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On the Sparre-Andersen Risk Models

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Abstract

This thesis develops several strategies for calculating ruin-related quantities for a variety of extended risk models. We focus on the Sparre-Andersen risk model, also known as the renewal risk model. The idea of arbitrary distribution for the waiting time between claim payments arose in the 1950's from the collective risk theory, and received many extensions and modifications in recent years. Our goal is to tackle model assumptions that are either too relaxed for traditional methods to apply, or so complicated that elaborate algebraic tools are needed to obtain explicit solutions.

In Chapter 2, we consider a Lévy risk process and a Sparre-Andersen risk process with Parisian ruin in the presence of a constant dividend barrier. We demonstrate that with few exceptions, ruin occurs with certainty. Generalizations to certain dependent risk processes are discussed. We also provide a reinsurance contract in which the certainty of ruin can be avoided.

In Chapter 3, we investigate a class of Sparre-Andersen risk processes in which the inter-claim time is rational-distributed. A key property of the rational class is derived, which allows for direct derivation of an integro-differential equation satisfied by a probability concerning the maximum surplus. The solution is constructed using a set of linearly independent functions, one of which is obtained by a standard technique through a defective renewal equation while the rest are obtained via a homogeneous equation. The necessary boundary conditions are presented. We also provide examples involving rational claim sizes as well as an application to the total dividends paid under a threshold strategy.

In Chapter 4, we extend an exponential-combination dependence structure to an Erlang-combination for the Sparre-Andersen risk models in presence of diffusion. A set of tools are developed for establishing certain integro-differential equations in Gerber–Shiu analysis. This new technique lifts previous constraint on the multiplicities of parameters of the inter-claim times. We then illustrate applications of these equations under a variety of special dependence models. Results are compared with existing literature, including the diffusion-free cases.

Finally, in Chapter 5, we collect various results and provide conclusions. We also give an outline of potential future research.

Keywords: Diffusion process; Gerber–Shiu discounted penalty function; Integro-differential equation; Laplace transform; Parisian ruin; Sparre-Andersen risk model

Summary for Lay Audience

In ruin theory, the uncertainty faced by an insurance company is often described by a collective risk model. Under the classical assumptions, there are two sources of uncertainty: The first is the irregularity of *when* a claim would occur and the second is the unpredictability of *how much* a payment would be. We are interested in a wide range of quantifiable risk measures—the likelihood of ruin, the time of ruin and the severity of ruin, etc.

Miscellaneous extensions to the classical model are desired to better depict real-world phenomena. For instance, one may consider a more general distribution for inter-claim times; a dependence structure between inter-claim times and claim sizes; or a perturbation in premium income. These extensions aim to improve alignment with observations and require dedicated tools to provide actuaries with explicit solutions. Each model studied in this thesis is based on one or more aforementioned extensions. In particular, we focus on the following three different aspects of such extensions.

In the first article (Chapter 2), we consider a strategy in which the company pays out dividend whenever its surplus attains a constant level. We explore conditions that lead to certain ruin, and those that render ruin impossible.

In the second article (Chapter 3), we look at the company's maximum revenue and study the boundary behavior of related risk measures. The structure of rational distributions is revealed by utilizing integral transforms. An application to the expected total dividend is discussed as well.

In the third article (Chapter 4), we investigate a general dependence model under perturbation and develop a set of algebraic tools for analyses on complex dependence structure. This enables us to build a general type of equations, which can then be evaluated under various special cases.

These additional considerations introduce some common obstacles: The distributional assumptions are either too relaxed or too complicated to be handled by traditional methods. In response to these issues, we adopt different strategies and develop new techniques. Our goal is to generalize the well-known results to their fullest potential while making exciting new discoveries down the road.

Co-authorship Statement

This thesis consists of materials based on three jointly authored research articles. The first article is co-authored with my supervisor Dr. Kristina P. Sendova and Dr. Chen Yang at the Economics and Management School of Wuhan University. In particular, Dr. Yang contributed on the topic of a spectrally negative Lévy process (see Theorem 2.3.1). This article is published in the journal *Statistics & Probability Letters*. The second article is published in the *Journal of Computational and Applied Mathematics*. The third article will be submitted for publication in the near future. The last two articles are co-authored with my supervisor.

I certify that I am the lead author for all these articles. I would like to thank Dr. Sendova for her crucial contributions on suggesting research topics and recommending potential solutions.

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To my parents
—the two eternal beacons of light

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Chapter 1

Introduction

When modeling the surplus evolution of an insurance company, one often starts with a stochastic process illustrated by Figure 1.1. Here, we observe that the company's surplus increases from its initial position u according to a constant slope, starting at time $t = 0$. The dynamic surplus is then followed by three claim payments, represented by three downward dashed lines, occurred at times V_1, $V_1 + V_2$ and $V_1 + V_2 + V_3$, respectively. Let us denote these claim payments by Y_1, Y_2 and Y_3, so that the inter-claim times and the claim sizes can be represented by sequences of random variables $\{V_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$, respectively.

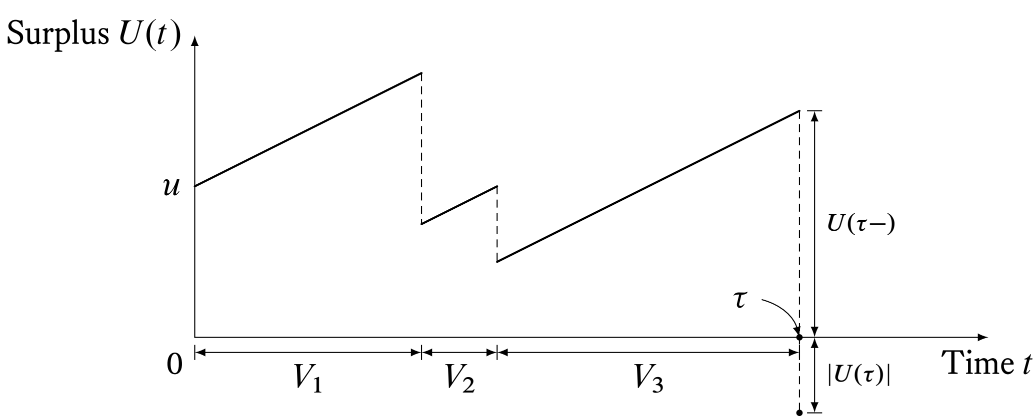


Figure 1.1: A surplus process in the event of ruin.

We also observe that the last claim payment Y_3 in Figure 1.1 is so large that it causes the company's surplus to drop below zero. This event corresponds to the company no longer having sufficient funds to operate, and hence is *ruined*. Due to the intrinsic randomness of the inter-claim times and the claim sizes, ruin is not necessarily a certain event. Therefore, it is crucial for the company to investigate how likely ruin would occur. This likelihood is formerly known as the *probability of ruin*, and is one of many risk measures pivotal to the company's business.

In the unfortunate event of ruin, other key quantities of interest include the *surplus immediately before ruin* and the *deficit at ruin*, represented by $U(\tau-)$ and $|U(\tau)|$ in Figure 1.1. Careful studies on these two quantities reveal how severe the impact of ruin is to the company, and they may assist with the preparation of certain contingency plans.

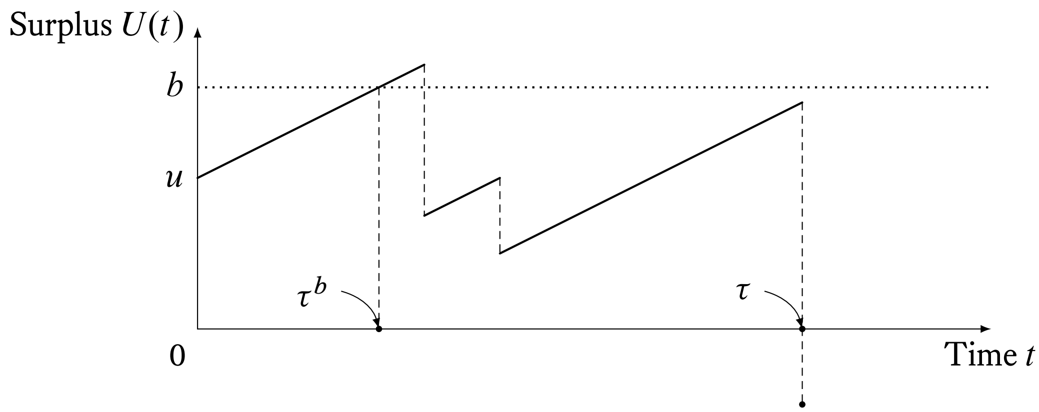


Figure 1.2: A surplus process with an up-crossing through level b.

Another related quantity of interest is shown in Figure 1.2, where a predetermined level b is drawn as a dotted line. We are interested in the event of the company's surplus crossing *above* b before ruin. This event turns out to be closely linked to the maximum revenue of the company. It also provides quantifiable measures that can be used to determine the dividend, if any, from the company to its shareholders.

1.1 Basic model description and notation

To introduce the *classical compound Poisson model* (also known as *classical risk model*) formally, let $Y, Y_1, Y_2, \ldots$ be independent and identically distributed positive random variables. They represent the successive individual claim amounts. These random variables are assumed to have common cumulative distribution function

$$F_Y(y) = P(y) = \mathbb{P}[Y \leqslant y]$$

with $P(0) = 0$, probability density function $p(y)$ with

$$P(y) = \int_0^y p(x)\,\mathrm{d}x, \quad y \geqslant 0,$$

and Laplace transform

$$\tilde{p}(s) = \int_0^\infty \mathrm{e}^{-sy}\, p(y)\,\mathrm{d}y = \int_0^\infty \mathrm{e}^{-sy}\,\mathrm{d}P(y), \quad \mathrm{Re}(s) \geqslant 0.$$

Furthermore, let the total number of claims up to time $t \geqslant 0$, denoted by $N(t)$ and independent of $Y_1, Y_2, \ldots$, be a *Poisson process* with rate parameter $\lambda > 0$. Consequently (see Ross 1996, Sections 2.1 and 2.2), the respective inter-claim time random variables $V_1, V_2, \ldots$ are independent and exponentially distributed with mean $1/\lambda$, and are independent of $Y_1, Y_2, \ldots$

The *aggregate-claim process* (also known as *aggregate loss*) is defined by

$$S(t) = \sum_{j=1}^{N(t)} Y_j, \quad t \geqslant 0,$$

where $S(t) = 0$ if $N(t) = 0$. This is a *compound Poisson process* (see Ross 1996, Section 2.5), which consequently has stationary and independent increments. Also, for fixed $t > 0$ we have

$$\mathbb{E}[S(t)] = \mathbb{E}[N(t)]\,\mathbb{E}[Y] = \lambda t \mathbb{E}[Y].$$

For an insurer's *surplus process*, denote $u \geqslant 0$ to be the initial capital and $c > 0$ the premium rate. To maintain a positive loading, we require the collected premiums to be greater than the expected claim payments; that is, we require $ct > \lambda t \mathbb{E}[Y]$ at any given moment $t > 0$. Thus, we may write

$$c = (1 + \theta)\lambda \mathbb{E}[Y],$$

where $\theta > 0$ is the relative security loading. The surplus process $\{U(t) : t \geqslant 0\}$ can therefore be expressed as

$$U(t) = u + ct - S(t), \quad t \geqslant 0.$$

Furthermore, the *time of (ultimate) ruin* is defined as

$$\tau = \inf\{\, t \geqslant 0 : U(t) < 0 \,\},$$

where $\tau = +\infty$ if ruin does not occur in finite time. If ruin does occur, then $|U(\tau)|$ is the *deficit at ruin* and $U(\tau-)$ is the *surplus immediately before ruin*. These two quantities are usually dependent. Their sum, $|U(\tau)| + U(\tau-)$, represents the claim that causes ruin.

Another important quantity connected to the time of ruin is the *probability of ultimate ruin* defined by

$$\psi(u) = \mathbb{P}[\,\tau < \infty \mid U(0) = u\,] = 1 - \varphi(u), \quad u \geqslant 0,$$

where $\varphi(u) = \mathbb{P}[\,\tau = \infty \mid U(0) = u\,]$ represents the *probability of ultimate survival*. When results are obtained regarding the probability of ruin, they are not entirely realistic due to the simple assumptions of the model. Nevertheless, this quantity provides a measure of the riskiness of the portfolio.

We now define the *Gerber–Shiu expected discounted penalty function*

$$m(u) = \mathbb{E}\big[\,\mathrm{e}^{-\delta\tau} w\big(U(\tau-), |U(\tau)|\big)\mathbb{1}(\tau < \infty) \;\big|\; U(0) = u\,\big], \quad u \geqslant 0.$$

This function was first introduced by Gerber & Shiu (1998). Here, the constant $\delta \geqslant 0$ is interpreted as the force of interest, or the variable of a (defective) Laplace transform; the penalty function $w(x_1, x_2)$ for $x_1 \geqslant 0$ and $x_2 > 0$ is a nonnegative function of the surplus immediately before ruin x_1 and the deficit at ruin x_2; and $\mathbb{1}(E)$ is the indicator function of an event E.

Initially, the function was intended as a tool for analyzing the expected discounted penalty as a function of the surplus x_1 and the deficit x_2. It has, though, a much broader meaning and serves to recover a number of quantities of special interest in ruin theory. These include the probability of ultimate ruin, the Laplace transform of the time to ruin, the joint and marginal distributions and moments of the surplus immediately before ruin and the deficit at ruin, etc. More specifically,

- probability of ruin:
  $\delta = 0$, $w(x_1, x_2) \equiv 1$;
- (defective) joint and marginal moments of the surplus and deficit:
  $\delta = 0$, $w(x_1, x_2) = x_1^k x_2^l$, where k and l are nonnegative integers;
- (defective) moments of the discounted deficit:
  $w(x_1, x_2) = x_2^l$, where l is a nonnegative integer;
- (defective) joint distribution of the surplus and deficit:
  $\delta = 0$, $w(x_1, x_2) = \mathbb{1}_{(-\infty,x]\times(-\infty,y]}(x_1, x_2)$, where x, y are nonnegative real numbers;
- (defective) distribution of the claim causing ruin:
  $\delta = 0$, $w(x_1, x_2) = \mathbb{1}_{(-\infty,z]}(x_1 + x_2)$, where z is a nonnegative real number;
- (defective) trivariate Laplace transform of the time to ruin, the surplus and the deficit:
  $w(x_1, x_2) = \mathrm{e}^{-sx_1 - rx_2}$, and the marginal transforms are derived by setting some of δ, s and r equal to zero.

It is desirable to express quantities of interest, such as some of the listed above, in terms of known quantities. A potential approach is to find an explicit expression for the Gerber–Shiu function and then investigate its particular cases.

Let b be fixed real number and define the *time of first up-crossing* through level b as

$$\tau^b = \inf\{ t \geqslant 0 : U(t) \geqslant b \}.$$

See Figure 1.2 for an illustration. Due to the positive loading condition, the surplus process has a positive drift towards $+\infty$. Thus, the random quantity τ^b is finite almost surely. We define

$$\chi(u;b) = \mathbb{P}[\, \tau^b < \tau \mid U(0) = u \,].$$

It follows that

$$\chi(u;\infty) = \lim_{b\to\infty} \chi(u;b) = \varphi(u).$$

Finally, we define

$$\xi(u;b) = \mathbb{P}\left[\, \tau < \infty,\ \sup_{0\leqslant t\leqslant \tau} U(t) \leqslant b \;\middle|\; U(0) = u \,\right]$$

and observe that

$$\xi(u;\infty) = \psi(u) = m(u)\big|_{\delta=0, w\equiv 1}.$$

1.2 Extensions to the classical compound Poisson model

The classical risk model for the surplus process assumes a Poisson arrival for the claim payments, as well as independence between the inter-claim times and the claim sizes. In this thesis, we will work on several classes of *Sparre-Andersen risk models* (also known as *renewal risk models*) and their extensions. The renewal risk models are generalizations to the classical risk model, where more general distributions for the inter-claim times are allowed.

We have some common goals in the following chapters: We wish to relax particular assumptions on the inter-claim times $\{V_i\}_{i=1}^{\infty}$ and to generalize the known results in current literature to their fullest potential. As a result, we face many difficulties along the way. These complications are overcome by using different strategies in the next three chapters, leading to exciting new discoveries. In particular, we consider the following extensions to the renewal risk models:

1. Suppose that the company adopts a constant dividend barrier strategy, in which all its premium income exceeding the barrier is paid to the shareholders. In this dividend model,

ruin is shown to be *certain* under various assumptions on the inter-claim times, including Poisson, generalized Erlang(n) as well as phase-type arrivals of the claims. How far does this statement reach if we consider *arbitrary* inter-claim times? Is ruin still a certain event if there is dependence between the inter-claim times and the claim sizes?

2. In the presence of a constant dividend barrier for the surplus process, let us consider a common rational distribution for the inter-claim times. Is it possible to derive an explicit solution for the distribution of the maximum surplus? What are the properties of related functions at the level boundary? If a dividend strategy is adopted, then how can we apply the solutions when evaluating dividend?

3. Assume an additional diffusion perturbation (see Figure 1.3 for an illustration), and a general Erlang-combination dependence structure between the inter-claim times and the claim sizes. We wish to study the time of ruin, the surplus immediately before ruin and the deficit at ruin collectively. What are the necessary tools for establishing elaborate integro-differential equations for these quantities? Do these equations recover existing ones under much simpler dependence structures?

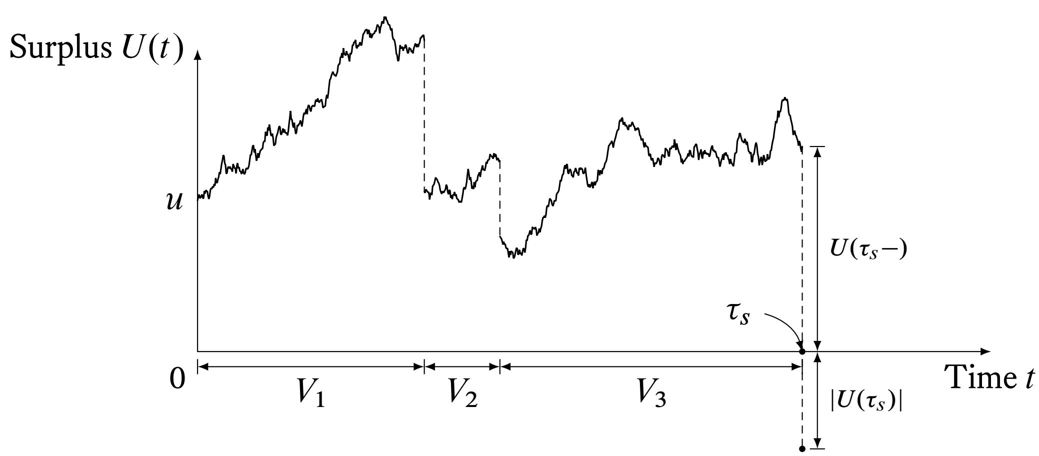


Figure 1.3: A surplus process perturbed by a Wiener diffusion. Here, the time of ruin is denoted with an additional subscript "s" to indicate that ruin is caused by a claim rather than the diffusion. If ruin is caused by the diffusion, then the time of ruin is denoted by τ_d. Due to continuous oscillation of the Wiener diffusion, we have $U(\tau_d-) = |U(\tau_d)| = 0$ if $\tau_d < \infty$.

We mention here that answers to some of the questions above are known, provided that the common distribution of $\{V_i\}_{i=1}^{\infty}$ is exponential, Erlang(n), generalized Erlang(n) and phase-type. These well-established techniques rely heavily on the *strong Markov property* of the surplus process. Since they often involve a type of state-transitioning argument, they do not carry over when the surplus process fails to behave like a Markov chain. This observation presents a major obstacle in our analyses. So we adopt different strategies to overcome this hurdle.

We shall outline some of the strategies used in the subsequent chapters.

- In Chapter 2, we utilize the Borel–Cantelli lemma and provide probabilistic arguments under the constant dividend barrier model. Such technique has advantages over differential equations, since it allows the underlying distribution to be arbitrary—including *discontinuous* distributions. In fact, this strategy is so versatile that we can use it to study *Parisian ruin* beyond the regular ruin. Also, since the conditions are stated in terms of marginal distributions, our results have wide applications to dependence extensions.
- In Chapter 3, we strategically apply Laplace transforms to uncover hidden structures under the rational arrival model. We also provide detailed boundary condition analyses and observe surprising non-smooth behavior at the level boundary for certain functions.
- In Chapter 4, we develop a new set of tools to handle parameters with multiplicities under the Erlang-combination dependence model perturbed by diffusion. These tools differ from applications of the martingale theory, in that they are purely algebraic manipulations. Thus, the new tools are able to provide explicit expressions for the desired integro-differential equations.

Chapter 2

Dividend barrier strategy: Proceed with caution

2.1 Introduction

The idea of Parisian barrier options is proposed by Chesney et al. (1997). This type of options allows the owner to keep the option even when the price of the underlying asset is in the *knock-out region* (or *red zone*), unless the price stays in that region long enough. Likewise, the concept of *Parisian ruin* introduced to ruin theory by Dassios & Wu (2008) allows the surplus process to stay below zero within a pre-fixed period of time. Under this framework, Dassios & Wu (2008) obtain the Parisian ruin probability under a Cramér–Lundberg model with exponential claims. More recently, Loeffen et al. (2013) derive an elegant expression for the Parisian ruin probability for a class of Lévy risk models, and Czarna & Palmowski (2013) demonstrate the optimality of a constant barrier dividend strategy.

In this chapter, we show that an absorbing barrier leads to Parisian ruin *almost surely* for a large class of risk models—including the general Lévy risk models and most of the Sparre-Andersen risk models; that is, the probability of finite-time ruin is 100% for various risk models, with a few exceptions. These exceptions are observed for some special Sparre-Andersen risk models, and key properties of these special Sparre-Andersen risk models are presented as a condition (Theorem 2.3.3, Case 3) under which the Parisian ruin probability is reduced to zero.

The rest of this chapter is organized as follows. Section 2.2 provides the mathematical basis and formal definitions. Then, we study the Parisian ruin probability in Section 2.3, and present the results as Theorems 2.3.1 and 2.3.3. Some generalizations of Theorem 2.3.3 to a dependent surplus model are discussed in Section 2.3.3. In Section 2.4, we investigate a Sparre-Andersen risk process with inter-claim times that follow a K_n distribution. We show in Theorem 2.4.1 and Corollary 2.4.1.1 that when the inter-claim times are K_n, ruin is *always* certain in the presence of a constant dividend barrier. This unifies and generalizes the results obtained in Lin et al. (2003) and Li & Garrido (2004*a*). We then present a numerical comparison between the times to ruin (with and without the barrier), to show that the constant barrier strategy should be avoided if possible. Finally, we discuss in Section 2.5 a possible strategy the insurer may take to reduce the probability of ruin when dividend is paid.

2.2 Preliminaries

The *classical ruin model* or *Cramér–Lundberg model* is based on insurance company's surplus process:

$$U(t) = u + ct - \sum_{j=1}^{N(t)} Y_j, \quad t \geqslant 0,$$

where $u \geqslant 0$ is the initial capital, c is the constant premium rate, $\{N(t) : t \geqslant 0\}$ is a (homogeneous) Poisson process with rate $\lambda > 0$, and $\{Y_i\}_{i=1}^{\infty}$—independent of $\{N(t) : t \geqslant 0\}$—is a sequence of independent and identically distributed (i.i.d.) positive random variables (r.v.'s) with finite mean $0 < \mathbb{E}[Y_1] < \infty$. To have a positive drift in the surplus process, it suffices to require the *positive loading condition*: $c = (1+\theta)\,\lambda\,\mathbb{E}[Y_1] > \lambda\,\mathbb{E}[Y_1]$, where $\theta > 0$ is the relative security loading.

Two generalizations about the classical ruin model are of particular interest. We note that the component $ct - \sum_j Y_j$ is nothing more than a compound Poisson process with drift. One may replace this expression by a *spectrally negative Lévy process* $\{X(t) : t \geqslant 0\}$, or, replace the counting process $\{N(t) : t \geqslant 0\}$ by a general *renewal process*. The former generalization includes the classical ruin model, the *perturbed ruin model* by a Brownian motion, the *gamma risk model* and the *α-stable risk model*. The latter is known as the *renewal risk model* or *Sparre-Andersen risk model* (Andersen 1957), and may further be generalized to introduce dependence between the claim sizes $\{Y_i\}_{i=1}^{\infty}$ and the process $\{N(t) : t \geqslant 0\}$.

2.2.1 Model settings for a spectrally negative Lévy process

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geqslant 0}, \mathbb{P})$ denote a filtered probability space, on which a spectrally negative Lévy process $\{X(t) : t \geqslant 0\}$ is defined. Moreover, we shall assume that $\{X(t) : t \geqslant 0\}$ does not have monotone sample paths. The *Laplace exponent* of $\{X(t) : t \geqslant 0\}$ is defined as a function $\Psi(\cdot)$ satisfying

$$\mathrm{e}^{\Psi(s)} = \mathbb{E}\big[\mathrm{e}^{sX(1)}\big] \quad \text{for } s \in \mathbb{C},\ \mathrm{Re}(s) \geqslant 0.$$

It is well-known that $\Psi(s)$ has a unique decomposition (Doney 2007, pp. 95–96, Eqs. (9.2.1)–(9.2.3)):

$$\begin{aligned}\Psi(s) &= \log\Big(\mathbb{E}\big[\mathrm{e}^{sX(1)}\big]\Big)\\ &= \gamma s + \frac{\sigma^2}{2}s^2 + \int_{(-\infty,0)} \big(\mathrm{e}^{sx} - 1 - sx\mathbb{1}_{\{x>-1\}}\big)\,\mathrm{d}\nu(x),\end{aligned}$$

where $\gamma \in \mathbb{R}$ and $\sigma \geqslant 0$ are constants, $\mathbb{1}_{\{\cdot\}}$ is the indicator function, and ν is a *Lévy measure* satisfying $\nu\big([0,\infty)\big) = 0$ and $\int_{-\infty}^{0}\big(1 \wedge x^2\big)\,\mathrm{d}\nu(x) < \infty$. Then the insurer's surplus process, driven by $\{X(t) : t \geqslant 0\}$, is given by

$$U(t) := u + X(t), \quad t \geqslant 0, \quad u \in \mathbb{R}. \tag{2.1}$$

Note that we allow the initial capital to be negative, since assuming Parisian ruin allows the risk process to start with negative initial value. The positive loading condition is given by $\mathbb{E}[X(1)] = \Psi'(0+) > 0$. We shall denote the conditional probability and conditional expectation given $U(0) = u$ as $\mathbb{P}_u[\cdot]$ and $\mathbb{E}_u[\cdot]$, respectively.

2.2.2 Model settings for a Sparre-Andersen risk process

Let $V, V_1, V_2, V_3, \ldots$ be nonnegative and independent inter-claim time r.v.'s. The r.v.'s V and $V_2, V_3, \ldots$ are assumed to be i.i.d. with common cumulative distribution function (c.d.f.)

$$K(t) = \mathbb{P}[V \leqslant t] = 1 - \bar{K}(t), \quad t \geqslant 0,$$

while $K_1(t) = \mathbb{P}[V_1 \leqslant t] = 1 - \bar{K}_1(t)$, the c.d.f. of V_1, may be different from $K(t)$. We emphasize here that the V_j's *need not have probability density functions* (p.d.f.'s). We define a continuous-time counting process $\{N(t) : t \geqslant 0\}$ by

$$N(t) := \sup\Bigg\{n : \sum_{j=1}^{n} V_j \leqslant t\Bigg\}, \quad t \geqslant 0,$$

which is called a *general* (or *delayed*) *renewal process*. We assume $0 < \mathbb{E}[V] < \infty$ and set $\lambda := 1/\mathbb{E}[V]$. The parameter λ is the *rate* of $\{N(t) : t \geqslant 0\}$. When the V_j's are i.i.d. exponential r.v.'s with mean $1/\lambda$, $\{N(t) : t \geqslant 0\}$ reduces to a (homogeneous) Poisson process with rate λ.

Let $Y, Y_1, Y_2, Y_3, \ldots$ be positive and independent claim-size r.v.'s. The r.v.'s Y and $Y_2, Y_3, \ldots$ are assumed to be i.i.d. with common c.d.f.

$$P(y) = \mathbb{P}[Y \leqslant y] = 1 - \bar{P}(y), \quad y > 0,$$

with $P(0) = 0$ to avoid noncontributing claims, while $P_1(y) = \mathbb{P}[Y_1 \leqslant y] = 1 - \bar{P}_1(y)$, the c.d.f. of Y_1, may be different from $P(y)$, but again $P_1(0) = 0$. Here, the Y_j's *need not have p.d.f.'s either*. We also assume that $0 < \mathbb{E}[Y] < \infty$, and that $\{Y_i\}_{i=1}^{\infty}$ is independent of $\{N(t) : t \geqslant 0\}$. The insurer's surplus process is thus modeled by

$$U(t) := u + ct - \sum_{j=1}^{N(t)} Y_j, \quad t \geqslant 0, \quad u \in \mathbb{R}, \tag{2.2}$$

where $c = (1 + \theta)\,\lambda\,\mathbb{E}[Y]$ is the constant premium rate and $\theta > 0$ is the relative security loading. Again, the initial capital u may be negative.

2.2.3 Constant dividend barrier and Parisian ruin

Now, suppose that dividends are paid to the shareholders under a constant barrier strategy, and let us assume a constant level $b \geqslant u$. The surplus process modified by this barrier strategy will be denoted as $\{U_b(t) : t \geqslant 0\}$. More specifically, let $\{L_b(t) : t \geqslant 0\}$ be the cumulative dividends process, then

$$L_b(t) = \Big[\sup_{0 \leqslant s \leqslant t} U(s) - b\Big]_+, \quad t \geqslant 0.$$

Hence,

$$U_b(t) := U(t) - L_b(t), \quad t \geqslant 0, \tag{2.3}$$

where $\{U(t) : t \geqslant 0\}$ is given either by model (2.1) or by model (2.2).

Let $p \geqslant 0$ be a constant *Parisian clock*. Define the *time of Parisian ruin* under this barrier strategy as

$$\tau_p^b := \inf\big\{t > p : t - g_b(t) > p\big\}, \tag{2.4}$$

where $g_b(t) := \sup\{ s \in [0,t] : U_b(s) \geqslant 0 \}$. Here, we use the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$, where $\emptyset$ represents the empty set. Note that τ_p^b is a *stopping time*. We put

$$\tau_p := \tau_p^\infty$$

to be the time of Parisian ruin of the original risk process $\{U(t) : t \geqslant 0\}$. The ruin-related quantity studied in the next section is the *Parisian ruin probability*:

$$\psi(u; b, p) := \mathbb{P}_u\big[\tau_p^b < \infty\big] = \mathbb{P}\big[\, \tau_p^b < \infty \mid U_b(0) = u \,\big]. \tag{2.5}$$

2.3 The Parisian ruin probability $\psi(u; b, p)$

2.3.1 Results for a spectrally negative Lévy process

Theorem 2.3.1. *Consider the spectrally negative Lévy risk model under a constant dividend barrier given by* (2.1) *and* (2.3). *Then* $\psi(u; b, p) \equiv 1$.

