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A Groundwork for A Logic of Objects

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Abstract

The history of philosophy is rich with theories about objects; theories of object kinds, their nature, the status of their existence, etc. In recent years philosophical logicians have attempted to formalize some of these theories, yielding many fruitful results. This thesis intends to add to this tradition in philosophical logic by developing a second-order formal system that may serve as a groundwork for a multitude of theories of objects (e.g. concrete and abstract objects, impossible objects, fictional objects, and others). Through the addition of what we may call *sortal* quantifiers (i.e. quantifiers that bind individual variables ranging over objects of three unique sorts), a groundwork for a logic that captures concrete and non-concrete objects will be developed. We then extend this groundwork by the addition of a single new operator and the modal operators of a Priorian temporal logic. From this extension, our formal system can represent and define concrete, abstract, fictional, and impossible objects as well as formally axiomatize informal theories of them.

Keywords

Logic, formal-system, object, Meinong, Parsons, Zalta, extant, depictable, sentential, syntax, semantics, representation, sortal, sortal-quantifier, many-sorted-logic, free-logic, modal-logic, temporal-logic, axiomatization, function, structure, model, theory, deduction, completeness, soundness.

Summary for Lay Audiences

First order languages pick out individual objects with constant symbols (e.g. c, d, c_{10}, e) and variables (e.g. x, y, v_{10}, z). Constants *name* individual objects and variables are *assigned* to individual objects (where assigning some object to a variable is a similar process to determining what a pronoun like 'it' denotes in natural languages). The objects in our formal system are 'described' using predicate symbols (e.g. B, L, P_2, S). If a constant b names Bertrand Russell and the predicate P indicates a philosopher, then Pb is interpreted as, 'Bertrand Russell is a philosopher'. Now, what happens when we want our formal language to represent an object like a square circle? We could name the square circle s, and indicate squareness and circularity with the predicates S and C (respectively) and have Ss & Cs mean (roughly) 'the square circle is both square and circular', but notice that, as a matter of fact, square things are not circular and circular things are not square. The square circle is impossible for this reason. This also means that to adequately represent the square circle in our formal language it would be implied that Cs & $\neg Cs$ (where the '$\neg$' symbol is read 'it is not the case that'). This is to say that 'the square circle is circular and it's not the case that the square circle is circular'. This is a contradiction and it essentially ruins our formal language by ensuring that the logic of it can prove everything. It is the aim of this thesis project to develop a formal language and logic that can represent impossible objects like the square circle, but others too that are of interest to philosophers. By adding a few new symbols, we can save ourselves from contradiction and keep our logic useful. Ideally, all important kinds of objects will be representable in the proposed language as well as important statements about them. From this formal groundwork, theories of objects can be formalized.

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Table of Contents

Chapter 1

Introduction

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The history of philosophy is rich with theories about objects; theories of object kinds, their nature, the status of their existence, etc. In recent years philosophical logicians have attempted to formalize some of these theories, yielding many fruitful results. My thesis intends to add to this tradition in philosophical logic by developing a second-order logical system that may serve as a groundwork for a multitude of theories of objects (e.g. concrete and abstract objects, impossible objects, fictional objects, and others). Through the addition of what we may call *sortal* quantifiers (i.e. quantifiers that bind individual variables ranging over objects of three unique sorts), a groundwork for a logic that captures concrete and non-concrete objects will be developed. We will then extend this groundwork by the addition of a single novel operator and the modal operators of a Priorian temporal logic. From this extension, our formal system can represent and define concrete, abstract, fictional, and impossible objects as well as formally axiomatize informal theories of them.

From the first development (and application) of formal systems that could speak of *all*, *some*, and *a single* object, determining the *truth* of such talk was of the utmost importance. Because truth was of the utmost importance, an ability to deal with true statements about *fictional* objects was of utmost importance as, clearly, certain natural language statements speak of fictional entities, where, because formal logics translate many of these kinds of natural language statements, formal statements also speak of fictional entities. In considering these fictional entities, some philosophers - Gottlob Frege for instance - would treat any statement involving them as not subject to truth or falsity. Frege, instead, treated statements referencing fictional entities as sources of "aesthetic delight"[1]. For Frege, it didn't matter whether or not a name like 'Odysseus' referred to a real world

[1] Frege, 1892, pg 42.

entity, since Odysseus appears in a work of art. Bertrand Russell, a contemporary of Frege, had a different approach to fictional entities. Russell would treat names like ‘Odysseus’ (where it is important to note here that Russell treated all names this way, fictional or otherwise) as a shorthand for a grouping of descriptions that applied to the entity that was Odysseus[2]. These descriptions included, *the subject of Homer’s ‘Odyssey’*, *the son of Laertes*, *the person Socrates found wily*, etc. Since, should one speak of Odysseus they would be suggesting that there existed an entity that was *the subject of Homer’s ‘Odyssey’*, *the son of Laertes*, *the person Socrates found wily*, etc, and there was not an object that possessed these descriptive properties, Russell regarded such statements as false.

I find Frege and Russell’s solutions unsatisfying. Both thinkers are attempting to formally account for ordinary language sentences involving fictional entities without a formal account of the ordinary language semantics assumed in their use. In ordinary language conversations about fictional entities, sentences involving them are perfectly meaningful and are often considered straightforwardly true. Note though, that out of a spirit of pluralism, I only suggest that Frege and Russell’s solutions are unsatisfying. Frege’s solution is a solution that has certain applications, as is Russell’s (where Russell’s solution resulted in one of the more monumental of innovations in logic). I contend though, that a formal semantics that captures ordinary language semantics is also needed. That said, there have been very few developments in logic since Frege and Russell (save for some aspects of what are called ‘free logics’ in semantics and some syntactical innovations put forward by Edward Zalta and others) that have the ability to deal with the problem of fictional objects. Further, impossible objects (i.e. objects possessing self-contradictory properties) like the square circle, that are clearly of a different kind of object from the kind of object that Odysseus is, are not differentiated at all (and not given the proper formal treatment therefore). This lack of a more systematic delineation (in logic) of object *kinds* is another issue that this thesis intends to remedy. Although certain semantics deal with fictional and real entities, abstract and concrete entities,

[2] Russell, 1905a, pg 491.

etc., these entities are often grouped into one of only two nebulous categories[3]. More stratification and modality is required, I contend.

To gain a better understanding of the disconnect between formal semantics and ordinary language semantics, consider the aforementioned property of Odysseus, that being, *the person Socrates found wily*. To anyone who has read *Lesser Hippias*, it would seem clear that Socrates accepted this of Odysseus[4] and anyone considering what Socrates said, would take a statement like 'Socrates found Odysseus wily' to be true. However, on Frege's view we would not know how to deal with such a statement as Frege only makes clear that there is no truth or falsity in regard to statements referring to fictional entities. Frege fails to inform us of what to do in cases of fictional entities relating to non-fictional ones. With Russell the statement would be straightforwardly false as it would break down into a series of conjunctions one of which suggests there exists an object possessing all of the properties sufficient for being Odysseus. Since there is no such object, this conjunct would be false and, therefore, so would the conjunction. Standard free logics tell us that *if Odysseus exists, then he is wily (and Socrates believes this to be the case)* where certain other logics of fictional objects suggest that Odysseus bears different singular and relational properties (or, at least, bears them *differently*) than Socrates, meaning Odysseus is not in the class of objects that exemplify wily*ness*, but bears this property in some other way[5].

The above attempts at dealing with fictional objects do not track ordinary language discussions of these same objects and so, relative to ordinary language semantics, cannot be correct. The underlying issue is that these attempts maintain that either no such objects exist (in any sense) or they provide these objects with a qualified status of existence. Therefore, fictional objects are not in the domains or relevant relations of any semantic structure. This results in statements like 'Socrates found Odysseus wily' turning out false because Odysseus

[3] For example in Zalta, 1983 all object kinds are either concrete or abstract. Free logic semantics treat objects as either existing or not existing. Parsons, 1978 has more delineation (for example, possible/not-possible, complete/incomplete, and consistent/contradictory objects are informally defined), but, by Parsons' admission he does not violate the law of noncontradiction (Parsons, 75 – Footnote 21), and so a particular conception of *impossible* object is absent.
[4] Plato, 2016, 371e.
[5] *Cf.* Parsons and Zalta.

cannot feature in a tuple with Socrates at all, or cannot first be said to feature in the domain of discourse, and then be said to relate to Socrates in the relevant way. Intuitively, however, we all accept that we (us extant persons) can be related to non-extant objects (in intentional ways and not). I can listen to Mr. Smith filibuster and I can be made perplexed by an impossible object like the square circle. In both cases I may be listening to and made perplexed by things that do not exist in space and time, but I do not seem to be listening to *nothing*; nor do I appear to be made perplexed by *nothing*. Common sense allows us to understand (and ordinary language use captures) such relations, but formal languages do not (and I think they should).

To remedy this, I intend to complete a project that develops a logical notation and semantics that make more expressible our natural language statements as formal statements. My project involves both syntactic and semantic innovations of what are known as *many sorted logics*, *modal logics*, and the aforementioned *free logics*. My project requires that I go beyond what is allowable in modal logics as well as free logics. This is so, as these logics tend to only allow us to talk of things that may exist, where they are not fit to deal with things that can definitely not exist in any context but are spoken of by people all the same (i.e. impossible objects). I will, therefore, briefly elaborate on such innovations.

My syntax will include the addition of the above mentioned sortal operators to be used in quantifying objects that are fictional, impossible, abstract, etc. In the case of non-extant entities bearing non-contradictory properties, such objects will be quantified as existing in an 'initially-depictable-order' (i.e. they may be thought of, imagined, and fleshed out in fictional works, but do not exist in the physical world). In the case of entities bearing contradictory properties (like, *squareness* and *circleness* simultaneously), such objects will be quantified in an 'exclusively-sentential-order' (i.e. they are not able to be represented imagistically and do not exist in the physical world - hence are only able to be represented linguistically/descriptively). Impossible objects are not just exclusively sentential however. Traditionally, impossible objects are described as such because to completely describe them is to violate the law of non-contradiction. An additional unary operator that allows for the violating of this law will be included in the system's vocabulary therefore. The third sortal quantifier will bind variables ranging over objects that are concrete, in an 'extant-order'. Lastly, in later chapters, the system will be extended to include temporal operators representing points of time in the past and the future. All of these innovations will allow for non-extant objects to be spoken of and formulated in logical languages as well as modelled, but not lead to contradiction in derivation systems.

The semantics will be a *multi-domain, positive, free-logic semantics*. I call this semantics 'multi-domain' as, when interpreting the statements of my augmented language, the domain (*viz*. the set of objects that we are talking about) consists of an overarching domain (named 'the sentential domain'), a subset of that domain (named 'the domain of depictables) and finally, a subset of the domain of depictables (named 'the domain of extants'). The semantics involves 'free-logic' as I have included unique domains that account for names that do not denote extant objects. Finally, the logic is 'positive' as, in the case of sentences speaking of non-extant objects, these objects will be accounted for in the sentential domain (and possibly in the domain of depictables). Names that denote such elements refer to *something* then, and are rendered true or false on appropriate interpretations. So, in the case of a sentence like 'Homer Simpson is bald', on this semantics, we can determine such a statement true since the name 'Homer Simpson' denotes an element in the *domain of depictables*, and this element is in the extension of *people who are bald*.

Note then, that the above proposed logic allows for non-extant objects to exist but doesn't just assert their existence by fiat. There is the appropriate delineation required, and so we do not have Homer Simpson existing as an object the same way that, say, Justin Trudeau does. Further, the logic is neutral as to just what realizes something like a Homer Simpson or the square circle, and only commits to the idea that something functions to make us laugh or perplex us, but is amenable to categorization as a psychological construction perhaps, or a Platonic particular, or just a bunch of descriptions on the page, etc. I think a logic such as this is ideal then, in that it solves the above problems of philosophical logic, but doesn't imply any undue ontological constraints (some of which are themselves counter-intuitive).

The remainder of this thesis will progress according to the following structure, over five chapters. Our next chapter (chapter two) will present a minimal, but sufficient, informal theory of objects and object kinds, as well as an account of the logics that have attempted to represent these object kinds. Chapter three will provide the reader with an introduction to the object theory of Alexius Meinong, explain why Meinong's theory is relevant to this project, and detail some logics that capture Meinong's theory. In chapter four we will lay out the above mentioned groundwork (with the vocabulary required to represent impossible objects), and detail some of the system's more important theorems and applications. In chapter five we prove important meta-results regarding the groundwork. Lastly, in chapter six, we extend the groundwork to a temporal logic that allows for the representation of object kinds relevant in philosophy and then tie up some loose ends. We move now, to a discussion of object kinds.

Chapter 2

Object Kinds and Object Representation in Modal, Free, and Many-Sorted Logics

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§2.1. Objects and Object Kinds

In this brief chapter we provide an informal discussion of objects, some of the kinds of objects that there are, how various logical systems have represented these objects, and provide some critique of these representations. We start the discussion with the concept of *object*. By *object* we mean,

> *any entity o that exemplifies a property (or properties) and determines instantiations of those properties as those only exemplified by o (where no other instantiated properties are exemplified by o).*

It is assumed uncontroversial that properties are able to be exemplified, instantiated, and individuated. So, when properties are exemplified, instantiated, and individuated, something must serve the function of exemplification, instantiation, and individuation. That function is an object. Because talk of objects invariably involves the exemplification and instantiation of properties as well as their individuation, at the very least, a definition of *object* should include these concepts. We leave the informal definition of *object* as basic as the one above though, in order that we allow for a sufficient understanding of what is meant when we speak of objects, but not put any undue constraints on metaphysicians who may find our proposed logic useful. In fact, as an aside, an avoidance of undue metaphysical constraints and an openness to pluralism will be a running theme of this project.

By 'kind' we mean, the subsets of the set of objects determined by certain properties. By 'object kind', we mean the subset of objects determined by the modes in which these objects may exist. Lastly, by 'mode of existence', we mean any of the properties *exists in space and time*, *is depictable imagistically*, *is representable verbally*, *is not described in a way that violates the law of non-contradiction*, and *is causally efficacious*. So, if an object exists in space and time, it exists in that mode, and by that fact, is of a *concrete* kind. If an object is able to be represented imagistically and verbally, but doesn't exist in space and time, it exists in the former two modes and is of a *non-concrete* kind.

The object kinds that this thesis concerns itself with are those that appear perennially in philosophical discussions. They are concrete objects, non-concrete objects, abstract objects (often considered the counterparts of concrete objects), fictional objects, impossible objects, and vague objects. As seen above, concrete objects are those that are said to exist in space and time (where being in space and time is a mode of existence). That said, it is also implied above that objects need not exist in just one mode. In fact, it is assumed that objects in space and time are both depictable imagistically and representable verbally. Further, objects that are depictable are assumed to be representable verbally. Perhaps we can better convey the aforementioned dynamics formally. If we treat the sets E, D, and S as the sets of objects that *exist in space and time*, *are depictable imagistically*, and *are representable verbally*, respectively, our informal dynamics translates to the formal claim $E \subseteq D \wedge D \subseteq S$ (hence $E \subseteq S$). We could discuss in detail the dynamics that exist between the other object kinds, but that will be made clear in the chapters to follow. For now, it suffices to say that all objects of all kinds exist in some combination of modes E, D, and S, where all objects of all kinds exist, at least, in the mode represented by S. We will, from here on, refer to elements of E as *extant*, elements of D as *depictable*, and elements of S as *sentential*.

We will close this passage on object kinds by informally explicating each kind according to the above modes. Each definition is meant to capture ordinary language descriptions of objects of these kinds. Note however that this thesis does not treat the following definitions as canonical. Instead, we only seek to define our object terms in a way that captures their ordinary use. We do this out of a spirit of pluralism (a pluralism that suggests such definitions *ought* to be represented formally). We define each object kind thus

2.c. Concrete:	x is concrete *iff* x exists in space and time.
2.d. Non-concrete:	x is non-concrete *iff* it is not the case that x is concrete.
2.a. Abstract:	x is abstract *iff* x is non-concrete and not causally efficacious.
2.f. Fictional:	x is fictional *iff* x is non-concrete, not abstract, and has never been (nor will ever be) concrete.

2.i. Impossible:	x is impossible *iff* x is not depictable and to completely describe x would be to violate the law of non-contradiction.
2.v. Vague:	x is vague *iff* x is impossible due to being described as both identical to and not identical to itself.

The properties mentioned in the above definitions are often treated as essential to being an object of the respective kind. They are essential too, therefore, to philosophical theories of (or involving) such objects. But, it will be argued, these properties are not actually captured by many of the logics that intend (or at least purport) to represent objects of these kinds. It would be of use therefore, to have access to a logic that both formally captures the above definitions and is able to formally represent objects of the kinds defined above.

§2.2. Representing Object Kinds Formally

The remainder of this chapter will consist of a survey of each object kind and the formal systems that capture them. How these objects are represented formally, how they are modelled, and some critique of these maneuvers will be our focus. We start with concrete objects, where our discussion of them will be brief. It is assumed here that classical predicate logics represent concrete objects adequately, and therefore so do any extensions of these logics. If to be the variable x, bound by the quantifier $\exists x$, is to be (tacitly) concrete, then as long as the domains of any relevant structures accommodate this assumption, we do not run into any issues of representation. Should a statement like,

$$\exists x(x = t)$$

mean to convey the idea that t is concrete, then nothing further is required by way of formalism. Further, formulas featuring concrete objects can be reasoned with deductively without contradiction, as they are just formal representations of objects in the physical world, translated into standard first order languages. Should contradictions arise in derivation here, it would be an error of proper representation, or possibly the informal theory is itself inconsistent, but it would not be a shortcoming of the formal system *per se*.

Where we would run into problems would be in attempting to represent objects of other kinds in these systems. All objects would identify with some concrete object, after all, leaving all objects concrete. To represent an object t as non-concrete would result in,

$$\neg\exists x(x = t)$$

where, presumably, we want t to exist as an object of some kind but, on standard semantics, this implies,

$$\exists x(x = t)$$

which is a contradiction. Classical first order predicate logics can represent objects of different kinds but, as we have shown, they cannot do so through existentialization. The concept of *sort* would need to be introduced, either by extending the logic to a many-sorted one, or through the addition of more complexity to each formula (*viz*. by the introduction of additional predicates indicating the intended sort). We will move to a discussion of these innovations then, in order that we may show that concreteness and non-concreteness can be represented.

