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Abelian Integral Method and its Application

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Abstract

Oscillation is a common natural phenomenon in real world problems. The most efficient mathematical models to describe these cyclic phenomena are based on dynamical systems. Exploring the periodic solutions is an important task in theoretical and practical studies of dynamical systems.

Abelian integral is an integral of a polynomial differential 1-form over the real ovals of a polynomial Hamiltonian, which is a basic tool in complex algebraic geometry. In dynamical system theory, it is generalized to be a continuous function as a tool to study periodic solutions in planar dynamical systems. The zeros of Abelian integral and their distributions provide the number of limit cycles and their locations.

In this thesis, we apply the Abelian integral method to study limit cycles bifurcating from the periodic annuli for some hyperelliptic Hamiltonian systems. For two kinds of quartic hyperelliptic Hamiltonian systems, the periodic annulus is bounded by either a homoclinic loop connecting a nilpotent saddle, or a heteroclinic loop connecting a nilpotent cusp to a hyperbolic saddle. For a quintic hyperelliptic Hamiltonian system, the periodic annulus is bounded by a more degenerate heteroclinic loop, which connects a nilpotent saddle to a hyperbolic saddle. We bound the number of zeros of the three associated Abelian integrals constructed on the periodic structure by employing the combination technique developed in this thesis and Chebyshev criteria. The exact bound for each system is obtained, which is three. Our results give answers to the open questions whether the sharp bound is three or four. We also study a quintic hyperelliptic Hamiltonian system with two periodic annuli bounded by a double homoclinic loop to a hyperbolic saddle, one of the periodic annuli surrounds a nilpotent center. On this type periodic annulus, the exact number of limit cycles via Poincaré bifurcation, which is one, is obtained by analyzing the monotonicity of the related Abelian integral ratios with the help of techniques in polynomial boundary theory. Our results give positive answers to the conjecture in a previous work.

We also extend the methods of Abelian integrals to study the traveling waves in two weakly dissipative partial differential equations, one is a perturbed generalized BBM equation and the other is a cubic-quintic nonlinear, dissipative Schrödinger equation. The dissipative partial differential equations (PDEs) are reduced to singularly perturbed ordinary differential equation (ODE) systems. On the associated critical manifold, the Abelian integrals are constructed globally on the periodic structure of the related Hamiltonians. The existence of solitary, kink and periodic waves and their coexistence are established by tracking the vanishment of the Abelian integrals along the homoclinic loop, heteroclinic loop and periodic orbits. Our method is novel and easily applied to solve real problems compared to the variational analysis.

Keywords: ODE, PDE, Abelian integral, limit cycle, traveling wave, weak Hilbert's 16th problem.

Summary for Lay Audience

Periodic motions appear in almost all natural and engineering dynamical systems. Determining the number of periodic solutions and their locations plays an important role in solving real world problems, in particular on stability and bifurcations of the system. It is important to determine what may cause oscillation and what may destroy oscillation, and what affects the period and amplitude of oscillation. However, it is not easy to determine all possible locations, periods and amplitudes even for the oscillations in two-dimensional dynamical systems. In this thesis, we apply an integral defined on a family of continuous ovals as a bifurcation function to study the oscillation phenomena in some perturbed temporal or temporal-spatial dynamical systems, and obtain new results which solve open problems in the existing literature.

Co-Authorship Statement

This integrated-article thesis is based on 5 papers co-authored with Dr. Pei Yu. The article version of Chapter 2 has been published in *Journal of Differential Equations*; The article version of Chapter 5 has been published in *Discrete and Continuous Dynamical System-ser B*. The paper based on chapter 4 was accepted by the *Bulletin des Sciences Mathématiques*. Papers based on Chapters 3 and 6 have been submitted for publication. Dr. Pei Yu provided assistance with the theoretical understanding of Abelian integrals, gave the PhD candidate lots of guidance and feedback on these papers. The final drafts were revised by Dr. Pei Yu.

To my family for their endless love,
encouragements and support.

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Chapter 1

Introduction

1.1 Oscillation in differential dynamical systems

In natural science and engineering, researchers usually use mathematical tools to model real world problems, or build mathematical models for conducting more detailed research, for example, predicting what may happen under some exact controls, growing or decaying? Among various kinds of nonlinear phenomena in real world, the most common one is oscillation, such as the heart beating [71], the periodic outbreak of the influenza [52], the business cycle in economics [8], vibration of machines [47], the protein oscillation in bacteria [36], the periodic pattern induced by spatial variation [35] and so on. The most efficient mathematical models to describe these cyclic phenomena are based on dynamical systems, including Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs). Therefore, exploring the periodic solutions is an important task in the study of dynamical systems, particularly on determining the number of periodic solutions and their locations, as well as investigating what may cause oscillation and what may destroy oscillation, and what affects the period and amplitude of oscillation. However, it is not easy to determine all possible locations, periods and amplitudes even for the oscillations in two-dimensional dynamical systems.

For two-dimensional ODE systems, an isolated periodic orbit is called a limit cycle. The Poincaré-Bendixson theorem [27] is usually used to prove the existence of limit cycles. However, it is not a trivial task to construct more than one compact regions for finding more than one limit cycle. The non-existence of limit cycles in two-dimensional systems can be verified by showing non-vanishment of the divergence (Bendixson-Dulas criterion [27]) or by showing non-existence of critical points in a related region. Another way to find limit cycles is to construct Poincaré map near the center-focus singularity, for which the linearization matrix has a pair of purely imaginary eigenvalues by introducing trigonometric transformation into the two-dimensional system. Then n small limit cycles, surrounding the singularity, will arise in a very small neighborhood of the singularity when the Liapunov coefficients (or focus values) v_i, $i = 0, 1, 2, \cdots, n$, satisfy

$$|v_0| \ll |v_1| \ll \cdots \ll |v_n|, \quad \text{and} \quad v_{i-1}v_i < 0. \tag{1.1}$$

This is called Hopf bifurcation and the associated singularity is called a Hopf critical point, see [31]. The Liapunov coefficients can be computed via symbolic algorithms with the aid of computer algebra systems, such as Maple. However, it is difficult to study the algebraic variety of the Liapunov coefficients and further analyze the condition given in (1.1), see [31]. Even though there exist other methods for studying limit cycles, they are applicable to planar systems with strict restrictions [27]. So far, there does not exist general methodology which can be used to determine large limit cycles in general planar systems without special restrictions.

For higher dimensional ODE systems, periodic motions are often detected by studying Hopf bifurcation in the related two-dimensional systems projected on center manifold, see [31]. The center manifold is usually defined locally, implying that the periodic motion detected for the original higher dimensional system has small amplitude. In numerical studies, researchers have indeed found larger amplitude periodic motions by choosing parameter values in a neighborhood of the Hopf bifurcation curve [53]. But, it is still unclear how a periodic solution emerging from a Hopf bifurcation becomes a large amplitude oscillation. In particular, it is very interesting to note that slow-fast motions can occur in some higher dimensional systems, which cannot be investigated by the standard singular perturbation theory. This phenomenon was first discovered in [72]. There exist four conditions ensuring the existence of slow-fast motion, see [69, 72],
(i) there exists at least one equilibrium solution;
(ii) there exists a saddle-node bifurcation or a transcritical bifurcation at an intersection of the two equilibrium solutions;
(iii) there is a Hopf bifurcation which occurs from one of the equilibrium solutions; and
(iv) there exists a "window" between the Hopf bifurcation point and the saddle-node/transcritical bifurcation point in which oscillations continuously exist.
The criteria have successfully been applied to find slow-fast oscillations in many kinds of biological models, such as HIV model [67, 69, 72] for modeling the Viral Blips, Tritrophic Food Chain Model [66] and biologically relevant organic reactions [68]. Higher dimensional ODE systems can also exhibit oscillations via Bogdanov-Takens (BT) bifurcation, which occurs when the matrix of linearized system has a double-zero eigenvalue [39]. The number of limit cycles can be investigated by studying a reduced two-dimensional system of the form, which is the projected system on the center manifold,

$$\dot{x} = y, \quad \dot{y} = -g(x) + f(x)y, \tag{1.2}$$

where $g(x)$ and $f(x)$ are polynomials. System (1.2) has a simple form, but it is really a very useful and interesting model. It appears not only in the study of local bifurcation in higher dimensional ODE systems, but also in modeling various kinds of oscillations, such as spring vibration [47], wind-induced vibrations to tall buildings [1]. In Newton mechanics, $g(x)$ is called restoring term and $f(x)$ damping term. System (1.2) can also appear in the study of traveling waves of PDEs, such as studying the fronts in Fisher-Kolmogorov equation [50], the solitary waves in dissipative KdV equation [19] and perturbed generalized BBM equation [12], etc. The limit cycles arising in (1.2) correspond to the periodic wave trains for the related PDEs, and the existence of a limit cycle is one neccessary condition on the existence of a traveling front with a tail. It also has theoretical significance to study limit cycles in system (1.2) because it is related to the famous Hilbert's 16th problem.

However, it is difficult to study the limit cycles in system (1.2) even though it has a simple form. For convenience, we call system (1.2) type (m, n) if $(\deg(g), \deg(f)) = (m, n)$. There exist no efficient methods to study the number and location of limit cycles in system (1.2), except for the Hopf bifurcation method of studying small limit cycles. When the damping term $f(x)y$ is relatively weaker than the restoring term $g(x)$, Dumortier and Li investigated the number of limit cycles and their locations for system (1.2) with $m = 3$ and $n = 2$ in a series of papers [20, 21, 22, 23], in which the exact bifurcation diagrams are obtained. For example, they proved that there exist at most five limit cycles in system (1.2) of Duffing oscillator type, and in each potential component there are at most two limit cycles [23]. For types $(4, 3)$ and $(5, 4)$ system (1.2), few results have been reported on the maximal number of limit cycles, but on some upper and lower bounds [7, 56, 57, 62, 63, 64, 74]. It is still open on the exact bound of the maximal number of limit cycles in different subsystems of types $(4, 3)$ and $(5, 4)$ system (1.2). In the first part of this thesis, we will apply the Abelian integral method with the help of some techniques in geometry and symbolic computation to study the limit cycles in system (1.2), without being restricted to small limit cycles.

PDEs can model spatiotemporal oscillations in real world problems by studying periodic traveling waves, which are significant as the one-dimensional equivalent of spiral waves and target patterns in two-dimensional space. In solving real world problems, certain relatively weak influences are unavoidable due to the existence of uncertainty and higher order correction to the originally mathematical models, for example in describing the shallow water waves in nonlinear dissipative media [16] and dispersive media [37]. In other words, it is more realistic to consider perturbed models. For example, in explaining wave motions on a liquid layer over an inclined plane, Topper and Kawahara [59] derived the following equation,

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0. \tag{1.3}$$

When the inclined plane is relatively long and the surface tension is relatively weak, the diffusion term and the 4th-order dispersion term in the above equation may be treated as weak terms. Therefore, (1.3) can be put in the form of

$$u_t + uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.4}$$

where ε is a small positive perturbation parameter ($0 < \varepsilon \ll 1$), see [48]. When the diffusion u_{xx} and the dissipation u_{xxxx} terms vanish, (1.4) is reduced to the classical KdV equation, and so (1.4) is called perturbed KdV equation. However, the weakly dissipative terms broke the integrable structure of the original model, causing more challenge to analyze. For the weakly dissipative PDE models, many works were concerned on the existence of solitary and kink waves and their stability, such as solitary waves in perturbed generalized BBM equation [12] and perturbed KdV equations [19, 46, 48], kink waves in perturbed defocing KdV equation [13] and perturbed Sine-Gordon equation [18]. There exist some open problems, for example, can the solitary (kink) and periodic waves coexist in the dissipative equation? What is the amplitude of the coexisting periodic wave? Moreover, how many periodic waves with different amplitudes can coexist? And what are the exact parameter conditions for these physical phenomena to occur? These open problems arise from all kinds of PDEs with weakly multiple dissipations. In the second part of this thesis, we focus on the above problems for a perturbed generalized BBM equation with degeneracy and a cubic-quintic nonlinear Schrödinger equation with multiple dissipations. The main mathematical tool used is the Abelian integral theory, which will be introduced in next sections.

1.2 Abelian integrals

In mathematics, an Abelian integral, named after the Norwegian mathematician Niels Henrik Abel, is an integral of an algebraic function in the form,

$$\int_{x_0}^{x_1} \mathcal{R}(w, x)dx, \tag{1.5}$$

where $\mathcal{R}(w, x)$ is some rational function in variables w and x that are related by an algebraic equation

$$F(w, x) = a_0(x)w^n + a_1(x)w^{n-1} + \cdots + a_n(x) = 0,$$

where the coefficients $a_i(x)$ are polynomials in x, $i = 0, 1, \cdots, n$. In complex geometry, the polynomial $F(w, x)$ defines a compact Riemann surface F, which is an n-sheeted covering of the Riemann sphere. Abelian integral provides a way mapping an algebraic curve into Abelian varieties. Multi-dimensional generalizations of the theory of Abelian integrals form the subject matter of algebraic geometry and the theory of complex manifolds in modern mathematics. An extended version of Abelian integral, as a continuous function, is an integral of a polynomial differential 1-form,

$$\omega = Q(x, y)dx - P(x, y)dy \tag{1.6}$$

over the real ovals of a polynomial Hamiltonian $H(x, y)$, given by

$$\mathcal{A}(h) = \oint_{\Gamma_h} Q(x, y)dx - P(x, y)dy, \quad h \in J, \tag{1.7}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials of degree $n \geq 2$, $H(x, y)$ is a polynomial of degree $m + 1$ and has at least one family of ovals $\{\Gamma_h\}$, which are parameterized by $\{(x, y) : H(x, y) = h,\ h \in J\}$ where J is an open interval. An oval is a smooth simple closed curve which is free of critical points of $H(x, y)$. The endpoints of J correspond to the critical points or a non-simple closed curve in the level set of $H(x, y)$. $\mathcal{A}(h)$ is usually a multivalued function on J, in the sense that there might exist several ovals lying on the same level set $\{(x, y) : H(x, y) = h, h \in J\}$. The extended version of Abelian integral is connected to the prominent mathematician David Hilbert and his 16th Problem. We recall that Hilbert's 16th problem [34] considers the maximal number of limit cycles and their distributions in two-dimensional polynomial systems,

$$\dot{x} = F(x, y), \quad \dot{y} = G(x, y), \tag{1.8}$$

where $F(x, y)$ and $G(x, y)$ are polynomials with $\max(\deg(F), \deg(G)) = n$. Let $\mathbb{H}(n)$ denote the maximal number of limit cycles of system (1.8). This problem is still not completely solved even for quadratic polynomial systems (i.e., for the simplest case n = 2). Mathematicians have dedicated a lot to this open problem for more than one century. However, we only know that $\mathbb{H}(n)$ is finite up to now, see the relatively new monograph [60]. Many theories and methodologies have been developed for solving the problem, and a lot of good results on the lower bounds for $\mathbb{H}(n)$ have been obtained, such as $\mathbb{H}(2) \geq 2$, $\mathbb{H}(3) \geq 13$ and $\mathbb{H}(4) \geq 28$, see recent new publications [2, 49]. In order to overcome the difficulty in solving the Hilbert's 16th problem, researchers have tried to study the relative weakened problems or weaker versions

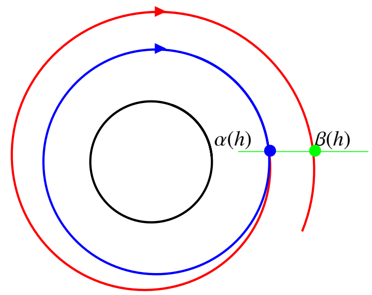

Figure 1.1: Poincaré map of system (1.9)

of the problem, for example, studying limit cycles arising from certain special bifurcations, or focusing on systems with simper forms.

Anorld's version of the Hilbert's 16th problem [3] connects the Abelian integral given in (1.7) to studying limit cycles in the following special form of system (1.8),

$$\dot{x} = H_y(x, y) + \varepsilon P(x, y), \qquad \dot{y} = -H_x(x, y) + \varepsilon Q(x, y), \tag{1.9}$$

where $P(x, y)$, $Q(x, y)$ and $H(x, y)$ are given in (1.7), $\varepsilon > 0$ is sufficiently small. The ovals $\Gamma_h = \{(x, y) : H(x, y) = h,\ h \in J\}$ are periodic orbits of system $(1.9)_{\varepsilon=0}$. Such a family of periodic orbits is called a periodic annulus $\{\Gamma_h\}$. The endpoints of J correspond to the boundaries of periodic annulus $\{\Gamma_h\}$, which may be centers, saddle loops, polycycle or infinity in dynamical system language. System (1.9) is called a perturbed Hamiltonian system or a near-hamiltonian system, and it is, in geometric language, a deformation of a polynomial Hamiltonian with a polynomial differential 1-form ω given in (1.6),

$$dH(x, y) + \varepsilon\omega.$$

Under perturbation ($\varepsilon \neq 0$), most periodic orbits of system $(1.9)_{\varepsilon=0}$ are broken and only a finite number of periodic orbits persist as isolated closed orbits (**limit cycles**) of system (1.9). This is usually called **Pioncaré bifurcation** or limit cycles bifurcating from periodic annulus. The tools to study the limit cycles via Pioncaré bifurcation in system (1.9) are described below.

Definition 1.2.1 *Let* $(\alpha(h), 0)$ *denote the intersection point of* Γ_h *with the positive x-axis,* $\Gamma_{h,\varepsilon}$ *be the positive orbit of* (1.9) *starting from the point* $(\alpha(h), 0)$ *at time* $t = 0$*, and* $(\beta(h, \varepsilon), 0)$ *the first intersection point of* $\Gamma_{h,\varepsilon}$ *with the positive x-axis at time* $t = t^*(\varepsilon)$*. The* **Pioncaré map** *is defined by* $\mathbb{P}_\varepsilon : \alpha(h) \to \beta(h)$ *as the first return map of the solutions of* (1.9)*, see Figure 1.1. The corresponding difference is the* **displacement function** *defined by*

$$\Delta(h) = \beta(h) - \alpha(h), \quad h \in J.$$

The displacement function has a representation as a power series in the variable ε:

$$\Delta(h) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \cdots,$$

which is convergent for small ε. $M_k(h)$ is called the kth-order Pioncaré-Pontryagin function or Melnikov function of order k. The relationship between $\mathcal{A}(h)$, $M_1(h)$ and the number of limit cycles of system (1.9) was discussed in many monographs, see [15, 27, 31, 60]. We recommend the relatively new one [60] for reference.

Theorem 1.2.2 (Poincaré-Pontryagin-Andronov Theorem) *The following statements hold:*

(I)
$$M_1(h) = \mathcal{A}(h) \quad \text{for } h \in J.$$

(II) *If there exists $h^* \in J$ such that $\mathcal{A}(h^*) = 0$ and $\mathcal{A}'(h^*) \neq 0$, then system* (1.9) *has a unique hyperbolic limit cycle $\Gamma_{h^*,\varepsilon}$ bifurcating from Γ_{h^*} such that $\Gamma_{h^*,\varepsilon} \to \Gamma_{h^*}$ as $\varepsilon \to 0$.*

(III) *If $\mathcal{A}(h^*) = \mathcal{A}'(h^*) = \mathcal{A}''(h^*) = \cdots = \mathcal{A}^{(k-1)}(h^*) = 0$, and $\mathcal{A}^{(k)}(h^*) \neq 0$, then system* (1.9) *has at most k limit cycles bifurcating from Γ_{h^*}.*

(IV) *The total number (counting multiplicity) of the limit cycles of system* (1.9) *bifurcating from the annulus $\{\Gamma_h\}$ is bounded by the maximum number of isolated zeros (taking into account their multiplicities) of the Abelian integral $\mathcal{A}(h)$ for $h \in J$.*

Therefore, the zeros of $\mathcal{A}(h)$ provide the information on the persisting limit cycles of system (1.9) in the sense of the first order Poincaré bifuraction when ε is sufficiently small. Studying the maximal number of zeros of $\mathcal{A}(h)$ denoted by $\mathbb{Z}(m, n)$ is so-called the **weak Hilbert's 16th problem** or Anorld-Hilbert's 16th problem [3, 60], which has produced most of results on the study of the Hilbert's 16th problem. We note that $\mathbb{Z}(n + 1, n)$ gives a lower bound of $\mathbb{H}(n)$. However, the weak version of the Hilbert's 16th problem is still very difficult to solve, and so far only the case $(m, n) = (3, 2)$ has been completely solved, see [14] and references therein. Note that the maximum number $\mathbb{Z}(m, n)$ must be a uniform bound with respect to all possible Hamiltonian $H(x, y)$ with all possible families of ovals Γ_h and arbitrary polynomial 1-forms ω. Khovansky and Varchenko [38, 61] proved that $\mathbb{Z}(m, n)$ is bounded by a constant, however, gave no information and clues on $\mathbb{Z}(m, n)$.

Definition 1.2.3 *The maximal number (counting multiplicity) of the limit cycles of system* (1.9) *bifurcating from the annulus $\{\Gamma_h\}$ for small ε is called* **annulus cyclicity** *of* $\{\Gamma_h\}$.

Note that the claim (IV) in Theorem 1.2.2 implies that the annulus cyclicity of $\{\Gamma_h\}$ is exactly $\mathbb{Z}(m, n)$. It should be also noted that the proof of Theorem 1.2.2 is based on the application of the Implicit Function Theorem to the displacement map. Therefore, the displacement map should be analytic. Hence, one should assume that the parameter space and the annulus are compact. Therefore, it is not confusing that the claim (IV) holds for $h \in [h_1 + \sigma, h_2 - \sigma] \subset J := (h_1, h_2)$, where σ is sufficiently small. As $\mathbb{Z}(m, n) < +\infty$ [38, 61] and this total number is the uniform bound, we take the maximum of this number as $\sigma \to 0$, then get the cyclicity $\mathbb{Z}(m, n)$ of the period annulus $\{\Gamma_h\}$. The number $\mathbb{Z}(m, n)$ can include the number of limit cycles bifurcating from the elementary center and the hyperbolic saddle homoclinic loop, because the displacement function is analytic at these two kinds of boundaries, while it cannot be extended to any other kinds of polycycles, because it is unknown if the displacement map is analytic on these boundaries, see [60].

Abelian integral is a very efficient and powerful mathematical tool for studying limit cycles in certain subclasses of system (1.9). We choose Abelian integral as a tool in this thesis to study the limit cycles and establish the existence of traveling waves. There are several classical methods for studying the zeros of Abelian integral (1.7) associated with system (1.9) under certain assumptions.

1.3 Zeros of the Abelian integral near the boundaries of $\{\Gamma_h\}$

One efficient method to detect the zeros of $\mathcal{A}(h)$ is to study the asymptotic expansions of $\mathcal{A}(h)$ on the boundaries of one periodic annulus $\{\Gamma_h\}$. As stated above, the boundaries of $\{\Gamma_h\}$ can be an elementary center, nilpotent center, homoclinic loop passing through a (hyperbolic or nilpotent) saddle, or a cusp, heteroclinic loop or polycycle connecting (hyperbolic or nilpotent) saddles, cusps, or a more degenerate singularity. Suppose that the orientation of the orbits of (1.9) is clockwise, and h_1 and h_2 are the left and right endpoints of J, respectively. Then the Hamiltonian $H(x, y)$ at h_1 and h_2 define the inner and outer boundaries of $\{\Gamma_h\}$, given by $\Gamma_{h_1} = \{(x, y)|H(x, y) = h_1\}$ and $\Gamma_{h_2} = \{(x, y)|H(x, y) = h_2\}$, respectively.

Suppose that Γ_{h_2} is a homoclinic loop passing through a hyperbolic saddle at the origin. Without loss of generality, we may further assume the polynomial Hamiltonian has the normal form,

$$H(x, y) = h_2 + \frac{\lambda}{2}(y^2 - x^2) + \sum_{i+j\geq 3} h_{ij}x^i y^j, \quad \lambda > 0.$$

Roussarie [51] proved that

$$\mathcal{A}(h) = \sum_{j\geq 0}(c_{2j}(h - h_2)^j + c_{2j+1}(h - h_2)^{j+1} \ln(|h - h_2|), \tag{1.10}$$

for $0 < -(h - h_2) \ll 1$, with the formulas for the first four coefficients c_i ($i = 0, 1, 2, 3$) given in [29], and the fifth one was only obtained under a very strict assumption [58]. We note that the coefficients c_{2j+1} are called local coefficients because they are some local quantities near the saddle, not related to the global homoclinic loop, see [29]. We will use the symbols c_i to denote the coefficients of the asymptotic expansions of $\mathcal{A}(h)$ near all kinds of boundaries, such as homoclinic loops and heteroclinic loops. It should be noted that they have different formulas for different homoclinic loops and heteroclinic loops.

When Γ_{h_2} is a homoclinic loop passing through a nilpotent cusp at the origin, the polynomial Hamiltonian can be reduced to the normal form by some linear transformation,

$$H(x, y) = h_2 + \frac{y^2}{2} + \sum_{i+j\geq 3} h_{ij}x^i y^j, \quad h_{30} < 0.$$

Han *et al.* [33] proved that $\mathcal{A}(h)$ has the following asymptotic expansions,

$$\mathcal{A}(h) = c_0 + B_{00}c_1|h - h_2|^{\frac{5}{6}} + (c_2 + bc_1)(h - h_2) + B_{10}c_3|h - h_2|^{\frac{7}{6}} - \frac{1}{11}B_{00}c_4|h - h_2|^{\frac{11}{6}} + O((h - h_2)^2), \tag{1.11}$$

for $0 \leq -(h-h_2) \ll 1$, and

$$\mathcal{A}(h) = c_0 + B_{00}^* c_1 (h-h_2)^{\frac{5}{6}} + (c_2 + b^* c_1)(h-h_2) + B_{10}^* c_3 (h-h_2)^{\frac{7}{6}} + \frac{1}{11} B_{00}^* c_4 (h-h_2)^{\frac{11}{6}} + O((h-h_2)^2), \tag{1.12}$$

for $0 \leq h-h_2 \ll 1$, where b, b^*, B_{00}, B_{00}^*, B_{10} and B_{10}^* are some constants. The formulas for the first five coefficients c_j ($j = 0, 1, 2, 3, 4$) are obtained by the methods developed in [33]. We note that c_1, c_3 and c_4 are local coefficients.

For a homoclinic loop Γ_{h_2} passinng through a nilpotent saddle at the origin, the polynomial Hamiltonian has the normal form,

$$H(x,y) = h_2 + \frac{y^2}{2} + \sum_{i+j\geq 3} h_{ij} x^i y^j,$$

with $h_{30} = 0$ and $h_{40} < \frac{h_{21}^2}{2}$. Han *et al.* [30] obtained the asymptotic expansion of $\mathcal{A}(h)$ and the exact formulas for the first six coefficients,

$$\begin{aligned} \mathcal{A}(h) = \;& c_0 + c_1|h-h_2|^{\frac{3}{4}} + c_2(h-h_2)\ln|h-h_2| + c_3(h-h_2) + c_4|h-h_2|^{\frac{5}{4}} + c_5|h-h_2|^{\frac{7}{4}} \\ & + c_6(h-h_2)^2\ln|h-h_2| + O((h-h_2)^2), \end{aligned} \tag{1.13}$$

for $0 < -(h-h_2) \ll 1$. In the expansion, c_1, c_2, c_4, c_5 and c_6 are local coefficients.

When the outer boundary of $\{\Gamma_h\}$ is a heteroclinic loop Γ_{h_2} connecting a hyperbolic saddle and a nilpotent cusp, Sun *et al.* [55] gave the asymptotic expansion of $\mathcal{A}(h)$ near Γ_{h_2} and provided a method to compute the coefficients,

$$\begin{aligned} \mathcal{A}(h) = \;& c_0 + c_1|h-h_2|^{\frac{5}{6}} + c_2(h-h_2)\ln|h-h_2| + c_3(h-h_2) + c_4|h-h_2|^{\frac{7}{6}} \\ & + c_5|h-h_2|^{\frac{11}{6}} + c_6(h-h_2)^2\ln|h-h_2| + O((h-h_2)^2), \end{aligned} \tag{1.14}$$

for $0 < -(h-h_2) \ll 1$. For a heteroclinic loop Γ_{h_2} connecting a hyperbolic saddle and a nilpotent saddle, the asymptotic expansion of $\mathcal{A}(h)$ near Γ_{h_2} was first investigated by Asheghi *et al.* [6]. The idea of computing the local coefficients is based on the method given in [70],

$$\begin{aligned} \mathcal{A}(h) \;=\;& c_0 + c_1|h-h_2|^{\frac{3}{4}} + c_2(h-h_2)\ln|h-h_2| + c_3(\delta)(h-h_2) \\ & + c_4|h-h_2|^{\frac{5}{4}} + c_5|h-h_2|^{\frac{7}{4}} + c_6(h-h_2)^2\ln|h-h_2| + O((h-h_2)^2), \end{aligned} \tag{1.15}$$

for $0 < -(h-h_2) \ll 1$.

The inner boundary of $\{\Gamma_h\}$ is usually an elementary center or a nilpotent center. $\mathcal{A}(h)$ has the following expansion near an elementary center, see [32],

$$\mathcal{A}(h) = \sum_{i\geq 0} b_i (h-h_1)^{i+1}, \tag{1.16}$$

for $0 < h-h_1 \ll 1$ and the coefficients b_i can be obtained by using the Maple program given in [32]. For the asymptotic expansions of $\mathcal{A}(h)$ near other inner and outer boundaries, a survey paper is referred [26].

We note that the coefficients in the asymptotic expansions of $\mathcal{A}(h)$ are some linear functions of the coefficients of $P(x,y)$ and $Q(x,y)$. For convenience, we denote by $\eta \in \mathbb{R}^N$ the vector composed of the coefficients of $P(x, y)$ and $Q(x, y)$, and we use the symbol $\mathcal{A}(h, \eta)$ when ever it is needed. The asymptotic expansions of $\mathcal{A}(h)$ with its coefficients can be utilized for identifying zeros of $\mathcal{A}(h)$ near the boundaries. We present a criterion to study the zeros near the boundaries for a periodic annulus $\{\Gamma_h\}$, which has an elementary center and a saddle homoclinic loop to be the inner and outer boundaries, respectively.

Theorem 1.3.1 *Consider system* (1.9) *and the asymptotic expansions* (1.10) *and* (1.16) *for the Abelian integral given in* (1.7). *If there exists an* $\eta_0 \in \mathbb{R}^N$ *satisfying*

$$\begin{aligned} b_0(\eta_0) = b_1(\eta_0) = \cdots = b_{k-1}(\eta_0) = 0,\ b_k(\eta_0) \neq 0, \\ c_0(\eta_0) = c_1(\eta_0) = \cdots = c_{l-1}(\eta_0) = 0,\ c_l(\eta_0) \neq 0, \end{aligned} \tag{1.17}$$

and

$$\operatorname{rank}\left[\frac{\partial(b_0, b_1, \cdots, b_{k-1}, c_0, c_1, \cdots, c_{l-1})}{\partial \eta}\right] = k + l, \tag{1.18}$$

then $\mathcal{A}(h, \eta)$ *can have* $k + l + \frac{1-\text{sgn}(\mathcal{A}(h_1+\varepsilon_1,\eta_0)\mathcal{A}(h_2-\varepsilon_2,\eta_0))}{2}$ *zeros for some* η *near* η_0, k *zeros of which near* $h = h_1$ *in* $(h_1, h_1 + \varepsilon_1)$, l *zeros near* $h = h_2$ *in* $(h_2 - \varepsilon_2, h_2)$, $\frac{1-\text{sgn}(\mathcal{A}(h_1+\varepsilon_1,\eta_0)\mathcal{A}(h_2-\varepsilon_2,\eta_0))}{2}$ *zero in* $(h_1 + \varepsilon_1, h_2 - \varepsilon_2)$, *where* ε_1 *and* ε_2 *are some sufficiently small parameters. Therefore, system* (1.9) *can have* $k + l + \frac{1-\text{sgn}(\mathcal{A}(h_1+\varepsilon_1,\eta_0)\mathcal{A}(h_2-\varepsilon_2,\eta_0))}{2}$ *limit cycles for some* (ε, η) *near* $(0, \eta_0)$, k *limit cycles of which near the center* Γ_{h_1}, l *limit cycles of which are near the homoclinic loop* Γ_{h_2}, *and* $\frac{1-\text{sgn}(\mathcal{A}(h_1+\varepsilon_1,\eta_0)\mathcal{A}(h_2-\varepsilon_2,\eta_0))}{2}$ *limit cycle is inside the* l *limit cycles, surrounding the* k *limit cycles.*

Theorem 1.3.1 is a different version of bifurcation theorem firstly proposed in [28]. A simpler proof of an equivalent theorem is given in Chapter 6 of this thesis. Many good results on the number of limit cycles for certain systems of (1.9) have been obtained by investigating the coefficients of the asymptotic expansions of $\mathcal{A}(h)$ via a variation or generalization of Theorem 1.3.1, see [26, 58].

1.4 Bounding the number of zeros of $\mathcal{A}(h)$

It is rather difficult to identify the exact bound of $\mathbb{Z}(m, n)$. Researchers have tried to get some smaller upper bounds for approximating $\mathbb{Z}(m, n)$. In 2010, a historical result was reported by Binyamini, Novikov and Yakovenko [10],

$$\mathbb{Z}(n + 1, n) \leq 2^{2^{Poly(n)}},$$

where $Poly(n)$ represents an explicit, polynomially growing term with the exponent not exceeding 61. However, this upper bound is believed to be too much larger than the real ones. There are some methods to bound the number of zeros of Abelian integrals: the method based on Picárd-Fuchs equation [20, 21, 22, 23], the method based on the Argument Principle [15], the averaging method [11], the method by using Chebyshev property [25, 45], and the method based on complexification of the Abelian integrals [44]. However, each method is only applicable to system (1.7) with certain strict assumptions.

Picárd-Fuchs equation method is applicable to the following special system of (1.9), with $P = 0$, $Q = f(x)y$ and the Hamiltonian $H(x, y) = \frac{y^2}{2} + \int g(x)dx$,

$$\dot{x} = y, \qquad \dot{y} = -g(x) + \varepsilon f(x)y, \tag{1.19}$$

which is a generalized polynomial Liénard system of (1.2) with a weak damping term $\varepsilon f(x)y$. The related Abelian integral is given in the simper form,

$$\mathbb{I}(h) = \oint_{\Gamma_h} f(x)ydx = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \cdots + \alpha_n I_n(h),$$

where $f(x) = \sum_{i=0}^{n} \alpha_i x^i$, and

$$I_i(h) = \oint_{\Gamma_h} x^i ydx, \quad i = 0, 1, \cdots, n, \tag{1.20}$$

which are elementary Abelian integrals, to generate the full Abelian integral $\mathbb{I}(h)$. They are called generating elements of $\mathbb{I}(h)$. System (1.19) has a simple form, however it still plays a very important role in studying limit cycles of general planar systems obtained from modifying (1.19), see [40, 49]. Further, system (1.19) can be applied to model real world oscillating phenomena, see [17].

We note that studying the bound on the number of zeros of $\mathbb{I}(h)$ can be regarded as a combination of Smale's 13th problem [54] and Anorld's weak version. In fact, Smale's 13th problem is another weaker version of the Hilbert's 16th problem, which restricts the Hilbert's 16th problem to the classical polynomial Liénard system,

$$\ddot{x} + f(x)\dot{x} + x = 0,$$

where $f(x)$ is a polynomial of degree n, see [43] for the recent progress reports. Suppose that the polynomials $g(x)$ and $f(x)$ in (1.19) have the degrees m and n, system (1.19) is usually called type (m, n). Let $\mathbb{Z}_L(m, n)$ denote the maximal number of zeros of $\mathbb{I}(h)$ for (1.19) of type (m, n), where L represents Liénard system. There exist few methods to identify $\mathbb{Z}_L(m, n)$. Most interests are focused on $\mathbb{Z}_L(m, n)$ with smaller n, such as $n = 2, 3$.

If $\mathbb{I}(h)$ can be decoupled, the Picárd-Fuchs equation for $I_i(h)$ can be constructed and given by

$$G(h)\frac{d}{dh}\begin{pmatrix} I_0(h) \\ I_1(h) \\ \vdots \\ I_n(h) \end{pmatrix} = \mathbb{M}\begin{pmatrix} I_0(h) \\ I_1(h) \\ \vdots \\ I_n(h) \end{pmatrix}, \tag{1.21}$$

where $G(h)$ is a polynomial with a degree less than n, $\mathbb{M}$ is an $(n+1)\times(n+1)$ matrix depending on the Hamiltonian. Based on Picárd-Fuchs equation, the Riccati equation and a related geometric curve can be constructed. The core problem is to determine the intersection of a line and a related curve on bounding the zeros of $\mathbb{I}(h)$. This idea is inspired from algebraic geometry theory. It has been successfully applied by Dumortier and Li [20, 21, 22, 23] to identify $\mathbb{Z}_L(3, 2)$ for system (1.19) of degree 3, and the exact values of $\mathbb{Z}_L(3, 2)$ for five different topological portraits were obtained. It was proved that two is the sharp bound on $\mathbb{Z}_L(5, 4)$ if the quintic symmetric system (1.19) has a unique periodic annulus bounded by a heteroclinic loop [4, 5, 73]. For these systems, the related Abelian integral $\mathbb{I}(h)$ has three generating elements and the dimension of Picárd-Fuchs equations is three, which make it feasible to conduct the geometric analysis. However, it becomes much more difficult when system (1.19) is non-symmetric or has degree equal or larger than 4, implying that $\mathbb{I}(h)$ has more than three generating elements and the dimensions of the Picárd-Fuchs equation system and Ricatti equations are higher, which make it intractable in determining the intersection of the related planes and surfaces. To the best of our knowledge, very fewer results have been reported for $\mathbb{I}(h)$ with higher dimensions by Picárd-fuchs equation method.

It has been shown that the Chybeshev criterion proposed in [25, 45] can be applied to bound $\mathbb{Z}_L(m, n)$ for Abelian integrals with more than three generating elements. Chebyshev criterion is a generalized version of linear independence of analytic functions, see [25, 45].

Definition 1.4.1 *Suppose $s_0(x), s_1(x), \cdots$ and $s_{n-1}(x)$ are analytic functions on a real open interval Ω.*

(A) *The continuous Wronskian of $\{s_0(x), s_1(x), \ldots, s_{i-1}(x)\}$ for $x \in \Omega$ is*

$$W[s_0(x), s_1(x), \ldots, s_{i-1}(x)] = \begin{vmatrix} s_0(x) & s_1(x) & \cdots & s_{i-1}(x) \\ s_0'(x) & s_1'(x) & \cdots & s_{i-1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ s_0^{(i-1)}(x) & s_1^{(i-1)}(x) & \cdots & s_{i-1}^{(i-1)}(x) \end{vmatrix},$$

where $s_i^{(j)}(x)$ is the jth order derivative of $s_i(x)$, $j \geq 2$.

(B) *The set $\{s_0(x), s_1(x), \ldots, s_{n-1}(x)\}$ is called a Chebyshev system if any nontrivial linear combination,*

$$\kappa_0 s_0(x) + \kappa_1 s_1(x) + \cdots + \kappa_{n-1} s_{n-1}(x),$$

has at most $n-1$ isolated zeros on Ω. Note that $W[s_0(x), s_1(x), \ldots, s_{n-1}(x)] \neq 0$ is one sufficient condition assuring $\{s_0(x), s_1(x), \ldots, s_{n-1}(x)\}$ forms a Chebyshev system.

(C) *The ordered set $\{s_0(x), s_1(x), \ldots, s_{n-1}(x)\}$ is called extended complete Chebyshev system (*ECT*-system) if for each $i \in \{1, 2, \cdots, n\}$ any nontrivial linear combination,*

$$\kappa_0 s_0(x) + \kappa_1 s_1(x) + \cdots + \kappa_{i-1} s_{i-1}(x),$$

has at most $i-1$ zeros with multiplicities counted.

Let the Hamiltonian for the unperturbed system (1.19),

$$H(x, y) = \frac{y^2}{2} + \int g(x)dx := \frac{y^2}{2} + V(x),$$

satisfy $xV'(x) > 0$ and $V(0) = 0$. There exists a family of closed ovals $\{\Gamma_h\} \subseteq \{(x, y) | H(x, y) = h,\ h \in J\}$ surrounding the origin $(0, 0)$, where $J = (0, h^*)$ and $h^* = H(\partial\{\Gamma_h\})$. The projection of $\{\Gamma_h\}$ on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$. There is an analytic involution $z = z(x)$ for all $x \in (x_l, x_r)$ defined by

$$V(x) = V(z(x)). \tag{1.22}$$

Let

$$\mathbb{I}_i(h) = \oint_{\Gamma_h} \xi_i(x) y^{2n^*-1} dx \ \text{ for } h \in (0, h^*), \tag{1.23}$$

where $n^* \in \mathbb{N}$ and $\xi_i(x)$ is an analytic function on (x_l, x_r), $(i = 0, 1, \ldots, n-1)$. Further, define

$$s_i(x) := \frac{\xi_i(x)}{V'(x)} - \frac{\xi_i(z(x))}{V'(z(x))}. \tag{1.24}$$

Then we have

Theorem 1.4.2 ([25]) *Consider the integrals* (1.23) *and the functions* (1.24). $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{n-1}\}$ *is an* ECT *system on* $(0, h^*)$ *if* $n^* > n-2$ *and* $\{s_0, s_1, \cdots, s_{n-1}\}$ *is an* ECT *system on* $(x_l, 0)$ *or* $(0, x_r)$.

Theorem 1.4.3 ([45]) *Consider the integrals* (1.23) *and the functions* (1.24). *If the following conditions hold:*
(a) $W[s_0, s_1, \ldots, s_i]$ *does not vanish on* $(0, x_r)$ *for* $i = 0, 1, \cdots, n-2$*;*
(b) $W[s_0, s_1, \ldots, s_{n-1}]$ *has* k *zeros on* $(0, x_r)$ *with multiplicities counted; and*
(c) $n^* > n + k - 2$,
then any nontrivial linear combination of $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{n-1}\}$ *has at most* $n + k - 1$ *zeros on* $(0, h^*)$ *with multiplicities counted. In this case, we call* $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{n-1}\}$ *a Chebyshev system with accuracy* k *on* $(0, h^*)$.

Theorems 1.4.2 and 1.4.3 have been applied to bound $\mathbb{Z}_L(4,3)$ for type $(4,3)$ system (1.19), see [7, 56, 57, 62, 63, 64, 74]. However, only an upper bound of $\mathbb{Z}_L(4,3)$ was obtained for each system investigated in these papers. It is still unknown what is the exact bound of $\mathbb{Z}_L(4,3)$. On the other hand, it contains an algebraic problem on bounding $\mathbb{Z}_L(4,3)$ by applying Chebyshev criteria. The algebraic analysis is based on some symbolic computation for larger semi-algebraic systems, for which it is difficult to conduct real root classification. Therefore, only the Abelian integral $\mathbb{I}(h)$ with four generating elements have been analyzed, even though Chebyshev criteria may be applicable theoretically for $\mathbb{I}(h)$ with arbitrary generating elements. Efficient computation methods need to be developed or combining other techniques with the Chebyshev criteria to identify $\mathbb{Z}_L(4,3)$.

Research interests have focused on Abelian integrals with two generating elements for system (1.19) in the form of

$$\mathbb{I}(h) = a_0 I_0(h) + a_1 I_1(h),$$

and

$$\mathbb{I}(h) = a_0 \oint_{\Gamma_h} f_1(x) y dx + a_1 \oint_{\Gamma_h} f_2(x) y dx,$$

where $f_1(x)$ and $f_2(x)$ are polynomials in x without parameters. If $\mathbb{I}(h)$ has at most one zero, in other words, the two generating elements form a Chebyshev system, then the ratios (if they are well defined) of two generating elements,

$$\mathbb{R}_1(h) = \frac{I_1(h)}{I_0(h)} \tag{1.25}$$

and

$$\mathbb{R}_2(h) = \frac{\oint_{\Gamma_h} f_2(x) y dx}{\oint_{\Gamma_h} f_1(x) y dx}, \tag{1.26}$$

are monotonic. In 1996, Li and Zhang [41] gave a criterion to determine the monotonicity of the ratio $\mathbb{R}_2(h)$, which is certainly applicable to the ratio $\mathbb{R}_1(h)$. Sixteen years later, a new criterion was developed in [42] to determine the monotonicity of the ratio $\mathbb{R}_1(h)$.

Theorem 1.4.4 ([42]) *Consider the Abelian integral $I_1(h)$ and $I_0(h)$ for system* (1.19)*. If*

$$x(h) + z(h)$$

is monotonic on $h \in J$, then the ratio $\mathbb{R}_1(h)$ is monotonic on J, where $z(x)$ is the involution defined by (1.22) *satisfying $V(x(h)) = V(z(h)) = h$.*

It seems that Theorem 1.4.4 is much easier to be used compared to other methods such as Chebyshev criterion [25, 45] and the direct analysis [41]. However, the problem becomes hard when $H(x, y)$ contains a parameter, because the analysis needs root classification for a two-dimensional parametric-semi-algebraic system, which is usually quite difficult. Therefore, it needs to combine efficient symbolic computation with the monotonicity analysis.

1.5 Abelian integral method applied to PDEs

As discussed above, it is more realistic to investigate perturbed models when PDEs are used to model real world problems. One technique to deal with the perturbation problem is to reduce the PDE to a singularly perturbed ODE (including higher dimension Hamiltonian systems) by introducing wave transform and successive derivatives, for example, see [12, 13, 18, 19, 46, 48] and references therein. In these works the Fenichel's criterion [24] is applied to assure the existence of the invariant manifold, and then the problem is reduced to a problem with regular perturbation on the manifold. Then the vanishment of the Abelian integral (usually called Melnikov integral) along the homoclinic (heteroclinic) loop assures the existence of solitary (kink) wave. This technique has been successfully established for solving the existence of solitary and kink waves. For example, the perturbed KdV equation (1.4) is reduced to the following regular perturbation problem on the related slow manifold,

$$\dot{u} = y, \quad \dot{y} = u(1 - u) + \varepsilon(\alpha_0 + \alpha_1 u)y,$$

where α_0 and α_2 are expressed in terms of the original equation coefficients. The condition assuring the vanishment of the Abelian integral along the homoclinic loop establishes the existence of a solitary wave. However, the Abelian integral is not extended on the whole integrable structure, but was only estimated along the homoclinic loop or heteroclinic loop, see [12, 13, 18, 19, 46, 48]. We construct the Abelian integral based on the whole periodic structure of the Hamiltonian for the regular perturbation problem on the critical manifold. This allows us to further examine the unknown problems on the periodic traveling waves and their coexistence with solitary and kink waves for the weakly dissipative equations. This methodology and Abelian integral theory will be fully utilized to establish the existence of periodic waves for two kinds of dissipative PDEs in Chapters 5 and 6.

1.6 Contributions and outline of the thesis

In this thesis, we study limit cycles and traveling waves in some nonlinear ODEs and PDEs by Abelian integral method. There are three main contributions. (1) We propose a combination technique with the Chebyshev criterion to deal with a parametric set of Abelian integrals. With this new technique, we obtain the exact bound on the number of limit cycles bifurcating from the periodic annuli of three kinds of Hamiltonian systems, which solves three open questions in the area. (2) We introduce the boundary polynomial theory in computer algebra to analyze the monotonicity of ratios of two Abelian integrals, and prove that there exists at most one limit cycle bifurcating from the periodic annulus surrounding a nilpotent center. The result gives a positive answer to the two conjectures in [65]. (3) The third contribution is the extension of applying Abelian integral theory to solve some PDEs. The existence and coexistence problems on solitary, kink and periodic waves are solved by analyzing the Abelian integrals in the frame work of singular perturbation.

The outline of this thesis is described below.

In Chapter 2, we study the exact bound on the number of limit cycles bifurcating the periodic annuli for two hyperelliptic Hamiltonian systems of degree four. The Abelian integrals are analyzed by the combination technique developed in this thesis and the Chebyshev criterion. We provide a rigorous proof to show that the exact bound is three for both systems.

