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Enriched Derivators

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Abstract

In homotopical algebra, the theory of derivators provides a convenient abstract setting for computing with homotopy limits and colimits. In enriched homotopy theory, the analogues of homotopy (co)limits are weighted homotopy (co)limits. In this thesis, we develop a theory of derivators and, more generally, prederivators enriched over a monoidal derivator $\mathcal{E}$. In parallel to the unenriched case, these $\mathcal{E}$-prederivators provide a framework for studying the constructions of enriched homotopy theory, in particular weighted homotopy (co)limits.

As a precursor to $\mathcal{E}$-(pre)derivators, we study $\mathcal{E}$-categories, which are categories enriched over a bicategory $\mathcal{P}rof(\mathcal{E})$ associated to $\mathcal{E}$. We prove a number of fundamental results about $\mathcal{E}$-categories, which parallel classical results for enriched categories. In particular, we prove an $\mathcal{E}$-category Yoneda lemma, and study representable maps of $\mathcal{E}$-categories.

In any $\mathcal{E}$-category, we define notions of weighted homotopy limits and colimits. We define $\mathcal{E}$-derivators to be $\mathcal{E}$-categories with a number of further properties; in particular, they admit all weighted homotopy (co)limits. We show that the closed $\mathcal{E}$-modules studied by Groth, Ponto and Shulman give rise to associated $\mathcal{E}$-derivators, so that the theory of $\mathcal{E}$-(pre)derivators captures these examples. However, by working in the more general context of $\mathcal{E}$-prederivators, we can study weighted homotopy (co)limits in other settings, in particular in settings where not all weighted homotopy (co)limits exist.

Using the $\mathcal{E}$-category Yoneda lemma, we prove a representability theorem for $\mathcal{E}$-prederivators. We show that we can use this result to deduce representability theorems for closed $\mathcal{E}$-modules from representability results for their underlying categories.

Keywords: Enriched derivators, homotopy theory, enriched categories, weighted homotopy limits and colimits

Summary for Lay Audience

Modern homotopy theory has its origins in geometry, in particular in the study of homotopical properties of geometric objects. For familiar shapes, these are properties that are unaffected by continuous deformations, such as stretching or shrinking. For example, the number of connected components of a shape is a homotopical property, while properties like length, curvature and dimension are not.

A number of technical difficulties arise in studying the homotopy theory of shapes, and a large amount of abstract machinery has been developed to overcome these difficulties. This abstraction has led to widespread applications for homotopy theoretic methods throughout modern mathematics.

Homotopy colimits are important homotopy theoretic constructions. Classically, these provide a means of gluing shapes together to make new shapes with prescribed properties. In practice, working with homotopy colimits is technical, and requires that we keep track of large amounts information. Derivators provide an elegant framework for packaging this information, allowing us to carry out computations which might be intractable or impossible with other methods.

In a number of settings, homotopical information is available intrinsically in the form of an enrichment. In this context, the analogues of homotopy colimits are called weighted homotopy colimits. In this thesis, we define and study enriched derivators, which provide a framework for computing with weighted homotopy colimits and other constructions from enriched homotopy theory.

Acknowledgements

Contents

Chapter 1

Introduction

Derivators

The theory of derivators is one of several current approaches to homological and homotopical algebra. Derivators were introduced independently in [19] and [17], along with similar theories developed in [25] and [8]. To motivate their definition, we begin by recalling a well-known deficiency of another approach to homotopical algebra, the theory of triangulated categories.

Triangulated categories are a prominent and important axiomatisation of stable homotopy theory. See [30, 35] for a comprehensive introduction. Given any stable model category or stable quasicategory, its homotopy category is naturally triangulated. In general, information is lost when we pass to the homotopy category; however, in many situations, enough is retained that the original homotopy theory can be studied, and computations can be made, using only the underlying triangulated homotopy category. For example, if $\mathcal{T}$ is a triangulated category, we may study homotopy cofibre sequences in $\mathcal{T}$. In particular, any morphism in $\mathcal{T}$ has a homotopy cofibre. However, this construction cannot be made into a functor of the form $\mathcal{T}^{[1]} \to \mathcal{T}$, where $\mathcal{T}^{[1]}$ is the category of arrows in $\mathcal{T}$. When $\mathcal{T} = \mathrm{Ho}(\mathcal{M})$ is the homotopy category of a stable model category, this problem reflects the fact that the homotopy cofibre construction in a model category induces a functor $\mathrm{Ho}(\mathcal{M}^{[1]}) \to \mathrm{Ho}(\mathcal{M})$, rather than an functor $\mathrm{Ho}(\mathcal{M})^{[1]} \to \mathrm{Ho}(\mathcal{M})$.

This problem can be addressed by considering the derivator associated to $\mathcal{M}$, rather than only the homotopy category. Introductions to derivators can be found in [11, 12, 14, 15],

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and the first chapter of [4]. The essential idea is to consider a family of categories, one for each small category A, rather than a single category. In the case of a model category $\mathcal{M}$, this corresponds to keeping track of the family of homotopy categories $\mathrm{Ho}(\mathcal{M}^A)$, for every $A \in \mathbf{Cat}$, rather than the single homotopy category $\mathrm{Ho}(\mathcal{M})$, where these homotopy categories are formed with respect to pointwise weak equivalences. (Note that, in general, it is not obvious that these homotopy categories are locally small. See [2] for a proof.) Moreover, for any functor $u : A \to B$, we keep track of the (derived) pullback functor $u^* : \mathrm{Ho}(\mathcal{M}^B) \to \mathrm{Ho}(\mathcal{M}^A)$. It is possible to show that this functor has both adjoints. Using these, we can study homotopy limits and colimits in $\mathcal{M}$ and, in particular, if $\mathcal{M}$ is pointed we can recover the homotopy cofibre functor $\mathrm{Ho}(\mathcal{M}^{[1]}) \to \mathrm{Ho}(\mathcal{M})$.

We now discuss derivators in more detail. We can break the definition into two steps. First, a **prederivator** (see Definition 2.1.1) is a 2-functor $\mathcal{D} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$. We denote its values as follows:

Given a functor $u : A \to B$, we call the functor $u^* : \mathcal{D}(B) \to \mathcal{D}(A)$ the **pullback functor along** u. If u^* has a left adjoint $u_! : \mathcal{D}(A) \to \mathcal{D}(B)$, we call this functor the **homotopy left Kan extension along** u. Dually, if u^* has a right adjoint $u_* : \mathcal{D}(A) \to \mathcal{D}(B)$, we call this functor the **homotopy right Kan extension along** u. In the case of the unique map $\mathrm{p} : A \to [\mathbf{0}]$ to the terminal category, we call the right and left homotopy Kan extensions **homotopy limits** and **homotopy colimits** respectively. **Derivators** are prederivators that satisfy four additional axioms, which we recall in Definition 2.1.16. In particular, if $\mathcal{D}$ is a derivator, one axiom, **Der 3**, implies that any functor $u : A \to B$ admits both a left and right homotopy Kan extension in $\mathcal{D}$. Another axiom, **Der 4**, gives a formula for calculating homotopy Kan extensions.

We have already alluded to the fact that any model category $\mathcal{M}$ gives rise to a derivator $\mathcal{H}\mathrm{o}(\mathcal{M})$, whose value at $A \in \mathbf{Cat}$ is the homotopy category $\mathrm{Ho}(\mathcal{M}^A)$. This is the main theorem of [2]. Similarly, given any complete and cocomplete quasicategory Q, we can form a derivator $\mathcal{H}\mathrm{o}(Q)$, whose value at $A \in \mathbf{Cat}$ is $\mathrm{Ho}(Q^{NA})$, where NA denotes the nerve of A, and $\mathrm{Ho}(Q^{NA})$ denotes the homotopy category of the quasicategory Q^{NA}. See [33] for a proof that this defines a derivator. Thus, the passage from either a model category or a (co)complete quasicategory to its homotopy category factors through an associated derivator.

We have seen that the derivator associated to a stable model category retains more information than the triangulated homotopy category. However, derivators do not retain all of the homotopical information that is available in model categories or quasicategories. See [43, Section 2.5] for a discussion of what information is lost. Thus, derivators cannot be thought of as a replacement for model categories or quasicategories. However, in certain settings, there are advantages to working with derivators rather than these other models of homotopy theory, where carrying around all of the available information can result in technical difficulties. For example, working in a derivator rather than a model category does away with the need to manage fibrant and cofibrant replacements and zigzags of weak equivalences. At the derivator level, we only have access to information that is homotopically meaningful, and homotopy invariant.

Derivators retain enough information to define homotopy Kan extensions, in particular homotopy limits and colimits, using universal properties, and provide enough tools to carry out elegant formal computations. Since these universal properties characterise homotopy limits and colimits in other models of homotopy theory, results we prove in derivators must hold, in particular, in model categories and quasicategories. Thus, derivators provide a convenient abstract setting in which we can manage homotopy coherence and compute with homotopy Kan extensions. This also carries over to morphisms of derivators: to prove, for example, that a given left Quillen functor commutes with a particular homotopy limit, it suffices to show this for the associated derivator map. This perspective on derivators is developed and exemplified in [13, 14, 16].

Actions of derivators

In this thesis, we study enrichment of prederivators and derivators, and develop formal methods for studying the constructions of enriched category theory (see [27]) in homotopical settings. In particular, just as derivators are a tool for studying homotopy limits and colimits, enriched derivators provide a setting for studying **weighted homotopy limits and colimits**, which are the enriched analogue.

There are a number of approaches to enriched homotopy theory. Simplicial enrichments, in particular, are well-studied and fundamentally important. Simplicial model categories were introduced in [37], and, for any model category, constructions of simplicial mapping spaces were defined and studied in [7]. See [21] for a textbook treatment. Weighted homotopy limits and colimits in simplicial model categories are studied in [9]. For a survey of other

approaches to enrichment in homotopy theory, and a unified treatment of weighted homotopy limits and colimits, see [39].

The properties of derivators that make them convenient for studying other aspects of homotopy theory also make them an elegant setting for studying enrichment. For example, managing cofibrant and fibrant replacement in a model category can vastly complicate proofs that are relatively straightforward on the derivator level. This is an important advantage when it comes to studying enrichment, since we often want to verify lists of coherence conditions, and this can become difficult or impossible if we have to keep track of cofibrant and fibrant replacements. We give the following relevant example. In [23] it was conjectured that, for any monoidal model category $\mathcal{M}$, its homotopy category $\mathrm{Ho}(\mathcal{M})$ is naturally a central algebra over the homotopy category of simplicial sets $\mathrm{Ho}(\mathbf{sSet})$. All of this, except for the centrality condition, was shown using model categorical methods in [23]. However, the final coherence condition was only successfully checked in [3], using the associated derivator $\mathcal{H}\mathrm{o}(\mathcal{M})$.

We will now outline some relevant previous work on enrichment in derivators. A **monoidal derivator** $\mathcal{E}$ is a derivator equipped with a tensor product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ that is coherently unital and associative, and which, in an appropriate sense, preserves homotopy colimits in both variables. Note that, for any category A, the tensor product induces a monoidal structure on the category $\mathcal{E}(A)$. Monoidal derivators are studied in [3, 13]; we recall the definition in Section 3.3. Given any monoidal model category $\mathcal{M}$, its associated derivator $\mathcal{H}\mathrm{o}(\mathcal{M})$ is monoidal.

Given a monoidal derivator $\mathcal{E}$, an **action** of $\mathcal{E}$ on a derivator $\mathcal{D}$ is a coherently associative and unital map $\otimes : \mathcal{E} \times \mathcal{D} \to \mathcal{D}$, which preserves homotopy colimits in both variables. See Definition 3.4.1. We call a derivator $\mathcal{D}$ an $\mathcal{E}$-**module** if it is equipped with an $\mathcal{E}$-action. A fundamentally important result, proved in [3], is that any derivator $\mathcal{D}$ has a unique $\mathcal{H}\mathrm{o}(\mathbf{sSet})$-module structure. This theorem is conceptually important, but it also has practical implications for calculating homotopy colimits. See [36, Section 7] for such an application; using the $\mathcal{H}\mathrm{o}(\mathbf{sSet})$-action, homotopy colimits in an arbitrary derivator can be computed from homotopy colimits in $\mathcal{H}\mathrm{o}(\mathbf{sSet})$. This is true in general: computations involving homotopy colimits in an $\mathcal{E}$-module can be reduced to computations in $\mathcal{E}$. This approach is used to characterise stable derivators in [16].

Derivator **two-variable adjunctions** are studied in [13]. We recall the definition in Section 3.2. An $\mathcal{E}$-module $\mathcal{D}$ is called a **closed** $\mathcal{E}$-**module** if the $\mathcal{E}$-action $\otimes : \mathcal{E} \times \mathcal{D} \to \mathcal{D}$ is

part of a two-variable adjunction. For example, if $\mathcal{M}$ is a monoidal model category and $\mathcal{N}$ is an $\mathcal{M}$-enriched model category, then the derivator $\mathcal{H}o(\mathcal{N})$ is a closed $\mathcal{H}o(\mathcal{M})$-module. As a part of the structure, a closed $\mathcal{E}$-module is equipped with two additional maps:

$$\mathrm{map}_{\mathcal{D}}(-,-) : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}$$

$$\lhd : \mathcal{D} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{D}$$

Using the first of these maps, closed $\mathcal{E}$-modules have a notion of mapping objects, which take values in $\mathcal{E}$. For this reason, closed $\mathcal{E}$-modules are called $\mathcal{E}$-enriched derivators in [15, 16]; however, we will reserve this terminology for a different, though related, concept. Given a closed $\mathcal{E}$-module $\mathcal{D}$, the map $\otimes : \mathcal{E} \times \mathcal{D} \to \mathcal{D}$ can be used to define weighted homotopy colimits in $\mathcal{D}$, and $\lhd : \mathcal{D} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{D}$ can be used to define weighted homotopy limits. Examples of weighted homotopy colimits include the pullback functors in $\mathcal{D}$, and ordinary left Kan extensions. In [15, 16], computations with weighted homotopy colimits are reduced to computations in $\mathcal{E}$ with the corresponding weights.

Enriched prederivators and derivators

In this work, we establish an alternative approach to enrichment in derivators, which incorporates the closed $\mathcal{E}$-modules of [15, 16], as well as a number of other examples. In particular, our framework allows us to study enrichment of general prederivators, rather than being restricted to derivators. Moreover, using this approach, we can formulate local definitions of weighted homotopy limits and colimits, which agree with the global definitions in [15, 16] when all weighted homotopy (co)limits exist. This allows us to study weighted homotopy (co)limits in a broader range of settings, including in situations where only certain weighted homotopy (co)limits exist.

We develop the theory of enriched derivators in a series of steps, starting with the concept of $\mathcal{E}$-categories. We may then add extra structure to obtain $\mathcal{E}$-prederivators and, finally, $\mathcal{E}$-derivators. We will now outline this process.

Any monoidal derivator $\mathcal{E}$ gives rise to an associated bicategory, which we denote by $\mathrm{Prof}(\mathcal{E})$ and call the **bicategory of profunctors in** $\mathcal{E}$. This bicategory is defined in [13], and we recall its definition in Remark 3.5.3. See [32] for basic bicategorical definitions. An $\mathcal{E}$-**category** $\underline{\mathcal{A}}$ is defined, in Definition 4.1.1, to be a category enriched over the bicategory $\mathrm{Prof}(\mathcal{E})$. In particular, this includes the following data:

- For each small category A, a (large) set of objects $\mathcal{A}_0(A)$.

- For any two objects $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$, an object $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \in \mathcal{E}(A^{\mathrm{op}} \times B)$.

In addition, $\underline{\mathcal{A}}$ is equipped with notions of **composition** and **units**, subject to natural axioms that express associativity and unitality of composition. In this way, the definition of $\mathcal{E}$-categories is analogous to the familiar definition of enriched categories in [27]; in fact, for any category A, an $\mathcal{E}$-category $\underline{\mathcal{A}}$ gives rise to an $\mathcal{E}([\mathbf{0}])$-category $\underline{\mathcal{A}}(A)$, which we describe in Remark 4.1.4 and Remark 4.3.9.

Our development of the basic theory of $\mathcal{E}$-categories mirrors the classical development of enriched category theory. For example, given an $\mathcal{E}$-category $\underline{\mathcal{A}}$, a category A and an object $X \in \mathcal{A}_0(A)$, the mapping objects induce a **representable** $\mathcal{E}$-category morphism

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{A^{\mathrm{op}}}$$

where $\underline{\mathcal{E}}^{A^{\mathrm{op}}}$ is an $\mathcal{E}$-category associated to the **shifted prederivator** $\mathcal{E}^{A^{\mathrm{op}}}$ of Example 2.1.6. This $\mathcal{E}$-category is described in Theorem 4.1.10; note, in particular, that for any category B, the set $\mathcal{E}_0^{A^{\mathrm{op}}}(B)$ is the set of objects in the category $\mathcal{E}(A^{\mathrm{op}} \times B)$.

Representable maps play a vital role in the theory of $\mathcal{E}$-categories, and in our study of $\mathcal{E}$-prederivators and $\mathcal{E}$-derivators. In part, this is a consequence of Theorem 4.2.1, the $\mathcal{E}$-category Yoneda lemma:

Theorem. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, let A be a category, and let $X \in \mathcal{A}_0(A)$. Let $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{A^{\mathrm{op}}}$ be an $\mathcal{E}$-category map. We have a natural bijection:*

$$\mathcal{E}\text{-}\mathbf{Cat}(\underline{\mathcal{A}}, \underline{\mathcal{E}}^{A^{\mathrm{op}}})(\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -), F) \cong \mathcal{E}(A^{\mathrm{op}} \times A)(h_A, FX)$$

Here $\mathcal{E}\text{-}\mathbf{Cat}(\underline{\mathcal{A}}, \underline{\mathcal{E}}^{A^{\mathrm{op}}})$ is the hom-category in the 2-category of $\mathcal{E}$-categories, which is defined in Lemma 4.1.5. The object $h_A \in \mathcal{E}(A^{\mathrm{op}} \times A)$ is called the **identity profunctor**; we recall its definition in Definition 3.5.1. It is the unit object in a monoidal structure on $\mathcal{E}(A^{\mathrm{op}} \times A)$, which we describe in Section 3.5.

We define $\mathcal{E}$-prederivators in Definition 5.1.1. An $\mathcal{E}$-category $\underline{\mathcal{A}}$ is called an $\mathcal{E}$-**prederivator** if, among other conditions, it is equipped with a notion of pullback along functors. In particular, given a functor $u : A \to B$ and an object $X \in \mathcal{A}_0(B)$, we have an object $u^*X \in \mathcal{A}_0(A)$. We show, in Theorem 5.1.10, that any $\mathcal{E}$-prederivator $\underline{\mathcal{A}}$ gives rise to a prederivator $\mathcal{A}$, which we call the **prederivator induced by** $\underline{\mathcal{A}}$.

In Definition 5.3.1, we define weighted homotopy limits and colimits in $\mathcal{E}$-categories. Given an $\mathcal{E}$-category $\underline{\mathcal{A}}$, categories A and B, and objects $X \in \mathcal{A}_0(A)$ and $W \in \mathcal{E}(A^{\mathrm{op}} \times B)$, the **homotopy colimit of X weighted by** W, if it exists, is an object $W \otimes_A X \in \mathcal{A}_0(B)$. This object must represent the $\mathcal{E}$-category map below:

$$\underline{\mathcal{A}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)} \underline{\mathcal{E}}^{A^{\mathrm{op}}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}A^{\mathrm{op}}}(W,-)} \underline{\mathcal{E}}^{B^{\mathrm{op}}}$$

Thus, for any category C and any $Z \in \mathcal{A}_0(C)$, the weighted homotopy colimit is equipped with isomorphisms

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_A X, Z) \cong \widetilde{\mathrm{map}}_{\mathcal{E}A^{\mathrm{op}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z))$$

in $\mathcal{E}(B^{\mathrm{op}} \times C)$, which are $\mathcal{E}$-**natural** in Z (see Definition 4.1.3). This is a local definition for the weighted homotopy colimit; if, given an object $X \in \mathcal{A}_0(A)$, the weighted homotopy colimit of X exist for all possible weights, then we can obtain a global characterisation in the manner of [15, 16]. Specifically, the weighted homotopy colimits assemble into an $\mathcal{E}$-category map $- \otimes_A X$, which forms part of the following $\mathcal{E}$-category **adjunction** (see Section 4.2):

$$\underline{\mathcal{E}}^{A^{\mathrm{op}}} \underset{\widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)}{\overset{- \otimes_A X}{\rightleftarrows}} \underline{\mathcal{A}} \qquad \bot$$

In Definition 5.3.6, we define $\mathcal{E}$-**derivators**. These are $\mathcal{E}$-prederivators that, in particular, admit all weighted homotopy limits and colimits. We show, in Theorem 5.3.10, that if $\underline{\mathcal{A}}$ is an $\mathcal{E}$-derivator, the induced prederivator $\mathcal{A}$ is a derivator.

In Theorem 5.3.7, we show that any closed $\mathcal{E}$-module $\mathcal{D}$ gives rise to an $\mathcal{E}$-derivator $\underline{\mathcal{D}}$. In this way, the theory of closed $\mathcal{E}$-modules is encompassed by the theory of $\mathcal{E}$-prederivators and $\mathcal{E}$-derivators. However, in general, $\mathcal{E}$-prederivators need not admit all weighted homotopy limits and colimits: for instance, in Example 5.3.9, we discuss the prederivator of compact objects in a triangulated derivator, which we show admits certain, but not all, $\mathcal{H}o(\mathbf{Spt})$-weighted homotopy colimits. In this way, working with $\mathcal{E}$-prederivators gives us the flexibility to study natural examples of enriched homotopy theories that are not captured by closed $\mathcal{E}$-modules. Moreover, even if an $\mathcal{E}$-prederivator $\underline{\mathcal{A}}$ does not admit all weighted homotopy (co)limits, if it admits enough, we can still manipulate these as in [15, 16]. In particular, we can carry out

computations with weighted homotopy (co)limits by doing computations with the weights in $\mathcal{E}$.

Working in the 2-category of $\mathcal{E}$-prederivators has other advantages. For example, the $\mathcal{E}$-category Yoneda lemma of Theorem 4.2.1 is extremely useful. We give an application of the Yoneda lemma in Theorem 5.2.6, in which we prove the following representability theorem for $\mathcal{E}$-prederivator maps:

Theorem. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator. An $\mathcal{E}$-category morphism $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}$ is representable if and only if the $\mathcal{E}([\mathbf{0}])$-functor*

$$F : \underline{\mathcal{A}}([\mathbf{0}]) \to \underline{\mathcal{E}}([\mathbf{0}])$$

is representable as an $\mathcal{E}([\mathbf{0}])$-functor.

The $\mathcal{E}([\mathbf{0}])$-functor $F : \underline{\mathcal{A}}([\mathbf{0}]) \to \underline{\mathcal{E}}([\mathbf{0}])$ that appears in this theorem is defined in Remark 4.1.4. In this way, even if we are only interested in studying concepts that can be phrased on the level of unenriched derivators, it can be beneficial to work in the 2-category of $\mathcal{H}o(\mathbf{sSet})$-prederivators, particularly since derivators associated to model categories and quasicategories are naturally $\mathcal{H}o(\mathbf{sSet})$-enriched.

Organisation

In Chapter 2, we survey the elementary theory of prederivators and derivators. Most of the content of this section can be found in [11, 12, 13]. We recall the basic definitions and a number of important examples in Section 2.1. In Section 2.2, we discuss preservation of homotopy Kan extensions and study adjunctions between derivators. In Section 2.3, we recall some simple aspects of the theory of two-variable derivator maps. Finally, in Section 2.4, we recall the definitions of pointed and triangulated derivators. We also recall some important results from the theory of triangulated categories, which we use in Chapter 3 and Chapter 5 to study enriched triangulated derivators.

In Chapter 3, we discuss actions of monoidal derivators. First, in Section 3.1, we recall the definitions of ends and coends in a derivator; these play an essential role in the theory of derivator two-variable adjunctions, which we recall in Section 3.2. In Section 3.3 and Section 3.4, we recall the definitions of monoidal derivators and their actions. Much of the material in these first sections can be found in [13]; beyond this point, unless otherwise spec-

ified, the results are new. In Section 3.5 and Section 3.6, we collect a number of coherence results for structure arising from the action of a monoidal derivator. These results provide important ingredients for the development of $\mathcal{E}$-categories and $\mathcal{E}$-prederivators in Chapter 4 and Chapter 5. We also use them, in Section 3.7, to study cotensors in closed $\mathcal{E}$-modules. In particular, in Proposition 3.7.5, we prove that a left adjoint between closed $\mathcal{E}$-modules preserves tensors if and only if its right adjoint preserves cotensors. In Section 3.8, we use this result to prove Theorem 3.8.3, a representability theorem for triangulated closed modules over triangulated monoidal derivators.

In Chapter 4, we introduce $\mathcal{E}$-categories. In Section 4.1 we develop their basic theory and give a number of examples; in particular, in Theorem 4.1.10, we prove that any closed $\mathcal{E}$-module gives rise to an associated $\mathcal{E}$-category. We prove the $\mathcal{E}$-category Yoneda lemma, Theorem 4.2.1, in Section 4.2, and use this to study $\mathcal{E}$-category adjunctions. In Section 4.3, we study monoidal morphisms, and prove, in Proposition 4.3.4, that we may transfer enrichment along monoidal adjunctions.

In Chapter 5, we introduce $\mathcal{E}$-prederivators and $\mathcal{E}$-derivators. We study $\mathcal{E}$-prederivators in Section 5.1, which we introduce in Definition 5.1.1. In Section 5.2, we define the 2-category of $\mathcal{E}$-prederivators, and prove Theorem 5.2.6, a representability theorem for $\mathcal{E}$-prederivator maps. In Section 5.3, we define weighted homotopy limits and colimits in an $\mathcal{E}$-category, and give the definition of $\mathcal{E}$-derivators in Definition 5.3.6. In Theorem 5.3.7, we show that the $\mathcal{E}$-category associated to a closed $\mathcal{E}$-module is an $\mathcal{E}$-derivator. Finally, in Theorem 5.3.10, we show that any $\mathcal{E}$-derivator induces a derivator.

Notation. We will use the following notation throughout:

- We will write $[\mathbf{n}] = \{0 \to 1 \to \cdots \to n\}$ for the ordinal number $n + 1$ regarded as a category. In particular, $[\mathbf{0}]$ is the terminal category.

- We will denote small categories with upright font. For example, $\mathrm{A}, \mathrm{B}, \ldots \in \mathbf{Cat}$.

- Given categories A and B, we will denote the symmetry isomorphism for the product by $\sigma : \mathrm{A} \times \mathrm{B} \xrightarrow{\cong} \mathrm{B} \times \mathrm{A}$.

- Given categories A and B, we will write $\mathrm{p_A} : \mathrm{A} \times \mathrm{B} \to \mathrm{B}$ for the canonical projection onto B. We will also write $\mathrm{p_A} : \mathrm{B} \times \mathrm{A} \to \mathrm{B}$.

  More generally, we will write $\mathrm{p_A}$ for any map projecting away the category A. For example, we also write $\mathrm{p_A} : \mathrm{A} \to [\mathbf{0}]$ for the canonical map to the terminal category.

- When we have multiple copies of the same category appearing in a product, for example A in the product $A \times A$, we will write $A = A_1 = A_2$ to keep track of the maps σ and p_A. In this example, the two projections are $p_{A_1} : A_1 \times A_2 \to A_2$ and $p_{A_2} : A_1 \times A_2 \to A_1$.

Chapter 2

Derivators

In this chapter we recall the elementary theory that we will need in later chapters. We begin, in Section 2.1, with the definition of the 2-category **Der** of derivators. This section includes a number of important definitions and basic results; in particular, we introduce homotopy Kan extensions and homotopy exact squares. In Section 2.2, we discuss preservation of homotopy Kan extensions and study adjunctions in **Der**. In Section 2.3, we look at some aspects of the theory of two-variable maps; we will return to this in Chapter 3 once we have recalled the theory of ends and coends. Finally, in Section 2.4, we briefly discuss pointed and triangulated derivators; in later chapters, these will provide important examples of enriched derivators. In this section we also recall some basic results about triangulated categories, which play a significant role in the theory of triangulated derivators. Much of the material in this chapter can be found in [11]. There are also useful introductions to basic derivator theory in [12], [13, Section 2] and [4, Chapter 1].

2.1 Prederivators and derivators

This section contains the definitions and basic elements of the theory of derivators. Most of the material in the section can be found in [11, Chapter 1,2]. We begin with the definition of prederivators, and develop the language that we require to state the additional axioms that define derivators. These are defined towards the middle of the section, in Definition 2.1.16. We also introduce a number of examples, some of which we will revisit repeatedly.

We will denote the 2-category of small categories by **Cat**, and the 2-category of large cate-

gories by **CAT**. We take the same size conventions as [31].

Definition 2.1.1. A **prederivator** is a 2-functor $\mathcal{D} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$. We denote its values as follows:

Definition 2.1.2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be prederivators. We call a **pseudonatural transformation** $F : \mathcal{D}_1 \to \mathcal{D}_2$ a **morphism of prederivators**. Explicitly, this consists of functors $F : \mathcal{D}_1(A) \to \mathcal{D}_2(A)$ for any category A, and natural isomorphisms

for any functor $u : A \to B$. This data must satisfy the following equalities:

1. For any category A, we have:

2. For any composable maps $A \xrightarrow{u} B \xrightarrow{v} C$, we have:

$$
\begin{array}{ccc}
\mathcal{D}_1(\mathrm{C}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{C}) \\
\downarrow{\scriptstyle v^*} & \ \ \gamma\ \cong & \downarrow{\scriptstyle v^*} \\
\mathcal{D}_1(\mathrm{B}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{B}) \\
\downarrow{\scriptstyle u^*} & \ \ \gamma\ \cong & \downarrow{\scriptstyle u^*} \\
\mathcal{D}_1(\mathrm{A}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{A})
\end{array}
\qquad = \qquad
\begin{array}{ccc}
\mathcal{D}_1(\mathrm{C}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{C}) \\
\downarrow{\scriptstyle (v\circ u)^*} & \ \ \gamma\ \cong & \downarrow{\scriptstyle (v\circ u)^*} \\
\mathcal{D}_1(\mathrm{A}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{A})
\end{array}
$$

3. For any natural transformation $\kappa : u \Rightarrow v$, we have:

$$
\begin{array}{ccc}
\mathcal{D}_1(\mathrm{B}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{B}) \\
{\scriptstyle v^*}\Big(\xleftarrow{\ \kappa^*\ }\Big){\scriptstyle u^*} & \ \ \gamma\ \cong & \downarrow{\scriptstyle u^*} \\
\mathcal{D}_1(\mathrm{A}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{A})
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathcal{D}_1(\mathrm{B}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{B}) \\
\downarrow{\scriptstyle v^*} & \ \ \gamma\ \cong\ {\scriptstyle v^*}\Big(\xleftarrow{\ \kappa^*\ }\Big){\scriptstyle u^*} \\
\mathcal{D}_1(\mathrm{A}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{A})
\end{array}
$$

Definition 2.1.3. Given prederivator maps $F, G : \mathcal{D}_1 \to \mathcal{D}_2$, a **modification** $\theta : F \Rightarrow G$ consists of natural transformations

$$
\mathcal{D}_1(\mathrm{A}) \ \ \underset{G}{\overset{F}{\Rightarrow}}\theta\ \ \mathcal{D}_2(\mathrm{A})
$$

such that, for any functor $u : \mathrm{A} \to \mathrm{B}$, we have:

$$
\begin{array}{ccc}
\mathcal{D}_1(\mathrm{B}) & \underset{G}{\overset{F}{\Rightarrow}}\theta & \mathcal{D}_2(\mathrm{B}) \\
\downarrow{\scriptstyle u^*} & \ \ \gamma\ \cong & \downarrow{\scriptstyle u^*} \\
\mathcal{D}_1(\mathrm{A}) & \xrightarrow{\ G\ } & \mathcal{D}_2(\mathrm{A})
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathcal{D}_1(\mathrm{B}) & \xrightarrow{\ F\ } & \mathcal{D}_2(\mathrm{B}) \\
\downarrow{\scriptstyle u^*} & \ \ \gamma\ \cong & \downarrow{\scriptstyle u^*} \\
\mathcal{D}_1(\mathrm{A}) & \underset{G}{\overset{F}{\Rightarrow}}\theta & \mathcal{D}_2(\mathrm{A})
\end{array}
$$

Prederivators, morphisms and modifications form a 2-category **PDer**. Given prederivators $\mathcal{D}_1$ and $\mathcal{D}_2$, we write $\underline{\mathrm{Hom}}(\mathcal{D}_1, \mathcal{D}_2)$ for the category of morphisms from $\mathcal{D}_1$ to $\mathcal{D}_2$.

Example 2.1.4. Given any (large) category $\mathcal{C}$, we can form its **represented prederivator**:

$$y(\mathcal{C}) : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$$
$$\mathrm{A} \mapsto \mathcal{C}^{\mathrm{A}}$$

We may also form the **constant prederivator**:

$$c(\mathcal{C}) : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$$
$$\mathrm{A} \mapsto \mathcal{C}$$

In analogy with representable prederivators, for any prederivator $\mathcal{D}$, we call $\mathcal{D}([\mathbf{0}])$ the **underlying category** of $\mathcal{D}$, and for any $u : \mathrm{A} \to \mathrm{B}$, we call $u^* : \mathcal{D}(\mathrm{B}) \to \mathcal{D}(\mathrm{A})$ the **restriction** or **pullback functor along** u.

Example 2.1.5. Given any model category $\mathcal{M}$, we can form a prederivator

$$\mathcal{H}\mathrm{o}(\mathcal{M}) : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$$
$$\mathrm{A} \mapsto \mathrm{Ho}(\mathcal{M}^{\mathrm{A}})$$

where the homotopy category $\mathrm{Ho}(\mathcal{M}^{\mathrm{A}})$ is formed with respect to the pointwise weak equivalences in $\mathcal{M}^{\mathrm{A}}$. Note that $\mathcal{M}^{\mathrm{A}}$ may not carry a model structure in general; the fact that the localisation can still be formed without moving to a larger universe follows from [2]. Moreover, given model categories $\mathcal{M}$ and $\mathcal{N}$, and a left Quillen functor $F : \mathcal{M} \to \mathcal{N}$, the derived functors induce a prederivator map $\mathbb{L}F : \mathcal{H}\mathrm{o}(\mathcal{M}) \to \mathcal{H}\mathrm{o}(\mathcal{N})$. Similarly, any right Quillen functor $G : \mathcal{N} \to \mathcal{M}$, induces a prederivator map $\mathbb{R}G : \mathcal{H}\mathrm{o}(\mathcal{N}) \to \mathcal{H}\mathrm{o}(\mathcal{M})$.

Similarly, given a quasicategory Q, we have a prederivator

$$\mathcal{H}\mathrm{o}(Q) : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$$
$$\mathrm{A} \mapsto \mathrm{Ho}(Q^{N\mathrm{A}})$$

where $N\mathrm{A}$ is the nerve of A, and $\mathrm{Ho}(Q^{N\mathrm{A}})$ is the homotopy category of the quasicategory $Q^{N\mathrm{A}}$.

Example 2.1.6. For any prederivator $\mathcal{D}$ and any category J, we can form the **shifted**

prederivator:

$$\mathcal{D}^{\mathrm{J}} : \mathbf{Cat}^{\mathrm{op}} \rightarrow \mathbf{CAT}$$

$$\mathrm{A} \mapsto \mathcal{D}(\mathrm{J} \times \mathrm{A})$$

Moreover, any functor $u : \mathrm{J} \rightarrow \mathrm{K}$ induces a prederivator map $u^* : \mathcal{D}^{\mathrm{K}} \rightarrow \mathcal{D}^{\mathrm{J}}$, with component at A given by

$$(u \times \mathrm{A})^* : \mathcal{D}(\mathrm{K} \times \mathrm{A}) \rightarrow \mathcal{D}(\mathrm{J} \times \mathrm{A}),$$

and any natural transformation $\kappa : u \Rightarrow v$ induces modification $\omega^* : u^* \Rightarrow v^*$. These organise into a 2-functor:

$$\mathbf{Cat}^{\mathrm{op}} \times \mathbf{PDer} \rightarrow \mathbf{PDer}$$

$$(\mathrm{J}, \mathcal{D}) \quad \mapsto \quad \mathcal{D}^{\mathrm{J}}$$

Remark 2.1.7. Given a prederivator map $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$, and a functor $u : \mathrm{J} \rightarrow \mathrm{K}$, the structure isomorphisms of F induce a modification:

$$
\begin{array}{ccc}
\mathcal{D}_1^{\mathrm{K}} & \xrightarrow{\ F\ } & \mathcal{D}_2^{\mathrm{K}} \\
{\scriptstyle u^*}\downarrow & {\scriptstyle \gamma}\;\Swarrow_{\cong} & \downarrow{\scriptstyle u^*} \\
\mathcal{D}_1^{\mathrm{J}} & \xrightarrow[\ F\ ]{} & \mathcal{D}_2^{\mathrm{J}}
\end{array}
$$

Moreover, given $\theta : F \Rightarrow G$, the modification condition lifts to an equality:

$$
\begin{array}{ccc}
\mathcal{D}_1^{\mathrm{K}} \;{\theta\Downarrow}\; & \mathcal{D}_2^{\mathrm{K}} \\
{\scriptstyle u^*}\downarrow \quad {\scriptstyle \gamma}\Swarrow_{\cong} \quad \downarrow{\scriptstyle u^*} \\
\mathcal{D}_1^{\mathrm{J}} \xrightarrow[G]{} \mathcal{D}_2^{\mathrm{J}}
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathcal{D}_1^{\mathrm{K}} \xrightarrow{\ F\ } \mathcal{D}_2^{\mathrm{K}} \\
{\scriptstyle u^*}\downarrow \quad {\scriptstyle \gamma}\Swarrow_{\cong} \quad \downarrow{\scriptstyle u^*} \\
\mathcal{D}_1^{\mathrm{J}} \;{\theta\Downarrow}\; \mathcal{D}_2^{\mathrm{J}}
\end{array}
$$

Often it will be convenient to state and prove results at the level of shifted prederivators. For example, rather than prove a result about the component $F : \mathcal{D}_1(\mathrm{A}) \rightarrow \mathcal{D}_2(\mathrm{A})$ of a prederivator map F, it may be more convenient to work with the shifted map $F : \mathcal{D}_1^{\mathrm{A}} \rightarrow \mathcal{D}_2^{\mathrm{A}}$ in **PDer**.

Definition 2.1.8. Given any prederivator $\mathcal{D}$, we may define its **opposite** prederivator:

$$\mathcal{D}^{\mathrm{op}} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$$
$$\mathrm{A} \mapsto \mathcal{D}(\mathrm{A}^{\mathrm{op}})^{\mathrm{op}}$$

Example 2.1.9. For any category $\mathcal{C}$, we have a canonical isomorphism $y(\mathcal{C})^{\mathrm{op}} \cong y(\mathcal{C}^{\mathrm{op}})$, induced by the isomorphisms $(\mathcal{C}^{\mathrm{A}^{\mathrm{op}}})^{\mathrm{op}} \cong (\mathcal{C}^{\mathrm{op}})^{\mathrm{A}}$. Similarly, for any model category $\mathcal{M}$, we have $\mathcal{H}\mathrm{o}(\mathcal{M})^{\mathrm{op}} \cong \mathcal{H}\mathrm{o}(\mathcal{M}^{\mathrm{op}})$.

Example 2.1.10. For any prederivator $\mathcal{D}$ and any category J, we have $(\mathcal{D}^{\mathrm{J}})^{\mathrm{op}} = (\mathcal{D}^{\mathrm{op}})^{\mathrm{J}^{\mathrm{op}}}$.

Definition 2.1.11. Let $\mathcal{D}$ be a prederivator, and let A be a category. Suppose we have a map $f : a \to b$ in A, classified by a natural transformation

$$[\mathbf{0}] \quad \overset{a}{\underset{b}{\Downarrow f}} \quad \mathrm{A}.$$

Applying $\mathcal{D}$ to this diagram, we obtain

$$\mathcal{D}([\mathbf{0}]) \quad \overset{a^*}{\underset{b^*}{\Downarrow f^*}} \quad \mathcal{D}(\mathrm{A}).$$

For any object $X \in \mathcal{D}(\mathrm{A})$, write X_a for $a^* X$, and write $X_f : X_a \to X_b$ for the component of f^* at X. These assignments define a functor:

$$\mathrm{dia}_{\mathrm{A}}(X) : \mathrm{A} \to \mathcal{D}([\mathbf{0}])$$
$$a \mapsto X_a$$

We call this the **underlying diagram of** X. This construction induces a functor $\mathrm{dia}_{\mathrm{A}} : \mathcal{D}(\mathrm{A}) \to \mathcal{D}([\mathbf{0}])^{\mathrm{A}}$. Similarly, the underlying diagram functors of the shifted prederivators $\mathcal{D}^{\mathrm{J}}$ induce **partial underlying diagram functors** $\mathrm{dia}_{\mathrm{A}}^{\mathrm{J}} : \mathcal{D}(\mathrm{J} \times \mathrm{A}) \to \mathcal{D}(\mathrm{J})^{\mathrm{A}}$ for any J.

Definition 2.1.12. Let $\mathcal{D}$ be a prederivator, and let $u : \mathrm{A} \to \mathrm{B}$ be a functor. If the pullback functor $u^* : \mathcal{D}(\mathrm{B}) \to \mathcal{D}(\mathrm{A})$ has a left adjoint $u_! : \mathcal{D}(\mathrm{A}) \to \mathcal{D}(\mathrm{B})$, we call this map the **left**

(homotopy) Kan extension along u. If u^* admits a right adjoint $u_* : \mathcal{D}(A) \to \mathcal{D}(B)$, we call this the **right (homotopy) Kan extension** along u.

In particular, in the case of the unique map $\mathrm{p} : A \to [\mathbf{0}]$, we call these the **(homotopy) colimit** and **(homotopy) limit**, and denote them by:

$$\mathrm{hocolim} := \mathrm{p}_! : \mathcal{D}(A) \to \mathcal{D}([\mathbf{0}])$$

$$\mathrm{holim} := \mathrm{p}_* : \mathcal{D}(A) \to \mathcal{D}([\mathbf{0}])$$

Example 2.1.13. Given a category $\mathcal{C}$, homotopy Kan extensions in the representable prederivator $y(\mathcal{C})$ are exactly Kan extensions. If $\mathcal{M}$ is a model category, homotopy Kan extensions in $\mathcal{H}o(\mathcal{M})$ recover the familiar homotopy Kan extensions.

Definition 2.1.14. Let $\mathcal{D}$ be a prederivator, and suppose we have a natural transformation:

Suppose that each of the functors above admits left and right homotopy Kan extensions in $\mathcal{D}$. Then the square above induces canonical maps:

$$
\begin{array}{ccc}
\mathcal{D}(B) & & \\
\uparrow_{u_*} & \eta & \\
\mathcal{D}(A) \xleftarrow{u^*} \mathcal{D}(B) & & \\
\kappa_* := \quad v^* \uparrow \quad \kappa^* \quad \uparrow w^* & & \\
\mathcal{D}(J) \xleftarrow{z^*} \mathcal{D}(K) & & \\
\epsilon \quad \uparrow z_* & & \\
\mathcal{D}(J) & &
\end{array}
$$

These two transformations are **mates** of the natural transformation κ^* in the sense of [28, Section 2]. An introduction to mates can be found in [14, Appendix A]. It follows that $\kappa_!$ and κ_* are **conjugate**, as in [34, Chapter IV.7]. In particular, $\kappa_!$ is an isomorphism if and only if κ_* is. If this is the case, we say the square κ is $\mathcal{D}$-**exact**.

Remark 2.1.15. Given a commutative square, we can apply the constructions of Definition 2.1.14 to the identity transformation, taking the convention that the transformation goes from the top composite the bottom. Note that, even in this case, the direction of the 2-cell is important. For example, we have a commutative square:

$$
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow p & \text{id} & \downarrow p \\
[\mathbf{0}] & =\!=\!= & [\mathbf{0}]
\end{array}
$$

For any prederivator $\mathcal{D}$ admitting the relevant homotopy colimits we get a canonical map:

$$
\begin{array}{ccc}
\mathcal{D}(B) & \xrightarrow{p_!} & \\
\downarrow u^* & & \mathcal{D}([\mathbf{0}]) \\
\mathcal{D}(A) & \xrightarrow{p_!} &
\end{array}
$$

On the other hand, considering the identity as a map in the opposite direction

$$
\begin{array}{ccc}
\mathrm{A} & \xrightarrow{\ \ \mathrm{p}\ \ } & [\mathbf{0}] \\
\downarrow u & \overset{\mathrm{id}}{\swarrow} & \Vert \\
\mathrm{B} & \xrightarrow{\ \ \mathrm{p}\ \ } & [\mathbf{0}]
\end{array}
$$

and taking a prederivator $\mathcal{D}$ admitting homotopy limits, we get a canonical map:

$$
\begin{array}{ccc}
\mathcal{D}(\mathrm{B}) & \overset{\mathrm{p}_*}{\longrightarrow} & \\
\downarrow u^* & \Swarrow & \mathcal{D}([\mathbf{0}]) \\
\mathcal{D}(\mathrm{A}) & \underset{\mathrm{p}_*}{\longrightarrow} &
\end{array}
$$

For this reason, even if the 2-cell in a square is the identity, we will indicate its direction when we discuss $\mathcal{D}$-exact squares.

We now give the definition of derivator. Following [11], we list four derivator axioms **Der 1-4**. Other sources, for example [42], add a fifth derivator axiom, **Der 5**, which we recall in Definition 2.1.18. We will call prederivators that satisfy all five axioms strong derivators.

Definition 2.1.16. A prederivator $\mathcal{D}$ is a **derivator** if it satisfies the following axioms:

Der 1 $\mathcal{D}$ preserves coproducts. That is, the canonical map

$$
\mathcal{D}\left(\coprod_i \mathrm{A}_i\right) \to \prod_i \mathcal{D}(\mathrm{A}_i)
$$

is an equivalence. In particular, $\mathcal{D}(\emptyset) \simeq [\mathbf{0}]$.

Der 2 A map $f : X \to Y$ in $\mathcal{D}(\mathrm{A})$ is an isomorphism if and only if $f_a : X_a \to Y_a$ is an isomorphism in $\mathcal{D}([\mathbf{0}])$, for every $a \in \mathrm{A}$.

Der 3 Any functor $u : \mathrm{A} \to \mathrm{B}$ admits both a left and right Kan extension in $\mathcal{D}$.

Der 4 For any functor $u : \mathrm{A} \to \mathrm{B}$ and any $b \in \mathrm{B}$, the squares below are $\mathcal{D}$-exact:

$$
\begin{array}{ccc}
A \downarrow b & \xrightarrow{\ \mathrm{pr}\ } & A \\
{\scriptstyle p}\downarrow & \swarrow & \downarrow{\scriptstyle u} \\
[\mathbf{0}] & \xrightarrow[\ b\ ]{} & B
\end{array}
\qquad
\begin{array}{ccc}
b \downarrow A & \xrightarrow{\ \mathrm{pr}\ } & A \\
{\scriptstyle p}\downarrow & \nearrow & \downarrow{\scriptstyle u} \\
[\mathbf{0}] & \xrightarrow[\ b\ ]{} & B
\end{array}
$$

Thus, for any $X \in \mathcal{D}(A)$, we have canonical isomorphisms $\operatorname{hocolim}(\mathrm{pr}^* X) \cong (u_! X)_b$ and $(u_* X)_b \cong \operatorname{holim}(\mathrm{pr}^* X)$.

We denote the full sub-2-category of **PDer** on derivators by **Der**. A prederivator map between derivators will often be called a **derivator map**.

In Definition 2.1.16, the axioms **Der 1** and **Der 2** give conditions on the values of a prederivator that make them behave like the homotopy categories of diagram categories. We call a prederivator a **semiderivator** if it satisfies these two axioms.