Proof. Loeffen et al. (2013) show that under the positive loading condition $\mathbb{E}[X(1)] > 0$ we have

$$\psi(u; \infty, p) = 1 - \mathbb{E}[X(1)] \frac{\displaystyle\int_0^\infty W(u+z)\, z \, \mathrm{d}\mu_p(z)}{\displaystyle\int_0^\infty z \, \mathrm{d}\mu_p(z)}, \quad u \in \mathbb{R}, \tag{2.6}$$

where $\mathrm{d}\mu_p(z) = \mathbb{P}\big[X(p) \in \mathrm{d}z\big]$ is the *law* of $X(p)$, and $W(\cdot)$ is the *scale function* defined by its Laplace transform

$$\int_0^\infty \mathrm{e}^{-sx}\, W(x)\, \mathrm{d}x = \frac{1}{\Psi(s)}, \quad s > 0.$$

Now, assume $u < b$ and define the *first passage time* of level b as

$$\tau_b^+ = \inf\{\, t > 0 : X(t) > b \,\}.$$

Then by the *total probability theorem* and the *strong Markov property* we have

$$\psi(u; b, p) = \psi(b; b, p)\, \mathbb{P}_u\big[\tau_b^+ < \tau_p^b\big] + \mathbb{P}_u\big[\tau_p^b < \tau_b^+\big]. \tag{2.7}$$

Since $U_b(t) = U(t)$ for all $0 \leqslant t \leqslant \tau_b^+$, we have

$$\mathbb{P}_u\big[\tau_p^b < \tau_b^+\big] = \mathbb{P}_u\big[\tau_p < \tau_b^+\big]. \tag{2.8}$$

Thus, substituting (2.8) into (2.7) yields

$$\psi(u; b, p) = \psi(b; b, p)\,\mathbb{P}_u\big[\tau_b^+ < \tau_p\big] + \mathbb{P}_u\big[\tau_p < \tau_b^+\big]. \tag{2.9}$$

On the other hand, applying the total probability theorem to $\psi(u; \infty, p)$ produces

$$\psi(u; \infty, p) = \psi(b; \infty, p)\,\mathbb{P}_u\big[\tau_b^+ < \tau_p\big] + \mathbb{P}_u\big[\tau_p < \tau_b^+\big],$$

which implies

$$\mathbb{P}_u\big[\tau_b^+ < \tau_p\big] = \frac{1 - \psi(u; \infty, p)}{1 - \psi(b; \infty, p)}. \tag{2.10}$$

Thus, by combining (2.9) and (2.10) we obtain

$$\psi(u; b, p) = 1 - \big[1 - \psi(b; b, p)\big]\frac{1 - \psi(u; \infty, p)}{1 - \psi(b; \infty, p)}, \quad u < b. \tag{2.11}$$

It is worth noting that (2.11) holds when $u = b$ as well. So the function $\psi(\,\cdot\,; b, p)$ is left-continuous at b.

From (2.11), it is evident that $\psi(u; b, p) = 1$ follows if $\psi(b; b, p) = 1$ holds. The rest of the proof is to show $\psi(b; b, p) = 1$. To this end, define the function $\xi(x) := \psi(b - x; b, p)$ on $[0, \infty)$. Then,

$$\xi(x) = 1 - \big[1 - \psi(b; b, p)\big]\frac{\displaystyle\int_0^\infty W(b + z - x)\,z\,\mathrm{d}\mu_p(z)}{\displaystyle\int_0^\infty W(b + z)\,z\,\mathrm{d}\mu_p(z)}, \quad x \geqslant 0.$$

The Laplace transform of $\xi(x)$ is given by

$$\widetilde{\xi}(s) = \frac{1}{s} - \frac{\big[1 - \psi(b; b, p)\big]\displaystyle\int_0^\infty \mathrm{e}^{-sx}\int_0^\infty W(b + z - x)\,z\,\mathrm{d}\mu_p(z)\,\mathrm{d}x}{\displaystyle\int_0^\infty W(b + z)\,z\,\mathrm{d}\mu_p(z)}$$

for all $s > 0$. If we showed that

$$\lim_{s\to\infty}\left[s\int_0^\infty \mathrm{e}^{-sx}\int_0^\infty W(b + z - x)\,z\,\mathrm{d}\mu_p(z)\,\mathrm{d}x\right] = 0, \tag{2.12}$$

then $\lim_{s\to\infty}\big[s\,\widetilde{\xi}(s)\big] = 1$, and by the *initial value theorem* for Laplace transforms, we will obtain the desired result $\psi(b; b, p) = \xi(0) = \lim_{x\to 0+}\xi(x) = 1$.

To prove (2.12), notice that for each fixed $x > 0$, the integrand $s\,\mathrm{e}^{-sx}\int_0^\infty W(b+z-x)\,z\,\mathrm{d}\mu_p(z)$ converges to 0 point-wise as $s \to \infty$. Moreover, since $W(x)$ is strictly increasing in x, by *Tonelli's theorem* we have

$$\begin{aligned} s\int_0^\infty \mathrm{e}^{-sx}\int_0^\infty & W(b+z-x)\,z\,\mathrm{d}\mu_p(z)\,\mathrm{d}x \\ &= \int_0^\infty\int_0^\infty s\,\mathrm{e}^{-sx}\,W(b+z-x)\,\mathrm{d}x\,z\,\mathrm{d}\mu_p(z) \\ &= \int_0^\infty\int_0^{b+z} s\,\mathrm{e}^{-sx}\,W(b+z-x)\,\mathrm{d}x\,z\,\mathrm{d}\mu_p(z) \\ &= \int_0^\infty\int_0^{b+z} s\,\mathrm{e}^{-s(b+z-x)}\,W(x)\,\mathrm{d}x\,z\,\mathrm{d}\mu_p(z) \\ &\leqslant \int_0^\infty \left[1-\mathrm{e}^{-s(b+z)}\right] W(b+z)\,z\,\mathrm{d}\mu_p(z) \\ &< \int_0^\infty W(b+z)\,z\,\mathrm{d}\mu_p(z) = \frac{1-\psi(b;\infty,p)}{\mathbb{E}[X(1)]}\int_0^\infty z\,\mathrm{d}\mu_p(z) < \infty. \end{aligned}$$

Therefore, the *dominated convergence theorem* applies and (2.12) follows.

We conclude that $\xi(0+) = \psi(b;b,p) = 1$, and the proof is complete. □

2.3.2 Results for a Sparre-Andersen risk process

We begin with a concept introduced by Feller (1971, Section 5.2).

Definition 2.3.2. Suppose X is a given random variable. A point $x \in \mathbb{R}$ is said to be a *point of increase* of X if $\mathbb{P}[x-\epsilon < X \leqslant x+\epsilon] > 0$ for all $\epsilon > 0$. In particular, 0 is a point of increase of the inter-claim time random variable $V \geqslant 0$ if $\mathbb{P}[-\epsilon < V \leqslant \epsilon] = \mathbb{P}[V \leqslant \epsilon] = K(\epsilon) > 0$ for all $\epsilon > 0$.

Theorem 2.3.3. *Consider the Sparre-Andersen risk model under a constant dividend barrier given by* (2.2) *and* (2.3).

1. *If* $\overline{P}(b+cp) > 0$, *then* $\psi(u;b,p) = 1$.
2. *If* 0 *is a point of increase of* V, *then* $\psi(u;b,p) = 1$.
3. *If* $\overline{P}(b+cp) = 0$ *and if* 0 *is not a point of increase of* V, *then it is possible to choose a new premium rate* c^* *such that* $\psi^*(u;b,p) = 0$.

Proof. To prove Cases 1 and 2, it suffices to assume that $\{Y_i\}_{i=1}^{\infty}$ is an i.i.d. sequence, and so is $\{V_i\}_{i=1}^{\infty}$. In fact, assume that Y_1 or V_1 has different distribution. If the size of Y_1 does not cause Parisian ruin, then the surplus process renews and reduces to the i.i.d. case. Otherwise, Parisian ruin occurs cp amount of time after the time of the first large enough claim. In either scenario we have $\psi(u; b, p) = 1$ (provided that we have established this under the i.i.d. setting).

If $u < -cp$, then clearly $\psi(u; b, p) = 1$ since the surplus will not have enough time to reach level 0. It is also evident that $\psi(u; b, p) \geqslant \psi(b; b, p)$ for $u \leqslant b$. Thus, the assertion of Case 1 follows if $\psi(b; b, p) = 1$.

Proof of Case 1. Assume that $\{Y_i\}_{i=1}^{\infty}$ is i.i.d. To evaluate $\psi(b; b, p)$, we may condition on the size of the first claim and obtain

$$\begin{aligned} \psi(b; b, p) &= \mathbb{E}[\psi(b - Y_1; b, p)] \\ &= \mathbb{E}\big[\psi(b - Y_1; b, p)\, \mathbb{1}_{\{Y_1 \leqslant b+cp\}}\big] + \mathbb{P}[Y_1 > b + cp] \\ &\geqslant \psi(b; b, p)\, \mathbb{P}[Y_1 \leqslant b + cp] + \mathbb{P}[Y_1 > b + cp], \end{aligned}$$

which implies

$$\psi(b; b, p)\, \mathbb{P}[Y_1 > b + cp] \geqslant \mathbb{P}[Y_1 > b + cp].$$

Since $\mathbb{P}[Y_1 > b + cp] = \bar{P}(b + cp) > 0$, we conclude that $\psi(b; b, p) = 1$.

Proof of Case 2. We shall construct an event with positive probability leading to ruin. Firstly, we choose a fixed level $d > 0$ such that $\alpha := \mathbb{P}[Y_j > d] > 0$ for $j = 1, 2, \ldots$ This is possible since $\mathbb{P}[Y_j > 0] \equiv 1$. Set $\ell = \lceil (b + cp + 1)/d \rceil$, so that

$$\mathbb{P}\left[\sum_{j=1}^{\ell} Y_j > b + cp + 1\right] \geqslant \prod_{j=1}^{\ell} \mathbb{P}[Y_j > d] = \alpha^{\ell} > 0.$$

Next, choose $\epsilon = 1/(c\ell) > 0$ and then $\beta := \mathbb{P}[V_j \leqslant \epsilon] > 0$ for $j = 1, 2, \ldots$ Consider the event

$$A := \left\{ V_1 \leqslant \frac{1}{c\ell},\ V_2 \leqslant \frac{1}{c\ell},\ \ldots,\ V_\ell \leqslant \frac{1}{c\ell} \ \text{ and } \ \sum_{j=1}^{\ell} Y_j > b + cp + 1 \right\},$$

The occurrence of A implies that the insurer receives ℓ consecutive claims with total amount exceeding $b + cp + 1$ within $1/c$ amount of time, during which the growth in surplus does not exceed $c/c = 1$. The net drop is thus more than $b + cp$, and the surplus will not reach 0 after the Parisian clock. We see that the insurer is ruined. The probability of A satisfies

$$\mathbb{P}[A] \geqslant \beta^{\ell} \alpha^{\ell} > 0.$$

By the *second Borel–Cantelli lemma*, the same event will happen infinitely often with probability one. Therefore, $\psi(u; b, p) = 1$.

Proof of Case 3. Suppose that $K(\epsilon_0) = 0$ for some fixed $\epsilon_0 > 0$. Setting $c^* := (b + cp)/\epsilon_0$, after each payment the insurer's surplus will always recover to level b before the next claim. Since immediately before a claim the surplus is always at b, and the claim size never exceeds $b + cp$, the insurer can never be ruined. So the newly chosen c^* yields $\psi^*(u; b, p) = 0$. □

2.3.3 Generalization to dependent Sparre-Andersen models

So far we have studied the Sparre-Andersen risk model with barrier when the claim sizes and the inter-claim times are independent. However, there might be cases when the claim size depends on the inter-claim time or vice versa. Namely, assume that the pairs (V_1, Y_1), (V_2, Y_2), ... or that the pairs (Y_1, V_2), (Y_2, V_3), ... are independent and identically distributed vectors.

We shall briefly discuss a possible generalization of Theorem 2.3.3. If $\mathbb{P}[Y > b + cp] > 0$, it follows immediately that $\psi(u; b, p) \equiv 1$, since the proof of Theorem 2.3.3, Case 1 involves only the marginal distribution of Y. Analogously, the proof of Theorem 2.3.3, Case 3 involves only the marginal distributions of Y and V. We may therefore choose a new premium rate so that $\psi(u; b, p) = 0$ provided that $\mathbb{P}[Y > b + cp] = 0$ and 0 is not a point of increase of V.

It should be noted that these arguments are valid even if the first finite number of inter-claim times and claim sizes are differently distributed.

2.4 Illustration with K_n inter-claim times

The *strong Markov property* of a Lévy risk process, which is essential to the proof of Theorem 2.3.1, no longer holds for a general Sparre-Andersen risk process. However, we see by Theorem 2.3.3 that ruin is still certain under a fairly general setting.

To illustrate an application of Theorem 2.3.3, we shall assume that the inter-claim time distribution $K(t)$ has a p.d.f. $k(t)$ with *Laplace transform* $\widetilde{k}(s) := \int_0^\infty \mathrm{e}^{-st}\, k(t)\, \mathrm{d}t$, where $s \in \mathbb{C}$ and $\mathrm{Re}(s) \geqslant 0$. We shall further assume that $K(t)$ is in the K_n family of distributions; that is, $\widetilde{k}(s)$ is of the form

$$\widetilde{k}(s) = \frac{\lambda^* + s\,\beta(s)}{(\lambda_1 + s)^{n_1} \cdots (\lambda_m + s)^{n_m}}, \tag{2.13}$$

where $1 \leqslant m \leqslant n$, $\lambda_1, \ldots, \lambda_m$ are distinct positive constants, $n_1, \ldots, n_m$ are positive integers with $\sum_{i=1}^m n_i = n$, $\lambda^* = \prod_{i=1}^m \lambda_i^{n_i}$, $\beta(s) = \sum_{i=0}^{n-2} \beta_i\, s^i$ is a polynomial of degree $n - 2$ or less with real-valued coefficients, and $\lambda^* + s\,\beta(s)$ and $\prod_{i=1}^{m}(\lambda_i + s)^{n_i}$ do not share common factors. By the *partial fraction decomposition theorem*, we may rewrite expression (2.13) as

$$\widetilde{k}(s) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\alpha_{ij}\left(\frac{\lambda_i}{\lambda_i + s}\right)^j, \tag{2.14}$$

where α_{ij}, $i = 1,\ldots,m$, $j = 1,\ldots,n_i$, are real-valued constants. Noting that $[\lambda_i/(\lambda_i + s)]^j$ is the Laplace transform of an Erlang(j) density with mean j/λ_i, expression (2.14) implies that $k(t)$ is given by

$$k(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\alpha_{ij}\,\frac{\lambda_i^j\,t^{j-1}\,\mathrm{e}^{-\lambda_i t}}{(j-1)!}, \quad t \geqslant 0. \tag{2.15}$$

Necessarily, we have $\sum_{i=1}^{m}\sum_{j=1}^{n_i}\alpha_{ij} = 1$, although some α_{ij}'s may be negative.

By (2.13) and (2.15), the K_n family of distributions includes, as special cases, the exponential (when $n = 1$), the Erlang (when $m = 1$ and $\beta \equiv 0$), the generalized Erlang (when $\beta \equiv 0$), and the phase-type distributions, as well as mixtures of these.

In relation to a Sparre-Andersen risk model with finite dividend barrier, we have the following results.

Theorem 2.4.1. *For any K_n distribution $K(t)$, 0 is a point of increase of $K(t)$.*

Proof. Suppose, to the contrary, that 0 is not a point of increase of $K(t)$; that is, $K(\epsilon_0) = 0$ for some constant $\epsilon_0 > 0$.

Now, since $K(t)$ has a density $k(t)$ given by (2.15), the density $k(t)$ must be 0 for all $0 \leqslant t < \epsilon_0$. But clearly, the different Erlang densities form a set of linearly independent functions. It follows that the α_{ij}'s in (2.15) all must be 0, contradicting the fact that they sum up to 1. Therefore, 0 is a point of increase of $K(t)$. □

Corollary 2.4.1.1. *For the Sparre-Andersen model under constant dividend barrier with K_n inter-claim times, the probability of Parisian ruin is $\psi(u; b, p) \equiv 1$.*

Theorem 2.4.1 and Corollary 2.4.1.1—the latter being a consequence of Theorem 2.3.3—unify and generalize the results obtained in Lin et al. (2003) and Li & Garrido (2004*a*) regarding ruin probability.

Now that we see under the general K_n-inter-claim time assumption, ruin is certain in the presence of a constant dividend barrier. We shall provide further comparison between the two times of ruin τ_p^b and τ_p to show that the constant dividend strategy should be avoided.

Example 2.4.1 (A numerical illustration)**.** The assumptions on $K(t)$ and $P(y)$ in this example are taken from Li & Garrido (2004*a*, Section 5). Suppose that the inter-claim time distribution is generalized Erlang(2) with parameters λ_1 and λ_2, and that the claim-size distribution is exponential with mean $1/\beta$.

When $p = 0$, the *defective* Laplace transform of τ_0:

$$\mathbb{E}_u\big[\mathrm{e}^{-\delta\tau_0}\,\mathbb{1}_{\{\tau_0<\infty\}}\big],$$

and the *proper* Laplace transform of τ_0^b:

$$\mathbb{E}_u\big[\mathrm{e}^{-\delta\tau_0^b}\big], \quad 0 \leqslant u \leqslant b,$$

are known. For explicit formulas, see Dickson & Hipp (2001, pp. 339–340) and Li & Garrido (2004*a*, Section 5), respectively.

The probabilities of ruin can be recovered by setting $\delta = 0$ in both Laplace transforms. To calculate proper moments of the ruin times, differentiation with respect to δ can be used. For instance,

$$\mathbb{E}_u\big[(\tau_0^b)^n\big] = (-1)^n \frac{\partial^n}{\partial\delta^n}\mathbb{E}_u\big[\mathrm{e}^{-\delta\tau_0^b}\big]\Big|_{\delta=0}, \quad n = 1, 2, \ldots$$

We set $\lambda_1 = \lambda_2 = 1.0$, $\beta = 0.5$, $c = 1.1$ and $b = 10$. For each fixed $u = 0$, 5 and 10, we compute the theoretical ruin probabilities and means and variances of the ruin times. The results are shown in Table 2.1.

Table 2.1: Theoretical ruin probabilities and proper means and variances of τ_0^b and τ_0.

u	$\mathbb{P}_u[\tau_0^b < \infty]$	$\mathbb{E}_u[\tau_0^b]$	$\mathrm{var}_u[\tau_0^b]$	$\mathbb{P}_u[\tau_0 < \infty]$	$\mathbb{E}_u[\tau_0 \mid \tau_0 < \infty]$	$\mathrm{var}_u[\tau_0 \mid \tau_0 < \infty]$
0	1	19.669	1320.983	0.880	20.430	6400.154
5	1	45.079	2204.282	0.652	65.380	21399.865
10	1	51.255	2248.172	0.483	110.330	36399.575

When $p > 0$, no explicit formulas for the Laplace transforms are available. We use Monte Carlo simulation to obtain estimated probabilities and means. Since the algorithm must stop within finite time, we shall estimate quantities of the forms $\mathbb{P}_u[\tau_p^b \leqslant M]$, $\mathbb{E}_u[\tau_p^b \mid \tau_p^b \leqslant M]$, etc., where M is sufficiently large (for instance, $M = 30{,}000$). The results are shown in Table 2.2.

Table 2.2: Estimated ruin probabilities and proper means of ruin times, with standard errors in parentheses.

	Estimates when $p = 0$			
u	$\mathbb{P}_u[\tau_p^b \leqslant M]$	$\mathbb{E}_u[\tau_p^b \mid \tau_p^b \leqslant M]$	$\mathbb{P}_u[\tau_p \leqslant M]$	$\mathbb{E}_u[\tau_p \mid \tau_p \leqslant M]$
0	1.000 (0.000)	19.674 (0.115)	0.881 (0.001)	20.183 (0.249)
5	1.000 (0.000)	45.178 (0.149)	0.654 (0.001)	65.840 (0.463)
10	1.000 (0.000)	51.338 (0.150)	0.482 (0.001)	110.485 (0.600)

	Estimates when $p = 1$			
u	$\mathbb{P}_u[\tau_p^b \leqslant M]$	$\mathbb{E}_u[\tau_p^b \mid \tau_p^b \leqslant M]$	$\mathbb{P}_u[\tau_p \leqslant M]$	$\mathbb{E}_u[\tau_p \mid \tau_p \leqslant M]$
0	1.000 (0.000)	31.279 (0.156)	0.824 (0.001)	30.163 (0.308)
5	1.000 (0.000)	56.660 (0.182)	0.609 (0.001)	75.162 (0.492)
10	1.000 (0.000)	62.872 (0.184)	0.451 (0.001)	118.718 (0.619)

	Estimates when $p = 2$			
u	$\mathbb{P}_u[\tau_p^b \leqslant M]$	$\mathbb{E}_u[\tau_p^b \mid \tau_p^b \leqslant M]$	$\mathbb{P}_u[\tau_p \leqslant M]$	$\mathbb{E}_u[\tau_p \mid \tau_p \leqslant M]$
0	1.000 (0.000)	44.375 (0.199)	0.770 (0.001)	40.310 (0.367)
5	1.000 (0.000)	69.893 (0.221)	0.571 (0.001)	84.548 (0.525)
10	1.000 (0.000)	76.051 (0.222)	0.423 (0.001)	130.747 (0.660)

Notice that for the ruin probability of τ_p^b, each of the 9 parameter combinations of u and p leads to ruin before M. Hence, the margin of errors for these probability estimations are precisely 0 in this simulation study. These observations suggest that not only does ruin occur almost surely, it tends to occur *early* when there is a barrier.

One can also compare the $p = 0$ portion of Table 2.2 with Table 2.1 to conclude that the estimations are fairly accurate.

It should be evident by simply comparing values about τ_p^b with values about τ_p that a constant dividend barrier leads to early ruin in general. Also, increasing the Parisian clock would reduce the probability of ruin when the barrier is absent, but has no effect on the probability of ruin when the barrier is present.

2.5 A strategy to reduce the probability of ruin

To avoid almost surely ruin, we should modify the constant barrier strategy. One such modification is given by Lin & Pavlova (2006), who propose a threshold dividend strategy. Here, we shall present an alternative based on the results of Theorem 2.3.3.

Recall from Theorem 2.3.3 that if the conditions in Case 3 are satisfied, we may choose a *new* premium rate such that ruin will *never* occur. This suggests that when using the *original* premium rate, $\psi(u; b, p)$ does not necessarily equal 1. Our goal is to find possible ways to reproduce the conditions in Case 3.

For each claim payment, the requirement $\mathbb{P}[Y_j > b + cp] = 0$ is easily satisfied by introducing a policy limit $l \leqslant b + cp$; that is, by using $Y_j' := Y_j \wedge l$ instead of Y_j. For inter-claim times, however, the modification is more involved.

The idea is to delay each inter-claim time V_j by a fixed time horizon ϵ. This may be achieved by entering into a reinsurance contract, transferring some of the claim payments to the reinsurer and sharing part of the original premiums with the reinsurer. The details are as follows: The insurer first chooses a positive constant ϵ. Suppose that we are at $t = 0$ or at the moment immediately after a payment. During the next ϵ-length period, any claims issued are to be covered by the *reinsurer*. After the ϵ-period, the next first claim is to be covered by the insurer, and the above procedure repeats.

For example, if V_j is exponential with parameter λ, then the delayed inter-claim time V_j' follows a *shifted exponential distribution* with parameters ϵ and λ. This is due to the *memoryless property* of exponential distributions.

Here is another example. If V_j is Erlang(2) with parameter λ, then V_j' follows the same distribution as $\epsilon + Z$, where Z is a two-point mixture of an exponential and an Erlang(2) distributions, both with the same parameter λ. In general, V_j' is the *residual lifetime* r.v. plus ϵ and therefore satisfies $V_j' \geqslant \epsilon$. Hence, 0 is not a point of increase of V_j'.

The expected payment rate reduces from $\mathbb{E}[Y']/\mathbb{E}[V]$ to $\mathbb{E}[Y']/\mathbb{E}[V']$. Thus, it is reasonable to reduce the premium rate from c to $c' := c\,\mathbb{E}[V]/\mathbb{E}[V']$. Note that *the positive loading condition is maintained*; that is, we have $c' > \mathbb{E}[Y']/\mathbb{E}[V']$. As a result, a premium rate of $c - c'$ should be distributed to the reinsurer.

In order to demonstrate this dividend-reinsurance strategy, we shall provide a numerical example. Despite our effort, we find no literature dealing with such delayed inter-claim times (recall the result of Theorem 2.4.1, which eliminates the use of K_n-distribution). Also, it is evident that no phase-type inter-claim time satisfies the condition of Theorem 2.3.3, Case 3. Therefore, we use Monte Carlo simulations in place of analytical results.

Example 2.5.1 (A dividend-reinsurance strategy)**.** We set an absorbing barrier $b = 4$ and an initial capital $u = 1$. We assume that $V'_j = V_j + \epsilon$ and $Y'_j = Y_j \wedge 3$, where V_j is exponential with mean 1 and Y_j is exponential with mean 1/2. Consequently, $\mathbb{E}[V'_j] = 1 + \epsilon$ and $\mathbb{E}[Y'_j] = 0.49876$. Let $c' = 1/(1 + \epsilon)$, which is a little more than twice of the payment rate. The reinsurer's premium rate is given by $c - c' = 1 - c' = \epsilon/(1 + \epsilon)$, which also measures the transfer ratio of insurer's claim payments. Finally, we set $p = 0$, 1 and 2 in this simulation study.

Table 2.3: Parisian ruin probability estimates, with standard errors in parentheses.

		Estimate of $\mathbb{P}_u[\tau_p^b < (1+\epsilon)\, M]$		
ϵ	$\epsilon/(1+\epsilon)$	$p = 0$	$p = 1$	$p = 2$
0	0.0000	1.0000 (0.0000)	1.0000 (0.0000)	0.9935 (0.0008)
1	0.5000	0.9999 (0.0001)	0.9729 (0.0016)	0.8267 (0.0038)
2	0.6667	0.9959 (0.0006)	0.9509 (0.0022)	0.8038 (0.0040)
3	0.7500	0.9903 (0.0010)	0.9457 (0.0023)	0.7872 (0.0041)

It should be noted that one *cannot* obtain an estimate of $\psi(u; b, p)$, since the algorithm must stop within finite time. We shall instead estimate a quantity of the form $\mathbb{P}_u[\tau_p^b < (1 + \epsilon)\, M]$, where the upper bound M is sufficiently large (for instance, $M = 10{,}000$). When ϵ equals 0, 1, 2 and 3, the claim transfer ratio is 0.0%, 50.0%, 66.7% and 75.0%, respectively. The results are presented in Table 2.3. Note that the "$\epsilon = 0$ and $p = 2$" entry differs from the theoretical value, $\psi(u; b, p) = 1$, within 0.7%.

Chapter 3

Maximum surplus and R_n class of distributions with an application to dividends

3.1 Introduction

When insurance ruin models are discussed, the surplus process takes the center stage. A surplus process is usually expressed in the following form

$$U_c(t) = u + c\,t - S(t), \quad t \geqslant 0,$$

where the aggregate loss process $\{S(t) : t \geqslant 0\}$ is made up by two independent sequences of random variables: An inter-claim times' sequence $\{V_i\}_{i=1}^{\infty}$ and a claim sizes' sequence $\{Y_i\}_{i=1}^{\infty}$. One of the major quantities of interest is the time of ruin defined by

$$\tau_c = \inf\{\, t \geqslant 0 : U_c(t) < 0 \,\},$$

based on which the well-known *Gerber–Shiu function* (Gerber & Shiu 1998) is constructed:

$$m_c(u) = \mathbb{E}\big[\, \mathrm{e}^{-\delta \tau_c} w\big(U_c(\tau_c-), |U_c(\tau_c)|\big) \mathbb{1}(\tau_c < \infty) \;\big|\; U_c(0) = u \,\big], \quad u \geqslant 0.$$

In this chapter we will investigate the following quantities

$$\chi_c(u; b) = \mathbb{P}\big[\, \tau_c^b < \tau_c \;\big|\; U_c(0) = u \,\big]$$

and its counterpart

$$\xi_c(u;b) = 1 - \chi_c(u;b),$$

where b is a pre-specified barrier and τ_c^b is the time of first up-crossing defined by

$$\tau_c^b = \inf\{\, t \geqslant 0 : U_c(t) \geqslant b \,\}.$$

As Section 3.2 illustrates, the function $\xi_c(u;b)$, when viewed as a function of b, represents the distribution of the *maximum surplus before ruin*. Observe that the Gerber–Shiu function $m_c(u)$ essentially depends on only one stopping time, namely τ_c, while the function $\xi_c(u;b)$ involves an additional stopping time, namely τ_c^b. These two functions coincide (see Section 3.2 for more details) only in the special case where we set $\delta = 0$ and $w(x,y) \equiv 1$ for the former and set $b = \infty$ for the latter; that is,

$$m_c(u)\big|_{\delta=0,w\equiv 1} = \xi_c(u;\infty), \quad u \geqslant 0.$$

The classical ruin theory assumes that $\{V_i\}_{i=1}^{\infty}$ is a sequence of independent and exponentially distributed random variables, resulting in a compound Poisson process $\{S(t) : t \geqslant 0\}$ and from which explicit solution for the Gerber–Shiu function $m_c(u)$ can be found by solving a *defective renewal equation* (Gerber & Shiu 1998). The idea of arbitrary inter-claim time distribution first emerged in Andersen (1957). Decades later, Dickson & Hipp (2001) established an *integro-differential equation* for $m_c(u)$, where $\{V_i\}_{i=1}^{\infty}$ were assumed to be Erlang(2). The Erlang(2) framework has since been further extended to Erlang(n) by Li & Garrido (2004*b*), and later on to generalized Erlang(n) (i.e., convolution of n independent exponential distributions) by Li & Garrido (2004*a*) and Gerber & Shiu (2005). Afterwards, the literature branched into two different directions: (1) The K_n framework was introduced by Li & Garrido (2005*b*). Then, the Rational(n) (or R_n) framework was studied by Albrecher et al. (2012) (see also Asmussen & Albrecher 2010, Section XII.3) with the addition of a geometric Brownian diffusion; (2) The phase-type framework was studied by Ahn & Badescu (2007) and Song et al. (2010) (see also Asmussen & Albrecher 2010, Chapter VII for an exposition). These developments on the theory for $m_c(u)$ are perhaps better represented in terms of the *Laplace transform* of the inter-claim time distribution:

$$\frac{\lambda}{s+\lambda} \longrightarrow \frac{\lambda^2}{(s+\lambda)^2} \longrightarrow \frac{\lambda^n}{(s+\lambda)^n} \longrightarrow \prod_{i=1}^{n} \frac{\lambda_i}{s+\lambda_i} \nearrow K_n\colon \frac{g(s)}{\prod_{i=1}^{n}(s+\lambda_i)} \longrightarrow R_n\colon \frac{g(s)}{f(s)}$$

$$\prod_{i=1}^{n} \frac{\lambda_i}{s+\lambda_i} \searrow \text{Phase-type}$$

Here in the diagram, λ, λ_i are *positive* constants, and $f(s)$, $g(s)$ are polynomials with degrees n and $n-1$ (or less), respectively. Note that both the class of rational distributions and the class of phase-type distributions are *dense*, and so is the class of combinations of exponential distributions; that is, any continuous distribution supported on the positive real line can be approximated well using rational distributions, phase-type distributions or combinations of exponential distributions.

However, there is an asymmetric development on the theory for $\xi_c(u;b)$. In particular, early studies on the maximum surplus appeared in Li & Dickson (2006) under the Erlang(n) framework, which was followed by Li (2008), Li & Lu (2009) and Cheung & Landriault (2010) under the phase-type framework. Notice also that the function $\chi_c(u;b) = 1 - \xi_c(u;b)$ represents the probability that the surplus process attains level b prior to ruin, which falls into the area of a *two-sided exit problem*. For a comprehensive treatment for the two-sided exit problem, we refer to Landriault et al. (2017). Note, however, that Landriault et al. (2017) pose the *phase-type* assumption on the claim sizes $\{Y_i\}_{i=1}^{\infty}$, whilst in the aforementioned literature the distributional assumptions are made on the inter-claim times $\{V_i\}_{i=1}^{\infty}$ permitting consideration of *arbitrary* claim size distribution. To the best of our knowledge, there is no analogous extension from the theory for $m_c(u)$ to that for $\xi_c(u;b)$ under the R_n framework with arbitrary claim size distribution. Thus, the goal of this chapter is to complete the theory for the function $\xi_c(u;b)$ under the R_n framework. We shall do so by establishing an integro-differential equation (Theorem 3.3.2 and Corollary 3.3.2.1) from first principles—which will reveal a similar structure compared with the Gerber–Shiu function $m_c(u)$—and then presenting a modified version of the boundary conditions for $\chi_c(u;b)$ at $u = b-$ (Theorem 3.4.2). These boundary conditions will produce the solutions for $\chi_c(u;b)$ and $\xi_c(u;b)$ in a more straightforward manner.