Since many-sorted logics can be translated into first-order predicate logics, we will not discuss the latter but only the former. After all, what can be said in a many-sorted logic can be translated into first-order logic[6]. So, for ease of discussion we will use the language of many-sorted logic and we will assume that the more streamlined representation (from the more complex syntax and semantics) of many-sorted logics generate formulas and interpretations that have counterparts in standard first-order logics (where these counterparts are what would be presented should we attempt to capture the remaining object kinds in first-order logic). With a many sorted logic (abbreviated 'MSL') we can easily delineate concrete objects from non-concrete. We start by adding to a first order logic a set S whose elements are sorts (in our case let $S = \{c, d\}$ where c indicates the *concrete* sort and d indicates the *non-concrete* sort). Next, we add (in place of individual variables: v_0, v_1, ...) a stock of individual variables for each sort, *i.e.* $v^c{}_0$, $v^c{}_1$, ... ; $v^d{}_0$, $v^d{}_1$, ... , where variables $v^c{}_i$ indicate concrete objects and variables $v^d{}_i$ indicate non-concrete objects. In place of constants c_0, c_1, ... we add constants for each sort, *i.e.* $c^c{}_0$, $c^c{}_1$, ... ; $c^d{}_0$,

[6] A proof of this result can be found in Bell, DeVidi, and Solomon p. 119-120.

$c^d{}_1$, ... , where constants $c^c{}_i$ indicate concrete objects and constants $c^d{}_i$ indicate non-concrete objects. Lastly, to each of our predicate symbols P_0, P_1, ... , we add a signature $\langle s_1,...,s_n\rangle$ of sorts $s \in S$, that indicate for each predicate P_i (of some arity n) the sorts of any terms $t_1,\dots,t_n$, where $P_i t_1,\dots,t_n$ (in other words, $t_1,\dots,t_n$ are of the sorts $s_1,...,s_n$ respectively). We model sentences in our many sorted logic according to structures containing a domain for each sort (in our case a domain of concrete objects and a domain of non-concrete objects), where denotations, assignments, and tuples must be of the sort respective to the constants, variables, and predicates that they interpret.

Note that with the above MSL, we no longer treat non-concreteness as definable from concreteness, but instead as a primitive concept in its own right. However, if our sorts are disjoint (which, by 2.c. and 2.d., they should be), we can still capture this definability property with the following axiom (where x and y are variables of the relevant sort),

$$\forall x^d \neg \exists y^c (x^d = y^c)$$

but it isn't essential that we do this. What is essential however, is that we be able to represent the remaining object kinds in MSL, either by defining them according to the sorts that we already have, or by adding new sorts. It is likely the case that we will need additional sorts. For instance, if we added a sort e for *causally efficacious objects* and a sort t for *points in time*, we could define abstract and fictional objects according to 2.a. and 2.f. respectively. According to the following

Abstract(t^d) abbreviates $\neg\exists x^e(x^e = t^d)$

viz. the non-concrete object is not a causally efficacious object, hence abstract.

Fictional(u^d) abbreviates $\neg$Abstract(u^d) $\wedge \neg\exists z^t E u^d z^t$ (where 'Eu^dz^t' is read, u^d *was extant at time* z^t)

viz. the non-concrete object is not abstract and there is no point in time that the non-concrete object is extant. Hence, the non-concrete object is fictional.

However, augmenting our MSL in this way would only be of use if, ultimately, we could define each and every one of our object kinds. But, it should be clear at this point that if we are to represent impossible objects (formally) as violating the law of non-contradiction, then no additional sort is needed, as impossible objects are represented as some t where for some predicate P,

$$Pt \wedge \neg Pt$$

Simply adding to S a sort i for impossible objects would not just be superfluous, it would be counterintuitive (should there not be some additional feature of t that makes t self-contradictory). If we reason according to a consistent t, isn't t possible? Regardless of our intuitions, it would certainly be the case that our terms $t^i{}_0, \ldots, t^i{}_n$ would just be a subset of our terms $t^d{}_0, \ldots, t^d{}_n$ as, syntactically, they function in the exact same way. But, we do not necessarily want to declare impossible objects to be concrete or nonconcrete. However, if for each t^i, it would not be the case that for some predicate P

$$Pt^i \wedge \neg Pt^i$$

then t^i is just another term that is non-concrete. Alternatively, if each t^i actually violated the law of non-contradiction, the deduction system of our MSL would reduce theories involving impossible objects to absurdity. Since we are not intending our MSL to be paraconsistent, we want to avoid absurdity.

Clearly, impossible objects pose a problem for MSL. However, there are other logical systems that may be able to represent impossible objects. The concept of *possibility* is made explicit in modal logics (and by extension, *impossibility*), and free logics are those systems that are capable of modelling statements involving names that do not refer to any object (might the names of impossible objects just be non-denoting terms?). Perhaps modal or free logics might be of use, and so we will investigate them in the subsequent passages. We will start with modal logics.

Consider the modal logics that employ the possibility operator '$\Diamond$' and the necessity operator '$\Box$', and model statements featuring them according to a relation R on possible worlds $w \in W$. Since the informal concepts of possibility and necessity are explicitly captured in these systems, they represent impossible objects readily enough. Consider the square circle, where the predicate S indicates square*ness* and the predicate C indicates circle*ness*. We can represent the impossibility of the object $\iota x(Sx \wedge Cx)$ (*i.e.* the object x that is *square* and *circular*) thus

$$\neg\Diamond\exists y(y = [\iota x(Sx \wedge Cx)])$$

However, if we wanted to say that the square circle is an object of philosophical interest (which seems a patent truth) we would represent it thus (let *is of philosophical interest* be represented by the predicate 'P'),

$$P[\iota x(Sx \wedge Cx)]$$

But, with some logic we may derive

$$\exists y(y = [\iota x(Sx \wedge Cx)])$$

and so, by the standard axioms of modal logics stronger than K, we may derive

$$\Diamond\exists y(y = [\iota x(Sx \wedge Cx)])$$

which straightforwardly contradicts our account of the square circle as impossible. It seems we can either represent the square circle as impossible (but not talk about it) or talk about the square circle (but not as impossible). Notice too, that the square circle is impossible on this account not because $\iota x(Sx \wedge Cx)$ violates the law of noncontradiction but because there are no worlds $w \in W$, where $\exists y(y = [\iota x(Sx \wedge Cx)])$ obtains. But, if we were to amend our definition of the square circle in order to capture its inconsistent features, we would reduce our theory of the square circle to absurdity. In fact, because violating the law of noncontradiction is required, the standard definition of *impossible* cannot be met at all in modal logics. Straightforwardly, if we treated the square circle as $\iota x[(Sx \wedge \neg Sx) \wedge (Cx \wedge \neg Cx)]$ (thereby violating the law of noncontradiction), it would be implied that

$$\exists x[(Sx \wedge \neg Sx) \wedge (Cx \wedge \neg Cx)]$$

which obviously leads to contradiction and cannot be modelled relative to any possible world. And, this is the case with any theory involving impossible objects. From this we see that modal logics are not equipped to deal with impossible objects should we want to reason beyond just the impossibility of their existence or define them according to the standard definition of impossible. Perhaps free logics can achieve this?

Call a first order logic *free* (abbreviated 'FFOL') if among the elements of its vocabulary there is a predicate E! that indicates *existence*, and the structures of its semantics are defined in such a way that they include a (possibly empty) domain M and a (non-empty) overarching domain M' where $M \subseteq M'$. Consider a free logic to be *positive* (abbreviated 'PFFOL') if a term t that does not denote/is-assigned-to an element of M, may nevertheless result in a true formula featuring t if t denotes/is-assigned-to an element in M'. Since we are only interested in logics that may model sentences involving both extant and non-extant terms, we will focus just on PFFOL systems. One last thing to note, for any term t, E!t is true on a PFFOL structure *if and only if* the element $b \in M'$ that is the denotation/assignment of t is also a member of M. With PFFOL logics, sentences like the quasi-informal

$\exists x$(Sherlock-Holmes(x) $\wedge$ Plays-The-Violin(x))

would be true on a PFFOL structure where the object $b \in M'$ assigned to x is in the relation reserved just for Sherlock Holmes and in the relation of elements that play the violin. However, a statement like

$\exists x$(E!x $\wedge$ Sherlock-Holmes(x) $\wedge$ Plays-The-Violin(x))

would be false as, because Sherlock Holmes is fictional, it would not be the case that (for the element $b \in M'$ that is the denotation of x) $b \in M$.

From this it should be clear that PFFOL systems are more than capable of representing concrete and non-concrete objects (let E! indicate not just existence, but concrete existence). Further, it should be clear that if we were to extend PFFOL to a many-sorted logic with the right sorts, we could formally delineate abstract objects from fictional. However, there doesn't appear to be anything in the additional vocabulary and semantics of PFFOL systems that would allow us to represent theories involving impossible objects consistently (or model them). Again, if the law of non-contradiction is to be violated, then it does not matter whether say, for the element that is the square circle (call it 's'), $s \in M'$ but $s \notin M$, as it is impossible for both

$s \in \{x : x \text{ is square}\}$

and

$s \notin \{x : x \text{ is square}\}$

As well, we're not saved from absurdity just because the square circle is said to not exist, *viz*.

$\neg!Es \wedge (\text{Square}(s) \wedge \neg\text{Square}(s)) \wedge (\text{Circular}(s) \wedge \neg\text{Circular}(s))$ **

Since because we may still derive either

$\text{Square}(s) \wedge \neg\text{Square}(s)$

or

$\text{Circular}(s) \wedge \neg\text{Circular}(s)$

from **, any theory featuring ** reduces to absurdity. Alternatively, any theory representing the square circle, that doesn't define the square circle as self-contradictory is representing the square circle as a possible object therefore, hence is not actually representing the square circle. So it goes for any impossible object represented consistently, hence PFFOL (whether it be extended to a many sorted logic or not) cannot represent impossible objects as impossible.

It follows from what was argued above, that of all the likely candidates for capturing a logic of objects that both carefully delineates these objects and adequately represents them, none of these candidates were able to do this without additional formalism. And, even if they were, the multiple extensions and qualifications required would likely make these systems so cumbersome to work with, that they would not be of any practical use. However, in the end, it was the impossible object (and by implication the vague object) that was not able to be represented no matter how we augmented the systems. No introduction of new operators, predicates, sorts, etc. enabled impossible objects to be represented. However, this just means we require a formalism beyond what is already available, that is, a kind which allows for the representation of impossible objects, in which sentences involving them can be modeled and the system of deduction not rendered inconsistent. Such logics do in fact exist. These logics are logics referred to as 'Meinongian', named after Austrian philosopher Alexius Meinong. They attempt to capture Meinong's ontology, which included the existence of impossible objects like the square circle. We move now to the next chapter in which we discuss Meinong's ontology and two systems of logic that capture it.

Chapter 3

Alexius Meinong and The Meinongian Logics of Terence Parsons and Edward Zalta

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§3.1. The Object Theory of Alexius Meinong

I imagine that the mention of a logic of objects that makes a special distinction between fictional and impossible objects (as well as grants them the same status of existence assumed of concrete objects), evokes in the metaphysician, Alexius Meinong's theory of objects. Naturally, the question then arises, *is the proposed logic of this thesis a Meinongian one*? Not to get too far ahead of ourselves but the answer to that question is 'no', for reasons that will be made clear in what follows. However, before explaining why the proposed logic is non-Meinongian, some discussion of Alexius Meinong and his work is required, as is some discussion of the nature of Meinongian logic. This chapter then, will briefly outline the key facets of Meinong's theory of objects, describe the similarities and differences Meinong's theory bears to our own, and end by discussing the Meinongian logics of Terence Parsons and Edward Zalta.

Alexius Meinong's main contributions to theories of objects was his conception of *being* and the ways in which different kinds of objects have being. *Being*, for Meinong, is less a synonym for *existence*, and more a term indicating an object that is either of the physical world or abstract. All objects to Meinong had what was called *Außersien* (roughly, 'outside being'), but not all objects had *being*[7]. For example, my (physical) laptop sitting on my desk displaying an incomplete thesis has *being*, the (abstract) state of affairs in which my laptop sits on my desk displaying an incomplete thesis has *being*, but (my imagined) scenario in which my laptop sits on my desk displaying a complete thesis does *not* have *being* (just *outside being*). Meinong further delineates objects into

[7] Jacquette, 2015, see Preface and Chs. 4 and 5. As well, Marek, 2019 features a helpful table of object kinds on pg. 32 of the PDF version of the article.

complete and incomplete categories[8]. An object is complete if, for any possible property ***p***, it is able to be determined whether (or not) the object has property ***p***. An object is incomplete if it is *not* complete.

In regard to the ways in which objects might *be* (in addition to their having outside being), Meinong's key insight came to him while studying intentionality of mind, that is, the fact that human thought is *about* things. We can think about something like a golden mountain, thought Meinong, yet a golden mountain doesn't actually have being. However, since we cannot direct our attention towards that which doesn't exist, but we direct our attention towards the golden mountain, the golden mountain must exist, if only in a different manner as that of, say, Mount Everest. To Meinong, a mountain like Everest had *existence*, that is, it existed in space and time, but the golden mountain did not. Instead, the golden mountain *subsisted*, that is, it had outside-being and was consistent in terms of its properties, but it did not have being in a concrete or temporal context, only as an object of thought. Lastly, as was already mentioned objects like the square circle *only* have outside being, as they are not consistent in terms of their properties (hence do not subsist). That is, they are objects, but they are neither concrete, nor are they the object of our thoughts (where Meinong seems to treat such objects as necessarily phenomenological). Although all objects have outside being, only a smaller subset have outside being *only*. Something like the square circle cannot exist in space and time, nor be imagined, hence it cannot be an object of thought and has outside being only. Something like a number cannot exist in space and time, so it *subsists*, but a number is not merely an object of thought, so it has *being*. It's quite a complicated theory, and I've only really scratched the surface. But, from these concepts and one further principle of Meinongian object theory, we may proceed to our discussion of logics that capture Meinongian object theory, and therefore impossible objects.

Meinong's theory allows for any combination of properties to be exemplified by an object. For example, the single property *blue* permits of an object that is blue simpliciter, and the contradictory pair of properties *blue all over* and *green all over* permit an object that is simultaneously blue all over and green all over simpliciter.

[8] Jacquette, 2015, pg. 15.

Additionally, his theory permits the outside-being of any conceivable object. Most Meinongian theorists (and detractors) agree that these two theorems stem from the more fundamental (but unstated) principle

> for any possible combination of describable properties, there exists an object that exemplifies all and only those properties[9].

Call this the naïve object theory (abbreviated 'NOT'). For Meinongians, some approximation of NOT is essential to any theory of objects called 'Meinongian'.

The principle is not without problems however. For one, it conflicts with a quasi-tacit principle that our proposed logic abides by. And, now would be as good a time as any to present it. The principle is as follows,

> an object in which a property ***p*** is exemplified is describable as 'possessing property ***p***', but it isn't always the case that an object describable as 'possessing property ***p***', has property ***p*** exemplified by it.

which is incompatible with NOT. For example, our logic doesn't assume that the object described as square and circular actually has these properties (in fact, it is assumed that the object does not). This makes our proposed logic a non-Meinongian one. Of course, our logic's incompatibility with NOT only means an external inconsistency (maybe it is our principle that is flawed?). The bigger issue with NOT (believe many) is that it generates a theory of objects that is *internally* inconsistent. We will consider one such argument briefly. Since Bertrand Russell's repudiation of NOT is both brief and considered most definitive, we will discuss it.

Russell, an admitted admirer of some of Meinong's work (but detractor in regard to Meinong's object theory), was highly critical of NOT and famously, in 'On Denoting', argued that regardless of Meinong's distinguishing between *existence* and *subsistence* in consistent objects, NOT still implied an object in Meinong's ontology that possessed the properties gold*ness*, mountain*ness*, and exist*ness*. The object must then, by NOT, exist. But, because the object had no concrete being, it did not exist as well. This is a clear violation of the law of non-contradiction (and intolerable) thought Russell[10]. However, Russell need not have gone to the lengths he did

[9] Zalta, 1983, pg. 6.
[10] Russell, 1905a, pg. 483.

to prove that contradiction. He had already acknowledged the outside-being of the square circle, an object that violates the law of non-contradiction in and of itself. Regardless, Russell was a little hard on Meinong. Meinong's theory possessed enough room for nuance that there were ways of extending it to a consistent account of objects (impossible objects included). And, many Meinongians have formulated such theories. We close this chapter with a discussion of two of them.

§3.2. The Meinongian Logics of Terence Parsons and Edward Zalta

As mentioned, the logical system to be established in this thesis is non-Meinongian. But, that doesn't mean that it isn't Meinong inspired. Our system captures the concepts of *existence*, *subsistence*, and *outside-being* (essential to Meinongian theories of objects) but differs from Meinongian logics in that it does not satisfy the *naïve object theory* principle ('NOT'). Meinongian logics then, are those logical systems that satisfy NOT as a matter of necessity, where our system does not. The remaining passages of this chapter will deal predominantly with the Meinongian logics of Terence Parsons and Edward Zalta. Parson's and Zalta's logics both capture the most current developments in Meinongian logics (and logics of objects in general) as well as parallel, in important ways, the system I'm proposing. Some comparison and commentary would therefore be useful, in order to indicate both sources of inspiration and essential departures. I will start then with a survey of Parson's and Zalta's logics and then provide the commentary.

As stated, Parsons and Zalta's logics are Meinongian, and so must satisfy the principle NOT. Because of a need to satisfy NOT, for Parsons and Zalta, an impossible object like the square circle, must be an object that in fact has the properties square-*ness* and circle-*ness*. This leads to contradiction of course as a square object is non-circular and a circular object is non-square. A square circle is impossible for this reason, but more importantly, in derivation systems, the square circle (adequately defined) leads to explosion (i.e. anything can be proven) and sentences featuring the square circle cannot be modelled (the object would have to both be a member and not be a member of the set of square things and the set of circular things). These properties are problematic for any logician. Parsons and Zalta solve the problems above by creating second order systems capable of dealing with non-extant and impossible objects.

The main innovation of Parsons' system is its unique conception of *object* as well as *kinds* of properties. Parsons divides properties into two distinct kinds, *nuclear* and *extranuclear*. Informally, nuclear properties are those that constitute the object and extra-nuclear properties are those that are exemplified by the object but are not essential to it[11]. For example, the properties of square-*ness* and circle-*ness* are essential to the square circle, hence constitute it and are nuclear properties, but the property *is thought about by Parsons* is not essential, hence extranuclear. Formally, a nuclear property p (represented by the predicate P^n) is a function that maps a possible world w to a set of individuals of that world that have property p[12]. Objects (to Parsons) are not individuals but are sets of nuclear properties[13]. For example, the square circle is not an individual in some world in which square-*ness* and circle-*ness* are exemplified by it, but the set {square-*ness*, circle-*ness*}. An object o has a nuclear property p just in case $p \in o$. An extra-nuclear property P (represented by the predicate P^e) is a function that maps a possible world w onto subsets of the set of objects O[14]. An object o has an extranuclear property P just in case $o \in P(w)$. Lastly, an impossible object o is one where, in no world w is there an individual i whose nuclear properties are such that $o \subseteq i^c$ (where i^c represents the set of i's nuclear properties)[15].

As for Zalta's logic, the main innovation lies in his formalization of different *ways* in which objects bear properties. Zalta's system retains the concepts of *nuclear* and *extranuclear* (as well as the informal definitions of them). However, he does not represent these concepts as *kinds* of properties, but instead as *ways* in which objects

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[11] Parsons, 1975, pg 569.

[12] Parsons, 1978, pg 138.