The axiom **Der 3** is a completeness condition. **Der 4** allows us to calculate the underlying diagram of a homotopy Kan extension entirely in terms of homotopy limits and colimits. A semiderivator $\mathcal{D}$ is called a **left derivator** if it satisfies the parts of **Der 3** and **Der 4** that deal with left Kan extensions. **Right derivators** are defined dually. Note that this terminology agrees with the terminology of [16], but reverses the terminology of [3].

Remark 2.1.17. A square

$$
\begin{array}{ccc}
A & \xrightarrow{\ u\ } & B \\
{\scriptstyle v}\downarrow & \overset{\kappa}{\swarrow} & \downarrow{\scriptstyle w} \\
J & \xrightarrow[\ z\ ]{} & K
\end{array}
$$

is called **homotopy exact** if it is $\mathcal{D}$-exact for every derivator $\mathcal{D}$. By definition, every comma square of the form given in **Der 4** is homotopy exact. A complete characterisation of homotopy exact squares appears in [14, Section 3].

Definition 2.1.18. A prederivator $\mathcal{D}$ is called **strong** if it satisfies the following axiom:

Der 5 For each category A, the partial underlying diagram functor

$$
\operatorname{dia}_{[1]}^{A} : \mathcal{D}(A \times [1]) \to \mathcal{D}(A)^{[1]}
$$

is full and essentially surjective.

This axiom gives an important connection between the maps in a category $\mathcal{D}(A)$ and the objects in $\mathcal{D}(A \times [1])$. It is analogous to the triangulated category axioms (**TR1** and **TR3** in [35]) that allow us to extend a map in a triangulated category to a distinguished triangle, and to extend a commutative square to a map of distinguished triangles.

Example 2.1.19. A prederivator $\mathcal{D}$ is a (strong) derivator if and only if $\mathcal{D}^{\mathrm{op}}$ is.

Example 2.1.20. A representable prederivator $y(\mathcal{C})$ is a derivator if and only if $\mathcal{C}$ is complete and cocomplete. In this case, **Der 4** says that Kan extensions in $\mathcal{C}$ are computed pointwise. Any representable prederivator is strong, since all underlying diagram functors are equivalences.

The constant prederivator $c(\mathcal{C})$ is not a derivator for any $\mathcal{C} \neq [\mathbf{0}]$, since in this case **Der 1** fails.

Example 2.1.21 (Cisinski)**.** For any model category $\mathcal{M}$, the prederivator $\mathcal{H}o(\mathcal{M})$ is a strong derivator. For a general model category, the proof is technical; this is the main result of [2]. For a simpler proof in the case of combinatorial model categories, see [11, Section 1.3].

Example 2.1.22. Let Q be a quasicategory. The prederivator $\mathcal{H}o(Q)$ is strong. If Q is complete and cocomplete, then $\mathcal{H}o(Q)$ is a derivator. See [14, 33] for proofs of this fact.

Example 2.1.23. For any (strong) derivator $\mathcal{D}$ and any category A, the shifted prederivator $\mathcal{D}^A$ is a (strong) derivator. See [11, Section 1.3] for a proof.

Shifted derivators are extremely useful when proving general theorems about categories that arise as the values of derivators: if a statement holds for the underlying category of every derivator $\mathcal{D}$, then for any category A the statement must also be true for $\mathcal{D}(A)$, since this is the underlying category of $\mathcal{D}^A$. The following remark gives a simple example:

Remark 2.1.24. If $\mathcal{D}$ is a derivator, **Der 1** implies that homotopy products and coproducts coincide with products and coproducts in $\mathcal{D}([\mathbf{0}])$. By **Der 3**, then, $\mathcal{D}([\mathbf{0}])$ admits all products and coproducts. Using Example 2.1.23, it follows that $\mathcal{D}(A)$ has all products and coproducts, for any category A.

The following lemma is a simple but important consequence of **Der 2**:

Lemma 2.1.25. *Suppose we have prederivators* $\mathcal{D}_1$ *and* $\mathcal{D}_2$*, and a modification*

$$\mathcal{D}_1 \underset{G}{\overset{F}{\rightrightarrows}} \Downarrow\theta \quad \mathcal{D}_2.$$

*If $\mathcal{D}_2$ satisfies **Der 2**, then θ is an isomorphism if and only if the component on underlying categories is an isomorphism.*

Proof. For any category A, and any $X \in \mathcal{D}_1(A)$ consider the component $\theta_X : FX \to GX$ in $\mathcal{D}_2(A)$. By **Der 2**, this is an isomorphism if and only if, for every $a \in A$, the map $(\theta_X)_a : (FX)_a \to (GX)_a$ is an isomorphism in $\mathcal{D}_2([\mathbf{0}])$. But the modification condition for θ implies that the diagram below commutes:

$$
\begin{array}{ccc}
(FX)_a & \xrightarrow[\cong]{\gamma} & F(X_a) \\
{\scriptstyle (\theta_X)_a} \downarrow & & \downarrow {\scriptstyle \theta_{X_a}} \\
(GX)_a & \xrightarrow[\gamma]{\cong} & G(X_a)
\end{array}
$$

Thus, θ is an isomorphism if and only if θ_x is an isomorphism for every $x \in \mathcal{D}_1([\mathbf{0}])$. $\qquad\square$

2.2   Cocontinuous maps and adjunctions

We begin this section with a discussion of the interaction between derivator maps and homotopy Kan extensions, proving some simple results that we will need in Chapter 3 and Chapter 4. Other results along similar lines can be found in [12]. In the second half of the chapter we recall the basic theory of adjunctions from [11, Section 2]. This will form a basis for the discussion of two-variable adjunctions in Chapter 3.

Definition 2.2.1. Suppose we have derivators $\mathcal{D}_1$ and $\mathcal{D}_2$, and a morphism $F : \mathcal{D}_1 \to \mathcal{D}_2$. For any functor $u : A \to B$ we have a canonical transformation:

We say F **preserves the left homotopy Kan extension along** u if this map is an isomorphism. If F preserves all left homotopy Kan extensions, we say F is **cocontinuous**. We denote the full subcategory of $\underline{\mathrm{Hom}}(\mathcal{D}_1, \mathcal{D}_2)$ on the cocontinuous maps by $\underline{\mathrm{Hom}}_!(\mathcal{D}_1, \mathcal{D}_2)$.

Dually, we can define **continuous** maps; denote the category of these by $\underline{\mathrm{Hom}}_*(\mathcal{D}_1, \mathcal{D}_2)$.

We record the following fact, whose proof can be found in [11, Section 2]:

Lemma 2.2.2. *A derivator map $F : \mathcal{D}_1 \to \mathcal{D}_2$ is cocontinuous if and only if it preserves homotopy colimits.*

The following lemma and its dual can be found in [12, Section 3]:

Lemma 2.2.3. *Let $F, G : \mathcal{D}_1 \to \mathcal{D}_2$ be derivator maps, and let $\theta : F \Rightarrow G$ be a modification. Then, for any functor $u : A \to B$ and any $X \in \mathcal{D}_1(A)$, the diagram below commutes, where the vertical arrows are the canonical maps of Definition 2.2.1:*

$$
\begin{array}{ccc}
u_! F X & \xrightarrow{u_!(\theta_X)} & u_! G X \\
\downarrow & & \downarrow \\
F u_! X & \xrightarrow{\theta_{u_! X}} & G u_! X
\end{array}
$$

Proof. The commutativity of this diagram for any $X \in \mathcal{D}_1(A)$ expresses the equality of the two pasting diagrams below:

This follows immediately from the modification condition for θ. $\qquad\square$

Lemma 2.2.4. *Suppose we have a derivator map $F : \mathcal{D}_1 \to \mathcal{D}_2$ and a natural transformation:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ u\ \ } & B \\
\downarrow{\scriptstyle v} & \overset{\kappa}{\Swarrow} & \downarrow{\scriptstyle w} \\
J & \xrightarrow[\ \ z\ \ ]{} & K
\end{array}
$$

Consider the natural transformation $\kappa_! : v_! \circ u^* \Rightarrow z^* \circ w_!$ of Definition 2.1.14. For any $X \in \mathcal{D}_1(B)$, the diagram below commutes, where the vertical maps are induced by the canonical map in Definition 2.2.1, and the structure isomorphisms of F:

$$
\begin{array}{ccc}
v_! u^* F X & \xrightarrow{\ (\kappa_!)_{FX}\ } & z^* w_! F X \\
\downarrow & & \downarrow \\
F v_! u^* X & \xrightarrow[\ F(\kappa_!)_X\ ]{} & F z^* w_! X
\end{array}
$$

Proof. The commutative diagram above expresses the equality of certain pasting diagrams. Using the triangle equality to cancel instances of units and counits, we can reduce these to the following:

$$\mathcal{D}_1(\mathrm{B}) \xrightarrow{w_!} \mathcal{D}_1(\mathrm{K}) \xrightarrow{z^*} \mathcal{D}_1(\mathrm{J}) \xrightarrow{F} \mathcal{D}_2(\mathrm{J})$$

That these two diagrams are equal follows easily from the coherence conditions for the derivator map F, as in Definition 2.1.2. □

Definition 2.2.5. An **adjunction** between derivators is an adjunction in the 2-category **Der**, in the sense of [28, Section 2].

We will make regular use of the following lemma, which characterises derivator left adjoints:

Lemma 2.2.6. *A derivator map $F : \mathcal{D}_1 \to \mathcal{D}_2$ is a left adjoint if and only if it is cocontinuous and each component functor $F : \mathcal{D}_1(\mathrm{A}) \to \mathcal{D}_2(\mathrm{A})$ has a right adjoint $G : \mathcal{D}_2(\mathrm{A}) \to \mathcal{D}_1(\mathrm{A})$. Moreover, a derivator map is an equivalence if and only if it is a pointwise equivalence.*

Proof. A proof of this lemma can be found in [11, Section 2]. We give a brief outline. Suppose $F : \mathcal{D}_1 \to \mathcal{D}_2$ is a derivator map such that each component functor $F : \mathcal{D}_1(\mathrm{A}) \to \mathcal{D}_2(\mathrm{A})$ has a right adjoint $G : \mathcal{D}_2(\mathrm{A}) \to \mathcal{D}_1(\mathrm{A})$. Given $u : \mathrm{A} \to \mathrm{B}$, consider the following pasting diagram:

If these maps are isomorphisms for every $u : \mathrm{A} \to \mathrm{B}$, then these form the structure isomorphisms for a derivator map $G : \mathcal{D}_2 \to \mathcal{D}_1$, which is then right adjoint to F.

This transformation is conjugate to the canonical map in Definition 2.2.1. Thus, this is an isomorphism if and only if F is cocontinuous. Moreover, it is clearly an isomorphism when F is a pointwise equivalence, since in that case it is the pasting of three isomorphisms. □

Remark 2.2.7. A derivator left adjoint $F : \mathcal{D}_1 \to \mathcal{D}_2$ is an equivalence if and only if the underlying functor $F : \mathcal{D}_1([\mathbf{0}]) \to \mathcal{D}_2([\mathbf{0}])$ is an equivalence. This follows by applying Lemma 2.1.25 to the unit and counit of the adjunction.

Example 2.2.8. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and $F : \mathcal{M} \to \mathcal{N}$ be a left Quillen functor, with right adjoint $G : \mathcal{N} \to \mathcal{M}$. Then the derived functors induce an adjunction:

$$\mathcal{H}o(\mathcal{M}) \quad \underset{\mathbb{R}G}{\overset{\mathbb{L}F}{\rightleftarrows}} \quad \mathcal{H}o(\mathcal{N})$$

$$\perp$$

By Remark 2.2.7, a left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ induces an equivalence on derivators if and only if it is a Quillen equivalence.

We record one more fact, whose proof is in [11, Section 2]:

Lemma 2.2.9. *Let $\mathcal{D}$ be a derivator, and let $u : A \to B$ be a functor. Then the derivator map $u^* : \mathcal{D}^B \to \mathcal{D}^A$ is continuous and cocontinuous.*

Combining Lemma 2.2.6 and Lemma 2.2.9, for any functor $u : A \to B$ and any derivator $\mathcal{D}$, the map $u^* : \mathcal{D}^B \to \mathcal{D}^A$ admits both a left adjoint $u_! : \mathcal{D}^A \to \mathcal{D}^B$ and a right adjoint $u_* : \mathcal{D}^A \to \mathcal{D}^B$ in **Der**.

Remark 2.2.10. Given a derivator $\mathcal{D}$ and a functor $u : A \to B$, we can lift the canonical transformations of Definition 2.2.1 to a modification:

The map F is cocontinuous if and only if each of these modifications is an isomorphism.

Definition 2.2.11. Given prederivators $\mathcal{D}_1$ and $\mathcal{D}_2$, we can form a prederivator $\mathcal{H}om(\mathcal{D}_1, \mathcal{D}_2)$ as follows:

$$
\begin{aligned}
\mathcal{H}om(\mathcal{D}_1, \mathcal{D}_2) : \mathbf{Cat}^{\mathrm{op}} &\to \mathbf{CAT} \\
A &\mapsto \underline{\mathrm{Hom}}(\mathcal{D}_1, \mathcal{D}_2^A) \\
(u : A \to B) &\mapsto (u^* \circ - : \underline{\mathrm{Hom}}(\mathcal{D}_1, \mathcal{D}_2^B) \to \underline{\mathrm{Hom}}(\mathcal{D}_1, \mathcal{D}_2^A))
\end{aligned}
$$

Similarly, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are derivators, we have a prederivator:

$$\mathcal{H}\mathrm{om}_!(\mathcal{D}_1, \mathcal{D}_2) : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$$
$$\mathrm{A} \mapsto \underline{\mathrm{Hom}}_!(\mathcal{D}_1, \mathcal{D}_2^{\mathrm{A}})$$
$$(u : \mathrm{A} \to \mathrm{B}) \mapsto (u^* \circ - : \underline{\mathrm{Hom}}_!(\mathcal{D}_1, \mathcal{D}_2^{\mathrm{B}}) \to \underline{\mathrm{Hom}}_!(\mathcal{D}_1, \mathcal{D}_2^{\mathrm{A}}))$$

This is well-defined by Lemma 2.2.9.

Remark 2.2.12. If $\mathcal{D}_2$ is a derivator, then so is $\mathcal{H}\mathrm{om}(\mathcal{D}_1, \mathcal{D}_2)$. Given a functor $u : \mathrm{A} \to \mathrm{B}$, the Kan extensions along u in $\mathcal{H}\mathrm{om}(\mathcal{D}_1, \mathcal{D}_2)$ are given by postcomposition with $u_! : \mathcal{D}^{\mathrm{A}} \to \mathcal{D}^{\mathrm{B}}$ and $u_* : \mathcal{D}^{\mathrm{A}} \to \mathcal{D}^{\mathrm{B}}$. On the other hand, $\mathcal{H}\mathrm{om}_!(\mathcal{D}_1, \mathcal{D}_2)$ is a left derivator, but not a derivator in general. See [3, Section 5].

We end this section with an important theorem, describing the universal property of the derivator of spaces, which appears in [3].

Theorem 2.2.13 (Cisinski). *For any left derivator $\mathcal{D}$, the map*

$$\underline{\mathrm{Hom}}_!(\mathcal{H}\mathrm{o}(\mathbf{sSet}), \mathcal{D}) \to \mathcal{D}([\mathbf{0}]),$$

given by evaluation at the point $\Delta^0 \in \mathrm{Ho}(\mathbf{sSet})$, is an equivalence.

2.3 Two-variable maps

In this section, we review some aspects of two-variable derivator maps, primarily following [13]. These have a more complicated theory than single-variable maps, resulting in part from their external variants, which we discuss at the beginning of this section, and in part from the cancelling variants, which we will discuss in Chapter 3.

Given derivators $\mathcal{D}_1$ and $\mathcal{D}_2$, the product $\mathcal{D}_1 \times \mathcal{D}_2$ is formed pointwise: for any $\mathrm{A} \in \mathbf{Cat}$, we have

$$(\mathcal{D}_1 \times \mathcal{D}_2)(\mathrm{A}) = \mathcal{D}_1(\mathrm{A}) \times \mathcal{D}_2(\mathrm{A}).$$

Suppose we have a derivator map $\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$. For each category A, this map has a component functor of the form

$$\otimes : \mathcal{D}_1(\mathrm{A}) \times \mathcal{D}_2(\mathrm{A}) \to \mathcal{D}_3(\mathrm{A}).$$

From this, we can construct an **external** version of $\otimes$ as follows:

$$\widetilde{\otimes} : \mathcal{D}_1(A) \times \mathcal{D}_2(B) \to \mathcal{D}_3(A \times B)$$
$$(X, Y) \mapsto p_B^* X \otimes p_A^* Y$$

For a fixed object $X \in \mathcal{D}_1(A)$ or $Y \in \mathcal{D}_2(B)$, the external product induces derivator maps:

$$X \widetilde{\otimes} - : \mathcal{D}_2 \to \mathcal{D}_3^A$$

$$- \widetilde{\otimes} Y : \mathcal{D}_1 \to \mathcal{D}_3^B$$

Note that our notation for the external two-variable map differs from that in [13].

Remark 2.3.1. The external product lifts to a derivator map:

$$\widetilde{\otimes} : \mathcal{D}_1^A \times \mathcal{D}_2^B \xrightarrow{\ p_B^* \times p_A^* \ } \mathcal{D}_1^{A \times B} \times \mathcal{D}_2^{A \times B} \xrightarrow{\ \otimes \ } \mathcal{D}_3^{A \times B}$$

Given functors $u : A \to C$ and $v : B \to D$, the structure isomorphism of $\otimes$ induces a natural isomorphism:

$$
\begin{array}{ccc}
\mathcal{D}_1^C \times \mathcal{D}_2^D & \xrightarrow{\ \widetilde{\otimes}\ } & \mathcal{D}_3^{C \times D} \\
{\scriptstyle u^* \times v^*}\Big\downarrow & {\scriptstyle \gamma}\ \ \Swarrow \cong & \Big\downarrow{\scriptstyle (u \times v)^*} \\
\mathcal{D}_1^A \times \mathcal{D}_2^B & \xrightarrow[\ \widetilde{\otimes}\ ]{} & \mathcal{D}_3^{A \times B}
\end{array}
$$

Definition 2.3.2. A two-variable map $\otimes$ is called **cocontinuous** if, for every $X \in \mathcal{D}_1(A)$ and $Y \in \mathcal{D}_2(B)$, the maps

$$X \widetilde{\otimes} - : \mathcal{D}_2 \to \mathcal{D}_3^A$$

$$- \widetilde{\otimes} Y : \mathcal{D}_1 \to \mathcal{D}_3^B$$

are cocontinuous. Let $\underline{\mathrm{Hom}}_!(\mathcal{D}_1, \mathcal{D}_2; \mathcal{D}_3)$ denote the full subcategory of $\underline{\mathrm{Hom}}(\mathcal{D}_1 \times \mathcal{D}_2, \mathcal{D}_3)$ on the cocontinuous maps.

Remark 2.3.3. Given a two-variable map $\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$, its opposite is a map

$$\otimes^{\mathrm{op}} : \mathcal{D}_1^{\mathrm{op}} \times \mathcal{D}_2^{\mathrm{op}} \to \mathcal{D}_3^{\mathrm{op}}.$$

For any $X \in \mathcal{D}_1(A)$ we have an isomorphism

$$X \widetilde{\otimes^{\mathrm{op}}} - \cong (X \widetilde{\otimes} -)^{\mathrm{op}} : \mathcal{D}_2^{\mathrm{op}} \to (\mathcal{D}_3^{\mathrm{op}})^{A^{\mathrm{op}}}.$$

So $\otimes$ is cocontinuous if and only if $\otimes^{\mathrm{op}}$ is **continuous** in the obvious sense.

Remark 2.3.4. As in Remark 2.2.10, we may lift the isomorphisms associated to a cocontinuous two-variable map $\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$ to modifications. Given any functor $u : A \to B$, and any category J, we have a canonical map

$$
\begin{array}{ccccc}
\mathcal{D}_1^{J} \times \mathcal{D}_2^{A} & \xrightarrow{\mathcal{D}_1^{J} \times u_!} & \mathcal{D}_1^{J} \times \mathcal{D}_2^{B} & \xrightarrow{\widetilde{\otimes}} & \mathcal{D}_3^{J \times B} \\
 & {}^{\eta}\nearrow & \downarrow{\mathcal{D}_1^{J} \times u^*} \quad {}^{\gamma^{-1}}\nearrow \cong \quad (J \times u)^* \downarrow & & {}^{\epsilon}\nearrow \\
 & & \mathcal{D}_1^{J} \times \mathcal{D}_2^{A} \xrightarrow{\widetilde{\otimes}} \mathcal{D}_3^{J \times A} \xrightarrow{(J \times u)_!} \mathcal{D}_3^{J \times B}
\end{array}
$$

and a similar map where the Kan extension is in the first variable. The two-variable map $\otimes$ is cocontinuous if and only if each of these modifications is an isomorphism.

Lemma 2.3.5. *There is an equivalence of categories*

$$\underline{\mathrm{Hom}}(\mathcal{D}_1 \times \mathcal{D}_2, \mathcal{D}_3) \simeq \underline{\mathrm{Hom}}(\mathcal{D}_1, \mathcal{H}\mathrm{om}(\mathcal{D}_2, \mathcal{D}_3))$$

which restricts to an equivalence:

$$\underline{\mathrm{Hom}}_!(\mathcal{D}_1, \mathcal{D}_2; \mathcal{D}_3) \simeq \underline{\mathrm{Hom}}_!(\mathcal{D}_1, \mathcal{H}\mathrm{om}_!(\mathcal{D}_2, \mathcal{D}_3))$$

Proof. See [3, Section 5] for a complete proof. For convenience, we record the maps in both directions.

Given a map $\otimes \in \underline{\mathrm{Hom}}(\mathcal{D}_1 \times \mathcal{D}_2, \mathcal{D}_3)$, the corresponding map in $\underline{\mathrm{Hom}}(\mathcal{D}_1, \mathcal{H}\mathrm{om}(\mathcal{D}_2, \mathcal{D}_3))$ has components:

$$\mathcal{D}_1(A) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{D}_2, \mathcal{D}_3^A)$$

$$X \longmapsto X \widetilde{\otimes} -$$

For the inverse, suppose we have a map $\varphi \in \underline{\mathrm{Hom}}(\mathcal{D}_1, \mathcal{H}\mathrm{om}(\mathcal{D}_2, \mathcal{D}_3))$. For any $X \in \mathcal{D}_1(A)$,

φ gives us a map $\varphi(X) : \mathcal{D}_2 \to \mathcal{D}_3^A$. The morphism corresponding to φ in $\underline{\mathrm{Hom}}(\mathcal{D}_1 \times \mathcal{D}_2, \mathcal{D}_3)$ has components given by

$$\mathcal{D}_1(A) \times \mathcal{D}_2(A) \longrightarrow \mathcal{D}_3(A \times A) \xrightarrow{\ \delta^*\ } \mathcal{D}_3(A)$$

$$(X, Y) \longmapsto \varphi(X)(Y) \longmapsto \delta^*(\varphi(X)(Y))$$

where $\delta : A \to A \times A$ is the diagonal map. $\qquad\square$

Lemma 2.3.5 carries over to the following statement about two-variable maps. For a proof, see [13, Theorem 3.11].

Remark 2.3.6. Let $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ be prederivators. Consider the 2-functor below:

$$\mathcal{D}_1 \mathrel{\overline{\times}} \mathcal{D}_2 : \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$$

$$(A, B) \longmapsto \mathcal{D}_1(A) \times \mathcal{D}_2(B)$$

We may also form the following composite, where the first map takes a pair of categories A and B to their product $A \times B$:

$$\mathcal{D}_3 \circ \times : \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat}^{\mathrm{op}} \xrightarrow{\ \times\ } \mathbf{Cat}^{\mathrm{op}} \xrightarrow{\ \mathcal{D}_3\ } \mathbf{CAT}$$

The maps of Lemma 2.3.5 induce an equivalence of categories

$$\underline{\mathrm{Hom}}(\mathcal{D}_1 \times \mathcal{D}_2, \mathcal{D}_3) \simeq \mathrm{Psnat}(\mathcal{D}_1 \mathrel{\overline{\times}} \mathcal{D}_2, \mathcal{D}_3 \circ \times),$$

where the second category has objects given by the pseudonatural transformations from $\mathcal{D}_1 \mathrel{\overline{\times}} \mathcal{D}_2$ to $\mathcal{D}_3 \circ \times$, and maps given by modifications.

Explicitly, a prederivator map $\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$ corresponds to a pseudonatural transformation with component at (A, B) given as follows:

$$\mathcal{D}_1(A) \times \mathcal{D}_2(B) \longrightarrow \mathcal{D}_3(A \times B)$$

$$(X, Y) \longmapsto X \mathrel{\widetilde{\otimes}} Y$$

For any derivator $\mathcal{D}$, we can apply Theorem 2.2.13 and Lemma 2.3.5 to the left derivator $\mathcal{H}om_!(\mathcal{D}, \mathcal{D})$, to get an equivalence:

$$\underline{\mathrm{Hom}}_!(\mathcal{D}, \mathcal{D}) \simeq \underline{\mathrm{Hom}}_!(\mathcal{H}o(\mathbf{sSet}), \mathcal{H}om_!(\mathcal{D}, \mathcal{D})) \simeq \underline{\mathrm{Hom}}_!(\mathcal{H}o(\mathbf{sSet}), \mathcal{D}; \mathcal{D})$$

Under this equivalence, $\mathrm{id} \in \underline{\mathrm{Hom}}_!(\mathcal{D}, \mathcal{D})$ gives us a canonical cocontinuous map:

$$\otimes : \mathcal{H}o(\mathbf{sSet}) \times \mathcal{D} \to \mathcal{D}$$

This is the essentially unique cocontinuous map such that

$$\Delta^0 \widetilde{\otimes} - \cong \mathrm{id} : \mathcal{D} \to \mathcal{D}.$$

For the derivator $\mathcal{H}o(\mathcal{M})$ associated to a model category $\mathcal{M}$, this map agrees with the familiar action of $\mathbf{sSet}$ on $\mathcal{M}$, constructed, for example, in [23, Chapter 5]. We will discuss actions of derivators in detail in Chapter 3.

2.4 Pointed and triangulated derivators

In this section we include a brief overview of pointed, stable and triangulated derivators, most of which can be found in [11, Section 3,4]. These are the derivator analogues of pointed and stable model categories, and a number of the concepts that are important in that setting can also be developed in derivators, for example a theory of homotopy fibres and cofibres. Note that, in contrast to [11], we do not assume stable derivators are strong. Thus, the definition of stable derivator in [11] is what we call a triangulated derivator in Definition 2.4.6. To study triangulated derivators, we will use a number of results from the theory of triangulated categories, which we recall at the end of this section. Using these, we will be able to prove representability theorems for enriched triangulated derivators in Chapter 3 and Chapter 5.

Definition 2.4.1. We say a derivator $\mathcal{D}$ is **pointed** if the underlying category has a zero object $0 \in \mathcal{D}([\mathbf{0}])$.

If $\mathcal{D}$ is a pointed derivator, note that $\mathrm{p}_A^* 0 \in \mathcal{D}(A)$ is a zero object, for any category A.

Example 2.4.2. Let $\mathcal{M}$ be a model category. Then the derivator $\mathcal{H}o(\mathcal{M})$ is pointed if and only if the unique map from the initial object in $\mathcal{M}$ to the terminal object is a weak equivalence. In particular, if $\mathcal{M}$ is a pointed model category then $\mathcal{H}o(\mathcal{M})$ is pointed.

We have an analogue of Theorem 2.2.13 for pointed derivators:

Theorem 2.4.3 (Cisinski)**.** *For any pointed derivator $\mathcal{D}$, the map*

$$\underline{\mathrm{Hom}}_!(\mathcal{H}\mathrm{o}(\mathbf{sSet}_*), \mathcal{D}) \to \mathcal{D}([\mathbf{0}])$$

given by evaluation at $\mathrm{S}^0 \in \mathrm{Ho}(\mathbf{sSet}_*)$ is an equivalence.

For any pointed derivator $\mathcal{D}$, this universal property induces a canonical cocontinuous map

$$\wedge : \mathcal{H}\mathrm{o}(\mathbf{sSet}_*) \times \mathcal{D} \to \mathcal{D}.$$

This is the essentially unique cocontinuous map such that

$$\mathrm{S}^0 \; \widetilde{\wedge} - \cong \mathrm{id} : \mathcal{D} \to \mathcal{D}.$$

See [3] for a complete proof of this fact.

Definition 2.4.4. For any pointed derivator $\mathcal{D}$, the action of $\mathcal{H}\mathrm{o}(\mathbf{sSet}_*)$ on $\mathcal{D}$ induces the **suspension** map:

$$\Sigma := \mathrm{S}^1 \; \widetilde{\wedge} - : \mathcal{D} \to \mathcal{D}$$

We call a pointed derivator $\mathcal{D}$ **stable** if this map is an equivalence.

Given a pointed derivator, it is possible to define suspension in elementary terms, without appealing to the action of $\mathcal{H}\mathrm{o}(\mathbf{sSet}_*)$. See [11] for this approach. See [16] for a number of equivalent characterisations of stability.

Example 2.4.5. Given a pointed model category $\mathcal{M}$, the derivator $\mathcal{H}\mathrm{o}(\mathcal{M})$ is stable if and only if $\mathcal{M}$ is a stable model category.

Definition 2.4.6. Let $\mathcal{D}$ be a stable derivator. If in addition $\mathcal{D}$ is strong, we call $\mathcal{D}$ a **triangulated** derivator.

The following theorem is the motivation for the term triangulated derivator. **Der 5** is essential in its proof.

Theorem 2.4.7 (Groth, Maltsiniotis)**.** *Let $\mathcal{D}$ be a triangulated derivator. Then, for any category A, the category $\mathcal{D}(\mathrm{A})$ has a canonical triangulated structure, and for any $u : \mathrm{A} \to \mathrm{B}$, the functor $u^* : \mathcal{D}(\mathrm{B}) \to \mathcal{D}(\mathrm{A})$ is canonically exact.*

Moreover, suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are triangulated derivators, and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a derivator map that preserves initial objects and homotopy pushouts. Then, for each category A, the functor $F : \mathcal{D}_1(\mathrm{A}) \to \mathcal{D}_2(\mathrm{A})$ has a canonical exact structure. The same is true if F preserves terminal objects and homotopy pullbacks.

See [11] for a proof of this theorem, and for the definition of **homotopy pushouts**, which are called **cocartesian squares** in [11]. We will not recall the definition here: for our purposes, it suffices to note that all cocontinuous derivator maps preserve initial objects and homotopy pushouts.

We now recall some concepts from the theory of triangulated categories, which we will use to study triangulated derivators. See [30, 35] for more details. There is also a useful survey in the first section of [40].

Definition 2.4.8. Let $\mathcal{T}$ be a triangulated category that admits all coproducts. An object $x \in \mathcal{T}$ is called **compact** if the functor it represents

$$\mathcal{T}(x, -) : \mathcal{T} \to \mathbf{Ab}$$

preserves coproducts.

A triangulated category $\mathcal{T}$ is called **compactly generated** if it admits all coproducts and there is a set $\mathcal{C}$ of objects in $\mathcal{T}$ with the following properties:

1. Every object $x \in \mathcal{C}$ is compact.

2. If $y \in \mathcal{T}$ has the property that $\mathcal{T}(x, y) = 0$ for every $x \in \mathcal{C}$, then $y = 0$.

We call $\mathcal{C}$ a **set of compact generators for** $\mathcal{T}$.

Compactly generated triangulated categories are historically important and well-studied. However, the following generalisation, introduced in [29], retains several important properties of compactly generated triangulated categories, and admits many more examples:

Definition 2.4.9. A triangulated category $\mathcal{T}$ is called **perfectly generated** if it admits all coproducts and there is a set $\mathcal{P}$ of objects in $\mathcal{T}$ with the following properties:

1. Let $f_i : y_i \to z_i$ be a family of maps in $\mathcal{T}$. Suppose that, for every $x \in \mathcal{P}$ and every f_i, the map

$$\mathcal{T}(x, f_i) : \mathcal{T}(x, y_i) \to \mathcal{T}(x, z_i)$$

is surjective. Then, for every object $x \in \mathcal{P}$, the map

$$\mathcal{T}(x, \coprod f_i) : \mathcal{T}(x, \coprod y_i) \to \mathcal{T}(x, \coprod z_i)$$

is surjective.

2. If $y \in \mathcal{T}$ has the property that $\mathcal{T}(x, y) = 0$ for every $x \in \mathcal{P}$, then $y = 0$.

We call $\mathcal{P}$ a **set of perfect generators for** $\mathcal{T}$.

We will now record some important properties of perfectly generated triangulated categories. First, we recall two more definitions:

Definition 2.4.10. Let $\mathcal{T}$ be a triangulated category. A functor $F : \mathcal{T}^{\mathrm{op}} \to \mathbf{Ab}$ is called a **cohomological functor** if it takes any distinguished triangle in $\mathcal{T}$ to an exact sequence in $\mathbf{Ab}$.

Definition 2.4.11. Let $\mathcal{T}$ be a triangulated category that admits all coproducts. We say that $\mathcal{T}$ **satisfies Brown representability** if any cohomological functor $F : \mathcal{T}^{\mathrm{op}} \to \mathbf{Ab}$ that preserves products (that is, takes coproducts in $\mathcal{T}$ to products in $\mathbf{Ab}$) is representable.

Versions of the following theorem have been proven in a number of settings. The version we give below is proved in [29]:

Theorem 2.4.12. *Perfectly generated triangulated categories satisfy Brown representability.*

The following is a useful property of triangulated categories that satisfy Brown representability. A simple proof can be found in [35, Chapter 8].

Lemma 2.4.13. *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories, and suppose $\mathcal{T}$ satisfies Brown representability. An exact functor $F : \mathcal{T} \to \mathcal{S}$ has a right adjoint if and only if it preserves coproducts.*

We will now apply these theorems to triangulated derivators. A proof of the following lemma can be found in [22, Section 3]:

Lemma 2.4.14. *Let $\mathcal{D}$ be a triangulated derivator. Suppose $\mathcal{G}$ is a set of compact (resp. perfect) generators for $\mathcal{D}([\mathbf{0}])$. Then, for any category A, the set*

$$\mathcal{G}_{\mathrm{A}} = \{a_! x \mid a \in \mathrm{A}, x \in \mathcal{G}\}$$

is a compact (resp. perfect) generating set for $\mathcal{D}(A)$.

By Theorem 2.4.12 and Lemma 2.4.14, the following lemma applies to any triangulated derivator $\mathcal{D}_1$ whose underlying category is perfectly generated:

Proposition 2.4.15. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be triangulated derivators, and suppose $\mathcal{D}_1(A)$ satisfies Brown representability, for any category A. Then a derivator map $F : \mathcal{D}_1 \to \mathcal{D}_2$ has a right adjoint if and only if it is cocontinuous.*

Proof. Let A be a category. If $F : \mathcal{D}_1 \to \mathcal{D}_2$ is cocontinuous then the component functor $F : \mathcal{D}_1(A) \to \mathcal{D}_2(A)$ preserves coproducts, and, by Theorem 2.4.7, is exact. Thus, the result follows by Lemma 2.2.6 and Lemma 2.4.13. □

We conclude this section with an analogue of Theorem 2.2.13 for stable derivators:

Theorem 2.4.16 (Heller)**.** *For any stable derivator $\mathcal{D}$, the map*

$$\underline{\mathrm{Hom}}_!(\mathcal{H}\mathrm{o}(\mathbf{Spt}), \mathcal{D}) \to \mathcal{D}([\mathbf{0}])$$

given by evaluation at the sphere spectrum $\mathbb{S} \in \mathrm{Ho}(\mathbf{Spt})$ is an equivalence.

For derivators satisfying a stronger analogue of **Der 5**, this theorem is essentially proved in [20]; see [42, Section 9], however, for a proof that the derivator constructed in [20] is equivalent to $\mathcal{H}\mathrm{o}(\mathbf{Spt})$. See [6] for a similar proof that does not use **Der 5**.

Using this universal property, for any stable derivator $\mathcal{D}$ we obtain a canonical cocontinuous map

$$\wedge : \mathcal{H}\mathrm{o}(\mathbf{Spt}) \times \mathcal{D} \to \mathcal{D},$$

essentially unique such that

$$\mathbb{S} \,\widetilde{\wedge}\, - \,\cong\, \mathrm{id} : \mathcal{D} \to \mathcal{D}.$$

For the derivator $\mathcal{H}\mathrm{o}(\mathcal{M})$ associated to a stable model category $\mathcal{M}$, this map recovers the action constructed in [38].

Chapter 3

Actions of Monoidal Derivators

In this chapter we discuss the theory of modules over monoidal derivators. Closed modules play a particularly important role in Chapter 4 and Chapter 5, and in this chapter we give a number of examples. Moreover, we show that, given a closed module $\mathcal{D}$ over a symmetric monoidal derivator, the shifted derivators $\mathcal{D}^A$ and the opposite derivator $\mathcal{D}^{op}$ are also closed modules. In this way, we build a collection of examples, which, as we will show in Chapter 5, induce important examples of enriched derivators. In addition, we prove a number of results in this chapter that will contribute to our development of the theory in Chapter 4 and Chapter 5.

In order to study monoidal derivators and their actions, we recall the theory of ends and co-ends in Section 3.1, and use this to discuss derivator two-variable adjunctions in Section 3.2. In Section 3.3 and Section 3.4, we recall the definition of monoidal derivators and their actions. Up to this point, much of this material can be found in [13], although our presentation differs in a number of ways. Section 3.5 and Section 3.6 contain an in-depth discussion of the structure arising from the action of a monoidal derivator. The results we prove in these sections form the groundwork for a number of the results in Chapter 4 and Chapter 5. We also use the results of these sections in Section 3.7, which, given a closed $\mathcal{E}$-module $\mathcal{D}$, discusses the action on the opposite derivator $\mathcal{D}^{op}$. Finally, in Section 3.8, we prove a representability theorem for triangulated closed modules over triangulated monoidal derivators.

3.1   Ends and coends

In this section, we recall from [13, Section 5] the definition of (homotopy) ends and coends in a derivator, and prove a number of simple results that we will need in the later sections of Chapter 3 and in Chapter 4. The definition of ends and coends that we give here goes via the twisted arrow category. See [13, Appendix A] for a discussion of equivalent definitions.

Given a category A, recall that the **twisted arrow category** $tw(A)$ is the category of elements of the hom-functor $hom : A^{op} \times A \to \mathbf{Set}$. Explicitly, objects of $tw(A)$ are arrows $f : a \to b$ in A, and an arrow from $f : a \to b$ to $g : c \to d$ is a commutative square:

$$
\begin{array}{ccc}
a & \xleftarrow{\ h\ } & c \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
b & \xrightarrow[\ k\ ]{} & d
\end{array}
$$

We have a canonical map $(s,t) : tw(A) \to A^{op} \times A$. Taking the opposite of this map, we get:

$$(t^{op},\ s^{op}) : tw(A)^{op} \xrightarrow{(s,t)^{op}} A \times A^{op} \cong A^{op} \times A$$

Definition 3.1.1. For any derivator $\mathcal{D}$ and any category A, the **(homotopy) coend** over A is the composite

$$\int^A : \mathcal{D}^{A^{op} \times A} \xrightarrow{(t^{op},\ s^{op})^*} \mathcal{D}^{tw(A)^{op}} \xrightarrow{p_!} \mathcal{D}.$$

We will denote the right adjoint of this map by

$$\partial^A : \mathcal{D} \xrightarrow{p^*} \mathcal{D}^{tw(A)^{op}} \xrightarrow{(t^{op},\ s^{op})_*} \mathcal{D}^{A^{op} \times A}.$$

Dually, the **(homotopy) end** over A is the composite

$$\int_A : \mathcal{D}^{A^{op} \times A} \xrightarrow{(s,t)^*} \mathcal{D}^{tw(A)} \xrightarrow{p_*} \mathcal{D}$$

and its left adjoint will be denoted

$$\partial_A : \mathcal{D} \xrightarrow{p^*} \mathcal{D}^{tw(A)} \xrightarrow{(s,t)_!} \mathcal{D}^{A^{op} \times A}.$$

If $\mathcal{D} = y(\mathcal{C})$ is a represented derivator, these constructions recover the usual end and coend.

Definition 3.1.2. Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a derivator map, and let A be a category. We say F **preserves** ∂_A if the canonical pasting

$$
\begin{array}{ccccc}
\mathcal{D}_1 & \xrightarrow{\ p^*\ } & \mathcal{D}_1^{\mathrm{tw}(A)} & \xrightarrow{\ (s,t)_!\ } & \mathcal{D}_1^{A^{\mathrm{op}} \times A} \\
{\scriptstyle F}\downarrow & {\scriptstyle \gamma}\!\!\nearrow\,{\scriptstyle \cong} & {\scriptstyle F}\downarrow & \nearrow & {\scriptstyle F}\downarrow \\
\mathcal{D}_2 & \xrightarrow[\ p^*\ ]{} & \mathcal{D}_2^{\mathrm{tw}(A)} & \xrightarrow[\ (s,t)_!\ ]{} & \mathcal{D}_2^{A^{\mathrm{op}} \times A}
\end{array}
$$

is an isomorphism, where the second square is the modification of Remark 2.2.10. Note that this is the case if and only if F preserves the left Kan extension along (s,t). Similarly, we can define functors that preserve ends, coends and ∂^A.

Remark 3.1.3. Let $F, G : \mathcal{D}_1 \to \mathcal{D}_2$ be derivator maps, and let $\theta : F \Rightarrow G$ be a modification. Using Lemma 2.2.3, it follows immediately that θ respects the constructions of Definition 3.1.1. For example, given any category A and $X \in \mathcal{D}_1(A^{\mathrm{op}} \times A)$, the diagram below commutes, where the vertical arrows are the canonical maps, as in Definition 3.1.2:

$$
\begin{array}{ccc}
\int^A FX & \xrightarrow{\ \int^A \theta_X\ } & \int^A GX \\
\downarrow & & \downarrow \\
F \int^A X & \xrightarrow[\ \theta_{\int^A X}\ ]{} & G \int^A X
\end{array}
$$

Definition 3.1.4. Suppose we have a functor $u : A \to B$. This induces a functor on twisted arrow categories $\mathrm{tw}(u) : \mathrm{tw}(A) \to \mathrm{tw}(B)$ that makes the diagram below commute:

$$
\begin{array}{ccc}
\mathrm{tw}(A) & \xrightarrow{\ \mathrm{tw}(u)\ } & \mathrm{tw}(B) \\
{\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\
A^{\mathrm{op}} \times A & \xrightarrow[\ u^{\mathrm{op}} \times u\ ]{} & B^{\mathrm{op}} \times B
\end{array}
$$

Using this commutative diagram, each of the constructions of Definition 3.1.1 can be extended to act on the functor u. Since we use them repeatedly, we will describe ∂_u and $\int^u$ explicitly:

Definition 3.1.5. For any functor $u : A \to B$, and any derivator $\mathcal{D}$, we have the following modifications:

$$\partial_u \quad := \quad$$

$$
\begin{array}{ccc}
 & \mathcal{D}^{\text{tw}(B)} & \xrightarrow{\;(s,t)_!\;} & \mathcal{D}^{B^{\text{op}}\times B} \\
\nearrow^{p^*} & \downarrow^{\text{tw}(u)^*} & \Nearrow & \downarrow^{(u^{\text{op}}\times u)^*} \\
\mathcal{D} & & & \\
\searrow_{p^*} & \mathcal{D}^{\text{tw}(A)} & \xrightarrow[\;(s,t)_!\;]{} & \mathcal{D}^{A^{\text{op}}\times A}
\end{array}
$$

$$\int^u \quad := \quad$$

$$
\begin{array}{ccc}
\mathcal{D}^{B^{\text{op}}\times B} & \xrightarrow{\;(t^{\text{op}},\, s^{\text{op}})^*\;} & \mathcal{D}^{\text{tw}(B)^{\text{op}}} & \searrow^{p_!} \\
\downarrow^{(u^{\text{op}}\times u)^*} & & \downarrow^{(\text{tw}(u)^{\text{op}})^*} & \Nearrow \quad \mathcal{D} \\
\mathcal{D}^{A\times A^{\text{op}}} & \xrightarrow[\;(t^{\text{op}},\, s^{\text{op}})^*\;]{} & \mathcal{D}^{\text{tw}(A)^{\text{op}}} & \nearrow_{p_!}
\end{array}
$$

The non-identity modifications in these pastings are obtained as in Definition 2.1.14 and Remark 2.1.15. The transformations ∂^u and $\int_u$ are dual; that is, they can be obtained from these by moving to the opposite derivator $\mathcal{D}^{\text{op}}$.

Given any category C and $X \in \mathcal{D}(C)$, rather than use a subscript as for most modifications, we will denote the component of ∂_u at X by $\partial_u X$. We will denote the others similarly.

Note that $\int_u$ is dual to $\int^u$, and is a mate of ∂_u under the adjunctions $\partial_A \dashv \int_A$ and $\partial_B \dashv \int_B$. Using this, the following lemmas have analogues for each of the constructions of Definition 3.1.1. Rather than record each version explicitly, we state each in the case that we will use most frequently in subsequent sections.

Lemma 3.1.6. *Let $\mathcal{D}$ be a derivator, and let $\mathcal{D} \Downarrow \mathbf{Der}$ denote the category whose objects are arrows $F : \mathcal{D} \to \mathcal{D}'$ in $\mathbf{Der}$ and whose morphisms from F to F' are given by modifications:*

$$
\begin{array}{ccc}
 & \xrightarrow{\;F\;} & \mathcal{D}' \\
\nearrow & \Nearrow & \downarrow^{G} \\
\mathcal{D} & & \\
\searrow_{F'} & & \mathcal{D}''
\end{array}
$$

Then we have a functor $\partial_- : \mathbf{Cat}^{\mathrm{op}} \to \mathcal{D} \Downarrow \mathbf{Der}$, taking categories A to ∂_A and functors u to ∂_u.

