuous*, which Kant defines in this way: “The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. Space and time are *quanta continua* [continuous magnitudes], because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way that this part is again a space or a time” (CPR A169/B211). What Kant calls continuity is now known as density. The criterion of density is, as Kant expresses, for all points x and y, where $x > y$, there exists a point z such that $x > z > y$. However, Kant infers this property from the ability to construct a continuous line in space. This line cannot be equal to the continuity of the Real number line since there are points on the real line that are not constructible. Such numbers are the irrational numbers, or unending series of rational numbers. If we take ‘>’ to stand for the linear order ‘to the right of,’ then it is possible to construct a series or rational numbers that approach, say, π, but which can never reach it since this is a non-terminating series. As Freidman writes, “This sequence of rationals *converges* (to "something," as it were), but in the set Q of rational numbers (and even in the expanded set Q* of Euclidean-constructible numbers) there is no limit point it *converges to.* Such limit points are "missing" from a merely dense set such as the rationals. A truly continuous set contains "all" such limit points”

[53] “I take Kant's choice of language to be especially significant here, for his "fließende Größsen" is the standard German equivalent of Newton's "fluents.” This expression is used, for example, by the mathematician Abraham Kästner in his influential textbooks on analysis and mathematical physics. Kästner's analysis text attempts to develop the calculus from a "rigorous" standpoint that makes no appeal to infinitely small quantities. In this connection he develops a version of Newton's method of fluxions, and, what is more remarkable for a German author of this period, he argues that Newton's fluxions are in some respects dearer and more perspicuous than Leibniz's differentials. Further, he explicitly applauds Collin Maclaurin's attempt, in his monumental Treatise of Fluxions (1742), to develop the calculus on the basis of a kinematic conception of the limit operation.” (Friedman 1992, 75). Also see the recently translated ‘On Kästner’s Treatises’ (Kant 2014).

(Friedman 1992, 72). This is why motion, as a *continuous* magnitude, is introduced to allow the series to converge to a limit, without positing an indefinite progression. [54]

For both extensive and intensive magnitudes, the source of their density (infinite divisibility) and homogeneity is time as the form of inner sense. Kant writes, “Magnitudes of this sort can also be called **flowing** [*fließende*], since the synthesis (of the productive imagination) in their generation is a progress in time, the continuity of which is customarily designated by the expression "flowing" ("elapsing")” (CPR A170/B212). As discussed, the figurative synthesis of the imagination is an action of the understanding on the imagination, which produces a pure time through which appearances can be ordered objectively. All magnitude is the result of a synthesis according to principles, specifically, the application of the categories of quantity or quality. These principles produce constitutive connection, as opposed to dynamical conjunction.[55] Kant writes, “[connection] is the synthesis of a manifold of what **does not necessarily** belong **to each other**, as e.g., the two triangles into which a square is divided by the diagonal do not of themselves necessarily belong to each other, and of such a sort is the synthesis of the **homogeneous** in everything that can be considered **mathematically”** (CPR B201n). Continuous magnitudes are what result when the understanding represents appearance in space and time, that is, relates

[54] Terms such as ‘limit’ and ‘convergence,’ even ‘continuity,’ do not have the same meaning as they did in the eighteenth century. As Guicciardini explains, “it should be noted that Netwon, as all his contemporaries, had a rather intuitive concept of convergence – a concept that nowadays would be considered as unrigorous. For instance, he thought it sufficient to say that the binomial series can be applied when *x* is ‘small’” (Guicciardini 1999, 20). It is not until Augustin-Louis Cauchy that the concepts of limit, continuity, and convergence take rigorous form. For a discussion of Cauchy and Newton see (Grabiner 2005 81-84), and for a discussion of convergence see (ibid., 102-109) and (Friedman 1992, 72-73). As such, their use will be generally avoided, and when used, will be defined by their context.

[55] For Kant, conjunction is the synthesis of dynamical categories, which does not *produce* homogeneity or unity in the manifold. Instead, it works on the assumption of an a priori form of appearances produced by the former categories. It is “the synthesis of that which is manifold insofar as they necessarily belong to one another, as e.g., an accident belongs to some substance, or the effect to the cause - thus also as represented as unhomogeneous but yet as combined a priori, which combination, since it is not arbitrary, I call dynamical, since it concerns the combination of the existence of the manifold” (CPR B201n). This conjunction is either of concepts in intuition a priori, or of different appearances by means of their form (magnitude).

qualities of sensation by means of a common form. Though both extensive and intensive magnitudes are continuous, connection can only be established through the schema proper to the given category. And, because they have different schemata, they also have different ways of ordering the manifold of appearance.

Number, the schema of quantity, connects the manifold in a stepwise manner. Kant writes, “I call an extensive magnitude that in which the representation of the parts makes possible the representation of the whole (and therefore necessarily precedes the latter)” (CPR A162/B203). Returning to an example from the section on the imagination, Kant argues that it is impossible to represent a line in thought without drawing it, that is, “successively generating all its parts from one point, and thereby first sketching this intuition” (ibid.). Extensive magnitudes can be generated only in intuition, and as such, the synthesis must conform to the transcendental synthesis of the imagination, i.e., “it can only be cognized through *successive* synthesis (from part to part) in apprehension” [emphasis added] (CPR A163/B204). Three consequences are noteworthy, firstly, though the parts of intuition can be spatial or temporal, the synthesis itself takes the form of inner sense, and is temporal. The transcendental use of the understanding requires that the categories become time-determinations, including when intuition itself is taken as the matter in an a priori judgment, as it is in geometry (formal intuition). Even adding together seconds requires a succession of syntheses.

Secondly, the distinction between geometry and arithmetic arises precisely at this moment. As mentioned in 1.3, geometry is the science of pure space, and is capable of exhibiting its concepts a priori by constructing particular figures. This is possible only because the concepts, or their schemata, are sets of rules for the determination of extensive (spatial) magnitudes as such, and that compose any figure.[56] This is not the case for arithmetic because the primary operation of arithmetic is calculation, not construction. In much the same way that the *process* of construction is temporal, so is the process of counting or calculating; the difference is that calculation does not produce figures, or particular magnitudes, but mere magnitude in the form of numbers. The schema of number is what gives rise to magnitudes as collections of units, but the act of synthesis must be applied for each magnitude constructed. Numbers are concepts that represent a determinate set of the repeated act of synthesis. Unlike construction, which can take a magnitude to be arbitrarily large or small (as long as the axioms apply), calculation must proceed in the same way each time, since "it is only the synthesis of that which is homogeneous (of units) that is at issue here, the synthesis here can take place only in a single way" (CPR A164/B205). Lastly, the operation of calculating is what determines the *extensive* character of these magnitudes. For Kant, extension means 'composed according to the schema of number,' which is applied to a given intuition (pure or empirical). Whatever discrete spaces or times are given, they can be ordered in pure space and time, which is infinitely divisible. The understanding must successively stitch these appearances together in time and, as such, "every appearance as intuition is an extensive magnitude, as it can only be cognized through successive synthesis (from part to part) in apprehension" (CPR A163/B204). The synthesis of apprehension presents perceptions according to the subjective order of how these perceptions were acquired; this is the source of proceeding part by part. The understanding brings unity to the manifold by relating perceptions according to their form (i.e., objectively), so that the multitude of aggregates can be held in one consciousness. That is to say, the understanding represents the subjective order of apprehension to itself as the succession of

[56] Kant gives the example of constructing a triangle, and writes "If I say: "With three lines, two of which taken together are greater than the third, a triangle can be drawn," then I have here the mere function of the productive imagination, which draws the lines greater or smaller, thus allowing them to abut at any arbitrary angle" (CPR A164/B205).

appearances, which can be collected and enumerated because the form of each appearance is determined by a synthesis wherein sensations acts as matter, and are plotted along the form of pure succession. The iteration of this act of synthesis, assigning sensations a determinate ordinal value, is the general time-condition (schema) of number.

Compare this to the ‘Anticipations of perception,’ where Kant describes intensive magnitudes, which are also continuous, but synthesized according to different rules. One way of imagining the difference is this: extensive magnitudes are contiguous, and are given in a single synthesis, a sort of snapshot that can be surveyed, moving from one section to the next; intensive magnitudes are the levels of exposure and contrast, which can be *measured* in terms of aperture f-numbers and the time in a developer bath. In other words, it is the magnitude of reality in a synthesis as a kind of saturation. Kant also describes them in relation to apprehension; he writes, “Apprehension, merely by means of sensation, fills only an instant …As something in the appearance, the apprehension of which is not a successive synthesis, proceeding from the parts to the whole representation, it therefore has no extensive magnitude; the absence of sensation in the same moment would represent this as empty, thus = 0” (CPR A167/B209). The categories of quality – reality, negation, limitation – are based on a set of judgments that concern sensibility – affirmative, negative, infinite – since quality differentiates empirical from pure cognition.

The aim is not to ‘anticipate’ or represent actual quality a priori, rather, Kant argues that “there is something which can be cognized a priori in every sensation, as sensation in general (without a particular one being given)” (CPR A167/B209). The categories and the functions of judgment can only describe the various ways the understanding can represent quality without generating quality in intuition. Kant sets the upper and lower limits of sensation in general as reality (the real) and negation (absence); “the real, which corresponds to sensations in general, in opposition to the negation = 0, only represents something whose concept in itself contains a being, and does not signify anything except the synthesis in an empirical consciousness in general" (CPR A176/B217). This is a very intriguing definition, especially since it so closely resembles the kind of scholastic thinking that posits the existence of objects depending on the perfection of their concept, and which Kant argues against.[57] In the 'Postulates of empirical thinking in general,' he writes "In the mere concept of a thing no characteristic of its existence can be encountered at all" (CPR A225/B272). Kant avoids scholasticism or dogmatic rationalism by relating the concept of 'the real' to 'the synthesis in an empirical consciousness in general,' and not the thing itself. And to avoid ungrounded speculation Kant insists that it is not an actual empirical synthesis, but a 'synthesis in general.'

The aim of all synthesis is to produce unity in the manifold of perception. In this case it is not accomplished by adding units together *in* intuition (category of unity), but accomplished *as* the unity of the apprehension *itself*. The real stands for any act of empirical synthesis, which produces unity in the manifold by means of the transcendental synthesis of the imagination. This kind of synthesis does not begin with parts, but with the act of the understanding (through apprehension) that must logically precede the representation of parts. Kant explains that, "I call that magnitude which can only be apprehended as a unity, and in which multiplicity can only be represented through approximation to negation = 0, intensive magnitude. Thus every reality in the appearance has intensive magnitude, i.e., a degree" (CPR A168/B210). 'Negation = 0' refers to situations where no empirical synthesis as occurred, where there is no reference to an object because there is no unity in sensation (because there is no sensation, = 0). Reality is not so much a predicate of

[57] See the discussion of matter and form in 'On the amphiboly of the concepts of reflection…' (CPR A266/B322-A268/B324).

objects as it is a concept that represents how much ‘thinghood’ or objectivity a representation has with regard to its form in inner sense (time). At the end of the ‘Schematism’ section, Kant clarifies that, “Since time is only the form of intuition, thus of objects as appearances, that which corresponds to the sensation in these is the transcendental matter of all objects, as things in themselves (thinghood [*Sachheit*], reality)” (CPR A144/B183). So, the being that is contained in the concept ‘reality’ is not an empirical quality, but the a priori representation of how much of an appearance is ordered objectively, i.e., to what degree the appearance is ordered according to the categories or pure rules of the understanding. When Kant writes that *degree* is the schema of quality, he means that the representation of reality, negation, and limitation is only a measure of how much of an appearance is represented objectively. In this way, negation can only be thought and never be represented in intuition, since the understanding cannot represent the lack of its own activity to itself. This can only be given: “The quality of sensation is always merely empirical and cannot be represented *a priori* at all (e.g. colors, taste, etc.)” (CPR A175/B217).

This detour on the way to Kant’s use of the fluxional method of calculus was necessary, insofar as the concepts of the continuous and the discrete, the *fluent* and the *moment*, are connected in the schema degree. The magnitude of degree is not the same as the magnitude of number because the former measures the reality of any given part or unit, and not the accumulation of several units. For this reason, it is called *intensive* magnitude. The use of intensity does not carry the connotations of being ‘too much’ or ‘unbearable,’ but denotes that the degree of objective determination for an appearance, between the given and zero, and is the range within which this kind of measurement can be utilized. This interval is also what Kant calls a moment [*Moment*], where, just as in the ‘Transcendental aesthetic,’ one considers the matter or being in the concept to be the source of affection in sensibility.[58] Kant describes this in terms of 'moments' of gravity: "If one regards this *reality as cause* (whether of the sensation or of another reality in appearance, e.g., an alteration), then one calls the degree of reality as cause a "moment," e.g., the moment of gravity, because, indeed, the degree designates only that magnitude the apprehension of which is not successive but instantaneous." [emphasis added] (CPR A168-169/B210). There are two fundamental parallels between Kant's and Newton's use of moments. Firstly, that a moment is an instant of time, but an instant laden with information about the object at hand. Secondly, a moment can be infinitely subdivided into smaller parts that continuously reduce the instant toward zero, without ever being zero. Again, Kant makes the explicit parallel when describing the continuum of intensity between the given and zero. He writes,

> Accordingly, every sensation, thus also every reality in appearance, however small it may be, has a degree, i.e., an intensive magnitude, which can still always be diminished, and between reality and negation there is a continuous nexus of possible realities, and of possible smaller perceptions. Every color, e.g., red, has a degree, which, however small it may be, is never the smallest, and it is the same with warmth, with the moment of gravity, etc. (CPR A169/B211)

Where Newton's primary concerns were with mechanics, Kant includes other natural phenomena which are not amenable to analysis by extensive magnitudes alone. This is because there are variations in phenomena that do no depend on the amount of space. Kant explains that "an expansion that fills a space, e.g. warmth, and likewise every other reality (in appearance) can, without in the least leaving the smallest part of this space empty, decrease in degree infinitely, and nonetheless fill the space with this smaller degree just as well as another appearance does with a

[58] "The effect of an object on the capacity for representation, insofar as we are affected [*affiziert*] by it, is **sensation**. That intuition which is related to the object through sensation is called **empirical**. The undetermined object of an empirical intuition is called **appearance**" (CPR A20/B34).

larger one" (CPR A174/B216). These magnitudes can be considered fluents, since they can be described as the increase or decrease of variation over time. But Kant adds, against Newton's insistence on absolute space and time, that "the nature of our perceptions makes an explanation of his sort possible" (CPR A175/B216).[59]

The last paragraph of the 'Anticipations' is most telling of the degree to which Kant has removed Newton's fluents from the state of nature. He concludes by stating, again, that all sensation must be given a posteriori and that only the degree of reality can be cognized a priori. The final sentence, provides a clue as to *how* exactly magnitude, both intensive and extensive, are cognized. He writes, "It is remarkable that we can cognize *a priori* of all magnitudes in general only a single **quality**, namely continuity, but that in all quality (the real of appearances) we can cognize *a priori* nothing more than their intensive **quantity**, namely that they have a degree, and everything else is left to experience" (CPR A176/B218). The category of quality does not itself represent *qualities*, only an intensive quantity that is relative to the 'transcendental matter' of possible experience, i.e., the thing considered as it is in-itself.[60] The only quality that can be represented a priori is continuity, that is, the form of magnitude in general. The place of continuity as the (only) quality given a priori to the understanding is not explicitly dealt with, and this is the only passage in the *Critique* where Kant writes about continuity in this way. What is certain,

[59] Friedman compares space as, according to Newton, an affection and 'the sensorium of God,' to the critical view of space. For instance, "In particular, absolute space or pure extension is even an affection of God himself, since God is omnipresent or everywhere. God can thereby create matter or body (as something quite distinct from pure extension) by endowing certain determined regions of space with the conditions of mobility, impenetrability, and obedience to the laws of motion. God can do this anywhere in space, in virtue of his omnipresence, by his immediate thought and will, just as our souls can move our bodies by our immediate thought and will. It is essentially this doctrine which surfaces in Newton's well-known published statements, in the General Scholium to the *Principia* and the Queries the *Optics*, that space is the "sensorium" of God" (Friedman 2009, 36).
[60] This is also explicitly stated in 'The doctrine of pure reason,' which reads: "The only concept that represents this empirical content of appearances *a priori* is the concept of the **thing** in general, and the synthetic *a priori* cognition of this can never yield *a priori* more than the mere rule of the synthesis of that which perception may give *a posteriori*, but never the intuition of the real object, since this must necessarily be empirical" (CPR A720/B748).

however, is that calculus is only possible because of the magnitudes necessary to think perception, which can only exist transcendentally, that is, as a concept given form in intuition. As with every science, the objects of inquiry must possess a determinate relation to possible experience. For Kant, fluents and fluxions are akin to geometrical figures, in that both are pure sensible concepts or concepts that can be synthetically combined a priori. This is because the pure forms of intuition are the ground of a synthetic judgement, not any particular experience. Calculus becomes a species of kinematics, since the concepts of 'rate of change' or 'instantaneous velocity' are valid not only of empirical objects, but of motion generally. Just as geometry supposes the formal intuition of space (space as parts), kinematics supposes time as formal intuition. Similarly, just as *construction* is a temporal activity of giving determinate ratios to parts of space, so *drawing* a curve gives determinate ratios to moments of time. Geometry and kinematics are connected by arithmetic, as the discipline that deals with mere magnitude [*quantum*], which abstracts from the qualitative or empirical characteristics of an object only to measure its magnitude. Arithmetic still has objects (e.g. strokes or dots on a page), but the concept of magnitude can represent the relevant marks a priori and calculate number, and the ratio of number, through the pure form of time produced by the transcendental synthesis of the imagination. As Freidman writes:

> "These procedures [enumeration, measurement] are thus independent of the metric of time – of the consideration of time itself as a magnitude – and involve only the fact that our representations, whatever objects or content they may have, are temporally ordered. Arithmetic and algebra, therefore, depend only on time as a form of intuition: as the form of inner sense (CPR A33-34/B49-51). By contrast, to represent time as itself a magnitude (as "time-magnitude"), the representation of space must also be considered – presumably, in the pure theory of motion. It is this pure theory of motion that alone enables us to consider time as a

formal intuition: that is, as an object of intuition or as itself an intuition (CPR B160). (Friedman 1992, 133).

Friedman refers us to both the footnote on formal v. form of intuition, and the 'Transcendental exposition of the concept of time,' where Kant first argues that time is the form of inner sense, and that any activity of thought (pure or empirical) must take place in time. Succession and congruence come together in kinematics, since moments of time can be given extensive magnitude alongside parts of space, and together can represent not only fluents (continuous variation as a function of time), but fluxions as well (the ratio that describes variation at an instant).[61] But geometry also exhibits a similar combination of succession and congruence through construction: "motion, as description of a space, is a pure act of the successive synthesis of the manifold in outer intuition in general through productive imagination, and belongs not only to geometry but even to transcendental philosophy" (CPR B155n). The movement of points generates lines, and the construction of figures generates ratios. Geometry and kinematics complete one another, in that the formal intuition of one supposes the form of intuition of the other; the construction of spaces supposes time as iteration and succession, the fluxion of continuous motion supposes the contiguity and proportionality of space. This is the case because both *sciences stem from the same action of the understanding on sensibility*, or, that they are both *grounded in the pure forms of intuition*. And not only this, but the real objects of the mathematics that underly these sciences must also find their ultimate source and validation in the *possibility of experience*.

[61] Bressoud explains that it is with the calculus that the restriction against composing unlike units is lifted. "To Hellenic and Hellenistic philosophers, and by extension to the philosophers of the seventeenth century, ratios could only be ratios of like quantities: lengths to lengths, areas to areas, volumes to volumes. That fact is important in understanding how they thought about velocity. Velocity was never seen as a ratio because distance and time are incommensurable" (Bressoud 2019, 89). The way around this restriction is to introduce the necessary quantities, such as an additional length, so that the accumulated area can be measured against an accumulating length (accumulator function), such that, "Area/time ÷ length/time = length." Bressoud continues, "The scientists of the eighteenth century realized that as long as they were careful with units they could dispense with restricting ratios to like quantities" (ibid.).