[13] Parsons, 1978, pg 139.

[14] Parsons, 1978, pg 141.

[15] Parsons, 1978, pg 140.

bear their properties. In Zalta's system, an object doesn't bear a nuclear property *p* in the standard sense, it *encodes* *p* but it does bear extranuclear properties *P* in the standard way, i.e. it *exemplifies P*[16]. Formally, concrete objects *exemplify* all of their properties (represented symbolically in the standard way, i.e. *Px*) whereas non-extant (for Zalta, 'abstract') objects *encode* at least one property (represented symbolically as *xP*)[17].

How these unique systems represent impossible objects but get around the above mentioned problems of explosion and satisfiability is explained thus. Let S = *is square*, C = *is circular*, the iota symbol ι = *the object ___ such that*, and the lambda symbol λ = *the property of object(s) ___ such that*[18]. With these symbols, on Parsons' logic, we may define the square circle as

$\iota x(S^n x \wedge C^n x)$ (abbreviated *ISP*)[19]

Informally: the object that has the nuclear properties of square-*ness* and circle-*ness*.

Here we need not worry about contradictions arising from the fact that, in regard to individuals, a square object *x* implies *x* is non-circular and a circular object *x* implies *x* is non-square, as the square circle is a set of nuclear properties, not an individual. If the previous implications applied to objects, they would determine the square circle to be {square-*ness*, circle-*ness*, *not*-circle-*ness*, *not*-square-*ness*} (which is not the object that is the square circle, by definition). The assignment of *x* is just the object, {square-*ness*, circle-*ness*}. However, Parsons' system does include objects like {square-*ness*, circle-*ness*, *not*-circle-*ness*, *not*-square-*ness*} where, instead of categorizing these objects as *impossible*, he defines them as *contradictory*[20]. We may formally define this object as follows (where $\overline{P}^n$ indicates a negated nuclear property but isn't equivalent to $\neg P^n$)

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[16] Zalta, 1983, pg 12.

[17] Zalta, 1983, pg 18.

[18] Zalta, 1983, pg 18.

[19] Note that the term forming iota (*viz*. the definite description operator) is not used in Parsons 1975 or Parsons 1978. However, I think this use is permissible here as the operator allows for greater clarity (we see clearly that impossible objects like the square circle are composed of consistent properties) and it does not alter the nature of Parsons' logic in any way.

$\iota x(S^n x \wedge C^n x \wedge \bar{S}^n x \wedge \bar{C}^n x)$ (abbreviated *CSP*)

Informally: the object that has the nuclear properties of square-*ness* and circle-*ness* and has the negated nuclear properties of square-*ness* and circle-*ness*.

Here, we have no formal contradiction either, as there is no P^n such that from *CSP* we can derive $P^n x \wedge \neg P^n x$ (although, this is because a proviso like $\forall \bar{P}^n \forall P^n \forall x(\bar{P}^n x \leftrightarrow \neg P^n x)$ is not a tautology in Parson's system).

Zalta's system does have a proviso similar to $\forall \bar{P}^n \forall P^n \forall x(\bar{P}^n x \leftrightarrow \neg P^n x)$ as a tautology formulable in it (i.e. his proof theory includes the schema '$\forall x_1, \ldots, \forall x_n([\lambda v_1, \ldots, v_n,\ \boldsymbol{p}]x_1, \ldots, x_n \leftrightarrow \boldsymbol{p}(v_1, \ldots, v_n/x_1, \ldots, x_n)$[21]' that may take the form '$\forall x([\lambda v, \neg Pv]x \leftrightarrow \neg Px$'). But, as will be shown, his system avoids inconsistency due to its novel definition of predication. In Zalta's system, we may define the square circle thus

$\iota x(xS \wedge xC)$ (abbreviated *ISZ*)

Informally: the object that encodes the properties of square-*ness* and circle-*ness*.

As we can see, *ISZ* encodes square-*ness* and circle-*ness* but does not exemplify either. Granted, the square circle does exemplify *not*-square-*ness* and *not*-circle-*ness*, but to represent it as including these properties would still yield the consistent object

$\iota x(xS \wedge xC \wedge [\lambda y, \neg Sy]x \wedge [\lambda y, \neg Cy]x)$ (abbreviated *CSZ*)

Informally: the object that encodes the properties square-*ness* and circle-*ness* and exemplifies the properties *not*-square-*ness* and *not*-circle-*ness*.

Here the negated properties of the square circle do not imply a formal contradiction since (with some omitted derivation) neither $xS \wedge \neg Sx$ nor $xC \wedge \neg Cx$ are of the form $Px \wedge \neg Px$ or $xP \wedge \neg xP$.

Before ending this survey, it will be helpful to give a sketch of how sentences involving impossible and contradictory objects may be modelled *per* Parsons' and Zalta's systems. On these systems statements like

$\exists x(S^n x \wedge C^n x \wedge \bar{S}^n x \wedge \bar{C}^n x)$

Informally: there is an object that has the nuclear properties of square-*ness* and circle-*ness* and has the negated nuclear properties of square-*ness* and circle-*ness*.

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[20] Parsons, 1978, pg 140.

[21] Zalta, 1983, pg 28.

and

$\exists x(xS \wedge xC \wedge [\lambda y, \neg Sy]x \wedge [\lambda y, \neg Cy]x)$

Informally: there is an object that encodes the properties of *square-ness* and *circle-ness* and exemplifies the properties *not-square-ness* and *not-circle-ness*.

are satisfiable just in case (*per* Parsons) x is assignable to an object that has square-*ness*, circle-*ness*, negated square-*ness*, and negated circle-*ness* as members of it[22] and (*per* Zalta) x is assignable to an element that is in the set of encoded square and circular things and not in the set of exemplified square and circular things[23]. Since the domain of Parsons' structure may contain the element named by *CSP* and, on Zalta's structure, the element named by *CSZ* may be in the relation interpreting the encoded S (and not in the relation interpreting the exemplified S), both cases of satisfaction are possible in the respective systems. Now for the commentary

In terms of any criticism I might have, the best I can do is say, I find Parson's and Zalta's solutions to defining and modelling problematic object kinds, *slightly* unsatisfying. Otherwise, it must be noted that their respective systems are ingenious, novel, and most importantly, applicable to many theories of metaphysics. Although I would have liked to have seen more delineation of object kinds (Zalta treats all objects as either concrete or abstract for instance), the main issue I have is that *impossibility* on these logics is, once again, defined in a non-ordinary manner. That is, the main criterion for ordinary conceptions of impossibility (i.e. *the object, sufficiently described, leads to the violating of the law of non-contradiction*) is not met. The criterion is not met as there is no object x where for some property P, Px and $\neg Px$.

Now, this isn't to say that Parsons and Zalta's logics fail to do what Parsons and Zalta intended them to do. It is just that the views of impossible objects that they proffer do not capture the standard view of impossibility.

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[22] Parsons, 1975, pg 570.

[23] Zalta, 1983, pg 27.

I think a program in the logic of objects that delineates objects according to a greater variety of kinds as well as meets the criterion for impossibility mentioned above, while avoiding explosion and allowing for all informally true sentences to be modelled, is *also* a fruitful one. I think the reason why the impossibility criterion is difficult to meet is that logicians are trying to satisfy Meinong's NOT principle and so, as mentioned, we will not seek to do this. I should make clear however that I am not saying that objects like the square circle do not actually have these properties, but I do want my logic to remain neutral in this regard. Impossible objects are simply those things that, according to our linguistic practice and thought processes, can only be talked about but, because they are contradictory when described, cannot be depicted. We now move to our chapter in which such concepts are formalized.

Chapter 4

A Groundwork for A Logic of Objects (With Impossibility)

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Here I will present the proposed system, starting with the new operators. I will explain that certain of the operators (the sortal quantifiers) are interdefinable and provide further instructions for determining recursively what constitutes a formula using these new operators. I will also provide some useful abbreviations (most of which are common logical symbols often treated as primitive in other systems). Then, as is customary, I will present the semantics and deduction theory of the groundwork, ending the chapter with demonstrations of derivation and modelling. At various places the presentation of the system will be interrupted by some commentary on the philosophical aspects of the system itself. This is necessary but might make it difficult to get an overall picture of the system. For this reason, I will include just the presentation of the system, its syntax and semantics, and its deduction theory as an appendix at the end of the chapter.

Being as straightforward as it is, the groundwork itself will be presented along with the symbols required to define impossible objects. The simultaneous laying out of the groundwork and its partial application to a theory of impossible objects will both demonstrate the expressiveness of the system and save us from having to lay out the entire system all over again (just for the sake of a single new operator, definition, and abbreviation). The groundwork, then, is built from the vocabulary below, save for the operator '$\ulcorner\,\urcorner$' (remove $\ulcorner\,\urcorner$ as well as the rules for defining formulas whose main operator is $\ulcorner\,\urcorner$, and you have the groundwork vocabulary and syntax proper).

The groundwork itself can account for concrete and non-concrete objects. In the case of non-concrete entities bearing non-contradictory properties, such objects will be quantified as existing in a 'depictable-non-extant-order' - *i.e.* their properties may be imagined or depicted imagistically, according to some approximation (of sufficient accuracy). In the case of entities bearing incompatible properties (like, *squareness* and *circularness*), such objects cannot have properties like *squareness* and *circularness* approximated in *any sense* and so will be quantified in an 'exclusively sentential-order' (i.e. they are only able to be described linguistically/descriptively). It is important to understand however, that at this point, an object like the square circle, although informally impossible, is not formally impossible on the system. Being an object that cannot have its relevant properties depicted imagistically, it is only said to be *exclusively sentential*. To be impossible is to violate the law of non-contradiction, i.e. to have it be said of objects like the square circle ('s'), that there is some predicate P where $Ps \wedge \neg Ps$. Impossible objects, in general, cannot be formally defined as such with just the groundwork.

Lastly, a third quantifier will range over objects that are concrete, in an 'extant-order'. Note that, on this theory, objects that are extant are also depictable and sentential, and objects that are depictable are also sentential. These quantifiers will allow for non-concrete objects to be spoken of and represented in logical languages, but not lead to contradiction in derivation systems. Let '$\mathcal{GL}$' stand for the language of the groundwork and let '$\mathcal{IL}$' stand for the language of the groundwork capturing impossibility.

Considering its nebulous nature, a note on the informal concept of *depictability* seems to be in order. We will provide that here before moving on to our presentation of $\mathcal{IL}$. First, it is assumed that the concepts *extant* and *sentential* are easily enough understood and uncontroversial in their treatment as ways that objects may exist. To be extant is to just exist in space and time, at the current time (which implies concreteness). And, to be sentential is just to exist as an object that can be described verbally. It is assumed that all objects may be described verbally (more or less truthfully), hence all objects are sentential at least. *Depictable* is a different way of being, however. Perhaps the nature of *depictability* is intuitively clear to certain readers, but perhaps not for others. To elucidate then, consider Russell's account of *accuracy*, from 'Vagueness' - that is, the various degrees to which a representation approaches an isomorphism with its referent.

Russell points out that representations like "maps, charts, photographs, and catalogues" bear much more properties in common with their referents than linguistic representations of the same referents do[24]. A map of

[24] Russell, 1905b, pg. 89.

Hawaii, for example, has much more in common with the land mass that is Hawaii than the word ‘Hawaii’ does. The map shares the same outline, area (to scale), and orientation of sub regions (i.e. a region like Maui is south of a region like O’ahu on both Earth and the map), among other features. On the other hand, the verbal token ‘Hawaii’ has much fewer properties in common with the actual land mass. But, it isn’t necessarily the case that the token has no properties in common with the land mass. In fact, it is highly likely that they share some properties (for example, it could be that the dot of the first ‘i’ in the token shares the same shape as a small island in the region). We may conclude then, that most representations of objects share some property in common with the object represented. From these notions, we will define the concept of *depictability*.

Consider first that descriptions pick out properties of objects. The description ‘is square’ is meant to describe the purported square*ness* property of the relevant object, for instance. Descriptions map to properties in an informal sense then. With this in mind, we define ‘depictable’ as follows.

> An object is *depictable*, we will say, *iff* each subset of the set of descriptions true of it, when mapped to the relevant propert(ies), can have (in theory) all of the properties of any one of those subsets exemplified by at least one (not necessarily common) representation of the object.

Mount Everest can be imagined, painted, constructed to scale via computer modelling, and depicted according to a multitude of other types of representations, all of which possess properties in common with the mountain itself. Consider that even the token ‘Mount Everest’ could have some property of it map to a property of Mount Everest itself (like, for example, an angle measurement in the letter ‘M’ being the same as an angle measurement of the southernmost precipice of Everest). Here the singleton {*has features at an angle of n°*} contains a property mapped from (on this hypothesis) a true description of Mount Everest and has all of the properties that are members of it exemplified by *both* the representation ‘Mount Everest’ and Mount Everest itself. Mount Everest (in principle) can have each subset of described properties (where the descriptions in question are true of Everest) feature in a verbal or imagistic representation of it. Mount Everest is accordingly depictable.

An object like the square circle is not depictable however. The descriptions ‘is square’ and ‘is circular’ are true of the square circle and so feature in a subset of its set of descriptions. However, the properties of square*ness* and circle*ness* cannot both feature in any representation of the square circle. For example, any property of the token 'the square circle' that involves an arc negates square*ness* and any property of the token that involves a straight line or angle negates circle*ness*. That exhausts the types of features of the token 'the square circle' (or any textual representation of the square circle) that can be treated as square or circular. Additionally, no utterance of 'the square circle' is either square or circular. And, lastly, it goes without saying that we cannot draw, paint, imagine, etc. square*ness* and circle*ness* in one object. It follows that, because they imply at least one doubleton consisting of contradictory properties, the square circle and other impossible objects, are *not* depictable.

We now move to the presentation of $\mathcal{IL}$. The system is laid out thus (as an extension of John L Bell's presentation of first-order systems, with his permission):

4.0. Preliminaries:

Let λ be a function $\lambda : I \to \omega$, that maps indices in I to natural numbers in ω.

Let v be a function $v : \omega \to \omega$, that maps natural numbers to natural numbers in ω (not necessarily to themselves).

4.1. Vocabulary:

4.1.1. Vocabulary for A Standard Second-Order System

$v_0, v_1, \ldots$	*individual variables*
$V_0, V_1, \ldots$	*predicate variables of degree $v(n)$*
for each $i \in I$, a predicate symbol P_i of degree $\lambda(i)$	*predicate symbols*
for each $j \in J$ an individual constant c_j	*individual constants*
$=$	*equality symbol*
$\neg$	*logical operators: negation*
$\wedge$	*logical operators: conjunction*
$\exists$	*existential quantifier symbol*
(,), [,]	*punctuation symbols*

individual variables and constants are called *individual terms,* where t, u (possibly with subscripts) denote arbitrary individual terms.

let T, U (possibly with subscripts) denote arbitrary predicate constants and predicate variables.

4.1.2. $\mathcal{IL}$ Extension

E$\exists$	(this quantifier binds *extantial* objects, or a concrete object)
D$\exists$	(this quantifier binds *depictable* objects, or an object able to be represented imagistically)
S$\exists$	(this quantifier binds *sentential* objects, or a verbally representable object)
!E$\exists$	(this quantifier binds objects starting at *extantiality*, or a depictable, sentential, extant object)
!D$\exists$	(this quantifier binds objects starting at *depictability*, or a depictable, sentential, non-extant object)

!S∃	(this quantifier binds objects starting at *sententiality*, or an exclusively sentential object)
⌈ ⌉	(read as *it is said, and only said, that…*)

let **C**, **D** (possibly with subscripts) denote arbitrary individual constants and predicate constants.

let **X**, **Y** (possibly with subscripts) denote arbitrary individual variables and predicate variables.

let **V**, **W** (possibly with subscripts) denote variable and constant symbols of either kind.

4.2. Formulas

4.2.1. *Atomic formulas* of $\mathcal{L}$:= finite strings (of the basic symbols in 4.1.1.) either of the forms $T_i t_1,\ldots t_{\lambda(i)}$, $t = u$, or $T = U$

4.2.2. *Formulas* of $\mathcal{L}$ (or $\mathcal{L}$-*formulas*) := finite strings (of the basic symbols (i) - (vii)) defined in the following recursive manner:

(a) any *atomic formula* is a *formula*
(b.0) if $\boldsymbol{p}$, $\boldsymbol{q}$ are *formulas*, so also are $\neg\boldsymbol{p}$, $\boldsymbol{p} \wedge \boldsymbol{q}$, $\exists x\boldsymbol{p}$, $\exists X\boldsymbol{p}$ (where x is any variable v_n and X any variable V_n)
(c) a finite string of symbols is a formula exactly when it follows from finitely many applications of (a) and (b)

4.2.3. A *sentence* is a formula with no free variables.

($\mathcal{IL}$ Extension of 4.2.2.)

(b.1). If $\boldsymbol{p}$ is a formula, then $\ulcorner\boldsymbol{p}\urcorner$ is a formula.

(b.2). If $\boldsymbol{p}$ is a formula, then E$\exists x\boldsymbol{p}$, D$\exists x\boldsymbol{p}$, S$\exists x\boldsymbol{p}$ are formulas.

(b.3). If $\boldsymbol{p}$ is a formula, then !E$\exists x\boldsymbol{p}$, !D$\exists x\boldsymbol{p}$, !S$\exists x\boldsymbol{p}$ are formulas.

