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Equivariant cohomology for 2-torus actions and torus actions with compatible involutions

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Abstract

The Borel equivariant cohomology is an algebraic invariant of topological spaces with actions of a compact group that inherits a canonical module structure over the cohomology of the classifying space of the acting group. The study of syzygies in equivariant cohomology characterize in a more general setting the torsion-freeness and freeness of these modules by topological criteria. In this thesis, we study the syzygies for elementary 2-abelian groups (or 2-tori) in equivariant cohomology with coefficients over a field of characteristic two. We approach the equivariant cohomology theory by an equivalent approach using group cohomology which will allow us to distinguish syzygies via the exactness of the Atiyah-Bredon sequence using "shifted" and "virtual" subgroups that overcome the problem of the finiteness of the acting group. We apply this characterization to study the equivariant cohomology for locally standard actions of 2-tori and also for torus actions with compatible involution, that will generalize the equivariant formality for Hamiltonian torus actions on symplectic manifolds.

Keywords: equivariant cohomology, group actions, torus actions, syzygies, group cohomology, Hamiltonian actions, Weyl invariants, manifold with corners, big polygon spaces.

Summary for Lay Audience

In algebraic topology we study properties of spaces by algebraic means; the homotopy, homology or cohomology theories are algebraic invariants of the spaces that are preserved under continuous deformations. Another approach to study the properties of spaces is throughout their symmetries or group actions, which is known as the theory of transformation groups. The equivariant cohomology is an algebraic invariant of both the topology of the space and its given group action. This algebraic invariant will become a module over a polynomial ring in several interesting situations; for example, when involutions or reflections of the space are considered. In this thesis, we characterize topological and algebraic properties of the equivariant cohomology for these particular actions, we relate this results with the progress done by circle and torus actions, and we also provide applications on symplectic manifolds and manifold with corners, that generalize the notions of polytopes and polyhedra.

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Chapter 1

Introduction

Topology studies those invariant and properties of spaces that are preserved through continuous deformations. Many of these properties can be captured by algebraic means: *homotopy groups, homology, cohomology theories* and other tools from algebraic topology carry a lot of information about the space. On the other hand, a different approach to studying invariants or properties of spaces is through its symmetries; this led to the *theory of transformation groups*; more precisely, studying actions of topological groups on spaces. To relate these two main approaches, one can use the theory of equivariant cohomology as it captures both the topology of a space X and the action of a group G on X in terms of ordinary cohomology. This theory was formally introduced by Borel [Borel, 1960] in his transformation groups seminar, yet earlier constructions involving equivariant de Rham cohomology and equivariant differential forms were given by Cartan in [Cartan, 1950]. The equivariant cohomology has many applications in different fields in mathematics; namely, there are important results in algebraic topology, algebraic geometry, transformation groups, symplectic manifolds, K-theory, index theory and cobordism theory.

Following [Borel, 1960], let G be a compact Lie group, $EG \to BG$ a universal principal bundle for G and let X be a topological space with a continuous action of G or a *G-space*. Observe that the space $EG \times X$ is homotopy equivalent to X and the diagonal action of G on it is free. The equivariant cohomology of X with coefficients in a commutative ring R, denoted by $H_G^*(X;R)$, is defined as the singular cohomology of the space $X_G = (X \times EG)/G$ (known nowadays as the Borel construction or homotopy quotient of X). Observe that the projection map $EG \times X \to EG$ induces a map $X_G \to BG$ and thus a map of rings in cohomology $H^*(BG:R) \to H_G^*(X;R)$. Hence, $H_G^*(X;R)$ becomes canonically a $H^*(BG;R)$-module.

The equivariant cohomology is natural with respect to both G and X. Namely, any map $Y \to X$ between G-spaces X and Y which preserves the action induces a map of $H^*(BG;R)$-modules $H_G^*(X;R) \to H_G^*(Y;R)$. On the other hand, for any homomorphism of Lie groups $\alpha\colon G \to K$ and any K-space X, there is an induced action of G on X and a morphism of $H^*(BK;R)$-modules $H_K^*(X;R) \to H_G^*(X;R)$.

The equivariant cohomology for spaces with an action of a compact Lie group has been studied since 1960 from different approaches, and several tools have been developed to compute it. For example, let G denote a compact connected Lie group. The G-equivariant cohomology with real coefficients of a G-space X will be a module over the graded commutative polynomial ring $H^*(BG)$ concentrated in even degrees and it will be determined by the restriction to the action on the maximal torus T in G as the following result exhibits [Hsiang, 1975, Ch.III§1 Prop.1].

Theorem 1.1 (Reduction to torus actions)**.** *Let G be a compact connected Lie group, T be a maximal torus in G with normalizer N and $W = N/T$ be the corresponding Weyl group. Let X be a G-manifold, and consider cohomology with real coefficients. Then,*

- *There is an isomorphism of $H^*(BG)$-algebras natural in X*

$$H_G^*(X) \cong H_T^*(X)^W.$$

- *There is an isomorphism of $H^*(BT)$-algebras natural in X*

$$H_T^*(X) \cong H^*(BT) \otimes_{H^*(BG)} H_G^*(X).$$

On the other hand, torus actions appear naturally in other instances in mathematics; namely, in algebraic geometry (toric varieties), symplectic geometry (Hamiltonian actions) and combinatorics (moment angle complexes). Therefore, the study of equivariant cohomology for these actions has been an ongoing trend in mathematics during the last decades.

Earlier results for torus equivariant cohomology allow us to characterize the torsion of the equivariant cohomology for nice spaces (such as manifolds or CW-complexes) in terms of the space of fixed points. More precisely, let $\Bbbk$ be a field and let X be a T-space with fixed points X^T. The kernel of the map $\varphi\colon H_T^*(X;\Bbbk) \to H_T^*(X^T;\Bbbk)$ induced by the inclusion consists of the $H^*(BT;\Bbbk)$-torsion elements of $H_T^*(X;\Bbbk)$. Therefore, $H_T^*(X;\Bbbk)$ is torsion-free if and only if the map φ is injective.

Furthermore, it has also been of great interest to distinguish T-spaces whose equivariant cohomology is a free $H^*(BT;\Bbbk)$-module. Let $\Bbbk = \mathbb{Q}$ and T be a torus of dimension n. Let X be a T-space such that $H_T^*(X;\Bbbk)$ is a free module. For $x \in X$, let T_x denote the stabilizer subgroup of x in T. Then the finite filtration $X_0 = X^T \subseteq X_1 \subseteq \cdots \subseteq X_n = X$ given by $X_k = \{x \in X : \dim(T_x) \geq n-k\}$ induces a long exact sequence [Atiyah, 1974], [Bredon et al., 1974]

$$0 \to H_T^*(X;\Bbbk) \xrightarrow{\varphi} H_T^*(X_0;\Bbbk) \to H_T^{*+1}(X_1,X_0;\Bbbk) \to \cdots \to H_T^{*+n}(X_n,X_{n-1};\Bbbk) \qquad (1.0.1)$$

In particular, the exactness of the first terms of this sequence allows one to completely determine the T-equivariant cohomology of X in terms of the two spaces X^T and X_1, and in many situations, they are easier to handle.

For example, the *GKM*-method applies this sequence to compute the T-equivariant cohomology of toric complex projective varieties with finitely many fixed points and one-dimensional orbits [Goresky et al., 1997]. Moreover, [Allday et al., 2014] showed that the exactness of the sequence (1.0.1) is equivalent to $H_T^*(X;\Bbbk)$ being a reflexive module, that is, it is isomorphic to its double $H^*(BT;\Bbbk)$-dual. Furthermore, the authors related the exactness of such a sequence to algebraic properties of modules that lies in between torsion-freeness and freeness that we will refer to syzygies and we will discuss next.

For a finitely generated module M over a polynomial ring R in n indeterminates, we say that M is a j-th syzygy if it fits in an exact sequence

$$0 \to M \to F_1 \to \cdots \to F_j$$

of finitely generated free modules F_i, $1 \leq i \leq j$. The first syzygies are the torsion-free modules and the n-th syzygies correspond to the free modules by the *Hilbert Syzygy Theorem*.
As a consequence of Theorem 1.1, the syzygies can also be characterized for compact connected Lie group actions in terms of torus actions as the next theorem exhibit.

Theorem 1.2 ([Franz, 2016, Prop.4.2]). *Let X be a G-space with an action of a compact connected Lie group G. Suppose $\Bbbk$ is a field of characteristic zero. If $T \subseteq G$ is a maximal torus in G, then the G-equivariant cohomology of X is a j-th syzygy over $H^*(BG)$ if and only if $H_T^*(X)$ also is as a module over $H^*(BT)$.*

Therefore, the algebraic properties of the equivariant cohomology are determined by the torus actions for compact connected Lie group actions and cohomology with coefficients on a field of characteristic zero. Allday, Franz and Puppe have achieved results in syzygies in equivariant cohomology for torus actions. For example, the characterization of syzygies and the equivariant Poincaré duality [Allday et al., 2014], the quotient criterion for syzygies to locally standard torus actions [Franz, 2017] and the construction of the Big polygon spaces [Franz, 2015] whose equivariant cohomology is not free but it is a syzygy of order m for $m < n/2$ where n is the rank of the acting torus; these spaces will be discussed in Section 4.4 of this document.

It is interesting to consider what happens when the ground field has positive characteristic p. When p is odd, the syzygies in equivariant cohomology for actions of a p-torus G (or elementary p-abelian groups) are described in terms of torus equivariant cohomology by restricting to the polynomial part of $H^*(BG;\Bbbk)$ isomorphic to the cohomology of the classifying space of a torus [Allday et al.,]. On the other hand, the equivariant cohomology for the case $p=2$ is still a module over a polynomial ring but now the generators are sitting in degree 1 which requires a different treatment than the torus case. However, for a T-spaces X, in this thesis we proved a reduction to the subgroup $T_2 \subseteq T$ consisting of the order 2 elements in T (See section 4.2).

Theorem 1.3 (Reduction to 2-torus actions)**.** *Let $\Bbbk$ be a field of characteristic 2 and X be a T-space with an action of a torus T. Let T_2 be the maximal 2-torus in T. Then there is an isomorphism of $H^*(BT_2)$-algebras*

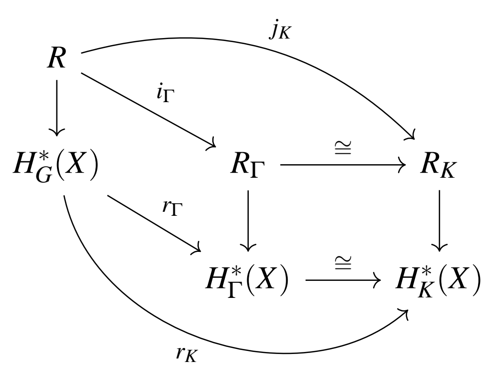
$$H^*_{T_2}(X) \cong H^*_T(X) \otimes_{H^*(BT)} H^*(BT_2)$$

and so $H^*_T(X)$ is a j-th syzygy over $H^*(BT)$ if and only if $H^*_{T_2}(X)$ is as a module over $H^*(BT_2)$.

This reduction theorem motivates a complete study of the equivariant cohomology for 2-torus actions analogous to the progress done on torus actions and p-torus actions by Allday-Franz-Puppe. The key of this reduction is that the cohomology of the classifying space BT_2 is free as a module over the cohomology of BT and it also applies for the case of p odd. As in the torus case, these particular actions appear naturally in other fields in mathematics; mainly, they will be induced by the action of the maximal 2-torus on the real locus of complex projective varieties, anti-symplectic involution on Hamiltonian symplectic manifolds and real moment angle complexes.

The main restriction of the characterization of syzygies in equivariant cohomology for 2-tori lies in three main facts; namely, the acting group is finite, the classifying space BG is not simply connected and its cohomology is not concentrated in even degrees only. To overcome all these issues, we implement an algebraic construction of the equivariant cohomology for 2-torus actions. Firstly, if G denotes a 2-torus, we use that there is a natural isomorphism of $H^*(BG)$-algebras $H_G^*(X) \cong Ext_G^*(\mathbb{k}, C^*(X))$ which arises from an isomorphism between spectral sequences. Secondly, under this algebraic approach to equivariant cohomology, there are multiplicative subgroups Γ of the group of units $\mathbb{k}[G]^\times$ which extend the notion of usual subgroups of G when $\mathbb{k}$ is infinite. These subgroups are strongly related to linear subspaces $K \subseteq H_1(BG)$, and they will allow us to overcome the issue of the finiteness of G. A Γ-equivariant cohomology $H_\Gamma^*(X)$ and a K-equivariant cohomology $H_K^*(X)$ are defined, and they are modules over $R_\Gamma = H_\Gamma^*(pt)$ and $R_K = H_K^*(pt)$ respectively. They generalize the topological construction of the Borel equivariant cohomology for 2-tori. In particular, the following theorem was proved.

Theorem 1.4. *Let G be a 2-torus of rank n, X be a G-space and $K \subseteq H_1(BG)$ be a linear subspace of G of dimension s. There is a subgroup Γ of the multiplicative units of $\mathbb{k}[G]$ of G isomorphic to a 2-torus of rank s and canonical isomorphisms $R_\Gamma \xrightarrow{\cong} R_K$ and $H_\Gamma^*(X) \xrightarrow{\cong} H_K^*(X)$ such that the diagram*



commutes (and the labelled maps are canonical).

Therefore, these constructions will imply that results from torus actions involving the *Atiyah-Bredon sequence, the equivariant homology, the equivariant Poincaré duality, the quotient criterion for syzygies* carry over to the 2-torus setting. Some details of Allday-Franz-Puppe's proof of the torus actions setting work analogously in the 2-torus case; however, there are also results that either do not carry over or need to be carefully checked.

We apply the 2-torus reduction theorem to spaces with some semi-direct product actions, motivated by the following situation appearing in a symplectic setting. Let M be a compact symplectic manifold with a Hamiltonian action of a torus T and an anti-symplectic involution $\tau\colon M \to M$ which is compatible with the torus action in the sense of that, for any $g \in T$, and $x \in M$, it holds that $g \cdot \tau(x) = \tau(g^{-1} \cdot x)$, In [Duistermaat, 1983, Thm. 3.1], it was proved that for the real locus M^τ of M and cohomology with coefficients over $\mathbb{F}_2$ there is an additive isomorphism

$$H^*(Y;\mathbb{F}_2) = \bigoplus_{i=1}^{N} H^{*-\frac{d_i}{2}}(F_i^\tau;\mathbb{F}_2). \tag{3}$$

where F_i is a connected component of M^T for $i = 1,\ldots,N$. Furthermore, in [Biss et al., 2004, Thm. A], an isomorphism in equivariant cohomology for the real locus of M was proved by Biss-Guillemin-Holm. Explicitly, the action of T on M induces an action of the maximal 2-torus on M^τ and the equivariant cohomology satisfies,

$$H^*_{T_2}(M^\tau;\mathbb{F}_2) = \bigoplus_{i=1}^{N} H_{T_2}^{*-\frac{d_i}{2}}(F_i^\tau;\mathbb{F}_2) \tag{4}$$

and an extension of this result was established in [Baird and Heydari, 2018, Thm.16] for non-abelian Lie groups. In particular, these results imply that $H^*_{T_2}(Y;\mathbb{F}_2)$ is free over $H^*(BT_2;\mathbb{F}_2)$.

To obtain further advances in this setting, in this thesis we examine whether this relation between the equivariant cohomology of a space and its real locus still holds in a more general situation relying only on the topology of the space. Indeed, in the case of a topological space X with a torus action and a compatible involution, this is equivalent to an action of the semidirect product $G = T \rtimes \langle\tau\rangle$, where τ acts on T by inversion. Therefore, we studied the equivariant cohomology $H^*_G(X;\mathbb{F}_2)$ as a module over $H^*(BG;\mathbb{F}_2)$. First of all, we showed that the cohomology ring $H^*(BG;\mathbb{F}_2)$ is canonically isomorphic to a polynomial algebra (Theorem 4.1.2) and thus we are allowed to characterize the syzygies for these particular actions. In particular, we also prove a reduction to 2-torus actions as we stated in one of our main results (Theorem 4.2.9).

Theorem 1.5. *Let X be a compact space with an action of a torus T and a compatible involution τ. Let $G = T \rtimes \langle \tau \rangle$ and $H \subseteq G$ be the subgroup consisting of all elements of order 2. Then $H^*_G(X;\mathbb{F}_2)$ is a k-th syzygy over $H^*(BG;\mathbb{F}_2)$ if and only if $H^*_H(X;\mathbb{F}_2)$ is a k-th syzygy over $H^*(BH;\mathbb{F}_2)$.*

And as a consequence, a characterization of syzygies for the real locus can be derived as we state in the following result (Theorem 4.3.5).

Theorem 1.6. *Let X be a path-connected finite-dimensional G-CW-Complex. If $H^*_G(X)$ is a j-th syzygy over $H^*(BG)$, then so is $H^*_{T_2}(X^\tau)$ as a module over $H^*(BT_2)$.*

This thesis is intended to be as self-contained as possible and it is organized as follows. In Chapter 2 we discuss the generalities of the equivariant cohomology, classical results in this area and several criteria for equivariant formality. Chapter 3 contains the main work of this thesis on the equivariant cohomology for 2-torus actions, the algebraic construction of the equivariant cohomology and the relation between shifted and virtual subgroups of 2-torus are established, also the analogous results of Allday-Franz-Puppe for the torus case are stated and proved. In Chapter 4 we study the equivariant cohomology for torus actions and a compatible involution, criteria for equivariant formality and syzygies in equivariant cohomology for spaces with these particular actions. Finally, we provide a proof for the reduction from torus actions to 2-torus actions in equivariant cohomology with $\mathbb{F}_2$-coefficients, and we present applications of these results to the syzygy order of the real locus and also using Franz's big polygon spaces to construct syzygies in equivariant cohomology for these particular actions of any order. Appendices A and B contain the background needed on fibrations, spectral sequences, regular sequences and syzygies.

Chapter 2

Generalities of equivariant cohomology

2.1 Group actions and principal G-bundles

Let X be a topological space and G be a topological group with identity element $e \in G$. We say that X is a **G-space** if there is a continuous action of G on X; that is, there is a continuous map $\cdot\colon G \times X \to X$ satisfying $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for any $g, h \in G$ and $x \in X$. A continuous map $f\colon X \to Y$ between G-spaces is a **G-map** if $f(g \cdot x) = g \cdot f(x)$ for any $x \in X$, $g \in G$. We define the **fixed point subspace** of X as the set $X^G = \{x \in X : g \cdot x = x, \ \forall g \in G\}$, and we say that the **action is trivial** if $g \cdot x = x$ for any $g \in G$, $x \in X$; or equivalently, $X^G = X$. We will denote by X/G the **orbit space** of the action, the element $G \cdot x = \{g \cdot x : g \in G\} \in X/G$ is called **the orbit of** x under G. Analogously, for a subset $A \subseteq X$, $G \cdot A = \{g \cdot a : g \in G, a \in A\}$. We say that A is **G-invariant** if $G \cdot A \subseteq A$. For $x \in X$, the **isotropy subgroup** of G at x is defined as the subgroup $G_x = \{g \in G : g \cdot x = x\}$ and we say that the **action is free** if $G_x = \{e\}$ for any $x \in X$. Observe that if the action is free, then the fixed point subspace X^G is empty.

We will assume during this document that G is a compact Hausdorff group and X is a Hausdorff space, and we will use the term "group" to denote a compact Hausdorff group unless otherwise specified. Similarly, a G-space will denote a Hausdorff topological space with an action of a group. Under our assumptions, the canonical map $\alpha_x\colon G/G_x \to G\cdot x$ is a homeomorphism. Given two G-spaces X and Y, we denote by $X\times_G Y$ where $X\times Y$ is a G- space with the diagonal action. Under extra assumptions on the group and the acting space, we have the following result [Hsiang, 1975, Thm.I.5].

Theorem 2.1.1 (Equivariant tubular neighbourhood theorem)**.** *Let G be a compact Lie group and let X be a G-space which is completely regular. For any $x\in X$ there is a subset $A\subseteq X$ such that A is G_x-invariant and there is a G-equivariant homeomorphism $G\cdot A\cong G\times_{G_x} A$ and it is a neighbourhood of the orbit space $G\cdot x$ which retracts onto it . When G is a Lie group and X is a manifold with a smooth action of G, A can be taken homeomorphic to the normal tangent space $T_xX/T_x(G\cdot x)$. In this case, $G\times_{G_x} A\to G\cdot x$ is a vector bundle.*

We say that two maps $f_0, f_1\colon X\to Y$ are G-**homotopic** if there is a homotopy $F\colon X\times I\to Y$ between f_0 and f_1 such that $F(-,t)$ is a G-map for every $t\in I$. Analogously, we say that X and Y are G-**homotopy equivalent** if the composites fg and gf are G-homotopic to the identity map for some G-maps $f\colon X\to Y$ and $g\colon Y\to X$. Let G and H be groups, X be a G-space and Y be both a G-space and a H-space. Suppose further that the actions on Y are compatible in the sense that $g\cdot(h\cdot y) = h\cdot(g\cdot y)$ for any $g\in G$, $h\in H$, $y\in Y$. Then there is a well defined action of H on the orbit space $X\times_G Y$. In particular, this construction satisfies the following properties,

Proposition 2.1.2.

1. *If $Y=\{*\}$, then $X\times_G Y = X/G$.*

2. *Let $Y=G$ and G act on Y by left multiplication. This action is compatible with the action of $H=G$ on itself given by $g\cdot g' = g'g^{-1}$. Therefore, Y is (G,G)-space and $X\times_G G$*

becomes a G-space; moreover, the projection $X \times G \to X$ induces a G-equivariant homeomorphism $X \times_G G \cong X$.

3. *Let Z be an H-space. There is a natural homeomorphism $(X \times_G Y) \times_H Z \cong X \times_G (Y \times_H Z)$.*

4. *Let Z be an H-space. Suppose that H is a subgroup of G, then $X \times_G G \times_H Z \cong X \times_H Z$. In particular, if Z is a G-homeomorphic into a point, we get that $X \times_G (G/H) \cong X/H$.*

Definition 2.1.3. A principal G-bundle $p\colon E \to B$ is a G-map between G-spaces, where G acts trivially on B and there is an open covering of B; namely, $\{U_\alpha : \alpha \in I\}$ and G-homeomorphisms $\varphi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times G$ making the diagram

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & U_\alpha & \end{array}$$

commutative, and $U_\alpha \times G$ has the action $g \cdot (b, g') = (b, g'g^{-1})$. Observe that this last condition implies that G acts freely on E and therefore $p : E \to B$ induces a homeomorphism $E/G \cong B$.

Recall that the principal G-bundles are in particular fiber bundles and hence fibrations.

Remark 2.1.4.

- Let X be a space, $p\colon E \to B$ be a principal G-bundle and let $f\colon X \to B$ be a continuous map. Then the pullback $f^*p\colon f^*E \to X$ is a principal G-bundle over X where f^*p is the projection on the second factor. Moreover, any two homotopic maps give rise to G-homeomorphic pullback bundles.

- Let $p\colon E \to B$ be a principal G-bundle and let X be a G-space. Then the G-equivariant map $E \times X \to E \times \{*\}$ descends into a map $q\colon E \times_G X \to E \times_G \{*\} \cong B$ which is a fiber bundle with fiber X and structure group G. This fiber bundle is called the associated bundle to $p : E \to B$ with fiber X.

- Let G be a Lie group and H be a closed subgroup. In this case the projection $G \to G/H$ is a principal H-bundle. In particular, for any principal H-bundle $p\colon E \to B$, the associated bundle $E \times_H G \to B$ is a principal G-bundle. On the other hand, for any principal G-bundle $p'\colon E' \to B'$, the quotient map $q : E' \to E'/H$ is a principal H-bundle and the map $E'/H \to B'$ is a fiber bundle with fiber G/H.

2.2 Classifying spaces

The contents of this section are classical and there are many sources where these topics are discussed; as a preferred reference, the results cited in this section can be found in [tom Dieck, 2008, §14.4]. We will also refer to the theory of fibrations and spectral sequences discussed in Appendix A. Let $[X,B]$ denote the set of maps between X and B up to homotopy, and $\mathcal{P}_G(X)$ the set of principal G-bundles over X up to G-homeomorphism; there is a classical result relating these two sets just defined.

Proposition 2.2.1. *Let $p : E \to B$ be a principal G-bundle with E contractible and let X be a CW-complex. Then any $E' \in \mathcal{P}_G(X)$ is of the form f^*E for a unique $f \in [X,B]$.*

A principal G-bundle satisfying this condition is called a **universal G-bundle**, and B is called the **classifying space of G**. Moreover, if G is a CW-complex with cellular operations and such a bundle exists, E and B can be taken to be CW-complexes. B is unique up to homotopy equivalence and E is unique up to G-homotopy equivalence. In [Milnor, 1956] it is shown that the universal bundle $EG \to BG$ exists and it is unique up to G-homotopy equivalence for any topological group G. The most commonly known examples of classifying spaces are the following.

Example 2.2.2.

1. Let $G = \mathbb{Z}$. Since G acts freely and properly discontinuously on $\mathbb{R}$ by translation, we get that $E\mathbb{Z} = \mathbb{R}$ and $B\mathbb{Z} = S^1$.

2. Let $G = S^1$. Let $S^\infty = \bigcup_{n \geq 0} S^{2n+1}$ be the colimit of the complex spheres. S^∞ is contractible and admits a free action of S^1 which arises from the scalar multiplication of S^1 in $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ for every $n \geq 0$. We obtain then that $ES^1 = S^\infty$ and $BS^1 = \mathbb{C}P^\infty$ is the infinite dimensional complex projective space. On the other hand, if we identify $\mathbb{Z}/p$ as the subgroup of S^1 consisting of the pth-roots of unity for $p \geq 2$, we get that $E\mathbb{Z}/p = S^1$, $B\mathbb{Z}/2 = \mathbb{R}P^\infty$ is the infinite dimensional real projective space and $B\mathbb{Z}/p = L_p^\infty$ is the infinite dimensional p-lens space for $p > 2$.

More generally, if G is a discrete group, then $BG = K(G,1)$ is the Eilenberg-Maclane space and EG is its universal covering.

Remark 2.2.3. Let $\alpha\colon H \to G$ be a continuous group homomorphism. First notice that the homomorphism α induces on G a structure of an H-space. For the universal H bundle $EH \to BH$, there is an associated bundle $EH \times_H G \to BH$ which is a principal G-bundle. Therefore, from Theorem 2.2.1, there is a unique map (up to homotopy) $B\alpha\colon BH \to BG$ that classifies the principal G bundle $EH \times_H G \to BH$. This construction is functorial with respect to group continuous homomorphism; namely, $B(id_G) = id_{BG}$ for the identity map $id_G\colon G \to G$, and for a composite $K \xrightarrow{\beta} H \xrightarrow{\alpha} G$ we have that $B(\alpha \circ \beta) = B(\alpha) \circ B(\beta)$. This follows from the isomorphisms of principal G-bundles $EG \times_G G \cong EG$ and $EK \times_K H \times_H G \cong EK \times_K G$ respectively.

In particular, for a closed subgroup $H \subseteq G$, there is a characterization of the canonical map $Bi\colon BH \to BG$ induced by the inclusion $i\colon H \to G$ given by the following result [tom Dieck, 2008, pp.348].

Proposition 2.2.4. *Let G be a compact Lie group and $H \subseteq G$ be a closed subgroup. In the sequence*

$$H \xrightarrow{i} G \xrightarrow{\pi} G/H \xrightarrow{j} BH \xrightarrow{Bi} BG,$$

each subsequence consisting of three consecutive objects is a fibration.

As we will see later, many fibrations that we will be interested come as a pullback or as an associated bundle of principal G-bundles where the base space is a classifying space. In this case, the action of the fundamental group of the base space on the cohomology of the fiber, as introduced in Appendix A, can be explicitly described.

Proposition 2.2.5. *Let G be a topological group, F be a left G-space and $EG \xrightarrow{\pi} BG$ be the universal principal G-bundle. Then the action of $\pi_1(BG)$ on the cohomology of F in the associated fiber bundle $F \to EG \times_G F \to BG$ is determined by the induced action of G on $H^*(F)$.*

Remark 2.2.6. Let now N be a closed normal subgroup of G, $K = G/N$ be the quotient group and $\beta : G \to K$ be the quotient map. Any action of G on a space X induces a canonical action of N on X. Moreover, K acts on the orbit space X/N via $k \cdot \overline{x} = \overline{gx}$ where g is such that $\beta(g) = k$. The action is well defined; namely, if h is such that $\beta(h) = k$, then $gh^{-1} \in \ker(\beta) = \mathrm{Im}(\alpha) = N$ and we get then $\overline{gx} = \overline{nhx} = \overline{hx}$. Furthermore, it can be shown that there is a homeomorphism $X/G \cong (X/N)/K$ via the map $[x]_G \mapsto [\overline{x}]_K$.

We will use this remark to show that there is a strong relation between group extensions and fibrations involving the classifying spaces, as it is illustrated in the following result [tom Dieck, 2008, pp.349].

Theorem 2.2.7. *Let $1 \to N \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ be a group extension of topological groups where N is closed in G. Then there is a fibration*

$$BN \to BG \to BK.$$

Moreover, if the sequence is split, the action of $\pi_1(BK)$ on BN is induced by the canonical action of K on N given by the isomorphism of G with the semidirect product $N \rtimes K$.

Remark 2.2.8. Suppose that there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N' & \xrightarrow{\alpha'} & G' & \longrightarrow & K' & \longrightarrow & 1 \end{array}$$

where the rows are group extensions. By naturality of the classifying space construction, we get a commutative diagram

$$\begin{array}{ccccc} BN & \longrightarrow & BG & \longrightarrow & BK \\ \downarrow & & \downarrow & & \downarrow \\ BN' & \longrightarrow & BG' & \longrightarrow & BK' \end{array}$$

between the classifying spaces and thus the vertical arrows induce a map of fibrations. In particular, we get a map of spectral sequences.

We finish this section with this important example.

Example 2.2.9. From Example 2.2.2(2) we have seen that $B\mathbb{Z}/2 = \mathbb{R}P^\infty$ and $BS^1 = \mathbb{C}P^\infty$. From Proposition 2.2.4 the inclusion $\mathbb{Z}/2 = \{\pm 1\} \to S^1$ induces a fibration

$$S^1/(\mathbb{Z}/2) \cong S^1 \to B\mathbb{Z}/2 \to BS^1.$$

We are going to compute explicitly the map $p\colon \mathbb{R}P^\infty \to \mathbb{C}P^\infty$ and the induced map in cohomology $p^*\colon H^*(\mathbb{C}P^\infty;\mathbb{F}_2) \to H^*(\mathbb{R}P^\infty;\mathbb{F}_2)$. Since $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[t]$ and $H^*(\mathbb{C}P^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[c]$ where $\deg(t) = 1$ and $\deg(c) = 2$ it is enough to determine the image of the generator c under p^*. Using the Gysin long exact sequence (Corollary A.10), we have a short exact sequence

$$H^2(BS^1) \xrightarrow{p^*} H^2(B\mathbb{Z}/2) \to H^1(BS^1).$$

Since $H^1(BS^1) = 0$ as this cohomology is concentrated in even degree, we have that the map $p^*\colon H^2(BS^1) \to H^2(B\mathbb{Z}/2)$ is an isomorphism and so $p^*(c) = t^2$.

2.3 The Borel construction

Let G denote a compact Hausdorff topological group (or just a group in our notation). From this point on, for a G-space X we will mean a topological space X which has the G-homotopy type of a G-CW-complex. Following [Allday and Puppe, 1993, §1.1], a G-CW-complex X is a colimit of spaces X_n obtained inductively as follows: Let $X_{-1} = \emptyset$. For each $n \in \mathbb{N}$ there is a set of indices I_n, a set of closed subgroups $\{H_i : i \in I_n\}$ and G-equivariant maps $f_i\colon G/H_i \times S^{n-1} \to X_{n-1}$ such that X_n is the pushout of the diagram

$$\begin{array}{ccc} \bigsqcup_{i\in I_n} G/H_i \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \bigsqcup_{i\in I_n} G/H_i \times D^n & \longrightarrow & X_n \end{array}$$

Here D^n is considered as a trivial G-space (and so S^{n-1}). If G is discrete, this definition is equivalent to a cellular action on a CW-complex.

Borel introduced the notion of equivariant cohomology as an extension of the ordinary cohomology for G-spaces [Borel, 1960] which deals with both the topology and the nature of the action. In this section we review his construction, known nowadays as the Borel construction, and we will discuss its main properties. We start with the following definition.

Definition 2.3.1. For a G-space X the **Borel construction** of X is defined as the quotient space $X_G = (EG \times X)/X$ where G acts on $EG \times X$ diagonally.

Observe that the Borel construction is well defined (up to homotopy) and it is independent of the choice of the model for EG.

Example 2.3.2.

- If $X = pt$ we get $pt_G = EG \times_G pt = BG$.
- Assume that G acts trivially on X, then $X_G \cong EG/G \times X = BG \times X$.

Important properties of the Borel construction are given in the following proposition (see [Hausmann, 2014, §7.3] for a reference of the following results).

Proposition 2.3.3.

(i) *For a G-map* $f\colon X \to Y$ *between G-spaces, there is an induced map* $f_G\colon X_G \to Y_G$ *between the respective Borel construction of* X *and* Y*. Furthermore, if* X *is G-homotopy equivalent to* Y*, then* X_G *is homotopy equivalent to* Y_G*.*

(ii) *The constant map* $X \to pt$ *induces a map* $p\colon X_G \to BG$ *which is a fiber bundle with fiber* X*; the map* p *coincides with the map induced by the projection* $EG \times X \to EG$*. If the action has a fixed point* $x \in X$*, the map* p *admits a section* $s_x\colon BG \to X_G$*. Furthermore, for any G-map* $f\colon X \to Y$ *the diagram*

$$\begin{array}{ccc} X_G & \xrightarrow{\quad f_G \quad} & Y_G \\ & {}_{p}\searrow \quad \swarrow_{p} & \\ & BG & \end{array}$$

is commutative.

(iii) *The quotient map* $EG \times X \to X_G$ *is a principal G-bundle which coincides with the pullback of the map* $p\colon X_G \to BG$ *over the universal principal bundle* $EG \to BG$*.*

(iv) *For any group homomorphism* $\phi\colon G \to G'$ *and any* G'*-space* X*, there is a map of fibrations*

$$\begin{array}{ccc} X_G & \longrightarrow & X_{G'} \\ \downarrow & & \downarrow \\ BG & \longrightarrow & BG' \end{array}$$

where the action of G *on* X *is the one induced by* ϕ*.*

Definition 2.3.4 (Equivariant cohomology)**.** Let X be a G-space. The G**-equivariant cohomology** of X with coefficients in a field $\Bbbk$, denoted by $H^*_G(X;\Bbbk)$, is defined as the singular cohomology of the Borel construction of X; that is,

$$H^*_G(X;\Bbbk) = H^*(X_G;\Bbbk).$$

Observe that for a trivial G-space X, $H^*_G(X;\Bbbk) \cong H^*(BG;\Bbbk) \otimes H^*(X;\Bbbk)$.

Proposition 2.3.5. *Let X be a G-space.*

(i) If G acts freely on X, $H^*_G(X;\Bbbk) \cong H^*(X/G;\Bbbk)$. Moreover, if X is taken to be a G-CW-complex, there is a homotopy equivalence $X_G \simeq X/G$.

(ii) Suppose that $\mathrm{char}(\Bbbk) = 0$ *and G acts locally freely on X that is, the stabilizer group G_x is finite for any $x \in X$. Then $H^*_G(X;\Bbbk) \cong H^*(X/G;\Bbbk)$.*

From Proposition 2.3.3 we have that any G-map $f\colon X \to Y$ induces a map in G-equivariant cohomology $f^*_G\colon H^*_G(Y;\Bbbk) \to H^*_G(X;\Bbbk)$. In particular, the canonical map $p\colon X \to pt$ induces a map of rings $p^*\colon H^*(BG;\Bbbk) \to H^*_G(X;\Bbbk)$. Therefore, the G-equivariant cohomology inherits a canonical $H^*(BG;\Bbbk)$-module structure and thus $H^*_G(X;\Bbbk)$ is a graded $H^*(BG;\Bbbk)$-module.

The assumption of $\dim_{\Bbbk} H^*(X;\Bbbk) < \infty$ is considered so the equivariant cohomology is a finitely generated module in several interesting cases as we state in the following proposition (see [Allday and Puppe, 1993, Prop 3.10.1]).

Proposition 2.3.6. *Let $\Bbbk$ be a field and let X be a G-space such that* $\dim_{\Bbbk} H^*(X;\Bbbk) < \infty$.

(a) If $G = T$ is a torus then $H^*_T(X;\Bbbk)$ is a finitely generated $H^*(BT;\Bbbk)$-module.

(b) If $G = (\mathbb{Z}/p)^n$ and $\Bbbk$ is a field of characteristic p, then $H^*_G(X;\Bbbk)$ is a finitely generated $H^*_G(X;\Bbbk)$-module.

Remark 2.3.7. The equivariant cohomology can be also defined for topological pairs (X,Y) where Y is an invariant subspace of X. The introduction of the equivariant cohomology for these pairs is useful for proving results as in the ordinary cohomological setting. For example, an equivariant version of the Mayer-Vietoris sequence, the Tautness property and the excision theorem arises from the ordinary version. More precisely, we have the following properties for the equivariant cohomology.

- **Excision:** Let $A,B \subseteq X$ be G-invariant subsets of a G-space X such that $int(A) \cup int(B) = X$. Then there is an isomorphism $H^*_G(A, A\cap B) \cong H^*_G(X,B)$ induced by the inclusion $(A, A\cap B) \to (X,B)$.

- **Mayer-Vietoris:** Let (X,Y) be an equivariant pair and suppose that $X = int(A) \cup int(B)$ and $Y = int(C) \cup int(D)$ for some subspaces $A,B \subseteq X$ and $C,D \subseteq Y$. There is a long exact sequence
$$\cdots \to H^{n-1}_G(A\cap B, C\cap D) \to H^n_G(X,Y) \to H^n_G(A,C) \oplus H^n_G(B,D) \to H^{n+1}_G(X,Y) \to \cdots$$

- **Tautness:** Let $Y \subseteq X$ be a taut subspace[1]. Then $H^*_G(X,Y) \cong \lim_U H^*(X,U)$ where U runs over all invariant neighbourhoods of Y in X.