2.2 Apagogic Proof and the Limits of Intuition

In section four of 'The discipline of pure reason,' Kant broaches the topic of proof in both mathematics and philosophy. Proofs are necessary in the sciences of pure reason inasmuch as they show what elements could be held *responsible* for the leaps made by synthetic judgments a priori.[62] Kant acknowledges that, "The proof, therefore, had to indicate at the same time the possibility of achieving synthetically and *a priori* a certain cognition of things which is not contained in the concept for them" (CPR A783/811). Constructions are not themselves proofs, rather, constructions demonstrate the validity of synthetic a priori judgments by showing that such judgments are true of experience, i.e. that a judgment could be true of an object. Proofs give validity to what Kant calls *transcendental propositions*, in contrast to empirical propositions which are judgments of experience. The distinction is this: empirical propositions are made possible by the transcendental conditions of experience, where transcendental propositions are judgments made at the level of concepts, without consideration of a priori or a posteriori intuition.[63] Recalling the distinction between philosophical and mathematical cognition, Kant explains that "Synthetic propositions that pertain to **things** in general, the intuition of which cannot be given *a priori*, are transcendental. Thus transcendental propositions can never be given through construction of concepts, but only in accordance with *a priori* concepts. They contain merely the rule in accordance with which a certain

[62] "The proof does not show, that is, that the given concept (e.g., of that which happens) leads directly to another concept (that of a cause), for such a transition would be a leap for which nothing could be held responsible; rather it shows that experience itself, hence the object of experience, would be impossible without such a connection" (CPR A783/B811).

[63] In his essay 'Transcendental Arguments: Genuine and Spurious," Jaakko Hintikka discusses the role of transcendental propositions in possible experience: "Thus a transcendental argument is for Kant one which shows the possibility of a certain type of synthetic knowledge *a priori* by showing how it is due to those activities of ours by means of which the knowledge in question is obtained. This is, I take it, what Kant means by saying that a transcendental proposition "makes possible the very experience which is its own ground of proof"" (Hintikka 1972, 275).

synthetic unity of that which cannot be intuitively represented *a priori* (of perceptions) should be sought empirically" (CPR A720-721/B748-749). The judgment, say, effects necessarily have causes, concerns only the "time-conditions in general" of any object, and proceeds from the concepts alone insofar as "the concept is a rule of the synthesis of perceptions, which are not pure intuitions and which therefore cannot be **given** *a priori*" (CPR A722/B750n).[64] Where transcendental propositions can only be cognized from concepts, that is, they concern only the rules for the synthesis of appearances, the validity of such propositions must be proved by means of an intuition that would be impossible *without* said connection of concepts. Such proofs proceed by constructing the necessary intuition, which is equally a demonstration that the proposed synthetic judgment does not contradict the principles of the understanding. For example, 'all quality must be represented as intensive magnitude' is a principle that "has the special property that it first makes possible its ground of proof, namely experience, and always must be presupposed in this" (CPR A737/B765). This method of proof *directly* demonstrates the objective validity of a proposition because the construction implicitly demonstrates its relation to objects of possible experience, i.e. it is possible for a sensible intuition to take such a form.[65]

The above method of proof is called *ostensive*, since it involves an ostensive or symbolic construction, insofar as the latter must be completed by an ostensive construction. In contrast, Kant

[64] This is the example he gives when discussing demonstrations of pure reason: "no one can have fundamental insight into the proposition "Everything that happens has its cause" from these given concepts alone. Hence it is not a dogma, although from another point of view, namely that of the sole field of its possible use, i.e., experience, it can very well be proved apodictically. But although it must be proved, it is called a **principle** and not a **theorem** because it has the special property that it first makes possible its ground of proof, namely experience, and must always be presupposed in this" (CPR A727/B765).

[65] In some way, direct proofs are circular because 1) principles of the understanding make experience possible, and 2) possible experience proves the validity of the principles of the understanding. Again, this is evidence of Maimon's universal antinomy of thought, insofar as objective validity means only complete determination by the understanding, in contrast to determination by sensation. In what follows, it will become more obvious that *rules* and *principles* play a larger role than intuitions in transcendental proofs. And that it is by means of *spontaneity* and *not passivity* that synthetic judgments amplify the content of concepts.

also describes an *apagogic* method of proof which demonstrates the validity of propositions by contradiction or "refutation of their opposites" (CPR A792/B820). There are various styles of proof by contradiction which include reductio ad absurdum, reductio ad impossible, reductio ad incommodum, which more or less are arguments from self-contradiction, falsity, and exception.[66] Paolo Mancosu acknowledges that 'proof by contradiction' is a nebulous term, but proposes that, "Minimally, it means a proof that starts from assuming as a premiss the negation of the proposition to be proved. From this premiss we then derive a falsity or, equivalently, a contradiction. We are then allowed to infer the proposition that had to be proved" (Mancosu 1996, 105). There is a long (mostly Aristotelian) history of mathematical philosophy that debates whether direct and apagogic proofs are equally scientific.[67] Kant contributes to this history by advocating for the use of direct proofs where possible, and treating apagogic proofs as a last resort in mathematics alone. He writes, "The direct or ostensive proof is, in all kinds of cognition, that which is combined with the conviction of truth and simultaneously with insight into its sources; the apagogic proof, on the contrary, can produce certainty, to be sure, but never comprehensibility of the truth in regard to its connection with the grounds of its possibility" (CPR A789-790/B817-818). In other words, direct proofs explain *why* something is true, where apagogic proofs explain only *that* it is true.

In 'The Antithetic of Pure Reason,' Kant's diatribe against apagogic proof, he shows it is impossible to prove transcendental propositions in the same manner one proves empirical propositions, because the former are posited independently of any given intuition, and can never

[66] For simplicity, the term 'apagogic proof' will encompass these various styles of proof. Later on, proofs by reductio ad absurdum will be explicitly discussed.
[67] In the seventeenth-century, the paradigm of science was deeply, if not completely, influenced by the Aristotle's *Posterior Analytics*, namely, that scientific knowledge must adequately demonstrate insight into the material, efficient, formal, and final causes of things. Where these causes could be easily identified in natural science, it is not obvious what are the material or final causes of geometrical demonstrations. Mancosu shows that direct proofs could be reconciled with Aristotelian science thus conceived, but this came at the expense of stripping apagogic demonstrations of their scientific status (see Mancosu 1996, 8-33, especially 24-28).

be an object of experience. For example, the first antinomy results from either affirming or denying "The world has a beginning in time, and in space it is also enclosed in boundaries." Since the presence or absence of such limits can never be given in an intuition, one must rely on apagogic arguments. The argument for the thesis begins by assuming the absence of a beginning, then establishing an infinite series of moments that precede the present. But, as Kant clarifies, "The true (transcendental) concept of infinity is that the successive synthesis of unity in the traversal of a quantum can never be completed" (CPR A432/B460). So, it is impossible that the present could exist because the infinite series of preceding moments could not be traversed. As for the boundaries in space, were there none, it would be impossible to imagine the world as the given totality of the parts of space, because totality is thought either by delimiting a quantum in advance (indeterminate) or synthesizing each part that composes the world (determinate). Since the former is assumed false, the latter must be true. But it is impossible for all parts of space to be simultaneously given since the synthesis of all parts can never be completed because the world is boundless. (CPR A428/B456). Such arguments are entirely sophistical because they do not demonstrate the reason why the indefinite synthesis of moments or spaces is impossible. It is impossible because each synthesis is a judgment effected by the understanding, and such judgments are made serially because the pure form of inner sense is time. At this point, it is evident that the beginning of the world is a contradictory concept. The pure form of inner sense is time divorced from the empirical series of perceptions. Any non-empirical judgments must conform to time as succession, but an eternal non-empirical succession proper to cognition and not the world. In the *Prolegomena*, Kant writes:

> §52c. The logical criterion of the impossibility of a [self-contradictory] concept consists in this, that if we presuppose it, two contradictory propositions both become false;

> consequently, as no middle between them is conceivable, nothing at all is though by that concept. The first two antinomies, which I call mathematical because they are concerned with the addition or division of the homogenous, are founded on such a contradictory concept; and hence I explain how it happens that both the thesis and antithesis of the two are false. (Kant 2001, 76)[68]

The concept is contradictory because it presumes a determinate form of intuition while it is impossible to be *given* as an intuition. It is true that Kant gives apagogic proof a proper place in mathematical demonstrations: "only in those sciences where it is impossible to substitute that which is subjective in our representations for that which is objective, namely the cognition of what is in the object" (CPR A791/B819). In other words, so long as an apagogic proof makes no pretention to prove anything about *appearances*, or *sensible* intuitions then they are permissible.

The unique capacity for such proofs to manipulate indefinite series of operations *without* running through each step of the progression not only has mathematical significance, but philosophical significance as well, insofar as it is a means of *utilizing intuitive relations against any intuitive figure*. For instance, take Archimedes' proof of Proposition 1 in *Measurement of a Circle*, which reads: "The area of any circle is equal to a right-angled triangle in which one of the

[68] Only the first two (mathematical) antimonies center on contradictory concepts. As Kant writes in §53., "In the first (the mathematical) class of antinomies the falsehood of the presupposition consists in representing in one concept something self-contradictory as if it were compatible (i.e., an appearance as an object in itself). But as to the second (the dynamical) class of antinomies, the falsehood of the presupposition consists in representing as contradictory what is compatible. Consequently, whereas in the first case the opposed assertions were both false, in this case, on the other hand, where they are opposed to one another by mere misunderstanding, they may both be true." (Kant 2001, 78). The mathematical antinomies are contraries, that is, they can both be false but cannot both be true. The dynamical antinomies are subcontraries, that is, they can both be true but cannot both be false. So the world cannot have and not have a beginning, but there can be some causes that are empirical and some that are free. Only contradictory propositions (All S is P, Some S is not P) can provide certainty. Kant makes this point in the *Jäsche Logic*: "When I prove a truth from its grounds I provide a direct proof for it, and when I infer the truth of a proposition from the falsehood of its opposite I provide an indirect one. If this latter is to have validity, however, the propositions must be opposed *contradictorily* or *diametraliter.* For two propositions opposed only as contraries *(contrarie opposita)* can both be false" (Kant 1992, 575).

sides about the right angle is equal to the radius, and the base [i.e. the other of the sides about the right angle] to the circumference [of the circle]" (Dijksterhuis 1987, 222). As Dijksterhuis notes, "The proof is furnished by means of the *indirect* method for dealing with infinite processes" [emphasis added] (ibid.). He explains that Archimedes, in all but one instance, employed the *compression method* (i.e. the method of exhaustion) for approximating the limit towards which an infinite process tends. In *Measurement*, the difference form of compression is employed, as opposed to the ratio form.[69] Both, however, work in the manner described above by Mancosu. The proof begins by assuming the area of the circle, A, is larger than the area of the triangle, T, whose height is the radius of the circle, and whose base is the circumference, i.e. $A > T$. Now, inscribe a regular polygon of n sides, whose area is P, within the circle (see Figure 2 from Bressoud 2019, 9). Since P is inscribed, $A > P$, but it is possible to construct another polygon with more sides (greater value of n) such that $A - P < A - T$. That is to say, though one cannot construct a polygon with infinitely many sides, it is in principle possible to reduce the difference between P and A to less than an assignable quantity. We began by assuming that T was smaller than A, but the area of the inscribed polygon whose circumference is less than that of the circle is greater than T. Therefore, $A - P < A - T$ since $T < P$ (the area of the polygon more closely approximates the area of the circle than does the area of the triangle). This contradicts the assumption that $A - T < A - P$, which is to say $T > P$ because no inscribed polygon can posses a circumference equal to the circle. So, the area of the triangle *cannot be smaller*, than that of the circle. The second step of the proof is to repeat this process by assuming that $T > A$, and to circumscribe a polygon about the

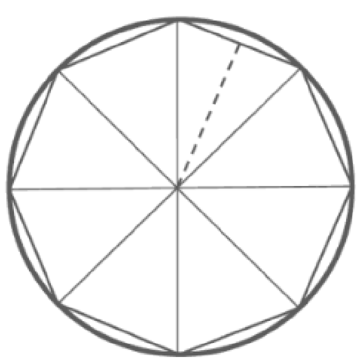

Figure 2

[69] I assume that the difference form is employed because the proof aims to demonstrate equality, which is expressed by the ratio 1:1. In this instance, the use of ratios can be dropped without any consequence.

circle, ever increasing the number of sides so that $P < T$, which contradicts the initial assumption again. This step proves that *T cannot be greater* than *A*, which, in combination with the result of the first step, places the value of *T* between any value larger or smaller than *A*.[70] Form this, one concludes that $T = A$, not directly from constructions, but indirectly by appealing to the *Archimedean axiom* of all (Euclidean) magnitudes.

In his essay ‘Leery Bedfellows: Newton and Leibniz on the Status of Infinitesimals,’ Richard T. W. Arthur defines the Archimedean axiom as: “If *a* and *b* are two line segments (or other continuous geometric quantities) with $a < b$, we can always find a (finite) number *n* such that $na > b$,” and shows how the Archimedean property of such quantities is common to *both* Newton’s fluxional method for approaching the calculus, and Leibniz’s analytical method (Arthur 2008, 18). Though this axiom derives its name from the demonstrations of Archimedes, it is much earlier formulated in Euclid’s *Elements*. Friedman points to Book V, Definition 4 which he paraphrases as, “two magnitudes *a* and *b have a ratio* if for some number *n*, *na* (that is, *a* added to itself *n* times) is greater than *b* – and vice versa” (Friedman 1992, 111n.28). Where Bressoud and Dijksterhuis cite Book X, Proposition 1, which reads: “If, from the greater of two unequal magnitudes (which are) laid out, (a part) greater than half is subtracted, and (if from) the remainder (a part) greater than half (is subtracted), and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude” (Euclid 2008, 282-283). [71] The axiom is vital in geometrical reasoning, since it is what establishes a functional

[70] Or, more succinctly, “the method of exhaustion proceeded by means of a double-reductio, effecting two different constructions of polygonal spaces, one circumscribing the given gradiform space, the other inscribed within it, to prove that the area of the given space could be neither greater than nor less than a certain quantity” (Levey 2008, 118).

[71] See (Bressoud 2019, 8). In his discussion of *Measurement*, Dijksterhuis cites Book XII Proposition 2 of Euclid’s *Elements*, which reads “Circles are to one another as the square on (their) diameters” (Euclid 2008, 473). The proof restates the proposition in terms of areas, namely if the squares on the diameter of two circles are not proportional, the larger circle “will be to some area either less than, or greater than” the smaller circle. From this, the areas of both

relationship between known and unknown magnitudes, and Friedman broaches this subject when discussing the capacity of geometrical figures to represent irrational magnitudes as incommensurable ratios between line segments (see 1.3n.32). It is also the axiom that allows Newton (and Leibniz) to imagine the coincidence of two points on a curve when their difference is less than any assignable magnitude, despite the impossibility of geometrically constructing this coincidence.

In his Method of Fluxions, Newton described the change in variables as a function of time, which is assumed to be continuously elapsing, as a way of rigorously founding the Archimedean operation of continually reducing the difference between two magnitudes. By positing fluents in nature, Newton also addresses the fact that each static geometrical construction can only be a snapshot of a process which fulfils itself independently of its representation.[72] The intuitive argument is that, as two point move closer together (or farther apart) their values must coincide at the moment where the two points coincide. Concerning the synthetic method as explicated in the *Principia*, Arthur writes, "when Newton comes to secure the foundations of his synthetic method in the *Method of First and Ultimate Ratios*, he appeals to Lemma 1, which is a synthetic version of the *Archimedean axiom*" (Arthur 2008, 9). The Lemma reads:

> Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal. (quoted in ibid.)

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circles are approximated by inscribing a square, constructing triangles on each of its sides, and obtaining a polygon with double the sides and greater area. The crux is the reduction, and eventual coincidence, of the difference between the area of the circle and that of the tringle. Here, Euclid cites Book X Proposition 1.

[72] This is also a response to Cavalieri's and Leibniz's use of infinitesimals which, if not totally devoid of content, were too abstracted from the objects of mathematics. This is what Guicciardini calls 'the ontological content of the method of fluxions' (Guicciardini 1999, 35).

The key phrase is '*their difference is less than any given quantity*,' since it reveals that the fluxional method is equally a method of reasoning by *reductio*. This formulation of the Archimedean Axiom is *synthetic* because it presumes to have insight into the rules immanent to "fluent geometrical figures, synthetic in the sense of flowing, increasing, staying constant, or decreasing continuously in time" (ibid., 14).[73] This is also true of Newton's analytic method, since, in *Method of Fluxions* he writes "And a ratio of equality is to be regarded as one which differs less from equality than any unequal ratio can be assigned" (quoted in Arthur 2008, 19). Arthur's commentary reads: "The principle appealed to here is this: If an inequality is such that its difference from a strict equality can be made smaller than any that can be assigned, it can be taken for an equality. Let us call this the *Principle of Unassignable Difference*. This principle, clearly, is the analytic equivalent of the chief synthetic axiom, Lemma 1 of the *Method of First and Ultimate Ratios*. And like that Lemma, it derives its warrant from the *Archimedean axiom*" (ibid.).[74] In any case, Newton aims to give apagogic reasoning a rigorous foundation by grounding continuous and flowing quantities in the heart of nature.[75]

[73] More formally, Arthur renders the synthetic version of the axiom as: "given two quantities whose difference D is less than some quantity a, we can always find a number n such that $n\mathrm{D} > a$, so that $c = a/n < \mathrm{D}$" (Arthur 2008, 18). Notice that reference to both time and motion are dropped in this instance.

[74]Against those that identify a discontinuity in Newton's methodology, the most notable is Colin Maclauren (an English advocate for fluxions), Arthur is clear that – "This common warrant underwrites the equivalence between the analytic and synthetic methods of fluxions, allowing the translatability of statements given in terms of "indivisibles" (i.e. infinitesimals) into fluxional terminology…" (Arthur 2008, 19).

[75] Guicciardini cites a number of extrinsic reasons why Newton might have insisted on a 'natural' as opposed to 'fictional' foundation for infinitesimals – "from the early 1670s, he was led to look for a restoration of an ancient knowledge in alchemy. Newton began, in the 1670s, to conceive himself as a man who belonged to a remnant of interpreters who could restore, through the deciphering of Biblical texts, the original natural philosophy and religion of mankind, revealed by God to Noah. He believed that this knowledge had passed, through the sages of Israel, to Egypt. He also maintained that it was corrupted there by priest and rulers. From time to time, enlightened interpreters were able to restore part of this lost wisdom. Mathematics has long enjoyed a close relationship with mysticism. For Newton 'mathematics was God's language'. It is striking that in the same years Newton began attributing to Jews, Egyptians and Pythagoreans a lost knowledge concerning alchemy, God *and* mathematics. It is plausible that in Newton's mind the restoration of the lost books of the ancient geometers of Alexandria was linked to his attempt to re-establish a *prisca sapientia* [pristine knowledge]" (Guicciardini 30-31).

This is unacceptable for Kant since all continuous magnitudes (extensive or intensive) have their source in the human faculty of sensation. Not only this, but all cognition is the result of a synthesis, a spontaneous act of the understanding. In section VII of 'The Antinomy of Pure Reason,' Kant writes that:

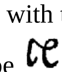
> For the **appearances**, in their apprehension, are themselves nothing other than an empirical synthesis (in space and time) and thus are given only **in this synthesis**. Now it does not follow at all that if the conditioned (in appearance) is given, then the synthesis constituting its empirical condition is thereby also given too and presupposed; on the contrary, this synthesis takes place for the first time in the regress, and never without it. (CPR A499/B527)[76]

That is to say, even if it is in principle possible to indefinitely subdivide moments, or indefinitely reduce the difference between two magnitudes, *the proof of their identity must demonstrate their literal coincidence* in intuition (i.e. the synthetic a priori conditions of experience). In direct and ostensive proofs, each element of the construction is determined *as much as possible* by the understanding, and as such, is a thoroughly objective representation of the transcendental proposition in question. However, this is impossible for infinite series because, not only is there no last term, *there is no point at which a member of the series becomes that toward which the series as a whole tends.* In her magnificent book, *Realizing Reason,* Danielle Macbeth traces the trajectory of mathematical practice from object-involving methods (namely, Euclidean geometry) to a fully conceptual, non-intuitive, *mathematical practice*. She marks a rift between the

[76] Or, as Kant writes in the *Dohna-Wundlacken Logic*: "The given multitude of all parts is finite or infinite, i.e., all the parts of a body taken together, and we can never do this, for every body is infinitely divisible, one would never be finished. Hence taking all the parts together is a *contradictio in adjecto*. – The result of all this is that one cannot make use of apagogic proofs in philosophy because here one cannot present them in intuition" (Kant 1992, 483).

mathematical techniques of the seventeenth and eighteenth centuries and the Kantian philosophy of mathematics. Concerning Zeno's paradox, and the temptation to identify the specific point at which Achilles passes the Tortoise, she writes that "the task is not to construct the limit – that is, actually to produce it using the familiar rules of arithmetic and basic algebra – but to *describe* the property an infinite sum must have if it is to converge to a limit. To understand the infinite, at least in these cases, requires a *conceptual* approach as contrasted with a computational one" (Macbeth 2014, 211). As elaborated in 1.3, Kant's understanding of arithmetic is wholly computational, in that any numbers or symbols used in calculation only mark the succession of syntheses in time. Moreover, he understands algebra to be a shorthand which aids in manipulating the proportions of a figure, and whose objective validity wholly depends on the final construction of a figure in intuition. This view is conservative even for the geometrical paradigm Kant is supposedly defending, especially since Descartes' development of analytic geometry was not only accompanied by, but facilitated by a revolution in *notation* that addressed some of the pitfalls of reducing every equation to a figure.