We shall highlight some key differences between the analysis for the Gerber–Shiu function $m_c(u)$ and for the function $\xi_c(u;b)$. As demonstrated by Albrecher et al. (2012), with the absence of a constant barrier b, an integro-differential equation for $m_c(u)$ holds for all $u \geqslant 0$. This implies that $m_c(u)$ itself must be sufficiently smooth. Moreover, Albrecher et al. (2012) show that $m_c(u)$ admits the asymptotic behavior $m_c(u) \sim C_\kappa e^{-\kappa u}$ as $u \to \infty$ for some $C_\kappa > 0$ and $\kappa > 0$. This implies that $m_c(u) > 0$ for all u sufficiently large. However, the most predominate feature of the function $\xi_c(u;b)$ is that it vanishes for all $u \geqslant b$. With the presence of a constant barrier b, an integro-differential equation holds only for $0 \leqslant u \leqslant b$ as illustrated by Theorem 3.3.2. This necessarily calls for additional analysis on the boundary conditions at $u = b-$, leading to a dedicated Section 3.4. When the barrier is present, we mention Lin et al. (2003, Eq.'s (2.6) and (2.8)) under the exponential framework and Li & Garrido (2004*a*, Eq.'s (15) and (16)) under the generalized Erlang(n) framework for the boundary analyses. Lastly, since the integro-differential equation satisfied by $m_c(u)$ has an *unbounded* domain, while the integro-differential equation satisfied by $\xi_c(u; b)$ has a *bounded* domain, the number and the type of solutions in the two cases differ. Namely, in the former case, a single solution converging to zero at infinity is required. In the latter case, a set of linearly independent solutions that are possibly diverging at infinity is needed.

Li & Garrido (2004*a*, Eq. (16)) show that $\xi_c(u; b)$ is in fact $(n-1)$-times continuously differentiable for all $u \geqslant 0$ under the generalized Erlang(n) framework. But, as illustrated by Theorem 3.4.2 and Example 3.5.1, the function $\xi_c(u; b)$ is merely continuous under the R_n framework in general. These observations suggest that the regularity conditions in Albrecher et al. (2012) are not straightforward to be verified and therefore it is not obvious that $\xi_c(u; b)$ satisfies a similar integro-differential equation as does $m_c(u)$. Instead, we provide a direct approach to establish such equation from first principles, and thus circumventing the martingale theory utilized by Albrecher et al. (2012).

The rest of this chapter is organized as follows. We introduce the notation and quantities of interest in Section 3.2 and give a short exposition on the R_n class, the K_n class and the phase-type distributions in Section 3.2.1. The main results are presented as Theorems 3.3.1, 3.3.2 and 3.3.6 in Section 3.3. In particular, Theorem 3.3.1 provides novel properties of the distributions in the R_n class. Theorems 3.3.2 and 3.3.6 focus on the probability of ruin without reaching a pre-specified level. This probability is of interest on its own right but may also be employed as an auxiliary quantity. We provide further analysis and examples in Sections 3.4 and 3.5, respectively. In particular, Theorem 3.4.2 in Section 3.4 deals with the boundary conditions at $u = b-$ and Example 3.5.1 in Section 3.5 displays the graphs of the aforementioned probabilities for various levels. In contrast to the smoothly vanishing probabilities as shown in Li & Dickson (2006, Eq. (2.7)), Figure 3.1 illustrates different behavior when the inter-claim times are general R_n and no longer Erlang-like. Finally, as an application, we implement the results in Section 3.3 in the analysis of the moments of the total dividends up to ruin in Section 3.6.

3.2 Preliminaries

We assume the inter-claim times $V_1, V_2, \ldots$ are independent and identically distributed (i.i.d.) nonnegative random variables (r.v.'s) with common cumulative distribution function (c.d.f.)

$K(t)$, probability density function (p.d.f.) $k(t)$ and Laplace transform $\mathcal{L}[k](s) = \widetilde{k}(s) = \int_0^\infty \mathrm{e}^{-st}\, k(t)\,\mathrm{d}t$. This induces a *renewal process* $\{N(t) : t \geqslant 0\}$, defined by $N(t) = \sup\big\{ n \in \mathbb{N} : \sum_{i=1}^{n} V_i \leqslant t \big\}$ for $t \geqslant 0$. Let the claim sizes $Y_1, Y_2, \ldots$ be i.i.d. positive r.v.'s with common c.d.f. $P(y)$, p.d.f. $p(y)$ and Laplace transform $\mathcal{L}[p](s) = \widetilde{p}(s) = \int_0^\infty \mathrm{e}^{-sy}\, p(y)\,\mathrm{d}y$. Also, assume that $\{Y_n\}_{n=1}^{\infty}$ is independent of $\{N(t) : t \geqslant 0\}$. We now define the (regular) *surplus process* $\{U_c(t) : t \geqslant 0\}$ to be

$$U_c(t) = u + c\,t - \sum_{i=1}^{N(t)} Y_i, \quad t \geqslant 0,$$

with a given initial condition $U_c(0) = u \geqslant 0$. It is convenient to use the notation $\mathbb{P}_u[\,\cdot\,] = \mathbb{P}[\,\cdot \mid U_c(0) = u\,]$ and $\mathbb{E}_u[\,\cdot\,] = \mathbb{E}[\,\cdot \mid U_c(0) = u\,]$.

The *time of ruin* is given by

$$\tau_c = \inf\{\, t \geqslant 0 : U_c(t) < 0 \,\}.$$

Evidently, if the premium rate $c \leqslant \mathbb{E}[Y_1]/\mathbb{E}[V_1]$, then the *probability of ruin*:

$$\psi_c(u) = \mathbb{P}_u[\tau_c < \infty]$$

is constant one. To avoid certain ruin in finite time, it suffices to require the *positive loading condition*: $c = (1+\theta)\,\mathbb{E}[Y_1]/\mathbb{E}[V_1] > \mathbb{E}[Y_1]/\mathbb{E}[V_1]$, under which $\psi_c(u) < 1$ for $u \geqslant 0$.

The *time of first up-crossing* is defined by

$$\tau_c^b = \inf\{\, t \geqslant 0 : U_c(t) \geqslant b \,\}, \quad \text{for } b > u.$$

If $b \leqslant u$, then we set $\tau_c^b = 0$. Under the positive loading condition, which we shall assume henceforth, we have $\mathbb{P}_u[\tau_c^b < \infty] \equiv 1$.

We will be investigating the following quantity:

$$\chi_c(u; b) = \mathbb{P}_u[\tau_c^b < \tau_c]. \tag{3.1}$$

This is the probability that the surplus process attains level b from initial capital u without first falling below zero. It is evident that $\chi_c(u; b) = 1$ for $b \leqslant u$. A related quantity is

$$\xi_c(u; b) = \mathbb{P}_u\Big[\, \tau_c < \infty,\ \sup_{0 \leqslant t \leqslant \tau_c} U_c(t) \leqslant b \,\Big]. \tag{3.2}$$

This is the probability that ruin occurs from initial capital u without the surplus process reaching level b prior to ruin. We observe that

$$\xi_c(u;\infty) = \psi_c(u),$$

and $\chi_c(u;\infty) = \varphi_c(u)$ yields the *probability of ultimate surviving*.

Notice that τ_c is a point of drop of the path $U_c(t)$ (provided that $\tau_c < \infty$). Since the path is right continuous with left limits and has discontinuity at each drop, we conclude that $U_c(t)$ does not attain its supremum for $0 \leqslant t \leqslant \tau_c$. This means that up-crossing does not occur prior to ruin even if $\sup_{0\leqslant t\leqslant \tau_c} U_c(t) = b$. As a result, the two events $\{\tau_c < \infty,\ \tau_c^b < \tau_c\}$ and $\{\tau_c < \infty,\ \sup_{0\leqslant t\leqslant\tau_c} U_c(t) > b\}$ coincide. It is evident that $\xi_c(u;b) = 0$ for $b \leqslant u$. Furthermore, the following relation holds:

$$\chi_c(u;b) + \xi_c(u;b) \equiv 1. \tag{3.3}$$

3.2.1 The R_n class versus the K_n class and the phase-type distributions

The purpose of this section is to elaborate about the R_n class of distributions. We shall demonstrate that the K_n class is a proper subclass of R_n, and that there exist R_n distributions which cannot be represented as phase-type distributions. According to Li & Garrido (2005*b*, Eq. (5)), if $k(t)$ belongs to the K_n class, then its Laplace transform is given by

$$\mathcal{L}[k](s) = \frac{g(s)}{\prod_{i=1}^{n}(s+\lambda_i)}, \tag{3.4}$$

where $\lambda_i > 0$, $i = 1, 2, \ldots, n$, and $g(s)$ is a polynomial of degree $n-1$ or less. The R_n class consists of distributions whose Laplace transforms can be expressed as rational functions, i.e., $\mathcal{L}[k](s) = g(s)/f(s)$, where $f(s)$ and $g(s)$ are polynomials with degrees n and $n-1$ (or less), respectively.

Theorem 3.2.1. *The K_n class of distributions is a proper subclass of the R_n class. Additionally, the R_n class is not contained in the class of phase-type distributions.*

Proof. The fact that $K_n \subseteq R_n$ is apparent by their definitions. To see that the K_n class is a proper subclass of the R_n class, consider the following density function $k(t)$ with Laplace transform

$$\mathcal{L}[k](s) = \frac{\frac{5}{3}(s^2+3)}{(s+1)\,[(s+1)^2+4]} = \frac{5+\frac{5}{3}s^2}{5+7s+3s^2+s^3}. \tag{3.5}$$

Clearly, $k(t)$ is of class R_3. But the denominator of its Laplace transform (3.5) has a pair of conjugate complex roots, so $k(t)$ is not of class K_3 according to definition (3.4).

We now demonstrate that the $k(t)$ described by (3.5) does not belong to the class of phase-type distributions, either. To this end, we cite the following fact: Horváth & Telek (2015, Theorem 3) states that if $k(t)$ were to be represented as a phase-type distribution with finite-dimensional transition matrix, then necessarily $k(t) > 0$ for all $t > 0$. However, by direct Laplace inversion of (3.5), we obtain

$$k(t) = \frac{5}{3}\,\mathrm{e}^{-t}\left[1 - \sin(2t)\right]\mathbb{1}(t \geqslant 0). \tag{3.6}$$

We observe that $k(t) = 0$ for $t = (4m + 1)\,\pi/4$, where m is any nonnegative integer. Therefore, $k(t)$ cannot be a phase-type distribution with finite-dimensional representation. □

3.3 Main results

The following result—Theorem 3.3.1—plays an essential role in the analysis of the structure of integro-differential equations. It is worth pointing out that a reverse version of Theorem 3.3.1 has been discovered as Orbán-Mihálykó & Mihálykó (2014, Theorems 8.4 and 8.5), where structure (3.7) can be deduced by assuming (3.9) holds for some polynomial $f(s)$. In conjunction with our result, this can be viewed as a *necessary and sufficient* condition.

Theorem 3.3.1. *Suppose that the Laplace transform of $k(t)$ is of the form*

$$\mathcal{L}[k](s) = \frac{g(s)}{f(s)} = \frac{b_0 + b_1\,s + \cdots + b_{n-1}\,s^{n-1}}{a_0 + a_1\,s + \cdots + a_{n-1}\,s^{n-1} + s^n}, \tag{3.7}$$

where $f(s) = \sum_{i=0}^{n} a_i s^i$ is a polynomial of degree n with leading coefficient $a_n = 1$ and $g(s) = \sum_{i=0}^{n-1} b_i s^i$ is a polynomial of degree $n-1$ or less with $b_0 = a_0$. Then the following collection of numbers $\{k(0), k'(0), \ldots, k^{(n-1)}(0)\}$ is uniquely determined by the two collections of numbers $\{a_1, a_2, \ldots, a_n\}$ and $\{b_0, b_1, \ldots, b_{n-1}\}$, and satisfies

$$\begin{cases} k(0) = b_{n-1}, \\ k'(0) + a_{n-1}\,k(0) = b_{n-2}, \\ k''(0) + a_{n-1}\,k'(0) + a_{n-2}\,k(0) = b_{n-3}, \\ \vdots \\ k^{(n-1)}(0) + a_{n-1}\,k^{(n-2)}(0) + \cdots + a_2\,k'(0) + a_1\,k(0) = b_0. \end{cases} \tag{3.8}$$

Furthermore,

$$\left[f\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)k\right](t) = a_0\,k(t) + a_1\,k'(t) + \cdots + a_{n-1}\,k^{(n-1)}(t) + k^{(n)}(t) \equiv 0. \tag{3.9}$$

Proof. To prove (3.8), we need to verify $\sum_{i=1}^{\ell} a_{n-i+1}\,k^{(\ell-i)}(0) = b_{n-\ell}$ for $\ell = 1, 2, \ldots, n$. The proof is inductive. We begin by demonstrating the result for $\ell = 1$. By the Initial Value Theorem (IVT) of Laplace transforms,

$$\begin{aligned} k(0) &= \lim_{t\to 0+} k(t) = \lim_{s\to\infty} s\,\mathscr{L}[k](s) \\ &= \lim_{s\to\infty} \frac{s\,g(s)}{f(s)} = \lim_{s\to\infty} \frac{b_0\,s + b_1\,s^2 + \cdots + b_{n-1}\,s^n}{a_0 + a_1\,s + \cdots + a_{n-1}\,s^{n-1} + s^n} = b_{n-1}. \end{aligned}$$

This proves the first equation in the system (3.8).

Next, for some $2 \leqslant \ell \leqslant n$, suppose that the previous $\ell - 1$ equations in system (3.8) are true. To complete the inductive proof, we need to show that the ℓth equation:

$$k^{(\ell-1)}(0) + a_{n-1}\,k^{(\ell-2)}(0) + \cdots + a_{n-(\ell-2)}\,k'(0) + a_{n-(\ell-1)}\,k(0) = b_{n-\ell} \tag{3.10}$$

is also true.

By the property of Laplace transforms of derivatives, we have

$$\begin{aligned} \mathscr{L}\big[k^{(\ell-1)}\big](s) &= s^{\ell-1}\,\mathscr{L}[k](s) - \sum_{i=0}^{\ell-2} s^{\ell-2-i}\,k^{(i)}(0) \\ &= \frac{s^{\ell-1}\,g(s)}{f(s)} - s^{\ell-2}\,k(0) - \cdots - k^{(\ell-2)}(0). \end{aligned}$$

By the IVT, we deduce

$$\begin{aligned} k^{(\ell-1)}(0) &= \lim_{t\to 0+} k^{(\ell-1)}(t) = \lim_{s\to\infty} s\,\mathscr{L}\big[k^{(\ell-1)}\big](s) \\ &= \lim_{s\to\infty} \frac{s^{\ell}\,g(s) - f(s)\left[s^{\ell-1}\,k(0) + \cdots + s\,k^{(\ell-2)}(0)\right]}{f(s)}. \end{aligned} \tag{3.11}$$

Expanding and regrouping the numerator of (3.11) in powers of s, the coefficient of s^{n+j} ($j = 1, \ldots, \ell - 1$) is given by

$$\underbrace{b_{n-(\ell-j)} - a_{n-(\ell-j-1)}\,k(0) - \cdots - a_n\,k^{(\ell-j-1)}(0)}_{(\ell-j+1)\text{ terms}}, \quad 1 \leqslant j \leqslant \ell - 1,$$

which equals 0 by the inductive hypothesis that the $(\ell - j)$th equation in (3.8) holds. Now, the coefficient of s^n is given by

$$\underbrace{b_{n-\ell} - a_{n-(\ell-1)}\, k(0) - \cdots - a_{n-1}\, k^{(\ell-2)}(0)}_{\ell \text{ terms}},$$

so (3.11) yields

$$k^{(\ell-1)}(0) = b_{n-\ell} - a_{n-(\ell-1)}\, k(0) - \cdots - a_{n-1}\, k^{(\ell-2)}(0).$$

After rearranging the terms, we derive (3.10) as required. This establishes the system of equations (3.8).

To prove (3.9), it suffices to prove $\mathcal{L}[f(\mathrm{d}/\mathrm{d}t)k](s) \equiv 0$ by the uniqueness theorem of Laplace transforms. To this end, we shall evaluate $\mathcal{L}[f(\mathrm{d}/\mathrm{d}t)k](s)$ directly as follows.

$$\begin{aligned}
\mathcal{L}\left[f\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)k\right](s) &= \mathcal{L}\big[a_0\, k + a_1\, k' + \cdots + a_{n-1}\, k^{(n-1)} + k^{(n)}\big](s) \qquad (3.12)\\
&= a_0\, \mathcal{L}[k](s) + a_1\, \mathcal{L}[k'](s) + \cdots + a_{n-1}\, \mathcal{L}\big[k^{(n-1)}\big](s) + \mathcal{L}\big[k^{(n)}\big](s)\\
&= a_0\, \mathcal{L}[k](s) + a_1\, s\, \mathcal{L}[k](s) - a_1\, k(0)\\
&\quad + \cdots + a_{n-1}\, s^{n-1}\, \mathcal{L}[k](s) - a_{n-1}\, s^{n-2}\, k(0) - \cdots - a_{n-1}\, k^{(n-2)}(0)\\
&\quad + s^n\, \mathcal{L}[k](s) - s^{n-1}\, k(0) - \cdots - s\, k^{(n-2)}(0) - k^{(n-1)}(0)\\
&= \big(a_0 + a_1\, s + \cdots + a_{n-1}\, s^{n-1} + s^n\big)\, \mathcal{L}[k](s)\\
&\quad - \big[a_1\, k(0) + a_2\, k'(0) + \cdots + a_{n-2}\, k^{(n-3)}(0) + a_{n-1}\, k^{(n-2)}(0) + k^{(n-1)}(0)\big]\\
&\quad - \big[a_2\, k(0) + a_3\, k'(0) + \cdots + a_{n-1}\, k^{(n-3)}(0) + k^{(n-2)}(0)\big]\, s\\
&\quad - \cdots - \big[a_{n-1}\, k(0) + k'(0)\big]\, s^{n-2} - k(0)\, s^{n-1} \qquad (3.13)\\
&= \big(a_0 + a_1\, s + \cdots + a_{n-1}\, s^{n-1} + s^n\big)\, \mathcal{L}[k](s)\\
&\quad - \big(b_0 + b_1\, s + \cdots + b_{n-2}\, s^{n-2} + b_{n-1}\, s^{n-1}\big)\\
&= f(s)\, \mathcal{L}[k](s) - g(s) \equiv 0
\end{aligned}$$

as needed. □

Theorem 3.3.2 below is a direct generalization of Li & Dickson (2006, Theorem 2.2), just as Orbán-Mihálykó & Mihálykó (2014, pp. 188–189) generalizes Li & Garrido (2005*b*). The reader is referred to Asmussen & Albrecher (2010, Section XII.3c) for a proof under the special case where $b_1 = b_2 = \cdots = b_{n-1} = 0$; that is, where the numerator of the Laplace transform of $k(t)$ is equal to a constant b_0. Note that the analysis of the boundary conditions at $u = b-$ is deferred to Theorems 3.4.1 and 3.4.2 in Section 3.4.

Theorem 3.3.2. *Suppose that the Laplace transform of $k(t)$ is given by (3.7). Then $\xi_c(u;b)$ satisfies the following integro-differential equation*

$$\left[f\left(-c\,\frac{\partial}{\partial u}\right)\xi_c\right](u;b) = \left[g\left(-c\,\frac{\partial}{\partial u}\right)\gamma_c\right](u;b), \quad 0 \leqslant u < b, \tag{3.14}$$

where

$$\gamma_c(u;b) = \int_0^u \xi_c(u-y;b)\,p(y)\,\mathrm{d}y + \bar{P}(u), \quad 0 \leqslant u < b, \tag{3.15}$$

and $\bar{P}(u) = 1 - P(u)$. Furthermore, $\xi_c(u;b) \equiv 0$ for all $u \geqslant b$.

Proof. By conditioning on the time t and the amount y of the first claim, we have for $u < b$ that

$$\begin{aligned}\xi_c(u;b) &= \int_0^{\frac{b-u}{c}} \left[\int_0^{u+ct} \xi_c(u+ct-y;b)\,p(y)\,\mathrm{d}y + \int_{u+ct}^{\infty} p(y)\,\mathrm{d}y\right] k(t)\,\mathrm{d}t \\ &= \int_0^{\frac{b-u}{c}} \gamma_c(u+ct;b)\,k(t)\,\mathrm{d}t = \frac{1}{c}\int_u^b \gamma_c(x;b)\,k\Big(\frac{x-u}{c}\Big)\,\mathrm{d}x.\end{aligned} \tag{3.16}$$

It is worth noting that identity (3.16) also holds for $u \geqslant b$ since $k(t) = 0$ for $t < 0$, and that the function $\gamma_c(u;b)$ given by (3.15) is indeed well-defined for all $u \geqslant 0$.

Differentiating (3.16) with respect to $u \geqslant 0$ multiple times, we obtain

$$\frac{\partial}{\partial u}\xi_c(u;b) = -\frac{1}{c^2}\int_u^b \gamma_c(x;b)\,k'\Big(\frac{x-u}{c}\Big)\,\mathrm{d}x - \frac{1}{c}\,\gamma_c(u;b)\,k(0),$$

$$\frac{\partial^2}{\partial u^2}\xi_c(u;b) = \frac{1}{c^3}\int_u^b \gamma_c(x;b)\,k''\Big(\frac{x-u}{c}\Big)\,\mathrm{d}x + \frac{1}{c^2}\,\gamma_c(u;b)\,k'(0) - \frac{1}{c}\,\frac{\partial}{\partial u}\gamma_c(u;b)\,k(0),$$

and more generally

$$\begin{aligned}\frac{\partial^\ell}{\partial u^\ell}\xi_c(u;b) = \frac{(-1)^\ell}{c^{\ell+1}}\int_u^b \gamma_c(x;b)\,k^{(\ell)}\Big(\frac{x-u}{c}\Big)\,\mathrm{d}x \\ - \sum_{i=0}^{\ell-1} \frac{(-1)^{\ell-1-i}}{c^{\ell-i}}\,k^{(\ell-1-i)}(0)\,\frac{\partial^i}{\partial u^i}\gamma_c(u;b).\end{aligned}$$

Thus, for $\ell = 0, 1, \ldots, n$, we have

$$\left(-c\,\frac{\partial}{\partial u}\right)^{\ell}\xi_c(u;b) = \frac{1}{c}\int_u^b \gamma_c(x;b)\,k^{(\ell)}\Big(\frac{x-u}{c}\Big)\,\mathrm{d}x + \sum_{i=0}^{\ell-1} k^{(\ell-1-i)}(0)\left(-c\,\frac{\partial}{\partial u}\right)^{i}\gamma_c(u;b). \tag{3.17}$$

Therefore, applying $f(-c\cdot\partial/\partial u)$ to $\xi_c(u;b)$ and employing results (3.8) and (3.9), we derive

$$
\begin{aligned}
&\left[f\left(-c\,\frac{\partial}{\partial u}\right)\xi_c\right](u;b)\\
&\quad= a_0\,\xi_c(u;b) + a_1\left(-c\,\frac{\partial}{\partial u}\right)\xi_c(u;b) + a_2\left(-c\,\frac{\partial}{\partial u}\right)^2\xi_c(u;b)\\
&\qquad+\cdots+a_{n-1}\left(-c\,\frac{\partial}{\partial u}\right)^{n-1}\xi_c(u;b)+\left(-c\,\frac{\partial}{\partial u}\right)^{n}\xi_c(u;b)\\
&\quad=\frac{1}{c}\int_u^b \gamma_c(x;b)\left[a_0\,k\Big(\frac{x-u}{c}\Big)\right]\mathrm{d}x\\
&\qquad+\frac{1}{c}\int_u^b \gamma_c(x;b)\left[a_1\,k'\Big(\frac{x-u}{c}\Big)\right]\mathrm{d}x + a_1\,k(0)\,\gamma_c(u;b)\\
&\qquad+\frac{1}{c}\int_u^b \gamma_c(x;b)\left[a_2\,k''\Big(\frac{x-u}{c}\Big)\right]\mathrm{d}x\\
&\qquad\quad+a_2\,k'(0)\,\gamma_c(u;b)+a_2\,k(0)\left(-c\,\frac{\partial}{\partial u}\right)\gamma_c(u;b)\\
&\qquad+\cdots+\frac{1}{c}\int_u^b \gamma_c(x;b)\left[a_{n-1}\,k^{(n-1)}\Big(\frac{x-u}{c}\Big)\right]\mathrm{d}x\\
&\qquad\qquad+\sum_{i=0}^{n-2}a_{n-1}\,k^{(n-2-i)}(0)\left(-c\,\frac{\partial}{\partial u}\right)^i\gamma_c(u;b)\\
&\qquad+\frac{1}{c}\int_u^b \gamma_c(x;b)\left[k^{(n)}\Big(\frac{x-u}{c}\Big)\right]\mathrm{d}x+\sum_{i=0}^{n-1}k^{(n-1-i)}(0)\left(-c\,\frac{\partial}{\partial u}\right)^i\gamma_c(u;b)\\
&\quad=0+\left[a_1\,k(0)+a_2\,k'(0)+\cdots+a_{n-1}\,k^{(n-2)}(0)+k^{(n-1)}(0)\right]\gamma_c(u;b)\\
&\qquad+\left[a_2\,k(0)+\cdots+a_{n-1}\,k^{(n-3)}(0)+k^{(n-2)}(0)\right]\left(-c\,\frac{\partial}{\partial u}\right)\gamma_c(u;b)\\
&\qquad+\cdots+\left[a_{n-1}\,k(0)+k'(0)\right]\left(-c\,\frac{\partial}{\partial u}\right)^{n-2}\gamma_c(u;b)+k(0)\left(-c\,\frac{\partial}{\partial u}\right)^{n-1}\gamma_c(u;b)\\
&\quad=b_0\,\gamma_c(u;b)+b_1\left(-c\,\frac{\partial}{\partial u}\right)\gamma_c(u;b)\\
&\qquad+\cdots+b_{n-2}\left(-c\,\frac{\partial}{\partial u}\right)^{n-2}\gamma_c(u;b)+b_{n-1}\left(-c\,\frac{\partial}{\partial u}\right)^{n-1}\gamma_c(u;b)\\
&\quad=\left[g\left(-c\,\frac{\partial}{\partial u}\right)\gamma_c\right](u;b).
\end{aligned}
$$

□

Corollary 3.3.2.1. *Suppose that the Laplace transform of* $k(t)$ *is given by* (3.7). *Then* $\chi_c(u;b)$ *satisfies the following homogeneous integro-differential equation*

$$\left[f\left(-c\,\frac{\partial}{\partial u}\right)\chi_c\right](u;b) = \left[g\left(-c\,\frac{\partial}{\partial u}\right)\zeta_c\right](u;b), \quad 0 \leqslant u < b, \tag{3.18}$$

where

$$\zeta_c(u;b) = \int_0^u \chi_c(u-y;b)\,p(y)\,\mathrm{d}y, \quad 0 \leqslant u < b. \tag{3.19}$$

Proof. Observe that by (3.3) we have $\xi_c(u;b) = 1 - \chi_c(u;b)$ and

$$\begin{aligned}\gamma_c(u;b) &= \int_0^u \left[1 - \chi_c(u-y;b)\right] p(y)\,\mathrm{d}y + 1 - P(u)\\ &= P(u) - \int_0^u \chi_c(u-y;b)\,p(y)\,\mathrm{d}y + 1 - P(u)\\ &= 1 - \zeta_c(u;b).\end{aligned}$$

By Eq. (3.14), we have

$$\left[f\left(-c\,\frac{\partial}{\partial u}\right)(1-\chi_c)\right](u;b) = \left[g\left(-c\,\frac{\partial}{\partial u}\right)(1-\zeta_c)\right](u;b).$$

By the linearity property of differential operators and noting that $f(-c\cdot\partial/\partial u)1 = g(-c\cdot\partial/\partial u)1 = 1$, we see that

$$1 - \left[f\left(-c\,\frac{\partial}{\partial u}\right)\chi_c\right](u;b) = 1 - \left[g\left(-c\,\frac{\partial}{\partial u}\right)\zeta_c\right](u;b),$$

which yields (3.18) after rearranging the terms. □

In order to solve for $\xi_c(u;b)$, we need to find all n linearly independent solutions for the unbounded case $b = \infty$. To this end, we shall transform Eq. (3.14) into a defective renewal equation. This is done by a series of lemmas listed below. The proofs of these lemmas are deferred to Section 3.A. See also Labbé et al. (2011, Theorem 6.1) for a very similar derivation. It is worth mentioning that these techniques are standard.

We shall henceforth assume that $\widetilde{k}(s)$ is given by (3.7). For notational convenience, we denote

$$\mathcal{A}(s) = f(-cs) \quad \text{and} \quad \mathcal{B}(s) = g(-cs). \tag{3.20}$$

Clearly, $\mathcal{A}(s)$ is a polynomial of degree n with leading coefficient $(-c)^n$ and $\mathcal{B}(s)$ is a polynomial of degree $n-1$ or less.

Lemma 3.3.3. *The Laplace transform* $\mathcal{L}[\xi_c](s;\infty) = \int_0^\infty \mathrm{e}^{-su}\,\xi_c(u;\infty)\,\mathrm{d}u$ *satisfies*

$$\big(\mathcal{A}(s) - \mathcal{B}(s)\,\widetilde{p}(s)\big)\,\mathcal{L}[\xi_c](s;\infty) = \frac{1-\widetilde{p}(s)}{s}\mathcal{B}(s) - \mathcal{P}_{n-1}(s), \tag{3.21}$$

where $\mathcal{P}_{n-1}(s)$ *is a polynomial of degree* $n-1$ *or less.*

To further transform Eq. (3.21), one key ingredient involves factoring

$$\mathcal{A}(s) - \mathcal{B}(s)\,\widetilde{p}(s) \quad \text{and} \quad \frac{1-\widetilde{p}(s)}{s}\mathcal{B}(s) - \mathcal{P}_{n-1}(s),$$

on the left-hand and right-hand sides of (3.21), respectively. We firstly locate the zeros on both sides of (3.21).

Lemma 3.3.4. *The generalized Lundberg's equation*

$$\widetilde{k}(-cs)\,\widetilde{p}(s) = 1 \tag{3.22}$$

has exactly n *roots with nonnegative real parts counting multiplicities, which may be enumerated as*

$$\begin{gathered} t_1 = t_2 = \cdots = t_{k_1} = \rho_1, \quad t_{k_1+1} = t_{k_1+2} = \cdots = t_{k_1+k_2} = \rho_2, \\ \cdots\cdots\cdots\cdots \\ t_{k_1+k_2+\cdots+k_{\ell-1}+1} = t_{k_1+k_2+\cdots+k_{\ell-1}+2} = \cdots = t_{k_1+k_2+\cdots+k_{\ell-1}+k_\ell} = \rho_\ell, \end{gathered} \tag{3.23}$$

where $\sum_{i=1}^{\ell} k_i = n$ *and* $\rho_1, \rho_2, \ldots, \rho_\ell$ *are distinct complex roots with* $\operatorname{Re}\rho_i \geqslant 0$, $i = 1, 2, \ldots, \ell$.

With the zeros located, we are now ready to factor the two sides of (3.21). To this end, we shall use the notions of *divided differences* and *translation transforms*.

More specifically, let $\varphi(z)$ be an arbitrary complex function. The divided difference of φ is defined recursively by $\varphi[t_1] = \varphi(t_1)$, $\varphi[t_1, t_2] = \big(\varphi[t_2] - \varphi[t_1]\big)/(t_2 - t_1)$, …, and more generally, $\varphi[t_1, \ldots, t_{i-2}, t_{i-1}, t_i] = \big(\varphi[t_1, \ldots, t_{i-2}, t_i] - \varphi[t_1, \ldots, t_{i-2}, t_{i-1}]\big)/(t_i - t_{i-1})$, etc. The translation transform of φ is defined by $T_s\varphi(x) = \int_x^\infty \mathrm{e}^{-s(y-x)}\,\varphi(y)\,\mathrm{d}y$. Some properties of these two are presented in Section 3.B. For instance, Proposition 3.B.2 enables multiple identical arguments t_j to appear in divided differences, and identity (3.65) relates translation transforms to divided differences.