2.4 The localization theorem

The localization theorem is a powerful tool in equivariant cohomology that has no analogue in the setting of ordinary singular cohomology. It relates the equivariant cohomology of a space with the equivariant cohomology of its fixed point subspace. The first approach of this result is due to [Borel, 1960, Ch.IV] and the reference used for the results cited in this section is [Mukherjee, 2005, §1.3].

[1] A subspace Y is taut in X if the canonical map $\lim_{Y\subseteq U} H^*(U) \to H^*(Y)$ is an isomorphism, the limit is taken over all the neighbourhoods of Y in X. If Y is compact, or Y is closed and X paracompact, then Y is taut in X

Definition 2.4.1. Let X be a G-space and $x \in X$. The inclusion $G_x \to G$ of the stabilizer subgroup induces a map $j_x\colon R \to R_x$ where $R = H^*(BG;\Bbbk)$ and $R_x = H^*(BG_x;\Bbbk)$. If $S \subseteq R$ is a multiplicative subset, we define the **subspace of cohomological S-fixed points** as the set

$$X^S = \{x \in X : S \cap \ker(j_x^*) = \emptyset\}.$$

Let $f \in H^*(BG;\Bbbk)$ and set $X^f = \{x \in X : j_x^*(f) = 0\}$. Notice that $X \setminus X^f$ is open by the equivariant tubular neighbourhood theorem (Theorem 2.1.1); therefore, X^S is closed since $X^S = \bigcap_{s \in S} X^s$. Moreover, X^S is also a G-invariant subspace of X as $j_x = j_{g \cdot x}$. The next example illustrates why this subspace is called the cohomological S-fixed point set.

Example 2.4.2. Consider $S = R \setminus \{0\}$ and assume $X^G \neq \emptyset$. For any $x \in X^G$, j_x is an isomorphism since $G_x = G$ and the map $p^*\colon H^*(BG;\Bbbk) \to H_G^*(X;\Bbbk)$ is injective by Proposition 2.3.3-(ii). This implies that $x \in X^S$ and thus X^S contains all fixed points of X under the action of G.

In general, X^S could contain more points than fixed points, but in specific cases, these spaces coincide as we illustrate in the following proposition.

Proposition 2.4.3. *Let $G = (S^1)^n$ be an n-dimensional torus , X be a G-space and assume that* $\mathrm{char}(\Bbbk) = 0$. *For any subtorus $K \subseteq G$, let $PK \subseteq R = H^*(BG;\Bbbk)$ be the prime ideal* $\ker(j_K^*\colon R \to H^*(BK;\Bbbk))$ *where j_K^* is the map induced by the inclusion $K \to G$. Then $X^S = X^K$ where $S = R \setminus PK$.*

In particular, from this proposition, when $K = G$ we obtain $X^S = X^G$ where $S = R \setminus \{0\}$.

For any R-module M, $S^{-1}M$ denotes the localization with respect to S, it is defined as the set of pairs $(s,m) \in S \times M$ under the equivalence relation $(s,m) \sim (s',m')$ if and only if there is a $t \in S$ such that $tsm' = ts'm$. The localization $S^{-1}M$ inherits a $S^{-1}R$-module structure and it is isomorphic to $S^{-1}R \otimes_R M$. With this notation, we can state the most general version of the localization theorem for singular cohomology.

Theorem 2.4.4 (Localization theorem)**.** *Let G be a compact Lie group and X be a compact G-CW complex. Set R and $S \subseteq R$ as above. Then the inclusion map $X^S \to X$ induces a map $H^*_G(X;\Bbbk) \to H^*_G(X^S;\Bbbk)$ which becomes an isomorphism after localization by S, that is, the map*

$$S^{-1}H^*_G(X;\Bbbk) \to S^{-1}H^*_G(X^S;\Bbbk)$$

is an isomorphism.

There are many sources where a proof of this version of the localization theorem can be found. See, for instance, [Hsiang, 1975, Thm.III.1], [tom Dieck, 1987, Ch.III.3] or [Allday and Puppe, 1993, Thm.3.1.6].

Now we will discuss some consequences of the localization theorem. Notice that the fact that $r\colon H^*_G(X;\Bbbk) \to H^*_G(X^S;\Bbbk)$ is an isomorphism after localization under S we get a characterization on the kernel and image of this map. In fact, if $x \in \ker(r)$ there is an element $s \in S$ such that $s \cdot x = 0$, that is $\ker(r)$ is S-torsion. On the other hand, for any $y \in H^*_G(X^G;\Bbbk)$ there is an element $t \in S$ such that $t \cdot y \in \operatorname{Im}(r)$.

Also, there is a concrete version of the localization theorem for torus actions and finite cyclic group actions.

Theorem 2.4.5 (Localization theorem for p-torus)**.** *Let $G = (S^1)^n$ or $(\mathbb{Z}/p)^n$ for some $n \geq 1$. Let X be a compact G-CW-complex and denote by $R = H^*(BG;\Bbbk)$ and $S = R \setminus \{0\}$. Then the inclusion $X^G \to X$ induces an isomorphism*

$$S^{-1}H^*_G(X;\Bbbk) \to S^{-1}H^*_G(X^G;\Bbbk).$$

of $S^{-1}R$-modules where the cohomology is considered with coefficients in a field of characteristic zero if $G = (S^1)^n$ or characteristic p if $G = (\mathbb{Z}/p)^n$.

In particular, we obtain that $\ker(\varphi\colon H^*_G(X;\Bbbk) \to H^*_G(X^G;\Bbbk))$ is the R-submodule consisting of torsion elements. Therefore, we can state the following corollary with the same assumptions as those in Theorem 2.4.5.

Corollary 2.4.6.

- *$X^G \neq \emptyset$ if and only if $H^*_G(X;\Bbbk)$ is not a torsion $H^*(BG)$-module.*
- *The canonical map $H^*_G(X) \to H^*_G(X^G)$ is injective if and only if $H^*_G(X)$ is a torsion-free $H^*(BG)$-module.*

The assumptions on X are necessary for the theorem to be true; for example, let $G = S^1$ act freely on $X = EG \cong \mathbb{C}P^\infty$ with the canonical action. Then $X^G = \emptyset$, $H^*_G(X;\Bbbk) \cong H^*(BG;\Bbbk)$ and the inclusion $X^G \to X$ induces the zero map in cohomology.

2.5 Equivariant formality

In this section, we introduce the notion of equivariant formality that intrinsically relates the $H^*(BG;\Bbbk)$-module structure of $H^*_G(X;\Bbbk)$ to the geometric and topological properties of the space and its fixed point subspace. These results are classical in the development of the theory of equivariant cohomology and the first approaches to this notion are due to [Borel, 1960]. However, the use of the term *equivariant formality* has its first appearance in [Goresky et al., 1997] although other authors rather use the terminology *totally non-homologous to zero* or *cohomology extension of the fiber* to avoid confusion with the concept of *formal spaces* in rational homotopy theory, since the equivariant formality is not the equivariant extension of this notion. We will also assume that EG is path-connected (e.g if G is a CW-complex).

Definition 2.5.1. Let X be a G-space. We say that X is **G-equivariantly formal** over $\Bbbk$, or X has a formal action of G (if the ground field $\Bbbk$ is understood) if the map

$$r_G\colon H^*_G(X;\Bbbk) \to H^*(X;\Bbbk)$$

is surjective, where r is the map induced in cohomology by the inclusion of the fiber in the fiber bundle $X \to X_G \to BG$.

Notice that if $g \in G$ and $l_g \colon X \to X$ denotes the map $l_g(x) = g \cdot x$, then for any $z \in EG$ we have that $i_z \circ l_g = i_{g \cdot z}$ where i_z denotes the inclusion of the fiber $i_z \colon X \to X_G$; that is, it is the map given by $i_z(x) = [z,x] \in EG \times_G X$. In particular, for any $a \in H^*_G(X;\mathbb{k})$, we obtain that $l_g^* r(a) = l_g^* i_z^*(a) = (i_z l_g)^*(a) = (i_{g\cdot z})^*(a) = r(a)$. Therefore, the image of r is contained in the G-invariant elements of the cohomology of the fiber, that is $\mathrm{Im}(r) \subseteq H^*(X;\mathbb{k})^G$. From Proposition 2.2.4, the action of $\pi_1(BG)$ on the cohomology of the fiber coincides with the above and if the map r is surjective, then every element is invariant under this action. So we can summarize this fact in the following consequence of equivariant formality.

Proposition 2.5.2. *Let X be a G-space with a formal action of G. In the fiber bundle $X \to X_G \to BG$, $\pi_1(BG)$ acts trivially on the cohomology of the fiber.* □

If X is G-equivariantly formal, or equivalently, the map r_G admits a section, from the Leray-Hirsch theorem (Theorem A.8) we have an isomorphism

$$H^*_G(X;\mathbb{k}) \cong H^*(BG;\mathbb{k}) \otimes H^*(X;\mathbb{k})$$

of $H^*(BG;\mathbb{k})$-modules. In particular, if $H^*(X;\mathbb{k})$ is finitely generated, we have that $H^*_G(X;\mathbb{k})$ is a free $H^*(BG;\mathbb{k})$-module. We can now state the following characterization of equivariant formality using the theory of spectral sequences discussed in Appendix A (Theorems A.11 and A.12).

Proposition 2.5.3. *Let G be a compact Lie group and X be a G-space such that* $\dim_{\mathbb{k}}(X,\mathbb{k}) < \infty$. *The following are equivalent.*

(a) X is G-equivariantly formal.

(b) In the fibration $X \to X_G \to BG$, $\pi_1(BG)$ acts trivially on the cohomology of the fiber and the spectral sequence degenerates at E_2.

(c) The action of $\pi_1(BG)$ on the cohomology of the fiber is trivial and $H^*_G(X;\Bbbk)$ is a free $H^*(BG;\Bbbk)$-module.

The localization theorem can be used to find a useful criterion to determine whether or not a G-space is equivariantly formal by relating the singular cohomology of the space with the one of the fixed point subspace. First we recall the following topological invariant of a space X.

Definition 2.5.4. Let X be a topological space. Define the ***i*-th Betti number** of X as the integer $b_i(X) = \dim_{\Bbbk} H^i(X;\Bbbk)$, and the **Betti sum** of X as

$$b(X) = \sum_i b_i(X) \in \mathbb{N} \cup \{\infty\}.$$

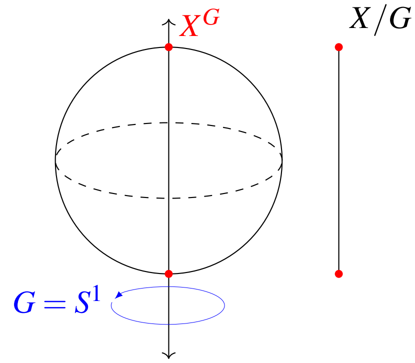
The following result ([Allday and Puppe, 1993, Thm.3.10.4]) is extremely useful to characterize equivariant formality for some particular group actions.

Proposition 2.5.5 (Criterion for equivariant formality)**.** *Let X be a compact G-CW-complex space where G is a torus of rank n (resp. p-torus of rank n with p any prime), and let $\Bbbk = \mathbb{Q}$ (resp. $\Bbbk = \mathbb{F}_p$). X is G-equivariantly formal over $\Bbbk$ if and only if $b(X) = b(X^G)$.*

As a corollary of this characterization and the localization theorem, we have the following criterion for equivariant formality for torus and p-torus actions and $\Bbbk = \mathbb{Q}$ or $\Bbbk = \mathbb{F}_p$ respectively.

Corollary 2.5.6. *Let X be a G-space such that $b(X) < \infty$. Then X is G-equivariantly formal if and only if $H^*_G(X;\Bbbk) = H^*(BG;\Bbbk) \otimes H^*(X)$ as $H^*(BG;\Bbbk)$-modules.*

Example 2.5.7. Let $G = S^1$ and $X = S^2$. Consider the action of G on X given by the rotation along the vertical axis. Thus X^G consists of two points: the north and the south pole.



Notice that the orbit space X/G is contractible and thus it does not provide any information about either the topology of the space or the nature of the action. On the other hand, by considering the equivariant cohomology $H^*_G(X)$, we can see that it is a free $H^*(BG)$-module as $b(X^G) = 2 = b(X)$, and from the criterion for equivariant formality we get an isomorphism of $H^*(BG)$-modules.

$$H^*_{S^1}(S^2;\mathbb{Q}) \cong H^*(BS^1;\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(S^2;\mathbb{Q}).$$

Remark 2.5.8. The equivariant formality property is inherited by restriction to subgroups. Let X be a G-space and $H \subseteq G$ be a closed subgroup. Notice that X becomes an H-space by restriction of the G-action. Since the restriction map r_G factors through r_H, if X is G-equivariantly formal, then X is H-equivariantly formal by looking at the surjectivity of the restriction maps.

Chapter 3

Equivariant cohomology for 2-torus actions

3.1 Algebraic construction of the equivariant cohomology

In this section, we define the equivariant homology and cohomology for 2-torus action analogous to the case of torus actions [Allday et al., 2014]. Introducing the equivariant homology will lead to an equivariant extension of the Poincaré duality for spaces that satisfies this property at the ordinary cohomology level. To achieve this, we need to use a purely algebraic construction of the equivariant cohomology using the normalized singular chain complex of a G-space as a G-module. The approach that we use is from the group cohomology point of view and it was also discussed in [Allday and Puppe, 1993, Ch.I §1.2-1.3]. Recall that by convention, a chain complex C_* is considered as a cochain complex negatively graded, so the differential on C_* is a map of degree 1. We also denote the cohomology of the complex C^* by $H^*(C)$.

We first will quickly review some fundamental facts of group cohomology; a preferred reference is the classical book [Brown, 1982]. Let $\Bbbk$ denote a field and G be a finite group. Let $P_* \to$

$\Bbbk$ be a projective resolution of $\Bbbk$ as $\Bbbk[G]$-module. For any $\Bbbk[G]$-module M we define the **group cohomology of G with coefficients in M** as the cohomology of the cochain complex $\mathrm{Hom}_{\Bbbk[G]}(P_*, M)$; that is

$$H^*_{grp}(G;M) = H^*(\mathrm{Hom}_{\Bbbk[G]}(P_*, M)) = \mathrm{Ext}^*_{\Bbbk[G]}(\Bbbk, M)$$

Recall that this is well defined (does not depend on the chosen resolution) as any two projective resolutions are chain homotopic. Group cohomology carries a cup product in the following sense. Given two $\Bbbk[G]$-modules M, N there is a map

$$H^p(G,M) \otimes H^q(G,N) \xrightarrow{\cup} H^{p+q}(G, M \otimes N)$$

which is associative, graded commutative and natural with respect to $\Bbbk[G]$-maps [Brown, 1982, Ch. V §2]. If we set $M = N = \Bbbk$, there is an isomorphism of graded rings

$$H^*_{grp}(G, \Bbbk) \cong H^*(BG; \Bbbk)$$

between the group cohomology (with the cup product defined above) and the singular cohomology of the classifying space [Adem and Milgram, 2013, Ch.II.Thm.4.4]

Group cohomology can be also defined with coefficients over a cochain complex M^* of $\Bbbk[G]$-modules as follows. Given a projective resolution $P_* \to \Bbbk$ we consider the double complex $D_{*,*} = \mathrm{Hom}_{\Bbbk[G]}(P_*, M^*)$ with differential $d(f) = d_M \circ f - (-1)^{|f|} f \circ d_P$ where $|f|$ denotes the degree of f. We define the **cohomology of G with coefficients in M^*** (some authors define it as the hypercohomology of G) as the cohomology of the total complex $Tot(D_{*,*})$, that is

$$\mathbb{H}(G, M^*) = H^*(\mathrm{Hom}_{\Bbbk[G]}(P_*, M^*)).$$

The hypercohomology is natural with respect to M^*; namely, for any chain map $f\colon M^* \to N^*$ there is an induced map $f^*\colon \mathbb{H}^*(G;M^*) \to \mathbb{H}^*(G;N^*)$. Moreover, if f is a weak equivalence, then there is an isomorphism $\mathbb{H}^*(G,M^*) \cong \mathbb{H}^*(G,N^*)$. In particular, if M^* has a $\Bbbk[G]$-differential graded algebra structure with a map $m\colon M^* \otimes M^* \to M^*$, then the cohomology $\mathbb{H}^*(G,M^*)$ inherits a graded algebra structure by composing the cup product with the multiplication map of M^*

$$\mathbb{H}^*(G,M^*)\otimes\mathbb{H}^*(G,M^*)\xrightarrow{\cup}\mathbb{H}^*(G,M^*\otimes M^*)\xrightarrow{m^*}\mathbb{H}^*(G,M^*).$$

In some cases, the hypercohomology can be explicitly described as the next result shows.

Proposition 3.1.1.

- *Let M^* be a complex of trivial $\Bbbk[G]$-modules. Then $\mathbb{H}^*(G,M^*)\cong H^*_{grp}(G)\otimes H^*(M)$.*
- *Let M^* be a complex of free $\Bbbk[G]$-modules. Then $\mathbb{H}^*(G,M^*)\cong H^*(M^G)$ where M^G denotes the complex $M^G=\mathrm{Hom}_{\Bbbk[G]}(\Bbbk,M^*)$.*

Proof. Following [Brown, 1982, Ch.VII.§5], for the first assertion, let P_* be a free resolution of $\Bbbk$ as $\Bbbk[G]$-module and denote by $P_G=P_*\otimes_G\Bbbk$. By definition, we have that $H_*(P_G)$ is the group homology $H_*(G)$ of G . Now If M^* is a complex of trivial $\Bbbk[G]$-modules, there is an isomorphism of double complexes $\mathrm{Hom}_G(P_*,M^*)\cong\mathrm{Hom}_{\Bbbk}(P_G,M^*)$. Using the universal coefficient theorem, we have an isomorphism $\mathbb{H}^*(G,M^*)\cong\mathrm{Hom}(H_*(G),H^*(M))$. The statement follows from the fact that the $\Bbbk$-dual of the group homology is isomorphic to the group cohomology for finite G.

To prove the second assertion, We start by filtering the double complex $\mathrm{Hom}_G(P_*,M^*)$ by column degree, and we have a convergent spectral sequence

$$E_1^{p,q}=H^q(\mathrm{Hom}_{\Bbbk[G]}(P_*,M^p))\Rightarrow\mathbb{H}^*(G,M^*).$$

Notice that $E_1^{p,q}=H^q_{grp}(G,M^p)$, and as each M^p is a free $\Bbbk[G]$-module by assumption, we can write $M^p=\Bbbk[G]\otimes V_p=\mathrm{Ind}^G_{\{e\}}V_p$ for some $\Bbbk$-vector space[1]. From the Shapiro's Lemma in group cohomology, and since $\mathrm{Ind}^G_{\{e\}}V_p\cong\mathrm{Coind}^G_{\{e\}}V_p$, we obtain that $H^q_{grp}(G,M^p)\cong H^q_{grp}(\{e\},V_p)$. Therefore, $E_1^{p,q}=0$ if $q>0$ and $E_1^{p,0}=(M^p)^G$; this implies that the spectral sequence collapses in the E_2-page and we have that $H^*(M^G)\cong\mathbb{H}^*(G,M^*)$. □

[1]For any subgroup $H\subseteq G$ and an H-module M, the induced G-module Ind^G_H is defined as $\Bbbk[G]\otimes_H M$. Similarly, $\mathrm{Coind}^G_H=\mathrm{Hom}_H(\Bbbk[G],M)$.

Since X is a G-CW complex by our assumptions, the chain and cochain complexes $C_*(X)$ and $C^*(X)$ are $\mathbb{k}[G]$-modules by extension of the cellular action on X. We have the following properties of the cohomology of G with coefficients in the cochain complex $C^*(X)$, that we will refer as the hypercohomology of X and it will be denoted by $\mathbb{H}^*(G, C^*(X))$. The cup product of the group cohomology defined at the beginning of this section, and the usual cup product defined in the cochain complex $C^*(X)$ induce on $\mathbb{H}^*(G, C^*(X))$ an algebra structure [Brown, 1982, Ch.X §4]. The main properties of this construction are the following [Brown, 1982, Ch. VII].

Proposition 3.1.2. *Let X be a G-CW complex. Then,*

- *If G acts freely on X, then* $\mathbb{H}^*(G, C^*(X)) \cong H^*(X/G)$.
- *If X is a contractible free G-space, then* $C_*(X)$ *is a free resolution of* $\mathbb{k}$ *as* $\mathbb{k}[G]$*-module.*
- *Any G-map* $f\colon X \to Y$ *where Y is a G-CW complex induces a map of algebras* $\mathbb{H}^*(G, C^*(Y)) \xrightarrow{f_G^*} \mathbb{H}(G, C^*(X))$. *If f induces an isomorphism in cohomology,* f_G^* *is a ring isomorphism.*

From this result we get the following remarks.

Remark 3.1.3. The total space of the universal bundle $EG \to BG$ can be taken as a free G-CW complex. Thus $C_*(EG)$ provides a free resolution of $\mathbb{k}$ as $\mathbb{k}[G]$-module and

$$H^*(BG) \cong H^*(EG/G) \cong \mathbb{H}^*(G, C^*(EG)) \cong \mathbb{H}^*(G, C^*(pt)) \cong \mathbb{H}^*(G; \mathbb{k})$$

by Proposition 3.1.2. In particular, the constant map $X \to pt$ induces a map $\mathbb{H}^*(G, C^*(pt)) \to \mathbb{H}^*(G, C^*(X))$ of algebras which induces on the hypercohomology of X a structure of $H^*(BG)$-module.

As we have seen, the hypercohomology shares similar properties with the G-equivariant cohomology for G-spaces. In fact, they are isomorphic $H^*(BG)$-modules as we will show next.

Proposition 3.1.4. *There is an isomorphism of $H^*(BG)$-modules $H^*_G(X) \cong \mathbb{H}^*(G, C^*(X))$ natural in X.*

Proof. It follows from the commutative diagram

$$\begin{array}{ccc} EG\times X & \longrightarrow & X \\ \downarrow & & \downarrow \\ EG\times pt & \longrightarrow & pt \end{array}$$

where the horizontal maps are homotopy equivalences. The top map induces an isomorphism $\mathbb{H}(G, C^*(X)) \cong \mathbb{H}(G, C^*(EG\times X)) \cong H^*(X_G)$ by Proposition 3.1.2. □

Using this identification, we can give an explicit construction of the differential of the double complex in the case when G is a 2-torus of rank r and $\Bbbk$ is a field of characteristic two as we will state in the following result (Compare with [Allday and Puppe, 1993, Prop.1.3.23]).

Theorem 3.1.5. *Let $G = \langle g_1, \ldots, g_r\rangle$ be a 2-torus and let M^* be a cochain complex of $\Bbbk[G]$-modules. There is an isomorphism $R := H^*(BG) \cong \Bbbk[t_1, \ldots, t_r]$ and an isomorphism of R-modules*

$$\mathbb{H}^*(G; M^*) \cong H^*(C^*_G(M^*))$$

such that the double complex $C^*_G(M) = R\otimes M^*$ has a differential given by $d(p\otimes m) = p\otimes d(m) + \sum_{i=1}^r t_i p\otimes (e+g_i)m$.

Proof. First we will assume that $G = \mathbb{Z}/2 = \langle g\rangle$. Recall that the standard resolution P_* of $\Bbbk$ as $\Bbbk[G]$-module is given by $P_i = \Bbbk[G]e_i$ where the differential on P_* is the $\Bbbk[G]$-linear extension of $\delta(e_i) = (e+g)e_{i-1}$ (we set $e_i = 0$ for $i < 0$). And the map $\varepsilon\colon P_0 \to \Bbbk$ is given by $\varepsilon(a_e e + a_g g)e_0 = a_e + a_g$. This resolution induces the isomorphism $R \cong \Bbbk[t]$ where t is dual to e_1. Consider the coalgebra $H_*(BG) \cong \bigoplus_{i=0}^r \Bbbk w_i$ dual to R where w_i is dual to t^i. We define the complex $Q_i = H_i(BG)\otimes P_i \cong \Bbbk[G]w_i$ with differential $d(w_i\otimes x) = w_{i-1}\otimes (e+g)\cdot x$ which induces a free resolution of $\Bbbk$ as $\Bbbk[G]$-module. From Proposition 3.1.4 and the independence of the choice of the resolution, we obtain that the equivariant cohomology of M^* can be computed as the cohomology of the double complex

$$\begin{aligned}\mathrm{Hom}_G(Q_*, M^*) &= \mathrm{Hom}_G(H_*(BG)\otimes \Bbbk[G], M^*)\\ &\cong \mathrm{Hom}(H_*(BG), \mathrm{Hom}_G(\Bbbk[G], M^*))\\ &\cong \mathrm{Hom}(H_*(BG), M^*) \cong H^*(BG)\otimes M^*\end{aligned}$$

This isomorphism of bigraded modules is extended to an isomorphism of double complexes by translating the differential from the leftmost term. In fact, the isomorphism

$$\phi\colon \mathrm{Hom}_G(H_p(BG)\otimes \Bbbk[G], M^q) \to H^p(BG)\otimes M^q$$

is given by $\phi(f) = t^p \otimes f(w_p \otimes e)$ for $f\colon H_p(BG)\otimes \Bbbk[G] \to M_q$. We get that $\phi(\partial f) = t^p \otimes d_M(f(w_p\otimes e)) + t^{p+1}\otimes (e+g)f(w_p\otimes e)$, and that ∂ is a map of $H^*(BG)$-modules.

Now we will prove the general case. If $G = G_1\times\cdots\times G_r = \langle g_1,\dots,g_r\rangle$ is a 2-torus of rank r, there are isomorphisms $\Bbbk[G]\cong \Bbbk[G_1]\otimes\cdots\Bbbk[G_r]$ and $H^*(BG)\cong \Bbbk[t_1,\dots,t_r]$ as algebras. If P^i_* denotes the resolution of $\Bbbk$ as $\Bbbk[G_i]$-module, the standard resolution of $\Bbbk$ as $\Bbbk[G]$-module is given by $P_* = P^1_*\otimes\cdots\otimes P^r_*$. Let $\Lambda_p = \{I = (k_1,\dots,k_r)\in\mathbb{N}^r : \sum_{i=1}^r k_i = p\}$. Notice that the set of elements $\{e_I = e^1_{k_1}\otimes\cdots\otimes e^r_{k_r} : I\in\Lambda_p\}$ form a $\Bbbk[G]$ basis for P_p where $P^i_p = \Bbbk[G_i]e^i_p$ for $1\le i\le r$. Under these identifications, the differential on each P^i_* is given by $d(e^i_p) = \tau_i e^i_{p-1}$ with $\tau_i = e+g_i$, and over P_* is the map $d(e_I) = \sum_{i=1}^r \tau_i e_{I_i}$ where $I_i = (k_1,\dots,k_i-1,\dots,k_r)$ for all integer p and $I\in\Lambda_p$. Similarly to the base case, if $w^i_k\in H_k(BG_i)$ denotes the dual element to $t^k_i\in H^k(BG_i)$. we have that $H_p(BG)\cong \Bbbk w_I$ where $w_I = w^1_{k_1}\otimes\cdots\otimes w^r_{k_r}$ and $I\in\Lambda_p$. As in the base case, there is an isomorphism of double complexes $\varphi\colon \mathrm{Hom}_{\Bbbk[G]}(H_*(BG)\otimes P_*, M^*)\cong H^*(BG)\otimes M^*$ that maps $f\colon H_p(BG)\otimes P_p\to M^q$ to $\sum_{I\in\Lambda_p} t^I\otimes f(w_I\otimes e)$ where $t^I = t_1^{k_1}\cdots t_r^{k_r}$ is dual to w_I. By translating the differential on the left term into the right term; we have that

$$\begin{aligned}\varphi(\partial f) &= \varphi(d_M \circ f) + \sum_{I \in \Lambda_p} t^I \otimes f(d(w_I \otimes e)) \\ &= \varphi(d_M \circ f) + \sum_{I \in \Lambda_{p+1}} t^I \otimes \sum_{i=1}^{r} \tau_i f(w_{I_i} \otimes e) \\ &= \varphi(d_M \circ f) + \sum_{i=1}^{r} \sum_{I \in \Lambda_{p+1}} t_i t^{I_i} \otimes \tau_i f(w_{I_i} \otimes e) \\ &= \varphi(d_M \circ f) + \sum_{i=1}^{r} t_i \sum_{I \in \Lambda_p} t^I \otimes \tau_i f(w_I \otimes e)\end{aligned}$$

This implies that for any $p \otimes m \in R \otimes M^*$, the differential is given by $d(p \otimes m) = p \otimes dm + \sum_{i=1}^{r} t_i p \otimes \tau_i m$, □

We call this complex the singular Cartan model of M^* and it will be denoted by $C_G^*(M)$. Notice that the singular Cartan model is a module over $R = H^*(BG)$ and the differential is a map of R-modules. This construction is natural with respect to maps of cochain complexes; that is, for any map $f\colon N^* \to M^*$ of chain complexes there is an induced map of double complexes $f\colon C_G^*(N) \to C_G^*(M)$ and thus a map in cohomology. Furthermore, the Cartan model is independent from the chosen generators; more precisely, if $G = \langle g_1, \dots, g_r \rangle = \langle g'_1, \dots, g'_r \rangle$ are two choices of generators, there is a homotopy equivalence between the two respective Cartan models $C_G^*(M^*) \simeq C_G^*(M^*)'$ arising from the independence of the projective resolution in group cohomology. Finally, if M^* is a differential graded $\Bbbk[G]$-algebra, there is a natural product on the singular Cartan model that makes it into a differential graded algebra and it can be explicitly described as we state in the following proposition.

Proposition 3.1.6. *Let M^* be a differential graded $\Bbbk[G]$-algebra. The singular Cartan model $C_G^*(M)$ is a differential graded algebra with product given by $(t^I \otimes m) \cup (t^J \otimes n) = t^{I+J} \otimes m \cdot g^I n$ where $g^I = g_1^{k_1} \cdots g_r^{k_r}$ and $I = (k_1, \dots, k_r) \in \Lambda_p$. This product is natural with respect both G and M.*

Proof. Following [Brown, 1982, Ch.V], the hypercohomology $\mathbb{H}^*(G;M)$ inherits a cup product which is independent of the chosen resolution of $\Bbbk$ as $\Bbbk[G]$-module. It can be explicitly constructed in the following way: if P_* is such a resolution, and $\Delta\colon P_* \to P_* \otimes P_*$ is a chain map of $\Bbbk[G]$-modules where $P_* \otimes P_*$ has the diagonal action of G, for $u, v \in \operatorname{Hom}_G(P_*, M)$ then $u \cup v \in \operatorname{Hom}(P_*, M)$ is given by the composite

$$P_* \xrightarrow{\Delta} P_* \otimes P_* \xrightarrow{u \otimes v} M \otimes M \xrightarrow{\cdot} M$$

and it satisfies $d(u \cup v) = du \cup v + u \cup dv$. Let us start with the base case $G = \langle g \rangle$ and P_* is the standard resolution, for $e_n \in P_n$, we have[2] that $\Delta(e_n) = \sum_{k+l=n} e_k \otimes g^k e_l$. This follows from the commutativity of this map with the differentials; namely,

$$\begin{aligned} d\Delta(e_n) &= d\left(\sum_{k+l=n} e_k \otimes g^k e_l\right) = \sum_{k+l=n} \tau e_{k-1} \otimes g^k e_l + \sum_{k+l=n} e_k \otimes g^k \tau e_{l-1} \\ &= \sum_{k+l=n-1} e_k \otimes g^{k+1} e_l + \sum_{k+l=n-1} g e_k \otimes g^{k+1} e_l + \sum_{k+l=n-1} e_k \otimes g^k e_l + \sum_{k+l=n-1} e_k \otimes g^{k+1} e_l \\ &= \sum_{k+l=n-1} e_k \otimes g^k e_l + g \cdot \sum_{k+l=n-1} e_k \otimes g^k e_l = \Delta(\tau e_{n-1}) = \Delta(d(e_n)) \end{aligned}$$

Furthermore, for elements $t^p \otimes m, t^q \otimes n \in H^*(BG) \otimes M^*$ which correspond to the elements $u(e_p) = m, v(e_q) = n$ (and zero otherwise) $\in \operatorname{Hom}_G(P_*, M^*)$ respectively, we have then that $(u \cup v)(e_{p+q}) = \sum_{k+l=p+q} u(e_k) \cdot v(e_l) = u(e_p) \cdot g^p v(e_q) = m \cdot g^p n$, $(u \cup v)(e_n) = 0$ for $n \neq p+q$ and so $(u \cup v)$ corresponds to the element $t^{p+q} \otimes m \cdot g^p n$. Now we proceed to prove the general case. Let $G = \langle g_1, \ldots, g_r \rangle$ and $P_* = P_*^1 \otimes \cdots \otimes P_*^r$ be the standard resolution of $\Bbbk$ as $\Bbbk[G]$-module. For a basis element $e_I \in \Lambda_p$, $\Delta(e_I) = \Delta^1(e_{k_1}) \otimes \cdots \otimes \Delta^r(e_{k_r})$ (up to a suitable permutation of its terms) where $I = (k_1, \ldots, k_r)$ and $\Delta^i : P_*^i \to P_*^i \otimes P_*^i$. As before, for $u = t^I \otimes m$ and $v = t^J \otimes n \in H^*(BG) \otimes M^*$, if $I + J$ denotes the componentwise sum of the entries of each r-tuple, then $(u \cup v)(e_K) = 0$ if $K \neq I + J$ and $(u \cup v)(e_{I+J}) = u(e_I) \cdot g_1^{k_1} \cdots g_r^{k_r} v(e_J) = m \cdot g^I n$; in other words, $(t^I \otimes m) \cup (t^J \otimes n) = t^{I+J} \otimes m \cdot g^I n$ as desired. □

[2] This formula can be found as an exercise in [Brown, 1982, p.108].

Definition 3.1.7. Let X be a G space where G is a 2-torus. By setting $M = C^*(X)$ in the theorem above, we have a double complex $C^*_G(C^*(X))$ and it will be denoted by $C^*_G(X)$. This complex is called the **singular Cartan model** of X and there is an isomorphism of $H^*(BG)$-modules $H^*_G(X) \cong H^*(C^*_G(X))$ by the above result and Proposition 3.1.4.

Notice that there is a first quadrant spectral sequence

$$E_1 = R \otimes H^*(X) \Rightarrow H^*_G(X; \Bbbk). \tag{3.1.1}$$

In [Allday and Puppe, 1993, Cor. B.2.4] the authors showed that there is a homotopy equivalence of DG-modules over R

$$C^*_G(X) = R \otimes C^*(X) \to R \otimes H^*(X) \tag{3.1.2}$$

where the differential d on the right hand side can't be explicitly described; however, sometimes it is convenient to replace the singular Cartan model by the free and finitely generated model $R \otimes H^*(X)$. For instance, equation (3.1.2) shows that $H^*_G(X)$ is a finitely generated R-module if $H^*(X)$ is finitely dimensional over $\Bbbk$. This model is called the **minimal Hirsh-Brown model** of the G-space X. We can also state the following consequence of considering this model.

Proposition 3.1.8. *Let X be a finite dimensional G-CW complex such that the equivariant cohomology $H^*_G(X)$ is free as an R-module. Then X is G-equivariantly formal.*

Proof. It is enough to check that $\operatorname{rank}_R H^*_G(X) = \dim_{\Bbbk} H^*(X)$ and then by Corollary 2.5.6 we have that X is equivariantly formal. In fact, let M be the model $H^*(X) \otimes R$ which is a free R-module and let us consider $\Bbbk$ as an R-module via the augmentation map $R \xrightarrow{\varepsilon} \Bbbk$ given by $t_i \mapsto 0$. Therefore, we have that $M \otimes_R \Bbbk \cong H^*(X)$ as chain complexes where the differential on the right hand side is the zero map.
As M is free over R, applying the algebraic Eilenberg-Moore spectral sequence [Eilenberg and Moore, 1966], [Smith, 1967] we have a spectral sequence

$$E_2^{*,*} = \operatorname{Tor}^R_*(H^*(M), \Bbbk) \Rightarrow H^*(M \otimes_R \Bbbk)$$

since we are assuming $H^*(M) = H^*_G(X)$ to be free over R, the spectral sequence collapses (it is concentrated in the zeroth column) and $E_2^{0,*} = H^*_G(X) \otimes_R \Bbbk = E_\infty^{0,*} \cong H^*(M \otimes_R \Bbbk) = H^*(X)$. This implies that $\operatorname{rank}_R H^*_G(X) = \dim_\Bbbk H^*(X)$. □

Remark 3.1.9. Recall that in Proposition 2.5.3 we proved that X is G-equivariantly formal if and only if $H^*_G(X)$ is a free R-module and the local coefficient system in the fibration $X \to X_G \to BG$ is trivial. However, for 2-torus actions, we just showed that the freeness of the equivariant cohomology is sufficient for X being equivariantly formal.

Now we will construct a dual to the equivariant cohomology as it will allow to extend the Poincaré duality for ordinary cohomology into the equivariant setting. Roughly speaking, the equivariant homology will be the dual of the singular Cartan model constructed in Theorem 3.1.5.

Definition 3.1.10. Let M^* be a cochain complex of $\Bbbk[G]$-modules. The ***G*-equivariant homology** of M^* is defined as the homology of the chain complex

$$C_*^G(M) = \operatorname{Hom}_R(C_G^*(M), R)$$

and it will be denoted by $H_*^G(M)$.

It is easy to check that the equivariant homology is contravariant with respect to M. That is, if $f\colon M^* \to N^*$ is a map of $\Bbbk[G]$-cochain complexes, then there is an induced map $f_*^G\colon H_*^G(N) \to H_*^G(M)$ of R-modules. Furthermore, it is also contravariant with respect to G; that is, for any group homomorphism $G' \to G$ between 2-tori and for any $\Bbbk[G]$-chain complex M_*, there is a map $H_*^G(M) \to H_*^{G'}(M)$ as $H^*(BG)$-modules. Finally, If $M = C^*(X)$ for some G-space X, then we denote the G-equivariant homology of X by $H_*^G(X)$.

In general, the G-equivariant homology is neither the homology of the Borel construction nor the group homology with coefficients in a cochain complex. For example, set $X = pt$ and then $H_*^G(X) \cong R$ but $H_*(X_G)$ is torsion as the homology is negatively graded by convention.