The potency of analytic geometry was not due to reason's determination of its own limits, as Kant would have it. It was due to both a notation and a method that did not depend on intuition to adequately represent its objects. Following the example of Michel Serfati, the Cartesian exponent (e.g. a^2) allowed for the disarticulation of substance (the base) and relation (the exponent) (Serfati 2010a, 115). It is since possible to represent the operation of squaring an arbitrary expression (be it a single term or whole equation), which was impossible in the cossic notation that preceded Descartes.[77] In addition to this, the post-cossic symbolism of François Viète operated according to

[77] Serfati examines the notation of both Diophantus (a third century mathematician) and Michael Stifel (a fifteenth century monk and mathematician). The flaw with the cossic notation (in this case it is Stifel's symbols) is that both a quantity, , and its square,  (and its cube  too), are expressed by one symbol only. The 'square' symbol can be used only with regard to a single term, and cannot be used to express the square of a complex one, like the equation

two sets of operations, which tackle the two *species* of objects, namely, mathematical/symbolic and physical/geometrical objects. Viète's Analytical Art begins by translating an arithmetical or geometrical problem into his system of symbols, then one transforms the equation according to the rules of the algebra, lastly, the transformed equation is given an arithmetical or geometrical interpretation.[78] Macbeth highlights that the first stage "takes one out of a particular domain of enquiry, either arithmetical or geometrical, into a purely formal system of uninterpreted signs that are to be manipulated according to rules laid out in advance, and only in the last stage…are the signs again provided an interpretation" (Macbeth 2014, 126).[79] Beginning in the *Regulae*, Descartes transformed this empty formalism into "une seul description sur le terrain, à propos d'une réalité que lui-même envisageait pourtant sous deux aspects irréductibles, mathématique et physique" (Serfati 1997, 210). These two advancements over the previous notation, in addition to their mnemonic value, qualitatively altered the capacity to represent mathematical objects. In other words, with new notation came the possibility to discern the implicit order that governed a given figure and represent this order in its full arbitrariness. Serfati emphasizes that, "Above all, the

1ʒ + 1 ꝛ + 2. One could only multiply the equation by itself, and compute the result. What seems to be a notational restriction actually is a representational restriction, since the content of thought is formulated through these symbols. The inability to substitute one term for another keeps the operation of squaring tied to the geometrical notion of constructing a square, instead of developing an analogous operation where an arbitrary term can be subject to the same rule, regardless of its intuitive sense. As Serfati explains, this kind of computation "requires the calculator to use various memorizing methods, and thus to appeal to elements of meaning foreign to the symbolic system. Actually, such "square", since it is unrepresentable in the cossic system is not capable of being individuated, that is, it cannot be *objectified*. In other words, it is *inconceivable* as an object in the system" (Serfati 2010a, 114). For a detailed explication of the cossic system, see (Serfati 1997, 184-193).

[78] The three stages are named *zetetics*, *poristics*, and *exegetics* respectively.

[79] Macbeth explains that "the arithmetical operations that are applied to numbers in the Analytical Art are essentially different from those applied to geometrical figures. In arithmetic, as Viète understands arithmetic, one calculates with numbers, each calculation taking numbers to yield numbers; in geometry (again, as Viète understands it) one constructs using figures, and in the cases of multiplication and division, and in the geometrical analogue of the taking of roots, the result of a construction is a different sort of figure from that which began (or even a different sort of entity altogether, namely, a ratio)" (Macbeth 2014, 125-126). She also notes, in the Eudoxian theory of proportions, as employed by Euclid and Archimedes, proportion is univocal, that is, it makes no such distinction between ratio and number. Descartes reintroduces the generality of proportion, not be reverting to the geometrical paradigm of the ancients, but by reifying relations and treating them as proper mathematical objects.

advent of symbolism appears not as a mere "change of notation" against a (supposedly) unchanged mathematical background, but on the contrary, as a decisive conceptual revolution, in particular with regard to the creation of objects" (Serfati 2010a, 108). It is true that Descartes set determinate boundaries on what counts as a well-formed equation (for instance, the base (substance) of a term must be accompanied a numerical relation, not one that is arbitrary), and that he did not explicitly design his algebra as a means of breaking with the past (Serfati 1997, 237-241). However, he introduced a system of symbols capable of producing well-formed equations that *did not inherently* lend themselves to geometrical representation, because the objects of algebra could equally be produced by combinatorial means alone.[80]

Kant overlooked this link between symbolism and invention, between notation and cognition, because the need to completely and objectively determine the representations of cognition is incompatible with the algebraic paradigm introduced by the calculus. The demand is to grasp the rules according to which appearances are ordered, not the individual elements. This cannot be done by successively constructing intuitions, since rules only reveal themselves as relations that range over any number of arbitrary objects.[81] According to Marcelo Dascal:

> C'est alors que la pensée se sert d'un raccourci : au lieu de fixer le regard sur chaque élément séparément pour effectuer ensuite la fusion de tous les éléments et obtenir ainsi l'idée complexe, elle « ferme les yeux » aux éléments, renonce à un mode séquentiel de perception, ignore les détails et effectue un saut directement vers la fusion ou synthèse de l'idée complexe. Ce faisant, elle procède en *aveugle*, car, elle ne procède pas par l'examen de chaque chiffre, sa

[80] This is especially true of Newton's use of complex exponents (e.g. $a^{x+1/x}$), or Leibniz's development of transcendental curves which could not be resolved into any geometrical representation (Serfati 2013, 84-89).
[81] Or in Leibniz's own words, "And anyone who is not satisfied with this can be shown in the manner of Archimedes that the error is less than any assignable quantity and cannot be given by any construction" (L 546).

> reconnaissance, l'évocation du nombre correspondant, et leur union dans l'idée du nombre total. (Dascal 1978, 206)

This is precisely the program of Leibniz's *blind* or *symbolic thought*, which animates his combinatorial approach to the calculus, and bolsters his commitment to continuity in both mathematics and metaphysics. This is also the model of cognition that inspires Maimon to emphasize the importance of the rules according to which appearances arise over constructing the objects to which such appearances belong.

Chapter 3

3.1 Real Definitions between Finite and Infinite Cognition

No matter the shortcomings of Kantian philosophy, it is impossible to return to the pre-critical utopia of a world without representation. Leibniz would have agreed that the human mind is finite, and that human reason could at most possess partial, *symbolic* knowledge (as in mathematics), though never possess adequate, *intuitive* knowledge of any thing whatsoever.[82] In his essay 'Meditations on Knowledge, Truth, and Ideas' (1684), adequacy of knowledge is a third criterion added by Leibniz to Descartes' already famous criteria of clarity and distinctness. Where clarity is knowledge sufficient to distinguish one concept from another, and distinctness is knowledge sufficient for enumerating the marks or predicates of a concept, Leibniz explains that, "when every ingredient that enters into a distinct concept is itself known distinctly, or when analysis is carried through to the end, knowledge is adequate. I am not sure that a perfect example of this can be given by man, but our concept of numbers approaches it closely" (L 292). Adequate knowledge of a thing can only be obtained by specifying the components of its concept, and likewise specifying the components of the components. The set of determinations that compose a complex concept cannot all be given to the human mind at once. Leibniz gives the example of a thousand-sided polygon (chiliagon). Though one easily grasps the words 'thousand-sided' and 'polygon,' the individual determinations denoted by 'thousand-sided' are not each explicitly comprehended. However, the vagueness or generality of the words might facilitate a judgment that would

[82] For instance, see this passage from 'Meditations on Knowledge, Truth, and Ideas' (1684): "We know [an idea is true] when we resolve a concept into its necessary elements or into other concepts whose possibility is known, and we know that there is nothing incompatible in them…Whenever our knowledge is adequate, we have a priori knowledge of a possibility, for if we have carried out the analysis to the end and no contradiction has appeared, the concept is obviously possible. Whether men will ever be able to carry out a perfect analysis of concepts, that is, to reduce their thoughts to the *first possibles* or to irreducible concepts, or (what is the same thing) to the absolute attributes of God themselves or the first causes and the final end of things, I shall not now venture to decide" (L 293).

otherwise be impossible (e.g. solving for the area of a chiliogon by constructing triangles whose bases are the length of each side, and whose height the apothem). Accordingly, the process of using signs to serially specify the marks of a concept would not be adequate or intuitive knowledge, but symbolic knowledge.

"Such thinking," writes Leibniz, "I usually call *blind* or *symbolic*; we use it in algebra and in arithmetic, and indeed almost everywhere." (ibid.). As discussed in 1.2, Kant believes that this process would be indefinite, since concepts can always be further specified. This is a process made possible by time as the form of inner sense. He also explains that the resultant concept would not be adequate to the thing since it remains a representation and not a thoroughly determinate intuition. Adequate knowledge is for this reason *problematic* not symbolic, because it can never be "the concept of an object, but rather the problem, unavoidably connected with the limitation of our sensibility, of whether there may not be objects entirely exempt from the intuition of our sensibility, a question that can only be given the indeterminate answer that since sensible intuition does not pertain to all things without distinction room remains for more and other objects" (CPR A288/B344).[83] Symbolic cognition does not admit that adequate knowledge is possible for humans, only that it is not *in principle* impossible (i.e. it is a difference of degree not kind). Maimon explains that at the end of this infinite progression, the symbolic and the intuitive would coincide, but that this is both impossible for us and not the point: "So symbolic cognition extends to infinity (*qua materia*), as with, for example, a circle viewed as a polygon with infinitely many sides, the asymptotes of a curved line, and the like. Although we cannot think the infinite as an object [of intuition] this is besides the point here, since we do not make use of infinity to think the

[83] Adequate knowledge includes all determinations beyond what can be gleaned by sensible intuition, which would not only produce a thoroughly determinate representation, but this panoptic view would be indistinguishable from the thing itself. For this reason, the thing-in-itself is not a concept but a thought-being (*noumenon*) which acts only as the limit of sensible intuition. This is consistent with our treatment of it in section 2.1 and in 1.1n.14 and 16.

object, but merely to think the way it arises" (ETP 273-274). Symbolic cognition takes only the formal component of any object (its concept or rule) and thinks it independently of any matter, including pure intuition.[84] Symbolic cognition supplements the finite mind's incapacity to ever grasp a thoroughly (matter and form) determinate object.

Intuitive knowledge is possible only for a mind whose judgment and ratiocination can produce thoroughly determinate representations, that is, whose intuition is intellectual. [85] Kant and Leibniz agree that intuitive knowledge is only ever analytic and a priori; all connections are necessary connections, and nothing can be discovered that is not already known. But this capacity does not belong to human minds, which require the mediation of concepts and sensible intuitions in all acts of cognition. Synthetic cognition is properly finite since it must discover and ratify all non-analytic connections. For Kant, synthetic a priori propositions can be discovered and proved by constructing the concept in intuition. The matter is more complicated for Leibniz. The concept of each substance or thing expresses all of creation from a particular vantage point, however, a synoptic view is reserved for God alone. In the 'Discourse on Metaphysics,' he explains that "when we well consider the connection of things, it can be said that there are at all times in the soul of Alexander traces of all that has happened to him and marks of all that will happen to him and even traces of all that happens in the universe, though it belongs only to God to know them all" (L 308). An adequate concept of Alexander would contain all such marks and traces, and a distinct knowledge of the components of each mark and trace. For God or some infinite, intuitive intellect, these are analytic a priori connections. For finite, discursive intellects, these connections must be

[84] This is what Maimon means when he writes "As a result, an object of symbolic cognition is: a form, or way, of thinking an object of intuition, that is itself treated as an object (but not of intuition)" (ETP 272).
[85] As a prelude to what will follow, Leibniz's intuitive knowledge is what Kant will call 'intellectual intuition,' or a kind of non-sensible cognition whose representations are things as they are considered in-themselves. Maimon will reconcile these two uses of the term intuition by positing intellectual intuition as a regulative ideal for human cognition.

amassed and collected by synthetic means and *might come to be* analytic at the end of the process (whether or not this is possible is another matter). Against Kant, Leibniz maintains that this can be done by *symbolic* and not necessarily intuitive cognition. In a letter to Edmond Mariotte (1676), Leibniz takes the characteristic of algebraic equations as the model for such a process. He writes that "les noms sont des especes des characteres. Effectivement l'algebre ne vous sçauroit donner au bout du compte que des caracteres, sçavoir la valeur d'une lettre exprimée par quelques autres lettres ; mais cela suffit pour entendre la chose même. Et les definitions en font de même. Puisqu'une equation en effect n'est qu'une espece de la definition" (Leibniz 1926, 271 ; also quoted in Dascal 1987, 77n.23). In short, the finite human mind is presented with words, sounds, or any number of corporeal traces by means of which ideas are communicated or expressed. And it is according to these signs alone that the human mind can define and understand the concepts of things.[86] Though there can be no adequate symbolic definition of a concept, Leibniz introduces the radical possibility of symbolically constructing *real definitions*, or formulating the rules according to which objects arise, even if such a process is indefinite.[87]

The distinction between nominal definitions (*definitiones quid nominis*) and real definitions (*definitiones quid rei*) was introduced by Giovanni Saccheri in 1697 to clarify the definitions of, well, definitions; namely, what is entailed by the definitions that appear in the works of Euclid. As

[86] It is true that Leibniz explored the possibility of memory or recollection without signs. This, however, is a kind of reflection is of no use to cognition. As Dascal notes, "It remains therefore true that, although there may be some 'pure' character-less thought of this type, thought which involves reasoning, the only kind which interests philosophy, science – in short, every systematic and developed for of thought – cannot exist without characters" (Dascal 1987, 56). He takes this oscillation between 'pure' memory and reasoned cognition as evidence that Leibniz was still grappling with the relation between inadequate human knowledge and the world. The immediacy of memory is later replaced with the ideas of expression and harmony, and the principle of sufficient reason and law of continuity that accompany these ideas.

[87]According to Francesco Bellucci: "a nominal definition gives us only *an* enumeration of component marks, while a real definition gives us the *unique* enumeration of the *primitive* component marks. The former shows us *a* possible definition of the term, whereas the latter exhibits the *unique* definition of that term that alone prove [*sic*] the possibility of the thing defined" (Bellucci 2013, 337).

T. L. Heath summarizes, “the former are only intended to explain the meaning that is to be attached to a given term, whereas the latter, besides declaring the meaning of a word, affirm at the same time the existence of the thing defined or, in geometry, the possibility of constructing it” (Heath 1908, 144-145). Existence is affirmed either by postulating it, or by means of an actual construction. As mentioned in the previous chapter (1.4), Kant employs this terminology when he explains that mathematical constructions must demonstrate the objective validity of the thing constructed, i.e. that the object must be possible. Mathematical definitions are real definitions because they supply the rules necessary to construct concepts in pure intuition, which must take the form of possible appearances insofar as pure intuition is ‘the mere possibility of external appearances.’[88] Recall that real definitions entail construction, and “to **construct** a concept means to exhibit *a priori* the intuition corresponding to it…completely *a priori*, without having had to borrow the pattern for it from any experience” (CPR A713/B741, see 1.2). For Kant, real definitions are restricted to the concepts of mathematics insofar as they are *arbitrary,* that is, all the component marks sufficiently represent the object, and that this sufficiency “is not derived from anywhere else and thus in need of a proof, which would make the supposed definition incapable of standing at the head of all judgments about an object” (CPR A727/B755). Such a concept ‘stands at the head’ of all judgments because by manipulating the construction, one can amplify the intension of a concept beyond what is entailed by the nominal definition (i.e. the collection of marks sufficient for distinguishing the concept from others), regardless of whether there is a corresponding sensible intuition in experience. Yet, it is exactly this *intuitive immediacy*

[88] Though nominal definitions play a less pivotal role in Kant’s philosophical edifice, he does discuss them in his logical works. For example, “By mere *definitions of names,* or *nominal definitions,* are to be understood those that contain the meaning that one wanted arbitrarily to give to a certain name, and which therefore signify only the logical essence of their object, or which serve merely for distinguishing it from other objects” (Kant 1992, 634). He explains that they begin from attributes of an object, and collect them under a common name rather than begin from the a priori possibility of the object. This is why, according to Kant, no empirical concept can be given a real definition (CPR A728/756).

that obtains between definitions and constructions that produces the discord between sensible intuition and pure intuition, the given appearances of actual experience and the pure constructions of possible experience (see 1.4). And not only this, the misplaced emphasis on the activity of the understanding elides the continuity between the rule-governed synthesis of both the understanding and the imagination.

In his essay ‘Kant (vs. Leibniz, Wolff and Lambert) on real definitions in geometry,’ Jeremy Heis identifies the link between real definitions, constructions, and proof in Kant’s philosophy of mathematics. He explains, “[Kant’s] argument that the construction must be *immediately* contained in the definition thus depends essentially on proofs’ requiring construction” (Heis 2014, 610). As explicated above, the construction of a non-empirical concept proves the objective validity of its definition, or what is the same, proves the validity of certain synthetic a priori judgments. However, the link between the specific operations, steps, or procedures, and the combination of marks or predicates of the definition cannot be proved, because “the proof that a concept can be constructed would itself already require that the concept be constructed” (ibid., 610-611). Heis is right to be sceptical of the notion of intuitive immediacy, not because it is vague, but because this notion elides the difference between the imagination as an empirical faculty that produces appearances and as a transcendental faculty that is determined to schematize (i.e. order intuitions according to the pure concepts of the understanding). This difference was sketched in terms of Kantian construction in 1.4. In Maimon’s terms, this is the difference between the imagination as the *faculty of fictions* and the imagination as the *faculty of sensations.* More than a terminological dispute, this distinction allows Maimon to reduce the stock of cognitive faculties. In place of Kant’s model, which goes: empirical synthesis of the imagination – transcendental synthesis of the imagination – transcendental unity of the understanding, Maimon removes the middle term and treats it as a mere heuristic where the understanding can demonstrate the objective validity of a concept (i.e. that it can be given form in space or time). The understanding cannot, however, ground the definition of a concept by means of this procedure, because the reality of the set of marks or predicates that compose the definition must inform the transcendental synthesis of the imagination (i.e. the definition of a concept must logically precede its schema). How can this be? How is it then possible to amplify the connections between concepts? Maimon does not deny the importance of synthetic judgments, in fact, he thinks that they are necessary for discursive (finite) cognition. He instead denies that the relation between concepts and their intuitive construction-procedures (schemata) cannot be grounded in possible experience, because possible experience already assumes this very correspondence. Maimon gives the example that, "in the judgement that the straight line is the shortest line between two points we have an apodictic cognition of a correspondence between two rules that the understanding prescribes to itself for the construction [*Bildung*] of a certain line: (**being straight** and **being the shortest**).We do not comprehend why these two must be combined in one subject, but it is enough that we have insight into the possibility of this correspondence (in so far as they are both *a priori*)" (ETP 55).