Lemma 3.3.5. *Define* $\omega(s) = \prod_{i=1}^{\ell}(s-\rho_i)^{k_i}$, *where* $\rho_1, \rho_2, \ldots, \rho_\ell$ *are the distinct roots of Eq.* (3.22). *Then*

$$\mathcal{A}(s) - \mathcal{B}(s)\,\widetilde{p}(s) = \omega(s)\left\{(-c)^n + T_s\left[\sum_{j=1}^{n}(-1)^{n-j}\,\mathcal{B}[t_1, t_2, \ldots, t_j]\,T_{t_j}T_{t_{j+1}}\cdots T_{t_n}p\right](0)\right\}, \tag{3.24}$$

and

$$\frac{1-\widetilde{p}(s)}{s}\mathcal{B}(s)-\mathcal{P}_{n-1}(s)=-\omega(s)\,T_s\Bigg[\sum_{j=1}^{n}(-1)^{n-j}\,\mathcal{B}[t_1,t_2,\dots,t_j]\,T_{t_j}\,T_{t_{j+1}}\cdots T_{t_n}\overline{P}\Bigg](0), \tag{3.25}$$

where $t_1, t_2, \dots, t_n$ *are the enumerated roots* (3.23).

Theorem 3.3.6. *The function* $\xi_c(u;\infty)$ *satisfies the following defective renewal equation*

$$\xi_c(u;\infty)=\phi\int_0^u \xi_c(u-x;\infty)\,h(x)\,\mathrm{d}x+\phi\,\overline{H}(u),\quad u\geqslant 0, \tag{3.26}$$

where

$$\phi=\frac{(-1)^{n-1}}{c^n}\sum_{i=1}^{\ell}\frac{1}{(k_i-1)!}\frac{\partial^{k_i-1}}{\partial s^{k_i-1}}\Big[\frac{(s-\rho_i)^{k_i}\,\mathcal{B}(s)}{\omega(s)}\,T_s\overline{P}(0)\Big]\Big|_{s=\rho_i}, \tag{3.27}$$

$$h(x)=\frac{(-1)^{n-1}}{\phi\,c^n}\sum_{i=1}^{\ell}\frac{1}{(k_i-1)!}\frac{\partial^{k_i-1}}{\partial s^{k_i-1}}\Big[\frac{(s-\rho_i)^{k_i}\,\mathcal{B}(s)}{\omega(s)}\,T_s\,p(x)\Big]\Big|_{s=\rho_i},\quad x\geqslant 0, \tag{3.28}$$

$$\overline{H}(u)=\frac{(-1)^{n-1}}{\phi\,c^n}\sum_{i=1}^{\ell}\frac{1}{(k_i-1)!}\frac{\partial^{k_i-1}}{\partial s^{k_i-1}}\Big[\frac{(s-\rho_i)^{k_i}\,\mathcal{B}(s)}{\omega(s)}\,T_s\overline{P}(u)\Big]\Big|_{s=\rho_i},\quad u\geqslant 0, \tag{3.29}$$

and $\rho_1, \rho_2, \dots, \rho_\ell$ *are the distinct roots of* (3.22) *and* $\omega(s)=\prod_{i=1}^{\ell}(s-\rho_i)^{k_i}$.

Proof. By Lemmas 3.3.3 and 3.3.5, the Laplace transform $\mathcal{L}[\xi_c](s;\infty)$ satisfies

$$\big\{(-c)^n+T_s\widehat{h}(0)\big\}\,\mathcal{L}[\xi_c](s;\infty)=-T_s\widehat{H}(0), \tag{3.30}$$

where

$$\widehat{h}(x)=\sum_{j=1}^{n}(-1)^{n-j}\,\mathcal{B}[t_1,t_2,\dots,t_j]\,T_{t_j}\,T_{t_{j+1}}\cdots T_{t_n}\,p(x),$$

$$\widehat{H}(x)=\sum_{j=1}^{n}(-1)^{n-j}\,\mathcal{B}[t_1,t_2,\dots,t_j]\,T_{t_j}\,T_{t_{j+1}}\cdots T_{t_n}\overline{P}(x).$$

Since $\int_0^\infty \widehat{h}(x)\,\mathrm{d}x=T_0\widehat{h}(0)=\widehat{H}(0)$, using Leibniz's rule for divided differences, we derive

$$\widehat{h}(x)=(\mathcal{B}\eta_x)[t_1,t_2,\dots,t_n]=\sum_{i=1}^{\ell}\frac{1}{(k_i-1)!}\frac{\partial^{k_i-1}}{\partial s^{k_i-1}}\Big[\frac{(s-\rho_i)^{k_i}\,\mathcal{B}(s)}{\omega(s)}\,T_s\,p(x)\Big]\Big|_{s=\rho_i},$$

$$\widehat{H}(x)=(\mathcal{B}\overline{\eta}_x)[t_1,t_2,\dots,t_n]=\sum_{i=1}^{\ell}\frac{1}{(k_i-1)!}\frac{\partial^{k_i-1}}{\partial s^{k_i-1}}\Big[\frac{(s-\rho_i)^{k_i}\,\mathcal{B}(s)}{\omega(s)}\,T_s\overline{P}(x)\Big]\Big|_{s=\rho_i},$$

where $\eta_x(s) = T_s\, p(x)$ and $\overline{\eta}_x(s) = T_s\, \overline{P}(x)$.

We divide both sides of Eq. (3.30) by $(-c)^n$, then rearrange the terms so one $\mathcal{L}[\xi_c](s;\infty)$ is kept on the left-hand side. We normalize the convolution term on the right-hand side by factoring out the constant

$$\phi = \frac{-1}{(-c)^n} \int_0^\infty \widehat{h}(x)\,\mathrm{d}x = \frac{(-1)^{n-1}}{c^n}\, \widehat{H}(0),$$

which yields (3.27). Finally, we arrive at

$$\mathcal{L}[\xi_c](s;\infty) = \phi\, \mathcal{L}[\xi_c](s;\infty)\, T_s h(0) + T_s \overline{H}(0), \tag{3.31}$$

where the functions $h(x) = -\phi^{-1}\,(-c)^{-n}\,\widehat{h}(x)$ and $\overline{H}(x) = -\phi^{-1}\,(-c)^{-n}\,\widehat{H}(x)$. These calculations yield (3.28) and (3.29), respectively. Direct Laplace inversion of (3.31) then yields Eq. (3.26) and the proof is complete. □

Corollary 3.3.6.1. *The function $\chi_c(u;\infty)$ satisfies the following defective renewal equation*

$$\chi_c(u;\infty) = \phi \int_0^u \chi_c(u-x;\infty)\, h(x)\,\mathrm{d}x + 1 - \phi, \quad u \geqslant 0, \tag{3.32}$$

where ϕ and $h(x)$ are given by (3.27) and (3.28), respectively.

Proof. Substitute identity (3.3) into Eq. (3.26) and then use the fact that $\overline{H}(u) = \int_u^\infty h(x)\,\mathrm{d}x$. After rearranging the equation, we derive Eq. (3.32). □

Remark 3.3.1. The solutions to the defective renewal equations (3.26) and (3.32) are well-known. In fact, their Laplace transforms are

$$\widetilde{\xi}_c(s;\infty) = \frac{\phi\, \widetilde{\overline{H}}(s)}{1 - \phi\, \widetilde{h}(s)}$$

and

$$\widetilde{\chi}_c(s;\infty) = \frac{1-\phi}{s - \phi\, s\, \widetilde{h}(s)},$$

respectively. Thus, the functions $\xi_c(u;\infty)$ and $\chi_c(u;\infty)$ can be obtained by Laplace inversion.

Note that the function $\chi_c(u;b)$ satisfies a *homogeneous* integro-differential equation (3.18). Thus, it is easier to find other linearly independent solutions for Eq. (3.18) than for Eq. (3.14).

3.4 Analyses for a homogeneous integro-differential equation and the boundary conditions

The purpose of this section is to analyze a homogeneous integro-differential equation, which shall produce the rest of the $n-1$ linearly independent solutions needed to construct $\chi_c(u;b)$ and $\xi_c(u;b)$. This is realized in Theorem 3.4.1 and the succeeding remark. However, the standard boundary conditions for $\chi_c(u;b)$ at $u = b-$ are difficult to use under the R_n framework. We realize then that $\chi_c(u;b)$ is a linear combination of the n solutions and seek the linear coefficients instead. The system of equations satisfied by these coefficients is derived from the boundary conditions satisfied by $\chi_c(u;b)$ at $u = b-$. This is established in Theorem 3.4.2.

When $b = \infty$, Corollary 3.3.6.1 provides *one* solution for the function satisfying Eq. (3.18) in Corollary 3.3.2.1. We now seek $n-1$ other linearly independent solutions. To this end, consider the following *homogeneous* integro-differential equation

$$\left[\mathcal{A}\left(\frac{\partial}{\partial u}\right)v\right](u) = \left[\mathcal{B}\left(\frac{\partial}{\partial u}\right)w\right](u), \quad u \geqslant 0, \tag{3.33}$$

where

$$w(u) = \int_0^u v(u-y)\, p(y)\, \mathrm{d}y, \quad u \geqslant 0.$$

Taking derivatives of w at 0 yields

$$w^{(l)}(0) = \sum_{r=1}^{l} v^{(r-1)}(0)\, p^{(l-r)}(0), \quad l = 0, 1, 2, \dots, \tag{3.34}$$

with the convention that $\sum_{r=1}^{0} = 0$. The relation (3.34) between the derivatives of w at 0 and those of v at 0 is more transparent in the following matrix form

$$\begin{pmatrix} w(0) \\ w'(0) \\ w''(0) \\ \vdots \\ w^{(n-3)}(0) \\ w^{(n-2)}(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ p(0) & 0 & 0 & \dots & 0 & 0 & 0 \\ p'(0) & p(0) & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ p^{(n-4)}(0) & p^{(n-5)}(0) & p^{(n-6)}(0) & \dots & 0 & 0 & 0 \\ p^{(n-3)}(0) & p^{(n-4)}(0) & p^{(n-5)}(0) & \dots & p(0) & 0 & 0 \end{pmatrix} \begin{pmatrix} v(0) \\ v'(0) \\ v''(0) \\ \vdots \\ v^{(n-3)}(0) \\ v^{(n-2)}(0) \\ v^{(n-1)}(0) \end{pmatrix}$$

We remark that under the generalized Erlang(n) framework, the derivatives of $p(y)$ at $y = 0$ are not needed. This can be seen from Eq. (3.36) below, as $B_1 = B_2 = \cdots = B_{n-1} = 0$ for generalized Erlang(n) inter-claim times.

Let $v_1, v_2, \ldots, v_{n-1}$ be $n-1$ solutions satisfying

$$v_r^{(l)}(0) = \mathbb{1}(r = l+1), \quad 1 \leqslant r \leqslant n-1,\ 0 \leqslant l \leqslant n-1.$$

Then

$$w_r^{(l)}(0) = \mathbb{1}(r \leqslant l)\, p^{(l-r)}(0), \quad 1 \leqslant r \leqslant n-1,\ 0 \leqslant l \leqslant n-2.$$

We now apply Laplace transforms on both sides of Eq. (3.33) to obtain

$$\mathcal{A}(s)\,\widetilde{v}(s) - \sum_{j=1}^{n} s^{j-1} \sum_{l=0}^{n-j} A_{l+j}\, v^{(l)}(0) = \mathcal{B}(s)\,\widetilde{w}(s) - \sum_{j=1}^{n-1} s^{j-1} \sum_{l=0}^{n-1-j} B_{l+j}\, w^{(l)}(0).$$

Rearranging and substituting the solutions $v_1, v_2, \ldots, v_{n-1}$, we obtain

$$\big(\mathcal{A}(s) - \mathcal{B}(s)\,\widetilde{p}(s)\big)\,\widetilde{v}_r(s) = -d_r(s), \quad r = 1, 2, \ldots, n-1, \tag{3.35}$$

where

$$d_r(s) = \sum_{j=0}^{n-r} d_{r,j}\, s^j$$

is a polynomial of degree at most $n-r$, and

$$d_{r,j} = -A_{r+j} + \sum_{l=r}^{n-2-j} B_{1+j+l}\, p^{(l-r)}(0), \quad r = 1, 2, \ldots, n-1,\ j = 0, 1, \ldots, n-r. \tag{3.36}$$

Here, the empty sums $\sum_{l=r}^{r-1}$ and $\sum_{l=r}^{r-2}$ are understood as 0 when $j = n-r-1$ and $j = n-r$, respectively. We see that under the R_n framework where $\mathcal{B}(s)$ is a non-constant polynomial, the derivatives of $p(y)$ at $y = 0$ are needed to identify the polynomials $d_r(s)$ for $r = 1, 2, \ldots, n-1$. In other words, for a general rational *inter-claim time* distribution, the higher-order derivatives of the *claim size* density function contribute to the solutions for $\xi_c(u; b)$ and $\chi_c(u; b)$ regarding the distribution of the maximum surplus.

Applying (3.24) in Lemma 3.3.5, we may rewrite the rth equation of (3.35) as

$$\left\{(-c)^n + T_s\left[\sum_{j=1}^{n}(-1)^{n-j}\,\mathcal{B}[t_1, t_2, \ldots, t_j]\, T_{t_j} T_{t_{j+1}} \cdots T_{t_n} p\right](0)\right\} \widetilde{v}_r(s) = -\frac{d_r(s)}{\omega(s)}.$$

Multiplying both sides by $(-1)^{n-1}/c^n$ and rearranging the terms yield

$$\widetilde{v}_r(s) = \phi\, \widetilde{h}(s)\, \widetilde{v}_r(s) + \phi\, \frac{(-1)^{n-1}}{\phi\, c^n}\, \frac{d_r(s)}{\omega(s)}, \tag{3.37}$$

where the constant ϕ and the function h are given by (3.27) and (3.28) in Theorem 3.3.6, respectively. According to the theory of partial fraction decomposition, we derive

$$\frac{(-1)^{n-1}}{\phi\, c^n} \frac{d_r(s)}{\omega(s)} = \sum_{i=1}^{\ell} \sum_{j=1}^{k_i} \frac{C_{r,i,j}}{(s-\rho_i)^j},$$

where

$$C_{r,i,j} = \frac{1}{(k_i - j)!} \frac{(-1)^{n-1}}{\phi\, c^n} \frac{\partial^{k_i-j}}{\partial s^{k_i-j}} \left[\frac{(s-\rho_i)^{k_i} d_r(s)}{\omega(s)} \right] \bigg|_{s=\rho_i} \tag{3.38}$$

for $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, k_i$.

The above derivations yield the following result:

Theorem 3.4.1. *Let $v_r(u)$ be a solution of (3.33) satisfying $v_r^{(l)}(0) = \mathbb{1}(r = l+1)$, $r = 1, 2, \ldots, n-1$. Then the function $v_r(u)$ satisfies the following defective renewal equation*

$$v_r(u) = \phi \int_0^u v_r(u-x)\, h(x)\, \mathrm{d}x + \phi \sum_{i=1}^{\ell} \left(\sum_{j=1}^{k_i} \frac{C_{r,i,j}}{(j-1)!} u^{j-1} \right) \mathrm{e}^{\rho_i u}, \quad u \geqslant 0,$$

where the constants $C_{r,i,j}$ are given by (3.38).

Proof. Apply Laplace inversion on both sides of Eq. (3.37) and use the fact that

$$\int_0^\infty \mathrm{e}^{-su}\, u^{j-1}\, \mathrm{e}^{\rho u}\, \mathrm{d}u = \frac{(j-1)!}{(s-\rho)^j}.$$

□

Remark 3.4.1. Analogous to the functions $\xi_c(u;\infty)$ and $\chi_c(u;\infty)$ in Remark 3.3.1, the functions $v_r(u)$ are obtained by solving (3.37) and then inverting the Laplace transforms

$$\widetilde{v}_r(s) = \frac{1}{1 - \phi\, \widetilde{h}(s)} \frac{(-1)^{n-1}}{c^n} \frac{d_r(s)}{\omega(s)}$$

for $r = 1, 2, \ldots, n-1$, respectively. Observe that the functions $v_r(u)$ are *unbounded* for $u \geqslant 0$.

For a general finite barrier $b < \infty$, it is evident that the solution $\chi_c(u;b)$ to the homogeneous integro-differential equation (3.18) is in the form of the following linear combination

$$\chi_c(u;b) = c_0\, \varphi_c(u) + c_1\, v_1(u) + \cdots + c_{n-1}\, v_{n-1}(u), \tag{3.39}$$

where c_0, c_1, ..., c_{n-1} are constants depending on b and $\varphi_c(u) = \chi_c(u;\infty)$ is obtained from Corollary 3.3.6.1. It follows that $\xi_c(u;b)$ is a linear combination of the solution $\psi_c(u) =$

$\xi_c(u;\infty)$ to the non-homogeneous equation and the solutions $\varphi_c(u)$, $v_1(u)$, ..., $v_{n-1}(u)$ to the homogeneous equation; that is,

$$\xi_c(u;b) = 1-\chi_c(u;b) = \psi_c(u) + (1-c_0)\,\varphi_c(u) - c_1\,v_1(u) - \cdots - c_{n-1}\,v_{n-1}(u). \quad (3.40)$$

Moreover, we denote

$$\zeta_c(u) := \zeta_c(u;\infty) = \int_0^u \varphi_c(u-y)\,p(y)\,\mathrm{d}y, \quad u \geqslant 0.$$

Then by linearity we have

$$\zeta_c(u;b) = c_0\,\zeta_c(u) + c_1\,w_1(u) + \cdots + c_{n-1}\,w_{n-1}(u). \quad (3.41)$$

Observe that all functions $\varphi_c(u)$, $\zeta_c(u)$, $v_r(u)$, $w_r(u)$ for $r = 1, 2, \ldots, n-1$ are known. It remains to determine the constants $c_0, c_1, \ldots, c_{n-1}$ by establishing a system of equations from the boundary conditions satisfied by $\chi_c(u;b)$. In the following derivations, the various derivatives evaluated at $u = b$ are to be understood as evaluated at $u = b-$.

Theorem 3.4.2. *The constants $c_0, c_1, \ldots, c_{n-1}$ satisfy the following linear system of equations*

$$\varphi_c(b)\,c_0 + \sum_{r=1}^{n-1} v_r(b)\,c_r = 1, \quad (3.42)$$

and

$$\Big[\varphi_c^{(\ell)}(b) + \sum_{i=0}^{\ell-1} \frac{(-1)^{\ell-1-i}}{c^{\ell-i}}\,k^{(\ell-1-i)}(0)\,\zeta_c^{(i)}(b)\Big]\,c_0 + \sum_{r=1}^{n-1}\Big[v_r^{(\ell)}(b) + \sum_{i=0}^{\ell-1} \frac{(-1)^{\ell-1-i}}{c^{\ell-i}}\,k^{(\ell-1-i)}(0)\,w_r^{(i)}(b)\Big]\,c_r = \frac{(-1)^{\ell-1}}{c^{\ell}}\,k^{(\ell-1)}(0) \quad (3.43)$$

for $\ell = 1, 2, \ldots, n-1$.

Remark 3.4.2. When the inter-claim times follow an Erlang(n) distribution, the density function satisfies $k(0) = k'(0) = k''(0) = \cdots = k^{(n-2)}(0) = 0$, and thus the system of equations (3.42) and (3.43) implies Eq. (2.7) of Li & Dickson (2006). However, if $k(t)$ is a general R_n density, then $\chi_c(u;b)$ and $\xi_c(u;b)$ will generally *not* have vanishing derivatives at $u = b$.

Proof of Theorem 3.4.2. Substituting the two identities $\xi_c(u;b) = 1-\chi_c(u;b)$ and $\gamma_c(u;b) = 1-\zeta_c(u;b)$ into Eq. (3.16), then rearranging the terms, we obtain

$$\chi_c(u;b) = 1 - K\Big(\frac{b-u}{c}\Big) + \frac{1}{c}\int_u^b \zeta_c(x;b)\,k\Big(\frac{x-u}{c}\Big)\,\mathrm{d}x. \quad (3.44)$$

Evaluating both sides of (3.44) at $u = b$ and using (3.39), we derive the first equation (3.42) in the linear system.

Analogous to the proof of Theorem 3.3.2, we now differentiate (3.44) w.r.t. u and obtain

$$\begin{aligned}\frac{\partial^\ell}{\partial u^\ell}\chi_c(u;b) = \frac{(-1)^\ell}{c^{\ell+1}}\int_u^b \zeta_c(x;b)\,k^{(\ell)}\Big(\frac{x-u}{c}\Big)\,\mathrm{d}x \\ + \frac{(-1)^{\ell-1}}{c^\ell}\,k^{(\ell-1)}\Big(\frac{b-u}{c}\Big) - \sum_{i=0}^{\ell-1}\frac{(-1)^{\ell-1-i}}{c^{\ell-i}}\,k^{(\ell-1-i)}(0)\,\frac{\partial^i}{\partial u^i}\zeta_c(u;b)\end{aligned}$$

for all integer $\ell \geqslant 1$. Now, evaluating both sides at $u = b$ and using both (3.39) and (3.41), we derive

$$\begin{aligned}c_0\,\varphi_c^{(\ell)}(b) + \sum_{r=1}^{n-1} c_r\,v_r^{(\ell)}(b) = \frac{(-1)^{\ell-1}}{c^\ell}\,k^{(\ell-1)}(0) \\ - \sum_{i=0}^{\ell-1}\frac{(-1)^{\ell-1-i}}{c^{\ell-i}}\,k^{(\ell-1-i)}(0)\Big[c_0\,\zeta_c^{(i)}(b) + \sum_{r=1}^{n-1} c_r\,w_r^{(i)}(b)\Big].\end{aligned}$$

After rearranging the terms, this is simplified to (3.43) in the linear system. □

3.5 Illustrations with specific claim-size distributions

3.5.1 Exponential claims

We will demonstrate that our results recover the known results on ruin probabilities. Suppose that claim sizes follow an exponential distribution with parameter β; that is, $p(y) = \beta\,\mathrm{e}^{-\beta y}$ for $y \geqslant 0$. Then Lundberg's equation (3.22) simplifies to

$$(s+\beta)\,f(-cs) - \beta\,g(-cs) = 0. \tag{3.45}$$

We find

$$T_s p(x) = \frac{\beta\,\mathrm{e}^{-\beta x}}{s+\beta} \quad\text{and}\quad T_s\overline{P}(x) = \frac{\mathrm{e}^{-\beta x}}{s+\beta}.$$

So

$$\phi = \frac{(-1)^{n-1}}{c^n}\sum_{i=1}^{\ell}\frac{1}{(k_i-1)!}\,\frac{\partial^{k_i-1}}{\partial s^{k_i-1}}\Big[\frac{(s-\rho_i)^{k_i}\,\mathcal{B}(s)}{(s+\beta)\,\omega(s)}\Big]\Big|_{s=\rho_i},$$

$$h(x) = \beta\,\mathrm{e}^{-\beta x}, \quad x \geqslant 0, \qquad\text{and}\qquad \overline{H}(u) = \mathrm{e}^{-\beta u}, \quad u \geqslant 0.$$

If, in addition, the roots ρ_i are all simple, then the constant ϕ further simplifies to

$$\phi = \frac{(-1)^{n-1}}{c^n} \sum_{i=1}^{n} \frac{\mathcal{B}(\rho_i)}{(\rho_i + \beta) \prod_{j \neq i} (\rho_i - \rho_j)} = 1 - \frac{\kappa}{\beta},$$

where $-\kappa$ is the unique negative root of Lundberg's equation (3.45). The constant κ is also known as the *adjustment coefficient*. This simple constant ϕ under exponential claims is a well-known result, see, for instance, Asmussen & Albrecher (2010, p. 156, Theorem 2.2, the constant π_+).

It follows that

$$\frac{1}{1 - \phi\, \widetilde{h}(s)} = \frac{s + \beta}{s + \kappa}$$

and

$$\widetilde{\psi}_c(s) = \frac{\phi\, \widetilde{\widetilde{H}}(s)}{1 - \phi\, \widetilde{h}(s)} = \frac{\phi}{s + \kappa}, \quad \widetilde{v}_r(s) = \frac{(-1)^{n-1}}{c^n} \frac{(s + \beta)\, d_r(s)}{(s + \kappa)\, \omega(s)}$$

for $r = 1, 2, \ldots, n-1$. This implies that the ruin probability is $\psi_c(u) = \phi\, \mathrm{e}^{-\kappa u}$ for $u \geqslant 0$. See Asmussen & Albrecher (2010, p. 156, the paragraph immediately before Remark 2.3) for the same identity.

3.5.2 Rational claims

More generally, suppose that the claim-size density $p(y)$ is of class R_m with Laplace transform

$$\mathcal{L}[p](s) = \frac{Q_{m-1}(s)}{Q_m(s)} = \frac{Q_{m-1}(s)}{(s + \beta_1)(s + \beta_2) \cdots (s + \beta_m)},$$

where $Q_{m-1}(s)$ is a polynomial of degree $m-1$ or less, $Q_{m-1}(0) = Q_m(0) = \beta_1 \beta_2 \cdots \beta_m$, and

$$\mathbb{E}[Y] = \frac{Q_m'(0) - Q_{m-1}'(0)}{\beta_1 \beta_2 \cdots \beta_m}.$$

Furthermore, assume that the Lundberg's equation

$$Q_m(s)\mathcal{A}(s) - Q_{m-1}(s)\mathcal{B}(s) = 0$$

has $n + m$ distinct roots ρ_i, $i = 1, 2, \ldots, n$ and $-\kappa_j$, $j = 1, 2, \ldots, m$ with $\mathrm{Re}(\rho_i) \geqslant 0$ and $\mathrm{Re}(\kappa_j) > 0$. Then

$$Q_m(s)\mathcal{A}(s) - Q_{m-1}(s)\mathcal{B}(s) = (-c)^n\, \omega(s)\, \varsigma(s),$$

where $\varsigma(s) = \prod_{j=1}^{m}(s+\kappa_j)$.

Thus, we derive

$$\begin{aligned}\kappa_1\,\kappa_2\cdots\kappa_m &= \lim_{s\to 0}\frac{Q_m(s)\mathcal{A}(s) - Q_{m-1}(s)\mathcal{B}(s)}{(-c)^n\,\omega(s)}\\ &= \frac{Q_m'(0)\mathcal{A}(0) + Q_m(0)\mathcal{A}'(0) - Q_{m-1}'(0)\mathcal{B}(0) - Q_{m-1}(0)\mathcal{B}'(0)}{(-c)^n\,\omega'(0)}\\ &= \frac{a_0\,\beta_1\,\beta_2\cdots\beta_m}{\rho_2\,\rho_3\cdots\rho_n}\,\frac{c\,\mathbb{E}[V]-\mathbb{E}[Y]}{c^n}.\end{aligned}$$

Here, we have used the convention that $\rho_1 = 0$ is one of the simple roots. Therefore, we obtain the identity

$$\frac{\kappa_1\,\kappa_2\cdots\kappa_m}{\beta_1\,\beta_2\cdots\beta_m} = \frac{a_0\left(c\,\mathbb{E}[V]-\mathbb{E}[Y]\right)}{c^n\,\rho_2\,\rho_3\cdots\rho_n}. \tag{3.46}$$

In conjunction with

$$\begin{aligned}\left.\frac{(s-\rho_i)\mathcal{B}(s)}{\omega(s)}\right|_{s=\rho_i} &= \left.\left(\frac{Q_m(s)\mathcal{A}(s)}{Q_{m-1}(s)\prod_{j\neq i}(s-\rho_j)} - \frac{(-c)^n\,(s-\rho_i)\,\varsigma(s)}{Q_{m-1}(s)}\right)\right|_{s=\rho_i}\\ &= \frac{\mathcal{A}(\rho_i)/\widetilde{p}(\rho_i)}{\prod_{j\neq i}(\rho_i-\rho_j)},\end{aligned}$$

we derive

$$\phi = 1 - \frac{\kappa_1\,\kappa_2\cdots\kappa_m}{\beta_1\,\beta_2\cdots\beta_m},\quad \widetilde{h}(s) = \frac{Q_m(s)-\varsigma(s)}{\phi\,Q_m(s)},\quad \frac{1}{1-\phi\,\widetilde{h}(s)} = \frac{Q_m(s)}{\varsigma(s)}.$$

3.5.3 A numerical illustration

Example 3.5.1 (A numerical example). Let the inter-claim time density $k(t)$ be given by (3.6) with Laplace transform (3.5). Then

$$k(0) = \frac{5}{3},\quad k'(0) = -5,\quad k''(0) = \frac{25}{3}.$$

One may directly verify that system (3.8) and identity (3.9) hold. Here, we observe $\mathbb{E}[V] = -\mathcal{L}[k]'(0) = 7/5 = 1.4$.

Assume that the claim-size density $p(y)$ and its Laplace transform are

$$p(y) = \left(\frac{3}{2}\,\mathrm{e}^{-y} - \mathrm{e}^{-2y}\right)\mathbb{1}(y \geqslant 0)\quad\text{and}\quad \mathcal{L}[p](s) = \frac{2+\frac{1}{2}\,s}{2+3\,s+s^2},$$

respectively. Note that $\mathbb{E}[Y] = -\mathcal{L}[p]'(0) = 5/4 = 1.25$.