Now we assume that $b(X) < \infty$. We have that $C^*_G(X)$ is homotopy equivalent to the twisted tensor product $H^*(X)\widetilde{\otimes}R$ as in Equation (3.1.2) and then $C^G_*(X)$ is homotopy equivalent to $H_*(X)\otimes R$ since $\mathrm{Hom}_R(-,R)$ preserves homotopy equivalences. The latter is a bounded below graded module over R by our assumptions on X. This will avoid convergence problems on spectral sequences . In particular, there is a convergent spectral sequence

$$E_1 = R\widetilde{\otimes}H_*(X;\mathbb{k}) \Rightarrow H^G_*(X;\mathbb{k}) \tag{3.1.3}$$

analogous to (3.1.1). Also, there is a universal coefficient theorem for equivariant (co)homology that we will state in the next proposition; the proof is derived in the same way as [Allday et al., 2014, Prop. 3.5] which uses the following fact: Let M be a differential graded R-module which is free over R. Then there is a spectral sequence with E_2-term given by $E_2 = \mathrm{Ext}_R(H^*(M),R)$ converging to $H^*(\mathrm{Hom}_R(M,R))$.

Proposition 3.1.11 (Universal coefficients theorem)**.** *Let X be a G-space. There are spectral sequences natural in X*

$$E_2^p = \mathrm{Ext}^p_R(H^*_G(X),R) \Rightarrow H^G_*(X)$$

$$E_2^p = \mathrm{Ext}^p_R(H^G_*(X),R) \Rightarrow H^*_G(X)$$

Combining this result and Proposition 3.1.8 we can state the following consequence.

Corollary 3.1.12. *Let X be a G-space. Then X is G-equivariantly formal if and only if the equivariant homology $H^G_*(X)$ is a free R-module.*

Proof. If $H^*_G(X)$ is a free R-module, the Universal coefficient spectral sequence collapses at E_2 and we have that $H^G_*(X) \cong \mathrm{Hom}_R(H^*_G(X),R)$ is the dual of a free module and so it is free. The converse statement holds in a similar fashion. □

3.2 Equivariant Poincaré duality and equivariant Euler class

Let M be a connected closed orientable manifold of dimension n, there is a canonical isomorphism $H^*(M;\Bbbk) \to H_{n-*}(M;\Bbbk)$ given by taking the cap product with the orientation class $\sigma \in H_n(M;\Bbbk)$ for any field $\Bbbk$. This is a well-known result in algebraic topology and it is widely known as the Poincaré duality theorem for closed orientable manifolds. The aim of this section is to extend the Poincaré duality theorem to the equivariant cohomology for 2-torus actions and a field $\Bbbk$ of characteristic two that satisfies similar properties to the non-equivariant version. The main results arising from this construction are analogous to [Allday et al., 2014, §3] where the torus case is treated.

We first consider a generalized notion of the (non-equivariant) Poincaré duality in a topological setting, as we state in the following definition.

Definition 3.2.1. Let $\Bbbk$ be a field. A topological space X is called a **$\Bbbk$-Poincaré duality space** ($\Bbbk$-PD space) of formal dimension n, if it satisfies the following conditions:

- $\dim_{\Bbbk} H^*(X;\Bbbk) < \infty$.
- $H^i(X;\Bbbk) = 0$ for $i > n$ and $H^n(X;\Bbbk) \cong \Bbbk$.
- There is a distinguished homology class $\sigma \in H_n(X;\Bbbk)$, called *orientation*, such that the pairing

$$H^j(X;\Bbbk) \times H^{n-j}(X;\Bbbk) \to \Bbbk$$

given by $(x,y) \mapsto \langle x \cup y, \sigma \rangle$ is non-degenerate for $0 \leq j \leq n$. Here the form $\langle \cdot, \cdot \rangle$ is the map induced by evaluating a chain on a cochain.

Notice that this induces an isomorphism $H^j(X;\Bbbk) \cong H_{n-j}(X;\Bbbk)$ (given by the map $x \mapsto x \cap \sigma$). Or in other words, $H^*(X;\Bbbk) \cong \mathrm{Hom}_{\Bbbk}(H^*(X;\Bbbk),\Bbbk)$ is an isomorphism of $\Bbbk$-vector spaces of degree $-n$.

Let G be a 2-torus and $\Bbbk$ a field of characteristic two as usual. Observe that the orientation class $\sigma \in H_n(X;\Bbbk)$ is invariant under the G-action induced in homology. This follows from the fact that the trivial action of G on $H_n(X;\mathbb{F}_2)$ extends to the action on $H_n(X;\mathbb{F}_2)\otimes_{\mathbb{F}_2}\Bbbk \cong H_n(X;\Bbbk)$ by extension of scalars.

The lifting of the orientation σ to $H^*_G(X)$ will follow from the spectral sequence (3.1.3) as $E_2^{-n,0} \cong H_n(X)^G$ is the only term of degree $-n$ and σ is invariant under the G-action. This leads to the equivariant extension of Poincaré duality by considering the equivariant cap product

$$C^*_G(X) \times C^G_*(X) \xrightarrow{\cap} C^G_*(X)$$

given by $(T \cap \gamma)(S) = \gamma(T \cup S)$ for $T,S \in C^*_G(X)$ and $\gamma \in C^G_*(X)$. The cup product in the singular Cartan model $C^*_G(X)$ is given as in Proposition 3.1.6 by $(p\otimes x)\cup(q\otimes y) = pq\otimes(x\cup g^\alpha y)$ for some multi-index $\alpha \in \mathbb{N}^r$.

Proposition 3.2.2 (G-equivariant Poincaré duality)**.** *Let M be a G-space that is also a $\Bbbk$-Poincaré duality space of formal dimension n. Let $[\sigma] \in H_n(M)$ denote its orientation class. There is a class $[\sigma]_G \in H^G_n(X)$ which is a lifting of $[\sigma]$ under the canonical map $H^G_*(X) \to H_*(X)$, and there is a natural isomorphism $PD_M\colon H^*_G(M) \to H^G_{n-*}(M)$ (with respect to both G and X) given by the cap product with $[\sigma]_G$.*

Proof. Notice that $1\otimes H_n(X)^G$ is the only term of degree $-n$ in the E_2-page of the spectral sequence (3.1.3) and so $\sigma \in H_n(X)^G$ corresponds to a non zero class $[\sigma]_G \in H^G_n(X)$. To prove the second assertion, first notice that $H^G_*(X)$ is isomorphic to the cohomology of the double complex $\mathrm{Hom}_\Bbbk(C^*(X),R)$ and so there is a spectral sequence with E_1-term $E_1 = \mathrm{Hom}_\Bbbk(H^q(X),R^p)$ converging to $H^G_*(X)$ filtering by R-degree. Similarly, the spectral sequence for the singular Cartan model of X obtained by R-degree filtering is given by $E_1 = R_p\otimes H^q(X)$. Let $\sigma_G \in C^G_n(X)$ be a representative of $[\sigma]_G$, and let φ be the map $C^*_G(X) \to C^G_*(X)$ induced by the cap product with σ_G. Such a map induces a map between spectral sequences given by the R-linear extension of $\varphi(1\otimes x)(1\otimes y) = ((1\otimes x)\cap\sigma_G)(1\otimes xy) = (\sigma_G)((1\otimes x)\cup(1\otimes y)) = \sigma_G(1\otimes(x\cup g^0y)) = (xy)(\sigma) = (x\cap\sigma)(y)$ for $x,y\in H^*(X)$ and thus it corresponds to the R-linear extension of the non-equivariant Poincaré duality on $H^*(X)$ which is an isomorphism. Therefore, the E_1-page of these spectral sequences are isomorphic and so $H_G^*(X)$ is isomorphic to $H_*^G(X)$ via the cap product with $[\sigma]_G$. Naturality of the equivariant Poincaré duality map follows from both the naturality of the non-equivariant map and the spectral sequences. □

There is also an equivariant Poincaré-Lefschetz duality that involves relative cohomology. The proof is similar to the above results as it is an extension of the duality in ordinary cohomology.

Proposition 3.2.3. *Let M be a G space which is also a $\Bbbk$-Poincaré duality space of formal dimension n. Let $N\subseteq M$ be a closed G-invariant subspace. There are isomorphisms of R-modules $pd_1\colon H_G^{n-*}(N)\to H_*^G(M,M\setminus N)$ and $pd_2\colon H_G^{n-*}(M,N)\to H_*^G(M\setminus N)$ which fit in a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_G^{n-*}(M,N) & \longrightarrow & H_G^{n-*}(M) & \longrightarrow & H_G^{n-*}(N) & \longrightarrow & H_G^{n+1-*}(M,N) & \longrightarrow \\
& \downarrow{\scriptstyle pd_2} & & \downarrow{\scriptstyle PD_M} & & \downarrow{\scriptstyle pd_1} & & \downarrow{\scriptstyle pd_2} & \\
\cdots \longrightarrow & H_*^G(M\setminus N) & \longrightarrow & H_*^G(M) & \longrightarrow & H_*^G(M,M\setminus N) & \longrightarrow & H_{*-1}^G(M\setminus N) & \longrightarrow
\end{array}$$

where the horizontal exact sequences arise from the equivariant long exact sequence associated to the pair (M,N) and $(M,M\setminus N)$ respectively.

If M is a closed G-manifold then it satisfies Proposition 3.2.2. Let $N\subseteq M$ be a closed G-invariant submanifold of M of dimension n and write $j_*^G\colon H_*^G(N)\to H_*^G(M)$ as the map induced in equivariant homology by the inclusion $j\colon N\to M$; similarly, $j_G^*\colon H_G^*(M)\to H_G^*(N)$ denotes the map induced in equivariant cohomology. Consider the composite

$$\nu_{N,M}\colon H_G^*(N)\xrightarrow{PD_N} H_{n-*}^G(N)\xrightarrow{j_*^G} H_{n-*}^G(M)\xrightarrow{PD_M^{-1}} H_G^{m-n+*}(M)\xrightarrow{j_G^*} H_G^{m-n+*}(N).$$

Under this construction, we introduce the following definition.

Definition 3.2.4. The G-**equivariant Euler class** of N with respect to M denoted by $e_G(N \subseteq M)$ is defined as the cohomology class $\nu_{N,M}(1) \in H_G^{m-n}(N)$.

Two important properties of the equivariant Euler class are the following.

- Naturality: Let G and K be 2-tori and $\alpha\colon K \to G$ be a group homomorphism. For any G-manifold M and G-invariant submanifold N it holds that $\alpha_G^*(e_G(N \subseteq M)) = e_K(N \subseteq M)$.
- Multiplicativity: Suppose that $G = K \times L$, $N \subseteq M$ is K-invariant and $N' \subseteq M'$ is L-invariant submanifolds of M and M' respectively. Then $e_G(N \times N' \subseteq M \times M') = e_K(N \subseteq M)e_L(N' \subseteq M')$.

The second property will follow from the next proposition.

Proposition 3.2.5 (Künneth theorem for equivariant homology)**.** *Let $G = K \times L$ be a decomposition of 2-subtori which induces an isomorphism of algebras $R \cong R_K \otimes R_L$ where $R_K = H^*(BK)$ and $R_L = H^*(BL)$. For any K-manifold M and L-manifold N, there is an isomorphism of R-modules*

$$\times\colon H_*^K(M) \otimes H_*^L(N) \to H_*^G(M \times N).$$

Moreover, this isomorphism satisfies $[M]_K \times [N]_L = [M \times N]_G$.

Proof. Let $\pi_M\colon M \times N \to M$ and $\pi_N\colon M \times N \to N$ be the projections. The non-equivariant Künneth theorem follows from the homotopy equivalence $\phi\colon C^*(M) \otimes C^*(N) \to C^*(M \times N)$ given by $\phi(a \otimes b) = \pi_M^*(a) \cdot \pi_N^*(b)$. Such a map induces on the singular Cartan model a homotopy equivalence of R-algebras $\phi_G\colon C_K^*(M) \otimes C_L^*(N) \to C_G^*(M \times N)$ given by $\phi_G((p \otimes a) \otimes (q \otimes b)) = pq \otimes \pi_M^*(a) \cdot \pi_N^*(b)$. This induces a quasi-isomorphism $C_*^G(M \times N) \to \mathrm{Hom}_R(C_K^*(M) \otimes C_L^*(N), R)$ by dualizing the above map. Then it only remains to prove that the latter complex is quasi-isomorphic to $C_*^K(M) \otimes C_*^L(N)$.

Filtering these complexes by R-degree, we get spectral sequences

$$E_1 = R \otimes H_*(M) \otimes H_*(N) \cong R \otimes \mathrm{Hom}_{\Bbbk}(H^*(M), \Bbbk) \otimes \mathrm{Hom}_{\Bbbk}(H^*(N), \Bbbk) \Rightarrow H_*^K(M) \otimes H_*^L(N)$$

and

$$E_1' = R \otimes \mathrm{Hom}_{\Bbbk}(H^*(M) \otimes H^*(N), \Bbbk) \Rightarrow H^*(\mathrm{Hom}_R(C_K^*(M) \otimes C_L^*(N), R)).$$

There is a quasi-isomorphism $f \colon E_1 \to E_1'$ induced by the map

$$\mathrm{Hom}_{\Bbbk}(C^*(M), \Bbbk) \otimes \mathrm{Hom}_{\Bbbk}(C^*(M), \Bbbk) \to \mathrm{Hom}_{\Bbbk}(C^*(M) \otimes C^*(N), \Bbbk)$$

which is itself a quasi-isomorphism because of the assumptions on M and N their homology and cohomology are finitely dimensional over $\Bbbk$. Then f induces an isomorphism between the E_1-terms of the spectral sequences above and so the R-modules $\mathrm{Hom}_R(C_K^*(M) \otimes C_L^*(N), R)$ and $C_*^K(M) \otimes C_*^L(N)$ are quasi-isomorphic. This fact, combined with the dual of ϕ_G, provide an isomorphism of R-modules

$$\times \colon H_*^K(M) \otimes H_*^L(N) \to H_*^G(M \times N).$$

Finally, the isomorphism $H_*(M) \otimes H_*(N) \to H_*(M \times N)$ maps $[M] \times [N]$ to $[M \times N]$ and this implies that the elements $[M]_K \times [N]_L$ and $[M \times N]_G$ both restrict to $[M \times N]$ under the canonical map $H_*^G(M \times N) \to H_*(M \times N)$. $\square$

Example 3.2.6. Let $G = G_1 \times G_2$ be a 2-torus of rank 2 where $G_1 = \{1, g\}, G_2 = \{1, \tau\}$. Let G act on $\mathbb{C}$ where g acts as the multiplication by -1 and τ as the complex conjugation. Let x, w denote the generators of $H^*(BG_1)$ and $H^*(BG_2)$ dual to g and τ respectively. Then the equivariant Euler class $e_G(0 \subseteq \mathbb{C}) = \alpha x^2 + \beta xw + \gamma w^2 \in H^2(BG)$. Let $K = \{1, s\}$ and t be the generator of $H^*(BK)$. Consider the following cases

- Let s act on $\mathbb{C}$ in the same fashion as g and let $j_1 : K \to G$ be the map sending s to g, then the induced map in cohomology is given by $j_1^*(x) = t$ and $j_2^*(w) = 0$. Notice that $e_K(0 \subseteq \mathbb{C}) = t^2$ since g acts non-trivially in both components of $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}$ and the Euler class is multiplicative. From the naturality of the Euler class we get $\alpha t^2 = j_1^*(e_G(0 \subseteq \mathbb{C})) = e_{G_1}(0 \subseteq \mathbb{C}) = t^2$; therefore, $\alpha = 1$.

- Let s act on $\mathbb{C}$ in the same fashion as τ. As before, the map $j_2 : K \to G$ sending s to τ induces the map in cohomology mapping x to 0 and w to t. In this case, $e_K(0 \subseteq \mathbb{C}) = 0$ since τ acts trivially on one real factor of $\mathbb{C}$. Therefore, by naturality, we obtain $\gamma = 0$
- Finally, let s act on $\mathbb{C}$ as $g\tau$, and $j_3 : K \to G$ sends s to (g, τ) and, in cohomology, both x, w are sent to t. Since s acts trivially on one real factor of $\mathbb{C}$, $e_K(0 \subseteq \mathbb{C}) = 0$ and by naturality, $t^2 + \beta t^2 = j_3^*(e_G(0 \subseteq \mathbb{C})) = e_K(0 \subseteq \mathbb{C}) = 0$. Therefore, $\beta = 1$.

So we have proved that $e_G(0 \subseteq \mathbb{C}) = x(x+w)$.

3.3 The Atiyah-Bredon sequence

In this section we construct an Atiyah-Bredon sequence for 2-torus actions analogously to [Allday et al., 2014] in the torus case; their proofs carry over to the 2-torus case and are imitated with slight modifications to our settings. For the torus case, if X is a T-equivariantly formal over a field of characteristic zero, where T is a torus, in [Chang and Skjelbred, 1974] the authors showed that there is an exact sequence

$$0 \to H_T^*(X) \to H_T^*(X^T) \to H_T^{*+1}(X_1, X^T) \tag{3.3.1}$$

where X_1 is the union of all T-orbits of dimension at most 1. Furthermore, [Atiyah, 1974] and [Bredon et al., 1974] proved independently that there is an extended exact sequence

$$0 \to H_T^*(X) \to H_T^*(X^T) \to H_T^{*+1}(X_1, X^T) \to \cdots \to H_T^{*+r}(X, X_{r-1}) \to 0$$

where X_i is the union of all T-orbits of dimension at most i. The authors [Allday et al., 2014] studied the "Atiyah-Bredon" sequence in a more general setting; more precisely, if a space is not T-equivariantly formal, then a partial exactness of the sequence turns into algebraic properties of the T-equivariant cohomology; in fact, the syzygy modules in commutative algebra [3] play an important role in the equivariant cohomology theory.

For the 2-torus case, we start by constructing a filtration of a G-space by its orbits as we state in the following definition.

Definition 3.3.1. Let X be a G-space. The ***G*-equivariant skeleton** X_i of X is the union of orbits of size at most 2^i for $-1 \leq i \leq r$. In particular, $X_{-1} = \emptyset$, $X_0 = X^G$ and $X_r = X$. This induces a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_r = X$$

This filtration is called the ***G*-orbit filtration** of X.

These subsets satisfy the following properties.

Proposition 3.3.2. *Let X be a G-space. Then each of the subset X_i is closed in X for any $-1 \leq i \leq r$. Moreover, if $b(X) < \infty$, then $b(X_i) < \infty$.*

Proof. Note that $X_i = \{x \in X : \operatorname{rank}(G_x) \geq r - i\}$. To see that this subset is closed, it is enough to check that for any $x \in X$, there is a neighbourhood U of x such that for any $y \in U$, $G_y \subseteq g^{-1}G_x g$ for some $g \in G$. This will follow from the equivariant tubular neighbourhood theorem (Theorem 2.1.1); in fact, for a given $x \in X$, there is a neighbourhood of the form $U = G \times_{G_x} A$ for some G_x-invariant subset A of X. We consider the map $f\colon U \to G \cdot x$ as the composite of the projection $G \times_{G_x} A \to G/G_x$ and the homeomorphism $G/G_x \cong G \cdot x$. This map sends $y = [g,a]$ to $g \cdot x$ for $y \in U$; moreover, the map f is G-equivariant and so for any $h \in G_y$ we get $h \cdot f(y) = f(y)$; in other words, $h \cdot (g \cdot x) = g \cdot x$. This implies that $g^{-1}hg \in G_x$, or equivalently, $h \in g^{-1}G_x g$.

The second part of the proposition follows by noticing that $X_i = \bigcup\{X^H : H \subseteq G,\ \operatorname{rank}(H) \geq i\}$, $b(X^H) < \infty$ (this is a consequence of the localization theorem 2.4.5) and there are finitely many subgroups occurring in G. ☐

[3] See Appendix B for a discussion of this notion

Recall that an R-module is said to be Cohen-Macaulay if $\operatorname{depth}_R(M) = \dim_R(M)$ (See Appendix B). Now we will show that for a G-space X, the G-equivariant homology and cohomology of the pairs (X_i, X_{i-1}) arising from the G-equivariant skeleton filtration are either zero or Cohen-Macaulay over $R = H^*(BG)$.

Proposition 3.3.3. *The R-modules $H^*_G(X_i, X_{i-1})$ and $H^G_*(X_i, X_{i-1})$ are either zero or Cohen-Macaulay of dimension $r-i$ for $0 \leq i \leq r$.*

Proof. Let $0 \leq i \leq r$ such that $X_{i-1} \neq X_i$ and set $Y = X_i \setminus X_{i-1}$. We first assume that for every $x \in Y$, $G_x = H$ for some fixed 2-subtorus H of rank $r-i$. Let K be a 2-torus complement of H so that $G = H \times K$. We have then an isomorphism of algebras $R \cong R_H \otimes R_K$ where $R_H = H^*(BH)$ and $R_K = H^*(BK)$. Now we use the Tautness property for equivariant cohomology and excision to get

$$H^*_G(X_i, X_{i-1}) \cong \operatorname{colim}_U H^*_G(X_i, U) \cong \operatorname{colim}_U H^*_G(Y, Y \cap U)$$

where limit is taken over all G-invariant open neighbourhoods U of X_{i-1} in X_i. Now from Remark 2.2.6 we have that $Y_G \cong BH \times Y_K$ since H acts trivially on Y. This implies that

$$\begin{aligned}
\operatorname{colim}_U H^*_G(Y, Y \cap U) &\cong R_H \otimes \operatorname{colim}_U H^*_K(Y, Y \cap U) \\
&\cong R_H \otimes \operatorname{colim}_U H^*(Y/K, Y \cap U/K) \\
&\cong R_H \otimes H^*(X_i/K, X_{i-1}/K)
\end{aligned}$$

using that K acts freely on Y. Now observe $H^*(X_i/K, X_{i-1}/K)$ might not be a trivial R_K-module; in fact, it depends on the splitting of the exact sequence

$$1 \to H \to G \to K \to 1$$

if it were a trivial R_K-module, this will imply that $H^*_G(X_i, X_{i-1})$ is a Cohen-Macaulay module over R of dimension $\dim R_H = r-i$. In the case of a non-trivial action, we may consider the finite filtration of $H^*_G(X_i, X_{i-1}) \cong R_H \otimes H^*(X_i/K, X_{i-1}/K)$ using that $H^*(X_i/K, X_{i-1}/K)$ is finite dimensional over $\Bbbk$. In fact, we set $F_p = \bigoplus_{k\geq p} R_H \otimes H^k(X_i/K, X_{i-1}/K)$ so that $F_0 = H^*_G(X_i, X_{i-1})$ and $F_p = 0$ for every $p \geq n$ and some $n > 0$. Such a filtration consists of successive quotients that are free over R_H with trivial action of R_K. Thus, each of these quotients is a Cohen-Macaulay R-module of dimension $r-i$. By the long exact sequence of the Ext functor associated to the short exact sequence

$$0 \to F_{p+1} \to F_p \to F_p/F_{p+1} \to 0$$

we conclude that each module of the filtration is a Cohen-Macaulay R-module of dimension $r-i$; in particular, $H^*_G(X_i, X_{i-1})$ is a Cohen-Macaulay R-module of dimension $r-i$.

Now we proceed to prove the general case. Write $X_i \setminus X_{i-1} = \bigcup_{k=1}^n Y_k$ as the disjoint union of finitely many spaces Y_i such that for every $x \in Y_k$, $G_x = H_k$ for some fixed 2-subtorus H_k of rank $r-i$. Then by the Mayer-Vietoris long exact sequence for equivariant cohomology (See remark 2.3.7) we have an isomorphism of R-modules

$$H^*_G(X_i, X_{i-1}) = \bigoplus_{k=1}^{n} H^*_G(X_i, X_i \setminus Y_k)$$

Therefore, by applying the previous case on each factor, we have that $H^*_G(X_i, X_{i-1})$ is the direct sum of Cohen-Macaulay R-modules of dimension $r-i$ and hence $H^*_G(X_i, X_{i-1})$ is also a Cohen-Macaulay ring of dimension $r-i$.

The statement for G-equivariant homology follows from the Universal Coefficient Theorem (Theorem 3.1.11) and the characterization of Cohen-Macaulay modules by the Ext modules given in Proposition B.7. □

Recall that the orbit filtration of X leads to a filtration of $C^G_*(X)$ and a spectral sequence converging to $H^G_*(X)$. In particular, we get the next result.

Corollary 3.3.4. *The spectral sequence*

$$E^1_{*,p} = H^G_*(X_p, X_{p-1}) \Rightarrow H^G_*(X).$$

arising from the orbit filtration degenerates at E^1.

Proof. For any p, the differential $d^1\colon E^1_{*,p} \to E^1_{*-1,p}$ is a map between Cohen-Macaulay modules of dimension $r-p$ and $r-p+1$ respectively . Therefore, by Proposition B.8 we have that this map is zero. Analogously, one obtains that any higher differential $d^k\colon E^k_{*,p} \to E^k_{*-k,p-k+1}$ is zero for $k \geq 1$ and thus the spectral sequence degenerates at E^1. □

Proposition 3.3.3 and 3.3.4 lead to the following theorem.

Theorem 3.3.5. *For any G-space X, the spectral sequence arising from the orbit filtration $E_1^p = H^*_G(X_p, X_{p-1}) \Rightarrow H^*_G(X)$ and the spectral sequence from the universal coefficient theorem $E_2^p = \mathrm{Ext}^p_R(H^G_*(X), R) \Rightarrow H^*_G(X)$ are naturally isomorphic from the E_2-page on.*

Proof. The given proof in [Allday et al., 2014, Thm. 4.8] for the torus case is purely algebraic and it carries over to the 2-torus case in the same fashion. □

Definition 3.3.6. For any G-space X the E_1 page of the cohomology spectral sequence associated to the G-orbit filtration of X leads to a sequence

$$H^*_G(X_0) \to H^*_G(X_1, X_0)[1] \to \cdots \to H^*_G(X_r, X_{r-1})[r]$$

which will be called the **G-Atiyah-Bredon sequence** of X and this complex will be denoted by $AB^*_G(X)$. The **augmented G-Atiyah-Bredon sequence** is the complex

$$0 \to H^*_G(X) \to H^*_G(X_0) \to H^*_G(X_1, X_0)[1] \to \cdots \to H^*_G(X_r, X_{r-1})[r]$$

obtained by connecting the canonical map induced by the inclusion $X_0 \hookrightarrow X$ to the G-Atiyah-Bredon sequence. The augmented sequence will be denoted by $\overline{AB}^*_G(X)$ and we set $\overline{AB}^{-1}_G(X) = H^*_G(X)$

As a consequence of Theorem 3.3.5, we get that the cohomology of this complex can be described in the following corollary.

Corollary 3.3.7. *$H^i(AB^*_G(X)) \cong \mathrm{Ext}^i_R(H^G_*(X), R)$ for any $i \geq 0$.*

As an immediate consequence we have that if two G-spaces X and Y are G-homotopic then $H^*(AB^*_G(X)) \cong H^*(AB^*_G(Y))$. The first terms of the augmented Atiyah-Bredon sequence induce a map $H^*_G(X) \to H^0(AB^*_G(X))$. From Theorem 3.3.5 it follows that this map coincides with the canonical map $H^*_G(X) \to \mathrm{Hom}_R(H^G_*(X), R)$. Therefore, the Chang-Skjelbred sequence (3.3.1) $0 \to H^*_G(X) \to H^*_G(X_0) \to H^*(X_1, X_0)$ is exact if and only if the canonical map $H^*_G(X) \to \mathrm{Hom}_R(H^G_*(X), R)$ is an isomorphism.

In [Allday et al., 2014, Lem 5.6] the authors constructed the augmented Atiyah-Bredon sequence for torus action in a very remarkable way. Although the proof was made for the torus case, it holds in general for any filtration of a topological space X, as we state in the following result.

Lemma 3.3.8. *Let X be a topological space with a filtration $X_{-1} = \emptyset \subseteq X_0 \subseteq \cdots \subseteq X$. Then the sequence*

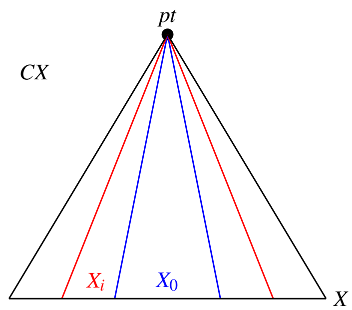
$$0 \to H^*(X) \to H^*(X_0) \to H^*(X_1, X_0) \to \cdots \tag{3.3.2}$$

(up to a degree shift) can be obtained as the E_1-term of a spectral sequence converging to 0. The analogous statement in equivariant cohomology holds if X is a G-space and the filtration consists of G-invariant subspaces of X.

Proof. Let $Y = C(X) = (X \times [0,1])/(x,0) \sim (x',0)$ be the cone over X. Then the relative cohomology $H^*(Y, pt) = 0$ where pt denotes the apex of the cone. Consider the filtration $Y_i = X \cup C(X_i)$ for $i \geq -1$, and so we get a filtration of Y

$$\emptyset \subseteq Y_{-1} = X \subseteq Y_0 = X \cup C(X_0) \subseteq \cdots \subseteq Y$$

as can be pictured out in the following figure



Such a filtration induces a spectral sequence with E_1-term given by $E_1^p = H^*(Y_{p-1}, Y_{p-2}, pt) \Rightarrow H^*(Y, pt) = 0$. Such a spectral sequence gives rise to a sequence

$$0 \to H^*(Y_{-1}, pt) \to H^{*+1}(Y_0, Y_{-1}, pt) \to H^{*+2}(Y_1, Y_0, pt) \to \cdots \tag{3.3.3}$$

notice that $H^*(Y_{-1}, pt) \cong H^*(X)$, $H^*(Y_0, Y_{-1}, pt) \cong H^*(X \cup C(X_0), X \cup pt) \cong H^*(X_0)$ and $H^*(Y_i, Y_{i-1}, pt) = H^*(X \cup C(X_i), X \cup C(X_{i-1}), pt) \cong H^*(X_i, X_{i-1})$. Therefore, sequence (3.3.3) coincides with the sequence (3.3.2) up to a degree shift. □

Proposition 3.3.9. *The augmented Atiyah-Bredon sequence can be obtained as the E_1 page of a spectral sequence converging to 0. If this sequence is exact everywhere but possible except at two adjacent terms, then it is exact everywhere. This statement also holds for the localization of the Atiyah-Bredon sequence with respect to any multiplicative subset $S \subseteq H^*(BG)$.*

3.4 Shifted and virtual subgroups of 2-torus

Unlike the torus case where there are infinitely many subtori for a fixed torus T, in the 2-torus case we need to introduce the notion of shifted and virtual subgroups to be able to get analogous results from the torus case into the 2-torus case. The shifted subgroups were widely discussed in [Allday and Puppe, 1991], (and the references therein) and the virtual subgroups were treated in [Allday et al.,]. We will only refer to some properties that will be useful for our purposes.

Let G be a 2-torus of rank r and $\mathbb{k}$ be a field of characteristic 2. Choose generators $G = \langle g_1, \dots, g_r \rangle$ and let $\tau_i = e + g_i \in \mathbb{k}[G]$. Then $\mathbb{k}[G]$ is isomorphic to an exterior algebra in n-variables and so it is a local ring with maximal ideal $\mathfrak{m} = \langle \tau_1, \dots, \tau_r \rangle$. We also denote the multiplicative subgroup of units in this ring by $\mathbb{k}[G]^\times$. Consider an element $u \in \mathbb{k}[G]^\times$ that can be written in the form $u = e + \sum_{i=1}^r \alpha_i \tau_i \bmod \mathfrak{m}^2$ for some $(\alpha_1, \dots . \alpha_r) \in \mathbb{k}^r$. We say that u is a non-trivial unit if $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{k}^r$ is non-zero. In this case, u generates a subgroup of $\mathbb{k}[G]^\times$ of order 2.

Proposition 3.4.1. *[Allday and Puppe, 1991, Thm.2.2] Let $1 \leq s \leq r$ be an integer and let $u_1, \dots, u_s$ be non-trivial units in $\mathbb{k}[G]$ such that $u_j = e + \sum_{i=1}^r \alpha_i^j \tau_i \bmod \mathfrak{m}^2$. If the vectors $\alpha_j = (\alpha_1^j, \dots, \alpha_r^j)$ are linearly independent in $\mathbb{k}^r$, then the set $\{u_1, \dots, u_s\}$ generate a subgroup $\Gamma(u_1, \dots, u_s)$ of $\mathbb{k}[G]^\times$ isomorphic to a 2-torus of rank s. Moreover, the canonical map of rings $i_\Gamma \colon \mathbb{k}[\Gamma] \to \mathbb{k}[G]$, induced by the inclusion, is injective where $\Gamma = \Gamma(u_1, \dots, u_s)$.*

Definition 3.4.2. The group $\Gamma \subseteq K[G]^\times$ satisfying the conditions of the above proposition is called a **shifted subgroup** of G of rank s and we write $\Gamma \subseteq^{\mathfrak{s}} G$.

Notice that $\mathbb{k}[G]$ is a $\mathbb{k}$-algebra generated by the set $\{\tau_1, \dots, \tau_r\}$, that is, any element in $w \in \mathbb{k}[G]$ can be written as $w = p(\tau_1 \dots, \tau_r)$ where $p \in \mathbb{k}[x_1, \dots, x_r]$. Moreover, any linearly independent $\mathbb{k}$-linear combination of the set $\{\tau_1, \dots, \tau_r\}$ modulo $\mathfrak{m}^2$ will also give rise to a generating set of $\mathbb{k}[G]$, as the next lemma will exhibit.

Lemma 3.4.3. *Let $\sigma_j \in \mathbb{k}[G]$ be elements such that $\sigma_j = \sum_{i=1}^r \alpha_i^j \tau_i \bmod \mathfrak{m}^2$ and $\alpha_i^j \in \mathbb{k}$ for $j = 1, \dots, r$. Suppose that the matrix $A = (\alpha_i^j) \in GL_{\mathbb{k}}(r)$, then the set $\{\sigma_1, \dots, \sigma_r\}$ generate $\mathbb{k}[G]$ as $\mathbb{k}$-algebra.*

Proof. Let S be the $\mathbb{k}$-algebra generated by $\{\sigma_1, \dots, \sigma_r\}$ and denote by $R = \mathbb{k}[G]$. Then it is enough to show that $\tau_i \in S$ for $i = 1, \dots, r$, or equivalently, that the map $S \to R$ induced by the inclusion is surjective. Firstly, for any positive integer n consider the set $\Lambda_n = \{I \in (i_1,\dots,i_r) \in \mathbb{N}^r : \sum_{j=1}^{r} i_j = n\}$, and for $I \in \Lambda_n$ define $\tau_I = \tau_1^{i_1}\cdots\tau_r^{i_r}$; analogously σ_I is defined. Since $\tau_i^2 = \sigma_i^2 = 0$, for $n > r$ we have that $\tau_I = \sigma_I = 0$. Recall that $\mathfrak{m}$ is the maximal ideal of $\Bbbk[G]$ generated by $\{\tau_1,\dots,\tau_r\}$, similarly, we denote by $\mathfrak{n}$ the maximal ideal of S generated by $\{\sigma_1,\dots,\sigma_r\}$. Under this notation, we have that $\mathfrak{m}^n$ (resp. $\mathfrak{n}^n$) is the ideal generated by $\{\tau_I : I \in \Gamma_n\}$ (resp. $\{\sigma_I : I \in \Gamma_n\}$) and so $\mathfrak{n}^n \subseteq \mathfrak{m}^n$. Now notice that $\tau_j = \sum_{i=1}^{r} \beta_i^j \sigma_i \mod \mathfrak{m}^2$ where $B = (\beta_i^j) = A^{-1}$, this implies that for $I \in \Gamma_n$, $\tau_I = \sum_{J\in\Gamma_n} \beta_J \sigma_J \mod \mathfrak{m}^{n+1}$ for some elements $\beta_J \in \Bbbk$. In particular, for $n = r$, we have that $\Gamma_n = \{I = (1,\dots,1)\}$, and $\tau_I = \tau_1 \dots \tau_r = \det(B)\sigma_I$, thus $\mathfrak{m}^r = \mathfrak{n}^r$. The proof finishes by reverse induction by proving that $\mathfrak{m}^n = \mathfrak{n}^n$ for all n as a consequence of the Nakayama's Lemma as the ideal $\mathfrak{m}$ is nilpotent. □

Using this lemma, we can guarantee the existence of complementary shifted subgroups as we show in the next result.

Proposition 3.4.4. *Let $\Gamma \subseteq^{\mathfrak{s}} G$ be a shifted subgroup of rank s. There is a shifted subgroup Γ' of rank $r-s$ such that the canonical map $\Bbbk[\Gamma]\otimes\Bbbk[\Gamma'] \to \Bbbk[G]$ is an isomorphism of algebras. In particular, this implies that $\Bbbk[G]$ is a free $\Bbbk[\Gamma]$-module.*

Proof. Let us suppose that Γ is generated by elements $\{u_1,\dots,u_s\}$ where $u_j = e + \sum_{i=1}^{r} \alpha_i^j \tau_i \mod \mathfrak{m}^2$. Consider the $\Bbbk$-vector space $V = \operatorname{span}(\tau_1,\dots,\tau_r)$ and set $\sigma_j = \sum_{i=1}^{r} \alpha_i^j \tau_i \in V$. Since the elements $\{\sigma_1,\dots,\sigma_s\}$ are linearly independent over $\Bbbk$, we can extend the set to a basis $\{\sigma_1,\dots,\sigma_s,\sigma_{s+1},\dots,\sigma_r\}$ of V. We define the elements $u_j = e + \sigma_j$ for $s+1 \le j \le r$ and let Γ' be the shifted subgroup of G generated by these elements. The canonical maps i_Γ and i'_Γ induce a map $\varphi\colon \Bbbk[\Gamma]\otimes\Bbbk[\Gamma'] \to \Bbbk[G]$ given by $\varphi(a\otimes b) = i_\Gamma(a) i_{\Gamma'}(b)$. We will now see that this map is an isomorphism of algebras. Since $\Bbbk[G]$ is an algebra generated by $\{\tau_1,\dots,\tau_r\}$, to show that φ is surjective we only need to observe that $\tau_i \in im(\varphi)$ by Lemma 3.4.3. The isomorphism follows then because as $\Bbbk$-vector spaces, both $\Bbbk[\Gamma]\otimes\Bbbk[\Gamma']$ and $\Bbbk[G]$ have dimension 2^r. □

Now consider $\Bbbk$ as a trivial $\Bbbk[G]$-module. Recall that the standard resolution $P_* \to \Bbbk$ satisfies $P_1 = \bigoplus_{i=1}^r \Bbbk[G]e_i$, $P_0 = \Bbbk[G]e_0$ and the differential is the map $d\colon P_1 \to P_0$ given by $d(e_i) = \tau_i e_0$ as discussed in the proof of Theorem 3.1.5 . The augmentation map $\varepsilon\colon P_0 \to \Bbbk$ is the norm map $\varepsilon(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g$. Under this resolution, we have that $R := H^*(BG;\Bbbk) = \mathrm{Ext}_{\Bbbk[G]}(\Bbbk,\Bbbk) \cong \Bbbk[t_1,\dots,t_r]$ where the variables t_i are dual to the generators e_i under the functor $\mathrm{Hom}_{\Bbbk[G]}(\_,\Bbbk)$ and the differential induced by the multiplication by τ_i. In this case, we will say that t_i is dual to τ_i

Let $\Gamma = \Gamma(u_1,\dots,u_s) \subseteq^{\mathfrak{s}} G$. Any $\Bbbk[G]$-module M is also a $\Bbbk[\Gamma]$-module via the map i_Γ. In particular, when $M = \Bbbk$, there is a standard resolution $Q_* \to \Bbbk$ as $\Bbbk[\Gamma]$-module where the differential d_Γ is induced by the multiplication by the elements $(e+u_j)$. In this case, we denote by R_Γ or $H^*(B\Gamma;\Bbbk)$ the ring $\mathrm{Ext}_{\Bbbk[\Gamma]}(\Bbbk,\Bbbk)$ and it is isomorphic to the polynomial ring $\Bbbk[y_1,\dots,y_s]$ where each y_j is dual to u_j. Under this notation, we can compute the map induced in cohomology by the inclusion $\Gamma \to \Bbbk[G]$ as the next proposition exhibits.