Take as another example, the relationship between the intuition of a circle, its definition, and its construction procedure. In Book 1, Definition 15 of the *Elements,* Euclid defines the circle thus: "A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another" (Euclid 2008, 6). Maimon (perhaps here following Wolff) would agree that this is a nominal definition, since it does not convey whether such a thing can be constructed.[89] The genetic component of this definition is found in Postulate 3 of the same

[89] See (Heis 2014, 612) and (ETP 50).

book, which reads: “And to draw a circle with any center and radius” (ibid., 7). For Kant, the postulate is unnecessary since the definition already contains the rule for inscribing the circle, because the understanding must represent its concepts to itself in intuition; one cannot think a circle without describing or generating it in thought (CPR B154). In a letter to Herz he explains, that for any drawing of a circle, “I can demonstrate all the properties of a circle just as well on it, presupposing the (so-called nominal) definition, which is in fact a real definition, even if this circle is not at all like one drawn by rotating a straight line attached to a point. I assume that the points of the circumference are equidistant from the centre point. The proposition ‘to inscribe a circle’ is a practical corollary of the definition (or so-called postulate), which could not be demanded at all if the possibility – yes, the very sort of possibility of the figure – were not already given in the definition” (quoted in Heis 2014, 611). Kant’s point is that, regardless of how the actual intuition was constructed, the concept of circle is already possible because it contains a determinate construction procedure (i.e. the rotation of a line about one of its endpoints). Maimon decomposes the argument accordingly: some intuitive figure is given to consciousness; the understanding subsumes this intuition under the concept that most appropriately accounts for the pure spatial relations of the intuition; this concept (circle) has a determinate construction procedure which is governed by a rule (i.e. the rotation of the line); when the rule that determines the construction is compared to the definition of the circle, they are found to be identical; a line rotated about one of its endpoints remains identical with itself, that is to say, at every position it radiates outwards and is equal to itself in all other positions; hence, the relationship between the construction procedure (i.e. rule) and the definition is immediate and a priori (ETP 43). Kant would not say it is analytic because the time-determinate that corresponds to the concept is not contained in the concept. This is exactly the gap that Maimon aims to interrogate. Against Kant, Maimon maintains that this immediacy is not *grounded* in possible experience, but is grounded in something prior to the transcendental synthesis of the imagination (which gives concepts intuitive form). Maimon concludes that:

> This identity is not given by experience; experience only gives something that is represented absolutely and this allows what cannot be comprehended in itself (the forms and categories) to be comprehended. The matter of intuition, what is directly related to an object, makes the form of intuition comprehensible; that is to say, it makes comprehensible both the forms of intuition with all their possible connections and relations, and also the pure concepts of the understanding or forms of thought, which do not relate directly to an object but only indirectly, by means of the categories. (ETP 43-44)

Experience cannot be the ground of immediacy because the construction itself does not explain *why* the abstract definition 'all radii are equal' can be generated by means of rotation. To adapt Maimon's words to this particular example, 'what determines the faculty of judgment to think the rule-governed rotation as corresponding to the rule of the understanding itself, and to think each particular member of this sequence as corresponding to each particular member of the of the rue of the understanding?' (ETP 54). In other words, one cannot take for granted that concepts find their way into sensibility, since it is impossible to supply a *reason* for the correspondence of these two faculties.

3.2 Cognition without Construction: The Real Ground of Synthesis

In the previous chapter, Maimon's universal antinomy of thought was introduced as a means of emphasizing the way that constructions legislate the activity of the imagination. His argument turns on the fact that constructions can never be as determinate as intuitions, not because the imagination cannot produce determinate representations, but because the *understanding* cannot produce such representations. In an article for the *Berlin Journal for Enlightenment*, Maimon parses the cognitive dynamics of construction in a way that highlights Kant's equivocal use of the imagination. He writes,

> The understanding prescribes a rule for the productive imagination to produce a space enclosed by three lines; the imagination obeys and constructs a triangle but sees that at the same time three angles are forced on it, something the understanding certainly did not ask for. At this point the understanding becomes cunning: it learns to see into [*einsehen*] the previously unknown connection between three sides and three angles, although its ground is still unknown to it. It thus makes a virtue out of a necessity, adopting an imperious manner and saying: 'a triangle must have three angles', as if it were itself the lawgiver here, when in fact it has to obey a completely unknown law-giver. (Maimon 2010, 247)

Such unintended determinations are not only given to the understanding in experience, but are also gleaned in a priori constructions. This is uncontroversial, and is in essence the mechanism of synthetic a priori judgments according to Kant. However, the source of these implicit determinations is not the understanding, precisely because they are not intended and not cognized priori to the productive (transcendental) synthesis of the imagination. In the case of both empirical experience and pure constructions, "the judgments themselves, as forms or ways of thinking these objects, are in both cases *a posteriori*" and differ only according to the *source* of the synthesis (ETP 76).

This difference is modal. As Kant specifies in 'The Postulates of Empirical Experience,' modality "does not assert of a concept anything other than the action of the cognitive faculty through which it is generated" (CPR A234/B287). The objects of cognition can be either possible, actual, or necessary.[90] In a concise passage, he explicates the various relations characteristic of each modality: "if it is merely connected in the understanding with the formal conditions of experience, its object is called possible; if it is in connection with perception (sensation, as the matter of the senses), and through this determined by means of the understanding, then the object is actual; and if it is determined through the connection of perceptions in accordance with concepts, then the object is called necessary" (CPR A234/B286). But this overestimates the role of the understanding in actuality, especially since he has conceded that "Synthesis in general is, as we shall subsequently see, the mere effect of the imagination, of a blind though indispensable function of the soul, without which we would have no cognition at all, but of which we are seldom even conscious" (CPR A78/B103). According to Maimon, actuality differs significantly from possibility because the introduction of sensibility also introduces another, *passive* synthesis that is not performed by the understanding, nor by the imagination determined by the understanding.[91] The sufficient criterion for actual experience is not that a representation is thoroughly determinate, as Kant might have it (Kant 1992, 597). It might very well be the case that all sensible intuitions are individual representations to which no other determinations can be added, but this is not what *convinces* us (epistemologically) that an object is actual. Sensible intuitions alone cannot convince the mind that it is completely determinate, because "all of these determinations can only be known

[90] The modalities of possibility, actuality, and necessity only have empirical significance because they name the particular relations that the cognitive faculty of thinking can have with an object. They are modalities of transcendental logic, not pure logic, and answer the concern that, "if the categories are not to have a merely logical significance and analytically express the form of thinking, but are to concern things and their possibility, actuality, an necessity, then they must pertain to possible experience and its synthetic unity, in which alone objects of cognition are given" (CPR A219/B267).

[91] Kant recognizes that a second faculty is introduced when actually cognizing an object, but he does not remark on the passive synthesis of the imagination except to say that "The synthesis of a manifold, however, (whether it be given empirically or *a priori*) first brings forth a cognition, which to be sure may initially still be raw and confused, and thus in need of analysis; yet the synthesis alone is that which properly collects the elements for cognitions and unifies them into a certain content; it is therefore the first thing to which we have to attend if we wish to judge about the first origin of our cognition" (CPR A77/B103).

a posteriori from experience, not *a priori*, so that I can only be convinced that it is *omni modo determinatum* by means of an experience continuing to infinity (but that is impossible), and so an *ens omni modo determinatum* is merely an idea" (ETP 102). But the fact that the understanding is not the author of this synthesis requires no explication. Maimon proposes another definition of actuality, namely: "actuality is certainly that within which I perceive a synthesis; however, this synthesis does not proceed in accordance with the laws of the understanding (the determinable and the determination), but merely in accordance with the laws of the imagination" (ibid.). Synthetic judgments are not made possible by the action of the understanding on the imagination, as Kant would have it; they are made possible by the imagination which, for no explicit or intelligible reason, assigns determinate, intuitive forms to concepts.[92]

In a counter-intuitive manner, Maimon defends the centrality of synthetic judgments not by grounding all cognition in the pure forms of intuition that facilitate such judgments, but by demonstrating that synthetic cognition is primarily a constraint on finite cognition, and that the cognitive brunt is distributed across two distinct syntheses. Synthetic cognition supplies the context wherein the understanding can compile and parse the various rules necessary to produce or construct such objects, without being the ground of such connections. David R. Lachterman calls this the *evidentiary axis* of construction; "That something 'worldly' answers to a noncontradictory concept cannot be ascertained by analysis of the concept on its own"

[92] The 'intuitive forms' at bottom refer to the categories, insofar as they are the logical functions of judgment incarnated in inner sense. "The same understanding, therefore, and indeed by means of the very same actions through which it brings the logical form of a judgment into concepts by means of the analytical unity, also brings a transcendental content into its representations by means of the synthetic unity of the manifold in intuition in general, on account of which they are called pure concepts of the understanding that pertain to objects *a priori*; this can never be accomplished by universal logic" (CPR A79/B105). The categories are nothing but the general time-determinations that might order any number of intuitions. This would imply that the schemata of all other concepts are composed of categories, which is a plausible position. The pure concepts that Maimon occasionally refers to are the concepts of reflection, which are reciprocally defined and are not used to cognize objects (e.g. Identity and Difference, Agreement and Opposition, Inner and Outer, Matter and Form). Since such concepts are only employed reflexively, they do not require intuitive form, and as such are what guide purely logical, analytic thought.

(Lachterman 1992, 498). For Maimon, these connections also have reality beyond the domain of experience because once they are established, they come to define the concept and can be thought independently of any intuition as mere relations. The understanding forms concepts by means of specification, adding determinations to more general concepts. Such general concepts are 'determinable' since specific determinations are predicated of them. Maimon gives the example of 'straight line' to prove his point: "in the synthesis of a straight line, 'line' is the determinable and can be thought both in itself as well as with other determinations (**crooked**), whereas 'being straight' is the determination and cannot be thought in itself in the absence of something it can determine" (ETP 99). This synthesis is one-sided, since the determination needs the determinable in order to be thought, but not vice versa. As such the understanding thinks from general to specific, since it can think the ground of each determination.[93]

Constructions lose their central place as what guarantee the reality of concepts, since actual experience implores the understanding to amplify concepts and affix newly discovered determinations to a subject. From there it treats these determinations as constitutive of the definition of the concept because they abide an occult yet real ground. Moreover, the understanding enlists a set of concepts that are *not subject to synthesis,* which it employs in order to bring unity to concepts themselves. Kant calls these the concepts of reflection (identity-difference, agreement-opposition, inner-outer, matter-form or determinable-determination), which in Maimon's hands are the concepts without which even purely logical representations are without unity, and as such,

[93] For an intuitive intellect, each determination is in itself possible since dependence runs in both directions. The determinable can be thought intuitively, that is, it is possible in itself. For the discursive intellect, such determinables must be given in intuition since they do not originate with the finite mind. Maimon writes: "For an infinite understanding everything is in itself fully determined because it thinks all possible real relations [*Real-Verhältnisse*] between the ideas as their principles [*Principien*]…For the infinite understanding, concepts are judgements of the possibility of things, and judgements are conclusions as to the necessity of things, deduced from the former; for a finite understanding, concepts are also judgements of the possibility of things, but they are in a one-sided synthesis" (ETP 86n.1).

would not be representations.[94] These concepts are not subject to synthesis because they are *relational* concepts, whose synthesis "is reciprocal so that neither of the parts of the synthesis can be thought without the other, as for example with cause and effect where each of the parts is both determinable (by the other) and determination (of the other) at the same time" (ETP 99). Maimon's list of relational concepts exceeds Kant's list of reflective ones, since it also includes the categories of relation (substance-accident, cause-effect, reciprocal determination), but it is not for this reason any less general since the form of the relation itself is absolutely a priori. Moreover, it is the first set of concepts, identity and difference, that sits at the bottom of all others since, as Kant acknowledges, "we ought to call these concepts concepts of comparison (*conceptus comparationis*)" (CPR A262/B318). Comparison requires the interplay of identity and difference, which Maimon illustrates when he writes: "So if A and B are completely identical, there is in this case no manifold. There is therefore no comparison, and consequently no consciousness (and also no consciousness of identity). But if they are completely different, then there is no unity and, once again, no comparison, and consequently also no consciousness, not even consciousness of this difference, since, considered objectively, difference is just a lack of identity (even though, considered subjectively, it is a unity, or relation of objects to one another)" (ETP 16). There is no comparison in the formula A=A because the purely logical object, without any determinations, is identical only to itself. The condition of *real* thought involves a process of differentiation, where the adequate concept of a thing is assembled according to its relations to what it is not.[95]

[94] Kant distinguishes between logical reflection and transcendental reflection. The former is "the state of mind in which we first prepare ourselves to find out the subjective conditions under which we can arrive at concepts," where the latter "is the consciousness of the relation [*Verhältnisses*] of given representations to our various sources of cognition, through which alone their relation among themselves can be correctly determined" (CPR A260/B317). Maimon emphasizes the logical content of relational concepts because it is this pure cognition of relation that is at stake in symbolic cognition.

[95] In a hefty note, Maimon identifies the relation between reality and difference: "The form of identity refers to an *objectum logicum*, i.e. to an undetermined object, because every object in general is identical with itself. By contrast, the form of difference refers only to a real object, because it presupposes determinable objects (in that an *objectum*

Relational concepts are less concepts than they are 'universal forms of our thought,' since even the logical functions of judgment presuppose a minimal notion of comparison, as in the case of distinguishing one unity from another, or affirming one predicate and not another, etc. This is not to say that all concepts are innately in the mind, or that the mind could ever posses an adequate concept of an object. Rather, it means only that there exists a minimum of unity that the mind must contribute if it is to translate actual experience into the relations that compose real objects. This is the revised notion of synthesis that Maimon proposes: "the original (the objective) necessarily determines the copy (the subjective) with respect to existence, but not the reverse, even though at times there is no way for us to have cognition of the original other than by means of the copy, as for instance we have cognition of the category from out of a determinate temporal sequence. The latter is thus the ideal ground of the former, but the former the real ground of the latter" (ETP 135-136). It is the above 'universal forms of our thought' that allow the mind to abstract from its own activity (even if this is only thought as a problem), and pass from the ideal to the real at the point where spontaneity = 0.

To use terms more familiar to Leibniz, signs and symbols express relations between the components of concepts by means of the syntactic or combinatorial relations that they exhibit in space and time. And to use terms more explicitly related to symbolic cognition, the formal component of an actual object must be thought in itself, that is "symbolically, i.e. not of determined objects, but of objects in general" (ETP 69). The imagination gives an intuition in which the process of genesis has already been completed, which gives only the fact that certain concepts

logicum cannot be different from an *objectum logicum*, i.e. from itself). So the form of identity is the form of all thought in general (as well as the form of the merely logical); whereas the form of difference is the form of all real [*reellen*] thought, and consequently an object of transcendental philosophy" (ETP 345-346n.9).

accompany one another, but not the reason why.[96] For the understanding, these concepts are nothing but the set of rules the mind must observe if it wishes to decide whether such an object is possible. If the understanding is to ascribe any certainty (i.e. a priori connection) to this set of rules, "we must assume the (for us) synthetic connection between the subject and the predicate must have an inner ground so that if we, for example, had insight into [*einsehen*] the true essence of a straight line, and accordingly could define it, then this synthetic proposition would follow analytically" (ETP 61). Otherwise the contiguity of determinations in the appearance could never yield an a priori connection, only a probable one, i.e., the connection would hold only for the individual and not for the concept in general.[97] Lachterman calls this the *operational axis* of constructions, namely, the movement from *symbolic* cognition (grasping the formal in the actual) to *ostensive* construction (grounding the possible in the real).[98] For this reason, Maimon writes: "I freely admit that the synthesis of imagination must have an inner ground, i.e., an understanding

[96] Maimon seems to have revealed a cognitive pressure point which will later be exploited by the specular form of late capitalism: "Even during their leisure time, consumers must orient themselves according to the unity of production. The active contribution which Kantian schematism still expected of subjects – that they should, from the first, relate sensuous multiplicity to fundamental concepts – is denied to the subject by industry. It purveys schematism as its first service to the customer. According to Kantian schematism, a secret mechanism within the psyche preformed immediate data to fit them into the system of pure reason. That secret has now been unraveled" (Horkheimer & Adorno 2002, 98).

[97] About this correspondence between the definitions of concepts and their intuitive form as rules, Maimon writes "we do not in fact have any insight [*einsehen*] into the ground of this correspondence, but we are not for all that any the less convinced of the *factum* itself" (ETP 54).

[98] Lachterman writes about the relation of the symbolic to the ostensive: "On the one hand, if symbols transcribe the operations of pure thought there are no intrinsic limits on the combinatorial possibilities of pure thought itself; there are, however, no guarantees that any arbitrary symbolic formation (array of signs) will correspond to a "real" object or state of affairs (see, again, the case of $\sqrt{-2}$) . On the other hand, there is no mechanical method for the discovery of just those symbolic equations whose solutions *are* constructible, that is, do yield "real" objects such as loci or rational numbers… At all events, the evidence assembled so far points univocally to the conclusion that construction, far from being merely an "extrinsic criterion," is for Maimon the heart of the matter. As first symbolic, and then ostensive, imaginative construction allows us to traffic between the region of pure thought and the domain of the intuitive" (Lachterman 1992, 507). The quip in the second to last sentence is directed at Martial Gueroult, who argues that Maimon liberates the principle of determinability from construction, in favour of difference as the mark of real objects (Gueroult 1929, 50) and (ETP 345n.9). Though difference is the mark of 'transcendental philosophy,' for us finite minds it is not enough to pass from difference *materialiter* to difference *formailer* by means of reflection alone. The *reality* of real definitions comes from the ineffable connection between rules and the determinate way they guide objects to arise in intuition, which is the crux of construction.

that is acquainted with the inner essence of gold has to construct its concept of gold so that these properties must necessarily follow from the essence; nevertheless, for us this synthesis will always remain a mere synthesis of the imagination" (ETP 103). It remains a synthesis of the imagination because the finite human mind cannot think the sufficient real ground of any object, or what is the same, a thoroughly determinate concept from which all formal and material determinations would follow analytically. This is the universal antinomy of thought in general, where "the solution rests on this: the understanding can and must be considered in two opposed ways. 1) As an absolute understanding (unlimited by sensibility and its laws). 2) As our understanding, in accordance with its limitation. So the understanding can and must think its objects according to two opposed laws" (ETP 227).

Now it is perhaps easier to demonstrate the consequences that symbolic thought has for Kant's construction-centric philosophy of mathematics. Constructions cannot ground transcendental propositions, let alone synthetic a priori judgments that exploit the forms of intuition. At most, constructions exhibit the a priori determinations that compose the definition of a concept, which are for us synthetic but supposed analytic. Constructions are only ever the forms of *possible* objects because possibility signifies at most the *formal* completeness of a concept, and never its *material* completeness. Maimon distinguishes these when he states "that there is an obvious difference between the totality of conditions by means of which an object of intuition is thought, and the totality of the intuitions themselves that are subsumed under these conditions" (ETP 76). No matter how determinate the concept, or how complete the rule, the understanding can only serially and sequentially construct a finite number of pure intuitions. And such intuitions are always *general* or *universal*, never attaining the specificity of a *factum*. This is the meaning of Maimon's repeated use of the phrase 'the understanding can only think intuitions as *arising*, and not as having *arisen*,' and its variations–

> The understanding can only think objects as flowing [*fliessend*][99] (with the exception of the forms of judgement, which are not objects). The reason for this is that the business of the understanding is nothing but thinking, i.e. producing unity in the manifold, which means that it can only think an object by specifying the way it arises or the rule by which it arises: this is the only way that the manifold of an object be brought under the unity of the rule, and consequently the understanding cannot think an object as having already arisen but only as arising, i.e. as flowing [*fliessend*]. (ETP 33)[100]

Inversely, and to the same degree, the faculty of intuition is determined by rules without comprehending them. Maimon continues, "the faculty of intuition (that certainly **conforms to rules** but does not **comprehend rules**) can only represent the manifold itself, and not any rule or unity in the manifold; so it must think its objects as already having arisen not as [being in the process of] arising" (ibid. 33-34). Discursive cognition is caught between these two modes of representation without being able to reconcile them. In the latter case, the understanding must

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[99] Maimon has Newtonian fluents in mind here. Just after this quote, Maimon discusses the *Introduction to the Quadrature of Curves* and explains that the understanding cannot think the rules that are exhibited by motion "unless this relation that can only be thought in intuition [motion] is related to its elements," i.e. its differentials (ETP 35). Fluents are ultimately the rule of the understanding as represented by means of construction, "in intuition the line precedes the movement of a point within it; on the other hand, in the concept it is exactly the reverse" (ibid.). The rule that guides movement precedes the intuition of motion, but the flowing quantities that compose motion are exhibited in intuition by the understanding.

[100] On the next page Maimon gives this very illuminating example: "For example, the understanding thinks a determined (although not individual) triangle by means of a relation of magnitudes [*Grössen-Verhältnis,* i.e. proportion or ratio] between two of its sides (their position being given and thus unalterable), through which the position and magnitude of the third side is also determined. The understanding thinks this rule all at once; but because the rule contains merely the universal relation of the sides (according to any arbitrarily adopted unit), the magnitude of the sides (according to a determinate unit) still remains undetermined. But in the construction of this triangle the magnitude of the sides can only be presented as determined; so, in this case, there is a determination that was not contained in the rule and that is necessarily attached to intuition; this determination can be different in different constructions even when the rule or relation is kept the same. Consequently, in view of all the possible constructions, this triangle must never be thought by the understanding as having arisen, but rather as arising, i.e. flowing" (ETP 34).

represent to itself a passive synthesis of the imagination, one it did not effect. But, as a spontaneous faculty, this cannot be done. The same issue was discussed above with respect to degree and how the understanding can only represent quality to itself as intensive quantity (2.1). Maimon writes that passivity/receptivity “is only an idea that we can approach by means of ever diminishing consciousness, but can never reach because the complete absence of consciousness = 0 and so cannot be a modification of the cognitive faculty” (ETP 168). Here, the understanding is caught *in medias res*, between the two limit concepts of a ‘primitive,’ passive consciousness and an ‘absolute,’ intuitive consciousness as the minima and maxima of representation. In addition, it must ask itself ‘by what right can I assume the passive synthesis of the imagination is governed by a rule?’ (*quid juris*). As remarked in 1.2n.19, a jurist must ask a second question, *quid facti?,* or, ‘what is the fact or case subject to litigation?’ For Kant, this fact is *experience* as empirical cognition; that “we make use of a multitude of empirical concepts without objection from anyone, and take ourselves to be justified in granting them a sense and supposed signification even without any deduction, because we have experience ready at hand to prove their objective reality” (CPR A84/B116). This is not the fact in need of explanation for Maimon, because he doubts the supposed role of the understanding in *actual* experience. Instead, it is the fact that the imagination imposes a synthesis on the understanding from without; that the imagination is now a conduit of the understanding, now a conduit of some *unknown* origin. That is to say, what in the imagination can account for its bi-unilateral determinability, what element grounds “each state that it can reach, without distinction, i.e. a determinable but undetermined state”? (ETP 352). Constructions of the understanding must give way to differentials of sensation.