Proof. It is easy to check that forming the twisted arrow category preserves identities and composition. The result then follows by the functoriality of mates, as in [28, Section 2]. □

Lemma 3.1.7. *Suppose we have a natural transformation:*

$$
A \underset{v}{\overset{u}{\rightrightarrows}} B \quad \Downarrow \kappa
$$

For any derivator $\mathcal{D}$, any category C, and any $X \in \mathcal{D}(C)$ we have a commutative square:

$$
\begin{array}{ccc}
\partial_A X & \xrightarrow{\;\partial_u X\;} & (u^{\mathrm{op}} \times u)^* \partial_B X \\
\downarrow{\scriptstyle \partial_v X} & & \downarrow{\scriptstyle (u^{\mathrm{op}} \times \kappa)^*_{\partial_B X}} \\
(v^{\mathrm{op}} \times v)^* \partial_B X & \xrightarrow[(\kappa^{\mathrm{op}} \times v)^*_{\partial_B X}]{} & (u^{\mathrm{op}} \times v)^* \partial_B X
\end{array}
$$

Proof. Given the natural transformation κ, we can form a functor $\mathrm{tw}(u,v) : \mathrm{tw}(A) \to \mathrm{tw}(B)$ given by:

$$
\mathrm{tw}(A) \longrightarrow \mathrm{tw}(B)
$$

$$
\begin{array}{ccc}
a & & u(a) \\
\downarrow{\scriptstyle f} & \longmapsto & \downarrow{\scriptstyle u(f)} \\
 & & u(b) \\
\downarrow & & \downarrow{\scriptstyle \kappa_b} \\
b & & v(b)
\end{array}
$$

This makes the diagram below commute:

$$
\begin{array}{ccc}
\mathrm{tw}(A) & \xrightarrow{\;\mathrm{tw}(u,v)\;} & \mathrm{tw}(B) \\
\downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\
A^{\mathrm{op}} \times A & \xrightarrow[u^{\mathrm{op}} \times v]{} & B^{\mathrm{op}} \times B
\end{array}
$$

Moreover, we have natural transformations $\text{tw}(u, \kappa) : \text{tw}(u) \Rightarrow \text{tw}(u, v)$ and $\text{tw}(\kappa, v) : \text{tw}(v) \Rightarrow \text{tw}(u, v)$, which satisfy the following equalities:

$$
\begin{array}{ccc}
\begin{array}{c}
\text{tw}(A) \xRightarrow{\text{tw}(u,\kappa)} \text{tw}(B) \\
\text{(s,t)} \downarrow \qquad \qquad \downarrow \text{(s,t)} \\
A^{\text{op}} \times A \xrightarrow{u^{\text{op}} \times v} B^{\text{op}} \times B
\end{array}
& = &
\begin{array}{c}
\text{tw}(A) \xrightarrow{\text{tw}(u)} \text{tw}(B) \\
\text{(s,t)} \downarrow \qquad \qquad \downarrow \text{(s,t)} \\
A^{\text{op}} \times A \xRightarrow{u^{\text{op}} \times \kappa} B^{\text{op}} \times B
\end{array}
\end{array}
$$

(top-left diagram: $\text{tw}(A) \xrightarrow{\text{tw}(u)} \text{tw}(B)$ over $\text{tw}(u,v)$ with modification $\text{tw}(u,\kappa)$; arrows $u^{\text{op}} \times v$)

(top-right diagram: $u^{\text{op}} \times u$ over $u^{\text{op}} \times v$ with modification $u^{\text{op}} \times \kappa$)

$$
\begin{array}{ccc}
\begin{array}{c}
\text{tw}(A) \xRightarrow{\text{tw}(\kappa,v)} \text{tw}(B) \\
\text{(s,t)} \downarrow \qquad \qquad \downarrow \text{(s,t)} \\
A^{\text{op}} \times A \xrightarrow{u^{\text{op}} \times v} B^{\text{op}} \times B
\end{array}
& = &
\begin{array}{c}
\text{tw}(A) \xrightarrow{\text{tw}(v)} \text{tw}(B) \\
\text{(s,t)} \downarrow \qquad \qquad \downarrow \text{(s,t)} \\
A^{\text{op}} \times A \xRightarrow{\kappa^{\text{op}} \times v} B^{\text{op}} \times B
\end{array}
\end{array}
$$

(bottom-left diagram: $\text{tw}(v)$ over $\text{tw}(u,v)$ with modification $\text{tw}(\kappa,v)$)

(bottom-right diagram: $v^{\text{op}} \times v$ over $u^{\text{op}} \times v$ with modification $\kappa^{\text{op}} \times v$)

These squares give rise to modifications, as in Definition 2.1.14. Pasting these to the modification

$$
\begin{array}{ccc}
\begin{array}{c}
\mathcal{D} \xrightarrow{p^*} \mathcal{D}^{\text{tw}(B)} \\
\text{tw}(u)^* \xRightarrow{\text{tw}(u,\kappa)^*} \text{tw}(u,v)^* \\
\mathcal{D} \xrightarrow{p^*} \mathcal{D}^{\text{tw}(A)}
\end{array}
& = &
\begin{array}{c}
\mathcal{D} \xrightarrow{p^*} \mathcal{D}^{\text{tw}(B)} \\
\text{tw}(v)^* \xRightarrow{\text{tw}(\kappa,v)^*} \text{tw}(u,v)^* \\
\mathcal{D} \xrightarrow{p^*} \mathcal{D}^{\text{tw}(A)}
\end{array}
\end{array}
$$

gives the following equality:

This is precisely the equality we require. □

Lemma 3.1.8. *For any category* A *and any derivator* $\mathcal{D}$*, there is a canonical isomorphism:*

Proof. We have an isomorphism $\mathrm{tw}(A^{\mathrm{op}}) \xrightarrow{\cong} \mathrm{tw}(A)$ that makes the diagram below commute:

This induces the required isomorphism:

□

Lemma 3.1.9 (Fubini Theorem for Derivators)**.** *For any categories* A *and* B*, and any derivator* $\mathcal{D}$*, there are canonical isomorphisms:*

$$\begin{array}{ccc}
\mathcal{D}^{A^{op} \times A \times B^{op} \times B} & & \\
\cong \downarrow & \cong \Rightarrow & \int^A \int^B \\
\mathcal{D}^{(A \times B)^{op} \times (A \times B)} & \xrightarrow{\int^{A \times B}} & \mathcal{D} \\
\cong \uparrow & \cong \Rightarrow & \\
\mathcal{D}^{B^{op} \times B \times A^{op} \times A} & & \int^B \int^A
\end{array}$$

Proof. We have an isomorphism $\mathrm{tw}(A \times B) \xrightarrow{\cong} \mathrm{tw}(A) \times \mathrm{tw}(B)$, which makes the diagram below commute:

$$\begin{array}{ccc}
\mathrm{tw}(A \times B) & \xrightarrow{\cong} & \mathrm{tw}(A) \times \mathrm{tw}(B) \\
{\scriptstyle (s,t)} \downarrow & & \downarrow {\scriptstyle (s,t) \times (s,t)} \\
(A \times B)^{op} \times (A \times B) & \xrightarrow{\cong} & A^{op} \times A \times B^{op} \times B
\end{array}$$

As in the proof of Lemma 3.1.8, this induces the desired isomorphisms. $\qquad\square$

Notation 3.1.10. In light of Lemma 3.1.8 and Lemma 3.1.9, we will suppress instances of the symmetry isomorphism $\sigma^* : \mathcal{D}(A \times B) \xrightarrow{\cong} \mathcal{D}(B \times A)$ from our notation as much as possible. This will be convenient when there are multiple parameters indexing a shifted derivator: for example, we may write $\int^A : \mathcal{D}^{A \times B \times A^{op}} \to \mathcal{D}^B$ without ambiguity.

On the other hand, if multiple copies of the same category appear in the index of a shifted derivator, we will often use different subscripts to denote the same category. For example, if we denote $A = A_1 = A_2 = A_3$ then the notation $\int^{A_{1,3}} : \mathcal{D}^{A_1^{op} \times A_2 \times A_3} \to \mathcal{D}^{A_2}$ is unambiguous.

Remark 3.1.11. Using the pasting properties of mates, it is easy to show that any derivator map $F : \mathcal{D}_1 \to \mathcal{D}_2$ respects the isomorphism in Lemma 3.1.9. That is, given any object $X \in \mathcal{D}_1(A^{op} \times A \times B^{op} \times B)$, the diagram below commutes:

$$\int^{A} \int^{B} FX \xrightarrow{\cong} \int^{A \times B} FX$$

$$\int^{A} F \int^{B} X$$

$$F \int^{A} \int^{B} X \xrightarrow{\cong} F \int^{A \times B} X$$

The vertical maps are the canonical morphisms, as in Definition 3.1.2.

The following lemma is technical, but the isomorphism it provides is extremely important in the theory of monoidal derivators. In particular, it induces the unit isomorphisms in the bicategory of Remark 3.5.3 associated to any monoidal derivator.

Proposition 3.1.12. *For any derivator $\mathcal{D}$, and any category $A = A_i$, we have an isomorphism:*

Proof. We must show that the composite

$$\mathcal{D}^{A} \xrightarrow{(A \times p)^{*}} \mathcal{D}^{A \times \mathrm{tw}(A)} \xrightarrow{(A \times (s, t))_{!}} \mathcal{D}^{A \times A^{\mathrm{op}} \times A} \xrightarrow{((t^{\mathrm{op}}, s^{\mathrm{op}}) \times A)^{*}} \mathcal{D}^{\mathrm{tw}(A^{\mathrm{op}})^{\mathrm{op}} \times A} \xrightarrow{(p \times A)_{!}} \mathcal{D}^{A}$$

is isomorphic to the identity.

This isomorphism is constructed in the proof of [13, Lemma B.1]. We will give an outline of the proof.

Define the category P_A to be the following pullback. In [13, Lemma B.1], this square is shown to be homotopy exact:

$$\begin{array}{ccc}
\mathrm{P_A} & \xrightarrow{\ \ m_A\ \ } & A \times \mathrm{tw}(A) \\
{\scriptstyle n_A}\downarrow & \quad\lrcorner \qquad \overset{id}{\Longleftarrow} & \downarrow{\scriptstyle A\times(s,t)} \\
\mathrm{tw}(A^{\mathrm{op}})^{\mathrm{op}} \times A & \xrightarrow[\ (t^{\mathrm{op}},s^{\mathrm{op}})\times A\ ]{} & A \times A^{\mathrm{op}} \times A
\end{array}$$

The category $\mathrm{P_A}$ has objects given by pairs of composable maps $a \xrightarrow{g} b \xrightarrow{f} c$ in A, with a morphism from $a_1 \xrightarrow{g_1} b_1 \xrightarrow{f_1} c_1$ to $a_2 \xrightarrow{g_2} b_2 \xrightarrow{f_2} c_2$ consisting of a commutative diagram:

$$\begin{array}{ccc}
a_1 & \xrightarrow{\ \alpha\ } & a_2 \\
{\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\
b_1 & \xleftarrow{\ \beta\ } & b_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
c_1 & \xrightarrow{\ \gamma\ } & c_2
\end{array}$$

The functor m_A takes $a \xrightarrow{g} b \xrightarrow{f} c$ to the pair $(a, b \xrightarrow{f} c)$, and n_A takes $a \xrightarrow{g} b \xrightarrow{f} c$ to $(a \xrightarrow{g} b, c)$. There is a natural transformation

$$\begin{array}{ccccc}
\mathrm{P_A} & \xrightarrow{\ \ m_A\ \ } & A \times \mathrm{tw}(A) & \xrightarrow{\ A\times p\ } & A \\
{\scriptstyle n_A}\downarrow & & & & \| \\
\mathrm{tw}(A^{\mathrm{op}})^{\mathrm{op}} \times A & & \overset{\vartheta_A}{\Longleftarrow} & & \| \\
{\scriptstyle p\times A}\downarrow & & & & \| \\
A & =\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= & & & A
\end{array}$$

with component at the object $a \xrightarrow{g} b \xrightarrow{f} c$ given by $a \xrightarrow{f\circ g} c$. In [13, Lemma B.1], this square is also shown to be homotopy exact, so we obtain a canonical isomorphism:

$$\mathcal{D}^A \xrightarrow{(A\times p)^*} \mathcal{D}^{A\times tw(A)} \xrightarrow{(A\times(s,\,t))_!} \mathcal{D}^{A\times A^{op}\times A} \xrightarrow{((t^{op},\,s^{op})\times A)^*} \mathcal{D}^{tw(A^{op})^{op}\times A} \xrightarrow{(p\times A)_!} \mathcal{D}^A$$

$$\cong$$

$$\mathcal{D}^{P_A}$$

with m^*, $n_!$, $\cong$

Remark 3.1.13. Given any category $A = A_i$, the pasting diagrams below are equal, for any derivator $\mathcal{D}$:

$$\mathcal{D}^{A_1^{op}\times A_2} \xrightarrow{\partial_{A_{3,4}}} \mathcal{D}^{A_1^{op}\times A_2\times A_3^{op}\times A_4} \xrightarrow{\int^{A_{1,4}}} \mathcal{D}^{A_3^{op}\times A_2} \xrightarrow{\int^{A_{3,2}}} \mathcal{D}$$

with $\cong$

$$\mathcal{D}^{A_1^{op}\times A_2} \xrightarrow{\partial_{A_{3,4}}} \mathcal{D}^{A_1^{op}\times A_2\times A_3^{op}\times A_4} \xrightarrow{\int^{A_{1,4}}} \mathcal{D}^{A_3^{op}\times A_2} \xrightarrow{\int^{A_{3,2}}} \mathcal{D}$$

$$\int^{A_{3,2}}, \quad \mathcal{D}^{A_1^{op}\times A_4} \xrightarrow{\int^{A_{1,4}}} \mathcal{D}$$

with $\cong$

This is shown in the proof of [13, Lemma B.5]. Dually, the pasting diagrams below are also equal:

$$\mathcal{D} \xrightarrow{\partial_{A_{1,2}}} \mathcal{D}^{A_1^{op}\times A_2} \xrightarrow{\partial_{A_{3,4}}} \mathcal{D}^{A_1^{op}\times A_2\times A_3^{op}\times A_4} \xrightarrow{\int^{A_{1,4}}} \mathcal{D}^{A_3^{op}\times A_2}$$

with $\cong$

$$\begin{array}{ccc}
& \mathcal{D}^{A_1^{op} \times A_2} & \xrightarrow{\partial_{A_{3,4}}} & \\
\partial_{A_{1,2}} \nearrow & \Big\downarrow \cong & \mathcal{D}^{A_1^{op} \times A_2 \times A_3^{op} \times A_4} & \xrightarrow{\int^{A_{1,4}}} \\
\mathcal{D} \xrightarrow{\partial_{A_{3,4}}} \mathcal{D}^{A_3^{op} \times A_4} & \xrightarrow{\partial_{A_{1,2}}} & \Big\downarrow \cong & \mathcal{D}^{A_3^{op} \times A_2}
\end{array}$$
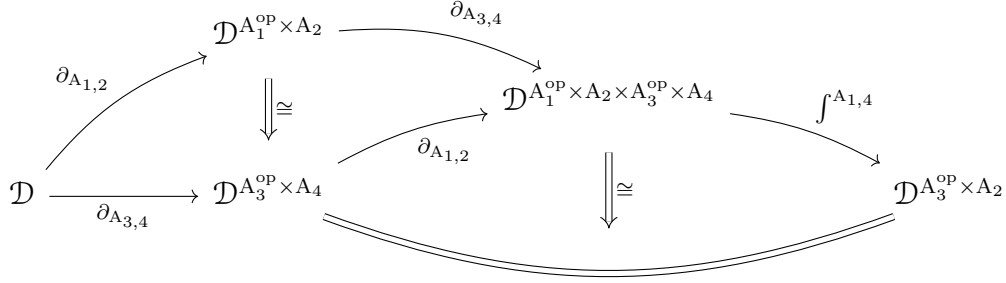
Remark 3.1.14. Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a derivator map. Using Lemma 2.2.4, it follows that F respects the isomorphism of Proposition 3.1.12. That is, for any category $A = A_i$, and any $X \in \mathcal{D}_1(A_1)$, the diagram below commutes, where the horizontal maps are the canonical morphisms of Definition 3.1.2:

$$\begin{array}{ccc}
\int^{A_{1,2}} \partial_{A_{2,3}} FX & \longrightarrow \int^{A_{1,2}} F\partial_{A_{2,3}} X & \longrightarrow F \int^{A_{1,2}} \partial_{A_{2,3}} X \\
& \cong \searrow \quad \xrightarrow{} FX \xleftarrow{} \quad \cong &
\end{array}$$

3.2 Two-variable adjunctions

In this section, we recall the definition and some basic properties of two-variable adjunctions between derivators, which were introduced and studied in [13]. Our presentation differs slightly from [13]. In particular, we begin with Theorem 3.2.2, which collects all of the structure present in a derivator two-variable adjunction, and then point out that this structure is uniquely determined by far less. This inverts the approach of [13], but has the advantage of fixing notation from the outset.

Before we give the derivator analogue, we recall the definition of a two-variable left adjoint between categories. Given categories $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, a functor $\otimes : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ is called a **two-variable left adjoint** if, for each $x \in \mathcal{C}_1$ and each $y \in \mathcal{C}_2$, the functors $x \otimes - : \mathcal{C}_2 \to \mathcal{C}_3$ and $- \otimes y : \mathcal{C}_1 \to \mathcal{C}_3$ have right adjoints. Equivalently, there are functors

$$\triangleright : \mathcal{C}_2^{op} \times \mathcal{C}_3 \to \mathcal{C}_1$$
$$\triangleleft : \mathcal{C}_3 \times \mathcal{C}_1^{op} \to \mathcal{C}_2$$

and natural isomorphisms:

$$\mathcal{C}_1(x, y \triangleright z) \cong \mathcal{C}_3(x \otimes y, z) \cong \mathcal{C}_2(y, z \triangleleft x)$$

We now give the derivator analogue. Note that the following is not the definition given in [13], but is equivalent to it by [13, Lemma 8.8].

Definition 3.2.1. We call a derivator map $\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$ a **two-variable left adjoint** if, for any categories A and B, and any $X \in \mathcal{D}_1(A)$ and $Y \in \mathcal{D}_2(B)$, the derivator maps

$$X \widetilde{\otimes} - : \mathcal{D}_2 \to \mathcal{D}_3^A$$

$$- \widetilde{\otimes} Y : \mathcal{D}_1 \to \mathcal{D}_3^B$$

have right adjoints.

The following theorem highlights the importance of Proposition 3.1.12. We will use this theorem repeatedly, especially in subsequent sections of Chapter 3.

Theorem 3.2.2. *Suppose we have a cocontinuous two-variable map*

$$\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$$

and two continuous derivator maps

$$\rhd : \mathcal{D}_2^{\mathrm{op}} \times \mathcal{D}_3 \to \mathcal{D}_1$$
$$\lhd : \mathcal{D}_3 \times \mathcal{D}_1^{\mathrm{op}} \to \mathcal{D}_2.$$

The following are equivalent:

1. *For any $X \in \mathcal{D}_1(A)$, the map $X \widetilde{\otimes} - : \mathcal{D}_2 \to \mathcal{D}_3^A$ has a right adjoint given by*

$$\mathcal{D}_3^A \xrightarrow{- \widetilde{\lhd} X} \mathcal{D}_2^{A \times A^{\mathrm{op}}} \xrightarrow{\int_{A^{\mathrm{op}}}} \mathcal{D}_2$$

   and for any $Y \in \mathcal{D}_2(B)$, the map $- \widetilde{\otimes} Y : \mathcal{D}_1 \to \mathcal{D}_3^B$ has a right adjoint given by

$$\mathcal{D}_3^B \xrightarrow{Y \widetilde{\rhd} -} \mathcal{D}_1^{B^{\mathrm{op}} \times B} \xrightarrow{\int_B} \mathcal{D}_1.$$

2. *For any $X \in \mathcal{D}_1(A)$, the map $- \widetilde{\lhd} X : \mathcal{D}_3 \to \mathcal{D}_2^{A^{\mathrm{op}}}$ has a left adjoint given by*

$$\mathcal{D}_2^{A^{\mathrm{op}}} \xrightarrow{X \widetilde{\otimes} -} \mathcal{D}_3^{A \times A^{\mathrm{op}}} \xrightarrow{\int^{A^{\mathrm{op}}}} \mathcal{D}_3$$

and for any $Z \in \mathcal{D}_3(C)$, the map $Z \widetilde{\lhd^{\mathrm{op}}} - : \mathcal{D}_1 \to (\mathcal{D}_2^{\mathrm{op}})^{C^{\mathrm{op}}}$ has a right adjoint given by

$$(\mathcal{D}_2^{\mathrm{op}})^{C^{\mathrm{op}}} \xrightarrow{\; - \widetilde{\rhd} Z \;} \mathcal{D}_1^{C^{\mathrm{op}} \times C} \xrightarrow{\int_C} \mathcal{D}_1.$$

3. *For any $Y \in \mathcal{D}_2(B)$, the map $Y \widetilde{\rhd} - : \mathcal{D}_3 \to \mathcal{D}_1^{B^{\mathrm{op}}}$ has a left adjoint given by*

$$\mathcal{D}_1^{B^{\mathrm{op}}} \xrightarrow{\; - \widetilde{\otimes} Y \;} \mathcal{D}_2^{B^{\mathrm{op}} \times B} \xrightarrow{\int^B} \mathcal{D}_3$$

and for any $Z \in \mathcal{D}_3(C)$, the map $- \widetilde{\rhd^{\mathrm{op}}} Z : \mathcal{D}_2 \to (\mathcal{D}_1^{\mathrm{op}})^{C^{\mathrm{op}}}$ has a right adjoint given by

$$(\mathcal{D}_1^{\mathrm{op}})^{C^{\mathrm{op}}} \xrightarrow{\; Z \widetilde{\lhd} - \;} \mathcal{D}_2^{C \times C^{\mathrm{op}}} \xrightarrow{\int_{C^{\mathrm{op}}}} \mathcal{D}_2.$$

Proof. Given maps $\otimes, \rhd$ and $\lhd$ as in the statement of the theorem, consider the maps below:

$$\lhd^{\mathrm{op}} : \mathcal{D}_3^{\mathrm{op}} \times \mathcal{D}_1 \to \mathcal{D}_2^{\mathrm{op}}$$
$$\otimes^{\mathrm{op}} : \mathcal{D}_1^{\mathrm{op}} \times \mathcal{D}_2^{\mathrm{op}} \to \mathcal{D}_3^{\mathrm{op}}$$
$$\rhd : \mathcal{D}_2^{\mathrm{op}} \times \mathcal{D}_3 \to \mathcal{D}_1$$

By Remark 2.3.3, $\lhd^{\mathrm{op}}$ is cocontinuous and $\otimes^{\mathrm{op}}$ is continuous, so this triple satisfies the hypotheses of the theorem. Condition (2) for the triple $(\otimes, \rhd, \lhd)$ is exactly condition (1) for the new triple $(\lhd^{\mathrm{op}}, \otimes^{\mathrm{op}}, \rhd)$, (3) for $(\otimes, \rhd, \lhd)$ is (2), and (1) for $(\otimes, \rhd, \lhd)$ is (3). Thus, to prove the theorem, it suffices to prove that (1) implies (2).

So, suppose we have maps $(\otimes, \rhd, \lhd)$ as in the statement of the theorem, that satisfy condition (1). Given $X \in \mathcal{D}(A)$, consider the map

$$\mathcal{D}_2^{A^{\mathrm{op}}} \xrightarrow{\; X \widetilde{\otimes} - \;} \mathcal{D}_3^{A \times A^{\mathrm{op}}} \xrightarrow{\int^{A^{\mathrm{op}}}} \mathcal{D}_3.$$

By (1), this has a right adjoint given by the composite

$$\mathcal{D}_3 \xrightarrow{\; \partial^{A_{2,3}^{\mathrm{op}}} \;} \mathcal{D}_3^{A_2 \times A_3^{\mathrm{op}}} \xrightarrow{\; - \widetilde{\lhd} X \;} \mathcal{D}_2^{A_2 \times A_3^{\mathrm{op}} \times A_1^{\mathrm{op}}} \xrightarrow{\int_{A_{1,2}^{\mathrm{op}}}} \mathcal{D}_2^{A_3^{\mathrm{op}}},$$

where $A = A_1 = A_2 = A_3$ as in Notation 3.1.10. Since $\lhd$ is continuous, $- \widetilde{\lhd} X$ commutes with $\partial^{A^{\mathrm{op}}}$, so the composite above is isomorphic to:

$$\mathcal{D}_3 \xrightarrow{\; - \widetilde{\lhd} X \;} \mathcal{D}_2^{A_1^{\mathrm{op}}} \xrightarrow{\; \partial^{A_{2,3}^{\mathrm{op}}} \;} \mathcal{D}_2^{A_1^{\mathrm{op}} \times A_2 \times A_3^{\mathrm{op}}} \xrightarrow{\int_{A_{1,2}^{\mathrm{op}}}} \mathcal{D}_2^{A_3^{\mathrm{op}}}$$

By Proposition 3.1.12, this composite is isomorphic to $- \widetilde{\triangleleft} X$.

Let $Z \in \mathcal{D}_3(\mathrm{C})$, and consider the map $Z \widetilde{\widetilde{\triangleleft}^{\mathrm{op}}} - : \mathcal{D}_1 \to (\mathcal{D}_2^{\mathrm{op}})^{\mathrm{C}^{\mathrm{op}}}$. The component of this map at A is the opposite of the component of $Z \widetilde{\triangleleft} - : \mathcal{D}_1^{\mathrm{op}} \to \mathcal{D}_2^{\mathrm{C}}$ at A^{op}:

$$Z \widetilde{\triangleleft} - : \mathcal{D}_1(\mathrm{A})^{\mathrm{op}} \to \mathcal{D}_2(\mathrm{C} \times \mathrm{A}^{\mathrm{op}})$$

Given $X \in \mathcal{D}_1(\mathrm{A})$ and $Y \in \mathcal{D}_2(\mathrm{C} \times \mathrm{A}^{\mathrm{op}})$, we have a string of natural isomorphisms:

$$\mathcal{D}_2(\mathrm{C} \times \mathrm{A}^{\mathrm{op}})(Y, Z \widetilde{\triangleleft} X) \cong \mathcal{D}_3(\mathrm{C})(\textstyle\int_{\mathrm{A}^{\mathrm{op}}} X \widetilde{\otimes} Y, Z)$$
$$\cong \mathcal{D}_3(\mathrm{C} \times \mathrm{A} \times \mathrm{A}^{\mathrm{op}})(X \widetilde{\otimes} Y, \partial^{\mathrm{A}^{\mathrm{op}}} Z)$$
$$\cong \mathcal{D}_1(\mathrm{A})(X, \textstyle\int_{\mathrm{C} \times \mathrm{A}^{\mathrm{op}}} Y \widetilde{\triangleright} \partial^{\mathrm{A}^{\mathrm{op}}} Z)$$

The first follows from the description of the left adjoint to $- \widetilde{\triangleleft} X$ which we have just proved; the second is by definition of $\partial^{\mathrm{A}^{\mathrm{op}}}$; the third follows from the description of the right adjoint to $- \widetilde{\otimes} Y$.

Since $\triangleright$ is continuous, $Y \widetilde{\triangleright} -$ commutes with $\partial^{\mathrm{A}^{\mathrm{op}}}$. Applying the Fubini theorem of Lemma 3.1.9 to the end $\int_{\mathrm{C} \times \mathrm{A}^{\mathrm{op}}}$, we can use Proposition 3.1.12 to cancel $\int_{\mathrm{A}^{\mathrm{op}}}$ with $\partial^{\mathrm{A}^{\mathrm{op}}}$. This leaves us with the desired description of the right adjoint to $Z \widetilde{\widetilde{\triangleleft}^{\mathrm{op}}} -$:

$$(\mathcal{D}_2^{\mathrm{op}})^{\mathrm{C}^{\mathrm{op}}} \xrightarrow{- \widetilde{\triangleright} Z} \mathcal{D}_1^{\mathrm{C}^{\mathrm{op}} \times \mathrm{C}} \xrightarrow{\int_{\mathrm{C}}} \mathcal{D}_1 \qquad\qquad \square$$

The theorem below is a consequence of the results in [13, Section 9]:

Theorem 3.2.3. *A map $\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$ is a two-variable left adjoint if and only if we can construct derivator maps*

$$\triangleright : \mathcal{D}_2^{\mathrm{op}} \times \mathcal{D}_3 \to \mathcal{D}_1$$
$$\triangleleft : \mathcal{D}_3 \times \mathcal{D}_1^{\mathrm{op}} \to \mathcal{D}_2$$

such that $\otimes$, $\triangleleft$ and $\triangleright$ satisfy the equivalent conditions of Theorem 3.2.2.

In light of Theorem 3.2.3, we will call a triple of derivator maps as in Theorem 3.2.2 a **two-variable adjunction**. We denote them by $(\otimes, \triangleright, \triangleleft) : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$.

It follows that Theorem 3.2.2 gives a complete description of the various adjunctions that arise from a 2-variable adjunction. We will use these descriptions repeatedly, especially

in Section 3.4 and Section 3.7.

Remark 3.2.4. Given a two-variable adjunction $(\otimes, \triangleright, \triangleleft) : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$, the proof of Theorem 3.2.2 implies that $(\triangleleft^{\mathrm{op}}, \otimes^{\mathrm{op}}, \triangleright) : \mathcal{D}_3^{\mathrm{op}} \times \mathcal{D}_1 \to \mathcal{D}_2^{\mathrm{op}}$ and $(\triangleright^{\mathrm{op}}, \triangleleft, \otimes^{\mathrm{op}}) : \mathcal{D}_2 \times \mathcal{D}_3^{\mathrm{op}} \to \mathcal{D}_1^{\mathrm{op}}$ are also two-variable adjunctions. These are the **cycled** two-variable adjunctions of [13, Section 9].

We finish this section with the definition of the cancelling variant of a two-variable derivator map:

Definition 3.2.5. Given a two-variable map $\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$, categories A and B, and objects $X \in \mathcal{D}_1(\mathrm{A}^{\mathrm{op}})$ and $Y \in \mathcal{D}_2(\mathrm{B})$, we will use the following notation for the **cancelling tensor product** that appears in Theorem 3.2.2:

$$X \otimes_{\mathrm{A}} - : \mathcal{D}_2^{\mathrm{A}} \xrightarrow{\; X \widetilde{\otimes} - \;} \mathcal{D}_3^{\mathrm{A}^{\mathrm{op}} \times \mathrm{A}} \xrightarrow{\; \int^{\mathrm{A}} \;} \mathcal{D}_3$$

$$- \otimes_{\mathrm{B}} Y : \mathcal{D}_1^{\mathrm{B}^{\mathrm{op}}} \xrightarrow{\; - \widetilde{\otimes} Y \;} \mathcal{D}_2^{\mathrm{B}^{\mathrm{op}} \times \mathrm{B}} \xrightarrow{\; \int^{\mathrm{B}} \;} \mathcal{D}_3$$

Together these maps induce a derivator map:

$$\otimes_{\mathrm{A}} : \mathcal{D}_1^{\mathrm{A}^{\mathrm{op}}} \times \mathcal{D}_2^{\mathrm{A}} \to \mathcal{D}_3$$

Suppose we have a functor $u : \mathrm{A} \to \mathrm{B}$. Pasting the isomorphism from Remark 2.3.1 to the map $\int^u$ of Definition 3.1.5 gives the following canonical modification:

3.3 Monoidal derivators

In this brief section we recall the definition of monoidal derivators, which can be found in [10, 13]. The definition is a direct analogue of the familiar definition of monoidal category.

Definition 3.3.1. Let $\mathcal{E}$ be a prederivator. We say $\mathcal{E}$ is a **monoidal prederivator** if it is equipped with a product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, and a unit object $\mathbb{1} \in \mathcal{E}([\mathbf{0}])$, together with isomorphisms:

$$
\begin{array}{ccc}
 & \xrightarrow{-\widetilde{\otimes}\mathbb{1}} & \\
\mathcal{E} & \cong \| \rho & \mathcal{E} \\
 & \cong \| \lambda & \\
 & \xrightarrow[\mathbb{1}\widetilde{\otimes}-]{} &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{E} \times \mathcal{E} \times \mathcal{E} & \xrightarrow{\mathcal{E}\times\otimes} & \mathcal{E} \times \mathcal{E} \\
\downarrow{\scriptstyle \otimes\times\mathcal{E}} & \overset{\alpha}{\underset{\cong}{\nearrow}} & \downarrow{\scriptstyle \otimes} \\
\mathcal{E} \times \mathcal{E} & \xrightarrow[\otimes]{} & \mathcal{E}
\end{array}
$$

These must satisfy the familiar coherence conditions, as in [10] and [27, Chapter 1.1]. We say that $\mathcal{E}$ is a **symmetric monoidal prederivator** if, in addition, we have a natural isomorphism

$$
\begin{array}{ccc}
\mathcal{E} \times \mathcal{E} & \xrightarrow[\cong]{\sigma} & \mathcal{E} \times \mathcal{E} \\
 & \underset{\tau}{\overset{\cong}{\Longleftarrow}} & \\
{\scriptstyle \otimes} \searrow & & \swarrow {\scriptstyle \otimes} \\
 & \mathcal{E} &
\end{array}
$$

where σ is the canonical twist. This map must also satisfy coherence conditions, as in [10] and [27, Chapter 1.4].

Remark 3.3.2. To give a monoidal structure on a prederivator $\mathcal{E}$ is equivalent to giving a factorisation of the 2-functor $\mathcal{E} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ through the 2-category of monoidal categories. Similarly, a symmetric monoidal structure on $\mathcal{E}$ corresponds to a factorisation through the 2-category of symmetric monoidal categories. See [10] for details.

Definition 3.3.3. Let $\mathcal{E}$ be a derivator, equipped with the structure of a monoidal prederivator. We say that $\mathcal{E}$ is a **monoidal derivator** if the product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is cocontinuous. If, in addition, the product is a two-variable left adjoint, we call $\mathcal{E}$ a **closed monoidal derivator**.

If the underlying monoidal structure is symmetric, then we call $\mathcal{E}$ a **symmetric monoidal derivator**.

Example 3.3.4. If $\mathcal{E}$ is a monoidal derivator, then so is the shifted derivator $\mathcal{E}^A$, for any category A. The product $\otimes : \mathcal{E}^A \times \mathcal{E}^A \to \mathcal{E}^A$ is given by the shifted product on $\mathcal{E}$. See [13, Example 3.24].

Example 3.3.5. For any monoidal model category $\mathcal{M}$, the associated derivator $\mathcal{H}o(\mathcal{M})$ is a closed monoidal derivator. See [13, Example 3.23] and [3, Proposition 6.1]. In particular, the derivator of spaces $\mathcal{H}o(\mathbf{sSet})$ is monoidal, as is the derivator of spectra $\mathcal{H}o(\mathbf{Spt})$. Note that the monoidal structure on $\mathcal{H}o(\mathbf{Spt})$ can be constructed directly from the monoidal structure on $\mathcal{H}o(\mathbf{sSet})$, without going via a monoidal model category of spectra. See [20, Section 10] for this approach.

3.4 Closed actions of monoidal derivators

In this section we recall the definition of modules over a monoidal derivator $\mathcal{E}$, and give several examples. These are studied in [10, 13, 16]. Closed $\mathcal{E}$-modules — that is, those for which the $\mathcal{E}$-action is a two-variable left adjoint — are of particular importance to us. These provide archetypal examples of $\mathcal{E}$-prederivators, and we will revisit them repeatedly in Chapter 4 and Chapter 5. We recall the definition in Definition 3.4.7.

Definition 3.4.1. Let $\mathcal{E}$ be a monoidal derivator. A **(left) action** of $\mathcal{E}$ on a derivator $\mathcal{D}$ is given by a cocontinuous map $\otimes : \mathcal{E} \times \mathcal{D} \to \mathcal{D}$, together with isomorphisms:

These must satisfy coherence conditions, as in [10] and [24, Section 1], which are analogous to those satisfied by a monoidal product. If $\mathcal{D}$ is equipped with an $\mathcal{E}$-action, we call $\mathcal{D}$ an **$\mathcal{E}$-module**.

Example 3.4.2. Any symmetric monoidal derivator $\mathcal{E}$ is a left (and right) module over itself.

Definition 3.4.3. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be $\mathcal{E}$-modules. A derivator map $F : \mathcal{D}_1 \to \mathcal{D}_2$ is called an **$\mathcal{E}$-module morphism**, or is said to **preserve tensors**, if it is equipped with an isomorphism:

$$\begin{array}{ccc}
\mathcal{E} \times \mathcal{D}_1 & \xrightarrow{\;\mathcal{E} \times F\;} & \mathcal{E} \times \mathcal{D}_2 \\
\Big\downarrow{\scriptstyle \otimes} & \quad \varphi \quad \cong & \Big\downarrow{\scriptstyle \otimes} \\
\mathcal{D}_1 & \xrightarrow{\quad F \quad} & \mathcal{D}_2
\end{array}$$

This isomorphism must satisfy the following coherence conditions, as in [10]:

1. For any category A, and any $X \in \mathcal{D}_1(A)$ and $Y, Z \in \mathcal{E}(A)$, the diagram below must commute:

$$\begin{array}{ccc}
(Z \otimes Y) \otimes FX & \xrightarrow{\quad \alpha \quad} & Z \otimes (Y \otimes FX) \\
& & \Big\downarrow{\scriptstyle Z \otimes \varphi} \\
{\scriptstyle \varphi} \Big\downarrow & & Z \otimes F(Y \otimes X) \\
& & \Big\downarrow{\scriptstyle \varphi} \\
F((Z \otimes Y) \otimes X) & \xrightarrow{\quad F(\alpha) \quad} & F(Z \otimes (Y \otimes X))
\end{array}$$

2. For any $X \in \mathcal{D}_1(A)$, the diagram below must commute:

$$\begin{array}{ccc}
\mathbb{1} \widetilde{\otimes} F(X) & \xrightarrow{\quad \varphi \quad} & F(\mathbb{1} \widetilde{\otimes} X) \\
& {\scriptstyle \lambda} \searrow \qquad \swarrow {\scriptstyle F(\lambda)} & \\
& FX &
\end{array}$$

Definition 3.4.4. Let $F, G : \mathcal{D}_1 \to \mathcal{D}_2$ be $\mathcal{E}$-module morphisms. A modification $\beta : F \Rightarrow G$ is called an **$\mathcal{E}$-module modification**, or is said to **respect tensors**, if the diagram below commutes, for any category A, and any $X \in \mathcal{D}_1(A)$ and $Y \in \mathcal{E}(A)$:

$$\begin{array}{ccc}
Y \otimes F(X) & \xrightarrow{\;Y \otimes \beta_X\;} & Y \otimes G(X) \\
{\scriptstyle \varphi} \Big\downarrow & & \Big\downarrow {\scriptstyle \varphi} \\
F(Y \otimes X) & \xrightarrow{\;\beta_{Y \otimes X}\;} & G(Y \otimes X)
\end{array}$$

Definition 3.4.5. Given two $\mathcal{E}$-modules $\mathcal{D}_1$ and $\mathcal{D}_2$, let $\underline{\mathrm{Hom}}^{\mathcal{E}}(\mathcal{D}_1, \mathcal{D}_2)$ denote the category whose objects are $\mathcal{E}$-module maps from $\mathcal{D}_1$ to $\mathcal{D}_2$, and whose morphisms are $\mathcal{E}$-module modifications. Similarly, let $\underline{\mathrm{Hom}}_!^{\mathcal{E}}(\mathcal{D}_1, \mathcal{D}_2)$ denote the full subcategory of $\underline{\mathrm{Hom}}^{\mathcal{E}}(\mathcal{D}_1, \mathcal{D}_2)$ on the cocontinuous $\mathcal{E}$-module maps.

Example 3.4.6. Using the universal property of $\mathcal{H}\mathrm{o}(\mathbf{sSet})$ described in Theorem 2.2.13, we can see that any derivator $\mathcal{D}$ has a unique module structure over $\mathcal{H}\mathrm{o}(\mathbf{sSet})$. The action

$$\otimes : \mathcal{H}\mathrm{o}(\mathbf{sSet}) \times \mathcal{D} \to \mathcal{D}$$

is the map described at the end of Section 2.3; that is, it is the essentially unique cocontinuous map such that

$$\Delta^0 \;\widetilde{\otimes}\; - \cong \mathrm{id} : \mathcal{D} \to \mathcal{D}.$$

The coherent structure isomorphisms can be constructed using the universal property. By similar arguments, any cocontinuous derivator map $F : \mathcal{D}_1 \to \mathcal{D}_2$ is a $\mathcal{H}\mathrm{o}(\mathbf{sSet})$-module morphism.

Similarly, for any stable derivator $\mathcal{D}$, the map

$$\wedge : \mathcal{H}\mathrm{o}(\mathbf{Spt}) \times \mathcal{D} \to \mathcal{D}$$

described at the end of Section 2.4 is a unique $\mathcal{H}\mathrm{o}(\mathbf{Spt})$-action on $\mathcal{D}$, and any cocontinuous map between stable derivators is a $\mathcal{H}\mathrm{o}(\mathbf{Spt})$-module map.

Definition 3.4.7. An $\mathcal{E}$-module $\mathcal{D}$ is called a **closed $\mathcal{E}$-module** if the action $\otimes : \mathcal{E} \times \mathcal{D} \to \mathcal{D}$ is a two-variable left adjoint. In this case, we will denote the associated maps given by Theorem 3.2.3 as follows:

$$\mathrm{map}_{\mathcal{D}}(-,-) : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}$$

$$\lhd : \mathcal{D} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{D}$$

We refer to $\otimes$ as the **tensor product** and $\lhd$ as the **cotensor product** on $\mathcal{D}$. We will call $\mathrm{map}_{\mathcal{D}}(-,-)$ the **mapping space** or **mapping object** morphism for $\mathcal{D}$.

Example 3.4.8. Any closed symmetric monoidal derivator $\mathcal{E}$ is a closed module over itself. In this case, the cotensor product can be given in terms of the mapping space:

$$\lhd : \mathcal{E} \times \mathcal{E}^{\mathrm{op}} \xrightarrow[\cong]{\sigma} \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \xrightarrow{\mathrm{map}_{\mathcal{E}}(-,-)} \mathcal{E}$$

To verify this description of $\lhd$, we can apply Theorem 3.2.2 and check that the composite above has the desired properties. This is immediate once we observe that, for any category A and any $X \in \mathcal{E}(A)$, the symmetry isomorphism τ and the isomorphism of Lemma 3.1.8

induce an isomorphism between the cancelling products

$$X \otimes_{\mathrm{A}^{\mathrm{op}}} - : \mathcal{E}^{\mathrm{A}^{\mathrm{op}}} \to \mathcal{E}$$
$$- \otimes_{\mathrm{A}} X : \mathcal{E}^{\mathrm{A}^{\mathrm{op}}} \to \mathcal{E}$$

of Definition 3.2.5. This isomorphism is discussed in greater detail in Lemma 3.5.7.

Example 3.4.9. Let $\mathcal{D}$ be a triangulated derivator. Since $\mathcal{D}$ is stable, it has a unique $\mathcal{H}\mathrm{o}(\mathbf{Spt})$-module structure, by Example 3.4.6. Using the explicit construction of the action in [3, 6, 20], we can check that, for any category A and any $X \in \mathrm{Ho}(\mathbf{Spt}^{\mathrm{A}})$, the map

$$X \,\tilde{\wedge}\, - : \mathcal{D} \to \mathcal{D}^{\mathrm{A}}$$

has a right adjoint. On the other hand, for any category B and any $Y \in \mathcal{D}(\mathrm{B})$, since $\mathrm{Ho}(\mathbf{Spt})$ is compactly generated, Proposition 2.4.15 implies that the cocontinuous map

$$- \,\tilde{\wedge}\, Y : \mathcal{H}\mathrm{o}(\mathbf{Spt}) \to \mathcal{D}^{\mathrm{B}}$$

has a right adjoint. Thus, any triangulated derivator $\mathcal{D}$ has a unique closed $\mathcal{H}\mathrm{o}(\mathbf{Spt})$-module structure. See [5, Appendix A.3] and [15, Section 4.4] for more details.

Example 3.4.10. Let $\mathcal{M}$ be a monoidal model category and let $\mathcal{N}$ be an $\mathcal{M}$-enriched model category. See [18] for a discussion of enriched model categories. By [13, Example 3.23], the derivator $\mathcal{H}\mathrm{o}(\mathcal{N})$ is a closed $\mathcal{H}\mathrm{o}(\mathcal{M})$-module. See also [15, Examples 4.75].

Example 3.4.11. For any model category $\mathcal{M}$, by Example 3.4.6, the derivator $\mathcal{H}\mathrm{o}(\mathcal{M})$ is a $\mathcal{H}\mathrm{o}(\mathbf{sSet})$-module. It follows from [3, Section 6] that this action is closed. Mapping spaces can be constructed using cosimplicial frames, as in [23, Chapter 5].

Example 3.4.12. Given any $\mathcal{E}$-module $\mathcal{D}$, and any category A, the shifted derivator $\mathcal{D}^{\mathrm{A}}$ has an $\mathcal{E}$-module structure given by

$$\otimes : \mathcal{E} \times \mathcal{D}^{\mathrm{A}} \xrightarrow{\mathrm{p}^* \times \mathcal{D}^{\mathrm{A}}} \mathcal{E}^{\mathrm{A}} \times \mathcal{D}^{\mathrm{A}} \xrightarrow{\otimes} \mathcal{D}^{\mathrm{A}},$$

where $\otimes : \mathcal{E}^{\mathrm{A}} \times \mathcal{D}^{\mathrm{A}} \to \mathcal{D}^{\mathrm{A}}$ is the action on $\mathcal{D}$ shifted by A. Note that, in terms of the original action on $\mathcal{D}$, this map is simply the external tensor product

$$\tilde{\otimes} : \mathcal{E} \times \mathcal{D}^{\mathrm{A}} \to \mathcal{D}^{\mathrm{A}}.$$

The maps α and λ are inherited from $\mathcal{D}$.

Moreover, if $\mathcal{D}$ is a closed $\mathcal{E}$-module, then so is $\mathcal{D}^A$, by [13, Example 8.15]. The associated maps from Theorem 3.2.3 are given by

$$\mathrm{map}_{\mathcal{D}^A}(-,-) : (\mathcal{D}^{\mathrm{op}})^{A^{\mathrm{op}}} \times \mathcal{D}^A \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{D}}(-,-)} \mathcal{E}^{A^{\mathrm{op}} \times A} \xrightarrow{\int_A} \mathcal{E}$$

$$\lhd : \mathcal{D}^A \times \mathcal{E}^{\mathrm{op}} \xrightarrow{\mathcal{D}^A \times \mathrm{p}^*} \mathcal{D}^A \times (\mathcal{E}^{\mathrm{op}})^A \xrightarrow{\lhd} \mathcal{D}^A,$$

where $\lhd : \mathcal{D}^A \times (\mathcal{E}^{\mathrm{op}})^A \to \mathcal{D}^A$ is the shifted map associated to $\mathcal{D}$. To see this directly, apply Theorem 3.2.2 to these three maps.

Lemma 3.4.13. *Let $\mathcal{D}$ be an $\mathcal{E}$-module and let $u : A \to B$ be a functor. Then the derivator map*

$$u^* : \mathcal{D}^B \to \mathcal{D}^A$$

has a canonical $\mathcal{E}$-module morphism structure. Moreover, for any natural transformation $\kappa : u \Rightarrow v$, the modification $\kappa^* : u^* \Rightarrow v^*$ is an $\mathcal{E}$-module modification.

Proof. Using the description from Example 3.4.12 of the $\mathcal{E}$-action on the shifted derivator as an external tensor product, we require an isomorphism:

$$
\begin{array}{ccc}
\mathcal{E} \times \mathcal{D}^B & \xrightarrow{\mathcal{E} \times u^*} & \mathcal{E} \times \mathcal{D}^A \\
\widetilde{\otimes} \downarrow & \overset{\varphi}{\underset{\cong}{\Longleftarrow}} & \downarrow \widetilde{\otimes} \\
\mathcal{D}^B & \xrightarrow{u^*} & \mathcal{D}^A
\end{array}
$$

We take the isomorphism of Remark 2.3.1 to be this structure isomorphism. The coherence conditions of Definition 3.4.3 follow from the fact that the structure isomorphisms α and λ for the $\mathcal{E}$-module structure on $\mathcal{D}$ are modifications.

Given a natural transformation $\kappa : u \Rightarrow v$, we need to check that $\kappa^* : u^* \Rightarrow v^*$ satisfies Definition 3.4.4. This follows from Axiom 3 of Definition 2.1.2, for the prederivator map $\otimes : \mathcal{E} \times \mathcal{D} \to \mathcal{D}$. $\qquad\square$

3.5   The cancelling tensor product

In this section, building on work in [13], we revisit the cancelling tensor product of Definition 3.2.5, associated to any $\mathcal{E}$-module. In Proposition 3.5.2, we recall that this product is associative and unital, with coherence induced by the original action. In particular, for any monoidal derivator $\mathcal{E}$, we obtain a bicategory $\mathrm{Prof}(\mathcal{E})$, which plays a vital role as the enriching bicategory in Chapter 4 and Chapter 5. For basic bicategory definitions see [32]. We conclude this section by showing that cocontinuous $\mathcal{E}$-module maps and $\mathcal{E}$-module modifications respect the cancelling tensor. The results in this section, in particular the list of commutative diagrams that we collect here, will be used as we develop the theory of $\mathcal{E}$-categories in Chapter 4.

Definition 3.5.1. Let $\mathcal{E}$ be a monoidal derivator. For any category A, consider the map

$$\partial_A : \mathcal{E}([\mathbf{0}]) \to \mathcal{E}(A^{\mathrm{op}} \times A)$$

of Definition 3.1.1. Denote the image of $\mathbb{1} \in \mathcal{E}([\mathbf{0}])$ under this map by $h_A \in \mathcal{E}(A^{\mathrm{op}} \times A)$. In [15], this object is called the **Yoneda bimodule** or the **identity profunctor**. Similarly, given a functor $u : A \to B$, consider the modification ∂_u of Definition 3.1.5. Denote the component of this map at $\mathbb{1}$ by

$$h_u : h_A \to (u^{\mathrm{op}} \times u)^* h_B$$

in $\mathcal{E}(A^{\mathrm{op}} \times A)$.