Proposition 3.4.5. *Let $\Gamma \subseteq^{\mathfrak{s}} G$ be a shifted subgroup of G of rank s. Suppose that $\Gamma = \Gamma(u_1,\dots,u_s)$ where $u_j = e + \sum_{i=1}^r \alpha_i^j \tau_i \bmod \mathfrak{m}^2$ for $j = 1,\dots,s$. Then the map $i_\Gamma^*\colon R \to R_\Gamma$ induced in cohomology is given by $i_\Gamma^*(t_i) = \sum_{j=1}^s \alpha_i^j y_j$.*

Proof. Write $u_j = e + \sum_{i=1}^r(\alpha_i^j + \sum_{k<i} c_{ki}^j \tau_k)\tau_i$ where $\alpha_j \in \Bbbk$ and $c_{ki}^j \in \Bbbk[G]$. The standard resolution Q_* of $\Bbbk$ as trivial $\Bbbk[\Gamma]$-module is given by

$$\cdots \xrightarrow{d_\Gamma} \bigoplus_{j=1}^{s} \Bbbk[\Gamma] f_i \xrightarrow{d_\Gamma} \Bbbk[\Gamma] f_0 \to \Bbbk$$

where $d_\Gamma(f_j) = (e+u_j)f_0$. On the other hand, the map $i_\Gamma\colon \Bbbk[\Gamma] \to \Bbbk[G]$ induces a map of resolutions $i_\Gamma\colon Q_* \to P_*$ where $i_\Gamma(f_0) = e_0$ and $i_\Gamma(f_j) = \sum_{i=1}^r(\alpha_i^j + \sum_{k<i} c_{ki}^j \tau_k)e_i$ for $j = 1,\dots,s$.

In fact, this map makes the diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \bigoplus_{i=1}^{r} \Bbbk[G]e_i & \xrightarrow{d} & \Bbbk[G]e_0 & \longrightarrow & \Bbbk \\
 & & {\scriptstyle i_\Gamma}\uparrow & & {\scriptstyle i_\Gamma}\uparrow & & \| \\
\cdots & \longrightarrow & \bigoplus_{j=1}^{s} \Bbbk[\Gamma]f_i & \xrightarrow{d_\Gamma} & \Bbbk[\Gamma]f_0 & \longrightarrow & \Bbbk
\end{array}$$

commute. To compute the map $i_\Gamma^*\colon R \to R_\Gamma$ in cohomology, we need to look at the map $\mathrm{Hom}_{\Bbbk[G]}(P_*, \Bbbk) \to \mathrm{Hom}_{\Bbbk[\Gamma]}(Q_*, \Bbbk)$ at the level H^1. Let $\mathrm{Hom}_{\Bbbk[G]}(P_1, \Bbbk) \cong \bigoplus_{i=1}^{r} \Bbbk t_i$ where t_i is dual to e_i in the sense that $\mathrm{Hom}_{\Bbbk[G]}(\Bbbk[G]e_i, \Bbbk) \cong \Bbbk t_i$, and let $\mathrm{Hom}_{\Bbbk[\Gamma]}(Q_*, \Bbbk) \cong \bigoplus_{j=1}^{s} \Bbbk y_j$ where y_j is dual to f_j. Then we have that $i_\Gamma^*(t_i) = \sum_{j=1}^{s} \lambda_j^i y_j$ where

$$\begin{aligned}
\lambda_j^i &= (i_\Gamma^*)(t_i)(f_j) = t_i(i_\Gamma)(f_j) \\
&= t_i\left(\sum_{l=1}^{r} (\alpha_l^j + \sum_{k<l} c_{kl}^j \tau_k) e_l\right) = (\alpha_i^j + \sum_{k<i} c_{ki}^j \tau_k) \cdot 1 = \alpha_i^j
\end{aligned}$$

since $\tau_k \cdot 1 = 0$ in $\Bbbk$. Finally, we have that $i_\Gamma^*(t_i) = \sum_{j=1}^{s} \alpha_i^j y_j$. □

Definition 3.4.6. Let M^* be a cochain complex of $\Bbbk[G]$-modules. For a shifted subgroup $\Gamma \subseteq^s G$, we define the **Γ-equivariant cohomology** of M as the cohomology of the bicomplex $\mathrm{Hom}_{\Bbbk[G]}(Q_*, M^*)$ where $Q_* \to \Bbbk$ is a free resolution of $\Bbbk$ as $\Bbbk[\Gamma]$-module and we will denote it by $H_\Gamma^*(M)$. In particular, when X is a G-space, we take $M = C^*(X)$ in the above construction and we denote by $H_\Gamma^*(X)$ the Γ-equivariant cohomology of X. The canonical map $X \to pt$ makes $H_\Gamma^*(X)$ into an R_Γ-module.

As a Corollary of Proposition 3.4.5, we have that the structure of R_Γ-module is independent of the representative of the class of the generators $u_1, \ldots, u_s$ of Γ modulo $\mathfrak{m}^2$.

Corollary 3.4.7. *Let $\Gamma_1 = \Gamma(u_1, \ldots, u_s)$ and $\Gamma_2 = \Gamma(v_1, \ldots, v_s)$ be shifted subgroups of G where $u_j - v_j \in \mathfrak{m}^2$ for all $j = 1, \ldots, s$. For any G-space X, there are canonical isomorphisms*

$\iota\colon R_{\Gamma_1} \to R_{\Gamma_2}$ and $\phi\colon H^*_{\Gamma_1}(X) \to H^*_{\Gamma_2}(X)$ such that the diagram

$$\begin{array}{ccccc} R_G & \xrightarrow{i^*_{\Gamma_1}} & R_{\Gamma_1} & \longrightarrow & H^*_{\Gamma_1}(X) \\ \| & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \phi} \\ R_G & \xrightarrow{i^*_{\Gamma_2}} & R_{\Gamma_2} & \longrightarrow & H^*_{\Gamma_2}(X) \end{array}$$

commutes.

Proof. By Proposition 3.4.4, we can choose a shifted subgroup $\Gamma' \subseteq^{\mathfrak{s}} G$ which is complementary to both Γ_1 and Γ_2, thus there are isomorphism of algebras $\Bbbk[G] \cong \Bbbk[\Gamma_1] \otimes \Bbbk[\Gamma'] \cong \Bbbk[\Gamma_2] \otimes \Bbbk[\Gamma']$. Using the change of ring properties for the Ext functor, we have a canonical isomorphism

$$\mathrm{Ext}_{\Bbbk[\Gamma_i]}(\Bbbk, C^*(X)) \cong \mathrm{Ext}_{\Bbbk[G]}(\Bbbk[\Gamma'], C^*(X))$$

for $i = 1, 2$. Since the latter term does not depend on Γ_1 or Γ_2, we have that $R_{\Gamma_1} \cong R_{\Gamma_2}$ and $H^*_{\Gamma_1}(X) \cong H^*_{\Gamma_2}(X)$. The commutativity of the diagram follows from the naturality of the Ext functor. □

Notice that similar to the algebraic construction of the equivariant cohomology given in Section 3.1, there is a free differential graded R_Γ-module $C^*_\Gamma(X) = R_\Gamma \otimes C^*(X)$ such that $H^*_\Gamma(X) \cong H^*(C^*_\Gamma(X))$ as R_Γ-modules.

Now we will discuss another construction which will be related to the shifted subgroups.

Definition 3.4.8. Let $\Bbbk$ be a field of characteristic 2 and G be a 2-torus of rank r. Consider the $\Bbbk$-vector space $L_G = H_1(BG; \Bbbk)$ of dimension r. A **virtual subgroup** K of G is defined as a linear subspace of L, and we say that K is a virtual subgroup of rank j if $\dim_{\Bbbk} K = j$. In this case we write $K \subseteq^{\mathfrak{v}} G$ and $\operatorname{rank} K = j$. If $K' \subseteq^{\mathfrak{v}} G$ is such that $K \oplus K' = L_G$, we say that K' is complementary to K. Notice that a complementary subgroup is generally not unique.

Remark 3.4.9. There is a correspondence from the set of subgroups of G to the set of virtual subgroups of G given by $K \mapsto j_K(H_1(BK))$ where $j_K\colon H_*(BK) \to H_*(BG)$ is the map induced by the inclusion. In particular, if $\Bbbk = \mathbb{F}_2$, this is a one-to-one correspondence between the subgroups G' of G and the virtual subgroups of G. In fact, if $H^*(BG) \cong \mathbb{F}_2[t_1, \dots, t_r]$ where t_i is dual to $\tau_i = e + g_i$, for any subgroup $K = \langle k \rangle$ of rank 1, we have that $j_K(L_K) = \sum_{i=1}^r \varepsilon_i t_i^*$ where $k = g_1^{\varepsilon_1} \cdots g_r^{\varepsilon_r}$. This follows from Proposition 3.4.5 by considering the virtual subgroup $\Gamma(k)$ as $k = (e+\tau_1)^{\varepsilon_1} \cdots (e+\tau_r)^{\varepsilon_r} = e + \sum_{i=1}^r \varepsilon_i \tau_i \mod \mathfrak{m}^2$. The claim for a subgroup of any rank follows by the Künneth theorem. This shows that the subgroups of G and linear subspaces of L_G are uniquely determined by the vectors $(\varepsilon_1, \dots, \varepsilon_r) \in \mathbb{F}_2^r$.

Recall that for a vector space V over any field $\Bbbk$ and a subspace $W \subseteq V$, the annihilator of W in V is the subspace $W^\perp$ of $V^* = \mathrm{Hom}(V, \Bbbk)$ consisting of the linear maps $f \in V^*$ such that $f(W) = 0$.

Let $R = H^*(BG)$ and let K be a virtual subgroup of G. We construct the R-algebra $R_K = R/(K^\perp)$ where $(K^\perp)$ is the ideal generated by $K^\perp \subseteq H^1(BG)$, and there is a canonical map $j_K \colon R \to R_K$. Then for any G-space X, we define the differential graded module over R_K

$$C_K^*(X) = R_K \otimes_R C_G^*(X)$$

where the differential on $R_K \otimes_R C_G^*(X)$ is the standard differential of a tensor product (the differential on R_K is set to be zero). Notice that $C_K^*(X)$ is free as a graded R_K-module.

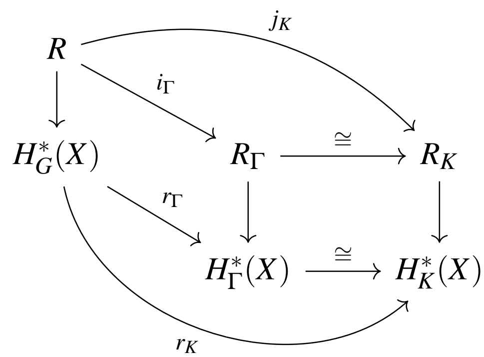
Definition 3.4.10. The K**-equivariant cohomology** of the G-space X is the graded R_K-module,

$$H_K^*(X; \Bbbk) = H_*(C_K^*(X)).$$

Now we will relate the shifted and virtual subgroups of G which will give rise to a relation between the respective equivariant cohomologies of X. In fact, let $V_G = \mathfrak{m}/\mathfrak{m}^2$ be the $\Bbbk$-vector space spanned by $\{\tau_1, \dots, \tau_r\}$, and let $L_G = H_1(BG)$ be spanned by $\{t_1^*, \dots, t_r^*\}$ dual to the variables $t_i \in H^1(BG)$. The map $\Phi_G \colon V_G \to L_G$ which sends τ_i to t_i^* is a natural isomorphism. In fact, for any group homomorphism $h \colon G \to G'$, the induced maps $\tilde{h} \colon V_G \to V_{G'}$ and $h_* \colon L_G \to L_{G'}$ satisfy $\phi_{G'} \circ \tilde{h} = h_* \circ \phi_G$ after a choice of generators in G and G'. In the case where $G = G'$

and h is an isomorphism, the commutativity of the diagram shows that ϕ_G is independent of the set of generators chosen. Using this identification, we have the following Theorem.

Theorem 3.4.11. *Let X be a G-space and $K \subseteq^{\mathfrak{v}} G$ be a virtual subgroup of G of rank s. There is a shifted subgroup $\Gamma \subseteq^{\mathfrak{s}} G$ of G of rank s and canonical isomorphisms $R_\Gamma \xrightarrow{\cong} R_K$ and $H^*_\Gamma(X) \xrightarrow{\cong} H^*_K(X)$ such that the diagram*



is commutative

Proof. Let $\{v_1, \dots, v_s\}$ be a basis of K. Write $v_j = \sum_{i=1} \alpha_i^j t_i^*$ and define the elements $u_j = e + \sum_{i=1}^r \alpha_i^j \tau_j \in \Bbbk[G]$. Then $\Gamma = \Gamma(u_1, \dots, u_s)$ is a shifted subgroup of rank s. We will show now that $\ker(i_\Gamma^* \colon R \to R_\Gamma) = (K^\perp)$. In fact, since i_Γ^* is a map between polynomial rings generated in degree 1, it is enough to show that $\ker(i_\Gamma^* \colon H^1(BG) \to H^1(B\Gamma)) = K^\perp$. By Proposition 3.4.5, we know that this map is given by A^T where A is the matrix $A = (\alpha_i^j) \in \Bbbk^{r \times s}$ under the standard basis. Therefore, $\ker(i_\Gamma^*) = \ker(A^T) = \mathrm{Im}(A)^\perp = K^\perp$. Therefore, $R/(K^\perp) \cong R_\Gamma$ and the map i_Γ^* coincides with the projection j_K under this canonical isomorphism.

Now choose $N \subseteq^{\mathfrak{v}} G$ complementary to K. Let $\Gamma' \subseteq^{\mathfrak{s}} G$ be the shifted subgroup of G arising from N. Then we have an isomorphism of $\Bbbk$-algebras $\Bbbk[\Gamma] \otimes \Bbbk[\Gamma'] \cong \Bbbk[G]$ by Proposition 3.4.4. Let $Q_* \to \Bbbk$, Q'_* and P be the standard resolutions of $\Bbbk$ as $\Bbbk[\Gamma]$, $\Bbbk[\Gamma']$ and $\Bbbk[G]$-module respectively. Denote by $\widetilde{P_*} = Q_* \otimes Q'_*$ the induced resolution of $\Bbbk$ as $\Bbbk[\Gamma] \otimes \Bbbk[\Gamma'] = \Bbbk[G]$-module. Since the G-equivariant cohomology is independent from the chosen resolution, there is a chain equivalence of differential graded R-modules between the singular Cartan model $C^*_G(X)$ and the complex $\widetilde{C^*_G(X)}$ where the differential is the one induced by the resolution $\widetilde{P}_*$; that is, $d(p\otimes x) = p\otimes d(x) + \sum_{i=1}^r y_i p\otimes \sigma_i p$ where $R_\Gamma \cong \Bbbk[y_1,\dots,y_s]$ and y_i is dual to $\sigma_i = u_i + e$. Therefore, we have a chain equivalence of R_K-modules

$$C^*_K(X) = R_K\otimes_R C^*_G(X) \simeq R_K\otimes_R \widetilde{C^*_G(X)} = R_K\otimes_R (R\tilde{\otimes} C^*(X)) \cong R_\Gamma\otimes C^*(X) = C^*_\Gamma(X). \quad \square$$

In particular, if $K = \ker(H^1(BG)\to H^*(BG'))^\perp = V_{G'}^\perp$ for some 2-subtorus G', then the K-equivariant cohomology defined above and the usual G'-equivariant cohomology coincide.

Now we will look at some applications of these constructions; firstly, analogous to the torus actions case. [Allday et al., 2002, Prop.3.5], we can state the following result.

Proposition 3.4.12. *Let X be a G space and $\Bbbk$ be an infinite field of characteristic 2. Then X is G-equivariantly formal if and only if it is K-equivariantly formal for any virtual subgroup $K\subseteq^{\mathfrak{v}} G$ of rank 1.*

Proof. The direct implication follows from Remark 2.5.8 . Conversely, suppose that X is not G-equivariantly formal. If G does not act trivially on the cohomology of X, there is an element $g\in G$ that does not act trivially on $H^*(X)$ and thus X is not formal with respect to the subgroup $K = \langle g\rangle$. Let us assume then that G acts trivially on the cohomology of X. Then we have a spectral sequence

$$E_2 = H^*(BG)\otimes H^*(X) \Rightarrow H^*_G(X)$$

and as X is not G-equivariantly formal, we find $k\geq 2$ minimum so that $d_k\colon E_k^{0,q}\to E_k^{r,q+1-k}$ is non-zero. Write $d(a) = \sum_{i=1}^l p_i\otimes a_i$ where $\{a_1,\dots,a_l\}$ is a $\Bbbk$-basis of $H^{q+1-k}(X)$ and p_i is a homogeneous polynomial of degree k. By naturality of the spectral sequences, it is enough to find $K\subseteq^v G$ such that $r_K(p_1)\neq 0$ where $r_K\colon H^*(BG)\to H^*(BK)$ is the map induced by the inclusion. In fact, let $(\lambda_1,\dots,\lambda_r)\in\Bbbk^r$ not zero such that $p_1(\lambda_1,\dots,\lambda_r)\neq 0$; this is possible as $\Bbbk$ is infinite. Consider the virtual subgroup $K = \langle\lambda_1 t_1^* + \cdots + \lambda_r t_r^*\rangle$ of $H_1(BG)$. Then the map $r_K\colon R\to R_K$ satisfies that $r_K(p_1)$ is non zero and thus X is not K-equivariantly formal. $\square$

Definition 3.4.13. Let X be a G-space and K a virtual subgroup of G. For any $x \in X$ we define the virtual isotropy subgroup of x in K as

$$K_x = K \cap V_{G_x}^{\perp}$$

and the orbit filtration of G with respect to K by

$$X_i = \{x \in X : \operatorname{rank} K_x \geq \operatorname{rank} K - i\}$$

Remark 3.4.14. We can also define the K-fixed subspace $X^K \subseteq X$ as the space $X^K = \{x \in X : K_x = K\} = \{x \in X : V_{G_x} \subseteq K^{\perp}\}$. Notice that if $PK = \ker(R \to R_K)$ and $S = R \setminus PK$, then $X^S = X^K$ and by the localization theorem for the G-equivariant cohomology, there is an isomorphism of $S^{-1}R$-modules $S^{-1}H_G^*(X) \cong S^{-1}H_G^*(X^K)$.

On the other hand, we also construct an augmented K-Atiyah-Bredon sequence and it will be denoted by $\overline{AB}_K^*(X)$. Moreover, the results from the previous section involving the orbit filtration and the Atiyah-Bredon complex carry over K-equivariant cohomology (Proposition 3.3.3, Corollary 3.3.4, Theorem 3.3.5, Corollary 3.3.7 and Lemma 3.3.9).

To prove a characterization of syzygies for 2-torus actions which was also given in [Allday et al., , Thm. 5.1]. we state the following Lemma proved in [Allday et al., , Lem.4.4] which will allow an extension of the ground field of the mentioned characterization.

Lemma 3.4.15. *Let $R = \Bbbk[t_1, \ldots, t_r]$ be a polynomial ring over any field $\Bbbk$. Let M be an R-module and let u be an indeterminate so $M(u)$ is an $R(u)$-module. For any $1 \leq j \leq r$, M is a j-th syzygy over R if and only if $M(u)$ also is as a module over $R(u)$.*

Now we proceed to prove the characterization of syzygies

Theorem 3.4.16. *Let G be a 2-torus, X be a G-space and $\Bbbk$ be a field of characteristic 2. Write $R = H^*(BG; \Bbbk)$. The following conditions are equivalent for any $1 \leq j \leq r$*

1. *The augmented Atiyah-Bredon sequence* $\overline{AB}^*_G(X)$ *is exact at* $H^*_G(X_i, X_{i-1})$ *for all* $-1 \leq i \leq j-2$.

2. *The restriction map* $H^*_G(X) \to H^*_K(X)$ *is surjective for all virtual subgroups* K *of* G *of rank* $r-j$.

3. $H^*_G(X)$ *is free over all subrings* $R_{K'} \subseteq R$ *where* K' *is a virtual subgroup of rank* j.

4. $H^*_G(X)$ *is a* j*-th syzygy over* R.

Proof.

(2) $\Leftrightarrow$ (3) First we will assume that $K \subseteq G$ which will serve as a motivation to the general case. Let L be a complementary subgroup so that $G \cong K \times L$. There is a homeomorphism $X_G \cong (X_K)_L$ and so a fiber bundle $X_K \to X_G \to BL$. As in Theorem A.11, we have that the restriction map $H^*_G(X) \to H^*_K(X)$ is surjective if and only if the spectral sequence $E_2 = H^*(BL; H^*_K(X)) \to H^*_G(X)$ degenerates at this page, or equivalently, $H^*_G(X)$ is a free R_L-module which proves the equivalence for regular subgroups of G. Now we suppose that $K \subseteq^{\mathfrak{v}} G$ and K' is a complement to K. By choosing subgroups $\Gamma, \Gamma' \subseteq^s G$ corresponding to K, K' respectively; it can be shown that there is a chain equivalence of R'_Γ-modules

$$C^*_G(X) \simeq C^*_{\Gamma'}(C^*_\Gamma(X)),$$

and in particular, there is a spectral sequence $E_2 = H^*(B\Gamma'; H^*_\Gamma(X)) \Rightarrow H^*_G(X)$

(4) $\Rightarrow$ (3) By assumption, there is an exact sequence $0 \to H^*_G(X) \to F_1 \to \cdots \to F_j$ where F_i is a free R-module for $1 \leq i \leq j$. Since $R \cong R_K \otimes R_{K'}$, then each F_i is also a free $R_{K'}$-module. As $\operatorname{rank} K' = j$, by the Hilbert Syzygy Theorem (which also holds for non finitely generated modules) we conclude that $H^*_G(X)$ is a free $R_{K'}$-module.

(1) $\Rightarrow$ (4) For $j=1$, the assumption translates into the exactness of the sequence $0 \to H^*_G(X) \to H^*_G(X^G)$. Therefore, $H^*_G(X)$ is a first syzygy since $H^*_G(X^G)$ is a free R-module. Now assume that $j \geq 2$ and consider a finitely generated free resolution of $H^G_*(X)$

$$F_{j-1} \to \cdots \to F_0 \to H^G_*(X) \to 0$$

The exactness of the Atiyah-Bredon complex for $-1 \leq i \leq j-2$ implies that $\mathrm{Hom}_R(H^G_*(X),R) = H^*_G(X)$ and so $\mathrm{Ext}^i_R(H^G_*(R),R) = 0$ for $i > 0$ by Corollary 3.3.7. Therefore, dualizing the above resolution, we get an exact sequence

$$0 \to H^*_G(X) \to F^*_0 \to \cdots \to F^*_{j-1}$$

which is equivalent to $H^*_G(X)$ being a jth-syzygy.

(3) $\Rightarrow$ (1). If condition (3) holds for a field $\Bbbk$, then it also holds for $\mathbb{F}_2$. By Lemma 3.4.15 we have that this condition is also true for $\mathbb{F}_2(u)$ and so we can suppose without loss of generality that $\Bbbk$ is an infinite field, and this will imply (1) for any field as it is independent from the ground field. Firstly, for any $K \subseteq^{\mathfrak{v}} G$ and any $1 \leq j \leq r$, we denote by $P(K,j)$ the statement (3) $\Rightarrow$ (1); that is, if $H^*_K(X)$ is free over all subrings $R_{K'} \subseteq R_K$ for any virtual subgroup $K' \subseteq K$ of rank j, then the augmented Atiyah-Bredon sequence $\overline{AB}^*_K$ is exact at the position i for $-1 \leq i \leq j-2$. We will prove that $P(K,j)$ holds for any K and j by induction; that is, we assume that $P(K',j')$ holds for any subgroup $K' \subseteq K$ with $\operatorname{rank} K' < \operatorname{rank} K$ and $j' < j$, and then we will show that $P(K,j)$ holds.

We denote by X^K_i the i-th skeleton of the orbit filtration of X with respect to K. Let $n = \operatorname{rank} K$ and let $K_1, \ldots, K_m$ be the isotropy subgroups occurring for elements $x \in X^K_{j-1}$. Notice that for any t, $\operatorname{rank} K_t \geq n-(j-1) = n-j+1$ and $\dim V_t \leq j-1$ where $V_t = \ker(H^1(BK) \to H^1(BK_t))$. Choose a subspace $W \subseteq H^1(BK)$ which is transverse to all V_t and $\dim W \geq n-j+1$, and we define S to be the multiplicative subset of R_K generated by $S \setminus \{0\}$. Then for any $x \in X \setminus X^K_{j-1}$, $\operatorname{rank} K_x < n-j+1$ and so $\dim V_x \geq j$;

in particular, $W \cap V_x \neq \{0\}$, and this implies that $S^{-1}H^*_K(X_i, X_{i-1}) = 0$ for $i \geq j$ and thus $S^{-1}H^i(AB_K(X)) = 0$ as well. On the other hand, by induction hypothesis over $P(K, j-1)$, we have that $S^{-1}H^i(AB^*_K(X)) = 0$ for $i < j-2$. Proposition 3.3.9 implies that the localized Atiyah-Bredon sequence is exact everywhere. Therefore, to finish the proof we will show that the localization map $H^{j-2}(AB_K(X)) \to S^{-1}H^{j-2}(AB_K(X))$ is injective.

Let $a \in S$ and without loss of generality we might assume that $a \in W$. We will show that the multiplication by a is injective. Consider $K_a = \mathrm{Ann}_K(a) \subseteq^{\mathfrak{v}} K$. We claim that $X_i^K = X_i^{K_a}$ for $i \leq j-1$. Since $\operatorname{rank} K_a = n-1$, it is enough to show that $\operatorname{rank}(K_a)_x = \operatorname{rank} K_x - 1$. In fact, since $(K_a)_x = K_x \cap K_a$, we have that $\operatorname{rank}(K_a)_x = \operatorname{rank} K_x + \operatorname{rank} K_a - \operatorname{rank}(K_x + K_a)$. As we chose W transverse to all V_x, we have that $V_x \cap \operatorname{span}(a) = \{0\}$ and thus $K_x + K_a = K$. This shows that $\operatorname{rank}(K_a)_x = \operatorname{rank} K_x - 1$.

Now we can choose $L \subseteq^{\mathfrak{v}} K$ to be a complement of K_a in K. Notice that $\operatorname{rank} K_a = n-1$, $\operatorname{rank} L = 1$ and we have a canonical isomorphism of rings $R_L \cong \Bbbk[a]$. There is a chain equivalence $C^*_{K_a}(X) \cong C^*_K(X) \otimes_{\Bbbk[a]} \Bbbk$ of R_{K_a}-modules; using the universal coefficient theorem there is a short exact sequence

$$0 \to H^i_k(X) \otimes_{\Bbbk[a]} \Bbbk \to H^i_{K_a}(X) \to \mathrm{Tor}_1^{\Bbbk[a]}(H^{i+1}_K(X), \Bbbk) \to 0$$

by hypothesis $H^*_K(X)$ is free over $\Bbbk[a] = R_L \subseteq R_K$ and so the Tor term vanishes and $H^*_{K_a}(X) \cong H^*_K(X) \otimes_{\Bbbk[a]} (X)$. Similarly, we have an isomorphism $H^*_{K_a}(X_i^K, X_{i-1}^K) \cong H^*_K(X_i^K, X_{i-1}^K) \otimes_{\Bbbk[a]} \Bbbk$ for $i \leq j-2$. Using again the universal coefficient theorem we have a short exact sequence

$$0 \to H^{j-3}(AB^*_K(X)) \otimes_{\Bbbk[a]} \Bbbk \to H^{j-3}(AB^*_{K_a}(X)) \to \mathrm{Tor}_1^{\Bbbk[a]}(H^{j-2}_K(X), \Bbbk) \to 0$$

As $H^*_K(X) \to H^*_{K'}(X)$ is surjective for all subgroup $K' \subseteq K$ of rank $n-j$ by assumption, then $H^*_{K_a}(X) \to H^*_{K''}(X)$ is surjective for all subgroup $K'' \subseteq K_a$ of rank $(n-1)-(j-1)$.

Thus by induction hypothesis on $P(K_a, j-1)$, we have that $H^{j-3}(AB^*_{K_a}(X)) = 0$. In particular, the Tor term in the above sequence vanishes and we conclude that a is not a zero divisor in $H^{j-2}_K(X)$ for any $a \in S$ and so the localization map $H^{j-2}(AB_K(X)) \to S^{-1}H^{j-2}(AB_K(X)) = 0$ is injective. □

We finish this section with an analogous result to [Allday et al., 2014, Prop 5.2] that bounds the possible syzygy orders of the equivariant cohomology for Poincaré Duality spaces.

Proposition 3.4.17. *Let X be a $\Bbbk$-Poincaré duality space which is also a G-space where G is a 2-torus of rank r. If $H^*_G(X)$ is a j-th syzygy with $j \geq r/2$, then $H^*_G(X)$ is a free module.*

Proof. Let $M = H^*_G(X)$. As M is a j-th syzygy, Theorem 3.4.16 implies that $H^*(AB^*_G(X)) = 0$ for $i \leq j-2$. On the other hand, M also admits a regular sequence $(x_1, \ldots, x_j)$ of length j by Theorem B.10. Then we have that $\operatorname{Pr}\dim(M) = \operatorname{Pr}\dim(M/(x_1, \ldots, x_j)M) - j \leq r - j$ where $\operatorname{Pr}\dim(M)$ denotes the projective dimension of M. This implies that $\operatorname{Ext}^i_R(M, R) = 0$ for $i \geq r-j$. From the Poincaré duality isomorphism (Theorem 3.2.2) we have that $H^*_G(X) \cong H^G_*(X)[-n]$ and thus $0 = \operatorname{Ext}^*_R(M[n], R) \cong \operatorname{Ext}^i_R(H^G_*(X), R) \cong H^i(AB^*_G(X))$ by Corollary 3.3.7. Therefore, the Atiyah-Bredon sequence of X is exact at all but two adjacent positions, so it is exact everywhere by Corollary 3.3.9. Thus we get that $H^*_G(X)$ is a free R-module again by Theorem 3.4.16. □

3.5 The quotient criterion for 2-torus actions on manifolds with corners

The approach of this section is analogous to [Franz, 2017, §3]. We will study the equivariant cohomology of a space with a 2-torus action as a syzygy module using approximations of the action by the subgroups and their respective fixed point subspaces. This approach will be useful to determine the equivariant cohomology of manifolds whose orbit space carries a structure of manifold with corners.

Let R denote a polynomial ring in r variables over some field $\Bbbk$ and write $\dim R$ for the Krull dimension of R. Recall that we say that a finitely generated R-module M is a j-th syzygy over R if there is an exact sequence

$$0 \to M \to F_1 \to \cdots \to F_j$$

with finitely generated free R-modules $F_1, \dots, F_j$. In particular, M is a j-th syzygy over R if and only if

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \min(j, \operatorname{depth}_{R_\mathfrak{p}} R_\mathfrak{p})$$

for all prime ideals $\mathfrak{p} \lhd R$ (Theorem B.10).

Now let $G = (\mathbb{Z}/2)^r$ be a 2-torus of rank r, $\Bbbk$ be a field of characteristic 2 and let $R = H^*(BG) = \Bbbk[t_1, \dots, t_r]$. Let $K \subseteq G$ be a 2-subtorus and set $L = G/K$. We denote by $R_K = H^*(BK)$ and $R_L = H^*(BL)$. Recall that the projection map $G \to L$ induces a graded algebras morphism $R_L \to R$ which makes R canonically an R_L-module. If we identify L with some 2-torus complementary to K in G, we get a non-canonical isomorphism $G \cong K \times L$ and $R \cong R_K \otimes R_L$ and then R is a free R_L-module. We will discuss an analogous construction for the virtual subgroups introduced in the last section; firstly, we will introduce the following definition.

Definition 3.5.1. Let $K \subseteq^{\mathfrak{v}} G$ be a virtual subgroup of G. We say that K **acts trivially** on X if $V_{G_x} \subseteq K^\perp$ for every $x \in X$.

This definition is motivated by the situation arisen from the usual subgroups of G; namely,if a subgroup $K \subseteq G$ acts trivially on X, then $K \subseteq G_x$ for any $x \in G$. This is equivalent to $V_{G_x} \subseteq V_K$ where $V_K = \ker(H^1(BG) \to H^1(BK))$.

Lemma 3.5.2. *Let X be a G-space and $K \subseteq^{\mathfrak{v}} G$, and let $L \subseteq^{\mathfrak{v}} G$ be a complementary subgroup to K. Suppose that K acts trivially on X, then*

(i) $H^*_K(X) \cong R_K \otimes H^*(X)$ *as R_K-modules*

(ii) *There are isomorphisms of R-algebras*

$$H^*_G(X) \cong R \otimes_{R_L} H^*_L(X) \cong R_K \otimes H^*_L(X).$$

The first isomorphism is canonical and the second one depends on the splitting $K \oplus L \cong H_1(BG)$.

(iii) $\operatorname{depth}_R H^*_G(X) = \operatorname{depth}_{R_L} H^*_L(X) + \operatorname{rank} K.$

Proof. To prove (i), let $G_X = \bigcap_{x\in X} G_x$ and choose a decomposition $G \cong G_X \times G' = \langle g_1,\dots,g_k\rangle \times \langle g_{k+1},\dots,g_r\rangle$ which leads to an isomorphism $R \cong R_{G_X} \otimes R_{G'} = \Bbbk[t_1,\dots,t_r]$. Since K acts trivially on X, we have that $V_{G_x} \subseteq K^{\perp}$ for any $x \in X$. This implies that $V_{G_X} = V_{\bigcap_{\in X} G_x} = \sum_{x\in X} V_{G_x} \subseteq K^{\perp}$, or equivalently, $K \subseteq \operatorname{span}(t_1^*,\dots,t_k^*)$. Let $\{v_1,\dots,v_s\}$ be a basis of K, then $v_j = \sum_{i=1}^k \alpha_i^j t_i^*$ and so we have an associated shifted subgroup of G, $\Gamma(u_1,\dots,u_s)$ where $u_j = e + \sum_{i=1}^k \alpha_i^j \tau_i$ and τ_i acts trivially on X for $i = 1,\dots,k$. If $\Gamma'(u_{s+1},\dots,u_r)$ denotes a shifted subgroup associated to L, the isomorphism $\Bbbk[G] \cong \Bbbk[\Gamma] \otimes \Bbbk[\Gamma']$ of rings induces an isomorphism $R \cong \Bbbk[y_1,\dots,y_s] \otimes \Bbbk[y_{s+1},\dots,y_r]$ and a chain equivalence $C^*_G(X) \simeq (R_\Gamma \otimes R'_\Gamma) \otimes C^*(X)$ where the differential on the right hand side is given by $d(p\otimes\sigma) = p \otimes d(\sigma) + \sum_{i=1}^r y_i p \otimes (e+u_i)\cdot\sigma$. Since $(e+u_i)$ acts trivially on $C^*(X)$ for $i = 1,\dots,s$, we have then that $C^*_G(X) \simeq R_\Gamma \otimes (R_{\Gamma'} \otimes C^*(X)) \simeq R_\Gamma \otimes C_{\Gamma'}(X)$. This induces an isomorphism in cohomology $H^*_G(X) \cong R_K \otimes H^*_L(X)$ by Proposition 3.4.11.

To prove the second assertion, the canonical map $\Bbbk[G] \cong \Bbbk[\Gamma] \otimes \Bbbk[\Gamma'] \to \Bbbk[G] \otimes_{\Bbbk[\Gamma]} \Bbbk \cong \Bbbk[\Gamma']$ induces a map $H^*_{\Gamma'}(X) \cong H^*_L(X) \to H^*_G(X)$ of R_L-modules, and together with the canonical map $R \to H^*_G(X)$ induces a morphism

$$R \otimes_{R_L} H^*_L(X) \to H^*_G(X).$$

Combining the above maps with the isomorphism $R \cong R_K \otimes R_L$ we get a commutative diagram

$$\begin{array}{ccc} R \otimes_{R_L} H_L^*(X) & \longrightarrow & H_G^*(X) \\ \cong \downarrow & & \downarrow \cong \\ R_K \otimes R_L \otimes_{R_L} H_L^*(X) & \xrightarrow{\cong} & R_K \otimes H_L^*(X) \end{array}$$

To prove the last assertion, by (ii) we have isomorphisms

$$\mathrm{Ext}^i_R(H_G^*(X), R) = \mathrm{Ext}^i_R(R \otimes_{R_L} H_L^*(X), R) = R \otimes_{R_L} \mathrm{Ext}^i_{R_L}(H_L^*(X), R_L)$$

and thus $\mathrm{Ext}^i_R(H_G^*(X), R) = 0$ if and only if $\mathrm{Ext}^i_{R_L}(H_L^*(X), R_L) = 0$ because R is a free R_L-module. Therefore, the result follows from $\dim R = \dim R_L + \mathrm{rank}\, K$. □

From this result we may prove the following Lemma.