3.3 What is a Differential Anyway?

If 'differentials of sensation' are to meant to depose constructions as the centerpiece of transcendental philosophy, then it is evident what they must *not* be. It might be useful to run through some of these characteristics as a guide to what follows. Foremostly, differentials must address the role of constructions in the proof of transcendental propositions. Recall that the propositions themselves cannot be constructed since they are only the pure relations of thought, and as such, "Each must conduct his affair by means of a legitimate proof through the transcendental deduction of its grounds of proof, i.e., directly, so that one can see what his claim of reason has to say for itself" (CPR A794/B822). The 'deduction of the grounds of proof' is a concise way of referring to the manner in which it is possible to demonstrate the objective validity of a synthetic a priori judgment that proceeds from the concepts themselves. Any objectively valid (real) definition of a concept must 'immediately descend to the conditions of sensibility,' at which point the proposition is proved valid (or not) because the constructed form of appearance would have been *impossible* if the proposition were false.[101] So the ground of the proposition, the *reason why* it is valid resides not at the level of concepts (the definition) but at the level of intuition (i.e. possible experience). Concerning transcendental proofs, Vuillemin writes, "Possibilité de la synthèse et validité objective des concepts ne font qu'un : celle-ci renvoie à la possibilité de l'objet de l'expérience, celle-là à la possibilité de l'expérience même et principe suprême de l'idéalisme transcendantal consiste à affirmer l'identité de ces deux possibilités" (Vuillemin 1993, 55-56). If

[101] It is on the heels of discussing mathematical constructions (demonstrations) that Kant writes: "That this is also the case with all categories, however, and the principles spun out from them, is also obvious from this: That we cannot even give a real definition of a single one of them, i.e., make intelligible the possibility of their object, without immediately descending to conditions of sensibility, thus to the form of the appearances, to which, as their sole objects, they must consequently be limited, since, if one removes this condition, all significance, i.e., relation to the object, disappears, and one cannot grasp through an example what sort of thing is really intended by concepts of that sort" (CPR A240-241/B300).

differentials are to account for the connection between the understanding and sensibility (possible experience/appearances), it cannot resort to *ostensive* constructions, because such a 'solution' completely misconstrues the *factum* in need of legitimation. It might be impossible for discursive cognition to intuitively, and in one consciousness, grasp both the structure of cognition and the principles according to which (if any) such a structure is composed and operates.[102] In an emphatic passage, Kant precludes insight into the facets if cognition that are given, and not produced by cognition itself: "But for the peculiarity of our understanding, that it is able to bring about the unity of apperception *a priori* only by means of the categories and only through precisely this kind and number of them, a further ground may be offered just as little as one can be offered for why we have precisely these and no other functions for judgment or for why space and time are the sole forms of our possible intuition" (CPR B146).[103] This does not, however, inhibit the understanding from reckoning with its own facticity, so to speak. The mind might not be able to step outside of its own head, but it can certainly evaluate *how* it interfaces with the parts it has inherited from who-knows-where.

So, what is a differential? As mentioned in the previous chapter, the differential is in itself nothing yet the relation between two differentials in not nothing, regardless of how miniscule the

[102] In the B-preface to the *Critique*, Kant estimates the end and overall project of a systematic, critical metaphysics: "For pure speculative reason has this peculiarity about it, that it can and should measure its own capacity a according to the different ways for choosing the objects of its thinking, and also completely enumerate the manifold ways of putting problems before itself, so as to catalog the entire preliminary sketch of a whole system of metaphysics; because, regarding the first point, in *a priori* cognition nothing can be ascribed to the objects except what the thinking subject takes out of itself, and regarding the second, pure speculative reason is, in respect of principles of cognition, a unity entirely separate and subsisting for itself, in which, as in an organized body, every part exists for the sake of all the others as all the others exist for its sake, and no principle can be taken with certainty in **one** relation unless it has at the same time been investigated in its **thoroughgoing** [*durchgängigen*] relation to the entire use of pure reason" (CPR Bxxiii). This is also reprised in the final chapter, 'The history of pure reason.' Though such a panoptic view is possible, it is possible only as a science and never as a conscience.

[103]The history of German idealism is an attempt to prove this statement wrong; either as Fichte's self-positing I, as Schelling's self-generating Nature, or Hegel's self-determining Spirit. This attempt to represent the determinateness of representation, for the mind to get around the back of itself, begins with Maimon, as a regulative idea that at once enables thought to determine itself and forgo claims about thing themselves. For the influence of Maimon on Fichte (Atlas 1964, 316-320) and on Hegel (Bergman 1967, 248-251).

value of the relata may be. Recall Newton's use of the term 'moment,' which is a minute expanse of time wherein the value of a fluent may vary by an amount so small it is unassignable. Where the moment sets $t = 0$, the fluxion is a non-zero variation that allows the mathematician to calculate the rate of change at a point (i.e. where no time has passed). No matter how short the duration, a moment can always further be subdivided because as time continues to elapse the progression of differences will continually shrink in turn. Leibniz uses the differential in exactly the same manner, except that he does not ground the continuity of fluctuation in the continuous passage of time. Conversely and in a completely analytic fashion, Leibniz extrapolates the operation of forming series of finite differences (Δ) and series of finite sums of finite differences (Σ), where "*we see that a difference sequence is transformed into a sequence of an infinite number of infinitely small terms; these are called the* differentials. *A finite sum sequence is transformed into a sequence of an infinite number of infinitely large terms; these terms are called the* sums" (Bos 1974, 16).[104] That is to say, the symbol *d* denotes a variable though finite quantity that is assigned to another quantity, e.g. x, such that dx is *infinitesimally small* (i.e. $x = x - dx$), or, "can be taken as small as we wish" (L 543). This is what Leibniz terms the *syncategorematic* character of differentials, i.e. that they have meaning only by virtue of their connection to other finite though variable

[104] This is the result of Leibniz's engagement with the sums of infinite series, most notably with Pascal's arithmetical triangle and Leibniz's own harmonic triangle. The former is a particular sequence or scheme that can represent integers as the sums of other integers. Leibniz's innovation on this scheme was to construct a triangle of reciprocal integers that progressed not by summation, but by differences. As Serfati notes, "To a given sequence of integers x with general term x_n ($n > 1$), one associates two other sequences. First, the *differences-sequence* of x, denoted dx, the general term of which $(dx)_n$ is $(dx)_n = x_n - x_{n-1}$ ($n \geq 2$) and (conventionally) $(dx)_1 = x_1$ (1). Then, another sequence, the *sums-sequence* of x, denoted $\int x$, with general term $(\int x)_n$ defined, for $n \geq 1$, by relation $(\int x)_n = \sum_{1 \leq k \leq n} x_k$ From this one can prove that d and $\int$ are two reciprocal one-to-one mappings on the set $S = \mathrm{IR}^{\mathrm{IN}}$ of all the sequence of real numbers" (Serfati 2013, 77) The reversibility of these sequences (harmonic and arithmetical) was Leibniz's clue that the operations of summation and differentiation were reciprocal. For a brief account see (ibid., 73-78).

magnitudes.[105] Tzuchien Tho places this characteristic in-between the scholastic notions of the *categorematic* and the *hypercategorematic*. He writes,

> Briefly put, the categorematic infinite is the infinite as a number, a unity or mathematical entity. This categorematic infinite is contradictory once it is posed, since there will always be a number that can be added to produce a greater sum. The syncategorematic infinite is an infinite that posits that, insofar as there is no greatest number, there will always be a greater number than any given finite quantity… There is finally a hypercategorematic infinite that is invoked in this passage,[106] a "true infinite" which is "nothing but in the absolute." This is the infinite in its absolute sense, the infinity of God or an infinity without parts. (Tho 2012, 75).

Differentials, or infinitely small quantities, are not entities as are fluents, but are something of a syntactic element that signifies the threshold beneath which not difference can be assigned. The differential is the infinitesimally small difference that aids in mathematical practice without corresponding to a mathematical object, or as Leibniz puts it, "Nor does it matter whether there are such quantities in nature, for it suffices that they be introduced by a fiction, since they allow abbreviations of speech and thought in discovery as well as in demonstration" (quoted in Levey 124n.35). Already the inspiration of Leibniz's analytical method marks a significant departure from the foundations of time, motion, or any other modes of conscious experience. The objects of

[105] In 'Archimedes Infinitesimals and the Law of Continuity,' Samuel Levey gives some etymological context for the term. "On the traditional account, a categorematic term is one that *predicates*, that is, has reference or a semantic content of its own. By contrast, a term is syncategorematic when it predicates only in conjunction with other terms: it has no referent or semantic content of its own, but rather contributes to the meaning of sentence only by virtue of its links with other terms in the expressions to which it belongs. (*Syncategorematic* literally means 'jointly predicating'; its Latinate equivalent is *consignificantia*.) The distinction is not perfectly sharp independently of a given semantic theory, but it is easy to illustrate by examples. 'Apple', 'wise' and 'gold' are categorematic terms; 'if', 'some' and 'any' are syncategorematic" (Levey 2008, 108).

[106] Here is the passage in question, from the *New Essays on Human Understanding*: "Properly speaking, it is true that there are an infinity of things, that is to say that there are always more than can be assigned. But, there is nothing as an infinite number nor a line or other infinite quantity, if we take them for true totalities, as it is easy to show. The schools wanted, due to this, to admit a syncategorematic infinite, as they say, and not a categorematic infinite. The true infinite, in rigor, is nothing but in the absolute, that before any composition, and is not formed by the addition of parts" (quoted in Tho 2012, 75).

differential calculus are not 'things' in the empirical sense, but combinatorial tools that can be used to extend mathematical practice beyond the bounds of experience.

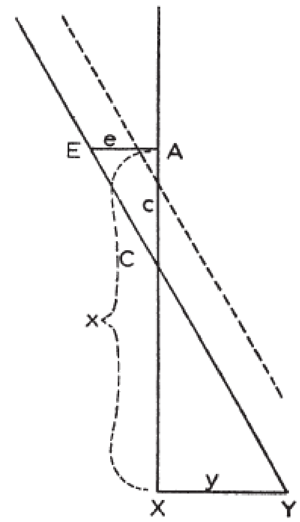

Figure 3

In a letter to Varignon [1702], Leibniz produces an intuitive demonstration of both the non-zero property of differentials and the autarchy of the differential quotient (i.e. $\frac{dy}{dx}$) with respect to the values of the given line segments. In the supplement to his letter, 'Justification of the Infinitesimal Calculus by that of Ordinary Algebra,' Leibniz attached Figure 3 (Leomker 1989 545). Leibniz succinctly lays out the parameters of the construction thus: "Let two straight lines *AX* and *EY* meet at *C*, and from points *E* and *Y* drop *EA* and *YX* perpendicular to the straight line *AX*. Call *AC, c* and *AE, e; AX, x* and *XY, y* [Figure 3]. Then since triangles *CAE* and *CXY are* similar, it follows that $(x-c)/y = c/e$." (ibid.). At this point, Leibniz instructs us to imagine that the line *EY* approaches *A*, whilst retaining the angle at *C* which is assumed other than 45° so that the proportions $c{:}e$ and $x;y$ are not 1:1. The three points *E, C*, and *A* will coincide and the line segment *c* will be reduced to 0. The result is that the preceding equation, $(x-c)/y = c/e$, will become $x/y = c/e$, however, the *ratio* c/e will not become nothing and will, in fact, preserve the ratio whose value is also the ratio of the radius *CX* and the tangent *XY* for a circle with center *C* (ibid., 546n.6). Moreover, if both *c* and *e* were taken to be zero in an absolute, not relative sense, the resultant equation would become $x/y = 0/0 = 1$, because *c* and *e* would possess the same value.[107] Leibniz concludes: "Hence *c* and *e* are not taken for zeros in this

[107] I will not comment on the problematic use of 0/0 = 1, but it can be argued that Leibniz is attempting to show that one cannot reject the argument because *c* and *e* individually become zero. If the persistence of the ratio is denied, then one would have to admit that both variables reach a common value "since one zero equals another" (L 545). As such, their ratio would be 1:1 which can only happen if the other angles of a right tringle are 45° which is contrary to the assumption.

algebraic calculus, except comparatively in relation to *x* and *y;* but *c* and *e* still have an algebraic relation to each other. And so they are treated as infinitesimals, exactly as are the elements which our differential calculus recognizes in the ordinates of curves for momentary increments and decrements" (ibid.). Differentials are 'undetermined yet determinable,' since, in relation to the magnitude from which they differ, they are nothing; and when the differentials are determined by their relation to one another, they remain in themselves variable.[108]

The most evident Kantian or Newtonian criticism is, 'The operative elements in your demonstration are space (figure), time (translation), and motion (both together). Despite the claim that constructions cannot ground synthetic a priori judgments, the supposed alternative can only be demonstrated by means of what it disavows!' And while this would be a valid criticism if all use of differentials required the construction of geometrical figures, this is not the case. As it will be more explicitly shown below, Leibniz is utilizing the geometrical construction as a heuristic and not a veritable proof of the consistency of infinitesimal quantities. And even in the extensive proof supplied in *De quadratura arithmetica* [1676], it is not the construction that is operative in the demonstration, but the Archimedes-style proof by exhaustion which skirts the usual two pronged approach (one to prove the value cannot be smaller than *x*, and one to prove it cannot be greater) in favour of a new definition of equality. But before abandoning the above demonstration,

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[108] This syncategorematic character of infinitesimals distinguishes Leibniz method from Newton's, and explain (in part) why Leibniz's method withstood the trials of history. Vuillemin argues : "D'abord, à la différence de l'algorithme newtonien, adapte aux calculs où les dérivations successives, choisie une fois pour toutes, comme c'est le cas en Dynamique pour l'estimation de la vitesse et de l'accélération, l'algorithme leibnizien désigne la variable par rapport à laquelle on différencie ; il est donc beaucoup plus général et il est le seul dont on puisse faire usage dans les équations aux dérivées partielles, quand la dérivation a lieu par rapport à plusieurs variables. En second lieu, si l'on considère par exemple une fonction composée $z = k\ (x)$, où $z = g(y)$ et $y = f(x)$, comparée à la notation newtonienne : $k'(x) = g'(y) \cdot f'(x)$…la notation leibnizienne : $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$ a l'avantage de mettre l'accent sur les quantités *x, y, z,* plutôt que sur leur lien fonctionnel explicite. Ce lien, indiquée par Newton, exprime une opération qui produit une quantité *y* à partir d'une autre, *x.* Au contraire, le symbolisme leibnizien traite les quantités indépendamment de leur genèse…les limites des quotients sont regardées comme si elles étaient des quotients actuels" (Vuillemin 1993, 33-34). In this passage, the primacy of independent variables and the autarchy of quotients are key.

it is interesting to note that Maimon uses an analogous demonstration in his *Essay* to explicate both the notion of the differential and its central place in divining the implicit order that guides the passive syntheses of the imagination.[109]

Chapter 7 of the *Essay on Transcendental Philosophy* concerns 'Magnitude,' specifically the relation between extensive and intensive magnitude. He begins the chapter with an uncontroversial statement, namely, "Magnitude is either plurality thought as unity or unity thought as plurality. In the former case the magnitude is extensive and in the latter, intensive" (ETP 120). This has already been established above (see 2.1). Extensive magnitudes proceed part by part because the schema of quantity (number) is the synthesis that orders sensations according to a pure succession. This is the way the understanding represents apprehension to itself, that is, according to the pure form of time as inner sense. As for intensive magnitudes, the schema of quality (degree) was revealed to be an 'intensive quantity' and not itself a quality. Unlike number, which thinks a plurality of syntheses as a unit(y), degree measures the objectivity of a representation, or the amount that representations are determined according to the understanding (transcendental unity of apperception); recall Kant's statement - "Since time is only the form of intuition, thus of objects as appearances, that which corresponds to the sensation in these is the transcendental matter of all objects, as things in themselves (thinghood [*Sachheit*], reality)" (CPR A144/B183). Each instant of apperception is a moment wherein the cause of reality is the thing as it is in-itself, or the objective correlate of the unity of apperception. Each moment is a unity unto itself and must be

[109] In fact, he summarizes the above demonstration before formulating his own – "consider a triangle; now move one side in relation to the opposite angle so that it always remains parallel to itself; do this until the triangle becomes an infinitely small [triangle] (a differential). The extensive magnitude of the sides then completely disappears and is reduced to their differentials but the relation of the sides always remains the same because it is not the relation of number to number with respect to one and the same unit, but the relation of unit to unit" (ETP 394-395n56). This also helps clarify Leibniz's argument as discussed in note 105.

put into relation with other moments if the degree of reality is to vary at all. It is at this point that Maimon begins to turn the Kantian edifice inside out.

It is evident that there is some kind of reversibility between extensive and intensive magnitudes, not only because they are both continuous (flowing) magnitudes, but also because they are differentiated only by the scope of unity and plurality, i.e. whether unity is predicated of plurality (extensive) or plurality is predicated of unity (intensive). As in Leibniz's above demonstration, the extensive magnitudes of *c* and *e* are reduced to their differentials, which are not *numbers* in the sense that there is no internal plurality, only the form of unity. This leads Maimon to conclude that, "Consequently, the intensive magnitude (the quality of the quantum) is in this case the differential of the extensive quantities, and the extensive quantities are the integral of the intensive magnitude" (ETP 395n.56).[110] It is this claim that Maimon defends with his own intuitive demonstration (see Figure 4, ETP 395). Here is his lucid description in full:

> So let us take a [right-angled] triangle *abc* whose adjacent and opposite sides *ab*, *bc* are equal to one another. Let us further assume that one side *bc* moves relative to the opposite angle *bac* in such a way that it always remains parallel with itself, until it becomes *df* which I assume to be

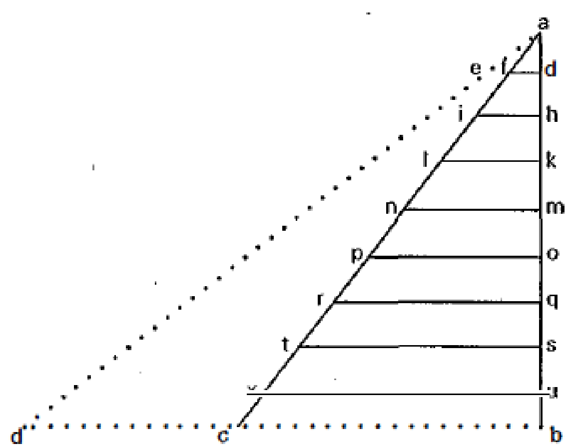

Figure 4

> infinitely small. Consequently *ad* and *af* also become infinitely small, as does the triangle as a whole. The relation of *af* to *ad* or *df*, always remains just the same, namely it is equal to $\sqrt{2}$:1. So, it is not a relation of number to number, since I have assumed both to be infinitely small,

[110] The translators remark that this note originally appeared after note 50 on pages 394-396. As such, they preserved the original pagination, even though the not is printed on pages 205-206 of the English edition (ETP 394n.32).

omni dabili minora [Smaller than any(thing) given. In this case: smaller than any given number. Translators' Note]; and consequently it cannot be expressed by any number in relation to any unit, but only by the relation of one unit to another unit, i.e. this relation does not hold between the lines in so far as they are measurable, but merely in so far as they are determined by their quality (by their position). As a result, they are not extensive but intensive magnitudes; with intensive magnitudes, the representation of the parts does not make possible the representation of the whole, but the reverse; because they have no parts, their magnitude can only be grasped by comparing it [the magnitude] as a whole with other wholes, for example by comparing *df* with *de*. (ETP 395-396n.56)

Maimon formulates this demonstration in terms of extensive and intensive magnitudes, and shows how the reduction of the former to their differentials, or sensation = 0, can be represented as the relation between two unit(ie)s with no internal parts. Likewise, two unit(ie)s which are not themselves composed of parts can only be represented together if they are put into relation with one another. The relation obtains even when the terms are in themselves nothing. Simon Duffy explains that, "The infinitely small can legitimately be predicated of the quality of a sensible representation because the a priori rule of the understanding that determines the differential in mathematical cognition can be applied to our understanding of the relation between quality and quantity in sensible representation" (Duffy 2014, 236-237). The syncategorematic character of differentials is not an abstract property of magnitude since it determines the role of the differential in synthetic judgments. The relation of extensive and intensive is isomorphic to the relation of possibility and actuality. The concept, as universal, contains within itself a multitude of mutually exclusive determinations (e.g. triangle contains right-angle, obtuse-angled, and equilateral) which, when constructed, cannot be represented all at once in the same figure. The possible triangle possesses *inner* unity where the actual triangle, given as an appearance, possesses only *outer* unity. If each of the three tringles are given, it is only by abstracting from the determinateness of the appearances that all three can be compared, and can be thought together under some concept (ETP 121-123). But extensive magnitudes only *become* intensive at the point where the difference constitutive of plurality is reduced to zero. Likewise, intensive magnitudes *become* extensive when their unity or identity is formulated in terms of infinitesimally small differences, which establishes continuity between discrete units of magnitude. As François Duchesneau explains the significance of infinitesimals "has to do with extending relational properties pertaining to infinite continuous series and thereby subverting the apparent discreteness of perceptually isolated states" (Duchesneau 2008, 242). The common principle in both of the above demonstrations, and the one that captures this interplay between identity and difference, is what Leibniz calls the *law of continuity*.