4.2.3. Form($\mathcal{IL}$) := $\{\boldsymbol{p} : \boldsymbol{p}$ is a formula of $\mathcal{IL}\}$.

(Abbreviations)

4.2.4. if $\boldsymbol{p}$ and $\boldsymbol{q}$ are formulas, then

$\boldsymbol{p} \vee \boldsymbol{q}$	abbreviates	$\neg(\neg\boldsymbol{p} \wedge \neg\boldsymbol{q})$
$\boldsymbol{p} \rightarrow \boldsymbol{q}$	abbreviates	$\neg\boldsymbol{p} \vee \boldsymbol{q}$
$\boldsymbol{p} \leftrightarrow \boldsymbol{q}$	abbreviates	$(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\boldsymbol{q} \rightarrow \boldsymbol{p})$
$\forall X\boldsymbol{p}$	abbreviates	$\neg\exists X\neg\boldsymbol{p}$
$\forall x\boldsymbol{p}$	abbreviates	$\neg\exists x\neg\boldsymbol{p}$
E$\forall x\boldsymbol{p}$,	abbreviates	$\neg$E$\exists x\neg\boldsymbol{p}$
D$\forall x\boldsymbol{p}$,	abbreviates	$\neg$D$\exists x\neg\boldsymbol{p}$
S$\forall x\boldsymbol{p}$,	abbreviates	$\neg$S$\exists x\neg\boldsymbol{p}$
!E$\forall x\boldsymbol{p}$,	abbreviates	$\neg$!E$\exists x\neg\boldsymbol{p}$
!D$\forall x\boldsymbol{p}$,	abbreviates	$\neg$!D$\exists x\neg\boldsymbol{p}$
!S$\forall x\boldsymbol{p}$,	abbreviates	$\neg$!S$\exists x\neg\boldsymbol{p}$
$\boldsymbol{p}([\ni x\boldsymbol{q}(x)])$	abbreviates	$\exists x(\boldsymbol{q}(x) \wedge \boldsymbol{p}(x))$

$\boldsymbol{p}([\iota x\boldsymbol{q}(x)])$	abbreviates	$\exists x[(\boldsymbol{q}(x) \wedge \forall y(\boldsymbol{q}(y) \rightarrow y = x)) \wedge \boldsymbol{p}(x)]$
$\mathrm{E}t$	abbreviates	$\mathrm{E}\exists x(x = t)$
$\mathrm{D}t$	abbreviates	$\mathrm{D}\exists x(x = t)$
$\mathrm{S}t$	abbreviates	$\mathrm{S}\exists x(x = t)$
${}^{s}Tt$	abbreviates	$\ulcorner Tt \wedge \neg Tt \urcorner \wedge\ !\mathrm{S}\exists x(x = t)$
${}^{s}It$	abbreviates	$\ulcorner \neg t = t \urcorner \wedge\ !\mathrm{S}\exists x(x = t)$

Before moving on, some commentary in regard to the novel quantifiers, $\mathrm{E}\exists x\boldsymbol{p}$, $!\mathrm{E}\exists x\boldsymbol{p}$, $\mathrm{D}\exists x\boldsymbol{p}$, $!\mathrm{D}\exists x\boldsymbol{p}$, …, etc., is in order. Some may be anticipating that, considering the theories and semantics likely to follow from $\mathcal{GL}$, the symbols E, !E, D, !D, S, !S, etc. may just as easily be treated as sortals instead of appendages of existential quantifiers. In this event, the system can then be translated into a many-sorted logic. That $\mathcal{GL}$ is translatable into a many-sorted logic may very well be the case. However, should this be the case, the presentation of $\mathcal{GL}$ above would function as one identical to a many-sorted logic, but one that I suggest is far more economical. If $\mathcal{GL}$ were to be translated to a many-sorted logic we would need define a set S of sortals (where E, D, S $\in S$). We would need an additional stock of variables for each sortal (*i.e.* $v^{\mathrm{E}}{}_{0}$, $v^{\mathrm{E}}{}_{1}$, ... ; $v^{\mathrm{D}}{}_{0}$, $v^{\mathrm{D}}{}_{1}$, ... ; $v^{\mathrm{S}}{}_{0}$, $v^{\mathrm{S}}{}_{1}$, ...). And finally, for each $i \in I$, predicates T_i and individual constants c_i would require a signature s and $\langle s_1,...,s_{\lambda(i)}\rangle$ respectively (i.e. a formal indicator that c_i is of the sort $s \in S$ and a formal indicator that, for each $t_1,...t_{\lambda(i)}$ where $T_it_1,...t_{\lambda(i)}$, t_1, ..., $t_{\lambda(i)}$ are of the sort(s) s_1, ..., $s_{\lambda(i)} \in S$, respectively). Clearly, these additions to the language mean more definitions will need to be added to the deduction system and semantics. $\mathcal{GL}$ on the other hand includes just an additional six symbols where the deduction system and semantics are defined in virtually the same way as in standard second order systems. So, if $\mathcal{GL}$ functions identically to a standard many-sorted logic, it does so with far less machinery. Alternatively, if $\mathcal{GL}$ cannot be translated into a many-sorted logic, then it simply stands as a logic of its own.

4.3. Interdefinability

4.3.1. $\forall y(!\mathrm{E}\exists x(x = y) \quad \leftrightarrow \quad [(\mathrm{E}\exists x(x = y) \wedge \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$

$!\mathrm{E}c$ abbreviates $!\mathrm{E}\exists x(x = c)$

(Let 4.3.1. be more generally labelled 'Ax. !E∃')

4.3.2. $\forall y(!\mathrm{D}\exists x(x = y) \quad \leftrightarrow \quad [(\neg\mathrm{E}\exists x(x = y) \wedge \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$

$!\mathrm{D}c$ abbreviates $!\mathrm{D}\exists x(x = c)$

(Let 4.3.1. be more generally labelled 'Ax. !D∃')

4.3.3. $\forall y(!\exists \mathrm{S}x(x = y) \leftrightarrow [(\neg \mathrm{E}\exists x(x = y) \wedge \neg \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$

$!\mathrm{S}c$ abbreviates $!\mathrm{S}\exists x(x = c)$

(Let 4.3.1. be more generally labelled 'Ax. !S∃')

Informally, 4.3.1. says, an object starting at extantiality is defined as an extant, depictable, and sentential object. 4.3.2. says, an object starting at depictability is defined as a non-extant but depictable and sentential object. 4.3.3. says, an object starting at sententiality is defined as a non-extant and non-depictable, but sentential object.

4.4. Concrete, Non-Concrete, Impossible (and sententializing variants)

concrete(t)	abbreviates	$!\mathrm{E}t$
non-concrete(t)	abbreviates	$!\mathrm{D}t$
T-impossible(t)	abbreviates	${}^{s}Tt$
I-impossible(t)	abbreviates	${}^{s}It$
Impossible(t)	abbreviates	$(\exists X)X\text{-}impossible(t) \vee I\text{-}impossible(t)$
^{s}*Concrete*(t)	abbreviates	$\ulcorner !\mathrm{E}t \wedge \neg !\mathrm{E}t \urcorner \wedge !\mathrm{S}\exists x(x = t)$
^{s}*Non-concrete*(t)	abbreviates	$\ulcorner !\mathrm{D}t \wedge \neg !\mathrm{D}t \urcorner \wedge !\mathrm{S}\exists x(x = t)$
^{s}*T-Impossible*(t)	abbreviates	$\ulcorner {}^{s}Tt \wedge \neg {}^{s}Tt \urcorner \wedge !\mathrm{S}\exists x(x = t)$
^{s}*I-Impossible*(t)	abbreviates	$\ulcorner {}^{s}It \wedge \neg {}^{s}It \urcorner \wedge !\mathrm{S}\exists x(x = t)$
^{s}*Impossible*(t)	abbreviates	$\ulcorner (\exists X)X\text{-}impossible(t) \wedge \neg(\exists X)X\text{-}impossible(t) \urcorner \wedge !\mathrm{S}\exists x(x = t)$ $\vee$ $\ulcorner I\text{-}impossible(t) \wedge \neg I\text{-}impossible(t) \urcorner \wedge !\mathrm{S}\exists x(x = t)$

4.5. Structure

4.5.1. $\boldsymbol{\mathcal{M}} = (E, D, S, \mathrm{Sent}(\boldsymbol{\mathcal{IL}}), \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{V}}, \{R_i : i \in I\}, \{e_j : j \in J\}, R\ulcorner\urcorner, \boldsymbol{\mathcal{R}})$

$\boldsymbol{\mathcal{M}}$ denotes the structure, i.e. a undecuple. $\boldsymbol{\mathcal{M}}$ consists of a nonempty domain, S, which is a set with a well-defined subset D having a well-defined subset E.

What follows is a theory for defining the domain of $\mathcal{M}$. Its employment, it must be said, is the choice of the logician's. I suspect, however, that the theory captures an uncontroversial aspect of objects (uncontroversial, at least, to those who accept that descriptions exist and that they may be true of certain objects and not others), and so is tacitly true of any domain populated by any objects a formal theory is meant to capture. More importantly however, the theory provides an answer to the anticipated question, *what exactly are the elements of the domain of your structure (especially the impossible elements)*? We will define E, D, and S in accordance with the following theory, inspired by Terence Parsons definition of object[25] (by way of some preliminary information).

D1. Let A be an arbitrary set and $\mathbf{P}(A)$ be the power set of A, then

$$\mathbf{P}^+(A) \quad := \quad \mathbf{P}(A) - \{\varnothing\}$$

D2. Let f be an arbitrary function, then

$$\mathrm{DOM}f := \{x : \exists y \in (x, y) \in f\} \text{ and } \mathrm{RAN}f := \{y : \exists x \in (x, y) \in f\}.$$

D3. Define OB: $\mathcal{V} \to \mathbf{P}^+(\boldsymbol{\mathcal{D}})$ to be a function (call it the 'function of objects'). The function of objects maps the class $\mathcal{V}$ to $\mathbf{P}^+(\boldsymbol{\mathcal{D}})$, where $\mathcal{V}$ is the class of all elements o (where 'o' is an element name, possibly with subscripts) and $\boldsymbol{\mathcal{D}}$ is the set of all descriptions d (where 'd' is a description, possibly with subscripts) and $OB(o)$ is the set of descriptions for an element o that completely describes o.

D4. Let $\mathcal{V} \times OB$ be the Cartesian product of V and OB so that for each $o \in \mathcal{V}$, we define OB_o such that,

$$OB_o \quad = \quad \{(o, y) : (o, (o, y)) \in \mathcal{V} \times OB\}$$

Informally, OB_o (for any $o \in \mathcal{V}$) is a subset of OB (more specifically, OB_o is a function, but this has yet to be proven). The condition on our definition of each OB_o ensures that the only element that features in the domain of any OB_o is o itself (where this will be proven below).

D5. Let O be the set of OB_o (for each $o \in \mathcal{V}$).

L1. For each $OB_o \in O$, OB_o is a function.

Proof:
$$\begin{aligned} OB_o \in O \quad &\Rightarrow \quad OB_o = \{(o, y) : (o, (o, y)) \in \mathcal{V} \times OB\} && \text{on D5} \\ &\Rightarrow \quad (o, y) \in OB_o \quad \Rightarrow \quad (o, y) \in OB && \text{on D4} \\ &\Rightarrow \quad OB_o \subseteq OB && \end{aligned}$$

[25] For Parsons, objects are represented as sets of nuclear properties. In a similar vein, I represent objects as sets, more precisely, graphs (or, even more precisely, as functions) consisting of a single two-tuple, itself consisting of an element and the set of descriptions accurate of the element. This leaves the definition of the object nebulous (by design), but at the very least an entity, of which we can say, determines certain descriptions true of itself and other descriptions not true of itself.

Where, because $(x, y) \in OB_o \Rightarrow (x, y) \in OB$ and because OB is a function, should it be the case that, for any x, y_1, y_2,

$$(x, y_1), (x, y_2) \in OB_o \qquad \text{and} \qquad y_1 \neq y_2$$

then

$$(x, y_1), (x, y_2) \in OB \qquad \text{and} \qquad y_1 \neq y_2$$

and we see that OB cannot be a function. This is a contradiction, hence

$$(x, y_1), (x, y_2) \in OB_o \qquad \Rightarrow \qquad y_1 = y_2$$

which proves that OB_o is a function. □

*Call each function of O an 'object function'.

T1. For each $OB_o \in O$, $\forall y, x \in \mathrm{DOM}OB_o$,

$$x = y \qquad \textit{and} \qquad OB_o(x) = OB_o(y)$$

Proof: (on D4) for each $x \in \mathrm{DOM}OB_o$ $\quad x = o$

which means, for any $x, y \in \mathrm{DOM}OB_o$, $\quad x = o = y$

hence $x = y$ and because OB_o is a function (on L1), $OB_o(x) = OB_o(y)$ □

Informally, theorem T1 states that each object function OB_o contains just a single element in its domain and a single element in its range. By D4, the element in the domain of any OB_o is the object OB_o is indexed to and the element in its range is the set of descriptions that completely describe the object that OB_o is indexed to. Each OB_o is essentially a function that determines a complete and accurate description of itself.

*note that since each object function's domain contains just the object it's indexed to, there should be no confusion should we shorten '$OB_o(o)$' to '$OB(o)$'. Note too that for any OB_o, OB_o is essentially a restriction function f on OB that restricts OB to $\{o\}$. Since

$$f[\{o\}] = \{(x, y) : (x, y) \in OB \wedge x \in \{o\}\} = \{(o, a)\} \text{ (where } a \text{ is the set of descriptions of } o)$$

and

$$OB_o = \{(o, y) : (o, (o, y)) \in \mathcal{V} \times OB\} = \{(o, a)\} \text{ (where } a \text{ is the set of descriptions of } o),$$

we have it that,

$$OB_o = f[\{o\}]$$

So, this is another way of looking at object functions.

Here are some examples of object functions and their images:

OB(The Square Circle) = {square, non-square, circular, non-circular, …}

and

OB(Homer Simpson) = {the-father-from-The-Simpsons, bald, non-concrete, …}

and

OB(Mount Everest) = {the-tallest-mountain-on-earth, treacherous, concrete, …}

Note that descriptive terms may be positive or negative. A variable ^{+}d indicates a positive descriptive term, *viz*. a descriptive term *not* prefixed with a 'non' (e.g. 'square') and ^{-}d indicates a negative term, *viz*. a descriptive term negated with a 'non', 'not', etc. (e.g. 'non-square'). Lastly, let $d(o_1, \ldots, o_n)$ indicate a described relation where o_1 is described as bearing relation d to objects up to and including o_n and all objects (save for o_1) up to and including o_n are described as bearing relation d to o_1 (and all objects o_k bear relation d to $o_1, \ldots, o_{k-1}, o_{k+1}, \ldots o_n$). So, if $d(o_1, o_2)$ is the description 'the father of', $d(o_1) \in OB(o_2)$ and $d(o_2) \in OB(o_1)$. One instance of the previous case might be, 'the father of Justin Trudeau' $\in$ OB(Pierre Trudeau) where conversely, 'Pierre Trudeau is the father of' $\in$ OB(Justin Trudeau). Here is our last definition.

D6. For each $d \in \mathcal{D}$, $R_d := \{(OBo_1, \ldots, OBo_n) \in O^n : ({}^{+}d(\ldots, o_n) \in OB(o_1) \wedge \ldots \wedge {}^{+}d(o_1, \ldots, o_{n-1}) \in OB(o_n))$

$\wedge$

$({}^{-}d(\ldots, o_n) \notin OB(o_1) \wedge \ldots \wedge {}^{-}d(o_1, \ldots, o_{n-1}) \notin OB(o_n))\}$

As an example of a D6 relation, let $R_{is\text{-}the\text{-}father\text{-}of}$ be the set of ordered pairs (x, y) where x is described as being the father of y and y is described as having x as a father of him/her. Since

'is the father of Justin Trudeau' $\in$ OB(Pierre Trudeau) $\wedge$ 'Pierre Trudeau is the father of' $\in$ OB(Justin Trudeau)

$\wedge$

'is *not* the father of Justin Trudeau' $\notin$ OB(Pierre Trudeau) $\wedge$ 'Pierre Trudeau is *not* the father of' $\notin$ OB(Justin Trudeau)

We have it that

$(OB_{\text{Pierre-Trudeau}}, OB_{\text{Justin-Trudeau}}) \in R_{is\text{-}the\text{-}father\text{-}of}$

We may now define E, D, and S as follows (refer to this definition as 'D7')

$S \in \mathbf{P}^{+}(O)$

$D := \{ OB_o \in S : (\text{'concrete'} \in OB(o) \vee \text{'non-concrete'} \in OB(o)) \wedge \text{for any } d \in D, {}^{-}d, {}^{+}d \notin OB(o)\}$

$E := \{ OB_o \in D : \text{‘concrete’} \in OB(o) \wedge \text{‘non-concrete’} \notin OB(o) \wedge \text{for any } d \in D, {}^{-}d, {}^{+}d \notin OB(o)\}$

To address a potential metaphysical concern, note that each domain (i.e. S, D, E) may contain object functions OB_o despite the fact that functions are traditionally understood to be abstract objects and therefore, are not extant objects. However, despite our presenting these functions set theoretically, they are to be treated as representations of objects (of different kinds) that realize the function in question, where it is the represented object that is in our domain.

For clarification, take for example, an extant object like Mount Everest. Clearly Mount Everest functions as an object that makes descriptions like ‘the-tallest-mountain-on-earth’, ‘treacherous’, etc. true descriptions of it. $OB_{Mount\text{-}Everest}$ represents the object in S that functions to make the aforementioned set of descriptions a complete description of Mount Everest, i.e. $OB_{Mount\text{-}Everest}$ represents Mount Everest itself. The nature of an object like Homer Simpson is not so easily determined, and hence what realizes the function $OB_{Homer\text{-}Simpson}$ is not either. However, whatever the object that Homer Simpson is (platonic particular, descriptions in a television script, animation cells, etc.) something functions to make the set of descriptions including ‘bald’, ‘non-concrete’, ‘the father from the Simpsons’, etc. a description of Homer Simpson. That *something* is what we represent with $OB_{Homer\text{-}Simpson}$ where we can certainly depict such an object but have not determined it to be extant. The same can be said for The Square Circle, save for the fact that (regardless of what object The Square Circle is) it functions to make the set of descriptions containing ‘square’ and ‘non-square’ a description of it, hence what realizes $OB_{The\text{-}Square\text{-}Circle}$ is not able to be depicted as a square and circular object (as a single representation), and is therefore exclusively sentential.

Lastly, it should be pointed out that the theory of objects above in no way factors in satisfaction conditions for formulas and theories of $\mathcal{IL}$. The main role of the theory is establishing, to which domain, objects belong. A secondary role of the theory is determining certain constraints put on other facets of the structure in order to accommodate the odd nature of impossible objects. Other than the roles of answering the metaphysical question, determining domain membership, and accommodating impossible objects, the above theory serves no other purpose in $\mathcal{IL}$. This might seem counterintuitive considering that each element comes prepackaged with a (at least) countable set of descriptions. One may be tempted to ask, why not define the relations on them too? The answer is, the principle established in the previous chapter, that is,

> an object in which a property $\boldsymbol{p}$ is exemplified is describable as 'possessing property $\boldsymbol{p}$', but it isn't always the case that an object describable as 'possessing property $\boldsymbol{p}$', has property $\boldsymbol{p}$ exemplified by it.

If the latter conjunct of this principle were false, then we could not have impossible objects. But, as was previously argued for, an eliminativism like this is not tenable. We move now to a discussion of the remaining elements of the above structure.