Lemma 3.5.3. *Let $\mathfrak{p} \lhd R$ be a prime ideal and $K \subseteq^{\mathfrak{v}} G$ be a virtual subgroup such that $(K^{\perp}) = \mathfrak{q}$ where $\mathfrak{q}$ is the prime ideal generated by $\mathfrak{p} \cap H^1(BG)$. Let $L \subseteq^{v} G$ be a complementary subgroup to K. Then for any $i \geq 0$ we have*

$$\mathrm{Ext}^i_{R_\mathfrak{p}}(H_G^*(X)_\mathfrak{p}, R_\mathfrak{p}) = 0 \Leftrightarrow \mathrm{Ext}^i_{R_L}(H_L^*(X^K), R_L) = 0$$

Proof. Since X is a finite G-CW-complex, by the Localization Theorem (see Remark 3.4.14) and Lemma 3.5.2 we have isomorphisms

$$H_G^*(X)_\mathfrak{p} \cong H_G^*(X^K)_\mathfrak{p} \cong (R \otimes_{R_L} H_G^*(X^K))_\mathfrak{p} \cong R_\mathfrak{p} \otimes_{R_L} H_G^*(X^K),$$

and as in the proof of Lemma 3.5.2 we get

$$\mathrm{Ext}^i_{R_\mathfrak{p}}(H_G^*(X)_\mathfrak{p}, R_\mathfrak{p}) \cong R_\mathfrak{p} \otimes_{R_L} \mathrm{Ext}^i_{R_L}(H_L^*(X^K), R_L)$$

To conclude, it is enough to show that $R_\mathfrak{p}$ is a faithfully flat R_L-module.
Indeed, flatness of $R_\mathfrak{p}$ follows from the facts that R is a free R_L-module and the localization is exact. To prove that it is faithfully flat, we just check that the localization map

$$(R \otimes_{R_L} N) \cong R_K \otimes N \to (R_K \otimes N)_\mathfrak{p} \cong R_\mathfrak{p} \otimes_{R_L} N$$

is injective for any R_L-module N.

Let $a \notin \mathfrak{p}$, write $a = \sum_i a_i \otimes u_i$ where $a_i \in R_K$ and $\{u_i\}$ is a homogeneous basis for R_L and $u_0 = 1$. Since $a \notin \mathfrak{p}$, almost all $a_i = 0$ and we can suppose that $a_0 \neq 0$. Therefore for any non-zero $x \in R_K \otimes N$, $a \cdot x \neq 0$ which implies that the localization map is injective. □

With these results at hand, we proceed to prove the following result.

Lemma 3.5.4.

(i) For any subgroup $K \subseteq^{\mathfrak{v}} G$,

$$\operatorname{depth}_R H_G^*(X^K) \geq \operatorname{depth}_R H_G^*(X)$$

(ii) $H_G^*(X)$ is a j-th syzygy over R if and only if

$$\operatorname{depth}_{R_L} H_L^*(X^K) \geq \min(j, \operatorname{rank} L)$$

for any subgroup $K \subseteq^{\mathfrak{v}} G$ with complementary subgroup $L \subseteq^{\mathfrak{v}} G$.

Proof. To prove statement (i), let $\mathfrak{p} = \ker(H^*(BG) \to H^*(BK))$ and let $i < \operatorname{depth} H_G^*(X)$. It is enough to show that $\operatorname{Ext}_R^{r-i}(H_G^*(X^K), R) = 0$; by Lemma 3.5.2-(iii) this is equivalent to check that $\operatorname{Ext}_{R_L}^{\operatorname{rank} L-(i-\operatorname{rank} K)}(H_L^*(X^K), R_L) = 0$. It will follow from Lemma 3.5.3 as

$$\operatorname{Ext}_{R_L}^{\operatorname{rank} L-(i-\operatorname{rank} K)}(H_L^*(X^K), R_L) = 0 \Leftrightarrow (\operatorname{Ext}_R^{r-i}(H_G^*(X), R))_{\mathfrak{p}} = 0.$$

The latter term is zero as $i < \operatorname{depth} H_G^*(X)$. Now we proceed to prove the equivalence in (ii).

$\Leftarrow$: Let $\mathfrak{p} \lhd R$ be a prime ideal and set $\mathfrak{q}, K$ and L as in Lemma 3.5.3. Notice that for any $i > \max(\operatorname{rank} L - j, 0)$, we have that $\operatorname{Ext}_{R_L}^i(H_L^*(X), R_L) = 0$; therefore,

$$\operatorname{Ext}_{R_{\mathfrak{p}}}^i(H_G^*(X)_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$$

by Lemma 3.5.3. Since $\dim R_{\mathfrak{p}} \geq \operatorname{rank} L$, the above equality also holds for all $i > \max(\bar{r} - j, 0)$ where $\bar{r} = \dim R_{\mathfrak{p}}$. This means that

$$\operatorname{depth}_{R_{\mathfrak{p}}} H^*(X)_{\mathfrak{p}} \geq \min(j, \bar{r}),$$

that is, $H_G^*(X)$ is a j-th syzygy by Theorem B.10.

$\Rightarrow$: For a given subgroup $K \subseteq^{\mathfrak{v}} G$, consider $\mathfrak{p} = \ker(H^*(BG) \to H^*(BK))$, in this case, $\operatorname{rank} L = \dim R_{\mathfrak{p}}$. Therefore by Proposition B.10.

$$\operatorname{depth}_{R_{\mathfrak{p}}} H_G^*(X)_{\mathfrak{p}} \geq \min(j, \operatorname{rank} L),$$

as before, for $i > \max(\operatorname{rank} L - j, 0)$

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(H_G^*(X)_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$$

and by Lemma 3.5.3

$$\operatorname{Ext}^i_{R_L}(H_L^*(X), R_L) = 0$$

that is, $\operatorname{depth}_{R_L} H^*(X^K) \geq \min(j, \operatorname{rank} L)$ as desired. □

Now we can prove the following results.

Proposition 3.5.5. *Let X be a G-manifold and $j \geq 0$. Then $H^*(X)$ is a j-th syzygy if and only if $H^i(AB_L^*(X^K)) = 0$ for any subgroup K occurring as an isotropy subgroup in X where $L = G/K$ and $i > \max(\operatorname{rank} L - j, 0)$.*

Proof. As we discussed in the proof of Theorem 3.4.16, $H^*(X)$ is a j-th syzygy with coefficients over a field $\Bbbk$ if and only if it is over the field $\mathbb{F}_2$. We will show that condition (ii) of Lemma 3.5.4 is satisfied for any subgroup $K \subseteq G$. Let K be a subgroup of G. Then X^K is a closed submanifold of X by the tubular neighbourhood theorem. For a connected component $Y \subseteq X^K$, there is a principal orbit G/G_x where $x \in Y$, so $K \subseteq G_x$ as subgroup. Set $K' = G_x$. $L' = G/K'$ and write $\operatorname{rank} K' = \operatorname{rank} K + k$ for some integer $k \geq 0$. Using Lemma 3.5.3-(iii) and 3.5.4 we get

$$\begin{aligned}\operatorname{depth}_{R_L} H_L^*(Y) &\geq \operatorname{depth}_{R_L} H_L^*(X^{K'}) \\ &= \operatorname{depth}_{R'_L} H_{L'}^*(X^{K'}) + k \\ &\geq \min(j, \operatorname{rank} L') + k \geq \min(j, \operatorname{rank} L).\end{aligned}$$

Finally, observe that

$$\operatorname{depth}_{R_L} H_L^*(X^K) = \min\{i : \operatorname{Ext}_{R_L}^{\operatorname{rank} L - i}(H_L^*(X^K), R_L) \neq 0\} = \min\{i : H^{\operatorname{rank} L - i}(AB_L^*(X^K)) \neq 0\}$$

by Corollary 3.3.7 and the Poincaré duality isomorphism $H_*^G(X) \cong H_G^*(X)$. Then

$$\operatorname{depth}_{R_L} H_L^*(X^K) \geq \min(j, \operatorname{rank} L) \Leftrightarrow H^i(AB_L^*(X^K)) = 0 \text{ for all } i > \max(\operatorname{rank} L - j, 0). \quad \square$$

The next Theorem is one of the main results of this section.

Theorem 3.5.6. *If $H_G^*(X)$ is a j-th syzygy over R then so is $H_L^*(X^K)$ over R_L for any subgroup $K \subseteq^{\mathfrak{d}} G$ and complementary subgroup $L \subseteq^{\mathfrak{d}} G$. Furthermore, If K is a subgroup of G, then L can be canonically identified with the quotient G/K.*

Proof. Let $K \subseteq^{\mathfrak{d}} G$, $Y = X^K$ and L be a complementary subgroup to K. We will show that the condition of Lemma 3.5.4 holds for $H_L^*(Y)$. Let $K' \subseteq^{\mathfrak{d}} L$ and choose a complementary subgroup $L' \subseteq^{\mathfrak{d}} L$ of K' in L. Notice that $K_0 = K \oplus K'$ is a complementary subgroup of L' in G, and we have that $Y^{K'} = (X^K)^{K'} = X^{K_0}$. On the other hand, there is an isomorphism $H_{L'}^*(Z) \cong H_{L,L'}^*(Z)$ where $H_{L,L'}^*(Z)$ denotes the cohomology of the complex $C_{L,L'}^*(Z) = R_L/(\operatorname{Ann}_L(L')) \otimes_{R_L} C_L^*(Z)$ for any G-space Z; in fact, this follows from the ring isomorphism $R_L/(\operatorname{Ann}_L(L')) \cong R_{L'}$. Notice that $H_{L.L'}^*(Z)$ is canonically a $R_{L,L'}$-module where $R_{L,L'} = R_L/(\operatorname{Ann}_L(L'))$.

Applying then Lemma 3.5.4 to the subgroup $K_0 \subseteq^{\mathfrak{d}} G$, we get

$$\operatorname{depth}_{R_{L,L'}} H_{L'}^*(Y^{K'}) = \operatorname{depth}_{R_{L'}} H_{L'}^*(X^{K_0}) \geq \min(j, \operatorname{rank} L')$$

showing that $H_L^*(Y)$ is a j-th syzygy over R_L by Lemma 3.5.4. $\square$

As a Corollary we have the following result.

Corollary 3.5.7. *Suppose that $H_G^*(X)$ is a torsion-free $H^*(BG)$-module. Let $K \subseteq^{\mathfrak{d}} G$ and $Z \subseteq X^K$ be a connected component. Then $Z \cap X^G \neq \emptyset$.*

Proof. Let L be a complementary subgroup to K. By Theorem 3.5.6 we have that $H_L^*(X^K)$ is a torsion-free $H^*(BL)$-module. As $H_L^*(X^K) = \bigoplus_{Z \subseteq X^K} H_L^*(Z)$, where the index runs over all the connected components of X^K, we get that $H^*_L(Z)$ is also a torsion-free $H^*(BL)$-module. This implies that $Z^L = Z \cap X^G \neq \emptyset$ by Corollary 2.4.6. □

Now we will relate this construction to G-manifolds M whose orbit space M/G inherits a structure of manifold with corners. We first quickly review the definition of manifolds with corners.

Definition 3.5.8. Let M be a topological manifold with boundary of dimension n. We say that M **is a manifold with corners** if M has an atlas $\{(U_i, \varphi_i)\}$ where $\varphi_i \colon U_i \to V_i$ is a homeomorphism of U_i onto an open subset $V_i \subseteq R_{n,r} = [0,\infty)^r \times \mathbb{R}^{n-r}$ for some fixed $r \leq n$, and the map $\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a diffeomorphism for all i, j.

For any $z = (x,y) \in \mathbb{R}_{n,r}$, we define c_z as the number of zero coordinates of x in $[0,\infty)^r$. If M is a manifold with corners, then c_m is well-defined for any $m \in M$. We say that F is a facet of M if F is the closure of a connected component of the subspace $M_1 = \{m \in M \colon c_m = 1\}$. Notice that F is a $(n-1)$-dimensional submanifold with boundary of ∂M and $\bigcup_{F \text{ facet}} F = \partial M$. Moreover, any finite intersection of facets $\bigcap_{i=1}^k F_i$ is either empty, or a disjoint union of submanifolds of M of codimension k. A connected component of such an intersection is called a face of M of codimension k. Notice that any face of M of codimension k is the closure of a connected component of the subspace $M_k = \{m \in M : c_m = k\}$.

Remark 3.5.9. Any manifold with corners becomes a filtered space by setting $X_i = \bigcup_{k \leq i} M_{r-i}$, so X_i consists of all faces of codimension at least $r-i$. In particular, $X_0 = M_r$ and $X_r = M$.

A face of $\mathbb{R}_{r,n} = [0,\infty)^r \times \mathbb{R}^{n-r}$ of codimension k is the subspace $A_I = \{(x,y) \in \mathbb{R}_{r,n} : x_i = 0 \text{ for } i \notin I\}$ for some $I \subseteq \{1,\dots,r\}$, $|I| = k$. In particular, the facets of $R_{r,n}$ are the subspaces $A^j = \{(x,y) \in \mathbb{R}_{r,n} : x_j = 0\}$. We say that M is a nice manifold with corners if for any $m \in M$ there is a neighbourhood $U \subseteq M$ and an open set $V \subseteq \mathbb{R}_{n,r}$ and a homeomorphism $\varphi \colon U \to V$ such that $\varphi^{-1}(V \cap A^j) = U \cap F$ for any facet F of M and any $1 \leq j \leq r$.

Now we will relate this construction to 2-torus actions. Let G be a 2 torus of rank r. There is a standard action G on the space $\mathbb{R}^n$ with $n \geq r$. In fact, identifying $\mathbb{Z}/2 = \{\pm 1\}$ we have an action of G on $\mathbb{R}^n$ given by

$$(g_1, \dots, g_r) \cdot (x_1, \dots, x_r, x_{r+1}, \dots, x_n) = (g_1 x_1, \dots, g_r x_r, x_{r+1}, \dots, x_n)$$

and thus the quotient space $\mathbb{R}^n/G \cong \mathbb{R}_{r,n} = [0,\infty)^r \times \mathbb{R}^{n-r}$ is a manifold with corners. More generally, we have the following definition.

Definition 3.5.10. Let X be a G-manifold of dimension n. A **G-standard chart** (U, φ) of $x \in X$ is a G-invariant open neighbourhood U of x in X and a G-equivariant homeomorphism $\varphi\colon U \to V$ on some G-invariant open set $V \subseteq \mathbb{R}^n$ (with the standard action of G on $\mathbb{R}^n$). We say that X is a **locally standard G-manifold** (or that the G-action is locally standard) if X has an atlas $\{(U_i, \varphi_i)\}$ consisting of standard charts such that the change of coordinates $\varphi_i \circ \varphi_j^{-1}\colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a G-equivariant diffeomorphism for all i, j.

In particular, the quotient space X/G is a manifold with corners. Any face $P \subseteq X/G$ is a topological manifold with boundary ∂P. Let $\pi\colon X \to X/G$ be the quotient map, and we identify X^G with its image in X/G. For any subspace $A \subseteq X/G$ we write $\pi^{-1}(A) = X^A$. Notice that all points in X lying over the interior of P have a common isotropy group G_P. We denote by $G_P^* = G/G_P$ which is isomorphic to a complementary 2-torus of G_P in G. Also, we write $\operatorname{rank} P = \operatorname{rank} G_P^*$. Observe that $X^P = \pi^{-1}(P)$ is a connected component of X^{G_P} and the set $X^{P \setminus \partial P} = X^P \setminus X^{\partial P}$ is the open subset of X^P where G_P^* acts freely. This implies that

$$H_{G_P^*}(X^P, X^{\partial P}) \cong H^*(P, \partial P). \tag{3.5.1}$$

For two faces P, Q of X/G we write $P \subseteq_1 Q$ if $P \subseteq Q$ and $\operatorname{rank} Q = \operatorname{rank} P + 1$. If $F \subseteq_1 X/G$, X^F is a closed submanifold of X of codimension 1 and we denote by N_F its normal bundle. For any face $P \subseteq F$, consider $E_{F.P}$ the vector bundle over $X^{P \setminus \partial P}$ obtained as the pullback of N_F under the inclusion of $X^{P \setminus \partial P}$ on X^F. Under the inclusion $(X^P, X^{\partial P}) \to (E_{F,P}, E_{F,P}^0)$, the equivariant

Thom class of $E_{F,P}$ induces a class in the equivariant cohomology $e_{F,P} \in H^1_G(X^P, X^{\partial P})$. Finally, we denote by $t_{F,P} \in H^*(BG_p)$ the restriction of $e_{F,P}$ under the inclusion $(pt, \emptyset) \to (X^P, X^{\partial P})$. Notice that $t_{F,P}$ is the equivariant Euler class of the G_P-equivariant line bundle over a point which corresponds to the generator of $H^*(BG_P)$ in $H^1(BG_P)$.

Remark 3.5.11. For any face P of X/G, consider the $\Bbbk$-algebra $R_P = H^*(BG_P)$. Notice that P is a connected component of the intersection $\bigcap_{P \subseteq F \subseteq_1 X/G} F$ and thus X^P is a connected component of the intersection $\bigcap_{P \subseteq F \subseteq_1 X/G} X^F$. Moreover, for any point $x \in X^{P \setminus \partial P}$, there is an isomorphism $G_P \cong \prod_{P \subseteq F \subseteq_1 X/G} G_F$ by looking at a standard chart of x in X. This implies that $t_{F,P}$ is a basis for the vector space $H^1(BG_P)$ which extends to an isomorphism of algebras

$$R_P \cong \Bbbk[t_{F,P} : P \subseteq F \subseteq_1 X/G]$$

If $P \subseteq Q$, $G_Q \subseteq G_P$ and we have a canonical map $\rho_{PQ} \colon R_P \to R_Q$. It follows from the naturality of the Euler class and the above remark that $\rho_{PQ}(t_{F,P}) = t_{F,Q}$ if $Q \subseteq F$ and 0 otherwise. Now we will proceed to prove the following lemma.

Lemma 3.5.12. *Let P be a face in X/T.*

(i) The composition

$$\phi_P \colon H^*(P, \partial P) \xrightarrow{\cong} H^*_{G^*_P}(X^P, X^{\partial P}) \to H^*_G(X^P, X^{\partial P})$$

induces a map $\psi_P \colon H^*(P, \partial P) \otimes R_P \to H^*_G(X^P, X^{\partial P})$ which is an isomorphism of graded vector spaces.

(ii) If $P \subseteq_1 Q$ the following diagram

$$\begin{array}{ccc} H^*(P, \partial P) \otimes R_P & \xrightarrow{\psi_P} & H^*_G(X^P, X^{\partial P}) \\ \downarrow{\scriptstyle \delta \otimes \rho_{PQ}} & & \downarrow{\scriptstyle \delta} \\ H^{*+1}(Q, \partial Q) \otimes R_Q & \xrightarrow{\psi_Q} & H^{*+1}_G(X^Q, X^{\partial Q}) \end{array}$$

is commutative where δ is the connecting homomorphism arisen from the cohomology long exact sequence of the triple $(Q, \partial Q, \partial Q \setminus (P \setminus \partial P))$.

Proof. To prove the first claim, notice that the map $\phi_P\colon H^*(P,\partial P)\xrightarrow{\cong} H^*_{G^*_P}(X^P,X^{\partial P})\to H^*_G(X^P,X^{\partial P})$ is the composite of the isomorphism (3.5.1) and the map in equivariant cohomology induced by the canonical projection $G\to G^*_P=G/G_P$. Suppose that $F_1,\dots,F_k$ are the facets containing P. Using (i), we can define a map

$$\psi_P\colon H^*(P,\partial P)\otimes R_P\to H^*_G(X^P,X^{\partial P})$$

by setting $\psi_P(\alpha\otimes t^{m_1}_{F_1,P}\cdots t^{m_k}_{F_k,P})=\phi_P(\alpha)e^{m_1}_{F_1,P}\cdots e^{m_k}_{F_k,P}$. On the other hand, similar to Lemma 3.5.2(i), we have an isomorphism of algebras

$$\rho\colon H^*_G(X^P,X^{\partial P})\to R_P\otimes H_{G^*_P}(X^P,X^{\partial P})\to H^*(P,\partial P)\otimes R_P$$

by choosing a splitting of $G=G_P\times G^*_P$. In particular, for $e_{F,P}\in H^1(X^P,X^{\partial P})$, we have that $\rho(e_{F,P})\in H^0(P,\partial P)\otimes(R_P)_1\oplus H^1(P,\partial P)\otimes(R_p)_0\cong(R_P)_1\oplus H^1(P,\partial P)$. As $t_{F,P}$ is the restriction of $e_{F,P}$ to R_P we have then that $\rho(e_{F,P})=t_{F,P}+a_F$ for some $a_F\in H^1(P,\partial P)$. Using this computation we get that for $\alpha\in H^*(P,\partial P)$ it holds that

$$\begin{aligned}
\rho\circ\psi_P(\alpha\otimes t^{m_1}_{F_1,P}\cdots t^{m_k}_{F_k,P}) &= \rho(\phi_P(\alpha)e^{m_1}_{F_1,P}\cdots e^{m_k}_{F_k,P})\\
&=(\alpha\otimes 1)\rho(e_{F_1,P})^{m_1}\cdots\rho(e_{F_k,P})^{m_k}\\
&=(\alpha\otimes 1)(1\otimes t_{F_1,P}+a_{F_1}\otimes 1)^{m_1}\cdots(1\otimes t_{F_k,P}+1\otimes a_{F_k})^{m_k}\\
&=\alpha\otimes(t^{m_1}_{F_1,P}\cdots t^{m_k}_{F_k,P})+S
\end{aligned}$$

where S consists of sums of terms in $H^*(P,\partial P)\otimes H^*(R_P)$ whose elements in the second factor are of degree lower than $m_1+\cdots+m_k$; therefore, we obtained that $\rho\circ\psi_P$ is bijective and so is ψ_P.

Finally, to prove (iii), we need to check that $\delta\psi_P(\alpha\otimes t^{m_1}_{F_1,P}\cdots t^{m_k}_{F_k,P})=\psi_Q(\delta(\alpha)\otimes\rho_{PQ}(t^{m_1}_{F_1,P}\otimes t^{m_k}_{F_k,P}))$. As the maps ϕ_P,ϕ_Q arise from natural constructions, they commute with δ. Furthermore, since $\rho_{PQ}(t_{F,P})$ is either $t_{F,Q}$ if $Q\subseteq F$ or zero otherwise, we only need to prove that $\delta(\beta e_{F,P})$ is either $\delta(\beta)e_{F,Q}$ if $Q\subseteq F$ or zero otherwise. Recall that δ arises from the connecting homomorphism $\delta\colon H^*(P,\partial P)\cong H^*(\partial Q,\partial Q\setminus(P\setminus\partial P))\to H^*(Q,\partial Q)$ which induces the map

$$\delta\colon H_G^*(X^P,X^{\partial P})\cong H_G^*(X^{\partial Q},X^{\partial Q\setminus(P\setminus\partial Q)})\to H_G^{*+1}(X^Q,X^{\partial Q}).$$

In the case $P\subseteq Q$, by the Thom isomorphism theorem we have isomorphisms $H_G^{*-1}(X^{P\setminus\partial P})\cong H_G^*(X^P,X^{\partial P})$ and $H_G^{*-1}(X^{Q\setminus\partial Q})\cong H_G^*(X^Q,X^{\partial Q})$ induced by the multiplication by $e_{F,P}$ and $e_{F,Q}$ respectively. As both $e_{F,P}$ and $e_{F,Q}$ are restrictions of the equivariant Euler class of the normal bundle N^F, we have that $\delta(\beta e_{F,P})=\delta(\beta)e_{F,Q}$. In the second case, we have that $e_{F,P}$ is then the restriction of the Euler class of the normal bundle of X^P in X^Q as $P\subseteq_1 Q$. By the Thom-Gysin exact sequence we have that δ vanishes precisely in the multiples of $e_{F,P}$. □

For a face P of X/G, the filtration by faces leads to a spectral sequence with E_1-term given by

$$E_1^{p,q}=\bigoplus_{\substack{Q\subseteq P\\ \operatorname{rank}Q=i}} H^{p+q}(Q,\partial Q)\Rightarrow H^*(P)$$

the columns of this spectral sequence give rise to a complex that will be denoted by $B^i(P)$. This complex will be related to the Atiyah-Bredon sequence constructed in Section §3.3 and it will provide a criterion to the syzygies in G-equivariant cohomology as it is shown in the following theorem which is analogous to [Franz, 2017, Thm.1.3] for the torus case.

Theorem 3.5.13. *Let X be a G-manifold with a locally standard action of a 2-torus G. Then $H_G^*(X)$ is a j-th syzygy over $H^*(BG;\Bbbk)$ if and only if for any face P of the manifold with corners $M=X/G$ we have that $H^i(B^*(P))=0$ for any $i>\max(\operatorname{rank}P-j,0)$*

Proof. Let Q be a face of X/G. We define the element $t_Q=\prod_{Q\subseteq F\subseteq_1 X/G}t_{F,Q}\in R_Q$. These elements induce an isomorphism of vector spaces $R_Q\cong\bigoplus_{Q\subseteq P}R_Pt_P$. On the other hand, by Lemma 3.5.12 there is an isomorphism of vector spaces $H^*(Q,\partial Q)\otimes R_Q\to H_G^*(X^Q,X^{\partial Q})$ compatible with the differentials. We have then an isomorphism

$$\bigoplus_{Q:\operatorname{rank}Q=i} H^*(Q,\partial Q)\otimes R_Q\cong\bigoplus_{Q:\operatorname{rank}Q=i} H_G^*(X^Q,X^{\partial Q}).\tag{3.5.2}$$

Noticing that the i-th equivariant skeleton of X is given by $X^i = \bigcup_{\substack{P \\ \operatorname{rank} P = i}} X^P = \bigcup_{\substack{P \\ \operatorname{rank} P = i+1}} X^{\partial P}$, we see that the last term of (3.5.2) is the i-th term of the Atiyah-Bredon sequence $AB_G^i(X)$ and so there is an isomorphism (with an appropriate degree shift)

$$\bigoplus_{\substack{Q \\ \operatorname{rank} Q = i}} \bigoplus_{\substack{P \\ Q \subseteq P}} H^*(Q, \partial Q) \otimes R_P t_P \cong \bigoplus_{P \subseteq X/G} B^i(P) \otimes R_P t_P \cong AB_G^i(X)$$

compatible with the differentials. Therefore, $H^i(AB_G^*(X)) = \bigoplus_{P \subseteq X/G} H^i(B^*(P)) \otimes R_P t_P$.

From Proposistion 3.5.5, we have that $H_G^*(X)$ is a j-th syzygy if and only if $H^i(AB_{G_P^*}(X^P)) = 0$ for all faces P and $i > \max(\operatorname{rank} P - j, 0)$. The isomorphism above shows that this condition is equivalent to the vanishing of $H^i(B^*(P))$ for all P and $i > \max(\operatorname{rank} P - j, 0)$. □

We will use this criterion to construct syzygies in G-equivariant cohomology for 2-torus actions. The dimension of a manifold with a locally standard action of a 2-torus is constrained to the rank of the torus. In fact, if $G = (\mathbb{Z}/2)^r$ and X is a G-manifold with a locally standard action of G and $X^G \neq \emptyset$, then $\dim X \geq r$. In fact, if the action is locally standard, then X^G is a submanifold of codimension at least r and there can not be any fixed points if $\dim X < r$.

Example 3.5.14. If X is a manifold with a locally standard action of $\mathbb{Z}/2$, then the orbit space $M = X/G$ is a manifold with boundary. Conversely, any manifold with boundary can be realized as the orbit space of the manifold $X = (M \sqcup M)/\partial M$ with the involution induced by the map $M \sqcup M \to M \sqcup M$ that swaps factors. The action is locally standard on X as it can be seen as the reflection along the hyperplane where ∂M lies and so $X^G = \partial M$.

Theorem 3.5.13 translates in this case on the statement that X is G-equivariantly formal if and only if the map $H^*(\partial M) \to H^{*+1}(M, \partial M)$ is surjective, or equivalently, the map $H^*(M) \to H^*(\partial M)$ induced by the inclusion is injective. For example, if $M = S^1 \times [0,1]$ is a cylinder (see Fig.4.1), then the map $H^*(M) \to H^*(\partial M)$ is injective and so the manifold X is G-equivariantly formal. It is easily seen that X is homeomorphic to the torus $S^1 \times S^1$ and the involution is given by the axis reflection on one S^1 factor. Notice that $4 = b(X) = b(X^G) = b(\partial M)$ so the conclusion of Theorem 3.5.13 agrees with the one from Theorem 2.5.5.

On the other hand, M does not need to be orientable; for example, if M is the Mobius strip, then M can be realized as the orbit space of a Klein bottle X (see Fig.4.2). Moreover, the map induced in cohomology by the inclusion $\partial M \to M$ is the zero map and thus Theorem 3.5.13 implies that $H^*_G(X)$ is not equivariantly formal. This also agrees with the Betti number criterion as $3 = b(X) \neq b(X^G) = b(\partial M) = 2$.

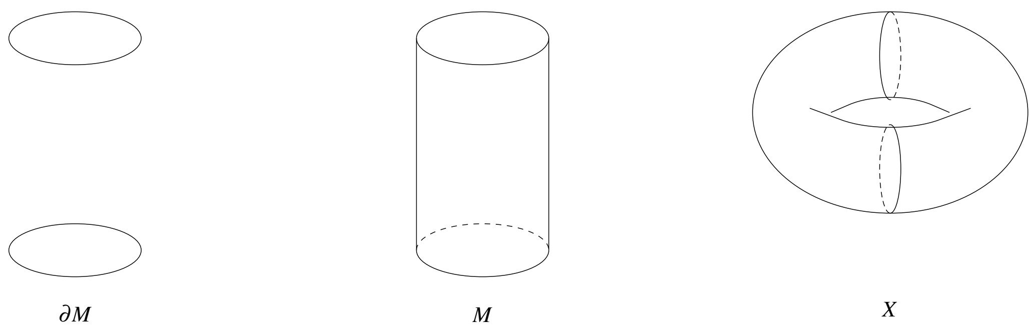


Fig.4.1 - Orientable Manifold with boundary M

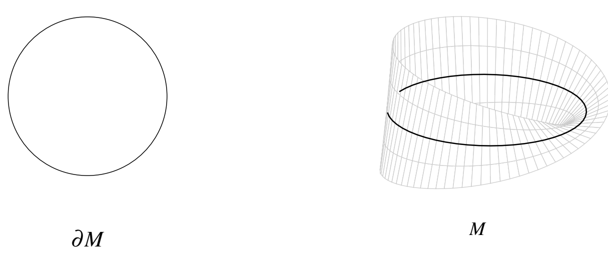


Fig 4.2 - Non-orientable manifold with boundary M

Remark 3.5.15. More generally, following the constructions presented in [Lü and Masuda, 2008] and [Yu et al., 2012], under some conditions a manifold with corner M can be realized (up to homeomorphism) as the quotient space X/G of a manifold X with a locally standard action of a suitable 2-torus G. In fact, let M be a nice manifold with corners and let $\mathcal{F} = \{F_\alpha\}$ denote the set of all facets of F. If $r = \max\{i : \bigcap_{\gamma\in\Gamma,\, |\Gamma|=i} F_\gamma \neq \emptyset\}$, then M is locally diffeomorphic to $\mathbb{R}^n \times [0,\infty)^r$ for some integer $n \geq 0$, but not to $\mathbb{R}^n \times [0,\infty)^{r+1}$.

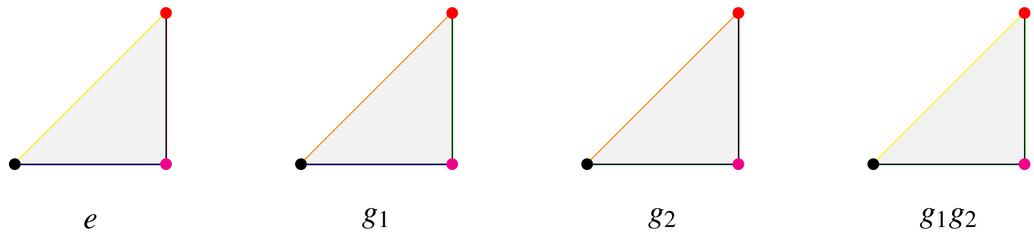
Suppose further that there is a function $\lambda : \mathcal{F} \to G = (\mathbb{Z}/2)^r$ such that the set $\{\lambda(F_i) : i \in I\}$ is linearly independent over $\mathbb{Z}/2$ whenever $\bigcap_{i\in I} F_i \neq \emptyset$. As M is a nice manifold with corners, any face P is realized as the intersection of the facets that contain P; namely, $P = \bigcap_{P\subseteq F} F$. Set G_P be the subgroup of G generated by $\{\lambda(F) : P \subseteq F\}$. Consider the space $X = (M \times G)/\sim$ where $(x,g) \sim (x,h)$ if and only if $x \in P$ and $gh \in G_p$. X is locally diffeomorphic to $\mathbb{R}^{n+r}$ and the canonical action of G in X given by $g\cdot[x,h] = [x,gh]$ is locally standard and there is a homeomorphism $X/G \cong M$.

We will say that X is obtained from M by *the unfolding construction*. The following examples will illustrate this construction.

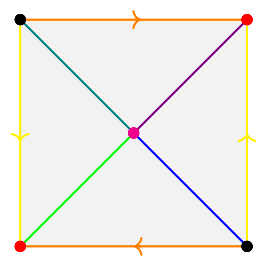
Let $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ and let M be a nice manifold with corners locally diffeomorphic to $\mathbb{R}^n \times [0,\infty)^2$. Then the face lattice of M consists of facets F (of rank 1) and faces P of rank 0. Set X be the manifold with a locally standard action of G constructed above. From Theorem 3.5.13 we have the following cases.

- $H^*_G(X)$ is a 2-nd syzygy (or equivariantly formal) if and only if for any facet F the map $H^*(\partial F) \to H^{*+1}(F,\partial F)$ is surjective and the sequence $\bigoplus_P H^*(P) \to \bigoplus_F H^{*+1}(F,\partial F) \to H^{*+2}(M,\partial M) \to 0$ is exact at the second and third position.

  Example 3.5.16. Let M be the 1-simplex $\{(x_1,x_2) \in \mathbb{R}_2, x_1+x_2 \leq 1, x_1,x_2 \geq 0\}$. Then we take $\tilde{X} = M \times G$ which is homeomorphic to 4 copies of M each labeled by an element of $g \in G$. At the same time, we label the facets of M by the elements g_1, g_2, g_1g_2 so they are pairwise linearly independent. The manifold $X = \tilde{X}/\sim$ is then constructed by gluing together points of the same color as shown below.



Since the facets of M are acyclic, and so is M, it is easy to check that the G-equivariant cohomology of X is a free module by looking at the complex $B^*(P)$ described above. This can be also seen directly from the construction of the manifold X as we obtain that X is diffeomorphic to $\mathbb{R}P^2$ as the next picture illustrates.



where the action of G on X is given by the reflection along the main diagonals on the square. Therefore, X^G consists of 3 points and thus $b(X) = b(X^G) = 3$; which confirms the result obtained from the quotient criterion.

- $H^*_G(X)$ is a 1-st syzygy if and only if for any facet F the map $H^*(\partial F) \to H^{*+1}(F, \partial F)$ is surjective and the sequence $\bigoplus_P H^*(P) \to \bigoplus_F H^{*+1}(F, \partial F) \to H^{*+2}(M, \partial M) \to 0$ is exact only at the third position. If the latter holds, the sequence is exact also at the second position. In fact, by Theorem 3.4.17, we have that any 1-st syzygy (or torsion-free) module is a 2-nd syzygy (or free) module as $\operatorname{rank} G = 2$.

Example 3.5.17. To construct a space whose equivariant cohomology is torsion-free but not free, we need to consider an action of a 2-torus of rank at least 3. Following [Franz, 2015, Lemma 7.1], let us start with the following manifold with corners

$$M = \{(u,z) \in (\mathbb{R}^2 \times \mathbb{R}_+)^3 : |u_i|^2 + |z_i|^2 = 1, u_1 + u_2 + u_3 = 0\}$$

and $i = 1,2,3$, where $\mathbb{R}_+$ denotes the non-negative real numbers. Then M is a smooth manifold with corners locally diffeomorphic to $[0,\infty)^3 \times \mathbb{R}$. The projection $M \to (\mathbb{R}^2)^3$ of the first component induces a homeomorphism between M and the subspace of $(\mathbb{R}^2)^3$ consisting of these triples (u_1,u_2,u_2) such that $\max\{|u_1|,|u_2|,|u_3|\} \leq 1$ and $u_1+u_2+u_3 = 0$. The latter space describes the configuration of triangles (including degenerate triangle) in $\mathbb{R}^2$ with sides of length at most 1. Therefore, M is homeomorphic to the intersection of a 6-dimensional ball with a linear subspace of codimension 2 and thus M is topologically a 4-dimensional ball. In particular, $\partial M \cong S^3$ and $H^*(M,\partial M) \cong \widetilde{H}^*(S^4)$. Now we will look at the face decomposition of M.

- M has exactly one face P of rank zero. Namely, it is given by those elements $(u,z) \in M$ such that $z_i = 0$, and then $u_i \in S^1$ for all i. Since one of the $u's$ entries depends on the other two, P can be identified with the manifold

  $$P = \{(u_1,u_2,u_3) \in (S^1)^3 : u_1+u_2+u_3 = 0\} = \{(x,y) \in S^1 \times S^1 : |x-y| = 1\}$$

  P then is the configuration space of equilateral triangles in $\mathbb{R}^2$ with one vertex in the origin and two over the circle. Each of these configurations is determined by a rotation of any of the pairs $(1,e^{i\pi/3})$ or $(1,e^{-i\pi/3})$. In particular, this implies that $P \cong S^1 \sqcup S^1$. Thus we have that $H^0(P) \cong H^1(P) = \mathbb{k} \oplus \mathbb{k}$ and it is zero in any other degree.