Proposition 3.5.2. *Let $\mathcal{E}$ be a monoidal derivator. For any categories* A *and* B *and any object* $X \in \mathcal{E}(A^{\mathrm{op}} \times B)$, *the unit isomorphisms* λ *and* ρ *of Definition 3.3.1 induce isomorphisms:*

$$\lambda : h_B \otimes_B X \xrightarrow{\cong} X$$
$$\rho : X \otimes_A h_A \xrightarrow{\cong} X$$

Similarly, given additional categories C *and* D, *and* $Y \in \mathcal{E}(B^{\mathrm{op}} \times C)$ *and* $Z \in \mathcal{E}(C^{\mathrm{op}} \times D)$, *the associativity isomorphism* α *induces an isomorphism*

$$\alpha : (Z \otimes_C Y) \otimes_B X \xrightarrow{\cong} Z \otimes_C (Y \otimes_B X)$$

in $\mathcal{E}(A^{\mathrm{op}} \times D)$. *More generally, there are analogous isomorphisms* α *and* λ *for any given*

$\mathcal{E}$-module $\mathcal{D}$.

These satisfy the following coherence conditions:

1. *Given categories* A, B, C, D *and* E, *and objects* $X \in \mathcal{D}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$, $Y \in \mathcal{E}(\mathrm{B}^{\mathrm{op}} \times \mathrm{C})$, $Z \in \mathcal{E}(\mathrm{C}^{\mathrm{op}} \times \mathrm{D})$ *and* $W \in \mathcal{E}(\mathrm{D}^{\mathrm{op}} \times \mathrm{E})$, *the diagram below commutes:*

$$
\begin{array}{ccc}
((W \otimes_{\mathrm{D}} Z) \otimes_{\mathrm{C}} Y) \otimes_{\mathrm{B}} X & \xrightarrow{\ \alpha\, \otimes_{\mathrm{B}} X\ } & (W \otimes_{\mathrm{D}} (Z \otimes_{\mathrm{C}} Y)) \otimes_{\mathrm{B}} X \\
\Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha} \\
(W \otimes_{\mathrm{D}} Z) \otimes_{\mathrm{C}} (Y \otimes_{\mathrm{B}} X) & & W \otimes_{\mathrm{D}} ((Z \otimes_{\mathrm{C}} Y) \otimes_{\mathrm{B}} X) \\
& \!\!\!\!\!\!\!\!\!\xrightarrow[\ \ \alpha\ \ ]{}\ \ W \otimes_{\mathrm{D}} (Z \otimes_{\mathrm{C}} (Y \otimes_{\mathrm{B}} X)) \xleftarrow[\ \ W \otimes_{\mathrm{D}} \alpha\ \ ]{} &
\end{array}
$$

2. *Given* $X \in \mathcal{D}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$ *and* $Y \in \mathcal{E}(\mathrm{B}^{\mathrm{op}} \times \mathrm{C})$, *the diagram below commutes:*

$$
\begin{array}{ccc}
(Y \otimes_{\mathrm{B}} \mathrm{h_B}) \otimes_{\mathrm{B}} X & \xrightarrow{\ \ \alpha\ \ } & Y \otimes_{\mathrm{B}} (\mathrm{h_B} \otimes_{\mathrm{B}} X) \\
& {\scriptstyle \rho\, \otimes_{\mathrm{B}} X}\searrow \quad \swarrow {\scriptstyle Y \otimes_{\mathrm{B}} \lambda} & \\
& Y \otimes_{\mathrm{B}} X &
\end{array}
$$

Proof. This is proved in [13, Theorem 5.9] in the case of $\mathcal{D} = \mathcal{E}$, and the proof in the general case carries over unchanged. We recall the construction of the maps α, λ and ρ.

Given $X \in \mathcal{D}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$, the map $\lambda : \mathrm{h_B} \otimes_{\mathrm{B}} X \xrightarrow{\ \cong\ } X$ is the component at X of the modification below:

Here we have $B = B_1 = B_2 = B_3$, as in Notation 3.1.10, and the unlabelled isomorphisms come from the cocontinuity of $\otimes$ and Proposition 3.1.12. For $X \in \mathcal{E}(A^{op} \times B)$, the isomorphism ρ is obtained similarly.

For $X \in \mathcal{D}(A^{op} \times B)$, $Y \in \mathcal{E}(B^{op} \times C)$ and $Z \in \mathcal{E}(C^{op} \times D)$, the associativity isomorphism from $(Z \otimes_C Y) \otimes_B X = \int^B(\int^C (Z \widetilde{\otimes} Y) \widetilde{\otimes} X)$ to $(Z \otimes_C Y) \otimes_B X = \int^C Z \widetilde{\otimes} \int^B (Y \widetilde{\otimes} X)$ is given by the following composite:

$$\int^B(\int^C(Z\widetilde{\otimes}Y)\widetilde{\otimes}X) \xrightarrow{\;\cong\;} \int^B\int^C(Z\widetilde{\otimes}Y)\widetilde{\otimes}X \xrightarrow{\;\cong\;} \int^C\int^B(Z\widetilde{\otimes}Y)\widetilde{\otimes}X$$
$$\downarrow{\scriptstyle\int^C\int^B \alpha}$$
$$\int^C\int^B Z\widetilde{\otimes}(Y\widetilde{\otimes}X)$$
$$\downarrow{\scriptstyle\cong}$$
$$\int^C Z\widetilde{\otimes}\int^B(Y\widetilde{\otimes}X)$$

The unlabelled isomorphisms follow by the cocontinuity of $\otimes$ and Lemma 3.1.9. Note that, just as λ and ρ are modifications, the map α above is the component of a modification between certain derivator maps. $\qquad\square$

Remark 3.5.3. Proposition 3.5.2 can be rephrased as follows, which is how it appears in [13, Theorem 5.9]. For any monoidal derivator $\mathcal{E}$, the following data forms a bicategory $\mathcal{P}rof(\mathcal{E})$:

- The objects of $\mathcal{P}rof(\mathcal{E})$ are small categories.

- Given categories A and B, the hom-category from A to B is $\mathcal{E}(B^{op} \times A)$.

- Composition is given by the cancelling product:

$$\otimes_B : \mathcal{E}(B^{op} \times A) \times \mathcal{E}(C^{op} \times B) \to \mathcal{E}(C^{op} \times A)$$

- For any category A, the identity on A is given by

$$h_A \in \mathcal{E}(A^{op} \times A).$$

We call $\mathcal{P}rof(\mathcal{E})$ the **bicategory of profunctors** associated to $\mathcal{E}$. This is the terminology of [13]; it is called the **bicategory of bimodules** in [15].

Notation 3.5.4. From this point onwards, we will take the convention that the cancelling tensor cancels the outside variables, as in Remark 3.5.3:

$$\otimes_B : \mathcal{E}(B^{\mathrm{op}} \times C) \times \mathcal{E}(A^{\mathrm{op}} \times B) \to \mathcal{E}(A^{\mathrm{op}} \times C)$$

Example 3.5.5. Given any monoidal derivator $\mathcal{E}$ and any category $A = A_i$, Proposition 3.5.2 implies that the shifted derivator $\mathcal{E}^{A^{\mathrm{op}} \times A}$ is monoidal, with unit given by $h_A \in \mathcal{E}(A^{\mathrm{op}} \times A)$, and tensor given by the cancelling tensor product:

$$\otimes_{A_{1,4}} : \mathcal{E}^{A_1^{\mathrm{op}} \times A_2} \times \mathcal{E}^{A_3^{\mathrm{op}} \times A_4} \to \mathcal{E}^{A_3^{\mathrm{op}} \times A_2}$$

If $\mathcal{E}$ is closed monoidal then so is $\mathcal{E}^{A^{\mathrm{op}} \times A}$. In this case, Theorem 3.2.2 provides a description of the required adjoints. Note, however, that if $\mathcal{E}$ is symmetric monoidal it does not follow that $\mathcal{E}^{A^{\mathrm{op}} \times A}$ is symmetric; see Lemma 3.5.7 for a description of the structure induced on shifted derivators by the symmetry isomorphism from $\mathcal{E}$.

Remark 3.5.6. In addition to the commutative diagrams of Proposition 3.5.2, for any $\mathcal{E}$-module $\mathcal{D}$, any categories A, B and C, and any objects $X \in \mathcal{D}(A^{\mathrm{op}} \times B)$ and $Y \in \mathcal{E}(B^{\mathrm{op}} \times C)$, the diagram below commutes:

In the case of $\mathcal{D} = \mathcal{E}$, in light of Remark 3.5.3, this is a special case of the coherence theorem for bicategories, as in [32]. In general, the commutativity of this diagram follows from the commutative diagrams in Proposition 3.5.2. The proof of this is essentially the same as the proof for monoidal categories in [26].

Lemma 3.5.7. *Let $\mathcal{E}$ be a symmetric monoidal derivator. For any categories A, B and C, and any $X \in \mathcal{E}(B \times A^{\mathrm{op}})$ and $Y \in \mathcal{E}(C \times B^{\mathrm{op}})$, the symmetry isomorphism of Definition 3.3.1 induces a map*

$$\tau : \sigma^* Y \otimes_B \sigma^* X \xrightarrow{\cong} \sigma^*(X \otimes_{B^{\mathrm{op}}} Y)$$

that satisfies the following coherence conditions:

1. *Given categories A, B, C and D, and objects $X \in \mathcal{E}(B \times A^{\mathrm{op}})$, $Y \in \mathcal{E}(C \times B^{\mathrm{op}})$ and $Z \in \mathcal{E}(D \times C^{\mathrm{op}})$, the diagram below commutes:*

$$(\sigma^*Z \otimes_{\mathrm{C}} \sigma^*Y) \otimes_{\mathrm{B}} \sigma^*X \xrightarrow{\ \alpha\ } \sigma^*Z \otimes_{\mathrm{C}} (\sigma^*Y \otimes_{\mathrm{B}} \sigma^*X)$$

$$\downarrow{\scriptstyle \tau \otimes_{\mathrm{B}} \sigma^*X} \qquad\qquad \downarrow{\scriptstyle \sigma^*Z \otimes_{\mathrm{C}} \tau}$$

$$\sigma^*(Y \otimes_{\mathrm{C}^{\mathrm{op}}} Z) \otimes_{\mathrm{B}} \sigma^*X \qquad\qquad \sigma^*Z \otimes_{\mathrm{C}} \sigma^*(X \otimes_{\mathrm{B}^{\mathrm{op}}} Y)$$

$$\downarrow{\scriptstyle \tau} \qquad\qquad \downarrow{\scriptstyle \tau}$$

$$\sigma^*(X \otimes_{\mathrm{B}^{\mathrm{op}}} (Y \otimes_{\mathrm{C}^{\mathrm{op}}} Z)) \xleftarrow{\ \sigma^*(\alpha)\ } \sigma^*((X \otimes_{\mathrm{B}^{\mathrm{op}}} Y) \otimes_{\mathrm{C}^{\mathrm{op}}} Z)$$

2. *Given $X \in \mathcal{E}(\mathrm{B} \times \mathrm{A}^{\mathrm{op}})$, the diagram below commutes*

$$\sigma^*X \otimes_{\mathrm{A}} \mathrm{h}_{\mathrm{A}} \xrightarrow{\ \cong\ } \sigma^*X \otimes_{\mathrm{A}} \sigma^*\mathrm{h}_{\mathrm{A}^{\mathrm{op}}}$$

$$\downarrow{\scriptstyle \rho} \qquad\qquad \downarrow{\scriptstyle \tau}$$

$$\sigma^*X \xleftarrow{\ \sigma^*(\lambda)\ } \sigma^*(\mathrm{h}_{\mathrm{A}^{\mathrm{op}}} \otimes_{\mathrm{A}^{\mathrm{op}}} X)$$

where the isomorphism along the top is induced by the canonical isomorphism

$$\mathrm{h}_{\mathrm{A}} \xrightarrow{\ \cong\ } \sigma^*\mathrm{h}_{\mathrm{A}^{\mathrm{op}}},$$

an instance of Lemma 3.1.8 for ∂_{A}.

3. *Given $X \in \mathcal{E}(\mathrm{B} \times \mathrm{A}^{\mathrm{op}})$, the diagram below commutes:*

$$\mathrm{h}_{\mathrm{B}} \otimes_{\mathrm{B}} \sigma^*X \xrightarrow{\ \cong\ } \sigma^*\mathrm{h}_{\mathrm{B}^{\mathrm{op}}} \otimes_{\mathrm{B}} \sigma^*X$$

$$\downarrow{\scriptstyle \lambda} \qquad\qquad \downarrow{\scriptstyle \tau}$$

$$\sigma^*X \xleftarrow{\ \sigma^*(\rho)\ } \sigma^*(X \otimes_{\mathrm{B}^{\mathrm{op}}} \mathrm{h}_{\mathrm{B}^{\mathrm{op}}})$$

Proof. For any categories J and K, the symmetry isomorphism τ induces an isomorphism:

$$\mathcal{E}^{\mathrm{J}} \times \mathcal{E}^{\mathrm{K}} \xrightarrow{\ \widetilde{\otimes}\ } \mathcal{E}^{\mathrm{J} \times \mathrm{K}}$$

$$\downarrow{\scriptstyle \sigma} \qquad {\scriptstyle \tau} \nearrow{\scriptstyle \cong} \qquad \downarrow{\scriptstyle \sigma^*}$$

$$\mathcal{E}^{\mathrm{K}} \times \mathcal{E}^{\mathrm{J}} \xrightarrow{\ \widetilde{\otimes}\ } \mathcal{E}^{\mathrm{K} \times \mathrm{J}}$$

Taking $K = J^{op}$, we get an induced map on the cancelling tensor products, using the isomorphism of Lemma 3.1.8:

The map we require can be obtained as a shifted version of this modification. The coherence conditions follow from the coherence conditions on the original symmetry isomorphism $\tau : \otimes \circ \sigma \Rightarrow \otimes$. $\qquad\square$

Remark 3.5.8. If $\mathcal{E}$ is a symmetric monoidal derivator, then Lemma 3.5.7 endows the maps σ^* with the structure of an isomorphism between the bicategory $\mathrm{Prof}(\mathcal{E})$ and its opposite $\mathrm{Prof}(\mathcal{E})^{op}$; that is, the bicategory with the same objects, and with hom-category from A to B given by $\mathcal{E}(A^{op} \times B)$. See [32] for basic bicategorical definitions.

Explicitly, this map takes each small category $A \in \mathrm{Prof}(\mathcal{E})$ to its opposite $A^{op} \in \mathrm{Prof}(\mathcal{E})^{op}$, and the action on hom-categories is given by:

$$\sigma^* : \mathcal{E}(B^{op} \times A) \xrightarrow{\cong} \mathcal{E}(A \times B^{op})$$

We will now turn our attention to the interaction of $\mathcal{E}$-module maps and $\mathcal{E}$-module modifications with the cancelling tensors.

Lemma 3.5.9. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be $\mathcal{E}$-modules, and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a cocontinuous $\mathcal{E}$-module morphism. Then for any categories A, B and C, and any $X \in \mathcal{D}_1(A^{op} \times B)$ and $Y \in \mathcal{E}(B^{op} \times C)$, the isomorphism φ of Definition 3.4.3 induces an isomorphism*

$$\varphi : Y \otimes_B FX \xrightarrow{\cong} F(Y \otimes_B X)$$

in $\mathcal{D}_2(A^{op} \times C)$. These satisfy the following coherence conditions, which are exact analogues of those in Definition 3.4.3:

1. *For any categories A, B, C and D, and any $X \in \mathcal{D}_1(A^{op} \times B)$, $Y \in \mathcal{E}(B^{op} \times C)$ and $Z \in \mathcal{E}(C^{op} \times D)$, the diagram below commutes:*

$$(Z \otimes_{\mathrm{C}} Y) \otimes_{\mathrm{B}} FX \xrightarrow{\ \alpha\ } Z \otimes_{\mathrm{C}} (Y \otimes_{\mathrm{B}} FX)$$

with vertical maps φ on the left, $Z \otimes_{\mathrm{C}} \varphi$ on the right to $Z \otimes_{\mathrm{C}} F(Y \otimes_{\mathrm{B}} X)$, then φ to

$$F((Z \otimes_{\mathrm{C}} Y) \otimes_{\mathrm{B}} X) \xrightarrow{\ F(\alpha)\ } F(Z \otimes_{\mathrm{C}} (Y \otimes_{\mathrm{B}} X))$$

2. *For any $X \in \mathcal{D}_1(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$, the diagram below commutes:*

$$\mathrm{h}_{\mathrm{B}} \otimes_{\mathrm{B}} F(X) \xrightarrow{\ \varphi\ } F(\mathrm{h}_{\mathrm{B}} \otimes_{\mathrm{B}} X)$$

with λ and $F(\lambda)$ to FX.

Proof. Given $X \in \mathcal{D}_1(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$ and $Y \in \mathcal{E}(\mathrm{B}^{\mathrm{op}} \times \mathrm{C})$, the map $\varphi : Y \otimes_{\mathrm{B}} FX \xrightarrow{\cong} F(Y \otimes_{\mathrm{B}} X)$ is given by the composite

$$\int^{\mathrm{B}} (Y \widetilde{\otimes} FX) \xrightarrow{\ \int^{\mathrm{B}} \varphi\ } \int^{\mathrm{B}} F(Y \widetilde{\otimes} X) \xrightarrow{\ \cong\ } F \int^{\mathrm{B}} (Y \widetilde{\otimes} X),$$

where the first map is induced by the structure isomorphism of Definition 3.4.3, and the second follows by the cocontinuity of F.

To see that the diagrams commute, we can rewrite φ as above, and α and λ as in the proof of Proposition 3.5.2. Using Remark 3.1.3 and Remark 3.1.11, the first diagram reduces to $\int^{\mathrm{B}} \int^{\mathrm{C}}$ applied to the diagram below, in $\mathcal{D}_2(\mathrm{B}^{\mathrm{op}} \times \mathrm{B} \times \mathrm{C}^{\mathrm{op}} \times \mathrm{C} \times \mathrm{A}^{\mathrm{op}} \times \mathrm{D})$:

$$(Z \widetilde{\otimes} Y) \widetilde{\otimes} FX \xrightarrow{\ \alpha\ } Z \widetilde{\otimes} (Y \widetilde{\otimes} FX)$$

with vertical maps φ on the left, $Z \widetilde{\otimes} \varphi$ on the right to $Z \widetilde{\otimes} F(Y \widetilde{\otimes} X)$, then φ to

$$F((Z \widetilde{\otimes} Y) \widetilde{\otimes} X) \xrightarrow{\ F(\alpha)\ } F(Z \widetilde{\otimes} (Y \widetilde{\otimes} X))$$

This commutes by the first axiom of Definition 3.4.3. Similarly, using Remark 3.1.14, the second diagram reduces to the second axiom of Definition 3.4.3. $\qquad\square$

Lemma 3.5.10. *Let $F, G : \mathcal{D}_1 \to \mathcal{D}_2$ be cocontinuous $\mathcal{E}$-module maps, and let $\beta : F \Rightarrow G$ be an $\mathcal{E}$-module modification. Then for any categories A, B and C, and any $X \in \mathcal{D}_1(A^{\mathrm{op}} \times B)$ and $Y \in \mathcal{E}(B^{\mathrm{op}} \times C)$, the diagram below commutes:*

$$
\begin{array}{ccc}
Y \otimes_B F(X) & \xrightarrow{Y \otimes_B \beta_X} & Y \otimes_B G(X) \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\
F(Y \otimes_B X) & \xrightarrow[\beta_{Y \otimes_B X}]{} & G(Y \otimes_B X)
\end{array}
$$

Proof. Let $X \in \mathcal{D}_1(A^{\mathrm{op}} \times B)$ and $Y \in \mathcal{E}(B^{\mathrm{op}} \times C)$. The diagram we want to commute may be rewritten as follows, using the definition of φ from the proof of Lemma 3.5.9:

$$
\begin{array}{ccc}
\int^B (Y \widetilde{\otimes} FX) & \xrightarrow{\int^B (Y \widetilde{\otimes} \beta_X)} & \int^B (Y \widetilde{\otimes} GX) \\
\downarrow{\scriptstyle \int^B \varphi} & & \downarrow{\scriptstyle \int^B \varphi} \\
\int^B F(Y \widetilde{\otimes} X) & \xrightarrow[\int^B \beta_{Y \widetilde{\otimes} X}]{} & \int^B G(Y \widetilde{\otimes} X) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
F \int^B (Y \widetilde{\otimes} X) & \xrightarrow[\beta_{\int^B (Y \widetilde{\otimes} X)}]{} & G \int^B (Y \widetilde{\otimes} X)
\end{array}
$$

The square at the top commutes by (the external version) of the $\mathcal{E}$-module modification condition of Definition 3.4.4. The second square commutes by Remark 3.1.3. $\qquad \square$

Remark 3.5.11. In this section, we have proved a number of coherence results for the cancelling tensor product, resulting from the analogous coherence for the internal or external tensor product. On the other hand, suppose we have an $\mathcal{E}$-module $\mathcal{D}$, categories A and B, and $X \in \mathcal{D}(A)$ and $Y \in \mathcal{E}(B)$. Using the isomorphism $Y \otimes_B \partial_B X \cong Y \widetilde{\otimes} X$, coherence results for the cancelling tensor product also imply the analogous results for the external product, and, using Remark 2.3.6, the internal product.

3.6   The maps $\otimes_u$ and h_u

In this section, we investigate the action of ∂ and $\int$ on morphisms. Specifically, given a functor $u : A \to B$, we revisit the maps ∂_u and $\int^u$ of Definition 3.1.5, and the associated maps h_u of Definition 3.5.1 and $\otimes_u$ of Definition 3.2.5. We conclude this section with a lemma describing the coherence between the maps h_u and $\otimes_u$ in any $\mathcal{E}$-module. We will need this result, and others in this section, frequently in Chapter 5. In particular, we use the results in this section in the proof of Theorem 5.1.14, showing that any closed $\mathcal{E}$-module induces an $\mathcal{E}$-prederivator.

Consider the commutative diagram of Definition 3.1.4:

$$
\begin{array}{ccccc}
\mathrm{tw}(A) & \xrightarrow{(s,t)} & A^{\mathrm{op}} \times A & \xrightarrow{A^{\mathrm{op}} \times u} & A^{\mathrm{op}} \times B \\
{\scriptstyle \mathrm{tw}(u)} \downarrow & & & & \downarrow {\scriptstyle u^{\mathrm{op}} \times B} \\
\mathrm{tw}(B) & \xrightarrow[\quad(s,t)\quad]{} & & & B^{\mathrm{op}} \times B
\end{array}
\tag{3.1}
$$

By [13, Lemma 5.4], for any derivator $\mathcal{D}$, the commutative diagram (3.1) induces an isomorphism:

$$
\begin{array}{ccccccc}
\mathcal{D}^{A^{\mathrm{op}} \times B} & \xrightarrow{(u^{\mathrm{op}} \times B)_!} & \mathcal{D}^{B^{\mathrm{op}} \times B} & \xrightarrow{\quad (t^{\mathrm{op}},\, s^{\mathrm{op}})^* \quad} & \mathcal{D}^{\mathrm{tw}(B)^{\mathrm{op}}} & & \\
& {\scriptstyle \eta} \nearrow & \downarrow {\scriptstyle (u^{\mathrm{op}} \times B)^*} & & \downarrow {\scriptstyle (\mathrm{tw}(u)^{\mathrm{op}})^*} & {\scriptstyle p_!} \searrow & \\
& & \mathcal{D}^{A^{\mathrm{op}} \times B} \xrightarrow{(A^{\mathrm{op}} \times u)^*} & \mathcal{D}^{A^{\mathrm{op}} \times A} \xrightarrow{(t^{\mathrm{op}},\, s^{\mathrm{op}})^*} & \mathcal{D}^{\mathrm{tw}(A)^{\mathrm{op}}} & \xrightarrow[\quad p_! \quad]{} & \mathcal{D}
\end{array}
$$

Here the unlabelled 2-cell is the canonical modification of Remark 2.1.15. Thus, for any $X \in \mathcal{D}(A^{\mathrm{op}} \times B)$, we have a canonical isomorphism:

$$
\int^A (A^{\mathrm{op}} \times u)^* X \xrightarrow{\ \cong\ } \int^B (u^{\mathrm{op}} \times B)_! X
\tag{3.2}
$$

Note that, if we paste the counit

$$
\begin{array}{ccc}
\mathcal{D}^{B^{op}\times B} & & \\
(u^{op}\times B)^* \downarrow & \xRightarrow{\ \epsilon\ } & \\
\mathcal{D}^{A^{op}\times B} & \xrightarrow{(u^{op}\times B)_!} & \mathcal{D}^{B^{op}\times B}
\end{array}
$$

onto the diagram above, then we obtain the modification $\int^u$ of Definition 3.1.5. Thus, for any $Y \in \mathcal{D}(B^{op} \times B)$, we may factor $\int^u Y$ as follows:

$$
\int^u Y : \int^A (A^{op}\times u)^*(u^{op}\times B)^*Y \xrightarrow{\ \cong\ } \int^B (u^{op}\times B)_!(u^{op}\times B)^*Y \xrightarrow{\ \int^B \epsilon\ } \int^B Y \qquad (3.3)
$$

Similarly, the commutative diagram (3.1) induces an isomorphism

$$
\begin{array}{ccccc}
& \xrightarrow{p^*} & \mathcal{D}^{tw(B)} & \xrightarrow{\quad (s,t)_! \quad} & \mathcal{D}^{B^{op}\times B} \\
\mathcal{D} & & \downarrow tw(u)^* & \Longrightarrow & \downarrow (u^{op}\times B)^* \\
& \xrightarrow{p^*} & \mathcal{D}^{tw(A)} & \xrightarrow{(s,t)_!} \mathcal{D}^{A^{op}\times A} \xrightarrow{(A^{op}\times u)_!} & \mathcal{D}^{A^{op}\times B}
\end{array}
$$

where the nonidentity 2-cell is the canonical modification of Definition 2.1.14. The component of this map at $Z \in \mathcal{D}([\mathbf{0}])$ gives a canonical isomorphism:

$$
(A^{op} \times u)_! \partial_A Z \xrightarrow{\ \cong\ } (u^{op} \times B)^* \partial_B Z \qquad (3.4)
$$

As above, if we paste the unit

$$
\begin{array}{ccc}
\mathcal{D}^{A^{op}\times A} & \xrightarrow{(A^{op}\times u)_!} & \mathcal{D}^{A^{op}\times B} \\
 & \xRightarrow{\ \eta\ } & \downarrow (A^{op}\times u)^* \\
 & & \mathcal{D}^{A^{op}\times A}
\end{array}
$$

onto this diagram, then we obtain ∂_u of Definition 3.1.5. Thus, for any $Z \in \mathcal{D}([\mathbf{0}])$, we may factor $\partial_u Z$ as follows:

$$
\partial_u Z : \partial_A Z \xrightarrow{\ \eta\ } (A^{op}\times u)^*(A^{op}\times u)_!\partial_A Z \xrightarrow{\ \cong\ } (A^{op}\times u)^*(u^{op}\times B)^*\partial_B Z
$$

Finally, suppose $\mathcal{D}$ is an $\mathcal{E}$-module, and consider the map

$$
\begin{array}{ccc}
\mathcal{E}^{B^{op}} \times \mathcal{D}^{B} & \xrightarrow{\ \otimes_{B}\ } & \\
\downarrow{\scriptstyle (u^{op})^{*} \times u^{*}} & {\scriptstyle \otimes_{u}}\!\!\!\nearrow\!\!\!\nearrow & \mathcal{D} \\
\mathcal{E}^{A^{op}} \times \mathcal{D}^{A} & \xrightarrow{\ \otimes_{A}\ } &
\end{array}
$$

of Definition 3.2.5. For any objects $X \in \mathcal{D}(B)$ and $Y \in \mathcal{E}(A^{op})$, the isomorphism (3.2) above induces a canonical isomorphism $Y \otimes_{A} u^{*}X \xrightarrow{\cong} (u^{op})_{!}Y \otimes_{B} X$ as follows:

$$
\int^{A}(Y \widetilde{\otimes} u^{*}X) \xrightarrow{\cong} \int^{A}(A^{op} \times u)^{*}(Y \widetilde{\otimes} X) \xrightarrow{\cong} \int^{B}(u^{op} \times B)_{!}(Y \widetilde{\otimes} X) \tag{3.5}
$$
$$
\downarrow{\scriptstyle \cong}
$$
$$
\int^{B}((u^{op})_{!}Y \widetilde{\otimes} X)
$$

The first isomorphism in this composite is the structure isomorphism for $Y \widetilde{\otimes} -$, the second is an instance of (3.2), and the final isomorphism follows by the cocontinuity of $\otimes$. Using this isomorphism (3.5) and our description of $\int^{u}$ in (3.3), for any $X \in \mathcal{D}(B)$ and $Z \in \mathcal{E}(B^{op})$, we can factor $Z \otimes_{u} X$ as follows:

$$
Z \otimes_{u} X : (u^{op})^{*}Z \otimes_{A} u^{*}X \xrightarrow{\cong} (u^{op})_{!}(u^{op})^{*}Z \otimes_{B} X \xrightarrow{\ \epsilon\, \otimes_{B}X\ } Z \otimes_{B} X \tag{3.6}
$$

Remark 3.6.1. Let $u : A \to B$ be a functor. For any closed $\mathcal{E}$-module $\mathcal{D}$, and any $X \in \mathcal{D}(B)$, the canonical isomorphism

$$
\begin{array}{ccc}
\mathcal{E}^{A^{op}} & \xrightarrow{\ (u^{op})_{!}\ } & \mathcal{E}^{B^{op}} \\
{\scriptstyle -\otimes_{A}u^{*}X}\!\searrow & {\scriptstyle \cong}\!\!\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle -\otimes_{B}X} \\
& & \mathcal{D}
\end{array}
$$

of (3.5) is conjugate to the isomorphism

$$
\begin{array}{ccc}
\mathcal{E}^{A^{op}} & \xleftarrow{\ (u^{op})^{*}\ } & \mathcal{E}^{B^{op}} \\
{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{D}}(u^{*}X,-)}\!\nwarrow & {\scriptstyle \cong}\!\!\nearrow\!\!\!\nearrow\,{\scriptstyle \gamma} & \uparrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{D}}(X,-)} \\
& \mathcal{D} &
\end{array}
$$

induced by the structure isomorphism for $\mathrm{map}_{\mathcal{D}}(-,-)$. This appears in the proof of [13, Theorem 9.1], but it can also be checked directly. Using this fact and the description of $\otimes_u$ in (3.6), for any category C, and any $Y \in \mathcal{D}(\mathrm{C})$, the diagram below commutes:

$$
\begin{array}{ccc}
(u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{D}}(X,Y) \otimes_{\mathrm{A}} u^* X & \xrightarrow{\ \gamma\, \otimes_{\mathrm{A}} u^* X\ } & \widetilde{\mathrm{map}}_{\mathcal{D}}(u^* X, Y) \otimes_{\mathrm{A}} u^* X \\
{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{D}}(X,Y) \otimes_u X} \Big\downarrow & & \Big\downarrow {\scriptstyle \epsilon} \\
\widetilde{\mathrm{map}}_{\mathcal{D}}(X,Y) \otimes_{\mathrm{B}} X & \xrightarrow{\quad\ \epsilon\ \quad} & Y
\end{array}
$$

Remark 3.6.2. Let $u : \mathrm{A} \to \mathrm{B}$ be a functor, and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a derivator map. Using Lemma 2.2.4, it follows that F respects $\int^u$ and ∂_u. For example, in the case of ∂_u, this means that the diagram below commutes, for any $X \in \mathcal{D}_1([\mathbf{0}])$:

$$
\begin{array}{ccc}
\partial_{\mathrm{A}} F X & \xrightarrow{\hspace{6cm}} & F \partial_{\mathrm{A}} X \\
{\scriptstyle \partial_u F X} \Big\downarrow & & \Big\downarrow {\scriptstyle F \partial_u X} \\
(u^{\mathrm{op}} \times u)^* \partial_{\mathrm{B}} F X & \xrightarrow{\quad} (u^{\mathrm{op}} \times u)^* F \partial_{\mathrm{B}} X \xrightarrow[\cong]{\quad} & F (u^{\mathrm{op}} \times u)^* \partial_{\mathrm{B}} X
\end{array}
$$

Similarly, suppose we have $\mathcal{E}$-modules $\mathcal{D}_1$ and $\mathcal{D}_2$, and a cocontinuous $\mathcal{E}$-module morphism $F : \mathcal{D}_1 \to \mathcal{D}_2$. For any categories A, B, C and D, any functor $u : \mathrm{A} \to \mathrm{B}$, and objects $X \in \mathcal{D}_1(\mathrm{C}^{\mathrm{op}} \times \mathrm{B})$ and $Y \in \mathcal{E}(\mathrm{B}^{\mathrm{op}} \times \mathrm{D})$, the diagram below commutes:

$$
\begin{array}{ccc}
(u^{\mathrm{op}} \times \mathrm{D})^* Y \otimes_{\mathrm{A}} (\mathrm{C}^{\mathrm{op}} \times u)^* F X & \xrightarrow{\ \cong\ } & (u^{\mathrm{op}} \times \mathrm{D})^* Y \otimes_{\mathrm{A}} F(\mathrm{C}^{\mathrm{op}} \times u)^* X \\
& & \Big\downarrow {\scriptstyle \varphi} \\
{\scriptstyle Y \otimes_u F X} \Big\downarrow & & F\big((u^{\mathrm{op}} \times \mathrm{D})^* Y \otimes_{\mathrm{A}} (\mathrm{C}^{\mathrm{op}} \times u)^* X\big) \\
& & \Big\downarrow {\scriptstyle F(Y \otimes_u X)} \\
Y \otimes_{\mathrm{B}} F X & \xrightarrow{\quad\quad \varphi \quad\quad} & F(Y \otimes_{\mathrm{B}} X)
\end{array}
$$

The map φ is the canonical isomorphism of Lemma 3.5.9.

Example 3.6.3. Let $\mathcal{D}$ be a closed $\mathcal{E}$-module and let $X \in \mathcal{D}(\mathrm{A})$. Using Proposition 3.5.2, the derivator map $- \otimes_{\mathrm{A}} X : \mathcal{E}^{\mathrm{A}^{\mathrm{op}}} \to \mathcal{D}$ is an $\mathcal{E}$-module map. Applying Remark 3.6.2, for any functor $v : \mathrm{B} \to \mathrm{C}$, and any objects $Y \in \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{C})$ and $Z \in \mathcal{E}(\mathrm{C}^{\mathrm{op}} \times \mathrm{D})$, we obtain the following commutative diagram:

$$((v^{\mathrm{op}} \times \mathrm{D})^* Z \otimes_{\mathrm{B}} (\mathrm{A}^{\mathrm{op}} \times v)^* Y) \otimes_{\mathrm{A}} X \xrightarrow{\ \alpha\ } (v^{\mathrm{op}} \times \mathrm{D})^* Z \otimes_{\mathrm{B}} ((\mathrm{A}^{\mathrm{op}} \times v)^* Y \otimes_{\mathrm{A}} X)$$

$$\downarrow (Z \otimes_v Y) \otimes_{\mathrm{A}} X \qquad\qquad \cong \downarrow$$

$$(v^{\mathrm{op}} \times \mathrm{D})^* Z \otimes_{\mathrm{B}} v^*(Y \otimes_{\mathrm{A}} X)$$

$$\downarrow Z \otimes_v (Y \otimes_{\mathrm{A}} X)$$

$$(Z \otimes_{\mathrm{C}} Y) \otimes_{\mathrm{A}} X \xrightarrow{\ \ \ \alpha\ \ \ } Z \otimes_{\mathrm{C}} (Y \otimes_{\mathrm{A}} X)$$

Suppose we have a functor $u : \mathrm{A} \to \mathrm{B}$. The following lemma shows that the isomorphisms of Proposition 3.1.12 respect the canonical morphism $\int^{\mathrm{A}_{1,2}} \partial_{\mathrm{A}_{2,3}} \Rightarrow \int^{\mathrm{B}_{1,2}} \partial_{\mathrm{B}_{2,3}}$ induced by u. We use this to prove Lemma 3.6.5, which expresses the coherence between h_u and $\otimes_u$ in an $\mathcal{E}$-module.

Proposition 3.6.4. *For any derivator $\mathcal{D}$, and any category $\mathrm{A} = \mathrm{A}_i$, consider the isomorphism*

$$\mathcal{D}^{\mathrm{A}_1} \xrightarrow[\partial_{\mathrm{A}_{2,3}}]{} \mathcal{D}^{\mathrm{A}_1 \times \mathrm{A}_2^{\mathrm{op}} \times \mathrm{A}_3} \xrightarrow[\cong]{\int^{\mathrm{A}_{1,2}}} \mathcal{D}^{\mathrm{A}_3}$$

of Proposition 3.1.12. Given a functor $u : \mathrm{A} \to \mathrm{B}$, the pasting diagram
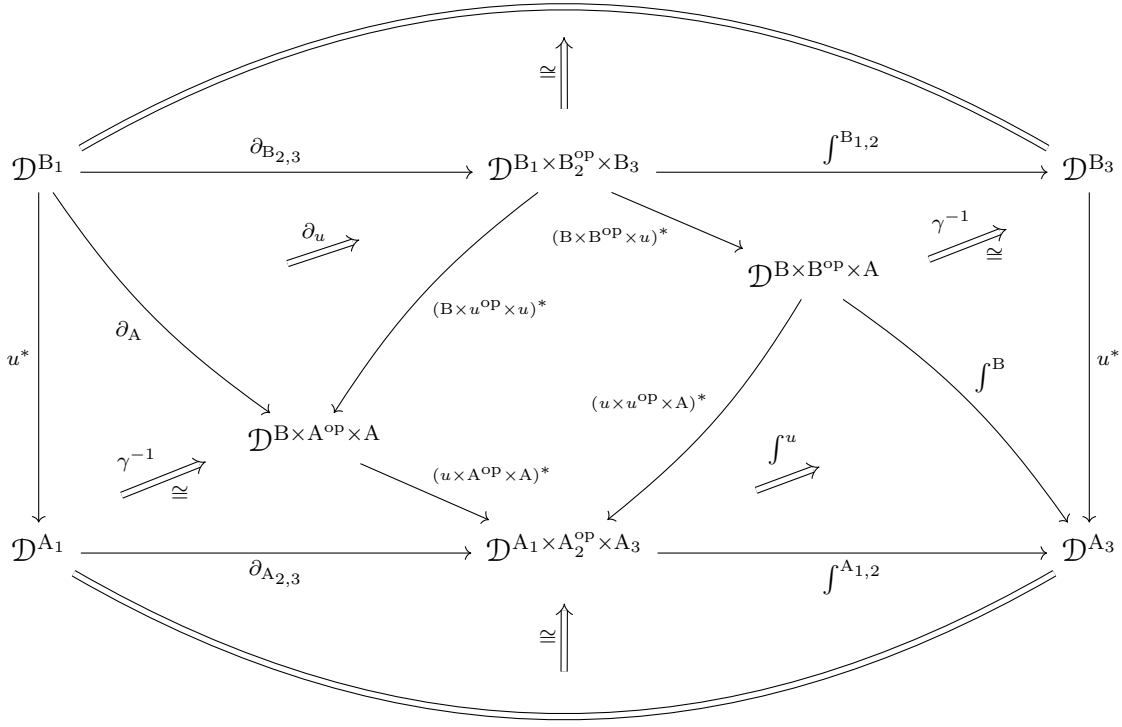
reduces to id_{u^*}.

Proof. Expanding the maps ∂_u and $\int^u$ using Definition 3.1.5, the large rectangle in the pasting diagram above reduces to the following:
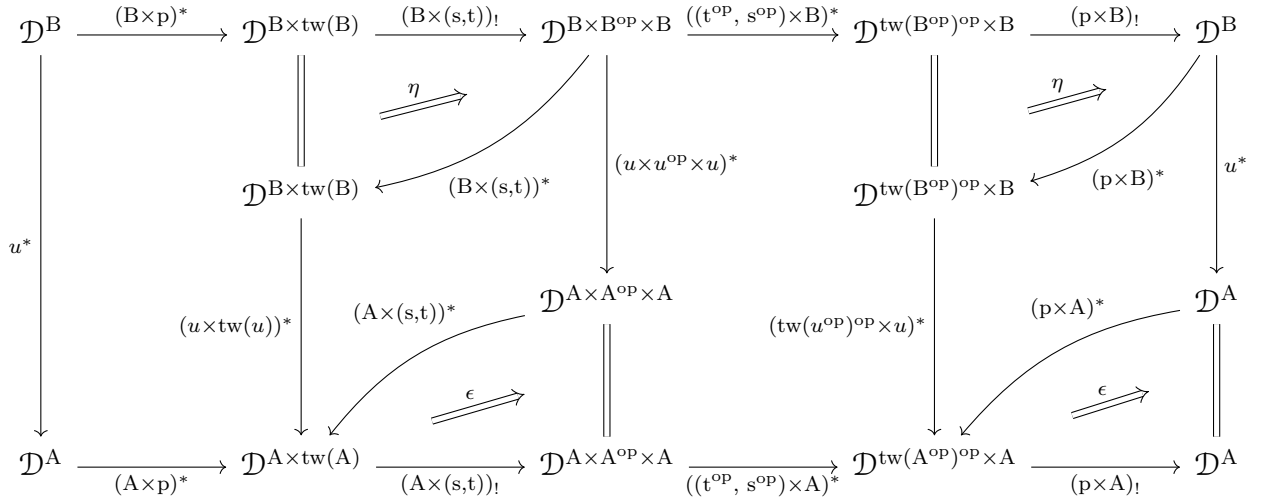
Recall the construction of the isomorphisms $\int^{A_{1,2}} \partial_{A_{2,3}} \cong \mathrm{id}$ and $\int^{B_{1,2}} \partial_{B_{2,3}} \cong \mathrm{id}$ described

in Proposition 3.1.12, in particular the pullbacks P_A and P_B. The functor $u : A \to B$ induces a map $P_u : P_A \to P_B$ making the diagram below commute:

$$
\begin{array}{c}
\text{(3.7)}
\end{array}
$$

Explicitly, this functor takes an object $a \xrightarrow{g} b \xrightarrow{f} c$ in P_A to $u(a) \xrightarrow{u(g)} u(b) \xrightarrow{u(f)} u(c)$ in P_B. If we paste the isomorphisms

$$(n_A)_! \circ (m_A)^* \cong ((t^{op}, s^{op}) \times A)^* \circ (A \times (s, t))_!$$

$$(n_B)_! \circ (m_B)^* \cong ((t^{op}, s^{op}) \times B)^* \circ (B \times (s, t))_!$$

onto the top and bottom of the pasting diagram above, then the commutative diagram (3.7) allows us to simplify and obtain the following:

We may now paste the remaining isomorphisms onto the top and bottom of this rectangle. The resultant pasting diagram reduces easily to the identity, using the fact that the two pasting diagrams below are equal:

$$
\begin{array}{ccccc}
P_A & \xrightarrow{m_A} & A \times tw(A) & \xrightarrow{A \times p} & A \\
\downarrow{\scriptstyle n_A} & & & & \| \\
tw(A^{op})^{op} \times A & & \overset{\vartheta_A}{\Longleftarrow} & & A \\
\downarrow{\scriptstyle p \times A} & & & & \| \\
A & =\!=\!=\!=\!=\!=\!= & & & A \\
& & & & \downarrow{\scriptstyle u} \\
& & & & B
\end{array}
\qquad
\begin{array}{ccccc}
P_A & & & & \\
\downarrow{\scriptstyle P_u} & & & & \\
P_B & \xrightarrow{m_B} & B \times tw(B) & \xrightarrow{B \times p} & B \\
\downarrow{\scriptstyle n_B} & & & & \| \\
tw(B^{op})^{op} \times B & & \overset{\vartheta_B}{\Longleftarrow} & & \| \\
\downarrow{\scriptstyle p \times B} & & & & \| \\
B & =\!=\!=\!=\!=\!=\!= & & & B
\end{array}
$$

To see this, note that the component of the first at the object $a \xrightarrow{g} b \xrightarrow{f} c$ in P_A is $u(f \circ g)$, and the component of the second is $u(f) \circ u(g)$. □

Lemma 3.6.5. *Let $\mathcal{D}$ be an $\mathcal{E}$-module, let $u : A \to B$ be a functor, and let $X \in \mathcal{D}(B)$. The diagram below commutes:*

$$
\begin{array}{ccc}
h_A \otimes_A u^*X & \xrightarrow{h_u \otimes_A u^*X} & (u^{op} \times u)^* h_B \otimes_A u^*X \\
& & \downarrow{\scriptstyle (B^{op} \times u)^* h_B \otimes_u X} \\
\downarrow{\scriptstyle \lambda} & & (B^{op} \times u)^* h_B \otimes_B X \\
& & \downarrow{\scriptstyle \cong} \\
u^*X & \xleftarrow{\;u^*(\lambda)\;} & u^*(h_B \otimes_B X)
\end{array}
$$

Proof. To see that this diagram commutes, use the definition of λ in Proposition 3.5.2, and the definitions of h_u and $\otimes_u$, to rewrite the diagram in terms of ∂_u and $\int^u$. Using cocontinuity and pseudonaturality of $\otimes$, we can then pull the instances of ∂_u, $\int^u$ and u^* out of the tensor product. From here, the commutativity follows immediately from Proposition 3.6.4. □

3.7 Cotensors in closed $\mathcal{E}$-modules

In this section, we consider cotensors in a closed $\mathcal{E}$-module $\mathcal{D}$. If $\mathcal{E}$ is symmetric monoidal, we show that these are part of a closed $\mathcal{E}$-action on the opposite derivator $\mathcal{D}^{\mathrm{op}}$. The main result of this section is Proposition 3.7.5, which proves a derivator analogue of the well-known fact that a left adjoint preserves tensors if and only if its right adjoint preserves cotensors. In particular, this result implies that, for any $X \in \mathcal{D}(\mathrm{A})$, the morphism $\widetilde{\mathrm{map}}_{\mathcal{D}}(X, -) : \mathcal{D} \to \mathcal{E}^{\mathrm{A}^{\mathrm{op}}}$ preserves cotensors. This is the content of Example 3.7.7. Throughout this section, let $\mathcal{E}$ be a symmetric monoidal derivator.

Example 3.7.1. If $\mathcal{D}$ is a closed $\mathcal{E}$-module, then $\mathcal{D}^{\mathrm{op}}$ is a (right) $\mathcal{E}$-module via the map

$$\lhd^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D}^{\mathrm{op}}.$$

See [16, Section 4]. The structure isomorphisms are induced, using the adjunctions of Theorem 3.2.2, by the coherent isomorphisms for the cancelling tensor of Proposition 3.5.2. Using the symmetry isomorphism for $\mathcal{E}$, this right $\mathcal{E}$-module structure induces a left $\mathcal{E}$-module structure. The $\mathcal{E}$-module structure is closed, with the required adjoints given by Theorem 3.2.2.

Remark 3.7.2. Let $\mathcal{D}$ be a closed $\mathcal{E}$-module, let A be a category, and let $X \in \mathcal{D}(\mathrm{A})$. Using the descriptions of the various adjoints in Theorem 3.2.2, we have an isomorphism

$$\widetilde{\mathrm{map}}_{\mathcal{D}^{\mathrm{op}}}(X, -) \cong \widetilde{\mathrm{map}}_{\mathcal{D}}(-, X) : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{A}}.$$

Definition 3.7.3. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be closed $\mathcal{E}$-modules. We say that a derivator map $G : \mathcal{D}_1 \to \mathcal{D}_2$ **preserves cotensors** if its opposite $G^{\mathrm{op}} : \mathcal{D}_1^{\mathrm{op}} \to \mathcal{D}_2^{\mathrm{op}}$ is an $\mathcal{E}$-module map for the $\mathcal{E}$-module structures of Example 3.7.1. Similarly, a modification between cotensor-preserving maps is said to **respect cotensors** if its opposite respects tensors.