Chapter 3 is concerned on the cyclicity of the periodic annulus in a quintic Hamiltonian system. The undamped system is hyperelliptic, non-symmetric with a degenerate heteroclinic loop, which connects a hyperbolic saddle to a nilpotent saddle. The annulus cyclicity is obtained, which is three, by studying the associated Abelian integral with the combination technique. This result provides a positive answer to the open question whether the annulus cyclicity is three or four. For completeness, the Hopf cyclicity is also derived for the smooth and non-smooth damping terms. When the smooth polynomial damping term has degree n, we first introduce a transformation based on the involution of the Hamiltonian, and then analyze the coefficients involved in the bifurcation function to show that the Hopf cyclicity is $\left[\frac{2n+1}{3}\right]$. Further, for piecewise smooth polynomial damping with a switching manifold on the y-axis, we consider the damping terms to have degrees l and n, respectively, and prove that the Hopf cyclicity of the origin is $\left[\frac{3l+2n+4}{3}\right]$ ($\left[\frac{3n+2l+4}{3}\right]$) when $l \geq n$ ($n \geq l$).

In Chapter 4, we study the monotonicity of the ratio of the Abelian integrals, $\oint_{\gamma_i(h)} y dx$ and $\oint_{\gamma_i(h)} xy dx$, in an interval, where $i = 1, 2$, and $\gamma_i(h)$ is a compact component of some hyperelliptic curves with genus two. The monotonicity implies that there exists at most one limit cycle bifurcating from a periodic annulus surrounding a nilpotent center. Our results give positive answers to the two conjectures proposed by Wang *et al.* [65].

In Chapter 5, we investigate a generalized BBM equation with weak backward diffusion, dissipation and Marangoni effects. The analysis is reduced to a regular perturbation problem on a critical manifold, and the Abelian integral is constructed on the whole periodic structure. Main attention is focused on periodic and solitary waves on a manifold via studying the number of zeros of some linear combination of Abelian integrals. The uniqueness of the periodic waves is established when the equation contains one coefficient in backward diffusion and dissipation terms, by showing that the Abelian integrals form a Chebyshev set. The monotonicity of the wave speed is proved, and moreover the upper and lower bounds of the limiting wave speeds are obtained. Especially, when the equation involves Marangoni effect due to imposed weak thermal gradients, it is shown that at most two periodic waves can exist. The exact conditions are obtained for the existence of one and two periodic waves as well as for the coexistence of one solitary and one periodic waves. The analysis is mainly based on the Chebyshev criterion and asymptotic expansion of Abelian integrals near the solitary and singularity.

In Chapter 6, we study the cubic-quintic nonlinear Schrödinger equation that involves dissipative terms. Due to the existence of the dissipative term, the Hamiltonian structure of the equation is destroyed, which makes the equation more challenging to analyze. Under the framework of singular perturbation theory, we first restrict our equation on a normally hyperbolic manifold M_ε associated with the slow system, derived from the original cubic-quintic nonlinear Schrödinger equation, to obtain a planar dynamical system. This allows us to further examine the dynamics of the equation by constructing the Abelian integral on the whole periodic structure of the Hamiltonian. Some interesting results are obtained, such as the existence of periodic and kink waves, in particular, the coexistence and uniqueness of periodic and kink waves, and the least upper bound on the number of isolated periodic waves.

Conclusions and future work are drawn in Chapter 7. Note that each chapter of this thesis is self-contained.

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Chapter 2

Limit cycles in two perturbed hyperelliptic Hamiltonian systems of degree 4

2.1 Introduction

Hilbert's 16th problem [14] asks for the maximal number of limit cycles and their distribution for a polynomial planar vector field of degree n. It is extremely difficult and still unsolved even for $n = 2$. In order to reduce the difficulty, general polynomial systems are restricted to the following perturbed Hamiltonian systems,

$$\dot{x} = H_y(x, y) + \varepsilon p(x, y), \qquad \dot{y} = -H_x(x, y) + \varepsilon q(x, y), \tag{2.1}$$

where $p(x, y)$ and $q(x, y)$ are polynomials of degree $n \geq 2$, ε is sufficiently small, $H(x, y)$ is a polynomial of degree $n + 1$ which has at least one family of closed orbits denoted by Γ_h for the unperturbed system $(2.1)_{\varepsilon=0}$, parameterized by $\{(x, y)|H(x, y) = h,\ h \in J\}$, where J is an open interval. The perturbations destroy integrability and most periodic orbits of $(2.1)_{\varepsilon=0}$ become spirals. Only a finite number of isolated closed orbits with small deformation persist as limit cycles of (2.1). The main idea for studying the "persisting limit cycles" is to investigate the zeros of Poincaré map or return map on the periodic annulus. Hence, the "persisting limit cycles" generated by perturbation is usually called Poincaré bifurcation. When the perturbation parameter ε is close to zero, the return map is approximated by the following Abelian integral,

$$A(h) = \oint_{\Gamma_h} q(x, y)dx - p(x, y)dy, \quad h \in J. \tag{2.2}$$

The zeros of $A(h)$ correspond to the number of the persisting limit cycles of system (2.1) in the sense of first order Poincaré bifuraction, see [19]. Studying the maximal number of zeros of $A(h)$ is called weak Hilbert's 16th problem and was proposed by Anorld [1]. In fact, most of results on Hilbert's 16th problem were obtained from studying system (2.1).

However, the weak version is still very difficult, and up to now only the case $n = 2$ has been completely solved, see a unified proof in [7] and references therein. A much weaker case is defined by $H(x, y) = \frac{y^2}{2} + \int g(x)dx$, $p = 0$ and $q = f(x)y$, for which the perturbed Hamiltonian system is given by

$$\dot{x} = y, \qquad \dot{y} = -g(x) + \varepsilon f(x)y, \tag{2.3}$$

which has a simpler form of the Abelian ingral,

$$\mathbb{I}(h) = \oint_{\Gamma_h} f(x) y dx.$$

Note that the form of system (2.3) includes, as a special form, the classical Liénard system,

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{2.4}$$

and Smale proposed to study the maximal number of limit cycles of system (2.4) as one of the mathematical problems for the $21th$ century [23].

Although system (2.3) has a simple form, it is important in studying the weak Hilbert's 16th problem and has many applications in the real world. System (2.3) has been used to generate a new perturbed Hamiltonian system by replacing the first equation in system (2.3) with $\dot{x} = y(y^2 - a^2)$, and the zeros of $\mathbb{I}(h)$ for system (2.3) play an important role in studying the zeros of Abelian integral of the new system. In fact, the best result of 13 limit cycles for cubic systems was obtained by such a transform [20]. Also, it is noted that system (2.3) often appears in studying local bifurcations such as Bogdanov-Takens bifurcation with higher codimension [34], and often occurs in many applications [8].

For convenience, we call system (2.3) type (m, n) if $g(x)$ and $f(x)$ are of degrees m and n, respectively. Type $(m, m-1)$ implies that the degree of the perturbation is the same as that of the unperturbed system. Dumortier and Li [9, 10, 11, 12] obtained the sharp bounds on the number of zeros of the corresponding Abelian integrals for five cases of system (2.3) with type $(3, 2)$, for which 5 is the sharp bound on the number of isolated zeros of Abelian integrals when the unperturbed system has a figure eight loop, while 2 is the sharp bound for the saddle-loop case. The main tools used in their study are Picárd-Fuchs equations and Ricatti equations in algebraic geometry, which tansferred the problem to studying the intersections of the related line with a curve. For type $(5, 4)$ of system (2.3) with symmetry, the perturbation still has three terms, and the dimension of Picárd-Fuchs equations is the same as that of type $(3, 2)$. It has been proved that 2 is the sharp bound for the cases of heteroclinic loop [2, 3, 28, 35] and for double homoclinic loop (corresponding to each bounded periodic annulus) [5]. The methods used there include Picárd-Fuchs equations and Chebyshev criterion [13, 22]. The latter is a generalization of Li and Zhang's criterion [21] for determining the Chebyshev property of two Abelian integrals. The advantage of using the criterion is to change the complicated geometric study to a purely algebraic analysis.

A difficulty will arise if the Hamiltonian has degree more than 4 without symmetry, implying that there will be more than three generating elements in $\mathbb{I}(h)$. Thus, the Picárd-Fuchs equations and Ricatti equations have higher dimensions, which increases difficulty in investigating the intersection of the related plane and surface. Many results have been obtained for the least upper bounds on the number of zeros of $\mathbb{I}(h)$ by Chybeshev criterion, [6, 26, 27, 33, 31, 32, 36] for type $(4, 3)$, [4] for type $(5, 4)$ without symmetry and [18, 24, 29] for type $(7, 6)$ with symmetry. However, it is noted that the upper bounds, obtained in almost all above mentioned results, are not the exact upper bounds or sharp bounds. Therefore, the sharp bounds are still open, even for $\mathbb{I}(h)$ of type $(4, 3)$ except one case to be discussed below. The type $(4, 3)$ of system (2.3) is the following perturbed Hamiltonian system of degree 4,

$$\dot{x} = y, \quad \dot{y} = \mu x(x-1)(x-\alpha)(x-\beta) + \varepsilon(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) y, \tag{2.5}$$

where $\mu = \pm 1$, $(\alpha, \beta) \in \mathbb{R}^2$. The unperturbed system (2.5) has 11 cases according to the outside boundaries of the periodic annulus, determined by the values of α and β, see [34]. We would not list all 11 cases of the topological classification except the following cases that have results on zeros of Abelian integrals:

(I) a cusp-saddle cycle ($\alpha = 1$ and $\beta = -\frac{2}{3}$, $\mu = -1$),

(II) a nilpotent-saddle loop ($\alpha = \beta = 1, \mu = 1$),

(III) a saddle loop surrounding a nilpotent center ($\alpha = \beta = 0, \mu = 1$),

(IV) a saddle loop with a cusp outside but near the saddle ($\alpha = -1, \beta < -1, \mu = 1$).

For the cases (I), (II) and (III), it has been proved that 4, 4 and 5 are respectively their least upper bounds. However, only 3 zeros have been obtained for the three cases, see the reports in [6, 26, 31, 32, 33, 36]. The bounds on the number of zeros of Abelian integrals for the cases (I) and (II) were first investigated in [6, 33], in which the verification of Chebyshev property was based on numerical computation. It was pointed out in [32] that 3 was not reliable to be considered as the sharp bound and it was claimed in [32, 36] that a least upper bound on the number of zeros of the Abelian integral is 4 based upon interval analysis and symbolic computation. Therefore, whether 3 or 4 is the sharp bound for the two cases (I) and (II) is still unknown.

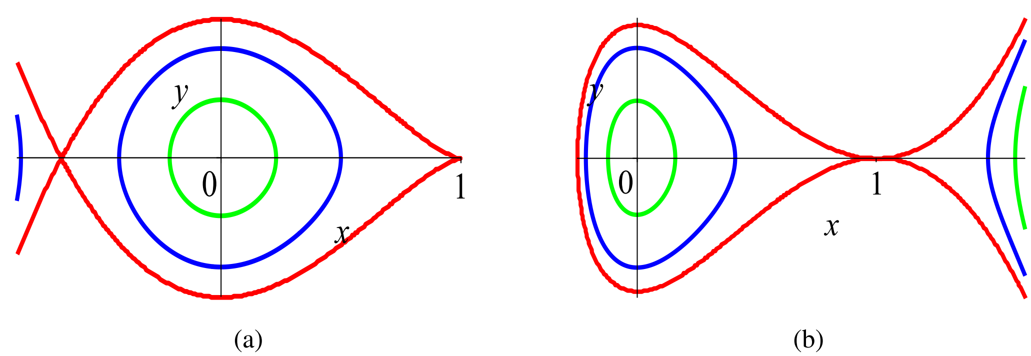

Figure 2.1: Phase portraits of system (2.5) showing (a) a cusp-saddle cycle (red color) for $\alpha = 1, \beta = -\frac{2}{3}, \mu = -1$, and (b) a nilpotent-saddle loop (red color) for $\alpha = \beta = 1, \mu = 1$.

For case (I), the corresponding Hamiltonian of system $(2.5)_{\varepsilon=0}$ with $\alpha = 1, \beta = -\frac{2}{3}$ and $\mu = -1$ is

$$\mathcal{H}(x, y) = \frac{y^2}{2} + \frac{x^2}{3} - \frac{x^3}{9} - \frac{x^4}{3} + \frac{x^5}{5}. \tag{2.6}$$

The phase portrait corresponding to $\mathcal{H} = h$ for $h \in (0, \frac{4}{45})$ and $x \in (-\frac{2}{3}, 1)$, is given in Figure 2.1(a), showing a family of closed orbits Γ_h surrounded by a heteroclinic cycle $\Gamma_{\frac{4}{45}}$, connecting a hyperbolic saddle at $(-\frac{2}{3}, 0)$ and a nilpotent cusp of order 1 at $(1, 0)$. The corresponding Abelian integral is given by

$$\mathcal{A}(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h) + \alpha_3 I_3(h), \tag{2.7}$$

where

$$I_i(h) = \oint_{\Gamma_h} x^i y dx, \quad i = 0,\ 1,\ 2,\ 3. \tag{2.8}$$

For case (II), the Hamiltonian of system $(2.5)_{\varepsilon=0}$ with $\alpha = \beta = 1$ and $\mu = 1$ is

$$\mathcal{H}^*(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}. \tag{2.9}$$

The phase portrait corresponding to $\mathcal{H}^* = h$ for $h \in (0, \frac{1}{20})$ and $x \in (-\frac{1}{4}, 1)$, is depicted in Figure 2.1(b), indicating a family of closed orbits L_h surrounded by a homoclinic loop $L_{\frac{1}{20}}$, with a nilpotent saddle of order 1 at $(1, 0)$. Similarly, we obtain the Abelian integral for this case,

$$\mathcal{M}(h) = \alpha_0 \bar{I}_0(h) + \alpha_1 \bar{I}_1(h) + \alpha_2 \bar{I}_2(h) + \alpha_3 \bar{I}_3(h), \tag{2.10}$$

where

$$\bar{I}_i(h) = \oint_{L_h} x^i y dx, \quad i = 0,\ 1,\ 2,\ 3. \tag{2.11}$$

In this work, we will prove that the sharp bound on the number of zeros of $\mathcal{A}(h)$ and $\mathcal{M}(h)$ is 3. The main results are stated in the following two theorems.

Theorem 2.1.1 *The Abelian integral $\mathcal{A}(h)$ for Case (I) of system* (2.5) *has at most* 3 *zeros on* $(0, \frac{4}{45})$ *for all possible* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4$*, and this is the sharp bound.*

Theorem 2.1.2 *The Abelian integral $\mathcal{M}(h)$ for Case (II) of system* (2.5) *has at most* 3 *zeros on* $(0, \frac{1}{20})$ *for all possible* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4$*, and this is the sharp bound.*

The main mathematical tools that we will apply to prove the two theorems are the Chebyshev criterion and asymptotic property of the Abelian integrals. However, we will not directly apply the Chebyshev criterion to $\{I_i(h),\ i = 0, 1, 2, 3\}$ and $\{\bar{I}_i(h),\ i = 0, 1, 2, 3\}$, since that leads to an upper bound 4, see [6, 31, 32, 36]. Instead, we combine the generating elements $\{I_i(h)\}$ or $\{\bar{I}_i(h)\}$ $(i = 0, 1, 2, 3)$ and treat one perturbation parameter as a parameter in the algebraic Chebyshev systems. The range of this parameter is then bounded to yield a bounded 3-dimensional parameter set via three different combinations, on which a further analysis is given to exclude the possibility of 4 zeros of the Abelian integrals. Properly combing the generating elements plays a crucial role in obtaining the sharp bound. The detailed proof is only given for Theorem 2.1.1, since Theorem 2.1.2 can be similarly proved.

The rest of this chapter is organized as follows. In section 2, we present expansions of Abelian integrals near the centers and briefly introduce the Chebyshev criterion. The proof of Theorem 2.1.1 is given in section 3, and an outline for proving Theorem 2.1.2 is given in section 4. Conclusion is drawn in section 5.

2.2 Asymptotic expansions and Chebyshev criterion

The asymptotic expansions of Abelian integrals are proposed to study its zeros near the endpoints of the annuli, and these zeros correspond to limit cycles near the centers, homoclinic loops and heteroclinic loops, see a survey article [15]. In our work, we will use it to study the dynamics of the Abelian integrals on the whole periodic annulus.

2.2.1 Asymptotic expansions of $\mathcal{A}(h)$ and $\mathcal{M}(h)$ near the centers

Near the center $(x, y) = (0, 0)$, $\mathcal{A}(h)$ and $\mathcal{M}(h)$ have the following expansions (see [17]):

$$\mathcal{A}(h) = \sum_{i\geq 0} b_i h^{i+1} \qquad \text{and} \qquad \mathcal{M}(h) = \sum_{i\geq 0} \overline{b}_i h^{i+1}, \tag{2.12}$$

for $0 < h \ll 1$. The coefficients of b_i and $\overline{b}_i$ can be obtained by using the program developed in [17] as

$$\begin{aligned} b_0 &= \sqrt{6}\pi\,\alpha_0, \\ b_1 &= \tfrac{\sqrt{6}\pi}{32}\,(41\,\alpha_0 + 12\,\alpha_1 + 24\,\alpha_2) \\ b_2 &= \tfrac{\sqrt{6}\pi}{3072}(17017\alpha_0 + 5736\alpha_1 + 10320\alpha_2 + 2880\alpha_3) \end{aligned}$$

and

$$\begin{aligned} \overline{b}_0 &= 2\,\pi\,\alpha_0, \\ \overline{b}_1 &= \tfrac{\pi}{4}\,(21\,\alpha_0 + 12\,\alpha_1 + 4\,\alpha_2)\,, \\ \overline{b}_2 &= \frac{\pi}{32}\,(1379\,\alpha_0 + 872\,\alpha_1 + 440\,\alpha_2 + 160\,\alpha_3)\,. \end{aligned}$$

Using the expansions in (2.12), we can easily find the limit of the ratios of two integrals such as $\lim\limits_{h\to 0} \frac{I_2(h)}{I_0(h)}$. They will be used in our proof.

2.2.2 Chebyshev criterion

In this subsection, we prsent some results on Chebyshev criterion, which are needed for proving our main theorems.

Definition 2.2.1 *Suppose the analytic functions $l_0(x), l_1(x), \cdots$ and $l_{m-1}(x)$ are defined on a real open interval J.*

(A) *The continuous Wronskian of $\{l_0(x), l_1(x), \ldots, l_{i-1}(x)\}$ for $x \in J$ is*

$$W[l_0(x), l_1(x), \ldots, l_{i-1}(x)] = \begin{vmatrix} l_0(x) & l_1(x) & \cdots & l_{i-1}(x) \\ l_0'(x) & l_1'(x) & \cdots & l_{i-1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ l_0^{(i-1)}(x) & l_1^{(i-1)}(x) & \cdots & l_{i-1}^{(i-1)}(x) \end{vmatrix},$$

where $l_i^{(j)}(x)$ is the jth order derivative of $l_i(x)$, $j \geq 2$.

(B) *The set $\{l_0(x), l_1(x), \ldots, l_{m-1}(x)\}$ is called a Chebyshev system if any nontrivial linear combination,*

$$\kappa_0 l_0(x) + \kappa_1 l_1(x) + \cdots + \kappa_{m-1} l_{m-1}(x),$$

has at most $m-1$ isolated zeros on J. Note that $W[l_0(x), l_1(x), \ldots, l_{m-1}(x)] \neq 0$ is one sufficient condition assuring $\{l_0(x), l_1(x), \ldots, l_{m-1}(x)\}$ forms a Chebyshev system.

(C) *The ordered set* $\{l_0(x), l_1(x), \ldots, l_{m-1}(x)\}$ *is called an extended complete Chebyshev system (ECT-system) if for each* $i \in \{1, 2, \cdots, m\}$ *any nontrivial linear combination,*

$$\kappa_0 l_0(x) + \kappa_1 l_1(x) + \cdots + \kappa_{i-1} l_{i-1}(x),$$

has at most $i - 1$ *zeros with multiplicities counted.*

Let $H(x, y) = \mathbb{U}(x) + \frac{y^2}{2}$ be an analytic function. Assume there exists a punctured neighborhood $\mathcal{N}$ of the origin $(0, 0)$ foliated by closed curves $\Gamma_h \subseteq \{(x, y)|H(x, y) = h,\ h \in (0, h^*),\ h^* = H(\partial\mathcal{N})\}$. The projection of $\mathcal{N}$ on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$, and $x\mathbb{U}'(x) > 0$ for all $x \in (x_l, x_r)\backslash\{0\}$. $\mathbb{U}(x) = \mathbb{U}(z(x))$ defines an analytic involution $z = z(x)$ for all $x \in (x_l, x_r)$. Let

$$\mathbb{I}_i(h) = \oint_{\Gamma_h} \eta_i(x) y^{2s-1} dx \ \text{ for } \ h \in (0, h^*), \tag{2.13}$$

where $s \in \mathbb{N}$ and $\eta_i(x)$ are analytic functions on (x_l, x_r), $i = 0, 1, \ldots, m - 1$. Further, define

$$l_i(x) := \frac{\eta_i(x)}{\mathbb{U}'(x)} - \frac{\eta_i(z(x))}{\mathbb{U}'(z(x))}. \tag{2.14}$$

Then we have

Lemma 2.2.2 ([13]) *Consider the integrals* (2.13) *and the functions* (2.14). $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *is an* ECT *system on* $(0, h^*)$ *if* $s > m-2$ *and* $\{l_0, l_1, \cdots, l_{m-1}\}$ *is an* ECT *system on* $(x_l, 0)$ *or* $(0, x_r)$.

Lemma 2.2.3 ([22]) *Consider the integrals* (2.13) *and the functions* (2.14). *If the following conditions hold:*
(a) $W[l_0, l_1, \ldots, l_i]$ *does not vanish on* $(0, x_r)$ *for* $i = 0, 1, \cdots, m - 2$,
(b) $W[l_0, l_1, \ldots, l_{m-1}]$ *has* k *zeros on* $(0, x_r)$ *with multiplicities counted, and*
(c) $s > m + k - 2$,
then any nontrivial linear combination of $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *has at most* $m+k-1$ *zeros on* $(0, h^*)$ *with multiplicities counted. In this case, we call* $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *a Chebyshev system with accuracy* k *on* $(0, h^*)$.

2.3 Proof of Theorem 2.1.1

2.3.1 Partition of the parameter space

In this subsection, we divide the parameter space for $\mathcal{A}(h)$ to obtain a subset which is the only set for $\mathcal{A}(h)$ to might have 4 zeros on $(0, \frac{4}{45})$. We write $\mathcal{H}(x, y) = \frac{y^2}{2} + U(x)$. Then,

$$q(x, z) \doteq \frac{U(x) - U(z)}{x - z} = 0$$

which defines the involution $z(x)$, $x \in (0, 1)$ on the periodic annulus. We have the following result.

Lemma 2.3.1 *The following equations hold:*

$$8h^3 I_i(h) = \oint_{\Gamma_h} S_i(x) y^7 dx \equiv \widetilde{I}_i(h), \quad i = 0,\ 1,\ 2,\ 3,$$

where $S_i(x) = \frac{x^i g_i(x)}{354375(2+3x)^6(x-1)^9}$, *in which each polynomial* $g_i(x)$ *has degree* 15.

Proof First, multiplying $I_i(h)$ by $\frac{y^2+2U(x)}{2h} = 1$ yields

$$\begin{aligned}
8h^3 I_i(h) &= \oint_{\Gamma_h} (2U(x) + y^2)^3 x^i y dx \\
&= \oint_{\Gamma_h} 8\, x^i U^3(x) y dx + \oint_{\Gamma_h} 12\, x^i U^2(x) y^3 dx \\
&+ \oint_{\Gamma_h} 6\, x^i U(x) y^5 dx + \oint_{\Gamma_h} x^i y^7 dx,\ i = 0, 1, 2, 3.
\end{aligned} \tag{2.15}$$

Then applying Lemma 4.1 in [13] to (2.15) to increase the power of y in the first three integrals to 7 proves the lemma.

Without loss of generality, we assume that $\alpha_3 = 1$ when $\alpha_3 \neq 0$. Further, introduce the following combinations:

$$\begin{aligned}
\mathcal{I}_{23}(h) &= \oint_{\Gamma_h} \left(\alpha_2 x^2 + x^3\right) y dx, \\
\mathcal{I}_{13}(h) &= \oint_{\Gamma_h} \left(\alpha_1 x + x^3\right) y dx, \\
\mathcal{I}_{03}(h) &= \oint_{\Gamma_h} \left(\alpha_0 + x^3\right) y dx.
\end{aligned} \tag{2.16}$$

Then

$$\begin{aligned}
\mathcal{A}(h) &= \alpha_0 I_0(h) + \alpha_1 I_1(h) + \mathcal{I}_{23}(h) \\
&= \alpha_0 I_0(h) + \alpha_2 I_2(h) + \mathcal{I}_{13}(h) \\
&= \alpha_1 I_1(h) + \alpha_2 I_2(h) + \mathcal{I}_{03}(h).
\end{aligned}$$

The following lemma directly follows from Lemma 2.3.1.

Lemma 2.3.2 *The following equations hold:*

$$\begin{aligned}
8h^3 \mathcal{I}_{23}(h) &= \oint_{\Gamma_h} (\alpha_2 S_2(x) + S_3(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{23}(h), \\
8h^3 \mathcal{I}_{13}(h) &= \oint_{\Gamma_h} (\alpha_1 S_1(x) + S_3(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{13}(h), \\
8h^3 \mathcal{I}_{03}(h) &= \oint_{\Gamma_h} (\alpha_0 S_0(x) + S_3(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{03}(h).
\end{aligned}$$

Now, let

$$\begin{aligned} L_i(x) &= \Big(\frac{S_i}{U'}\Big)(x) - \Big(\frac{S_i}{U'}\Big)(z(x)), \\ \mathcal{L}_{i3}(x) &= \Big(\frac{\alpha_i S_i + S_3}{U'}\Big)(x) - \Big(\frac{\alpha_i S_i + S_3}{U'}\Big)(z(x)). \end{aligned} \tag{2.17}$$

Then

$$\begin{aligned} \frac{d}{dx}L_i(x) &= \frac{d}{dx}\Big(\frac{S_i}{U'}\Big)(x) - \frac{d}{dz}\Big[\Big(\frac{S_i}{U'}\Big)(z(x))\Big] \times \frac{dz}{dx}, \\ \frac{d}{dx}\mathcal{L}_{i3}(x) &= \frac{\partial}{\partial x}(\mathcal{L}_{i3}(x)) + \frac{\partial}{\partial z}(\mathcal{L}_{i3}(x)) \times \frac{dz}{dx}, \end{aligned}$$

where $\frac{dz}{dx} = -\frac{q_x(x,z)}{q_z(x,z)}$. Direct computations yield

$$\begin{aligned} W[L_0](x) &= \frac{(x-z)Q_0(x,z)}{118125\, xz(2+3\,x)^7(x-1)^{11}(2+3\,z)^7(z-1)^{11}}, \\ W[L_1](x) &= \frac{(x-z)Q_1(x,z)}{118125\,(2+3\,x)^7(x-1)^{11}(2+3\,z)^7(z-1)^{11}}, \\ W[L_0(x), L_1(x)] &= \frac{(x-z)^3 Q_{01}(x,z)}{13953515625x^2z^2(z-1)^{22}(3\,z+2)^{13}(x-1)^{22}(3\,x+2)^{13}P_0(x,z)}, \\ W[L_0(x), L_2(x)] &= \frac{(x-z)^3 Q_{02}(x,z)}{13953515625x^2z^2(z-1)^{22}(3\,z+2)^{13}(x-1)^{22}(3\,x+2)^{13}P_0(x,z)}, \\ W[L_1(x), L_2(x)] &= \frac{(x-z)^3 Q_{12}(x,z)}{13953515625x^2z^2(z-1)^{22}(3\,z+2)^{13}(x-1)^{22}(3\,x+2)^{13}P_0(x,z)}, \\ W[L_0(x), L_1(x), \mathcal{L}_{23}] &= \frac{(x-z)^6 M_1(x,z,\alpha_2)}{c^* x^3z^3(x-1)^{32}(3\,x+2)^{18}(z-1)^{32}(3\,z+2)^{18}P_0^3(x,z)}, \\ W[L_0(x), L_2(x), \mathcal{L}_{13}] &= \frac{(x-z)^6 M_2(x,z,\alpha_1)}{c^* x^3z^3(x-1)^{32}(3\,x+2)^{18}(z-1)^{32}(3\,z+2)^{18}P_0^3(x,z)\alpha_1}, \\ W[L_1(x), L_2(x), \mathcal{L}_{03}] &= \frac{(x-z)^6 M_3(x,z,\alpha_0)}{c^* x^3z^3(x-1)^{32}(3\,x+2)^{18}(z-1)^{32}(3\,z+2)^{18}P_0^3(x,z)\alpha_0}, \end{aligned} \tag{2.18}$$

where Q_0, Q_1, Q_{01}, Q_{02} and Q_{12} are polynomials of degree 34, 33, 66, 67 and 64, respectively, $c^* = 1648259033203125$, $z = z(x)$ is determined by $q(x,z) = 0$ and

$$P_0(x,z) = 9\,x^3 + 18\,x^2z + 27\,xz^2 + 36\,z^3 - 15\,x^2 - 30\,xz - 45\,z^2 - 5\,x - 10\,z + 15.$$

Applying Sturm's Theory to the resultant between $q(x,z)$ and $P_0(x,z)$ with respect to z shows that the resultant has no roots for $x \in (0,1)$, which implies that $P_0(x,z)$ does not vanish for $x \in (0,1)$. Hence, the Wronskians are well defined.

The following result indicates that we only need to discuss the case when $\alpha_3 \neq 0$.

Proposition 2.3.3 *When $\alpha_3 = 0$, $\mathcal{A}(h)$ has at most 2 zeros on $(0, \frac{4}{45})$.*

The proof of Proposition 2.3.3 relies on computing and verifying the non-vanishment of Wronskians $W[L_0]$, $W[L_0, L_1]$ and $W[L_0, L_1, L_2]$, and then the application of Lemma 2.2.3. Since the computation and verification are straightforward, we omit the proof here for briefness.

To prove Theorem 2.1.1, we need to show non-vanishing of certain numerators and denominators of the related Wronskians in (2.18) for $x \in (0,1)$. Taking the numerator $Q_{01}(x,z)$ of the Wronskian $W[L_0, L_1]$ for example, we only need to prove that the two-dimensional system $\{Q_{01}(x,z), q(x,z)\}$ does not vanish on $\{(x,z) | -\frac{2}{3} < z < 0 < x < 1\}$, because z in

$Q_{01}(x,z)$ is determined by $q(x,z) = 0$, and $z(x) \in (-\frac{2}{3}, 0)$ when $x \in (0,1)$. To do this, we apply triangular-decomposition and root isolating to $\{Q_{01}(x,z), q(x,z)\}$ to decompose the non-linear system into several triangular systems, and then isolate the roots of each triangular-decomposed system. Since all roots of these triangular systems are the roots of the original system $\{Q_{01}(x,z), q(x,z)\}$, we only need to check if these decomposed systems have roots on $\{(x,z) | -\frac{2}{3} < z < 0 < x < 1\}$. This idea has been successfully applied to determine the zeros of Abelian integrals, see [25, 26, 27, 36]. Instead of the triangular-decomposition method, one may also use the interval analysis [32], which computes two resultants between $Q_{01}(x,z)$ and $q(x,z)$ with respect to x and z, respectively, yielding several two dimensional regions. Finally, one verifies if $Q_{01}(x,z)$ vanishs on these regions by determining the intersection of the curves $Q_{01}(x,z)$ and $q(x,z)$, see [32] for details.

By applying the triangular-decomposition and root isolating to the numerators of the Wronskians, we obtain the following result.

Lemma 2.3.4 *Each of the Wronskians,* $W[L_0]$, $W[L_1]$, $W[L_0, L_1]$, $W[L_0, L_2]$ *and* $W[L_1, L_2]$, *does not vanish for* $x \in (0,1)$.

Next, we investigate the last three Wronskians in (2.18). Their numerators have the forms,

$$\begin{aligned} M_1(x,z,\alpha_2) &= \alpha_2\beta_2(x,z) - \beta_1(x,z), \\ M_2(x,z,\alpha_1) &= \alpha_1\gamma_2(x,z) - \gamma_1(x,z), \\ M_3(x,z,\alpha_0) &= \alpha_0\delta_2(x,z) - \delta_1(x,z), \end{aligned}$$

where β_1, β_2, γ_1, γ_2, δ_1 and δ_2 are polynomials of degrees 98, 97, 99, 97, 100 and 97, respectively. $M_i(x,z,\alpha_i) = 0$ defines three functions,

$$\alpha_2(x,z) = \frac{\beta_1(x,z)}{\beta_2(x,z)}, \ \alpha_1(x,z) = \frac{\gamma_1(x,z)}{\gamma_2(x,z)}, \ \alpha_0(x,z) = \frac{\delta_1(x,z)}{\delta_2(x,z)},$$

and their derivatives,

$$\begin{aligned} \overline{\alpha}_2(x,z) &= \frac{\partial\alpha_2(x,z)}{\partial x} + \frac{\partial\alpha_2(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\overline{\beta}_1(x,z)}{\overline{\beta}_2(x,z)}, \\ \overline{\alpha}_1(x,z) &= \frac{\partial\alpha_1(x,z)}{\partial x} + \frac{\partial\alpha_1(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\overline{\gamma}_1(x,z)}{\overline{\gamma}_2(x,z)}, \\ \overline{\alpha}_0(x,z) &= \frac{\partial\alpha_0(x,z)}{\partial x} + \frac{\partial\alpha_0(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\overline{\delta}_1(x,z)}{\overline{\delta}_2(x,z)}. \end{aligned}$$

The denominators $\beta_2(x,z)$, $\gamma_2(x,z)$, $\delta_2(x,z)$, $\overline{\beta}_2(x,z)$, $\overline{\gamma}_2(x,z)$ and $\overline{\delta}_2(x,z)$ do not vanish for $x \in (0,1)$, because they do not have common roots with $q(x,z)$ for $(x,z) \in (0,1) \times (-\frac{2}{3}, 0)$ by triangular-decomposition and root isolating. Hence, all of the functions $\alpha_i(x,z)$ and $\overline{\alpha}_i(x,z)$ $(i = 2,1,0)$ are well defined for $x \in (0,1)$.

We have the following lemma.

Lemma 2.3.5 (i) $\alpha_2(x, z(x))$ *is decreasing from* $(0, \frac{5}{2})$ *to a minimum* (x^*, α_2^*) *and then increasing to* $(1, -\frac{4}{3})$*;*

(ii) $\alpha_1(x, z(x))$ *is increasing from* $(0, -5)$ *to a maximum* $(x^\dagger, \alpha_1^\dagger)$ *and then decreasing to* $(1, -\frac{1}{3})$*;*

(iii) $\alpha_0(x, z(x))$ *is increasing from* $(0, 0)$ *to a maximum* $(x^\ddagger, \alpha_0^\ddagger)$ *and then decreasing to* $(1, \frac{2}{3})$*, where*

$$x^*, x^\dagger, x^\ddagger \in \underbrace{\left[\frac{108804604063}{137438953472}, \frac{3400143877}{4294967296}\right]}_{1/10^{10}}$$

and

$$\alpha_2^* \in \underbrace{\left[-\frac{64307\ldots30528}{32922\ldots64125}, -\frac{44191\ldots26125}{22624\ldots29984}\right]}_{1/10^6} \approx [-1.95327706, -1.95327692],$$

$$\alpha_1^\dagger \in \underbrace{\left[\frac{11707\ldots14473}{17176\ldots36512}, \frac{84772\ldots81309}{12437\ldots49248}\right]}_{1/10^7} \approx [0.06815643778, 0.06815645227],$$

$$\alpha_0^\ddagger \in \underbrace{\left[\frac{68092\ldots44807}{73774\ldots11552}, \frac{78889\ldots10781}{85472\ldots94528}\right]}_{1/10^7} \approx [0.9229843148, 0.9229843212].$$

Proof We only prove case (i), since the cases (ii) and (iii) can be proved similarly. A direct computation shows that

$$\lim_{x\to 0} \alpha_2(x, z(x)) = \frac{5}{2}, \qquad \lim_{x\to 1} \alpha_2(x, z(x)) = -\frac{4}{3}.$$

On $\{(x, z) | -\frac{2}{3} < z < 0 < x < 1\}$, $\overline{\beta}_1(x, z)$ and $q(x, z)$ have a unique common root $(x^*, z^*) \in D_0$, where

$$D_0 = \left[\frac{108804604063}{137438953472}, \frac{3400143877}{4294967296}\right] \times \left[-\frac{41095301255}{68719476736}, -\frac{82190602509}{137438953472}\right].$$

x^* is the unique simple zero of $\overline{\alpha}_2(x, z(x))$ by verifying that $\frac{d}{dx}\overline{\alpha}_2(x, z(x))$ has no zeros in $[\frac{108804604063}{137438953472}, \frac{3400143877}{4294967296}]$. Therefore, x^* is the unique critical point of $\alpha_2(x, z(x))$, and thus the monotonicity of $\alpha_2(x, z(x))$ on $(0, x^*) \bigcup (x^*, 1)$ can be easily determined by comparing the values of $\alpha_2(x, z(x))$ at $x = 0,\ x^*,\ 1$ as $\frac{5}{2}$, -1.953277, $-\frac{4}{3}$. Alternatively, using

$$\lim_{x\to 0^+} \overline{\alpha}_2(x, z(x)) = 0^- \quad \text{and} \quad \lim_{x\to 0^+} \frac{d}{dx}(\overline{\alpha}_2(x, z(x))) = -\frac{441}{40} < 0,$$

we know that $\alpha_2(x, z(x))$ is monotonically deceasing on $(0, x^*)$ and monotonically increasing on $(x^*, 1)$

It can be further shown that the resultant between $\frac{\partial \beta_i(x,z)}{\partial x}$ (for $i = 1, 2$) and $q(x, z)$ with respect to z has no roots in the interval $[\frac{108804604063}{137438953472}, \frac{3400143877}{4294967296}]$ by Sturm's Theorem. Hence, $\beta_i(x, z)$ $(i = 1, 2)$ reaches its maximal and minimum values at the boundaries of D_0. Direct computation gives

$$\begin{aligned}
\min_{D_0}\beta_1(x,z) &= -\frac{44292\ldots44113}{16265\ldots68512} \approx -2.723190846\times10^{13},\\
\max_{D_0}\beta_1(x,z) &= -\frac{13257\ldots78375}{48683\ldots66176} \approx -2.723190757\times10^{13},\\
\min_{D_0}\beta_2(x,z) &= \frac{98768\ldots92375}{70844\ldots04416} \approx 1.394165169\times10^{13},\\
\max_{D_0}\beta_2(x,z) &= \frac{52797\ldots66369}{37870\ldots34272} \approx 1.394165228\times10^{13}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\alpha_2^* = \alpha_2(x^*, z(x^*)) &\in \left[\frac{\min\limits_{D_0}\beta_1(x,z)}{\min\limits_{D_0}\beta_2(x,z}, \frac{\max\limits_{D_0}\beta_1(x,z)}{\max\limits_{D_0}\beta_2(x,z)}\right]\\
&= \underbrace{\left[-\frac{64307\ldots30528}{32922\ldots64125}, -\frac{44191\ldots26125}{22624\ldots29984}\right]}_{1/10^6}\\
&\approx [-1.95327706, -1.95327692].
\end{aligned}$$

Note in the above proof that the exact rational numbers are obtained from symbolic computation, demonstrating the accuracy of computation. It is also noted that the critical point $(x^*, \alpha_2(x^*, z(x^*)))$ divides the curve $\{(x, \alpha_2(x, z(x)))|0 < x < 1\}$ into two simple segments (curves). The points on the two curves correspond to the simple roots of $M_1(x, z(x), \alpha_2(x, z(x)))$, while x^* is a root of multiplicity 2. The following lemma follows from Lemma 2.3.5.

Lemma 2.3.6 *For $x \in (0,1)$, when α_2 belongs to the intervals $[\alpha_2^*, -\frac{4}{3})$, $[-\frac{4}{3}, \frac{5}{2})$ and $(-\infty, \alpha_2^*)\bigcup[\frac{5}{2}, +\infty)$, $W[L_0, L_1, \mathcal{L}_{23}]$ has 2, 1 and 0 roots with multiplicities counted, respectively.*

Combining Lemmas 2.3.4 and 2.3.6 and applying Lemma 2.2.3, we have the following result.

Proposition 2.3.7 *$\mathcal{A}(h)$ has at most 4, 3, 2 zeros in $(0, \frac{4}{45})$ when α_2 is located in the intervals $[\alpha_2^*, -\frac{4}{3})$, $[-\frac{4}{3}, \frac{5}{2})$, and $(-\infty, \alpha_2^*)\bigcup[\frac{5}{2}, +\infty)$, respectively.*

Similarly, we have

Proposition 2.3.8 *$\mathcal{A}(h)$ has at most 4, 3, 2 zeros in $(0, \frac{4}{45})$ when α_1 belongs to the intervals $(-\frac{1}{3}, \alpha_1^\dagger]$, $(-5, -\frac{1}{3}]$, and $(-\infty, -5]\bigcup(\alpha_1^\dagger, +\infty)$, respectively.*

Proposition 2.3.9 *$\mathcal{A}(h)$ has at most 4, 3, 2 zeros in $(0, \frac{4}{45})$ when α_0 is located in the intervals $(\frac{2}{3}, \alpha_0^\ddagger]$, $(0, \frac{2}{3}]$, and $(-\infty, 0]\bigcup(\alpha_0^\ddagger, +\infty)$, respectively.*

Define

$$D^\ddagger = \left\{(\alpha_0, \alpha_1, \alpha_2)|\alpha_0 \in \left(\frac{2}{3}, \alpha_0^\ddagger\right],\ \alpha_1 \in \left(-\frac{1}{3}, \alpha_1^\dagger\right],\ \alpha_2 \in \left[\alpha_2^*, -\frac{4}{3}\right)\right\}.$$

Then Propositions 2.3.7, 2.3.8 and 2.3.9 imply that

Proposition 2.3.10 *$\mathcal{A}(h)$ may have 4 zeros only if $(\alpha_0, \alpha_1, \alpha_2) \in D^\ddagger$.*

2.3.2 Non-existence of 4 zeros of $\mathcal{A}(h)$ on $D^{\ddagger}$

Finally, we prove that $\mathcal{A}(h)$ cannot have 4 zeros when $(\alpha_0, \alpha_1, \alpha_2) \in D^{\ddagger}$. First, we have

Lemma 2.3.11 *For $h \in (0, \frac{4}{45})$, the following hold:*

(1) *the generating element $I_0(h)$ is positive;*

(2) *the ratio $\frac{I_1(h)}{I_0(h)}$ is increasing from 0 to $\frac{2}{27}$;*

(3) *the ratio $\frac{I_2(h)}{I_0(h)}$ is increasing from 0 to $\frac{116}{891}$; and*

(4) *the ratio $\frac{I_3(h)}{I_0(h)}$ is increasing from 0 to $\frac{136}{3861}$.*

Proof By Green formula, $I_0(h) = \oint_{\Gamma_h} ydx = \iint_{\mathcal{D}} dxdy$, where $\mathcal{D}$ is the region bounded by Γ_h (periodic annulus), and therefore, $I_0(h) > 0$. The non-vanishing property of $W[L_0]$, $W[L_0, L_1]$, $W[L_0, L_2]$ and $W[L_0, L_3]$ proved in Lemma 2.3.4 implies that $\frac{I_1(h)}{I_0(h)}$, $\frac{I_2(h)}{I_0(h)}$ and $\frac{I_3(h)}{I_0(h)}$ are monotonic on $(0, \frac{4}{45})$. By the expansion of $\mathcal{A}(h)$ near $h = 0$, we have

$$\lim_{h\to 0} \frac{I_1(h)}{I_0(h)} = \lim_{h\to 0} \frac{I_2(h)}{I_0(h)} = \lim_{h\to 0} \frac{I_3(h)}{I_0(h)} = 0.$$

Taking the limit as $h \to \frac{4}{45}$ yields

$$\lim_{h\to \frac{4}{45}} \frac{I_1(h)}{I_0(h)} = \lim_{h\to \frac{4}{45}} \frac{\oint_{\Gamma_h} xydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{4}{45}}} xydx}{\oint_{\Gamma_{\frac{4}{45}}} ydx} = \frac{2}{27},$$

$$\lim_{h\to \frac{4}{45}} \frac{I_2(h)}{I_0(h)} = \lim_{h\to \frac{4}{45}} \frac{\oint_{\Gamma_h} x^2ydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{4}{45}}} x^2ydx}{\oint_{\Gamma_{\frac{4}{45}}} ydx} = \frac{116}{891},$$

and

$$\lim_{h\to \frac{4}{45}} \frac{I_3(h)}{I_0(h)} = \lim_{h\to \frac{4}{45}} \frac{\oint_{\Gamma_h} x^3ydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{4}{45}}} x^3ydx}{\oint_{\Gamma_{\frac{4}{45}}} ydx} = \frac{136}{3861}.$$

Proposition 2.3.12 $\mathcal{A}(h) > 0$ *for* $(\alpha_0, \alpha_1, \alpha_2) \in D^{\ddagger}$.

Proof When $(\alpha_0, \alpha_1, \alpha_2) \in D^{\ddagger}$, by the results obtained in Lemma 2.3.11, it is easy to show that

$$\alpha_0 + \alpha_2 \frac{I_2(h)}{I_0(h)} > \frac{2}{3} + \frac{116}{891}\alpha_2^* \geq \frac{2}{3} + \frac{116}{891} \times \left(-\frac{64307\ldots30528}{32922\ldots64125}\right) > \frac{2}{3} + \frac{116}{891} \times (-2) = \frac{362}{891},$$

and

$$\alpha_1 \frac{I_1(h)}{I_0(h)} > -\frac{1}{3} \times \frac{2}{27} = -\frac{2}{81}.$$

Then, using the results for $\frac{I_3(h)}{I_0(h)}$ and $I_0(h)$ in Lemma 2.3.11, we have, for $h \in (0, \frac{4}{45})$, that

$$\begin{aligned}\mathcal{A}(h) &= \Big[\Big(\alpha_0 + \alpha_2\frac{I_2(h)}{I_0(h)}\Big) + \alpha_1\frac{I_1(h)}{I_0(h)} + \frac{I_3(h)}{I_0(h)}\Big]I_0(h) \\ &> \Big(\frac{362}{891} - \frac{2}{81} + 0\Big)I_0(h) \\ &= \frac{340}{891}I_0(h) > 0.\end{aligned}$$

So $\mathcal{A}(h)$ has no zeros for $(\alpha_0, \alpha_1, \alpha_2) \in D^\ddagger$.

Proof of Theorem 2.1.1. Combining Propositions 2.3.3, 2.3.10 and 2.3.12 proves Theorem 2.1.1.

2.4 An outline of the proof for Theorem 2.1.2

Theorem 2.1.2 can be similarly proved as that for Theorem 2.1.1. Hence, we give an outline of the proof for Theorem 2.1.2. Similar to Propositions 2.3.3, 2.3.10 and Lemma 2.3.11, we have the following results.