Sent($\mathcal{IL}$) is a set containing the sentences of Form($\mathcal{IL}$). $\mathcal{C}$, $\mathcal{P}$, $\mathcal{V}$, are the sets of constant, predicate, predicate variable symbols (respectively) of $\mathcal{IL}$. $\{R_i : i \in I\}$ is a family of relations on S with the following condition,

for any $OB_o \in S$ and any $d \in D$

$$^{+}d, {}^{-}d \in OB(o) \quad \Rightarrow \quad R_d = R_i \text{ (for some } i \in I \text{ where } \lambda(i) = 1) \quad \text{and} \quad OB(o) \notin R_i$$

$\{e_j : j \in J\}$ is a family of designated elements of S with the following condition,

for any $OB_o \in S$ and any $d \in D$

$$^{+}d, {}^{-}d \in OB(o) \quad \Rightarrow \quad OB_o = e_j \quad \text{(for some } j \in J)$$

$R\ulcorner\urcorner$ is a relation on Sent($\mathcal{IL}$) with the following conditions,

for any sentence s, if $s \in R\ulcorner\urcorner$ for any structure $\mathcal{M}$, then *it is not the case that* $\mathcal{M}$ satisfies s (more on 'satisfaction in §4.7.).

and

for any $OB_o \in S$ and any $d \in D$,

if $d \neq$ 'concrete', 'non-concrete', 'impossible', 'abstract', 'fictional', or 'vague', then

$$^{+}d, {}^{-}d \in OB(o) \quad \Rightarrow \quad OB_o = e_j \text{ and } R_d = R_i \quad \Rightarrow \quad `P_i c_j \wedge \neg P_i c_j' \in R\ulcorner\urcorner$$

or

if $d =$ 'concrete', 'non-concrete', 'impossible', 'abstract', 'fictional', or 'vague', then (where $\boldsymbol{p}$ captures the relevant mode of existence)

$^{+}d, {}^{-}d \in OB(o) \quad \Rightarrow \quad OB_o = e_j \quad \Rightarrow \quad \text{‘}\boldsymbol{p}(c_j) \wedge \neg\boldsymbol{p}(c_j)\text{’} \in R\ulcorner\urcorner$

And lastly, let

$\mathbf{R}$ abbreviate $\{R_i : i \in I\}$

and

$\cup\mathbf{P}(S^n) := \mathbf{P}(S^1) \cup \mathbf{P}(S^2) \ldots$ for each $n \in \omega$

So that,

$\boldsymbol{\mathcal{R}} \subseteq \cup\mathbf{P}(S^n)$, with the constraint,

$\mathbf{R} \subseteq \boldsymbol{\mathcal{R}}$

To augment the structure, let (λ, J) be the *type* of $\boldsymbol{\mathcal{M}}$ (where *similar* structures are structures of the same *type)*. Although functors have not been included in $\boldsymbol{\mathcal{IL}}$, should they be, any n-place operation (denoted, ‘f: $S^n \rightarrow S$’) is an (n+1)-place single-valued relation on S. To close this section, let us put one final condition on the structure, namely, that any $\boldsymbol{\mathcal{IL}}$ structure satisfy every instance of the comprehension axiom scheme, *viz*.

$\exists X \forall x_1, \ldots, \forall x_n (X x_1, \ldots, x_n \leftrightarrow \boldsymbol{p}(x_1, \ldots, x_n))$

In essence then, should we satisfy all of the constraints listed above, our structure would meet all of the conditions required of a *faithful* Henkin structure. A Henkin structure is a second order structure $\boldsymbol{\mathcal{M}}$ where predicate variables range over a domain $\boldsymbol{\mathcal{R}}$, and is *faithful* if $\boldsymbol{\mathcal{M}}$ satisfies each instance of the comprehension schema. Henkin models will be of import when it comes time to prove certain meta-results of the system. The rationale behind choosing Henkin structures is the preference for completeness in addition to soundness. The ability to determine properties of the deductive system by virtue of semantic consequence in addition to properties of our structures by virtue of syntactic consequence (as opposed to just the latter) adds greater utility. Further, we may introduce certain constraints on our Henkin structures that allow them to behave as standard second-order semantic structures do (while leaving the semantics Henkin). For one example, we can achieve the same outcomes as a standard semantics if we limit our Henkin structures to those that are *full* (viz. $\boldsymbol{\mathcal{R}} = \cup\mathbf{P}(S^n)$), as our variables now range over all subsets of the domain. That said, standard second-order semantics *cannot* be augmented to feature the same properties that Henkin semantics do (either the variables range over all subsets of the domain or they range over a select few subsets and the semantics become Henkin). With Henkin structures grounding our semantics we have far more options in deciding how we want our semantics to relate to our syntax.

4.6. Interpretation

4.6.1. (Variable Assignment) Given the structure, $\boldsymbol{\mathcal{M}}$ of type (λ, J),

A-sequence := a countable sequence of elements of S (denoted, '$\boldsymbol{a} = (a_0, a_1, \ldots)$')

R-sequence := a countable sequence of elements of $\boldsymbol{\mathcal{R}}$ (denoted, '$\boldsymbol{r} = (R_0, R_1, \ldots)$') with the following constraint:

For each n, the nth R in $\boldsymbol{r}$ is of degree $v(n)$

4.6.2. (Interpreting the Symbols) Given $\boldsymbol{\mathcal{M}}$, $\boldsymbol{a}$, $\boldsymbol{r}$ (where we read '$\mathbf{V}^{(\mathcal{M}, a, r)}$' as the element of $\boldsymbol{\mathcal{M}}$ that $\mathbf{V}$ is interpreted-by/names/is-assigned),

Interpretation of $\boldsymbol{\mathcal{IL}}$ in $(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})$	:=	i) $P_i^{(\mathcal{M}, a, r)}$	=	R_i
		ii) $V_n^{(\mathcal{M}, a, r)}$	=	R_n
		iii) $c_j^{(\mathcal{M}, a, r)}$	=	e_i
		iv) $v_n^{(\mathcal{M}, a, r)}$	=	a_n

4.6.3. (Variant Assignment)

For $n \in \omega$, $b \in S$,

$[n|b]\boldsymbol{a} := (a_0, a_1, \ldots, a_{n-1}, b, a_{n+1}, \ldots)$

For $n \in \omega$, $S \in \boldsymbol{\mathcal{R}}$ (where S is of degree $v(n)$)

$[n|S]\boldsymbol{r} := (R_0, R_1, \ldots, R_{n-1}, S, R_{n+1}, \ldots)$

4.7. Satisfaction

I will start with the satisfaction conditions for formulas of the standard second order language, then move on to define conditions for formulas unique to $\boldsymbol{\mathcal{IL}}$. The conditions for the standard language are thus:

4.7.1. For $\boldsymbol{p} \in Form(\boldsymbol{\mathcal{IL}})$,

$\boldsymbol{a}$, $\boldsymbol{r}$ *satisfy* $\boldsymbol{p}$ *in* $\boldsymbol{\mathcal{M}}$ (denoted, '$\boldsymbol{\mathcal{M}} \vDash_{a,r} \boldsymbol{p}$') :=

4.7.1.1. for terms t, u,

$$\boldsymbol{\mathcal{M}} \vDash_{a,r} t = u \quad \Leftrightarrow \quad t^{(\mathcal{M}, a, r)} = u^{(\mathcal{M}, a, r)}$$

for predicates T, U,

$\mathcal{M} \vDash_{a,r} T = U \quad \Leftrightarrow \quad T^{(\mathcal{M},\, a,\, r)} = U^{(\mathcal{M},\, a,\, r)}$

4.7.1.2. for terms $t_1, \ldots, t_{\lambda(i)}$ and predicate T_i

$\mathcal{M} \vDash_{a,r} T_i\, t_1, \ldots, t_{\lambda(i)} \quad \Leftrightarrow \quad (t_1^{(\mathcal{M},\, a,\, r)}, \ldots, t_{\lambda(i)}^{(\mathcal{M},\, a,\, r)}) \in T_i^{(\mathcal{M},\, a,\, r)}$

4.7.1.3. $\mathcal{M} \vDash_{a,r} \neg \boldsymbol{p} \quad \Leftrightarrow \quad$ *it is not the case that* $\mathcal{M} \vDash_{a,r} \boldsymbol{p}$

4.7.1.4. $\mathcal{M} \vDash_{a,r} \boldsymbol{p} \wedge \boldsymbol{q} \quad \Leftrightarrow \quad \mathcal{M} \vDash_{a,r} \boldsymbol{p}$ and $\mathcal{M} \vDash_{a,r} \boldsymbol{q}$

4.7.1.5. $\mathcal{M} \vDash_{a,r} \exists V_n \boldsymbol{p} \quad \Leftrightarrow \quad$ for some $S \in \mathcal{R}$ of degree $v(n)$, $\mathcal{M} \vDash_{a\,[n|S]r}\, \boldsymbol{p}$

($\mathcal{IL}$ Extension)

4.7.2. $\mathcal{M} \vDash_{a,r} \ulcorner \boldsymbol{p} \urcorner \quad \Leftrightarrow \quad$ $\boldsymbol{p}$ contains free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and for some $\mathbf{C}_1 \in \mathcal{C} \cup \mathcal{P}, \ldots,$ $\mathbf{C}_n \in \mathcal{C} \cup \mathcal{P}$, $\mathbf{C}_1^{(\mathcal{M},\, a,\, r)} = \mathbf{X}_1^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $\mathbf{C}_n^{(\mathcal{M},\, a,\, r)} = \mathbf{X}_n^{(\mathcal{M},\, a,\, r)}$

and

$\boldsymbol{p}(\mathbf{X}_1, \ldots, \mathbf{X}_n / \mathbf{C}_1, \ldots, \mathbf{C}_n) \in R\ulcorner\urcorner$

or

$\boldsymbol{p}$ does not contain free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and $\boldsymbol{p} \in R\ulcorner\urcorner$

We read 4.7.2. as, *it is said, and only said, that* $\boldsymbol{p}$ is true *iff* $\boldsymbol{p}$ (with possible substitutions) is a member of $R\ulcorner\urcorner$. Here some explanation is required in order to clarify the notion of *saying and only saying* something. It would also be useful to elaborate on the nature of the operator $\ulcorner\ \urcorner$, and the fact that the process of satisfying formulas of form $\ulcorner \boldsymbol{p} \urcorner$ (with free variables in $\boldsymbol{p}$) is as unconventional as it is. As a preliminary, let the condition

$\boldsymbol{p}$ contains free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and for some $\mathbf{C}_1 \in \mathcal{C} \cup \mathcal{P}, \ldots, \mathbf{C}_n \in \mathcal{C} \cup \mathcal{P}$, $\mathbf{C}_1^{(\mathcal{M},\, a,\, r)} = \mathbf{X}_1^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $\mathbf{C}_n^{(\mathcal{M},\, a,\, r)} = \mathbf{X}_n^{(\mathcal{M},\, a,\, r)}$

and

$\boldsymbol{p}(\mathbf{X}_1, \ldots, \mathbf{X}_n / \mathbf{C}_1, \ldots, \mathbf{C}_n) \in R\ulcorner\urcorner$

be abbreviated 'CON'.

Syntactically/logically, $\ulcorner\ \urcorner$ simply blocks any further decomposition of formulas of form $\ulcorner \boldsymbol{p} \urcorner$ in order to allow for the violating of the law of noncontradiction through assuming/derivation of formulas of the form $\ulcorner \boldsymbol{q}(t) \wedge \neg \boldsymbol{q}(t) \urcorner$, but not allowing for the derivation of $\boldsymbol{q}(t) \wedge \neg \boldsymbol{q}(t)$ (from $\ulcorner \boldsymbol{q}(t) \wedge \neg \boldsymbol{q}(t) \urcorner$) which would necessitate the negation of any assumption implying $\ulcorner \boldsymbol{q}(t) \wedge \neg \boldsymbol{q}(t) \urcorner \rightarrow \boldsymbol{q}(t) \wedge \neg \boldsymbol{q}(t)$, hence lead to explosion. Semantically, $\ulcorner\ \urcorner$

captures an implicit *aspect of our informal statements* about impossible objects, and the condition CON captures an implicit *aspect of the way we speak informally* about impossible objects.

In regard to *our informal statements* about impossible objects, it is the case that in ordinary philosophical conversation we speak about impossible objects without fearing any interlocutor will halt the conversation on the grounds that *an impossible object is self-contradictory, hence our conversation is meaningless*. We may say, for example, that *the square circle is of philosophical interest* and have this statement (despite the implied contradiction) come off as completely meaningful. In cases like this, we converse as though we have agreed to treat the property terms 'square' and 'not-square' and 'circular' and 'not-circular' as functioning differently from all other properties used to describe the square circle (with these properties facilitating meaning despite their being contradictory).

What we observe of these conversations is that (in conversing) we do not speak as though the objects in question actually exemplify 'square-*ness*' and 'not-square-*ness*' and 'circle-*ness*' and 'not-circle-*ness*'. This would imply that some of our statements *are* meaningless and elicit objections. And yet, no objections are made. In our conversing, we do not speak as though impossible objects lead to inconsistency in any way, but we recognize that these objects are themselves contradictory and impossible (suggesting we are not simply ignorant of these facts). Another observation we may make is this - when we talk about impossible objects like the square circle (as impossible) we differentiate easily between contradictory properties that make the object impossible and contradictory properties that do not (where we allow attribution of the former kind but not the latter). As an example of this, asserting that *the square circle is square* at one point in the conversation and then asserting that *the square circle is not-square* at another point is perfectly acceptable, but asserting that *the square circle is interesting* at one point in the conversation and then asserting that *the square circle is not-interesting* at another point is not. The latter case will elicit, in more perceptive listeners, the pointing out of an inconsistency.

These observations suggest that, in ordinary conversation, there *is* the aforementioned special function granted to properties (that make impossible objects impossible), where this special function allows for meaning and consistency. In addition, this special function is regularly applied to the relevant contradictory properties and denied of all others. That said, the nature of this special function is decidedly nebulous. How might this function be realized we may ask. One way this function may be realized, is through the following tacit qualification:

> it is said (and it is agreed[26] that it is only said that) '___'. (Q)

Here, with an application of Q to a statement like

> *the square circle is square and not-square as well as interesting and not-interesting*

we see that the statement is elliptical for

> it is said (and it is agreed that it is only said that) '*the square circle is square and not-square' and the square circle is interesting and not-interesting.*

In the elliptical case, it is perfectly meaningful (and possibly true) to say that it is said (and only said that) *the square circle is square and not square*. But, it is meaningless (and false) to say that *the square circle is interesting and not interesting*. $\ulcorner\ \urcorner$ is the operator that makes explicit, in formal languages, the tacit Q in informal conversation.

In regard to *the way we speak informally* about impossible objects, CON captures a subtle limitation, in ordinary conversation, of how we interpret pronouns in statements featuring Q. Consider an application of Q to *it is square and not-square*. This combination yields the following

> it is said (and it is agreed that it is only said that) '*it is square and not-square'.* (QI)

QI is infelicitous (in an ordinary language sense) should the object that *it* stands in for not be named in a previous statement where this name (call it '**N**') features in a true stating of

> it is said (and it is agreed that it is only said that) '***N*** *is square and not-square'.* (QN)

[26] Where 'agreed' here means an informal, even tacit, agreement to not do more than describe the contradictory object. We may infer such an agreement by the facts that all parties understand the contradictory nature of the object in question, but speak of it as though it may feature in consistent conversations, nonetheless. The only way to do this is to assume such a qualification.

This is so as (because we cannot depict nor be made acquainted with impossible objects), if QN (or something equivalent) has not been stated, then *it* is uninterpreted in QI, and because of this lack of an interpretation, any interlocutor will be ignorant as to what *it* stands in for and will not know whether or not to apply Q to 'it is square and not square'. The average interlocutor will respond by asking, *what do you mean by it?* or by stating *nothing is both square and not square, that's impossible*. Here QI is not actually QI, but just 'it is square and not square'. In ordinary conversation, 'it is square and not square' cannot become QI until the object it stands in for is identified with a name **N**, where it is also established that '**N** is square and not square'. Obviously QN, if accepted in the conversation, will satisfy these conditions. In ordinary conversation then, we do not treat QI as true unless we treat QN (or something equivalent) as true - where conversely, if we treat QN as true (and establish that *it* stands in for the object that **N** names), then we treat QI as true. In ordinary conversation, QI is true just in case QN is true. Just to cover all of the bases, it should be mentioned that if any interlocutor were to actually accept QI without QN, then because the *it* in QI can stand in for any object, anything could be said (and only said to be) square and not square. This consequence is, as well, infelicitous. One final note, a formula like 'Xx' with free variables 'X' and 'x' would have a natural language analogue along the lines of 'That's the way it is'. So, the above rationale for pronouns applies, too, to indexicals picking out properties (i.e. cases where linguistic context is required to understand which property is being referred to), hence we extend the satisfaction conditions to include predicate variables, accordingly.

If we treat free variables as analogous to unquantified indexicals, a variable assignment as analogous to interpreting an indexical, and lastly, we treat CON as analogous to the stipulation that a previous statement be made naming an impossible object or adjective describing an impossible object, then since $\ulcorner \urcorner$ is analogous to Q, we see how CON formally mirrors our ordinary language treatment of indexicals standing in for impossible objects and their properties. Further, if CON is not stipulated, we run into the same problem formally, that we do informally, when we apply Q to a statement with an unquantified and unnamed pronoun. That is, if we simply assign to a variable x in $\ulcorner Px \wedge \neg Px \urcorner$ some object in the domain and then establish '$Px \wedge \neg Px$' $\in R\ulcorner\urcorner$, we would necessitate that $\ulcorner Px \wedge \neg Px \urcorner \rightarrow \forall x \ulcorner Px \wedge \neg Px \urcorner$ be universally valid in $\mathcal{IL}$. This is clearly an undesirable consequence. For this reason, to satisfy a formula $\ulcorner \boldsymbol{p} \urcorner$ with free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$, we must apply what we might call a quasi-Robinsonian constraint, *i.e.* we must first find a constant(s) $\mathbf{C}_1, \ldots, \mathbf{C}_n$ naming the object(s) that $\mathbf{X}_1, \ldots, \mathbf{X}_n$ stand(s) in for, and then show that '$\boldsymbol{p}(\mathbf{X}_1, \ldots, \mathbf{X}_n / \mathbf{C}_1, \ldots, \mathbf{C}_n)$' $\in R\ulcorner\urcorner$. We may now move on to the sections on satisfaction for quantified formulas and abbreviations.

4.7.3. $\mathcal{M} \vDash_{a,r} \exists v_n \boldsymbol{p}$ $\Leftrightarrow$ for some $b \in S$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} \mathrm{E}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ E is non-empty and for some $b \in E$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} \mathrm{D}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ D is non-empty and for some $b \in D$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} \mathrm{S}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ for some $b \in S$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

What is said in 4.7.3. is, a sentence positing an object of any *kind* is true *iff* some object in S assigned to v_n, allows for $\boldsymbol{p}$ to be satisfied. A sentence positing an extant object is true *iff* some object in E, assigned to v_n, allows for $\boldsymbol{p}$ to be satisfied, where the same can be said for D and S statements, save for the fact that objects are assigned from D, for $\mathrm{D}\exists v_n \boldsymbol{p}$ and S, for $\mathrm{S}\exists v_n \boldsymbol{p}$. Note lastly, since S is the overarching domain of any $\mathcal{IL}$ structure, elements of S that satisfy $\exists$ sentences, satisfy $\mathrm{S}\exists$ sentences and vice versa (in other words, sentences of the form $\mathrm{S}\exists v_n \boldsymbol{p}$ and $\exists v_n \boldsymbol{p}$ are equivalent).