Example 3.7.4. Let $\mathcal{D}$ be a closed $\mathcal{E}$-module and let $u : \mathrm{A} \to \mathrm{B}$ be a functor. Consider the derivator map $u^* : \mathcal{D}^{\mathrm{B}} \to \mathcal{D}^{\mathrm{A}}$. The opposite of this map is equal to

$$(u^{\mathrm{op}})^* : (\mathcal{D}^{\mathrm{op}})^{\mathrm{B}^{\mathrm{op}}} \to (\mathcal{D}^{\mathrm{op}})^{\mathrm{A}^{\mathrm{op}}}.$$

By Lemma 3.4.13, this is an $\mathcal{E}$-module map. Thus, $u^* : \mathcal{D}^{\mathrm{B}} \to \mathcal{D}^{\mathrm{A}}$ preserves cotensors as well as tensors.

Proposition 3.7.5. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be closed $\mathcal{E}$-modules, and suppose we have an adjunction:*

$$\mathcal{D}_1 \quad \overset{F}{\underset{R}{\rightleftarrows}} \quad \bot \quad \mathcal{D}_2$$

Then F preserves tensors if and only if R preserves cotensors.

Proof. If the left adjoint F preserves tensors, we will show that the right adjoint R preserves cotensors. Once we prove this, applying the result to the opposite adjunction

$$(\mathcal{D}_2)^{\mathrm{op}} \quad \overset{R^{\mathrm{op}}}{\underset{F^{\mathrm{op}}}{\rightleftarrows}} \quad \bot \quad (\mathcal{D}_1)^{\mathrm{op}}$$

proves the reverse implication as well.

Suppose F preserves tensors. Taking the opposite of the structure isomorphism in Definition 3.4.3, we require a coherent isomorphism:

$$
\begin{array}{ccc}
\mathcal{D}_2 \times \mathcal{E}^{\mathrm{op}} & \xrightarrow{R \times \mathcal{E}^{\mathrm{op}}} & \mathcal{D}_1 \times \mathcal{E}^{\mathrm{op}} \\
{\scriptstyle \triangleleft} \downarrow & \overset{\cong}{\nearrow} & \downarrow {\scriptstyle \triangleleft} \\
\mathcal{D}_2 & \xrightarrow{R} & \mathcal{D}_1
\end{array}
$$

Using Lemma 2.3.5, to give the desired isomorphism, we may equivalently provide isomorphisms

$$
\begin{array}{ccc}
\mathcal{D}_2 & \xrightarrow{R} & \mathcal{D}_1 \\
{\scriptstyle - \,\widetilde{\triangleleft}\, X} \downarrow & \overset{\cong}{\nearrow} & \downarrow {\scriptstyle - \,\widetilde{\triangleleft}\, X} \\
\mathcal{D}_2^{\mathrm{B}^{\mathrm{op}}} & \xrightarrow{R} & \mathcal{D}_1^{\mathrm{B}^{\mathrm{op}}}
\end{array}
$$

for each category B, and each $X \in \mathcal{E}(\mathrm{B})$, such that these isomorphisms organise into a modification in X. Take these to be conjugate to the isomorphism between left adjoints:

$$
\begin{array}{ccc}
\mathcal{D}_2 & \xleftarrow{\quad F \quad} & \mathcal{D}_1 \\
{\scriptstyle X \otimes_{\mathrm{B}^{\mathrm{op}}} -} \big\uparrow & \overset{\cong}{\underset{\varphi^{-1}}{\Swarrow}} & \big\uparrow {\scriptstyle X \otimes_{\mathrm{B}^{\mathrm{op}}} -} \\
\mathcal{D}_2^{\mathrm{B}^{\mathrm{op}}} & \xleftarrow{\quad F \quad} & \mathcal{D}_1^{\mathrm{B}^{\mathrm{op}}}
\end{array}
$$

This isomorphism is defined in Lemma 3.5.9. The description of the left adjoints follows from Theorem 3.2.2.

Let $u : \mathrm{A} \to \mathrm{B}$ be a functor. To see that the isomorphisms above form a modification in X, we need the pasting diagrams below to be equal:

We can check this equality by taking mates under the adjunctions $F(X \otimes_{\mathrm{B}^{\mathrm{op}}} -) \dashv R(-) \widetilde{\vartriangleleft} X$ and $u^* X \otimes_{\mathrm{A}^{\mathrm{op}}} F(-) \dashv R(- \widetilde{\vartriangleleft} u^* X)$.

Using the definition of the isomorphism $R(- \widetilde{\vartriangleleft} X) \cong R(-) \widetilde{\vartriangleleft} X$ above, and the fact that

is conjugate to the canonical isomorphism

the equality follows by the commutative diagram in Remark 3.6.2 for F.

The coherence conditions for R follow from the coherence conditions for F, once again using conjugacy. $\qquad\square$

Lemma 3.7.6. *Suppose we have closed $\mathcal{E}$-modules $\mathcal{D}_1$ and $\mathcal{D}_2$, and a pair of adjunctions*

such that the left adjoints F_1 and F_2 preserve tensors. Then a modification $\theta : F_1 \Rightarrow F_2$ respects tensors if and only if its conjugate $\vartheta : R_2 \Rightarrow R_1$ respects cotensors.

Proof. Let $\theta : F_1 \Rightarrow F_2$ be a modification. Suppose θ respects tensors. We will show that its conjugate $\vartheta : R_2 \Rightarrow R_1$ respects cotensors. As in the proof of Proposition 3.7.5, the converse follows from this result by taking opposites.

Let A be a category and let $X \in \mathcal{E}(A)$. We need to show that the pasting diagrams below are equal:

Taking conjugates, and using the definition of the isomorphisms $R_i(- \mathbin{\widetilde{\lhd}} X) \cong R_i(-) \mathbin{\widetilde{\lhd}} X$ from the proof of Proposition 3.7.5, these pasting diagrams are equal if and only if the pasting diagrams below are equal:

Since θ respects tensors, these two pasting diagrams are equal by Lemma 3.5.10. $\square$

Example 3.7.7. Let $\mathcal{D}$ be a closed $\mathcal{E}$-module, and let $X \in \mathcal{D}(A)$. The derivator map $- \otimes_A X : \mathcal{E}^{A^{op}} \to \mathcal{D}$ preserves tensors. To see this, note that the external component of the structure isomorphism from Definition 3.4.3 has the form $(W \widetilde{\otimes} Z) \otimes_A X \xrightarrow{\cong} W \widetilde{\otimes} (Z \otimes_A X)$, for any $Z \in \mathcal{E}(A^{op} \times B)$ and $W \in \mathcal{E}(C)$. This is given by the composite below:

$$\int^A (W \widetilde{\otimes} Z) \widetilde{\otimes} X \xrightarrow{\int^A \alpha} \int^A W \widetilde{\otimes} (Z \widetilde{\otimes} X) \xrightarrow{\cong} W \widetilde{\otimes} \int^A (Z \widetilde{\otimes} X)$$

The second isomorphism in the composite above comes from the cocontinuity of $\otimes$. The internal component of this map can be recovered using Remark 2.3.6. By Proposition 3.7.5, it follows that

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(X, -) : \mathcal{D} \to \mathcal{E}^{A^{op}}$$

preserves cotensors.

Example 3.7.8. Let $\mathcal{D}$ be a closed $\mathcal{E}$-module and let $u : A \to B$ be a functor. By Lemma 3.4.13 and Example 3.7.4, the map $u^* : \mathcal{D}^B \to \mathcal{D}^A$ preserves both tensors and cotensors. Thus, by Proposition 3.7.5, its left adjoint $u_! : \mathcal{D}^A \to \mathcal{D}^B$ preserves tensors, and its right adjoint $u_* : \mathcal{D}^A \to \mathcal{D}^B$ preserves cotensors.

3.8 A representability theorem for triangulated $\mathcal{E}$-modules

We begin this section with a characterisation of cocontinuous $\mathcal{E}$-module maps from $\mathcal{E}^{A^{op}}$ to a closed $\mathcal{E}$-module $\mathcal{D}$; using the results of Section 3.5, it is easy to see that any such map is isomorphic to $- \otimes_A X : \mathcal{E}^{A^{op}} \to \mathcal{D}$, for some object $X \in \mathcal{D}(A)$. When $\mathcal{E}$ is triangulated, we can combine this observation with Proposition 3.7.5, to prove a representability theorem for triangulated closed $\mathcal{E}$-modules. This is Theorem 3.8.3, the final result of this chapter.

Proposition 3.8.1. *Let* $\mathcal{D}$ *be an* $\mathcal{E}$*-module and let* A *be a category. The functor*

$$\underline{\mathrm{Hom}}_!^{\mathcal{E}}(\mathcal{E}^{A^{op}}, \mathcal{D}) \to \mathcal{D}(A),$$

given by evaluation at $h_A \in \mathcal{E}(A^{op} \times A)$, *is an equivalence.*

Proof. We claim that the functor

$$\mathcal{D}(A) \longrightarrow \underline{\mathrm{Hom}}^{\mathcal{E}}_!(\mathcal{E}^{A^{op}}, \mathcal{D})$$

$$X \longmapsto \quad - \otimes_A X$$

is an inverse equivalence for evaluation at h_A. To see this, we must show that both composites are naturally isomorphic to the identity.

Suppose we have an object $X \in \mathcal{D}(A)$. The isomorphisms

$$\lambda : h_A \otimes_A X \xrightarrow{\;\cong\;} X$$

of Proposition 3.5.2 are natural in X, so these provide one of the required isomorphisms.

For the other isomorphism, let $F : \mathcal{E}^{A^{op}} \to \mathcal{D}$ be a cocontinuous $\mathcal{E}$-module map. For any category B and any $Z \in \mathcal{E}(A^{op} \times B)$, consider the isomorphism

$$Z \otimes_A Fh_A \xrightarrow{\;\varphi\;} F(Z \otimes_A h_A) \xrightarrow{\;F(\rho)\;} FZ,$$

where φ is the map of Lemma 3.5.9, and ρ is the map of Proposition 3.5.2. This is a modification in Z; we must check that it respects tensors. It suffices to check this on the external product, but this follows from the first commutative diagram of Lemma 3.5.9, using Remark 3.5.11. Thus, we have an isomorphism

$$- \otimes_A Fh_A \xRightarrow{\;\cong\;} F$$

in $\underline{\mathrm{Hom}}^{\mathcal{E}}_!(\mathcal{E}^{A^{op}}, \mathcal{D})$. Finally, we need to show that this isomorphism is natural in F. That is, given cocontinuous $\mathcal{E}$-module maps $F, G : \mathcal{E}^{A^{op}} \to \mathcal{D}$ and an $\mathcal{E}$-module modification $\beta : F \Rightarrow G$, we need the diagram below to commute, for any $Z \in \mathcal{E}(A^{op} \times B)$:

$$
\begin{array}{ccccc}
Z \otimes_A Fh_A & \xrightarrow{\;\varphi\;} & F(Z \otimes_A h_A) & \xrightarrow{\;F(\rho)\;} & FZ \\
\Big\downarrow{\scriptstyle Z \otimes_A \beta_{h_A}} & & & & \Big\downarrow{\scriptstyle \beta_Z} \\
Z \otimes_A Gh_A & \xrightarrow[\;\varphi\;]{} & G(Z \otimes_A h_A) & \xrightarrow[\;G(\rho)\;]{} & GZ
\end{array}
$$

This commutes by Lemma 3.5.10. □

Definition 3.8.2. Let $\mathcal{D}$ be a closed $\mathcal{E}$-module, and let A be a category. We say a derivator map $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{A}}$ is **representable** if there is an object $X \in \mathcal{D}(\mathrm{A})$ and an isomorphism:

$$F \cong \widetilde{\mathrm{map}}_{\mathcal{D}}(-, X) : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{A}}$$

There is also a dual concept for maps of the form $F : \mathcal{D} \to \mathcal{E}^{\mathrm{A}^{\mathrm{op}}}$.

Theorem 3.8.3. *Let $\mathcal{E}$ be a symmetric monoidal derivator, let $\mathcal{D}$ be a closed $\mathcal{E}$-module, and let A be a category. Suppose that $\mathcal{E}$ and $\mathcal{D}$ are triangulated, and that, for any category C, the triangulated category $\mathcal{D}(\mathrm{C})$ satisfies Brown representability. Then a derivator map*

$$F : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{A}}$$

is representable if and only if it is continuous and preserves cotensors.

Proof. Note that a continuous, cotensor-preserving map from $\mathcal{D}^{\mathrm{op}}$ to $\mathcal{E}^{\mathrm{A}}$ takes homotopy colimits in $\mathcal{D}$ to homotopy limits in $\mathcal{E}^{\mathrm{A}}$, and tensors in $\mathcal{D}$ to cotensors in $\mathcal{E}^{\mathrm{A}}$. For the forward implication, let $X \in \mathcal{D}(\mathrm{A})$, and consider the derivator map

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(-, X) : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{A}}.$$

This map is continuous, and it preserves cotensors by Remark 3.7.2 and Example 3.7.7.

On the other hand, suppose we have a continuous, cotensor-preserving map $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{A}}$. By Proposition 2.4.15 (applied to F^{op}), this map must have a left adjoint $G : \mathcal{E}^{\mathrm{A}} \to \mathcal{D}^{\mathrm{op}}$. By Proposition 3.7.5, G preserves tensors. Thus, by Proposition 3.8.1, we have an isomorphism

$$G \cong G\mathrm{h}_{\mathrm{A}} \vartriangleleft_{\mathrm{A}} - : \mathcal{E}^{\mathrm{A}} \to \mathcal{D}^{\mathrm{op}}.$$

Here $\vartriangleleft_{\mathrm{A}}$ is the cancelling version of the $\mathcal{E}$-action on $\mathcal{D}^{\mathrm{op}}$ from Example 3.7.1. But we know that the right adjoint of this map is $\widetilde{\mathrm{map}}_{\mathcal{D}}(-, G\mathrm{h}_{\mathrm{A}}) : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{A}}$. Thus, this map $\widetilde{\mathrm{map}}_{\mathcal{D}}(-, G\mathrm{h}_{\mathrm{A}})$ must be isomorphic to F. □

Note that, by Theorem 2.4.12 and Lemma 2.4.14, the previous theorem applies to any triangulated derivator $\mathcal{D}$ whose underlying category is perfectly generated.

In particular, applying this theorem with $\mathcal{E} = \mathcal{H}\mathrm{o}(\mathbf{Spt})$, we get the following special case: given a triangulated derivator $\mathcal{D}$ whose underlying category is perfectly generated, a derivator

map

$$F : \mathcal{D}^{\mathrm{op}} \to \mathcal{H}o(\mathbf{Spt})$$

is representable if and only if it is continuous.

Chapter 4

$\mathcal{E}$-Categories

In this chapter, we study $\mathcal{E}$-categories, a precursor to the $\mathcal{E}$-prederivators and $\mathcal{E}$-derivators of Chapter 5. In Section 4.1 we develop the basic theory and give a number of examples; in particular, in Theorem 4.1.10, we prove that any closed $\mathcal{E}$-module gives rise to an associated $\mathcal{E}$-category. In Section 4.2, we prove a Yoneda lemma for $\mathcal{E}$-categories, and use this to study $\mathcal{E}$-category adjunctions. The Yoneda lemma is extremely useful, and we use it repeatedly as we develop the theory of $\mathcal{E}$-prederivators in Chapter 5. We end this chapter with Section 4.3, in which we study monoidal morphisms, and prove that enrichment can be transferred along a monoidal adjunction.

Throughout the rest of the thesis, unless otherwise specified, $\mathcal{E}$ will denote a closed symmetric monoidal derivator.

4.1 Basic definitions

In this section, we define $\mathcal{E}$-categories, and develop the basic theory that we will need in subsequent sections. Many of the results in this section are formally similar to the development of basic enriched category theory in [27]. Note, in fact, that an $\mathcal{E}$-category in the sense of Definition 4.1.1 is a $\mathcal{P}rof(\mathcal{E})$-category in the sense of [41]; that is, a category enriched over the bicategory $\mathcal{P}rof(\mathcal{E})$ of Remark 3.5.3. However, a number of fundamental constructions, such as the mapping space $\mathcal{E}$-morphisms of Proposition 4.1.11, are not definable over a general enriching bicategory. Thus, the majority of the results in this section do not follow from any general results that we know of in the bicategorical literature, so we include a complete

account of the theory, starting from the definition.

Definition 4.1.1. An $\mathcal{E}$-**category** $\underline{A}$ consists of the following data:

- For each small category A, a (large) set of objects $\mathcal{A}_0(A)$.

- For any two objects $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$, an object $\widetilde{\mathrm{map}}_\mathcal{A}(X, Y) \in \mathcal{E}(A^{\mathrm{op}} \times B)$.

- For any three objects $X \in \mathcal{A}_0(A)$, $Y \in \mathcal{A}_0(B)$ and $Z \in \mathcal{A}_0(C)$, a map

$$\circ : \widetilde{\mathrm{map}}_\mathcal{A}(Y, Z) \otimes_B \widetilde{\mathrm{map}}_\mathcal{A}(X, Y) \to \widetilde{\mathrm{map}}_\mathcal{A}(X, Z)$$

  in $\mathcal{E}(A^{\mathrm{op}} \times C)$, which we call **composition**.

- For every object $X \in \mathcal{A}_0(A)$, a map

$$j : h_A \to \widetilde{\mathrm{map}}_\mathcal{A}(X, X)$$

  in $\mathcal{E}(A^{\mathrm{op}} \times A)$, which we call the **unit**.

These must satisfy the following coherence conditions:

1. For any $X \in \mathcal{A}_0(A)$, $Y \in \mathcal{A}_0(B)$, $Z \in \mathcal{A}_0(C)$ and $W \in \mathcal{A}_0(D)$, the diagram below commutes:

2. For any $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$, the diagram below commutes:

$$h_B \otimes_B \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{\ j \otimes_B \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Y) \otimes_B \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$

$$\lambda \searrow \qquad \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xleftarrow{\ \circ\ }$$

3. For any $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$, the diagram below commutes:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \otimes_A h_A \xrightarrow{\ \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \otimes_A j\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \otimes_A \widetilde{\mathrm{map}}_{\mathcal{A}}(X,X)$$

$$\rho \searrow \qquad \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xleftarrow{\ \circ\ }$$

Here the maps α, λ and ρ are those of Proposition 3.5.2.

Definition 4.1.2. Let $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ be $\mathcal{E}$-categories. An $\mathcal{E}$-**morphism** $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ consists of the following data:

- For any category A, and any $X \in \mathcal{A}_0(A)$, an object $FX \in \mathcal{B}_0(A)$.

- For any objects $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$, a map

$$F : \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY)$$

in $\mathcal{E}(A^{\mathrm{op}} \times B)$.

These must satisfy the following coherence conditions:

1. For any $X \in \mathcal{A}_0(A)$, $Y \in \mathcal{A}_0(B)$ and $Z \in \mathcal{A}_0(C)$, the diagram below commutes:

$$
\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_B \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) & \xrightarrow{\ \circ\ } & \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z) \\
\downarrow{\scriptstyle F \otimes_B F} & & \downarrow{\scriptstyle F} \\
\widetilde{\mathrm{map}}_{\mathcal{B}}(FY,FZ) \otimes_B \widetilde{\mathrm{map}}_{\mathcal{B}}(FX,FY) & \xrightarrow{\ \circ\ } & \widetilde{\mathrm{map}}_{\mathcal{B}}(FX,FZ)
\end{array}
$$

2. For any $X \in \mathcal{A}_0(A)$, the diagram below commutes:

$$h_A \xrightarrow{\ j\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(X, X)$$

$$\downarrow F$$

$$h_A \xrightarrow{\ j\ } \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FX)$$

Definition 4.1.3. Let $F, G : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ be $\mathcal{E}$-morphisms. An $\mathcal{E}$-**natural transformation** $\beta : F \Rightarrow G$ consists of maps

$$\beta_X : h_A \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, GX)$$

in $\mathcal{E}(A^{\mathrm{op}} \times A)$, for every category A, and every $X \in \mathcal{A}_0(A)$. These must make the diagram below commute, for any $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$:

$$
\begin{array}{ccc}
h_B \otimes_B \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) & \xrightarrow{\ \beta_Y \otimes_B F\ } & \widetilde{\mathrm{map}}_{\mathcal{B}}(FY, GY) \otimes_B \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY) \\
\uparrow \lambda^{-1} & & \downarrow \circ \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) & & \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, GY) \\
\downarrow \rho^{-1} & & \uparrow \circ \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \otimes_A h_A & \xrightarrow{\ G \otimes_A \beta_X\ } & \widetilde{\mathrm{map}}_{\mathcal{B}}(GX, GY) \otimes_A \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, GX)
\end{array}
$$

The following construction can be carried out for categories enriched over any bicategory (see [41, Section 2]):

Remark 4.1.4. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, and let A be a category. It is immediate from Definition 4.1.1 that the set of objects $\mathcal{A}_0(A)$ are the objects of an $\mathcal{E}(A^{\mathrm{op}} \times A)$-enriched category, where $\mathcal{E}(A^{\mathrm{op}} \times A)$ is equipped with the monoidal structure induced by Example 3.5.5. Given objects $X, Y \in \mathcal{A}_0(A)$, the mapping object between them is given by $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \in \mathcal{E}(A^{\mathrm{op}} \times A)$, and units and composition are inherited from the $\mathcal{E}$-category structure. We will denote this $\mathcal{E}(A^{\mathrm{op}} \times A)$-enriched category by $\underline{\mathcal{A}}(A)$.

Denote the underlying category of this $\mathcal{E}(A^{\mathrm{op}} \times A)$-enriched category by $\mathcal{A}(A)$. To avoid confusion with the established meaning of the underlying category of a prederivator, we will call $\mathcal{A}(A)$ the **category induced by $\mathcal{A}$ at A.**

Explicitly, $\mathcal{A}(\mathrm{A})$ is defined as follows: objects are given by the set $\mathcal{A}_0(\mathrm{A})$, and, for any $X, Y \in \mathcal{A}_0(\mathrm{A})$, we have

$$\mathcal{A}(\mathrm{A})(X, Y) = \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{A})(\mathrm{h_A}, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)).$$

For any object $X \in \mathcal{A}_0(\mathrm{A})$, the identity map is given by the unit

$$j : \mathrm{h_A} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, X)$$

in $\mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{A})(\mathrm{h_A}, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, X))$. Given objects $X, Y, Z \in \mathcal{A}_0(\mathrm{A})$ and maps

$$f : \mathrm{h_A} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$$

$$g : \mathrm{h_A} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z)$$

their composite is given by the following:

$$\mathrm{h_A} \xrightarrow{\cong} \mathrm{h_A} \otimes_{\mathrm{A}} \mathrm{h_A} \xrightarrow{g \,\otimes_{\mathrm{A}} f} \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \otimes_{\mathrm{A}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \xrightarrow{\circ} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$$

The unnamed isomorphism is $\lambda^{-1} = \rho^{-1} : \mathrm{h_A} \to \mathrm{h_A} \otimes_{\mathrm{A}} \mathrm{h_A}$.

Similarly, any $\mathcal{E}$-morphism $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ induces an $\mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{A})$-functor $F : \underline{\mathcal{A}}(\mathrm{A}) \to \underline{\mathcal{B}}(\mathrm{A})$. The underlying functor of this $\mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{A})$-functor is called the **functor induced by F at A**. It takes any object $X \in \mathcal{A}(\mathrm{A})$ to $FX \in \mathcal{B}(\mathrm{A})$, and given a map

$$f : \mathrm{h_A} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$$

in $\mathcal{A}(\mathrm{A})$, its image is given by

$$\mathrm{h_A} \xrightarrow{f} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \xrightarrow{F} \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY).$$

Suppose we have $\mathcal{E}$-morphisms $F, G : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$. Any $\mathcal{E}$-natural transformation $\beta : F \Rightarrow G$ induces an $\mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{A})$-natural transformation

$$\underline{\mathcal{A}}(\mathrm{A}) \quad \overset{F}{\underset{G}{\Rrightarrow}}{\scriptstyle\beta} \quad \underline{\mathcal{B}}(\mathrm{A}).$$

For any $X \in \mathcal{A}(A)$, the component

$$\beta_X : h_A \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, GX)$$

may be thought of as a map in $\mathcal{B}(A)$. It follows immediately from the $\mathcal{E}$-naturality condition that these form a natural transformation between the functors induced by F and G, which we call the **natural transformation induced by β at A.**

Lemma 4.1.5. *$\mathcal{E}$-Categories, $\mathcal{E}$-morphisms and $\mathcal{E}$-natural transformations form a 2-category $\mathcal{E}$-**Cat**. Moreover, any category A induces a 2-functor:*

$$\mathcal{E}\text{-}\mathbf{Cat} \to \mathbf{CAT}$$
$$\underline{\mathcal{A}} \mapsto \mathcal{A}(A)$$

Proof. We need to define composition for $\mathcal{E}$-morphisms and $\mathcal{E}$-natural transformations. First, suppose we have $\mathcal{E}$-morphsims $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ and $G : \underline{\mathcal{B}} \to \underline{\mathcal{C}}$. The composite $G \circ F$ takes an object $X \in \mathcal{A}_0(A)$ to $GFX \in \mathcal{C}_0(A)$, and the action on mapping spaces is simply given by

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \xrightarrow{F} \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY) \xrightarrow{G} \widetilde{\mathrm{map}}_{\mathcal{C}}(GFX, GFY)$$

for any $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$.

Given $\mathcal{E}$-natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, their vertical composite $\beta \cdot \alpha : F \Rightarrow H$ has component at $X \in \mathcal{A}_0(A)$ given by the composite $\beta_X \circ \alpha_X$ in $\mathcal{B}(A)$. Note that an $\mathcal{E}$-natural transformation is an isomorphism if and only if each of its components is an isomorphism.

Given $\mathcal{E}$-morphisms $F, H : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ and $G, K : \underline{\mathcal{B}} \to \underline{\mathcal{C}}$, suppose we have $\mathcal{E}$-natural transformations $\alpha : F \Rightarrow H$ and $\beta : G \Rightarrow K$. Their horizontal composite $\beta \circ \alpha$ has component at $X \in \mathcal{A}_0(A)$ given by

$$\beta_{HX} \circ G(\alpha_X) = K(\alpha_X) \circ \beta_{FX}$$

in $\mathcal{C}(A)$.

It is routine to check that the composite of $\mathcal{E}$-morphisms is an $\mathcal{E}$-morphism, and that the vertical and horizontal composites above define $\mathcal{E}$-natural transformations. The 2-category axioms for this data follow from the corresponding facts for the 2-category **CAT**. $\square$

Example 4.1.6. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category. For each category A, suppose we have a set of

objects $\mathcal{B}_0(A) \subseteq \mathcal{A}_0(A)$. The **full sub-$\mathcal{E}$-category** of $\underline{\mathcal{A}}$ on these objects is the $\mathcal{E}$-category $\underline{\mathcal{B}}$ defined as follows:

- For each category A, the objects are $\mathcal{B}_0(A)$.

- For any two objects $X \in \mathcal{B}_0(A)$ and $Y \in \mathcal{B}_0(B)$, the mapping object is $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \in \mathcal{E}(A^{\mathrm{op}} \times B)$.

- Units and composition are inherited from $\underline{\mathcal{A}}$.

The $\mathcal{E}$-category axioms for $\underline{\mathcal{B}}$ follow immediately from the axioms for $\underline{\mathcal{A}}$. Note that, for any category A, the induced category $\mathcal{B}(A)$ is the full subcategory of $\mathcal{A}(A)$ on the objects $\mathcal{B}_0(A)$.

Definition 4.1.7. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category. We may form the **opposite** $\mathcal{E}$-category $\underline{\mathcal{A}}^{\mathrm{op}}$ as follows: for any category A, we define

$$\mathcal{A}_0^{\mathrm{op}}(A) = \mathcal{A}_0(A^{\mathrm{op}}),$$

and, given $X \in \mathcal{A}_0(A^{\mathrm{op}})$ and $Y \in \mathcal{A}_0(B^{\mathrm{op}})$, define

$$\widetilde{\mathrm{map}}_{\mathcal{A}^{\mathrm{op}}}(X, Y) = \sigma^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, X) \in \mathcal{E}(A^{\mathrm{op}} \times B),$$

where $\sigma : A^{\mathrm{op}} \times B \xrightarrow{\cong} B \times A^{\mathrm{op}}$ is the canonical isomorphism.

For any $X \in \mathcal{A}_0(A^{\mathrm{op}})$, the unit is given by the composite below

$$h_A \xrightarrow{\cong} \sigma^* h_{A^{\mathrm{op}}} \xrightarrow{\sigma^* j} \sigma^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, X)$$

where the unnamed isomorphism is an instance of Lemma 3.1.8, as in Lemma 3.5.7.

Similarly, given objects $X \in \mathcal{A}_0(A^{\mathrm{op}})$, $Y \in \mathcal{A}_0(B^{\mathrm{op}})$ and $Z \in \mathcal{A}_0(C^{\mathrm{op}})$, composition is as follows:

$$\sigma^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y) \otimes_B \sigma^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, X) \xrightarrow{\tau} \sigma^* (\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, X) \otimes_{B^{\mathrm{op}}} \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y))$$
$$\downarrow{\sigma^*(\circ)}$$
$$\sigma^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X)$$

Here τ is the canonical isomorphism of Lemma 3.5.7.

The $\mathcal{E}$-category axioms for $\underline{\mathcal{A}}^{\mathrm{op}}$ follow easily from the axioms for $\underline{\mathcal{A}}$, using the coherence of Lemma 3.5.7.

Remark 4.1.8. For any $\mathcal{E}$-category $\underline{\mathcal{A}}$, and any category A, we have an isomorphism of categories

$$\mathcal{A}^{\mathrm{op}}(\mathrm{A}) \cong \mathcal{A}(\mathrm{A}^{\mathrm{op}})^{\mathrm{op}}.$$

To see this, note that the objects of both categories are the same by definition. Moreover, given two objects $X, Y \in \mathcal{A}_0(\mathrm{A}^{\mathrm{op}})$, we have:

$$\begin{aligned}
\mathcal{A}^{\mathrm{op}}(\mathrm{A})(X, Y) &= \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{A})(\mathrm{h}_{\mathrm{A}}, \sigma^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)) \\
&\cong \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{A})(\sigma^* \mathrm{h}_{\mathrm{A}^{\mathrm{op}}}, \sigma^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, X)) \\
&\cong \mathcal{E}(\mathrm{A} \times \mathrm{A}^{\mathrm{op}})(\mathrm{h}_{\mathrm{A}^{\mathrm{op}}}, \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, X)) \\
&= \mathcal{A}(\mathrm{A}^{\mathrm{op}})(Y, X)
\end{aligned}$$

It is easy to check that this bijection respects identities and composition, using Lemma 3.5.7.

Remark 4.1.9. In the obvious way, we can extend the opposite $\mathcal{E}$-category construction of Definition 4.1.7 to $\mathcal{E}$-morphisms and $\mathcal{E}$-natural transformations. We obtain a 2-functor on $\mathcal{E}$-**Cat**, which preserves the direction of 1-cells but reverses the direction of 2-cells.

Theorem 4.1.10. *Let $\mathcal{D}$ be a closed $\mathcal{E}$-module, with action:*

$$\otimes : \mathcal{E} \times \mathcal{D} \to \mathcal{D}$$
$$\triangleleft : \mathcal{D} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{D}$$
$$\mathrm{map}_{\mathcal{D}}(-, -) : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}$$

Then there is an associated $\mathcal{E}$-category $\underline{\mathcal{D}}$ such that, for any category A, the induced category $\mathcal{D}(\mathrm{A})$ of Remark 4.1.4 recovers the value at A of the derivator $\mathcal{D}$.

Proof. The $\mathcal{E}$-category $\underline{\mathcal{D}}$ is defined as follows. For any category A, the set $\mathcal{D}_0(\mathrm{A})$ is the set of objects of $\mathcal{D}(\mathrm{A})$. Given $X \in \mathcal{D}_0(\mathrm{A})$ and $Y \in \mathcal{D}_0(\mathrm{B})$, the mapping object $\widetilde{\mathrm{map}}_{\mathcal{D}}(X, Y) \in \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$ is their image under the functor:

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(-, -) : \mathcal{D}(\mathrm{A})^{\mathrm{op}} \times \mathcal{D}(\mathrm{B}) \to \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$$

To describe composition and units, let $X \in \mathcal{D}(\mathrm{A})$, and consider the adjunction

$$\mathcal{E}^{A^{\mathrm{op}}} \underset{\widetilde{\mathrm{map}}_{\mathcal{D}}(X,-)}{\overset{-\,\otimes_A X}{\rightleftarrows}} \perp \mathcal{D},$$

described in Theorem 3.2.2. Given any $X \in \mathcal{D}(A)$, the unit

$$j : \mathrm{h}_A \to \widetilde{\mathrm{map}}_{\mathcal{D}}(X,X)$$

is adjunct under this adjunction to the canonical isomorphism

$$\lambda : \mathrm{h}_A \otimes_A X \to X$$

of Proposition 3.5.2. Similarly, given objects $X \in \mathcal{D}(A)$, $Y \in \mathcal{D}(B)$ and $Z \in \mathcal{D}(C)$, composition is adjunct to the following:

$$(\widetilde{\mathrm{map}}_{\mathcal{D}}(Y,Z) \otimes_B \widetilde{\mathrm{map}}_{\mathcal{D}}(X,Y)) \otimes_A X \xrightarrow{\ \alpha\ } \widetilde{\mathrm{map}}_{\mathcal{D}}(Y,Z) \otimes_B (\widetilde{\mathrm{map}}_{\mathcal{D}}(X,Y) \otimes_A X)$$

$$\downarrow \widetilde{\mathrm{map}}_{\mathcal{D}}(Y,Z)\,\otimes_B\,\epsilon$$

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(Y,Z) \otimes_B Y$$

$$\downarrow \epsilon$$

$$Z$$

To see that this data satisfies the axioms of Definition 4.1.1, we can replace each diagram that we need to commute by its adjunct. The first axiom then follows using the first commutative diagram of Proposition 3.5.2. The second and third axioms of Definition 4.1.1 follow using the second commutative diagram in Proposition 3.5.2.

For any category A, we still need to show that the induced category of Remark 4.1.4 recovers the value of $\mathcal{D}$ at A. Given $X, Y \in \mathcal{D}(A)$, consider the isomorphsims:

$$\mathcal{D}(A)(X,Y) \cong \mathcal{D}(A)(\mathrm{h}_A \otimes_A X, Y)$$
$$\cong \mathcal{E}(A^{\mathrm{op}} \times A)(\mathrm{h}_A, \widetilde{\mathrm{map}}_{\mathcal{D}}(X,Y))$$

Given a map $f : X \to Y$ in $\mathcal{D}(A)$, write $\tilde{f} : \mathrm{h}_A \to \widetilde{\mathrm{map}}_{\mathcal{D}}(X,Y)$ for the corresponding map in $\mathcal{E}(A^{\mathrm{op}} \times A)$. Thus, $\tilde{f}$ is the unique map making the diagram below commute:

$$\begin{array}{ccc}
h_A \otimes_A X & \xrightarrow{\quad \lambda \quad} & X \\
{\scriptstyle \tilde{f} \otimes_A X} \Big\downarrow & & \Big\downarrow {\scriptstyle f} \\
\widetilde{map}_{\mathcal{D}}(X,Y) \otimes_A X & \xrightarrow{\quad \epsilon \quad} & Y
\end{array}$$

We claim that the description of identity and composition in Remark 4.1.4 corresponds to identity and composition in $\mathcal{D}(A)$ under these bijections.

By definition, given any $X \in \mathcal{D}(A)$, we have $\widetilde{id}_X = j : h_A \to \widetilde{map}_{\mathcal{D}}(X,X)$. This shows that the bijection respects identities. To see that it respects composition, suppose we have composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{D}(A)$. Using the definition of composition from Remark 4.1.4, we need the diagram below to commute:

$$\begin{array}{ccc}
h_A \xrightarrow{\rho^{-1}} h_A \otimes_A h_A & \xrightarrow{\tilde{g} \otimes_A \tilde{f}} & \widetilde{map}_{\mathcal{D}}(Y,Z) \otimes_A \widetilde{map}_{\mathcal{D}}(X,Y) \\
& & \Big\downarrow {\scriptstyle \circ} \\
\xrightarrow{\quad \widetilde{(g \circ f)} \quad} & & \widetilde{map}_{\mathcal{D}}(X,Z)
\end{array}$$

Taking the adjunct under the adjunction $- \otimes_A X \dashv \widetilde{map}_{\mathcal{D}}(X,-)$, and using the definition of composition in $\underline{\mathcal{D}}$ given above, we can see that this commutes. $\qquad \square$

By Example 3.4.12, for any category J and any closed $\mathcal{E}$-module $\mathcal{D}$, the shifted derivator $\mathcal{D}^J$ is a closed $\mathcal{E}$-module. Thus, by Theorem 4.1.10, $\mathcal{D}^J$ gives rise to an associated $\mathcal{E}$-category. In particular, by Example 3.4.8, this is the case for $\mathcal{E}^J$. More generally, we will prove in Section 4.3 that, for any $\mathcal{E}$-category $\underline{\mathcal{A}}$ and any category J, we may form a shifted $\mathcal{E}$-category $\underline{\mathcal{A}}^J$.

Proposition 4.1.11. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, let A be a category, and let $X \in \mathcal{A}_0(A)$. The mapping objects in $\underline{\mathcal{A}}$ induce an $\mathcal{E}$-morphism:*

$$\widetilde{map}_{\mathcal{A}}(X,-) : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{A^{op}}$$

Proof. Let $X \in \mathcal{A}_0(A)$. On objects, the $\mathcal{E}$-morphism

$$\widetilde{map}_{\mathcal{A}}(X,-) : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{A^{op}}$$

takes $Y \in \mathcal{A}_0(B)$ to $\widetilde{map}_{\mathcal{A}}(X,Y) \in \mathcal{E}(A^{op} \times B)$. Given a further object $Z \in \mathcal{A}_0(C)$, we need

a map

$$\widetilde{\operatorname{map}}_{\mathcal{A}}(X,-) : \widetilde{\operatorname{map}}_{\mathcal{A}}(Y,Z) \to \widetilde{\operatorname{map}}_{\mathcal{E}^{\mathrm{A}^{\mathrm{op}}}}(\widetilde{\operatorname{map}}_{\mathcal{A}}(X,Y), \widetilde{\operatorname{map}}_{\mathcal{A}}(X,Z))$$

in $\mathcal{E}(\mathrm{B}^{\mathrm{op}} \times \mathrm{C})$. Consider the adjunction below:

Under this adjunction, take the structure map to be adjunct to composition:

$$\circ : \widetilde{\operatorname{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{B}} \widetilde{\operatorname{map}}_{\mathcal{A}}(X,Y) \to \widetilde{\operatorname{map}}_{\mathcal{A}}(X,Z)$$

To prove that this satisfies the axioms of Definition 4.1.2, replace the diagrams that we need to commute by their adjuncts, and use the description of units and composition in $\underline{\mathcal{E}}^{\mathrm{A}^{\mathrm{op}}}$ from Theorem 4.1.10. The second axiom of Definition 4.1.2 for $\widetilde{\operatorname{map}}_{\mathcal{A}}(X,-)$ corresponds precisely to the second axiom of Definition 4.1.1 for $\underline{\mathcal{A}}$. Similarly, the first axiom of Definition 4.1.2 corresponds to the first of Definition 4.1.1. □

Definition 4.1.12. Let $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{\mathrm{A}^{\mathrm{op}}}$ be an $\mathcal{E}$-morphism. We say that F is **representable** if there is an object $X \in \mathcal{A}_0(\mathrm{A})$ and an $\mathcal{E}$-natural isomorphism:

$$F \cong \widetilde{\operatorname{map}}_{\mathcal{A}}(X,-) : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{\mathrm{A}^{\mathrm{op}}}$$

Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, let A and B be categories, and let $X \in \mathcal{A}_0(\mathrm{A})$. As in Remark 4.1.4, consider the functor

$$\widetilde{\operatorname{map}}_{\mathcal{A}}(X,-) : \mathcal{A}(\mathrm{B}) \to \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$$

induced by the $\mathcal{E}$-morphism $\widetilde{\operatorname{map}}_{\mathcal{A}}(X,-)$. Given any map $f : h_{\mathrm{B}} \to \widetilde{\operatorname{map}}_{\mathcal{A}}(Y,Z)$ in $\mathcal{A}(\mathrm{B})$, write

$$\widetilde{\operatorname{map}}_{\mathcal{A}}(X,f) : \widetilde{\operatorname{map}}_{\mathcal{A}}(X,Y) \to \widetilde{\operatorname{map}}_{\mathcal{A}}(X,Z)$$

for its image in $\mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$. Using the description of $\widetilde{\operatorname{map}}_{\mathcal{A}}(X,-)$ from Proposition 4.1.11, this is the composite below:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{\lambda^{-1}} \mathrm{h_B} \otimes_\mathrm{B} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{f \otimes_\mathrm{B} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)} \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_\mathrm{B} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$
$$\downarrow \circ$$
$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z)$$

Given any object $Y \in \mathcal{A}_0(\mathrm{B}) = \mathcal{A}_0^{\mathrm{op}}(\mathrm{B^{op}})$, we will write

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(-,Y) := \widetilde{\mathrm{map}}_{\mathcal{A}^{\mathrm{op}}}(Y,-) : \underline{\mathcal{A}}^{\mathrm{op}} \to \underline{\mathcal{E}}^{\mathrm{B}}.$$

As above, any map $g : \mathrm{h_A} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z)$ in $\mathcal{A}(\mathrm{A})$ induces a map

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(g,Y) : \widetilde{\mathrm{map}}_{\mathcal{A}}(Z,Y) \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$

in $\mathcal{E}(\mathrm{A^{op}} \times \mathrm{B})$. Explicitly, this is the following composite:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Z,Y) \xrightarrow{\rho^{-1}} \widetilde{\mathrm{map}}_{\mathcal{A}}(Z,Y) \otimes_\mathrm{A} \mathrm{h_A} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(Z,Y) \otimes_\mathrm{A} g} \widetilde{\mathrm{map}}_{\mathcal{A}}(Z,Y) \otimes_\mathrm{A} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z)$$
$$\downarrow \circ$$
$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$

Remark 4.1.13. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, let A be a category, and let $f : \mathrm{h_A} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$ be a map in $\mathcal{A}(\mathrm{A})$. The unit axioms for $\underline{\mathcal{A}}$ imply that the diagram below commutes:

$$
\begin{array}{ccc}
\mathrm{h_A} & \xrightarrow{\quad j \quad} & \widetilde{\mathrm{map}}_{\mathcal{A}}(X,X) \\
{\scriptstyle j}\downarrow & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{A}}(X,f)} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Y) & \xrightarrow[\widetilde{\mathrm{map}}_{\mathcal{A}}(f,Y)]{} & \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)
\end{array}
$$

So, precomposing with j, we can recover the map f from $\widetilde{\mathrm{map}}_{\mathcal{A}}(X,f)$ or $\widetilde{\mathrm{map}}_{\mathcal{A}}(f,Y)$.

Remark 4.1.14. Suppose $\underline{\mathcal{D}}$ is the $\mathcal{E}$-category associated to a closed $\mathcal{E}$-module $\mathcal{D}$, and let $X \in \mathcal{D}(\mathrm{A})$ and $Y \in \mathcal{D}(\mathrm{B})$. Then, for any category C, the functors

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(X,-) : \mathcal{D}(\mathrm{C}) \to \mathcal{E}(\mathrm{A^{op}} \times \mathrm{C})$$
$$\widetilde{\mathrm{map}}_{\mathcal{D}}(-,Y) : \mathcal{D}(\mathrm{C})^{\mathrm{op}} \to \mathcal{E}(\mathrm{C^{op}} \times \mathrm{B}),$$

induced by the $\mathcal{E}$-morphisms $\widetilde{\mathrm{map}}_{\mathcal{D}}(X,-)$ and $\widetilde{\mathrm{map}}_{\mathcal{D}}(-,Y)$, recover the component functors

of the original derivator maps:

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(X, -) : \mathcal{D} \to \mathcal{E}^{A^{\mathrm{op}}}$$

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(-, Y) : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{B}$$

Using the descriptions above of $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -)$ and $\widetilde{\mathrm{map}}_{\mathcal{A}}(-, Y)$, we can give a useful characterisation of $\mathcal{E}$-naturality:

Lemma 4.1.15. *Let $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ be $\mathcal{E}$-categories, and let $F, G : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ be $\mathcal{E}$-morphisms. Suppose we have a collection of maps*

$$\beta_X : \mathrm{h}_A \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, GX),$$

for every category A, *and every $X \in \mathcal{A}_0(A)$. These maps form an $\mathcal{E}$-natural transformation $\beta : F \Rightarrow G$ if and only if the diagram below commutes, for any $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$:*

$$
\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) & \xrightarrow{\quad F \quad} & \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY) \\
{\scriptstyle G} \downarrow & & \downarrow {\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, \beta_Y)} \\
\widetilde{\mathrm{map}}_{\mathcal{B}}(GX, GY) & \xrightarrow[\widetilde{\mathrm{map}}_{\mathcal{B}}(\beta_X, GY)]{} & \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, GY)
\end{array}
$$

Proof. The diagram in Definition 4.1.3 expressing $\mathcal{E}$-naturality is the same as this diagram, once we expand $\widetilde{\mathrm{map}}_{\mathcal{B}}(FX, \beta_Y)$ and $\widetilde{\mathrm{map}}_{\mathcal{B}}(\beta_X, GY)$. $\qquad \square$

Using this characterisation, we can now prove the following two lemmas:

Lemma 4.1.16. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, let* A *be a category, and let $f : \mathrm{h}_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ be a map in $\mathcal{A}(A)$. Then the maps*

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(f, Z) : \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$$

are $\mathcal{E}$-natural in $Z \in \mathcal{A}_0(B)$.

Proof. Fix a map $f : \mathrm{h}_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ in $\mathcal{A}(A)$. By Lemma 4.1.15, we need the diagram below to commute, for any $Z \in \mathcal{A}_0(B)$ and $W \in \mathcal{A}_0(C)$:

5

5

First commutative diagram:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Z,W) \xrightarrow{\ \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,-)\ } \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z), \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,W))$$

with vertical left map $\widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)$ to
$$\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z), \widetilde{\mathrm{map}}_{\mathcal{A}}(X,W))$$

right vertical map $\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z),\widetilde{\mathrm{map}}_{\mathcal{A}}(f,W))$ and bottom map $\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(f,Z),\widetilde{\mathrm{map}}_{\mathcal{A}}(X,W))$ to
$$\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z), \widetilde{\mathrm{map}}_{\mathcal{A}}(X,W))$$

Under the adjunction

$$\mathcal{E}(\mathrm{B^{op}} \times \mathrm{C}) \underset{\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z),-)}{\overset{-\otimes_{\mathrm{B}}\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z)}{\rightleftarrows}} \perp \ \mathcal{E}(\mathrm{A^{op}} \times \mathrm{C})$$

this reduces to the square:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Z,W) \otimes_{\mathrm{B}} \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \xrightarrow{\ \circ\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,W)$$

left map $\widetilde{\mathrm{map}}_{\mathcal{A}}(Z,W)\otimes_{\mathrm{B}}\widetilde{\mathrm{map}}_{\mathcal{A}}(f,Z)$, right map $\widetilde{\mathrm{map}}_{\mathcal{A}}(f,W)$, bottom:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Z,W) \otimes_{\mathrm{B}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z) \xrightarrow{\ \circ\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(X,W)$$

Using the descriptions of $\widetilde{\mathrm{map}}_{\mathcal{A}}(f,Z)$ and $\widetilde{\mathrm{map}}_{\mathcal{A}}(f,W)$ given above, we can see that this diagram commutes. $\square$

Remark 4.1.17. For any $\mathcal{E}$-category $\underline{\mathcal{A}}$, and any category A, the assignment

$$y : \mathcal{A}(\mathrm{A})^{\mathrm{op}} \longrightarrow \mathcal{E}\text{-}\mathbf{Cat}(\underline{\mathcal{A}}, \underline{\mathcal{E}}^{\mathrm{A^{op}}})$$
$$X \mapsto \widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)$$

is functorial. Two $\mathcal{E}$-natural transformations are equal if and only if their components are equal, so the functoriality of y follows by the functoriality of the map

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(-,W) : \mathcal{A}(\mathrm{A})^{\mathrm{op}} \to \mathcal{E}(\mathrm{A^{op}} \times \mathrm{B}),$$

for any category B and any $W \in \mathcal{A}_0(\mathrm{B})$.