Proposition 2.4.1 *When $\alpha_3 = 0$, $\mathcal{M}(h)$ has at most 2 zeros on $(0, \frac{1}{20})$.*

Proposition 2.4.2 *$\mathcal{M}(h)$ may have 4 zeros only if $(\alpha_0, \alpha_1, \alpha_2) \in D^*$, where*

$$D^* = \Big\{(\alpha_0, \alpha_1, \alpha_2) | \alpha_0 \in [\alpha_0^\ddagger, -1),\ \alpha_1 \in (3, \alpha_1^\dagger],\ \alpha_2 \in [\alpha_2^*, -3)\Big\}$$

with

$$\begin{aligned}\alpha_2^* &\in \underbrace{\Big[-\frac{60508\cdots90001}{17339\cdots00000}, -\frac{55031\cdots00000}{15770\cdots80367}\Big]}_{1/10^9} \approx [-3.4896007790, -3.4896007772], \\ \alpha_1^\dagger &\in \underbrace{\Big[\frac{61748\cdots00000}{15770\cdots80367}, -\frac{23327\cdots84791}{59578\cdots00000}\Big]}_{1/10^9} \approx [3.9154855416, 3.9154855436], \\ \alpha_0^\ddagger &\in \underbrace{\Big[-\frac{23405\cdots41729}{16376\cdots00000}, -\frac{22538\cdots00000}{15770\cdots80367}\Big]}_{1/10^9} \approx [-1.4291710471, -1.4291710464].\end{aligned}$$

Lemma 2.4.3 *For $h \in (0, \frac{1}{20})$, the ratio $\frac{I_1(h)}{I_0(h)}$ increases from 0 to $\frac{1}{6}$, and the ratio $\frac{I_3(h)}{I_2(h)}$ increases from 0 to $\frac{19}{39}$.*

Proposition 2.4.4 *$\mathcal{M}(h) < 0$ when $(\alpha_0, \alpha_1, \alpha_2) \in D^*$.*

Proof When $(\alpha_0, \alpha_1, \alpha_2) \in D^*$, considering the above intervals expressed by fractions, $\frac{I_1(h)}{I_0(h)} \in (0, \frac{1}{6})$ and $\frac{I_3(h)}{I_2(h)} \in (0, \frac{19}{39})$ in Lemma 2.4.3, it is obvious that

$$\alpha_0 + \alpha_1\frac{I_1(h)}{I_0(h)} < 0 \ \text{ and } \alpha_2 + \frac{I_3(h)}{I_2(h)} < 0.$$

By Green formula, $I_i(h) = \oint_{\Gamma_h} y dx = \iint_{D^\dagger} x^i dxdy$, where $D^\dagger$ is the region (periodic annulus) surrounded by Γ_h, therefore, $I_0(h) > 0$ and $I_2(h) > 0$. Hence, we have

$$\mathcal{M}(h) = \Big(\alpha_0 + \alpha_1 \frac{I_1(h)}{I_0(h)}\Big) I_0(h) + \Big(\alpha_2 + \frac{I_3(h)}{I_2(h)}\Big) I_2(h) < 0.$$

So $\mathcal{M}(h)$ has no zeros when $(\alpha_0, \alpha_1, \alpha_2) \in D^*$.

Combining Propositions 2.4.1, 2.4.2 and 2.4.4 proves Theorem 2.1.2.

2.5 Conclusion

In this work, we have given a further investigation on the works [6, 31, 32, 33, 36] and proved that the sharp bound is 3 on the number of zeros of the Abelian integrals of system (2.3) for the cases with a cusp-saddle and a nilpotent-saddle loop. Previous works have obtained 3 zeros and an upper bound 4. Our approach narrows the parameters to a set which is the only set to possibly have 4 zeros of the Abelian integrals. Then we rule out the possibility of 4 zeros and thus proved the sharp bound to be 3. This completely solved the upper bound problem for the cases (I) and (II). For case (III), it has been shown in [26, 32] that a least upper bound is 5 but only 3 zeros have been obtained. We can apply the method developed in this chapter to investigate case (III) and to show that a least upper bound is 4. However, whether 4 is the sharp bound for case (III) needs a further study.

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Chapter 3

Annulus cyclicity and Hopf cyclicity in a damping quintic Hamiltonian system

3.1 Introduction

Periodic motions appear in almost all natural and engineering dynamical systems. Determining the number of periodic solutions and their locations plays an important role in the study of dynamical systems. For example, in chemical reactions [18], it is important to determine what may cause oscillation and what may destroy oscillation, and what affects the period and amplitude of oscillation. However, it is not easy to determine all possible locations, periods and amplitudes even for the oscillations in a two-dimensional reactor. The relative open problem in mathematics is the well-known Hilbert's 16th problem [24], which considers the maximal number of limit cycles, denoted by $\mathbb{H}(n)$, and their distribution in two-dimensional polynomial systems. This problem is still not completely solved even for quadratic polynomial systems (i.e., for the simplest case $n = 2$). Many theories and methodologies have been developed for solving the problem, and a lot of good results such as lower bounds on $\mathbb{H}(n)$ have been obtained, see a recent paper [1].

In order to overcome the difficulty in solving the Hilbertls 16th problem, researchers have tried to study the relative weakened problems or weaker versions of the problem, for example, studying limit cycles arising from certain special bifurcations, or focusing on systems with simper forms. Anorld's version of Hilbert's 16th problem [2] is equivalent to studying limit cycles by investigating the first-order Poincaré bifurcation of the following perturbed Hamiltonian system,

$$\dot{x} = H_y(x, y) + \varepsilon P(x, y), \qquad \dot{y} = -H_x(x, y) + \varepsilon Q(x, y), \tag{3.1}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials of degree $n \geq 2$, $\varepsilon > 0$ is sufficiently small, $H(x, y)$ is a polynomial of degree $n + 1$ and has at least one family of closed orbits. Suppose the ovals, parameterized by $\{(x, y)|H(x, y) = h,\ h \in J\}$ where J is an open interval, are periodic orbits of system $(3.1)_{\varepsilon=0}$, forming a periodic annulus denoted by $\{\Gamma_h\}$. The number of zeros of the Abelian integral,

$$A(h) = \oint_{\Gamma_h} Q(x, y)dx - P(x, y)dy, \quad h \in J,$$

estimates the zeros of the return map that is constructed on the periodic annulus $\{\Gamma_h\}$. Therefore, the zeros of $A(h)$ provide the information on the persisting limit cycles of system (3.1) in the sense of the first order Poincaré bifuraction when ε is sufficiently small, see [23]. Studying the zeros of $A(h)$ is so-called the weak Hilbert's 16th problem, which has produced most of results on the Hilbert's 16th problem. However, even the weak version of the Hilbert's 16th problem is still very difficult to solve, and so far only the case $n = 2$ has been completely solved, see [8] and references therein.

Smale [33] proposed a simple version of the Hilbert's 16th problem based on the classical polynomial Liénard system,

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{3.2}$$

where $f(x)$ is a polynomial of degree n. Lins *et al.* [27] proved that system (3.2) has at most $[\frac{n}{2}]$ limit cycles for $n = 1, 2$ and conjectured that the result is true for all $n \geq 1$. Li and Llibre [26] proved that the conjecture is true for $n = 3$. However, in 2007, Dumortier *et al.* [17] proved that there exist systems which have at least $[\frac{n}{2}] + 1$ limit cycles for even $n \geq 6$. Four years later, Maesschalck and Dumortier [11] proved that there exist systems that have at least $[\frac{n}{2}] + 2$ limit cycles for $n \geq 5$. Maesschalck and Huzak [12] proved the results to $n - 2$ for $n \geq 5$, which improved $[\frac{n}{2}] + 2$ for $n \geq 9$. In short, up to now, the sharp bound on the maximal number of limit cycles of (3.2) is still unknown and the conjecture of Lins *et al.* is still open for $n = 4$.

Recently, more interests have focused on the generalized Liénard system, which includes many various types of oscillators,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3.3}$$

where $g(x)$ and $f(x)$ are polynomials with degrees m and n, respectively, usually called type (m, n). In Newtonian mechanics, $f(x)$ is the damping term and $g(x)$ is the restoring or potential term. The sharp bound on the number of limit cycles of system (3.3) depends on the degrees m and n denoted by $\mathbb{H}_L(m, n)$, where L represents Liénard system. It is more difficult to determine $\mathbb{H}_L(m, n)$ for the generalized Liénard system (3.3) than that for system (3.2), because of the nonlinear restoring term $g(x)$. However, even when system (3.3) has a simple form, it still plays a very important role in studying limit cycles of general planar systems obtained from modifying (3.3), see [25, 32]. Further, system (3.3) can be applied to model real world oscillating phenomena, see [10].

There are two different ways to study $\mathbb{H}_L(m, n)$. One way is to consider the limit cycles via Poincaré bifurcation by assuming the damping term in the form of $-\varepsilon f(x)y$. Then system (3.3) becomes a special form of (3.1),

$$\dot{x} = y, \qquad \dot{y} = -g(x) + \varepsilon f(x)y, \tag{3.4}$$

with $P(x, y) = 0$, $Q(x, y) = f(x)y$ and the Hamiltonian $H(x, y) = \frac{y^2}{2} + \int g(x)dx$. The corresponding Abelian integral is in the simple form,

$$\mathbb{I}(h) = \oint_{\Gamma_h} f(x)ydx.$$

It is known that the sharp bound on the maximal number of zeros of $\mathbb{I}(h)$ on a periodic annulus is the annulus cyclicity (for a concrete system) by Poincaré-Pontryagin-Andronov Theorem [23]. The cyclicity is denoted by $\mathbb{Z}_L(m, n)$ with L representing the Liénard system. However, even for the simple form, it is not easy to determine the cyclicity $\mathbb{Z}_L(m,n)$, which was only completely determined for type $(m,m-1)$ with $m=2,3$, see [13, 14, 15, 16]. Type $(m,m-1)$ means that the perturbation term $-\varepsilon f(x)y$ and the restoring term $g(x)$ have the same degree. For type $(5,4)$ of system (3.4) with symmetry, since there are only three perturbation terms, the Picárd-Fuchs equation method can be applied. It has been proved that 2 is the sharp bound if the unperturbed system has a heteroclinic loop [3, 4, 37, 44]. It becomes much more difficult when system (3.4) is non-symmetric or has degree equal to or larger than 4, implying that $\mathbb{I}(h)$ has more than 3 generating elements. Thus, the dimensions of the Picárd-Fuchs equation system and Ricatti equations are higher, which makes it troublesome in determining the intersection of the related planes and surfaces. On the other hand, it has been shown that the Chybeshev criterion [19, 31] can be applied to bound $\mathbb{Z}_L(m,n)$ for Abelian integrals with more than 3 elements, see [6, 34, 35, 40, 41, 42, 45] for type $(4,3)$, but only an upper bound of $\mathbb{Z}_L(4,3)$ was obtained for each system investigated in these papers. Recently, Sun and Yu [38] improved the results by introducing a combination technique for two systems with a nilpotent singularity. There is no sharp bound reported for non-symmetric type $(5,4)$ systems.

Another way to study limit cycles of the generalized Liénard system (3.3) is to investigate the small limit cycles bifurcating from Hopf singularities. The exact bound on the maximal number of small limit cycles due to Hopf bifurcation is usually called Hopf cyclicity. For convenience, we denote the Hopf cyclicity of system (3.3) by $\mathbb{H}_L^s(m,n)$, where s represents small limit cycles. There are lots of results on $\mathbb{H}_L^s(m,n)$ which were obtained by computing Lyapunov coefficients. In 2006, it was proved by Yu and Han [43] that $\mathbb{H}_L^s(4,n)=\mathbb{H}_L^s(n,4)$ and $\mathbb{H}_L^s(5,n)=\mathbb{H}^s(n,5)$ for $n=10,11,12,13$, and $\mathbb{H}_L^s(6,n)=\mathbb{H}_L^s(n,6)$ for $n=5,6$. Other exact values of $\mathbb{H}_L^s(m,n)$ for some fixed values of m and n were summarized in Table 1 of [29]. We have noticed that these results were mainly obtained by studying $g(x)$ in the form of $g(x)=-x+\varepsilon g_m(x)$ with $\deg g_m(x)=m$. In other words, they perturb a linear center. When the degrees m and n are not fixed, two better lower bounds on $\mathbb{H}_L^s(m,n)$ were estimated in [29] and [22]. The averaging method of order 1, 2 or 3 was applied in [29] to system (3.3) by assuming

$$(g(x),f(x))=\Big(\sum_{k\ge1}\varepsilon^k g_m^k(x),\sum_{k\ge1}\varepsilon^k f_n^k(x)\Big),$$

while $g(x)=\bar{g}_m(x)+\varepsilon g_m(x)$ was taken in [22]. However, fewer results were reported on $\mathbb{H}_L^s(m,n)$ for arbitrary values of m or n. Up to now, we only know that, see [7, 30],

$$\begin{aligned}
&\mathbb{H}_L^s(m,n)=\frac{n}{2}\ \text{ if } g(x) \text{ is an odd degree polynomial;}\\
&\mathbb{H}_L^s(m,n)=\frac{n}{2}\ \text{ if } f(x) \text{ is an even degree polynomial;}\\
&\mathbb{H}_L^s(m,2n+1)=\left[\frac{m-2}{2}\right]+n\ \text{ if } f(x) \text{ is an odd degree polynomial;}\\
&\mathbb{H}_L^s(2m,2)=m \text{ if } g(x)=x+g_e(x)\ \text{ with } g_e(x)\ \text{ being an even degree polynomial.}
\end{aligned}$$

The above results were obtained with a strict assumption on $f(x)$ or $g(x)$. Moreover, in the last three decades, few results were obtained with similar restrictions on the damping and restoring terms. The main difficulty comes from analyzing the dimension of the related algebraic variety of the set of the Lyapunov coefficients.

For fixed $m = 2$, it was proved respectively by Han [20, 21], and Christopher and Lynch [9] that

$$\mathbb{H}_L^s(2,n) = \mathbb{H}_L^s(n,2) = \left[\frac{2n+1}{3}\right]$$

for all $n \geq 1$. When $m = 3$, it was proved in [9] that system (3.3) with $g(x) = -x(2+3x+4bx^2)$ has the Hopf cyclicity at the origin,

$$\mathbb{H}_L^s(3,n) = \mathbb{H}_L^s(n,3) = 2\left[\frac{3n+2}{8}\right], \quad 1 \leq n \leq 50.$$

Recently, Tian *et.al* [39] studied an equivalent system to the one in [9] by taking $g(x) = -x(x - 2x + ax^2)$, and proved that the Hopf cyclicity near the origin is

$$\mathbb{H}_L^s(3,n) = \left[\frac{3n+2}{4}\right] \ \text{ for } n \geq 1 \text{ if } a = \frac{8}{9}.$$

It should be noted that when $m \geq 3$, system (3.3) may have rich topological phase portraits due to the complicated topological phase portraits of the system $\ddot{x} + g(x) = 0$, for example, there may exist more than one singularity of focus type except the singularity at the origin. Therefore, $\mathbb{H}_L^s(3,n)$ $(n \geq 1)$ only includes the number of small limit cycles bifurcating from the origin.

For fixed $m \geq 4$, however, there are no results reported on the Hopf cyclicity for any type (m,n) of system (3.3) with arbitrary $n \geq 1$ due to the difficulty arising from stronger nonlinear restoring term $g(x)$.

In this chapter, we study a non-symmetric system (3.3) with $m = 5$. It has a unique singularity of centre-focus type and the undamped system ($f(x) \equiv 0$) has a unique periodic annulus. This periodic annulus is bounded by a non-symmetric heteroclinic loop connecting a degenerate singularity. We study the annulus cyclicity for $f(x) = \varepsilon \sum_{i=0}^{4} \alpha_i x^i$ with $\varepsilon > 0$ sufficiently small, for which the damping term $\varepsilon f(x) y$ has the same degree as that of the undamped system. We also study the small limit cycles near the origin and determine the Hopf cyclicity when the damping term is an nth degree smooth polynomial or a piecewise smooth polynomials. The system is given in the form of

$$\dot{x} = y, \quad \dot{y} = x(x-1)^3\Big(x + \frac{1}{2}\Big) + f(x)y, \tag{3.5}$$

where $f(x) = f_i(x)$ $(i = 1,2,3)$ with

$$\begin{aligned}
f_1(x) &= \varepsilon(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4), \\
f_2(x) &= \sum_{i=0}^{n} \alpha_i x^i, \\
f_3(x) &= \begin{cases} \displaystyle\sum_{i=0}^{n} \alpha_i^+ x^i, & x > 0, \\ \displaystyle\sum_{i=0}^{l} \alpha_i^- x^i, & x < 0, \end{cases}
\end{aligned}$$

where $\varepsilon > 0$ is sufficiently small, α_i and $\alpha_i^{\pm}$ are bounded parameters. The undamped system (3.5) is a Hamiltonian system with the Hamiltonian,

$$\mathcal{H}(x,y) = \frac{y^2}{2} + \frac{x^2}{4} - \frac{x^3}{6} - \frac{3x^4}{8} + \frac{x^5}{2} - \frac{x^6}{6}.$$

There is a family of closed orbits $\Gamma_h = \{(x,y)|\mathcal{H}(x,y) = h, h \in (0, \frac{1}{24})\}$, which forms a unique periodic annulus $\{\Gamma_h\}$ bounded by a degenerate heteroclinic loop, denoted by Γ^*. The heteroclinic loop connects a hyperbolic saddle $(-\frac{1}{2}, 0)$ and a nilpotent saddle $(1,0)$, see Figure 3.1. For $f(x) = f_1(x)$, the associated Abelian integral is given by

$$\mathcal{A}(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h) + \alpha_3 I_3(h) + \alpha_4 I_4(h), \tag{3.6}$$

where

$$I_i(h) = \oint_{\Gamma_h} x^i y dx, \quad i = 0,\ 1,\ 2,\ 3,\ 4. \tag{3.7}$$

As discussed above, the difficulty in studying the bifurcation of limit cycles of system (3.5) arises from the non-symmetry, stronger nonlinearity of $g(x)$ and the degeneracy, implying that one has to consider more than 3 generating elements in the Abelian integral for studying the annulus cyclicity, which requires more efficient computation in dealing with the damping terms and their independence for studying the Hopf cyclicity. In fact, system (3.5) with $f(x) = f_1(x)$ was first studied by Ashegh *et al.* [5], who claimed that there are at most three limit cycles bifurcating from the periodic annulus by analyzing the first order Poincaré bifurcation, and the three limit cycles can be obtained near the boundary of the annulus. The result implies that the cyclicity of the periodic annulus is three when $f(x) = f_1(x)$. However, the result was questionable because there exists a discrepancy between the symbolic computation and numerical analysis. As a matter of fact, two years later, Sun *et al.* [36] reconsidered the problem and provided a rigorous proof, but only an upper bound was obtained. For convenience, the results obtained in [5, 36] are summarized in the following theorem.

Theorem 3.1.1 ([5, 36]) *For system* (3.5) *with* $f(x) = f_1(x)$*,*

(i) *there exist no closed orbits enclosing three singularities* $(-\frac{1}{2}, 0)$*,* $(0,0)$ *and* $(1,0)$ *for all possible bounded parameters* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^5$*;*

(ii) $\mathcal{A}(h)$ *has at most four zeros in* $(0, \frac{1}{24})$*, and three zeros can be reached near the endpoints of the interval* $(0, \frac{1}{24})$*, implying that there are at most four limit cycles bifurcating from the periodic annulus for sufficiently small* $\varepsilon > 0$*, and three limit cycles can be obtained either near the singularity* $(0,0)$ *or near the heteroclinic loop.*

Therefore, it is still unknown whether the annulus cyclicity is three or four for system (3.5) when $f(x) = f_1(x)$. In this chapter, we will provide a rigorous proof to give a positive answer on the exact cyclicity. This result is stated in the following theorem.

Theorem 3.1.2 *For system* (3.5) *with* $f(x) = f_1(x)$*, the Abelian integral* $\mathcal{A}(h)$ *has at most three zeros in* $(0, \frac{1}{24})$ *for all possible* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^5$*, and this is the sharp bound, i.e., the cyclicity of the periodic annulus is three.*

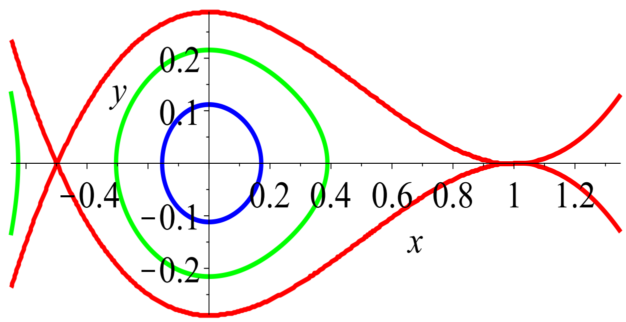

Figure 3.1: The phase portrait of the undamped system (3.5), showing that the heteroclinic loop (red color) connects a hyperbolic saddle $(-\frac{1}{2}, 0)$ and a nilpotent saddle $(1, 0)$.

In addition, we study the Hopf cyclicity of the unique center-focus singularity at the origin for different types of damping terms. In particular, for smooth dampings, we have the result on the Hopf cyclicity of the origin as follows.

Theorem 3.1.3 *For system* (3.5) *with* $f(x) = f_2(x)$, *the Hopf cyclicity of the origin is* $\left[\frac{2n+1}{3}\right]$.

When the damping term is a piecewise smooth polynomial in x of degree l and n, we have the following result.

Theorem 3.1.4 *For system* (3.5) *with* $f(x) = f_3(x)$, *the Hopf cyclicity of the origin is* $\left[\frac{3n+2l+4}{3}\right]$ *if* $n \geq l$ *or* $\left[\frac{3l+2n+4}{3}\right]$ *if* $l \geq n$.

The main mathematical tools that we will apply to prove Theorem 3.1.2 are asymptotic property and Chebyshev criterion of the Abelian integrals $\{I_i(h)\}_{i=0}^4$. We will introduce three combinations of related two Abelian integrals to obtain three new integral systems including a parameter, and then apply the Chebyshev criterion to the new systems. The range of each parameter in the integral system is then bounded via the algebraic property of the curves and number of zeros of the algebraic system. The algebraic system is derived from the ratio of two Wronskians. The ranges of three parameters give a bounded 3-dimensional parameter set on which the full Abelian integral may have four zeros. A further analysis is carried out to exclude the possibility of 4 zeros of the Abelian integrals. Properly combing the generating elements plays a crucial role in obtaining the sharp bound, since directly applying Chebyshev criterion fails [5, 36]. To prove Theorems 3.1.3 and 3.1.4, we properly utilize the potential in the undamped system (3.5) to define an involution, and then to introduce two transformations composed of trigonometric functions for the two components of the involution. This makes it possible to analyze the independence of the elements in algebraic variety, finally yielding the Hopf cyclicity for the smooth and non-smooth damping system (3.5).

The rest of this chapter is organized as follows. In section 2, we present some preliminaries, which contain certain new theories and methods on Poincaré bifurcation and Hopf bifurcation, and an extended Chebyshev criterion. We prove Theorems 3.1.2, 3.1.3 and 3.1.4 in sections 3, 4 and 5, respectively. Conclusion is drawn in section 6.

3.2 Chebyshev criterion and local bifurcation theory

3.2.1 Chebyshev criterion

In this subsection, we briefly present the Chebyshev criterion developed in [19, 31], which is one of the basic tools for proving our main results.

Definition 3.2.1 *Suppose* $s_0(x), s_1(x), \cdots$ *and* $s_{m-1}(x)$ *are analytic functions on a real open interval* Ω.

(A) *The continuous Wronskian of* $\{s_0(x), s_1(x), \dots, s_{i-1}(x)\}$ *for* $x \in \Omega$ *is*

$$W[s_0(x), s_1(x), \dots, s_{i-1}(x)] = \begin{vmatrix} s_0(x) & s_1(x) & \cdots & s_{i-1}(x) \\ s_0'(x) & s_1'(x) & \cdots & s_{i-1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ s_0^{(i-1)}(x) & s_1^{(i-1)}(x) & \cdots & s_{i-1}^{(i-1)}(x) \end{vmatrix},$$

where $s_i^{(j)}(x)$ *is the jth order derivative of* $s_i(x)$, $j \geq 2$.

(B) *The set* $\{s_0(x), s_1(x), \dots, s_{m-1}(x)\}$ *is called a Chebyshev system if any nontrivial linear combination,*

$$\kappa_0 s_0(x) + \kappa_1 s_1(x) + \cdots + \kappa_{m-1} s_{m-1}(x),$$

has at most $m-1$ *isolated zeros on* Ω. *Note that* $W[s_0(x), s_1(x), \dots, s_{m-1}(x)] \neq 0$ *is one sufficient condition assuring* $\{s_0(x), s_1(x), \dots, s_{m-1}(x)\}$ *to form a Chebyshev system.*

(C) *The ordered set* $\{s_0(x), s_1(x), \dots, s_{m-1}(x)\}$ *is called an extended complete Chebyshev* (ECT) *system if for each* $i \in \{1, 2, \cdots, m\}$ *any nontrivial linear combination,*

$$\kappa_0 s_0(x) + \kappa_1 s_1(x) + \cdots + \kappa_{i-1} s_{i-1}(x),$$

has at most $i-1$ *zeros with multiplicities counted.*

Let $H(x, y) = V(x) + \frac{y^2}{2}$ be an analytic function with $xV'(x) > 0$ and $V(0) = 0$. There exists a family of closed ovals $\{\Gamma_h\} \subseteq \{(x, y)|H(x, y) = h,\ h \in (0, h^*)\}$ surrounding the origin $(0, 0)$, where $h^* = H(\partial\{\Gamma_h\})$. The projection of $\{\Gamma_h\}$ on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$. $V(x) = V(z(x))$ defines an analytic involution $z = z(x)$ for all $x \in (x_l, x_r)$. Let

$$\mathbb{I}_i(h) = \oint_{\Gamma_h} \xi_i(x) y^{2n^*-1} dx \ \text{ for } h \in (0, h^*), \tag{3.8}$$

where $n^* \in \mathbb{N}$ and $\xi_i(x)$ is analytic in (x_l, x_r), $i = 0, 1, \dots, m-1$. Further, define

$$s_i(x) := \frac{\xi_i(x)}{V'(x)} - \frac{\xi_i(z(x))}{V'(z(x))}. \tag{3.9}$$

Then we have

Lemma 3.2.2 ([19]) *Consider the integrals* $\mathbb{I}_i$ *in* (3.8) *and the functions* s_i *in* (3.9). $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *is an* ECT *system in* $(0, h^*)$ *if* $n^* > m-2$ *and*
$\{s_0, s_1, \cdots, s_{m-1}\}$ *is an* ECT *system in* $(x_l, 0)$ *or* $(0, x_r)$.

Lemma 3.2.3 ([31]) *Consider the integrals* (3.8) *and the functions* (3.9). *If the following conditions hold:*
(a) $W[s_0, s_1, \ldots, s_i]$ *does not vanish in* $(0, x_r)$ *for* $i = 0, 1, \cdots, m-2$,
(b) $W[s_0, s_1, \ldots, s_{m-1}]$ *has* k *zeros in* $(0, x_r)$ *with multiplicities counted, and*
(c) $n^* > m + k - 2$,
then any nontrivial linear combination of $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *has at most* $m + k - 1$ *zeros in* $(0, h^*)$ *with multiplicities counted. In this case, we call* $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *a Chebyshev system with accuracy* k *in* $(0, h^*)$.

3.2.2 Hopf bifurcation theory for Liénard system

Computing and analyzing the Lyapunov coefficients of Poincaré map, which is locally constructed around a foci, is the classical method to study Hopf bifurcation of general planar differential systems. However, it is not an easy task for computing the Lyapunov coefficients, which are needed to analyze the algebraic varieties in order to determine Hopf cyclicity. For Liénard type system, one equivalent method to computing the Lyapunov coefficients was developed in [20, 21], which is summarized as follows.

Consider the system of the form,

$$\dot{x} = P(y) - F(x, \eta), \quad \dot{y} = -g(x), \tag{3.10}$$

where η is an n-dimensional parameter vector, $P(y)$, $F(x, \eta)$ and $g(x)$ are analytic satisfying $P'(0)g'(0) > 0$, $F(0, \eta) = g(0) = P(0) = 0$ and $F_x(0, \eta^*) = 0$ for some $\eta^* \in \mathbb{R}^n$. These assumptions assure that the origin is a center or focus of system (3.10) for η chosen from a very small neighborhood of η^*. Then one can construct the Poincaré map locally around the origin, which has the following expansion,

$$\mathcal{P}(r, \eta) = \sum_{i=1}^{\infty} v_i(\eta) r^i \;\text{ for } |r| \ll 1 \text{ and } |\eta - \eta^*| \ll 1, \tag{3.11}$$

where $v_i(\eta) \in C^\infty$. An isolated positive zero of $\mathcal{P}(r, \eta)$ near $r = 0$ corresponds to a small limit cycle of system (3.10) due to Hopf bifurcation. Therefore, it only needs to study the sharp upper bound on the maximal number of isolated positive zeros of $\mathcal{P}(r, \eta)$ for studying the Hopf cyclicity of a focus or a center. Particularly, we have the following expansion of the bifurcation function for the generalized Liénard system (3.10),

$$F(z(x), \eta) - F(x, \eta) = \sum_{i=1}^{\infty} B_i(\eta) x^i \;\text{ for } 0 < x \ll 1, \tag{3.12}$$

where $z(x)$ is the involution defined by the potential $G(x) = \int g(x)dx$ with $G(z(x)) = G(x)$. It was proved in [20, 21] that,

$$\begin{aligned}
v_1 &= N_1(B_1)B_1, \\
v_{2j} &= O(B_1, B_3, \cdots, B_{2j-1}), \\
v_{2j+1} &= N_{2j+1}(B_1)B_{2j+1} + O(B_1, B_3, \cdots, B_{2j-1}),
\end{aligned}$$

where $N_{2j+1}(B_1) \in C^\infty$. Therefore, we only need compute B_i and analyze its algebraic variety $\{B_i = 0\}$ for all $i \geq 1$ to study the Hopf cyclicity. The following Lemma [21] states the criterion.

Lemma 3.2.4 *Consider system* (3.10) *and the expansion* (3.12). *Suppose there exists* $k \geq 1$ *such that*

$$F(z(x), \eta) \equiv F(x, \eta),\ B_{2j+1} = 0$$

for $j = 0, 1, \cdots, k$ *and there exists some* $\eta^* \in \mathbb{R}^n$ *such that*

$$B_{2j+1}(\eta^*) = 0,\ j = 0, 1, \cdots, k,$$
$$\operatorname{rank}\left[\frac{\partial(B_1, B_3, \cdots, B_{2k+1})}{\partial \eta}\Big|_{\eta=\eta^*}\right] = k + 1.$$

Then the Hopf cyclicity of system (3.10) *at the origin is* k*.*

Liu and Han [28] extended the theory to study Hopf bifurcation of the following piecewise nonsmooth Liénard system,

$$(\dot{x}, \dot{y}) = \begin{cases} (P(y) - F^+(x, \eta), -g^+(x)),\ x > 0, \\ (P(y) - F^-(x, \eta), -g^-(x)),\ x < 0, \end{cases} \tag{3.13}$$

where η is an n dimensional parameter vector, $P(y)$, $g^{\pm}(x)$ and $F^{\pm}(x)$ are analytic and satisfy

$$P(0) = F^{\pm}(0, \eta) = g^{\pm}(0) = 0$$

and

$$(F_x^{\pm}(0, \eta^*))^2 - 4P'(0)(g^{\pm})'(0) < 0$$

for some $\eta^* \in \mathbb{R}^n$. Similarly, one can construct a Poincaré map expanded for η near η^* as

$$d(\rho, \eta) = v_1(\eta)\rho + v_2(\eta)\rho^2 + \cdots + v_j(\eta)\rho^j + \cdots,\quad 0 < \rho \ll 1,$$

and the bifurcation function similar to (3.12) is given by

$$F^-(z(x), \eta) - F^+(x, \eta) = \sum_{j=1}^{\infty} B_j(\eta)x^j, \tag{3.14}$$

for $0 < x \ll 1$, where $z(x)$ is the involution defined by $G^+(z(x)) = G^-(x)$ with $G^{\pm}(s) = \int g^{\pm}(s)ds$. Liu and Han [28] gave that

$$v_1(\eta) = W_1(\eta)B_1,$$
$$v_j(\eta) = W_j(\eta)B_j + O(|B_1, B_2, \cdots, B_{j-1}|),$$

where $W_j \in C^{\infty}$ and $W_j > 0$ for B_1 small. The following lemma gives the Hopf cyclicity.

Lemma 3.2.5 ([39]) *Let k positive integers satisfy* $r_1 < r_2 < \cdots < r_k$ *and form the ordered sequence* $\{r_i\}_{i=1}^k$*. If the following items are verified:*
(i) $B_j(\eta) \equiv 0$ *for* $0 \leq j < r_1$*;*
(ii) $B_j(\eta) = O(B_{r_1}, B_{r_2}, \cdots, B_{r_{s(j)}})$ *where* $r_{s(j)} = \max\{r_s < j\}$*;*
(i) *there exists some* η^* *such that* $B_{r_j}(\eta^*) = 0$ *for* $0 \leq j \leq k$ *and*

$$\operatorname{rank}\left[\frac{\partial(B_{r_1}, B_{r_2}, \cdots, B_{r_k})}{\partial \eta}\Big|_{\eta=\eta^*}\right] = k,$$

then the Hopf cyclicity of system (3.13) *at the origin is* $k - 1$*.*

3.3 Proof of Theorem 3.1.2

In this section, we prove Theorem 3.1.2. We divide the parameter space for $\mathcal{A}(h)$ to obtain a cube which is the only set for $\mathcal{A}(h)$ to might have 4 zeros on $h \in (0, \frac{1}{24})$. As it was shown in [36] that $I_1(h) \equiv I_2(h)$, $\mathcal{A}(h)$ is spanned by

$$\{I_0(h), I_1(h), I_3(h), I_4(h)\}.$$

Hence, it only needs to analyze the set $\{I_0(h), I_1(h), I_3(h), I_4(h)\}$.

We write $V(x) = \mathcal{H}(x,y) - \frac{y^2}{2}$. Then

$$v(x,z) := \frac{V(x) - V(z)}{x - z} = 0$$

defines the involution $z(x)$, $x \in (0,1)$ on the periodic annulus. We have the following result.

Lemma 3.3.1 *The following equations hold:*

$$8h^3 I_i(h) = \oint_{\Gamma_h} \rho_i(x) y^7 dx \equiv \widetilde{I}_i(h), \quad i = 0,\ 1,\ 3, 4,$$

where $\rho_i(x) = \frac{x^i g_i(x)}{22680(1+2x)^6(x-1)^{12}}$, *in which each polynomial* $g_i(x)$ *has degree* 18.

Proof First, multiplying $I_i(h)$ by $\frac{y^2+2V(x)}{2h} = 1$ yields

$$\begin{aligned} 8h^3 I_i(h) &= \oint_{\Gamma_h} (2V(x) + y^2)^3 x^i y dx \\ &= \oint_{\Gamma_h} 8\, x^i V^3(x) y dx + \oint_{\Gamma_h} 12\, x^i V^2(x) y^3 dx \\ &+ \oint_{\Gamma_h} 6\, x^i V(x) y^5 dx + \oint_{\Gamma_h} x^i y^7 dx,\ i = 0, 1, 3, 4. \end{aligned} \tag{3.15}$$

Then applying Lemma 4.1 in [19] to (3.15) to increase the power of y in the first three integrals to 7 proves the lemma.

Without loss of generality, we assume that $\alpha_4 = 1$ when $\alpha_4 \neq 0$. Further, introduce the following combinations:

$$\begin{aligned} \mathcal{I}_{34}(h) &= \oint_{\Gamma_h} \left(\alpha_3 x^3 + x^4\right) y dx, \\ \mathcal{I}_{14}(h) &= \oint_{\Gamma_h} \left(\alpha_1 x + x^4\right) y dx, \\ \mathcal{I}_{04}(h) &= \oint_{\Gamma_h} \left(\alpha_0 + x^4\right) y dx. \end{aligned} \tag{3.16}$$

Then

$$\begin{aligned} \mathcal{A}(h) &= \alpha_0 I_0(h) + \alpha_1 I_1(h) + \mathcal{I}_{34}(h) \\ &= \alpha_0 I_0(h) + \alpha_3 I_3(h) + \mathcal{I}_{14}(h) \\ &= \alpha_1 I_1(h) + \alpha_3 I_3(h) + \mathcal{I}_{04}(h). \end{aligned}$$

The following lemma directly follows from Lemma 3.3.1.

Lemma 3.3.2 *The following equations hold:*

$$
\begin{aligned}
8h^3\mathcal{I}_{34}(h) &= \oint_{\Gamma_h}(\alpha_3\rho_3(x)+\rho_4(x))y^7dx \stackrel{\triangle}{=} \widetilde{\mathcal{I}}_{34}(h),\\
8h^3\mathcal{I}_{14}(h) &= \oint_{\Gamma_h}(\alpha_1\rho_1(x)+\rho_4(x))y^7dx \stackrel{\triangle}{=} \widetilde{\mathcal{I}}_{14}(h),\\
8h^3\mathcal{I}_{04}(h) &= \oint_{\Gamma_h}(\alpha_0\rho_0(x)+\rho_4(x))y^7dx \stackrel{\triangle}{=} \widetilde{\mathcal{I}}_{04}(h).
\end{aligned}
$$

Now, let

$$
\begin{aligned}
&l_i(x) = \Big(\frac{\rho_i}{V'}\Big)(x) - \Big(\frac{\rho_i}{V'}\Big)(z(x)),\ i = 0,1,3,4.\\
&\mathcal{L}_{i4}(x) = \Big(\frac{\alpha_i\rho_i+\rho_4}{V'}\Big)(x) - \Big(\frac{\alpha_i\rho_i+\rho_4}{V'}\Big)(z(x)),\ i = 0,1,3.
\end{aligned}
\tag{3.17}
$$

Then

$$
\begin{aligned}
&\frac{d}{dx}l_i(x) = \frac{d}{dx}\Big(\frac{\rho_i}{V'}\Big)(x) - \frac{d}{dz}\left(\Big(\frac{\rho_i}{V'}\Big)(z(x))\right)\times\frac{dz}{dx},\\
&\frac{d}{dx}\mathcal{L}_{i4}(x) = \frac{\partial}{\partial x}(\mathcal{L}_{i4}(x)) + \frac{\partial}{\partial z}(\mathcal{L}_{i4}(x))\times\frac{dz}{dx},\ i = 0,1,3,
\end{aligned}
$$

where $\frac{dz}{dx} = -\frac{v_x(x,z)}{v_z(x,z)}$. A direct computation yields

$$
\begin{aligned}
W[l_0] &= \tfrac{(x-z)W_0(x,z)}{11340\,xz(2x+1)^7(x-1)^{15}(2z+1)^7(z-1)^{15}},\\
W[l_1] &= \tfrac{(x-z)W_1(x,z)}{11340\,(2x+1)^7(x-1)^{15}(2z+1)^7(z-1)^{15}},\\
W[l_0,l_1] &= \tfrac{(x-z)^3W_{01}(x,z)}{128595600\,x^2z^2(z-1)^{30}(2\,z+1)^{13}(x-1)^{30}(2\,x+1)^{13}w_0(x,z)},\\
W[l_0,l_3] &= \tfrac{(x-z)^3W_{03}(x,z)}{128595600\,x^2z^2(z-1)^{30}(2\,z+1)^{13}(x-1)^{30}(2\,x+1)^{13}w_0(x,z)},\\
W[l_0,l_4] &= \tfrac{(x-z)^3W_{04}(x,z)}{128595600\,x^2z^2(z-1)^{30}(2\,z+1)^{13}(x-1)^{30}(2\,x+1)^{13}w_0(x,z)},\\
W[l_1,l_3] &= \tfrac{(x-z)^3W_{13}(x,z)}{128595600\,(z-1)^{30}(2\,z+1)^{13}(x-1)^{30}(2\,x+1)^{13}w_0(x,z)},\\
W[l_0,l_1,l_3] &= \tfrac{(x-z)^6W_{013}(x,z)}{w^*x^3z^3(x-1)^{45}(2\,x+1)^{18}(z-1)^{45}(2\,z+1)^{18}w_0^3(x,z)},\\
W[l_0,l_1,\mathcal{L}_{34}] &= \tfrac{(x-z)^6\mathcal{W}_3(x,z,\alpha_3)}{w^*x^3z^3(x-1)^{45}(2\,x+1)^{18}(z-1)^{45}(2\,z+1)^{18}w_0^3(x,z)},\\
W[l_0,l_3,\mathcal{L}_{14}] &= \tfrac{(x-z)^6\mathcal{W}_1(x,z,\alpha_1)}{w^*x^3z^3(x-1)^{45}(2\,x+1)^{18}(z-1)^{45}(2\,z+1)^{18}w_0^3(x,z)},\\
W[l_1,l_3,\mathcal{L}_{04}] &= \tfrac{(x-z)^6\mathcal{W}_0(x,z,\alpha_0)}{w^*x^3z^3(x-1)^{45}(2\,x+1)^{18}(z-1)^{45}(2\,z+1)^{18}w_0^3(x,z)},
\end{aligned}
\tag{3.18}
$$

where $z = z(x)$ is the involution as defined by $v(x,z) = 0$, $w^* = 729137052000$, $w_0(x,z) = 2x+4z-3$, W_0, W_1, W_{01}, W_{03}, W_{04}, W_{13} and W_{013} are polynomials of degrees 40, 39, 78, 80, 81, 77 and 115, respectively, and the polynomials $\mathcal{W}_3$, $\mathcal{W}_2$ and $\mathcal{W}_1$ have the degrees 116, 118 and 119, respectively.

We claim that the Wronskians are well defined for $x \in (0,1)$, because $w_0(x,z)$ does not vanish for $x \in (0,1)$ by showing that the resultant between $v(x,z)$ and $w_0(x,z)$ with respect to z has no roots for $x \in (0,1)$.

The following result indicates that we only need to discuss the case when $\alpha_4 \neq 0$.

Proposition 3.3.3 *When* $\alpha_4 = 0$, $\mathcal{A}(h)$ *has at most* 2 *zeros in* $(0, \frac{1}{24})$.

The proof of Proposition 3.3.3 relies on symbolic computation for verifying the nonvanishment of Wronskians $W[l_0]$, $W[l_0, l_1]$ and $W[l_0, l_1, l_3]$ according to Lemma 3.2.3. Since the symbolic computation and verification are straightforward, we omit the proof here for briefness.

To prove Theorem 3.1.2, we need to show non-vanishing of certain denominators and numerators of the related Wronskians in (3.18) for $x \in (0, 1)$. Taking the numerator $W_{01}(x, z)$ of the Wronskian $W[l_0, l_1]$, for example, we only need to prove that the 2-dimensional system $\{W_{01}(x, z), v(x, z)\}$ does not vanish on $\{(x, z) | -\frac{1}{2} < z < 0 < x < 1\}$, because z in $W_{01}(x, z)$ is determined by $v(x, z) = 0$, and $z(x) \in (-\frac{1}{2}, 0)$ when $x \in (0, 1)$. To do this, we apply triangular-decomposition and root isolating to $\{W_{01}(x, z), v(x, z)\}$ to decompose the nonlinear system into several triangular systems, and then isolate the roots of each triangular-decomposed system. Since all roots of these triangular systems are the roots of the original system $\{W_{01}(x, z), v(x, z)\}$, we only need to check if these decomposed systems have roots on $\{(x, z) | -\frac{1}{2} < z < 0 < x < 1\}$. This idea has been successfully applied to determine the zeros of Abelian integrals, see [34, 35, 36, 38, 45]. Instead of the triangular-decomposition method, one may also use the interval analysis [41], which computes two resultants between $W_{01}(x, z)$ and $v(x, z)$ with respect to x and z, respectively, yielding several two dimensional regions. Finally, one verifies if $W_{01}(x, z)$ vanishs on these regions by determining the intersection points of the curves $W_{01}(x, z)$ and $v(x, z)$, see [41] for details.

By applying the triangular-decomposition and root isolating to the numerators of the Wronskians, we obtain the following result.

Lemma 3.3.4 *All of the Wronskians,* $W[l_0]$, $W[l_1]$, $W[l_0, l_1]$, $W[l_0, l_3]$, $W[l_0, l_4]$ *and* $W[l_1, l_3]$, *do not vanish for* $x \in (0, 1)$.

Next, we investigate the last three Wronskians in (3.18). Their numerators have the forms,

$$\begin{aligned}
\mathcal{W}_3(x, z, \alpha_3) &= \alpha_3 S_2(x, z) - S_1(x, z),\\
\mathcal{W}_1(x, z, \alpha_1) &= \alpha_1 S_2^{\ddagger}(x, z) - S_1^{\ddagger}(x, z),\\
\mathcal{W}_0(x, z, \alpha_0) &= \alpha_0 S_2^{*}(x, z) - S_1^{*}(x, z),
\end{aligned}$$

where S_1, S_2, $S_1^{\ddagger}$, $S_2^{\ddagger}$, S_1^{*} and S_2^{*}, are polynomials of degrees 116, 115, 118, 115, 119 and 115, respectively. $\mathcal{W}_i(x, z, \alpha_i) = 0$ (for $i = 3, 1, 0$) defines three functions,

$$\alpha_3(x, z) = \frac{S_1(x, z)}{S_2(x, z)}, \quad \alpha_1(x, z) = \frac{S_1^{\ddagger}(x, z)}{S_2^{\ddagger}(x, z)}, \quad \alpha_0(x, z) = \frac{S_1^{*}(x, z)}{S_2^{*}(x, z)},$$

and their derivatives,

$$\begin{aligned}
\widetilde{\alpha}_3(x, z) &= \frac{\partial \alpha_3(x, z)}{\partial x} + \frac{\partial \alpha_3(x, z)}{\partial z} \times \frac{dz}{dx} = \frac{\widetilde{S}_1(x, z)}{\widetilde{S}_2(x, z)},\\
\widetilde{\alpha}_1(x, z) &= \frac{\partial \alpha_1(x, z)}{\partial x} + \frac{\partial \alpha_1(x, z)}{\partial z} \times \frac{dz}{dx} = \frac{\widetilde{S}_1^{\ddagger}(x, z)}{\widetilde{S}_2^{\ddagger}(x, z)},\\
\widetilde{\alpha}_0(x, z) &= \frac{\partial \alpha_0(x, z)}{\partial x} + \frac{\partial \alpha_0(x, z)}{\partial z} \times \frac{dz}{dx} = \frac{\widetilde{S}_1^{*}(x, z)}{\widetilde{S}_2^{*}(x, z)}.
\end{aligned}$$

Each of the denominators $S_2(x,z)$, $S_2^{\ddagger}(x,z)$, $S_2^{*}(x,z)$, $\widetilde{S}_2(x,z)$, $\widetilde{S}_2^{\ddagger}(x,z)$ and $\widetilde{S}_2^{*}(x,z)$ does not vanish for $x \in (0,1)$, because they do not have common roots with $v(x,z)$ for $(x,z) \in (0,1) \times (-\frac{1}{2}, 0)$, verified by triangular-decomposition and root isolating. Hence, all of the functions $\alpha_i(x,z)$ and $\widetilde{\alpha}_i(x,z)$ $(i = 3,1,0)$ are well defined for $x \in (0,1)$.

We have the following lemma.