4.7.4. $\mathcal{M} \vDash_{a,r} !\mathrm{E}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ E is non-empty and for some $b \in E$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} !\mathrm{D}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ D is non-empty and for some non-empty $X \subseteq D$, $X \cap E = \varnothing$

and

for some $b \in D$ (where $b \notin E$), $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} !\mathrm{S}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ for some $b \in S$ (where $b \notin E$ and $b \notin D$), $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

What is said in 4.7.4. is, a sentence positing an object that starts at extantiality is true *iff* some object in E, assigned to v_n, allows for $\boldsymbol{p}$ to be satisfied. The same can be said for !D and !S statements, save for the fact that objects are assigned from D (but not E) for $!\mathrm{D}\exists v_n \boldsymbol{p}$, and S (but not D and S) for $!\mathrm{S}\exists v_n \boldsymbol{p}$.

4.7.5. for a term t

$$\mathcal{M} \vDash_{a,r} !\mathrm{E}t \quad \Leftrightarrow \quad \mathcal{M} \vDash_{a,r} !\mathrm{E}\exists x(x = t)$$

$$\mathcal{M} \vDash_{a,r} !\mathrm{D}t \quad \Leftrightarrow \quad \mathcal{M} \vDash_{a,r} !\mathrm{D}\exists x(x = t)$$

$$\mathcal{M} \vDash_{a,r} !\mathrm{S}t \quad \Leftrightarrow \quad \mathcal{M} \vDash_{a,r} !\mathrm{S}\exists x(x = t)$$

What is said in 4.7.5. is, statements of the form $!\mathrm{E}t$, $!\mathrm{D}t$, and $!\mathrm{S}t$ are satisfied *iff* what they abbreviate (*i.e.* $!\mathrm{E}\exists x(x = t)$, $!\mathrm{D}\exists x(x = t)$, and $!\mathrm{S}\exists \mathrm{x}(x = t)$ respectively) are satisfied.

4.7.6. for a term t

$$\mathcal{M} \vDash_{a,r} {}^{s}P_{i}t \quad \Leftrightarrow \quad \mathcal{M} \vDash_{a,r} \ulcorner P_{i}t \wedge \neg P_{i}t \urcorner \text{ and } \mathcal{M} \vDash_{a,r} !\mathrm{S}t$$

$$\mathcal{M} \vDash_{a,r} {}^{s}V_{i}t \quad \Leftrightarrow \quad \mathcal{M} \vDash_{a,r} \ulcorner V_{i}t \wedge \neg V_{i}t \urcorner \text{ and } \mathcal{M} \vDash_{a,r} !\mathrm{S}t$$

What is said in 4.7.6. is, an atomic formula whose predicate is sententialized is satisfied *iff* it's true that the sententialized object is *only* in the sentential domain and it's true that the sententializing predicate P, said and only said to relate, and not relate, to the object t, is a formula in $R\ulcorner\,\urcorner$. We now take a moment to prove a meta-property of the model theory, that is:

4.7.7. $\mathcal{M} \vDash_{a,r} \forall v_{n}\boldsymbol{p} \quad \Leftrightarrow \quad$ for all $b \in S$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

Proof (let '*it is not the case that*' be abbreviated 'Not'):

We assume here that a double negative makes a positive (call this metarule 'DN') and 'for some, not' is equivalent to 'not, for all' (call this metarule 'QN').

$\mathcal{M} \vDash_{a,r} \forall v_{n}\boldsymbol{p}$	$\Leftrightarrow$	$\mathcal{M} \vDash_{a,r} \neg\exists v_{n}\neg\boldsymbol{p}$	(on 4.2.5.)
	$\Leftrightarrow$	Not $\mathcal{M} \vDash_{a,r} \exists v_{n}\neg\boldsymbol{p}$	(on 4.7.1.3.)
	$\Leftrightarrow$	Not, for some $b \in S$ $\mathcal{M} \vDash_{[\mathrm{n}|\mathrm{b}]a,r} \neg\boldsymbol{p}$	(on 4.8.2.)
	$\Leftrightarrow$	Not, for some $b \in S$, NOT $\mathcal{M} \vDash_{[\mathrm{n}|\mathrm{b}]a,r} \boldsymbol{p}$	(on 4.7.1.3.)
	$\Leftrightarrow$	Not, Not, for all $b \in S$ $\mathcal{M} \vDash_{[\mathrm{n}|\mathrm{b}]a,r} \boldsymbol{p}$	(QN)
	$\Leftrightarrow$	for all $b \in S$ $\mathcal{M} \vDash_{[n|b]a,r} \boldsymbol{p}$	(DN)

This concludes the proof.

Note that virtually the same proof can be carried out to achieve the following:

4.7.8. $\mathcal{M} \vDash_{a,r} \mathrm{E}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ if E is non-empty, then for all $b \in E$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} \mathrm{D}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ if D is non-empty, then for all $b \in D$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} \mathrm{S}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ for all $b \in S$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

4.7.9. $\mathcal{M} \vDash_{a,r} !\mathrm{E}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ if E is non-empty, then for all $b \in E$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} !\mathrm{D}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ if D is non-empty and for some non-empty $X \subseteq D$, $X \cap E = \varnothing$, then for all $b \in D$ (where $b \notin E$), $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} !\mathrm{S}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ for all $b \in S$ (where $b \notin E$ and $b \notin C$), $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

4.7.10. We say that an $\mathcal{IL}$ formula $\boldsymbol{p}$ is *satisfiable* if for some $\mathcal{IL}$ structure $\mathcal{M}$ and variable assignments $\boldsymbol{a}$, $\boldsymbol{r}$, $\mathcal{M} \vDash_{a,r} \boldsymbol{p}$.

4.7.11. We say that an $\mathcal{IL}$ formula $\boldsymbol{p}$ is *valid* if for some $\mathcal{IL}$ structure $\mathcal{M}$ and all variable assignments $\boldsymbol{a}$, $\boldsymbol{r}$, $\mathcal{M} \vDash_{a,r} \boldsymbol{p}$.

4.7.12. We say that an $\mathcal{IL}$ formula $\boldsymbol{p}$ is *universally valid* (‘$\varnothing \vDash \boldsymbol{s}$’ or ‘$\vDash \boldsymbol{s}$’)if for all $\mathcal{IL}$ structures $\mathcal{M}$, $\mathcal{M} \vDash \boldsymbol{p}$.

4.7.13. For any $\Gamma \subseteq \mathrm{Sent}(\mathcal{IL})$ and any $\mathcal{IL}$ structures $\mathcal{M}$, we say that $\mathcal{M}$ is a *model* of Γ (‘$\mathcal{M} \vDash \Gamma$’) if, for each $\boldsymbol{s} \in \Gamma$, $\mathcal{M} \vDash \boldsymbol{s}$.

4.7.14. For any $\Gamma \subseteq \mathrm{Sent}(\mathcal{IL})$ and any $\boldsymbol{s} \in \mathrm{Sent}(\mathcal{IL})$, we say that Γ *entails* $\boldsymbol{s}$ (‘$\Gamma \vDash \boldsymbol{s}$’) if, for all $\mathcal{IL}$ structures $\mathcal{M}$,

$$\text{if} \quad \mathcal{M} \vDash \Gamma, \quad \text{then} \quad \mathcal{M} \vDash \boldsymbol{s}$$

4.8. Natural Deduction of $\mathcal{IL}$.

4.8.1. Inference Rules (where ‘⊳’ is read ‘from what preceded, infer…’ and open assumptions are sentences of $\mathcal{IL}$[27])

Reiteration (R)

$\boldsymbol{p}$
⊳ $\boldsymbol{p}$

Conjunction Introduction (∧I)

$\boldsymbol{p}$
$\boldsymbol{q}$
⊳ $\boldsymbol{p} \wedge \boldsymbol{q}$

Conjunction Elimination (∧E)

$\boldsymbol{p} \wedge \boldsymbol{q}$
⊳ $\boldsymbol{p}$
or
$\boldsymbol{p} \wedge \boldsymbol{q}$
⊳ $\boldsymbol{q}$

Conditional Introduction (→I)

$\underline{\boldsymbol{p}}$
$\boldsymbol{q}$
⊳ $\boldsymbol{p} \rightarrow \boldsymbol{q}$

Conditional Elimination (→E)

$\boldsymbol{p} \rightarrow \boldsymbol{q}$
$\boldsymbol{p}$
⊳ $\boldsymbol{q}$

Negation Introduction (¬I)

$\underline{\boldsymbol{p}}$
$\boldsymbol{q}$
$\neg\boldsymbol{q}$
⊳ $\neg\boldsymbol{p}$

Negation Elimination (¬E)

$\underline{\neg\boldsymbol{p}}$
$\boldsymbol{q}$
$\neg\boldsymbol{q}$
⊳ $\boldsymbol{p}$

[27] Again, since we’re trying to do as little ‘reinventing of any wheels’ as possible, the form of these inference rules is fairly standard. Said form should be familiar, at least, to anyone who has read introductory logic texts like Bergmann *et. al.*’s *The Logic Book*, Arthur’s *Natural Deduction*, and others.

Disjunction Introduction (∨I)

$\quad \boldsymbol{p}$
$\rhd \quad \boldsymbol{p} \vee \boldsymbol{q}$
or
$\quad \boldsymbol{q}$
$\rhd \quad \boldsymbol{p} \vee \boldsymbol{q}$

Disjunction Elimination (∨E)

$\quad \boldsymbol{p} \vee \boldsymbol{q}$
$\qquad \underline{\boldsymbol{p}}$
$\qquad \boldsymbol{r}$
$\qquad \underline{\boldsymbol{q}}$
$\qquad \boldsymbol{r}$
$\rhd \quad \boldsymbol{r}$

Biconditional Introduction (↔I)

$\qquad \underline{\boldsymbol{p}}$
$\qquad \boldsymbol{q}$
$\qquad \underline{\boldsymbol{q}}$
$\qquad \boldsymbol{p}$
$\rhd \quad \boldsymbol{p} \leftrightarrow \boldsymbol{q}$

Biconditional Elimination (↔E)

$\quad \boldsymbol{p} \leftrightarrow \boldsymbol{q}$
$\quad \boldsymbol{p}$
$\rhd \quad \boldsymbol{q}$
or
$\quad \boldsymbol{p} \leftrightarrow \boldsymbol{q}$
$\quad \boldsymbol{q}$
$\rhd \quad \boldsymbol{p}$

Universal Introduction (∀I)

$\quad \boldsymbol{p}(\mathbf{C/X})$
$\rhd \quad \forall \mathbf{X}\boldsymbol{p}$

With the conditions:
i) **C** does not occur in an open assumption.
ii) **C** does not occur in $\forall \mathbf{X}\boldsymbol{p}$.

Universal Elimination (∀E)

$\quad \forall \mathbf{X}\boldsymbol{p}$
$\rhd \quad \boldsymbol{p}(\mathbf{C/X})$

Existential Introduction (∃I)

$\quad \boldsymbol{p}(\mathbf{C/X})$
$\rhd \quad \exists \mathbf{X}\boldsymbol{p}$

Existential Elimination (∃E)

$\quad \exists \mathbf{X}\boldsymbol{p}$
$\qquad \underline{\boldsymbol{p}(\mathbf{C/X})}$
$\qquad \boldsymbol{q}$
$\rhd \quad \boldsymbol{q}$

With the conditions:
i) **C** does not occur in an open assumption.
ii) **C** does not occur in $\exists \mathbf{X}\boldsymbol{p}$.
iii) **C** does not occur in $\boldsymbol{q}$.

!-Extant Universal Introduction (!E∀I)

$\quad \underline{!\mathrm{E}c} \qquad$ Aux !E∀I
$\quad \ldots$
$\quad \boldsymbol{p}(c/x)$

$\rhd \;\; !\mathrm{E}\forall x\boldsymbol{p}$

With the conditions:
i) *c* does not occur in an open assumption outside the scope of Aux !E∀I.
ii) *c* does not occur in $!\mathrm{E}\forall x\boldsymbol{p}$.

!-Extant Universal Elimination (!E∀E)

$\quad !\mathrm{E}\forall x\boldsymbol{p}$
$\quad !\mathrm{E}c$
$\rhd \quad \boldsymbol{p}(c/x)$

!-Extant Existential Introduction (!E∃I)

$\quad \boldsymbol{p}(c/x)$
$\quad !\mathrm{E}c$
$\rhd \quad !\mathrm{E}\exists x\boldsymbol{p}$

!-Extant Existential Elimination (!E∃E)

$\quad !\mathrm{E}\exists x\boldsymbol{p}$
$\qquad \underline{\boldsymbol{p}(c/x) \wedge !\mathrm{E}c}$
$\qquad \boldsymbol{q}$
$\rhd \quad \boldsymbol{q}$

With the conditions:
i) *c* does not occur in an open assumption.
ii) *c* does not occur in $!\mathrm{E}\exists x\boldsymbol{p}$.
iii) *c* does not occur in $\boldsymbol{q}$.

!-Depictable Universal Introduction (!D∀I)

$\underline{!Dc\quad\quad}$ Aux !D∀I
…
$\boldsymbol{p}(c/x)$

⊳ $!D\forall x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux !D∀I.
ii) c does not occur in $!D\forall x\boldsymbol{p}$.

!-Depictable Universal Elimination (!D∀E)

$!D\forall x\boldsymbol{p}$
$!Dc$
⊳ $\underline{\boldsymbol{p}(c/x)}$

!-Depictable Existential Elimination (!D∃E)

$!D\exists x\boldsymbol{p}$
$\underline{\boldsymbol{p}(c/x) \wedge !Dc}$
$\boldsymbol{q}$
⊳ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in $!D\exists x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

!-Depictable Existential Introduction (!D∃I)

$\boldsymbol{p}(c/x)$
$!Dc$
⊳ $!D\exists x\boldsymbol{p}$

!-Sentential Universal Introduction (!S∀I)

$\underline{!Sc\quad\quad}$ Aux !S∀I
…
$\boldsymbol{p}(c/x)$

⊳ $!S\forall x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux !S∀I.
ii) c does not occur in $!S\forall x\boldsymbol{p}$.

!-Sentential Universal Elimination (!S∀E)

$!S\forall x\boldsymbol{p}$
$!Sc$
⊳ $\boldsymbol{p}(c/x)$

!-Sentential Existential Elimination (!S∃E)

$!S\exists x\boldsymbol{p}$
$\underline{\boldsymbol{p}(c/x) \wedge !Sc}$
$\boldsymbol{q}$
⊳ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in $!S\exists x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

!-Sentential Existential Introduction (!S∃I)

$\boldsymbol{p}(c/x)$
$!Sc$
⊳ $!S\exists x\boldsymbol{p}$

Extant Universal Introduction (E∀I)

$\underline{Ec\quad\quad}$ Aux E∀I
…
$\boldsymbol{p}(c/x)$

⊳ $E\forall x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux E∀I.
ii) c does not occur in $E\forall x\boldsymbol{p}$.

Extant Universal Elimination (E∀E)

$E\forall x\boldsymbol{p}$
Ec
⊳ $\boldsymbol{p}(c/x)$

Extant Existential Introduction (E∃I)

$\boldsymbol{p}(c/x)$
E*c*
▻ E∃*x**p***

Extant Existential Elimination (E∃E)

E∃*x**p***
$\underline{\boldsymbol{p}(c/x) \wedge \mathrm{E}c}$
$\boldsymbol{q}$
▻ $\boldsymbol{q}$

With the conditions:
i) *c* does not occur in an open assumption.
ii) *c* does not occur in E∃*x**p***.
iii) *c* does not occur in $\boldsymbol{q}$.

Depictable Universal Introduction (D∀I)

$\underline{\mathrm{D}c\quad}$ Aux D∀I
…
$\boldsymbol{p}(c/x)$

▻ D∀*x**p***

With the conditions:
i) *c* does not occur in an open assumption outside the scope of Aux D∀I.
ii) *c* does not occur in D∀*x**p***.

Depictable Universal Elimination (D∀E)

D∀*x**p***
D*c*
▻ $\boldsymbol{p}(c/x)$

Depictable Existential Introduction (D∃I)

$\boldsymbol{p}(c/x)$
D*c*
▻ D∃*x**p***

Depictable Existential Elimination (D∃E)

D∃*x**p***
$\underline{\boldsymbol{p}(c/x) \wedge \mathrm{D}c}$
$\boldsymbol{q}$
▻ $\boldsymbol{q}$

With the conditions:
i) *c* does not occur in an open assumption.
ii) *c* does not occur in D∃*x**p***.
iii) *c* does not occur in $\boldsymbol{q}$.

Sentential Universal Introduction (S∀I)

$\underline{\mathrm{S}c\quad}$ Aux S∀I
…
$\boldsymbol{p}(c/x)$

▻ S∀*x**p***

With the conditions:
i) *c* does not occur in an open assumption outside the scope of Aux S∀I.
ii) *c* does not occur in S∀*x**p***.

Sentential Universal Elimination (S∀E)

S∀*x**p***
S*c*
▻ $\boldsymbol{p}(c/x)$

Sentential Existential Introduction (S∃I)

$\boldsymbol{p}(c/x)$
S*c*
▻ S∃*x**p***

Sentential Existential Elimination (S∃E)

S∃*x**p***
$\underline{\boldsymbol{p}(c/x) \wedge \mathrm{S}c}$
$\boldsymbol{q}$
▻ $\boldsymbol{q}$

With the conditions:
i) *c* does not occur in an open assumption.
ii) *c* does not occur in S#∃*x**p***.
iii) *c* does not occur in $\boldsymbol{q}$.

Identity Introduction (=I)
⊳ $\forall \mathbf{X}(\mathbf{X} = \mathbf{X})$

Identity Elimination (=E)
$\mathbf{C} = \mathbf{D}$
$\boldsymbol{p}$
⊳ $\boldsymbol{p}(\mathbf{C}/\mathbf{D})$
Or
$\mathbf{C} = \mathbf{D}$
$\boldsymbol{p}$
⊳ $\boldsymbol{p}(\mathbf{D}/\mathbf{C})$

Note that we include the rules of the extended natural deduction system (i.e De Morgan, Transposition, quantifier negation, etc.) but, since the same results are derivable from all of the above (and are found in any logic textbook), we do not present them here. We do include the following axioms unique to $\mathcal{IL}$ and second-order logics however.

Default Sententiality (DS)
⊳ $\forall x(!\mathrm{E}x \vee [!\mathrm{D}x \vee !\mathrm{S}x])$
(informally: all objects start, at least, at sententiality)

No Proof (NP)
⊳ $\ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$
(informally: if we can say, but only say that ***s***, then ***s*** does not obtain)

Extensionality (Ex.)
⊳ $\forall X \forall Y(X = Y \leftrightarrow \forall x(Xx \leftrightarrow Yx))$
(informally: If two predicates are identical, then any object relating to the one, relates to the other and vice-versa)

Comprehension (Comp.)
⊳ $\exists X \forall x_1, \ldots, \forall x_n(Xx_1,\ldots,x_n \leftrightarrow \boldsymbol{p}(x_1,\ldots,x_n))$

(informally: This is the axiom scheme of comprehension *i.e.* any sequence of objects that satisfy some formula $\boldsymbol{p}$, relate to some predicate X and vice-versa)

Lastly, definitions 4.3.1. – 4.3.3. are axioms of $\mathcal{IL}$.