2
2

Lemma 4.1.18. *Let $F : \underline{A} \to \underline{B}$ be an $\mathcal{E}$-morphism. The maps*

$$F : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY)$$

are $\mathcal{E}$-natural in both $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$.

Proof. Fix $X \in \mathcal{A}_0(A)$. We will prove that the maps

$$F : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY)$$

are $\mathcal{E}$-natural in $Y \in \mathcal{A}_0(B)$; $\mathcal{E}$-naturality in the first variable is dual. Given $Y \in \mathcal{A}_0(B)$ and $Z \in \mathcal{A}_0(C)$, we need the diagram below to commute:

Under the adjunction

this corresponds exactly to the first axiom of Definition 4.1.2 for the $\mathcal{E}$-morphism F. $\qquad\square$

We end this section by proving that cocontinuous $\mathcal{E}$-module maps, and $\mathcal{E}$-module modifications between them, induce $\mathcal{E}$-morphisms and $\mathcal{E}$-natural transformations, using the coherence results at the end of Section 3.5.

Proposition 4.1.19. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be closed $\mathcal{E}$-modules. Then any cocontinuous $\mathcal{E}$-module map $F : \mathcal{D}_1 \to \mathcal{D}_2$ induces an $\mathcal{E}$-morphism*

$$F : \underline{\mathcal{D}}_1 \to \underline{\mathcal{D}}_2$$

on the associated $\mathcal{E}$-categories of Theorem 4.1.10. Moreover, for any category A, the functor $F : \mathcal{D}_1(A) \to \mathcal{D}_2(A)$ induced by this $\mathcal{E}$-morphism is the component at A of the original derivator map.

Proof. The $\mathcal{E}$-morphism $F : \underline{\mathcal{D}}_1 \to \underline{\mathcal{D}}_2$ takes any object $X \in \mathcal{D}_1(A)$ to its image under the derivator map, $FX \in \mathcal{D}_2(A)$. Let $X \in \mathcal{D}_1(A)$ and $Y \in \mathcal{D}_1(B)$, and consider the adjunction

$$\mathcal{E}^{A^{\mathrm{op}}} \underset{\widetilde{\mathrm{map}}_{\mathcal{D}_2}(FX,-)}{\overset{-\otimes_A FX}{\rightleftarrows}} \mathcal{D}_2.$$

Under this adjunction, define the structure map

$$F : \widetilde{\mathrm{map}}_{\mathcal{D}_1}(X, Y) \to \widetilde{\mathrm{map}}_{\mathcal{D}_2}(FX, FY)$$

to be adjunct to the composite below:

$$\widetilde{\mathrm{map}}_{\mathcal{D}_1}(X, Y) \otimes_A FX \xrightarrow{\varphi} F(\widetilde{\mathrm{map}}_{\mathcal{D}_1}(X, Y) \otimes_A X) \xrightarrow{F(\epsilon)} FY$$

Here the map φ is the canonical isomorphism of Lemma 3.5.9.

To prove that this satisfies the axioms of Definition 4.1.2, we replace the diagrams that need to commute by their adjuncts. Using the description of units and composition in $\underline{\mathcal{D}}_2$, from Theorem 4.1.10, the first axiom of Definition 4.1.2 follows from the first commutative diagram of Lemma 3.5.9, and the second axiom follows from the second commutative diagram.

To see that the functor of Remark 4.1.4 recovers the original functor $F : \mathcal{D}_1(A) \to \mathcal{D}_2(A)$, suppose we have a map $f : X \to Y$ in $\mathcal{D}_1(A)$. Using Theorem 4.1.10, we need to check that its image under the prederivator map F is adjunct to the composite below (up to the isomorphism $\lambda : \mathrm{h}_A \otimes_A FX \xrightarrow{\cong} FX$):

$$\mathrm{h_A} \xrightarrow{\ j\ } \widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,X) \xrightarrow{\ \widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,f)\ } \widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,Y) \xrightarrow{\ F\ } \widetilde{\mathrm{map}}_{\mathcal{D}_2}(FX,FY)$$

This is immediate, using the definition of the unit in $\underline{\mathcal{D}}_2$, from Theorem 4.1.10, and the second commutative diagram of Lemma 3.5.9. $\qquad\square$

Lemma 4.1.20. *Let $F, G : \mathcal{D}_1 \to \mathcal{D}_2$ be cocontinuous $\mathcal{E}$-module maps between closed $\mathcal{E}$-modules. Then any $\mathcal{E}$-module modification*

induces an $\mathcal{E}$-natural transformation between the associated $\mathcal{E}$-morphisms of Proposition 4.1.19.

Proof. Suppose we have categories A and B, and objects $X \in \mathcal{D}_1(A)$ and $Y \in \mathcal{D}_1(B)$. We need to check that the diagram of Lemma 4.1.15 commutes; equivalently, we may consider its adjunct under the adjunction below:

That diagram commutes, using the description of F and G from Proposition 4.1.19, and the fact that β respects the cancelling tensor product, as in Lemma 3.5.10. $\qquad\square$

Remark 4.1.21. Suppose $F : \mathcal{D}_1 \to \mathcal{D}_2$ is a continuous map between closed $\mathcal{E}$-modules. If F preserves cotensors then the dual of Proposition 4.1.19 implies that F induces an $\mathcal{E}$-category map $F : \underline{\mathcal{D}}_1 \to \underline{\mathcal{D}}_2$. Given a map F that is both continuous and cocontinuous, and preserves tensors and cotensors, there are two potentially distinct enrichments on F. Without coherence between the isomorphisms that express the preservation of tensors and cotensors, these two induced enrichments need not agree.

4.2 The Yoneda lemma and adjunctions for $\mathcal{E}$-categories

We begin this section with Theorem 4.2.1, an $\mathcal{E}$-category analogue of the Yoneda lemma. As is the case in other settings, this result is extremely useful, and we will use it repeatedly in

our discussion of enriched derivators in Chapter 5. We also use it immediately to prove Theorem 4.2.5, which gives a convenient characterisation of adjunctions in the 2-category $\mathcal{E}$-**Cat**, reminiscent of the familiar result for enriched categories in [27, Chapter 1]. We finish this section with Proposition 4.2.7, showing that an adjunction between closed $\mathcal{E}$-modules whose left adjoint preserves tensors induces an $\mathcal{E}$-category adjunction. This result is important for the proof, in Chapter 5, that closed $\mathcal{E}$-modules induce enriched derivators.

Theorem 4.2.1 (Yoneda lemma for $\mathcal{E}$-categories). *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, let A be a category, and let $X \in \mathcal{A}_0(\mathrm{A})$. Let $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{\mathrm{A^{op}}}$ be an $\mathcal{E}$-morphism. Then we have a natural bijection:*

$$\mathcal{E}\text{-}\mathbf{Cat}(\underline{\mathcal{A}}, \underline{\mathcal{E}}^{\mathrm{A^{op}}})(\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -), F) \cong \mathcal{E}(\mathrm{A^{op}} \times \mathrm{A})(\mathrm{h_A}, FX)$$

Proof. Let $\beta : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) \Rightarrow F$ be an $\mathcal{E}$-natural transformation. This determines a map in $\mathcal{E}(\mathrm{A^{op}} \times \mathrm{A})$ as follows:

$$\mathrm{h_A} \xrightarrow{\ j\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(X, X) \xrightarrow{\ \beta_X\ } FX$$

On the other hand, given $f : \mathrm{h_A} \to FX$ in $\mathcal{E}(\mathrm{A^{op}} \times \mathrm{A})$, we can construct the following family of maps, one for each $Y \in \mathcal{A}_0(\mathrm{B})$:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \xrightarrow{\ F\ } \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(FX, FY) \xrightarrow{\ \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(f, FY)\ } \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\mathrm{h_A}, FY)$$

$$\cong \Big\downarrow \varrho$$

$$FY$$

Here the isomorphism $\varrho : \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\mathrm{h_A}, -) \xrightarrow{\cong} \mathrm{id}$ is conjugate to the inverse of the unit isomorphism $\rho^{-1} : \mathrm{id} \xrightarrow{\cong} - \otimes_{\mathrm{A}} \mathrm{h_A}$. Thus, for any $Z \in \mathcal{E}(\mathrm{A^{op}} \times \mathrm{B})$, the diagram below commutes:

$$
\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\mathrm{h_A}, Z) \otimes_{\mathrm{A}} \mathrm{h_A} & \xrightarrow{\ \epsilon\ } & \\
\Big\downarrow \rho & & Z \\
\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(\mathrm{h_A}, Z) & \xrightarrow{\ \varrho\ } &
\end{array}
\tag{4.1}
$$

We claim that this map ϱ is $\mathcal{E}$-natural in $Z \in \mathcal{E}^{A^{op}}(B)$, in which case the composite above is $\mathcal{E}$-natural in $Y \in \mathcal{A}_0(B)$, by Lemma 4.1.16 and Lemma 4.1.18.

To see this, let $Z \in \mathcal{E}(A^{op} \times B)$ and $W \in \mathcal{E}(A^{op} \times C)$. By Lemma 4.1.15, we need the diagram below to commute:

$$\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(Z, W) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, -)} \widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, Z), \widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, W))$$

with diagonal map $\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(\varrho, W)$ and vertical map $\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, Z), \varrho)$ to $\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, Z), W)$.

Taking adjuncts under the adjunction

$$\mathcal{E}^{B^{op}} \underset{\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, Z), -)}{\overset{-\otimes_B \widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, Z)}{\rightleftarrows}} \mathcal{E}^{A^{op}}, \qquad \perp$$

this corresponds to the square below, using the description of $\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, -)$ given in Proposition 4.1.11:

$$\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(Z, W) \otimes_B \widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, Z) & \xrightarrow{\circ} & \widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(h_A, W) \\
\downarrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(Z,W) \otimes_B \varrho} & & \downarrow{\scriptstyle \varrho} \\
\widetilde{\mathrm{map}}_{\mathcal{E}A^{op}}(Z, W) \otimes_B Z & \xrightarrow{\epsilon} & W
\end{array}$$

We can expand the vertical arrows in this diagram using the equation $\varrho = \epsilon \circ \rho^{-1}$ from (4.1). Using the description of composition in $\underline{\mathcal{E}}^{A^{op}}$ from Theorem 4.1.10, we can then see that this diagram commutes.

Thus, we have well-defined functions in both directions. We will now show that they are mutually inverse.

First, suppose we have a map $f : h_A \to FX$ in $\mathcal{E}(A^{op} \times A)$. We need to show that the diagram below commutes:

$$\mathrm{h_A} \xrightarrow{j} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,X) \xrightarrow{F} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(FX,FX) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(f,FX)} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(\mathrm{h_A},FX)$$

with the diagonal map f to FX and ϱ the vertical map.

Using Remark 4.1.13, this diagram is equal to the diagram below:

$$
\mathrm{h_A} \xrightarrow{\tilde{f}} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(\mathrm{h_A},FX)
$$
with diagonal f and vertical ϱ to FX.
$$\tag{4.2}$$

Here $\tilde{f}$ is the map corresponding to $f : \mathrm{h_A} \to FX$, as in the proof of Theorem 4.1.10. That is, $\tilde{f}$ makes the diagram below commute:

$$
\begin{array}{ccc}
\mathrm{h_A} \otimes_{\mathrm{A}} \mathrm{h_A} & \xrightarrow{\lambda} & \mathrm{h_A} \\
\downarrow{\tilde{f} \otimes_{\mathrm{A}} \mathrm{h_A}} & & \downarrow{f} \\
\widetilde{\mathrm{map}}_{\mathcal{D}}(\mathrm{h_A},FX) \otimes_{\mathrm{A}} \mathrm{h_A} & \xrightarrow{\epsilon} & FX
\end{array}
$$

Using this commutative diagram, and (4.1), it is easy to see that (4.2) commutes, using the fact that $\lambda = \rho : \mathrm{h_A} \otimes \mathrm{h_A} \to \mathrm{h_A}$.

On the other hand, suppose we have an $\mathcal{E}$-natural transformation $\beta : \widetilde{\mathrm{map}}_{\mathcal{A}}(X,-) \Rightarrow F$. Given any $Y \in \mathcal{A}_0(\mathrm{B})$, we need the diagram below to commute:

$$
\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{F} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(FX,FY) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(\beta_X,FY)} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(X,X),FY)
$$
with the map $\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(j,FY)$ down to $\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A^{op}}}(\mathrm{h_A},FY)$, then ϱ down to FY, and the diagonal β_Y from $\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$ to FY.

This diagram commutes, using Lemma 4.1.15 for β, the definition of the structure map $\widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)$ from Proposition 4.1.11, the commutative diagram (4.1), and the third axiom of Definition 4.1.1 for $\underline{\mathcal{A}}$.

Thus, these functions form a bijection

$$\mathcal{E}\text{-}\mathbf{Cat}(\underline{\mathcal{A}}, \underline{\mathcal{E}}^{\mathrm{A^{op}}})(\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -), F) \cong \mathcal{E}(\mathrm{A^{op}} \times \mathrm{A})(\mathrm{h_A}, FX).$$

We still need to check that the bijection is natural in both X and F; note that both sides are indeed functorial in X and F, by Remark 4.1.17.

It is easy to check that this bijection is natural in F. On the other hand, suppose we have a map $f : X \to Y$ in $\mathcal{A}(\mathrm{A})$. Naturality in the first variable amounts to the commutativity of the diagram below, for any $F \in \mathcal{E}\text{-}\mathbf{Cat}(\underline{\mathcal{A}}, \underline{\mathcal{E}}^{\mathrm{A^{op}}})$, and any $\mathcal{E}$-natural map $\beta : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) \Rightarrow F$:

$$
\begin{array}{ccccc}
\mathrm{h_A} & \xrightarrow{j} & \widetilde{\mathrm{map}}_{\mathcal{A}}(X, X) & \xrightarrow{\beta_X} & FX \\
\downarrow{\scriptstyle j} & & & & \downarrow{\scriptstyle F(f)} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Y) & \xrightarrow[\widetilde{\mathrm{map}}_{\mathcal{A}}(f, Y)]{} & \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) & \xrightarrow[\beta_Y]{} & FY
\end{array}
$$

This diagram commutes, using Remark 4.1.13, and the naturality of the induced natural transformation β. $\qquad\square$

Remark 4.2.2. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category and let A be a category. Let $X \in \mathcal{A}_0(\mathrm{A})$, let $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{\mathrm{A^{op}}}$ be an $\mathcal{E}$-morphism, and suppose we have a map $f : \mathrm{h_A} \to FX$ in $\mathcal{E}(\mathrm{A^{op}} \times \mathrm{A})$. Consider the $\mathcal{E}$-natural map $\beta : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) \Rightarrow F$ determined by Theorem 4.2.1. The component of β at $Y \in \mathcal{A}_0(\mathrm{B})$, given in Theorem 4.2.1, can also be described as follows:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \xrightarrow{\rho^{-1}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \otimes_{\mathrm{A}} \mathrm{h_A} \xrightarrow{F \otimes_{\mathrm{A}} f} \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A^{op}}}}(FX, FY) \otimes_{\mathrm{A}} FX$$
$$\downarrow{\scriptstyle \epsilon}$$
$$FY$$

Corollary 4.2.3. *For any $\mathcal{E}$-category $\underline{\mathcal{A}}$ and any category A, the functor*

$$y : \mathcal{A}(\mathrm{A})^{\mathrm{op}} \to \mathcal{E}\text{-}\mathbf{Cat}(\underline{\mathcal{A}}, \underline{\mathcal{E}}^{\mathrm{A^{op}}})$$

of Remark 4.1.17 is fully faithful.

Proof. This map is faithful by Remark 4.1.13. To see that it is full, suppose we have objects $X, Y \in \mathcal{A}_0(\mathrm{A})$, and an $\mathcal{E}$-natural transformation

$$\beta : \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, -) \Rightarrow \widetilde{\mathrm{map}}_{\mathcal{A}}(X, -).$$

Consider the corresponding map $f : h_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ determined by Theorem 4.2.1:

$$f : h_A \xrightarrow{\;j\;} \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Y) \xrightarrow{\;\beta_Y\;} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$$

We may think of this as a map $f : X \to Y$ in $\mathcal{A}(A)$. Using Remark 4.2.2 and Proposition 4.1.11, the component of β at $Z \in \mathcal{A}_0(B)$ is the following composite:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \xrightarrow{\;\rho^{-1}\;} \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \otimes_A h_A \xrightarrow{\;\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z)\otimes_A f\;} \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \otimes_A \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$$
$$\downarrow_{\circ}$$
$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$$

But this is exactly $\widetilde{\mathrm{map}}_{\mathcal{A}}(f, Z)$. Thus, we have:

$$\beta = \widetilde{\mathrm{map}}_{\mathcal{A}}(f, -) : \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, -) \Rightarrow \widetilde{\mathrm{map}}_{\mathcal{A}}(X, -)$$

$\square$

Definition 4.2.4. An $\mathcal{E}$-category **adjunction** is an adjunction in the 2-category $\mathcal{E}$-**Cat**, in the sense of [28]. Thus, an $\mathcal{E}$-category adjunction consists of $\mathcal{E}$-categories $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$, a left adjoint $\mathcal{E}$-morphism $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ and a $\mathcal{E}$-morphism right adjoint $G : \underline{\mathcal{B}} \to \underline{\mathcal{A}}$, together with the counit $\mathcal{E}$-natural transformation $\epsilon : F \circ G \Rightarrow \mathrm{id}_{\underline{\mathcal{B}}}$ and the unit $\mathcal{E}$-natural transformation $\eta : \mathrm{id}_{\underline{\mathcal{A}}} \Rightarrow G \circ F$. These must satisfy the **triangle identities** $(G \circ \epsilon) \cdot (\eta \circ G) = \mathrm{id}_G$ and $(\epsilon \circ F) \cdot (F \circ \eta) = \mathrm{id}_F$.

Theorem 4.2.5. *The data of an adjunction in $\mathcal{E}$-**Cat** is equivalent to the following: a pair of $\mathcal{E}$-morphisms $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ and $G : \underline{\mathcal{B}} \to \underline{\mathcal{A}}$, and a family of isomorphisms*

$$\widetilde{\mathrm{map}}_{\mathcal{B}}(FX, Y) \cong \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY)$$

$\mathcal{E}$-natural in both $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{B}_0(B)$.

Proof. Let $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ and $G : \underline{\mathcal{B}} \to \underline{\mathcal{A}}$ be $\mathcal{E}$-morphisms, and suppose we have a family of maps as below, for each category A and each $X \in \mathcal{A}_0(A)$:

$$\eta_X : h_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GFX)$$

By Theorem 4.2.1, each map η_X determines an $\mathcal{E}$-natural transformation:

$$\begin{array}{ccc} & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(FX,-)} & \\ \underline{\mathcal{B}} & \Big\Downarrow \Omega_{X,-} & \underline{\mathcal{E}}^{\mathcal{A}^{\mathrm{op}}} \\ & G \quad \underline{\mathcal{A}} \quad \widetilde{\mathrm{map}}_{\mathcal{A}}(X,-) & \end{array}$$

Using Remark 4.2.2, the component of this map at $Y \in \mathcal{B}_0(\mathrm{B})$ is given by the composite

$$\Omega_{X,Y} : \widetilde{\mathrm{map}}_{\mathcal{B}}(FX,Y) \xrightarrow{G} \widetilde{\mathrm{map}}_{\mathcal{A}}(GFX,GY) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(\eta_X,GY)} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,GY).$$

We claim that the family $\eta_X : \mathrm{h}_{\mathrm{A}} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X,GFX)$ is $\mathcal{E}$-natural in $X \in \mathcal{A}_0(\mathrm{A})$ if and only if these maps $\Omega_{X,Y}$ are $\mathcal{E}$-natural in X.

First, suppose $\eta : \mathrm{id}_{\underline{\mathcal{A}}} \Rightarrow G \circ F$ is $\mathcal{E}$-natural. Using Lemma 4.1.18 for G, and the description of horizontal composition for $\mathcal{E}$-natural transformations in Lemma 4.1.5, it follows immediately that $\Omega_{X,Y}$ is $\mathcal{E}$-natural in X.

Conversely, suppose that for any category B, and any $Y \in \mathcal{B}_0(\mathrm{B})$, the map

$$\begin{array}{ccc} & F^{\mathrm{op}} \xrightarrow{\quad} \underline{\mathcal{B}}^{\mathrm{op}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(-,Y)} & \\ \underline{\mathcal{A}}^{\mathrm{op}} & \Big\Downarrow \Omega_{-,Y} & \underline{\mathcal{E}}^{\mathrm{B}} \\ & \widetilde{\mathrm{map}}_{\mathcal{A}}(-,GY) & \end{array}$$

is $\mathcal{E}$-natural. The commutative diagram of Lemma 4.1.15 that expresses this $\mathcal{E}$-naturality corresponds by adjointness to the following commutative diagram, for any $X \in \mathcal{A}_0(\mathrm{A})$ and $Z \in \mathcal{A}_0(\mathrm{C})$:

$$\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ,Y) \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ,Y) \otimes_{\mathrm{C}} F} & \widetilde{\mathrm{map}}_{\mathcal{B}}(FZ,Y) \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{B}}(FX,FZ) \\
\Big\downarrow{\scriptstyle G \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z)} & & \Big\downarrow{\scriptstyle \circ} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(GFZ,GY) \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z) & & \widetilde{\mathrm{map}}_{\mathcal{B}}(FX,Y) \\
\Big\downarrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{A}}(\eta_Z,GY) \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z)} & & \Big\downarrow{\scriptstyle G} \\
& & \widetilde{\mathrm{map}}_{\mathcal{A}}(GFX,GY) \\
& & \Big\downarrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{A}}(\eta_X,GY)} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(Z,GY) \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z) & \xrightarrow{\quad \circ \quad} & \widetilde{\mathrm{map}}_{\mathcal{A}}(X,GY)
\end{array}$$

In particular, take $X \in \mathcal{A}_0(A)$, $Z \in \mathcal{A}_0(C)$ and $Y = FZ \in \mathcal{B}_0(C)$. Precomposing with the composite

$$\widetilde{\operatorname{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\lambda^{-1}} h_C \otimes_C \widetilde{\operatorname{map}}_{\mathcal{A}}(X, Z) \xrightarrow{j \otimes_C \widetilde{\operatorname{map}}_{\mathcal{A}}(X, Z)} \widetilde{\operatorname{map}}_{\mathcal{B}}(FZ, FZ) \otimes_A \widetilde{\operatorname{map}}_{\mathcal{A}}(X, Z)$$

this commutative diagram reduces to the following:

But this is the diagram of Lemma 4.1.15, expressing the $\mathcal{E}$-naturality of η.

So the maps η_X are $\mathcal{E}$-natural in X if and only if the maps $\Omega_{X,Y}$ are $\mathcal{E}$-natural in X. Dually, a family of maps

$$\epsilon_Y : h_B \to \widetilde{\operatorname{map}}_{\mathcal{B}}(FGY, Y)$$

corresponds to maps, $\mathcal{E}$-natural in $X \in \mathcal{A}_0(A)$:

$$\Lambda_{X,Y} : \widetilde{\operatorname{map}}_{\mathcal{A}}(X, GY) \xrightarrow{F} \widetilde{\operatorname{map}}_{\mathcal{B}}(FX, FGY) \xrightarrow{\widetilde{\operatorname{map}}_{\mathcal{B}}(FX, \epsilon_Y)} \widetilde{\operatorname{map}}_{\mathcal{B}}(FX, Y)$$

These maps $\Lambda_{X,Y}$ are $\mathcal{E}$-natural in Y if and only if the maps ϵ_Y are $\mathcal{E}$-natural in Y.

Given this data, the equations $\Lambda_{X,Y} \circ \Omega_{X,Y} = \mathrm{id}$ and $\Omega_{X,Y} \circ \Lambda_{X,Y} = \mathrm{id}$ are equivalent to the triangle identities for ϵ and η. $\qquad \square$

Theorem 4.2.6. *Let $G : \underline{\mathcal{B}} \to \underline{\mathcal{A}}$ be an $\mathcal{E}$-morphism. Suppose that, for every category A and any object $X \in \mathcal{A}_0(A)$, there is an object $FX \in \mathcal{B}_0(A)$ representing the composite $\mathcal{E}$-morphism below:*

$$\underline{\mathcal{B}} \xrightarrow{G} \underline{\mathcal{A}} \xrightarrow{\widetilde{\operatorname{map}}_{\mathcal{A}}(X, -)} \underline{\mathcal{E}}^{A^{\mathrm{op}}}$$

Then there is a unique way to extend F to an $\mathcal{E}$-morphism $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$, such that F gives a left adjoint to $G : \underline{\mathcal{B}} \to \underline{\mathcal{A}}$.

Proof. On objects, the $\mathcal{E}$-morphism $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ takes $X \in \mathcal{A}_0(A)$ to the given object $FX \in \mathcal{B}_0(A)$. For any two objects $X \in \mathcal{A}_0(A)$ and $Z \in \mathcal{A}_0(C)$, we require the structure map

$$F : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ)$$

in $\mathcal{E}(A^{\mathrm{op}} \times C)$.

By assumption, for any $X \in \mathcal{A}_0(A)$, we have an isomorphism

$$\Omega_{X,Y} : \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, Y) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY),$$

$\mathcal{E}$-natural in $Y \in \mathcal{B}_0(B)$. By Theorem 4.2.1, this map is given by the composite

$$\Omega_{X,Y} : \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, Y) \xrightarrow{G} \widetilde{\mathrm{map}}_{\mathcal{A}}(GFX, GY) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(\eta_X, GY)} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY),$$

for a map $\eta_X : X \to GFX$, as in the proof of Theorem 4.2.5. Using these maps, we define the structure map as follows, for $X \in \mathcal{A}_0(A)$ and $Z \in \mathcal{A}_0(C)$:

$$F : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(X, \eta_Z)} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GFZ) \xrightarrow{\Omega_{X,FZ}^{-1}} \widetilde{\mathrm{map}}_{\mathcal{A}}(FX, FZ)$$

Note that this definition is necessary to make the diagram below commute:

$$
\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\ F\ } \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ) \xrightarrow{\ G\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(GFX, GFZ) \\
\end{array}
$$

$$\downarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(\eta_X, GFZ)} \qquad (4.3)$$

$$\xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(X, \eta_Z)} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GFZ)$$

If we show that F is indeed an $\mathcal{E}$-morphism, then this diagram will express $\mathcal{E}$-naturality for the maps $\eta_X : X \to GFX$. Thus, once we verify the axioms of Definition 4.1.2, it will follow that F is the unique left adjoint to G.

To verify the first axiom, note that the commutativity of the diagram (4.3) above implies that the diagram below commutes, for any $X \in \mathcal{A}_0(A)$, $Z \in \mathcal{A}_0(C)$ and $Y \in \mathcal{B}_0(B)$:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(-, GY)} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, GY), \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY))$$

(4.4)

The diagram (4.4) has the following structure:

- Top row: $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(-, GY)} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, GY), \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY))$
- Left vertical: $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{F} \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ)$
- $\widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(-, Y)} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, Y), \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, Y))$
- $\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, Y), \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, Y)) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, Y), \Omega_{X,Y})} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, Y), \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY))$
- Right vertical: $\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, GY), \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY)) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\Omega_{Z,Y}, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY))} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{B}}(\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, Y), \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GY))$

If we knew that F was an $\mathcal{E}$-morphism, this diagram would express the $\mathcal{E}$-naturality of the maps $\Omega_{X,Y}$ in X. The commutativity of this diagram follows by the same argument as in the proof of Theorem 4.2.5. It follows from the commutativity of the diagram (4.4) and the $\mathcal{E}$-naturality of $\widetilde{\mathrm{map}}_{\mathcal{A}}(-, \eta_Z)$, as in Lemma 4.1.16, that the composite

$$F : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(X, \eta_Z)} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GFZ) \xrightarrow{\Omega_{X,FZ}^{-1}} \widetilde{\mathrm{map}}_{\mathcal{A}}(FX, FZ)$$

also satisfies the would-be $\mathcal{E}$-naturality condition in X. That is, the diagram below commutes, for $X \in \mathcal{A}_0(\mathrm{A})$, $Z \in \mathcal{A}_0(\mathrm{C})$ and $W \in \mathcal{A}_0(\mathrm{D})$:

The diagram has the following structure:

- Top row: $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(-, W)} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{D}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, W), \widetilde{\mathrm{map}}_{\mathcal{A}}(X, W))$
- Left vertical: $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{F} \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ)$
- $\widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(-, FW)} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{D}}(\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, FW), \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FW))$
- Bottom row: $\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{D}}(\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, FW), \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FW)) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{D}}(F, \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FW))} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{D}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, W), \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FW))$
- Right vertical: $\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{D}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, W), \widetilde{\mathrm{map}}_{\mathcal{A}}(X, W)) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{D}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, W), F)} \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{D}}(\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, W), \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FW))$

Taking adjuncts, this diagram reduces to the first axiom of Definition 4.1.2 for F. Essentially, this is the proof of Lemma 4.1.18 in reverse.

The second axiom of Definition 4.1.2 for F follows easily from Remark 4.1.13 and the corresponding axiom for G. $\qquad\square$

Suppose we have a adjunction

$$\mathcal{D}_1 \quad \overset{F}{\underset{R}{\perp}} \quad \mathcal{D}_2$$
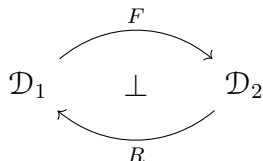
between closed $\mathcal{E}$-modules. By Proposition 3.7.5, F preserves tensors if and only if R preserves cotensors. If this is the case, then, by Proposition 4.1.19, F and R both induce $\mathcal{E}$-morphisms on the associated $\mathcal{E}$-categories. We will now show that these $\mathcal{E}$-morphisms form an adjunction:

Proposition 4.2.7. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be closed $\mathcal{E}$-modules, and suppose we have an adjunction:*

$$\mathcal{D}_1 \quad \overset{F}{\underset{R}{\perp}} \quad \mathcal{D}_2$$
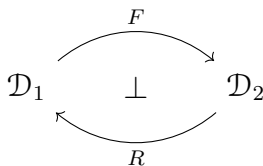
If F preserves tensors, then the induced $\mathcal{E}$-morphisms form an $\mathcal{E}$-category adjunction.

Proof. Let A be a category and let $X \in \mathcal{D}_1(A)$. Consider the isomorphism

$$
\begin{array}{ccc}
\mathcal{E}^{A^{\mathrm{op}}} & \xrightarrow{\ -\otimes_A X\ } & \mathcal{D}_1 \\
 & \overset{\varphi}{\underset{\cong}{\nearrow\!\!\!\!\nearrow}} & \ \downarrow F \\
 & \underset{-\otimes_A FX}{\searrow} & \mathcal{D}_2
\end{array}
$$
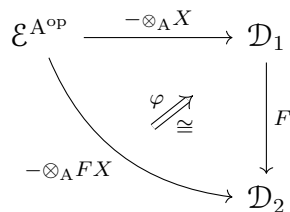
of Lemma 3.5.9. The conjugate of this map gives an isomorphism between the right adjoints:

$$\begin{array}{ccc}
\mathcal{E}^{A^{\mathrm{op}}} & \xleftarrow{\widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,-)} & \mathcal{D}_1 \\
& \overset{\cong}{\underset{\Omega}{\Longleftarrow}} & \Big\uparrow R \\
\widetilde{\mathrm{map}}_{\mathcal{D}_2}(FX,-) & & \mathcal{D}_2
\end{array}$$

By Lemma 3.7.6, and the dual of Lemma 4.1.20, this isomorphism induces an $\mathcal{E}$-natural isomorphism between the induced $\mathcal{E}$-category maps. Thus, we obtain an isomorphism

$$\Omega_{X,Y} : \widetilde{\mathrm{map}}_{\mathcal{D}_2}(FX,Y) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,RY),$$

$\mathcal{E}$-natural in $Y \in \mathcal{D}_2(\mathrm{B})$. By Theorem 4.2.6, there is a unique way to define an $\mathcal{E}$-morphism structure on F, such that F is a left adjoint to R. Specifically, for $X \in \mathcal{D}_1(\mathrm{A})$ and $Z \in \mathcal{D}_1(\mathrm{C})$, we have:

$$F : \widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,\eta_Z)} \widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,RFZ) \xrightarrow{\Omega_{X,FZ}^{-1}} \widetilde{\mathrm{map}}_{\mathcal{D}_2}(FX,FZ)$$

Here the map $\eta_Z : Z \to RFZ$ is obtained using Theorem 4.2.1, but it is easy to check that this is the unit of the unenriched adjunction. It follows that this composite is adjunct to

$$\widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,Y) \otimes_{\mathrm{A}} FX \xrightarrow{\varphi} F(\widetilde{\mathrm{map}}_{\mathcal{D}_1}(X,Y) \otimes_{\mathrm{A}} X) \xrightarrow{F(\epsilon)} FY$$

under the adjunction $- \otimes_{\mathrm{A}} FX \dashv \widetilde{\mathrm{map}}_{\mathcal{D}_2}(FX,-)$. Thus, the $\mathcal{E}$-morphism structure on F induced by Theorem 4.2.6 is the original structure of Proposition 4.1.19. $\qquad\square$

4.3 Transferring enrichments

In this section we study monoidal morphisms, starting with Lemma 4.3.2. This is an analogue of Lemma 3.5.9, for monoidal maps rather than module maps. In Proposition 4.3.4, we use this result to show that we can transfer enrichment along a monoidal adjunction. This construction has a number of applications, but in particular, we use it to define shifted $\mathcal{E}$-categories in Example 4.3.8.

Definition 4.3.1. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be monoidal derivators. A derivator map $F : \mathcal{E}_1 \to \mathcal{E}_2$ is called a **monoidal morphism** if it is equipped with an isomorphism

$$\xi : F\mathbb{1} \xrightarrow{\cong} \mathbb{1}$$

in $\mathcal{E}_2([\mathbf{0}])$, and an isomorphism:

$$
\begin{array}{ccc}
\mathcal{E}_1 \times \mathcal{E}_1 & \xrightarrow{\ F \times F\ } & \mathcal{E}_2 \times \mathcal{E}_2 \\
{\scriptstyle \otimes} \Big\downarrow & \underset{\cong}{\overset{\chi}{\nearrow}} & \Big\downarrow {\scriptstyle \otimes} \\
\mathcal{E}_1 & \xrightarrow[\ F\ ]{} & \mathcal{E}_2
\end{array}
$$

These isomorphisms must satisfy familiar coherence conditions, which precisely mirror the axioms for monoidal functors, as in [1].

Lemma 4.3.2. *Let $F : \mathcal{E}_1 \to \mathcal{E}_2$ be a cocontinuous monoidal morphism. For any categories* A, B *and* C, *and any $X \in \mathcal{E}_1(A^{\mathrm{op}} \times B)$ and $Y \in \mathcal{E}_1(B^{\mathrm{op}} \times C)$, the isomorphism χ of Definition 4.3.1 induces an isomorphism*

$$
\chi : F(Y \otimes_B X) \xrightarrow{\ \cong\ } FY \otimes_B FX
$$

in $\mathcal{E}_2(A^{\mathrm{op}} \times C)$. Similarly, the isomorphism ξ induces an isomorphism

$$
\xi : F\mathrm{h}_A \xrightarrow{\ \cong\ } \mathrm{h}_A
$$

in $\mathcal{E}_2(A^{\mathrm{op}} \times A)$, for any category A.

These satisfy the following coherence conditions:

1. *Given categories* A, B, C *and* D, *and objects $X \in \mathcal{E}_1(A^{\mathrm{op}} \times B)$, $Y \in \mathcal{E}_1(B^{\mathrm{op}} \times C)$ and $Z \in \mathcal{E}_1(C^{\mathrm{op}} \times D)$, the diagram below commutes:*

$$
\begin{array}{ccc}
F((Z \otimes_C Y) \otimes_B X) & \xrightarrow{\ F(\alpha)\ } & F(Z \otimes_C (Y \otimes_B X)) \\
{\scriptstyle \chi} \Big\downarrow & & \Big\downarrow {\scriptstyle \chi} \\
F(Z \otimes_C Y) \otimes_B FX & & FZ \otimes_C F(Y \otimes_B X) \\
{\scriptstyle \chi \otimes_B FX} \Big\downarrow & & \Big\downarrow {\scriptstyle FZ \otimes_C \chi} \\
(FZ \otimes_C FY) \otimes_B FX & \xrightarrow{\ \alpha\ } & FZ \otimes_C (FY \otimes_B FX)
\end{array}
$$

2. *Given $X \in \mathcal{E}(A^{\mathrm{op}} \times B)$, the diagram below commutes:*

$$F(X \otimes_A h_A) \xrightarrow{\quad \chi \quad} FX \otimes_A Fh_A$$

$$\downarrow{\scriptstyle F(\rho)} \qquad\qquad\qquad \downarrow{\scriptstyle FX \otimes_A \xi}$$

$$FX \xleftarrow{\quad \rho \quad} FX \otimes_A h_A$$

3. *Given $X \in \mathcal{E}(A^{op} \times B)$, the diagram below commutes:*

$$F(h_B \otimes_B X) \xrightarrow{\quad \chi \quad} Fh_B \otimes_B FX$$

$$\downarrow{\scriptstyle F(\lambda)} \qquad\qquad\qquad \downarrow{\scriptstyle \xi \otimes_B FX}$$

$$FX \xleftarrow{\quad \lambda \quad} h_B \otimes_B FX$$

Proof. Let $X \in \mathcal{E}_1(A^{op} \times B)$ and $Y \in \mathcal{E}_1(B^{op} \times C)$. The map $\chi : F(Y \otimes_B X) \xrightarrow{\cong} FY \otimes_B FX$ is given by the composite

$$F \int^B (Y \widetilde{\otimes} X) \xrightarrow{\quad \cong \quad} \int^B F(Y \widetilde{\otimes} X) \xrightarrow{\quad \int^B \chi \quad} \int^B (FY \widetilde{\otimes} FX),$$

where the first isomorphism is induced by the cocontinuity of F.

For any category A, the isomorphism $\xi : Fh_A \xrightarrow{\cong} h_A$ is given by the composite

$$F\partial_A \mathbb{1} \xrightarrow{\quad \cong \quad} \partial_A F\mathbb{1} \xrightarrow{\quad \partial_A \xi \quad} \partial_A \mathbb{1}$$

where the first isomorphism follows from the cocontinuity of F.

Each diagram in the statement corresponds to an axiom for the monoidal morphism F, as in Definition 4.3.1. The commutativity of each diagram can be reduced easily to the corresponding axiom, using essentially the same arguments as in the proof of Lemma 3.5.9.

$\square$

Remark 4.3.3. Let $F : \mathcal{E}_1 \to \mathcal{E}_2$ be a cocontinuous monoidal morphism. Then F induces a bicategory map

$$\mathcal{P}rof(F) : \mathcal{P}rof(\mathcal{E}_1) \to \mathcal{P}rof(\mathcal{E}_2)$$

between the bicategories of Remark 3.5.3. Explicitly, the map is the identity on objects, and, given categories A and B, the map on hom-categories is $F : \mathcal{E}_1(B^{op} \times A) \to \mathcal{E}_2(B^{op} \times A)$. The structure isomorphisms are χ and ξ of Lemma 4.3.2.

Proposition 4.3.4. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be closed symmetric monoidal derivators, and suppose we have an adjunction*

$$
\mathcal{E}_1 \underset{R}{\overset{F}{\rightleftarrows}} \bot \quad \mathcal{E}_2
$$

such that the left adjoint $F : \mathcal{E}_1 \to \mathcal{E}_2$ is monoidal. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}_2$-category. Then we can construct an $\mathcal{E}_1$-category with the same objects as $\underline{\mathcal{A}}$, and, for $X \in \mathcal{A}_0(A)$ and $Y \in \mathcal{A}_0(B)$, with mapping objects given by

$$
R \, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \in \mathcal{E}_1(A^{\mathrm{op}} \times B).
$$

Proof. Units and composition for the $\mathcal{E}_1$-enrichment are defined as follows. Given any $X \in \mathcal{A}_0(A)$, the unit

$$
j : \mathrm{h}_A \to R \, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, X)
$$

is adjunct to the composite

$$
F \mathrm{h}_A \xrightarrow{\ \xi\ } \mathrm{h}_A \xrightarrow{\ j\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(X, X),
$$

where ξ is the isomorphism of Lemma 4.3.2, and j is the unit for the $\mathcal{E}_2$-enrichment on $\underline{\mathcal{A}}$. Similarly, given $X \in \mathcal{A}_0(A)$, $Y \in \mathcal{A}_0(B)$ and $Z \in \mathcal{A}_0(C)$, composition is adjunct to the following, where χ is the isomorphism of Lemma 4.3.2:

$$
F(R \, \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \otimes_B R \, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)) \xrightarrow{\ \chi\ } FR \, \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \otimes_B FR \, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)
$$

$$
\downarrow{\scriptstyle \epsilon \otimes_B \epsilon}
$$

$$
\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \otimes_B \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)
$$

$$
\downarrow{\scriptstyle \circ}
$$

$$
\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)
$$

Using adjointness, and the commutative diagrams of Lemma 4.3.2, the $\mathcal{E}_1$-category axioms of Definition 4.1.1 reduce easily to the corresponding $\mathcal{E}_2$-category axioms for $\underline{\mathcal{A}}$. $\qquad\square$

Given a monoidal left adjoint $F : \mathcal{E}_1 \to \mathcal{E}_2$ as in Proposition 4.3.4, and an $\mathcal{E}_2$-category $\underline{\mathcal{A}}$, we will continue to denote the associated $\mathcal{E}_1$-category by $\underline{\mathcal{A}}$. This is partially justified by the following:

Remark 4.3.5. Let $F : \mathcal{E}_1 \to \mathcal{E}_2$ be a monoidal left adjoint, and let $\underline{\mathcal{A}}$ be an $\mathcal{E}_2$-category. Then for any category A, the induced category $\mathcal{A}(A)$, calculated with respect to the induced $\mathcal{E}_1$-enrichment of Proposition 4.3.4, agrees with that calculated using the original $\mathcal{E}_2$-enrichment. To see this, suppose we have $X, Y \in \mathcal{A}_0(A)$, and consider the isomorphisms:

$$\mathcal{E}_2(A^{\mathrm{op}} \times A)(h_A, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)) \cong \mathcal{E}_2(A^{\mathrm{op}} \times A)(F h_A, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y))$$
$$\cong \mathcal{E}_1(A^{\mathrm{op}} \times A)(h_A, R\,\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y))$$

It is easy to see that these isomorphisms preserve composition and units, using Lemma 4.3.2.

Proposition 4.3.6. *Let $\mathcal{E}$ be a monoidal derivator. Then for any category J, the map*

$$\partial_{\mathrm{J}} : \mathcal{E} \to \mathcal{E}^{\mathrm{J^{op}} \times \mathrm{J}}$$

is a monoidal morphism, where $\mathcal{E}^{\mathrm{J^{op}} \times \mathrm{J}}$ is considered with the monoidal structure of Example 3.5.5.

Proof. The structure isomorphisms for ∂_{J} are as follows. Define

$$\xi = \mathrm{id} : \partial_{\mathrm{J}} \mathbb{1} \to h_{\mathrm{J}}$$

in $\mathcal{E}(\mathrm{J^{op}} \times \mathrm{J})$, and define χ to be the composite below:

Here the first two isomorphisms follow from the cocontinuity of $\otimes$, as in Remark 2.3.4. The third is the isomorphism of Proposition 3.1.12. Using Remark 3.1.13, the pasting diagram above is equal to the one below:

$$\begin{array}{ccc}
\mathcal{E} \times \mathcal{E} & \xrightarrow{\mathcal{E} \times \partial_{J_{3,4}}} & \mathcal{E} \times \mathcal{E}^{J_3^{op} \times J_4} & \xrightarrow{\partial_{J_{1,2}} \times \mathcal{E}^{J_3^{op} \times J_4}} & \mathcal{E}_1^{J_1^{op} \times J_2} \times \mathcal{E}^{J_3^{op} \times J_4}
\end{array}$$

Thus, this diagram gives a second description of χ. Using these descriptions, and the definitions of λ and ρ from Proposition 3.5.2, it is easy to see that the diagrams below commute, for any $X \in \mathcal{E}(A)$:

$$\begin{array}{ccc}
\partial_J(X \widetilde{\otimes} \mathbb{1}) & \xrightarrow{\chi} & \partial_{J_{1,2}} X \otimes_{J_{1,4}} \partial_{J_{3,4}} \mathbb{1} \\
{\scriptstyle \partial_J(\rho)} \downarrow & & \| \\
\partial_J X & \xleftarrow{\rho} & \partial_{J_{1,2}} X \otimes_{J_{1,4}} h_{J_{3,4}}
\end{array}
\qquad
\begin{array}{ccc}
\partial_J(\mathbb{1} \widetilde{\otimes} X) & \xrightarrow{\chi} & \partial_{J_{1,2}} \mathbb{1} \otimes_{J_{1,4}} \partial_{J_{3,4}} X \\
{\scriptstyle \partial_J(\lambda)} \downarrow & & \| \\
\partial_J X & \xleftarrow{\lambda} & h_{J_{1,2}} \otimes_{J_{1,4}} \partial_{J_{3,4}} X
\end{array}$$

These express the unit axioms for ∂_J. It remains to show that the diagram below commutes, for any $X, Y, Z \in \mathcal{E}(A)$:

$$\begin{array}{ccc}
\partial_J((Z \otimes Y) \otimes X) & \xrightarrow{\partial_J(\alpha)} & \partial_J(Z \otimes (Y \otimes X)) \\
{\scriptstyle \chi} \downarrow & & \downarrow {\scriptstyle \chi} \\
\partial_{J_{3,2}}(Z \otimes Y) \otimes_{J_{3,6}} \partial_{J_{5,6}} X & & \partial_{J_{1,2}} Z \otimes_{J_{1,4}} \partial_{J_{5,4}}(Y \otimes_B X) \\
{\scriptstyle \chi \otimes_{J_{3,6}} \partial_{J_{5,6}} X} \downarrow & & \downarrow {\scriptstyle \partial_{J_{1,2}} Z \otimes_{J_{1,4}} \chi} \\
(\partial_{J_{1,2}} Z \otimes_{J_{1,4}} \partial_{J_{3,4}} Y) \otimes_{J_{3,6}} \partial_{J_{5,6}} X & \xrightarrow{\alpha} & \partial_{J_{1,2}} Z \otimes_{J_{1,4}} (\partial_{J_{3,4}} Y \otimes_{J_{3,6}} \partial_{J_{5,6}} X)
\end{array}$$

Note that, using the shifted derivator, we may assume $X, Y, Z \in \mathcal{E}([\mathbf{0}])$. Using the definition of α from Proposition 3.5.2, and Remark 3.1.3 and Remark 3.1.14, we can see that this diagram commutes. $\qquad\square$

Example 4.3.7. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category. For any category J, we may form an

$\mathcal{E}^{J^{\mathrm{op}} \times J}$-category $\underline{A}^{J}$ as follows. For any category A, we define

$$\mathcal{A}_0^{J}(A) = \mathcal{A}_0(J \times A).$$

Given $X \in \mathcal{A}_0^{J}(A)$ and $Y \in \mathcal{A}_0^{J}(B)$, define

$$\widetilde{\mathrm{map}}_{\mathcal{A}^J}(X, Y) = \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \in \mathcal{E}(J^{\mathrm{op}} \times J \times A^{\mathrm{op}} \times B).$$

Units and composition are inherited from the $\mathcal{E}$-category structure on $\underline{A}$, and the coherence conditions of Definition 4.1.1 carry over immediately. Note that, for any category A, we have an isomorphism

$$\mathcal{A}^{J}(A) \cong \mathcal{A}(J \times A) \tag{4.5}$$

between the category induced by the $\mathcal{E}^{J^{\mathrm{op}} \times J}$-category $\underline{A}^{J}$ at A, and the category induced by the $\mathcal{E}$-category $\underline{A}$ at $J \times A$.