We set $\theta = 12\%$ and $c = (1+\theta)\,\mathbb{E}[Y]/\mathbb{E}[V] = 1$. The Lundberg's equation (3.22) simplifies to

$$6\,s^5 + 5\,s^3 + 80\,s^2 + 9\,s = 0,$$

whose roots are

$$\rho_1 = 0, \quad \rho_2 \approx 1.162285 - 2.153018\,\mathrm{i}, \quad \rho_3 \approx 1.162285 + 2.153018\,\mathrm{i},$$
$$-\kappa_1 \approx -0.113315, \qquad -\kappa_2 \approx -2.211256.$$

Therefore,

$$\omega(s) = \prod_{i=1}^{3}(s-\rho_i) \approx s^3 - 2.324571\,s^2 + 5.986394\,s.$$

Furthermore,

$$T_s\,p(x) = \frac{3}{2}\frac{\mathrm{e}^{-x}}{s+1} - \frac{\mathrm{e}^{-2x}}{s+2}, \quad T_s\,\bar{P}(x) = \frac{3}{2}\frac{\mathrm{e}^{-x}}{s+1} - \frac{1}{2}\frac{\mathrm{e}^{-2x}}{s+2}.$$

So

$$\phi = \frac{(-1)^{3-1}}{c^3}\sum_{i=1}^{3}\frac{\mathcal{B}(\rho_i)}{\omega'(\rho_i)}\,T_{\rho_i}\,\bar{P}(0) = 1 - \frac{\kappa_1\,\kappa_2}{2} \approx 0.874716,$$

$$\begin{aligned} h(x) &= \frac{(-1)^{3-1}}{\phi\,c^3}\sum_{i=1}^{3}\frac{\mathcal{B}(\rho_i)}{\omega'(\rho_i)}\,T_{\rho_i}\,p(x) \\ &= \frac{2\,(1-\kappa_1)\,(1-\kappa_2)}{\kappa_1\,\kappa_2 - 2}\,\mathrm{e}^{-x} - \frac{2\,(2-\kappa_1)\,(2-\kappa_2)}{\kappa_1\,\kappa_2 - 2}\,\mathrm{e}^{-2x} \\ &\approx 1.227830\,\mathrm{e}^{-x} - 0.455660\,\mathrm{e}^{-2x}, \end{aligned}$$

and

$$\tilde{h}(s) = \frac{2 + (3-\kappa_1-\kappa_2)\,\phi^{-1}\,s}{(s+1)\,(s+2)}, \qquad \frac{1}{1-\phi\,\tilde{h}(s)} = \frac{(s+1)\,(s+2)}{(s+\kappa_1)\,(s+\kappa_2)}.$$

Combining with the fact that $\tilde{\tilde{H}}(s) = \left[1-\tilde{h}(s)\right]/s$, these calculations yield

$$\tilde{\psi}_c(s) = \frac{\phi\,\tilde{\tilde{H}}(s)}{1-\phi\,\tilde{h}(s)} = \frac{\kappa_1 + \kappa_2 - \frac{3}{2}\,\kappa_1\,\kappa_2 + \phi\,s}{(s+\kappa_1)\,(s+\kappa_2)}.$$

Direct Laplace inversion implies that

$$\begin{aligned} \psi_c(u) &= \frac{(2-3\,\kappa_1+\kappa_1^2)\,\kappa_2}{2\,(\kappa_2-\kappa_1)}\,\mathrm{e}^{-\kappa_1 u} - \frac{(2-3\,\kappa_2+\kappa_2^2)\,\kappa_1}{2\,(\kappa_2-\kappa_1)}\,\mathrm{e}^{-\kappa_2 u} \\ &\approx 0.881626\,\mathrm{e}^{-\kappa_1 u} - 0.006910\,\mathrm{e}^{-\kappa_2 u}, \end{aligned}$$

and

$$\varphi_c(u) = 1 - \psi_c(u) \approx 1 - 0.881626\,\mathrm{e}^{-\kappa_1 u} + 0.006910\,\mathrm{e}^{-\kappa_2 u}.$$

For the solutions $v_1(u)$ and $v_2(u)$, we compute

$$d_1(s) = \frac{47}{6} - 3\,s + s^2 \quad \text{and} \quad d_2(s) = -3 + s,$$

respectively. Substituting $d_1(s)$ and $d_2(s)$ into (3.37), we obtain

$$\begin{aligned}
\widetilde{v}_1(s) &= \frac{1}{1-\phi\,\widetilde{h}(s)}\,\frac{(-1)^{3-1}}{c^3}\,\frac{d_1(s)}{\omega(s)} = \frac{94 + 105\,s + 5\,s^2 + 6\,s^4}{9\,s + 80\,s^2 + 5\,s^3 + 6\,s^5},\\
\widetilde{v}_2(s) &= \frac{1}{1-\phi\,\widetilde{h}(s)}\,\frac{(-1)^{3-1}}{c^3}\,\frac{d_2(s)}{\omega(s)} = \frac{-36 - 42\,s + 6\,s^3}{9\,s + 80\,s^2 + 5\,s^3 + 6\,s^5}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
v_1(u) \approx{}& \tfrac{94}{9} - (0.156389 + 0.123400\,\mathrm{i})\,\mathrm{e}^{\rho_2 u} - (0.156389 - 0.123400\,\mathrm{i})\,\mathrm{e}^{\rho_3 u}\\
&- 9.198330\,\mathrm{e}^{-\kappa_1 u} + 0.066663\,\mathrm{e}^{-\kappa_2 u},\\
v_2(u) \approx{}& -4 + (0.259836 + 0.174805\,\mathrm{i})\,\mathrm{e}^{\rho_2 u} + (0.259836 - 0.174805\,\mathrm{i})\,\mathrm{e}^{\rho_3 u}\\
&+ 3.498276\,\mathrm{e}^{-\kappa_1 u} - 0.017947\,\mathrm{e}^{-\kappa_2 u}.
\end{aligned}$$

Eq.'s (3.42) and (3.43) yield

$$\begin{cases}
\varphi_c(b)\,c_0 + v_1(b)\,c_1 + v_2(b)\,c_2 = 1,\\
a_{1,0}\,c_0 + a_{1,1}\,c_1 + a_{1,2}\,c_2 = c^{-1}k(0),\\
a_{2,0}\,c_0 + a_{2,1}\,c_1 + a_{2,2}\,c_2 = -c^{-2}k'(0),
\end{cases}$$

where

$$\begin{aligned}
a_{1,0} &= \varphi_c'(b) + c^{-1}k(0)\,\zeta_c(b),\\
a_{1,r} &= v_r'(b) + c^{-1}k(0)\,w_r(b), \quad r = 1,2,\\
a_{2,0} &= \varphi_c''(b) + c^{-1}k(0)\,\zeta_c'(b) - c^{-2}k'(0)\,\zeta_c(b),\\
a_{2,r} &= v_r''(b) + c^{-1}k(0)\,w_r'(b) - c^{-2}k'(0)\,w_r(b), \quad r = 1,2,
\end{aligned}$$

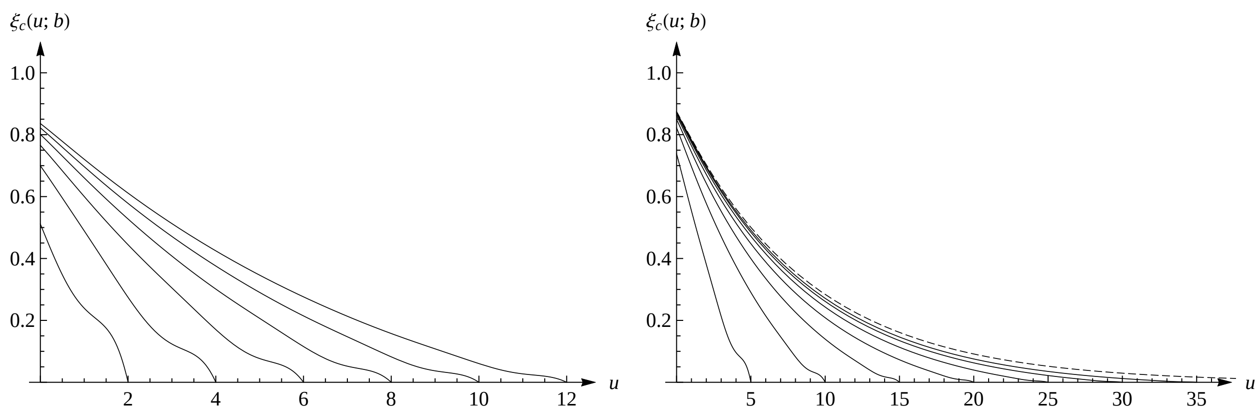
and

$$\begin{aligned}
\zeta_c(b) \approx{}& 1 - 1.024154\,\mathrm{e}^{-\kappa_1 b} + 0.024154\,\mathrm{e}^{-\kappa_2 b},\\
w_1(b) \approx{}& \tfrac{94}{9} + (0.003962 - 0.047560\,\mathrm{i})\,\mathrm{e}^{\rho_2 b} + (0.003962 + 0.047560\,\mathrm{i})\,\mathrm{e}^{\rho_3 b}\\
&- 10.685369\,\mathrm{e}^{-\kappa_1 b} + 0.233001\,\mathrm{e}^{-\kappa_2 b},\\
w_2(b) \approx{}& -4 - (0.000546 - 0.075023\,\mathrm{i})\,\mathrm{e}^{\rho_2 b} - (0.000546 + 0.075023\,\mathrm{i})\,\mathrm{e}^{\rho_3 b}\\
&+ 4.063821\,\mathrm{e}^{-\kappa_1 b} - 0.062729\,\mathrm{e}^{-\kappa_2 b}.
\end{aligned}$$

Therefore, the constants c_0, c_1 and c_2 are uniquely determined by the barrier b. We present the graphs of the function

$$\xi_c(u;b) = 1 - \chi_c(u;b) = 1 - c_0\,\varphi_c(u) - c_1\,v_1(u) - c_2\,v_2(u), \quad 0 \leqslant u \leqslant b$$

for various values of b in Figure 3.1.



(a) $\xi_c(u;b)$ for $b = 2, 4, \ldots, 12$.

(b) $\xi_c(u;b)$ for $b = 5, 10, \ldots, 35$ (as solid curves) and $\psi_c(u) = \xi_c(u;\infty)$ (as dashed curve).

Figure 3.1: Graphs of $\xi_c(u;b)$ for various levels b.

We remark here that there is no need for legends in Figures 3.1a and 3.1b, as the different barriers b can be inferred from the x-intercepts of the different solid curves. Figure 3.1a highlights the structure of $\xi_c(u;b)$ near $u = b$ for small values of b. In contrast to the behavior described by Eq. (2.7) of Li & Dickson (2006), each of the functions $\xi_c(u;b)$ here does *not* have vanishing derivatives, and therefore appears to vanish at $u = b$ in a non-smooth manner. Figure 3.1b emphasizes the asymptotic limiting behavior of $\xi_c(u;b)$ as $b \to \infty$, where the limit $\xi_c(u;\infty) = \psi_c(u)$ is shown as the dashed curve.

3.6 An application to total dividends under a threshold strategy

So far, we have studied the functions $\xi_c(u;b)$ and $\chi_c(u;b)$ extensively. Since these quantities are closely related to the distribution of the maximum surplus, they have applications to the evaluations of dividend under a *threshold strategy*. We shall explore one of the applications in this section. Note, in particular, the multiple appearances of the function $\chi_c(u;b)$ in Eq.'s (3.49)–(3.51).

Let $S(t) = \sum_{i=1}^{N(t)} Y_i$ be the aggregate loss. Under a threshold strategy, the surplus process is given by the following stochastic differential equation

$$\mathrm{d}U(t) = \begin{cases} c\,\mathrm{d}t - \mathrm{d}S(t), & U(t) < b, \\ (c-\alpha)\,\mathrm{d}t - \mathrm{d}S(t), & U(t) \geqslant b, \end{cases}$$

where the insurance company pays dividends at a rate $\alpha \in (0, c]$ to its shareholders and the initial condition is given by $U(0) = u \geqslant 0$. Let $\tau = \inf\{\, t \geqslant 0 : U(t) < 0 \,\}$ be the time of ruin.

We will need the notion of a *delayed surplus process* where the first inter-claim time random variable V_1 is allowed to follow a different distribution from the rest $V_2, V_3, \ldots$, but still keeping the independence property. Here, the word "delay" refers to a change in the distribution of V_1. We will choose a particular distribution for V_1 in the following delayed surplus process discussion.

Define the (conditional) random variable

$$V_u^b = \left(T_{N(\tau_c^b)+1} - \tau_c^b \,\middle|\, \tau_c^b < \tau \right), \quad u < b. \tag{3.47}$$

This is the time lapsed from the first up-crossing to the next claim arrival. See Figure 3.2 for an illustration. Let $K_u^b(t)$, $k_u^b(t)$ and $\mathcal{L}\left[k_u^b\right](s)$ be the c.d.f., p.d.f. and Laplace transform of V_u^b, respectively.

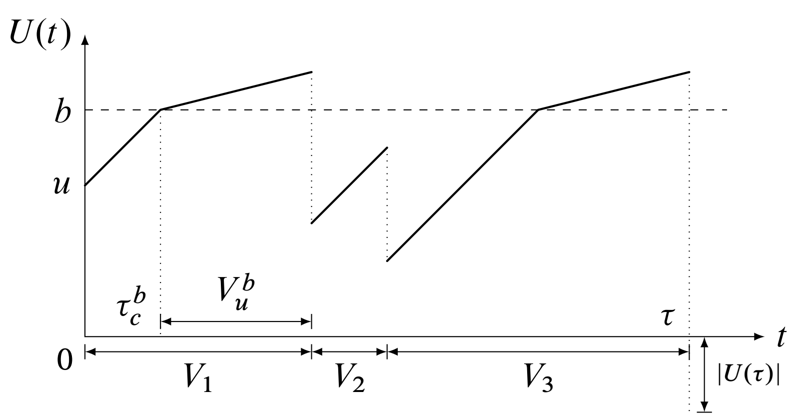


Figure 3.2: Constructing the random variable V_u^b from a regular surplus process.

Given $U(0) = u$, the (defective) joint p.d.f. of the deficit at ruin $|U(\tau)|$ and the time of ruin τ is denoted by $w_c(u, y, t)$, where $y > 0$ represents the amount of the deficit and $t > 0$ represents the value of the ruin time.

Finally, let $w_c^{u,b}(0, y, t)$ denote the (defective) joint p.d.f. of deficit at ruin and time to ruin of the delayed surplus process with *zero* initial capital; that is, a delayed surplus process with $V_1 = V_u^b$ in distribution and with $U(0) = 0$. As $u \to b-$, the limits $K_u^b \to K$, $k_u^b \to k$, $\mathcal{L}[k_u^b] \to \mathcal{L}[k]$ and $w_c^{u,b}(0, y, t) \to w_c(0, y, t)$ are obtained.

Let

$$D^b(t) = \int_0^{t\wedge\tau} \alpha\, \mathbb{1}_{\{U(s)\geqslant b\}}\, \mathrm{d}s, \quad t \geqslant 0, \alpha \in (0, c]$$

be the amount of accumulated (nominal) dividends by time t. We consider the nth moment

$$V^n(u; b) = \mathbb{E}_u[(\mathbb{D}^b)^n], \tag{3.48}$$

where

$$\mathbb{D}^b = D^b(\tau)\, \mathbb{1}_{\{\tau<\infty\}} = \begin{cases} D^b(\tau), & \tau < \infty, \\ 0, & \tau = \infty. \end{cases}$$

Due to potential delay, it is necessary to introduce the associated *delayed process* $\{U'(t) : t \geqslant 0\}$ with time of ruin τ' and

$$\mathbb{D}_u^b := D_u^b(\tau')\, \mathbb{1}_{\{\tau'<\infty\}}, \quad D_u^b(t) := \int_0^{t\wedge\tau'} \alpha\, \mathbb{1}_{\{U'(s)\geqslant b\}}\, \mathrm{d}s.$$
$$V_u^n(b; b) := \mathbb{E}[(\mathbb{D}_u^b)^n] = \mathbb{E}\big[\, (\mathbb{D}_u^b)^n \;\big|\; U'(0) = b \,\big].$$

In this notation: $V_u^n(b; b)$, the subscript u indicates the delay and its origin, the first argument in the parentheses indicates the initial capital of b, and the second indicates the barrier b.

We now present an application to the moments of total dividends under this threshold strategy. The proof is similar to that in Cheung et al. (2008, Section 2).

Theorem 3.6.1. *Suppose that $b \geqslant 0$ and that $n \geqslant 1$. For $u \geqslant 0$ and $V^n(u; b)$, we have*

$$V^n(u; b) = \chi_c(u; b)\, V_u^n(b; b), \quad 0 \leqslant u < b, \tag{3.49}$$

and

$$\begin{aligned} V^n(u; b) = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \int_0^\infty \int_0^b t^{n-j}\, \chi_c(b - y; b)\, V_{b-y}^j(b; b)\, w_{c-\alpha}(u - b, y, t)\, \mathrm{d}y\, \mathrm{d}t \\ + \alpha^n \int_0^\infty \int_b^\infty t^n\, w_{c-\alpha}(u - b, y, t)\, \mathrm{d}y\, \mathrm{d}t, \quad u \geqslant b. \end{aligned} \tag{3.50}$$

For $0 \leqslant u \leqslant b$ *and* $V_u^n(b;b)$, *we have*

$$V_u^n(b;b) = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \int_0^\infty \int_0^b t^{n-j}\, \chi_c(b-y;b)\, V_{b-y}^j(b;b)\, w_{c-\alpha}^{u,b}(0,y,t)\, \mathrm{d}y\, \mathrm{d}t$$
$$+ \alpha^n \int_0^\infty \int_b^\infty t^n\, w_{c-\alpha}^{u,b}(0,y,t)\, \mathrm{d}y\, \mathrm{d}t, \quad 0 \leqslant u \leqslant b. \tag{3.51}$$

Here, (1) $w_{c-\alpha}(u-b,\cdot,\cdot)$ *is the defective joint p.d.f. of* $|U_{c-\alpha}(\tau_{c-\alpha})|$ *and* $\tau_{c-\alpha}$ *from initial capital* $u-b$; (2) $w_{c-\alpha}^{u,b}(0,\cdot,\cdot)$ *is the defective joint p.d.f. of* $|U'_{c-\alpha}(\tau'_{c-\alpha})|$ *and* $\tau'_{c-\alpha}$ *from initial capital* 0, *with first inter-claim time distribution* K_u^b; *and* (3) $V^0(u;b) = V_u^0(b;b) = 1$.

Proof. When $u < b$, dividends are paid only if the first up-crossing through level b occurs before ruin. This event occurs with probability $\chi_c(u;b)$. After the up-crossing, the next claim arrival time will follow the distribution $K_u^b(t)$, while the subsequent inter-claim times follow the original distribution $K(t)$. The nth moment of the finite-time accumulated nominal dividends of this delayed risk process is then precisely given by $V_u^n(b;b)$. This proves Eq. (3.49).

For $u \geqslant b$, we shall focus on the instant when the surplus process first drops below level b. This has the same stochastic property as a regular surplus process dropping below 0 with premium rate of $c-\alpha$ and initial capital of $u-b$. By conditioning on the amount of the "deficit at ruin" y and the "time of ruin" t, two cases are possible: (1) The "deficit at ruin" $y \leqslant b$, so that the company is not ruined and the surplus process is renewed with an initial capital of $b-y$. The finite-time accumulated nominal dividends are thus given by $\alpha t + \mathbb{D}^b$. The expected nth moment is therefore

$$\mathbb{E}_{b-y}\big[(\alpha t + \mathbb{D}^b)^n\big] = \sum_{j=0}^{n} \binom{n}{j} (\alpha t)^{n-j} V^j(b-y;b)$$
$$= \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \times t^{n-j} \chi_c(b-y;b)\, V_{b-y}^j(b;b). \tag{3.52}$$

(2) The "deficit at ruin" $y > b$, so that the company is ruined. The finite-time accumulated nominal dividends is simply αt with nth moment $\alpha^n t^n$. This proves Eq. (3.50).

Repeat the same argument above to a delayed surplus process. Note that in this case, the instant when the surplus process first drops below level b has the same stochastic property as a delayed surplus process dropping below 0 with a premium rate of $c-\alpha$ and a *zero* initial capital. This is described by the joint p.d.f. $w_{c-\alpha}^{u,b}(0,y,t)$. The total dividends are given by $\mathbb{D}_u^b$. After this drop, the surplus process will be renewed, so the future dividends are given by $\mathbb{D}^b$. Thus, by replacing the appropriate terms in Eq. (3.52), we derive Eq. (3.51). □

3.A Proofs of some results

Proof of Lemma 3.3.3. Let $b = \infty$. Writing Eq. (3.14) as

$$\left[\mathcal{A}\left(\frac{\partial}{\partial u}\right)\xi_c\right](u;\infty) = \left[\mathcal{B}\left(\frac{\partial}{\partial u}\right)\gamma_c\right](u;\infty), \quad u \geqslant 0,$$

we apply Laplace transforms on both sides and directly evaluate as in (3.12) and (3.13) to obtain

$$\mathcal{A}(s)\,\mathcal{L}[\xi_c](s;\infty) + \mathcal{R}_{\mathcal{A}}(s) = \mathcal{B}(s)\,\mathcal{L}[\gamma_c](s;\infty) + \mathcal{R}_{\mathcal{B}}(s). \tag{3.53}$$

The function $\mathcal{R}_{\mathcal{A}}(s)$ is a polynomial of degree $n-1$ or less, whose coefficients depend on those of $\mathcal{A}(s)$ and the partial derivatives of $\xi_c(u;\infty)$ w.r.t. u at $u = 0$. Similarly, the function $\mathcal{R}_{\mathcal{B}}(s)$ is a polynomial of degree $n-2$ or less, whose coefficients depend on those of $\mathcal{B}(s)$ and the partial derivatives of $\gamma_c(u;\infty)$ w.r.t. u at $u = 0$.

By definition (3.15) of $\gamma_c(u;b)$ and the fact that $\widetilde{\overline{P}}(s) = \big(1-\widetilde{p}(s)\big)/s$, we derive

$$\mathcal{L}[\gamma_c](s;\infty) = \widetilde{p}(s)\,\mathcal{L}[\xi_c](s;\infty) + \frac{1-\widetilde{p}(s)}{s}. \tag{3.54}$$

Substituting expression (3.54) into Eq. (3.53) and rearranging the terms, we obtain

$$\big(\mathcal{A}(s) - \mathcal{B}(s)\,\widetilde{p}(s)\big)\,\mathcal{L}[\xi_c](s;\infty) = \frac{1-\widetilde{p}(s)}{s}\mathcal{B}(s) - \big(\mathcal{R}_{\mathcal{A}}(s) - \mathcal{R}_{\mathcal{B}}(s)\big).$$

Letting $\mathcal{P}_{n-1}(s) = \mathcal{R}_{\mathcal{A}}(s) - \mathcal{R}_{\mathcal{B}}(s)$, we arrive at (3.21) and the proof is complete. □

Proof of Lemma 3.3.4. We introduce an auxiliary parameter $\delta \geqslant 0$ and view (3.22) as a special case of the following equation

$$\widetilde{k}(\delta - cs)\,\widetilde{p}(s) = 1 \iff f(\delta - cs) - g(\delta - cs)\,\widetilde{p}(s) = 0. \tag{3.55}$$

Define

$$F(s) = f(\delta - cs) \quad \text{and} \quad G(s) = g(\delta - cs)\,\widetilde{p}(s).$$

Suppose that $\delta > 0$ for the moment. Let $R > 0$ be a sufficiently large constant and $\varGamma_R$ be the contour of the half-disk $\{\, s \in \mathbb{C} : \operatorname{Re} s \geqslant 0 \text{ and } |s| \leqslant R \,\}$. For $s = \mathrm{i}y$ on the imaginary axis, we have the following estimate

$$\left|\frac{1}{\widetilde{k}(\delta - \mathrm{i}cy)}\right| \geqslant \frac{1}{\widetilde{k}(\delta)} > 1 \geqslant |\widetilde{p}(\mathrm{i}y)|.$$

So $|G(\mathrm{i}y)| < |F(\mathrm{i}y)|$. Since $G(s)$ is bounded in modulus by a polynomial of degree $n-1$ or less on the right half-plane while $F(s)$ is a polynomial of degree n, for s on the right half-circle $\operatorname{Re} s \geqslant 0$ and $|s| = R$, we also have $|G(s)| < |F(s)|$. Therefore,

$$|G(s)| < |F(s)|, \quad s \in \Gamma_R.$$

By Rouché's theorem, $F(s) - G(s)$ and $F(s)$ have the same number of zeros in the interior of Γ_R. Since $F(s)$ has n zeros in the open right half-plane, Eq. (3.55) also has n roots with strictly positive real parts.

Finally, for $\delta = 0$, we apply Hurwitz's theorem to Eq. (3.55). Taking limits as $\delta \to 0+$, the roots of (3.55) converge to those of (3.22), and the limit roots all have nonnegative real parts. □

Proof of Lemma 3.3.5. Define

$$\Lambda_i(s) = \mathcal{A}[t_1, t_2, \ldots, t_i, s] - \mathcal{B}[t_1, t_2, \ldots, t_i, s]\, T_s p(0) + \sum_{j=1}^{i} (-1)^{i-j} \mathcal{B}[t_1, t_2, \ldots, t_j]\, T_s T_{t_j} T_{t_{j+1}} \cdots T_{t_i} p(0), \quad i = 1, 2, \ldots, n. \tag{3.56}$$

We show by induction that

$$L(s) = \mathcal{A}(s) - \mathcal{B}(s)\, \tilde{p}(s) = \Lambda_i(s) \prod_{j=1}^{i} (s - t_j), \quad i = 1, 2, \ldots, n. \tag{3.57}$$

Note that the equation $L(s) = 0$ is equivalent to the generalized Lundberg's equation (3.22), so $L(t_i) = 0$ for $i = 1, 2, \ldots, n$. For $i = 1$, the function $\Lambda_1(s)$ simplifies to

$$\begin{aligned}\Lambda_1(s) &= \mathcal{A}[t_1, s] - \mathcal{B}[t_1, s]\, T_s p(0) + \mathcal{B}[t_1]\, T_s T_{t_1} p(0) \\ &= \frac{\mathcal{A}(s) - \mathcal{A}(t_1)}{s - t_1} - \frac{\mathcal{B}(s) - \mathcal{B}(t_1)}{s - t_1} T_s p(0) + \mathcal{B}(t_1) \frac{T_{t_1} p(0) - T_s p(0)}{s - t_1}.\end{aligned}$$

Thus,

$$\begin{aligned}&(s - t_1)\, \Lambda_1(s) \\ &\quad = \mathcal{A}(s) - \mathcal{A}(t_1) - \mathcal{B}(s)\, T_s p(0) + \mathcal{B}(t_1)\, T_s p(0) + \mathcal{B}(t_1)\, T_{t_1} p(0) - \mathcal{B}(t_1)\, T_s p(0) \\ &\quad = \mathcal{A}(s) - \mathcal{B}(s)\, \tilde{p}(s) - \mathcal{A}(t_1) + \mathcal{B}(t_1)\, \tilde{p}(t_1) \\ &\quad = L(s) - L(t_1) \\ &\quad = L(s).\end{aligned}$$

So (3.57) holds for $i = 1$.

Suppose now that (3.57) holds for some i with $1 \leqslant i \leqslant n-1$. We shall prove that (3.57) also holds for $i+1$. It suffices to show that

$$(s - t_{i+1})\, \Lambda_{i+1}(s) = \Lambda_i(s).$$

Note that

$$\begin{aligned}
\Lambda_{i+1}(s) &= \mathcal{A}[t_1, t_2, \ldots, t_i, t_{i+1}, s] - \mathcal{B}[t_1, t_2, \ldots, t_i, t_{i+1}, s]\, T_s\, p(0) \\
&\quad + \sum_{j=1}^{i+1} (-1)^{i+1-j}\, \mathcal{B}[t_1, t_2, \ldots, t_j]\, T_s T_{t_j} T_{t_{j+1}} \cdots T_{t_i} T_{t_{i+1}}\, p(0) \\
&= \frac{\mathcal{A}[t_1, t_2, \ldots, t_i, s] - \mathcal{A}[t_1, t_2, \ldots, t_i, t_{i+1}]}{s - t_{i+1}} \\
&\quad - \frac{\mathcal{B}[t_1, t_2, \ldots, t_i, s] - \mathcal{B}[t_1, t_2, \ldots, t_i, t_{i+1}]}{s - t_{i+1}}\, T_s\, p(0) \\
&\quad + \sum_{j=1}^{i+1} (-1)^{i+1-j}\, \mathcal{B}[t_1, t_2, \ldots, t_j] \\
&\qquad \times \frac{T_{t_{i+1}} T_{t_j} T_{t_{j+1}} \cdots T_{t_i}\, p(0) - T_s T_{t_j} T_{t_{j+1}} \cdots T_{t_i}\, p(0)}{s - t_{i+1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(s - t_{i+1})\, \Lambda_{i+1}(s) &= \mathcal{A}[t_1, t_2, \ldots, t_i, s] - \mathcal{A}[t_1, t_2, \ldots, t_i, t_{i+1}] \\
&\quad - \mathcal{B}[t_1, t_2, \ldots, t_i, s]\, T_s\, p(0) + \mathcal{B}[t_1, t_2, \ldots, t_i, t_{i+1}]\, T_s\, p(0) \\
&\quad + \mathcal{B}[t_1, t_2, \ldots, t_i, t_{i+1}]\, T_{t_{i+1}}\, p(0) - \mathcal{B}[t_1, t_2, \ldots, t_i, t_{i+1}]\, T_s\, p(0) \\
&\quad + \sum_{j=1}^{i} (-1)^{i-j}\, \mathcal{B}[t_1, t_2, \ldots, t_j]\, T_s T_{t_j} T_{t_{j+1}} \cdots T_{t_i}\, p(0) \\
&\quad - \sum_{j=1}^{i} (-1)^{i-j}\, \mathcal{B}[t_1, t_2, \ldots, t_j]\, T_{t_{i+1}} T_{t_j} T_{t_{j+1}} \cdots T_{t_i}\, p(0) \\
&= \Lambda_i(s) - \Lambda_i(t_{i+1}).
\end{aligned}$$

Since t_{i+1} is still a zero of $\Lambda_i(s)$, we see that $(s - t_{i+1})\, \Lambda_{i+1}(s) = \Lambda_i(s)$ as required. Thus by taking $i = n$ we obtain

$$L(s) = \omega(s) \Big\{ \mathcal{A}[t_1, t_2, \ldots, t_n, s] - \mathcal{B}[t_1, t_2, \ldots, t_n, s]\, T_s p(0) + \sum_{j=1}^{n} (-1)^{n-j} \mathcal{B}[t_1, t_2, \ldots, t_j]\, T_s T_{t_j} T_{t_{j+1}} \cdots T_{t_n} p(0) \Big\}. \tag{3.58}$$

Similarly, for

$$R(s) = \frac{1 - \tilde{p}(s)}{s} \mathcal{B}(s) - \mathcal{P}_{n-1}(s) = -\mathcal{P}_{n-1}(s) + \mathcal{B}(s)\, \tilde{\tilde{\overline{P}}}(s),$$

note that $R(t_i) = 0$ for $i = 1, 2, \ldots, n$. We may simply replace $\mathcal{A}$ and p in (3.56) with $\mathcal{P}_{n-1}$ and $\overline{P}$, respectively, to derive

$$-R(s) = \omega(s) \Big\{ \mathcal{P}_{n-1}[t_1, t_2, \ldots, t_n, s] - \mathcal{B}[t_1, t_2, \ldots, t_n, s]\, T_s \overline{P}(0) + \sum_{j=1}^{n} (-1)^{n-j} \mathcal{B}[t_1, t_2, \ldots, t_j]\, T_s T_{t_j} T_{t_{j+1}} \cdots T_{t_n} \overline{P}(0) \Big\}. \tag{3.59}$$

Finally, recalling the degrees of the polynomials $\mathcal{A}(s)$, $\mathcal{B}(s)$ and $\mathcal{P}_{n-1}(s)$, we have

$$\mathcal{A}[t_1, t_2, \ldots, t_n, s] \equiv (-c)^n, \quad \mathcal{B}[t_1, t_2, \ldots, t_n, s] = \mathcal{P}_{n-1}[t_1, t_2, \ldots, t_n, s] \equiv 0.$$

Hence, expressions (3.58) and (3.59) reduce to (3.24) and (3.25), respectively. □

3.B Divided differences and translation transforms

We introduce the *divided difference* of a function $\varphi(x)$ defined by

$$\begin{aligned} \varphi[t_1] &= \varphi(t_1), \\ \varphi[t_1, t_2] &= \frac{\varphi[t_2] - \varphi[t_1]}{t_2 - t_1}, \\ &\cdots\cdots\cdots\cdots \\ \varphi[t_1, \ldots, t_{i-2}, t_{i-1}, t_i] &= \frac{\varphi[t_1, \ldots, t_{i-2}, t_i] - \varphi[t_1, \ldots, t_{i-2}, t_{i-1}]}{t_i - t_{i-1}}. \end{aligned} \tag{3.60}$$

Here are some properties of the divided difference:

Proposition 3.B.1. *Suppose that $t_1, t_2, \dots, t_i$ are distinct. Then*

$$\varphi[t_1, t_2, \dots, t_i] = \sum_{j=1}^{i} \frac{\varphi(t_j)}{\prod_{\substack{k=1\\k\neq j}}^{i} (t_j - t_k)}. \tag{3.61}$$

If we define $w_i(s) = (s - t_1)(s - t_2)\cdots(s - t_i)$, then (3.61) may be written in a more compact form

$$\varphi[t_1, t_2, \dots, t_i] = \sum_{j=1}^{i} \frac{\varphi(t_j)}{w_i'(t_j)}. \tag{3.62}$$

By (3.61) and (3.62), we see that the divided difference is *symmetric* in its arguments; that is, if $\pi\colon \{t_1, t_2, \dots, t_i\} \to \{t_1, t_2, \dots, t_i\}$ is a permutation, then

$$\varphi[t_1, t_2, \dots, t_i] = \varphi[\pi(t_1), \pi(t_2), \dots, \pi(t_i)].$$

If $t_1 = t_2$ in definition (3.60), the expression $\varphi[t_1, t_1]$ is to be interpreted as the limit

$$\varphi[t_1, t_1] = \lim_{t_2 \to t_1} \frac{\varphi[t_2] - \varphi[t_1]}{t_2 - t_1} = \varphi'(t_1),$$

provided that $\varphi(x)$ is differentiable. In general, if there are multiple identical arguments t_j on the left-hand side of (3.61), say $t_1 = t_2 = \cdots = t_{j_1} = r_1, t_{j_1+1} = t_{j_1+2} = \cdots = t_{j_1+j_2} = r_2$, etc., the corresponding divided difference shall be interpreted as the multivariate limit of the right-hand side of (3.61) as $t_1, t_2, \dots, t_{j_1} \to r_1, t_{j_1+1}, t_{j_1+2}, \dots, t_{j_1+j_2} \to r_2$, etc., jointly and distinctly. This limit exists as long as $\varphi(x)$ is sufficiently smooth.