Lemma 3.3.5 (i) *$\alpha_3(x,z(x))$ is decreasing from $(0, -\frac{3}{5})$ to a minimum $(x^{\dagger}, \alpha_3^{\dagger})$ and then increasing to $(1, -\frac{5}{2})$;*

(ii) *$\alpha_1(x,z(x))$ is increasing from $(0,0)$ to a maximum $(x^{\ddagger}, \alpha_1^{\ddagger})$ and then decreasing to $(1,2)$;*

(iii) *$\alpha_0(x,z(x))$ is decreasing from $(0,0)$ to a minimum (x^{*}, α_0^{*}) and then increasing to $(1, -\frac{1}{2})$, where*

$$x^{\dagger}, x^{\ddagger}, x^{*} \in \underbrace{\left[\frac{99571576491449}{140737488355328}, \frac{49785788245729}{70368744177664}\right]}_{1/10^{13}},$$

and

$$\alpha_3^{\dagger} \in \underbrace{\left[-\frac{44214\cdots 40352}{12885\cdots 98125}, -\frac{55559\cdots 02475}{16191\cdots 11584}\right]}_{1/10^{10}} \approx [-3.4312932408, -3.4312932406],$$

$$\alpha_1^{\ddagger} \in \underbrace{\left[\frac{31388\cdots 59375}{94249\cdots 82976}, \frac{16836\cdots 18911}{50552\cdots 20000}\right]}_{1/10^{10}} \approx [3.3303719012, 3.3303719013],$$

$$\alpha_0^{*} \in \underbrace{\left[-\frac{66176\cdots 98433}{72854\cdots 59040}, -\frac{53977\cdots 21875}{59424\cdots 65344}\right]}_{1/10^{11}} \approx [-0.90833169736, -0.90833169731].$$

Proof We only prove case (i), since other two cases (ii) and (iii) can be similarly proved. A direct computation shows that

$$\lim_{x\to 0} \alpha_3(x,z(x)) = -\frac{3}{5}, \qquad \lim_{x\to 1} \alpha_3(x,z(x)) = -\frac{5}{2}.$$

On $\{(x,z) | -\frac{1}{2} < z < 0 < x < 1\}$, $\widetilde{S}_1(x,z)$ and $v(x,z)$ have a unique common root $(x^{\dagger}, z^{\dagger}) \in D^{\dagger}$, where

$$\begin{aligned} D^{\dagger} &= \left[\frac{99571576491449}{140737488355328}, \frac{49785788245729}{70368744177664}\right] \\ &\quad \times \left[-\frac{32492637936074023}{72057594037927936}, -\frac{129970551744296065}{288230376151711744}\right]. \end{aligned}$$

$x^{\dagger}$ is the unique simple zero of $\widetilde{\alpha}_3(x,z(x))$ by verifying that $\frac{d}{dx}\widetilde{\alpha}_3(x,z(x))$ has no zeros on $[\frac{99571576491449}{140737488355328}, \frac{49785788245729}{70368744177664}]$. Therefore, $x^{\dagger}$ is the unique critical point of $\alpha_3(x,z(x))$, and thus the monotonicity of $\alpha_3(x,z(x))$ in $(0,x^{\dagger}) \bigcup (x^{\dagger}, 1)$ can be easily determined by comparing the values of $\alpha_3(x,z(x))$ at $x = 0$, $x^{\dagger}$ and 1 as $-\frac{3}{5}$, $-3.4312932408\cdots$ and $-\frac{5}{2}$, respectively. Alternatively, using

$$\lim_{x\to 0^+} \widetilde{\alpha}_3(x,z(x)) = 0^- \quad \text{and} \quad \lim_{x\to 0^+} \frac{d}{dx}(\widetilde{\alpha}_3(x,z(x))) = -\frac{147}{25} < 0,$$

we know that $\alpha_3(x,z(x))$ is monotonically deceasing in $(0,x^\dagger)$ and monotonically increasing in $(x^\dagger,1)$

It can be further shown that the resultant between $\frac{\partial S_i(x,z)}{\partial x}$ $(i=1,2)$ and $v(x,z)$ with respect to z has no roots over the interval

$$\left[\frac{99571576491449}{140737488355328}, \frac{49785788245729}{70368744177664}\right]$$

by Sturm's Theorem. Hence, $S_i(x,z)$ $(i=1,2)$ reaches its maximal and minimal values on the boundaries of $D^\dagger$. A direct computation yields

$$\begin{aligned}
\min_{D^\dagger} S_1(x,z) &= -\frac{21275\cdots15705}{13764\ldots10976} \approx -0.154565249238,\\
\max_{D^\dagger} S_1(x,z) &= -\frac{20094\ldots34375}{13000\ldots54592} \approx -0.154565249236,\\
\min_{D^\dagger} S_2(x,z) &= \frac{26631\ldots09375}{59119\ldots40896} \approx 0.045045770906,\\
\max_{D^\dagger} S_2(x,z) &= \frac{14098\ldots51625}{31297\ldots78144} \approx 0.045045770908.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\alpha_3^\dagger = \alpha_3(x^\dagger, z(x^\dagger)) &\in \left[\frac{\min_{D^\dagger} S_1(x,z)}{\min_{D^\dagger} S_2(x,z)}, \frac{\max_{D^\dagger} S_1(x,z)}{\max_{D^\dagger} S_2(x,z)}\right]\\
&= \underbrace{\left[-\frac{44214\cdots40352}{12885\cdots98125}, -\frac{55559\cdots02475}{16191\cdots11584}\right]}_{1/10^{10}}\\
&\approx [-3.4312932408, -3.4312932406].
\end{aligned}$$

Note in the above proof we have used symbolic computation which gives the exact expressions using rational numbers, demonstrating the accuracy of our analysis. It is also noted that the critical point $(x^\dagger, \alpha_3(x^\dagger, z(x^\dagger)))$ divides the curve $\{(x, \alpha_3(x,z(x)))|0 < x < 1\}$ into two simple segments (curves). Each point on the two segments corresponds to a simple root of $\mathcal{W}_3(x,z(x),\alpha_3(x,z(x)))$, while $x^\dagger$ is a root of multiplicity 2. The following lemma follows from Lemma 3.3.5.

Lemma 3.3.6 *For $x\in(0,1)$, when α_3 is located in the intervals $[\alpha_3^\dagger, -\frac{5}{2})$, $[-\frac{5}{2}, -\frac{3}{5})$ and $(-\infty, \alpha_3^\dagger) \bigcup [-\frac{3}{5}, +\infty)$, $W[l_0, l_1, \mathcal{L}_{34}]$ has 2, 1 and 0 roots with multiplicities counted, respectively.*

Combining Lemmas 3.3.4 and 3.3.6 and applying Lemma 3.2.3, we have the following result.

Proposition 3.3.7 *$\mathcal{A}(h)$ has at most 4, 3 and 2 zeros in $(0, \frac{1}{24})$ when α_3 belongs to the intervals $[\alpha_3^\dagger, -\frac{5}{2})$, $[-\frac{5}{2}, -\frac{3}{5})$, and $(-\infty, \alpha_3^\dagger) \bigcup [-\frac{3}{5}, +\infty)$, respectively.*

Similarly, we have

Proposition 3.3.8 *$\mathcal{A}(h)$ has at most 4, 3 and 2 zeros in $(0, \frac{1}{24})$ when α_1 is located in the intervals $(2, \alpha_1^{\ddagger}]$, $(0, 2]$, and $(-\infty, 0] \bigcup (\alpha_1^{\ddagger}, +\infty)$, respectively.*

Proposition 3.3.9 *$\mathcal{A}(h)$ has at most 4, 3 and 2 zeros in $(0, \frac{1}{24})$ when α_0 belongs to the intervals $[\alpha_0^*, -\frac{1}{2})$, $[-\frac{1}{2}, 0)$, and $(-\infty, \alpha_0^*) \bigcup [0, +\infty)$, respectively.*

Define

$$\mathcal{D} = \left\{(\alpha_0, \alpha_1, \alpha_3) | \alpha_0 \in \left[\alpha_0^*, -\frac{1}{2}\right), \ \alpha_1 \in \left(2, \alpha_1^{\ddagger}\right], \ \alpha_3 \in \left[\alpha_3^{\dagger}, -\frac{5}{2}\right)\right\}.$$

Then Propositions 3.3.7, 3.3.8 and 3.3.9 imply that

Proposition 3.3.10 *$\mathcal{A}(h)$ may have 4 zeros only if $(\alpha_0, \alpha_1, \alpha_3) \in \mathcal{D}$.*

Finally, we prove that $\mathcal{A}(h)$ cannot have 4 zeros when $(\alpha_0, \alpha_1, \alpha_3) \in \mathcal{D}$. First, we have

Lemma 3.3.11 *For $h \in (0, \frac{1}{24})$, the following hold:*

(1) *the generating element $I_0(h)$ is positive;*

(2) *the ratio $\frac{I_1(h)}{I_0(h)}$ is increasing from 0 to $\frac{1}{10}$;*

(3) *the ratio $\frac{I_3(h)}{I_0(h)}$ is increasing from 0 to $\frac{1}{28}$; and*

(4) *the ratio $\frac{I_4(h)}{I_0(h)}$ is increasing from 0 to $\frac{31}{1120}$.*

Proof By Green formula, $I_0(h) = \oint_{\Gamma_h} ydx = \iint_O dxdy$, where O represents the region bounded by Γ_h (a periodic annulus), and therefore, $I_0(h) > 0$. The non-vanishing property of $W[l_0]$, $W[l_0, l_1]$, $W[l_0, l_3]$ and $W[l_0, l_4]$ proved in Lemma 3.3.4 implies that $\frac{I_1(h)}{I_0(h)}$, $\frac{I_3(h)}{I_0(h)}$ and $\frac{I_4(h)}{I_0(h)}$ are monotonic in $(0, \frac{1}{24})$. By the expansion of $\mathcal{A}(h)$ near $h = 0$ (see the formulas given in [36]), we have

$$\lim_{h\to 0} \frac{I_1(h)}{I_0(h)} = \lim_{h\to 0} \frac{I_3(h)}{I_0(h)} = \lim_{h\to 0} \frac{I_4(h)}{I_0(h)} = 0.$$

Taking the limit as $h \to \frac{1}{24}$ yields

$$\lim_{h\to\frac{1}{24}} \frac{I_1(h)}{I_0(h)} = \lim_{h\to\frac{1}{24}} \frac{\oint_{\Gamma_h} xydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{1}{24}}} xydx}{\oint_{\Gamma_{\frac{1}{24}}} ydx} = \frac{1}{10},$$

$$\lim_{h\to\frac{1}{24}} \frac{I_3(h)}{I_0(h)} = \lim_{h\to\frac{1}{24}} \frac{\oint_{\Gamma_h} x^3ydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{1}{24}}} x^3ydx}{\oint_{\Gamma_{\frac{1}{24}}} ydx} = \frac{1}{28},$$

and

$$\lim_{h\to\frac{1}{24}} \frac{I_4(h)}{I_0(h)} = \lim_{h\to\frac{1}{24}} \frac{\oint_{\Gamma_h} x^4ydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{1}{24}}} x^4ydx}{\oint_{\Gamma_{\frac{1}{24}}} ydx} = \frac{31}{1120}.$$

Proposition 3.3.12 $\mathcal{A}(h) < 0$ *for* $(\alpha_0, \alpha_1, \alpha_3) \in \mathcal{D}$.

Proof When $(\alpha_0, \alpha_1, \alpha_3) \in \mathcal{D}$, by the results obtained in Lemma 3.3.11, it is easy to show that for $h \in (0, \frac{1}{24})$,

$$\begin{aligned}\mathcal{A}(h) &= \Big(\alpha_0 + \alpha_1\frac{I_1(h)}{I_0(h)} + \alpha_3\frac{I_3(h)}{I_0(h)} + \frac{I_4(h)}{I_0(h)}\Big)I_0(h)\\ &< \Big(\alpha_0 + \alpha_1\frac{I_1(h)}{I_0(h)} + \frac{I_4(h)}{I_0(h)}\Big)I_0(h)\\ &< \Big(-\frac{1}{2} + \alpha_1^{\ddagger} \times \frac{1}{10} + \frac{31}{1120}\Big)I_0(h)\\ &< 0.\end{aligned}$$

So $\mathcal{A}(h)$ has no zeros for $(\alpha_0, \alpha_1, \alpha_3) \in \mathcal{D}$.

Proof of Theorem 3.1.2. Combining Propositions 3.3.3, 3.3.10 and 3.3.12 proves Theorem 3.1.2.

3.4 Proof of Theorem 3.1.3

Taking the transformation, $\widetilde{y} = y - \int_0^x f_2(s)ds$, $\widetilde{x} = x$, reduces system (3.5) to the following form, after dropping the tilde,

$$\dot{x} = y - F(x,\delta), \quad y = x(x-1)^3\Big(x + \frac{1}{2}\Big), \tag{3.19}$$

where

$$F(x,\delta) = -\int_0^x f_2(s)ds = -\sum_{i=0}^{n}\frac{\alpha_i}{i+1}x^{i+1} := \sum_{i=1}^{N}\gamma_i x^i$$

with $N = n+1$, $\gamma_i = -\frac{1}{i}\alpha_{i-1}$ for $1 \le i \le N$, and $\delta = (\gamma_1, \cdots, \gamma_N) \in \mathbb{R}^N$.

In order to prove our main result on the Hopf cyclicity, we first introduce some known results.

Lemma 3.4.1 ([39]) *The following equalities hold for any constant* $\nu \in \mathbb{R}$,

1. $$\int_{-\pi}^{\pi} \sin^k(\theta+\nu)\sin(i\theta) = 0 \ \text{ if } i > k;$$

2. $$\int_{-\pi}^{\pi} \sin^k(\theta+\nu)\cos(i\theta) = 0 \ \text{ if } i > k;$$

3. $$\int_{-\pi}^{\pi} \sin^k(\theta+\nu)\sin(k\theta) = \frac{\pi}{2^{k-1}}\cos\Big(k\nu - \frac{k-1}{2}\pi\Big) \ \text{ if } k \in \mathbb{N}^+;$$

4.

$$\int_{-\pi}^{\pi} \cos^k(\theta+\nu)\sin(k\theta) = \frac{\pi}{2^{k-1}}\sin\Big(k\nu - \frac{k-1}{2}\pi\Big) \text{ if } k \in \mathbb{N}^+.$$

As discussed above, there exists an analytic involution $z(x)$ for the potential of the undamped system (3.5), defined on the periodic annulus by $v(x,z) = 2\,x^2+2\,xz+2\,z^2-3\,x-3\,z = 0$. Next, we introduce

$$x = \Theta(\theta) = \frac{1}{2} + \frac{\sqrt{3}}{2}\sin(\theta) + \frac{1}{2}\cos(\theta), \tag{3.20}$$

then

$$z = \Theta(-\theta) = \frac{1}{2} - \frac{\sqrt{3}}{2}\sin(\theta) + \frac{1}{2}\cos(\theta). \tag{3.21}$$

Let $J_k = \Theta^k(-\theta) - \Theta^k(\theta)$, and $K_k = \Theta^k(-\theta) + \Theta^k(\theta)$ for $k \in \mathbb{N}^+$. Then we have the following lemma, which establishes a key successful step in deriving the Hopf cyclicity.

Lemma 3.4.2 *For any $k \in \mathbb{N}^+$, we have*

$$J_k(\theta) = \sum_{i=1}^{k} c_{ki}\sin(i\theta), \quad K_k(\theta) = \sum_{i=1}^{k} \tilde{c}_{ki}\cos(i\theta), \tag{3.22}$$

where $c_{ki} = 0$ for i satisfying i mod $3 = 0$, $c_{kk} = -\frac{1}{2^{k-2}}\cos\left(\frac{k\pi}{6} - \frac{k-1}{2}\pi\right)$, $\tilde{c}_{kk} = \frac{1}{2^{k-2}}\sin\left(\frac{k\pi}{6} - \frac{k-1}{2}\pi\right)$.

Proof It is obvious that $\Theta(\theta)$ can be expressed as a linear combination of $\sin(i\theta)$ and $\cos(i\theta)$, $i = 1, 2, \cdots, k$. Then we have the formula (3.22) because $J_k(\theta)$ is an odd function and $K(\theta)$ an even one. By (3.20), we have

$$x = \Theta(\theta) = \frac{1}{2} + \sin\Big(\theta + \frac{\pi}{6}\Big).$$

By theory of Fourier series, we have

$$\begin{aligned} c_{kk} &= \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta^k(-\theta)\sin(k\theta)d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta^k(\theta)\sin(k\theta)d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta^k(-\theta)\sin(-k\theta)d(-\theta) - \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta^k(\theta)\sin(k\theta)d\theta \\ &= -\frac{2}{\pi}\int_{-\pi}^{\pi}\Theta^k(\theta)\sin(k\theta)d\theta \\ &= -\frac{2}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{s=0}^{k} C_k^s \frac{1}{2^{k-s}}\sin^s(\theta+\frac{\pi}{6})\Big)\sin(k\theta)d\theta. \end{aligned}$$

Applying Lemma 3.4.1, we obtain

$$c_{kk} = -\frac{2}{\pi}\int_{-\pi}^{\pi}\sin^k\Big(\theta+\frac{\pi}{6}\Big)\sin(k\theta)d\theta = -\frac{1}{2^{k-2}}\cos\Big(\frac{k\pi}{6} - \frac{k-1}{2}\pi\Big).$$

Similarly, we have

$$\begin{aligned}
\widetilde{c}_{kk} &= \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta^k(-\theta)\cos(k\theta)d\theta + \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta^k(\theta)\cos(k\theta)d\theta \\
&= -\frac{1}{\pi}\int_{-\pi}^{\pi}\Theta^k(-\theta)\cos(-k\theta)d(-\theta) + \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta^k(\theta)\cos(k\theta)d\theta \\
&= \frac{2}{\pi}\int_{-\pi}^{\pi}\Theta^k(\theta)\cos(k\theta)d\theta \\
&= \frac{2}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{s=0}^{k}C_k^s\frac{1}{2^{k-s}}\sin^s(\theta+\frac{\pi}{6})\Big)\cos(k\theta)d\theta.
\end{aligned}$$

It follows from Lemma 3.4.1 that

$$\widetilde{c}_{kk} = \frac{2}{\pi}\int_{-\pi}^{\pi}\sin^k\Big(\theta+\frac{\pi}{6}\Big)\cos(k\theta)d\theta = \frac{1}{2^{k-2}}\sin\Big(\frac{k\pi}{6}-\frac{k-1}{2}\pi\Big).$$

A direct computation shows that

$$\Theta\Big(u+\frac{2\pi}{3}\Big) = \Theta(-u).$$

Then

$$\Theta^k\Big(u+\frac{2\pi}{3}\Big) = \Theta^k(-u) \ \text{ for } k \in \mathbb{N}^+.$$

Further, for $i = 3j < k,\ j = 1, 2, \cdots, \left[\frac{k}{3}\right]$, we can show that

$$\begin{aligned}
\int_{-\pi}^{\pi}\Theta^k(\theta)\sin(3j\theta)d\theta &= \int_{-\frac{5\pi}{3}}^{\frac{\pi}{3}}\Theta^k\Big(u+\frac{2\pi}{3}\Big)\sin\Big(3j(u+\frac{2\pi}{3})\Big)du \\
&= \int_{-\frac{5\pi}{3}}^{\frac{\pi}{3}}\Theta^k\Big(u+\frac{2\pi}{3}\Big)\sin(3ju)du \\
&= \int_{-\frac{5\pi}{3}}^{\frac{\pi}{3}}\Theta^k(-u)\sin(3ju)du \\
&= \int_{-\pi}^{\pi}\Theta^k(-u)\sin(3ju)du.
\end{aligned}$$

Therefore,

$$c_{k,3j} = \frac{1}{\pi} \int_{-\pi}^{\pi} (\Theta^k(-u) - \Theta^k(\theta)) \sin(3ju) du = 0.$$

The proof is complete.

Proof of Theorem 3.1.3. It only needs to prove that the cyclicity of system (3.19) at the origin is $\left[\frac{2N-1}{3}\right]$.

By Lemma 3.2.4, we construct the following power series

$$F(z(x)) - F(x) = \sum_{k=1}^{N} \gamma_k (z^k - x^k) = \sum_{i \geq 1} B_i x^i \tag{3.23}$$

for $|x| \ll 1$, where $z(x)$ is the involution defined by $v(x, z) = 0$. We have two goals aiming at proving our result on the Hopf cyclicity. One goal is to prove the following relationship between the coefficients in $F(z(x)) - F(x)$,

$$B_{2j+1} = O(|B_1, B_3, \cdots, B_{2N^*+1}|), \quad j \geq N^* + 1, \tag{3.24}$$

where

$$N^* = N - \left[\frac{N}{3}\right] - 1 = \left[\frac{2N-1}{3}\right].$$

Another one is to show that

$$\operatorname{rank}\left[\frac{\partial(B_1, B_3, \cdots, B_{2N^*+1})}{\partial(\gamma_1, \gamma_2, \cdots, \gamma_N)}\right] = N^* + 1. \tag{3.25}$$

First, substituting the trigonometric transformations (3.20) and (3.21) into $F(z(x)) - F(x)$, we have

$$F(z(x)) - F(x) = \sum_{k=1}^{N} \gamma_k (\Theta^k(-\theta) - \Theta^k(\theta)) = \sum_{k=1}^{N} \gamma_k J_k(\theta) := \widetilde{F}(\theta). \tag{3.26}$$

By Lemma 3.4.2,

$$\widetilde{F}(\theta) = \sum_{k=1}^{N} \gamma_k J_k(\theta) = \sum_{k=1}^{N} \gamma_k \sum_{i=1}^{k} c_{ki} \sin(i\theta) = \sum_{i=1}^{N} \tilde{b}_i \sin(i\theta), \tag{3.27}$$

where

$$\tilde{b}_i = \sum_{k=1}^{N} \gamma_k c_{ki} \tag{3.28}$$

and

$$\tilde{b}_i = 0 \text{ for } i \text{ satisfying } i \bmod 3 = 0. \tag{3.29}$$

We have the following expansion of $\widetilde{F}(\theta)$ for θ near π,

$$\begin{aligned}\widetilde{F}(\theta) &= \sum_{i=1}^{N}\tilde{b}_i\sin(i\theta)=\sum_{i=1}^{N}\tilde{b}_i\sum_{j\geq 0}\cos(i\pi)\frac{(-1)^j}{(2j+1)!}i^{2j+1}(\theta-\pi)^{2j+1}\\ &= \sum_{i=1}^{N}\tilde{b}_i(-1)^i\sum_{j\geq 0}\frac{(-1)^j}{(2j+1)!}i^{2j+1}(\theta-\pi)^{2j+1}. \\ &= \sum_{j\geq 0}\frac{(-1)^j}{(2j+1)!}\tilde{B}_{2j+1}(\theta-\pi)^{2j+1},\end{aligned} \tag{3.30}$$

where

$$\widetilde{B}_{2j+1}=\sum_{i=1}^{N}(-1)^i i^{2j+1}\tilde{b}_i. \tag{3.31}$$

By (3.20), $\theta-\pi=x+O(x^2)$ for $|x|\ll 1$. The equalities (3.23) and (3.30) show that

$$B_{2j+1}=\frac{(-1)^j}{(2j+1)!}\widetilde{B}_{2j+1}+O(|\widetilde{B}_1,\widetilde{B}_3,\cdots,\widetilde{B}_{2j-1}|),\quad j\geq 0, \tag{3.32}$$

which implies that

$$\widetilde{B}_{2j+1}=\frac{(2j+1)!}{(-1)^j}B_{2j+1}+O(|B_1,B_3,\cdots,B_{2j-1}|),\quad j\geq 0. \tag{3.33}$$

Therefore, it only needs to prove the following result in order to reach our first goal.

$$\widetilde{B}_{2j+1}=O(|\widetilde{B}_1,\widetilde{B}_3,\cdots,\widetilde{B}_{2N^*+1}|),\quad \text{for } j\geq N^*+1. \tag{3.34}$$

Note from (3.27) and (3.29) that $\widetilde{F}(\theta)$ should be a linear collection of $N-\left[\frac{N}{3}\right]$ functions $\sin(i\theta)$, where $i\in\mathbb{S}$, which is the ordered sequence,

$$\mathbb{S}=\{1,2,\cdots,N\}/\{i \bmod 3=0\}=\{m_1,m_2,\cdots,m_{N^*+1}\}.$$

Note $N-\left[\frac{N}{3}\right]=N^*+1$, and so the equality (3.31) guarantes the matrix equation

$$(\widetilde{B}_1,\widetilde{B}_3,\cdots,\widetilde{B}_{2N^*+1})=((-1)^{m_1}\tilde{b}_{m_1},(-1)^{m_2}\tilde{b}_{m_2},\cdots,(-1)^{m_{N^*+1}}\tilde{b}_{m_{N^*+1}})\mathbb{M}_0, \tag{3.35}$$

where $\mathbb{M}_0$ is an $(N^*+1)\times(N^*+1)$ matrix with $\mathbb{M}_0[i,j]=m_i^{2j-1}$. A direct computation shows that

$$\det\mathbb{M}_0=\prod_{i=1}^{N^*+1}m_i\prod_{1\leq i<j\leq N^*+1}(m_j^2-m_i^2)\neq 0.$$

Then we have that

$$\widetilde{B}_{2j+1}=0\ \text{ for } 0\leq j\leq N^*$$

if and only if $\tilde{b}_i=0$ for $i=m_1,m_2,\cdots,m_{N^*+1}$, which implies that $\widetilde{F}(\theta)=0$ if and only if $\widetilde{B}_{2j+1}=0$ for all $0\leq j\leq N^*$. Thus, (3.34) holds.

Finally, we need to prove that

$$\operatorname{rank}\left[\frac{\partial(B_1, B_3, \cdots, B_{2N^*+1})}{\partial(\gamma_1, \gamma_2, \cdots, \gamma_N)}\right] = N^* + 1. \tag{3.36}$$

The equation (3.32) gives the following matrix equation,

$$(B_1, B_3, \cdots, B_{2N^*+1}) = (\widetilde{B}_1, \widetilde{B}_3, \cdots, \widetilde{B}_{2N^*+1})\mathbb{M}_1, \tag{3.37}$$

where

$$\mathbb{M}_1 = \begin{pmatrix} \frac{(-1)^0}{1!} & * & \cdots & * \\ & \frac{(-1)^1}{3!} & \cdots & * \\ & & \ddots & \vdots \\ \text{\Large 0} & & & \frac{(-1)^{N^*}}{(2N^*+1)!} \end{pmatrix}$$

and

$$\det \mathbb{M}_1 = \prod_{j=0}^{N^*} \frac{(-1)^j}{(2j+1)!} \neq 0.$$

It follows from (3.28) and Lemma 3.4.2 that

$$(\tilde{b}_1, \tilde{b}_2, 0, \tilde{b}_3, \tilde{b}_4, 0, \cdots, \tilde{b}_N) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \cdots, \gamma_N)\mathbb{M}_2,$$

where

$$\mathbb{M}_2 = \begin{pmatrix} c_{11} & 0 & 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & 0 & 0 & \cdots & 0 \\ c_{31} & c_{32} & 0 & 0 & \cdots & 0 \\ c_{41} & c_{42} & 0 & c_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ c_{N1} & c_{N2} & 0 & c_{N4} & \cdots & c_{NN} \end{pmatrix},$$

which is an $N \times N$ triangular matrix with the $3j$th column being zero for $j = 1, 2, \cdots, [\frac{N}{3}]$. Then we can delete all $3j$th columns and rows in $\mathbb{M}_2$ and assume $\alpha_{3j} = 0$ for $j = 1, 2, \cdots, [\frac{N}{3}]$. Therefore, we have the following result, by similarly using the ordered sequence $\mathbb{S}$,

$$(\tilde{b}_{m_1}, \tilde{b}_{m_2}, \cdots, \tilde{b}_{m_{N^*+1}}) = (\gamma_{m_1}, \gamma_{m_2}, \cdots, \gamma_{m_{N^*+1}})\mathbb{M}_3, \tag{3.38}$$

where

$$\mathbb{M}_3 = \begin{pmatrix} c_{11} & 0 & 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & 0 & 0 & \cdots & 0 \\ c_{41} & c_{42} & c_{44} & 0 & \cdots & 0 \\ c_{51} & c_{52} & c_{54} & c_{55} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ c_{m_{N^*+1},1} & c_{m_{N^*+1},2} & c_{m_{N^*+1},4} & c_{m_{N^*+1},5} & \cdots & c_{m_{N^*+1},m_{N^*+1}} \end{pmatrix},$$

with

$$\begin{aligned} \det \mathbb{M}_3 &= \prod_{k=m_1,m_2,\cdots,m_{N^*+1}} c_{kk} = \prod_{k=m_1,m_2,\cdots,m_{N^*+1}} -\frac{1}{2^{k-2}} \cos\Big(\frac{k\pi}{6} - \frac{k-1}{2}\pi\Big) \\ &= 3^{\frac{N^*}{2}} 2^{-N^*} \prod_{k=m_1,m_2,\cdots,m_{N^*+1}} -\frac{1}{2^{k-2}} \neq 0 \end{aligned}$$

by Lemma 3.4.2.

Combining (3.35), (3.37) and (3.38) completes the proof for (3.36). With the reached two goals, we have proved our result on the Hopf cyclicity by Lemma 3.2.4. Therefore, system (3.19) has the Hopf cyclicity $\left[\frac{2N-1}{3}\right]$, so it is $\left[\frac{2n+1}{3}\right]$ for system (3.5) with $f(x) = f_2(x)$. This completes the proof of Theorem 3.1.3.

3.5 Proof of Theorem 3.1.4

We only prove Theorem 3.1.4 for the case $l \geq n$, as the other case $l \leq n$ can be similarly proved. Like (3.19), we can rewrite system (3.5) as

$$\dot{x} = y - F^{\pm}(x, \delta), \quad y = x(x-1)^3\Big(x + \frac{1}{2}\Big), \tag{3.39}$$

where

$$F^{-}(x,\delta) = -\int_0^x \sum_{i=0}^{l} \alpha_i^- s^i ds = -\sum_{i=0}^{l} \frac{1}{i+1}\alpha_i^- x^{i+1} := \sum_{i=1}^{L} \gamma_i^- x^i$$

and

$$F^{+}(x,\delta) = -\int_0^x \sum_{i=0}^{n} \alpha_i^+ s^i ds = -\sum_{i=0}^{n} \frac{1}{i+1}\alpha_i^+ x^{i+1} := \sum_{i=1}^{N} \gamma_i^+ x^i$$

with $L = l + 1$, $N = n + 1$, $\gamma_i^{\pm} = -\frac{1}{i}\alpha_{i-1}^{\pm}$ for $i \geq 1$, and

$$\delta = (\gamma_1^-, \cdots, \gamma_L^-, \gamma_1^+, \cdots, \gamma_N^+) \in \mathbb{R}^{N+L}.$$

Proving Theorem 3.1.4 is equivalent to showing that the Hopf cyclicity of system (3.39) is $\left[\frac{3L+2N-1}{3}\right]$.

We have the bifurcation function of system (3.39) for $0 < x \ll 1$, given by

$$\begin{aligned} F(z(x)) - F(x) &= F^-(z(x)) - F^+(x) = \sum_{k=1}^{L} \gamma_k^- z^k(x) - \sum_{k=1}^{N} \gamma_k^+ x^k \\ &= \sum_{k=1}^{L} \left[b_k(z^k(x) - x^k) + e_k(z^k(x) + x^k) \right], \end{aligned} \tag{3.40}$$

where

$$b_k = \begin{cases} \frac{\gamma_k^- + \gamma_k^+}{2}, & 1 \leq k \leq N, \\ \frac{\gamma_k^-}{2}, & N < k \leq L, \end{cases} \qquad e_k = \begin{cases} \frac{\gamma_k^- - \gamma_k^+}{2}, & 1 \leq k \leq N, \\ \frac{\gamma_k^-}{2}, & N < k \leq L. \end{cases}$$

We again use the transformations (3.20) and (3.21) to obtain with Lemma 3.4.2,

$$\begin{aligned} F(z(x)) - F(x) &= \sum_{k=1}^{L} b_k J_k(\theta) + e_k K_k(\theta) \\ &= \sum_{k=1}^{L} \Big(b_k \sum_{i=1}^{k} c_{ki} \sin(i\theta) + e_k \sum_{i=0}^{k} \tilde{c}_{ki} \cos(i\theta)\Big), \\ &= \sum_{i=1}^{N} \tilde{b}_i \sin(i\theta) + \sum_{i=0}^{N} \tilde{e}_i \cos(i\theta) + \sum_{i=N+1}^{L} \gamma_i^- R_i(\theta) := \widetilde{F}(\theta), \end{aligned} \tag{3.41}$$

where

$$\tilde{b}_i = \sum_{k=i}^{L} b_k c_{ki}, \quad \tilde{e}_0 = \sum_{k=1}^{L} e_k \tilde{c}_{k0}, \quad \tilde{e}_i = \sum_{k=i}^{L} e_k \tilde{c}_{ki},$$

for $1 \le i \le N$ and

$$R_i(\theta) = \frac{1}{2} \sum_{j=N+1}^{i} (c_{ij} \sin(j\theta) + \tilde{c}_{ij} \cos(j\theta)),$$

for $N+1 \le i \le L$, with $\tilde{b}_i = 0$ if i mod $3 = 0$ for $1 \le i \le N$. We have

$$\tilde{e}_0 = -\sum_{i=1}^{N} \tilde{e}_i - \sum_{i=N+1}^{L} \frac{\gamma_i^-}{2} \sum_{j=N+1}^{i} \tilde{c}_{ij}$$

by $\widetilde{F}(\pi) = 0$. Then

$$\widetilde{F}(\theta) = \sum_{i \bmod 3 \neq 0}^{1 \le i \le N} \tilde{b}_i \sin(i\theta) + \sum_{i=1}^{N} \tilde{e}_i(\cos(i\theta) - 1) + \sum_{i=N+1}^{L} \gamma_i^- \widetilde{R}_i(\theta), \tag{3.42}$$

with

$$\widetilde{R}_i(\theta) = \frac{1}{2} \sum_{j=N+1}^{i} (c_{ij} \sin(j\theta) + \tilde{c}_{ij}(\cos(j\theta) - 1)), \ N+1 \le i \le L.$$

We define the parameter vector

$$\mathbf{v} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_4, \tilde{b}_5, \tilde{b}_7, \cdots, \tilde{e}_1, \cdots, \tilde{e}_N, \gamma_{N+1}^-, \gamma_{N+2}^-, \cdots, \gamma_L^-),$$

which has the dimension $L + N - \left[\frac{N}{3}\right]$. Thus, we can show that

$$\text{rank}\left[\frac{\partial \mathbf{v}}{\partial(b_1, b_2, \cdots, b_N, e_1, e_2, \cdots, e_N, \gamma_{N+1}^-, \cdots, \gamma_L^-)}\right] = L + N - \left[\frac{N}{3}\right]$$

by a similar proof for (3.38). Then

$$\text{rank}\left[\frac{\partial \mathbf{v}}{\partial(\gamma_1^+, \cdots, \gamma_N^+, \gamma_1^-, \cdots, \cdots, \gamma_L^-)}\right] = L + N - \left[\frac{N}{3}\right]. \tag{3.43}$$

$\widetilde{F}(\theta)$ can be expanded near $\theta = \pi$ as below,

$$\widetilde{F}(\theta) = \widetilde{B}_1(\theta - \pi) + \widetilde{B}_2(\theta - \pi)^2 + \widetilde{B}_3(\theta - \pi)^3 + \cdots, \tag{3.44}$$

where each coefficient $\widetilde{B}_i$ is a linear combination of the entries of $\mathbf{v}$. Let s be the maximal number of linear independent coefficients in (3.44), $S = \{r_1, r_2, \cdots, r_s\}$ with $r_i < r_{i+1}$ and these coefficients be denoted by

$$\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_s}.$$

Obviously, these independent coefficients can be determined one by one by taking $\widetilde{B}_{r_1}$ to be the first nonzero coefficient in (3.44) and $\widetilde{B}_{r_j}$ the first one that independent of $\widetilde{B}_{r_1}, \cdots, \widetilde{B}_{r_{j-1}}$, up to the sth one. Then $s \le L + N - \left[\frac{N}{3}\right]$ and

$$\widetilde{B}_j = LC_j(\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_j^*}), \tag{3.45}$$

where $j \notin S$ and $j < r_s$ with $r_j^* = \max\{r_i \in S | r_i < j\}$, LC_j denotes a linear combination, and $\widetilde{B}_j = O(\widetilde{B}_{r_1}, \cdots, \widetilde{B}_{r_s})$ for $j > r_s$. There exists an $s \times \left(L + N - \left[\frac{N}{3}\right]\right)$ matrix $\mathcal{M}_1$ such that,

$$(\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_s})^T = \mathcal{M}_1 \mathbf{v}^T, \tag{3.46}$$

In the following, we prove the claim,

$$\operatorname{rank}\mathcal{M}_1 = L + N - \left[\frac{N}{3}\right],$$

which only need prove all $\widetilde{B}_{r_j} = 0$, $j = 1, 2, \cdots, s$, if and only if $\mathbf{v} = 0$. By definition of $\widetilde{B}_{r_j}$, $j = 1, 2, \cdots, s$, it only need to prove $\widetilde{F}(\theta) \equiv 0$ if and only if $\mathbf{v} = 0$.

The elements in (3.42), $\{\sin i\theta, \cos j\theta - 1, \widetilde{R}_u(\theta)\}$ with $1 \le i \le N$, $i \bmod 3 \ne 0$, $1 \le j \le N$ and $N + 1 \le u \le L$ is a Chebyshev system of dimension $L + N - \left[\frac{N}{3}\right]$. Then $\widetilde{F}(\theta) \equiv 0$ if and only if $\mathbf{v} = 0$.

Suppose

$$F^-(z(x)) - F^+(x) = B_1 x + B_2 x^2 + \cdots + B_j x^j + \cdots, \quad 0 < x \ll 1. \tag{3.47}$$

By (3.20), $\theta - \pi = x + O(x^2)$ for $|x| \ll 1$. We substitute it into (3.47) and compare the coefficients to get

$$B_j = \widetilde{B}_j + \widetilde{E}_i(\widetilde{B}_1, \widetilde{B}_2, \cdots, \widetilde{B}_{j-1}), \quad j = 1, 2, \cdots, \tag{3.48}$$

where $\widetilde{E}_i$ is a linear function. Then, by (3.45) and (3.48), we have

$$B_j = O(\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_j^*}) \text{ for } j \notin S \tag{3.49}$$

and

$$B_{r_j} = \widetilde{B}_{r_j} + \widetilde{E}^*_{r_j}(\widetilde{B}_1, \widetilde{B}_2, \cdots, \widetilde{B}_{r_{j-1}}) \text{ for } r_j \in S \tag{3.50}$$

which further imply

$$\widetilde{B}_{r_j} = B_{r_j} + E_{r_j}(B_{r_1}, B_{r_2}, \cdots, B_{r_{j-1}}) \text{ for } r_j \in S, \tag{3.51}$$

where $\widetilde{E}^*_{r_j}$ and E_{r_j} are linear functions. Then combining (3.49) and (3.51) yields

$$B_j = O(B_{r_1}, B_{r_2}, \cdots, B_{r_j^*})$$

for $j \notin S$. The reminder of the proof is to show

$$\operatorname{rank}\left[\frac{\partial(B_{r_1}, B_{r_2}, \cdots, B_{r_s})}{\partial(\gamma_1^+, \cdots, \gamma_N^+, \gamma_1^- \cdots, \gamma_L^-)}\right] = L + N - \left[\frac{N}{3}\right]. \tag{3.52}$$

By (3.50), we have

$$\operatorname{rank}\left[\frac{\partial(B_{r_1}, B_{r_2}, \cdots, B_{r_s})}{\partial(\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_s})}\right] = L + N - \left[\frac{N}{3}\right], \tag{3.53}$$

and so

$$\operatorname{rank}\left[\frac{\partial(\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_s})}{\partial \mathbf{v}}\right] = \operatorname{rank}\mathcal{M}_1 = L + N - \left[\frac{N}{3}\right] \tag{3.54}$$

by (3.46). Combining (3.43), (3.53) and (3.54), we have shown that (3.52) holds. Therefore, the Hopf cyclicity of system (3.39) is $L + N - \left[\frac{N}{3}\right] - 1 = \left[\frac{3L+2N-1}{3}\right]$. This completes the proof of Theorem 3.1.4.

3.6 Conclusion

In this chapter, we conduct a further study on a Liénard system and give a rigorous proof to the open question remained in [5, 36]. We prove that the cyclicity of the periodic annulus of the Hamiltonian is 3 by showing the sharp bound to be 3 on the maximal number of zeros of the associated Abelian integral. The annulus cyclicity can be extended to the elementary center because the displacement map is analytic for $h = 0$. The non-symmetry and degeneracy of the system causes much difficulty in the computation and analysis for the Poincaré bifuraction, as well as in the study of Hopf bifurcation. We have obtained the Hopf cyclicity as $[\frac{2n+1}{3}]$ when the damping term is a smooth polynomial with an arbitrary degree n. The involution determined by the annulus is well utilized, based on which a transformation composed of trigonometric functions is introduced, which provides a tool to overcome the difficulty in analysis and computation. However, it is not easy to find such kind of a transformation for the involution in a general undamped Liénard system. It is even unknown if there exists such a transformation for the involution of a Hamiltonian. In this chapter, we find such a transform for our system that belongs to hyperelliptic Hamiltoinian. We have also studied the Hopf cyclicity of the origin when the damping term is a non-smooth polynomial with the switching manifold at the y-axis, having respectively degrees l and n, and proved that the Hopf cyclicity is $\left[\frac{3l+2n+4}{3}\right]$ $\left(\left[\frac{3n+2l+4}{3}\right]\right)$ when $l \geq n$ ($n \geq l$).

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Chapter 4

The monotonicity of ratios of some Abelian integrals

4.1 Introduction and main results

The well-known weak Hilbert's 16th problem [1] asks for the maximal number of isolated zeros of the following Abelian integral,

$$I(h) = \oint_{\Gamma_h} f(x,y)dx - g(x,y)dy, \quad h \in \Sigma,$$

where $f(x,y)$ and $g(x,y)$ are polynomials of degree m, $\{\Gamma_h\}$ is a family of ovals of the level set $\{H(x,y) = h, h \in \Sigma\}$, where Σ represents an interval and the Hamiltonian $H(x,y)$ is an $(n+1)$th-degree polynomial. This open problem is extremely difficult, and researchers choose some simpler forms of $H(x,y)$, $f(x,y)$ and $g(x,y)$ to study, see [12] for a relatively new survey work.

Suppose that $H(x,y) = y^2 + P_{n+1}(x)$, where $P_{n+1}(x)$ is a polynomial of degree $n+1$. In this case, the Abelian integral is usually called elliptic integral if $[\frac{n}{2}] < 2$, and hyperelliptic integral if $[\frac{n}{2}] \geq 2$. When $n = 2$, Petrov [15] proved that the sharp bound on the maximal number of zeros of $I(h)$ is $m-1$ for arbitrary m. When $n = 3$, Dumortier and Li [4, 5, 6, 7] obtained the sharp bound on the maximal number of zeros of $I(h)$ for $m \leq 3$. Later, it was proved that the bound is linearly dependent on m, see [16, 24] and references therein. $I(h)$ is a hyperelliptic integral if $n > 3$, and it is rather difficult to find an upper bound on the maximal number of zeros of such an $I(h)$. However, it is still very interesting and important to study the hyperelliptic integrals with $m = 2$. In this respect, if the ratio of two Abelian integrals,

$$\frac{\oint_{\Gamma(h)} ydx}{\oint_{\Gamma(h)} xydx},$$

is monotonic, in other words, the integral set $\{\oint_{\Gamma(h)} ydx, \oint_{\Gamma(h)} xydx\}$ is a Chebyshev system, then $I(h) = a_0 \oint_{\Gamma(h)} ydx + a_1 \oint_{\Gamma(h)} xydx$ has at most one isolated zero for any a_0 and a_1. We note that Gavrilov and Iliev [8] proved the ratio of the two complete hyperelliptic integrals of the first kind, associated with the hyperelliptic curve $H(x, y) = y^2 + P_5(x)$,

$$\frac{\oint_{\Gamma(h)} \frac{1}{y} dx}{\oint_{\Gamma(h)} \frac{x}{y} dx},$$

is monotonic.

In [13], Li and Zhang first provided a criterion to determine the monotonicity of the ratio of two Abelian integrals. 15 years later, the criterion was generalized by Grau *et al.* [9] to bound the number of zeros of a linear combination of m Abelian integrals with $m > 2$. The new criterion was first applied to bound the number of zeros of $I(h)$ for quartic damping Hamiltonian systems in [2, 11, 17, 19]. Later, Liu and Xiao [14] established a new criterion to determine the monotonicity of $\oint_{\Gamma(h)} ydx / \oint_{\Gamma(h)} xydx$, where $\{\Gamma(h)\}$ are the ovals defined by $y^2 + \Phi(x) = h$, $\Phi(x)$ is an analytic function and has a local minimum at the center type singularity of the corresponding Hamiltonian system. Using this new method, they gave the sufficient and necessary conditions for monotonicity of $\oint_{\Gamma(h)} ydx / \oint_{\Gamma(h)} xydx$ on the hyperelliptic closed curves defined by $\{(x, y) | y^2 + P_5(x) = h\}$.

In [20], Wang *et al.* studied the ratio of $\oint_{\Gamma(h)} ydx$ and $\oint_{\Gamma(h)} xydx$ on the hyperelliptic curves, given by

$$H(x, y) = y^2 + \int x(x-\alpha)(x-\beta)(x-\gamma)(x-1)dx \tag{4.1}$$

and

$$H^*(x, y) = y^2 - \int x(x-\alpha)(x-\beta)(x-\gamma)(x-1)dx. \tag{4.2}$$

The corresponding Hamiltonian systems are respectively described by

$$\dot{x} = -2y, \quad \dot{y} = x(x-\alpha)(x-\beta)(x-\gamma)(x-1) \tag{4.3}$$

and

$$\dot{x} = 2y, \quad \dot{y} = x(x-\alpha)(x-\beta)(x-\gamma)(x-1), \tag{4.4}$$

where $0 \le \alpha \le \beta \le \lambda \le 1$.

Moreover, in [20], the authors gave a complete classification of hyperelliptic curves and investigated the monotonicity of the ratios of $\oint_{\Gamma(h)} ydx$ and $\oint_{\Gamma(h)} xydx$ on these hyperelliptic curves. A number of good results were obtained in [20]. In particular, when $\lambda < 1$ and $\alpha = \beta = 0$, system (4.3) is reduced to

$$\dot{x} = -2y, \quad \dot{y} = x^3(x-\lambda)(x-1) \tag{4.5}$$

with the Hamiltonian function,

$$H(x, y) = y^2 + \frac{\lambda}{4}x^4 - \frac{1+\lambda}{5}x^5 + \frac{1}{6}x^6. \tag{4.6}$$

When $0 < \alpha < 1$ and $\beta = \lambda = 0$, system (4.3) becomes

$$\dot{x} = -2y, \quad \dot{y} = x(x-\alpha)(x-1)^3 \tag{4.7}$$

with the Hamiltonian

$$\mathcal{H}(x,y) = y^2 + \frac{\alpha}{2}x^2 - \frac{1+3\alpha}{3}x^3 + \frac{3+3\alpha}{4}x^4 - \frac{3+\alpha}{5}x^5 + \frac{1}{6}x^6. \tag{4.8}$$

The two compact components of the level sets of $H(x,y) = h$ and $\mathcal{H}(x,y) = h$ surrounding the nilpotent point $(0,0)$ of system (4.5) and the nilpotent point $(1,0)$ of system (4.7), are denoted by $\gamma_1(h)$ and $\gamma_2(h)$, as shown in Figures 4.1(a) and 4.1(b), respectively. It should be pointed out that it is more difficult to analyze the bifurcation and related problems for these degenerate cases.

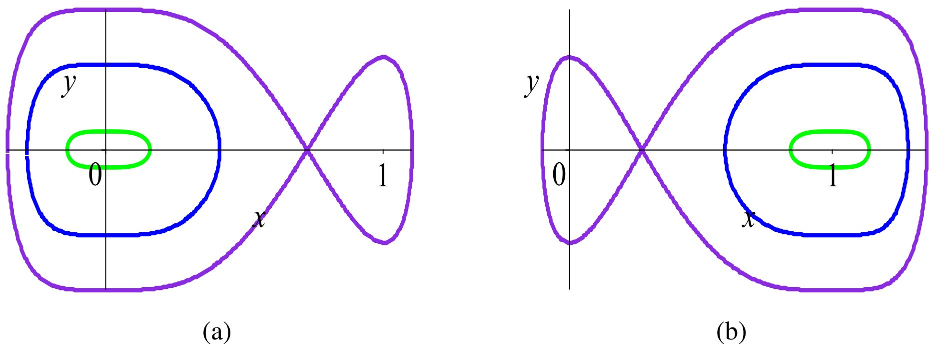

Figure 4.1: The level set of the Hamiltonians: (a) for system (4.5) and (b) for system (4.7).