The clearest and most explicit formulation of the law of continuity is found in a short piece submitted to the *Nouvelles de la république des lettres* in 1687. It reads: "*When the difference between two instances in a given series or that which is presupposed can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought or in their results must of necessity also be diminished or become less than any given quantity whatever*" (L 351).[111] In a later document from 1701, Leibniz recalls this formulation of the law and restates it as "If any continuous transition is proposed that finishes in a certain limiting case, then it is permissible to formulate a common reasoning which includes that final limiting case" (Levey 2008, 129n.44). This greatly resembles Lemma 1 in Newton's *Method of First and Ultimate Ratios,* but differs in two significant ways. Firstly, Newton specifically frames his

[111] Leibniz explains that this depends on a more fundamental principle, that of *general order,* which is summarized: "as the data are ordered, so the unknowns are ordered also" (L 351).

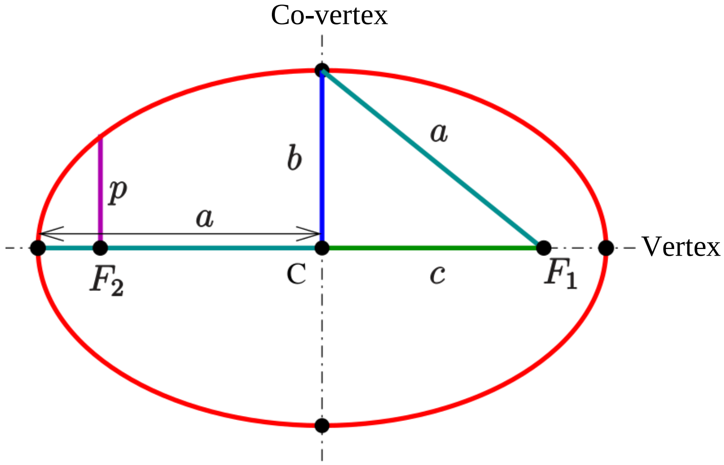

Figure 5

Lemma within 'any finite time' where Leibniz refrains from this language.[112] Secondly, and more significantly, the principle of continuity was posited to reconcile the syntactic operations of notation with the corresponding transformations of real objects. In the same letter, Leibniz gives the example that an ellipse becomes a parabola when one of its foci is indefinitely distanced from the other along the fixed major axis.[113] For a horizontal ellipse, the horizontal distance, *c*, between the focus, F_n, and the origin is proportional to the vertical width of the minor axis (2*b*, where *b* is the semi-minor axis), which is described by the Pythagorean theorem ($a^2 + b^2 = c^2$), where a is the semi-major axis or the hypotenuse on the right-triangle - vertex, origin, co-vertex (see Figure 5).[114] The eccentricity, *e*, of the ellipse is given by the ratio *c*/*a*. As *c* increases, eccentricity tends toward 1, which is to say that *c* and *a* tend towards equality and the ellipse tends toward a parabola. The advantage of this method is that "all the geometric theorems which are proved for the ellipse in general can be applied to the parabola by considering it as an ellipse one of whose foci is infinitely far removed from the other, or (to avoid the term 'infinite')

[112] Just below the above phrasing, Leibniz writes, "Or to put it more commonly, *when two instances or data approach each other continuously, so that one at last passes over into the other, it is necessary for their consequences or results (or the unknown) to do so also*" (L 351). This belies the aim of the law of continuity, which is not to ground the principle in any sort of movement in time and space, but to ensure that the relations established at the level of symbols express the relations between things.

[113]This is the first example to follow the law of continuity: "We know that a given ellipse approaches a parabola as much as is wished, so that the difference between ellipse and parabola becomes less than any given difference, when the second focus of the ellipse is withdrawn far enough from the first focus, for then the radii from that distant focus differ from parallel lines by an amount as small as can be desired" (L 352).

[114] "File:Ellipse-param.svg." *Wikimedia Commons, the free media repository*. 19 Jun 2017, 06:17 UTC. 17 Mar 2020, 20:06 <https://commons.wikimedia.org/w/index.php?title=File:Ellipse-param.svg&oldid=248322110>. [modified].

as a figure which differs from some ellipse by less than any given difference" (ibid., 352). However, this will never occur by means of iterating the translation of the focus, since there exists an ever-smaller difference between c and a; the ellipse remains an ellipse.

This is precisely the situation that the law of continuity aimed to remedy. The same problem was described above in relation to Archimedes' demonstration by means of exhaustion in *Measurement*, where the difference between a circle and a polygon of n sides can be made smaller than any given difference (2.2). As Serfati writes, "In the exhaustion procedure one deals with approximations; one considers the areas of the regular polygon inscribed in a circle and circumscribing it. The areas, easily calculable, of these very polygons approach that of the circle insofar as the difference may become "smaller than any given number". But the polygons remain polygons, and the circle, a circle…This is what the introduction of the new Leibnizian infinitesimals allowed for" (Serfati 2010b, 25). Put simply, the law of continuity allows for the properties of an infinite progression to pass into the result or *terminus* of the progression; "in Leibnizian terms, the *terminus inclusivus* transfers its properties to the *terminus exclusivus*" (ibid., 12). Continuity supplements the methods of construction and iteration by changing registers. From this vantage, the diversity of representations does not entail any borders between things, and the notions of equality and identity have significance up to a threshold of an infinitesimal difference. As Leibniz expresses in his letter to Varignon, the law of continuity allows us to consider "equality as the limit of inequality," and in the supplement to the letter adds that "although these terminations are excluded, that is, are not included in any rigorous sense in the variables which they limit, they nevertheless have the same properties as if they were included in the series, in accordance with the language of infinites and infinitesimals, which takes the circle, for example, as a regular polygon with an infinite number of sides" (L 544, 546). This is what Levey calls Leibniz's new principle of equality, namely, "if, for any n>0, the difference $|x - y|$ is less than $1/n$, then x = y" (Levey 2008, 113). However, this remains only a *convention* or a *fiction* since the ostensive and direct representation of these objects excludes the possibility of making the leap from infinitely similar to the same. Synthesis marks the incapacity of consciousness to constitute the reality it represents to itself, so all the cases "in which I take equality as a particular case of inequality, rest as a special case of motion, parallelism as a case of convergence, etc., assuming not that the difference of magnitudes which become equal is already zero but that it is in the act of vanishing", cannot be thought intuitively but only symbolically (L 546). To put this differently: *there is no experience of continuity* for finite minds, only the continuity gleaned through symbols. As Vuillemin writes, "Puisque nous sommes des substances finies, il nous faut un substitut au défaut de notre vision et ce substitut est l'algorithme. Dieu nous parle. Mais l'imagination fait alors partie de ce qu'il y a de plus essentiel dans le raisonnement. On ne peut saisir l'infini actuel…ni le ramener à des processus intuitivement indéfinis : il n'est accessible qu'à travers des systèmes finis de symboles" (Vuillemin 1993, 44). There is no experience of continuity because continuity comes only after the last instance of an infinite progression, or is thought as an infinitesimal difference. Vuillemin continues, "l'algorithme imite les choses comme le temps l'éternité ; il tend à les rejoindre à la limite, comme un raisonnement successif peut faire pure une intuition simultanée" (ibid., 45). Contrary to adequate or intuitive knowledge, experience is the domain of the *contiguum*. Any demonstration of the law of continuity cannot be phenomenal since actual experience consists of discrete, given individuals and possible experience consists of concepts that differ from one another, and so partition the domain of representation according to the intension of different universals. "In mathematics, the principle of continuity cannot deploy itself except at the symbolic level," remarks Serfati (2010b, 20). And against the domain of philosophical experience, "the combinatorial “form” of the property remains unchanged, be it in terms of sequences or of curves. Thus, the notation plays here the role of pivot, of pole, of paradigm” (ibid.). The above demonstrations offered by Leibniz and Maimon fall short of a rigorous proof of the validity of infinitesimals because their synthetic method appeals to the law of continuity precisely where it needs to be proved. The proper foundation of continuity was an afterthought for Leibniz, since the use of both differentials and fluxions in mathematical practice had been more than fruitful. The use of infinitesimals (in whatever form) did not require a firm philosophical foundation to work; *and this is exactly why calculus is a point of inflection for philosophy* for both Leibniz and Maimon.

Chapter 4

4.1 Intuition Banished: Apagogic Proof against Ostensive Construction

Many commentators have noted the importance of a recently published treatise by Leibniz titled *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis* [1676] [On the arithmetical quadrature of the circle, the ellipse, and the hyperbola. A corollary is a trigonometry without tables] (Leibniz 1993).[115] As Leibniz's longest mathematical work, it ranges over an immense amount of material that far exceeds the scope of this section (due to a lack of both space and expertise). Yet, there is a consensus that Proposition 6 (of 51) occupies more of a central place in Leibniz's account of continuity than it does in *De quadrature* itself. This proposition, according to Levey, "articulates a general technique for finding the quadrature of any continuous curve that contains no point of inflection and no point with a vertical tangent," and in doing so offers a 'rigorous' demonstration of infinitesimal quantities; rigorous because the infinitesimals are nevertheless finite magnitudes, and so obey Archimedes' principle (i.e. the Eudoxian theory of proportion) (Levey 2008, 116). Before he commences, Leibniz announces that this proposition is "most thorny," that is, it involves constructing a series of nested relations between two curves; namely, "it is demonstrated in fastidious detail that the construction of certain rectilinear and polygonal step spaces can be pursued to such a degree that they differ from one another or from curves by a quantity smaller than any given, which is something that is most often [simply] assumed by other authors," all in order to secure a foundation for infinitesimals (quoted in Arthur 2008, 20n.6). In fact, Leibniz himself did not find much value in the demonstration since it served only to retroactively validate a method that had already proved

[115] Some of the sources that stress the importance of this essay, and on which this section depends, are (Arthur 2008), (Knobloch 2002), (Knobloch 2008), (Levey 2008), (Tho 2012). Since I do not have access to the French edition by Marc Parmentier, I rely on the accounts of *De quadratura* provided in the named sources.

its worth. In the scholium to the proposition, he admits “I would gladly have omitted this proposition because nothing is more alien to my mind than those scrupulous minutiae of certain authors in which there is more ostentation than reward, for they consume time as if on certain ceremonies, include more labor than insight, and envelop the origins of discoveries in blind night, which often seems to me more prominent than the discoveries themselves” (quoted in Levey 2008, 125n.36). Even if such a demonstration is only meant to appease the sceptics, it nevertheless demonstrates something essential about the methods and principles of discovery, or what is the same, the act or power of invention.[116] This demonstration also makes apparent the logic that underlies Leibniz’s differentials: it is an Archimedean-style reductio argument that *does not rely on the same synthetic heuristics as before.* The following is a truncated account of the demonstration in Prop. 6. For a detailed account and commentary, see the sources listed in note 115.

We will first consider Figure 6 (Arthur 2008, 21). Leibniz wishes to prove that the difference between the area of a step-space[117] beneath a curve, and the actual area under the curve, can be reduced to ‘a quantity smaller than any give one’ (Leibniz 1993, 21). He begins with a semi-circle and aims to construct a curve (D) from a number of relations on the semi-circle. He does this by obtaining points ($_nT$) at the intersection of tangents of the semi-circle and the y-axis. The curve D is plotted along another series of points ($_nD$) that are obtained at the intersection of the perpendiculars (distance to y-axis) and ordinates (distance to x-axis) that run through the points of tangency ($_nC$) on the semicircle. Subsequent information is given by the secants of the semi-circle

[116] The first phrase is borrowed from the concluding line of the scholium to Prop. 6; “I do not deny that it is in the interest of geometry to have the very methods and principles of discovery rigorously demonstrated, so I thought I must yield somewhat to received opinions” (Levey 2008, 125n.36). The second from Leibniz’s defense of his differential method over the Method of Fluxions : “Il est vrai qu'il [Newton’s work] se sert d'autres caractères: mais comme la caractéristique même est, pour ainsi dire, une grande part de l'act d'inventer, je crois que les notres donnent plus d'ouverture” (Leibniz 1694, 307).

[117] This term translates *spatium rectilineum gradiforme* (Leibniz 1993, 18).

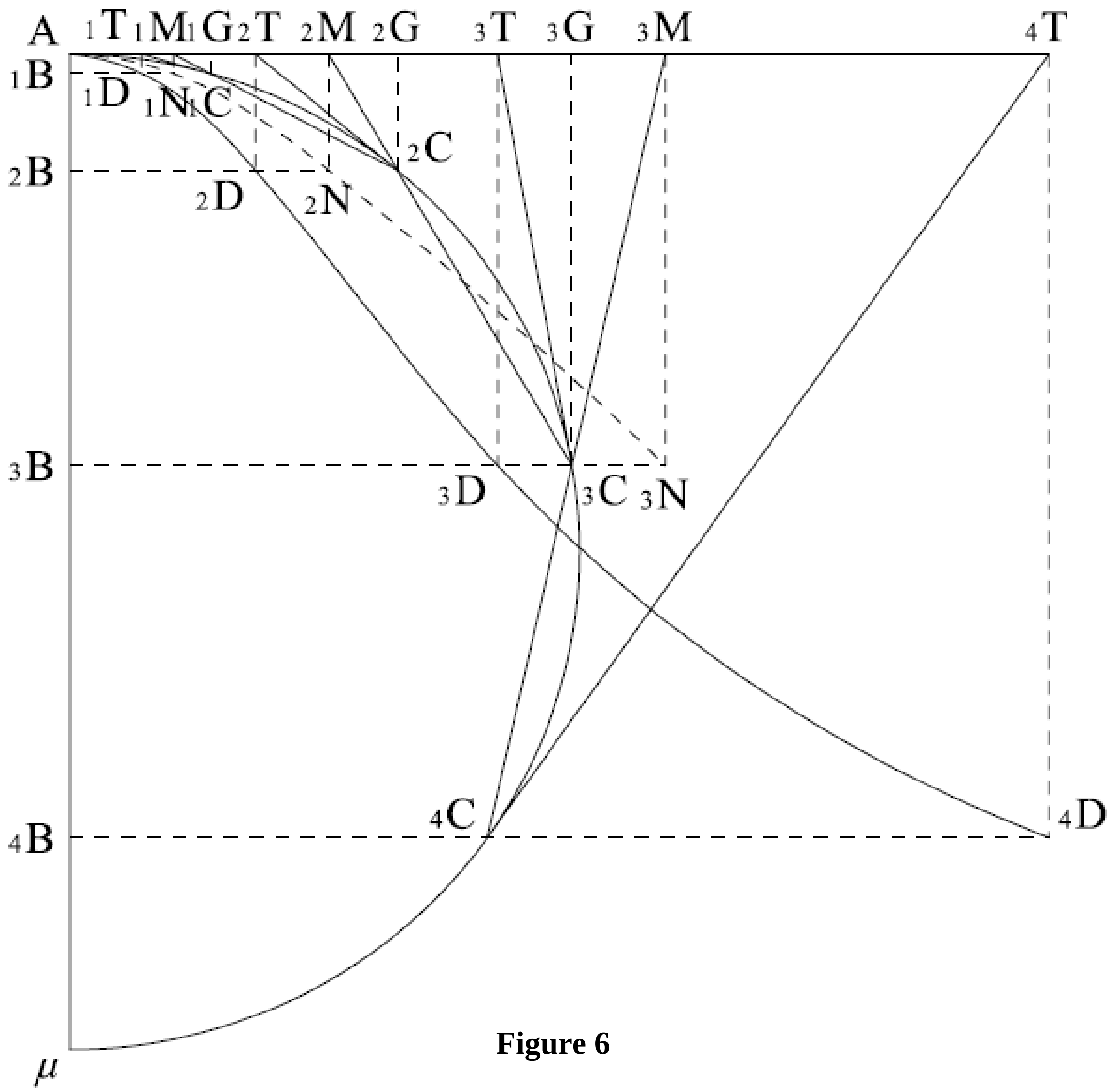

Figure 6

and their intersection of the y-axis ($_nM$). A further set of points ($_nN$) is obtained at the intersection of same ordinates as before (through both $_nD$ and $_nC$), and the perpendiculars from $_nM$. These same perpendiculars intersect with the curve D and produce a set of points ($_nF$).

Now consider Figure 7 (Knobloch 2002, 64). Here, two sets of rectangles are constructed. The first are *elementary* rectangles, whose bases are along B and whose height is measured by points $_nN$, which approximate the area under D (e.r. in Figure 7). The sum of all elementary triangles produces the area E, which is also called the gradiform space. The second are *complimentary* rectangles, whose bases are along the ordinate of $_1D$ and height measured by the subsequent points

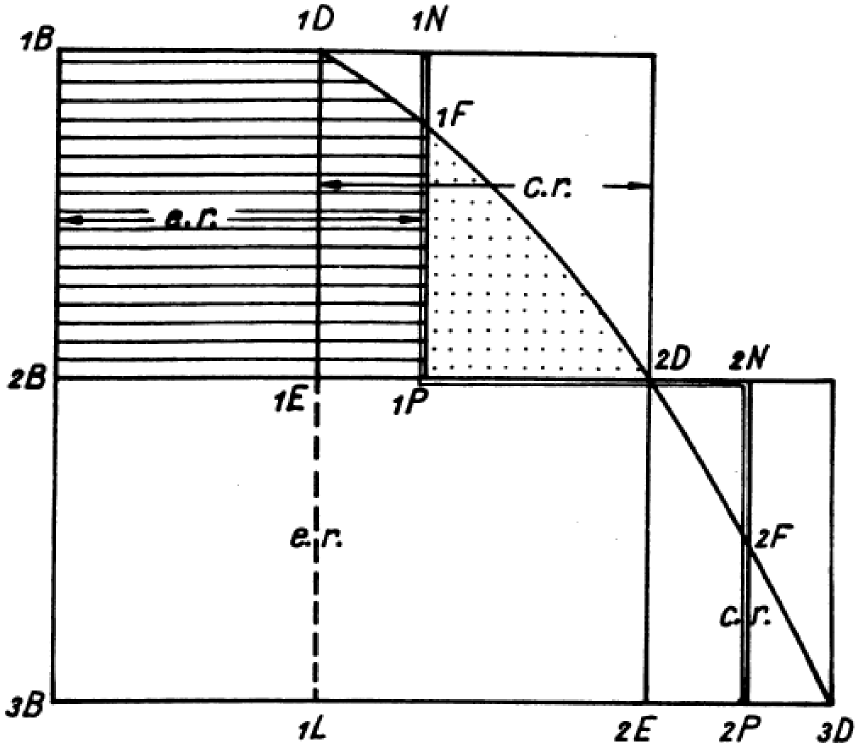

Figure 7

on D (c.r. in Figure 7). Leibniz claims that the difference between the real area under the curve (M) and the approximate area (E) given by elementary rectangles is always less than the area of the corresponding complimentary rectangle (C). Here is the proof according to Knobloch, "The common part ${}_1B\ {}_2B\ {}_1P\ {}_1F\ {}_1D$ is subtracted from both areas, giving $|{}_1B\ {}_2B\ {}_2D\ {}_1D - {}_1B\ {}_2B\ {}_1P\ {}_1N| = |{}_1D\ {}_1N\ {}_1F - {}_1F\ {}_1P\ {}_2D| < {}_1D\ {}_1E\ {}_2D$. The two threelinear areas do not overlap and lie within the complementary rectangle ${}_1D\ {}_1E\ {}_2D$. Hence even the sum of their areas is smaller than the area of this complementary rectangle" (ibid.). Arthur writes this inequality as $|M - E| < C$, which holds for all curvilinear areas and the elementary rectangles that approximate the area (Arthur 2008, 23). This difference between the area and its approximation, $|M - E|$, is called the 'step-space' (the line ${}_1N\ {}_1P\ {}_2D\ {}_2N\ {}_2P$ runs through the center of this space). The complementary rectangles are constructed to circumscribe the two threelinear spaces (${}_1D\ {}_1N\ {}_1F$ which belongs to E, and ${}_1F\ {}_1P\ {}_2D$ which belongs to M) that constitute the difference $|M - E|$. The sum of the complementary rectangles, C, is less than the area of a rectangle whose base is the difference between the greatest and smallest ordinate (e.g. ${}_1L\ {}_3D$) and whose height is the difference between greatest and smallest perpendiculars (e.g. ${}_1B\ {}_3B$) which is also the greatest height of any elementary rectangle (e.g. ${}_3B\ {}_3D$). This height is denoted by h_m, and the resulting equation is $C < {}_1L\ {}_3D \cdot h_m$, which, because the step-space in enclosed by the complimentary rectangles, establishes the relation $|M - E| < C < {}_1L\ {}_3D \cdot h_m$. Leibniz remarks that the greatest height h_m, "even though it is greater than, or at any rate not less than, any of the other intervals assumed, can nevertheless be assumed smaller than any assigned quantity; for however small it is assumed to be, others can be assumed still smaller" (ibid.n.8). As such, for whatever magnitude is assigned to h_m, C will be a smaller magnitude still, and the difference between the area under the curve and its approximation, $|M - E|$, will be less than any assignable difference. Q.E.D.