Proposition 3.B.2. *Suppose that $\Omega \subset \mathbb{C}$ is a simply connected domain and that $\varphi(z)$ is holomorphic in Ω. If*

$$t_1 = t_2 = \cdots = t_{k_1} = r_1, \quad t_{k_1+1} = t_{k_1+2} = \cdots = t_{k_1+k_2} = r_2,$$
$$\cdots\cdots\cdots\cdots$$
$$t_{k_1+k_2+\cdots+k_{\ell-1}+1} = t_{k_1+k_2+\cdots+k_{\ell-1}+2} = \cdots = t_{k_1+k_2+\cdots+k_{\ell-1}+k_\ell} = r_\ell,$$

where $k_1 + k_2 + \cdots + k_\ell = n$ and $r_1, r_2, \dots, r_\ell$ are distinct numbers in Ω, then

$$\varphi[t_1, t_2, \dots, t_n] = \sum_{i=1}^{\ell} \frac{1}{(k_i - 1)!} \frac{\mathrm{d}^{k_i - 1}}{\mathrm{d}z^{k_i - 1}} \frac{(z - r_i)^{k_i}\, \varphi(z)}{\omega(z)}\bigg|_{z = r_i}, \tag{3.63}$$

where $\omega(z) = (z - r_1)^{k_1}(z - r_2)^{k_2} \cdots (z - r_\ell)^{k_\ell}$.

We also introduce the *translation transform* of $\varphi(x)$ defined by

$$T_s\varphi(x) = \int_x^\infty \mathrm{e}^{-s(y-x)}\,\varphi(y)\,\mathrm{d}y, \tag{3.64}$$

provided that the integral is well-defined. Note that setting $x = 0$ in (3.64):

$$T_s\varphi(0) = \mathcal{L}[\varphi](s) = \widetilde{\varphi}(s)$$

yields the Laplace transform of φ, and

$$T_rT_s\varphi(x) = \frac{T_s\varphi(x) - T_r\varphi(x)}{r-s}, \quad T_rT_s\varphi(0) = \frac{\widetilde{\varphi}(s) - \widetilde{\varphi}(r)}{r-s}.$$

Again, the expression $T_s^2\varphi(x) = T_sT_s\varphi(x)$ is to be understood as

$$T_s^2\varphi(x) = \lim_{r\to s}\frac{T_s\varphi(x) - T_r\varphi(x)}{r-s} = -\frac{\partial}{\partial s}T_s\varphi(x).$$

Generalization to multiple translation transforms is possible.

In relation to divided difference, if we write $\eta_x(s) = T_s\varphi(x)$, then

$$T_{t_i}\cdots T_{t_2}T_{t_1}T_s\varphi(x) = (-1)^i\,\eta_x[s, t_1, t_2, \dots, t_i]. \tag{3.65}$$

By the symmetry property of divided differences, this shows that the translation transform is *commutative*. So if π is any permutation, then

$$T_{t_i}\cdots T_{t_2}T_{t_1}\varphi(x) = T_{\pi(t_i)}\cdots T_{\pi(t_2)}T_{\pi(t_1)}\varphi(x).$$

Chapter 4

An algebraic approach to a class of perturbed renewal risk models with dependence

4.1 Introduction

A Wiener diffusion was introduced as a perturbation to the classical compound Poisson risk model first by Gerber (1970). Many researchers have contributed for the Poisson arrival of the claims and the presence of diffusion. For instance, Dufresne & Gerber (1991) for the probability of ruin, Wang & Wu (2000) for the distributions of the maximum surplus before ruin and the deficit at ruin. With introduction of the *Gerber–Shiu expected discounted penalty function* (Gerber & Shiu 1998), there are Tsai & Willmot (2002) and more recently Liu & Zhang (2015).

Further extensions to the classical risk model perturbed by diffusion are investigated. For example, it is possible to consider a Sparre-Andersen risk process (Andersen 1957, also known as a renewal risk process) in which the inter-claim time distribution is not constrained to exponential. These studies include Li & Garrido (2005*a*, with generalized Erlang(n) times) and Song et al. (2010, with phase-type times). These models assume independence between the inter-claim times and the claim sizes. Although independence simplifies the computations for many quantities of interest, it may not be suitable for modeling catastrophic events such as earthquakes and hurricanes. More importantly, these analyses rely heavily on the *strong Markov property* of the risk process. This approach does not apply if the risk process no longer has stationary and independent increments, which is implied by our model (4.2).

Some first attempts to characterize a dependence structure between the inter-claim times and the claim sizes are found in Boudreault et al. (2006, with Poisson arrival and exponential-weighted mixture dependence), Cossette et al. (2010, with Poisson arrival and Farlie–Gumbel–Morgenstern (FGM) copula), Willmot & Woo (2012, with Erlang inter-claim time combination) and Chadjiconstantinidis & Vrontos (2014, with Erlang arrival and FGM copula). But these models lack diffusion perturbation.

Zhang & Yang (2011) treat a compound Poisson risk model with both extensions: A diffusion and an FGM copula dependence structure. Zhang et al. (2012, Eq. (4)) further extend the dependence structure to an *exponential inter-claim time combination*. We refer Zhang (2014, p. 249) for an exposition of some special dependence cases derived from this structure—they include exponential-weighted mixture, FGM copula with exponential marginals, and generalized FGM copula with Poisson arrival.

However, there is a notable limitation with the structure by Zhang et al. (2012). Namely, the marginal distribution of the inter-claim times is constrained to be generalized Erlang(n) with *strictly distinct* rate parameters. This prohibits its extension for the diffusion-free cases of Willmot & Woo (2012) and Chadjiconstantinidis & Vrontos (2014). Although conjectured by Zhang et al. (2012, Section 5), there are no published proofs dealing with rate parameters which have multiplicities. So the two mentioned diffusion-free cases have no diffusion-present counterparts in the literature.

In this chapter, we extend the exponential inter-claim time combination to an Erlang inter-claim time combination, and give an affirmative answer to the conjecture by Zhang et al. (2012). The rest of this chapter is organized as follows. In Section 4.2, we give detailed descriptions on the dependence structure and introduce our main tools. The main tools consist of the Bessel polynomials and a potential measure of the Wiener diffusion. In Section 4.3, we tackle the multiplicities of the rate parameters in a systematic and organized way and present our results as a series of propositions. These arguments are purely algebraic and they build up to the main result, Theorem 4.3.1, of this chapter. Finally, we demonstrate a variety of applications of Theorem 4.3.1 in Section 4.4 under special dependence structures. These include Theorem 4.4.1 for the independence case, Theorem 4.4.2 and Example 4.4.1 for the FGM copula case, and Theorem 4.4.3 for the exponential-weighted mixture case. Each result is compared with its diffusion-free counterpart, revealing the generality of our main result.

We do not perform the full Gerber–Shiu analysis but instead emphasize on deriving the desired integro-differential equations in this chapter. The analysis that follows involves locating the roots of a generalized Lundberg's equation, establishing a defective renewal equation by applying Laplace transforms on the derived integro-differential equation, and solving the defective renewal equation with Laplace inversion. As this is the standard practice, we refer the aforementioned papers and references therein for the corresponding detailed procedures.

4.2 Preliminaries

Consider the following renewal risk process perturbed by an independent diffusion

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i + \sigma W(t), \quad t \geqslant 0, \tag{4.1}$$

where $u \geqslant 0$ is the initial capital and c is the premium rate. We assume that the aggregate-claim process $S(t) = \sum_{i=1}^{N(t)} Y_i$ is a compound renewal process, where $\{N(t) : t \geqslant 0\}$ is a (renewal) counting process. The claim sizes $\{Y_i\}_{i=1}^{\infty}$ are independent and identically distributed (i.i.d.) random variables with common probability density function (p.d.f.) $f_Y(y)$, cumulative distribution function (c.d.f.) $F_Y(y)$ and survival function $\overline{F}_Y(y) = 1 - F_Y(y)$. The inter-claim times $\{V_i\}_{i=1}^{\infty}$ are i.i.d. random variables with common p.d.f. $f_V(t)$, c.d.f. $F_V(t)$ and survival function $\overline{F}_V(t) = 1 - F_V(t)$. Finally, $\{W(t) : t \geqslant 0\}$ is a standard Wiener process and $\sigma > 0$ is the constant volatility of the diffusion.

We shall make the dependence assumption on the joint p.d.f. between the two random variables V and Y as follows

$$f_{V,Y}(t, y) = \sum_{i \geqslant 1} \sum_{j=1}^{k_i} \frac{\lambda_i^j t^{j-1} \mathrm{e}^{-\lambda_i t}}{(j-1)!} f_{ij}(y), \tag{4.2}$$

where there are *finitely many* distinct constants $\lambda_1, \lambda_2, \ldots$, and their corresponding constants $k_1, k_2, \ldots$ are positive integers. We also assume that each of $f_{ij}(y)$ is a function that does not depend on t. This is the dependence structure introduced by Willmot & Woo (2012, Eq.'s (12) and (23)), but now with an additional diffusion perturbation.

The Gerber–Shiu expected discounted penalty function is defined by

$$m(u) = \mathbb{E}\big[\, \mathrm{e}^{-\delta\tau} w\big(U(\tau-), |U(\tau)|\big) \mathbb{1}(\tau < \infty) \,\big|\, U(0) = u \,\big], \quad u \geqslant 0, \tag{4.3}$$

where $\delta \geqslant 0$ is the force of interest and

$$\tau = \inf\{\, t \geqslant 0 : U(t) \leqslant 0 \,\}$$

is the time of ruin. The function $w(x_1, x_2)$ is a nonnegative penalty function of two variables, where x_1 is the surplus immediately before ruin and x_2 is the deficit at ruin. Finally, $\mathbb{1}(\,\cdot\,)$ is the indicator function.

Since ruin can be caused either by a downward jump from a claim or by the oscillation of the Wiener process, the function $m(u)$ can be decomposed as

$$m(u) = m_s(u) + m_d(u), \quad u \geqslant 0,$$

where the subscript s indicates aggregate-claim as the source of ruin and the subscript d indicates diffusion as the source of ruin. More specifically, we have

$$m_s(u) = \mathbb{E}\big[\, \mathrm{e}^{-\delta\tau} w\big(U(\tau-), |U(\tau)|\big) \mathbb{1}(\tau < \infty, U(\tau) < 0) \;\big|\; U(0) = u \,\big] \tag{4.4}$$

and

$$\begin{aligned} m_d(u) &= \mathbb{E}\big[\, \mathrm{e}^{-\delta\tau} w\big(U(\tau-), |U(\tau)|\big) \mathbb{1}(\tau < \infty, U(\tau) = 0) \;\big|\; U(0) = u \,\big] \\ &= w(0,0)\, \mathbb{E}\big[\, \mathrm{e}^{-\delta\tau} \mathbb{1}(\tau < \infty, U(\tau) = 0) \;\big|\; U(0) = u \,\big]. \end{aligned} \tag{4.5}$$

Note that by setting $\delta = 0$ and $w(x_1, x_2) \equiv 1$, we recover the probabilities of ruin

$$\begin{aligned} \psi(u) &= m(u)\big|_{\delta=0,w\equiv 1} &&= \mathbb{P}[\,\tau < \infty \mid U(0) = u\,], \\ \psi_s(u) &= m_s(u)\big|_{\delta=0,w\equiv 1} &&= \mathbb{P}[\,\tau < \infty,\ U(\tau) < 0 \mid U(0) = u\,], \\ \psi_d(u) &= m_d(u)\big|_{\delta=0,w\equiv 1} &&= \mathbb{P}[\,\tau < \infty,\ U(\tau) = 0 \mid U(0) = u\,]. \end{aligned}$$

It is convenient to denote

$$W_{-c}(t) = -ct - \sigma W(t), \quad t \geqslant 0, \tag{4.6}$$

so that $\{W_{-c}(t) : t \geqslant 0\}$ is a Wiener process with initial position 0, drift $-c$ and volatility σ. The surplus process may then be rewritten as

$$U(t) = u - W_{-c}(t) - S(t), \quad t \geqslant 0.$$

Let $\overline{W}_{-c}(t) = \sup_{0 \leqslant s \leqslant t} W_{-c}(s)$ be the running supremum of $\{W_{-c}(t) : t \geqslant 0\}$ and define the first hitting time of level $u \geqslant 0$ of the process $\{W_{-c}(t) : t \geqslant 0\}$ by

$$\tau_u = \inf\{\, t \geqslant 0 : W_{-c}(t) = u \,\}. \tag{4.7}$$

It is well-known (Borodin & Salminen 2002, p. 295, with appropriate re-parametrization) that

$$\mathbb{E}\big[\mathrm{e}^{-q\tau_u}\big] = \exp\left\{-\left(\sqrt{\frac{2q}{\sigma^2}+\frac{c^2}{\sigma^4}}+\frac{c}{\sigma^2}\right)u\right\},$$

where q is a constant. We also introduce the notation

$$\eta_{q,+} = \eta_q + \frac{c}{\sigma^2} \quad \text{and} \quad \eta_{q,-} = \eta_q - \frac{c}{\sigma^2}, \quad \text{with} \quad \eta_q = \sqrt{\frac{2q}{\sigma^2}+\frac{c^2}{\sigma^4}}. \tag{4.8}$$

So that

$$\mathbb{E}\big[\mathrm{e}^{-q\tau_u}\big] = \exp\{-u\eta_{q,+}\}. \tag{4.9}$$

We shall introduce the Bessel polynomials (Krall & Frink 1949, p. 101, Eq. (3)) defined by

$$\begin{aligned}\pi_r(x) &= \sum_{j=0}^{r} \frac{(r+j)!}{(r-j)!\,j!}\left(\frac{x}{2}\right)^j \qquad (4.10)\\ &= 1 + r(r+1)\frac{x}{2} + \cdots + \frac{(2r-1)!}{(r-1)!}\left(\frac{x}{2}\right)^{r-1} + \frac{(2r)!}{r!}\left(\frac{x}{2}\right)^r\end{aligned}$$

for $r = 0, 1, 2, \ldots$ For convenience, we list the first few Bessel polynomials here: $\pi_0(x) = 1$, $\pi_1(x) = 1 + x$, $\pi_2(x) = 1 + 3x + 3x^2$, $\pi_3(x) = 1 + 6x + 15x^2 + 15x^3$.

Remark 4.2.1. Although the standard notation for the Bessel polynomials is $y_r(x)$ for $r = 0, 1, 2, \ldots$, we use $\pi_r(x)$ here to avoid confusion and reserve y to represent quantities that are related to claim sizes.

Define the following potential measure

$$\mathcal{P}(u, \mathrm{d}x, \mathrm{d}y) = \mathbb{E}\Big[\mathrm{e}^{-\delta V}\mathbb{1}\big(\overline{W}_{-c}(V) < u, W_{-c}(V) \in \mathrm{d}x, Y \in \mathrm{d}y\big)\Big] \tag{4.11}$$

for $u \geqslant 0$, $x < u$ and $y > 0$ (see also Zhang & Yang 2011, p. 1192, Eq. (4.2)). Our goal is to derive an explicit expression for the density of $\mathcal{P}(u, \mathrm{d}x, \mathrm{d}y)$; that is, we seek an expression $p(u, x, y)$ such that

$$\mathcal{P}(u, \mathrm{d}x, \mathrm{d}y) = p(u, x, y)\,\mathrm{d}y\,\mathrm{d}x, \quad u \geqslant 0,\ x < u,\ y > 0.$$

By conditioning on V and using the fact that $\{W_{-c}(t) : t \geqslant 0\}$ and (V, Y) are independent, we may rewrite $\mathcal{P}(u, \mathrm{d}x, \mathrm{d}y)$ as

$$\begin{aligned}\mathcal{P}(u, \mathrm{d}x, \mathrm{d}y) &= \mathbb{E}\Big[\,\mathbb{E}\big[\mathrm{e}^{-\delta V}\mathbb{1}\big(\overline{W}_{-c}(V) < u, W_{-c}(V) \in \mathrm{d}x, Y \in \mathrm{d}y\big)\big|V\big]\Big] \\ &= \mathbb{E}\Big[\,\mathrm{e}^{-\delta V}\mathbb{1}\big(\overline{W}_{-c}(V) < u, W_{-c}(V) \in \mathrm{d}x\big)\cdot \mathbb{P}\big[Y \in \mathrm{d}y\big|V\big]\Big] \\ &= \int_0^\infty \mathrm{e}^{-\delta t}\mathbb{1}\big(\overline{W}_{-c}(t) < u, W_{-c}(t) \in \mathrm{d}x\big)\cdot f_{Y|V}(y|t)\,\mathrm{d}y\cdot f_V(t)\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\delta t}\mathbb{1}\big(\overline{W}_{-c}(t) < u, W_{-c}(t) \in \mathrm{d}x\big) f_{V,Y}(t,y)\,\mathrm{d}y\,\mathrm{d}t. \end{aligned} \tag{4.12}$$

4.3 Main results

We begin this section by stating the integro-differential equations satisfied by the Gerber–Shiu functions $m_s(u)$ and $m_d(u)$. Note that we assume the dependence structure (4.2) and assume the presence of diffusion. In order to validate the higher-order differentiation, we impose certain regularity conditions on the array of functions $f_{ij}(y)$ and the penalty function $w(x_1, x_2)$ (see also Li & Garrido 2005*a*, p. 163 and Appendix A). Namely, it suffices to require $f_{ij}(y)$ and $w(x_1, x_2)$ to be sufficiently smooth. These sufficient conditions can also be inferred from Proposition 4.3.6 directly.

Theorem 4.3.1. *Suppose that the joint p.d.f. of V and Y is of the form*

$$f_{V,Y}(t,y) = \sum_{i\geqslant 1}\sum_{j=1}^{k_i} \frac{\lambda_i^j t^{j-1}\mathrm{e}^{-\lambda_i t}}{(j-1)!} f_{ij}(y),$$

where the finitely many constants $\lambda_1, \lambda_2, \dots$ are distinct and $k_1, k_2, \dots$ are positive integers. Denote $n = \sum_{i\geqslant 1} k_i$ and assume the functions f_{ij} and w are $2n$-times continuously differentiable. Then the Gerber–Shiu functions $m_s(u)$ and $m_d(u)$ satisfy the following integro-differential equations

$$\begin{aligned}&\Big[\prod_{i\geqslant 1}\Big(\lambda_i + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)^{k_i}\Big] m_s(u) \\ &\quad = \sum_{i\geqslant 1}\sum_{j=1}^{k_i}\lambda_i^j\Big[\Big(\lambda_i + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)^{k_i - j}\prod_{\substack{l\geqslant 1\\ l\neq i}}\Big(\lambda_l + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)^{k_l}\Big]\gamma_{ij}(u)\end{aligned} \tag{4.13}$$

and

$$\left[\prod_{i\geqslant 1}\left(\lambda_i+\delta-c\mathcal{D}-\tfrac{\sigma^2}{2}\mathcal{D}^2\right)^{k_i}\right]m_d(u)$$
$$=\sum_{i\geqslant 1}\sum_{j=1}^{k_i}\lambda_i^j\left[\left(\lambda_i+\delta-c\mathcal{D}-\tfrac{\sigma^2}{2}\mathcal{D}^2\right)^{k_i-j}\prod_{\substack{l\geqslant 1\\ l\neq i}}\left(\lambda_l+\delta-c\mathcal{D}-\tfrac{\sigma^2}{2}\mathcal{D}^2\right)^{k_l}\right]\zeta_{ij}(u), \tag{4.14}$$

respectively, where $u\geqslant 0$, $\mathcal{D}=\partial/\partial u$ *is the differential operator and*

$$\gamma_{ij}(u)=\int_0^u m_s(u-y)f_{ij}(y)\,\mathrm{d}y+\int_u^\infty w(u,y-u)f_{ij}(y)\,\mathrm{d}y,$$
$$\zeta_{ij}(u)=\int_0^u m_d(u-y)f_{ij}(y)\,\mathrm{d}y$$

for $i\geqslant 1$ *and* $1\leqslant j\leqslant k_i$.

Remark 4.3.1. If $k_i=1$ for all $i\geqslant 1$; that is, if every λ_i has multiplicity one, then the integro-differential equations (4.13) and (4.14) reduce to

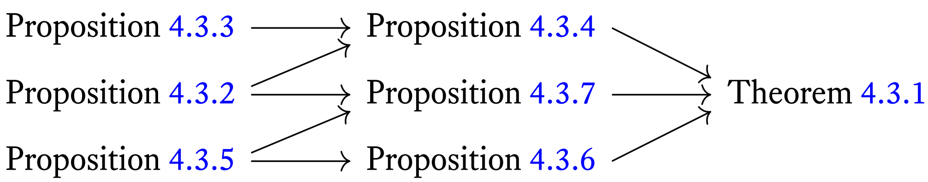
$$\left[\prod_{i\geqslant 1}\left(\lambda_i+\delta-c\mathcal{D}-\tfrac{\sigma^2}{2}\mathcal{D}^2\right)\right]m_s(u)=\sum_{i\geqslant 1}\lambda_i\left[\prod_{\substack{l\geqslant 1\\ l\neq i}}\left(\lambda_l+\delta-c\mathcal{D}-\tfrac{\sigma^2}{2}\mathcal{D}^2\right)\right]\gamma_i(u)$$

and

$$\left[\prod_{i\geqslant 1}\left(\lambda_i+\delta-c\mathcal{D}-\tfrac{\sigma^2}{2}\mathcal{D}^2\right)\right]m_d(u)=\sum_{i\geqslant 1}\lambda_i\left[\prod_{\substack{l\geqslant 1\\ l\neq i}}\left(\lambda_l+\delta-c\mathcal{D}-\tfrac{\sigma^2}{2}\mathcal{D}^2\right)\right]\zeta_i(u),$$

respectively, where we define $\gamma_i(u)=\gamma_{i1}(u)$ and $\zeta_i(u)=\zeta_{i1}(u)$ for simplicity. Variations of these reduced equations for simple λ_i can be found in Zhang & Yang (2011, Theorems 1 and 2), Zhang et al. (2014, Eq.'s (3.10) and (3.9)) and Zhang (2014, Theorems 4 and 3).

The proof of Theorem 4.3.1 is quite involved and is built on a series of propositions. So we defer the proof to the end of this section. Each of the propositions below represents a necessary component to establishing the integro-differential equations. In particular, Proposition 4.3.4 provides analytical expression for $p(u,x,y)$ under (4.2), which leads to the two *integral equations* satisfied by $m_s(u)$ and $m_d(u)$. The properties of the components within the integral equations are revealed by Propositions 4.3.6 and 4.3.7. This allows us to transform the integral equations into the desired integro-differential equations. Moreover, Propositions 4.3.2, 4.3.3 and 4.3.5 provide tools and shortcuts for the proofs of the aforementioned propositions. These results and the relation between them can be summarized in the following illustration.



Here, the arrows demonstrate how the results are connected. They do not represent implications.

The first result reveals the structure of the higher-order derivatives of an exponential function, whose exponent is a linear combinations of $\eta_{q,+}$, $\eta_{q,-}$ and η_q. This result is subsequently used in Propositions 4.3.4 and 4.3.7.

Proposition 4.3.2. *Let A, B and C be any constants that do not depend on q and let $D = A + B + C$. Define the following exponential function*

$$\mathcal{E}(q) = \mathcal{E}(q; A, B, C) := \exp\{-A\eta_{q,+} - B\eta_{q,-} - C\eta_q\},$$

where $\eta_{q,+}$, $\eta_{q,-}$ and η_q are given by (4.8). Then for $k = 1, 2, \dots,$ we have

$$(-1)^k \frac{\partial^k}{\partial q^k}\mathcal{E}(q) = \frac{D^k}{\sigma^{2k}}\,\eta_q^{-k}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big)\mathcal{E}(q), \tag{4.15}$$

and

$$(-1)^{k-1} \frac{\partial^{k-1}}{\partial q^{k-1}}\big[\eta_q^{-1}\mathcal{E}(q)\big] = \frac{D^{k-1}}{\sigma^{2k-2}}\,\eta_q^{-k}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big)\mathcal{E}(q), \tag{4.16}$$

where $\pi_0(x), \pi_1(x), \dots$ are the Bessel polynomials given by (4.10).

Remark 4.3.2. Note that $D = 0$ is a *removable singularity* in both (4.15) and (4.16). So these two identities also hold if $D = 0$.

Proof. We shall prove Eq. (4.15) by induction. We begin by observing that

$$\frac{\partial}{\partial q}\eta_{q,+} = \frac{\partial}{\partial q}\eta_{q,-} = \frac{\partial}{\partial q}\eta_q = \frac{1}{\sigma^2}\eta_q^{-1}.$$

Thus, by the chain rule we have

$$(-1)\frac{\partial}{\partial q}\exp\{-A\eta_{q,+} - B\eta_{q,-} - C\eta_q\} = \frac{D}{\sigma^2}\,\eta_q^{-1}\exp\{-A\eta_{q,+} - B\eta_{q,-} - C\eta_q\}. \tag{4.17}$$

Since $\pi_0(x) = 1$, this establishes Eq. (4.15) for the basis case $k = 1$.

Next, we assume that Eq. (4.15) holds for some $k = 1, 2, \ldots$, and demonstrate that it holds for $k + 1$ as well. To this end, note that $\eta_q^{-k}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big)$ is a polynomial in η_q^{-1} with degree $2k - 1$, and the powers of η_q^{-1} less than k all vanish. In addition, we observe that

$$(-1)\frac{\partial}{\partial q}\eta_q^{-j} = \frac{j}{\sigma^2}\eta_q^{-j-2} \tag{4.18}$$

for $j = 1, 2, \ldots$ Taking the negative derivative of Eq. (4.15) with respect to q yield

$$\begin{aligned}(-1)^{k+1}\frac{\partial^{k+1}}{\partial q^{k+1}}\mathcal{E}(q) &= \frac{D^k}{\sigma^{2k}}\mathcal{E}(q) \times (-1)\frac{\partial}{\partial q}\big[\eta_q^{-k}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big)\big] \\ &\quad + \frac{D^k}{\sigma^{2k}}\eta_q^{-k}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big) \times (-1)\frac{\partial}{\partial q}\mathcal{E}(q) \\ &= \frac{D^{k+1}}{\sigma^{2k+2}}\mathcal{E}(q) \\ &\quad \times \left\{-\frac{\sigma^2}{D}\frac{\partial}{\partial q}\big[\eta_q^{-k}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big)\big] + \eta_q^{-k-1}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big)\right\}.\end{aligned} \tag{4.19}$$

We evaluate the expression in the curly brackets of (4.19) using (4.18) and obtain

$$\begin{aligned}&\frac{\sigma^2}{D} \times (-1)\frac{\partial}{\partial q}\sum_{j=0}^{k-1}\frac{(k+j-1)!}{(k-j-1)!\,j!}\frac{D^{-j}}{2^j}\eta_q^{-k-j} + \eta_q^{-k-1}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big) \\ &= \sum_{j=0}^{k-1}\frac{(k+j)!}{(k-j-1)!\,j!}\frac{D^{-j-1}}{2^j}\eta_q^{-k-j-2} + \eta_q^{-k-1}\pi_{k-1}\big(D^{-1}\eta_q^{-1}\big) \\ &= \eta_q^{-(k+1)}\left[\sum_{j=0}^{k-1}\frac{(k+j)!}{(k-j-1)!\,j!}\frac{D^{-j-1}}{2^j}\eta_q^{-j-1} + \pi_{k-1}\big(D^{-1}\eta_q^{-1}\big)\right] \\ &= \eta_q^{-(k+1)}\left[\sum_{l=1}^{k}\frac{(k+l-1)!}{(k-l)!\,(l-1)!}\frac{D^{-l}}{2^{l-1}}\eta_q^{-l} + \sum_{l=0}^{k-1}\frac{(k+l-1)!}{(k-l-1)!\,l!}\frac{D^{-l}}{2^l}\eta_q^{-l}\right] \\ &= \eta_q^{-(k+1)}\Bigg[1 + \sum_{l=1}^{k-1}\frac{(k+l-1)! \times 2l + (k+l-1)! \times (k-l)}{(k-l)!\,l!}\frac{D^{-l}}{2^l}\eta_q^{-l} \\ &\qquad + \frac{(2k-1)!}{(k-1)!}\frac{D^{-k}}{2^{k-1}}\eta_q^{-k}\Bigg]\end{aligned}$$

$$= \eta_q^{-(k+1)}\left[1+\sum_{l=1}^{k-1}\frac{(k+l)!}{(k-l)!\,l!}\frac{D^{-l}}{2^l}\eta_q^{-l}+\frac{(2k-1)!\times 2k}{(k-1)!\times k}\frac{D^{-k}}{2^k}\eta_q^{-k}\right]$$
$$= \eta_q^{-(k+1)}\sum_{l=0}^{k}\frac{(k+l)!}{(k-l)!\,l!}\frac{D^{-l}}{2^l}\eta_q^{-l}$$
$$= \eta_q^{-(k+1)}\pi_k\big(D^{-1}\eta_q^{-1}\big). \tag{4.20}$$

Substituting the result (4.20) back into (4.19), we deduce

$$(-1)^{k+1}\frac{\partial^{k+1}}{\partial q^{k+1}}\mathcal{E}(q) = \frac{D^{k+1}}{\sigma^{2k+2}}\mathcal{E}(q)\times \eta_q^{-(k+1)}\pi_k\big(D^{-1}\eta_q^{-1}\big).$$

This completes the inductive step and thus Eq. (4.15) is proved for all $k = 1, 2, \ldots$

To prove Eq. (4.16), we multiply both sides of Eq. (4.17) by σ^2/D and take the negative derivative $k-1$ times, which yield

$$\frac{\sigma^2}{D}\times(-1)^k\frac{\partial^k}{\partial q^k}\mathcal{E}(q) = (-1)^{k-1}\frac{\partial^{k-1}}{\partial q^{k-1}}\big[\eta_q^{-1}\mathcal{E}(q)\big].$$

Therefore, Eq. (4.16) follows by multiplying Eq. (4.15) by σ^2/D. This completes the proof. □

Proposition 4.3.3. *Let $q > 0$ be a constant. For $u \geqslant 0$ and $x < u$, we have*

$$\int_0^\infty \mathbb{1}\big(\overline{W}_{-c}(t) < u, W_{-c}(t)\in \mathrm{d}x\big)\,\mathrm{e}^{-qt}\,\mathrm{d}t$$
$$= \begin{cases} \dfrac{1}{\sigma^2}\eta_q^{-1}\big(\mathrm{e}^{-x\eta_{q,+}} - \mathrm{e}^{-2u\eta_q + x\eta_{q,-}}\big)\,\mathrm{d}x, & 0\leqslant x < u,\\[2mm] \dfrac{1}{\sigma^2}\eta_q^{-1}\big(\mathrm{e}^{x\eta_{q,-}} - \mathrm{e}^{-2u\eta_q + x\eta_{q,-}}\big)\,\mathrm{d}x, & x < 0. \end{cases} \tag{4.21}$$

Proof. Let $\mathbf{e}_q$ be an exponential random variable with mean $1/q$, which is independent of the diffusion $\{W_{-c}(t) : t \geqslant 0\}$. It is well-known (Borodin & Salminen 2002, p. 252, Eq. (1.2.6)) that

$$\mathbb{P}\big[\overline{W}_{-c}(\mathbf{e}_q) \geqslant u, W_{-c}(\mathbf{e}_q)\in \mathrm{d}x\big] = \frac{q}{\sigma^2}\eta_q^{-1}\mathrm{e}^{-cx/\sigma^2+(-2u+x)\eta_q}\,\mathrm{d}x$$
$$= \frac{q}{\sigma^2}\eta_q^{-1}\mathrm{e}^{-2u\eta_q+x\eta_{q,-}}\,\mathrm{d}x \tag{4.22}$$

for $u \geqslant 0$ and $x < u$. Furthermore, Borodin & Salminen (2002, p. 250, Eq. (1.0.5)) state that

$$\mathbb{P}\big[W_{-c}(\mathbf{e}_q)\in \mathrm{d}x\big] = \frac{q}{\sigma^2}\eta_q^{-1}\mathrm{e}^{-cx/\sigma^2-|x|\eta_q}\,\mathrm{d}x = \begin{cases} \dfrac{q}{\sigma^2}\eta_q^{-1}\mathrm{e}^{-x\eta_{q,+}}\,\mathrm{d}x, & 0\leqslant x < u,\\[2mm] \dfrac{q}{\sigma^2}\eta_q^{-1}\mathrm{e}^{x\eta_{q,-}}\,\mathrm{d}x, & x < 0. \end{cases} \tag{4.23}$$

Since

$$\int_0^\infty \mathbb{1}\big(\overline{W}_{-c}(t) < u, W_{-c}(t) \in \mathrm{d}x\big)\, q\mathrm{e}^{-qt}\,\mathrm{d}t = \mathbb{P}\big[\overline{W}_{-c}(\mathbf{e}_q) < u, W_{-c}(\mathbf{e}_q) \in \mathrm{d}x\big]$$
$$= \mathbb{P}\big[W_{-c}(\mathbf{e}_q) \in \mathrm{d}x\big] - \mathbb{P}\big[\overline{W}_{-c}(\mathbf{e}_q) \geqslant u, W_{-c}(\mathbf{e}_q) \in \mathrm{d}x\big],$$

the desired result follows by subtracting (4.22) from (4.23) and then dividing by q. □

We now combine the results from Propositions 4.3.2 and 4.3.3, and derive an explicit expression for the density $\not p(u,x,y)$ under the dependence structure (4.2). Note that the regularity conditions for $f_{ij}(y)$ are not needed here.