- M has three faces of rank 1. Namely, Q_{12}, Q_{13} and Q_{23} where Q_{ij} consists of the pairs $(u,z) \in M$ such that $z_i = z_j = 0$. We identify Q_{ij} with the manifold with boundary

  $$Q = \{(x,y) \in S^1 \times S^1 : |x-y| \leq 1\}$$

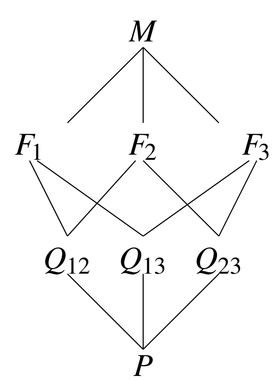
In terms of configuration spaces, this consists of isosceles triangles with one vertex in the origin, two over the circle and whose base is of length at most 1 (Here we allow the degenerate triangle). We can show that there is a homeomorphism $Q \cong S^1 \times I$ given by a rotation of the pairs $(1, e^{it\pi/3}) \in Q$ where $-1 \leq t \leq 1$. Computing the relative cohomology $H^*(Q,P)$ of the cylinder relative to the boundary we see that $H^1(Q,P) \cong H^2(Q,P) \cong \Bbbk$ and it is zero in other degrees.

- M has three facets (of rank 2). Namely, F_1, F_2, F_3 where F_i consists of the pairs $(u,z) \in M$ such that $z_i = 0$. We identify F_i with the manifold with corners

$$F = \{(x,y) \in S^1 \times D_2 : |x - y| \leq 1\}$$

This space describes the configuration of triangles with one side of length 1, and two of length at most 1. Each of these configurations is determined by a rotation of the pairs $(1, se^{it\pi/3}) \in F$ where $0 \leq s \leq 1$ and $-1 \leq t \leq 1$. Then F is homeomorphic to $S^1 \times I \times I \cong S^1 \times D_2$. Looking at the relative cohomology $H^*(F, \partial F)$ of the solid torus with respect to its boundary (the torus) we find that $H^2(F, \partial F) \cong H^3(F, \partial F) \cong \Bbbk$ and it is zero in other degrees.

The face lattice of M is then



Let X be the G-manifold that realizes M as a manifold with corners using the *unfolding* construction. Then the G-equivariant cohomology of X is a first syzygy but not a second syzygy as we will see using the quotient criterion. It is a first syzygy as the maps

$$H^*(P) \to H^{*+1}(Q,P)$$

$$H^*(Q_j,P) \oplus H^*(Q_k,P) \to H^{*+1}(F_i,\partial F_i),$$

$$\bigoplus_{i=1}^{3} H^*(F_i,\partial F_i) \to H^{*+1}(M,\partial M)$$

are surjective as it can be seen by using the explicit computation of these groups mentioned above. On the other hand, $H_G^*(X)$ is not a second syzygy as the sequence

$$\bigoplus_{j=1}^{3} H^*(Q_j,\partial Q_j) \to \bigoplus_{i=1}^{3} H^{*+1}(F_i,\partial F_i) \to H^{*+2}(M,\partial M) \to 0$$

is not exact at the second position; that is, $H^2(B^*(M)) \neq 0$. In fact, the complex $B^*(M)$ takes the form

$$\Bbbk^3 \to \Bbbk^3 \to 0 \to 0$$

when $* = 1$. The map $\Bbbk^3 \to \Bbbk^3$ is given by $(a,b,c) = (a+b, a+c, b+c)$ which is of rank 2 and then $H^2(B^*(M)) \neq 0$.

We constructed a 4-dimensional manifold X with an action of $G = (\mathbb{Z}/2)^3$ such that the equivariant cohomology $H_G^*(X)$ is torsion-free but not free as $H^*(BG)$-module. The generalization of this space will be discussed more deeply in section 4.4 which discusses the Big Polygon spaces introduced in [Franz, 2015]. In that section, we will construct syzygies of higher order in equivariant cohomology there. The space X obtained in Example 3.5.17 realizes the smallest possible dimension where a manifold with a locally standard action of a 2-torus G whose equivariant cohomology is torsion-free but not free exists. As we previously discussed, if $\operatorname{rank} G \leq 2$ then free is equivalent to torsion-free in equivariant cohomology, so the minimal example should occur when $\operatorname{rank} G = 3$. On the other hand, for torus manifolds the G-equivariant cohomology is free if and only if it is torsion-free, therefore $\dim X > 3$.

Chapter 4

Equivariant cohomology for torus actions and a compatible involution

Let T be a torus of rank r and X be a T-space. An involution on X is a continuous map $\tau\colon X \to X$ such that the composite of τ with itself is the identity map on X. We say that τ is a compatible involution for T if for any $g \in T$ and $x \in X$ we have that $\tau(g \cdot x) = g^{-1} \cdot \tau(x)$. Such a situation was studied by [Biss et al., 2004] for symplectic manifolds with Hamiltonian torus actions and a compatible anti-symplectic involution; the later was extended by [Baird and Heydari, 2018] to Hamiltonian actions for non-abelian Lie groups. In this chapter, we will study the equivariant cohomology of such spaces from a purely topological setting and we will see that it will be canonically related to 2-torus actions for cohomology with coefficients in a field $\Bbbk$ of characteristic 2.

4.1 Cohomology of $B((S^1)^n \rtimes \mathbb{Z}/2)$

Let X be a topological space with an involution τ. Observe that τ induces a well defined action of the group $\mathbb{Z}/2 = \{id, \tau\}$ on X. Conversely, an action of $\mathbb{Z}/2$ on X gives rise to an involution $\tau\colon X \to X$. In this case, we will use any of these equivalent concepts when is convenient and sometimes the $\mathbb{Z}/2$-equivariant cohomology of X will be called τ-equivariant cohomology and denoted by $H^*_\tau(X;\Bbbk)$.

To study the equivariant cohomology of a T-space X with a compatible involution τ , we notice that there is a well defined action of the group $G = T \rtimes_\tau \mathbb{Z}/2$ given by the semidirect product of T and $\mathbb{Z}/2$ where $\mathbb{Z}/2$ acts on T by inversion, that is, $\tau(g) = g^{-1}$ for $g \in T$. Conversely, a given action of this group G on X induces an action of T on X and a compatible involution τ. We can summarize it in the following remark.

Remark 4.1.1. Let X be a topological space. X is a G-space if and only if it is a T-space with a compatible involution τ.

In view of this remark, the equivariant cohomology of a T-space with a compatible involution can be realized as the G-equivariant cohomology by considering the induced action of the semidirect product $S^1 \rtimes \mathbb{Z}/2$. As we said in the beginning of this chapter, we will consider now cohomology with coefficients over $\Bbbk = \mathbb{F}_2$ since it will make the cohomology of the classifying space $B\tau$ (and any 2-torus in general) a polynomial ring. Even though the underlying topological space of G is homeomorphic to $T \times \mathbb{Z}/2$, we can not say that BG is homeomorphic to $BT \times B\tau$. However, as the G-equivariant cohomology of X is a module over $H^*(BG)$, we will show that under our assumptions there is a canonical isomorphism of algebras $H^*(BG) \cong H^*(BT) \otimes H^*(B\tau)$ and thus $H^*(BG)$ will be a polynomial ring in $(r+1)$ variables.

To compute the cohomology of G, the short exact sequence

$$1 \to T \to G \to \mathbb{Z}/2 \to 1$$

yields a fibration of classifying spaces

$$BT \to BG \to B\mathbb{Z}/2$$

by Theorem 2.2.7. Since $\pi_1(B\mathbb{Z}/2) = \mathbb{Z}/2$, by Proposition 2.2.5 the action of $\pi_1(B\mathbb{Z}/2)$ on $H^*(BT)$ is induced by the action of $\mathbb{Z}/2$ on BT. As $\mathbb{Z}/2$ acts on T by $\tau(g) = g^{-1}$, the action on $H^*(BT)$ is given by multiplication of each generator c_i by -1=1 . Therefore, $\pi_1(B\mathbb{Z}/2)$ acts trivially on the cohomology $H^*(BT)$. Using the Serre spectral sequence, we can compute explicitly the cohomology $H^*(BG)$ as we show in the following result.

Theorem 4.1.2. *There is a unique graded algebra isomorphism*

$$H^*(BG) \cong H^*(B\mathbb{Z}/2) \otimes H^*(BT) \cong \mathbb{F}_2[w, c_1, \ldots, c_n]$$

such that

- *The canonical maps $i^*\colon H^*(BG) \to H^*(BT)$, $p^*\colon H^*(B\mathbb{Z}/2) \to H^*(BG)$ induced by the inclusion $i\colon T \to G$ and projection $p\colon G \to \mathbb{Z}/2$ coincide with the canonical restriction map $\mathbb{F}_2[w, c_1, \ldots, c_n] \to \mathbb{F}_2[c_1, \ldots, c_n]$ and inclusion map $\mathbb{F}_2[w] \to \mathbb{F}_2[w, c_1, \ldots, c_n]$ respectively.*

- *There is an algebra homomorphism $\varphi : H^*(BT) \to H^*(BG)$, such that $i^* \circ \varphi$ is the identity over $H^*(BT)$ and it coincides with the canonical inclusion $\mathbb{F}_2[c_1, \ldots, c_n] \to \mathbb{F}_2[w, c_1 \ldots, c_n]$.*

- *The map $j^*\colon H^*(BG) \to H^*(B\mathbb{Z}/2)$ induced by the inclusion $j\colon 1 \times \mathbb{Z}/2 \to G$ coincides with the canonical projection $\mathbb{F}_2[w, c_1, \ldots, c_n] \to \mathbb{F}_2[w]$.*

Proof. Consider the fiber bundle

$$BT \to BG \to B\mathbb{Z}/2.$$

Since $\pi_1(B\mathbb{Z}/2) = \pi_0(\mathbb{Z}/2)$ acts trivially on the cohomology of $H^*(BT)$ by Theorem 2.2.7, the E_2 page of the associated Serre spectral sequence is given by

$$E_2^{p,q} \cong H^p(B\mathbb{Z}/2; H^q(BT)) \Rightarrow H^*(BG)$$

Therefore, by the universal coefficient theorem we have an $\mathbb{F}_2$-algebras isomorphism

$$E_2 \cong H^*(B\mathbb{Z}/2) \otimes H^*(BT) \cong \mathbb{F}_2[w, c_1, \ldots, c_n],$$

and thus, the differential d_2 depends only on the values on the generators w and c_i because it is a derivation. Observe that $d_2(w) = 0$ since w lies on the horizontal axis of the spectral sequence; on the other hand, $d_2(c_i) = 0$ for all $i = 1, \ldots, n$ since $d_2(c_i) \in E_2^{2,1} = H^2(B\mathbb{Z}/2) \otimes H^1(BT) = 0$. It follows that $d_2 = 0$, implying that $E_3 \cong E_2$. Now we consider the differential d_3; as before, it is determined by $d_3 : E_3^{0,2} \to E_3^{3,0}$. In this case, we have $d_3(c_i) = \alpha_i w^3$ with either $\alpha_i = 0$ or $\alpha_i = 1$.

The sub-extension

$$\begin{array}{ccccc} T & \longrightarrow & T \rtimes \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \\ \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & 1 \rtimes \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \end{array}$$

induces a map of spectral sequences $E_s^{p,q} \to \widetilde{E}_s^{p,q}$, where $\widetilde{E}_2 \cong H^*(B\mathbb{Z}/2)$ is the $\widetilde{E}_2$ page of the spectral sequence associated to the bottom exact sequence. By the naturality of the spectral sequences we have then a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i=1}^{n} \mathbb{F}_2 c_i \cong E_3^{0,2} & \xrightarrow{d_3} & E_3^{3,0} \cong F_2 w \\
\downarrow & & \downarrow = \\
0 = \widetilde{E}_3^{0,2} & \xrightarrow{\widetilde{d}_3} & \widetilde{E_3}^{3,0} = \mathbb{F}_2 w
\end{array}$$

which implies that $d_3 = 0$ since the right vertical arrow is the identity map and $\widetilde{d_3} = 0$.

Notice that for $r \geq 4$, we have that $E_r^{r,3-r} = 0$ and the differential $d_r : E_r^{0,2} \to E_r^{r,3-r}$ also is. Therefore, the spectral sequence collapses at E_2 and this implies that

$$E_\infty \cong E_2 \cong H^*(B\mathbb{Z}/2) \otimes H^*(BT) \cong \mathbb{F}_2[w, c_1, \dots, c_r] \cong H^*(BG)$$

Recall that the above isomorphism is a graded $\mathbb{F}_2$-vector space isomorphism; however, since $H^*(BT)$ is a finitely generated polynomial algebra, we can choose a multiplicative section $\tilde{\varphi} : H^*(BT) \to H^*(BG)$ the surjective map of $H^*(BG) \to H^*(BT)$ induced by the canonical map $\mathbb{F}_2[w, c_1, \dots, c_r] \to \mathbb{F}_2[c_1, \dots, c_r]$ which maps w to zero. Therefore, from the Leray-Hirsch Theorem that such map together with the canonical map $p^* \colon H^*(B\mathbb{Z}/2) \to H^*(BG)$ gives rise to an isomorphism of graded $H^*(B\mathbb{Z}/2)$-algebras

$$\tilde{\theta} : H^*(B\mathbb{Z}/2) \otimes H^*(BT) \to H^*(BG)$$

given by $\tilde{\theta}(\alpha \otimes \beta) = p^*(\alpha)\tilde{\varphi}(\beta)$.

Under this isomorphism, the canonical map induced by the inclusion $j : B(1 \times \mathbb{Z}/2) \to BG$ might satisfy $j(c_i) = w^2$ for some $1 \leq i \leq r$. In that case, we consider the section $\varphi(c_i) = \tilde{\varphi}(c_i) + w^2$ if $\tilde{\varphi}(c_i) = w^2$ and $\varphi(c_i) = \tilde{\varphi}(c_i)$ if $\tilde{\varphi}(c_i) = 0$. As discussed before, it induces an isomorphism

$$\theta : H^*(B\mathbb{Z}/2) \otimes H^*(BT) \to H^*(BG);$$

furthermore, such a section is unique since it is determined by the condition $j^*\varphi = 0$, and thus it makes the isomorphism θ unique as well. Therefore, we have that the composition

$$j^*\theta\colon H^*(B\mathbb{Z}/2)\otimes H^*(BT) \to H^*(BG) \to H^*(B\mathbb{Z}/2)$$

coincides with the canonical restriction map

$$\mathbb{F}_2[w, c_1, \dots, c_n] \to \mathbb{F}_2[w].$$

Now notice that the composition

$$H^*(BT) \xrightarrow{\varphi} H^*(BG) \xrightarrow{i^*} H^*(BT)$$

where i^* is induced by the inclusion $T \to G$ is the identity on $H^*(BT)$ since $i^*(w) = 0$ and φ was constructed as a section of this map. Therefore, we have that the maps

$$H^*(BT) \xrightarrow{\varphi} H^*(BG) \text{ and } H^*(BG) \xrightarrow{i^*} H^*(BT)$$

coincides with the canonical inclusion and restriction

$$\mathbb{F}_2[c_1, \dots, c_r] \to \mathbb{F}_2[w, c_1 \dots, c_r]$$

and

$$\mathbb{F}_2[w, c_1, \dots, c_n] \to \mathbb{F}_2[c_1, \dots, c_n]$$

respectively.

Using a similar argument over the composition, $H^*(B\mathbb{Z}/2) \xrightarrow{p^*} H^*(BG) \xrightarrow{j^*} H^*(B\mathbb{Z}/2/2)$ which is the identity over $\mathbb{F}_2[w]$, we conclude that the map

$$H^*(B\mathbb{Z}/2) \xrightarrow{p^*} H^*(BG)$$

coincides with the canonical inclusion

$$\mathbb{F}_2[w] \to \mathbb{F}_2[w, c_1, \dots, c_n]. \qquad \square$$

Now we will study the algebraic properties of the G-equivariant cohomology as a module over $H^*(BG)$. We start with the following result.

Lemma 4.1.3. *Let X be a G-space. There is an isomorphism of $\Bbbk$-algebras*

$$H^*_G(X) \cong H^*_\tau(X_T).$$

Proof. Let EG be the total space of the universal G-bundle, and consider EG as a model for ET. The involution τ induces an involution on the Borel construction X_T given by $\tau([z,x]_T) = [\tau \cdot z, \tau(x)]$ for $[z,x]_T \in X_T \cong EG \times_T X$. This induces a free action of $\mathbb{Z}/2$ on X_T and thus by Proposition 2.3.5 there is a homotopy equivalence $(X_T)_\tau \simeq X_T/(\mathbb{Z}/2)$. Moreover, the canonical projection $EG \times X \to X_G$ is T-invariant and induces a map $X_T \to X_G$ which is also τ-invariant, giving rise to a homeomorphism $X_T/(\mathbb{Z}/2) \cong X_G$. Combining these results, we have a homotopy equivalence $X_G \simeq (X_T)_\tau$ which induces the desired isomorphism in cohomology. □

Using this Lemma we can now prove the following result.

Theorem 4.1.4. *Let X be a G-space and assume that X is T-equivariantly formal. Then X is G-equivariantly formal if and only if the Borel construction X_T is τ-equivariantly formal.*

Proof. Suppose first that X is G-equivariantly formal, then using Theorem 4.1.2 and Lemma 4.1.3 we get

$$\begin{aligned} H^*_\tau(X_T) \cong H^*_G(X) &\cong H^*(BG) \otimes H^*(X) \\ &\cong H^*(B\tau) \otimes H^*(BT) \otimes H^*(X) \\ &\cong H^*(B\tau) \otimes H^*(X_T) \end{aligned}$$

and so X_T is τ-equivariantly formal. Reversing the above sequence of isomorphisms, the converse statement holds. However, we need to be careful with the $H^*(BG)$-module structure of $H^*_G(X)$ and the $H^*(B\tau)$-module structure of $H^*_\tau(X_T)$. From the diagram

$$\begin{array}{ccc} H^*_G(X) & \xrightarrow{\cong} & H^*_\tau(X_T) \\ \uparrow & & \uparrow \\ H^*(BG) & \xrightarrow{\cong} & H^*_\tau(BT) \\ \uparrow & & \uparrow \\ H^*(B\tau) & = & H^*(B\tau) \end{array}$$

and the canonical isomorphism $H^*(BG) \cong H^*(B\tau) \otimes H^*(BT)$ constructed in Theorem 4.1.2, the $H^*(B\tau)$-module structure on $H^*_\tau(X_T)$ coincides with the restriction of the $H^*(BG)$-module structure on $H^*_G(X)$ to the action of those elements of the form $\alpha \otimes 1 \in H^*(B\tau) \otimes H^*(BT) \cong H^*(BG)$. □

Now we will apply this theorem to a category of spaces introduced by [Hausmann et al., 2005] called *conjugation spaces*; this category includes examples such as complex Grassmannian, toric manifolds, polygon spaces and some symplectic manifolds. We review the definition and main properties of these spaces; however, we will omit details of the proofs and we will give the appropriate citation when necessary.

Definition 4.1.5. Let X be a space with an involution τ and denote by X^τ the subspace of fixed points under the involution τ. We say that X is a conjugation space if it satisfies the following conditions

- $H^{2k+1}(X) = 0$ for all $k \geq 0$.
- There is a degree-halving additive isomorphism $\kappa\colon H^{2*}(X) \to H^*(X^\tau)$.
- There is an additive section $\sigma\colon H^*(X) \to H^*_\tau(X)$ of the restriction map $\rho\colon H^*_\tau(X) \to H^*(X)$ (Definition 2.5.1).
- For any $m \geq 0$ and $a \in H^{2m}(X)$ the conjugation equation:

$$r \circ \sigma(a) = \kappa(a)w^n + p_m(w)$$

is satisfied. Here r is the map in τ-equivariant cohomology induced by the inclusion $X^\tau \to X$, $H^*(B\tau) \cong \mathbb{F}_2[w]$ and p_m is a polynomial in w with coefficients in $H^*(X^\tau)$ of degree less than m.

Notice that the existence of σ is equivalent to X being τ-equivariantly formal. The main properties of conjugation spaces are the following.

Proposition 4.1.6. *[Hausmann et al., 2005, §3]*

1. *κ and σ are ring homomorphisms.*
2. *κ and σ are unique. That is, if κ' and σ' are also maps that make X into a conjugation space, then $\kappa = \kappa'$ and $\sigma = \sigma'$.*
3. *The conjugation structure is natural. More precisely, if X and Y are conjugation spaces with conjugation structure (κ_1, σ_1) and (κ_2, σ_2) respectively, then for any G-equivariant map $f\colon Y \to X$ there are commutative diagrams,*
$$\begin{array}{ccc} H^{2*}(X) & \xrightarrow{f^*} & H^{2*}(Y) \\ \downarrow{\kappa_1} & & \downarrow{\kappa_2} \\ H^*(X^\tau) & \xrightarrow{(f^\tau)^*} & H^*(Y^\tau) \end{array} \qquad \begin{array}{ccc} H^*(X) & \xrightarrow{f^*} & H^*(Y) \\ \downarrow{\sigma_1} & & \downarrow{\sigma_2} \\ H^*_\tau(X) & \xrightarrow{f^*_G} & H^*_\tau(Y) \end{array}$$
where f^τ denotes the restriction of f to the fixed point subspaces $Y^\tau \to X^\tau$.

These spaces also satisfy very interesting properties; for our purposes, we only use the following result.[Hausmann et al., 2005, Thm.7.5].

Theorem 4.1.7. *Let X be a conjugation space with conjugation τ. Suppose that a torus T acts on X and that the action is compatible with τ. Then X_T is a conjugation space where the conjugation on X_T is the one induced by τ as in the proof of Lemma 4.1.3.*

This theorem shows that X_T is τ-equivariantly formal. Then immediately from Theorem 4.1.4 and that X is T-equivariantly formal by definition as $H^{odd}(X) = 0$ we obtain the following result.

Corollary 4.1.8. *Let X be a T-space which is also a conjugation space with a compatible involution τ. Then X is G-equivariantly formal.*

4.2 Reduction to 2-torus actions

Let $H = T_2 \times \mathbb{Z}/2 \cong (\mathbb{Z}/2)^{n+1}$ denote the maximal 2-torus subgroup in $G = T \rtimes \mathbb{Z}/2$ where $T_2 \leq T$ is the maximal 2-torus subgroup in T. Recall that $H^*(BH) \cong \mathbb{F}_2[t_1, \ldots, t_n, w]$ inherits a structure of $H^*(BG)$-module induced by the inclusion $H \hookrightarrow G$. In order to study this structure, we need to compute explicitly the induced map in cohomology by the inclusion; firstly, we state the following lemma.

Lemma 4.2.1. *The map $i^* \colon H^*(BG) \to H^*(BH)$ induced by the inclusion $i \colon H \to G$ is given by $i^*(c_i) = t_i^2 + t_i w$ for all $1 \leq i \leq n$ and $i^*(w) = w$.*

Proof. By theorem 4.1.2 we can assume that $n = 1$ and so $H^*(BG) \cong \mathbb{F}_2[c, w]$ and $H^*(BH) \cong \mathbb{F}_2[t, w]$. Notice that the statement $i^*(w) = w$ is clear as it follows from the map induced by the inclusion of $\mathbb{Z}/2$ into the second factor of $S^1 \rtimes \mathbb{Z}/2$ which factors through $H^*(BH)$, Now write $i^*(c) = \alpha t^2 + \beta t w + \gamma w^2$ for $\alpha, \beta, \gamma \in \mathbb{F}_2$. As before, the inclusion of $\mathbb{Z}/2$ in the first and second factor of G show that $\alpha = 1$ and $\gamma = 0$ (compare also with Example 2.2.9). To compute β, we consider the inclusion of G into $SO(3)$ by identifying G with the set of matrices $\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix} \in SO(3)$ where $A \in O(2)$. Recall that $H^*(BSO(3)) \cong \mathbb{F}_2[\omega_2, \omega_3]$ where $|\omega_i| = i$ for $i = 2, 3$ and the inclusion $H \to SO(3)$ induces the map $\phi \colon H^*(BSO(3)) \to H^*(BH)$ given by $\phi(\omega_2) = t^2 + tw + w^2$ and $\phi(\omega_3) = t^2 w + w t^2$. Since ϕ factors through i^*, this implies that $\beta = 1$ and so $i^*(c) = t^2 + tw$. □

In Theorem 4.1.2 we showed that there is an isomorphism of graded algebras $H^*(B(S^1 \rtimes \mathbb{Z}/2)) \cong H^*(B(S^1 \times \mathbb{Z}/2))$; however, Lemma 4.2.1 implies that the structure of $\mathcal{A}_2$-algebras[1] is not the same, as we state in the following corollary.

Corollary 4.2.2. *Let $G = (S^1)^n \rtimes \mathbb{Z}/2$ and $\tilde{G} = (S^1)^n \times \mathbb{Z}/2$ for $n \geq 1$. The mod2-cohomology of the classifying spaces BG and $B\tilde{G}$ are isomorphic as $\mathbb{F}_2$-algebras but not as $\mathcal{A}_2$-algebras.*

Proof. For $c \in H^2(BG)$ generator of degree 2, write $Sq^1(c) = \alpha cw + \beta w^3$ for $\alpha, \beta \in \mathbb{F}_2$. Let H be the maximal 2-torus in G. By naturality of the Steenrod operations, we have that $i^*(Sq^1(c)) = Sq^1(i^*(c))$ where $i \colon H \to G$ is the inclusion. Therefore, $\alpha(t^2w + tw^2) + \beta w^3 = Sq^1(t^2 + tw) = t^2w + wt^2$ by Lemma 4.2.1 and so $\alpha = 1, \beta = 0$, here t is the generator in $H^1(BH)$ which restricts from c. On the other hand, a similar argument applied to the inclusion $j \colon H \to \tilde{G}$ shows that $Sq^1(c) = 0$ as $j^*(c) = t^2$. □

Proposition 4.2.3. *$H^*(BH)$ is a free module of rank 2^n over $H^*(BG)$; moreover, it is freely generated by the elements of the form $t_1^{\varepsilon_1} t_2^{\varepsilon_2} \cdots t_n^{\varepsilon_n}$ where $\varepsilon_i \in \{0,1\}$ for all i, and the canonical multiplicative structure of $H^*(BH)$ as an $\mathbb{F}_2$-algebra induces an $H^*(BG)$-algebra structure completely determined by multiplication of the elements of this basis.*

Proof. Let $[n] = \{1,2,\ldots n\}$, $\Lambda = \{I : [n] \to \{0,1\}\}$ and denote by $t_I = t_1^{I(1)} \cdots t_n^{I(n)}$. Notice that for $i = 1,\ldots,n$, we have that $t_i^2 = c_i \cdot 1 + w \cdot t_i$ and this implies that any $p(t_1,\ldots,t_n,w) \in H^*(BH)$ can be written as a linear combination $\sum_{I \in \Lambda} P_I \cdot t_I$ for some $P_I \in H^*(BG)$. This shows that the elements $t_I : I \in \Lambda$ generate $H^*(BH)$ as $H^*(BG)$-module. It only remains to show that these elements are linearly independent over $H^*(BG)$. Without loss of generality, suppose that there is a homogeneous equation

$$\sum_{I \in \Lambda} P_I \cdot t_I = 0$$

[1] $\mathcal{A}_2$ denotes the Steenrod squares algebra $\mathbb{F}_2[Sq^i : i \geq 0]$

of degree m where $P_I \in H^*(BG)$. For any $I \in \Lambda$, let $\chi(I) = \sum_{i=1}^n I(i)$ so $\deg(t_I) = \chi(I)$ and $\deg(P_I) = m - \chi(I)$. Also, consider the set $\Gamma_I = \{\sigma = (\sigma_1, \ldots, \sigma_{n+1}) \in \mathbb{N}^{n+1} : \sum_{j=1}^n 2\sigma(j) + \sigma(n+1) = m - \chi(I)\}$. We can write $P_I = \sum_{\sigma \in \Gamma_I} \alpha_\sigma c_\sigma w^{\sigma(n+1)}$ where $c_\sigma = c_1^{\sigma(1)} \cdots c_n^{\sigma(n)}$ for some $\alpha_\sigma \in \mathbb{F}_2$. Therefore,

$$P_I \cdot t_I = \sum_{\sigma \in \Gamma_I} \alpha_\sigma t_{I+\sigma} (t+w)^\sigma w^{\sigma(n+1)}$$

where $t_{I+\sigma} = t_1^{I(1)+\sigma(1)} \cdots t_n^{I(n)+\sigma(n)}$ and $(t+w)^\sigma = (t_1+w)^{\sigma(1)} \cdots (t_n+w)^{\sigma(n)}$. We claim that in the sum

$$\sum_{I \in \Lambda} P_I \cdot t_I = \sum_{I \in \Lambda} \sum_{\sigma \in \Gamma_I} \alpha_\sigma t_{I+\sigma} (t+w)^\sigma w^{\sigma(n+1)}$$

there are no common terms. In fact, if $t_{I+\sigma}(t+w)^\sigma w^{\sigma(n+1)} = t_{\tilde{I}+\tilde{\sigma}}(t+w)^{\tilde{\sigma}} w^{\tilde{\sigma}(n+1)}$ for some $I, \tilde{I} \in \Lambda$ and $\sigma \in \Gamma_I$, $\tilde{\sigma} \in \Gamma_{\tilde{I}}$, then $\sigma = \tilde{\sigma}$ and so $I = \tilde{I}$ as $H^*(BH)$ is a unique factorization domain. Finally, this implies that $\alpha_\sigma = 0$ for all $\sigma \in \Gamma_I$ and $I \in \Lambda$ and so we obtain that $P_I = 0$ for all $I \in \Lambda$. □

Using now that $H^*(BH)$ is a free $H^*(BG)$-module we can prove the following result.

Lemma 4.2.4. *In the fibration $G/H \to BH \to BG$, $\pi_1(BG)$ acts trivially on the cohomology of G/H and there is an isomorphism of $H^*(BG)$-modules*

$$H^*(BH) \cong H^*(BG) \otimes H^*(G/H).$$

Proof. We first show that the local coefficient system is trivial. Consider the fibration $G/H \to BH \to BG$ which can be realized as the associated fiber bundle $EG \times_G G/H \to BG$. Therefore, it follows that the action of $\pi_1(BG)$ on G/H is induced by the action of $\pi_0(G)$ on G/H by Proposition 2.2.5. Since $\pi_0(G) = \{[(1,e)], [(1,\tau)]\}$ consists of two connected components, it is enough to show that the multiplication on the left by $(1,\tau)$ on G/H induces the identity map on the cohomology $H^*(G/H)$. More precisely, for any coset $(g,e)H \in G/H$ we have that $(1,\tau)(g,e)H = (g^{-1},e)H$ and thus such a map coincides with the map $T \to T$ sending $g \mapsto g^{-1}$

as $G/H \simeq T$. The induced map in cohomology is given the multiplication by -1, which is the identity on $H^*(T) \cong H^*(G/H)$ as we are considering cohomology with coefficients in a field of characteristic two. Then we can apply the Eilenberg-Moore spectral sequence associated to the fibration $G/H \to BH \to BG$ with E_2-term given by

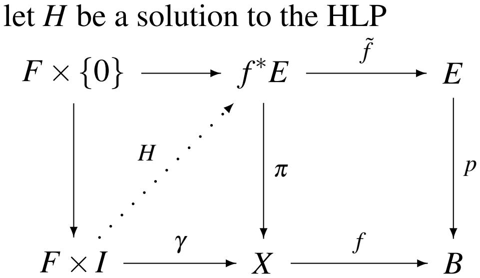
$$E_2 = \mathrm{Tor}_{H^*(BG)}(\mathbb{k}, H^*(BH)) \Rightarrow H^*(G/H).$$

Since $H^*(BH)$ is a free $H^*(BG)$-module by Lemma 4.2.3, the spectral sequence collapses and there is an isomorphism $H^*(G/H) \cong \mathbb{k} \otimes_{H^*(BG)} H^*(BH)$. This implies that the canonical map $H^*(BH) \to H^*(G/H)$ is surjective and thus there is an isomorphism of $H^*(BG)$-modules $H^*(BH) \cong H^*(BG) \otimes H^*(G/H)$ by the Leray-Hirsch theorem. □

Another situation where the action is trivial arises from the pullback of fiber bundles, as we state in the following result.

Lemma 4.2.5. *Let $F \to E \to B$ be a fiber bundle where $\pi_1(B)$ acts trivially on the cohomology of the fiber F. Then for any path-connected space X and any map $f : X \to B$, $\pi_1(X)$ acts trivially on the cohomology of the fiber F in the pullback bundle $F \to f^*E \to X$.*

Proof. Consider $f : (X, x_0) \to (B, b_0)$ as a map of pointed spaces, then $\tilde{f} : (f^*E, p) = (X \times_B E, (x_0, z_0)) \to (E, z_0)$ is the projection on the second component and $z_0 \in E_{b_0}$. Note that $(f^*E)_{x_0} = \{x_0\} \times E_{b_0}$ and therefore $\tilde{f}$ induces the identity map on cohomology $H^*(F) \to H^*(F)$. Now let $[\gamma] \in \pi_1(X, x_0)$ and let H be a solution to the HLP



Then, $G = \tilde{f} \circ H$ is a solution to the HLP

$$\begin{array}{ccc} F \times \{0\} & \longrightarrow & E \\ \downarrow & \overset{G}{\nearrow} & \downarrow p \\ F \times I & \xrightarrow{f \circ \gamma} & B \end{array}$$

By assumption, $g_{f\circ\gamma} = G|_{F\times\{1\}}$ acts trivially in the cohomology of the fiber. Let $h_\gamma = H|_{F\times\{1\}}$. Then we get,

$$g_{f\circ\gamma} = \tilde{f} \circ h_\gamma$$

which implies that h_γ^* is the identity map on the cohomology $H^*(F)$. □

Proposition 4.2.6. *Let X be a G-space, which is also an H-space under the restriction of the action. The fibration $G/H \to X_G \to X_H$ induces a canonical isomorphism*

$$H_H^*(X) \cong H_G^*(X) \otimes_{H^*(BG)} H^*(BH)$$

of $H^*(BH)$-algebras natural in X, where $H^*(BH)$ acts only on the right factor of $H_G^*(X) \otimes_{H^*(BG)} H^*(BH)$.

Proof. Notice that the fibration $G/H \to X_H \to X_G$ arises as a pullback bundle of the fibration $G/H \to BH \to BG$. Therefore, the E_2-term of the associated Serre spectral sequence is given by

$$E_2^{p,q} = H^p(X_G; H^q(G/H)) \cong H^p(X_G) \otimes H^q(G/H).$$

since $\pi_1(X_G)$ acts trivially on the cohomology of G/H by Lemmas 4.2.4 and 4.2.5 .

Notice that the only possible non-zero differential in the Serre spectral sequence is induced by the map $d_2 : E_2^{0,1} \to E_2^{2,0}$ because $H^*(G/H)$ is generated by $H^1(G/H)$. Recall that $X_H \to X_G$

can be realized as a pullback of the fibration $BH \to BG$, hence the map of fibrations

$$\begin{array}{ccccc} G/H & \longrightarrow & X_H & \longrightarrow & X_G \\ \| & & \downarrow & & \downarrow \\ G/H & \longrightarrow & BH & \longrightarrow & BG \end{array}$$

induces a map between the respective spectral sequences giving, by naturality, a commutative diagram

$$\begin{array}{ccc} H^1(G/H) & \xrightarrow{\tilde{d_2}} & H^2(BG) \\ \cong\downarrow & & \downarrow \\ H^1(G/H) & \xrightarrow{d_2} & H^2(X_G) \end{array}$$

and again, from Lemma 4.2.4, $\tilde{d_2} = 0$ and therefore $d_2 = 0$. We obtain that the spectral sequence collapses at the page E_2 and we get an isomorphism of $\mathbb{Z}/2$-vector spaces.

$$H^*_H(X) \cong H^*_G(X) \otimes H^*(G/H)$$

Moreover, from the Leray-Hirsch theorem, we get an isomorphism $H^*_H(X) \cong H^*_G(X) \otimes H^*(G/H)$ as $H^*_G(X)$-modules; and again from Lemma 4.2.4 we get isomorphisms

$$\begin{aligned} H^*_H(X) &\cong H^*_G(X) \otimes H^*(G/H) \\ &\cong H^*_G(X) \otimes_{H^*(BG)} (H^*(BG) \otimes H^*(G/H)) \\ &\cong H^*_G(X) \otimes_{H^*(BG)} H^*(BH) \end{aligned}$$

as $H^*_G(X)$-modules. Furthermore, the collapsing of the Eilenberg-Moore spectral sequence arisen from the pullback diagram, shows that the canonical maps $H^*_G(X) \to H^*_H(X)$ and $H^*(BH) \to H^*_H(X)$ of $H^*(BG)$-algebras induces a natural isomorphism $H^*_G(X) \otimes_{H^*(BG)} H^*(BH) \to H^*_H(X)$ as $H^*(BG)$-algebras. Finally, the last claim follows from the naturality of such an isomorphism by considering the map induced by the projection $X \to pt$. □

As an immediate consequence we have the following result.

Corollary 4.2.7. *Let X be a G-space such that $b(X) < \infty$. Then X is G-equivariantly formal if and only if $H^*_G(X)$ is a free $H^*(BG)$-module.*

Proof. By Proposition 2.5.3 it is enough to show that $\pi_1(BG)$ acts trivially on the cohomology of X. If $H^*_G(X) \cong H^*(BG) \otimes H^*(X)$, by Proposition 4.2.6 we have that

$$H^*_H(X) \cong H^*(BH) \otimes H^*(X).$$

Corollary 2.5.6 we have that X is H-equivariantly formal; therefore $H \cong \pi_1(BH)$ acts trivially on the cohomology on X. Since the map $BH \to BG$ induces a surjective map $H \cong \pi_1(BH) \to \pi_0(G) \cong \pi_1(BG)$, and the action of $\pi_1(BH)$ on X factors through $\pi_1(BG)$, we can conclude that $\pi_1(BG)$ acts trivially on the cohomology of X and so X is G-equivariantly formal. □

Now we can prove the main result of this section which is analogous to the reduction from compact connected Lie group actions to torus actions (Theorem 1.1). However, we first need the following lemma; here we will use the characterization of syzygies given in Theorem B.10.

Lemma 4.2.8. *Let R,S be algebras over a field $\Bbbk$ such that S be a free finitely generated R-module. Let A be an S-algebra and B an R-algebra such that $A \cong B \otimes_R S$ as S-modules. Then A is a j-th syzygy over S if and only if B is a j-th syzygy over R.*

Proof. Suppose that B is a j-th syzygy over R. Then there is an exact sequence of free R-modules

$$0 \to B \to F_1 \to \cdots \to F_j$$

Since S is a free R-module by assumption, tensoring the above sequence with S yields to an exact sequence

$$0 \to B \otimes_R S \to F_1 \otimes_R S \to \cdots \to F_j \otimes_R S.$$

Then $A = B \otimes_R S$ is a j-th syzygy over S as each $F_i \otimes_R S$ is a free S-module.