Let

$$I_0(h) = \oint_{\gamma_1(h)} y dx, \quad I_1(h) = \oint_{\gamma_1(h)} xy dx, \quad \mathcal{I}_0(h) = \oint_{\gamma_2(h)} y dx \quad \text{and } \mathcal{I}_1(h) = \oint_{\gamma_2(h)} xy dx.$$

It has been proved in [20] that

Theorem 4.1.1 (i) $\frac{I_1(h)}{I_0(h)}$ *is monotonic in* $(0, H(\lambda, 0))$ *for* $\lambda \in (0, \frac{2}{3}]$*, and* (ii) $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ *is monotonic in* $(\mathcal{H}(1,0), \mathcal{H}(\alpha,0))$ *for* $\alpha \in [\frac{1}{3}, 1)$.

Note that no answers are given in [20] for part (i) of Theorem 4.1.1 when $\lambda \in (\frac{2}{3}, 1)$ and for part (ii) when $\alpha \in (0, \frac{1}{3})$. Instead, the authors proposed the following conjecture.

Conjecture: (i) $\frac{I_1(h)}{I_0(h)}$ *is monotonic in* $(0, H(\lambda, 0))$ *for* $\lambda \in (\frac{2}{3}, 1)$*, and* (ii) $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ *is monotonic in* $(\mathcal{H}(1,0), \mathcal{H}(\alpha,0))$ *for* $\alpha \in (0, \frac{1}{3})$.

The aim of this chapter is to give a positive answer to the above conjecture. Our main results are given in the following two theorems.

Theorem 4.1.2 $\frac{I_1(h)}{I_0(h)}$ *is monotonic in* $(0, H(\lambda, 0))$ *for* $\lambda \in (\frac{2}{3}, 1)$.

Theorem 4.1.3 $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ *is monotonic in* $(\mathcal{H}(1,0), \mathcal{H}(\alpha,0))$ *for* $\alpha \in (0, \frac{1}{3})$.

Combing Theorems 4.1.1, 4.1.2 and 4.1.3 shows that $\frac{I_1(h)}{I_0(h)}$ is monotonic in $(0, H(\lambda, 0))$ when $\lambda \in (0, 1)$, and $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ is monotonic for $h \in (\mathcal{H}(1,0), \mathcal{H}(\alpha,0))$ when $\alpha \in (0\,1)$. Thus, any non-trivial linear combination, $a_0 I_0 + a_1 I_1$ (or $a_0\mathcal{I}_0 + a_1\mathcal{I}_1$), has at most one zero. By the Poincaré theorem (see [10]), system (4.5) (or system (4.7)) perturbed by $(a_0 + a_1 x)y$ (or $(\alpha_0 + \alpha_1 x)y$) can have at most one limit cycle, which can be reached.

Theorems 4.1.2 and 4.1.3 will be proved in Sections 4.2 and 4.3, respectively. The proofs are based on the criterion given in [14] and theory on the boundary of polynomial algebraic systems [22, 23]. The techniques developed in this chapter greatly simplify the analysis and can be applied to other types of differential equations.

4.2 Proof of Theorem 4.1.2

Let

$$\Phi(x) = H(x, y) - y^2 = \frac{\lambda}{4}\, x^4 - \frac{1+\lambda}{5}\, x^5 + \frac{1}{6}\, x^6.$$

It is not difficult to prove that $\Phi'(x)x > 0$ and there exist two analytic functions $\mu(h)$ and $\nu(h)$ satisfying

$$\Phi(\mu(h)) \equiv \Phi(\nu(h)) \equiv h, \quad a_\lambda < \mu(h) < 0 < \nu(h) < \lambda,$$

where $\lambda \in (-0.43708017\cdots, 0)$ with $\Phi(a_\lambda) = \Phi(\lambda)$. Further, define the function

$$U(h) := \mu(h) + \nu(h).$$

Then, in the following, we prove that $U'(h)$ does not vanish for $h \in (0, H(\lambda, 0))$ when $\lambda \in (\frac{2}{3}, 1)$, and thus the conclusion is true by using the criterion in [14].

$\Phi(\nu) = h$ and $\Phi'(\nu) > 0$ imply that $\nu'(h) > 0$ in $(0, H(\lambda, 0))$. Therefore, $\nu(h)$ has an inverse function $h = h^{-1}(\nu)$, which is substituted into $\mu(h)$ to yield $\mu(h) = \mu(\nu)$, where $\mu(\nu)$ is defined by $\Phi(\mu) - \Phi(\nu) = 0$, satisfying $a_\lambda < \mu < 0 < \nu < \lambda$. Factorizing $\Phi(\mu) - \Phi(\nu)$ gives $-\frac{\nu-\mu}{60} q(\nu, \mu, \lambda)$, where

$$\begin{aligned} q(\nu, \mu, \lambda) = {} & 12\,(\lambda + 1)(\mu^4 + \mu^3\nu + \mu^2\nu^2 + \mu\,\nu^3 + \nu^4) - 15\lambda(\mu + \nu)(\mu^2 + \nu^2) \\ & -10(\mu + \nu)(\mu^2 + \nu^2 + \mu\nu)(\mu^2 + \nu^2 - \mu\nu). \end{aligned}$$

In fact, $\mu(\nu)$ is determined by $q(\mu, \nu, \lambda)$. Hence,

$$U'(h) = \left[\frac{d\mu}{d\nu} + 1\right]\nu'(h) = \left[-\frac{q_\mu(\nu, \mu, \lambda)}{q_\nu(\nu, \mu, \lambda)} + 1\right]\nu'(h) = 2(\mu - \nu)\,\frac{U_1(\nu, \mu, \lambda)}{U_2(\nu, \mu, \lambda)}\nu'(h),$$

where

$$\begin{aligned} U_1(\nu, \mu, \lambda) = {} & 6(\lambda + 1)(3\,\mu^2 + 4\,\mu\,\nu + 3\,\nu^2) - 15\,\lambda\,(\mu + \nu) - 10\,(2\mu^3 + 3\,\mu^2\nu + 3\,\mu\,\nu^2 + 2\,\nu^3), \\ U_2(\nu, \mu, \lambda) = {} & 12\,\lambda\,(4\,\mu^3 + 3\,\mu^2\nu + 2\,\mu\,\nu^2 + \nu^3) - 15\,\lambda\,(3\,\mu^2 + 2\,\mu\,\nu + \nu^2) \\ & -10\,(5\mu^4 + 4\,\mu^3\nu + 3\,\mu^2\nu^2 + 2\,\mu\,\nu^3 + \nu^4) + 12\,(4\,\mu^3 + 3\,\mu^2\nu + 2\,\mu\,\nu^2 + \nu^3). \end{aligned}$$

It is sufficient to prove $U_i(\nu, \mu, \lambda) \neq 0$ for $i = 1, 2$ on

$$D: \left\{(\nu, \mu, \lambda)\middle|\kappa_\lambda < \mu < 0 < \nu < \lambda,\ \frac{2}{3} < \lambda < 1\right\}.$$

Computing the resultant between U_2 and q with respect to ν gives

$$r_0 = -1296000000000\mu^{12}(\mu - 1)^4(\lambda - \mu)^4,$$

which has no zeros on D. Therefore, U_2 and q have no common roots on D, which implies that $U_2(\nu,\mu,\lambda) \neq 0$ on D.

Similarly, computing the resultant between U_1 and q with respect to μ and ν respectively, we obtain

$$r_1(\nu,\lambda) = g(\nu,\lambda) \quad \text{and} \quad r_2(\mu,\lambda) = g(\mu,\lambda), \tag{4.9}$$

where $g(\omega,\lambda)$ is a polynomial, given by

$$\begin{aligned} g(\omega,\lambda) = & -54000\omega^4(\lambda+1)[64\lambda^2\omega^4(81\lambda^4 - 162\lambda^3\omega + 261\lambda^2\omega^2 - 300\lambda\omega^3 + 125\omega^4) \\ & -16\lambda\omega^3(648\lambda^5 - 567\lambda^4\omega + 990\lambda^3\omega^2 - 951\lambda^2\omega^3 - 900\lambda\omega^4 + 875\omega^5) \\ & +8\omega^2(648\lambda^6 + 1620\lambda^5\omega - 1521\lambda^4\omega^2 + 2790\lambda^3\omega^3 - 5997\lambda^2\omega^4 + 1800\lambda\omega^5 + 1000\omega^6) \\ & -12\omega\,(270\lambda^6 + 747\lambda^5\omega - 60\lambda^4\omega^2 + 801\lambda^3\omega^3 - 1860\lambda^2\omega^4 - 1268\lambda\omega^5 + 1600\omega^6) \\ & +18\,(225\lambda^6 + 210\lambda^5\omega + 303\lambda^4\omega^2 + 40\lambda^3\omega^3 - 676\lambda^2\omega^4 - 880\lambda\omega^5 + 928\omega^6) \\ & -27\,(325\,\lambda^5 - 140\,\lambda^4\omega + 332\,\lambda^3\omega^2 - 480\,\lambda^2\omega^3 - 336\,\lambda\,\omega^4 + 384\,\omega^5) \\ & +2\,(2025\,\lambda^4 - 1620\,\lambda^3\omega + 2592\,\lambda^2\omega^2 - 5184\,\lambda\,\omega^3 + 2592\,\omega^4)]. \end{aligned}$$

Taking $\nu = \frac{\lambda}{1+t}$ and $\lambda = \frac{3}{4} + \frac{1}{4(1+s)}$ yields

$$g(\nu,\lambda) = \frac{3375(4+3s)^8(8+7s)}{10456576(1+s)^{15}(1+t)^{12}}\, g^*(t,s),$$

where all coefficients of $g^*(t,s)$ are positive with $g^*(0,0) = 1600$. Therefore, $g^*(t,s) > 0$ on $\{(t,s)|t \in (0,+\infty), s \in [0,+\infty]\}$, which implies that U_1 and q have no common roots on

$$D_1 : \left\{(\nu,\mu,\lambda)|\kappa_\lambda < \mu < 0 < \nu < \lambda,\ \frac{3}{4} \leq \lambda < 1\right\}.$$

Hence, $U_1(\nu,\mu,\lambda) \neq 0$ on D_1, and so we have

Proposition 4.2.1 $\frac{I_1(h)}{I_0(h)}$ *is monotonic in* $(0, H(\lambda,0))$ *for* $\lambda \in [\frac{3}{4}, 1)$.

The remaining task is to investigate the problem on the region:

$$D\backslash D_1 = \left\{(\nu,\mu,\lambda)|\kappa_\lambda < \mu < 0 < \nu < \lambda,\ \frac{2}{3} < \lambda < \frac{3}{4}\right\}.$$

We will apply the following techniques in polynomial theory, see [22, 23] for more details.

Let k be a field, $x_1 \prec x_2 \prec \cdots \prec x_n$ be n ordered variables and $R(x) = k[x_1,\cdots,x_n]$ be the polynomial ring on k. The greatest variable x_i in $f(x_1,\cdots,x_i)$ is called its main variable, denoted by $mvar(f)$. The coefficient of the main variable of f is called leading coefficient, denoted by $lc(f)$.

Definition 4.2.2 (Semi-Algebraic Systems) *A semi-algebraic system* (SAS *for short*) *is a conjunctive polynomial formula of the following form:*

$$\text{SAS},\ \ \text{S}: \begin{cases} p_1(x_1,\cdots,x_n) = 0,\cdots,p_s(x_1,\cdots,x_n) = 0, \\ g_1(x_1,\cdots,x_n) \geq 0,\cdots,g_r(x_1,\cdots,x_n) \geq 0, \\ g_{r+1}(x_1,\cdots,x_n) > 0,\cdots,g_t(x_1,\cdots,x_n) > 0, \\ h_1(x_1,\cdots,x_n) \neq 0,\cdots,h_m(x_1,\cdots,x_n) \neq 0, \end{cases} \tag{4.10}$$

where $n, s \geq 1$, $t \geq r \geq 0$, $m \geq 0$, all $p_i, g_i, h_i \in R(x)$ are polynomials with integer coefficients.

An SAS is called a parametric SAS if $s < n$ (s indeterminates are viewed as independent variables and the other $n-s$ indeterminates are treated as parameters, denoted by $u = (x_{s+1}, \cdots, x_n)$). An SAS is usually denoted by $[F, N, P, H]$, where $F = [p_1, \cdots, p_s]$, $N = [g_1, \cdots, g_r]$, $P = [g_{r+1}, \cdots, g_t]$ and $H = [h_1, \cdots, h_m]$.

There exist several well-known methods, such as the Ritt-Wu method, Gröbner basis method and subresultant method [3, 18, 21], which enable us to transform an SAS (4.10) equivalently to one or more triangular semi-algebraic systems: $\mathrm{T}_1, \cdots, \mathrm{T}_l$ in the form of

$$\mathrm{T_j}: \begin{cases} f_1^j(u, x_1) = 0, f_2^j(u, x_1, x_2) = 0, f_3^j(u, x_1, x_2, x_3) = 0, \cdots, f_s^j(x_1, \cdots, x_s) = 0, \\ g_1(x_1, \cdots, x_n) \geq 0, \cdots, g_r(x_1, \cdots, x_n) \geq 0, \\ g_{r+1}(x_1, \cdots, x_n) > 0, \cdots, g_t(x_1, \cdots, x_n) > 0, \\ h_1(x_1, \cdots, x_n) \neq 0, \cdots, h_m(x_1, \cdots, x_n) \neq 0, \end{cases} \tag{4.11}$$

where $\{f_1^j(u, x_1), f_2^j(u, x_1, x_2), f_3^j(u, x_1, x_2, x_3), \cdots, f_s^j(x_1, x_2, x_3, \cdots, x_s)\}$ is a triangular set, or a normal ascending chain.

Let $\mathrm{dis}(f_i)$ denote the discriminant of a polynomial f_i with respect to x_i, $\mathrm{res}(\cdot, \diamond, x_j)$ denote the Sylvester resultant of $\cdot$ and $\diamond$ with respect to x_j, and $\gcd(f_1, f_2, \cdots, f_i)$ denote the greatest common factor of $f_1, f_2, \cdots, f_i$.

Definition 4.2.3 (Border Polynomial of a Triangular System) *Consider the parametric triangular semi-algebraic system* (4.11)*:* $\mathrm{T_j}$*. For convenience,* f_i^j *is denoted by* f_i (*only for this definition*)*. The following polynomial is called border polynomial of* (4.11)*:*

$$\mathrm{B_{T_j}} = lc(f_1)\mathrm{dis}(f_1) \prod_{2 \leq i \leq s} \mathrm{res}(lc(f_i)\mathrm{dis}(f_i); f_{i-1}, \cdots, f_1) \prod_{1 \leq j \leq t} \mathrm{res}(g_j; f_s, \cdots, f_1) \prod_{1 \leq k \leq m} \mathrm{res}(h_k; f_s, \cdots, f_1),$$

where

$$\mathrm{res}(*; f_i, \cdots, f_1) = \mathrm{res}(\cdots(\mathrm{res}(\underline{\mathrm{res}(*, f_i, x_i)}, f_{i-1}, x_{i-1}), \cdots), f_1, x_1).$$

For two TSA: $\mathrm{T_j}$ and $\mathrm{T_{\tilde{j}}}$, let

$$r_i^{j\tilde{j}} = \gcd(\mathrm{res}(f_i^j; f_i^{\tilde{j}}, f_{i-1}^{\tilde{j}}, \cdots, f_1^{\tilde{j}}), \mathrm{res}(f_i^{\tilde{j}}; f_i^j, f_{i-1}^j, \cdots, f_1^j)), \ 1 \leq i \leq s,$$

and

$$C_{j\tilde{j}} = \gcd(r_1^{j\tilde{j}}, \cdots, r_s^{j\tilde{j}}).$$

Definition 4.2.4 (Border Polynomial of SAS) *If a parametric* SAS S *is transformed equivalently to regular* TSAs $\{\mathrm{T}_1, \cdots, \mathrm{T}_l\}$*, then*

$$\mathrm{B_S} = \prod_{1 \leq j \leq \tilde{j} \leq l} C_{j\tilde{j}} \prod_{j=1}^{l} \mathrm{B_{T_j}}$$

is called the border polynomial of S.

Lemma 4.2.5 ([22, 23]) *The number of distinct real solutions of the* **SAS** S *is invariant in each connected component of the complement of* $\mathrm{B_S} = 0$ *in* R^{n-s}*.*

Remark 4. 2.1 When the parameter values satisfy the boundary $\mathrm{B_S} = 0$, it is usually called degenerate case, for which it should be analyzed by other methods, see [22, 23].

Based on the above described idea, Yang and Xia [22, 23] developed a practical method for computing the border polynomial of S, which has been included into the computer algebra system – Maple.

To complete the proof of Theorem 4.1.2, we construct the following **SAS**

$$\text{SAS}, \quad \mathrm{S_A}: \begin{cases} q(\nu,\mu,\lambda)=0, \quad q(\lambda,\kappa, \quad \lambda)=0, \quad U_1(\nu,\mu,\lambda)=0, \\ \nu>0, \quad -\mu>0, \quad \lambda-\nu>0, \quad \mu-\kappa>0, \quad -\kappa>0 \end{cases} \tag{4.12}$$

which has no roots. Computing its border polynomial we obtain

$$\begin{aligned} \mathrm{B_{S_A}} = &\left(\lambda^9 - \tfrac{5}{2}\lambda^8 + \tfrac{41641\,\lambda^7}{24592} + \tfrac{15855\,\lambda^6}{6148} - \tfrac{693\,\lambda^5}{212} + \tfrac{575721\,\lambda^4}{196736} - \tfrac{4279635\,\lambda^3}{1573888} - \tfrac{395847\,\lambda^2}{196736} + \tfrac{1563705\,\lambda}{393472} - \tfrac{133407}{98368}\right) \\ &\times\left(\lambda^8 - \tfrac{121\,\lambda^7}{38} + \tfrac{337543\,\lambda^6}{57456} - \tfrac{443339\,\lambda^5}{57456} + \tfrac{792185\,\lambda^4}{76608} - \tfrac{443339\,\lambda^3}{57456} + \tfrac{337543\,\lambda^2}{57456} - \tfrac{121\,\lambda}{38} + 1\right) \\ &\times(\lambda^6 - \tfrac{4}{3}\lambda^5 - \tfrac{13\,\lambda^4}{48} + \tfrac{2939\,\lambda^3}{864} - \tfrac{13\,\lambda^2}{48} - \tfrac{4}{3}\lambda + 1)\left(\lambda^2 - \tfrac{7}{4}\lambda + 1\right)\left(\lambda^2 + \tfrac{\lambda}{2} + 1\right) \\ &\times(\lambda+1)\left(\lambda - \tfrac{4}{3}\right)\left(\lambda - \tfrac{3}{2}\right)\left(\lambda - \tfrac{2}{3}\right)\left(-\tfrac{3}{4} + \lambda\right)\lambda. \end{aligned}$$

It can be shown that $\mathrm{B_{S_A}} = 0$ has a unique root, $\lambda^* = 0.70513143\cdots$, in $(\frac{2}{3}, \frac{3}{4})$, which is actually the root of the first factor of $\mathrm{B_{S_A}}$. Therefore, the complement of $\mathrm{B_{S_A}} = 0$ restricted to $\lambda \in (\frac{2}{3}, \frac{3}{4})$ is $(\frac{2}{3}, \lambda^*) \cup (\lambda^*, \frac{3}{4})$.

By Lemma 2.1, the number of zeros of (4.12) is invariant for $\lambda \in (\frac{2}{3}, \lambda^*)$ and for $\lambda \in (\lambda^*, \frac{3}{4})$. Therefore, we may choose a $\lambda \in (\lambda^*, \frac{3}{4})$ to investigate if $U_1(\nu,\mu,\lambda)$ vanishes on

$$D_2 = \left\{(\nu,\mu,\lambda)|\kappa_\lambda < \mu < 0 < \nu < \lambda, \lambda^* < \lambda < \frac{3}{4}\right\},$$

and take a $\lambda \in (\frac{2}{3}, \lambda^*)$ to investigate if $U_1(\nu,\mu,\lambda)$ vanishes on

$$D_3 = \left\{(\nu,\mu,\lambda)|\kappa_\lambda < \mu < 0 < \nu < \lambda, \frac{2}{3} < \lambda < \lambda^*\right\}.$$

First, taking $\lambda = \frac{72}{100} \in (\lambda^*, \frac{3}{4})$ and substituting it into (4.9) yield

$$\begin{aligned} r_2(\mu, \tfrac{72}{100}) = \; & \tfrac{10336\,x^8}{5} - \tfrac{5333376\,x^7}{625} + \tfrac{5061371904\,x^6}{390625} - \tfrac{79435137024\,x^5}{9765625} + \tfrac{250937411616\,x^4}{244140625} + \tfrac{190180493568\,x^3}{244140625} \\ & - \tfrac{50040379584\,x^2}{244140625} + \tfrac{4225074048\,x}{48828125} - \tfrac{442158912}{9765625}, \end{aligned}$$

which has no roots on $(-\frac{9}{25}, 0) = (-0.36, 0)$, while

$$\kappa_\lambda = \kappa_{\frac{72}{100}} \in \left[-\frac{92651}{262144}, -\frac{46325}{131072}\right] \approx [-0.3534355164, -0.3534317017]$$

when $\lambda = \frac{72}{100}$. Hence, $r_2(\mu, \frac{72}{100}) \neq 0$ on $(\kappa_{\frac{72}{100}}, 0)$ which implies that U_1 and q have no common roots on D_2, and so $U_1(\nu,\mu,\lambda) \neq 0$ on D_2. Therefore, the following result holds.

Proposition 4.2.6 *$\frac{I_1(h)}{I_0(h)}$ is monotonic in $(0, H(\lambda, 0))$ for $\lambda \in (\lambda^*, \frac{3}{4})$.*

Next, we choose a value of $\lambda \in (\frac{2}{3}, \lambda^*)$ to investigate if $U_1(\nu, \mu, \lambda)$ vanishes on D_3. Taking $\lambda = \frac{7}{10} \in (\frac{2}{3}, \lambda^*)$ and substituting it into (4.9) give

$$r_1\Big(\nu, \frac{7}{10}\Big) = r(\nu) \qquad \text{and} \qquad r_2\Big(\mu, \frac{7}{10}\Big) = r(\mu), \tag{4.13}$$

where

$$r(\omega) = 2120\,\omega^8 - \tfrac{43248\,\omega^7}{5} + \tfrac{8172924\,\omega^6}{625} - \tfrac{26278668\,\omega^5}{3125} + \tfrac{46507833\,\omega^4}{31250} + \tfrac{7359912\,\omega^3}{15625} - \tfrac{1901151\,\omega^2}{15625} + \tfrac{157437\,\omega}{3125} - \tfrac{64827}{2500}.$$

By applying Sturm's Theorem, we obtain that $r_1(\nu, \frac{7}{10})$ has a unique root $\nu_1 \in [0, \frac{7}{10}]$, and $r_2(\mu, \frac{7}{10})$ has a unique root $\mu_1 \in [\kappa_{\frac{7}{10}}, 0]$. By real root isolating,

$$\nu_1 \in \left[\frac{57315}{131072}, \frac{114631}{262144}\right] \approx [0.4372787476, 0.4372825623],$$
$$\mu_1 \in \left[-\frac{86919}{262144}, -\frac{43459}{131072}\right] \approx [-0.3315696716, -0.3315658569],$$

where $\kappa_{\frac{7}{10}} = -0.34603108\cdots$. Therefore, if $U_1(\nu, \mu, \frac{7}{10})$ and $q_1(\nu, \mu, \frac{7}{10})$ have a common root on D_3 with $\lambda = \frac{7}{10}$, the root must be in the region defined by

$$\widetilde{D}: \left[\frac{57315}{131072}, \frac{114631}{262144}\right] \times \left[-\frac{86919}{262144}, -\frac{43459}{131072}\right].$$

In the following, we will prove that $U_1(\nu, \mu, \frac{7}{10})$ and $q(\nu, \mu, \frac{7}{10})$ have no common roots by showing that $q(\nu, \mu, \frac{7}{10}) \neq 0$ on $\widetilde{D}$.

The resultant $\mathrm{res}(\frac{\partial q}{\partial \nu}, \frac{\partial q}{\partial \mu}, \mu)$ has no roots on $\left[\frac{57315}{131072}, \frac{114631}{262144}\right]$ by Sturm's Theorem, implying that there is no maximal or minimum value inside $\widetilde{D}$. Thus, the maximal and minimum values of $q(\nu, \mu, \frac{7}{10})$ are reached on the boundary of $\widetilde{D}$. However, a direct computation shows that $q(\nu, \mu, \frac{7}{10}) > 0$ on the boundary of $\widetilde{D}$, indicating that both the maximal and minimum values of $q(\nu, \mu, \frac{7}{10})$ are positive. Hence, $q(\nu, \mu, \frac{7}{10}) \neq 0$ on $\widetilde{D}$, leading to $U_1(\nu, \mu, \frac{7}{10}) \neq 0$ on $\widetilde{D}$. The above discussion gives the following proposition.

Proposition 4.2.7 *$\frac{I_1(h)}{I_0(h)}$ is monotonic in $(0, H(\lambda, 0))$ for $\lambda \in (\frac{2}{3}, \lambda^*)$.*

The rest of this section is to prove $U_1(\nu, \mu, \lambda^*) \neq 0$ on

$$D_4 = \{(\nu, \mu, \lambda) | \kappa_\lambda < \mu < 0 < \nu < \lambda, \lambda = \lambda^*\}.$$

Recall that λ^* is the root of the first factor of $\mathrm{B}_{\mathrm{S_A}}$, denoted by

$$w(\lambda) = \lambda^9 - \tfrac{5}{2}\lambda^8 + \tfrac{41641\,\lambda^7}{24592} + \tfrac{15855\,\lambda^6}{6148} - \tfrac{693\,\lambda^5}{212} + \tfrac{575721\,\lambda^4}{196736} - \tfrac{4279635\,\lambda^3}{1573888} - \tfrac{395847\,\lambda^2}{196736} + \tfrac{1563705\,\lambda}{393472} - \tfrac{133407}{98368}. \tag{4.14}$$

By computation and Sturm's Theorem, we can show that the resultant $\mathrm{res}(r_2, w, \lambda)$ has a unique zero $\mu_1^* = -0.34794635\cdots$ in $(-1, 0)$, and the resultant $\mathrm{res}(p(\kappa, \lambda), w, \lambda)$ has a unique zero $\kappa_{\lambda^*} = -0.34794635\cdots$ in $(-1, 0)$.

If $\mu_1^* = \kappa_{\lambda^*}$, then $r_2(\mu, \lambda^*)$ has no roots in the interval $(\kappa_{\lambda^*}, 0)$, implying that there are no common roots of $U_1(\nu, \mu, \lambda^*)$ and $q(\nu, \mu, \lambda^*)$ on D_4. Thus, $U_1(\nu, \mu, \lambda^*) \neq 0$ on D_4. In fact, it is true that $\mu_1^* = \kappa_{\lambda^*}$, because $\mathrm{res}(r_2, w, \lambda)$ and $\mathrm{res}(p(\kappa, \lambda), w, \lambda)$ have only one common factor, given by

$$\begin{aligned} cf = {} & 2477123436544\,\mu^{18} - 22294110928896\,\mu^{17} + 88660409909248\,\mu^{16} \\ & -208881731469312\,\mu^{15} + 339714939006976\,\mu^{14} - 439074987073536\,\mu^{13} \\ & +503505982218240\,\mu^{12} - 516729993408000\,\mu^{11} + 456405726382272\,\mu^{10} \\ & -350788104117504\,\mu^{9} + 238040277061248\,\mu^{8} - 133699127790168\,\mu^{7} \\ & +58621983725097\,\mu^{6} - 20391635790324\,\mu^{5} + 3828815281827\,\mu^{4} \\ & +1375428475098\,\mu^{3} - 983422004562\,\mu^{2} + 354781508544\,\mu - 142379421192, \end{aligned}$$

which has a unique root $(= -0.34794635\cdots)$ in $(-1, 0)$, while other factors of $\mathrm{res}(r_2, w, \lambda)$ and $\mathrm{res}(p, w, \lambda)$ have no common roots in $(-1, 0)$. Thus, we have

Proposition 4.2.8 $\frac{I_1(h)}{I_0(h)}$ *is monotonic in the interval* $(0, H(\lambda, 0))$ *for* $\lambda = \lambda^*$.

Combining Propositions 4.2.1, 4.2.6, 4.2.7 and 4.2.8, we have proved Theorem 4.1.2.

4.3 Proof of Theorem 4.1.3

In this section, we prove Theorem 4.1.3 corresponding to system (4.7). We simply transform system (4.7) to (4.5), and then the proof for Theorem 4.1.2 also works for Theorem 4.1.3. To achieve this, taking $x = -\widetilde{x} + 1$, $y = \widetilde{y}$ and $dt = -d\tau$, into system (4.7), we obtain a new system,

$$\dot{x} = -2y, \quad \dot{y} = x^3(x - (1 - \alpha))(x - 1) \tag{4.15}$$

by dropping the tilde, with the Hamiltonian function $\widetilde{\mathcal{H}}(x, y) = y^2 + \frac{1-\alpha}{4}x^4 - \frac{1+1-\alpha}{5}x^5 + \frac{1}{6}x^6$. The center level set $\gamma_2(h)$ of $\mathcal{H}(x, y) = h$ has been transformed to the origin, which is exactly the same as $\gamma_1(h)$. So it is obvious that system (4.15) is exactly the same as system (4.5) if denoting $1 - \alpha = \lambda$, and $\mathcal{I}_0(h) = I_0(h)$, $\mathcal{I}_1(h) = I_1(h)$. Therefore, $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ on $(\mathcal{H}(1, 0), \mathcal{H}(\alpha, 0))$ has the same monotonicity as $\frac{I_1(h)}{I_0(h)}$ on $(0, H(\lambda, 0))$. The proof of Theorem 4.1.3 is complete.

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Chapter 5

Periodic traveling waves in a generalized BBM equation with weak backward diffusion and dissipation terms

5.1 Introduction

Traveling waves in nonlinear wave equations can model many nonlinear complex phenomena in physics, chemistry, biology, mechanics, optics, etc. The wave profiles of long waves in shallow water with different conditions can be modeled by the famous Korteweg-de Vries (KdV) [26], Benjamin-Bona-Mahony [4], the Green-Naghdi [15] and Camassa-Holm [6] equations. In solving real world problems, certain relatively weak influences due to the existence of uncertainty or perturbation are unavoidable, for example in describing the shallow water waves in nonlinear dissipative media [8] and dispersive media [23]. In other words, one should add certain type of small terms in modelling the problems. Topper and Kawahara [42] studied the wave motions on a liquid layer over an inclined plane and established the following Partial Differential Equation (PDE),

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{5.1}$$

for which the wave motion is assumed depending only on the gradient direction. When the inclined plane is relatively long and the surface tension is relatively weak, the u_{xx} and u_{xxxx} terms are relatively small, and the following equation is more appropriate for describing the real situation,

$$u_t + uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{5.2}$$

where $0 < \varepsilon \ll 1$ represents small perturbations to the system. When $\varepsilon = 0$, the backward diffusion (u_{xx}) and dissipation (u_{xxxx}) vanish and (5.2) becomes the classical KdV equation [26], and so (5.2) is usually called a perturbed KdV equation. The KdV equation has played an important role in describing various physical problems, and many researchers have studied this equation and particularly paid attention to solitary and periodic waves. In 1993, Derks and Gils [9] discussed the uniqueness of traveling waves of equation (5.2). A year later, Ogawa [37] studied the existence of solitary and periodic waves of equation (5.2).

When the Marangoni effect is considered on the surface of a thin layer, additional nonlinearity in the form of $(uu_x)_x$ appears, see [12, 20]. For this model, Velarde *et.al.* [44] showed the consistent way of incorporating the Marangoni effect (heating the liquid layer from the air side) into the one-way long-wave assumption and derived the following equation:

$$u_t + 2\alpha_1 uu_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (uu_x)_x = 0, \tag{5.3}$$

which contains the nonlinear term $(uu_x)_x$ due to the Marangoni effect, describing the opposite to the Bénard convection [36, 45]. For the sake of completeness here, we notice that different cases by setting some parameters $\alpha_i = 0$ in Eq. (5.3) have been considered in many works, for example [10, 24, 27, 28]. In particular, Mansour [32] studied the existence of solitary waves in Eq. (5.3) with all small non-vanishing parameters α_2, α_4 and α_5, and in addition established the existence of solitary waves for the following equation [33].

$$u_t + \alpha_1 u^2 u_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (uu_x)_x = 0.$$

It has been noted that the solitary waves and the traveling waves with periodic spatial profiles are very sensitive to weak external influence. For example, stationary periodic patterns in thermal convection may not be observed in a weakly windy circumstance [5]. However, the weak Marangoni effect may destabilize the waves [21], and different perturbations may generate different dynamics of systems, leading to, for example, breaking the periodic traveling waves, changing its stability and yielding quasi-periodic motions on invariant tori, etc. One efficient way to deal with such problems is to apply bifurcation techniques from the view point of dynamical systems by taking the weak external effects as perturbations, and many good results have been obtained for certain nonlinear wave problems, see [13, 17, 22, 49].

In general, perturbations to a dynamical system may be classified into three types: periodic or quasi-periodic forcing, singular perturbation and regular perturbation. When a PDE is perturbed by quasi-periodic forcing terms, one method developed to investigate dynamics (quasi-periodic motion on some invariant tori) of the system is based on an infinite dimensional KAM theory. This theory is an extension of the well-known classical KAM theory, which was established by Kolmogorov [25], Anorld [2] and Moser [35]. It asserts that the majority of tori is persistent under perturbations if the Kolmogorov non-degenerate condition is satisfied.

When a perturbed system can be reduced to a singularly perturbed system, the first question is about the existence of traveling wave solutions of the system. There are lots of publications on this topic, such as singularly perturbed KdV equations [3, 16, 31, 37, 41], the perturbed dispersive-dissipative equations and reaction-diffusion systems [1, 30, 50, 43]. One classical method to deal with singular perturbations is to apply Fenichel's theory (e.g. see [11]), which assures the existence of an invariant manifold and then the problem is reduced to a regular perturbed system on this manifold, see [3, 16, 31, 37, 41]. In these cases, the perturbation always has only one or two terms with lower degrees on the invariant manifold, see above mentioned references and also the works of Derks and Gils [9] and Ogawa [37, 38, 39].

However, very few problems can be directly reduced to regularly perturbed systems. Thus, perturbations are usually not restricted on manifolds. Moreover, there exist fewer mathematical tools which can be used to study the dynamics of perturbed systems, and yet, the analysis and computation based on these approaches are difficult to be used for proving the existence of periodic traveling waves. Thus, when Zhou *et al* studied the Burgers-Huxley equation [53] and

Burgers-Fisher equation [54], they assumed that one coefficient in the equation and the wave speed are small so that these two small terms can be treated as two perturbations, which greatly simplifies the analysis and the proof on the existence of periodic waves [53, 54]. In general, if three or more perturbation terms are involved, the analysis becomes much more difficult.

After the works of Derks and Gils [9] and Ogawa [37], in 2014 Yan *et al.* [48] investigated the perturbed generalized KdV equation,

$$u_t + (u^n)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0.$$

When $\varepsilon = 0$ and $n = 2$, the above equation is reduced to the classical KdV equation,

$$u_t + (u^2)_x + u_{xxx} = 0.$$

Yan *et al.* [48] proved that there exists one periodic wave by choosing some wave speed c for sufficiently small $\varepsilon > 0$. However, the uniqueness of the periodic wave is still open.

Another well-known model describing the propagation of surface water waves in a uniform channel is the Benjamin-Bona-Mahony (BBM) equation,

$$u_t + uu_x - u_{xxt} = 0.$$

This model describes surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in harmonic crystals, see [34]. Due to its wide applications and rich dynamics, researchers have developed many different forms of BBM equations which are usually called generalized BBM equations, see [40, 47, 52] and the reference therein. Wazwaz studied the following generalized BBM equation [47],

$$(u^m)_t + (u^n)_x + (u^l)_{xxx} = 0, \tag{5.4}$$

and found its compaction of dispersive structures.

More recently, Chen *et al.* [7] investigated a perturbed generalized BBM equation,

$$(u^2)_t + (u^3)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{5.5}$$

and established the existence of solitary waves and uniqueness of periodic waves. Both of the works [7] and [48] studied the perturbation problems restricted on manifolds, by using geometric singular perturbation theory. In [7], the authors applied Picárd-Fuchs equations to determine the existence of periodic waves, and developed a good approach to prove that the dominating factor of the Melnikov function is monotonic, see Lemma 4.10 in [7]. Using the same approach, they also proved that the perturbed generalized defocusing mKdV equation,

$$u_t - u^2u_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0,$$

has a unique periodic wave. However, this approach failed to deal with the unperturbed equation having a nilpotent saddle or more degenerate cusp, corresponding to $m > 2$ in (5.4). This is because, taking $m = 3$ for example, one needs to consider more terms in Lemma 4.10 in [7] in order to find some combination of the terms in order to prove the monotonicity of the dominating part of the Melnikov function. However, in general this is very difficult in higher degenerate cases, which is similar to dealing with the cases when more than two perturbation terms are involved in the equations.

In this chapter, we study the BBM equation (5.4) for $m = 3$, $n = 4$ and $l = 1$ with two different kinds of weak dissipative effects P_1 and P_2, described by

$$(u^3)_t + (u^4)_x + u_{xxx} + \varepsilon P_i = 0, \quad i = 1, 2, \tag{5.6}$$

where

$$P_1 = u_{xx} + u_{xxxx}, \quad P_2 = ((\alpha_0 + \alpha_1 u + \alpha_2 u^2)u_x)_x, \tag{5.7}$$

in which α_0, α_1 and α_2 are bounded parameters. P_1 describes the weak second and fourth derivative diffusions without Marangoni effect. P_2 describes a generalized Marangoni effect. The unperturbed problem has a more degenerate singularity which is a nilpotent saddle. Especially, for the case with weak Marangoni effect P_2, the problem is not restricted on a manifold but is reduced to a regular problem with more parameters. It can be seen from our reduction of the problem with P_1 that the weak Marangoni effect P_2 is equivalent to the compound dissipative-Marangoni effect $\gamma_0 u_{xx} + \gamma_1 u_{xxxx} + \gamma_2(uu_x)_x$. The main mathematical tools we will use in this chapter are based on the relatively new theory of weak Hilbert's 16th problem and bifurcation theory.

The rest of this chapter is organized as follows. In section 2, we give a reduction analysis and state our main results. In section 3, we present some perturbation theories and derive a special form of Abelian integral, also called Melnikov function, for periodic and solitary waves. It will be shown that our method without using Picárd-Fuchs Equation is more effective compared to that used in the existing works. In section 4, we study the problem with perturbation P_1 by applying the Chebyshev criterion [14]. In section 5, we investigate the problem with perturbation P_2 and obtain the conditions on the existence of periodic waves. In particular, we derive the exact conditions on the existences of one and two periodic waves. Further, we establish a criterion on the coexistence of one solitary wave and one unique periodic wave. Conclusion is drawn in section 6.

5.2 Main results

In this section, we present our main results for system (5.6). First, we consider system (5.6) with perturbation P_1, that is,

$$(u^3)_t + (u^4)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0. \tag{5.8}$$

Taking $\xi = x - ct$ into (5.8) yields

$$-3cu^2(\xi)u'(\xi) + 4u^3(\xi)u'(\xi) + u'''(\xi) + \varepsilon(u''(\xi) + u''''(\xi)) = 0.$$

Then, integrating this equation and omitting the integral constant, we obtain

$$-cu^3 + u^4 + u''(\xi) + \varepsilon(u'(\xi) + u'''(\xi)) = 0. \tag{5.9}$$

Further introducing the transformations $\xi = \frac{\tau}{c^{\frac{3}{2}}}$ and $u = c\mu$ into (5.9) yields

$$-\mu^3(\tau) + \mu^4(\tau) + \mu''(\tau) + \varepsilon(c^{-\frac{3}{2}}u'(\tau) + c^{\frac{3}{2}}u'''(\tau)) = 0. \tag{5.10}$$

Similarly, system (5.6) with perturbation P_2 can be transformed to

$$-\mu^3(\tau) + \mu^4(\tau) + \mu''(\tau) + \varepsilon(a_0 + a_1\mu(\tau) + a_2\mu^2(\tau))\mu'(\tau) = 0, \tag{5.11}$$

where $a_0 = c^{-\frac{3}{2}}\alpha_0$, $a_1 = c^{-\frac{1}{2}}\alpha_1$ and $a_2 = c^{\frac{1}{2}}\alpha_2$. Because $\alpha_i's$ are independent, we will use $a_i's$ in our analysis for convenience.

Correspondingly, the unperturbed system of (5.10) (with $\varepsilon = 0$) is given by $-\mu^3(\tau)+\mu^4(\tau)+\mu''(\tau) = 0$, which is equivalent to the system,

$$\begin{aligned} \frac{d\mu}{d\tau} &= \nu, \\ \frac{d\nu}{d\tau} &= \mu^3 - \mu^4, \end{aligned} \tag{5.12}$$

which has a hyperelliptic Hamiltonian function, given by

$$\mathcal{H}(\mu, \nu) = \frac{\nu^2}{2} - \frac{\mu^4}{4} + \frac{\mu^5}{5}, \tag{5.13}$$

satisfying $\mathcal{H}(1, 0) = -\frac{1}{20}$ and $\mathcal{H}(0, 0) = \mathcal{H}(\frac{5}{4}, 0) = 0$. The function $\mathcal{H} = h$ for $h \in (-\frac{1}{20}, 0)$ and $\mu \in (0, \frac{5}{4})$, depicted in Figure 5.1, shows a family of closed orbits surrounded by a homoclinic loop Γ_0, with a nilpotent saddle of order 1 at the origin.

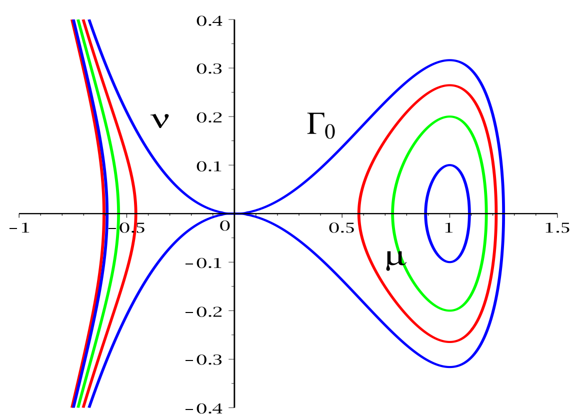

Figure 5.1: The portrait of system (5.12). The periodic orbits and homolcinc loop correspond to the periodic waves and the solitary wave of equation $(5.6)_{\varepsilon=0}$.

In order to state our main results clearly and systematically, we use the following notations: Γ_h denoting the closed curve defined by $\mathcal{H}(\mu, \nu) = h$; $\mu(\tau, h)$ representing the closed orbit of system (5.12) corresponding to Γ_h; $\mu(\tau, \varepsilon, h, c(\varepsilon, h))$ being the periodic wave of system (5.10) near Γ_h under the condition $c = c(\varepsilon, h)$; $\mu(\tau, \varepsilon, h)$ denoting the traveling wave of system (5.11) near Γ_h. Our main results are given in the following Theorem 5.2.1 and Theorem 5.2.2.

Theorem 5.2.1 *For the perturbed BBM equation* (5.6) *with perturbation* P_1*, the following hold.*

(i) *For any sufficiently small $\varepsilon > 0$ and any $h \in (-\frac{1}{20}, 0)$, there exists a smooth function $c(\varepsilon, h)$ in ε and h such that system* (5.6) *has one unique isolated periodic wave in a sufficiently small neighborhood of Γ_h, given by $u = c\mu(\tau, \varepsilon, h, c(\varepsilon, h))$, satisfying*

$$\lim_{\varepsilon\to 0} \mu(\tau, \varepsilon, h, c) = \mu(\tau, h),$$
$$\tfrac{\partial}{\partial\tau}\mu(0, \varepsilon, h, c) = \tfrac{\partial^2}{\partial\tau^2}\mu(0, \varepsilon, h, c) = 0, \quad \tfrac{\partial^3}{\partial\tau^3}\mu(0, \varepsilon, h, c) < 0,$$

and

$$\lim_{\varepsilon\to 0} c(\varepsilon, h) = c(h),$$

where $c(h)$ is a monotonically increasing function in h satisfying $1 < c(h) < \left(\frac{39}{25}\right)^{\frac{1}{3}}$.

(ii) *For any sufficiently small $\varepsilon > 0$, there exists a critical wave speed $c = (\frac{39}{25})^{\frac{1}{3}} + O(\varepsilon)$ such that system* (5.6) *has one solitary wave in a sufficiently small neighborhood of Γ_0.*

Theorem 5.2.2 *For any sufficiently small $\varepsilon > 0$, the perturbed BBM equation* (5.6) *with perturbation P_2 has at most two isolated periodic waves. More precisely, we have the following results.*

(i) *The Abelian integral $\mathcal{M}(h)$ given in* (5.25) *has one unique simple zero for any $h^* \in (-\frac{1}{20}, 0)$ if and only if*

$$a_1 = \lambda \in \left(-\infty, -\frac{5}{3}\right] \bigcup \left[-\frac{10}{11}, +\infty\right) \text{ and } a_0 = -\kappa(h^*, \lambda),$$

where $\kappa(h, \lambda)$ is defined in (5.45)*. Therefore, for any sufficiently small $\varepsilon > 0$, system* (5.6) *has one unique isolated periodic wave $u = c\mu(\tau, \varepsilon, h^*)$, in sufficiently small neighborhood of any closed curve Γ_{h^*} by taking $a_1 = \lambda + O(\varepsilon)$ and $a_0 = -\kappa(h^*, \lambda) + O(\varepsilon)$, satisfying*

$$\lim_{\varepsilon\to 0} \mu(\tau, \varepsilon, h^*) = \mu(\tau, h^*),$$
$$\tfrac{\partial}{\partial\tau}\mu(0, \varepsilon, h^*) = \tfrac{\partial^2}{\partial\tau^2}\mu(0, \varepsilon, h^*) = 0, \quad \tfrac{\partial^3}{\partial\tau^3}\mu(0, \varepsilon, h^*) < 0.$$

(ii) *The Abelian integral $\mathcal{M}(h)$ has exactly two simple zeros h_1 and h_2 if and only if*

$$a_1 = \lambda \in \left(-\frac{5}{3}, -\frac{10}{11}\right) \text{ and } a_0 = -\kappa(h_1, \lambda) = -\kappa(h_2, \lambda),$$

where

$$\kappa(h_1, \lambda) = \kappa(h_2, \lambda) \in \left(\min_{h\in(-\frac{1}{20}, 0)} \{\kappa(h, \lambda)\},\ \min\left\{\kappa\left(-\frac{1}{20}, \lambda\right), \kappa(0, \lambda)\right\}\right),$$

under which, for any sufficiently small $\varepsilon > 0$, system (5.6) *has exactly two isolated periodic waves $u_1 = c\mu(\tau, \varepsilon, h_1)$ and $u_2 = c\mu(\tau, \varepsilon, h_2)$, in sufficiently small neighborhoods of the closed curves Γ_{h_1} and Γ_{h_2} by choosing $a_1 = \lambda^* + O(\varepsilon)$ and $a_0 = -\kappa(h_1, \lambda) + O(\varepsilon)$, satisfying*

$$\lim_{\varepsilon\to 0} \mu(\tau, \varepsilon, h_i) = \mu(\tau, h_i),$$
$$\tfrac{\partial}{\partial\tau}\mu(0, \varepsilon, h_i) = \tfrac{\partial^2}{\partial\tau^2}\mu(0, \varepsilon, h_i) = 0, \quad \tfrac{\partial^3}{\partial\tau^3}\mu(0, \varepsilon, h_i) < 0, \quad i = 1, 2.$$

(iii) *The Abelian integral $\mathcal{M}(h)$ has a unique zero at $h = 0$ if and only if*

$$a_0 = -\frac{5}{6}a_1 - \frac{25}{33}, \tag{5.14}$$

and further, under (5.14), *$\mathcal{M}(h)$ has a unique simple zero in $(-\frac{1}{20}, 0)$ if and only if $a_1 = \lambda^{**} \in (-\frac{16}{11}, -\frac{10}{11})$. Therefore, for any sufficiently small $\varepsilon > 0$, system* (5.6) *can have a solitary wave by taking $a_0 = -\frac{5}{6}a_1 - \frac{25}{33} + O(\varepsilon)$, and coexistence of a solitary wave and a unique periodic wave by choosing $a_0 = -\frac{5}{6}\lambda^{**} - \frac{25}{33} + O(\varepsilon)$ and $a_1 = \lambda^{**} + O(\varepsilon)$.*

Before further analysis, in the next section we will present some definitions and lemmas in perturbation theory, which are needed for proving Theorems 5.2.1 and 5.2.2.