4.8.2. Proof.

A *proof* (alternatively *derivation*) in $\mathcal{IL}$ of $\boldsymbol{p}$ from Γ (where, $\boldsymbol{p} \in \text{Sent}(\mathcal{IL})$ and $\Gamma \subseteq \text{Sent}(\mathcal{IL})$) consists of a series:

$$
\begin{array}{ll}
1. & \Gamma \\
 & \vdots \\
 & \overline{} \\
m. & \boldsymbol{q}_1 \\
 & \ldots \\
n. & \boldsymbol{q}_n
\end{array}
$$

where 'Γ . . .' is a list of the sentences of Γ (where Γ is possibly empty), $\boldsymbol{p} = \boldsymbol{q}_n$, $\boldsymbol{q}_1 - \boldsymbol{q}_n$ are $\mathcal{IL}$-formulas, $\boldsymbol{q}_n$ can be derived by application of some rule of inference to formulas on lines $i < n$, and $\boldsymbol{q}_n$ falls only under the assumptions of 'Γ . . .'.

4.8.2.0. $\boldsymbol{p}$ *is provable from* Γ (denoted '$\Gamma \vdash \boldsymbol{p}$') *iff* there is a proof of $\boldsymbol{p}$ from Γ

4.8.2.1. Γ is *consistent* (in $\mathcal{IL}$) *iff* for no $\mathcal{IL}$-formula $\boldsymbol{p}$, $\Gamma \vdash \boldsymbol{p}$ and $\Gamma \vdash \neg\boldsymbol{p}$

4.8.2.2. $\varnothing \vdash \boldsymbol{p}$ is abbreviated $\vdash \boldsymbol{p}$

4.8.2.3. $\vdash \boldsymbol{p}$ indicates that $\boldsymbol{p}$ is a *theorem*

4.9. Theorems (where x and y are individual variables)

4.9.0.

4.9.0.1. $\vdash \text{E}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.2. $\vdash \text{D}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.3. $\vdash \text{S}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.4. $\vdash !\text{E}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.5. $\vdash !\text{D}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.6. $\vdash !\text{S}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

Proof: Each of theorems 4.9.0.1. - 4.9.0.6. follow from the fact that in any case of existentialization (for example, in $E\exists x\boldsymbol{p}(x)$) it follows by the relevant existential elimination rule (where some c is an arbitrary witness for x, i.e. $p(c)$) that $\exists x\boldsymbol{p}(x)$ is derived by existential introduction.

4.9.1. $\vdash \forall x S\exists y(x = y)$

(informally: all objects are sentential)

Proof: by DS, x either starts at extantiality, depictability, or sententiality. In any of those cases, sententiality is implied.

4.9.2. $\vdash \forall x(E\exists y(x = y) \rightarrow [D\exists y(x = y) \wedge S\exists y(x = y)])$

(informally: if x is extant, then x is depictable and sentential)

Proof: assume x is extant. By 4.9.1. x is sentential. If x is not depictable then (by 4.3.1.) x cannot start at extantiality. Further, because x is extant (by 4.3.3.), x cannot start at sententiality either. Since x neither starts at extantiality nor starts at sententiality, by DS, x must start at depictability and (by 4.3.2.) x is depictable. This is a contradiction, hence x is depictable.

4.9.3. $\vdash \forall x(D\exists y(x = y) \rightarrow S\exists y(x = y))$

(informally: if x is depictable, then x is sentential)

Proof: an immediate consequence of 4.9.1.

4.9.4. $\vdash \forall x[\neg(!Ex \wedge !Dx) \wedge \neg(!Dx \wedge !Sx) \wedge \neg(!Ex \wedge !Sx)]$

Informally: all objects can start in just one order.

Proof: Assume $!Ex \wedge !Dx$. By 4.3.1., $!Ex$ implies $E\exists y(y = x)$ and by 4.3.2., $!Dx$ implies $\neg E\exists y(y = x)$, a contradiction. The same logic applies to $!Dx \wedge !Sx$ and $!Ex \wedge !Sx$.

§4.10. A Demonstration of Derivation and of Modeling

Before closing this chapter, I would like to provide a simple demonstration of how an informal argument may be represented in the language of $\mathcal{IL}$ (and its conclusion derived) and a demonstration of how a sentence involving an impossible object can be represented in $\mathcal{IL}$ and satisfied on an $\mathcal{IL}$ structure. To demonstrate how derivation in $\mathcal{IL}$ natural deduction works, we will translate Anselm's ontological argument into $\mathcal{IL}$ and derive the existence of God. To demonstrate how modelling works in $\mathcal{IL}$, we will satisfy a formal statement positing the existence of a square circle. We start with the translation of Anselm's argument and the derivation of the existence of God (on Anselm's assumptions).

We present the informal version of Anselm's argument as it appears (verbatim) in the article pertaining to it in the *Internet Encyclopedia of Philosophy*. The argument is thus,

1. It is a conceptual truth (or, so to speak, true by definition) that God is a being than which none greater can be imagined (that is, the greatest possible being that can be imagined).
2. God exists as an idea in the mind.
3. A being that exists as an idea in the mind and in reality is, other things being equal, greater than a being that exists only as an idea in the mind.
4. Thus, if God exists only as an idea in the mind, then we can imagine something that is greater than God (that is, a greatest possible being that does exist).
5. But we cannot imagine something that is greater than God (for it is a contradiction to suppose that we can imagine a being greater than the greatest possible being that can be imagined.)
6. Therefore, God exists.

To get the preliminaries out of the way, for mnemonic purposes, we index each predicate to a symbol indicative of the property it picks out and we replace each constant c_i naming an object in the above proof with the initial of the object named. Let constant g name God and P_G indicate the *greater than* relation (i.e. ___*is greater than*___). The assumptions of the argument consist of two principles and two formulas positing the existence of God as an idea and the existence of something that exists in reality (statements 1, 3, 2, and (tacit in) 4 respectively). We define our set of assumptions Γ thus

$$\Gamma := \{\neg(\exists v_1)P_G v_1 g,\ (\mathrm{E}\forall v_2)(!\mathrm{D}\forall v_3)P_G v_2 v_3,\ (\mathrm{D}\exists v_4)(v_4 = g),\ (\mathrm{E}\exists v_5)(v_5 = v_5)\}$$

And carry out the proof as follows

1.	$\neg(\exists v_1)P_G v_1 g$	Assumption
2.	$(\mathrm{E}\forall v_2)(!\mathrm{D}\forall v_3)P_G v_2 v_3$	Assumption
3.	$(\mathrm{D}\exists v_4)(v_4 = g)$	Assumption
4.	$(\mathrm{E}\exists v_5)(v_5 = v_5)$	Assumption
5.	$\neg(\mathrm{E}\exists v_6)(v_6 = g)$	Aux. Assumption for ¬E
6.	$(c = c) \wedge (\mathrm{E}\exists v_7)(v_7 = c)$	Aux. Assumption for E∃E
7.	$\neg(\exists v_1)P_G v_1 g$	Aux. Assumption for ¬E
8.	$(\mathrm{E}\forall v_2)(!\mathrm{D}\forall v_3)P_G v_2 v_3$	2 R
9.	$(\mathrm{E}\exists v_7)(v_7 = c)$	6 ∧E
10.	$(!\mathrm{D}\forall v_3)P_G c v_3$	8, 9 E∀E

11. $\neg(\mathrm{E}\exists v_6)(v_6 = g) \wedge (\mathrm{D}\exists v_4)(v_4 = g)$ 5, 3 $\wedge$I

12. $(\mathrm{S}\exists v_8)(v_8 = g)$ Th. 4.9.1

13. $[\neg(\mathrm{E}\exists v_6)(v_6 = g) \wedge (\mathrm{D}\exists v_4)(v_4 = g)] \wedge (\mathrm{S}\exists v_8)(v_8 = g)$ 11, 12 $\wedge$I

14. $(!\mathrm{D}\exists v_9)(v_9 = g) \leftrightarrow ([\neg(\mathrm{E}\exists v_6)(v_6 = g) \wedge (\mathrm{D}\exists v_4)(v_4 = g)] \wedge (\mathrm{S}\exists v_8)(v_8 = g))$ Ax. 4.3.2.

15. $(!\mathrm{D}\forall v_3)P_Gcv_3$ 10 R

16. $(!\mathrm{D}\exists v_9)(v_9 = g)$ 14, 13 $\leftrightarrow$E

17. P_Gcg 15, 16 D$\forall$E

18. $(\exists v_1)P_Gv_1g$ 17 $\exists$I

19. $\neg(\exists v_1)P_Gv_1g$ 1 R

20. $(\exists v_1)P_Gv_1g$ 7-19 $\neg$E

21. $(\exists v_1)P_Gv_1g$ 4, 6-20 E$\exists$E

22. $\neg(\exists v_1)P_Gv_1g$ 1 R

23. $(\mathrm{E}\exists v_6)(v_6 = g)$ 5-22 $\neg$E $\square$

We see here that Anselm's argument is valid. However, the soundness of the argument is suspect. And, it is question begging. Simply evoking God as an entity that is greater than all things presupposes the reality of the entity to be proven real (or else all real things would be greater than the non-real God). If Anselm were being careful and rigorous, he would have made the qualified claim, *if God exists in reality, then God is that which is greater than all things*. But, that would have only resulted in the conclusion, *if God exists in reality, then God exists in reality*, where this is no proof of existence at all. Soundness is likely not a possibility anyhow, as it has been pointed out that the assumptions Anselm makes are prone to inconsistency. Christopher Viger has argued that putting no restrictions on P_G implies that the set of *all things God is greater than* is U-{God} (this set, Viger labels 'Ω'). Since the Russellian set $\mathbb{R}$ (i.e. $\{x : x \notin x\}$) is in U and not identified with God, $\mathbb{R} \in \Omega$. Hence God is greater than $\mathbb{R}$, and therefore something exists that is both a member of itself and not a member of itself[28]. This

[28] Viger, 2003.

contradiction is derivable from Anselm's initial assumptions, hence his set of assumptions are inconsistent. Question begging and inconsistency aside though, Anselm's argument did provide decent fodder for exercising the natural deduction system of $\mathcal{IL}$. We end this demonstration of derivation then, and move to our demonstration of modelling.

Take the sentence, *there is an object that is simultaneously square and circular*, as our example. In the language of $\mathcal{IL}$ we may represent this sentence as

$$\exists v_1({}^{s}P_s v_1 \wedge {}^{s}P_c v_1)$$

where again, for mnemonic purposes, our predicates P_i have been indexed to italicized lower-case initials of the properties they represent, and '*s*' replaces the constant c_i naming *the square circle*. Now, let our structure be $\mathcal{M}$ (with arbitrary assignments $\boldsymbol{a}$ and $\boldsymbol{r}$) and the following conditions

(Let $OB_{The\text{-}Square\text{-}Circle}$ be abbreviated by $\boxdot$)

i.	$\boxdot \in S$	and	$\boxdot \notin D$	and	$\boxdot \notin E$
ii.	$P_s{}^{(\mathcal{M},\, a,\, r)}$	=	$\{(x) \in S^1: x \text{ is square}\}$		
iii.	$P_c{}^{(\mathcal{M},\, a,\, r)}$	=	$\{(x) \in S^1: x \text{ is circular}\}$		
iv.	$s^{(\mathcal{M},\, a,\, r)}$	=	$\boxdot$		
v.	'$P_s s \wedge \neg P_s s$' $\in R\ulcorner\urcorner$		and	'$P_c s \wedge \neg P_c s$' $\in R\ulcorner\urcorner$	
vi.	$[1\vert\boxdot]\boldsymbol{a}$	=	the variant individual variable assignment where $a_1 = \boxdot$.		
vii.	$[2\vert\boxdot]\boldsymbol{a}$	=	the variant individual variable assignment where $a_2 = \boxdot$.		

Here is the demonstration,

$\mathcal{M} \vDash_{a,\, r} \exists v_1({}^{s}P_s v_1 \wedge {}^{s}P_c v_1)$ $\Leftrightarrow$ for some $b \in S$, $\mathcal{M} \vDash_{[1|b]a,\, r} {}^{s}P_s v_1 \wedge {}^{s}P_c v_1$

$\Leftrightarrow$ $\mathcal{M} \vDash_{[1|\boxdot]a,\, r} {}^{s}P_s v_1$ and $\mathcal{M} \vDash_{[1|\boxdot]a,\, r} {}^{s}P_c v_1$ (where $b = \boxdot$)

$\Leftrightarrow$ $\mathcal{M} \vDash_{[1|\boxdot]a,\, r} \ulcorner P_s v_1 \wedge \neg P_s v_1 \urcorner$ and $\mathcal{M} \vDash_{[1|\boxdot]a,\, r} !Sv_1$

and

$\mathcal{M} \vDash_{[1|\boxdot]a,\, r} \ulcorner P_c v_1 \wedge \neg P_c v_1 \urcorner$ and $\mathcal{M} \vDash_{[1|\boxdot]a,\, r} !Sv_1$

$\Leftrightarrow$ $\ulcorner P_s v_1 \wedge \neg P_s v_1 \urcorner$ contains free variables $v_1, \ldots, v_n$ and for some $c_1 \in \mathcal{C}, \ldots, c_n \in \mathcal{C}$, $c_1{}^{(\mathcal{M},\, a,\, r)} = v_1{}^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $c_n{}^{(\mathcal{M},\, a,\, r)} = v_n{}^{(\mathcal{M},\, a,\, r)}$ and '$P_s s \wedge \neg P_s s$' $\in R\ulcorner\urcorner$

and

$\mathcal{M} \vDash_{[1|\boxdot]a,\, r}$ $!\mathrm{S}\exists v_2(v_2 = v_1)$

and

$\ulcorner P_c v_1 \wedge \neg P_c v_1 \urcorner$ contains free variables $v_1, \ldots, v_n$ and for some $c_1 \in \mathcal{C}, \ldots, c_n \in \mathcal{C}$, $c_1^{(\mathcal{M},\, a,\, r)} = v_1^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $c_n^{(\mathcal{M},\, a,\, r)} = v_n^{(\mathcal{M},\, a,\, r)}$ and '$P_c s \wedge \neg P_c s$' $\in R \ulcorner \urcorner$

and

$\mathcal{M} \vDash_{[1|\boxdot]a,\, r}$ $!\mathrm{S}\exists v_2(v_2 = v_1)$

$\Leftrightarrow$ $\ulcorner P_s v_1 \wedge \neg P_s v_1 \urcorner$ contains free variables $v_1, \ldots, v_n$ and for some $c_1 \in \mathcal{C}, \ldots, c_n \in \mathcal{C}$, $c_1^{(\mathcal{M},\, a,\, r)} = v_1^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $c_n^{(\mathcal{M},\, a,\, r)} = v_n^{(\mathcal{M},\, a,\, r)}$ and '$P_s s \wedge \neg P_s s$' $\in R \ulcorner \urcorner$

and

for some $b \in S$ ($b \notin E$ and $b \notin D$), $\mathcal{M} \vDash_{[1|\boxdot]a,\, r}$ $v_2 = v_1$

and

$\ulcorner P_c v_1 \wedge \neg P_c v_1 \urcorner$ contains free variables $v_1, \ldots, v_n$ and for some $c_1 \in \mathcal{C}, \ldots, c_n \in \mathcal{C}$, $c_1^{(\mathcal{M},\, a,\, r)} = v_1^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $c_n^{(\mathcal{M},\, a,\, r)} = v_n^{(\mathcal{M},\, a,\, r)}$ and '$P_c s \wedge \neg P_c s$' $\in R \ulcorner \urcorner$

and

for some $b \in S$ ($b \notin E$ and $b \notin D$), $\mathcal{M} \vDash_{[1|\boxdot]a,\, r}$ $v_2 = v_1$

(1) $\Leftrightarrow$ $\ulcorner P_s v_1 \wedge \neg P_s v_1 \urcorner$ contains free variables $v_1, \ldots, v_n$ and for some $c_1 \in \mathcal{C}, \ldots, c_n \in \mathcal{C}$, $c_1^{(\mathcal{M},\, a,\, r)} = v_1^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $c_n^{(\mathcal{M},\, a,\, r)} = v_n^{(\mathcal{M},\, a,\, r)}$ and '$P_s s \wedge \neg P_s s$' $\in R \ulcorner \urcorner$

and

(2) $v_2^{(\mathcal{M},\, [2|\boxdot][1|\boxdot]a,\, r)} = v_1^{(\mathcal{M},\, [2|\boxdot][1|\boxdot]a,\, r)}$ (where $b = \boxdot$)

and

(3) $\ulcorner P_c v_1 \wedge \neg P_c v_1 \urcorner$ contains free variables $v_1, \ldots, v_n$ and for some $c_1 \in \mathcal{C}, \ldots, c_n \in \mathcal{C}$, $c_1^{(\mathcal{M},\, a,\, r)} = v_1^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $c_n^{(\mathcal{M},\, a,\, r)} = v_n^{(\mathcal{M},\, a,\, r)}$ and '$P_c s \wedge \neg P_c s$' $\in R \ulcorner \urcorner$

and

(4) $v_2^{(\mathcal{M},\, [2|\boxdot][1|\boxdot]a,\, r)} = v_1^{(\mathcal{M},\, [2|\boxdot][1|\boxdot]a,\, r)}$ (where $b = \boxdot$)

By conditions iv, vi, and vii, $v_2^{(\mathcal{M},\, [2|\boxdot][1|\boxdot]a,\, r)} = a_2 = \boxdot = s^{(\mathcal{M},\, a,\, r)} = \boxdot = a_1 = v_1^{(\mathcal{M},\, [2|\boxdot][1|\boxdot]a,\, r)}$. So, by conditions iv, v, vi, and vii, we have it that (1), (2), (3), and (4) obtain. From this it follows that '$\exists v_1({}^{s}P_s v_1 \wedge {}^{s}P_c v_1)$' is satisfied. Thus, the demonstration of modelling is concluded, as is the chapter.

Appendix: Syntax, Semantics, and Deduction Theory of $\mathcal{IL}$

4.0. Preliminaries:

Let λ be a function $\lambda : I \rightarrow \omega$, that maps indices in I to natural numbers in ω.

Let ν be a function $\nu : \omega \rightarrow \omega$, that maps natural numbers to natural numbers in ω (not necessarily to themselves).

4.1. Vocabulary:

4.1.1. Vocabulary for A Standard Second-Order System

$v_0, v_1, \ldots$	*individual variables*
$V_0, V_1, \ldots$	*predicate variables of degree* $\nu(n)$
for each $i \in I$, a predicate symbol P_i of degree $\lambda(i)$	*predicate symbols*
for each $j \in J$ an individual constant c_j	*individual constants*
$=$	*equality symbol*
$\neg$	*logical operators*: *negation*
$\wedge$	*logical operators: conjunction*
$\exists$	*existential quantifier symbol*
(,), [,]	*punctuation symbols*

* individual variables and constants are called *individual terms,* where t, u (possibly with subscripts) denote arbitrary individual terms.