Conversely, suppose that A is a j-th syzygy over S. We will show that B is a j-th syzygy over R by proving that every R-regular sequence of length at most j is also B-regular (see Theorem B.10 for this equivalence). Let $(r_1,\dots,r_k)$ be a R-regular sequence with $k \le j$. We claim that $(\bar{r_1},\dots,\bar{r_k})$ is an S-regular sequence where $\bar{r_k} = r_k \cdot 1_S \in S$. First, let $\sigma_1,\dots,\sigma_n$ be a R-basis of S. Let us check that the multiplication by $\bar{r_k}$ on $S/(\bar{r_1},\dots,\bar{r_{k-1}})$ is injective, In fact, if $\bar{r_k}s = \sum_{i=1}^{k-1} \bar{r_i}s_i$ and writing $s_i = \sum_{l=1}^{n} \lambda_l^i \sigma_l$ where $\lambda_l^i \in R$ we obtain

$$\begin{aligned}\sum_{l=1}^{n} r_k \lambda_l \sigma_l = \bar{r_k}s &= \sum_{i=1}^{k-1} \bar{r_i}s_i \\ &= \sum_{i=1}^{k-1}\sum_{l=1}^{n} r_i \lambda_l^i \sigma_l \\ &= \sum_{l=1}^{n}\left(\sum_{i=1}^{k-1} r_i \lambda_l^i\right)\sigma_l\end{aligned}$$

This implies that $r_k \lambda_l = \sum_{i=1}^{k-1} r_i \lambda_l^i$, and thus $\lambda_l = 0$ as $(r_1,\dots,r_k)$ is a R-regular sequence; therefore, we obtain that $s = 0$. This shows that $(\bar{r_1},\dots,\bar{r_k})$ is a S-regular sequence and thus it is a A-regular sequence.

It remains to show that $(r_1,\dots,r_k)$ is a B-regular sequence. As above, let us consider the multiplication map $B/(r_1,\dots,r_{k-1}) \xrightarrow{r_k} B/(r_1,\dots,r_{k-1})$ and let $b \in B$ be such that $r_k \cdot b = \sum_{i=1}^{k-1} r_i \cdot b_i$. We can write then

$$\begin{aligned}\bar{r_k}\cdot(b\otimes 1_S) &= (r_k \cdot b \otimes 1_S) \\ &= \left(\sum_{i=1}^{k-1} r_i \cdot b_i\right)\otimes 1_S \\ &= \sum_{i=1}^{k-1} \bar{r_i}\cdot(b_i \otimes 1_S)\end{aligned}$$

Since $(\bar{r_1},\dots,\bar{r_k})$ is a A-regular sequence, we conclude that $b\otimes 1_S = 0$, and so $b = 0$ as S is a free R-algebra. □

Combining Proposition 4.2.6 and Lemma 4.2.8 we have the following result.

Theorem 4.2.9. *Let X be a G-space. $H_G^*(X)$ is a j-th syzygy over $H^*(BG)$ if and only if $H_H^*(X)$ is a j-th syzygy over $H^*(BH)$.* □

Also from Proposition 3.4.17 we obtain immediately the next corollary.

Corollary 4.2.10. *Let X be a $\mathbb{k}$-Poincaré duality space with an action of G. If $H_G^*(X)$ is a j-th syzygy for $j \geq (n+1)/2$ then $H_G^*(X)$ is free over $H^*(BG)$.* □

In particular, $H_G^*(X)$ is a free $H^*(BG)$-module if and only if $H_H^*(X)$ is a free $H^*(BH)$-module. As a corollary from Theorem 4.2.9 and the criterion for equivariant formality (Proposition 2.5.5) we get the next result.

Corollary 4.2.11. *Let X be a path-connected finite dimensional G-CW-complex. Recall that $b(X) = \sum_i \dim_{\mathbb{F}_2} H^i(X)$ denotes the Betti sum of X and suppose that $b(X) < \infty$. Then X is G-equivariantly formal over $\mathbb{F}_2$ if and only if $b(X) = b(X^H)$.* □

Theorem 4.2.9 shows that the H-equivariant cohomology of X is determined by the G-equivariant cohomology of X. As in the case for compact connected Lie groups and maximal torus for rational coefficients, we can also describe the G-equivariant cohomology of X in terms of the Weyl invariants of the H-equivariant cohomology of X. Recall that the Weyl group of H in G is defined as the quotient $W = N_G(H)/H$ where $N_G(H)$ denotes the normalizer of H in G. We first prove the following proposition.

Proposition 4.2.12. *Let $W = N_G(H)/H$ be the Weyl group of H in G. Then $W \cong (\mathbb{Z}/2)^n$ and there is an isomorphism of algebras $H^*(BG) \cong H^*(BH)^W$ where the action on the cohomology of $H^*(BH)$ is induced by the conjugation action of W on H.*

Proof. Write $H = \langle (g_1, e), \ldots, (g_n, e), (1, \tau) \rangle$ where $g_i^2 = 1$ in the i-th factor S^1 of T. We claim that $N_G(H) \cong (\mathbb{Z}/4)^n \rtimes \mathbb{Z}/2$ where $(\mathbb{Z}/4)^n = \langle \theta_1, \ldots, \theta_n \rangle$ is generated by elements $\theta_i^2 = g_i$

and $\mathbb{Z}/2$ acts on $\mathbb{Z}/4$ by inversion. Notice that for any $(g,\sigma) \in G$ where $g \in T$ and $\sigma \in \langle \tau \rangle$, (g,σ) commutes with every element in H of the form (g_i, e) and so we only need to restrict to looking at the conjugation of the element $(1,e) \in H$ by (g,σ). Namely, if $(g,\sigma) \in N_G(H)$ we have that $(g,\sigma)(1,\tau)(g^\sigma,\sigma) = (g^2,\tau) \in H$ and thus we get $g \in \langle \theta_1,\dots,\theta_n \rangle$. This implies that $W \cong (\mathbb{Z}/2)^n$ is generated by the cosets $(\theta_i,e)H$ for $i = 1,\dots,n$.

Recall that for any topological group the map induced in the cohomology of the classifying space by the conjugation of a fixed element is the identity map [Adem and Milgram, 2013, Ch.II Thm.1.9], and this shows that $i^*(H^*(BG)) \subseteq H^*(BH)^W$. It only remains to check the reverse inclusion to finish the proof. We now proceed to compute the induced action on the cohomology of $H^*(BH) \cong \Bbbk[t_1,\dots,t_n,w]$ where the variables t_i are dual to $e+g_i$ and w is to $e+\tau$ in $\mathbb{F}_2[H]$ as discussed in Section 3.4. Fix $i \in \{1,\dots,n\}$, notice that any $(\theta_i,e)H \in W$ acts trivially on the generators $(g_j,e) \in H$; on the other hand, we have that $(\theta_i,e)H \cdot (1,\tau) = (\theta_i,e)(1,\tau)(\theta_i g_i,\tau) = (g_i,\tau)$. Using a similar approach as in the proof of Proposition 3.4.5, we see that the induced map φ_i by the action of $(\theta_i,e)H$ on the cohomology ring $\mathbb{F}_2[t_1,\dots,t_n,w]$ is given by $\varphi_i(t_j) = t_j$ for $j \neq i$, $\varphi_i(t_i) = t_i + w$ and $\varphi_i(w) = w$.

By Proposition 4.2.3, consider an element $P = \sum_{I\in\Lambda} P_I t_I \in H^*(BH)^W$ where $P_I \in H^*(BG)$ are uniquely determined. We will show that $P_I = 0$ if $I(k) \neq 0$ for some $1 \leq k \leq n$. Suppose that $I \in \Lambda$ is such that $I(k) \neq 0$, then $\varphi_k(P_I t_I) = P_I t_I + w P_I t_{I_k}$ where $I_k(j) = I(j)$ if $j \neq k$ and $I_k(k) = 0$. Under this notation, we have that $\varphi_k(t_{I_k}) = t_{I_k}$ and then the equation $P = \varphi_k(P)$ implies that $P_{I_k} + wP_I = P_{I_k}$ and so $P_I = 0$ as desired. □

Let X be a G-space. Notice that the actions of W on H induce an action on X_H as well as on equivariant cohomology. This action is given by $n \cdot [z,x] = [f_n(z), n\cdot x]$ where the action on the first factor is the one induced on EG by the conjugation of n. It is well defined as $n \cdot [hz,hx] = [nhn^{-1} f_n(z), (nh)(n^{-1}n)\cdot x] = n \cdot [z,x]$. Furthermore, since the conjugation action is trivial on both $H^*(BG)$ and $H^*(X)$, it follows that it also is on $H^*_G(X)$. In particular, this shows that $i^*(H^*_G(X)) \subseteq H^*_H(X)^W$. Actually, the isomorphism of Proposition 4.2.12 can be extended to a natural isomorphism in equivariant cohomology by using the Eilenberg-Moore spectral sequence. We summarize it in the following result.

Theorem 4.2.13. *Let X be a G-space, H the maximal 2-torus in G and W the Weyl group of H in G. Suppose that G acts trivially on the cohomology of X. Then there is a natural isomorphism of $H^*(BG)$-algebras*

$$H^*_G(X) \cong H^*_H(X)^W$$

induced by the inclusion $H \to G$.

Theorem 4.2.9 is a particular case of a more general situation as follows, first we introduce the following definition motivated by the approach done by [Baird and Heydari, 2018].

Definition 4.2.14. Let G be a compact group and let H be a closed subgroup of G. We say that the pair (G,H) has the **free extension property** over $\Bbbk$ or (G,H) is a **free extension pair** if the map $H^*(BH;\Bbbk) \to H^*(G/H;\Bbbk)$ is surjective.

Notice that the last statement is equivalent to the local coefficient system being trivial and the degeneracy of the Serre spectral sequence associated to the fibration $G/H \to BH \to BG$ and thus $H^*(BH)$ becomes a free $H^*(BG)$-module. A similar approach as in Proposition 4.2.6 shows that for any G-space X, there is a natural isomorphism of $H^*(BH;\Bbbk)$-algebras

$$H^*_H(X;\Bbbk) \cong H^*_G(X;\Bbbk) \otimes_{H^*(BG;\Bbbk)} H^*(BH).$$

For example, if $G = T$ is a torus and $H = T_2$ is the maximal 2-torus in T, then (T,T_2) has the free extension property over a field of characteristic two. Furthermore, if $G = U(n), SU(n), O(n)$ or $SO(n)$ and H is the unique (up to conjugation) maximal 2-torus in G, then the pair (G,H) satisfies the free extension property over a field of characteristic two [Baird and Heydari, 2018, Prop.6-Prop.8].

Using these notions, we can generalize the previous results into the next corollary.

Corollary 4.2.15. *Let (G,H) be a free extension pair over $\mathbb{F}_2$ where H is a 2-torus. For any G-space X and $1 \leq j \leq n$, $H^*_G(X)$ is a j-th syzygy over $H^*(BG)$ if and only if $H^*_H(X)$ also is over $H^*(BH)$. In particular, for torus actions, X is T-equivariantly formal if and only if it is T_2-equivariantly formal.*

This result shows that, considering coefficients over a field of characteristic 2, the equivariant cohomology for actions of tori, (special) unitary or orthonormal groups are closely related to the equivariant cohomology for the restriction to 2-torus actions.

Example 4.2.16. Let $X = \mathbb{R}P^n$ and $G = SO(n+1)$. Denote by G_2 the maximal 2-torus in G. Consider the canonical transitive action of G on S^n which descends to an action of G on $\mathbb{R}P^n$. As this action is transitive, we have that $X = G \cdot x$ for a chosen point $x \in \mathbb{R}P^n$ and thus $H^*_G(X) \cong H^*_G(G \cdot x) \cong H^*(BG_x)$. Notice that $G_x \cong SO(n) \rtimes \mathbb{Z}/2 \cong O(n)$ and $H^*_G(X) \cong H^*(BO(n))$. On the other hand, by Corollary 4.2.15 we have that X is G-equivariantly formal as it is G_2-equivariantly formal (X^{G_2} is a discrete space of $n+1$-points and thus $b(X) = b(X^{G_2})$). Finally we get an isomorphism of $H^*(BSO(n+1))$-modules

$$H^*(BO(n)) \cong H^*(BSO(n+1)) \otimes H^*(\mathbb{R}P^n)$$

where the $H^*(BG)$-module structure on $H^*(BO(n))$ is induced by the inclusion $O(n) \to SO(n+1)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix}$.

On the other hand, there is no apparent relation between equivariant formality for a group G and its maximal 2-torus when the ground field is changed as we illustrate in the following example.

Example 4.2.17. There exist T-spaces X and Y such that X is T-equivariantly formal over $\mathbb{Q}$ but not over $\mathbb{F}_2$, and Y is T-equivariantly formal over $\mathbb{F}_2$ but not over $\mathbb{Q}$. Firstly, let $X = SO(4)$

be the special orthogonal group. Identify $T = S^1$ with the special orthogonal group $SO(2)$. For $g \in T$, and $A \in SO(4)$ consider the action

$$g \cdot A = \begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix} A \begin{pmatrix} I_2 & 0 \\ 0 & g^{-1} \end{pmatrix}$$

In fact, if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{ij} \in \mathbb{R}^{2\times 2}$, then $g \cdot A = \begin{pmatrix} gA_{11} & gA_{12}g^{-1} \\ A_{21} & A_{22}g^{-1} \end{pmatrix}$. If $A \in X^T$, then we can easily check that $A_{11} = A_{22} = 0$, $\det(A_{12})\det(A_{21}) = 1$ and $A_{12}, A_{21} \in O(2)$. Furthermore, A_{12} commutes with all matrices in $SO(2)$ and this implies that $A_{12} \in SO(2)$ and so is A_{21}.

Summarizing, we have that the fixed point subspace $X^T = X^{T_2}$ consists of all matrices $A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}$ such that $A_{12}, A_{21} \in SO(2)$ and thus $X^T \cong SO(2) \times SO(2) = S^1 \times S^1$. In this case $b^{\mathbb{Q}}(X^T) = 4$. There is a homeomorphism $X \cong S^3 \times \mathbb{R}P^3$ [Hatcher, 2002, p.294] which implies that $b^{\mathbb{Q}}(X) = 4$ and thus X is T-equivariantly formal over $\mathbb{Q}$. On the other hand, $b^{\mathbb{F}_2}(X^{T_2}) = 4$ but $b^{\mathbb{F}_2}(X) = 8$; then X is not T-equivariantly formal over $\mathbb{F}_2$.

Secondly, consider the space $Y = S^1$ with the action of $T = S^1$ given by $g \cdot y = g^2 y$. With this action, Y is not T-equivariantly formal over $\mathbb{Q}$ as the action is locally free and so $H_T^*(Y;\mathbb{Q}) \cong H^*(Y/S^1;\mathbb{Q}) = H^*(pt;\mathbb{Q})$ (Proposition 2.3.5). On the other hand, the induced action of T_2 on Y is trivial and thus Y is T_2-equivariantly formal over $\mathbb{F}_2$, and so it is with respect to the T-action.

4.3 Equivariant cohomology for the real locus

In this section, we generalize the notion of spaces with a torus action and a compatible involution to a large class of groups, and we study the equivariant cohomology for the fixed point subspace under the compatible involution to generalize Theorem 1.5 from the introduction into a topological setting. We first introduce this notion motivated by the case when X is a complex variety and the involution is induced by the complex conjugation.

Definition 4.3.1. Let X be a space with involution τ. The real locus of X is defined as the fixed point subspace X^τ.

Let G be a compact group, X be a G-space and τ be an involution on X. We say that τ is a **compatible involution** on X if there is an action of $\mathbb{Z}/2 = \langle \tau \rangle$ on G such that $\tau(g \cdot x) = \tau(g) \cdot \tau(x)$ for any $g \in G$, $x \in X$. Similarly to the discussion at the beginning of this chapter, the condition of compatibility is equivalent to an action of the group $G_\tau = G \rtimes_\tau \mathbb{Z}/2$ on X. In particular, the group G^τ acts on the real locus X^τ as we state in the following remark.

Remark 4.3.2. Let X be a G-space with a compatible involution τ. The action of G on X induces a natural action of G^τ on the real locus X^τ. In fact, let $g \in G^\tau$ and $X \in X^\tau$. Then we have that $g \cdot x \in X^\tau$ as

$$\tau(g \cdot x) = \tau(g) \cdot \tau(x) = g \cdot x.$$

Definition 4.3.3. Let X be a G_τ-space and H be a τ-invariant subgroup of G. We say that (G,H) is a τ-free extension if both (G,H) and (G^τ, H^τ) are free extensions.

With this notation introduced, we can proceed to prove the following theorem for generalized syzygies over a Noetherian ring.

Theorem 4.3.4. *Let G be a compact group and let X be a G-space with a compatible involution τ. Suppose that there is 2-torus H in G such that τ acts trivially on H and (G,H) is a τ-free extension For any splitting $H_\tau \cong H \times L$ and for any integer $j \geq 1$, if $H^*_{G_\tau}(X)$ is a j-th syzygy over $H^*(BG_\tau)$, then so is $H^*_{G^\tau}(X^L)$ as a module over $H^*(BG^\tau)$. In particular, X^L is the real locus of X.*

Proof. As H is a 2-torus, (G,H) is a free extension if and only if (G_τ, H_τ) is. In fact, it follows from the commutativity of the diagram

$$\begin{array}{ccc} H^*(BH_\tau) & \longrightarrow & H^*(G_\tau/H_\tau) \\ \downarrow & & \downarrow \\ H^*(BH) & \longrightarrow & H^*(G/H) \end{array}$$

where the map $H^*(G_\tau/H_\tau) \to H^*(G/H)$ is an isomorphism and $H^*(BH_\tau) \to H^*(BH)$ is surjective. If X is a j-th syzygy over $H^*(BG_\tau)$, it follows from Corollary 4.2.15 that it also is as a module over $H^*(BH_\tau) \cong H^*(B(H \times \tau))$. We can use now the tools for syzygies for 2-torus actions discussed in §3.5. In fact, from Theorem 3.5.6 applied to the subgroup $L \subseteq H_\tau$, we obtain that $H^*_{H_\tau/L}(X^L) \cong H^*_H(X^L)$ is a j-th syzygy over $H^*(B(H_\tau/L)) \cong H^*(BH^\tau)$. Again by Corollary 4.2.15, we get that X is a j-th syzygy over $H^*(BG^\tau)$. □

This theorem applies, for instance, to the groups $G = T \rtimes (\mathbb{Z}/2)^n$ for any $n \geq 0$ where each involution $\tau \in (\mathbb{Z}/2)^n$ acts by inversion on T; in particular, this generalizes torus actions and torus actions with compatible involutions by considering H as the maximal 2-torus in G. It also applies to $SO(n)$ with the canonical τ-action that makes the isomorphism $SO(n) \rtimes_\tau \mathbb{Z}/2 \cong O(n)$; in this case, H is the maximal 2-torus in $SO(n)$. In particular, by Theorem 4.2.9, a lot of information of the G-equivariant cohomology comes by considering 2-torus actions and $\mathbb{F}_2$-coefficients and a topological generalization of Theorem 1.5 where Hamiltonian torus actions on symplectic manifolds (equivalently T-equivariantly formal) is discussed; since we are weakening the geometrical assumptions, we need to strengthen our hypothesis by assuming G-equivariantly formality on our spaces.

Theorem 4.3.5. *Let $G = T \rtimes \mathbb{Z}/2$ and X be a path-connected finite-dimensional G-CW-Complex where T is a torus. If $H^*_G(X)$ is a j-th syzygy over $H^*(BG)$, then so is $H^*_{T_2}(X^\tau)$ as a module over $H^*(BT_2)$. In particular, if X is G-equivariantly formal, then the real locus X^τ is T_2-equivariantly formal.*

Example 4.3.6. Let X be a T-space. Suppose X is also a conjugation space (Definition 4.1.5) with a compatible conjugation τ. Then from Theorem 4.3.5 and Corollary 4.1.8 we have that the real locus X^τ is T_2-equivariantly formal.

The assumptions of Theorem 4.3.5 can not be weakened. For example, If X is a G-space such that it is simultaneously T-equivariantly formal and τ-equivariantly formal, it is not necessary true that X is G-equivariantly formal or so its real locus X^τ is T_2-equivariantly formal as the next example exhibits.

Example 4.3.7. let $X = \{(u,z) \in \mathbb{C} \times \mathbb{R} : |u|^2 + |z|^2 = 1\} = S^2$, let $T = S^1$ act on X by $g \cdot (u,z) = (gu,z)$; more precisely, by scalar multiplication in the first factor. Let τ be the involution $\tau(u,z) = (\bar{u}, -z)$ which is compatible with the torus action. Notice that $X^T = \{(0,1),(0,-1)\} \cong S^0$ and $X^\tau = \{(-1,0),(1,0)\} \cong S^0$. Therefore, the action of T_2 on X^τ is the multiplication by ± 1 and thus it is a free T_2-space, this implies that its T_2-equivariant cohomology is not free over $H^*(BT_2)$. On the other hand, $H_T^*(X)$ is a free $H^*(BT)$-module since X and X^T have the same Betti sum.

One of the main issues of this example is that $X^G = \emptyset$, even assuming $X^G \neq \emptyset$ a counterexample can be found and its construction is motivated by [Su, 1964, Sec. 5]. We start by recalling the following well-known construction of topological spaces.

Definition 4.3.8. Let $f : X \to Y$ be a G-map between G-spaces X and Y. The mapping cylinder is defined as the G-space $M_f = (X \times [0,1]) \sqcup Y / \sim$ where $(x,1) \sim f(x)$, with the action given by $g \cdot (x,t) = (gx,t)$ for $(x,t) \in X \times [0,1]$ and the regular action on Y; notice that it is well defined at the points of the form $(x,1)$ since f is a G-map.

The space M_f is G-homotopic to Y, and therefore $H^*(M_f) \cong H^*(Y)$. Also, the fixed point subspace $(M_f)^G \cong M_{f^G}$ where $f^G \colon X^G \to Y^G$. Now let $g : X \to Z$ be a G-map and M_g the respective mapping cylinder, then the space $M_{f,g} = M_f \cup_{X\times\{0\}} M_g$ has cohomology groups fitting in the long exact sequence

$$0 \to H^0(M_{f,g}) \to H^0(Y) \oplus H^0(Z) \to H^0(X) \to H^1(M_{f,g}) \to \cdots$$

following from the Mayer-Vietoris long exact sequence. Moreover, $M_{f,g}$ becomes a G-space and $(M_{f,g})^G \cong M_{f^G,g^G}$. In particular, we have

Proposition 4.3.9. *Let m,n,r be different integers, $h: S^m \to S^n$ a map between spheres and consider $f = h \times id\colon S^m \times S^r \to S^n \times S^r$ and $g\colon S^m \times S^r \to S^m$ the projection. Then $H^*(M_{f,g})$ is free over $\mathbb{Z}/2$ where a copy of $\mathbb{Z}/2$ happens in degree $0, n, m+r+1, n+r$ and it is zero otherwise. In particular, $b(M_{f,g}) = 4$.*

Example 4.3.10. Let $X = S^3 \subseteq \mathbb{C}^2$, $Y = S^5 \subseteq \mathbb{C}^3$ and $Z = S^9 \subseteq \mathbb{C}^4$. Let $T = S^1$ act on X and Y by scalar multiplication on the first component , and let T act on Z by scalar multiplication on the first and second components and trivially otherwise. Then $X^T = S^1$, $Y^T = S^3$ and $Z^T = S^5$. Let τ act on X and Y as the complex conjugation on the first component respectively, and on Z as the complex conjugation on the first and second component and multiplication by -1 on the other components. Then $X^\tau = S^2$, $Y^\tau = S^4$ and $Z^\tau = S^1$. Note that the induced action of $T_2 \subseteq T$ is free on Z^τ.

Let $f: X\times Z \to Y \times Z$ be the map $i \times id$ where i is the inclusion $i(u,z) = (u,z,0)$, and $g: X\times Z \to X$ the projection. Consider the T-space $M = M_{f,g}$ and the induced action of τ on M becomes a compatible involution. Then $b(M) = b(M^T) = b(M^{T_2}) = b(M^\tau) = 4$ from Proposition 4.3.9, but $b(M^G) = b((M^\tau)^{T_2}) = 2$.

Example 4.3.11. Let $X = S^3$, $Y = S^2$ and $h: X \to Y$ be the Hopf map. This map can be explicitly presented as $h(u,z) = (2u\bar{z}, |u|^2 - |z|^2)$ where S^3 is seen as the unit sphere in $\mathbb{C}^2$ and S^2 as the unit sphere in $\mathbb{C}\times\mathbb{R}$. Let $T = S^1$ act on S^3 and S^2 as the complex multiplication in the first component respectively, and τ be the involution on S^3 and S^2 given by the complex conjugation in the first component respectively. Then τ is compatible with the torus action and $X^T \cong S^1$, $X^\tau \cong S^2$, $Y^T \cong S^0$ and $Y^\tau \cong S^1$. Now let $Z = S^5$ be the unit sphere in $\mathbb{C}^3$, let T act on Z by multiplication in the first component and τ be the involution on Z given by the complex conjugation in the first component, and multiplication by -1 in the second and third component; then $Z^T \cong S^3$ and $Z^\tau \cong S^0$, notice that the action of the 2-torus $T_2 \subseteq T$ on Z^τ is free.

Let $M = M_{f,g}$ be the construction of Proposition 4.3.9 , then $b(M) = b(M^T) = b(M^\tau) = 4$ and thus M is T-equivariantly formal; nevertheless, M^τ is not T_2-equivariantly formal since $b((M^\tau)^{T_2}) = 2 < b(M^\tau)$.

We can summarize these examples into the following proposition.

Proposition 4.3.12. *There is a topological space M with an action of a torus T and a compatible involution τ such that $M^G \neq \emptyset$, M is T-equivariantly formal and $\mathbb{Z}/2$-equivariantly formal, but the real locus M^τ is not T_2-equivariantly formal with respect to the induced action of the 2-torus $T_2 \subseteq T$ on M^τ.* □

4.4 Application: Big polygon spaces

The big polygon spaces provide remarkable examples of the study of equivariant cohomology of T-spaces since their equivariant cohomology is not free but they realize all other possible syzygy orders. Their non-equivariant and T-equivariant cohomology was determined by [Franz, 2015] where an upper bound for their syzygy order was conjectured which was proved later in [Franz and Huang, 2019]. These spaces generalize chain spaces and polygon spaces studied in different contexts by [Farber and Fromm, 2013] and [Hausmann, 2014] for instance. The real analogue of these spaces is also studied by [Puppe, 2018] in the case of 2-torus actions and cohomology with $\mathbb{F}_2$-coefficients.

Let a,b be positive integers and $M=(S^{2a+2b-1})^n \subseteq (\mathbb{C}^a\times\mathbb{C}^b)^n$, $l=(l_1,\dots,l_n)\in\mathbb{R}^n$ such that $l_i>0$ for all i. Consider

$$X=\left\{(u,z)\in M:\sum_{i=1}^n l_iu_i=0\right\}$$

and let $T=(S^1)^n$ acts on X by $g\cdot(u,z)=(u,gz)$, where $gz=(g_1z_1,\dots,g_nz_n)\in(\mathbb{C}^b)^n$, g_iz_i is the scalar multiplication in $\mathbb{C}^b$ for any $i=1,\dots,n$, and set τ as the complex conjugation in all variables. Then X is a T-space with a compatible involution and therefore it inherits an action of $G=T\rtimes\mathbb{Z}/2$; furthermore, X is an orientable compact connected T-manifold of dimension $(2a+2b-1)n-2a$ [Franz, 2015, Lemma 2.1]. Following the notation of [Farber and Fromm, 2013], the space of polygons of dimension d is defined as

$$E_d(l)=\{(u_1,\dots,u_n)\in(S^{d-1})^n:\sum_{i=1}^n l_iu_i=0\}$$

we obtain that $X^T=E_{2a}(l)$, $X^G=E_a(l)$. Moreover, the real locus X^τ is the real big polygon space studied in [Puppe, 2018].

We now focus on discussing the syzygy order of $H^*_G(X)$ over $H^*(BG)$, the first result is the following.

Proposition 4.4.1. *$H^*_G(X)$ is not free over $H^*(BG)$. In fact, $H^*_G(X)$ is not a j-th syzygy for $j\geq(n+1)/2$.*

Proof. For the first statement, we will use that $X^G=X^H$ and that the Betti sum of X^G is strictly less than the Betti sum of X to conclude that $H^*_G(X)$ is not free over $H^*(BG)$ (Corollary 4.2.11). The integer cohomology of X is free and its Betti sum is $b(X)=2^n$ as shown in [Franz, 2015, Prop.3.3]. On the other hand, when $a>2$, the $\mathbb{F}_2$-cohomology of the space $X^G=E_a(l)$ is isomorphic to a quotient of an exterior algebra on n-generators by a non-trivial ideal [Farber and Fromm, 2013, Prop. 4.2] and so $b(X^G)<2^n$. The same bound holds when $a=2$ by using that $E_1(l)\cong S^1\times E_1(l)/SO(2)$ and the computation of the Betti sum as in [Farber, 2008, §1.9]. This shows that $b(X^H)<b(X)$ and thus X is not G-equivariantly formal

by Corollary 4.2.11. The last assertion of the proposition follows directly by applying Corollary 4.2.10 as X is a compact manifold and it satisfies Poincaré duality for $\mathbb{F}_2$-cohomology. □

Now we restrict to the equilateral case of $l = (1,1,\dots,1) \in \mathbb{R}^n$ and $n = 2m+1$. Under these assumptions we have that $H_T^*(X)$ is an m-th syzygy over $H^*(BT)$ but not an $(m+1)$-st syzygy [Franz, 2015, Prop 5.1] with coefficients in a field of characteristic 0. We will prove that this condition still holds when we consider the G-equivariant cohomology of X and $\mathbb{F}_2$-coefficients, then from Theorem 4.3.5 we will give an alternative proof to [Puppe, 2018, Cor. 3.17] where it is computed the syzygy order of the equilateral real big polygon spaces. By Lemma 4.2.9 we can restrict to study the action of the 2-torus $H = T_2 \times \mathbb{Z}/2$ where T_2 denotes the 2-subtorus of T and $\mathbb{Z}/2 \cong \{id, \tau\}$. In order to compute the H-equivariant cohomology of X, we will use the equivariant Poincaré-Alexander-Lefschetz duality for 2-torus actions. Namely, let $\iota : M \setminus X \to M$ be the inclusion and let ι_*^H be the induced map in equivariant homology. For simplicity set $d = 2a+2b-1$; Theorem 3.2.3 implies that there is a short exact sequence

$$0 \to \operatorname{coker} i_*^H[nd] \to H_H^*(X) \to \ker i_*^H[nd-1] \to 0. \tag{4.4.1}$$

For any subset $J \subseteq \{1,2,\dots,n\}$, write $J^c = \{1,\dots,n\} \setminus J$ and $J \cup j = J \cup \{j\}$, and we define $l(J) = \sum_{j\in J} l_J$. We say that J is short if $l(J) < l(J^c)$. In our case with $l = (1,1,\dots,1)$, J is short if and only if $|J| \leq m$. Also, we define the manifolds

$$V_J = \{(u,z) \in M : \forall j \notin J \; (u_j, z_j) = *\}$$

$$W_J = \{(u,z) \in M : \forall j,k \notin J, \; u_j = u_k, z_j = z_k = 0\}$$

where $* \in S^{2a+2b-1} \cap (\mathbb{C}^a \times \{0\})$ is a chosen base point. Notice that V_J is homeomorphic to a product of $|J|$ d-spheres and $W_J \cong V_J \times S^{2a-1}$; these homeomorphism imply that $V_J \subseteq W_J$, $\dim V_J = |J|d$ and $\dim W_J = |J|d + (2a-1)$. Let $[V_J], [W_J]$ be their respective homological orientation classes and $[V_J]_H$, $[W_J]_H$ their equivariant lifting. Then $H_*(M)$ is free with basis

$\{[V_J] : J \subseteq \{1,\ldots,n\}\}$ and $H_*(M\setminus X)$ is free with basis $\{[V_J],[W_J] : J \text{ short}\}$ [Franz, 2015, Lem. 3.2].

Analogously to [Franz, 2015, Lem. 4.5, Prop.4.6] we have

Proposition 4.4.2.

(i) $H_*^H(M)$ *is a free* $H^*(BH)$*-module with basis* $\{[V_J]_H,\ J\subseteq\{1,\ldots,n\}\}$.

(ii) $H_*^H(M\setminus X)$ *is a free* $H^*(BH)$*-module with basis* $\{[V_J]_H,\ [W_J]_H,\ J \text{ short }\}$.

(iii) $\iota_*^H([V_J]_H) = [V_J]_H$ *and* $\iota_*^H([W_J]_H) = \sum_{j\notin J} w_j^b(w_j+w)^b[V_{J\cup j}]_H$.

Proof. Notice that $b(M) = b(M^H)$ as $M^H \cong (S^{2a-1})^n$; thus M is H-equivariantly formal and we obtain that the restriction map

$$H_*^H(M) \to H_*(M)$$

which is the edge homomorphism of the homological spectral sequence (3.1.3) with E_2-page given by $E_2 = H_*(M)\otimes H^*(BH)$ and converging to $H_*^H(M)$ is surjective as the basic elements $[V_J]$ have a lifting in $H_H^*(M)$. Therefore, the spectral sequence collapses and so $\{[V_J]_H,\ J\subseteq \{1,\ldots,n\}\}$ is a basis of $H_*^H(M)$ over $H^*(BH)$ as in the Leray-Hirsch Theorem (Theorem A.8) proving then (i). The proof of (ii) follows in a similar fashion.

To prove (iii), we will use the H-equivariant Euler class (Definition 3.2.4) since it will help to compute explicitly the map ι_*^H on the generators of $H_*^H(M\setminus X)$. Let $K = K_1\times K_2$ where $K_1 = \{1,g\}, K_2 = \{1,\tau\}$, g denotes the action induced by multiplication by -1 and τ the complex conjugation in $\mathbb{C}$, and let x,w denote the generators of $H^*(BK_1)$ and $H^*(BK_2)$ respectively. From Example 3.2.6 we have that $e_K(0\subseteq\mathbb{C}) = x(x+w)$. Let S denote the unit sphere in $\mathbb{C}^a\times\mathbb{C}^b$, let τ be the complex conjugation on $\mathbb{C}^a\times\mathbb{C}^b$ and g the multiplication by -1 on $\mathbb{C}^b$. Set K_1, K_2 and K as before. Then $e_K(S^{K_1}\subseteq S) = e_K(\mathbb{R}^a\times\{0\}\subseteq\mathbb{R}^a\times\mathbb{C}^b) = e_K(0\subseteq\mathbb{C}^b)$. From the above computation and the multiplicativity of the Euler class we get that $e_K(S^{K_1} \subseteq S) = x^b(x+w)^b$; or equivalently, $[S^{K_1}]_K = x^b(x+w)^b[S]_K$.

Now the proof for the case of the torus action on the big polygon space found in [Franz, 2015, Lem.4.5] can be imitated in our situation to show that (iii) holds. Firstly, the identity $i_*^H([V_J]_H) = [V_J]_H$ follows from the naturality of the equivariant homology, that is, from the commutative diagram

$$\begin{array}{ccc} H_*^H(V_J) & & \\ \downarrow & \searrow & \\ H_*^H(M \setminus X) & \xrightarrow{\iota_*^H} & H_*^H(M) \end{array} .$$

To compute $i_*^H([W_J])$, we need to "enlarge" the acting group. For $J \subseteq [n]$, define τ_J the involution on M given by the complex conjugation on the variables $u_j : j \in J$ and $z_j : j \in J$, and write $\sigma_J = \tau_{J^c}$. Set $H_J = T_2 \times \tau_J \times \sigma_J$ and $H \to H_J$ the map induced by the identity on T_2 and the map which sends τ to (τ_J, σ_J). Thus we get a map $H^*(BH_J) = \Bbbk[t_1,\dots,t_n,w_\tau,w_\sigma] \to H^*(BH) = \Bbbk[t_1,\dots,t_n,w]$ sending w_τ and w_σ to w and it is the identity in the other variables. Moreover, we have maps in equivariant homology

$$H_*^{H_J}(M) \to H_*^H(M)$$

Notice that the H_J-action on M induces an action of H on M; such an action coincides with the initial action of H on M described at the beginning of the section. Also, we have similar restriction maps for the H_J-invariant submanifolds $X, M \setminus X \subseteq M$.

Let $\tilde{M} = M \cap (\mathbb{C}^a \times \{0\})^n \cong (S^{2a-1})^n$. For $J \subseteq [n]$, let Δ_J be the inclusion of S^{2a-1} into the factors $j \in J$ of $\tilde{M}$. Notice that there is a homeomorphism $W_J \cong V_J \times \Delta_{J^c}$; moreover, such homeomorphism yields to an equivariant decomposition $H_J = (K_J \times \tau_J) \times (K_{J^c} \times \sigma_J)$ where $K_J \subseteq T_2$ is the 2-subtorus of non-trivial factors in the position $j \in J$. Therefore, by the Künneth theorem for equivariant homology (Theorem 3.2.5) we have that

$$[W_J]_{H_J} = [V_J]_{K_J \times \tau_J} \times [\Delta_{J^c}]_{K_{J^c} \times \sigma_J}$$

By naturality, we have that

$$i_*^{H_J}([W_J]_{H_J}) = i_*^{K_J\times\tau_J}([V_J]_{K_J\times\tau_J}) \times i_*^{K_{J^c}\times\sigma_J}([\Delta_{J^c}]_{K_{J^c}\times\sigma_J}) \tag{4.4.2}$$

As above, it is straightforward to check that $i_*^{K_J\times\tau_J}([V_J]_{K_J\times\tau_J}) = [V_J]_{K_J\times\tau_J}$, so it only remains to compute the last term of (4.4.2). Without loss of generality we can assume that $J=\emptyset$, so $\Delta_{J^c} = \Delta$ is the diagonal of $\tilde{M}$, $\sigma_J = \tau$, τ_J is trivial and $H_J = H$. So we need to compute $i_*^H([\Delta]_H)$. Since in $H_*(\tilde{M})$ we have that $[\Delta] = \sum_{j=1}^n [\Delta_j]$ and $\tilde{M}$ is H-equivariantly formal, we have then in equivariant homology that $[\Delta]_H = \sum_{j=1}^n [\Delta_j]_H$. Consider the inclusion $K_1 \to H$ into the j-th factor of T_2, denote this group by K_j. This map induces in cohomology an identification of x with t_j. Observe that $\Delta_j = V_j^{T_2} = V_j^{K_j}$ and thus $[\Delta_j]_H = [V_j^{K_j}]_{K_j\times\tau}$. We obtain by naturality of the Euler class and the above computation that $[\Delta_j]_H = t_j^b(t_j+w)^b[V_j]_H$. Finally, this implies that

$$i_*^H([\Delta]_H) = \sum_{j=1}^n t_j^b(t_j+w)^b[V_j]_H \tag{4.4.3}$$

For the general case, using this computation, for any J we have again by (4.4.2) that

$$\begin{aligned} i_*^{H_J}([W_J]_{H_J}) &= i_*^{K_J\times\tau_J}([V_J]_{K_J\times\tau_J}) \times i_*^{K_{J^c}\times\sigma_J}([\Delta_{J^c}]_{K_{J^c}\times\sigma_J}) \\ &= [V_J]_{K_J\times\tau_J} \times \sum_{j\notin J} t_j^b(t_j+w_\sigma)^b[V_j]_{K_{J^c}\times\sigma_J} \\ &= \sum_{j\notin J} t_j^b(t_j+w_\sigma)^b[V_{J\cup\{j\}}]_{H_J} \end{aligned}$$

The computation for the H-equivariant cohomology follows by naturality and using the restriction map $H^*(BH_J) \to H^*(BH)$ with maps w_σ to w.