This is not exactly an argument by means of exhaustion, since it is lacking the *double* reductio that specifies the value or ratio in question. Instead, Leibniz utilizes a *single* reductio argument in order to vanish the difference between discrete quantities. Levey explains that "Leibniz has, in effect, integrated the two sides of the classical double reduction by fashioning a step figure [|M – E|] that neither circumscribes nor is inscribed within the gradiform space [E] but nonetheless converges on it as a limit. The two sides of the underlying logic of the ancient method are correspondingly integrated in the new principle of equality" (Levey 2008, 118).[118] The ancient logic of exhaustion is put to use in a totally new context, and to a totally new end. Where Newton implicitly appeals to a synthetic form of the Archimedean principle, Leibniz explicitly appeals to its analytic form. Recall that, in 'Leery Bedfellows,' Arthur identifies Lemma 1[119] in Newton's

[118] Knobloch also remarks that "the step figure is neither an inscription nor a circumscription, rather something in between" (Knobloch 2002, 63).
[119] "Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal" (quoted in Arthur 2008, 9).

Method of First and Ultimate Ratios as a synthetic version of the Archimedean principle. The analytic version of the principle is logically identical, except that it does not appeal to any intuitive foundations (time, space, motion): "If an inequality is such that its difference from a strict equality can be made smaller than any that can be assigned, it can be taken for an equality" (Arthur 2008, 19). This is also the principle of equality that underpins Leibniz's differential calculus.

At the end of his essay, 'Archimedes, Infinitesimals and the Law of Continuity,' Levey brings together all of these argumentative threads when he juxtaposes the relationship of infinitesimals and the Archimedean principle to the relationship between limiting cases and the law of continuity. He takes the familiar case of a regular *n* sided polygon, which as *n* increases, the difference between the areas of the two figures is less than any assignable magnitude. The passage from polygon to circle occurs because 1) the Archimedean principle secures a smaller difference; and 2) the areas of both figures can, for example, be calculated as a series of triangles because the law of continuity treats the circle as an ideal limiting case of the series that produces smaller differences. The new principle of equality, or as Arthur writes, the principle of unassignable difference, is what sits between the intuitive contiguum and the ideal continuum. The single reductio argument acts as *proof of this transcendental proposition*, but proceeds in a way that was impossible by means of direct proof. As Leibniz himself writes at the beginning of Prop. 6, the proof "brings about only this that two spaces of which one passes into the other if we progress infinitely, approach each other to a difference which is smaller than any arbitrary assigned difference, *even then when the number of inscriptions remains only finite*" [emphasis added] (quoted in Knobloch 2002, 62). The nail in the coffin of ostensive construction is finite though miniscule. The constructed magnitude is finite, but the infinitesimal acts as a covert appendage that eludes experience, knitting together the difference between things. Levey is explicit on this point –

> Leibniz's demonstration of Prop. 6 is 'rigorous' in the modern sense of involving only finite quantities; it makes no reference to infinite or infinitely small values. And it is specifically the new *Archimedean principle of equality* that allows this. No direct construction of the area of the quadrilineal by means of a single step space would be possible without representing the step space as composed of infinitely many infinitely small (narrow) rectangles. But with the new principle of equality in play, it suffices to show that any given claim of finite inequality between the two areas can be proved false by some particular finite construction, even if there is no single finite construction that at once gives the quadrature of the curve exactly. No 'ultimate construction' lying at the limit is required. (Levey 2008, 118).

This knot of the Archimedean principle, the law of continuity, and the principle of equality ramify throughout Leibniz's work and reveal a blind and symbolic cognitive capacity unbound by possible experience. More than this, Leibniz's principle of equality symbolizes the integration of both *synthesis* and *analysis*, such that what is synthetic from one side is analytic from the other. Let us examine this. Levey gives a rough formalization of the two principles and the law, respectively: "Recall again the [Archimedean] Principle: for any quantities $x, y > 0$, if $x > y$, there is a natural number n such that $ny > x$. And this yielded the new principle of equality as a limit of differences: if for any n, $< |x - y| < \frac{1}{n}$, then $x = y$" (Levey 2008, 131-132). And just below this, he writes: "Let x and y be "what is given" or what is "presupposed," and let $f(x)$ and $f(y)$ be "what follows" or "is sought." The Law then says that as the difference $|x - y|$ becomes smaller than $1/\varepsilon$ for any $\varepsilon > 0$, the corresponding difference $|f(x) - f(y)|$ likewise becomes smaller than any given quantity" (ibid.). From this it is apparent that the Archimedean principle and the law of continuity each compose half of the principle of equality, where the former supplies the *rule* and the latter supplies the *intuition*. Judgments that progresses from the rule to the instance are synthetic, and ones that progress from instance to rule are analytic. Vuillemin explains this in genealogical terms, "soit on remonte d'un individu à un ancêtre, soit qu'on commence par le tronc pour en faire voire les rameaux" (Vuillemin 1993, 9).[120] However, as finite creatures it is impossible to know the *reason* for the truth of a rule; no matter the reason for the reciprocity between identity and difference, the reality of the rule must precede the proof of its reality. Conversely, an analytic judgment that begins from the necessary connection of several determinations can be ostensively demonstrated, but the rule according to which they were assembled is absent because the intuition is merely given. Already in the *Port-Royal Logic* of 1662, synthetic principles are divorced from any direct proof because, according to Kant, while it is possible to demonstrate the certainty of the principle by apagogic means, the direct proof must appeal to the notion of intuitive immediacy (i.e. until it is put into relation to an object, the principle is mere phantasy) (ibid.). Leibniz's trichotomic deduction of continuity skirts the limits of both sensibility and the understanding, where the threshold of cognition recedes into a vanishing, fictional difference.

Maimon makes this parallax explicit when he distinguishes between the *symbolically* infinitely small and the *intuitively* infinitely small. The former is the differential as thought by symbolic cognition, and "signifies a state that a quantum approaches ever closer to, but that it could never reach without ceasing to be what it is, so we can view it as in this state merely

[120] Pappus' definitions of analysis and synthesis are still relevant here. "**Analysis** then takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis…But in **synthesis**, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what were before antecedents, and successively connecting them one with the other, we arrive finally at the construction of what was sought" (Heath 1908, 138). The issue is that the effect contains an innumerable set of antecedents, so the complete condition of the individual is impossible for us. Only partial analyses can be performed. Descartes also notes a similar asymmetry, where the effect is less perfect than the cause. As creatures, any metaphysical purists must begin analytically. This is justification for the ontological argument (Vuillemin 1993, 27).

symbolically" (ETP 351n.15). He gives examples that rely on the law of continuity, e.g. the *n* sided polygon and the ellipse with an eccentricity of 1, in which continuity is *not* intuitively represented. He explains, "The only reason we are nevertheless able to designate these states (that quanta can never reach) is because they are limit concepts, i.e. a merely symbolic infinitely small" (ETP 352). The latter is the differential as it is in itself, which is to say, a view which intuitively presents all determinations according to the rules that account for each and every determination. In itself, the differential "signifies every state in general that a quantum can reach; here the infinitely small does not so much fail to be a quantum at all as it fails to be a determined quantum" (ibid., 351). Here the differential would be nothing less than *the real definition* of an object. These two valences of the differential do not prove a contradiction at the core of Maimon's thought; together they describe the capacity of finite, discursive cognition to exceed its own inextricable limits. Hernán Pringe clarifies the controversy in Maimon scholarship by advocating for such a reading: "As many scholars have already pointed out, Maimon calls "differential" not only the rule of the generation of the sensible, but also the element or smallest unit of the sensible. The inconsistency that this double characterization seems to imply…may be avoided by noting that Maimon distinguishes two different perspectives from which the operations of the mind should be considered [ETP 81-82, 376-377]. What from the subjective perspective is considered an element of the sensible, is from the objective one its rule of generation" (Pringe 2018, 39). It is this *regulative* and *methodological* use of differentials that facilitates the transit from sensation to reason because, for any appearance that has already arisen, the understanding must posit a differential of sensation that is the occult element that is passively ordered in perception.[121] The

[121] With regard to the intuitively infinitely small as real, Maimon remarks that, "This way of considering it is also useful for resolving the question, *quid juris?* because the pure concepts of the understanding or categories are never directly related to intuitions, but only to their elements, and these are ideas of reason concerning the way these intuitions arise; it is through the mediation of these ideas that the categories are related to the intuitions themselves.

differential as a *fiction* is an artefact of the combinatorial procedure that far outsrips possible experience, and according to Leibniz, "les characters bien choisis ont cela de merveilleux, qu'ils laissent pour ainsi dire les marques des pensées sur le papier; et nous donnent le moyen d'estre infallibles" (Leibniz 1926, 271). In the introduction to Maimon's *Essay*, Nick Midgley express the stakes of Maimon's project: "Maimon takes the calculus to show that understanding does not lie in intuition but in getting behind intuition to grasp its production… It is in leaving behind the domains of pure and empirical intuition that we move towards understanding the real" (ETP liv). Symbolic cognition worth its name must dissociate the synthetic relations constitutive of cognition from their contingent expression in space and time.

4.2 Ideas of the Understanding, Ideas of Reason

By way of introduction to this section, it is perhaps time to broach the question of *notation* or the *literal inscription* of symbols in mathematical practice. In the preceding chapter, symbolic cognition was explicated as a method that allowed a finite mind to think objects that exceed the bounds of experience, which are nonetheless real. Such objects are denoted by a sign that conforms to the strictures of intuition in an arbitrary way, namely, that there is no *given* or *natural* connection between the sign what it signifies. Thomas Hobbes attempted to argue against the "scab of symbols" on Euclidean geometry by appealing to the arbitrary relation between names and definitions (Mancosu 1996, 87). Arbitrariness can be taken in two senses according to Dascal: "the claim that a definition is arbitrary may mean either (a) that the *relation* between the *definiendum* and the *definiens* is arbitrary, i.e., that the same concept (represented by the *definiens)* might have

Just as in higher mathematics we produce the relations of different magnitudes themselves from their differentials, so the understanding (admittedly in an obscure way) produces the real relations of qualities themselves from the real relations of their differentials" (ETP 355).

been connected to other names *(definiendo)* or vice-versa; or else (b) that the combination of concepts which constitutes the *definiens* is itself an 'arbitrary' combination, i.e. that it is not subject to any constraints or principles" (Dascal 1987, 61). The second sense was addressed by the theory of real definitions, in that a number of determinations that present themselves in intuition can be appended to a more general, determinable concept. In principle, this process completes itself when a concept is thoroughly determinate and coincides with the thing it defines. However, the first sense of arbitrariness intervenes because the finite mind must make use of signs either when the concept becomes sufficiently complex, or if the concept defines an object that cannot be encountered in the domain of experience. Leibniz's primary insight was that cognition is not intuitive and that one should not hold out for the ultimate revelation of a thoroughly determinate representation.[122] Dascal remarks on this insight at the end of *La Semiologie de Leibniz*: "au lieu de se limiter à envisager les signes dans leur fonction communicative, il tourne son attention vers leurs fonctions cognitives ; et au lieu de les concevoir simplement comme des auxiliaires psychotechniques de la pensée, il a vu jusqu'à leur attribuer un rôle constitutif dans toute activité mental supérieure" (Dascal 1978, 223). This is most thoroughly realized in his project *Analysis Situs* [1679], where he lays the foundations for a science not only of magnitude but of *situation* or *relation* as well.[123]

In short, Leibniz aims to recast geometry in light of his law of continuity which displaces strict equality with a rigorous kind of *similarity*. 'On Analysis Situs' unexpectedly begins by identifying

[122] It is true that Leibniz thought there were primitive or simple concepts which could be directly symbolized, and on which the whole edifice of symbolic cognition would rest. This is his position in 'Meditations on Truth, Knowledge and Ideas' (L 292). But in his mathematical works, Leibniz develops a model of symbolic cognition that is constitutive of cognition. Signs are no longer utilized to organize the mental economy of ideas, but actually constitute our access to certain objects. See especially (Leibniz 1926, 269-271).

[123] Leibniz explains his rationale for this name: "I like to call it Analysis Situs, because it explains situation directly and immediately, so that, even if the figures are not drawn, they are portrayed to the mind through symbols; and whatever the empirical imagination understands from the figures, this calculus derives by exact calculation from the symbols" (L 257).

the quantitative nature of algebra, arithmetic, and geometry which all operate under a similar paradigm. Magnitude assumes an ultimate measure, usually in the form of a scale or set of units. Here, “Quantity can be grasped only when the things are actually present together or when some intervening thing can be applied to both. But quality presents something to the mind which can be known in a thing separately and can then be applied to the comparison of two things without actually bringing the two together either immediately or through the mediation of a third object as a measure” (ibid., 255). Where quantities are equal when their magnitudes are the same, qualities are similar when their forms are the same.[124] Leibniz gives a new formulation to his law of continuity which states that things with the same qualitative form, despite their quantitative inequality, cannot be distinguished: “I use this new axiom: “things which cannot be distinguished through their determinants (or through data adequate to define them) cannot be distinguished at all, since all other properties arise from these data” (L 256). This new axiom makes explicit the significance of relations over individuals, to the point that the individuals *can be analyzed without construction*. In a statement that far exceeds the consensus of his time, Leibniz concludes that “Therefore this calculus of situation which I propose will contain a supplement to sensory imagination and perfect it, as it were. It will have applications hitherto unknown not only in geometry but also in the invention of machines and in the descriptions of the mechanisms of nature” (L 257).

This project demanded a specific kind of notation, one which was not common to either algebra or geometry.[125] In fact, the reason why there had been so little progress in this direction was

[124] “Besides quantity, figure in general includes also quality or form. And as those figures are equal whose magnitude is the same, so those are similar whose form is the same… Thus a true geometric analysis ought not only consider equalities and proportions which are truly reducible to equalities but also similarities 'and, arising from the combination of equality and similarity, congruences” (L 254-255).

[125] Leibniz had great success applying the concept of *situs* to algebra. In a lengthy paper, Serfati shows that Leibniz developed a notation not only for representing *situs*, but for summations that preserve the particular pattern inherent in equations. The two numbers in 21 represent first the rank of the equation in a series of equations, and second the

"undoubtedly, the fact that no Characters directly representing themselves have yet been discovered. For without characters, it is hard to disentangle oneself amid a multitude and confusion of things" (Dascal 1987, 170). Leibniz set himself to this task in *Characteristica Geometrica* [1679]. Though this attempt at a new geometry was ultimately a failure, Leibniz remain convicted of two things (which were only later vindicated):[126] 1) that the spatial relations constitutive of geometry were in truth *conceptual relations* that could equally be expressed by non-spatial means. This has consequences that exceed geometry, or mathematics even: "there is something still greater underlying this project, for we will be able to express, by means of these characters, the true definitions of all that be longs to geometry, and everywhere to pursue the analysis until reaching the principles, i.e., until reaching the axioms and postulates" (Dascal 1987, 169). 2) That determinate features of the notation itself afford cognition the capacity to *qualitatively* change itself: "The more precision the characters have, that is, the more relations of the things they exhibit, the more useful they are. And when they exhibit all the relations of the things among themselves, in the way the arithmetical characters used by me do, then there is nothing in the thing which cannot be grasped through the characters" (ibid., 167). Though Maimon does not remark on the question of notation, he agrees that the space and time are not absolute forms. The forms of intuition express the underlying conceptual relations that the understanding must represent to itself. In grasping the symbols, one grasps the idea since "we can pass from a consideration of the

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rank of the unknown within the equation (Serfati 2001, 169). The term 11.20.30 would then combine the first unknown from the first equation, the coefficient from the second equation, and the coefficient of the third. Now, the equation 10.21.30 + 10.20.31 + 11.20.30 uses this notation to find a common structure, namely, that the sum of the second numbers in each term equal 1. Leibniz could denote all such terms that had this same structure by writing $1\dot{1}.2\dot{0}.30$ which in contemporary notation would be $\sum_{I+j+k=1} a_{1i}a_{2i}a_{3k}$ (ibid.,189). No matter the values of *i*, *j*, and *k* so long as they sum to 1, the equation has the same structure.

[126] Two mathematicians in particular cite Leibniz explicitly: Herman Grassmann, who explicitly makes use of *situs* and wrote a monograph on *Characteritica* (Vuillemin 1993, 36); and Abraham Robinson, who rehabilitated infinitesimal quantities in the wake of set theory, and developed the field of non-standard analysis (Tho 2012, 76-79)

relations in the expression to a knowledge of the corresponding properties of the thing expressed" (L 207). Moreover, Maimon commits heresy against Kantian philosophy when he invents the term *ideas of the understanding* to describe the role of differentials in cognition.

One of the greatest lessons of the first *Critique* is that not all concepts can be related to the forms of possible appearances. If such concepts have any validity at all, it can only be subjective validity. Most often when Kant uses this term, he is referring to the contingent order of apprehension, or the psychological habits of association (CPR B140-142). But problematic concepts, which are logically consistent though cannot be applied to experience, issue from the mind and are thought-entities. They are nothing, but a determinate kind of nothing. He writes "a concept without an object, like the *noumena*, which cannot be counted among the possibilities although they must not on that ground be asserted to be impossible (*ens rationis*)" (CPR A290/B347). This is not to say they are the residue or refuse of cognition. It is perhaps not too much of an exaggeration to say that the faculty of reason essentially depends on a single problematic concept: the unconditioned. Kant deduces this concept in the same way he deduces the categories. Just as the logical functions of judgment exhausted all of the ways of composing and asserting propositions, the three forms of syllogism – categorical, hypothetical, disjunctive – in (Aristotelean) logic all appeal to an unconditioned premise.[127] Similarly, the pure rules of inference are represented in inner sense, "whose schema is provided in general by logic in the three formal species of syllogisms, just as the categories find their logical schema in the four functions of all judgments" (CPR A406/B432). This premise gives the whole domain of the syllogism, which is restricted by a condition in the conclusion, e.g. $P \to Q, Q \to R, \therefore P \to R$, where the relation between

[127] "There will be as many concepts of reason as there are species of relation represented by the understanding by means of the categories; and so we must seek an **unconditioned**, **first**, for the **categorical** synthesis in a **subject**, **second** for the **hypothetical** synthesis of the members of a **series**, and **third** for the **disjunctive** synthesis of the parts in a **system**" (CPR A323/B379).

Q and *R* is restricted by the condition *P*. The logical universality [*universalitas*] of the domain corresponds to the concept of allness [*universitas*] in intuition, i.e. the totality of possible conditions or determinations that can restrict the domain.