Example 4.3.8. Let $\underline{A}$ be an $\mathcal{E}$-category, let J be a category, and consider the $\mathcal{E}^{J^{\mathrm{op}} \times J}$-category $\underline{A}^{J}$ of Example 4.3.7. By Proposition 4.3.4 and Proposition 4.3.6, we may transfer the $\mathcal{E}^{J^{\mathrm{op}} \times J}$-enrichment of $\underline{A}^{J}$ along the adjunction

$$\mathcal{E} \quad \xrightarrow{\partial_J} \quad \underset{\int_J}{\overset{\perp}{\rightleftarrows}} \quad \mathcal{E}^{J^{\mathrm{op}} \times J}.$$

Thus, we obtain the **shifted** $\mathcal{E}$-category $\underline{A}^{J}$. Explicitly, for any category A, we have $\mathcal{A}_0^{J}(A) = \mathcal{A}_0(J \times A)$, and, given $X \in \mathcal{A}_0^{J}(A)$ and $Y \in \mathcal{A}_0^{J}(B)$, we have

$$\widetilde{\mathrm{map}}_{\mathcal{A}^J}(X, Y) = \int_J \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \in \mathcal{E}(A^{\mathrm{op}} \times B).$$

Remark 4.3.9. For any $\mathcal{E}$-category $\underline{A}$ and any categories J and A, Remark 4.3.5 and the isomorphism (4.5) give rise to an isomorphism

$$\mathcal{A}^{J}(A) \cong \mathcal{A}(J \times A),$$

where $\mathcal{A}^{J}(A)$ is the category induced by the $\mathcal{E}$-category $\underline{A}^{J}$ at A, and $\mathcal{A}(J \times A)$ is the category induced by the $\mathcal{E}$-category $\underline{A}$ at $J \times A$. Thus, taking $A = [\mathbf{0}]$ and using Remark 4.1.4, the induced category $\mathcal{A}(J)$ is canonically $\mathcal{E}([\mathbf{0}])$-enriched, as well as $\mathcal{E}(J^{\mathrm{op}} \times J)$-enriched.

Chapter 5

Enriched Derivators

In this chapter, we study additional structure and properties that we can ask for in an $\mathcal{E}$-category. In Section 5.1, we study $\mathcal{E}$-prederivators, which we introduce in Definition 5.1.1. These are $\mathcal{E}$-categories equipped with a notion of pullback along functors; we show in Theorem 5.1.10 that these pullbacks are part of a prederivator structure on the induced categories of Remark 4.1.4. In Section 5.2, we show that $\mathcal{E}$-morphisms and $\mathcal{E}$-natural transformations between $\mathcal{E}$-prederivators induce prederivator maps and modifications. Using this, and the Yoneda lemma of Theorem 4.2.1, we prove a representability theorem for $\mathcal{E}$-prederivators in Theorem 5.2.6. Finally, in Section 5.3, we define weighted homotopy limits and colimits in an $\mathcal{E}$-category, and use these to define $\mathcal{E}$-derivators in Definition 5.3.6. We show, in Theorem 5.3.7, that the $\mathcal{E}$-category associated to a closed $\mathcal{E}$-module is an $\mathcal{E}$-derivator. Finally, in Theorem 5.3.10, we show that any $\mathcal{E}$-derivator induces a derivator.

5.1 $\mathcal{E}$-Prederivators

In this section, we introduce $\mathcal{E}$-prederivators, which are $\mathcal{E}$-categories endowed with extra structure. In particular, given a functor $u : A \to B$ and an $\mathcal{E}$-prederivator $\underline{A}$, we are able to form a pullback along u in $\underline{A}$. These pullbacks form part of a prederivator structure on the categories induced by $\underline{A}$. We record this fact in Theorem 5.1.10. We also show, in Proposition 5.1.12, that the mapping objects in an $\mathcal{E}$-prederivator induce prederivator maps. Finally, in Theorem 5.1.14, we show that, given a closed $\mathcal{E}$-module $\mathcal{D}$, the associated $\mathcal{E}$-category $\underline{\mathcal{D}}$ is an $\mathcal{E}$-prederivator.

Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, and suppose we have a functor $u : \mathrm{A} \to \mathrm{B}$. For $\underline{\mathcal{A}}$ to be an $\mathcal{E}$-prederivator, we want $\underline{\mathcal{A}}$ to have a notion of pullback along u. Suppose we have an object $X \in \mathcal{A}_0(\mathrm{B})$. Using Lemma 3.4.13 and Proposition 4.1.19, we can form the following $\mathcal{E}$-morphism:

$$\underline{\mathcal{A}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)} \underline{\mathcal{E}}^{\mathrm{B^{op}}} \xrightarrow{(u^{\mathrm{op}})^*} \underline{\mathcal{E}}^{\mathrm{A^{op}}}$$

The pullback of X along u should be an object $u^*X \in \mathcal{A}_0(\mathrm{A})$ that represents this $\mathcal{E}$-morphism. Dually, this same object should also represent the $\mathcal{E}$-morphism below:

$$\underline{\mathcal{A}}^{\mathrm{op}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(-,X)} \underline{\mathcal{E}}^{\mathrm{B}} \xrightarrow{u^*} \underline{\mathcal{E}}^{\mathrm{A}}$$

This motivates the following definition:

Definition 5.1.1. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category. We call $\underline{\mathcal{A}}$ an $\mathcal{E}$-**prederivator** if we have the following data:

- For any functor $u : \mathrm{A} \to \mathrm{B}$ and any $X \in \mathcal{A}_0(\mathrm{B})$, an object $u^*X \in \mathcal{A}_0(\mathrm{A})$.

- For any functors $u : \mathrm{A} \to \mathrm{B}$ and $v : \mathrm{C} \to \mathrm{D}$, and any $X \in \mathcal{A}_0(\mathrm{B})$ and $Y \in \mathcal{A}_0(\mathrm{D})$, an isomorphism

  $$\gamma^{u,v} : (u^{\mathrm{op}} \times v)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{\ \cong\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, v^*Y)$$

  in $\mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{C})$.

These must satisfy the following axioms:

1. For any category A and any $X \in \mathcal{A}_0(\mathrm{A})$, we have $\mathrm{id}^*X = X \in \mathcal{A}_0(\mathrm{A})$. Given an additional category B, and any $Y \in \mathcal{A}_0(\mathrm{B})$, we have:

   $$\gamma^{\mathrm{id},\mathrm{id}} = \mathrm{id} : \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$

   Moreover, given composable functors $\mathrm{A} \xrightarrow{u} \mathrm{B} \xrightarrow{v} \mathrm{C}$, and $X \in \mathcal{A}_0(\mathrm{C})$ we have $(v \circ u)^*X = u^*v^*X$, and given additional functors $\mathrm{D} \xrightarrow{w} \mathrm{E} \xrightarrow{z} \mathrm{F}$, and $Y \in \mathcal{A}_0(\mathrm{F})$, the diagram below commutes:

$$(u^{\mathrm{op}} \times w)^*(v^{\mathrm{op}} \times z)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{(u^{\mathrm{op}} \times w)^* \gamma^{v,z}} (u^{\mathrm{op}} \times w)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(v^*X, z^*Y)$$

$$\|$$

$$((v \circ u)^{\mathrm{op}} \times (z \circ w))^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$

$$\gamma^{u,w}$$

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*v^*X, w^*z^*Y)$$

$$\|$$

$$\gamma^{v \circ u, z \circ w} \qquad \widetilde{\mathrm{map}}_{\mathcal{A}}((v \circ u)^*X, (z \circ w)^*Y)$$

2. For any $u : \mathrm{A} \to \mathrm{B}$ and $X \in \mathcal{A}_0(\mathrm{B})$, the maps

$$\gamma^{u,\mathrm{id}} : (u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Y)$$

are $\mathcal{E}$-natural in $Y \in \mathcal{A}_0(\mathrm{C})$. For any $v : \mathrm{C} \to \mathrm{D}$ and $Y \in \mathcal{A}_0(\mathrm{D})$, the maps

$$\gamma^{\mathrm{id},v} : (\mathrm{A}^{\mathrm{op}} \times v)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, v^*Y)$$

are $\mathcal{E}$-natural in $X \in \mathcal{A}_0(\mathrm{A})$.

3. For any functor $u : \mathrm{A} \to \mathrm{B}$ and any $X \in \mathcal{A}_0(\mathrm{B})$, the diagram below commutes:

$$\mathrm{h}_{\mathrm{A}} \xrightarrow{\mathrm{h}_u} (u^{\mathrm{op}} \times u)^* \mathrm{h}_{\mathrm{B}} \xrightarrow{(u^{\mathrm{op}} \times u)^* j} (u^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,X)$$

$$\gamma^{u,u}$$

$$j \qquad \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, u^*X)$$

Here h_u is the map from Definition 3.5.1.

4. For any categories A, B, C and D, any functor $v : \mathrm{B} \to \mathrm{C}$, and any $X \in \mathcal{A}_0(\mathrm{A})$, $Y \in \mathcal{A}_0(\mathrm{C})$ and $Z \in \mathcal{A}_0(\mathrm{D})$, the diagram below commutes:

$$(v^{\mathrm{op}} \times \mathrm{D})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{B}} (\mathrm{A}^{\mathrm{op}} \times v)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{\ \gamma^{v,\mathrm{id}} \, \otimes_{\mathrm{B}} \gamma^{\mathrm{id},v}\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(v^*Y,Z) \otimes_{\mathrm{B}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,v^*Y)$$

$$\downarrow \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_v \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \qquad\qquad \downarrow \circ$$

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \xrightarrow{\quad\circ\quad} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z)$$

Here $\otimes_v$ is the map from Definition 3.2.5.

Remark 5.1.2. In Definition 5.1.1 Axiom 2, the $\mathcal{E}$-naturality conditions amount to the following:

1. For any categories A, B, C and D, any functor $u : \mathrm{A} \to \mathrm{B}$, and any $X \in \mathcal{A}_0(\mathrm{B})$, $Y \in \mathcal{A}_0(\mathrm{C})$ and $Z \in \mathcal{A}_0(\mathrm{D})$, the diagram below commutes:

$$(u^{\mathrm{op}} \times \mathrm{D})^* (\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)) \xrightarrow{\ \cong\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{C}} (u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$

$$\downarrow (u^{\mathrm{op}} \times \mathrm{D})^*(\circ) \qquad\qquad \downarrow \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{C}} \gamma^{u,\mathrm{id}}$$

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{C}} \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X,Y)$$

$$\downarrow \circ$$

$$(u^{\mathrm{op}} \times \mathrm{D})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z) \xrightarrow{\quad \gamma^{u,\mathrm{id}} \quad} \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X,Z)$$

2. For any categories A, B, C and D, any functor $w : \mathrm{C} \to \mathrm{D}$, and any $X \in \mathcal{A}_0(\mathrm{A})$, $Y \in \mathcal{A}_0(\mathrm{B})$ and $Z \in \mathcal{A}_0(\mathrm{D})$, the diagram below commutes:

$$(A^{\mathrm{op}} \times w)^*(\widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{B}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)) \xrightarrow{\cong} (B^{\mathrm{op}} \times w)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Y,Z) \otimes_{\mathrm{B}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$

$$\downarrow {\scriptstyle (A^{\mathrm{op}} \times w)^*(\circ)} \qquad\qquad\qquad \downarrow {\scriptstyle \gamma^{\mathrm{id},w} \,\otimes_{\mathrm{B}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)}$$

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, w^*Z) \otimes_{\mathrm{B}} \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y)$$

$$\downarrow {\scriptstyle \circ}$$

$$(A^{\mathrm{op}} \times w)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z) \xrightarrow[\gamma^{\mathrm{id},w}]{} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, w^*Z)$$

This follows, using adjointness, from the $\mathcal{E}$-naturality conditions in the form described in Lemma 4.1.15.

Example 5.1.3. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator, and let $\mathcal{L} \subseteq \mathcal{A}_0(A)$ be a set of objects in $\mathcal{A}_0([\mathbf{0}])$. For any category A, consider the set

$$\mathcal{B}_0(A) = \{X \in \mathcal{A}_0(A) \mid a^*X \in \mathcal{L} \ \forall a \in A\}.$$

Note that $\mathcal{B}_0([\mathbf{0}]) = \mathcal{L}$. As in Example 4.1.6, we may consider the full sub-$\mathcal{E}$-category $\mathcal{B}$ on these objects. We claim that this $\mathcal{E}$-category $\mathcal{B}$ is an $\mathcal{E}$-prederivator.

To see this, let $u : A \to B$ be a functor and let $X \in \mathcal{B}_0(B)$; we need to give an object $u^*X \in \mathcal{B}_0(A)$. Consider the object $u^*X \in \mathcal{A}_0(A)$. Using Axiom 1 of Definition 5.1.1, it follows that this object u^*X is in $\mathcal{B}_0(A)$. We take this to be the required object. The structure isomorphisms γ are also inherited from $\mathcal{A}$. The $\mathcal{E}$-prederivator axioms for $\mathcal{B}$ follow from the axioms for $\mathcal{A}$. We call $\mathcal{B}$ the **maximal sub-$\mathcal{E}$-prederivator on $\mathcal{L}$**.

Given an $\mathcal{E}$-prederivator, we want to show that the induced categories of Remark 4.1.4 organise into a prederivator. We begin to give the required structure in the following lemma; the final statement appears in Theorem 5.1.10.

Lemma 5.1.4. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator and let $u : A \to B$ be a functor. Then the assignment*

$$\mathcal{A}_0(B) \longrightarrow \mathcal{A}_0(A)$$
$$X \ \mapsto \ u^*X$$

extends to a functor $u^* : \mathcal{A}(B) \to \mathcal{A}(A)$ on the induced categories.

Proof. Let $f : h_B \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ be a map in $\mathcal{A}(B)$. For any category C and any object $Z \in \mathcal{A}_0(C)$, consider the composite below:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*Y, Z) \xrightarrow{(\gamma^{u,\mathrm{id}})^{-1}} (u^{\mathrm{op}} \times C)^*\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \xrightarrow{(u^{\mathrm{op}} \times C)^*\widetilde{\mathrm{map}}_{\mathcal{A}}(f, Z)} (u^{\mathrm{op}} \times C)^*\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$$
$$\downarrow{\gamma^{u,\mathrm{id}}}$$
$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Z)$$

By Lemma 4.1.16, this map is $\mathcal{E}$-natural in $Z \in \mathcal{A}_0(C)$. Using Corollary 4.2.3, we define $u^*f : h_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, u^*Y)$ to be the unique map in $\mathcal{A}(A)$ representing this $\mathcal{E}$-natural transformation. That is, u^*f is the unique map that makes the diagram below commute, for any category C and any $Z \in \mathcal{A}_0(C)$:

$$
\begin{array}{ccc}
(u^{\mathrm{op}} \times C)^*\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) & \xrightarrow{(u^{\mathrm{op}} \times C)^*\widetilde{\mathrm{map}}_{\mathcal{A}}(f, Z)} & (u^{\mathrm{op}} \times C)^*\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \\
\Big\downarrow{\gamma^{u,\mathrm{id}}} & & \Big\downarrow{\gamma^{u,\mathrm{id}}} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*Y, Z) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*f, Z)} & \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Z)
\end{array}
\tag{5.1}
$$

Functoriality of this construction follows by the uniqueness, using Remark 4.1.17. $\square$

Remark 5.1.5. In the proof of Lemma 5.1.4, to define $u^* : \mathcal{A}(B) \to \mathcal{A}(A)$ and prove it is functorial, it suffices to use the Yoneda lemma for the category $\mathcal{A}(A)$, rather than the $\mathcal{E}$-category Yoneda lemma of Theorem 4.2.1. However, Theorem 4.2.1 implies that the diagram (5.1) commutes, and we will use this repeatedly in the rest of the chapter, which is why we used the $\mathcal{E}$-category Yoneda lemma rather than the unenriched Yoneda lemma in this proof.

Remark 5.1.6. Suppose we have an $\mathcal{E}$-prederivator $\underline{\mathcal{A}}$ and a functor $u : A \to B$. Given a map $f : h_B \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ in $\mathcal{A}(B)$, the map $u^*f : h_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, u^*Y)$ of Lemma 5.1.4 can be explicitly described as follows:

$$h_A \xrightarrow{h_u} (u^{\mathrm{op}} \times u)^*h_B \xrightarrow{(u^{\mathrm{op}} \times u)^*f} (u^{\mathrm{op}} \times u)^*\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \xrightarrow{\gamma^{u,u}} \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, u^*Y)$$

This follows from Axioms 1 and 3 of Definition 5.1.1, using Remark 4.1.13.

In the proof of Lemma 5.1.4, the action of $u^* : \mathcal{A}(B) \to \mathcal{A}(A)$ on maps is defined representably using the $\mathcal{E}$-natural isomorphisms $\gamma^{u,\mathrm{id}}$. However, the explicit description in Remark 5.1.6, implies that u^* can also be defined using the dual isomorphisms $\gamma^{\mathrm{id},u}$:

Remark 5.1.7. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator, let $u : A \to B$ be a functor, and let $f : h_B \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ be a map in $\mathcal{A}(B)$. Then $u^*f : h_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, u^*Y)$ makes the diagram below commute, for any category C and any object $Z \in \mathcal{A}_0(\mathrm{C})$:

$$
\begin{array}{ccc}
(\mathrm{C^{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X) & \xrightarrow{(\mathrm{C^{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, f)} & (\mathrm{C^{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y) \\
\downarrow{\scriptstyle \gamma^{\mathrm{id}, u}} & & \downarrow{\scriptstyle \gamma^{\mathrm{id}, u}} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, u^*X) & \xrightarrow[\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, u^*f)]{} & \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, u^*Y)
\end{array}
$$

Note that this property characterises u^*f, by Corollary 4.2.3. We can check this equality using Remark 4.1.13, and the explicit description of u^*f from Remark 5.1.6.

We now give the corresponding results for natural transformations.

Lemma 5.1.8. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator, and let*

$$
A \; \underset{v}{\overset{u}{\rightrightarrows}} \; \Big\Downarrow {\scriptstyle \kappa} \; B
$$

be a natural transformation. This induces a canonical natural transformation

$$
\mathcal{A}(A) \; \underset{v^*}{\overset{u^*}{\leftleftarrows}} \; \Big\Downarrow {\scriptstyle \kappa^*} \; \mathcal{A}(B)
$$

between the functors of Lemma 5.1.4.

Proof. Let $X \in \mathcal{A}_0(B)$. We need to define the component of κ^* at X. For any category C and any object $Z \in \mathcal{A}_0(\mathrm{C})$, consider the composite below:

$$
\widetilde{\mathrm{map}}_{\mathcal{A}}(v^*X, Z) \xrightarrow{(\gamma^{v, \mathrm{id}})^{-1}} (v^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{(\kappa^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)} (u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)
$$
$$
\downarrow{\scriptstyle \gamma^{u, \mathrm{id}}}
$$
$$
\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Z)
$$

This map is $\mathcal{E}$-natural in $Z \in \mathcal{A}_0(\mathrm{C})$, by Lemma 3.4.13 and Lemma 4.1.20. Thus, by Corollary 4.2.3, we may define $\kappa_X^* : \mathrm{h}_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, v^*X)$ to be the unique map in $\mathcal{A}(\mathrm{A})$ representing this $\mathcal{E}$-natural transformation. That is, κ_X^* is the unique map that makes the diagram below commute, for any category C and any $Z \in \mathcal{A}_0(\mathrm{C})$:

$$
\begin{array}{ccc}
(v^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) & \xrightarrow{\ (\kappa^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z)\ } & (u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \\
\Big\downarrow{\gamma^{v,\mathrm{id}}} & & \Big\downarrow{\gamma^{u,\mathrm{id}}} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(v^*X, Z) & \xrightarrow[\ \widetilde{\mathrm{map}}_{\mathcal{A}}(\kappa_X^*, Z)\ ]{} & \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Z)
\end{array}
$$

Suppose we have a map $f : \mathrm{h}_B \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ in $\mathcal{A}(\mathrm{B})$. To check naturality, we need to show that $v^*(f) \circ \kappa_X^*$ and $\kappa_Y^* \circ u^*(f)$ represent the same $\mathcal{E}$-natural transformation. This follows easily from the definitions of u^* and v^* in Lemma 5.1.4. $\qquad\square$

As in Remark 5.1.7, we also have a dual description of the map κ_X^*:

Remark 5.1.9. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator, let $u, v : \mathrm{A} \to \mathrm{B}$ be functors, let $\kappa : u \Rightarrow v$ be a natural transformation, and let $X \in \mathcal{A}_0(\mathrm{B})$. For any category C and any $Z \in \mathcal{A}_0(\mathrm{C})$, the diagram below commutes:

$$
\begin{array}{ccc}
(\mathrm{C}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X) & \xrightarrow{\ (\mathrm{C}^{\mathrm{op}} \times \kappa)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z,X)\ } & (\mathrm{C}^{\mathrm{op}} \times v)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X) \\
\Big\downarrow{\gamma^{\mathrm{id},u}} & & \Big\downarrow{\gamma^{\mathrm{id},v}} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, u^*X) & \xrightarrow[\ \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, \kappa_X^*)\ ]{} & \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, v^*X)
\end{array}
$$

To check this, we can use Remark 4.1.13; the diagram above commutes if and only if the diagram below commutes:

$$
\begin{array}{ccccc}
\widetilde{\mathrm{map}}_{\mathcal{A}}(v^*X, v^*X) & \xrightarrow{(\gamma^{v,\mathrm{id}})^{-1}} & (v^{\mathrm{op}} \times \mathrm{A})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, v^*X) & \xrightarrow{(\kappa^{\mathrm{op}} \times \mathrm{A})^*} & (u^{\mathrm{op}} \times \mathrm{A})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, v^*X) \\
\Big\uparrow{j} & & & & \Big\downarrow{\gamma^{u,\mathrm{id}}} \\
\mathrm{h}_A & & & & \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, v^*X) \\
\Big\downarrow{j} & & & & \Big\uparrow{\gamma^{\mathrm{id},v}} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, u^*X) & \xrightarrow[(\gamma^{\mathrm{id},u})^{-1}]{} & (\mathrm{A}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, X) & \xrightarrow[(\mathrm{A}^{\mathrm{op}} \times \kappa)^*]{} & (A^{\mathrm{op}} \times v)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, X)
\end{array}
$$

Using Axioms 1 and 3 of Definition 5.1.1, we can reduce the commutativity of this diagram to the commutativity of the diagram below:

$$
\begin{array}{ccc}
\mathrm{h_A} & \xrightarrow{\ \mathrm{h}_u\ } & (u^{\mathrm{op}} \times u)^*\mathrm{h_B} \\
\downarrow{\scriptstyle \mathrm{h}_v} & & \downarrow{\scriptstyle (u^{\mathrm{op}} \times \kappa)^*} \\
(v^{\mathrm{op}} \times v)^*\mathrm{h_B} & \xrightarrow[(\kappa^{\mathrm{op}} \times v)^*]{} & (u^{\mathrm{op}} \times v)^*\mathrm{h_B}
\end{array}
$$

This diagram commutes by Lemma 3.1.7.

Theorem 5.1.10. *Any $\mathcal{E}$-prederivator $\underline{\mathcal{A}}$ induces a prederivator $\mathcal{A} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$, defined as follows:*

Here u^* is the functor of Lemma 5.1.4, and κ^* is the natural transformation of Lemma 5.1.8.

Proof. We need to check that this assignment is 2-functorial. This follows from the corresponding fact for $\mathcal{E}$, using Axiom 1 of Definition 5.1.1. To illustrate how this works, we will prove that $\mathcal{A}$ preserves composition of functors.

Suppose we have composable functors $\mathrm{A} \xrightarrow{u} \mathrm{B} \xrightarrow{v} \mathrm{C}$, and an object $X \in \mathcal{A}_0(\mathrm{C})$. By definition, we have $(v \circ u)^*X = u^*v^*X$. Suppose we have a map $f : \mathrm{h_C} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ in $\mathcal{A}(\mathrm{C})$. We want to show that $(v \circ u)^*f = u^*v^*f$. By definition of the pullback functors in Lemma 5.1.4, we must show that the diagram below commutes, for any $Z \in \mathcal{A}_0(\mathrm{D})$:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}((v \circ u)^*X, Z) \xrightarrow{(\gamma^{v \circ u, \mathrm{id}})^{-1}} ((v \circ u)^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$$

Diagram:

$\widetilde{\mathrm{map}}_{\mathcal{A}}((v \circ u)^*X, Z) \xrightarrow{(\gamma^{v \circ u, \mathrm{id}})^{-1}} ((v \circ u)^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$

$\downarrow (\gamma^{u,\mathrm{id}})^{-1}$

$(u^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(v^*X, Z)$

$\downarrow ((v \circ u)^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(f,Z)$

$((v \circ u)^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z)$

$\downarrow \gamma^{v \circ u, \mathrm{id}}$

$\widetilde{\mathrm{map}}_{\mathcal{A}}((v \circ u)^*Y, Z)$

$\uparrow \gamma^{u,\mathrm{id}}$

$(u^{\mathrm{op}} \times \mathrm{D})^*(\gamma^{v,\mathrm{id}})^{-1} \downarrow$ applied to $(u^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(v^*X, Z)$

$(u^{\mathrm{op}} \times \mathrm{D})^*(v^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \qquad (u^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(v^*Y, Z)$

$\xrightarrow{(u^{\mathrm{op}} \times \mathrm{D})^*(v^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(f,Z)}$

$\uparrow (u^{\mathrm{op}} \times \mathrm{D})^*(\gamma^{v,\mathrm{id}})$

$(u^{\mathrm{op}} \times \mathrm{D})^*(v^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z)$

This commutes by Axiom 2 of Definition 5.1.1. $\qquad\qquad\qquad\qquad\square$

We will refer to the prederivator $\mathcal{A}$ of Theorem 5.1.10 as the prederivator **induced** by $\underline{\mathcal{A}}$.

Lemma 5.1.11. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator. The opposite $\mathcal{E}$-category $\underline{\mathcal{A}}^{\mathrm{op}}$ carries a natural $\mathcal{E}$-prederivator structure, such that the induced prederivator $\mathcal{A}^{\mathrm{op}}$ is the opposite of the prederivator induced by $\underline{\mathcal{A}}$.*

Proof. Let $u : \mathrm{A} \to \mathrm{B}$ be a functor and let $X \in \mathcal{A}_0^{\mathrm{op}}(\mathrm{B}) = \mathcal{A}_0(\mathrm{B}^{\mathrm{op}})$. Define u^*X with respect to $\underline{\mathcal{A}}^{\mathrm{op}}$ to be $(u^{\mathrm{op}})^*X \in \mathcal{A}_0(\mathrm{A}^{\mathrm{op}})$ with respect to $\underline{\mathcal{A}}$.

Given functors $u : \mathrm{A} \to \mathrm{B}$ and $v : \mathrm{C} \to \mathrm{D}$, and $X \in \mathcal{A}_0(\mathrm{B}^{\mathrm{op}})$ and $Y \in \mathcal{A}_0(\mathrm{D}^{\mathrm{op}})$, we define $\gamma^{u,v}$ with respect to $\underline{\mathcal{A}}^{\mathrm{op}}$ to be the isomorphism

$$(u^{\mathrm{op}} \times v)^*\sigma^*\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, X) = \sigma^*(v \times u^{\mathrm{op}})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, X) \xrightarrow{\sigma^*(\gamma^{v^{\mathrm{op}}, u^{\mathrm{op}}})} \sigma^*\widetilde{\mathrm{map}}_{\mathcal{A}}((v^{\mathrm{op}})^*Y, (u^{\mathrm{op}})^*X)$$

with respect to $\underline{\mathcal{A}}$. It is easy to check that this data satisfies Definition 5.1.1, using the corresponding facts for $\underline{\mathcal{A}}$, and the respect of σ^* for h_u and $\otimes_u$. $\qquad\square$

Proposition 5.1.12. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator, let A and B be categories, and let $X \in \mathcal{A}_0(\mathrm{A})$ and $Y \in \mathcal{A}_0(\mathrm{B})$. For any category C, consider the functors*

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) : \mathcal{A}(\mathrm{C}) \to \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{C})$$

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(-, Y) : \mathcal{A}(\mathrm{C})^{\mathrm{op}} \to \mathcal{E}(\mathrm{C}^{\mathrm{op}} \times \mathrm{B})$$

induced by the representable $\mathcal{E}$-morphisms $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -)$ and $\widetilde{\mathrm{map}}_{\mathcal{A}}(-, Y)$. These form the components of prederivator maps

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathcal{E}^{\mathrm{A}^{\mathrm{op}}}$$

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(-, Y) : \mathcal{A}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{B}}$$

with structure isomorphisms given by the maps γ of Definition 5.1.1.

Proof. Let $u : \mathrm{C} \to \mathrm{D}$ be a functor, and let $X \in \mathcal{A}_0(\mathrm{A})$ and $Y \in \mathcal{A}_0(\mathrm{B})$. The structure isomorphism for $\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathcal{E}^{\mathrm{A}^{\mathrm{op}}}$ has component at $Z \in \mathcal{A}_0(\mathrm{D})$ given by:

$$\gamma^{\mathrm{id}, u} : (\mathrm{A}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{A}}(X, u^* Z)$$

The structure isomorphism for $\widetilde{\mathrm{map}}_{\mathcal{A}}(-, Y) : \mathcal{A}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{B}}$ has component at $Z \in \mathcal{A}_0(\mathrm{D})$ given by:

$$\gamma^{u, \mathrm{id}} : (u^{\mathrm{op}} \times \mathrm{B})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{A}}(u^* Z, Y)$$

We will outline a proof for the second map $\widetilde{\mathrm{map}}_{\mathcal{A}}(-, Y)$. The other is analogous.

First, we need to check that the maps $\gamma^{u, \mathrm{id}}$ are natural in $Z \in \mathcal{A}(\mathrm{D})^{\mathrm{op}}$. That is, for any map $f : \mathrm{h}_{\mathrm{D}} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, W)$ in $\mathcal{A}(\mathrm{D})$, we need the diagram below to commute:

$$
\begin{array}{ccc}
(u^{\mathrm{op}} \times \mathrm{B})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(W, Y) & \xrightarrow{(u^{\mathrm{op}} \times \mathrm{B})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(f, Y)} & (u^{\mathrm{op}} \times \mathrm{B})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y) \\
\downarrow{\gamma^{u, \mathrm{id}}} & & \downarrow{\gamma^{u, \mathrm{id}}} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(u^* W, Y) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(u^* f, Y)} & \widetilde{\mathrm{map}}_{\mathcal{A}}(u^* Z, Y)
\end{array}
$$

This holds by the commutativity of the diagram (5.1) in Lemma 5.1.4.

We now need to check that $\gamma^{u, \mathrm{id}}$ satisfies the axioms of Definition 2.1.2. Axioms 1 and 2 follow immediately from Axiom 1 of Definition 5.1.1. Finally, given a natural transformation κ, Axiom 3 of Definition 2.1.2 follows from the definition of κ^* in Lemma 5.1.8. $\qquad \square$

Remark 5.1.13. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator. The prederivator maps of Proposition 5.1.12 are the external components of a prederivator map

$$\mathrm{map}_{\mathcal{A}}(-, -) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{E}.$$

Given a category A and objects $X \in \mathcal{A}^{\mathrm{op}}(A)$ and $Y \in \mathcal{A}(A)$, we have

$$\mathrm{map}_{\mathcal{A}}(X,Y) = \delta^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X,Y) \in \mathcal{E}(A),$$

where $\delta : A \to A \times A$ is the diagonal map, as in Lemma 2.3.5.

Theorem 5.1.14. *Let $\mathcal{D}$ be a closed $\mathcal{E}$-module. The associated $\mathcal{E}$-category $\underline{\mathcal{D}}$ has a canonical $\mathcal{E}$-prederivator structure, such that the induced prederivator recovers the original $\mathcal{D}$.*

Proof. Let $u : A \to B$ be a functor, and let $X \in \mathcal{D}(B)$. To satisfy Definition 5.1.1, we require an object $u^* X \in \mathcal{D}(A)$. We take this object to be the image of X under the functor $u^* : \mathcal{D}(B) \to \mathcal{D}(A)$, coming from the prederivator structure on $\mathcal{D}$.

Given functors $u : A \to B$ and $v : C \to D$, and $X \in \mathcal{D}(B)$ and $Y \in \mathcal{D}(D)$, the required isomorphism

$$\gamma^{u,v} : (u^{\mathrm{op}} \times v)^* \widetilde{\mathrm{map}}_{\mathcal{D}}(X,Y) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{D}}(u^* X, v^* Y)$$

is induced by the structure isomorphisms of the prederivator map

$$\mathrm{map}_{\mathcal{D}}(-,-) : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E},$$

as in Remark 2.3.1. With these definitions, once we show that we do indeed obtain an $\mathcal{E}$-prederivator, it is immediate that the induced prederivator must recover the original prederivator. We will now check the axioms of Definition 5.1.1.

Axiom 1 follows immediately from Axioms 1 and 2 of Definition 2.1.2. For Axiom 2, suppose we have functors $u : A \to B$ and $v : C \to D$, and objects $X \in \mathcal{D}(B)$ and $Y \in \mathcal{D}(D)$, and consider the maps below:

$$\gamma^{u,\mathrm{id}} : (u^{\mathrm{op}} \times \mathrm{E})^* \widetilde{\mathrm{map}}_{\mathcal{D}}(X,Z) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{D}}(u^* X, Z)$$

$$\gamma^{\mathrm{id},v} : (\mathrm{E}^{\mathrm{op}} \times v)^* \widetilde{\mathrm{map}}_{\mathcal{D}}(Z,Y) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{D}}(Z, v^* Y)$$

We must show that these maps are $\mathcal{E}$-natural in $Z \in \mathcal{D}(E)$. We will prove this for the first map $\gamma^{u,\mathrm{id}}$; the proof for $\gamma^{\mathrm{id},v}$ is dual. First, note that $\gamma^{u,\mathrm{id}}$ is the component of a modification in Z. Moreover, by Example 3.7.4 and Example 3.7.7, the source and target of this modification preserve cotensors. Thus, if we can prove that $\gamma^{u,\mathrm{id}}$ respects cotensors, $\mathcal{E}$-naturality will follow by the dual of Lemma 4.1.20. Therefore, given any object $W \in \mathcal{E}(C)$, we need the diagram below to commute:

$$(u^{\mathrm{op}} \times \mathrm{C}^{\mathrm{op}} \times \mathrm{E})^* \widetilde{\mathrm{map}}_{\mathcal{D}}(X, Z \,\widetilde{\lhd}\, W) \xrightarrow{\;\cong\;} (u^{\mathrm{op}} \times \mathrm{C}^{\mathrm{op}} \times \mathrm{E})^* \widetilde{\mathrm{map}}_{\mathcal{E}}(W, \widetilde{\mathrm{map}}_{\mathcal{D}}(X, Z))$$

with vertical map $\gamma^{u,\mathrm{id}}$ on the left and, on the right, a composite through

$$\widetilde{\mathrm{map}}_{\mathcal{E}}(W, (u^{\mathrm{op}} \times \mathrm{E})^* \widetilde{\mathrm{map}}_{\mathcal{D}}(X, Z))$$

via $\widetilde{\mathrm{map}}_{\mathcal{E}}(W, \gamma^{u,\mathrm{id}})$ down to

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(u^* X, Z \,\widetilde{\lhd}\, W) \xrightarrow{\;\cong\;} \widetilde{\mathrm{map}}_{\mathcal{E}}(W, \widetilde{\mathrm{map}}_{\mathcal{D}}(u^* X, Z))$$

This diagram commutes; it expresses the fact that the isomorphism

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(X, Z \,\widetilde{\lhd}\, W) \cong \widetilde{\mathrm{map}}_{\mathcal{E}}(W, \widetilde{\mathrm{map}}_{\mathcal{D}}(X, Z))$$

of Example 3.7.7 is a modification in X. We will now consider Axiom 3 of Definition 5.1.1. Given a functor $u : \mathrm{A} \to \mathrm{B}$ and $X \in \mathcal{D}(\mathrm{B})$, the diagram in Axiom 3 commutes if and only if its adjunct diagram below commutes:

$$\mathrm{h}_{\mathrm{A}} \otimes_{\mathrm{A}} u^* X \xrightarrow{\mathrm{h}_u \otimes_{\mathrm{A}} u^* X} (u^{\mathrm{op}} \times u)^* \mathrm{h}_{\mathrm{B}} \otimes_{\mathrm{A}} u^* X \xrightarrow{(u^{\mathrm{op}} \times u)^* j \otimes_{\mathrm{A}} u^* X} (u^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{D}}(X, X) \otimes_{\mathrm{A}} u^* X$$

with $\gamma^{u,u} \otimes_{\mathrm{A}} u^* X$ down to

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(u^* X, u^* X) \otimes_{\mathrm{A}} u^* X$$

then ϵ down to $u^* X$, and the diagonal map λ from the top-left to $u^* X$.

Here we have used the definition of the unit $j : \mathrm{h}_{\mathrm{A}} \to \widetilde{\mathrm{map}}_{\mathcal{D}}(u^* X, u^* X)$, from Theorem 4.1.10, to simplify the left hand branch of the diagram. Using Lemma 3.6.5 and Remark 3.6.1, this diagram reduces to the definition of the unit $j : \mathrm{h}_{\mathrm{A}} \to \widetilde{\mathrm{map}}_{\mathcal{D}}(X, X)$, and so it commutes.

Finally, consider Axiom 4 of Definition 5.1.1. Once again, we can show the required diagram commutes by taking adjuncts, and using the definition of composition in $\underline{\mathcal{D}}$, from Theorem 4.1.10. The diagram we obtain commutes, using Example 3.6.3 and Remark 3.6.1. $\quad\square$

Remark 5.1.15. Let $\mathcal{D}$ be a closed $\mathcal{E}$-module, let A and B be categories, and let $X \in \mathcal{D}(\mathrm{A})$ and $Y \in \mathcal{D}(\mathrm{B})$. Then the prederivator maps

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(X, -) : \mathcal{D} \to \mathcal{E}^{\mathrm{A}^{\mathrm{op}}}$$

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(-, Y) : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{B}}$$

of Proposition 5.1.12 recover the original prederivator maps coming from the closed $\mathcal{E}$-module structure. The same is true for $\mathrm{map}_{\mathcal{D}}(-, -) : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}$ of Remark 5.1.13.

5.2 The 2-category of $\mathcal{E}$-prederivators

In the first half of this section, we show that $\mathcal{E}$-morphisms and $\mathcal{E}$-natural transformations respect the extra structure on $\mathcal{E}$-prederivators. In light of this, in Definition 5.2.3, we take these to be the 1-cells and 2-cells in the 2-category of $\mathcal{E}$-prederivators. It follows immediately that the Yoneda lemma of Theorem 4.2.1 carries over to $\mathcal{E}$-prederivators, and we use this in the second half of this section to prove Theorem 5.2.6, a representability theorem for $\mathcal{E}$-morphisms of the form $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}$. In Corollary 5.2.7, we apply this theorem to the $\mathcal{E}$-prederivator $\underline{\mathcal{D}}$ associated to a closed $\mathcal{E}$-module $\mathcal{D}$ to show that representability for a derivator map $F : \mathcal{D} \to \mathcal{E}$ can be deduced from representability theorems for the underlying category $\mathcal{D}([\mathbf{0}])$.

Proposition 5.2.1. *Let $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ be $\mathcal{E}$-prederivators, and let $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ be an $\mathcal{E}$-morphism between the underlying $\mathcal{E}$-categories. For any functor $u : \mathrm{A} \to \mathrm{B}$ and any $X \in \mathcal{A}_0(\mathrm{B})$, we have a canonical isomorphism:*

$$\phi^u : u^* F X \xrightarrow{\cong} F u^* X$$

These are the structure isomorphisms for a prederivator map $F : \mathcal{A} \to \mathcal{B}$, with component at A given by the induced functor $F : \mathcal{A}(\mathrm{A}) \to \mathcal{B}(\mathrm{A})$. Moreover, this is the unique prederivator map structure on these functors with the property that, for any $X \in \mathcal{A}_0(\mathrm{A})$ and $Y \in \mathcal{A}_0(\mathrm{B})$, the map

$$F : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY)$$

is a modification in both variables.

Proof. Let $u : \mathrm{A} \to \mathrm{B}$ be a functor and let $X \in \mathcal{A}_0(\mathrm{B})$. For any category C and any object $Z \in \mathcal{A}_0(\mathrm{C})$, consider the composite below:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Z) \xrightarrow{(\gamma^{u,\mathrm{id}})^{-1}} (u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{(u^{\mathrm{op}} \times \mathrm{C})^* F} (u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ)$$

$$\downarrow^{\gamma^{u,\mathrm{id}}}$$

$$\widetilde{\mathrm{map}}_{\mathcal{B}}(u^*FX, FZ)$$

By Lemma 4.1.18 this map is $\mathcal{E}$-natural in $Z \in \mathcal{A}_0(\mathrm{C})$. Thus, by Theorem 4.2.1, this determines a unique map

$$\phi^u_X : \mathrm{h}_\mathrm{A} \to \widetilde{\mathrm{map}}_{\mathcal{B}}(u^*FX, Fu^*X)$$

in $\mathcal{B}(\mathrm{A})$. Using Remark 4.2.2, the map ϕ^u is characterised by the fact that, for any category C and any $Z \in \mathcal{A}_0(\mathrm{C})$, the diagram below commutes:

$$
\begin{array}{ccc}
(u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) & \xrightarrow{\;\;(u^{\mathrm{op}} \times \mathrm{C})^* F\;\;} & (u^{\mathrm{op}} \times \mathrm{C})^* \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ) \\
\downarrow^{\gamma^{u,\mathrm{id}}} & & \downarrow^{\gamma^{u,\mathrm{id}}} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Z) & & \\
\downarrow^{F} & & \\
\widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^*X, FZ) & \xrightarrow[\;\;\widetilde{\mathrm{map}}_{\mathcal{B}}(\phi^u_X, FZ)\;\;]{} & \widetilde{\mathrm{map}}_{\mathcal{B}}(u^*FX, FZ)
\end{array}
$$

Dually, for any category C and any $Z \in \mathcal{A}_0(\mathrm{C})$, consider the following composite:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, u^*X) \xrightarrow{(\gamma^{\mathrm{id},u})^{-1}} (\mathrm{C}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X) \xrightarrow{(\mathrm{C}^{\mathrm{op}} \times u)^* F} (\mathrm{C}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, FX)$$

$$\downarrow^{\gamma^{\mathrm{id},u}}$$

$$\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, u^*FX)$$

This map is $\mathcal{E}$-natural in $Z \in \mathcal{A}_0(\mathrm{C})$, so, by Theorem 4.2.1, it determines a map

$$\psi^u_X : \mathrm{h}_\mathrm{A} \to \widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^*X, u^*FX)$$

in $\mathcal{B}(\mathrm{A})$, which is unique such that the diagram below commutes, for any category C and any $Z \in \mathcal{A}_0(\mathrm{C})$:

$$\begin{array}{ccc}
(\mathrm{C}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X) & \xrightarrow{\;\;(\mathrm{C}^{\mathrm{op}} \times u)^* F\;\;} & (\mathrm{C}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, FX) \\
\downarrow{\scriptstyle \gamma^{\mathrm{id},u}} & & \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, u^* X) & & \downarrow{\scriptstyle \gamma^{\mathrm{id},u}} \\
\downarrow{\scriptstyle F} & & \\
\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, Fu^* X) & \xrightarrow[\;\widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, \psi_X^u)\;]{} & \widetilde{\mathrm{map}}_{\mathcal{B}}(FZ, u^* FX)
\end{array}$$

We claim that ψ_X^u is inverse to ϕ_X^u, and that the isomorphisms ϕ^u organise into structure isomorphisms for F. Note that once these facts are established, the commutative diagrams above express the fact that

$$F : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y) \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FY)$$

is a modification in both variables. The uniqueness statement in the theorem follows from this observation.

We will start by showing that ϕ^u satisfies the conditions of Definition 2.1.2. First, we must show that the maps ψ_X^u are natural in $X \in \mathcal{A}(\mathrm{B})$. Let $f : \mathrm{h}_\mathrm{B} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ be a map in $\mathcal{A}(\mathrm{B})$. To show that the naturality condition $Fu^* f \circ \phi_X^u = \phi_Y^u \circ u^* Ff$ is satisfied, we may equivalently show that the diagram below commutes, for any category C and any $Z \in \mathcal{A}_0(\mathrm{C})$:

$$\begin{array}{ccccc}
\widetilde{\mathrm{map}}_{\mathcal{A}}(u^* Y, Z) & \xrightarrow{\;F\;} & \widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^* Y, FZ) & \xrightarrow{\;\widetilde{\mathrm{map}}_{\mathcal{B}}(\phi_Y^u, FZ)\;} & \widetilde{\mathrm{map}}_{\mathcal{B}}(u^* FY, FZ) \\
& & \downarrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^* f, FZ)} & & \downarrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{B}}(u^* Ff, FZ)} \\
& & \widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^* X, FZ) & \xrightarrow[\;\widetilde{\mathrm{map}}_{\mathcal{B}}(\phi_X^u, FZ)\;]{} & \widetilde{\mathrm{map}}_{\mathcal{B}}(u^* FX, FZ)
\end{array}$$

This diagram commutes, using the definition of ϕ^u above, and the commutative diagram (5.1) that defines u^*.

To verify Axiom 1 of Definition 2.1.2, for any category A and any $X \in \mathcal{A}_0(\mathrm{A})$, we must show that the unit $j : \mathrm{h}_\mathrm{A} \to \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FX)$ satisfies the defining property of ϕ_X^{id}. This is immediate once we unravel the definitions.

Similarly, suppose we have composable functors $\mathrm{A} \xrightarrow{\;u\;} \mathrm{B} \xrightarrow{\;v\;} \mathrm{C}$, and let $X \in \mathcal{A}_0(\mathrm{C})$. To

verify Axiom 2 of Definition 2.1.2, we must check that $\phi^u_{v^*X} \circ u^*\phi^v_X$ satisfies the defining property of $\phi^{v\circ u}_X$. That is, we need the diagram below to commute, for any category D and any $Z \in \mathcal{A}_0(\mathrm{D})$:

$$
\begin{array}{ccc}
((v \circ u)^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z) & \xrightarrow{\ ((v\circ u)^{\mathrm{op}}\times \mathrm{D})^*F\ } & ((v \circ u)^{\mathrm{op}} \times \mathrm{D})^*\widetilde{\mathrm{map}}_{\mathcal{B}}(FX,FZ) \\
\downarrow{\gamma^{v\circ u,\mathrm{id}}} & & \downarrow{\gamma^{v\circ u,\mathrm{id}}} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*v^*X,Z) & & \widetilde{\mathrm{map}}_{\mathcal{B}}(u^*v^*FX,FZ) \\
\downarrow{F} & & \uparrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(u^*\phi^v_X,FZ)} \\
\widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^*v^*X,FZ) & \xrightarrow[\ \widetilde{\mathrm{map}}_{\mathcal{B}}(\phi^u_{v^*X},FZ)\ ]{} & \widetilde{\mathrm{map}}_{\mathcal{B}}(u^*Fv^*X,FZ)
\end{array}
$$

This diagram commutes, using Axiom 1 of Definition 5.1.1, the defining property of ϕ^u given above, and the commutative diagram (5.1).