□

Let $R = H^*(BH) = \mathbb{F}_2[t_1,\ldots,t_n,w]$ and write $y_j = t_j(t_j+w)$. We will use the Koszul resolution of $L = R/(y_1^b,\ldots,y_n^b)$ analogous to [Franz, 2015, §5] to identify $H_H^*(X)$ with the Koszul syzygies appearing in the resolution as described in Appendix B. Notice that the sequence $(t_1,\ldots,t_n,w)$ is regular in R, and so the sequence $(t_1+w,\ldots,t_n+w)$ is regular. By Proposition B.3 it follows that the sequence $(y_1^b,\dots,y_n^b)$ is regular and so Proposition B.12 implies that there is a free resolution of L given by

$$0\to\bigwedge^n N\xrightarrow{\delta_n}\bigwedge^{n-1}N\to\cdots\to\bigwedge^2 N\xrightarrow{\delta_2}N\xrightarrow{\delta_1}R\to L\to 0$$

where N is the graded free R-module with basis $\{v_1,\dots,v_n\}$ each of degree $2b$. We denote by $N^\vee$ the R-dual of N generated by $\{\widetilde{v}_1,\dots,\widetilde{v}_n\}$ dual to the basis of N each of degree $-2b$. The map δ_k is given by $\delta_k(v_{j_1}\wedge\cdots\wedge v_{j_k})=\sum_{i=1}^k y_{j_i}v_{j_1}\wedge\cdots\wedge\widehat{v_{j_i}}\wedge\cdots\wedge v_{i_k}$. Set $K_k=\operatorname{Im}(\delta_k)[-2bk]$ the k-th Koszul syzygy of L. The degree shift is made so K_k is generated by elements of degree 0. Recall that by the self-duality of the Koszul resolution, K_k can be identified with the image of the map $\operatorname{Im}(d_{n-k+1})[2b(n-k)]$ where $d_k=\delta_k^\vee\colon\bigwedge^{k-1}N^\vee\to\bigwedge^k N^\vee$, where $d_k(\widetilde{v}_J)=\sum_{i\notin J}y_i\widetilde{v}_{J\cup j}$. We are identifying the basis of $\bigwedge^k N^\vee$ with the set of elements $\widetilde{v}_J=\widetilde{v}_{j_1}\wedge\cdots\wedge\widetilde{v}_{j_k}$ for $J=\{j_1<\cdots<j_k\}\subseteq\{1,\dots,n\}$. In particular, K_{k+1} and K_k can be identified with the kernel and image of the map d_{n-k+1} up to a degree shift.

We will use this construction to prove the main result of this section.

Proposition 4.4.3. *Let $n=2m+1$, $m\geq 1$. The H-equivariant cohomology of the big polygon space*

$$X=\left\{(u,z)\in M=(S^{2a+2b-1})^n:\sum_{i=1}^n u_i=0\right\}$$

is an m-th syzygy but not an $m+1$-st syzygy.

Proof. The proof is similar to the case of the torus action [Franz, 2015, Prop. 5.1]. First we compute the kernel and cokernel of $\iota_*^H\colon H_*^H(M\setminus X)\to H_*^H(M)$ (induced by the inclusion) in the short exact sequence (4.4.1)

$$0\to\operatorname{coker}\iota_*^H[nd]\to H_H^*(X)\to\ker\iota_*^H[nd-1]\to 0. \tag{4.4.4}$$

From Proposition 4.4.2 (i), (ii) we have that $H^*_H(M\setminus X)\cong\bigoplus_{|J|\leq m}(R\cdot[V_J]_H\oplus R\cdot[W_J]_H)$ and $H^*_H(M)\cong\oplus_{J\subseteq[n]}R\cdot[V_J]_H$ as R-modules. By Proposition 4.4.2 (iii), the kernel of the map

$$\iota^H_*\colon\bigoplus_{|J|\leq m}R\cdot[V_J]_H\oplus\bigoplus_{|J|<m}R\cdot[W_J]_H\to H^H_*(M)$$

is the free R-submodule of $H^H_*(M\setminus X)$ generated by the elements $[W_J]_H-\sum_{j\in J}y_j^b[V_{J\cup j}]_H$ where $|J|<m$ since $\iota^H_*([V_J]_H)=[V_J]_H$ and $\iota^H_*([W_J]_H)=\sum_{j\notin J}t_j^b(t_j+w)^b[V_{J\cup j}]_H$. On the other hand, the map

$$\iota^H_*\colon\bigoplus_{|J|=m}R\cdot[W_J]_H\to\bigoplus_{|J|=m+1}R\cdot[V_J]\subseteq H^H_*(M)$$

can be identified with the map d_{m+1} in the Koszul resolution of $L=R/(y_1^b,\dots,y_n^b)$ described above whose kernel is the Koszul syzygy K_{m+2}. So we obtain that

$$\ker(\iota^H_*)\cong\bigoplus_{|J|<m}R[-|J|d-\bar{d}]\oplus K_{m+2}[-md-\bar{d}+2]$$

The degree shifts follow from the fact that $\dim W_J=|J|d+\bar{d}$ and $\dim V_J=|J|d$ and the convention that the Koszul syzygies are generated in degree 0.

Similarly, we can see that $\mathrm{Im}(\iota^H_*)\cong\bigoplus_{|J|\leq m}R\cdot[V_J]_H\oplus\mathrm{Im}(d_{m+1})$ and thus

$$\mathrm{coker}(\iota^H_*)=H^H_*(M)/\,\mathrm{Im}(\iota^H_*)\cong\bigoplus_{|J|>m+1}R\cdot[V_J]_H\oplus\mathrm{coker}(d_{m+1})$$

Notice that from the Koszul resolution it follows that $\mathrm{coker}(d_{m+1})\cong\mathrm{Im}(d_{m+2})=K_m$ the m-th Koszul syzygy of L. Summarizing, we obtained that

$$\mathrm{coker}(\iota^H_*)\cong\bigoplus_{|J|>m+1}R[-|J|d]\oplus K_m[-(m+1)d].$$

and thus both $\ker(\iota^H_*)$ and $\mathrm{coker}(\iota^H_*)$ are m-th syzygies. To finish the proof, it is enough to show that the sequence (4.4.4) splits. This will follow from the following lemma [Puppe, 2018, Lem.3.12] and using the singular Cartan model as a free R-model for equivariant cohomology.

□

Lemma 4.4.4. *Let* $0 \to A \xrightarrow{\alpha} B \to C \to 0$ *be a short exact sequence of free differential graded* R*-modules such that* $H_*(A)$ *and* $H_*(B)$ *are free* R*-modules. Then the exact sequence* $0 \to \operatorname{coker}(\alpha_*) \to H_*(C) \to \ker(\alpha_*) \to 0$ *splits.*

As an immediate corollary of Proposition 4.4.3 and Theorem 4.3.5, we obtain the next result.

Corollary 4.4.5. *The equivariant cohomology of the real big polygon space* X^τ *under the action of the* 2*-torus* T_2 *is a* m*-th syzygy but not an* $m+1$*-st syzygy*

Appendix A

Fibrations and spectral sequences

In this section we recall the definition of fiber bundles, fibrations and spectral sequences, together with remarkable results and properties of these constructions. The references followed are [Mimura and Toda, 1991] and [Davis and Kirk, 2001].

Definition A.1. Let E, B be topological spaces with B connected. A **fiber bundle** over B, with total space E and fiber F is an open covering $\{U_\alpha : \alpha \in I\}$ of B and a continuous map $p\colon E \to B$ satisfying that for any α there exists a map $\varphi_\alpha\colon p^{-1}(U_\alpha) \to F \times U_\alpha$ such that the diagram

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & U_\alpha & \end{array}$$

is commutative.

Note that for any $b \in B$ there is a homeomorphism $p^{-1}(b) \cong F$; moreover, for any $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$, the map $\varphi_\alpha \varphi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is given by $(b,x) \mapsto (b, t_{\alpha\beta}(b) \cdot x)$, where $t_{\alpha\beta} : B \to K$ is a continuous map called transition function and K is a group acting on F called the structure group of the fiber bundle. A principal G-bundle $p\colon E \to B$ is a fiber bundle with fiber G and structure group G acting on itself by $g \cdot g' = g' g^{-1}$.

Definition A.2. Let X be a topological space. The **fundamental groupoid** of X is the category $\pi(X)$ whose objects are the elements of X and the set of morphisms between two objects $x,y \in X$ is the set $\mathrm{mor}(x,y) = \{f \colon I \to X : f(0) = x, f(1) = y\}/\simeq$ where $\simeq$ denotes the homotopy relation relative to end points.

A **system of local coefficients** on X is a contravariant functor $\mathcal{A} \colon \pi(X) \to$ **AbGrp** from the fundamental groupoid of X into the category of abelian groups. In other words, for any $x \in X$, $\mathcal{A}(x) = \mathcal{A}_x$ is an abelian group , and for any equivalence class of paths $f \colon I \to X$, $\mathcal{A}(f) \colon \mathcal{A}_{f(1)} \to \mathcal{A}_{f(0)}$ is a group homomorphism.
When $\mathcal{A}_x = A$ for any $x \in X$ and $\mathcal{A}(f)$ is the identity map on A, we say that the local system of coefficients $\mathcal{A}$ is trivial.

Definition A.3. Let X be a topological space and let $\mathcal{A}$ be a system of local coefficients on X. Define

$$C_n(X;\mathcal{A}) = \left\{\sum g_i \otimes \sigma_i : \sigma_i \colon \Delta^n \to X, g_i \in A_{\sigma_i(v_0)}\right\} \subseteq \bigoplus_{x\in X} A_x \otimes_{\mathbb{Z}} C_n(X)$$

where Δ^n is the standard n-simplex, $v_0 = (1,0,\ldots,0) \in \Delta^n$ and $C_n(X)$ is the free abelian group of n-chains in X. We make $C_n(X;\mathcal{A})$ into a chain complex by setting $d_n : C_n(X : \mathcal{A}) \to C_{n-1}(X;\mathcal{A})$ as $d_n(g \otimes \sigma) = \sum_{i=1}^{n}(-1)^i g \otimes \delta_i\sigma + \mathcal{A}(\lambda_\sigma)(g) \otimes \delta_0\sigma$, where $\delta_i\sigma$ denotes the restriction of σ to the $(n-1)$-simplex as the face across the vertex v_i, and λ_σ is the path from $\sigma(v_1)$ to $\sigma(v_0)$ given by $\lambda_\sigma(t) = \sigma(tv_0 + (1-t)v_1)$.

We define the **homology of X with local coefficients** $\mathcal{A}$, as the homology of the chain complex $C_n(X;\mathcal{A})$; namely, $H_*(X;\mathcal{A}) := H_*(C(X;\mathcal{A}))$. Analogously, we can define the cohomology with local coefficients $H^*(X;\mathcal{A})$. Note that when $\mathcal{A}$ is a trivial system of local coefficients, the (co)homology with local coefficients agrees with the ordinary singular (co)homology.

Definition A.4. Let E, B be topological spaces. A continuous map $p \colon E \to B$ is a **fibration** if it has the homotopy lifting problem with respect to any space X; that is, for any commutative diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f_0} & E \\ \Big\downarrow i_0 & \overset{\tilde{F}}{\nearrow} & \Big\downarrow p \\ X \times [0,1] & \xrightarrow{F} & B \end{array}$$

there is a homotopy $\tilde{F} : X \times [0,1] \to E$ that makes the diagram commute.

In particular, if $p \colon E \to B$ is a fiber bundle over a paracompact space, it is also a fibration.

Now let $p' \colon E' \to B'$ be another fibration. A map of fibrations is a pair of continuous maps $f : B \to B'$ and $\tilde{f} : E \to E'$ so that $p' \circ \tilde{f} = f \circ p$.

As in the case of fiber bundles, for any continuous map $f \colon X \to B$, the pullback f^*E of a fibration is a fibration over X, and the projection $f^*E \to E$ together with f induces a map of fibrations. An important property that fibrations satisfy is given by the following theorem [Davis and Kirk, 2001, §6.13].

Theorem A.5 (Homotopy long exact sequence associated to a fibration)**.** *Let $p : E \to B$ be a fibration, $b \in B$ and set $F = p^{-1}(b)$. Then there is a long exact sequence in homotopy*

$$\cdots \to \pi_n(F,f) \to \pi_n(E,e) \to \pi_n(B,b) \to \pi_{n-1}(F,b) \to \cdots$$
$$\cdots \to \pi_1(B,b) \to \pi_0(F,f) \to \pi_0(E,e) \to \pi_0(B,b).$$

where $f \in F$, $e \in E$ and $b \in B$ are the appropriate base points. It is important to remark that the exactness at the level of π_0 refers to the exactness as pointed spaces, in the sense that the kernel of the map (pre-image of the base point) is equal to the image.

In particular, from the fibration of the universal G-bundle $EG \to BG$, we have that $\pi_n(G) \cong \pi_{n+1}(BG)$ for $n \geq 0$.

The previous theorem is independent of the chosen base point as all the spaces F_b are homotopy equivalent. This implies that F is well-defined up to homotopy equivalence if B is path-connected. It will follow from the following remark.

Remark A.6. Let $\gamma\colon I \to B$ be a path in B. We associate a map $h_\gamma\colon F_{\gamma(0)} \to F_{\gamma(1)}$ satisfying

- If $\gamma \sim \gamma'$ (homotopy equivalent relative to end points) then $h_\gamma \sim h_{\gamma'}$ (homotopic maps).
- If γ is the constant path $h_\gamma \sim id_F$.
- $h_{\gamma * \gamma'} \sim h_\gamma \circ h_{\gamma'}$ as long as the concatenation of paths $\gamma * \gamma'$ is defined.

In particular, these properties imply that h_γ is a homotopy equivalence. Therefore; for any $b \in B$ we have a function

$$\pi_1(B,b) \to \{[f]\colon F \to F, [f] \text{ is the homotopy equivalence class of } f\}.$$

Furthermore, $\pi_1(B,b)$ defines a system of local coefficients $\mathcal{A}$ on B by setting $\mathcal{A}_b^q = H^q(F;R)$ where R is a ring and $\mathcal{A}^q([\gamma]) = h_\gamma^*$ for $[\gamma] \in \pi_1(B,b)$ and any $q \in \mathbb{N}$. In this case, the cohomology with respect to this system of local coefficients will be denoted by $H^*(B;\mathcal{H}^q(F;R))$. Observe that there is a canonical action of $\pi_1(B,b)$ on $H^q(F_b;R)$, and we say that $\pi_1(B,b)$ *acts trivially* on the cohomology of the fiber if this action is trivial, or equivalently, the system of local coefficients is trivial for every q.

Remark A.7. If B is simply connected, or if $E \to B$ is a fiber bundle with path connected structure group G, then the system of local coefficients is trivial.

When in a fiber bundle the system of local coefficients is trivial, we have the following theorem [Mimura and Toda, 1991, Ch.III.§4] due to Leray and Hirsch who proved it independently in 1940's. We say that a space X is of *finite cohomology type* if $H^*(X;R)$ is a finitely generated free R-module in each degree. For example, if $R = \Bbbk$ is a field and X is compact or a finite CW-complex, it is of finite cohomology type.

Theorem A.8 (Leray-Hirsch Theorem)**.** *Let $F \to E \to B$ be a fiber bundle such that either F is of finite cohomology type each degree, or $\pi_1(B)$ acts trivially in the cohomology of the fiber and*

B is of finitely cohomology type. Assume that the map $i^*\colon H^*(E;R) \to H^*(F;R)$ induced by the inclusion of the fiber $i\colon F \to E$ admits an additive section $\theta\colon H^*(F;R) \to H^*(E;R)$. Then the map

$$\phi\colon H^*(B;R) \otimes H^*(F;R) \to H^*(E;R)$$

given by $\phi(b \otimes x) = \pi^*(b) \cdot \theta(x)$ is an isomorphism of $H^*(B;R)$-modules.

It is important to remark that if R is a field, the above map θ exists if and only if i^* is surjective.

Powerful computational tools in homological algebra are the spectral sequences. In algebraic topology, the Serre spectral sequence and the Eilenberg-Moore spectral sequence arise from fibrations. We will finish then this section with the fundamental theorems of spectral sequences for fibrations. Here $H^*(X)$ will denote the singular cohomology $H^*(X;R)$ for any space X and a fixed commutative ring R.

Theorem A.9 (Cohomology Serre spectral sequence)**.** *Let $p\colon E \to B$ be a fibration with fiber F. There exists a sequence $(E_r^{p,q}, d_r)$, $r = 2, 3, \ldots$ of bigraded chain complexes of R-modules, and a sequence $\{D^{k,n-k}\}$ of subgroups of $H^n(E)$ satisfying the following properties.*

1. *There is a product*
$$E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}.$$

2. *d_r is a homomorphism of R-modules of bidegree $(r, 1-r)$, that is, $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying*
$$d_r \circ d_r = 0 \;\text{ and }\; d_r(xy) = d_r(x)y + (-1)^{p+q} x d_r(y)$$
such that the cohomology $H^*(E_r^{p,q})$ with respect to d_r is isomorphic to $E_{r+1}^{p,q}$, and such isomorphism preserves the product.

3. *There is an isomorphism* $\varphi\colon E_2^{p,q} \to H^p(B;\mathcal{H}^q(F))$ *such that* $\varphi(a\cdot b) = (-1)^{p'q}\varphi(a)\cup\varphi(b)$*. The product on the left-hand side term of is given in 1, and the product on the right-hand side is the one arising from the cohomology cup product.*

4. *For* $r > \max(p, q+1)$*, there is an isomorphism* $E_r^{p,q} \cong E_{r+1}^{p,q}$ *and this module is denoted by* $E_\infty^{p,q}$*.*

5. *The product in* $H^*(E)$ *induces a product*

$$D^{p,q} \otimes D^{p',q'} \to D^{p+p',q+q'}.$$

6. $D^{p,q} \supseteq D^{p+1,q-1}, D^{p,q}/D^{p+1,q-1} \cong E_\infty^{p,q}$ *where the isomorphism preserves the product.*

7. $H^n(E) = D^{0,n}$*.*

8. *A map of fibrations* $\tilde{f}\colon E \to E'$*,* $f\colon B \to B'$ *induces maps* $f_r\colon E_r' \to E_r$ *and* $f_D\colon D' \to D$ *which preserve products, commute with* d_r*,* $(f_r)^* = f_{r+1}^*$*, and* $(f_D)^* = f_\infty$*. Also,* f_2 *and* $f_D\colon (D')^{0,n} \to D^{0,n}$ *are induced by* f*.*

In this case, we write $H^*(B;\mathcal{H}^q(F)) \Rightarrow H^*(E)$ *and we say that the spectral sequence converges to* $H^*(E)$*. In the case where, for some* $r \geq 2$*,* $d_s = 0$ *for* $s \geq r$*, then* $E_r \cong E_\infty$ *and we say that the spectral sequence degenerates at* E_r*. Finally if* E_r *is concentrated in a single row or column, then the spectral sequence degenerates and we say that it collapses at the r-th page.*

See [Mimura and Toda, 1991, Ch.III.Thm.2.10] for a reference of this result. As an important corollary the Gysin long exact sequence [Mimura and Toda, 1991, Ch.III.Thm.2.10] can be derived.

Corollary A.10. *Let* $p\colon E \to B$ *be a fibration with fiber* F *satisfying* $H^*(F) \cong H^*(S^n)$ *for some* $n \geq 1$ *and suppose that* $\pi_1(B)$ *acts trivially on the cohomology of* F*. Then there is a long exact* *sequence*

$$\cdots \to H^k(B) \xrightarrow{p^*} H^k(E) \to H^{k-n}(B) \xrightarrow{q} H^{k+1}(B) \xrightarrow{p^*} \cdots$$

where q is the multiplication by an element $e \in H^{n+1}(B)$ called the Euler class of B.

The map $i^*\colon H^*(E) \to H^*(F)$ induced by the inclusion, it is closely related to the degeneracy of the spectral sequence as the next result states [Mimura and Toda, 1991, Ch.III.Thm. 4.4].

Theorem A.11. *Let $F \to E \to B$ be a fibration of path-connected spaces. The system of local coefficients is trivial and the Serre spectral sequence degenerates at E_2 if and only if the edge homomorphism $i^*\colon H^*(E) \to H^*(F)$ is surjective.*

We finish this section with another spectral sequence that arises from a fibration. An approach due to Eilenberg-Moore resulted in a tool that allows approximating the cohomology of the fiber F from knowledge of the cohomology of E and B. Usually in the literature, the assumption of B being simply connected is imposed in the hypothesis; however, the existence of the spectral sequence and the (non-strongly) convergence of it still holds when the local coefficient system is trivial as we state in the following theorem [Eilenberg and Moore, 1966],[Smith, 1967].

Theorem A.12 (Cohomology Eilenberg-Moore spectral sequence)**.** *Let $F \to E \to B$ be a fibration of connected spaces and assume that $\pi_1(B)$ acts trivially on the cohomology of the fiber. Let $f\colon X \to B$ be a continuous map and let f^*E denote the total space of the pullback bundle. There is a spectral sequence with E_2 term given by*

$$E_2 = \mathrm{Tor}_{H^*(B;\Bbbk)}(H^*(X;\Bbbk), H^*(E;\Bbbk))$$

that converges to $H^*(f^*E;\Bbbk)$. In particular, when $X = pt$, then $f^*E \cong F$ and we get a spectral sequence with E_2-term

$$E_2 = \mathrm{Tor}^{*,*}_{H^*(B;\Bbbk)}(\Bbbk, H^*(E;\Bbbk))$$

converging to $H^*(F;\Bbbk)$. Here cohomology with coefficients over a field $\Bbbk$ is being considered.

Appendix B

Regular sequences and syzygies

In this section, we include some generalities on the theory of Noetherian rings and modules over them. The main reason to restrict our study to this special setting is that the equivariant cohomology that we have been discussed in this document is a module over a polynomial ring (with field coefficients). Therefore, some of the algebraic properties derived from modules over Noetherian ring will be reflected over the equivariant cohomology. Most of the proofs of the results stated in this chapter will be omitted and can be found in the main references we are following [Bruns and Herzog, 1998] and [Bruns and Vetter, 2006].

Throughout this section R will denote a Noetherian ring satisfying either of the following properties:

(i) R is a local ring (i.e. R has a unique maximal ideal $\mathfrak{m}$).

(ii) R is an $\mathbb{N}$-graded ring and $\mathfrak{m}$ is the ideal consisting of homogeneous elements of positive degree.

The assumptions over R are needed to be able to use Nakayama's lemma. In the case (ii), all R-modules are assumed $\mathbb{N}$-graded.

Let M be an R-module. An element $x \in R$ is said to be M-regular if for any $m \in M$ we have that $xm = 0$, then $m = 0$; that is, x is not a zero divisor en M. We say that x is regular if it is R-regular.

Definition B.1. A sequence of elements $x = (x_1, \ldots, x_n) \in R^n$ is said to be an M-regular sequence if x_i is $M/(x_1, \ldots, x_{i-1})$-regular for $1 \leq i \leq n$ and $M/xM \neq 0$. A regular sequence is a R-regular sequence.

Example B.2. If $R = \Bbbk[x_1, \ldots, x_n]$, $(x_1, \ldots, x_n)$ is a regular sequence.

As a consequence of our assumptions over R, and the Nakayama's Lemma we have the following properties [Bruns and Herzog, 1998, Prop.1.1.6, Prob.1.1.10]

Proposition B.3. *Let M be a R-module and $x = (x_1, \ldots, x_n)$ an M-regular sequence. We have*

1. *Any permutation of x is an M-regular sequence.*
2. *If $(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ is an M-regular sequence, then $(x_1, \ldots, x_i y_i, \ldots, x_n)$ is an M-regular sequence.*
3. *$x^k = (x_1^k, \ldots, x_n^k)$ is an M-regular sequence for all $k \geq 1$.*

An M-sequence $x = (x_1, \ldots, x_k)$ is called maximal (resp. maximal in an ideal I), if $(x_1, \ldots, x_k, x_{k+1})$ is not an M-regular sequence for any $x_{k+1} \in R$ (resp. any $x_{k+1} \in I$). Now we will illustrate that all maximal regular sequences in a given ideal I have the same length.

Recall that the annihilator of M is defined as the ideal in R, $\operatorname{Ann}(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}$. We state the following proposition [Bruns and Herzog, 1998, pp.9]

Proposition B.4. *Let M, N be R-modules. Let $x = (x_1, \ldots, x_n)$ be an M-regular sequence and set $I = \operatorname{Ann}(N)$. Then*

1. *If I contains an M-regular element, then* $\mathrm{Hom}_R(N,M) = 0$.

2. *Suppose that $x_i \in \mathrm{Ann}(N)$ for $1 \le i \le n$. Then*

$$\mathrm{Hom}_R(N, M/(x_1,\ldots,x_n)M) \cong \mathrm{Ext}^n_R(N,M).$$

Let I be an ideal such that $IM \neq M$. Let $(x_1,\ldots,x_n)$ be a maximal regular sequence in I. Set $N = R/I$ and so $\mathrm{Ann}(N) = I$. From the previous proposition, since I contains a $M/(x_1,\ldots,x_k)$-regular element for $0 \le k < n$, we have that

$$\mathrm{Ext}^k_R(R/I,M) = \mathrm{Hom}_R(R/I, M/(x_1,\ldots,x_k)) = 0$$

and

$$\mathrm{Ext}^n_R(R/I,M) \cong \mathrm{Hom}_R(R/I, M/xM) \neq 0$$

Therefore, we have that

$$n = \min\{k : \mathrm{Ext}^k_R(R/I,M) \neq 0\}$$

is the length of a maximal M-sequence of elements in I.

Definition B.5. For an ideal I define $\mathrm{depth}_I(M) = \min\{k : \mathrm{Ext}^k_R(R/I,M) \neq 0\}$, if $IM = M$,then $\mathrm{depth}_I(M) = \infty$ and $\mathrm{Ext}^k_R(R/I,M) = 0$ for all $k \ge 0$. Finally define

$$\mathrm{depth}(M) = \mathrm{depth}_{\mathfrak{m}}(M) = \min\{k : \mathrm{Ext}^k_R(R/\mathfrak{m},M) \neq 0\}$$

Recall that for any ring R, $\dim R$ denotes the Krull dimension of R. It is defined as the supremum of the lengths of all chains of prime ideals in R. For an R-module M, we define $\dim M = \dim(R/\mathrm{ann}(M))$. It can be shown [Bruns and Herzog, 1998, Prop.1.2.12] that the following relation holds.

Proposition B.6. *Let M be a finitely generated R-module. Then* $\mathrm{depth}(M) \le \dim(M)$.

We say that M is a Cohen-Macaulay module if $\operatorname{depth}(M) = \dim(M)$. The following result [Eisenbud, 2005, Prop.A1.16] is a characterization dimension, depth and Cohen-Macaulay property for modules over a polynomial ring.

Proposition B.7. *Let R be a polynomial ring in n-indeterminate over a field $\Bbbk$ and M be an R-module.*

1. $\dim M = \max\{i : \operatorname{Ext}_R^{n-i}(M,R) \neq 0\}$.
2. $\operatorname{depth} M = \min\{i : \operatorname{Ext}_R^{n-i}(M,R) \neq 0\}$.
3. *M is a Cohen-Macaulay module of dimension j if and only if $\operatorname{Ext}_R^{n-i}(M,R) = 0$ for all $i \neq j$.*

The submodules of a Cohen-Macaulay module can be also characterized as we state in the following proposition [Allday and Puppe, 1993, Cor.A.6.16.].

Proposition B.8. *Let M be a Cohen-Macaulay R-module of dimension j. Then any non-zero submodule of M has dimension j.*

In particular, this implies that any map of R-modules $f\colon N \to M$ is zero if $\dim N < \dim M$. From now R will denote a polynomial ring in n indeterminate over a field $\Bbbk$ otherwise specified. Let M be a finitely generated R-module. The Hilbert syzygy theorem states that M is a free R-module if and only if there is an exact sequence

$$0 \to M \to F_1 \to \cdots \to F_n$$

of finitely generated free R-modules F_i. It is also possible to characterize torsion freeness through exact sequences. In fact, M is a torsion-free R-module if and only if it is a submodule of a finitely generated free module F; in other words, there is M fits into the exact sequence $0 \to M \to F$.

The aim of this section is to characterize the intermediate notion between free modules and torsion-free modules. We first state the following definition.

Definition B.9. Let R be a Noetherian ring and let M be a finitely generated R-module. We say that M is a j-th syzygy if there is an exact sequence

$$0 \to M \to F_1 \to \cdots \to F_j$$

of finitely generated free R-modules F_i.

The next result ([Bruns and Vetter, 2006, §16.E]) is a characterization of the j-th syzygy modules.

Theorem B.10. *Let M be a finitely generated R-module. Denote by $M^\vee = \operatorname{Hom}_R(M,R)$. The following statements are equivalent.*

(a) M is a j-th syzygy.

(b) For all prime ideals $\mathfrak{p} \subseteq R$, $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \min(j, \operatorname{depth}_{R_\mathfrak{p}} R_\mathfrak{p})$.

(c) Every R-sequence of length at most j is also an M-sequence.

(d)
- *If $j = 1$. M is torsion free.*
- *If $j = 2$. M is reflexive, that is, the canonical map $M \to M^{\vee\vee}$ is an isomorphism.*
- *If $j \geq 3$. M is reflexive and $\operatorname{Ext}^i_R(M^\vee, R) = 0$ for $i = 1, \ldots, j-2$.*

A way of constructing syzygies of a given order is using the Koszul complexes. We will review its construction and we mainly focus on its relation with regular sequences. The references followed are [Bruns and Herzog, 1998, §1.6] and [Weibel, 1995, §4.5].

An important example of a commutative graded R-algebra is the following

Example B.11. Let M be an R-module. The tensor product $T(M)$ is the graded R-algebra given by $T(M)_n = M^{\otimes n}$ with product given by $a \cdot b = a \otimes b$; However, such a product might not be commutative graded. If we consider the ideal I generated by elements of the form $x \otimes x \in T(M)$, the graded exterior algebra is defined as $\bigwedge M = T(M)/I$ and the induced product by $a \wedge b$. In this case, we have that $a \wedge b = (-1)^{|b||a|} b \wedge a$. Moreover, if M is a free R-module and $\{e_i : i \in I\}$ is a basis, then $(\bigwedge M)_n = \bigwedge^n M$ is generated by the elements $e_J = e_{j_1} \wedge \cdots e_{j_n}$ such that $J = \{j_1 < \cdots < j_n\} \subseteq I$. If M is free or rank m, then $\bigwedge^n M = 0$ for $n > m$.

Now we construct the Koszul complex for an R-module M and an R-linear map $f \cdot M \to R$. Consider the assignment

$$(v_1, \dots, v_n) \mapsto \sum_{i=1}^{n} (-1)^{i+1} f(v_i) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots v_n$$

which defines an alternating n-linear map $M^n \to \bigwedge^{n-1} M$. This map induces a linear map $d_f^n \colon \bigwedge^n M \to \bigwedge^{n-1} M$ and we get a graded linear morphism $df \colon \bigwedge M \to \bigwedge M$ satisfying

- $d_f \circ d_f = 0$.
- $d_f(x) \wedge y + (-1)^{|x|} x \wedge d_f(y)$.

The Koszul complex $K(f)$ is the chain complex

$$\cdots \to \bigwedge^{n} M \xrightarrow{df} \bigwedge^{n-1} M \to \cdots \to \bigwedge^{2} M \xrightarrow{df} M \xrightarrow{f} R \to 0.$$

The Koszul complex satisfies many interesting properties; however, for our purposes, we will focus on its applications to graded rings and graded algebras over a field $\Bbbk$.

Let R be a graded ring. For any $x \in R$, x is a regular element if and only if the sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ is exact, where (x) denotes the ideal generated by x. The latter sequence is equivalent to say that $R \xrightarrow{x} R \to R/(x)$ is a free resolution of $R/(x)$ as R-module. In fact, this resolution may be obtained as a Koszul complex in the following way: Let M be the free R-module generated by an element m of degree $|x|$. Consider the R-linear map $f \colon M \to R$

given by $f(m) = x$, then the Koszul complex $K(f)$ in this case is given by the chain complex $0 \to M \xrightarrow{f} R \to 0$. The degree shift on M is done so the map f is a map of degree 0. Since $\mathrm{Im}(f) = (x)$, the Koszul complex give rise to an exact sequence $0 \to M \xrightarrow{x} R \to R/(x) \to 0$ which coincides with the above sequence by identifying $M \cong R$ as ungraded rings.

Now we can take a look at the general case. Let $(x_1, \ldots, x_n)$ be a regular sequence of homogeneous elements in R and M be an R-module of rank n generated by elements $v_1, \ldots, v_n$. We regard M as a graded R-module by setting $|v_i| = |x_i|$. Consider the R-linear map $f \colon M \to R$ defined in the basis elements as $f(v_i) = x_i$. Then the differential $d_f = \delta$ in the Koszul complex $K(f) = K(x_1, \ldots, x_n)$

$$\delta_k \colon \bigwedge^k M \to \bigwedge^{k-1} M$$

given by $\delta_k(v_{j_1} \wedge \cdots \wedge v_{j_k}) = \sum_{i=1}^k (-1)^{i+1} x_{j_i} v_{j_1} \wedge \cdots \wedge \widehat{v_{j_i}} \wedge \cdots \wedge v_{i_k}$.

The main result out of this construction is summarized in the following result [Bruns and Herzog, 1998, Cor 1.6.14].

Proposition B.12. *Let $(x_1, \ldots, x_n)$ be a sequence in R. The Koszul complex $K(x_1, \ldots, x_n)$ provides a free resolution of the R-module $R/(x_1, \ldots, x_n)$ if and only if $(x_1, \ldots, x_n)$ is a regular sequence.*

The complex $K(x_1, \ldots, x_n)$ will be called the Koszul resolution of $R/(x_1, \ldots, x_n)$. From the exactness of the Koszul resolution, we have that the R-module $K_k = \mathrm{Im}(\delta_k) = \ker(\delta_{k-1})$ is a k-th syzygy, and it is called the k-th Koszul syzygy of $R/(x_1, \ldots, x_n)$.

The k-th Koszul syzygy also arises from the dual of the Koszul resolution $K(x_1, \ldots, x_n)$ given explicitly in the following way. Let $\{\widetilde{v_1}, \ldots, \widetilde{v_n}\}$ be a basis of $M^\vee = \mathrm{Hom}_R(M, R)$ dual to the basis $\{v_1, \ldots, v_n\}$. This basis induces a basis for the exterior algebra $\widetilde{v}_J = \widetilde{v_{j_1}} \wedge \cdots \widetilde{\wedge v_{j_k}}$ dual to v_J. So we have an isomorphism $(\bigwedge M^\vee) \cong (\bigwedge M)^\vee$. Under this identification, the map

$d_k = \delta_k^\vee \colon \bigwedge^{k-1} M^\vee \to \bigwedge^k M^\vee$ is given by $d_k(\widetilde{v_{j_1}} \wedge \cdots \wedge \widetilde{v}_{j_{k-1}}) = \sum_{i=1}^n x_i \widetilde{v}_{j_1} \wedge \cdots \wedge \widetilde{v}_{j_{k-1}} \wedge \widetilde{v}_i$. If we write $J = \{j_1 < \ldots < j_{k-1}\}$ then the map can be written as $d_k(\widetilde{v}_J) = \sum_{i \notin J} (-1)^{(J,i)} x_i \widetilde{v}_{J \cup i}$ where (J,i) is the number of elements $j \in J$ such that $j > i$. Let us denote the dual complex by $K^\vee(x_1, \ldots, x_n)$.

Now consider the isomorphism $\phi_n \colon \bigwedge^n M \to R$ such that $\phi_n(v_1 \wedge \cdots \wedge v_n) = 1$. This isomorphism induces an isomorphism $\phi_k \colon \bigwedge^k M \to \bigwedge^{n-k} M^\vee$ by setting $\phi_k(v)(w) = \phi_n(v \wedge w)$. We have then an isomorphism $\phi : K(x_1, \ldots, x_n) \to K^\vee(x_1, \ldots, x_n)$, that fits into a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \bigwedge^n M & \xrightarrow{\delta_n} & \bigwedge^{n-1} M & \xrightarrow{\delta_{n-1}} & \cdots & \longrightarrow & \bigwedge^2 M & \xrightarrow{\delta_2} & M & \xrightarrow{\delta_1} & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \\
0 & \longrightarrow & R & \xrightarrow{d_1} & M^\vee & \xrightarrow{d_2} & \cdots & \longrightarrow & \bigwedge^{n-2} M^\vee & \xrightarrow{d_{n-1}} & \bigwedge^{n-1} M^\vee & \xrightarrow{d_n} & \bigwedge^n M^\vee & \longrightarrow & 0
\end{array}$$

Therefore, we can identify the k-th Koszul syzygy $K_k = \mathrm{Im}(d_{n-k+1}) = \ker(d_{n-k+2})$.

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