Proposition 4.3.4. *Suppose that the joint p.d.f. of V and Y is of the form*

$$f_{V,Y}(t,y) = \sum_{i\geqslant 1}\sum_{j=1}^{k_i} \frac{\lambda_i^j t^{j-1}\mathrm{e}^{-\lambda_i t}}{(j-1)!} f_{ij}(y).$$

Then the measure $\mathcal{P}(u,\mathrm{d}x,\mathrm{d}y)$ defined by (4.11) has the explicit density $\not p(u,x,y)$ given by

$$\not p(u,x,y) = \sum_{i\geqslant 1}\sum_{j=1}^{k_i} \frac{\lambda_i^j f_{ij}(y)}{(j-1)!\,\sigma^{2j}}\eta_q^{-j}\Big[-(2u-x)^{j-1}\pi_{j-1}\big((2u-x)^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{-2u\eta_q + x\eta_{q,-}}$$
$$+ |x|^{j-1}\pi_{j-1}\big(|x|^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{-|x|\eta_{q,\mathrm{sgn}(x)}}\Big]\Big|_{q=\lambda_i+\delta} \tag{4.24}$$

for $u \geqslant 0$, $x < u$ and $y > 0$. Here, we interpret $\mathrm{e}^{-|0|\eta_{q,\mathrm{sgn}(0)}} = 1$.

Proof. Substituting the joint p.d.f. $f_{V,Y}(t,y)$ into (4.12), we derive

$$\mathcal{P}(u,\mathrm{d}x,\mathrm{d}y) = \sum_{i\geqslant 1}\sum_{j=1}^{k_i} \frac{\lambda_i^j f_{ij}(y)}{(j-1)!}\,\mathrm{d}y \int_0^\infty \mathbb{1}\big(\overline{W}_{-c}(t) < u, W_{-c}(t) \in \mathrm{d}x\big)\, t^{j-1}\mathrm{e}^{-(\lambda_i+\delta)t}\,\mathrm{d}t$$
$$= \sum_{i\geqslant 1}\sum_{j=1}^{k_i} \frac{\lambda_i^j f_{ij}(y)}{(j-1)!}\,\mathrm{d}y$$
$$\times \Big(-\frac{\partial}{\partial q}\Big)^{j-1} \int_0^\infty \mathbb{1}\big(\overline{W}_{-c}(t) < u, W_{-c}(t) \in \mathrm{d}x\big)\,\mathrm{e}^{-qt}\,\mathrm{d}t\Big|_{q=\lambda_i+\delta}$$

where we can apply Proposition 4.3.3 to obtain

$$= \sum_{i\geqslant 1}\sum_{j=1}^{k_i} \frac{\lambda_i^j f_{ij}(y)}{(j-1)!\,\sigma^2}\,\mathrm{d}y \\ \times (-1)^{j-1}\frac{\partial^{j-1}}{\partial q^{j-1}}\Big[\eta_q^{-1}\mathrm{e}^{-|x|\eta_{q,\operatorname{sgn}(x)}} - \eta_q^{-1}\mathrm{e}^{-2u\eta_q + x\eta_{q,-}}\Big]\,\mathrm{d}x\Big|_{q=\lambda_i+\delta}. \tag{4.25}$$

Next, with the help of (4.16) in Proposition 4.3.2, we derive

$$(-1)^{j-1}\frac{\partial^{j-1}}{\partial q^{j-1}}\Big[\eta_q^{-1}\mathrm{e}^{-|x|\eta_{q,\operatorname{sgn}(x)}}\Big] = \frac{|x|^{j-1}}{\sigma^{2j-2}}\,\eta_q^{-j}\,\pi_{j-1}\big(|x|^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{-|x|\eta_{q,\operatorname{sgn}(x)}},$$

and

$$(-1)^{j-1}\frac{\partial^{j-1}}{\partial q^{j-1}}\Big[\eta_q^{-1}\mathrm{e}^{-2u\eta_q+x\eta_{q,-}}\Big] = \frac{(2u-x)^{j-1}}{\sigma^{2j-2}}\,\eta_q^{-j}\,\pi_{j-1}\big((2u-x)^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{-2u\eta_q+x\eta_{q,-}}.$$

Substituting these resulting derivatives back into (4.25), rearranging the terms and factoring out $\mathrm{d}y\,\mathrm{d}x$, it follows

$$\mathcal{P}(u,\mathrm{d}x,\mathrm{d}y) \\ = \sum_{i\geqslant 1}\sum_{j=1}^{k_i}\frac{\lambda_i^j f_{ij}(y)}{(j-1)!\,\sigma^{2j}}\eta_q^{-j}\Big[-(2u-x)^{j-1}\pi_{j-1}\big((2u-x)^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{-2u\eta_q+x\eta_{q,-}} \\ + |x|^{j-1}\pi_{j-1}\big(|x|^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{-|x|\eta_{q,\operatorname{sgn}(x)}}\Big]\Big|_{q=\lambda_i+\delta}\,\mathrm{d}y\,\mathrm{d}x.$$

This completes the proof of (4.24). □

The next result links the first- and second-order derivatives of a Bessel polynomial of degree k to a Bessel polynomial of degree $k-1$. This result provides shortcuts for Propositions 4.3.6 and 4.3.7.

Proposition 4.3.5. *The Bessel polynomials satisfy the following recursive differential equations*

$$\frac{\partial^2}{\partial x^2}\big[x^k\pi_k(x^{-1}z^{-1})\big] - 2z\,\frac{\partial}{\partial x}\big[x^k\pi_k(x^{-1}z^{-1})\big] = -2kz\big[x^{k-1}\pi_{k-1}(x^{-1}z^{-1})\big] \tag{4.26}$$

and

$$\frac{\partial^2}{\partial x^2}\left[x^{k+1}\pi_k(x^{-1}z^{-1})\right] - 2z\,\frac{\partial}{\partial x}\left[x^{k+1}\pi_k(x^{-1}z^{-1})\right] = -2(k+1)z\left[x^k\pi_{k-1}(x^{-1}z^{-1})\right] \tag{4.27}$$

for $k = 1, 2, \ldots$

Proof. We verify Eq. (4.26) directly. Note that

$$x^k\pi_k(x^{-1}z^{-1}) = \sum_{j=0}^{k}\frac{(k+j)!}{(k-j)!\,j!}\frac{z^{-j}}{2^j}x^{k-j}.$$

Thus, we have

$$\frac{\partial}{\partial x}\left[x^k\pi_k(x^{-1}z^{-1})\right] = \sum_{j=0}^{k-1}\frac{(k+j)!}{(k-j-1)!\,j!}\frac{z^{-j}}{2^j}x^{k-j-1}$$

and

$$\frac{\partial^2}{\partial x^2}\left[x^k\pi_k(x^{-1}z^{-1})\right] = \sum_{j=0}^{k-2}\frac{(k+j)!}{(k-j-2)!\,j!}\frac{z^{-j}}{2^j}x^{k-j-2}. \tag{4.28}$$

We apply change of variable for the summing index and factor out $2z$ in Eq. (4.28) to obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left[x^k\pi_k(x^{-1}z^{-1})\right] - 2z\,\frac{\partial}{\partial x}\left[x^k\pi_k(x^{-1}z^{-1})\right] \\
&\quad = 2z\sum_{j=1}^{k-1}\frac{(k+j-1)!}{(k-j-1)!\,(j-1)!}\frac{z^{-j}}{2^j}x^{k-j-1} - 2z\sum_{j=0}^{k-1}\frac{(k+j)!}{(k-j-1)!\,j!}\frac{z^{-j}}{2^j}x^{k-j-1} \\
&\quad = -2kzx^{k-1} - 2z\sum_{j=1}^{k-1}\left[\frac{(k+j)!}{(k-j-1)!\,j!} - \frac{(k+j-1)!}{(k-j-1)!\,(j-1)!}\right]\frac{z^{-j}}{2^j}x^{k-j-1} \\
&\quad = -2kzx^{k-1} - 2kz\sum_{j=1}^{k-1}\frac{(k+j-1)!}{(k-j-1)!\,j!}\frac{z^{-j}}{2^j}x^{k-j-1} \\
&\quad = -2kz\left[x^{k-1}\pi_{k-1}(x^{-1}z^{-1})\right]
\end{aligned}$$

as required. The proof for Eq. (4.27) is analogous and is thus omitted. □

The integral equations satisfied by $m_s(u)$ and $m_d(u)$ both contain expressions involving $I_1(u)$, $I_2(u)$, and $I_3(u)$ in the next proposition. Hence, their properties under certain differential operators are collected here as a separate result. This serves to simplify the proof of Theorem 4.3.1. The recursive identity (4.26) in Proposition 4.3.5 is used.

Proposition 4.3.6. *Let q be a constant and let k be a positive integer. Suppose that $\gamma(u)$ is a generic $2k$-times continuously differentiable function that does not depend on q. Then*

$$\left(\frac{\partial^2}{\partial u^2}+\frac{2c}{\sigma^2}\frac{\partial}{\partial u}-\frac{2q}{\sigma^2}\right)^k\left[I_1(u;q,k,\gamma)+I_2(u;q,k,\gamma)\right]=(k-1)!\,(-2\eta_q)^k\gamma(u), \quad (4.29)$$

$$\left(\frac{\partial^2}{\partial u^2}+\frac{2c}{\sigma^2}\frac{\partial}{\partial u}-\frac{2q}{\sigma^2}\right)^k I_3(u;q,k,\gamma)=0, \quad (4.30)$$

where

$$I_1(u;q,k,\gamma)=\int_0^u (u-v)^{k-1}\pi_{k-1}\big((u-v)^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{-(u-v)\eta_{q,+}}\,\gamma(v)\,\mathrm{d}v, \quad (4.31)$$

$$I_2(u;q,k,\gamma)=\int_u^\infty (v-u)^{k-1}\pi_{k-1}\big((v-u)^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{(u-v)\eta_{q,-}}\,\gamma(v)\,\mathrm{d}v, \quad (4.32)$$

$$I_3(u;q,k,\gamma)=\int_0^\infty (u+v)^{k-1}\pi_{k-1}\big((u+v)^{-1}\eta_q^{-1}\big)\,\mathrm{e}^{-u\eta_{q,+}-v\eta_{q,-}}\,\gamma(v)\,\mathrm{d}v. \quad (4.33)$$

Proof. We prove (4.29) and (4.30) by induction. For the basis case $k=1$, since $\pi_0(x)=1$, the polynomials of u in the integrands are simply constant one. Thus, we see that

$$\left(\frac{\partial}{\partial u}+\eta_{q,+}\right)I_1(u;q,1,\gamma)=\gamma(u), \quad (4.34)$$

$$\left(\frac{\partial}{\partial u}-\eta_{q,-}\right)I_2(u;q,1,\gamma)=-\gamma(u). \quad (4.35)$$

Applying the operator $(\partial/\partial u-\eta_{q,-})$ to Eq. (4.34) and $(\partial/\partial u+\eta_{q,+})$ to Eq. (4.35), we obtain

$$\left(\frac{\partial^2}{\partial u^2}+\frac{2c}{\sigma^2}\frac{\partial}{\partial u}-\frac{2q}{\sigma^2}\right)I_1(u;q,1,\gamma)=\gamma'(u)-\eta_{q,-}\gamma(u),$$

$$\left(\frac{\partial^2}{\partial u^2}+\frac{2c}{\sigma^2}\frac{\partial}{\partial u}-\frac{2q}{\sigma^2}\right)I_2(u;q,1,\gamma)=-\gamma'(u)-\eta_{q,+}\gamma(u).$$

Note that $\eta_{q,+}+\eta_{q,-}=2\eta_q$. Summing up the two equations above yields

$$\left(\frac{\partial^2}{\partial u^2}+\frac{2c}{\sigma^2}\frac{\partial}{\partial u}-\frac{2q}{\sigma^2}\right)\left[I_1(u;q,1,\gamma)+I_2(u;q,1,\gamma)\right]=-2\eta_q\gamma(u),$$

which proves Eq. (4.29) for $k=1$. Furthermore, observe that

$$\left(\frac{\partial}{\partial u}+\eta_{q,+}\right)I_3(u;q,1,\gamma)=0. \quad (4.36)$$

Applying the operator $(\partial/\partial u - \eta_{q,-})$ to Eq. (4.36) yields

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right) I_3(u;q,1,\gamma) = 0,$$

which proves Eq. (4.30) for $k = 1$.

Now, assume that (4.29) and (4.30) hold for some $k \geqslant 1$. We shall prove that they also hold for $k+1$. To this end, first observe that the integrands in (4.31), (4.32) and (4.33) are all of the form

$$\text{Polynomial}(u;v,q,k-1) \times \text{Exponent}(u;v,q) \times \gamma(v).$$

To ease the computation of derivatives, it is worth introducing the following six auxiliary integrals

$$I_{i,1}(u;q,k+1,\gamma) = \int \left[\frac{\partial}{\partial u}\text{Polynomial}_i(u;v,q,k)\right] \text{Exponent}_i(u;v,q)\,\gamma(v)\,\mathrm{d}v$$

and

$$I_{i,2}(u;q,k+1,\gamma) = \int \left[\frac{\partial^2}{\partial u^2}\text{Polynomial}_i(u;v,q,k)\right] \text{Exponent}_i(u;v,q)\,\gamma(v)\,\mathrm{d}v$$

for $i = 1, 2$ and 3, with the corresponding integral bounds. We now apply (4.26) to obtain

$$\begin{aligned}
I_{1,2}(u;q,k+1,\gamma) - 2\eta_q I_{1,1}(u;q,k+1,\gamma) &= -2k\eta_q I_1(u;q,k,\gamma),\\
I_{2,2}(u;q,k+1,\gamma) + 2\eta_q I_{2,1}(u;q,k+1,\gamma) &= -2k\eta_q I_2(u;q,k,\gamma),\\
I_{3,2}(u;q,k+1,\gamma) - 2\eta_q I_{3,1}(u;q,k+1,\gamma) &= -2k\eta_q I_3(u;q,k,\gamma)
\end{aligned}$$

for $k = 1, 2, \ldots$

It is readily seen that

$$\left(\frac{\partial}{\partial u} + \eta_{q,+}\right) I_1(u;q,k+1,\gamma) = \frac{(2k)!}{2^k k!}\eta_q^{-k}\gamma(u) + I_{1,1}(u;q,k+1,\gamma), \tag{4.37}$$

$$\left(\frac{\partial}{\partial u} - \eta_{q,-}\right) I_2(u;q,k+1,\gamma) = -\frac{(2k)!}{2^k k!}\eta_q^{-k}\gamma(u) + I_{2,1}(u;q,k+1,\gamma), \tag{4.38}$$

and

$$\left(\frac{\partial}{\partial u} + \eta_{q,+}\right) I_3(u;q,k+1,\gamma) = I_{3,1}(u;q,k+1,\gamma). \tag{4.39}$$

Again, we apply the operator $(\partial/\partial u - \eta_{q,-})$ to Eq. (4.37) to obtain

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right) I_1(u;q,k+1,\gamma) \\
&\quad = \frac{(2k)!}{2^k k!}\eta_q^{-k}\gamma'(u) + \frac{(2k-1)!}{2^{k-1}(k-1)!}\eta_q^{-k+1}\gamma(u) \\
&\qquad + I_{1,2}(u;q,k+1,\gamma) - \eta_{q,+} I_{1,1}(u;q,k+1,\gamma) \\
&\qquad - \frac{(2k)!}{2^k k!}\eta_q^{-k}\eta_{q,-}\gamma(u) - \eta_{q,-} I_{1,1}(u;q,k+1,\gamma) \\
&\quad = \frac{(2k)!}{2^k k!}\eta_q^{-k}\gamma'(u) + \frac{(2k)!}{2^k k!}\eta_q^{-k}(\eta_q - \eta_{q,-})\gamma(u) \\
&\qquad + I_{1,2}(u;q,k+1,\gamma) - 2\eta_q I_{1,1}(u;q,k+1,\gamma) \\
&\quad = \frac{(2k)!}{2^k k!}\eta_q^{-k}\gamma'(u) + \frac{(2k)!}{2^k k!}\eta_q^{-k}(\eta_q - \eta_{q,-})\gamma(u) - 2k\eta_q I_1(u;q,k,\gamma). \qquad (4.40)
\end{aligned}$$

Similarly, we apply the operator $(\partial/\partial u + \eta_{q,+})$ to Eq. (4.38) to obtain

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right) I_2(u;q,k+1,\gamma) \\
&\quad = -\frac{(2k)!}{2^k k!}\eta_q^{-k}\gamma'(u) + \frac{(2k)!}{2^k k!}\eta_q^{-k}(\eta_q - \eta_{q,+})\gamma(u) - 2k\eta_q I_2(u;q,k,\gamma). \qquad (4.41)
\end{aligned}$$

Note that

$$(\eta_q - \eta_{q,-}) + (\eta_q - \eta_{q,+}) = 0.$$

Thus, summing up (4.40) and (4.41) yields

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right)\big[I_1(u;q,k+1,\gamma) + I_2(u;q,k+1,\gamma)\big] \\
&\qquad\qquad = -2k\eta_q\big[I_1(u;q,k,\gamma) + I_2(u;q,k,\gamma)\big].
\end{aligned}$$

Applying the inductive assumption, we deduce

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right)^{k+1}\big[I_1(u;q,k+1,\gamma) + I_2(u;q,k+1,\gamma)\big] = k!\,(-2\eta_q)^{k+1}\gamma(u)$$

as required. This proves (4.29) for all $k = 1, 2, \ldots$

Finally, in a completely analogous fashion, we apply the operator $(\partial/\partial u - \eta_{q,-})$ to Eq. (4.39) to obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right) & I_3(u;q,k+1,\gamma) \\ &= I_{3,2}(u;q,k+1,\gamma) - \eta_{q,+} I_{3,1}(u;q,k+1,\gamma) - \eta_{q,-} I_{3,1}(u;q,k+1,\gamma) \\ &= I_{3,2}(u;q,k+1,\gamma) - 2\eta_q I_{3,1}(u;q,k+1,\gamma) \\ &= -2k\eta_q I_3(u;q,k,\gamma).\end{aligned}$$

Thus, it follows from the inductive assumption that

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right)^{k+1} I_3(u;q,k+1,\gamma) = 0.$$

This proves (4.30) for all $k = 1, 2, \ldots$, and hence completes the proof. □

The integral equation satisfied by $m_d(u)$ contains extra terms involving $\mathrm{e}^{-\delta\tau_u}$, which needs additional treatment. This is done partially in the next proposition. Note that the recursive identity (4.27) in Proposition 4.3.5 is used.

Proposition 4.3.7. *Let q be a constant and let k be a positive integer. Then*

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right)^{k} (-1)^{k-1}\frac{\partial^{k-1}}{\partial q^{k-1}}\exp\{-u\eta_{q,+}\} = 0. \tag{4.42}$$

Proof. There is subtlety in the following inductive proof. By (4.15) in Proposition 4.3.2, we observe that

$$(-1)^{k-1}\frac{\partial^{k-1}}{\partial q^{k-1}}\exp\{-u\eta_{q,+}\} = \frac{u^{k-1}}{\sigma^{2k-2}}\,\eta_q^{-k+1}\pi_{k-2}\big(u^{-1}\eta_q^{-1}\big)\exp\{-u\eta_{q,+}\}$$

holds for $k \geqslant 2$. Therefore, we need to verify (4.42) for both $k = 1$ and $k = 2$ before the inductive step.

For $k = 1$, we have the identity $(\partial/\partial u + \eta_{q,+})\exp\{-u\eta_{q,+}\} = 0$, so

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right)\exp\{-u\eta_{q,+}\} = \left(\frac{\partial}{\partial u} - \eta_{q,-}\right)\left(\frac{\partial}{\partial u} + \eta_{q,+}\right)\exp\{-u\eta_{q,+}\} = 0.$$

Next, for $k = 2$, we have

$$\left(\frac{\partial}{\partial u} + \eta_{q,+}\right)\left[\frac{u}{\sigma^2}\eta_q^{-1}\exp\{-u\eta_{q,+}\}\right] = \frac{1}{\sigma^2}\eta_q^{-1}\exp\{-u\eta_{q,+}\},$$

and hence

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right)\left[\frac{u}{\sigma^2}\eta_q^{-1}\exp\{-u\eta_{q,+}\}\right] = -\frac{2}{\sigma^2}\exp\{-u\eta_{q,+}\}.$$

Applying the operator again establishes the basis case $k = 2$.

For the inductive step, assume that Eq. (4.42) holds for some $k \geqslant 2$. We have

$$\left(\frac{\partial}{\partial u} + \eta_{q,+}\right)\left[\frac{u^k}{\sigma^{2k}}\eta_q^{-k}\pi_{k-1}\left(u^{-1}\eta_q^{-1}\right)\exp\{-u\eta_{q,+}\}\right]$$
$$= \frac{\eta_q^{-k}}{\sigma^{2k}}\exp\{-u\eta_{q,+}\}\frac{\partial}{\partial u}\left[u^k\pi_{k-1}\left(u^{-1}\eta_q^{-1}\right)\right].$$

We then apply the operator $(\partial/\partial u - \eta_{q,-})$ to obtain

$$\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right)\left[\frac{u^k}{\sigma^{2k}}\eta_q^{-k}\pi_{k-1}\left(u^{-1}\eta_q^{-1}\right)\exp\{-u\eta_{q,+}\}\right]$$
$$= \frac{\eta_q^{-k}}{\sigma^{2k}}\exp\{-u\eta_{q,+}\}\left[\frac{\partial^2}{\partial u^2}\left[u^k\pi_{k-1}\left(u^{-1}\eta_q^{-1}\right)\right] - 2\eta_q\frac{\partial}{\partial u}\left[u^k\pi_{k-1}\left(u^{-1}\eta_q^{-1}\right)\right]\right]$$

and (4.27) implies

$$= -\frac{2k}{\sigma^2}\left[\frac{u^{k-1}}{\sigma^{2k-2}}\eta_q^{-k+1}\pi_{k-2}\left(u^{-1}\eta_q^{-1}\right)\exp\{-u\eta_{q,+}\}\right]$$
$$= -\frac{2k}{\sigma^2}\left[(-1)^{k-1}\frac{\partial^{k-1}}{\partial q^{k-1}}\exp\{-u\eta_{q,+}\}\right].$$

Using the inductive assumption shows that Eq. (4.42) also holds for $k + 1$. This finally completes the proof. □

We are now ready to prove Theorem 4.3.1 presented in the beginning of this section.

Proof of Theorem 4.3.1. We first prove (4.13). By conditioning on the time t of the first claim, the position x of $W_{-c}(t)$ and the amount y of the first claim, we have

$$m_s(u) = \int_{t=0}^{\infty}\int_{x=-\infty}^{u}\int_{y=0}^{u-x} \mathrm{e}^{-\delta t} m_s(u-x-y)$$
$$\times \mathbb{1}\left(\overline{W}_{-c}(t) < u, W_{-c}(t) \in \mathrm{d}x\right) f_{V,Y}(t,y)\,\mathrm{d}y\,\mathrm{d}t$$
$$+ \int_{t=0}^{\infty}\int_{x=-\infty}^{u}\int_{y=u-x}^{\infty} \mathrm{e}^{-\delta t} w\left(u-x, y-(u-x)\right)$$
$$\times \mathbb{1}\left(\overline{W}_{-c}(t) < u, W_{-c}(t) \in \mathrm{d}x\right) f_{V,Y}(t,y)\,\mathrm{d}y\,\mathrm{d}t$$

where we recall (4.12) to obtain

$$
\begin{aligned}
&= \int_{-\infty}^{u} \int_{0}^{u-x} m_s(u-x-y)\, p(u,x,y)\, \mathrm{d}y\, \mathrm{d}x \\
&\quad + \int_{-\infty}^{u} \int_{u-x}^{\infty} w\big(u-x, y-(u-x)\big)\, p(u,x,y)\, \mathrm{d}y\, \mathrm{d}x.
\end{aligned}
$$

Substituting the explicit expression for $p(u,x,y)$ given by Proposition 4.3.4, we obtain

$$
\begin{aligned}
m_s(u) = \sum_{i\geqslant 1} \sum_{j=1}^{k_i} & \frac{\lambda_i^j \eta_q^{-j}}{(j-1)!\,\sigma^{2j}} \\
& \times \int_{-\infty}^{u} \Big[-(2u-x)^{j-1} \pi_{j-1}\big((2u-x)^{-1}\eta_q^{-1}\big)\, \mathrm{e}^{-2u\eta_q + x\eta_{q,-}} \\
& \qquad + |x|^{j-1} \pi_{j-1}\big(|x|^{-1}\eta_q^{-1}\big)\, \mathrm{e}^{-|x|\eta_{q,\operatorname{sgn}(x)}} \Big] \gamma_{ij}(u-x)\, \mathrm{d}x \Big|_{q=\lambda_i+\delta}.
\end{aligned}
$$

Let us focus on the integral in one of the summands above and perform change of variable $v = u - x$. Observe that

$$
\begin{aligned}
&\int_{-\infty}^{u} (2u-x)^{j-1} \pi_{j-1}\big((2u-x)^{-1}\eta_q^{-1}\big)\, \mathrm{e}^{-2u\eta_q + x\eta_{q,-}}\, \gamma_{ij}(u-x)\, \mathrm{d}x \\
&\qquad = \int_{0}^{\infty} (u+v)^{j-1} \pi_{j-1}\big((u+v)^{-1}\eta_q^{-1}\big)\, \mathrm{e}^{-u\eta_{q,+} - v\eta_{q,-}}\, \gamma_{ij}(v)\, \mathrm{d}v \\
&\qquad = I_3(u; q, j, \gamma_{ij})
\end{aligned}
$$

according to (4.33),

$$
\begin{aligned}
&\int_{-\infty}^{0} (-x)^{j-1} \pi_{j-1}\big((-x)^{-1}\eta_q^{-1}\big)\, \mathrm{e}^{x\eta_{q,-}}\, \gamma_{ij}(u-x)\, \mathrm{d}x \\
&\qquad = \int_{u}^{\infty} (v-u)^{j-1} \pi_{j-1}\big((v-u)^{-1}\eta_q^{-1}\big)\, \mathrm{e}^{(u-v)\eta_{q,-}}\, \gamma_{ij}(v)\, \mathrm{d}v \\
&\qquad = I_2(u; q, j, \gamma_{ij})
\end{aligned}
$$

according to (4.32) and

$$
\begin{aligned}
&\int_{0}^{u} x^{j-1} \pi_{j-1}\big(x^{-1}\eta_q^{-1}\big)\, \mathrm{e}^{-x\eta_{q,+}}\, \gamma_{ij}(u-x)\, \mathrm{d}x \\
&\qquad = \int_{0}^{u} (u-v)^{j-1} \pi_{j-1}\big((u-v)^{-1}\eta_q^{-1}\big)\, \mathrm{e}^{-(u-v)\eta_{q,+}}\, \gamma_{ij}(v)\, \mathrm{d}v \\
&\qquad = I_1(u; q, j, \gamma_{ij})
\end{aligned}
$$

according to (4.31). These observations lead to

$$m_s(u) = \sum_{i\geqslant 1}\sum_{j=1}^{k_i} \frac{\lambda_i^j \eta_q^{-j}}{(j-1)!\,\sigma^{2j}} \big[I_1 + I_2 - I_3\big](u;q,j,\gamma_{ij})\Big|_{q=\lambda_i+\delta}. \tag{4.43}$$

Now we apply Proposition 4.3.6 to obtain

$$\begin{aligned}
&\Big(\lambda_i + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)^j \big[I_1 + I_2 - I_3\big](u;\lambda_i+\delta,j,\gamma_{ij}) \\
&\quad = \frac{\sigma^{2j}}{(-2)^j}\Big(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\Big)^j \big[I_1 + I_2 - I_3\big](u;q,j,\gamma_{ij})\Big|_{q=\lambda_i+\delta} \\
&\quad = (j-1)!\,\sigma^{2j}\,\eta_q^j\,\gamma_{ij}(u)\big|_{q=\lambda_i+\delta}.
\end{aligned}$$

Therefore, the desired integro-differential equation (4.13) is proved by applying the operator

$$\prod_{i\geqslant 1}\Big(\lambda_i + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)^{k_i}$$

on both sides of (4.43).