Finally, suppose we have functors $u, v : \mathrm{A} \to \mathrm{B}$ and a natural transformation $\kappa : u \Rightarrow v$. To check Axiom 3 of Definition 2.1.2, we must verify the equality $F(\kappa^*_X) \circ \phi^u_X = \phi^v_X \circ \kappa^*_{FX}$, for any $X \in \mathcal{A}_0(\mathrm{B})$. Equivalently, we may show that the diagram below commutes, for any category C and any $Z \in \mathcal{A}_0(\mathrm{C})$:

$$
\begin{array}{ccccc}
\widetilde{\mathrm{map}}_{\mathcal{A}}(v^*X,Z) & \xrightarrow{F} & \widetilde{\mathrm{map}}_{\mathcal{B}}(Fv^*X,FZ) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(\phi^v_X,FZ)} & \widetilde{\mathrm{map}}_{\mathcal{B}}(v^*FX,FZ) \\
& & \downarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(F(\kappa^*_X),FZ)} & & \downarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(\kappa^*_{FX},FZ)} \\
& & \widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^*X,FZ) & \xrightarrow[\widetilde{\mathrm{map}}_{\mathcal{B}}(\phi^u_X,FZ)]{} & \widetilde{\mathrm{map}}_{\mathcal{B}}(u^*FX,FZ)
\end{array}
$$

This diagram commutes, using the definition of ϕ^u given above, and the definition of κ^* from Lemma 5.1.8.

For any functor $u : \mathrm{A} \to \mathrm{B}$ and any $X \in \mathcal{A}_0(\mathrm{B})$, it remains to show that ψ^u_X is inverse to ϕ^u_X. To do this, note that we may describe ψ^u_X and ϕ^u_X explicitly as follows:

$$\phi^u_X : \mathrm{h_A} \xrightarrow{\ j\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, u^*X) \xrightarrow{(\gamma^{u,\mathrm{id}})^{-1}} (u^{\mathrm{op}} \times \mathrm{A})^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, u^*X) \xrightarrow{\ (u^{\mathrm{op}} \times \mathrm{A})^* F\ }$$

$$(u^{\mathrm{op}} \times \mathrm{A})^* \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, Fu^*X)$$

$$\downarrow \gamma^{u,\mathrm{id}}$$

$$\widetilde{\mathrm{map}}_{\mathcal{B}}(u^*FX, Fu^*X)$$

$$\psi^u_X : \mathrm{h_A} \xrightarrow{\ j\ } \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, u^*X) \xrightarrow{(\gamma^{\mathrm{id},u})^{-1}} (\mathrm{A}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, X) \xrightarrow{\ (\mathrm{A}^{\mathrm{op}} \times u)^* F\ }$$

$$(\mathrm{A}^{\mathrm{op}} \times u)^* \widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^*X, FX)$$

$$\downarrow \gamma^{\mathrm{id},u}$$

$$\widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^*X, u^*FX)$$

Using the description of composition in $\mathcal{B}(\mathrm{A})$ from Remark 4.1.4, we see that $\psi^u_X \circ \phi^u_X = \mathrm{id}$, using Axiom 3 of Definition 5.1.1, and Axiom 2 in the form given in Remark 5.1.2. Similarly, $\phi^u_X \circ \psi^u_X = \mathrm{id}$, using Axiom 4 of Definition 5.1.1. In both cases, we also use the axioms for F from Definition 4.1.2. $\qquad\square$

Lemma 5.2.2. *Let $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ be $\mathcal{E}$-prederivators, and $F, G : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ be $\mathcal{E}$-morphisms. Then any $\mathcal{E}$-natural transformation*

$$\underline{\mathcal{A}} \quad \overset{F}{\underset{G}{\Rrightarrow \beta}} \quad \underline{\mathcal{B}}$$

induces a modification $\beta : F \Rightarrow G$ between the prederivator maps $F, G : \mathcal{A} \to \mathcal{B}$ of Proposition 5.2.1.

Proof. Suppose we have a functor $u : \mathrm{A} \to \mathrm{B}$ and an object $X \in \mathcal{A}(\mathrm{B})$. We need to show that the following diagram in $\mathcal{B}(\mathrm{A})$ commutes:

$$\begin{array}{ccc} u^*FX & \xrightarrow{\ u^*\beta_X\ } & u^*GX \\ {\scriptstyle \phi^u_X}\downarrow & & \downarrow{\scriptstyle \phi^u_X} \\ Fu^*X & \xrightarrow[\ \beta_{u^*X}\ ]{} & Gu^*X \end{array}$$

Equivalently, we may show that the diagram below commutes, for any category C and any $Z \in \mathcal{A}_0(C)$:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Z) \xrightarrow{G} \widetilde{\mathrm{map}}_{\mathcal{B}}(Gu^*X, GZ) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(\beta_{u^*X}, GZ)} \widetilde{\mathrm{map}}_{\mathcal{B}}(Fu^*X, GZ)$$

with vertical maps $\widetilde{\mathrm{map}}_{\mathcal{B}}(\phi_X^u, GZ)$ and

$$\widetilde{\mathrm{map}}_{\mathcal{B}}(u^*GX, GZ) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{B}}(u^*\beta_X, GZ)} \widetilde{\mathrm{map}}_{\mathcal{B}}(u^*FX, GZ)$$

Using the $\mathcal{E}$-naturality of β, in the form given in Lemma 4.1.15, the definition of $u^*\beta_X$ from Lemma 5.1.4, and the definition of ϕ_X^u from Proposition 5.2.1, we can see that this diagram commutes. $\square$

In light of Proposition 5.2.1 and Lemma 5.2.2, $\mathcal{E}$-morphisms and $\mathcal{E}$-natural transformations are the appropriate 1-cells and 2-cells for the 2-category of $\mathcal{E}$-prederivators:

Definition 5.2.3. The **2-category of $\mathcal{E}$-prederivators** is the 2-category $\mathcal{E}$-**PDer**, with objects given by $\mathcal{E}$-prederivators, 1-cells given by $\mathcal{E}$-morphisms and 2-cells given by $\mathcal{E}$-natural transformations.

Remark 5.2.4. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category. By Definition 5.2.3, the identity $\mathcal{E}$-morphism $\mathrm{id}_{\underline{\mathcal{A}}}$ induces an equivalence between any two $\mathcal{E}$-prederivator structures on $\underline{\mathcal{A}}$. Thus, being an $\mathcal{E}$-prederivator is a property of an $\mathcal{E}$-category, rather than extra structure. See Example 5.3.2 for a related discussion.

With this definition, Theorem 4.2.1 carries over immediately to an $\mathcal{E}$-prederivator version:

Corollary 5.2.5. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator, let A be a category, let $X \in \mathcal{A}_0(A)$, and let $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{A^{\mathrm{op}}}$ be an $\mathcal{E}$-morphism. We have a natural bijection:*

$$\mathcal{E}\text{-}\mathbf{PDer}(\underline{\mathcal{A}}, \underline{\mathcal{E}}^{A^{\mathrm{op}}})(\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -), F) \cong \mathcal{E}(A^{\mathrm{op}} \times A)(\mathrm{h}_A, FX)$$

We finish this section with an application of Corollary 5.2.5. The following theorem reduces the representability of an $\mathcal{E}$-morphism F to the representability of its underlying $\mathcal{E}([\mathbf{0}])$-functor of Remark 4.1.4:

Theorem 5.2.6. *Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator. An $\mathcal{E}$-morphism $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}$ is representable if and only if the $\mathcal{E}([\mathbf{0}])$-functor*

$$F : \underline{\mathcal{A}}([\mathbf{0}]) \to \underline{\mathcal{E}}([\mathbf{0}])$$

is representable as an $\mathcal{E}([\mathbf{0}])$-functor.

Proof. Let $F : \underline{\mathcal{A}} \to \underline{\mathcal{E}}$ be an $\mathcal{E}$-morphism. If F is represented by an object $X \in A_0([\mathbf{0}])$, then the $\mathcal{E}$-natural isomorphism

induces an isomorphism between the corresponding $\mathcal{E}([\mathbf{0}])$-functors.

On the other hand, suppose the $\mathcal{E}([\mathbf{0}])$-functor $F : \underline{\mathcal{A}}([\mathbf{0}]) \to \underline{\mathcal{E}}([\mathbf{0}])$ is representable. That is, there is an object $X \in \mathcal{A}_0([\mathbf{0}])$ and an $\mathcal{E}([\mathbf{0}])$-natural isomorphism

By the (weak) Yoneda for enriched categories (see [27, Section 1.9]), this map β uniquely determines a map

$$f : \mathbb{1} \to FX$$

in $\mathcal{E}([\mathbf{0}])$. But, by Theorem 4.2.1, this map f uniquely determines an $\mathcal{E}$-natural transformation

Moreover, the $\mathcal{E}([\mathbf{0}])$-natural transformation induced by this $\mathcal{E}$-natural transformation is the original $\mathcal{E}([\mathbf{0}])$-natural transformation β.

The $\mathcal{E}$-natural map $\bar{\beta} : \widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) \Rightarrow F$ is an isomorphism if and only if each component is an isomorphism; that is, if and only if the induced modification

$$
\begin{array}{ccc}
 & \overset{\widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)}{\longrightarrow} & \\
\mathcal{A} & \Downarrow & \mathcal{E} \\
 & \underset{F}{\longrightarrow} &
\end{array}
$$

is an isomorphism. Since $\mathcal{E}$ satisfies **Der 2**, this is the case if and only if the natural transformation on underlying categories

$$
\begin{array}{ccc}
 & \overset{\widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)}{\longrightarrow} & \\
\mathcal{A}([\mathbf{0}]) & \Downarrow & \mathcal{E}([\mathbf{0}]) \\
 & \underset{F}{\longrightarrow} &
\end{array}
$$

is an isomorphism, by Lemma 2.1.25. (Note that this step is where we need $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator, rather than a general $\mathcal{E}$-category.) This natural transformation is an isomorphism, since it underlies the original $\mathcal{E}([\mathbf{0}])$-natural isomorphism β. $\qquad\square$

Corollary 5.2.7. *Let $\mathcal{D}$ be a closed $\mathcal{E}$-module. A derivator map*

$$
F : \mathcal{D} \to \mathcal{E}
$$

is representable in the sense of Definition 3.8.2 if and only if it is continuous and preserves cotensors, and the induced functor

$$
\mathcal{D}([\mathbf{0}]) \xrightarrow{\;F\;} \mathcal{E}([\mathbf{0}]) \xrightarrow{\;\mathcal{E}([\mathbf{0}])(\mathbb{1},-)\;} \mathbf{Set} \tag{5.2}
$$

is representable.

Proof. As in Theorem 3.8.3, the forward implication is immediate. For the reverse implication, suppose $F : \mathcal{D} \to \mathcal{E}$ is a continuous, cotensor-preserving map. Note that the cotensor product on the derivator $\mathcal{D}$ induces cotensors on the $\mathcal{E}([\mathbf{0}])$-category $\underline{\mathcal{D}}([\mathbf{0}])$, and these are preserved by the $\mathcal{E}([\mathbf{0}])$-functor $F : \underline{\mathcal{D}}([\mathbf{0}]) \to \underline{\mathcal{E}}([\mathbf{0}])$.

Consider the (unenriched) composite (5.2). Since $F : \underline{\mathcal{D}}([\mathbf{0}]) \to \underline{\mathcal{E}}([\mathbf{0}])$ preserves cotensors, it follows from [27, Theorem 4.85] that the $\mathcal{E}([\mathbf{0}])$-functor $F : \underline{\mathcal{D}}([\mathbf{0}]) \to \underline{\mathcal{E}}([\mathbf{0}])$ is representable if and only if this unenriched functor (5.2) is representable.

By Proposition 4.1.19, F induces an $\mathcal{E}$-morphism $F : \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ on the associated $\mathcal{E}$-prederivators. If (5.2) is representable, then Theorem 5.2.6 implies that this $\mathcal{E}$-morphism is representable. Thus, there is an object $X \in \mathcal{D}([\mathbf{0}])$, and an $\mathcal{E}$-natural isomorphism

$$
\begin{array}{ccc}
& \widetilde{\mathrm{map}}_{\mathcal{D}}(X,-) & \\
\underline{\mathcal{D}} & \cong \Downarrow & \underline{\mathcal{E}}. \\
& F &
\end{array}
$$

This $\mathcal{E}$-natural transformation induces the desired isomorphism between the induced derivator maps. $\qquad\square$

Corollary 5.2.7 allows us to improve slightly on Theorem 3.8.3, in the case of maps of the form $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$. Specifically, in the following theorem, we only require that the underlying category $\mathcal{D}([\mathbf{0}])$ satisfies Brown representability, rather than asking that $\mathcal{D}(C)$ satisfies Brown representability, for every category C.

Corollary 5.2.8. *Let $\mathcal{D}$ be a closed $\mathcal{E}$-module. Suppose that $\mathcal{E}$ and $\mathcal{D}$ are triangulated, and that $\mathcal{D}([\mathbf{0}])$ satisfies Brown representability. Then a derivator map*

$$
F : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}
$$

is representable in the sense of Definition 3.8.2 if and only if it is continuous and preserves cotensors.

Proof. Suppose $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ is a continuous, cotensor-preserving map, and consider the composite below:

$$
\mathcal{D}([\mathbf{0}])^{\mathrm{op}} \xrightarrow{\ F\ } \mathcal{E}([\mathbf{0}]) \xrightarrow{\ \mathcal{E}([\mathbf{0}])(\mathbb{1},-)\ } \mathbf{Ab}
$$

Since $F : \mathcal{D}([\mathbf{0}])^{\mathrm{op}} \to \mathcal{E}([\mathbf{0}])$ is an exact map of triangulated categories, this composite is a cohomological functor. Moreover, it takes coproducts in $\mathcal{D}([\mathbf{0}])$ to products in $\mathbf{Ab}$. Thus, since $\mathcal{D}([\mathbf{0}])$ satisfies Brown representability, this composite is representable. Thus, the result follows by Corollary 5.2.7. $\qquad\square$

5.3 $\mathcal{E}$-Derivators

In this section, we introduce $\mathcal{E}$-derivators and show that any closed $\mathcal{E}$-module gives rise to an $\mathcal{E}$-derivator. As a first step, we define $\mathcal{E}$-semiderivators in Definition 5.3.5, followed by $\mathcal{E}$-derivators in Definition 5.3.6. Just as derivators are, in particular, semiderivators that admit all homotopy limits and colimits, $\mathcal{E}$-derivators are $\mathcal{E}$-semiderivators that admit all weighted homotopy limits and colimits. However, in contrast to derivators, there are no further axioms we need to impose; in particular, we do not need any analogue of **Der 4**. Nonetheless, in Theorem 5.3.10, we prove that, for any $\mathcal{E}$-derivator $\underline{\mathcal{A}}$, the induced prederivator $\mathcal{A}$ is a derivator. On the other hand, in Theorem 5.3.7, we prove that the $\mathcal{E}$-prederivator associated to any closed $\mathcal{E}$-module is an $\mathcal{E}$-derivator.

We begin this section with the definitions of weighted homotopy limits and colimits. These are studied in [15, Section 4.5] and [16] in the context of $\mathcal{E}$-modules; if we apply Definition 5.3.1 in the $\mathcal{E}$-category associated to a closed $\mathcal{E}$-module, we recover these notions. However, in contrast to closed $\mathcal{E}$-modules, where all weighted homotopy limits and colimits always exist, in general $\mathcal{E}$-categories we may study particular weighted homotopy limits and colimits, in settings where all may not exist. In this way, it is preferable to work in the context of $\mathcal{E}$-categories and $\mathcal{E}$-prederivators, rather than restricting to closed $\mathcal{E}$-modules.

Definition 5.3.1. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-category, let A and B be categories, and let $X \in \mathcal{A}_0(\mathrm{A})$ and $W \in \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$. Consider the $\mathcal{E}$-morphism below:

$$\underline{\mathcal{A}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(X,-)} \underline{\mathcal{E}}^{\mathrm{A}^{\mathrm{op}}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A}^{\mathrm{op}}}}(W,-)} \underline{\mathcal{E}}^{\mathrm{B}^{\mathrm{op}}}$$

If this map is representable, we call the representing object the **homotopy colimit of X weighted by** W, and denote it by $W \otimes_{\mathrm{A}} X \in \mathcal{A}_0(\mathrm{B})$. Thus, $W \otimes_{\mathrm{A}} X$ is characterised by isomorphisms

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_{\mathrm{A}} X, Z) \cong \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A}^{\mathrm{op}}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)),$$

$\mathcal{E}$-natural in $Z \in \mathcal{A}_0(\mathrm{C})$.

Dually, given $Y \in \mathcal{A}_0(\mathrm{B})$ and $W \in \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$, the **homotopy limit of Y weighted by** W, if it exists, is an object $Y \lhd_{\mathrm{B}} W \in \mathcal{A}_0(\mathrm{A})$ representing the $\mathcal{E}$-morphism below:

$$\underline{\mathcal{A}}^{\mathrm{op}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(-,Y)} \underline{\mathcal{E}}^{\mathrm{B}} \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{B}}}(W,-)} \underline{\mathcal{E}}^{\mathrm{A}}$$

Let $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ be $\mathcal{E}$-categories and let $F : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ be an $\mathcal{E}$-morphism. Given $X \in \mathcal{A}_0(A)$ and $W \in \mathcal{E}(A^{\mathrm{op}} \times B)$, suppose that the weighted homotopy colimits $W \otimes_A X \in \mathcal{A}_0(B)$ and $W \otimes_A FX \in \mathcal{B}_0(B)$ both exist, and consider the composite below, for any category C, and any $Z \in \mathcal{A}_0(C)$:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_A X, Z) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{E}^{A^{\mathrm{op}}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)) \xrightarrow{\mathrm{map}_{\mathcal{E}^{A^{\mathrm{op}}}}(W, F)}$$

$$\widetilde{\mathrm{map}}_{\mathcal{E}^{A^{\mathrm{op}}}}(W, \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ))$$

$$\downarrow{\cong}$$

$$\widetilde{\mathrm{map}}_{\mathcal{B}}(W \otimes_A FX, FZ)$$

By Lemma 4.1.18, this map is $\mathcal{E}$-natural in Z. Thus, using Theorem 4.2.1, it determines a map

$$\phi : \mathrm{h}_B \to \widetilde{\mathrm{map}}_{\mathcal{B}}(W \otimes_A FX, F(W \otimes_A X))$$

in $\mathcal{B}(B)$, unique such that the diagram below commutes, for any category C and any object $Z \in \mathcal{A}_0(C)$:

$$
\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{E}^{A^{\mathrm{op}}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}^{A^{\mathrm{op}}}}(W, F)} & \widetilde{\mathrm{map}}_{\mathcal{E}^{A^{\mathrm{op}}}}(W, \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, FZ)) \\
\downarrow{\cong} & & \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_A X, Z) & & \downarrow{\cong} \\
\downarrow{F} & & \\
\widetilde{\mathrm{map}}_{\mathcal{B}}(F(W \otimes_A X), FZ) & \xrightarrow{\mathrm{map}_{\mathcal{B}}(\phi, FZ)} & \widetilde{\mathrm{map}}_{\mathcal{B}}(W \otimes_A FX, FZ)
\end{array}
$$

If this canonical map $\phi : W \otimes_A FX \to F(W \otimes_A X)$ is an isomorphism in $\mathcal{B}(B)$, then we say that F **preserves** the weighted homotopy colimit.

Example 5.3.2. Let $\underline{\mathcal{A}}$ be an $\mathcal{E}$-prederivator and let $u : A \to B$ be a functor. For any $X \in \mathcal{A}_0(B)$, and any $Z \in \mathcal{A}_0(C)$, consider the isomorphism below:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u^*X, Z) \cong (u^{\mathrm{op}} \times C)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$$

$$\cong (u^{\mathrm{op}} \times C)^* \widetilde{\mathrm{map}}_{\mathcal{E}^{B^{\mathrm{op}}}}(\mathrm{h}_B, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z))$$

$$\cong \widetilde{\mathrm{map}}_{\mathcal{E}^{B^{\mathrm{op}}}}((B^{\mathrm{op}} \times u)^* \mathrm{h}_B, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z))$$

In the composite above, the first and last maps are instances of $\gamma^{u,\mathrm{id}}$ for the $\mathcal{E}$-prederivators $\underline{A}$ and $\underline{\mathcal{E}}^{\mathrm{B}^{\mathrm{op}}}$. The second map is induced by the isomorphism $\varrho : \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{B}^{\mathrm{op}}}}(\mathrm{h_B}, -) \xrightarrow{\cong} \mathrm{id}$. In the proof of Theorem 4.2.1, this map is shown to be $\mathcal{E}$-natural; thus, the whole composite above is $\mathcal{E}$-natural in $Z \in \mathcal{A}_0(\mathrm{C})$.

Therefore, in any $\mathcal{E}$-prederivator $\underline{A}$ and for any $u : \mathrm{A} \to \mathrm{B}$, the object $u^*X \in \mathcal{A}_0(\mathrm{A})$ is the homotopy colimit of $X \in \mathcal{A}_0(\mathrm{B})$ weighted by $(\mathrm{B}^{\mathrm{op}} \times u)^*\mathrm{h_B} \in \mathcal{E}(\mathrm{B}^{\mathrm{op}} \times \mathrm{A})$. Moreover, given a second $\mathcal{E}$-prederivator $\underline{B}$ and an $\mathcal{E}$-morphism $F : \underline{A} \to \underline{B}$, the canonical map

$$\phi : (\mathrm{B}^{\mathrm{op}} \times u)^*\mathrm{h_B} \otimes_{\mathrm{B}} FX \to F((\mathrm{B}^{\mathrm{op}} \times u)^*\mathrm{h_B} \otimes_{\mathrm{B}} X)$$

is the isomorphism ϕ_X^u of Proposition 5.2.1. Thus, any $\mathcal{E}$-morphism between $\mathcal{E}$-prederivators preserves weighted homotopy colimits of this form.

Lemma 5.3.3. *Let $\underline{A}$ and $\underline{B}$ be $\mathcal{E}$-categories, and suppose we have an adjunction:*

Let $X \in \mathcal{A}_0(\mathrm{A})$, let $W \in \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$, and suppose the weighted homotopy colimit $W \otimes_{\mathrm{A}} X \in \mathcal{A}_0(\mathrm{B})$ exists. Then $F(W \otimes_{\mathrm{A}} X) \in \mathcal{B}_0(\mathrm{B})$ is the homotopy colimit of FX weighted by W. In particular, F preserves any weighted homotopy colimit that exists in $\underline{A}$.

Proof. Given $X \in \mathcal{A}_0(\mathrm{A})$ and $W \in \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B})$, we have the following string of isomorphisms, $\mathcal{E}$-natural in $Z \in \mathcal{B}_0(\mathrm{C})$:

$$\widetilde{\mathrm{map}}_{\mathcal{B}}(F(W \otimes_{\mathrm{A}} X), Z) \cong \widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_{\mathrm{A}} X, GZ)$$

$$\cong \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A}^{\mathrm{op}}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, GZ))$$

$$\cong \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A}^{\mathrm{op}}}}(W, \widetilde{\mathrm{map}}_{\mathcal{B}}(FX, Z))$$

Thus, $F(W \otimes_{\mathrm{A}} X) \in \mathcal{B}_0(\mathrm{B})$ has the defining property of $W \otimes_{\mathrm{A}} FX$, and it follows that F preserves the weighted homotopy colimit. $\square$

We will now begin to work towards the definition of $\mathcal{E}$-derivator. One property we desire of an $\mathcal{E}$-derivator is that the prederivator it induces should be a derivator. We observe below that this is already partway satisfied, for any $\mathcal{E}$-prederivator:

Proposition 5.3.4. *For any $\mathcal{E}$-prederivator $\underline{A}$, the induced prederivator $\mathcal{A}$ satisfies $\textbf{Der 2}$.*

Proof. Let A be a category, and let $f : h_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ be a map in $\mathcal{A}(A)$. We need to show that f is an isomorphism if and only if, for every object $a \in A$, the map $f_a : X_a \to Y_a$ is an isomorphism.

By Corollary 4.2.3, f is an isomorphism if and only if the map

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, f) : \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X) \to \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y)$$

is an isomorphism in $\mathcal{E}(\mathrm{C}^{\mathrm{op}} \times A)$, for every category C and every $Z \in \mathcal{A}_0(\mathrm{C})$.

For a fixed $Z \in \mathcal{A}_0(\mathrm{C})$, $\textbf{Der 2}$ for the derivator $\mathcal{E}^{\mathrm{C}^{\mathrm{op}}}$ implies that this map is an isomorphism if and only if, for every $a \in A$, the map

$$(\mathrm{C}^{\mathrm{op}} \times a)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, f) : (\mathrm{C}^{\mathrm{op}} \times a)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X) \to (\mathrm{C}^{\mathrm{op}} \times a)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y)$$

is an isomorphism in $\mathcal{E}(\mathrm{C}^{\mathrm{op}})$.

For a fixed $a \in A$, this map is an isomorphism if and only if

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, f_a) : \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X_a) \to \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y_a)$$

is an isomorphism, using the prederivator map structure for $\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, -)$.

Thus, f is an isomorphism if and only if, for every $a \in A$, every category C, and every $Z \in \mathcal{A}_0(\mathrm{C})$, the map $\widetilde{\mathrm{map}}_{\mathcal{A}}(Z, f_a) : \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, X_a) \to \widetilde{\mathrm{map}}_{\mathcal{A}}(Z, Y_a)$ is an isomorphism. By Corollary 4.2.3, this is true if and only if each map $f_a : X_a \to Y_a$ is an isomorphism in $\mathcal{A}([\mathbf{0}])$. $\square$

Definition 5.3.5. Let $\underline{A}$ be an $\mathcal{E}$-prederivator. We call $\underline{A}$ an $\mathcal{E}$-**semiderivator** if the induced prederivator $\mathcal{A}$ is a semiderivator; that is, $\mathcal{A}$ satisfies $\textbf{Der 1}$ and $\textbf{Der 2}$. Note that, by Proposition 5.3.4, this is the case if and only if $\mathcal{A}$ satisfies $\textbf{Der 1}$.

Definition 5.3.6. Let $\underline{A}$ be an $\mathcal{E}$-semiderivator. We say that $\underline{A}$ is a **left $\mathcal{E}$-derivator** if $\underline{A}$ admits all weighted homotopy colimits. Dually, we say that $\underline{A}$ is a **right $\mathcal{E}$-derivator** if $\underline{A}$ admits all weighted homotopy limits. We call $\underline{A}$ an $\mathcal{E}$-**derivator** if it is both a left and right $\mathcal{E}$-derivator.

Suppose $\underline{A}$ is a left $\mathcal{E}$-derivator, let A be a category, and let $X \in \mathcal{A}_0(A)$. For any object

$W \in \mathcal{E}(A^{\mathrm{op}} \times B)$, we have a family of isomorphisms

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_A X, Y) \cong \widetilde{\mathrm{map}}_{\mathcal{E}^{A^{\mathrm{op}}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)),$$

$\mathcal{E}$-natural in $Y \in \mathcal{A}_0(C)$. By Theorem 4.2.6, the weighted homotopy colimits organise into an $\mathcal{E}$-morphism $- \otimes_A X : \underline{\mathcal{E}}^{A^{\mathrm{op}}} \to \underline{\mathcal{A}}$, the left adjoint in an adjunction:

Thus, an $\mathcal{E}$-semiderivator $\underline{\mathcal{A}}$ is a left $\mathcal{E}$-derivator if and only if, for any category A and any $X \in \mathcal{A}_0(A)$, the $\mathcal{E}$-morphism

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, -) : \underline{\mathcal{A}} \to \underline{\mathcal{E}}^{A^{\mathrm{op}}}$$

has a left adjoint. Dually, an $\mathcal{E}$-semiderivator $\underline{\mathcal{A}}$ is a right $\mathcal{E}$-derivator if and only if, for any category B and any $Y \in \mathcal{A}_0(B)$, the $\mathcal{E}$-morphism

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(-, Y) : \underline{\mathcal{A}}^{\mathrm{op}} \to \underline{\mathcal{E}}^{B}$$

has a left adjoint.

Theorem 5.3.7. *Let $\mathcal{D}$ be a closed $\mathcal{E}$-module. The associated $\mathcal{E}$-prederivator $\underline{\mathcal{D}}$ is an $\mathcal{E}$-derivator.*

Proof. Since $\mathcal{D}$ is a derivator, the associated $\mathcal{E}$-prederivator $\underline{\mathcal{D}}$ is an $\mathcal{E}$-semiderivator. We need to show that $\underline{\mathcal{D}}$ admits all weighted homotopy limits and colimits.

Let A be a category and let $X \in \mathcal{D}(A)$. By Proposition 4.2.7, the adjunction

induces an adjunction between the corresponding $\mathcal{E}$-categories. Thus, $\underline{\mathcal{D}}$ admits all weighted homotopy colimits. Similarly, $\underline{\mathcal{D}}$ admits all weighted homotopy limits. $\qquad\square$

The proof of Theorem 5.3.7 justifies our notation for weighted homotopy limits and colimits: if $\mathcal{D}$ is a closed $\mathcal{E}$-module, given $X \in \mathcal{D}(A)$ and $W \in \mathcal{E}(A^{op} \times B)$, the weighted homotopy colimit $W \otimes_A X \in \mathcal{D}(B)$ is the image of W under the functor $- \otimes_A X : \mathcal{E}(A^{op} \times B) \to \mathcal{D}(B)$.

Remark 5.3.8. Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a cocontinuous $\mathcal{E}$-module map between closed $\mathcal{E}$-modules. Using the description of weighted homotopy colimits given in the proof of Theorem 5.3.7, it follows easily that the associated $\mathcal{E}$-morphism $F : \underline{\mathcal{D}}_1 \to \underline{\mathcal{D}}_2$ preserves all weighted homotopy colimits.

Example 5.3.9. We will now give an example of an enriched prederivator that admits some, but not all, weighted homotopy colimits.

Let $\mathcal{D}$ be a triangulated derivator. By Example 3.4.9, $\mathcal{D}$ is a closed $\mathcal{H}o(\mathbf{Spt})$-module, and so we can consider the associated $\mathcal{H}o(\mathbf{Spt})$-derivator $\underline{\mathcal{D}}$. Let $\underline{\mathcal{D}}^c$ be the maximal sub-$\mathcal{H}o(\mathbf{Spt})$-prederivator of $\mathcal{D}$ on the compact objects of $\mathcal{D}([\mathbf{0}])$, in the sense of Example 5.1.3. Thus, for any category A, the set $\mathcal{D}_0^c(A)$ consists of the objects $X \in \mathcal{D}(A)$ such that $X_a \in \mathcal{D}([\mathbf{0}])$ is compact for every $a \in A$.

We claim that an object $X \in \mathcal{D}(A)$ lies in $\mathcal{D}_0^c(A)$ if and only if the derivator map

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(X, -) : \mathcal{D} \to \mathcal{H}o(\mathbf{Spt}^{A^{op}})$$

is cocontinuous. By [12, Proposition 7.3] and [36, Theorem 7.13], this map is cocontinuous if and only if the functor on underlying categories

$$\widetilde{\mathrm{map}}_{\mathcal{D}}(X, -) : \mathcal{D}([\mathbf{0}]) \to \mathrm{Ho}(\mathbf{Spt}^{A^{op}})$$

preserves coproducts. By [18, Lemma 1.34], this functor preserves coproducts if and only if its left adjoint

$$- \wedge_A X : \mathrm{Ho}(\mathbf{Spt}^{A^{op}}) \to \mathcal{D}([\mathbf{0}])$$

takes the set of compact generators in $\mathrm{Ho}(\mathbf{Spt}^{A^{op}})$ to compact objects. By Lemma 2.4.14, $\{a_! \$ \mid a \in A^{op}\}$ is a set of compact generators for $\mathrm{Ho}(\mathbf{Spt}^{A^{op}})$, where $\$ \in \mathrm{Ho}(\mathbf{Spt})$ is the sphere spectrum. The isomorphism (3.4), applied to the map $a : [\mathbf{0}] \to A^{op}$, gives an isomorphism

$$a_! \$ \cong (A^{op} \times a)^* \partial_A \$ = (A^{op} \times a)^* h_A,$$

where, on the right hand side, we think of a as an object of A rather than A^{op}. Applying $- \wedge_A X$, we have

$$((A^{op} \times a)^* h_A) \wedge_A X \cong a^*(h_A \wedge_A X) \cong X_a.$$

This proves our claim.

Using this, suppose we have $X \in \mathcal{D}_0^c(A)$ and $W \in \text{Ho}(\mathbf{Spt}^{A^{\text{op}} \times B})$, and consider the composite

$$\underline{\mathcal{D}} \xrightarrow{\widetilde{\text{map}}_{\mathcal{A}}(X,-)} \underline{\mathcal{H}o}(\mathbf{Spt}^{A^{\text{op}}}) \xrightarrow{\widetilde{\text{map}}_{\mathcal{H}o(\mathbf{Spt}^{A^{\text{op}}})}(W,-)} \underline{\mathcal{H}o}(\mathbf{Spt}^{B^{\text{op}}}).$$

This composite is represented by $W \wedge_A X \in \mathcal{D}(B)$; thus, if the induced derivator map is cocontinuous, then $W \wedge_A X \in \mathcal{D}_0^c(B)$, and it follows that this object is the weighted homotopy colimit in $\underline{\mathcal{D}}^c$ as well as $\underline{\mathcal{D}}$. In particular, this is the case if

$$\widetilde{\text{map}}_{\mathcal{H}o(\mathbf{Spt}^{A^{\text{op}}})}(W,-) : \mathcal{H}o(\mathbf{Spt}^{A^{\text{op}}}) \longrightarrow \mathcal{H}o(\mathbf{Spt}^{B^{\text{op}}})$$

is cocontinuous; that is, if $(A^{\text{op}} \times b)^* W \in \text{Ho}(\mathbf{Spt}^{A^{\text{op}}})$ is compact for each $b \in B$. Thus, $\underline{\mathcal{D}}^c$ has homotopy colimits weighted by any pointwise compact object W.

However, in general, $\underline{\mathcal{D}}^c$ does not admit all weighted homotopy colimits. This follows from the following theorem, since the induced prederivator $\mathcal{D}^c$ in general does not admit all coproducts, so in particular it is not a derivator.

Theorem 5.3.10. *If $\underline{\mathcal{A}}$ is a left $\mathcal{E}$-derivator, then the induced prederivator $\mathcal{A}$ is a left derivator. Similarly, if $\underline{\mathcal{A}}$ is a right $\mathcal{E}$-derivator, then $\mathcal{A}$ is a right derivator, and if $\underline{\mathcal{A}}$ is an $\mathcal{E}$-derivator, then $\mathcal{A}$ is a derivator.*

Proof. We will show that, given a left $\mathcal{E}$-derivator $\underline{\mathcal{A}}$, the induced prederivator $\mathcal{A}$ is a left derivator. The corresponding result for right $\mathcal{E}$-derivators is dual, and the combination of these two results proves the corresponding result for $\mathcal{E}$-derivators.

Let $\underline{\mathcal{A}}$ be a left $\mathcal{E}$-derivator. By definition, $\mathcal{A}$ is a semiderivator, so we need only show that $\mathcal{A}$ admits all homotopy left Kan extensions, and prove the relevant part of **Der 4**.

We will start by showing that $\mathcal{A}$ admits homotopy left Kan extensions. Let $u : A \to B$ be a functor and consider the object $(u^{\text{op}} \times B)^* h_B \in \mathcal{E}(A^{\text{op}} \times B)$. For any object $X \in \mathcal{A}(A)$, denote the homotopy colimit of X weighted by this object as follows:

$$u_! X = (u^{\text{op}} \times B)^* h_B \otimes_A X \in \mathcal{A}(B)$$

For any category C, and any object $Z \in A_0(C)$, consider the isomorphisms below:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u_! X, Z) = \widetilde{\mathrm{map}}_{\mathcal{A}}((u^{\mathrm{op}} \times \mathrm{B})^* \mathrm{h}_{\mathrm{B}} \otimes_{\mathrm{A}} X, Z)$$

$$\cong \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{A}^{\mathrm{op}}}}((u^{\mathrm{op}} \times \mathrm{B})^* \mathrm{h}_{\mathrm{B}}, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z))$$

$$\cong \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{B}^{\mathrm{op}}}}(\mathrm{h}_{\mathrm{B}}, (u^{\mathrm{op}} \times \mathrm{C})_* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z))$$

$$\cong (u^{\mathrm{op}} \times \mathrm{C})_* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$$

The first isomorphism follows from the definition of the weighted homotopy colimit. The second is associated to the following adjunction, obtained using Proposition 4.2.7:

The final isomorphism $\varrho : \widetilde{\mathrm{map}}_{\mathcal{E}^{\mathrm{B}^{\mathrm{op}}}}(\mathrm{h}_{\mathrm{B}}, -) \xrightarrow{\cong} \mathrm{id}$ is shown to be $\mathcal{E}$-natural in the proof of Theorem 4.2.1. It follows that the entire composite is $\mathcal{E}$-natural in $Z \in \mathcal{A}_0(C)$. Denote this isomorphism by

$$\varpi^u : (u^{\mathrm{op}} \times \mathrm{C})_* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xrightarrow{\cong} \widetilde{\mathrm{map}}_{\mathcal{A}}(u_! X, Z).$$

We will now show that we can extend $u_!$ to a functor; the construction is similar to the definition of u^* in Lemma 5.1.4. Given a map $f : \mathrm{h}_{\mathrm{A}} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ in $\mathcal{A}(\mathrm{A})$, we define $u_! f : \mathrm{h}_{\mathrm{B}} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(u_! X, u_! Y)$, using Corollary 4.2.3, to be the unique map in $\mathcal{A}(\mathrm{B})$ that makes the diagram below commute, for any $Z \in \mathcal{A}_0(C)$:

$$
\begin{array}{ccc}
(u^{\mathrm{op}} \times \mathrm{C})_* \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) & \xrightarrow{(u^{\mathrm{op}} \times \mathrm{C})_* \widetilde{\mathrm{map}}_{\mathcal{A}}(f, Z)} & (u^{\mathrm{op}} \times \mathrm{C})_* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \\
\downarrow{\varpi^u} & & \downarrow{\varpi^u} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(u_! Y, Z) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(u_! f, Z)} & \widetilde{\mathrm{map}}_{\mathcal{A}}(u_! X, Z)
\end{array}
$$

As in Lemma 5.1.4, it is easy to check that this construction is functorial. It remains to show that it is indeed a left adjoint to u^*. We will do this directly, by providing the unit and counit of the adjunction, and showing that they satisfy the triangle identities.

Let $X \in \mathcal{A}(A)$. Using Corollary 4.2.3, we define $\eta_X : X \to u^* u_! X$ to be the unique map that makes the diagram below commute, for any category C and any $Z \in \mathcal{A}_0(C)$:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z) \xleftarrow{\quad \epsilon_{\widetilde{\mathrm{map}}_{\mathcal{A}}(X,Z)} \quad} (u^{\mathrm{op}} \times C)^* (u^{\mathrm{op}} \times C)_* \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)$$

with the downward arrow $(u^{\mathrm{op}} \times C)^* \varpi^u$ to

$$(u^{\mathrm{op}} \times C)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(u_! X, Z)$$

then $\gamma^{u, \mathrm{id}}$ down to

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u^* u_! X, Z)$$

and $\widetilde{\mathrm{map}}_{\mathcal{A}}(\eta_X, Z)$ closing the diagram.

To see that these maps form a natural transformation, let $f : \mathrm{h}_A \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ be a map in $\mathcal{A}(A)$. We need to check that we have $\eta_Y \circ f = u^* u_!(f) \circ \eta_X$. Equivalently, for any C and any $Z \in \mathcal{A}(C)$, we need to show that this diagram commutes:

$$
\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{A}}(u^* u_! Y, Z) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(u^* u_!(f), Z)} & \widetilde{\mathrm{map}}_{\mathcal{A}}(u^* u_! X, Z) \\
{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{A}}(\eta_Y, Z)} \downarrow & & \downarrow {\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{A}}(\eta_X, Z)} \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(f, Z)} & \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)
\end{array}
$$

Using the definitions of $u_!$ and η above, and the definition of u^* in Lemma 5.1.4, the commutativity of this diagram follows from the naturality of ϵ.

Similarly, for any $W \in \mathcal{A}(B)$, define $\epsilon_W : u_! u^* W \to W$ to be the unique map that makes the diagram below commute, for any category C and any $Z \in \mathcal{A}_0(C)$:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(W, Z) \xrightarrow{\quad \eta_{\widetilde{\mathrm{map}}_{\mathcal{A}}(W,Z)} \quad} (u^{\mathrm{op}} \times C)_* (u^{\mathrm{op}} \times C)^* \widetilde{\mathrm{map}}_{\mathcal{A}}(W, Z)$$

with the downward arrow $(u^{\mathrm{op}} \times C)_* \gamma^{u, \mathrm{id}}$ to

$$(u^{\mathrm{op}} \times C)_* \widetilde{\mathrm{map}}_{\mathcal{A}}(u^* W, Z)$$

then ϖ^u down to

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u_! u^* W, Z)$$

and $\widetilde{\mathrm{map}}_{\mathcal{A}}(\epsilon_W, Z)$ closing the diagram.

By a similar argument to the one above, these maps are natural in $W \in \mathcal{A}(B)$.

It remains to verify the triangle identities for η and ϵ. For the first identity, given $X \in \mathcal{A}(A)$, we need to check that we have $\epsilon_{u_!X} \circ u_!(\eta_X) = \mathrm{id}_{u_!X}$. Equivalently, for any C and any $Z \in \mathcal{A}(C)$, we need to show that the composite below is equal to the identity:

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(u_!X, Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(\epsilon_{u_!X}, Z)} \widetilde{\mathrm{map}}_{\mathcal{A}}(u_!u^*u_!X, Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(u_!(\eta_X), Z)} \widetilde{\mathrm{map}}_{\mathcal{A}}(u_!X, Z)$$

Using the definitions of $u_!$, ϵ and η above, and the definition of u^* in Lemma 5.1.4, this follows from the triangle identity for the adjunction

$$\mathcal{E}(B^{\mathrm{op}} \times C) \underset{(u^{\mathrm{op}} \times C)_*}{\overset{(u^{\mathrm{op}} \times C)^*}{\rightleftarrows}} \perp \mathcal{E}(A^{\mathrm{op}} \times C).$$

The other triangle identity follows similarly.

Thus, the induced prederivator $\mathcal{A}$ admits all homotopy left Kan extensions. It remains to check the relevant part of **Der 4**. To do this, suppose we have a homotopy exact square:

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow{\scriptstyle v} & \swarrow{\scriptstyle \kappa} & \downarrow{\scriptstyle w} \\ J & \xrightarrow{z} & K \end{array}$$

We will show that the canonical natural transformation

$$\mathcal{A}(J) \xleftarrow{v_!} \mathcal{A}(A) \xleftarrow{u^*} \mathcal{A}(B)$$

is an isomorphism. Applying this to the relevant comma square in Definition 2.1.16 gives us the result.

Thus, for any object $Y \in \mathcal{A}(B)$, we must show that the composite

$$v_!u^*Y \xrightarrow{v_!u^*(\eta_Y)} v_!u^*w^*w_!Y \xrightarrow{v_!\kappa^*_{w_!Y}} v_!v^*z^*w_!Y \xrightarrow{\epsilon_{z^*w_!Y}} z^*w_!Y$$

is an isomorphism in $\mathcal{A}(J)$. By Corollary 4.2.3, this map is an isomorphism if and only if, for any category C and any $Z \in \mathcal{A}_0(C)$, the composite

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(z^*w_!Y, Z) \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(\epsilon_{z^*w_!Y}, Z)} \widetilde{\mathrm{map}}_{\mathcal{A}}(v_!v^*z^*w_!Y, Z)$$

$$\downarrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{A}}(v_!\kappa^*_{w_!Y}, Z)}$$

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(v_!u^*w^*w_!Y, Z)$$

$$\downarrow{\scriptstyle \widetilde{\mathrm{map}}_{\mathcal{A}}(v_!u^*(\eta_Y), Z)}$$

$$\widetilde{\mathrm{map}}_{\mathcal{A}}(v_!u^*Y, Z)$$

is an isomorphism in $\mathcal{E}(J^{\mathrm{op}} \times C)$. But, using the definitions of ϵ and η above, and the definition of κ^* in Lemma 5.1.8, this composite is isomorphic to the component at $\widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z) \in \mathcal{E}(B^{\mathrm{op}} \times C)$ of the natural transformation below:

$$
\begin{array}{ccc}
\mathcal{E}(J^{\mathrm{op}} \times C) \xleftarrow{(v^{\mathrm{op}} \times C)_*} \mathcal{E}(A^{\mathrm{op}} \times C) \xleftarrow{(u^{\mathrm{op}} \times C)^*} \mathcal{E}(B^{\mathrm{op}} \times C) \\
\\
\mathcal{E}(J^{\mathrm{op}} \times C) \xleftarrow{(z^{\mathrm{op}} \times C)^*} \mathcal{E}(K^{\mathrm{op}} \times C) \xleftarrow{(w^{\mathrm{op}} \times C)_*} \mathcal{E}(B^{\mathrm{op}} \times C)
\end{array}
$$

Since the square κ is homotopy exact, this natural transformation is an isomorphism. Thus, the original composite is also an isomorphism. $\qquad\square$

In light of Theorem 5.3.10, if $\underline{\mathcal{A}}$ is an $\mathcal{E}$-derivator, we will refer to the induced prederivator $\mathcal{A}$ as the **induced derivator**.

Remark 5.3.11. Let $\underline{\mathcal{A}}$ be a left $\mathcal{E}$-derivator. Given a category A and an object $X \in \mathcal{A}(A)$, the $\mathcal{E}$-morphism $- \otimes_A X : \underline{\mathcal{E}}^{A^{\mathrm{op}}} \to \underline{\mathcal{A}}$ induces functors

$$- \otimes_A X : \mathcal{E}(A^{\mathrm{op}} \times B) \to \mathcal{A}(B)$$

for any category B.

On the other hand, given an object $W \in \mathcal{E}(A^{\mathrm{op}} \times B)$, the weighted homotopy colimits induce a functor:

$$W \otimes_A - : \mathcal{A}(A) \to \mathcal{A}(B)$$

Given a map $f : \mathrm{h_A} \to \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Y)$ in $\mathcal{A}(\mathrm{A})$, its image $W \otimes_{\mathrm{A}} f$ in $\mathcal{A}(\mathrm{B})$ is the unique map making the diagram below commute, for any category C and any $Z \in \mathcal{A}_0(\mathrm{C})$:

$$
\begin{array}{ccc}
\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A}^{\mathrm{op}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(Y, Z)) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A}^{\mathrm{op}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(f, Z))} & \widetilde{\mathrm{map}}_{\mathcal{E}\mathrm{A}^{\mathrm{op}}}(W, \widetilde{\mathrm{map}}_{\mathcal{A}}(X, Z)) \\
\cong \downarrow & & \downarrow \cong \\
\widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_{\mathrm{A}} Y, Z) & \xrightarrow{\widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_{\mathrm{A}} f, Z)} & \widetilde{\mathrm{map}}_{\mathcal{A}}(W \otimes_{\mathrm{A}} X, Z)
\end{array}
$$

Together, the functors above induce a two-variable functor:

$$- \otimes_{\mathrm{A}} - : \mathcal{E}(\mathrm{A}^{\mathrm{op}} \times \mathrm{B}) \times \mathcal{A}(\mathrm{A}) \to \mathcal{A}(\mathrm{B})$$

Let $u : \mathrm{A} \to \mathrm{B}$ be a functor. From the descriptions of u^* and $u_!$ in Example 5.3.2 and Theorem 5.3.10, we can check that we have natural isomorphisms:

$$u^* \cong (\mathrm{B}^{\mathrm{op}} \times u)^* \mathrm{h_B} \otimes_{\mathrm{B}} - : \mathcal{A}(\mathrm{B}) \to \mathcal{A}(\mathrm{A})$$
$$u_! \cong (u^{\mathrm{op}} \times \mathrm{B})^* \mathrm{h_B} \otimes_{\mathrm{A}} - : \mathcal{A}(\mathrm{A}) \to \mathcal{A}(\mathrm{B})$$

Using the isomorphism $(\mathrm{A}^{\mathrm{op}} \times u)_! \mathrm{h_A} \cong (u^{\mathrm{op}} \times \mathrm{B})^* \mathrm{h_B}$ from Section 3.6, we can also describe $u_!$ as follows:

$$u_! \cong (\mathrm{A}^{\mathrm{op}} \times u)_! \mathrm{h_A} \otimes_{\mathrm{A}} - : \mathcal{A}(\mathrm{A}) \to \mathcal{A}(\mathrm{B})$$

In this way, we recover the formulas for $\mathcal{E}$-modules, given in [16] and [15].

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