This results in three concepts – Soul, World, God – which have no corresponding intuition. If these concepts are used empirically, they spur a indefinite regress from the conditioned phenomena up through the entire series of its conditions.[128] Though they operate in difference spheres, these concepts are all versions or classes of the unconditioned: "So the transcendental concept of reason is none other than that of the **totality of conditions** to a given conditioned thing. Now since the unconditioned alone makes possible the totality of conditions, and conversely the totality of conditions is always itself unconditioned, a pure concept of reason in general can be explained through the concept of the unconditioned, insofar as it contains a ground of synthesis for what is conditioned" (CPR A322/B379). As something that eludes experience, the unconditioned cannot be combined with any empirical concepts or relations, only those notions [*notio*] which have their source in the understanding. The term notion is mentioned only in the extensive taxonomy of the elements of cognition that he gives at the beginning of the 'Transcendental Dialectic.'

> The genus is **representation** in general (*repraesentatio*). Under it stands the representation with consciousness (*perceptio*). A **perception** that refers to the subject as a modification of its state is a **sensation** (*sensatio*); an objective perception is a **cognition** (*cognitio*). The latter is either an **intuition** or a **concept** (*intuitus vel conceptus*). The former is immediately related to the object and is singular; the latter is mediate, by means of a mark, which can be common to

[128] These spheres or classes are given by the kind of syllogism, "of which the first contains the absolute (unconditioned) unity of the thinking subject, the second the absolute unity of the series of conditions of appearance, the third the absolute unity of the condition of all objects of thought in general. The thinking subject is the object of psychology, the sum total of all appearances (the world) is the object of cosmology, and the thing that contains the supreme condition of the possibility of everything that can be thought (the being of all beings) is the object of theology" (CPR A334/B391).

> several things. A concept is either an **empirical** or a **pure concept**, and the pure concept, insofar as it has its origin solely in the understanding (not in a pure image of sensibility), is called *notio*. A concept made up of notions, which goes beyond the possibility of experience, is an **idea** or a concept of reason. (CPR A320/B377).

The above pure and problematic concepts are all composed of notions of logical relation, and as such are ideas of reason. But there seems to be a problem. If concepts are properly used in objective cognition, and ideas are concepts that go beyond the possibility of experience, then what is the proper use of ideas? Experience is a hierarchy of representations where the concepts and principles of the understanding bring *distributive* unity to intuitions. No concept or principle applies to the whole of experience. Instead, local connections describe the genesis of objects without concern for the larger unity among principles themselves. This is precisely why there are *in principle* an indefinite series of empirical causes that condition any one object.[129] Even if reason does not *constitute* the elements of experience (as both the mathematical categories and principles do), it nonetheless unifies and *regulates* the principles of experience at a *transcendental* level. "Thus reason really has as object only the understanding and its purposive application, and just as the understanding unites the manifold into an object through concepts, so reason on its side unites the manifold of concepts through ideas by positing a certain collective unity as the goal of the understanding's actions, which are otherwise concerned only with distributive unity" (CPR A644/B672). This collective unity is the unique contribution of reason without which cognition would be stuck in the world of a relentless mechanical causality.

[129] This is the subject of the section 'Antinomies of pure reason.' Any inference from the infinite series of conditions to the existence of an unconditioned thing is illegitimate, since as Kant explains: "Reason never relates directly to an object, but solely to the understanding and by means of it to reason's own empirical use, hence it does not create any concepts (of objects) but only orders them and gives them that unity which they can have in their greatest possible extension, i.e., in relation to the totality of series; the understanding does not look to this totality at all, but only to the connection through which series of conditions always come about according to concepts" (CPR A643/B671).

By introducing ideas of the understanding, Maimon means to communicate a similar point about the relation between intuition and the understanding. In the 'Short Overview,' he writes "I extend the sphere of the ideas (as well as the sphere of the antinomies arising from these ideas) much further because I maintain that they are to be found not only in metaphysics but also in physics, and even in the most self-evident of all sciences, mathematics" (ETP 227).[130] By this, Maimon means that the unity the understanding brings to intuition is *foreign to intuition*. This can be grasped with Maimon's use the terms formal and material completeness which is just the "difference between the totality of conditions by means of which an object of intuition is thought, and the totality of the intuitions themselves that are subsumed under these conditions" (ETP 76). On one hand, the ideas of reason give formal completeness to the understanding, i.e. the totality of conditions, and the real ground of synthetic determinations is a singular idea of reason: an infinite understanding. This idea is posited as the coincidence of absolute material and formal completeness, which we will never achieve. He writes "We assume an infinite understanding (at least as idea), for which the forms are at the same time objects of thought, or that produces out of itself all possible kinds of connections and relations of things (the ideas). Our understanding is just the same, only in a limited way. This idea is sublime and will, I believe (if it is carried through), overcome the greatest difficulties of this kind" (ETP 64-65).[131]

[130] This also helps explain why Maimon calls differentials physical points, as opposed to metaphysical ones. He explains that, "Considered in itself as a quality, every sensible representation must be abstracted from all quantity whether extensive or intensive. For example, the representation of the colour red must be thought without any finite extension, although not as a mathematical but rather as a physical point, or as the differential of an extension" (ETP 26-27).
[131] In relation to synthetic judgments, the infinite understanding is the inner analytic ground of connection: "The gathering together of these qualities is merely a synthesis of the imagination, based on their simultaneous coexistence in time and space (the assumption of an inner ground is and remains merely an assumption - that is, for us, although it must be admitted that for the infinite understanding the assertoric-synthetic propositions must be apodictic and the apodictic-synthetic propositions analytic); but not a synthesis of the understanding" (ETP 92-93).

On the other hand, the ideas of the understanding give material completeness to intuition, i.e. the totality of intuitions. Consider the construction of a circle, or even the sensible intuition of a circle. Even if one possesses the definition of a circle and its construction-procedure (schema) it is nevertheless impossible to construct or perceive an intuition adequate to the concept. This is because "its material completeness (completeness of the manifold) cannot be given in intuition, because only a finite number of equal lines can be drawn. So this concept is not a concept of the understanding to which an object corresponds, but only an idea of the understanding, something that we can come infinitely close to in intuition. by means of the successive addition of such lines, and consequently a limit concept [*Gränzbegrif*]" (ETP 75-76). This solidifies Maimon's scepticism about the application of concepts to intuitions. Kant takes for granted that appearances can be represented according to a priori concepts, but as we have seen, the *quid facti?* concerns the orderliness of appearances that is *passively* generated by sensibility. The *quid juris?* concerns the differentials of sensibility that are undetermined yet determinable by cognition. According to Maimon, "Differential magnitudes are ideas of the understanding since they are real objects determined by means of a priori conditions; but they cannot be constructed, i.e. presented in intuition, because they must be viewed in abstraction from all magnitude (because they are expressed merely by means of a function-relation that, as a numerical relation, is continuously variable)" (ETP 374n.31).[132] This functional relation is continuously variable because any determinate relation of numbers or magnitudes can be reduced to zero without also annihilating

[132] He also gives examples of ideas of the reason. "That is, the rule of the asymptotes is the following: every part of it must be closer to the curve than the preceding part, without ever reaching the curve. To relate this rule to every particular possible part comprises an idea of the understanding because the rule contains something **impracticable** (but nothing **impossible**) when related to every particular part. This is because it means the following: draw first part *a* according to the rule, then *b*, then *c*, and so on without ever stopping. By contrast, applied to all possible parts (which are assumed to be already drawn) this rule is an idea of reason because it contains something impossible in that it represents the totality of the parts as complete and not complete at the same time; consequently this totality does not signify an object (not even of an infinite understanding), but merely the approach to an object" (ETP 375-376).

the relation between differentials. It is a *purely conceptual relation* that unifies the manifold of intuition, one that is not determined by the unity of apperception or the pure forms of space and time.

Kant criticized Leibniz for intellectualizing intuitions.[133] Maimon reveals that the forms of intuition are not only intellectual, but abstracted from actual sensation; they are fictions. Fictions of the imagination are not fictions of the understanding. The term fiction has already been used to describe the status of infinitesimals in differential calculus. They are also regulative ideas of the understanding; ones that the understanding must employ when it represents the order of apprehension to itself as the order of apperception. Between the appearance and construction, and the passivity and activity of the understanding, must intervene an element that behaves *as if* it were governed by some rule. This idea allows the understanding to think the absence of its own activity, not to prove the nature of things themselves, but to imagine an original *presentation* that unified the manifold of sensation such that consciousness is affected at all. Presentation, or the constitutive synthesis of *qualities*, "is not representation, i.e. a mere making present of what is not [now] present, but rather presentation, i.e. the representation of what was previously not as [now] existing" (ETP 30). The disparate things that affect consciousness are not immediately given by a rule, but the manifold of sensation *as manifold* (the unity itself) can only be retroactively represented *as pure space or pure time*. Maimon argues that the pure forms of space and time are *ens imaginarium,* on which point Kant concurs (ETP 19 and A291/B348). But it is only the imagination determined by the understanding that produces these forms, so space as the concept

[133] "In a word, Leibniz intellectualized the appearances, just as Locke totally sensitivized the concepts of understanding in accordance with his system of noogony (if I am permitted this expression), i.e., interpreted them as nothing but empirical or abstracted concepts of reflection" (CPR A271/B327).

of being-outside-one-another, or *difference*, precedes space as form of intuition. The following passage brings all of this together:

> The forms of concepts in general are identity (unity in the manifold), but also difference, by means of which the manifold is thought as a manifold. For example, suppose that two triangles are given to me (they are determined differently and hence are two triangles and not one). I relate them to one another and notice that they are both triangles, i.e. that they are identical. The concept of triangle in general arises from this. So let us see what must necessarily follow from these forms or conditions of our consciousness. The difference in our perceptions, i.e. being outside one another in time and space, makes the forms of our sensibility necessary (I speak here as a Leibnizian, who treats time and space as universal undetermined concepts of reflection that must have an objective ground); in other words, the form of sensibility is a schema of this difference, and through it this difference is determined i.e. *a priori*; what is given *materialiter* as different, can also only be thought *formaliter* as different. The reason for this is that, although form precedes matter, i.e. our mode of representation (the constitution of our mind) determines the representation itself, it is nevertheless the reverse with respect to our consciousness, in other words, in this case consciousness of the form presupposes the matter (because unless something determined is given to us, we cannot attain consciousness of the form). Being outside one another in time and space has its ground in the difference between things, i.e. the imagination, which is the ape of the understanding, represents the things *a* and *b* as external to one another in time and space because the understanding thinks them as different. So this concept of the understanding is the imagination's guiding principle and it must not lose sight of it if its procedure is to be legitimate; but if it does lose sight of it, then it falls into fictions no longer subject to any rule of the understanding. The concept of **being**

> **different** is more universal than that of **being outside one another** because the latter applies only to intuitions whereas the former can be applied to concepts as well, i.e. everything that is different must be perceived in intuition in space and time, but not the reverse. (ETP 132-134)

The reverse does not hold because appearances have already arisen and conceal the principles of their genesis. The relations of space that were so essential for geometrical constructions can be reduced to the non-empirical concepts of reflection. From there it is a process of determining the concept synthetically which involves abstracting the pure relations from their determinate appearances. Lachterman calls this process of abstraction *symbolic construction*, which combines both evidentiary and operational axes of construction (Lachterman 1992, 505). The construction does not stand in for relations of space and time (as Kant uses the term); it composes pure conceptual relations according to the finite means available to cognition. Intuition plays no small role in this, since it is only by means of exhibiting the concepts in intuition that it is possible to observe other determinations that implicitly follow from a concept. Lachterman writes that "the discrete elements of a cognitive operation, "a," "b," "c," for example, are arrayed by the imagination in a spatial field and, via the latter, in a temporally indexed sequence, "a_1," "b_2," "c_3." This spatiotemporal array *is* intuitable or sensuous, and the space and time ingredients in it are schemata or *Bilde* of the pure concepts of *Auseinandersein* [being-outside-one-another] and *Folge* [sequential discreteness]" (ibid.). Truth is never grounded in intuition, nor proved by experience alone. Intuition always carries with it a presentiment of truth; "a presentiment that, I believe, must play no insignificant role in the power of invention [*Erfindungskraft*]" (ETP 70).

In a roundabout manner, this demonstrates that the spatio-temporal determinations of experience can be transposed into the realm of pure relations, because these determinations are produced from pure relations. Such relations are not grounded in space or time because the synthetic judgments made possible by a construction or an empirical intuition are grounded in the real definitions of objects. Discursive cognition is put onto a continuum with intellectual intuition, which it approaches asymptotically. The upshot of symbolic cognition is that problematic concepts, ones which exceed the bounds of intuition, *can be exhibited by other means*. Notation allows the mind to *observe* what *cannot be experienced.* And this is why Leibniz dedicated himself to inventing different kinds of symbols with different representational capacities.[134] Taken to the limit, notation exceeds its merely representational capacities and augments the cognitive capacities of a finite mind.

4.3 Reality and the Inflection of Mind – by way of conclusion

In 'Anti-Eureka,' Matt Hare and Ben Woodard note that "All writing is at least a partial autonomization of thinking, but this fact is quite banal. The task is rather to try and develop the process of formalization as a model of transcendental reflection wherein thought is revealed to thinking through a mapping of relations back into the thinkable, a process which always teeters between a romance of the unintelligible and an acceptance of reified artifactuality" (Hare & Woodard 2017, 13). The two extremes of symbolic cognition both dissociate thought from the brain. The trap of romanticism is that it encompasses the human mind and its cognitive artefacts within an impersonal nature that is the ultimate cause of all genesis (including the genesis of the mind itself). The other extreme is a kind of formalism that cannot grasp the contingency of its historical or empirical situation. If cognition can be completely formalized and subsist independently of the brain, it still must explain the genesis of any one notation or any singular set

[134] See the classic essay 'Leibniz, the Master-Builder of Mathematical Notations' by Florian Cajori (1925 412-429).

of principles. Neither Leibniz nor Maimon quite succumb to either temptation; as a result, symbolic cognition falls somewhere in between. Leibniz is committed to the combinatorial character of symbols, since these symbols express the God-given reality of their objects. Maimon is committed to the genesis of real objects while remaining within the bounds of finite cognition, since an infinite understanding is a regulative idea. Without God, the Leibnizian system falls apart, because real definitions ultimately resolve into the identity or essence of the thing. In the *Monadology* he writes “44. For if there is a reality in the essences or possibilities, or in the eternal truths as well, this reality must be founded on something existent and actual, and therefore in the existence of a necessary being, in whom essence includes existence or in whom it is enough to be possible in order to be actual” (L 647). And since the mind of God holds the ideas from which eternal truths are derived, without God there is no realty in any concept or any possibility. More than this, there is no world outside of the partial view each substance has of the others: “57. Just as the same city viewed from different sides appears to be different and to be, as it were, multiplied in perspectives, so the infinite multitude of simple substances, which seem to be so many different universes, are nevertheless only the perspectives of a single universe according to the different points of view of each monad” (L 648). Synthetic cognition amounts to archeology of the soul; it is only by means of an internal principle of change that the determinations already in my substance are revealed.

Maimon inverts the role that God typically plays in metaphysics. Instead of grounding the reality of objects, the idea of reason is an enabling condition for the subject to make itself real. In chapter 10 of the *Essay*, ‘On the I, Materialism, Idealism, Dualism, etc.’ Maimon aims to answer the question: “What am I [*ich*]?” The kernel of this chapter is his explicit rejection of the transcendental object, as that something outside of consciousness indicated by affections and sensations. This is not to say he is an idealist, since he argues that they too are committed to the existence of determinate ideas, where “The representations time and space, and what is determined by them, are only confused thoughts of the connections and relations of things to one another” (ETP 160). Though the regulative idea of an infinite intellect has a crucial place in Maimon’s system, some actual mind independent of the mind, which would be the real ground of the objects of cognition, is a fiction. As an idea of reason, it bounds the unending series of conditions for any object: “we see that the possibility of each and every thing presupposes the possibility of both a more general and a more particular thing; as a result, in the series of subordinated things to which the given belongs, both a progress and a regress to infinity pertain to the complete possibility of a thing: this makes the idea of an infinite understanding a necessary one” (ETP 248). This idea is regulative only because it presents the infinite intellect as a terminus, an end of the series not included within the series. Things change when it is paired with the differential as an idea of the understanding. The differential is precisely that element through which the finite passes into the infinite, not all at once, but piecemeal and over a lifetime. The differential acts as a lens through which the infinite is made immanent to consciousness. The idea of an infinite understanding not only has objective import, but subjective import as well. As Samuel Atlas writes:

> This possibility of reaching out to the infinite offers a positive basis for metaphysics. Since the human mind is aware of the continuous growth and development of its thought, as manifested in the continuous growth of science and mathematics, and since it is in possession of the idea of infinite development, it can conceive of the idea of an infinite mind by a process of eliminating its own restrictions. Thus metaphysics, the subject matter of which is the absolute and the infinite, is a possible science. The idea of God, which is only regulative for Kant,

becomes for Maimon a constitutive idea derived by a sort of transcendental deduction. (Atlas 1964, 96).

This is not to say that the idea is constitutive of the *object it represents*, but that it is constitutive of the *subject that thinks it.* Maimon states that, even if the I "cannot be determined as an object in itself, it can nevertheless be determinately thought as an object in its modifications, by dint of approaching a determined object ever more closely to infinity…the complete attainment of the latter is not merely an idea, but in fact contains a contradiction, since it is both an object and not an object at the same time" (ETP 164). In its many forms, inquiry orders, structures, and produces regularity in the mind.[135] It is in producing order that the mind wrestles with disorder; the mind instigates a dialectic of determination and indetermination. In a masterful reversal of Kantian philosophy, Maimon posits the infinite understanding as an idea that is at once *objectively regulative* and *subjectively constitutive*.

For all of its technical specificity, Maimon's project avoids being systematic.[136] Philosophy is not an outward conquest, pushing back the frontiers until the surface of the globe has been won through 'violent conquest.' Kant closes *Critique of Pure Reason* by setting a determinate end to inquiry, asking "whether or not that which many centuries could not accomplish might not be attained even before the end of the present one: namely, to bring human reason to full satisfaction

[135]In the opening to 'The Law of Mind,' Peirce pits continuity and regularity (synechism) against absolute chance (tychism): "I have begun by showing that *tychism* must give birth to an evolutionary cosmology, in which all regularities of nature and of mind are regarded as products of growth, and to a Schelling-fashioned idealism which holds matter to be mere specialized and partially dead mind" (Peirce 1892, 312-313). Instead of the mind of nature concretizing into regularities, Maimon might suggest that the finite mind of consciousness itself becomes more real as it becomes more determined.

[136] "According to my system (or non-system), on the other hand, while it is true that reason does not think of foreign conquests, but only of securing its legitimate possession, at the same time it finds that this possession is unlimited, so that it can never enjoy it all at once, but only little by little to infinity: these are, however, merely legitimate acquisitions and not at all violent conquests" (ETP 442-443).

in that which has always, but until now vainly, occupied its lust for knowledge" (CPR A855/B883). Maimon does not believe that there ever will be an end to inquiry, which fundamentally changes the stakes of philosophy. Something of an ethics is attached to the indefinite process of inquiry. Maimon writes that "We therefore have not only a method by means of which we can approach the idea ever more closely in construction, but also a practical rule by which we go into ourselves [*in uns selbst gehen*], as it were, or better, by which we, as such, attain ever greater reality" (ETP 165). Ostensive construction makes the object an artifice of mind; symbolic cognition allows the mind to make itself into an artifact of the image it has of itself.[137] On this reading, subjectivity does not presuppose its own reality. Subjectivity posits a differential through which it passes from fleeting and multitudinous sensations into its own reality. In both senses is the subject a being-of-thought (*noumenon*), because (according to Sylvain Zac) "tantôt le noumène est synonyme de Dieu, tantôt il est synonyme de la différentielle et il décrit une sorte de processus de nouménisation" (Zac 1988, 299). Noumenalization describes the transit between the limit-concepts of a myopic and a panoptic consciousness, the minima and maxima of representation. The subject crafts itself between the two states it can think but never reach. "So we start in the middle with our cognition of things and finish in the middle again" (ETP 350).

Getting behind intuition to grasp its genesis constitutes a world of subjects – Going into oneself one emerges "as a garden full of plants or as a pond full of fish. But each branch of the plant, each member of the animal, each drop of its humors, is also such a garden or such a pond" (L 650).

[137] In a contemporary context, Reza Negarestani proposes a highly nuanced and more thoroughly developed concept of intelligence that is similarly committed to crafting itself: "To recapitulate, artificial general intelligence is not the champion of technology but a thought that, through a positive disenchantment of itself and its contingent history, has been enabled to explore its possible realizations and realizabilities—whether in a social formation or a multi-agent system of machines—as part of a much broader program of self-artificialization through which thought restructures and repurposes itself as the artefact of its own ends to maintain and expand its intelligibility. Just as the practice of thinking is non-optional for a thought that intends to remain intelligible, the practice of artificialization is not optional; it is mandated by the autonomy of thought's ends and demands" (Negarestani 2018, 460).

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