Next, we prove (4.14). By the same conditioning method, we have

$$\begin{aligned}
m_d(u) &= \int_{t=0}^{\infty}\int_{x=-\infty}^{u}\int_{y=0}^{u-x} \mathrm{e}^{-\delta t} m_d(u-x-y) \\
&\qquad\qquad \times \mathbb{1}\big(\overline{W}_{-c}(t) < u, W_{-c}(t) \in \mathrm{d}x\big) f_{V,Y}(t,y)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + w(0,0)\,\mathbb{E}\big[\mathrm{e}^{-\delta\tau_u}\mathbb{1}(\tau_u < V)\big] \\
&= \int_{-\infty}^{u}\int_{0}^{u-x} m_d(u-x-y)\,\not{p}(u,x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + w(0,0)\,\mathbb{E}\big[\mathrm{e}^{-\delta\tau_u}\mathbb{1}(\tau_u < V)\big].
\end{aligned}$$

Proposition 4.3.4 then implies

$$\begin{aligned}
m_d(u) &= \sum_{i\geqslant 1}\sum_{j=1}^{k_i} \frac{\lambda_i^j \eta_q^{-j}}{(j-1)!\,\sigma^{2j}} \big[I_1 + I_2 - I_3\big](u;q,j,\zeta_{ij})\Big|_{q=\lambda_i+\delta} \\
&\quad + w(0,0)\,\mathbb{E}\big[\mathrm{e}^{-\delta\tau_u}\mathbb{1}(\tau_u < V)\big].
\end{aligned}$$

Thus, the summands $[I_1 + I_2 - I_3]$ can be handled in the same way by Proposition 4.3.6. We see that (4.14) will be proved if we justify the following identity

$$\Big[\prod_{i\geqslant 1}\Big(\lambda_i + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)^{k_i}\Big]\mathbb{E}\big[\mathrm{e}^{-\delta\tau_u}\mathbb{1}(\tau_u < V)\big] = 0. \tag{4.44}$$

To this end, we denote $\alpha_{ij} = \int_0^\infty f_{ij}(y)\,\mathrm{d}y$ and calculate

$$\begin{aligned}
\mathbb{E}\big[\mathrm{e}^{-\delta\tau_u}\mathbb{1}(\tau_u < V)\big] &= \mathbb{E}\big[\mathbb{E}\big[\mathrm{e}^{-\delta\tau_u}\mathbb{1}(\tau_u < V)\big|\tau_u\big]\big] \\
&= \mathbb{E}\Big[\mathrm{e}^{-\delta\tau_u}\int_{\tau_u}^\infty f_V(t)\,\mathrm{d}t\Big] \\
&= \sum_{i\geqslant 1}\sum_{j=1}^{k_i}\alpha_{ij}\mathbb{E}\Big[\mathrm{e}^{-\delta\tau_u}\int_{\tau_u}^\infty \frac{\lambda_i^j t^{j-1}\mathrm{e}^{-\lambda_i t}}{(j-1)!}\,\mathrm{d}t\Big] \\
&= \sum_{i\geqslant 1}\sum_{j=1}^{k_i}\alpha_{ij}\sum_{l=0}^{j-1}\frac{\lambda_i^l}{l!}\mathbb{E}\big[\tau_u^l\,\mathrm{e}^{-(\lambda_i+\delta)\tau_u}\big] \\
&= \sum_{i\geqslant 1}\sum_{j=1}^{k_i}\alpha_{ij}\sum_{l=0}^{j-1}\frac{\lambda_i^l}{l!}\times(-1)^l\frac{\partial^l}{\partial q^l}\mathbb{E}\big[\mathrm{e}^{-q\tau_u}\big]\Big|_{q=\lambda_i+\delta} \\
&= \sum_{i\geqslant 1}\sum_{j=1}^{k_i}\alpha_{ij}\sum_{l=0}^{j-1}\frac{\lambda_i^l}{l!}\times(-1)^l\frac{\partial^l}{\partial q^l}\exp\{-u\eta_{q,+}\}\Big|_{q=\lambda_i+\delta}.
\end{aligned} \tag{4.45}$$

For each fixed summing index $i \geqslant 1$ in (4.45), the index l satisfies $l \leqslant j-1 \leqslant k_i - 1$. We can then apply Proposition 4.3.7 to obtain

$$\frac{\sigma^{2k_i}}{(-2)^{k_i}}\left(\frac{\partial^2}{\partial u^2} + \frac{2c}{\sigma^2}\frac{\partial}{\partial u} - \frac{2q}{\sigma^2}\right)^{k_i}(-1)^l\frac{\partial^l}{\partial q^l}\exp\{-u\eta_{q,+}\}\Big|_{q=\lambda_i+\delta} = 0$$

for $l \leqslant k_i - 1$. Hence, every summand in (4.45) vanishes after being differentiated by the corresponding operator.

This proves the desired identity (4.44), and hence completes the proof. □

4.4 Applications in special cases

The dependence structure (4.2) poses no restrictions on the array of functions $f_{ij}(y)$ for $i \geqslant 1$ and $1 \leqslant j \leqslant k_i$, other than two trivial conditions: In fact, some functions $f_{ij}(y)$ can even be negative, as long as the resulting joint p.d.f. $f_{V,Y}(t,y) \geqslant 0$ for all $t \geqslant 0$ and $y \geqslant 0$, and that $\int_0^\infty\int_0^\infty f_{V,Y}(t,y)\,\mathrm{d}y\,\mathrm{d}t = \sum_{i\geqslant 1}\sum_{j=1}^{k_i}\int_0^\infty f_{ij}(y)\,\mathrm{d}y = 1$. The choices for $f_{ij}(y)$ are essentially limitless.

Theorem 4.3.1 handles such generality in the form of Eq.'s (4.13) and (4.14). But as a result, the two arrays of functions $\gamma_{ij}(u)$ and $\zeta_{ij}(u)$ carry the same degrees of freedom as does the array $f_{ij}(y)$. This implies that the right-hand sides of (4.13) and (4.14) cannot be simplified further in general, unless additional assumptions are made.

In this section, we demonstrate how Theorem 4.3.1 sometimes provides simpler forms of the two integro-differential equations for $m_s(u)$ and $m_d(u)$. In particular, we consider three special cases. Theorem 4.4.1 deals with independence between V and Y. This first case also has applications when V and Y are *dependent* in certain ways. The second case is the FGM-dependence model and is examined in Theorem 4.4.2 and further in Example 4.4.1. Finally, we study a special exponential-weighted dependence model in Theorem 4.4.3. In the first case, we can reduce the number of functions $\gamma_{ij}(u)$ needed to just one. In the last two cases, this number can be reduced to two.

4.4.1 The independence model

When V and Y are independent, the array of functions $\gamma_{ij}(u)$ can be reduced to a single function $\gamma(u)$ and the array of functions $\zeta_{ij}(u)$ can be reduced to a single function $\zeta(u)$. Moreover, the double sums in the right-hand sides of (4.13) and (4.14) can be compactified into a single operator. This is the content of Theorem 4.4.1 in the current subsection.

Despite the independence assumption, Theorem 4.4.1 turns out to be useful in certain *dependence* models as well. We shall illustrate its applications under an FGM copula in Example 4.4.1 and under an exponential-weighted dependence model in Theorem 4.4.3.

Theorem 4.4.1 (Rational inter-claim times with independent claim sizes). *Suppose that V and Y are independent and that V is R_n-distributed with Laplace transform given by*

$$\tilde{f}_V(s) = \frac{g(s)}{f(s)} = \frac{g(s)}{\prod_{i\geqslant 1}(\lambda_i + s)^{k_i}},$$

where $\lambda_1, \lambda_2, \ldots$ are distinct with $\mathrm{Re}(\lambda_i) > 0$, $k_1, k_2, \ldots$ are positive integers with $\sum_{i\geqslant 1} k_i = n$, and $g(s)$ is a polynomial with degree $n-1$ or less. Assume $f_Y(y)$ is a $2n$-times continuously differentiable density function. Then the Gerber–Shiu functions $m_s(u)$ and $m_d(u)$ satisfy the following integro-differential equations

$$\left[f\left(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\right)\right]m_s(u) = \left[g\left(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\right)\right]\gamma(u) \tag{4.46}$$

and

$$\left[f\left(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\right)\right]m_d(u) = \left[g\left(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\right)\right]\zeta(u), \tag{4.47}$$

respectively, where $u \geqslant 0$ *and*

$$\gamma(u) = \int_0^u m_s(u-y) f_Y(y)\,\mathrm{d}y + \int_u^\infty w(u, y-u) f_Y(y)\,\mathrm{d}y,$$
$$\zeta(u) = \int_0^u m_d(u-y) f_Y(y)\,\mathrm{d}y.$$

Remark 4.4.1. When the polynomial $g(s) = \prod_{i\geqslant 1} \lambda_i^{k_i}$ is a constant, the distribution of V becomes generalized Erlang(n). In this case, Theorem 4.4.1 reduces to Li & Garrido (2005*a*, Theorem 1). In particular, (4.46) and (4.47) agree with Li & Garrido (2005*a*, Eq.'s (4) and (5)). The proof by Li & Garrido (2005*a*) relies on a Markovian state-transition argument. However, this argument does not carry over with the dependence structure (4.2) in general, since the strong Markov property no longer holds. Instead, we shall derive (4.46) and (4.47) as simple consequences of (4.13) and (4.14), respectively.

We also refer Albrecher et al. (2012) for a different proof on certain variations of (4.46) and (4.47). Note that Albrecher et al. (2012) utilize the martingale theory, while our application of Theorem 4.3.1 here is purely algebraic. Of course, the true power of Theorem 4.3.1 is its capability of handling a variety of dependence structures.

Proof of Theorem 4.4.1. By partial fraction decomposition we derive

$$\tilde{f}_V(s) = \sum_{i\geqslant 1} \sum_{j=1}^{k_i} \frac{C_{ij}}{(\lambda_i + s)^j} = \sum_{i\geqslant 1} \sum_{j=1}^{k_i} \frac{\lambda_i^j}{(\lambda_i + s)^j} \times \frac{C_{ij}}{\lambda_i^j},$$

where

$$C_{ij} = \frac{1}{(k_i - j)!} \frac{\partial^{k_i - j}}{\partial s^{k_i - j}} \left[\frac{(\lambda_i + s)^{k_i} g(s)}{f(s)} \right] \Bigg|_{s=-\lambda_i} \tag{4.48}$$

for $i \geqslant 1$ and $1 \leqslant j \leqslant k_i$. Therefore, the joint p.d.f. of (V, Y) is given by

$$f_{V,Y}(t,y) = f_V(t) f_Y(y) = \sum_{i\geqslant 1} \sum_{j=1}^{k_i} \frac{\lambda_i^j t^{j-1} \mathrm{e}^{-\lambda_i t}}{(j-1)!} \times \frac{C_{ij}}{\lambda_i^j} f_Y(y).$$

We can now apply Theorem 4.3.1 to obtain

$$\left[f\left(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\right) \right] m_s(u) = \left[\prod_{i\geqslant 1} \left(\lambda_i + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\right)^{k_i} \right] m_s(u)$$
$$= \left[\sum_{i\geqslant 1} \sum_{j=1}^{k_i} \frac{f\left(\delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)}{\left(\lambda_i + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)^j} C_{ij} \right] \gamma(u),$$

where

$$\frac{f\left(\delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)}{\left(\lambda_i + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)^j} = \left(\lambda_i + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)^{k_i - j} \prod_{\substack{l \geqslant 1 \\ l \neq i}} \left(\lambda_l + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)^{k_l}$$

is a formal expression.

The Hermite interpolation theory (see, for instance, Mastroianni & Milovanović 2008, Section 1.3.5) states the following identity

$$\sum_{i \geqslant 1} \sum_{j=1}^{k_i} \frac{f(z)}{(\lambda_i + z)^j} C_{ij} \equiv g(z)$$

for all $z \in \mathbb{C}$. Hence, we have

$$\left[\sum_{i \geqslant 1} \sum_{j=1}^{k_i} \frac{f\left(\delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)}{\left(\lambda_i + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)^j} C_{ij}\right] \gamma(u) = \left[g\left(\delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\right)\right] \gamma(u)$$

as required. The proof for the function $m_d(u)$ is analogous and is thus omitted. □

4.4.2 The Farlie–Gumbel–Morgenstern copula model

Now we turn to the case where V and Y are dependent through an FGM copula. In a rather straightforward manner, we use Theorem 4.3.1 to extend the diffusion-free result from Chadjiconstantinidis & Vrontos (2014). This is the content of Theorem 4.4.2.

Theorem 4.4.2 (Erlang inter-claim times with FGM-dependent claim sizes). *Suppose that the joint p.d.f. of V and Y is governed by an FGM copula as follows*

$$f_{V,Y}(t,y) = f_V(t) f_Y(y) + \theta h_Y(y) f_V(t) \left[2\bar{F}_V(t) - 1\right],$$

where $-1 \leqslant \theta \leqslant 1$ is the dependence parameter and $h_Y(y) = f_Y(y)\left[2\bar{F}_Y(y) - 1\right]$. Furthermore, assume that V is Erlang(n, λ)-distributed and $f_Y(y)$ is a $(6n-2)$-times continuously differentiable density function. Then the Gerber–Shiu functions $m_s(u)$ and $m_d(u)$ satisfy the following integro-differential equations

$$\begin{aligned} &\left[\mathcal{A}_1(\mathcal{D})\right]^n \left[\mathcal{A}_2(\mathcal{D})\right]^{2n-1} m_s(u) \\ &\quad = \lambda^n \left[\mathcal{A}_2(\mathcal{D})\right]^{2n-1} \gamma_1(u) + 2\theta \left[\sum_{j=n}^{2n-1} \lambda^j \binom{j-1}{n-1} \left[\mathcal{A}_1(\mathcal{D})\right]^n \left[\mathcal{A}_2(\mathcal{D})\right]^{2n-1-j}\right] \gamma_2(u) \end{aligned} \tag{4.49}$$

and

$$\begin{aligned}
&\big[\mathcal{A}_1(\mathcal{D})\big]^n\big[\mathcal{A}_2(\mathcal{D})\big]^{2n-1} m_d(u)\\
&\quad = \lambda^n\big[\mathcal{A}_2(\mathcal{D})\big]^{2n-1}\zeta_1(u) + 2\theta\Bigg[\sum_{j=n}^{2n-1}\lambda^j\binom{j-1}{n-1}\big[\mathcal{A}_1(\mathcal{D})\big]^n\big[\mathcal{A}_2(\mathcal{D})\big]^{2n-1-j}\Bigg]\zeta_2(u),
\end{aligned} \tag{4.50}$$

respectively, where $u \geqslant 0$, $\mathcal{A}_1(\mathcal{D}) = \lambda + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2$, $\mathcal{A}_2(\mathcal{D}) = 2\lambda + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2$ *and*

$$\begin{aligned}
\gamma_1(u) &= \int_0^u m_s(u-y)\big[f_Y(y) - \theta h_Y(y)\big]\,\mathrm{d}y\\
&\quad + \int_u^\infty w(u, y-u)\big[f_Y(y) - \theta h_Y(y)\big]\,\mathrm{d}y,\\
\gamma_2(u) &= \int_0^u m_s(u-y)\,h_Y(y)\,\mathrm{d}y\\
&\quad + \int_u^\infty w(u, y-u)\,h_Y(y)\,\mathrm{d}y,\\
\zeta_1(u) &= \int_0^u m_d(u-y)\big[f_Y(y) - \theta h_Y(y)\big]\,\mathrm{d}y,\\
\zeta_2(u) &= \int_0^u m_d(u-y)\,h_Y(y)\,\mathrm{d}y.
\end{aligned}$$

Remark 4.4.2. On the one hand, if we let $n = 1$, then (4.49) and (4.50) reduces to Zhang & Yang (2011, Eq.'s (4.8) and (4.17)), respectively. On the other hand, if we let $\sigma \to 0$, then $m_d(u)$ ceases to exist, and (4.49) implies Chadjiconstantinidis & Vrontos (2014, Eq. (22)). Finally, if we let $n = 1$ and $\sigma \to 0$ simultaneously, then the model reduces to a diffusion-free compound Poisson process with FGM-dependent claim sizes, and (4.49) agrees with Cossette et al. (2010, Proposition 5.1).

Proof of Theorem 4.4.2. According to Chadjiconstantinidis & Vrontos (2014, p. 128, Eq. (6)), the joint p.d.f. of (V, Y) is given by

$$\begin{aligned}
f_{V,Y}(t,y) &= \frac{\lambda^n t^{n-1}\mathrm{e}^{-\lambda t}}{(n-1)!}\big[f_Y(y) - \theta h_Y(y)\big]\\
&\quad + \sum_{i=0}^{n-1}\frac{(2\lambda)^{n+i}t^{n+i-1}\mathrm{e}^{-2\lambda t}}{(n+i-1)!}\times\frac{\theta}{2^{n+i-1}}\binom{n+i-1}{n-1}h_Y(y)
\end{aligned}$$

$$= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \big[f_Y(y) - \theta h_Y(y) \big] + \sum_{j=n}^{2n-1} \frac{(2\lambda)^j t^{j-1} e^{-2\lambda t}}{(j-1)!} \times \frac{\theta}{2^{j-1}} \binom{j-1}{n-1} h_Y(y).$$

In order to apply Theorem 4.3.1, we simply set

$$\begin{aligned} \lambda_1 &= \lambda, \quad k_1 = n, \qquad f_{1n}(y) = f_Y(y) - \theta h_Y(y), \\ \lambda_2 &= 2\lambda, \quad k_2 = 2n-1, \quad f_{2j}(y) = \frac{\theta}{2^{j-1}} \binom{j-1}{n-1} h_Y(y) \end{aligned}$$

for $n \leqslant j \leqslant 2n-1$ and $f_{ij}(y) = 0$ elsewhere.

The desired results follow by collecting the terms. □

In principle, the proof of Theorem 4.4.2 can be extended to cover the case where V is R_n-distributed. However, the interaction between the density $f_V(t)$ and its corresponding survival function $\overline{F}_V(t)$ creates too many additional terms. This makes the differential operators in the integro-differential equations extremely tedious.

We demonstrate this idea using R_3-distributed inter-claim times in Example 4.4.1. Note the use of Theorem 4.4.1 from the previous independence model. We invite the readers to decide whether it is worthwhile to favor the two intermediate functions $\gamma_1(u)$ and $\gamma_2(u)$ over the relatively simpler differential operators in (4.13).

Example 4.4.1 (R_3 inter-claim times with FGM-dependent claim sizes)**.** Suppose V follows an R_3 distribution with Laplace transform given by

$$\tilde{f}_V(s) = \frac{g(s)}{f(s)} = \frac{g(s)}{(\lambda_1 + s)(\lambda_2 + s)^2},$$

where $\lambda_1 \neq \lambda_2$ and $g(s)$ is a polynomial with degree 2 or less. Then

$$f_V(t) = \lambda_1 e^{-\lambda_1 t} \frac{C_{11}}{\lambda_1} + \lambda_2 e^{-\lambda_2 t} \frac{C_{21}}{\lambda_2} + \lambda_2^2 t e^{-\lambda_2 t} \frac{C_{22}}{\lambda_2^2}$$

and

$$\overline{F}_V(t) = e^{-\lambda_1 t} \frac{C_{11}}{\lambda_1} + e^{-\lambda_2 t} \frac{\lambda_2 C_{21} + C_{22}}{\lambda_2^2} + \lambda_2 t e^{-\lambda_2 t} \frac{C_{22}}{\lambda_2^2}$$

for $t \geqslant 0$, where

$$C_{11} = \frac{g(-\lambda_1)}{(\lambda_2 - \lambda_1)^2}, \quad C_{21} = \frac{g'(-\lambda_2) - C_{22}}{\lambda_1 - \lambda_2}, \quad C_{22} = \frac{g(-\lambda_2)}{\lambda_1 - \lambda_2}.$$

We see that

$$\begin{aligned}f_V(t)\bar{F}_V(t) &= 2\lambda_1 \mathrm{e}^{-2\lambda_1 t}\frac{C_{11}^2}{2\lambda_1^2} + 2\lambda_2 \mathrm{e}^{-2\lambda_2 t}\frac{\lambda_2 C_{21}^2 + C_{21}C_{22}}{2\lambda_2^3} \\ &\quad + (2\lambda_2)^2\, t\mathrm{e}^{-2\lambda_2 t}\frac{2\lambda_2 C_{21}C_{22} + C_{22}^2}{4\lambda_2^4} + (2\lambda_2)^3\, t^2 \mathrm{e}^{-2\lambda_2 t}\frac{C_{22}^2}{8\lambda_2^4} \\ &\quad + (\lambda_1+\lambda_2)\mathrm{e}^{-(\lambda_1+\lambda_2)t}\frac{(\lambda_1+\lambda_2)\lambda_2 C_{11}C_{21} + \lambda_1 C_{11}C_{22}}{(\lambda_1+\lambda_2)\lambda_1\lambda_2^2} \\ &\quad + (\lambda_1+\lambda_2)^2\, t\mathrm{e}^{-(\lambda_1+\lambda_2)t}\frac{C_{11}C_{22}}{(\lambda_1+\lambda_2)\lambda_1\lambda_2}.\end{aligned}$$

Under an FGM copula for $f_{V,Y}(t,y)$, we have

$$f_{V,Y}(t,y) = f_V(t)\big[f_Y(y) - \theta h_Y(y)\big] + 2\theta h_Y(y) f_V(t)\bar{F}_V(t).$$

Therefore, provided with a sufficiently smooth density function $f_Y(y)$, we can apply Theorem 4.3.1 and derive the left-hand side of (4.13) as

$$\Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[\big[\mathcal{A}_1(\mathcal{D})\big]\big[\mathcal{A}_2(\mathcal{D})\big]^3\big[\mathcal{A}_3(\mathcal{D})\big]^2\Big] m_s(u), \tag{4.51}$$

where $\mathcal{A}_1(\mathcal{D}) = 2\lambda_1 + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2$, $\mathcal{A}_2(\mathcal{D}) = 2\lambda_2 + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2$ and $\mathcal{A}_3(\mathcal{D}) = \lambda_1 + \lambda_2 + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2$.

The right-hand side of (4.13) here is more complicated. Let $\gamma_1(u)$ and $\gamma_2(u)$ be defined as in Theorem 4.4.2. We recall the results of Theorem 4.4.1 and observe that the operators acting on $\gamma_1(u)$ can be compactified into a single operator. However, the operators acting on $\gamma_2(u)$ cannot be further simplified. Thus, the right-hand side of (4.13) becomes

$$\begin{aligned}&\Big[g\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[\big[\mathcal{A}_1(\mathcal{D})\big]\big[\mathcal{A}_2(\mathcal{D})\big]^3\big[\mathcal{A}_3(\mathcal{D})\big]^2\Big]\gamma_1(u) \\ &\qquad + 2\theta\Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\mathcal{A}_4(\mathcal{D})\gamma_2(u),\end{aligned} \tag{4.52}$$

where

$$\begin{aligned}\mathcal{A}_4(\mathcal{D}) &= \frac{C_{11}^2}{\lambda_1}\big[\mathcal{A}_2(\mathcal{D})\big]^3\big[\mathcal{A}_3(\mathcal{D})\big]^2 \\ &\quad + \frac{(\lambda_2 C_{21} + C_{22})C_{21}}{\lambda_2^2}\big[\mathcal{A}_1(\mathcal{D})\big]\big[\mathcal{A}_2(\mathcal{D})\big]^2\big[\mathcal{A}_3(\mathcal{D})\big]^2 \\ &\quad + \frac{(2\lambda_2 C_{21} + C_{22})C_{22}}{\lambda_2^2}\big[\mathcal{A}_1(\mathcal{D})\big]\big[\mathcal{A}_2(\mathcal{D})\big]\big[\mathcal{A}_3(\mathcal{D})\big]^2\end{aligned}$$

$$
\begin{aligned}
&+ \frac{C_{22}^2}{\lambda_2}\big[\mathcal{A}_1(\mathcal{D})\big]\big[\mathcal{A}_3(\mathcal{D})\big]^2 \\
&+ \frac{\big[\lambda_1 C_{22} + (\lambda_1+\lambda_2)\lambda_2 C_{21}\big]C_{11}}{\lambda_1\lambda_2^2}\big[\mathcal{A}_1(\mathcal{D})\big]\big[\mathcal{A}_2(\mathcal{D})\big]^3\big[\mathcal{A}_3(\mathcal{D})\big] \\
&+ \frac{(\lambda_1+\lambda_2)C_{11}C_{22}}{\lambda_1\lambda_2}\big[\mathcal{A}_1(\mathcal{D})\big]\big[\mathcal{A}_2(\mathcal{D})\big]^3.
\end{aligned}
$$

The integro-differential equation satisfied by $m_d(u)$ under the current R_3-FGM model can be established in the same fashion using operators $f(\delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2)$, $g(\delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2)$, $\mathcal{A}_1(\mathcal{D})$, $\mathcal{A}_2(\mathcal{D})$, $\mathcal{A}_3(\mathcal{D})$ and $\mathcal{A}_4(\mathcal{D})$. The left-hand side of this equation will be (4.51), but with $m_s(u)$ replaced by $m_d(u)$. The right-hand side of this equation will be (4.52), but with $\gamma_1(u)$ and $\gamma_2(u)$ replaced by $\zeta_1(u)$ and $\zeta_2(u)$, respectively, where $\zeta_1(u)$ and $\zeta_2(u)$ are defined as in Theorem 4.4.2.

There is one more subtlety in (4.51). Let us further assume $2\lambda_1 = \lambda_2$. In this case, since the operator $f(\delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2)$ already contains $(\lambda_2 + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2)^2$ as one of its product, we see that the operator $\mathcal{A}_1(\mathcal{D}) = 2\lambda_1 + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2$ is redundant. This means (4.51) can be further optimized as

$$
\Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[\big[\mathcal{A}_2(\mathcal{D})\big]^3\big[\mathcal{A}_3(\mathcal{D})\big]^2\Big]m_s(u), \quad \text{when } 2\lambda_1 = \lambda_2.
$$

This effectively reduces the order of differentiation by 2. Of course, the right-hand side (4.52) will undergo a corresponding simplification.

When $\lambda_1 = 2\lambda_2$, similar consideration applies as well. In this case, we optimize (4.51) as

$$
\Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[\big[\mathcal{A}_1(\mathcal{D})\big]\big[\mathcal{A}_2(\mathcal{D})\big]^2\big[\mathcal{A}_3(\mathcal{D})\big]^2\Big]m_s(u), \quad \text{when } \lambda_1 = 2\lambda_2.
$$

Again, the total order of differentiation is reduced by 2.

As a closing remark of this example, we can conclude that these calculations and optimizations require a case-by-case analysis for a general R_n-distributed inter-claim time.

4.4.3 The exponential-weighted mixture dependence model

Finally, we look at a special exponential-weighted mixture dependence structure given by

$$
f_{Y|V}(y|t) = \mathrm{e}^{-\beta t} f_1(y) + (1 - \mathrm{e}^{-\beta t}) f_2(y),
$$

where $\beta > 0$ and $f_1(y)$, $f_2(y)$ are two arbitrary density functions. When V is exponentially distributed and when the risk process is diffusion-free, this reduces to the model studied by

Boudreault et al. (2006). Zhang et al. (2012) add diffusion to Boudreault et al. (2006), but their model is constrained under the condition $k_i \equiv 1$.

With the help of Theorem 4.3.1, this constraint is now lifted. We can work with general R_n-distributed inter-claim times in the presence of diffusion. We illustrate one more application of Theorem 4.4.1 in this special dependence case. This is the content of Theorem 4.4.3 below.

Theorem 4.4.3 (Rational inter-claim times with exponential-weighted-dependent claim sizes)**.** *Suppose that the conditional p.d.f. of Y given V is defined by*

$$f_{Y|V}(y|t) = \mathrm{e}^{-\beta t} f_1(y) + (1 - \mathrm{e}^{-\beta t}) f_2(y),$$

where $\beta > 0$ and $f_1(y)$, $f_2(y)$ are two sufficiently smooth density functions. Furthermore, assume that V is R_n-distributed with Laplace transform given by

$$\tilde{f}_V(s) = \frac{g(s)}{f(s)} = \frac{g(s)}{\prod_{i\geqslant 1}(\lambda_i + s)^{k_i}},$$

where $\lambda_1, \lambda_2, \ldots$ are distinct with $\mathrm{Re}(\lambda_i) > 0$, $k_1, k_2, \ldots$ are positive integers with $\sum_{i\geqslant 1} k_i = n$, and $g(s)$ is a polynomial with degree $n-1$ or less. Then the Gerber–Shiu functions $m_s(u)$ and $m_d(u)$ satisfy the following integro-differential equations

$$\begin{aligned}
&\Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[f\Big(\beta + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big] m_s(u) \\
&\quad = \Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[g\Big(\beta + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big][\gamma_1(u) - \gamma_2(u)] \\
&\qquad + \Big[f\Big(\beta + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[g\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\gamma_2(u)
\end{aligned} \tag{4.53}$$

and

$$\begin{aligned}
&\Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[f\Big(\beta + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big] m_d(u) \\
&\quad = \Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[g\Big(\beta + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big][\zeta_1(u) - \zeta_2(u)] \\
&\qquad + \Big[f\Big(\beta + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[g\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\zeta_2(u),
\end{aligned} \tag{4.54}$$

respectively, where $u \geqslant 0$ and

$$\begin{aligned}
\gamma_i(u) &= \int_0^u m_s(u-y) f_i(y)\,\mathrm{d}y + \int_u^\infty w(u, y-u) f_i(y)\,\mathrm{d}y, \\
\zeta_i(u) &= \int_0^u m_d(u-y) f_i(y)\,\mathrm{d}y
\end{aligned}$$

for $i = 1, 2$.

Remark 4.4.3. If we set $n = 1$, $\lambda_1 = \lambda$ and let $\sigma \to 0$, then the polynomial $g(s) = \lambda$. In this case, (4.53) leads to Boudreault et al. (2006, p. 271, Eq. (13)) after applying Laplace transforms on both sides.

Proof of Theorem 4.4.3. According to the assumptions, the joint p.d.f. of (V, Y) is given by

$$\begin{aligned} f_{V,Y}(t,y) &= f_{Y|V}(y|t) f_V(t) \\ &= \mathrm{e}^{-\beta t} f_V(t)\big[f_1(y) - f_2(y)\big] + f_V(t) f_2(y) \\ &= \sum_{i \geqslant 1} \sum_{j=1}^{k_i} \frac{(\lambda_i + \beta)^j t^{j-1} \mathrm{e}^{-(\lambda_i+\beta)t}}{(j-1)!} \times \frac{C_{ij}}{(\lambda_i + \beta)^j} \big[f_1(y) - f_2(y)\big] \\ &\quad + \sum_{i \geqslant 1} \sum_{j=1}^{k_i} \frac{\lambda_i^j t^{j-1} \mathrm{e}^{-\lambda_i t}}{(j-1)!} \times \frac{C_{ij}}{\lambda_i^j} f_2(y), \end{aligned}$$

where the constants C_{ij} are given in (4.48).

We apply Theorem 4.3.1 to obtain the left-hand side of (4.13) as

$$\Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Big[f\Big(\beta + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big] m_s(u).$$

Then, the right-hand side of (4.13) becomes

$$\begin{aligned} &\Big[f\Big(\delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Bigg[\sum_{i \geqslant 1} \sum_{j=1}^{k_i} \frac{f\big(\beta + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\big)}{\big(\lambda_i + \beta + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\big)^j} C_{ij}\Bigg]\big[\gamma_1(u) - \gamma_2(u)\big] \\ &\quad + \Big[f\Big(\beta + \delta - c\mathcal{D} - \tfrac{\sigma^2}{2}\mathcal{D}^2\Big)\Big]\Bigg[\sum_{i \geqslant 1} \sum_{j=1}^{k_i} \frac{f\big(\delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\big)}{\big(\lambda_i + \delta - c\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\big)^j} C_{ij}\Bigg]\gamma_2(u). \end{aligned}$$

An application of Theorem 4.4.1 now simplifies this right-hand side as required. The proof for $m_d(u)$ is analogous and is thus omitted. □

Chapter 5

Conclusions and future work

We have investigated a variety of extensions to the renewal risk model and made exciting new discoveries along the way. We summarize our findings as follows.

Chapter 2. Under a constant dividend barrier strategy, ruin is *certain* for a spectrally negative Lévy process. Ruin is also *certain* for a renewal risk process where either the claim would arrive within infinitesimal intervals or its size would exceed the dividend barrier. However, when these two conditions are violated, we show how ruin may be cleverly avoided. We propose a reinsurance contract and investigate the behavior of the likelihood of ruin through a simulation study. Numerical results provide preliminary conclusions on the effectiveness of this contract.

Chapter 3. Under rational arrival of the claim payments, we establish an integro-differential equation on a *bounded* domain and derive detailed boundary conditions for the functions of interest. The resulting solutions generally vanish at the boundary level in a non-smooth manner. This contrasts the smoothly vanishing solutions known in the literature.

Chapter 4. Under an Erlang-combination dependence structure and in the presence of diffusion perturbation, we generalize the existing diffusion-free model successfully and build a variety of elaborate integro-differential equations for the corresponding Gerber–Shiu functions. These results are achieved by a new set of general algebraic techniques. We also demonstrate a variety of applications and provide informative simplifications to emphasize the patterns in special cases.

Finally, we would like to point out potential future work.

- Consider the delayed exponential inter-claim times in Chapter 2. Can we obtain *explicit* solution for the probability of ruin? What is the sensitivity of this probability with respect to the delay? Is there a more effective reinsurance contract to implement our results for reducing such likelihood?
- In the presence of a diffusion perturbation, what is the distribution for the maximum surplus? With the addition of an Erlang-combination dependence structure, what do the new boundary conditions look like? Do we expect to observe similar non-smooth behavior at the boundaries?
- Is it possible to allow dependence between the diffusion and the aggregate-claim process, on top of the dependence between the inter-claim times and the claim sizes? What new techniques are required to handle this complicated extension?

These intriguing questions open endless possibilities for future research to come.

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