5.3 Perturbation theory and analysis

Lemma 5.3.1 (Fenichel Criteria) *Consider the system*

$$\begin{aligned} \dot{x} &= f_1(x, y, \varepsilon), \\ \dot{y} &= \varepsilon f_2(x, y, \varepsilon), \end{aligned} \tag{5.15}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ and $0 < \varepsilon \ll 1$ is a real parameter, f_1 and f_2 are C^∞ on the set $V \times I$, $V \subseteq \mathbb{R}^{n+l}$, I is an open interval containing zero. Assume that for $\varepsilon = 0$, system (5.15) *has a compact normally hyperbolic manifold M_0 which is contained in the set $f_1(x, y, 0) = 0$. The manifold M_0 is said to be normally hyperbolic if the linearization of* (5.15) *at each point in M_0 has exactly* $\dim(M_0)$ *eigenvalues on the imaginary axis. Then, for any $0 < r < +\infty$, there exists a manifold M_ε for ε sufficiently small such that the following conclusions hold.*

(i) *M_ε is locally invariant under the flow of* (5.15).

(ii) *M_ε is C^r in x, y and ε.*

(iii) *$M_\varepsilon = \{(x, y) | x = h^\varepsilon(y)\}$ for some C^r function h^ε, and y in some compact set K.*

(iv) *There exist locally invariant stable and unstable manifolds $W_s(M_\varepsilon)$, $W_u(M_\varepsilon)$, that lie within $O(\varepsilon)$ of, and are diffeomorphic to $W_s(M_0)$ and $W_u(M_0)$.*

Definition 5.3.2 *Suppose $f_0(x), f_1(x), \ldots, f_{n-1}(x)$ are analytic functions on an real open interval J.*

(i) *The family of sets $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ is called a Chebyshev system provided that any nontrivial linear combination,*

$$k_0 f_0(x) + k_1 f_1(x) + \cdots + k_{n-1} f_{n-1}(x),$$

has at most $n - 1$ isolated zeros on J.

(ii) *An ordered set of n functions* $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ *is called a complete Chebyshev system (CT-system for short) provided any nontrivial linear combination,*

$$k_0 f_0(x) + k_1 f_1(x) + \cdots + k_{i-1} f_{i-1}(x),$$

has at most $i-1$ *zeros for all* $i = 1, 2, \cdots, n$. *Moreover it is called an extended complete Chebyshev system (ECT-system for short) if the multiplicities of zeros are taken into account.*

(iii) *The continuous Wronskian of* $\{f_0(x), f_1(x), \ldots, f_{k-1}(x)\}$ *at* $x \in \mathbb{R}$ *is*

$$W[f_0(x), f_1(x), \ldots, f_{k-1}(x)] = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{k-1} \\ f_0^{'}(x) & f_1^{'}(x) & \cdots & f_{k-1}^{'}(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix},$$

where $f'(x)$ *is the first order derivative of* $f(x)$ *and* $f^{(i)}(x)$ *is the ith order derivative of* $f(x)$, $i \geq 2$. *The definitions imply that if the function tuple* $\{f_0(x), f_1(x), \cdots, f_{k-1}(x)\}$ *is an ECT-system on J, then it is a CT-system on J, and thus a T-system on J. However the inverse is not true.*

Let $\mathrm{res}(f_1, f_2)$ denote the resultant of $f_1(x)$ and $f_2(x)$, where $f_1(x)$ and $f_2(x)$ are two univariate polynomials of x on rational number field Q. As it is known, $\mathrm{res}(f_1(x), f_2(x)) = 0$ if and only if $f_1(x)$ and $f_2(x)$ have at least one common root.

Let $\mathrm{res}(f, g, x)$ and $\mathrm{res}(f, g, z)$ denote respectively the resultants between $f(x, z)$ and $g(x, z)$ with respect to x and z, where $f(x, z)$ and $g(x, z)$ are two polynomials in $\{x, z\}$ on rational number field. $\mathrm{res}(f, g, x)$ is a polynomial in z and $\mathrm{res}(f, g, z)$ is a polynomial in x. About the relation between the common roots of two polynomials and their resultants, the following result can be found in many works on polynomial algebra, such as [46]. For completeness we give a short proof.

Lemma 5.3.3 ([46]) (i) *Let* (x_0, z_0) *be a common root of* $f(x, z)$ *and* $g(x, z)$. *Then* $\mathrm{res}(f, g, x_0) = 0$ *and* $\mathrm{res}(f, g, z_0) = 0$. *However, the inverse is not true.*

(ii) *Let* $\mathrm{res}(f, g, z)$ *have a unique real root on some open interval* (α, β), *and* $\mathrm{res}(f, g, x)$ *have a unique real root on some open interval* (γ, θ). *Then there exists at most one common real root of* $f(x, z)$ *and* $g(x, z)$ *on* $(\alpha, \beta) \times (\gamma, \theta)$.

Proof (ii) is obvious if (i) is true. So we only prove (i). A two-variable polynomial can be treated as one univariate polynomial of one variable with the other treated as a parameter. Taking $f(x, z) = f_z(x)$ and $g(x, z) = g_z(x)$ which are polynomials of x with parameter z. Let x_0 be the common root of $f_{z_0}(x)$ and $g_{z_0}(x)$. Then $\mathrm{res}(f_{z_0}(x), g_{z_0}(x)) = \mathrm{res}(f, g, z_0) = 0$, where z_0 is the common root of $f_{x_0}(z)$ and $g_{x_0}(z)$, and therefore, $\mathrm{res}(f_{x_0}(z), g_{x_0}(z)) = \mathrm{res}(f, g, x_0) = 0$.

Let $H(x, y) = A(x) + \frac{y^2}{2}$ be an analytic function. Assume there exists a punctured neighborhood $\mathcal{P}$ of the origin foliated by ovals $\Gamma_h \subseteq \{(x, y) | H(x, y) = h,\ h \in (0, h_0),\ h_0 = H(\partial\mathcal{P})\}$. The projection of $\mathcal{P}$ on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$. Under these assumptions it is easy to verify that $xA'(x) > 0$ for all $x \in (x_l, x_r)\backslash\{0\}$, and $A(x)$ has a zero of even multiplicity at $x = 0$ and there exists an analytic involution $z(x)$, defined by $A(x) = A(z(x))$ for all $x \in (x_l, x_r)$. Let

$$\mathcal{I}_i(h) = \oint_{\Gamma_h} f_i(x) y^{2s-1} dx \quad \text{for } h \in (0, h_0), \tag{5.16}$$

where $f_i(x)$, $i = 0, 1, \ldots, n-1$, are analytic functions on (x_l, x_r) and $s \in \mathbb{N}$. Further, define

$$l_i(x) := \frac{f_i(x)}{A'(x)} - \frac{f_i(z(x))}{A'(z(x))}.$$

Then we have

Lemma 5.3.4 ([14]) *Under the above assumption,* $\{\mathcal{I}_0, \mathcal{I}_1, \cdots, \mathcal{I}_{n-1}\}$ *is an* ECT *system on* $(0, h_0)$ *if* $\{l_0, l_1, \cdots, l_{n-1}\}$ *is an* ECT *system on* $(x_l, 0)$ *or* $(0, x_r)$ *and* $s > n - 2$.

Lemma 5.3.5 ([29]) *Under the above assumption, if the following conditions are satisfied:*
(i) $W[l_0, l_1, \ldots, l_i]$ *does not vanish on* $(0, x_r)$ *for* $i = 0, 1, \cdots, n-2$,
(ii) $W[l_0, l_1, \ldots, l_{n-1}]$ *has* k *zeros on* $(0, x_r)$ *with multiplicities counted, and*
(iii) $s > n + k - 2$,
then, any nontrivial linear combination of $\{\mathcal{I}_0, \mathcal{I}_1, \cdots, \mathcal{I}_{n-1}\}$ *has at most* $n + k - 1$ *zeros on* $(0, h_0)$ *with multiplicities counted. In this case,* $\{\mathcal{I}_0, \mathcal{I}_1, \cdots, \mathcal{I}_{n-1}\}$ *is called a Chebyshev system with accuracy* k *on* $(0, h_0)$*, where* $W[l_0, l_1, \ldots, l_i]$ *denotes the Wronskian of* $\{l_0, l_1, \ldots, l_i\}$.

Now we consider system (5.10), which can be rewritten in the form of

$$\begin{aligned} \frac{d\mu}{d\tau} &= \nu, \\ \frac{d\nu}{d\tau} &= \omega, \\ \varepsilon c^{\frac{3}{2}} \frac{d\omega}{d\tau} &= \mu^3 - \mu^4 - \omega - \frac{\varepsilon}{c^{\frac{3}{2}}}\nu. \end{aligned} \tag{5.17}$$

Introducing the time scaling $\sigma = \frac{\tau}{\varepsilon}$ into (5.17) yields

$$\begin{aligned} \frac{d\mu}{d\sigma} &= \varepsilon\nu, \\ \frac{d\nu}{d\sigma} &= \varepsilon\omega, \\ c^{\frac{3}{2}} \frac{d\omega}{d\sigma} &= \mu^3 - \mu^4 - \omega - \frac{\varepsilon}{c^{\frac{3}{2}}}\nu. \end{aligned} \tag{5.18}$$

When $\varepsilon > 0$, system (5.17) is equivalent to (5.18). System (5.17) is called the slow system, while system (5.18) is called the fast system.

The slow system (5.17) determines its critical manifold, which is a two-dimensional submanifold in $\mathbb{R}^3$:

$$M_0 = \left\{ (\mu, \nu, \omega) \in \mathbb{R}^3 \middle| \, \omega = \mu^3 - \mu^4 \right\}.$$

The Jacobian matrix of the fast system (5.18) restricted on M_0 is given by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^{-\frac{3}{2}}(3\mu^2-4\mu^3) & 0 & -c^{-3/2} \end{bmatrix}$$

which has three eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -c^{-3/2}$, with λ_1 and λ_2 being on the imaginary axis. Therefore, M_0 is normally hyperbolic. Consequently, it follows from Lemma 5.3.1 that for $\varepsilon > 0$ sufficiently small, there exists a two-dimensional submanifold M_ε in $\mathbb{R}^3$, which is invariant under the flow of system (5.17), within the Hausdorff distance ε of M_0.

Let

$$M_\varepsilon = \left\{(\mu,\nu,\omega)\in\mathbb{R}^3 : \omega = \mu^3-\mu^4+\eta(\mu,\nu,\varepsilon)\right\},$$

where $\eta(\mu,\nu,\varepsilon)$ is smooth in μ, ν and ε, satisfying $\eta(\mu,\nu,0)=0$, and expanded as

$$\eta(\mu,\nu,\varepsilon) = \varepsilon\eta_1(\mu,\nu) + O(\varepsilon^2). \tag{5.19}$$

Substituting (5.19) into the last equation of (5.17) and comparing its coefficients yield

$$\eta_1(\mu,\nu) = c^{\frac{3}{2}}\left((-3\mu^2+4\mu^3)\nu - c^{-3}\nu\right).$$

The dynamics of (5.17) on M_ε is determined by

$$\begin{aligned} \frac{d\mu}{d\tau} &= \nu, \\ \frac{d\nu}{d\tau} &= \mu^3-\mu^4+\varepsilon c^{\frac{3}{2}}\left((-3\mu^2+4\mu^3)\nu - c^{-3}\nu\right) + O(\varepsilon^2). \end{aligned} \tag{5.20}$$

For any $h\in(-\frac{1}{20},0)$, $\mathcal{H}(\mu,\nu)=h$ defines a periodic orbit Γ_h of (5.12) (or the system (5.20) with $\varepsilon=0$). Let $(\alpha(h),0)$ denote the intersection point of Γ_h and the positive μ-axis, T the period of Γ_h. Further, let $\Gamma_{h,\varepsilon}$ be the positive orbit of (5.20) starting from the point $(\alpha(h),0)$ at time $\tau=0$, and $(\beta(h,\varepsilon),0)$ the first intersection point of $\Gamma_{h,\varepsilon}$ with the positive μ-axis at time $\tau=\tau^*(\varepsilon)$. Let $\mathcal{H}^*(\mu,\nu)$ denote the small perturbation of $\mathcal{H}(\mu,\nu)$. Then the difference between the two points is given by

$$\begin{aligned} \mathcal{H}^*((\beta(h,\varepsilon),0)) - \mathcal{H}^*((\alpha(h),0)) &= \int_{\Gamma_{h,\varepsilon}} d\mathcal{H}^* = \int_{\Gamma_{h,\varepsilon}} (-\mu^3+\mu^4)d\mu + \nu d\nu \\ &= \int_0^{\tau^*(\varepsilon)} \left((-\mu^3+\mu^4)\nu - \nu((-\mu^3+\mu^4) - \varepsilon c^{\frac{3}{2}}(-3\mu^2\nu+4\mu^3\nu-c^{-3}\nu))\right)d\tau \\ &= \int_0^{\tau^*(\varepsilon)} \varepsilon\nu c^{\frac{3}{2}}\left((-3\mu^2+4\mu^3)\nu - c^{-3}\nu\right)d\tau = \varepsilon\int_0^{\tau^*(\varepsilon)} c^{-\frac{3}{2}}\left(c^3(-3\mu^2+4\mu^3)\nu^2-\nu^2\right)d\tau \\ &\triangleq \varepsilon F(h,\varepsilon). \end{aligned}$$

By continuousness theorem, we have

$$\lim_{\varepsilon\to0}\Gamma_{h,\varepsilon} = \Gamma_h, \quad \lim_{\varepsilon\to0} b(h,\varepsilon) = a(h), \quad \lim_{\varepsilon\to0}\tau^*(\varepsilon) = T,$$

and thus,

$$
\begin{aligned}
F(h,\varepsilon) &= \int_0^T c^{-\frac{3}{2}}\left(c^3(-3\mu^2+4\mu^3)\nu^2-\nu^2\right)d\tau+O(\varepsilon),\\
&= c^{-\frac{3}{2}}\int_{\Gamma_h}\left(c^3(-3\mu^2+4\mu^3)\nu-\nu\right)d\mu+O(\varepsilon)\\
&\triangleq c^{-\frac{3}{2}}M(h)+O(\varepsilon),
\end{aligned}
\tag{5.21}
$$

where $M(h)$ is called Abelian integral or Melnikov function, given by

$$
M(h)=\int_{\Gamma_h}\left(c^3(-3\mu^2+4\mu^3)\nu-\nu\right)d\mu=\int_{\Gamma_h}\left(-1+c^3(-3\mu^2+4\mu^3)\right)\nu d\mu. \tag{5.22}
$$

It has been noted that compared with the application of Picárd-Fuchs equation method (eg. see [7]) which is often used to derive the Abelian integral, here our approach developed above is much simpler.

Similarly, for system (5.11), we take $\mu'(\tau)=\nu$ and follow the above procedure to obtain the following regular perturbation problem which is not restricted on a manifold,

$$
\begin{aligned}
\frac{d\mu}{d\tau}&=\nu,\\
\frac{d\nu}{d\tau}&=\mu^3-\mu^4+\varepsilon(a_0+a_1\mu+a_2\mu^2)\nu.
\end{aligned}
\tag{5.23}
$$

Let $(\alpha^*(h),0)$ be the intersection point of Γ_h and the positive μ-axis, T the period of Γ_h, $\Gamma^*_{h,\varepsilon}$ the positive orbit of (5.23) starting from the point $(\alpha^*(h),0)$ at time $\tau=0$, and $(\beta^*(h,\varepsilon),0)$ the first intersection point of $\Gamma_{h,\varepsilon}$ with the positive μ-axis at time $\tau=\tau^*(\varepsilon)$. Then, the difference between the two points $(\alpha^*(h),0)$ and $(\beta^*(h,\varepsilon),0)$ can be expressed as

$$
\begin{aligned}
\mathcal{H}^*(\beta^*(h,\varepsilon),0)-\mathcal{H}^*(\alpha^*(h),0)&=\int_{\Gamma^*_{h,\varepsilon}}d\mathcal{H}^*\\
=\quad \varepsilon\int_{\Gamma_h}(a_0+a_1\mu+\mu^2)\nu d\mu+O(\varepsilon)&\triangleq\varepsilon\mathcal{M}(h)+O(\varepsilon^2),
\end{aligned}
\tag{5.24}
$$

where the Abelian integral $\mathcal{M}(h)$ is given by

$$
\mathcal{M}(h)=\int_{\Gamma_h}(a_0+a_1\mu+\mu^2)\nu d\mu. \tag{5.25}
$$

To investigate the existence of periodic and solitary waves for the two perturbation problems, we need study the zeros of the functions $\mathcal{H}^*(\beta(h,\varepsilon),0)-\mathcal{H}^*(\alpha(h),0)$ and $\mathcal{H}^*(\beta^*(h,\varepsilon),0)-\mathcal{H}^*(\alpha^*(h),0)$ and their distributions. It follows from (5.21) and (5.24) that it suffices to consider the Abelian integrals $M(h)$ and $\mathcal{M}(h)$.

5.4 Analysis of system (5.6) with perturbation P_1

In the section, we study system (5.6) with perturbation P_1. Based on the discussion in the previous sections, we need only study the Abelian integral $M(h)$. Let

$$
J_n(h)=\oint_{\Gamma_h}\mu^n\nu d\mu. \tag{5.26}
$$

Then

$$M(h) = c^3(-3J_2 + 4J_3) - J_0.$$

Lemma 5.4.1 *For $h \in (-\frac{1}{20}, 0)$, $J_0'(h) > 0$ and $J_0(h) > 0$.*

Proof It is easy to obtain

$$J_0'(h) = \oint_{\Gamma_h} \frac{d\mu}{\nu} = \oint_{\Gamma_h} \frac{\nu d\tau}{\nu} = \int_0^{T(h)} d\tau = T(h) > 0,$$

where $T(h)$ denotes the period of Γ_h.

Since $\nu \to 0$ as $h \to -\frac{1}{20}$, we have

$$J_0(-\frac{1}{20}) = \lim_{h\to -\frac{1}{20}} \oint_{\Gamma_h} \nu d\mu = \lim_{h\to -\frac{1}{20}} \int_0^T \nu^2 d\tau = 0.$$

which, together with $J_0'(h) > 0$, implies $J_0(h) > 0$ for $h \in (-\frac{1}{20}, 0)$.

It follows from Lemma 5.4.1 that the following ratio is well defined,

$$X(h) = \frac{-3J_2 + 4J_3}{J_0}. \tag{5.27}$$

Then

$$M(h) = J_0(c^3 X(h) - 1). \tag{5.28}$$

In the remaining of this section we mainly prove the following proposition, which is needed for proving Theorem 5.2.1.

Proposition 5.4.2 *For $h \in (-\frac{1}{20}, 0)$, $X'(h) < 0$. Moreover,*

$$\frac{25}{39} < X(h) < 1, \quad \lim_{h\to -\frac{1}{20}} X(h) = 1, \quad \lim_{h\to 0} X(h) = \frac{25}{39}.$$

To prove Proposition 5.4.2, we need the following lemmas.

Lemma 5.4.3 *Suppose $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ is the Beta function with $p > 0$ and $q > 0$. Then the following hold:*

$$J_0(0) = \sqrt{2}\left(\frac{5}{4}\right)^3 B\left(\frac{3}{2}, 3\right), \; J_2(0) = \sqrt{2}\left(\frac{5}{4}\right)^5 B\left(\frac{3}{2}, 5\right), \; J_3(0) = \sqrt{2}\left(\frac{5}{4}\right)^6 B\left(\frac{3}{2}, 6\right).$$

In addition,

$$\frac{J_2(0)}{J_0(0)} = \frac{25}{33}, \; \frac{J_3(0)}{J_0(0)} = \frac{625}{858}, \; \frac{J_3(0)}{J_2(0)} = \frac{25}{26}.$$

Proof Let $\mathcal{H} - \frac{v^2}{2} = 0$. Then we have

$$J_n(0) = \sqrt{2}\int_0^{\frac{5}{4}} \mu^{n+2}\sqrt{1-\tfrac{4}{5}\mu}\, d\mu.$$

Let $1 - \frac{4}{5}\mu = t$, and so $\mu = \frac{5}{4}(1-t)$, $d\mu = -\frac{5}{4}dt$. Then we obtain

$$J_n(0) = \sqrt{2}\int_0^1 \Big(\frac{5}{4}\Big)^{n+3}(1-t)^{n+2}t^{\frac{1}{2}}dt = \sqrt{2}\Big(\frac{5}{4}\Big)^{n+3}B\Big(\frac{3}{2}, n+3\Big),$$

which proves the first part of the lemma.

Next, it follows from

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \text{ and } \Gamma(s+1) = s\Gamma(s),$$

where $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}dx$ $(s>0)$ is the Gamma function, that

$$\begin{aligned}\frac{J_2(0)}{J_0(0)} &= \frac{(\frac{5}{4})^5 B(\frac{3}{2},5)}{(\frac{5}{4})^3 B(\frac{3}{2},3)} = \Big(\frac{5}{4}\Big)^2 \times \frac{\Gamma(\frac{3}{2})\Gamma(5)}{\Gamma(\frac{13}{2})} \times \frac{\Gamma(\frac{9}{2})}{\Gamma(\frac{3}{2})\Gamma(3)}\\ &= \Big(\frac{5}{4}\Big)^2 \times \frac{4\times 3\times\Gamma(3)\times\Gamma(\frac{9}{2})}{\frac{11}{2}\times\frac{9}{2}\times\Gamma(\frac{9}{2})\times\Gamma(3)} = \frac{25}{33}.\end{aligned}$$

Similarly, we obtain

$$\frac{J_3(0)}{J_0(0)} = \frac{625}{858} \text{ and } \frac{J_3(0)}{J_2(0)} = \frac{25}{26}.$$

Lemma 5.4.4 *The following rates at $h = -\frac{1}{20}$ hold:*

$$\frac{J_2(-\frac{1}{20})}{J_0(-\frac{1}{20})} = \lim_{h\to-\frac{1}{20}} \frac{J_2(h)}{J_0(h)} = 1,$$

$$\frac{J_3(-\frac{1}{20})}{J_0(-\frac{1}{20})} = \lim_{h\to-\frac{1}{20}} \frac{J_3(h)}{J_0(h)} = 1,$$

$$\frac{J_3(-\frac{1}{20})}{J_2(-\frac{1}{20})} = \lim_{h\to-\frac{1}{20}} \frac{J_3(h)}{J_2(h)} = 1.$$

Proof Let $\mu = r\cos\theta + 1$, $v = r\sin\theta$. Then $\mathcal{H} - h = 0$ becomes

$$\mathcal{F}(r,\rho) \triangleq \frac{r^5}{5}\cos^5\theta + \frac{3}{4}r^4\cos^4\theta + r^3\cos^3\theta + \frac{r^2}{2} - \rho^2 = 0,$$

where $\rho = (h+\frac{1}{20})^{\frac{1}{2}}$. Applying the implicit function theorem to $\mathcal{F}(r,\rho)$ at $(r,\rho) = (0,0)$, we can show that there exist a smooth function $r = \chi(\rho)$ and a small positive number δ, $0<\rho<\delta\ll 1$ such that $\mathcal{F}(\chi(\rho),\rho) = 0$, and $\chi(\rho)$ can be expanded as

$$\begin{aligned}\chi(\rho) \;=\;& \sqrt{2}\rho - 2\rho^2\cos^3\theta + \sqrt{2}\Big(-\frac{3}{2}\cos^4\theta + 5\cos^6\theta\Big)\rho^3\\ &+\Big(-\frac{4}{5}\cos^5\theta + 18\cos^7\theta - 32\cos^9\theta\Big)\rho^4 + O(\rho^5).\end{aligned} \tag{5.29}$$

Therefore,

$$J_n(h) = \oint_{\Gamma_h} \mu^n \nu d\mu = \iint_{int\Gamma_h} \mu^n d\mu d\nu = \int_0^{2\pi} d\theta \int_0^{\chi(\rho)} r^{n+1} \cos^n \theta dr. \tag{5.30}$$

Noticing $\rho = (h + \frac{1}{20})^{\frac{1}{2}}$ and substituting (5.29) into (5.30) yield

$$\begin{aligned}
J_0(h) &= 2\pi(h + \tfrac{1}{20}) + \tfrac{21\pi}{4}(h + \tfrac{1}{20})^2 + O((h + \tfrac{1}{20})^3),\\
J_1(h) &= 2\pi(h + \tfrac{1}{20}) + \tfrac{9\pi}{4}(h + \tfrac{1}{20})^2 + O((h + \tfrac{1}{20})^3),\\
J_2(h) &= 2\pi(h + \tfrac{1}{20}) + \tfrac{\pi}{4}(h + \tfrac{1}{20})^2 + O((h + \tfrac{1}{20})^3),\\
J_3(h) &= 2\pi(h + \tfrac{1}{20}) - \tfrac{3\pi}{4}(h + \tfrac{1}{20})^2 + O((h + \tfrac{1}{20})^3)
\end{aligned} \tag{5.31}$$

for $0 < h + \frac{1}{20} \ll 1$. Therefore,

$$\frac{J_i(-\frac{1}{20})}{J_j(-\frac{1}{20})} = \lim_{h \to -\frac{1}{20}} \frac{J_i(h)}{J_j(h)} = 1, \ i, \ j = 0, \ 1, \ 2, 3.$$

This completes the proof of Lemma 5.4.4.

Lemma 5.4.5 *$J_n(h) = \sum\limits_{i=0}^{n} (-1)^i C_n^i I_i(h)$, where $I_i(h) = \oint_{\widetilde{\mathcal{H}}=h} \widetilde{\mu}^i \widetilde{\nu} d\widetilde{\mu}$, in which $\widetilde{\mu} = -\mu + 1$, $\widetilde{\nu} = -\nu$, and*

$$\widetilde{\mathcal{H}}(\widetilde{\mu}, \widetilde{\nu}) = \mathcal{H}(1 - \widetilde{\mu}, -\widetilde{\nu}). \tag{5.32}$$

In particular,

$$\begin{aligned}
&J_0(h) = I_0(h),\\
&J_2(h) = I_2(h) - 2I_1(0) + I_0(h),\\
&J_3(h) = -I_3(h) + 3I_2(0) - 3I_1(h) + I_0(h).
\end{aligned} \tag{5.33}$$

Proof A direct computation shows that

$$\begin{aligned}
J_n(h) &= \oint_{\Gamma_h} \mu^n \nu d\mu = \oint_{\widetilde{\mathcal{H}}=h} (1 - \widetilde{\mu})^n (-\widetilde{\nu}) d(1 - \widetilde{\mu})\\
&= \oint_{\widetilde{\mathcal{H}}=h} \left(\sum_{i=0}^{n} C_n^i (-1)^i \widetilde{\mu}^i \widetilde{\nu} \right) d\widetilde{\mu} = \sum_{i=0}^{n} C_n^i (-1)^i \oint_{\widetilde{\mathcal{H}}=h} \widetilde{\mu}^i \widetilde{\nu} d\widetilde{\mu}\\
&= \sum_{i=0}^{n} (-1)^i C_n^i I_i(h).
\end{aligned}$$

Then substituting $n = 0, \ 2, \ 3$ respectively into $J_n(h)$ yields (5.33).

Lemma 5.4.6 *On $(-\frac{1}{20}, 0)$, $\frac{J_2(h)}{J_0(h)}$ is decreasing monotonically from 1 to $\frac{25}{33}$, and $\frac{J_3(h)}{J_2(h)}$ is decreasing monotonically from 1 to $\frac{25}{26}$.*

Proof By Lemmas 5.4.3 and 5.4.4, we need only prove that $\frac{J_2(h)}{J_0(h)}$ and $\frac{J_3(h)}{J_2(h)}$ are monotonic on the interval $(-\frac{1}{20}, 0)$, which implies that each of the linear combination $\alpha_1 J_0(h) + \alpha_2 J_2(h)$ and $\alpha_1^* J_2(h) + \alpha_2^* J_3(h)$ has at most one zero on $(-\frac{1}{20}, 0)$. Let

$$f_0(\widetilde{\mu}) = 1, \quad f_2(\widetilde{\mu}) = \widetilde{\mu}^2 - 2\widetilde{\mu} + 1, \quad f_3(\widetilde{\mu}) = -\widetilde{\mu}^3 + 3\widetilde{\mu}^2 - 3\widetilde{\mu} + 1.$$

By Lemma 5.4.5, we have

$$J_k(h) = \oint_{\widetilde{\mathcal{H}}=h} f_k(\widetilde{\mu})\widetilde{\nu}d\widetilde{\mu},\ k = 0,\ 1,\ 2,\ 3.$$

Then, for each $f_i(\widetilde{\mu})$, $i = 0, 2, 3$, set

$$l_i(\widetilde{\mu}) = \left(\frac{f_i}{A'}\right)(\widetilde{\mu}) - \left(\frac{f_i}{A'}\right)(z(\widetilde{\mu})), \tag{5.34}$$

where $z(\widetilde{\mu})$ is an analytic involution defined by $A(\widetilde{\mu}) = A(z(\widetilde{\mu}))$ on $(-\frac{1}{4}, 1)$, and

$$A(\widetilde{\mu}) = \widetilde{\mathcal{H}}(\widetilde{\mu}, \widetilde{\nu}) - \frac{\widetilde{\nu}^2}{2}. \tag{5.35}$$

Factorizing $A(\widetilde{\mu}) - A(z)$ gives $-\frac{1}{20}(\widetilde{\mu} - z)q(\widetilde{\mu}, z)$, where

$$q(\widetilde{\mu}, z) = 4\sum_{i=0}^{4} \widetilde{\mu}^i z^{4-i} - 15\sum_{i=0}^{3} \widetilde{\mu}^i z^{3-i} + 20\sum_{i=0}^{2} \widetilde{\mu}^i z^{2-i} - 10(\widetilde{\mu} + z).$$

In fact, $z(\widetilde{\mu})$ is defined implicitly by $q(\widetilde{\mu}, z)$. Therefore,

$$\frac{d}{d\widetilde{\mu}} l_i(\widetilde{\mu}) = \frac{d}{d\widetilde{\mu}}\left(\frac{f_i}{A'}\right)(\widetilde{\mu}) + \frac{d}{dz}\left(\left(\frac{f_i}{A'}\right)(z(\widetilde{\mu}))\right)\frac{dz}{d\widetilde{\mu}}$$

with

$$\frac{dz}{d\widetilde{\mu}} = -\frac{\partial q(\widetilde{\mu}, z)}{\partial \widetilde{\mu}} \bigg/ \frac{\partial q(\widetilde{\mu}, z)}{\partial z}.$$

Further, a direct computation shows that

$$\begin{aligned}
l_0(\widetilde{\mu}) &= \frac{(\widetilde{\mu} - z)\, P_1(\widetilde{\mu})}{\widetilde{\mu}\,(\widetilde{\mu} - 1)^3\, z\,(z - 1)^3},\\
W[l_0(\widetilde{\mu}), l_2(\widetilde{\mu})] &= \begin{vmatrix} l_0(\widetilde{\mu}) & l_2(\widetilde{\mu}) \\ l_0'(\widetilde{\mu}) & l_2'(\widetilde{\mu}) \end{vmatrix} = \frac{-\,(\widetilde{\mu} - z)^3\, P_2(\widetilde{\mu})}{\widetilde{\mu}^2 z^2\,(z - 1)^5\,(\widetilde{\mu} - 1)^5\, P_0(\widetilde{\mu})},
\end{aligned}$$

and

$$\begin{aligned}
l_2(\widetilde{\mu}) &= \frac{(x - z)\,(z + x - 1)}{(z - 1)\, z\,(x - 1)\, x},\\
W[l_2(\widetilde{\mu}), l_3(\widetilde{\mu})] &= \begin{vmatrix} l_2(\widetilde{\mu}) & l_3(\widetilde{\mu}) \\ l_2'(\widetilde{\mu}) & l_3'(\widetilde{\mu}) \end{vmatrix} = \frac{(\widetilde{\mu} - z)^3\, P_3(\widetilde{\mu})}{\widetilde{\mu}^2 z^2\,(z - 1)^2\,(\widetilde{\mu} - 1)^2\, P_0(\widetilde{\mu})},
\end{aligned}$$

where

$$
\begin{aligned}
P_0(\widetilde{\mu}) = &\ 4\sum_{i=0}^{3}(4-i)\widetilde{\mu}^i z^{3-i} - 15\sum_{i=0}^{2}(3-i)\widetilde{\mu}^i z^{2-i} + 20\widetilde{\mu} + 40\,z - 10,\\
P_1(\widetilde{\mu}) = &\ \sum_{i=0}^{3}\widetilde{\mu}^i z^{3-i} - 3\sum_{i=0}^{2}\widetilde{\mu}^i z^{2-i} + 3(\widetilde{\mu}+z) - 1,\\
P_2(\widetilde{\mu}) = &\ 8\left(4\widetilde{\mu}^4 + 3\widetilde{\mu}^3 z + 6\widetilde{\mu}^2 z^2 + 3\widetilde{\mu}z^3 + 4\,z^4\right)(\widetilde{\mu}+z)^3\\
&+ 20\,(\widetilde{\mu}+z)\left(83\,\widetilde{\mu}^2 + 97\,\widetilde{\mu}z + 83\,z^2\right) - (250\,\widetilde{\mu}^6 + 850\,\widetilde{\mu}^5 z\\
&+ 1398\,\widetilde{\mu}^4 z^2 - 1604\,\widetilde{\mu}^3 z^3 + 1398\,\widetilde{\mu}^2 z^4 + 850\,\widetilde{\mu}z^5 + 250\,z^6)\\
&+ (\widetilde{\mu}+z)\left(834\,\widetilde{\mu}^4 + 1683\,\widetilde{\mu}^3 z + 2026\,\widetilde{\mu}^2 z^2 + 1683\,\widetilde{\mu}z^3 + 834\,z^4\right)\\
&- (1531\,\widetilde{\mu}^4 + 3997\,\widetilde{\mu}^3 z + 5074\,\widetilde{\mu}^2 z^2 + 3997\,\widetilde{\mu}z^3 + 1531\,z^4)\\
&- 5(211\,\widetilde{\mu}^2 + 350\,\widetilde{\mu}z + 211\,z^2 + +360(\widetilde{\mu}+z\pounds\copyright - 50,\\
P_3(\widetilde{\mu}) = &\ 4\left(\widetilde{\mu}^2 + z^2\right)\left(4\,\widetilde{\mu}^2 + 7\,\widetilde{\mu}z + 4\,z^2\right) - (\widetilde{\mu}+z)\left(61\,\widetilde{\mu}^2 + 38\,\widetilde{\mu}z + 61\,z^2\right)\\
&+ 5(17\,\widetilde{\mu}^2 + 24\,\widetilde{\mu}z + 17\,z^2) - 50(\widetilde{\mu}+z) + 10.
\end{aligned}
$$

Now, computing the resultant of $z+\widetilde{\mu}-1$ and $q(\widetilde{\mu},z)$ with respect to z, we obtain

$$\mathrm{res}(z+\widetilde{\mu}-1, q, z,) = 4\widetilde{\mu}^4 - 8\widetilde{\mu}^3 + 6\widetilde{\mu}^2 - 2\widetilde{\mu} - 1,$$

which has no real roots on $(0,1)$. This implies that $l_2(\widetilde{\mu})$ does not vanish for $\widetilde{\mu} \in (0,1)$.

Similarly, computing the resultant of $P_i(\widetilde{\mu}, z)$ and $q(\widetilde{\mu}, z)$ with respect to z and applying the Sturm's Theorem, we can show that $P_i(\widetilde{\mu}, z)$ $(i = 0,\ 1,\ 2,\ 3)$ does not vanish for $\widetilde{\mu} \in (0,1)$. This means that $l_0(\widetilde{\mu})$, $W[l_0, l_2]$ and $W[l_2, l_3]$ do not vanish for $\widetilde{\mu} \in (0,1)$. By Lemma 5.3.3, we have shown that $\{J_0, J_2\}$ and $\{J_2, J_3\}$ are all Chebyshev systems, and therefore, $\frac{J_2(h)}{J_0(h)}$ and $\frac{J_3(h)}{J_0(h)}$ are all monotonic on $(-\frac{1}{20}, 0)$.

Based on the above results, the proof for Proposition 5.4.2 is straightforward.

Proof [For Proposition 5.4.2] Since $\frac{J_3(h)}{J_2(h)}$ is decreasing monotonically from 1 to $\frac{25}{26}$, it implies that $-3 + 4\frac{J_3(h)}{J_2(h)}$ is positive and decreasing monotonically from 1 to $\frac{11}{13}$. Because $\frac{J_2(h)}{J_0(h)}$ is also positive and decreasing monotonically, we obtain that $X(h) = \frac{-3J_2+4J_3}{J_0} = \frac{J_2}{J_0}(-3+\frac{4J_3}{J_2})$, which is decreasing monotonically on $(-\frac{1}{20}, 0)$. This implies that $X'(h) < 0$, and so

$$\frac{25}{39} = \lim_{h\to 0} X(h) < X(h) < \lim_{h\to -\frac{1}{20}} X(h) = 1.$$

Now, we are ready to prove Theorem 5.2.1.

Proof [For Theorem 5.2.1] By (5.28), we choose $c = c(h) = (X(h))^{-\frac{1}{3}}$ for each $h \in (-\frac{1}{20}, 0)$, then $M(h) = 0$. The monotonicity of $X(h)$ means that the zero h is unique, and $c(h)$ satisfies $c'(h) > 0$. Thus,

$$1 < c(h) < \left(\frac{39}{25}\right)^{\frac{1}{3}}, \quad \lim_{h\to -\frac{1}{20}} c(h) = 1, \quad \lim_{h\to 0} c(h) = \left(\frac{39}{25}\right)^{\frac{1}{3}}.$$

The above results, together with the implicit function theorem imply that choosing $c = c(h) + O(\varepsilon)$ leads to that $M(h) + O(\varepsilon)$ has a unique zero near h. This proves the first part of Theorem 5.2.1 since $\mathcal{H}(\beta(h,\varepsilon),0)-\mathcal{H}(\alpha(h),0) = \varepsilon(c^{-\frac{3}{2}}M(h)+O(\varepsilon))$. The second part of the theorem is the limit case of $M(h)$ as $h \to 0$, which can be proved similarly.

5.5 Analysis of system (5.6) with perturbation P_2

In this section, we study the BBM equation (5.6) with perturbation P_2. As discussed in sections 2 and 3, we need only consider the Abelian integral $\mathcal{M}(h)$. Using the same notation in (5.26), we have

$$\mathcal{M}(h) = a_0 J_0(h) + a_1 J_1(h) + a_2 J_2(h).$$

5.5.1 Asymptotic expansion of the Abelian integral

One efficient method for studying the weak Hilbert's 16th problem is to investigate the asymptotic expansions of Abelian integrals, see [18, 51, 19]. $\mathcal{M}(h)$ has the following expansion (see [51, 19]):

$$\begin{aligned}\mathcal{M}(h) = \; & c_0(\delta) + c_1(\delta)|h|^{\frac{3}{4}} + [c_2(\delta) + b_0^* c_1(\delta)]h \ln|h| \\ & + [c_3(\delta) + b_1^* c_1(\delta) + b_2^* c_2(\delta)]h + O((-h)^{\frac{5}{4}})\end{aligned} \tag{5.36}$$

for $0 < -h \ll 1$, where the coefficients $c_j(\delta)$ are obtained by using the methods and formulas developed in [19] as follows:

$$\begin{aligned}& c_0(\delta) = \tfrac{25\sqrt{2}}{72072}(715\, a_1 + 650\, a_2 + 858\, a_0), \quad c_1(\delta) = 4\widetilde{A}_0 a_0, \\ & c_2(\delta) = -\tfrac{\sqrt{2}}{2} a_1, \quad c_3(\delta) = 5\sqrt{2} a_2,\end{aligned}$$

with $\widetilde{A}_0 < 0$. Therefore, we obtain the expansions of J_0, J_1, J_2, J_3, J_0', J_1', J_2' and J_3' as follows:

$$\begin{aligned}J_0(h) &= \tfrac{25\sqrt{2}}{84} + 4\widetilde{A}_0|h|^{\frac{3}{4}} + 4\widetilde{A}_0 b_0^* h\ln(-h) + 4b_1^*\widetilde{A}_0 h + O((-h)^{\frac{5}{4}}), \\ J_1(h) &= \tfrac{125\sqrt{2}}{504} - \tfrac{\sqrt{2}}{2} h\ln|h| - \tfrac{\sqrt{2}}{2} b_2^* h + O((-h)^{\frac{5}{4}}), \\ J_2(h) &= \tfrac{625\sqrt{2}}{2772} + 5\sqrt{2}h + O((-h)^{\frac{5}{4}}), \\ J_3(h) &= \tfrac{15625\sqrt{2}}{72072} + \tfrac{25\sqrt{2}}{6} h + O((-h)^{\frac{5}{4}}),\end{aligned} \tag{5.37}$$

and

$$\begin{aligned}J_0'(h) &= -3\widetilde{A}_0(-h)^{-\frac{1}{4}} + 4\widetilde{A}_0 b_0^* \ln(-h) + 4\widetilde{A}_0 b_0^* + 4\widetilde{A}_0 b_1^* + h.o.t., \\ J_1'(h) &= -\tfrac{\sqrt{2}}{2}\ln(-h) - \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} b_2^* + h.o.t., \\ J_2'(h) &= 5\sqrt{2}h + h.o.t., \\ J_3'(h) &= \tfrac{25\sqrt{2}}{6} + h.o.t.,\end{aligned} \tag{5.38}$$

where $h.o.t.$ denotes higher order terms.

5.5.2 Existence of periodic waves

It follows from Lemma 5.4.5 that

$$J_i(h) = \oint_{\widetilde{\mathcal{H}}=h} f_i(\widetilde{\mu})\widetilde{\nu}d\widetilde{\mu} = \sum_{j=0}^{i}(-1)^j C_i^j I_j(h),$$

where $f_i(\widetilde{\mu}) = (-\widetilde{\mu}+1)^i$, $I_j(h) = \oint_{\widetilde{\mathcal{H}}=h} \widetilde{\mu}^j \widetilde{\nu} d\widetilde{\mu}$. Then we have

Lemma 5.5.1 $8\left(h+\frac{1}{20}\right)^3 I_j(h) = \oint_{\widetilde{\mathcal{H}}=h} f_j(\widetilde{\mu})\widetilde{\nu}^5 d\widetilde{\mu} \equiv \widetilde{I}_j(h)$, *where* $f_j(\widetilde{\mu}) = \widetilde{\mu}^j + G_j(\widetilde{\mu}) + \widetilde{G}_j(\widetilde{\mu})$, *with* $G_j(\widetilde{\mu}) = \frac{\widetilde{\mu}^j g_j(\widetilde{\mu})}{30(\widetilde{\mu}-1)^4}$ *and* $\widetilde{G}_j(\widetilde{\mu}) = \frac{\widetilde{\mu}^j \widetilde{g}_j(\widetilde{\mu})}{1500(\widetilde{\mu}-1)^8}$, *in which* $g_j(\widetilde{\mu})$ *and* $\widetilde{g}_j(\widetilde{\mu})$ *are polynomials in* $\widetilde{\mu}$.

Proof Multiplying $I_j(h)$ by $\frac{\widetilde{\nu}^2+2A(\widetilde{\mu})}{2(h+\frac{1}{20})} = 1$ yields

$$\begin{aligned} I_j(h) &= \oint_{\widetilde{\mathcal{H}}=h} \frac{2A(\widetilde{\mu})+\widetilde{\nu}^2}{2(h+\frac{1}{20})}\widetilde{\mu}^j\widetilde{\nu}d\widetilde{\mu} \\ &= \frac{1}{2(h+\frac{1}{20})}\left(\oint_{\widetilde{\mathcal{H}}=h} 2\widetilde{\mu}^j A(\widetilde{\mu})\widetilde{\nu}d\widetilde{\mu} + \oint_{\widetilde{\mathcal{H}}=h}\widetilde{\mu}^j\widetilde{\nu}^3 d\widetilde{\mu}\right), \quad i = 0,1,2,3. \end{aligned} \tag{5.39}$$

By Lemma 4.1 of [14] (with $k = 3$ and $F(\widetilde{\mu}) = 2\widetilde{\mu}^j A(\widetilde{\mu})$), we have

$$\oint_{\widetilde{\mathcal{H}}=h} 2\widetilde{\mu}^j A(\widetilde{\mu})\widetilde{\nu}d\widetilde{\mu} = \oint_{\widetilde{\mathcal{H}}=h} G_j(\widetilde{\mu})\widetilde{\nu}^3 d\widetilde{\mu}, \tag{5.40}$$

where $G_j(\widetilde{\mu}) = \frac{d}{3d\widetilde{\mu}}\frac{2\widetilde{\mu}^j A(\widetilde{\mu})}{A'(\widetilde{\mu})} = \frac{\widetilde{\mu}^j g_j(\widetilde{\mu})}{30(\widetilde{\mu}-1)^4}$ with

$$g_j(\widetilde{\mu}) = 4\,j\widetilde{\mu}^4 - 19\,j\widetilde{\mu}^3 + 4\widetilde{\mu}^4 + 35\,j\widetilde{\mu}^2 - 16\widetilde{\mu}^3 - 30\,j\widetilde{\mu} + 25\widetilde{\mu}^2 + 10\,j - 20\widetilde{\mu} + 10.$$

Substituting (5.40) into (5.39) and multiplying $\frac{2A(\widetilde{\mu})+\widetilde{\nu}^2}{2(h+\frac{1}{20})} = 1$ give

$$\begin{aligned} I_j(h) &= \frac{1}{2(h+\frac{1}{20})}\oint_{\widetilde{\mathcal{H}}=h}(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^3 d\widetilde{\mu} \\ &= \frac{1}{4(h+\frac{1}{20})^2}\oint_{\widetilde{\mathcal{H}}=h}(2A(\widetilde{\mu})+\widetilde{\nu}^2)(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^3 d\widetilde{\mu} \\ &= \frac{1}{4(h+\frac{1}{20})^2}\oint_{\widetilde{\mathcal{H}}=h} 2A(\widetilde{\mu})(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^3 d\widetilde{\mu} \\ &\quad + \frac{1}{4(h+\frac{1}{20})^2}\oint_{\widetilde{\mathcal{H}}=h}(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^5 d\widetilde{\mu}. \end{aligned} \tag{5.41}$$

Again by Lemma 4.1 of [14] (here $k = 5$ and $F(\widetilde{\mu}) = 2A(\widetilde{\mu})(\widetilde{\mu}^j + G_j(\widetilde{\mu}))$), we obtain

$$\int_{\widetilde{\mathcal{H}}=h} 2A(\widetilde{\mu})(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^3 d\widetilde{\mu} = \int_{\widetilde{\mathcal{H}}=h}\widetilde{G}_j(\widetilde{\mu})\widetilde{\nu}^5 d\widetilde{\mu}, \tag{5.42}$$

where $\widetilde{G}_j(\widetilde{\mu}) = \frac{d}{5d\widetilde{\mu}}\left(\frac{2A(\widetilde{\mu})(\widetilde{\mu}^j+G_j(\widetilde{\mu}))}{A'(\widetilde{\mu})}\right) = \frac{\widetilde{\mu}^j\widetilde{g}_j(\widetilde{\mu})}{1500(\widetilde{\mu}-1)^8}$, and $\widetilde{g}_j(\widetilde{\mu})$ is a lengthy polynomial and omitted here for brevity. Substituting (5.42) into (5.41) proves Lemma 5.5.1.

Without loss of generality, we assume that $a_1 = \lambda$ and $a_3 = 1$. Further, let

$$\mathcal{J}_1(h) = \oint_{\Gamma_h} \Big(\mu + \frac{1}{\lambda}\mu^2\Big)\nu d\mu. \tag{5.43}$$

Then $\mathcal{M}(h) = \alpha_0 J_0(h) + \lambda \mathcal{J}_1(h)$. By Lemma 5.5.1, we have

Lemma 5.5.2

$$8\Big(h + \frac{1}{20}\Big)^3 J_i(h) = \oint_{\widetilde{\mathcal{H}}=h} \widetilde{f_i}(\widetilde{\mu})\widetilde{\nu}^5 d\widetilde{\mu} \triangleq \widetilde{J_i}(h),$$

and

$$8\Big(h + \frac{1}{20}\Big)^3 \mathcal{J}_1(h) = \oint_{\widetilde{\mathcal{H}}=h} (\widetilde{f_1}(\widetilde{\mu}) + \frac{1}{\lambda}\widetilde{f_2}(\widetilde{\mu}))\widetilde{\nu}^5 d\widetilde{\mu} \triangleq \widetilde{\mathcal{J}}_1(h),$$

where $\widetilde{f_i}(\widetilde{\mu}) = \sum_{j=0}^{i} (-1)^j C_i^j f_j(\widetilde{\mu})$.

Now, let

$$L_i(\widetilde{\mu}) = \Big(\frac{\widetilde{f_i}}{A'}\Big)(\widetilde{\mu}) - \Big(\frac{\widetilde{f_i}}{A'}\Big)(z(\widetilde{\mu})),$$
$$\mathcal{L}_1(\widetilde{\mu}) = \Big(\frac{\widetilde{f_1} + \frac{1}{\lambda}\widetilde{f_2}}{A'}\Big)(\widetilde{\mu}) - \Big(\frac{\widetilde{f_2} + \frac{1}{\lambda}\widetilde{f_3}}{A'}\Big)(z(\widetilde{\mu})).$$

Then

$$\frac{d}{d\widetilde{\mu}}L_i(\widetilde{\mu}) = \frac{d}{d\widetilde{\mu}}\Big(\frac{f_i}{A'}\Big)(\widetilde{\mu}) - \frac{d}{dz}\left(\Big(\frac{f_i}{A'}\Big)(z(\widetilde{\mu}))\right) \times \frac{dz}{d\widetilde{\mu}},$$
$$\frac{d}{d\widetilde{\mu}}\mathcal{L}_i(\widetilde{\mu}) = \frac{\partial}{\partial\widetilde{\mu}}(\mathcal{L}_i(\widetilde{\mu})) + \frac{\partial}{\partial z}(\mathcal{L}_i(\widetilde{\mu})) \times \frac{dz}{d\widetilde{\mu}},$$

and we obtain

$$W[L_0](\widetilde{\mu}) = \frac{3(\widetilde{\mu}-z)Q_1(\widetilde{\mu},z)}{500\widetilde{\mu}z(\widetilde{\mu}-1)^{11}(z-1)^{11}},$$
$$W[L_0(\widetilde{\mu}), \mathcal{L}_1(\widetilde{\mu})] = \begin{vmatrix} L_0(\widetilde{\mu}) & \mathcal{L}_1(\widetilde{\mu}) \\ L_0'(\widetilde{\mu}) & \mathcal{L}_1'(\widetilde{\mu}) \end{vmatrix} = \frac{-(\widetilde{\mu}-z)^3 Q_2(\widetilde{\mu},z)}{250000\widetilde{\mu}^2 z^2 \lambda (z-1)^{22}(\widetilde{\mu}-1)^{22} P_0(\widetilde{\mu},z)},$$
$$W[L_0(\widetilde{\mu}), L_1(\widetilde{\mu})] = \begin{vmatrix} L_0(\widetilde{\mu}) & L_1(\widetilde{\mu}) \\ L_0'(\widetilde{\mu}) & L_1'(\widetilde{\mu}) \end{vmatrix} = \frac{-(\widetilde{\mu}-z)^3 Q_3(\widetilde{\mu},z)}{250000\widetilde{\mu}^2 z^2 (z-1)^{22}(\widetilde{\mu}-1)^{22} P_0(\widetilde{\mu},z)},$$

where $Q_1(\widetilde{\mu}, z)$ is a two-variate polynomial of degree 19, $Q_2(\widetilde{\mu}, z) = S_{12}(\widetilde{\mu}, z)\lambda - S_{11}(\widetilde{\mu}, z)$, in which $S_{11}(\widetilde{\mu}, z)$ and $S_{12}(\widetilde{\mu}, z)$ are two-variate polynomials of degree 41 and 40, respectively, and $Q_3(\widetilde{\mu}, z)$ is of degree 40.

Computing the resultant of Q_1 and q with respect to z, and applying Strum's Theorem, we can show that Q_1 and q have no common zeros for $\widetilde{\mu} \in (0, 1)$. Therefore, $W[L_0]$ does not vanish for $\widetilde{\mu} \in (0, 1)$.

Similarly, computing the resultant of S_{12} and q with respect to z, and applying Strum's Theorem, we can prove that S_{12} and q have no common zeros for $\widetilde{\mu} \in (0, 1)$. Therefore, S_{12} does not vanish for $\widetilde{\mu} \in (0, 1)$. Solving $S_{12}(\widetilde{\mu}, z)\lambda - S_{11}(\widetilde{\mu}, z) = 0$ gives

$$\lambda(\widetilde{\mu}, z) = \frac{S_{11}(\widetilde{\mu}, z)}{S_{12}(\widetilde{\mu}, z)}, \tag{5.44}$$

for which we have the following result.

Lemma 5.5.3 *$\lambda(\widetilde{\mu}, z)$ is monotonic for $\widetilde{\mu} \in (0,1)$, and $\lambda(\widetilde{\mu}, z) \in (-\frac{5}{3}, 0)$.*

Proof A direct computation shows that

$$\lambda'(\widetilde{\mu}, z) = \frac{\partial \lambda(\widetilde{\mu}, z)}{\partial \widetilde{\mu}} + \frac{\partial \lambda(\widetilde{\mu}, z)}{\partial z} \times \frac{dz}{d\widetilde{\mu}} = \frac{S_{21}(\widetilde{\mu}, z)}{S_{22}(\widetilde{\mu}, z)}.$$

Computing the corresponding resultant $\mathrm{res}(S_{22}, q, z)$ and applying Sturm's theorem, we can show that $S_{22}(\widetilde{\mu}, z)$ has no zeros for $\widetilde{\mu} \in (0,1)$. Similarly, computing the resultant $\mathrm{res}(S_{21}, q, z)$ and applying Strum's Theorem, we can prove that $\mathrm{res}(S_{21}, q, z)$ has a unique zero in $\left[\frac{40125}{65536}, \frac{80251}{131072}\right] \subseteq (0,1)$. Further, computing the resultant $\mathrm{res}(S_{21}, q, x)$ and applying Strum's Theory show that $\mathrm{res}(S_{21}, q, x)$ has three zeros for z in the three intervals:

$$\left[-\frac{64373}{262144}, -\frac{128745}{524288}\right], \left[-\frac{125503}{1048576}, -\frac{62751}{524288}\right] \text{ and } \left[-\frac{56117}{524288}, -\frac{112233}{1048576}\right].$$

Therefore, if S_{21} and q have common roots on $(-\frac{1}{4}, 0) \times (0,1)$, the roots must lie in one of the following three domains:

$$\begin{aligned}
D_1 &: \left[-\frac{64373}{262144}, -\frac{128745}{524288}\right] \times \left[\frac{40125}{65536}, \frac{80251}{131072}\right], \\
D_2 &: \left[-\frac{125503}{1048576}, -\frac{62751}{524288}\right] \times \left[\frac{40125}{65536}, \frac{80251}{131072}\right], \\
D_3 &: \left[-\frac{56117}{524288}, -\frac{112233}{1048576}\right] \times \left[\frac{40125}{65536}, \frac{80251}{131072}\right].
\end{aligned}$$

The resultant $\mathrm{res}(\frac{\partial q}{\partial \widetilde{\mu}}, \frac{\partial q}{\partial z}, z)$ has no zeros in $[\frac{40125}{65536}, \frac{80251}{131072}]$, implying that q reaches its extreme values on the boundary of D_i. By Sturm's Theorem, we know that the derivatives of the four functions obtained by restricting $q(x, z)$ on the four line segments of the boundary of D_i ($i = 1, 2, 3$) have no zeros. Therefore, $q(\widetilde{\mu}, z)$ gets its maximal and minimum values at the four vertexes on each D_i. A direct computation yields

$$\begin{aligned}
\max_{D_1} q(\widetilde{\mu}, z) &= \frac{48507113359773182621}{1180591620717411303424}, & \min_{D_1} q(\widetilde{\mu}, z) &= \frac{775590116374354945381}{18889465931478580854784}, \\
\max_{D_2} q(\widetilde{\mu}, z) &= -\frac{306099632527480469877479}{302231454903657293676544}, & \min_{D_2} q(\widetilde{\mu}, z) &= -\frac{19131393530721391794519}{18889465931478580854784}, \\
\max_{D_3} q(\widetilde{\mu}, z) &= -\frac{20507964148271211547259}{18889465931478580854784}, & \min_{D_3} q(\widetilde{\mu}, z) &= -\frac{328130217324045232444719}{302231454903657293676544}.
\end{aligned}$$

The minimum and maximum values have the same signs on each D_i. Hence, q and S_{21} have no common zeros on each D_i. Therefore, S_{21} does not vanish for $\widetilde{\mu} \in (0,1)$. This implies that $\lambda'(\widetilde{\mu}, z) \neq 0$ for $\widetilde{\mu} \in (0,1)$. Thus, $\lambda(\widetilde{\mu}, z)$ is monotonic for $\widetilde{\mu} \in (0,1)$, and so

$$\lim_{\widetilde{\mu} \to 0} = \lambda(\widetilde{\mu}, z) = -\frac{5}{3}, \;\; \lim_{\widetilde{\mu} \to 1} = \lambda(\widetilde{\mu}, z) = 0.$$

This completes the proof of Lemma 5.5.3.

Lemma 5.5.3 implies that when $\lambda \in (-\frac{5}{3}, 0)$, $W[L_0, \mathcal{L}_1]$ has a simple root for $\widetilde{\mu} \in (0,1)$, and $W[L_0, \mathcal{L}_1]$ has no roots for $\widetilde{\mu} \in (0,1)$ when $\lambda \in (-\infty, -\frac{5}{3}] \bigcup (0, +\infty)$. By Lemmas 5.3.3 and 5.3.4, we have the following result.

Lemma 5.5.4 *$\mathcal{M}(h)$ has at most two zeros (counting multiplicities) for $\lambda \in (-\frac{5}{3}, 0)$, and at most one zero (counting multiplicity) for $\lambda \in (-\infty, -\frac{5}{3}] \bigcup (0, +\infty)$.*

Let

$$\kappa(h) = \frac{\lambda J_1(h) + J_2(h)}{J_0(h)}. \tag{5.45}$$

Then

$$\mathcal{M}(h) = J_0(h)(a_0 + \kappa(h)).$$

Lemma 5.5.4 implies the following proposition.

Proposition 5.5.5 *The ratio $\kappa(h)$ is monotonic for $\lambda \in (-\infty, -\frac{5}{3}] \bigcup [0, +\infty)$.*

Lemma 5.5.6 *If $\kappa'(h)$ has zeros, they must be simple. Moreover, $\kappa'(h)$ has $2n+1$ simple zeros on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, and $2n$ simple zeros on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{10}{11}, 0)$.*

Proof Firstly, we give a short proof for the first assertion by using an argument of contradiction. Let h^* be a zero of $\kappa'(h)$ with l multiplicities, $l \geq 2$. Then there must exist an a_0 such that $a_0 + \kappa(h)$ has a zero at $h = h^*$ with $l+1$ (≥ 3) multiplicities. Because $J_0(h) > 0$, the relationship between $\mathcal{M}(h)$ and $a_0 + \kappa(h)$ implies that $\mathcal{M}(h)$ has a zero at $h = h^*$ with $l+1$ (≥ 3) multiplicities. This contradicts Lemma 5.5.4.

With the expansion of $J_i(h)$ near $h = -\frac{1}{20}$, given in (5.31), a direct computation shows that

$$\kappa'\Big(-\frac{1}{20}\Big) = \lim_{h \to -\frac{1}{20}} \kappa'(h) = \lim_{h \to -\frac{1}{20}} \frac{(\lambda J_1'(h) + J_2'(h))J_0(h) - (\lambda J_1(h) + J_2(h))J_0'(h)}{J_0^2(h)} = -\frac{3}{2}\lambda - \frac{5}{2}.$$

Further, using the expansions of $J_i(h)$ and $J_i'(h)$ near $h = 0$ given respectively in (5.37) and (5.38), we can prove that

$$\begin{aligned} \kappa'(0) &= \lim_{h \to 0} \kappa'(h) = \lim_{h \to 0} \frac{(\lambda J_1'(h) + J_2'(h))J_0(h) - (\lambda J_1(h) + J_2(h))J_0'(h)}{J_0^2(h)} \\ &= \operatorname{sign}(\widetilde{A}_0(11\lambda + 10))\,\infty. \end{aligned}$$

Because $\widetilde{A}_0 < 0$, it is obvious that $\kappa'(h)$ has different signs at the two endpoints of the interval $(-\frac{1}{20}, 0)$ if $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, and has the same sign at the two endpoints of $(-\frac{1}{20}, 0)$ if $\lambda \in (-\frac{10}{11}, 0)$. This completes the proof.

Proposition 5.5.7 *$\kappa'(h)$ has a unique simple zero on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, namely, $\kappa(h)$ decreases from $\kappa(-\frac{1}{20})$ to a minimum value and then increases to $\kappa(0)$ for any $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$.*

Proof By Lemma 5.5.6, for a fixed $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, if $\kappa'(h)$ has three or more than three simple zeros on $(-\frac{1}{20}, 0)$, then there must exist an a_0 such that $a_0 + \kappa(h)$ has at least three zeros. This implies that $\mathcal{M}(h)$ can have at least three zeros, which contradicts Lemma 5.5.4. Therefore, $\kappa'(h)$ has a unique zero on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$. The signs of $\kappa'(-\frac{1}{20})$ and $\kappa'(0)$ when $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$ determines the property of $\kappa(h)$.

Proposition 5.5.8 *$\kappa'(h)$ has no zeros on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{10}{11}, 0)$, that is, $\kappa(h)$ is monotonic on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{10}{11}, 0)$.*

Proof By Lemma 5.5.6, for a fixed $\lambda \in (-\frac{10}{11}, 0)$, if $\kappa'(h)$ has four or more than four zeros on $(-\frac{1}{20}, 0)$, then there must exist an a_0 such that $a_0 + \kappa(h)$ has at least three zeros counting multiplicities. This contradicts Lemma 5.5.4.

Next, we prove that $\kappa'(h)$ does not have two zeros. Suppose otherwise $\kappa'(h)$ has two zeros for $\lambda \in (-\frac{10}{11}, 0)$. We have known that $\kappa'(-\frac{1}{20}) < 0$, and $\kappa'(0) < 0$, which imply that $\kappa(h)$ is decreasing at the endpoints of the interval $(-\frac{1}{20}, 0)$. Further, for $\lambda \in (-\frac{10}{11}, 0)$, $\lambda + 1 = \kappa(-\frac{1}{20}) > \kappa(0) = \frac{5}{6}\lambda + \frac{25}{33}$. This clearly indicates that there must exist an a_0 such that $a_0 + \kappa(h)$ has at least three zeros. This contradicts Lemma 5.5.4, and so the proof is complete.

5.5.3 Coexistence of one solitary wave and one periodic wave

In this subsection, we will investigate the condition for the existence of one solitary wave, for which we need study the condition satisfying $\mathcal{H}(\beta^*(h, \varepsilon), 0) - \mathcal{H}(\alpha^*(h), 0) = \varepsilon\mathcal{M}(h) + O(\varepsilon^2) = 0$ at $h = 0$ (and so $\alpha^*(0) = 0$). Firstly, solving $\mathcal{M}(0) = \oint\limits_{\Gamma_0}(a_0 + \lambda\mu + \mu^2)\nu d\mu = 0$ gives

$$\lambda = -\frac{10}{11} - \frac{6}{5}a_0, \tag{5.46}$$

under which

$$\kappa\Big(-\frac{1}{20}\Big) = \frac{1}{11} - \frac{6}{5}a_0, \quad \kappa(0) = -a_0. \tag{5.47}$$

We need discuss two cases for λ. If $\lambda \in (-\infty, -\frac{5}{3}] \bigcup [-\frac{10}{11}, +\infty)$, then (5.46) yields $a_0 \in (-\infty, 0] \bigcup [\frac{5}{9}, +\infty)$, and $\kappa(h)$ is monotonic by Propositions 5.5.5 and 5.5.8. Therefore, $a_0 + \kappa(h)$ increases from 0 to $\frac{1}{11} - \frac{6}{5}a_0$ for $a_0 \in (-\infty, 0)$, and decreases from 0 to $\frac{1}{11} - \frac{6}{5}a_0$ for $a_0 \in [\frac{5}{9}, +\infty)$. Hence, $\mathcal{M}(h) = J_0(h)(a_0 + \kappa(h))$ has no zeros for $\lambda \in (-\infty, -\frac{5}{3}] \bigcup [-\frac{10}{11}, +\infty)$ under the condition (5.46).

If $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, then it follows from (5.46) that $a_0 \in (0, \frac{5}{9})$. Further, we divide the interval $(0, \frac{5}{9})$ into three parts by using $\kappa(-\frac{1}{20})$ and $\kappa(0)$.

(i) When $a_0 = \frac{5}{11}$, $\kappa(-\frac{1}{20}) = \kappa(0) = -a_0$. The property of $\kappa(h)$ given in Proposition 5.5.7 implies that $a_0 + \kappa(h) < a_0 + \kappa(-\frac{1}{20}) = a_0 + \kappa(0) = 0$ for $h \in (-\frac{1}{20}, 0)$. Thus, $\mathcal{M}(h) = J_0(h)(a_0 + \kappa(h))$ has no zeros.

(ii) When $a_0 \in (\frac{5}{11}, \frac{5}{9})$, $\kappa(-\frac{1}{20}) \le \kappa(0) = -a_0$. The property of $\kappa(h)$ given in Proposition 5.5.7 shows that $a_0 + \kappa(h) < a_0 + \kappa(0) = 0$ for $h \in (-\frac{1}{20}, 0)$. Thus, $\mathcal{M}(h) = J_0(h)(a_0 + \kappa(h))$ has no zeros.

(iii) When $a_0 \in (0, \frac{5}{11})$, $\kappa(-\frac{1}{20}) > \kappa(0) = -a_0$ and $a_0 + \kappa(-\frac{1}{20}) > a_0 + \kappa(0) = 0$. The property of $\kappa(h)$ given in Proposition 5.5.7 shows that in the h-κ plane, the graph of $a_0 + \kappa(h)$ decreases form the point $(-\frac{1}{20}, a_0 + \kappa(-\frac{1}{20}))$, passing through the h-axis at some $h = h^* \in (-\frac{1}{20}, 0)$, then to a minimum point and then increases to $(0, 0)$. This implies that $a_0 + \kappa(h)$ has a unique zero $h = h^*$ in $(-\frac{1}{20}, 0)$. Summarizing the above results gives the following proposition.

Proposition 5.5.9 *$\mathcal{M}(h)$ has a zero at $h = 0$ and anther zero at $h = h^* \in (-\frac{1}{20}, 0)$ if and only if $a_0 \in (0, \frac{5}{11})$ with $\lambda = -\frac{10}{11} - \frac{6}{5}a_0$.*

Finally, we prove Theorem 5.2.2.

Proof [For Theorem 5.2.2] With the above results, we have proved the first parts of Theorem 5.2.2 (i), (ii) and (iii), as Propositions 5.5.5 and 5.5.8 for (i), Proposition 5.5.7 for (ii) and Proposition 5.5.9 for (iii). The second parts of (i) (ii) and (iii) of Theorem 5.2.2 can be directly proved by applying the implicit function theorem.

5.6 Conclusion

In this chapter, we have used bifurcation theory to study the existence of periodic and solitary waves in a BBM equation under weak dissipative influences and Marangoni effect. A special transformation given in (5.32) is introduced so that the Chebyshev criteria can be applied to overcome the difficulty arising from higher-order degenerate singularities, and then the exact condition on the number of periodic waves is obtained for the case with regular multiple-parameter perturbations. Also, the condition on the coexistence of one solitary wave and one periodic waves is derived. The methodologies developed in this chapter include the reduction of three generating elements to special two ones, asymptotic expansion of Abelian integrals, and asymptotic analysis on the dominating part of the Abelian integrals. Combination of these methods is not only useful in the study of other types of wave equations, but also has potential to be generalized to consider perturbations on hyperelliptic Hamiltonian systems.

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Chapter 6

Singular perturbation approach to dynamics of a cubic-quintic nonlinear Schrödinger equation with weakly dissipative effects

6.1 Introduction

It is well-known that in solving real world problems, certain relatively weak influences or perturbations are unavoidable due to the existence of uncertainty and higher order correction to the original mathematical models. To better understand the dynamics of the model problem, these influences or perturbations should be involved in the mode, and corresponding perturbed models need to be studied carefully.

For perturbed equations, very often, two techniques are applied. One technique is based on the inverse scattering transformation (IST), which has been proved to be a powerful method to deal with many kinds of perturbed equations, such as the perturbed KdV, NLS and mKdV equations [17, 34]. Moreover, it turns out that combining the Lie group theory and homotopy theory with the IST method can solve these perturbed equations more efficiently [3, 23]. More detailed descriptions related to the perturbation theory for nearly integrable equations can be found in [26, 27]. The other technique to deal with perturbation problems is to reduce the partial differential equations into a singularly perturbed system of ordinary differential equations (including higher dimension Hamiltonian systems) by introducing wave transform and successive derivatives, see [4, 5, 6, 8, 30, 31, 32]. Along this direction, the Fenichel's criterion [9] is applied to assure the existence of an invariant manifold, and the underlying problem is then reduced to a regularly perturbed problem on the manifold.

The following generalized nonlinear Schrödinger equation with different perturbations $\mathcal{P}(u)$ has been studied in many classical works [2, 18, 20, 24, 25, 33, 36, 39] and some relatively new papers, such as [21, 37].

$$iu_t + u_{xx} + f(|u|^2)u = \varepsilon\mathcal{P}(u). \tag{6.1}$$

Generally, the perturbation includes three types: (i) $up(|u|^2)_x$ describing the Raman effect [39, 40], (ii) $i|u|^2u_x$ describing the self-steepening [24, 25, 40, 41], and (iii) iu_{xxx} describing the higher dispersion correction [18, 21, 37], as well as their linear combination of weakly various effects [40]. More detailed explanations on other various perturbations can be found in [35].

However, along this direction, there are some interesting problems need to be further studied. For example, can the solitary (kink) and periodic waves coexist in the perturbed equations? What is the amplitude of the coexisting periodic wave? How many periodic waves with different amplitudes can coexist? Furthermore, what are the exact parameter conditions for these physical phenomena to occur? To attack these problems, we consider the cubic-quintic nonlinear Schrödinger (CQNLS) equation with *weak Raman effect*, which represents the macroscopic wave propagation in Bose-Einstein condensates (BEC for short),

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \gamma_1|u|^2u + \gamma_2|u|^4u + V(x)u = \varepsilon\mathcal{P}, \tag{6.2}$$

where γ_1 and γ_2, represent the effects due to an intrinsic nonlinear resonance in the material, giving rise to strong two-photon absorption in optics; while in the BEC they take account of two-body and three-body interactions, and their signs determine whether the interactions are repulsive or attractive; $V(x)$ is an external potential and can be chosen as an arbitrary function or constant, to be specific, in our work, we take $V(x) \equiv \omega$, where ω is the propagation constant in the wave profile

$$u(x,t) = u(x)\mathbf{e}^{i\omega t}. \tag{6.3}$$

In particular, we take the weak dissipative perturbation $\mathcal{P} = \alpha_1 u_x + \alpha_2|u^2|u_x + \alpha_3 u_{xxx}$, where $\alpha_1,\ \alpha_2$ and α_3 are some constants.

We would like to point out that Yang and Kaup [39] studied the dynamics and stability of the solitary wave of the perturbed CQNLS equation by choosing $\mathcal{P} = \sum\limits_{k=0}^{n} p_k(|u|^2)\frac{\partial^k u}{\partial x^k}$ and setting the external potential $V(x) = 0$.

Our main interest in this work is to examine the dynamics of solitary waves, in particular, kink and periodic waves in nearly integrable equations with *nonzero external potential* $V(x)$. Our main tool is the Abelian integral associated with the reduced near-Hamiltonian systems, which is related to the well-known weak Hilbert's 16th problem [1]. We comment that our approach can be used to deal with many kinds of nearly integrable equations, including the equations with local and nonlocal distributed delays described by spatial-temporal convolution.

The rest of this chapter is organized as follows. Section 6.2 is devoted to the problem reduction. We provide a phase analysis for the unperturbed equation based on bifurcation theory of planar systems, and construct our tools on the periodic structure of the Hamiltonian. Section 6.3 consists of two subsections, in subsection 6.3.1, we focus on the condition on the existence of kink under perturbations and the periodic wave with different amplitudes, in particular, the coexistence of the kink and periodic waves. In subsection 6.3.2, we discuss the uniqueness of the periodic wave when the kink persists. In section 6.4, we show that there exist at most two periodic waves with different amplitudes under perturbations. Finally, concluding remarks are given in section 6.5.

6.2 Reduction of equation (6.2)

Recall that in our system, we take $V(x) = \omega$. Substituting the wave profile (6.3) into (6.2) and replacing $u(x)$ by ϕ, the real part reads

$$\phi_{xx} + \gamma_1\phi^3 + \gamma_2\phi^5 + \varepsilon(\alpha_1\phi_x + \alpha_2\phi^2\phi_x + \alpha_3\phi_{xxx}) = 0, \tag{6.4}$$

where we assume $0 < \varepsilon \ll 1$, $\frac{d\phi}{dx}$, $\frac{d^2\phi}{dx^2}$ and $\frac{d^3\phi}{dx^3}$ all approach 0 as $x \to +\infty$. This is a singularly perturbed ordinary differential equation.

Upon introducing $\frac{d\phi}{dx} = y$ and $\frac{dy}{dx} = w$, equation (6.4) can be written as a system of first order equations, the so-called *slow system* under the framework of geometric singular perturbation theory,

$$\begin{aligned} \frac{d\phi}{dx} &= y, \\ \frac{dy}{dx} &= w, \\ \varepsilon\alpha_3\frac{dw}{dx} &= w + \gamma_1\phi^3 + \gamma_2\phi^5 - \varepsilon\alpha_1 y - \varepsilon\alpha_2\phi^2 y. \end{aligned} \tag{6.5}$$

For $0 < \varepsilon \ll 1$, the rescaling $x = \varepsilon\xi$ yields the so-called *fast system*

$$\begin{aligned} \frac{d\phi}{d\xi} &= \varepsilon y, \\ \frac{dy}{d\xi} &= \varepsilon w, \\ \alpha_3\frac{dw}{d\xi} &= w + \gamma_1\phi^3 + \gamma_2\phi^5 - \varepsilon\alpha_1 y - \varepsilon\alpha_2\phi^2 y. \end{aligned} \tag{6.6}$$

We would like to point put that systems (6.5) and (6.6) are equivalent for $0 < \varepsilon \ll 1$. When $\varepsilon \to 0$ in (6.5) and (6.6), we obtain the limiting slow system

$$\frac{d\phi}{dx} = y, \quad \frac{dy}{dx} = w, \quad 0 = w + \gamma_1\phi^3 + \gamma_2\phi^5, \tag{6.7}$$

and the limiting fast system

$$\frac{d\phi}{d\xi} = 0, \quad \frac{dy}{d\xi} = 0, \quad \frac{dw}{d\xi} = w + \gamma_1\phi^3 + \gamma_2\phi^5. \tag{6.8}$$

Thus, the flow of system (6.7) is confined to the set

$$M_0 = \{(\phi, y, w) \in \mathbb{R}^3 \mid w + \gamma_1\phi^3 + \gamma_2\phi^5 = 0\},$$

which is the equilibrium set of (6.8). Under the framework of classical geometric singular perturbation theory, M_0 is called critical manifold or slow manifold.

Lemma 6.2.1 *For system* (6.8)*, the slow manifold M_0 is normally hyperbolic.*

Proof The slow manifold M_0 is precisely the set of equilibria of (6.8). The linearization of (6.8) at each point of $(\phi, y, w) \in M_0$ has two zero eigenvalues whose generalized eigenspace is the tangent space of the two-dimensional slow manifold M_0 of equilibria, and the other eigenvalue is 1. Thus, the slow manifold M_0 is normally hyperbolic.

Due to the normal hyperbolicity of M_0 (see [9] for a reasoning), one can expect that there exists a locally invariant manifold M_ε associated with system (6.5) that converges to M_0 as $\varepsilon \to 0$, in the form of

$$M_\varepsilon = \{(\phi, y, w) \in \mathbb{R}^3 \mid w = -\gamma_1\phi^3 - \gamma_2\phi^5 + \varepsilon\Theta(\phi, y, \varepsilon)\},$$

where the function $\Theta(\phi, y, \varepsilon)$ has the form

$$\Theta(\phi, y, \varepsilon) = \Theta_0(\phi, y) + \sum_{i=1}^{\infty} \varepsilon^i\,\Theta_i. \tag{6.9}$$

This allows us to detect the kink wave of (6.2) by tracking the heteroclinic loop of the ODEs on the invariant manifold M_ε. It then follows that, on M_ε, the third equation in system (6.5) becomes

$$\begin{aligned}&\varepsilon\alpha_3\Big[-(3\gamma_1\phi^2 + 5\gamma_2\phi^4)y + \varepsilon\Big(\frac{\partial\Theta}{\partial\phi}y + \varepsilon\frac{\partial\Theta}{\partial y}(-\gamma_1\phi^3 - \gamma_2\phi^5 + \varepsilon\Theta)\Big)\Big]\\ &= -\gamma_1\phi^3 - \gamma_2\phi^5 + \varepsilon\Theta + \gamma_1\phi^3 + \gamma_2\phi^5 - \varepsilon\alpha_1 y - \varepsilon\alpha_2\phi^2 y.\end{aligned}$$

Comparing the coefficient of ε on both sides, one has

$$\Theta_0(\phi, y) = (\alpha_1 + (\alpha_2 - 3\gamma_1\alpha_3)\phi^2 - 5\gamma_2\alpha_3\phi^4)y. \tag{6.10}$$

Then system (6.5) or (6.4) reduced to M_ε has the form,

$$\frac{d\phi}{dx} = y, \quad \frac{dy}{dx} = -\phi^3(\gamma_1 + \gamma_2\phi^2) + \varepsilon f(\phi^2)y, \tag{6.11}$$

where $f(\phi^2) = \sigma_0 + \sigma_1\phi^2 + \sigma_2\phi^4$ with $\sigma_0 = \alpha_1$, $\sigma_1 = \alpha_2 - 3\gamma_1\alpha_3$ and $\sigma_2 = -5\gamma_2\alpha_3$. There exists a Hamiltonian structure when $\varepsilon = 0$,

$$H(\phi, y) = \frac{y^2}{2} + \frac{\gamma_1\phi^4}{4} + \frac{\gamma_2\phi^6}{6}. \tag{6.12}$$

Recall the fact that for both integrable and near-integrable equations, a solitary wave solution of (6.2) corresponds to a homoclinic orbit of (6.11); a kink (or anti-kink) wave solution of (6.2) corresponds to a heteroclinic orbit (or so-called connecting orbit) of (6.11); and a periodic orbit (limit cycle) of (6.11) corresponds to an isolated periodic wave solution of (6.2). Thus, investigating all bifurcations of solitary, kink and periodic waves of (6.2) is equivalent to analyzing homoclinic loops, heteroclinic loops and limit cycles of (6.11), respectively.

Note that the phase orbits defined by the vector field of system $(6.11)_{\varepsilon=0}$ determine all traveling wave solutions of $(6.2)_{\varepsilon=0}$. We shall investigate the bifurcation of phase portraits of system $(6.11)_{\varepsilon=0}$ in (ϕ, y)-phase plane as the parameters are varied. Clearly, on the (ϕ, y)-phase plane, for system $(6.11)_{\varepsilon=0}$, there are three equilibria $E_0(0,0)$, $E_1(-\sqrt{-\gamma_1/\gamma_2}, 0)$ and $E_2(\sqrt{-\gamma_1/\gamma_2}, 0)$ if $\gamma_1\gamma_2 < 0$, and only one equilibrium $E_0(0,0)$ if $\gamma_1\gamma_2 > 0$.

Let $M(E_i)$ be the coefficient matrix of the linearized system of $(6.11)_{\varepsilon=0}$ at the equilibrium point $(\phi_i, 0)$. By theory of planar dynamical systems [11], the equilibrium point is a saddle if $\det M(E_i) < 0$, a center if $\det M(E_i) > 0$, and a cusp with zero Poincaré index if $\det M(E_i) = 0$. When the singular point is nilpotent, the method developed by Han *et al.* [15] can be applied to determine its type and order, yielding three types of bounded solutions for the unperturbed system, as shown in Figure 6.1. In this chapter, we focus on the case with kink wave and degenerate center, that is, case (a).

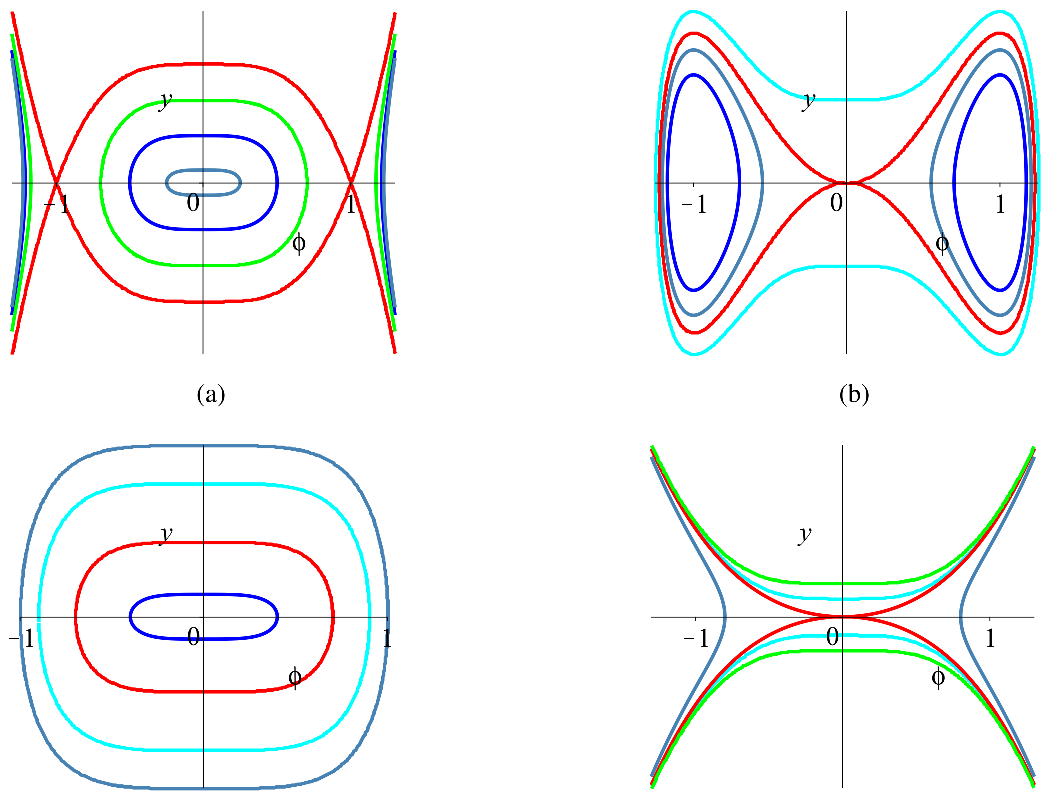

Figure 6.1: Phase portraits of system $(6.11)_{\varepsilon=0}$ for (a) $\gamma_1 > 0, \gamma_2 < 0$; (b) $\gamma_1 < 0, \gamma_2 > 0$; (c) $\gamma_1, \gamma_2 > 0$; and (d) $\gamma_1, \gamma_2 < 0$.

Upon introducing the following dimensionless re-scalings,

$$\phi = \sqrt{-\frac{\gamma_1}{\gamma_2}}\widetilde{\phi}, \quad y = -\frac{\gamma_1^{\frac{3}{2}}}{\gamma_2}\widetilde{y}, \quad x = \frac{\sqrt{-\gamma_2}}{\gamma_1}\tau, \quad \varepsilon = \frac{\gamma_2^2}{\sigma_2\gamma_1\sqrt{-\gamma_2}}\widetilde{\varepsilon},$$

system (6.11) reads (dropping the tilde for simplicity)

$$\phi' = y, \quad y' = \phi^3(\phi^2 - 1) + \varepsilon(a_0 + a_1\phi^2 + \phi^4)y, \tag{6.13}$$

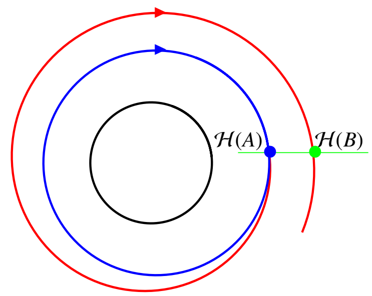

Figure 6.2: Poincaré map of system (6.13)

where $a_0 = \frac{\gamma_2^2\sigma_0}{\gamma_1^2\sigma_2}$ and $a_1 = -\frac{\gamma_2\sigma_1}{\gamma_1\sigma_2}$. The corresponding Hamiltonian is

$$\mathcal{H}(\phi, y) = \frac{y^2}{2} + \frac{\phi^4}{4} - \frac{\phi^6}{6}.$$

The phase portrait of system $(6.13)_{\varepsilon=0}$ is shown in Figure 6.1(a). There is a heteroclinic loop denoted by $L_{\frac{1}{12}}$ which connects the two hyperbolic saddles $E_1(-1, 0)$ and $E_2(1, 0)$ with the Hamiltonian quantity $\mathcal{H}(-1, 0) = \mathcal{H}(1, 0) = \frac{1}{12}$. A family of periodic orbits defined by $L_h = \{\mathcal{H}(\phi, y) = h,\ 0 < h < \frac{1}{12}\}$ surrounds the nilpotent center $E_0(0, 0)$. Investigating the periodic waves of (6.2) is equivalent to studying the isolated closed orbits of system (6.13).

First, we construct a displacement map on the periodic annulus of $(6.13)_{\varepsilon=0}$ (see Figure 6.2) given by

$$\mathbb{P}(h) = \mathcal{H}(B) - \mathcal{H}(A).$$

By Poincaré-Pontryagin Theorem [16], $\mathbb{P}(h) = \varepsilon\mathcal{A}(h, \eta) + O(\varepsilon^2)$, where $\mathcal{A}(h, \eta)$ is the following Abelian integral defined on the periodic annulus $\{L_h\}$,

$$\mathcal{A}(h, \eta) = \oint_{L_h} (a_0 + a_1\phi^2 + \phi^4)y\,d\phi, \tag{6.14}$$

where $\eta = (a_0, a_1)$, $h \in (0, \frac{1}{12})$. An isolated zero of $\mathcal{A}(h, \eta)$ corresponds to an isolated zero of $\mathbb{P}(h)$ for ε positive and sufficiently small, implying the existence of an isolated closed orbit (heteroclinic loop if $\mathcal{A}(\frac{1}{12}, \eta) = 0$) in system (6.13). Thus, our problem is reduced to studying the zeros of $\mathcal{A}(h, \eta)$. In the next two sections, we give a detailed analysis on the existence of zeros of $\mathcal{A}(h, \eta)$ and the condition such that $\mathcal{A}(\frac{1}{12}, \eta) = 0$, and investigate the uniqueness and distribution of the zeros of $\mathcal{A}(h, \eta)$ when $\mathcal{A}(\frac{1}{12}, \eta) = 0$. The solutions of the zeros provide the estimation on the amplitudes of the periodic waves of (6.2). We will also consider the maximal number of zeros of $\mathcal{A}(h, \eta)$ which determines the maximal number of periodic waves of (6.2).

6.3 Periodic and kink waves of (6.2)

We will use the asymptotic expansion of the Abelian integral $\mathcal{A}(h, \eta)$ defined in (6.14) near the heteroclinic loop $L_{\frac{1}{12}}$ and the center E_0, to find their zeros near the endpoints 0 and $\frac{1}{12}$, which characterizes the existence of the periodic waves and kinks. In particular, we will discuss the global existence and uniqueness of the periodic waves when the kink persists, in other words, the coexistence of periodic waves and kink.

6.3.1 Existence and coexistence of periodic and kink waves

Following the method developed by Han *et al.* [14], careful calculation gives

$$\mathcal{A}(h,\eta) = c_0(\eta) + c_1(\eta)\Big(h - \frac{1}{12}\Big)\ln\Big|h - \frac{1}{12}\Big| + c_2(\eta)\Big(h - \frac{1}{12}\Big) + \text{h.o.t.} \tag{6.15}$$

for $0 < -(h - \frac{1}{12}) \ll 1$, where h.o.t. denotes higher order terms. Further, applying the formulas given by Yang and Han [38] and Jiang and Han [22], we obtain

$$\mathcal{A}(h,\eta) = h^{\frac{3}{4}}(b_0(\eta) + b_1(\eta)h^{\frac{1}{2}} + b_2(\eta)h + \text{h.o.t.}) \tag{6.16}$$

for $0 < h \ll 1$. Here, the first four coefficients c_i and b_i can be further expressed by using the methods developed in [14, 38]:

$$\begin{aligned} c_0 &= J_{00}a_0 + J_{01}a_1 + J_{02}, \\ c_1 &= J_{10}a_0 + J_{11}a_1 + J_{12}, \\ c_2 &= J_{20}a_0 + J_{21}a_1 + J_{22}, \end{aligned} \tag{6.17}$$

where $J_{00} = \frac{3\sqrt{2}-3J^*}{8}$, $J_{01} = \frac{18\sqrt{2}-5J^*}{96}$, $J_{02} = \frac{39\sqrt{2}-7J^*}{512}$, $J_{10} = J_{11} = J_{12} = -\sqrt{2}$, $J_{20} = 0$, $J_{21} = 4J^*$, $J_{22} = 3J^* - 3\sqrt{2}$, and

$$\begin{aligned} b_0 &= \tfrac{8}{3}s_1a_0, \\ b_1 &= \tfrac{1}{5}s_2(16a_0 + 32a_1), \\ b_2 &= \tfrac{4}{63}s_1(15a_0 + 20a_1 + 24), \\ b_3 &= \tfrac{56}{405}s_2(55a_0 + 66a_1 + 72). \end{aligned} \tag{6.18}$$

Here, $J^* = -\sqrt{3}\operatorname{arcsinh}\big(\sqrt{2}\big)$, $s_1 = \frac{\pi^{\frac{3}{2}}\sqrt{2}}{(\Gamma(\frac{3}{4}))^2}$ and $s_2 = \frac{(\Gamma(\frac{3}{4}))^2\sqrt{2}}{\sqrt{\pi}}$.

The following result is a different version of bifurcation theorem in [13] for system (6.13), which will be used to identify the zeros of $\mathcal{A}(h,\eta)$. For readers' convenience, we state the criterion in a new form and provide a simpler proof.

Lemma 6.3.1 *Considering system* (6.13) *and the expansions* (6.15) *and* (6.16)*, it is supposed that there exists* $\eta_0 \in \mathbb{R}^N$ *such that*

$$\begin{aligned} c_0(\eta_0) = c_1(\eta_0) = \cdots = c_{l-1}(\eta_0) = 0,\ c_l(\eta_0) \neq 0, \\ b_0(\eta_0) = b_1(\eta_0) = \cdots = b_{k-1}(\eta_0) = 0,\ b_k(\eta_0) \neq 0 \end{aligned} \tag{6.19}$$

and

$$\operatorname{rank}\left[\frac{\partial(c_0, c_1, \cdots, c_{l-1}, b_0, b_1, \cdots, b_{k-1})}{\partial\eta}\right] = m + k. \tag{6.20}$$

Then $\mathcal{A}(h,\eta)$ can have $l + k + \frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta_0)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta_0))}{2}$ zeros for some η near η_0, l zeros of which are near $h = \frac{1}{12}$ in $(\frac{1}{12}-\varepsilon_2, \frac{1}{12})$, k zeros of which near $h = 0$ in $(0, \varepsilon_1)$, $\frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta_0)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta_0))}{2}$ zero in $(\varepsilon_1, \frac{1}{12}-\varepsilon_2)$, with ε_1 and ε_2 are positive and sufficiently small. Therefore, system (6.13) *can have $l+k+\frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta_0)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta_0))}{2}$ isolated periodic orbits for some (ε,η) near $(0,\eta_0)$, m isolated periodic orbits of which are near the heteroclinic loop $L_{\frac{1}{12}}$, k isolated periodic orbits of which near the center $E_0(0,0)$, and $\frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta_0)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta_0))}{2}$ isolated periodic orbit surrounds the center $E_0(0,0)$.*

Proof We only prove the case $l = 2$, and other cases can be proved similarly. By (6.19), we get

$$\mathcal{A}(h,\eta_0) = h^{\frac{3}{4}}\Big(b_k(\eta_0)h^{\frac{k}{2}} + O(h^{\frac{k+1}{2}})\Big), \quad 0 < h \ll 1,$$
$$\mathcal{A}(h,\eta_0) = c_2(\eta_0)\Big(h-\frac{1}{12}\Big) + O\Big(\Big(h-\frac{1}{12}\Big)^2 \ln\Big|h-\frac{1}{12}\Big|\Big), \ \ 0 < -\Big(h-\frac{1}{12}\Big) \ll 1.$$

Taking $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small, then $\mathcal{A}(h,\eta_0)$ has

$$\frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta_0)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta_0))}{2}$$

zero, which is either 0 or 1. When it is 1, the zero is denoted by $h_0^* \in (\varepsilon_1, \frac{1}{12}-\varepsilon_2)$. By (6.20) we know that c_0, c_1, c_2, b_0, $b_1, \cdots, b_{k-1}$ can be taken as free parameters. Next, we change the signs of them in turn to obtain the zeros of $\mathcal{A}(h,\eta)$. Let

$$\begin{aligned} &|c_0| \ll |c_1| \ll |c_2(\eta_0)|, \quad c_2(\eta_0)c_1 > 0, \quad c_1c_0 < 0,\\ &|b_0| \ll |b_1| \ll \cdots \ll |b_{k-1}| \ll |b_k(\eta_0)|, \quad b_k(\eta_0)b_{k-1} < 0, \quad b_{j-1}b_j < 0, \end{aligned} \tag{6.21}$$

for $j = 1, 2, \cdots, k-1$. Then we find 2 zeros of $\mathcal{A}(h,\eta)$ near $h = \frac{1}{12}$ inside $(\frac{1}{12}-\varepsilon_2, \frac{1}{12})$ and k zeros of $\mathcal{A}(h,\eta)$ near $h = 0$ inside $(0, \varepsilon_1)$.

Let $U(\eta_0) = \{\eta|(6.21) \text{ holds}\}$ be a subset of a neighborhood of η_0 with a very sufficiently small radius ε^*. Taking an $\eta \in U(\eta_0)$, then $\mathcal{A}(h,\eta) = A(h,\eta_0) + O(\varepsilon^*)$. Therefore,

$$\frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta))}{2} = \frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta_0)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta_0))}{2}$$

which implies that there exists $\frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta_0)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta_0))}{2}$ zero of $\mathcal{A}(h,\eta)$ near h_0^* (if it exists). This shows that $\mathcal{A}(h,\eta)$ can have $2 + k + \frac{1-\mathrm{sgn}(\mathcal{A}(\varepsilon_1,\eta_0)\mathcal{A}(\frac{1}{12}-\varepsilon_2,\eta_0))}{2}$ zeros for the case $l = 2$. The proof is complete.

In the following, the coefficients in the expansions of $\mathcal{A}(h,\eta)$ and Lemma 6.3.1 will be applied to determine the zeros of $\mathcal{A}(h,\eta)$. These zeros correspond to the periodic standing waves.

Solving $c_0(\eta) = c_1(\eta) = 0$ gives

$$\begin{cases} a_0 = \dfrac{1}{16}\,\dfrac{-101\,\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 315\,\sqrt{2}}{41\,\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 9\,\sqrt{2}} \approx 0.1626564078 := a_0^\dagger, \\ a_1 = -\dfrac{3}{16}\,\dfrac{185\,\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 153\,\sqrt{2}}{41\,\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 9\,\sqrt{2}} \approx -1.162656408 := a_1^\dagger. \end{cases}$$

Further, taking $\eta_0 = (a_0^\dagger, a_1^\dagger)$ yields

$$\begin{cases} c_2(\eta_0) = \dfrac{189\left(\operatorname{arcsinh}\left(\sqrt{2}\right)\right)^2 - 141\operatorname{arcsinh}\left(\sqrt{2}\right)\sqrt{6} - 216}{164\,\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 36\,\sqrt{2}} \approx -0.965646921, \\ b_0(\eta_0) = \dfrac{s_1}{6}\,\dfrac{-101\,\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 315\,\sqrt{2}}{41\,\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 9\,\sqrt{2}} \approx 2.274637142. \end{cases}$$

Therefore, taking $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ sufficiently small, one has $\mathcal{A}(\frac{1}{12} - \varepsilon_2, \eta_0) = c_2(\eta_0)(-\varepsilon_2) +$ h.o.t. > 0, $\mathcal{A}(\varepsilon_1, \eta_0) = \varepsilon_1^{\frac{3}{4}}(b_0(\eta_0) + \text{h.o.t.}) > 0$. Hence,

$$\frac{1 - \operatorname{sgn}(\mathcal{A}(\varepsilon_1, \eta_0)\mathcal{A}(\frac{1}{12} - \varepsilon_2, \eta_0))}{2} = 0.$$

It is not difficult to show that rank $\left[\frac{\partial(c_0,c_1)}{\partial(a_0,a_1)}\right] = 2$. Applying Lemma 6.3.1, and taking $\varepsilon > 0$ sufficiently small, one can show that system (6.13) has two isolated periodic orbits near the heteroclinic loop for some η inside a sufficiently small region enclosing η_0. Summarizing the above discussions, we have the following result.

Theorem 6.3.2 *Let $\varepsilon > 0$ be sufficiently small. Then equation* (6.2) *has two isolated periodic waves with different amplitudes near the kink wave $L_{\frac{1}{12}}$ for some (a_0, a_1) near $(a_0^\dagger, a_1^\dagger)$.*

Let $(l, k, 1)$ and $(l, k, 0)$ denote the distribution of the periodic waves. The distribution of the two periodic waves in Theorem 6.3.2 is $(2, 0, 0)$. Using the same procedure in proving Theorem 6.3.2, we can use Lemma 6.3.1 to prove the following theorem.

Theorem 6.3.3 *There exist some values of (a_0, a_1) such that equation* (6.2) *has two periodic waves, with the distributions* $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 0)$ *and* $(0, 0, 2)$.

Remark 6.3.4 *From the proof of Lemma* 6.3.1, *we know that $\mathcal{A}(\frac{1}{12}, \eta_0) = c_0(\eta_0) \neq 0$. Therefore, the kink wave $L_{\frac{1}{12}}$ is broken when the two periodic waves exist by Theorem* 6.3.2. *Similarly, the kink wave $L_{\frac{1}{12}}$ is broken when the two periodic waves exist with a distribution given in Theorem* 6.3.3.

Let $c_0 = 0$, we have $a_0 = \beta_1 a_1 + \beta_2$, where

$$\begin{aligned}\beta_1 &= \frac{5\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) - 27\sqrt{2}}{36\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 36\sqrt{2}} \approx -0.2308926097,\\ \beta_2 &= \frac{-7\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) - 39\sqrt{2}}{192\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 192\sqrt{2}} \approx -0.1057923643.\end{aligned} \tag{6.22}$$

Thus, fixing η as

$$\eta^* = (\beta_1 a_1 + \beta_2, a_1), \tag{6.23}$$

we have $\mathcal{A}(\frac{1}{12}, \eta^*) = 0$. Then the heteroclinic loop $L_{\frac{1}{12}}$ persists under an arbitrary perturbation. Solving $b_0 = 0$ gives

$$a_0 = 0 \;\text{ and }\; a_1 = \frac{21\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 117\sqrt{2}}{80\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) - 432\sqrt{2}} \approx -0.4581886118 := a_1^{**}.$$

Letting $\eta^{**} = (0, a_1^{**})$ and substituting η^{**} into c_1 and b_1 yield

$$c_1(\eta^{**}) \approx -0.7662370141 \quad \text{and} \quad b_1(\eta^{**}) \approx -3.513434941.$$

Therefore, taking $0 < \varepsilon_1, \varepsilon_2 \ll 1$, we have

$$\begin{aligned}\mathcal{A}(-\varepsilon_2, \eta^{**}) &= c_1(\eta^{**})(-\varepsilon_2)\ln|-\varepsilon_2| + \text{h.o.t.} < 0,\\ \mathcal{A}(\varepsilon_1, \eta^{**}) &= \varepsilon_1^{\frac{3}{4}}(b_1(\eta^{**})\varepsilon_1^{\frac{1}{2}} + \text{h.o.t.}) < 0.\end{aligned}$$

Hence, $\frac{1-\operatorname{sgn}(\mathcal{A}(h_1,\eta^{**})\mathcal{A}(h_2,\eta^{**}))}{2} = 0$. Taking b_0 as the unique free parameter in Lemma 6.3.1, we can show that there exist values of η near η^{**} such that system (6.13) has one isolated closed orbit in the neighborhood of the center $E_0(0,0)$. This means that the heteroclinic loop $L_{\frac{1}{12}}$ still exists for $c_0 = \mathcal{A}(\frac{1}{12}, \eta) = 0$. It then follows that

Theorem 6.3.5 *Let $\varepsilon > 0$ be sufficiently small. Then equation (6.2) can have one periodic wave in a neighborhood of the center $E_0(0,0)$ and the kink wave $L_{\frac{1}{12}}$ persists for some (a_0, a_1) near $(0, a_1^{**})$.*

Using the same approach described as above, we can prove the following theorem.

Theorem 6.3.6 *Taking ε positive and sufficiently small, there exist values of (a_0, a_1) such that equation (6.2) can have one periodic wave with the distribution $(1,0,0)$ or $(0,0,1)$ and the kink wave $L_{\frac{1}{12}}$ persists.*

Remark 6.3.7 *Theorems 6.3.5 and 6.3.6 imply that when the kink wave persists under the weakly dissipative influence, there exists a periodic wave, which can have a small amplitude or a large amplitude.*

6.3.2 Uniqueness of periodic wave when kink wave persists

In this subsection, we discuss the global existence and uniqueness of periodic wave when the kink wave persists. Rewrite the Hamiltonian function of system (6.13) as

$$\mathcal{H}(\phi, y) = \frac{y^2}{2} + \frac{\phi^4}{4} - \frac{\phi^6}{6} := \Psi(y) + \Phi(\phi) = h, \tag{6.24}$$

where $h \in (0, \frac{1}{12})$, $\phi(h) \in (-1, 1)$ and $y(h) \in (-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6})$. Then, the Abelian integral (6.14) can be rewritten as

$$\mathcal{A}(h, \eta) = \oint_{L_h} (a_0 + a_1\phi^2 + \phi^4) y d\phi = a_0 \mathcal{I}_0(h) + a_1 \mathcal{I}_1(h) + \mathcal{I}_2(h), \tag{6.25}$$

where

$$\mathcal{I}_i(h) = \oint_{L_h} \phi^{2i} y d\phi, \quad i = 0, 1, 2. \tag{6.26}$$

We have for $h \in (0, \frac{1}{12})$, $i = 0, 1, 2$,

$$\mathcal{I}_i(h) = \oint_{L_h} \phi^{2i} y d\phi = 4\int_0^{\phi^+(h)} \phi^{2i} y d\phi = 4\int_0^{\phi^+(h)} \phi^{2i} \frac{1}{6} \sqrt{12\,\phi^6 - 18\,\phi^4 + 72\,h}\, d\phi > 0,$$

where $\phi^+(h) > 0$ denotes the intersection point of the oval L_h with the positive ϕ-axis. Then the following ratios are introduced and well-defined on $h \in (0, \frac{1}{12})$,

$$\mathcal{P}(h) = \frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)} \text{ and } \mathcal{Q}(h) = \frac{\mathcal{I}_2(h)}{\mathcal{I}_0(h)}.$$

The following result can be now established.

Lemma 6.3.8 *With $h \in (0, \frac{1}{12})$, $\mathcal{P}(h)$ increases from $(0, 0)$ to $(\frac{1}{12}, -\beta_1)$, and $\mathcal{Q}(h)$ increases from $(0, 0)$ to $(\frac{1}{12}, -\beta_2)$, where β_1 and β_2 are given in (6.22).*

Proof For any $\phi(h) \in (0, 1)$ and $y(h) \in (0, \frac{\sqrt{6}}{6})$, there exist $z(h) = -\phi(h) < 0$ and $\widetilde{y}(h) = -y(h) < 0$ such that

$$\mathcal{H}(\phi(h), 0) = \mathcal{H}(z(h), 0) = \mathcal{H}(0, y(h)) = \mathcal{H}(0, \widetilde{y}(h)) = h.$$

Taking $f_1(\phi) = 1$, $f_2(\phi) = \phi^2$ and $g(y) = y$, we have

$$\xi(\phi) = \frac{f_2(\phi)\Phi'(z) - f_2(z)\Phi'(\phi)}{f_1(\phi)\Phi'(z) - f_1(z)\Phi'(\phi)} = \phi^2,$$
$$\zeta(y) = \frac{(g(\widetilde{y}) - g(y))\Psi'(\widetilde{y})\Psi'(y)}{g'(\widetilde{y})\Psi'(y) - g'(y)\Psi'(\widetilde{y})} = y^2,$$

which imply that $\xi'(\phi)\zeta'(y) = 4\phi(h)y(h) > 0$, so it follows from Li and Zhang (1996) that $\mathcal{P}'(h) > 0$. Taking $f_1(\phi) = 1$, $f_2(\phi) = \phi^4$ and $g(y) = y$, we obtain $\xi'(\phi)\zeta'(y) = 8\phi^3(h)y(h) > 0$, so it follows again from Li and Zhang (1996) that $\mathcal{Q}'(h) > 0$. The remaining part of the proof is to compute the limits of $P(h)$ and $Q(h)$ at the endpoints of the interval $(0, \frac{1}{12})$, which is straightforward by using the expansion of $\mathcal{A}(h)$ with the coefficients given in (6.17) and (6.18)

Recall that we have proved $\mathcal{A}(h,\eta^*)$ has a zero at $h = \frac{1}{12}$, where η^* is given in (6.23). We have a further result as given below.

Lemma 6.3.9 *$\mathcal{A}(h,\eta^*)$ has a unique zero $h^\dagger \in (0,\frac{1}{12})$, which increases from 0 to $\frac{1}{12}$ as a_1 decreases form $-\frac{\beta_2}{\beta_1}$ to $-a_1^\dagger$, where*

$$a_1^\dagger = \frac{555\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 459\sqrt{2}}{656\sqrt{3}\operatorname{arcsinh}\left(\sqrt{2}\right) + 144\sqrt{2}} \approx 1.162656408.$$

Proof Considering (6.23), we have

$$\mathcal{A}(h,\eta^*) = \oint_{L_h} a_1(\beta_1+\phi^2)yd\phi + \oint_{L_h}(\beta_2+\phi^4)yd\phi := a_1\mathcal{A}_1(h,\eta^*) + \mathcal{A}_2(h,\eta^*).$$

We first prove that the ratio $\frac{\mathcal{A}_2(h,\eta^*)}{\mathcal{A}_1(h,\eta^*)}$ is well defined for $h \in (0,\frac{1}{12})$ by showing that $\mathcal{A}_1(h,\eta^*) \neq 0$ for $h \in (0,\frac{1}{12})$. By the property and range of $\mathcal{P}(h)$ given in Lemma 6.3.8, we have

$$\mathcal{A}_1(h,\eta^*) = \mathcal{I}_0(h)(\beta_1 + \mathcal{P}(h)) < 0.$$

Let $\psi_1(\phi) = \frac{\alpha+\phi^2}{\Phi'(\phi)} - \frac{\alpha+z^2}{\Phi'(z)}$, $\psi_2(\phi) = \frac{\beta+\phi^4}{\Phi'(\phi)} - \frac{\beta+z^4}{\Phi'(z)}$, where $z = -\phi$, which is defined by $\Phi(\phi) = \Phi(z)$. $(\phi,0)$ and $(z,0)$ are respectively the right and left intersection points of the closed orbit $L_{h(\phi)}$ on the ϕ axis. Direct computation gives the Wronskian,

$$\begin{vmatrix} \psi_1(\phi) & \psi_2(\phi) \\ \psi_1'(\phi) & \psi_2'(\phi) \end{vmatrix} = \frac{8(\phi^4 + 2\beta_1\phi^2 - \beta_2)}{\phi^5(\phi^2-1)^2} \neq 0,\ \phi \in (0,1).$$

Therefore, $\{\psi_1,\psi_2\}$ forms a Chebyshev system, and so is $\{\mathcal{A}_1(h,\eta^*),\mathcal{A}_2(h,\eta^*)\}$ by the criterion in [10]. It follows that any non-trivial linear combination of $\mathcal{A}_1(h,\eta^*)$ and $\mathcal{A}_2(h,\eta^*)$ has at most one zero in $(0,\frac{1}{12})$. This also implies that the ratio $\frac{\mathcal{A}_2(h,\eta^*)}{\mathcal{A}_1(h,\eta^*)}$ is monotonic for $h \in (0,\frac{1}{12})$, see Figure 6.3. A direct computation yields

$$\lim_{h\to 0^+}\frac{\mathcal{A}_2(h,\eta^*)}{\mathcal{A}_1(h,\eta^*)} = \frac{\beta_2}{\beta_1} \approx 0.4581886118,$$
$$\lim_{h\to \frac{1}{12}^-}\frac{\mathcal{A}_2(h,\eta^*)}{\mathcal{A}_1(h,\eta^*)} = \frac{\beta_2 J_{10} + J_{12}}{\beta_1 J_{10} + J_{11}} = a_1^\dagger.$$

Therefore, for any a_1 belonging to $(-a_1^\dagger, -\frac{\beta_2}{\beta_1})$, $a_1 + \frac{\mathcal{A}_2(h,\eta^*)}{\mathcal{A}_1(h,\eta^*)}$ has a unique zero $h^\dagger \in (0,\frac{1}{12})$ with the location depending on a_1. Hence, the conclusion of the Lemma is true since $\mathcal{A}(h,\eta^*) = \mathcal{A}_1(h,\eta^*)\left(a_1 + \frac{\mathcal{A}_2(h,\eta^*)}{\mathcal{A}_1(h,\eta^*)}\right)$.

Lemma 6.3.9 implies that

Theorem 6.3.10 *There exists a unique periodic wave that coexists with the kink wave for parameter vector $\eta = \eta^* = (\beta_1 a_1 + \beta_2, a_1)$ with $a_1 \in (-a_1^\dagger, -\frac{\beta_2}{\beta_1})$, and the amplitude of the unique periodic wave increases as a_1 decreases.*

Remark 6.3.11 *We proved the monotonicity of the related ratios of some Abelian integrals in Lemmas 6.3.8 and 6.3.9 based on different criteria. The criterion given by Li and Zhang* [28] *for proving Lemma 6.3.8 is not applicable for proving Lemma 6.3.9.*

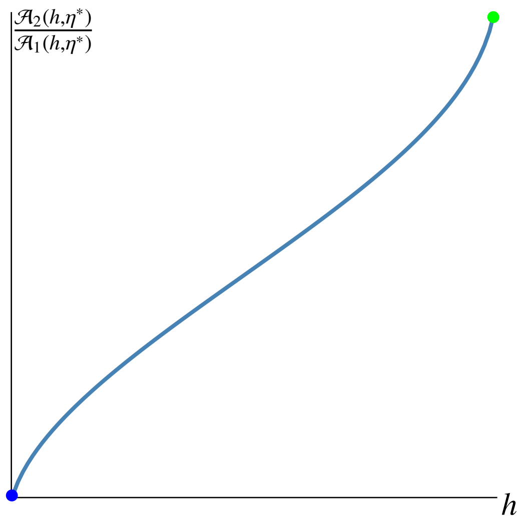

Figure 6.3: Simulation of $\frac{\mathcal{A}_2(h,\eta^*)}{\mathcal{A}_1(h,\eta^*)}$ on $h \in (0, \frac{1}{12})$.

6.4 The least upper bound on the number of periodic waves

In this section, we discuss the least upper bound on the number of periodic waves of equation (6.2) by applying the methods developed on the basis of geometry [7, 19] and algebra [10, 29]. We have written the Hamiltonian function of system (6.13) as the form in (6.24) with $h \in (0, \frac{1}{12})$, $\phi(h) \in (-1, 1)$ and $y(h) \in (-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6})$. And the Abelian integral (6.14) has been rewritten in the form of (6.26), with

$$\mathcal{I}_i(h) = \oint_{L_h} \phi^{2i} y d\phi > 0 \text{ for } i = 0, 1, 2.$$

Lemma 6.4.1 *For* $h \in (0, \frac{1}{12})$, $\frac{d}{dh}(\frac{\mathcal{Q}(h)}{\mathcal{P}(h)}) > 0$.

Proof By Lemma 6.3.8, $\frac{\mathcal{Q}(h)}{\mathcal{P}(h)}$ is well defined for $h \in (0, \frac{1}{12})$. Further, $\frac{\mathcal{Q}(h)}{\mathcal{P}(h)} = \frac{I_2(h)}{I_1(h)}$, which is monotonically increasing for $h \in (0, \frac{1}{12})$ by the method used in Lemma 6.3.8. Hence, the claim holds.

Then, by Lemma 6.3.8, letting $\mathcal{P} = \mathcal{P}(h)$, we have $h = \mathcal{P}^{-1}(\mathcal{P})$. Define the curve in the $(\mathcal{P}, \mathcal{Q})$-plane,

$$\Sigma = \left\{(\mathcal{P}, \mathcal{Q})(h) | \mathcal{Q}(\mathcal{P}) = \mathcal{Q}(\mathcal{P}^{-1}(\mathcal{P})) =: \psi(\mathcal{P}),\ h \in \left(0, \frac{1}{12}\right)\right\}. \tag{6.27}$$

By Lemma 6.4.1, $\psi'(\mathcal{P}) > 0$. It is not difficult to know that the number of zeros of $\mathcal{A}(h, \eta)$ equals the number of intersection points of the straight line,

$$\mathcal{L}: \quad a_0 + a_1 \mathcal{P} + \mathcal{Q} = 0, \tag{6.28}$$

and the curve Σ.

Lemma 6.4.2 $\lim\limits_{h \to 0^+} \frac{d^2 \mathcal{Q}}{d\mathcal{P}^2} > 0$.

Proof A direct calculation shows that

$$\begin{aligned}
\mathcal{P}'(h) &= \frac{\mathcal{I}_0(h)\mathcal{I}_1'(h)-\mathcal{I}_0'(h)\mathcal{I}_1(h)}{\mathcal{I}_0{}^2(h)}, \quad \mathcal{Q}'(h)=\frac{\mathcal{I}_0(h)\mathcal{I}_2'(h)-\mathcal{I}_0'(h)\mathcal{I}_2(h)}{\mathcal{I}_0{}^2(h)},\\
\frac{d^2\mathcal{Q}}{d\mathcal{P}^2} &= \frac{\mathcal{Q}''(h)\mathcal{P}'(h)-\mathcal{Q}'(h)\mathcal{P}''(h)}{\mathcal{P}^3(h)}\\
&= -\frac{\mathcal{I}_0(\mathcal{I}_2''\mathcal{I}_0\mathcal{I}_1'-\mathcal{I}_2''\mathcal{I}_0'\mathcal{I}_1-\mathcal{I}_0''\mathcal{I}_2\mathcal{I}_1'-\mathcal{I}_1''\mathcal{I}_0\mathcal{I}_2'+\mathcal{I}_1''\mathcal{I}_0'\mathcal{I}_2+\mathcal{I}_0''\mathcal{I}_1\mathcal{I}_2')}{(-\mathcal{I}_1'\mathcal{I}_0+\mathcal{I}_0'\mathcal{I}_1)^3}.
\end{aligned}$$

By (6.16) with the coefficients in (6.18), we obtain the expansions of $\mathcal{I}_i$, $\mathcal{I}_i'$ and $\mathcal{I}_i''$ for $i=0,1,2$, which are substituted into the above equation and then taking limit yields,

$$\lim_{h\to 0^+}\frac{d^2\mathcal{Q}}{d\mathcal{P}^2}=\frac{25\,\pi^4}{126\ (\Gamma\,(3/4))^8}\approx 3.801006321.$$

Lemma 6.4.3 *For a sufficiently small $\varepsilon^*>0$ and $h\in(\frac{1}{12}-\varepsilon^*,\frac{1}{12})$, $\frac{d^2\mathcal{Q}}{d\mathcal{P}^2}>0$.*

Proof By definition, one has

$$\lim_{h\to\frac{1}{12}}\mathcal{P}(h)=\frac{J_{01}}{J_{00}},\quad \lim_{h\to\frac{1}{12}}\mathcal{Q}(h)=\frac{J_{02}}{J_{00}},\quad \lim_{h\to\frac{1}{12}}\frac{\mathcal{Q}(h)}{\mathcal{P}(h)}=\frac{J_{02}}{J_{01}}.$$

Since $\frac{d\mathcal{Q}}{d\mathcal{P}}=\frac{\mathcal{Q}'(h)}{\mathcal{P}'(h)}=\frac{\mathcal{I}_0(h)\mathcal{I}_2'(h)-\mathcal{I}_0'(h)\mathcal{I}_2(h)}{\mathcal{I}_0(h)\mathcal{I}_1'(h)-\mathcal{I}_0'(h)\mathcal{I}_1(h)}$, and for $k=0,1,2$,

$$\begin{aligned}
\mathcal{I}_k'(h) &= J_{1k}\ln\Big(\frac{1}{12}-h\Big)+J_{1k}+J_{2k}+\cdots\\
&= \ln\Big(\frac{1}{12}-h\Big)\Bigg(J_{1k}+\frac{J_{1k}}{\ln(\frac{1}{12}-h)}+\frac{J_{2k}}{\ln(\frac{1}{12}-h)}+\cdots\Bigg),
\end{aligned}$$

we obtain

$$\lim_{h\to\frac{1}{12}}\frac{d\mathcal{Q}}{d\mathcal{P}}=\frac{J_{00}J_{12}-J_{10}J_{02}}{J_{00}J_{11}-J_{10}J_{01}}.$$

Let $\varepsilon^*>0$ be sufficiently small, and define

$$\Sigma^*=\Bigg\{(\mathcal{P},\mathcal{Q})(h)|\mathcal{Q}(\mathcal{P})=\mathcal{Q}(\mathcal{P}^{-1}(\mathcal{P})),\ h\in\Big(\frac{1}{12}-\varepsilon^*,\frac{1}{12}\Big)\Bigg\},$$

which is a very small arc of the curve Σ near $(\mathcal{P}^*,\mathcal{Q}^*)=(\mathcal{P}(\frac{1}{12}),\mathcal{Q}(\frac{1}{12}))=(\frac{J_{01}}{J_{00}},\frac{J_{02}}{J_{00}})$. The tangent line to Σ^* at the point $(\mathcal{P}^*,\mathcal{Q}^*)$, denoted by $\mathcal{L}^*$, is given by

$$\mathcal{L}^*:\ \ \mathcal{Q}-\mathcal{Q}^*=\frac{d\mathcal{Q}}{d\mathcal{P}}\Big|_{h=\frac{1}{12}}(\mathcal{P}-\mathcal{P}^*).$$

The vertical distance between the point $(\mathcal{P}(h),\mathcal{Q}(h))$ on the curve Σ^* and the point $(\mathcal{P},\mathcal{Q})=(\mathcal{P}(h),\mathcal{Q})$ on the tangent line $\mathcal{L}^*$ near $(\mathcal{P}^*,\mathcal{Q}^*)$ is

$$\begin{aligned}
\mathcal{Q}(h)-\mathcal{Q} &= \mathcal{Q}(h)-\mathcal{Q}^*-\frac{d\mathcal{Q}}{d\mathcal{P}}|_{h=\frac{1}{12}}(\mathcal{P}-\mathcal{P}^*)\\
&= \frac{1}{\mathcal{I}_0(h)}(\mathcal{I}_2(h)-\mathcal{Q}^*\mathcal{I}_0(h)-\frac{d\mathcal{Q}}{d\mathcal{P}}|_{h=\frac{1}{12}}(\mathcal{I}_1(h)-\mathcal{P}^*\mathcal{I}_0(h)))\\
&= \frac{\varrho}{\mathcal{I}_0(h)}\Bigg(\Big(\frac{1}{12}-h\Big)+O\Bigg(\Big(\frac{1}{12}-h\Big)^2\ln\Big(\frac{1}{12}-h\Big)\Bigg)\Bigg),
\end{aligned}$$

where

$$\begin{aligned}\varrho &= \frac{-J_0J_{11}J_{22}+J_0J_{12}J_{21}+J_1J_{10}J_{22}-J_1J_{12}J_{20}-J_2J_{10}J_{21}+J_2J_{11}J_{20}}{J_0J_{11}-J_1J_{10}}\\ &\approx 0.9656469152.\end{aligned}$$

We have shown $\mathcal{I}_0(h)>0$. Hence, $Q(h)-Q>0$ for $h\in(\frac{1}{12}-\varepsilon^*,\frac{1}{12})$. Therefore, the arc Σ^* is located above the tangent line $\mathcal{L}^*$, which implies that Σ^* is convex, and so $\frac{d^2Q}{d\mathcal{P}^2}>0$ on Σ^*.

Next, we consider $\mathcal{A}''(h,\eta)$. The equation $\mathcal{H}(\phi,y)=\frac{y^2}{2}+\frac{\phi^3}{4}-\frac{\phi^6}{5}=h$ implies that $\frac{\partial y}{\partial h}=\frac{1}{y}$. It follows that $\mathcal{I}_i''(h)=\frac{\partial}{\partial h}\mathcal{I}_i'(h)=\frac{\partial}{\partial h}\Big(\oint_{L_h}\frac{\phi^{2i}}{y}d\phi\Big)$ for $i=1,2$.

Lemma 6.4.4 *For $i=0,1,2$, we have*

(i) $2h\mathcal{I}_i(h)=\oint_{L_h}f_i^*(\phi)y^3d\phi,\quad 4h^2\mathcal{I}_i(h)=\oint_{L_h}f_i^{**}(\phi)y^5d\phi,$

(ii) $8h^3\mathcal{I}_i''(h)=\oint_{L_h}f_i(\phi)y^3d\phi:=\widehat{I_i}(h),$

where $f_i^*(\phi)$ and $f_i^{**}(\phi)$ are given later in the proof, and $f_i(\phi)=\frac{\phi^{2i}\mu_i(\phi)}{648(\phi-1)^6(\phi+1)^6}$ with

$$\begin{aligned}\mu_i(\phi)=\ &\left(64\,x^{12}-480\,x^{10}+1488\,x^8-2440\,x^6+2232\,x^4-1080\,x^2+216\right)i^3\\ &+\left(384x^{12}-2544x^{10}+7104x^8-10812x^6+9540x^4-4644x^2+972\right)i^2\\ &+\left(192\,x^{12}-944\,x^{10}+2020\,x^8-2582\,x^6+2394\,x^4-1674\,x^2+594\right)i\\ &+192\,x^{12}-944\,x^{10}+2020\,x^8-2582\,x^6+2394\,x^4-1674\,x^2+594.\end{aligned}$$

Proof (i) It is obvious that $\frac{2\Phi(\phi)+y^2}{2h}=1$ holds on each periodic orbit $L_h=\{H(\phi,y)=h\}$. Then, for $i=0,1,2,3$,

$$\mathcal{I}_i(h)=\frac{1}{2h}\oint_{L_h}(2\Phi(\phi)+y^2)\phi^{2i}yd\phi=\frac{1}{2h}\left(\oint_{L_h}2\phi^{2i}\Phi(\phi)yd\phi+\oint_{L_h}\phi^{2i}y^3d\phi\right).\tag{6.29}$$

By Lemma 4.1 in [10] (for this case $k=3$ and $F(\phi)=2\phi^{2i}\Phi(\phi)$), we have

$$\oint_{L_h}2\phi^{2i}\Phi(\phi)yd\phi=\oint_{L_h}G_i(\phi)y^3d\phi,$$

where $G_i(\phi)=\frac{d}{3d\phi}(\frac{2\phi^{2i}\Phi(\phi)}{\Phi'(\phi)})=\frac{\phi^{2i}(4i\phi^4+2\phi^4-10i\phi^2-3\phi^2+6i+3)}{18(\phi-1)^2(\phi+1)^2}$. Therefore,

$$\mathcal{I}_i(h)=\frac{1}{2h}\oint_{L_h}(\phi^{2i}+G_i(\phi))y^3d\phi:=\oint_{L_h}f_i^*(\phi)y^3d\phi,\tag{6.30}$$

with $f_i^*(\phi)=\phi^{2i}+G_i(\phi)=\frac{\phi^{2i}\left(4\,i\phi^4+20\,\phi^4-10\,i\phi^2-39\,\phi^2+6\,i+21\right)}{18(\phi-1)^2(\phi+1)^2}$. Further, using $\frac{2\Phi(x)+y^2}{2h}=1$ we have

$$\begin{aligned}\mathcal{I}_i(h)=&\frac{1}{2h}\oint_{L_h}\left(\frac{2\Phi(\phi)+y^2}{2h}\right)f_i^*(\phi)y^3d\phi\\ =&\frac{1}{4h^2}\oint_{L_h}(2\Phi(\phi)+y^2)(\phi^{2i}+G_i(\phi))y^3d\phi\\ =&\frac{1}{4h^2}\oint_{L_h}2\Phi(\phi)(\phi^{2i}+G_i(\phi))y^3d\phi+\frac{1}{4h^2}\oint_{L_h}(\phi^{2i}+G_i(\phi))y^5d\phi.\end{aligned}\tag{6.31}$$

Again by Lemma 4.1 in [10] (for this case $k = 5$ and $F(\phi) = 2\Phi(\phi)(\phi^{2i} + G_i(\phi))$, we obtain

$$\oint_{L_h} 2\Phi(\phi)(\phi^{2i} + G_i(\phi))y^3 d\phi = \oint_{L_h} \widetilde{G}_i(\phi)y^5 d\phi, \tag{6.32}$$

where $\widetilde{G}_i(\phi) = \frac{d}{5d\phi}(\frac{2\Phi(\phi)(\phi^{2i}+G_i(\phi))}{\Phi'(\phi)}) = \frac{\phi^{2i}\widetilde{g}_i(\phi)}{540(\phi-1)^4(\phi+1)^4}$ with

$$\begin{aligned}\widetilde{g}_i(\phi) =&16\, i^2\phi^8 + 88\, i\phi^8 - 80\, i^2\phi^6 + 40\,\phi^8 - 380\, i\phi^6 + 148\, i^2\phi^4 - 142\,\phi^6\\ &+ 628\, i\phi^4 - 120\, i^2\phi^2 + 213\,\phi^4 - 480\, i\phi^2 + 36\, i^2 - 162\,\phi^2 + 144\, i + 63.\end{aligned}$$

Substituting (6.32) into the last equation in (6.31) completes the proof of this part.

(ii) It follows from (6.31) and (6.32) that

$$\mathcal{I}_i(h) = \frac{1}{4h^2}\oint_{L_h}(\phi^{2i} + G_i(\phi) + \widetilde{G}_i(\phi))y^5 d\phi := \oint_{L_h} f_i^{**}(\phi)y^5 d\phi, \tag{6.33}$$

where $f_i^{**}(\phi) = \frac{\phi^{2i}g_i^{**}(\phi)}{540(\phi+1)^2(\phi-1)^2}$ with

$$\begin{aligned}g_i^{**}(\phi) =&16\, i^2\phi^8 + 208\, i\phi^8 - 80\, i^2\phi^6 + 640\,\phi^8 - 920\, i\phi^6 + 148\, i^2\phi^4 - 2512\,\phi^6\\ &+ 1528\, i\phi^4 - 120\, i^2\phi^2 + 3783\phi^4 - 1140\, i\phi^2 + 36\, i^2 - 2592\,\phi^2 + 324\, i + 693.\end{aligned}$$

Differentiating both sides of the first equation in part (i) with respect to h, we have

$$2\mathcal{I}_i(h) + 2h\mathcal{I}_i'(h) = 3\oint_{L_h} f_i^*(\phi)y d\phi. \tag{6.34}$$

Similarly, differentiating both sides of the second equation in (i), we obtain

$$8h\mathcal{I}_i(h) + 4h^2\mathcal{I}_i'(h) = 5\oint_{L_h} f_i^{**}(\phi)y^3 d\phi. \tag{6.35}$$

Then further differentiating both sides of (6.35) yields

$$8\mathcal{I}_i(h) + 16h\mathcal{I}_i'(h) + 4h^2\mathcal{I}_i''(h) = 15\oint_{L_h} f_i^{**}(\phi)y d\phi. \tag{6.36}$$

Now, combining (6.26), (6.34) and (6.36), we have

$$4h^2\mathcal{I}_i'' = \oint_{L_h}(15 f_i^{**}(\phi) - 24 f_i^*(\phi) + 8\phi^{2i})y d\phi. \tag{6.37}$$

Similar to the proof for part (i), multiplying both sides of (6.37) by $2h = 2\Phi(\phi) + y^2$, and applying Lemma 4.1 in [10] (in this case $k = 3$ and $F(\phi) = 2\Phi(\phi)(15 f_i^{**}(\phi) - 24 f_i^*(\phi) + 8\phi^{2i})$), we obtain

$$8h^3\mathcal{I}_i''(h) = \oint_{L_h} f_i(\phi)y^3 d\phi = \widehat{\mathcal{I}}_i(h).$$

Lemma 6.4.5 *$\mathcal{A}''(h,\eta)$ has at most 2 zeros for $h \in (0, \frac{1}{12})$ counting multiplicity.*

Proof We have $8h^3\mathcal{A}''(h,\eta) = a_0\widehat{I_0}(h) + a_1\widehat{I_1}(h) + \widehat{I_2}(h)$ by Lemma 6.4.4. Therefore, we only need to prove that $a_0\widehat{I_0}(h) + a_1\widehat{I_1}(h) + \widehat{I_2}(h)$ has at most two zeros for any possible values of a_0 and a_1. The following proof is based on Theorem B in (Grau *et al.*, 2011). Defining

$$l_i(\phi) := \Big(\frac{f_i}{\Phi'}\Big)(\phi) - \Big(\frac{f_i}{\Phi'}\Big)(-\phi), \quad i = 0, 1, 2. \tag{6.38}$$

A direct computation of the related Wronskian on $l_i(\phi)$ ($i = 0, 1, 2$) gives

$$W[l_1] = \frac{5w_1(\phi)}{324\ (\phi+1)^7\,(\phi-1)^7\,\phi}, \quad W[l_0, l_1] = \frac{5w_2(\phi)}{17496\,\phi^5\,(\phi+1)^{13}\,(\phi-1)^{13}},$$
$$W[l_0, l_1, l_2] = \frac{5w_3(\phi)}{5668704\,\phi^6\,(\phi+1)^{22}\,(\phi-1)^{22}},$$

where

$$\begin{aligned}
w_1(\phi) &= 32\phi^{10} - 200\phi^8 + 531\phi^6 - 765\phi^4 + 621\phi^2 - 243,\\
w_2(\phi) &= 2048\phi^{18} - 19456\phi^{16} + 81632\phi^{14} - 191952\phi^{12} + 253233\phi^{10}\\
&\quad - 118125\phi^8 - 159894\phi^6 + 295974\phi^4 - 188811\phi^2 + 45927,\\
w_3(\phi) &= 47710208\phi^{35} + 135790592\phi^{34} - 845021184\phi^{33} - 2340552704\phi^{32}\\
&\quad + 7033815040\phi^{31} + 18907004928\phi^{30} - 36326408192\phi^{29}\\
&\quad - 94266368000\phi^{28} + 129089712000\phi^{27} + 320235321728\phi^{26}\\
&\quad - 329219992672\phi^{25} - 765640649760\phi^{24} + 603892207740\phi^{23}\\
&\quad + 1259178557868\phi^{22} - 754467514569\phi^{21} - 1220180873751\phi^{20}\\
&\quad + 488325017682\phi^{19} - 12046198122\phi^{18} + 280465675443\phi^{17}\\
&\quad + 2337686705781\phi^{16} - 1198522735704\phi^{15} - 4519717615224\phi^{14}\\
&\quad + 1698938411814\phi^{13} + 5135082714090\phi^{12} - 1529485387164\phi^{11}\\
&\quad - 3973076491860\phi^{10} + 954545022774\phi^9 + 2131705287306\phi^8\\
&\quad - 413339142132\phi^7 - 766638284580\phi^6 + 118314060291\phi^5\\
&\quad + 169498323981\phi^4 - 19877664870\phi^3 - 19356852690\phi^2\\
&\quad + 1432233495\phi + 859340097.
\end{aligned}$$

Applying Sturm's Theorem to $w_1(\phi)$, $w_2(\phi)$ and $w_3(\phi)$, respectively, shows that they have no roots for $\phi \in (0, 1)$. By Theorem B in (Grau *et al.*, 2011), $\{\widehat{I_0}(h), \widehat{I_1}(h), \widehat{I_2}(h)\}$ forms a Chebyshev system, and therefore $a_0\widehat{I_0}(h) + a_1\widehat{I_1}(h) + \widehat{I_2}(h)$ has at most two zeros in $(0, \frac{1}{12})$ counting the multiplicity for all possible values (a_0, a_1).

Finally, we obtain our main result.

Theorem 6.4.6 *For all possible values of* $(a_0, a_1) \in \mathbb{R}^2$*,* $\mathcal{A}(h,\eta)$ *has at most two zeros counting multiplicity for* $h \in (0, \frac{1}{12})$*, and this is the sharp bound. Hence, system* (6.2) *can have at most two periodic waves, and this bound is sharp.*

Proof First, we show by contradiction that the curve Σ defined in (6.27) is globally convex for $h \in (0, \frac{1}{12})$. Σ is locally convex for $0 < h \ll 1$ and $0 < \frac{1}{12} - h \ll 1$ by Lemmas 6.4.2 and 6.4.3. Then Σ has $2n$ inflection points with $n \geq 0$. If $n \geq 1$, then there must exist constants $a_0^\star$ and $a_1^\star$ such that the straight line, $\mathcal{L}:\ a_0 + a_1\mathcal{P} + Q = 0$, cuts Σ at least four times (counting multiplicity) for $(a_0, a_1) = (a_0^\star, a_1^\star)$. This implies that $\mathcal{A}(h, \eta)$ has at least four zeros for $h \in (0, \frac{1}{12})$ for these $a_0^\star$ and $a_1^\star$. However, noticing $\mathcal{A}(0, \eta) \equiv 0$, it follows that $\mathcal{A}''(h, \eta)$ has at least three zeros (counting multiplicity) in $(0, \frac{1}{12})$ by the mean value theorem, which contradicts Lemma 6.4.5. Therefore, $n = 0$, which implies that Σ does not have inflection points and so the curve Σ is globally convex.

Since Σ, except the h, does not contain any varying parameter coefficients, it is a globally convex curve. However, the straight line $\mathcal{L}$ can be located anywhere in the $\mathcal{P}$-Q plane by varying a_0 and a_1, so we can choose different values of (a_0, a_1) such that $\mathcal{L}$ cuts Σ for zero, one or two times, but at most twice. For $h \in (0, \frac{1}{12})$, the number of zeros of $\mathcal{A}(h, \eta) = a_0\mathcal{I}_0(h) + a_1\mathcal{I}_1(h) + \mathcal{I}_2(h)$ is the number of the intersection points of the straight line $\mathcal{L}$ given in (6.28) and the curve Σ in the $(\mathcal{P}, Q)$-plane. Therefore, $\mathcal{A}(h, \eta)$ has at most two zeros in $(0, \frac{1}{12})$ and there indeed exist $a_0^\star$ and $a_1^\star$ such that $\mathcal{L}$ intersects Σ twice.

This completes the proof.

Remark 6.4.7 (*i*) *To find one upper bound of the number of zeros of $\mathcal{A}(h, \eta)$, the Chebyshev criterion* (*Theorem B,* [10] *and Theorem A,* [29]) *cannot be directly applied to consider $\{\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2\}$, because the second Wronskian of the criterion functions with respect to $\{\mathcal{I}_i, \mathcal{I}_j\}$ having any powers of y in $\mathcal{I}_0$, $\mathcal{I}_1$ and $\mathcal{I}_2$ always has one zero for $x \in (0, 1)$, $i, j = 0, 1, 2, i \neq j$.*

(*ii*) *We have combined the algebraic method* (*Chebyshev Criterion*) *and the geometric method to carefully analyze the zero bifurcation of the Abelian integral. Theorem* 6.4.6 *also presents a new result on the bound of the number of zeros of Abelian integral, which is related to Hilbert's 16th problem.*

6.5 Concluding remarks

In this work, we examine the dynamics of the cubic-quintic nonlinear Schrödinger equation involving weak dissipative terms, which is treated as a perturbed problem. A technical step is to reduce the perturbed partial differential equations into a singularly perturbed system of ordinary differential equations, from which the Fenichel's criterion can be applied to assure the existence of an invariant manifold, and the underlying problem is then further reduced to a regularly perturbed problem on the manifold. The Abelian integral is employed in our study as the main tool to establish the existence of kink and periodic waves, in particular, their coexistence. Furthermore, we have shown that there exist at most two periodic waves with different amplitudes for the equation under specific parameter conditions. This well solves the problem imposed in our introduction part and provides deep insights into the dynamics of the underlying equation. The strategy proposed in this chapter is a sophisticated combination of algebraic and geometric methods, which is more efficient and much simpler compared to other existing methods.

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Chapter 7

Conclusion and future work

7.1 Conclusion

In this thesis, we have studied the limit cycle bifurcation in some perturbed Hamiltonian systems and the existence of solitary, kink and periodic waves in some dissipative partial differentia equations by applying Abelian integral method.

We have studied the number of limit cycles bifurcating from the periodic annuli of two quartic hyperelliptic Hamiltonian systems, which have a nilpotent-saddle loop and a heteroclinic loop connecting a cusp to a hyperbolic saddle, respectively. Both of the related full Abelian integrals are decoupled into four generating elements, which are the elementary Abelian integrals generating the full Abelian integrals. We introduce three different combinations to the four generating elements and obtain three Abelian integrals including a parameter. Then each of the new parametric Abelian integrals has three generating elements. The Chebyshev criterion is applied to each parametric Abelian integral set, which gives the parameter partition based on the number of the zeros of each parametric Abelian integral. Thus, a cubic set of the three-dimensional parameter space is obtained, only on which the original full Abelian integral may have four zeros. A further analysis excludes the possibility of four zeros. Thus, the sharp bound on the maximal number of zeros of the associated full Abelian integrals is proved to be three for both systems. Therefore, there are at most three limit cycles bifurcating from the periodic annuli for the two perturbed quartic hyperelliptic Hamiltonian systems. Using the similar idea, we investigate a quintic Hamiltonian system with a more degenerate heteroclinic loop. We have proved that the sharp bound on the maximal number of zeros of the associated Abelian integral is three. The sharp bounds we obtained in this thesis give the answer to the open questions left in previous works that whether the sharp bound is three or four. The small limit cycles bifurcating from the origin of the quintic Hamiltonian system is also investigated for smooth and piecewise smooth polynomial damping terms. The Hopf cyclicities are obtained.

We extend the theory of Abelian integrals to the study on nonlinear wave equations. In solving real world problems, some dissipative effects should be included in nonlinear wave equations due to uncertain environments and external disturbing factors, such as shears in the layer of water, wind and temperatures. Thus, the integrability is destroyed and the study on the existence of solitary, kink and periodic waves becomes much challenging. We have investigated a nonlinear BBM equation and a Schrödinger equation with weakly external disturbing factors. The weakly dissipative equations are reduced into two singularly perturbed ODE systems by applying singular perturbation theory. For each system, the normally hyperbolic property of the equilibrium set of the fast system assures the related slow manifold, on which a regular perturbation problem is formed as a planar dynamical system. By studying the Abelian integral with various techniques, the existence of solitary, kink and periodic waves as well as their coexistence are established for both the perturbed BBM equation and the Schrödinger equation.

7.2 Future work

There are many interesting yet challenging problems that remain open and need future research.

As we discussed in Chapter 2, the quartic hyperelliptic Hamiltonian system has 12 topological portraits. We only identify $\mathbb{Z}_L(4,3)$ for the periodic annulus bounded by a nilpotent saddle homoclinic loop or a heteroclinic loop connececting a hyperbolic saddle to a nilpotent cusp. $\mathbb{Z}_L(4,3)$ for other periodic annuli has not been studied. We have applied our methods developed in Chapter 2 to the unperturbed system which has a nilpotent center and a saddle loop, and obtained a smaller upper bound to be four, which is better than that obtained in previous works. However, we could not determine the exact bound, whether it is four or three, since so far only three limit cycles are obtained. Our conjecture on the sharp bound is three and needs further investigation.

Through the study on the quintic Hamiltonian system in Chapter 3, we find that the degeneracy in the system leads to similar properties like those in the quadratic elliptic Hamiltonian system, such as that the dimension of first three Abelian integrals is two because of their linear dependance. Thus, an interesting question arises: What is the annulus cyclicity of the system if the damping is a full polynomial perturbation in the form of

$$Q(x,y)\frac{\partial}{\partial y} - P(x,y)\frac{\partial}{\partial x}, \tag{7.1}$$

where $\max(\deg(P(x,y)), \deg(Q(x,y))) = n$? Is it $n-1$? This is exactly the same as that in classical work [3]. We conjecture that the cyclicity is $n-1$ when the periodic annulus is perturbed by the full polynomial perturbation. We also conjecture that the cyclicity for Liénard type damping $(\sum_{i=0}^{n} \alpha_i x)y\frac{\partial}{\partial y}$ is $\left[\frac{2n+1}{3}\right]$. These conjectures are worth for future research and may need to develop new methodologies.

We have successfully applied the Abelian integral method to study the waves in some nonlinear PDEs. There exist many models involving other type dissipative effects, such as continuous delay dissipation in Camassa–Holm equation [1]. However, the problems on the existence of periodic waves and the coexistence of solitary and periodic waves are found to be challenging. Moreover, for the non-dissipative Camassa–Holm equation, the stability of small amplitude waves has not been investigated due to the complex structure of the Hamiltonian. Our method may be combined with the results in [2, 4] to attack stability problem of such systems.

In general, the stability of waves is an interesting and important research topic that is worth for further study. We have established the existence of solitary and kink waves, however we did not discuss the stability in this thesis. Studying the stability needs to analyze the point spectrum of the related differential operators when the dissipation is considered. It is a challenging task and need further investigation.

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Curriculum Vitae

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Selected Publications

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[2] **Xianbo Sun**, Pei Yu*, Determining the parameter range for Abelian integrals of a parametric Hamiltonian to achieve Chebyshev property.