** let T, U (possibly with subscripts) denote arbitrary predicate constants and predicate variables.

4.1.2. $\mathcal{IL}$ Extension

E$\exists$	(this quantifier binds an *extantial* object, or a concrete object)
D$\exists$	(this quantifier binds a *depictable* object, or an object able to be represented imagistically)
S$\exists$	(this quantifier binds a *sentential* object, or a verbally representable object)
!E$\exists$	(this quantifier binds an object starting at *extantiality*, or a depictable, sentential, extant object)
!D$\exists$	(this quantifier binds an object starting at *depictability*, or a depictable, sentential, non-extant object)
!S$\exists$	(this quantifier binds an object starting at *sententiality*, or an exclusively sentential object)
$\ulcorner\ \urcorner$	(read as *it is said, and only said, that…*)

*** let **C**, **D** (possibly with subscripts) denote arbitrary individual constants and predicate constants.

**** let **X**, **Y** (possibly with subscripts) denote arbitrary individual variables and predicate variables.

**** let **V**, **W** (possibly with subscripts) denote variable and constant symbols of either kind.

4.2. Formulas

4.2.1. *Atomic formulas* of $\mathcal{L}$:= finite strings (of the basic symbols (i) - (iv)) either of the forms $T_i t_1, \ldots t_{\lambda(i)}$, $t = u$, or $T = U$

4.2.2. *Formulas* of $\mathcal{L}$ (or $\mathcal{L}$-*formulas*) := finite strings (of the basic symbols (i) - (vii)) defined in the following recursive manner:

(a) any *atomic formula* is a *formula*

(b.0) if $\boldsymbol{p}$, $\boldsymbol{q}$ are *formulas*, so also are $\neg\boldsymbol{p}$, $\boldsymbol{p} \wedge \boldsymbol{q}$, $\exists x\boldsymbol{p}$, $\exists X\boldsymbol{p}$ (where x is any variable v_i and X any variable V_i)

(b.1). If $\boldsymbol{p}$ is a formula, then $\ulcorner\boldsymbol{p}\urcorner$ is a formula.

(b.2). If $\boldsymbol{p}$ is a formula, then $\mathrm{E}\exists x\boldsymbol{p}$, $\mathrm{D}\exists x\boldsymbol{p}$, $\mathrm{S}\exists x\boldsymbol{p}$ are formulas.

(b.3). If $\boldsymbol{p}$ is a formula, then $!\mathrm{E}\exists x\boldsymbol{p}$, $!\mathrm{D}\exists x\boldsymbol{p}$, $!\mathrm{S}\exists x\boldsymbol{p}$ are formulas.

(c) a finite string of symbols is a formula exactly when it follows from finitely many applications of (a) and (b)

4.2.3. Form($\mathcal{IL}$) := $\{\boldsymbol{p} : \boldsymbol{p}$ is a formula of $\mathcal{IL}\}$.

A *sentence* is a formula with no free variables.

(Abbreviations)

4.2.4. if $\boldsymbol{p}$ and $\boldsymbol{q}$ are formulas, then

$\boldsymbol{p} \vee \boldsymbol{q}$	abbreviates	$\neg(\neg\boldsymbol{p} \wedge \neg\boldsymbol{q})$
$\boldsymbol{p} \to \boldsymbol{q}$	abbreviates	$\neg\boldsymbol{p} \vee \boldsymbol{q}$
$\boldsymbol{p} \leftrightarrow \boldsymbol{q}$	abbreviates	$(\boldsymbol{p} \to \boldsymbol{q}) \wedge (\boldsymbol{q} \to \boldsymbol{p})$
$\forall X\boldsymbol{p}$	abbreviates	$\neg\exists X\neg\boldsymbol{p}$
$\forall x\boldsymbol{p}$	abbreviates	$\neg\exists x\neg\boldsymbol{p}$
$\mathrm{E}\forall x\boldsymbol{p}$,	abbreviates	$\neg\mathrm{E}\exists x\neg\boldsymbol{p}$
$\mathrm{D}\forall x\boldsymbol{p}$,	abbreviates	$\neg\mathrm{D}\exists x\neg\boldsymbol{p}$
$\mathrm{S}\forall x\boldsymbol{p}$,	abbreviates	$\neg\mathrm{S}\exists x\neg\boldsymbol{p}$
$!\mathrm{E}\forall x\boldsymbol{p}$,	abbreviates	$\neg!\mathrm{E}\exists x\neg\boldsymbol{p}$
$!\mathrm{D}\forall x\boldsymbol{p}$,	abbreviates	$\neg!\mathrm{D}\exists x\neg\boldsymbol{p}$
$!\mathrm{S}\forall x\boldsymbol{p}$,	abbreviates	$\neg!\mathrm{S}\exists x\neg\boldsymbol{p}$
$\boldsymbol{p}([\reflectbox{ι}x\boldsymbol{q}(x)])$	abbreviates	$\exists x(\boldsymbol{q}(x) \wedge \boldsymbol{p}(x))$
$\boldsymbol{p}([\iota x\boldsymbol{q}(x)])$	abbreviates	$\exists x[(\boldsymbol{q}(x) \wedge \forall y(\boldsymbol{q}(y) \to y = x)) \wedge \boldsymbol{p}(x)]$
$\mathrm{E}t$	abbreviates	$\mathrm{E}\exists x(x = t)$
$\mathrm{D}t$	abbreviates	$\mathrm{D}\exists x(x = t)$
$\mathrm{S}t$	abbreviates	$\mathrm{S}\exists x(x = t)$
${}^{s}Tt$	abbreviates	$\ulcorner Tt \wedge \neg Tt\urcorner \wedge\, !\mathrm{S}\exists x(x = t)$
${}^{s}It$	abbreviates	$\ulcorner\neg t = t\urcorner \wedge\, !\mathrm{S}\exists x(x = t)$

4.3. Interdefinability

Ax. !E∃ $\quad \forall y(!\mathrm{E}\exists x(x = y) \quad \leftrightarrow \quad [(\mathrm{E}\exists x(x = y) \wedge \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$

$!\mathrm{E}c$ abbreviates $!\mathrm{E}\exists x(x = c)$

Ax. !D∃ $\quad \forall y(!\mathrm{D}\exists x(x = y) \quad \leftrightarrow \quad [(\neg\mathrm{E}\exists x(x = y) \wedge \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$

$!\mathrm{D}c$ abbreviates $!\mathrm{D}\exists x(x = c)$

Ax. $!S\exists$	$\forall y(!\exists Sx(x = y)$	$\leftrightarrow$	$[(\neg E\exists x(x = y) \wedge \neg D\exists x(x = y)) \wedge S\exists x(x = y)])$

$!Sc$ abbreviates $!S\exists x(x = c)$

4.4. Concrete, Non-Concrete, Impossible (and sententializing variants)

concrete(t)	abbreviates	$!Et$
non-concrete(t)	abbreviates	$!Dt$
T-impossible(t)	abbreviates	${}^{s}Tt$
I-impossible(t)	abbreviates	${}^{s}It$
Impossible(t)	abbreviates	$(\exists X)X\textit{-impossible}(t) \vee I\textit{-impossible}(t)$
^{s}*Concrete*(t)	abbreviates	$\ulcorner !Et \wedge \neg !Et \urcorner \wedge !S\exists x(x = t)$
^{s}*Non-concrete*(t)	abbreviates	$\ulcorner !Dt \wedge \neg !Dt \urcorner \wedge !S\exists x(x = t)$
^{s}*T-Impossible*(t)	abbreviates	$\ulcorner {}^{s}Tt \wedge \neg {}^{s}Tt \urcorner \wedge !S\exists x(x = t)$
^{s}*I-Impossible*(t)	abbreviates	$\ulcorner {}^{s}It \wedge \neg {}^{s}It \urcorner \wedge !S\exists x(x = t)$
^{s}*Impossible*(t)	abbreviates	$\ulcorner (\exists X)X\textit{-impossible}(t) \wedge \neg(\exists X)X\textit{-impossible}(t) \urcorner \wedge !S\exists x(x = t)$ $\vee$ $\ulcorner I\textit{-impossible}(t) \wedge \neg I\textit{-impossible}(t) \urcorner \wedge !S\exists x(x = t)$

4.5. Structure

4.5.1. $\boldsymbol{\mathcal{M}} = (E, D, S, \mathrm{Sent}(\boldsymbol{\mathcal{IL}}), \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{V}}, \{R_i : i \in I\}, R\ulcorner\urcorner, \{e_j : j \in J\}, \boldsymbol{\mathcal{R}})$

4.6. Interpretation

4.6.1. (Variable Assignment) Given the structure, $\boldsymbol{\mathcal{M}}$ of type (λ, J),

A-sequence := a countable sequence of elements of S (denoted, '$\boldsymbol{a} = (a_0, a_1, \ldots)$')

R-sequence := a countable sequence of elements of $\boldsymbol{\mathcal{R}}$ (denoted, '$\boldsymbol{r} = (R_0, R_1, \ldots)$') with the following constraint:

For each n, the nth R in $\boldsymbol{r}$ is of degree $v(n)$

4.6.2. (Interpreting the Symbols) Given $\boldsymbol{\mathcal{M}}$, $\boldsymbol{a}$, $\boldsymbol{r}$ (where we read '$\mathbf{V}^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$' as the element of $\boldsymbol{\mathcal{M}}$ that $\mathbf{V}$ is interpreted-by/names/is-assigned),

Interpretation of $\boldsymbol{\mathcal{IL}}$ in $(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})$	:=	i) $P_i^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$	=	R_i
		ii) $V_n^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$	=	R_n
		iii) $c_j^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$	=	e_i
		iv) $v_n^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$	=	a_n

4.6.3. (Variant Assignment)

For $n \in \omega$, $b \in S$,

$[n|b]\boldsymbol{a} := (a_0, a_1, ..., a_{n-1}, b, a_{n+1}, ...)$

For $n \in \omega$, $S \in \boldsymbol{\mathcal{R}}$ (where S is of degree $v(n)$)

$[n|S]\boldsymbol{r} := (R_0, R_1, ..., R_{n-1}, S, R_{n+1}, ...)$

4.7. Satisfaction

4.7.1. For $\boldsymbol{p} \in Form(\boldsymbol{\mathcal{IL}})$,

$\boldsymbol{a}$, $\boldsymbol{r}$ *satisfy* $\boldsymbol{p}$ *in* $\boldsymbol{\mathcal{M}}$ (denoted, '$\boldsymbol{\mathcal{M}} \vDash_{a,r} \boldsymbol{p}$') :=

4.7.1.1. for terms t, u,

$\boldsymbol{\mathcal{M}} \vDash_{a,r} t = u \quad \Leftrightarrow \quad t^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})} = u^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$

for predicates T, U,

$\boldsymbol{\mathcal{M}} \vDash_{a,r} T = U \quad \Leftrightarrow \quad T^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})} = U^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$

4.7.1.2. for terms $t_1, \ldots, t_{\lambda(i)}$ and predicate T_i

$\boldsymbol{\mathcal{M}} \vDash_{a,r} T_i\, t_1, \ldots, t_{\lambda(i)} \quad \Leftrightarrow \quad (t_1^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}, \ldots, t_{\lambda(i)}^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}) \in T_i^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$

4.7.1.3. $\boldsymbol{\mathcal{M}} \vDash_{a,r} \neg\boldsymbol{p} \quad \Leftrightarrow \quad$ *it is not the case that* $\boldsymbol{\mathcal{M}} \vDash_{a,r} \boldsymbol{p}$

4.7.1.4. $\boldsymbol{\mathcal{M}} \vDash_{a,r} \boldsymbol{p} \wedge \boldsymbol{q} \quad \Leftrightarrow \quad \boldsymbol{\mathcal{M}} \vDash_{a,r} \boldsymbol{p}$ and $\boldsymbol{\mathcal{M}} \vDash_{a,r} \boldsymbol{q}$

4.7.1.5. $\boldsymbol{\mathcal{M}} \vDash_{a,r} \exists V_n\boldsymbol{p} \quad \Leftrightarrow \quad$ for some $S \in \boldsymbol{\mathcal{R}}$ of degree $v(n)$, $\boldsymbol{\mathcal{M}} \vDash_{a\,[n|S]\boldsymbol{r}} \boldsymbol{p}$

4.7.1. $\boldsymbol{\mathcal{M}} \vDash_{a,r} \ulcorner\boldsymbol{p}\urcorner \quad \Leftrightarrow \quad \boldsymbol{p}$ contains free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and for some $\mathbf{C}_1 \in \boldsymbol{\mathcal{C}} \cup \boldsymbol{\mathcal{P}}, \ldots,$ $\mathbf{C}_n \in \boldsymbol{\mathcal{C}} \cup \boldsymbol{\mathcal{P}}$, $\mathbf{C}_1^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})} = \mathbf{X}_1^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}, \ldots,$ and $\mathbf{C}_n^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})} = \mathbf{X}_n^{(\boldsymbol{\mathcal{M}}, \boldsymbol{a}, \boldsymbol{r})}$

and

$\boldsymbol{p}(\mathbf{X}_1, \ldots, \mathbf{X}_n / \mathbf{C}_1, \ldots, \mathbf{C}_n) \in R\ulcorner\urcorner$

or

$\boldsymbol{p}$ does not contain free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and $\boldsymbol{p} \in R\ulcorner\urcorner$

4.7.2. $\boldsymbol{\mathcal{M}} \vDash_{a,r} \exists v_n\boldsymbol{p} \quad \Leftrightarrow \quad$ for some $b \in S$, $\boldsymbol{\mathcal{M}} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\boldsymbol{\mathcal{M}} \vDash_{a,r} \mathrm{E}\exists v_n\boldsymbol{p} \quad \Leftrightarrow \quad E$ is non-empty and for some $b \in E$, $\boldsymbol{\mathcal{M}} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\boldsymbol{\mathcal{M}} \vDash_{a,r} \mathrm{D}\exists v_n\boldsymbol{p} \quad \Leftrightarrow \quad D$ is non-empty and for some $b \in D$, $\boldsymbol{\mathcal{M}} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\boldsymbol{\mathcal{M}} \vDash_{a,r} \mathrm{S}\exists v_n\boldsymbol{p} \quad \Leftrightarrow \quad$ for some $b \in S$, $\boldsymbol{\mathcal{M}} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

4.7.3. $\mathcal{M} \vDash_{a,r} !\mathrm{E}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ E is non-empty and for some $b \in E$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} !\mathrm{D}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ D is non-empty and for some non-empty $X \subseteq D$, $X \cap E = \varnothing$

and

for some $b \in D$ (where $b \notin E$), $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} !\mathrm{S}\exists v_n \boldsymbol{p}$ $\Leftrightarrow$ for some $b \in S$ (where $b \notin E$ and $b \notin D$), $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

4.7.4. for a term t

$\mathcal{M} \vDash_{a,r} !\mathrm{E}t$ $\Leftrightarrow$ $\mathcal{M} \vDash_{a,r} !\mathrm{E}\exists x(x = t)$

$\mathcal{M} \vDash_{a,r} !\mathrm{D}t$ $\Leftrightarrow$ $\mathcal{M} \vDash_{a,r} !\mathrm{D}\exists x(x = t)$

$\mathcal{M} \vDash_{a,r} !\mathrm{S}t$ $\Leftrightarrow$ $\mathcal{M} \vDash_{a,r} !\mathrm{S}\exists x(x = t)$

4.7.5. for a term t

$\mathcal{M} \vDash_{a,r} {}^{s}P_i t$ $\Leftrightarrow$ $\mathcal{M} \vDash_{a,r} \ulcorner P_i t \wedge \neg P_i t \urcorner$ and $\mathcal{M} \vDash_{a,r} !\mathrm{S}t$

$\mathcal{M} \vDash_{a,r} {}^{s}V_i t$ $\Leftrightarrow$ $\mathcal{M} \vDash_{a,r} \ulcorner V_i t \wedge \neg V_i t \urcorner$ and $\mathcal{M} \vDash_{a,r} !\mathrm{S}t$

4.7.6. $\mathcal{M} \vDash_{a,r} \forall v_n \boldsymbol{p}$ $\Leftrightarrow$ for all $b \in S$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

4.7.7. $\mathcal{M} \vDash_{a,r} \mathrm{E}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ if E is non-empty, then for all $b \in E$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} \mathrm{D}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ if D is non-empty, then for all $b \in D$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} \mathrm{S}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ for all $b \in S$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

4.7.8. $\mathcal{M} \vDash_{a,r} !\mathrm{E}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ if E is non-empty, then for all $b \in E$, $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} !\mathrm{D}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ if D is non-empty and for some non-empty $X \subseteq D$, $X \cap E = \varnothing$,
then for all $b \in D$ (where $b \notin E$), $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

$\mathcal{M} \vDash_{a,r} !\mathrm{S}\forall v_n \boldsymbol{p}$ $\Leftrightarrow$ for all $b \in S$ (where $b \notin E$ and $b \notin C$), $\mathcal{M} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

4.7.9. We say that an $\mathcal{IL}$ formula $\boldsymbol{p}$ is *satisfiable* if for some $\mathcal{IL}$ structure $\mathcal{M}$ and variable assignments $\boldsymbol{a}$, $\boldsymbol{r}$, $\mathcal{M} \vDash_{a,r} \boldsymbol{p}$.

4.7.10. We say that an $\mathcal{IL}$ formula $\boldsymbol{p}$ is *valid* if for some $\mathcal{IL}$ structure $\mathcal{M}$ and all variable assignments $\boldsymbol{a}$, $\boldsymbol{r}$, $\mathcal{M} \vDash_{a,r} \boldsymbol{p}$.

4.7.11. We say that an $\mathcal{IL}$ formula $\boldsymbol{p}$ is *universally valid* (‘$\varnothing \vDash \boldsymbol{s}$’ or ‘$\vDash \boldsymbol{s}$’) if for all $\mathcal{IL}$ structures $\mathcal{M}$, $\mathcal{M} \vDash \boldsymbol{p}$.

4.7.12. For any $\Gamma \subseteq \mathrm{Sent}(\mathcal{IL})$ and any $\mathcal{IL}$ structures $\mathcal{M}$, we say that $\mathcal{M}$ is a *model* of Γ (‘$\mathcal{M} \vDash \Gamma$’) if, for each $\boldsymbol{s} \in \Gamma$, $\mathcal{M} \vDash \boldsymbol{s}$.

4.7.13. For any $\Gamma \subseteq \mathrm{Sent}(\mathcal{IL})$ and any $\boldsymbol{s} \in \mathrm{Sent}(\mathcal{IL})$, we say that Γ *entails* $\boldsymbol{s}$ (‘$\Gamma \vDash \boldsymbol{s}$’) if, for all $\mathcal{IL}$ structures $\mathcal{M}$,

if $\mathcal{M} \vDash \Gamma$, then $\mathcal{M} \vDash \boldsymbol{s}$

4.8. Natural Deduction of $\mathcal{IL}$.

4.8.1. Inference Rules (where ‘$\rhd$’ is read ‘from what preceded, infer…’)

Reiteration (R)

p
▻ p

Conjunction Introduction (∧I)

p
q
▻ $p \wedge q$

Conjunction Elimination (∧E)

$p \wedge q$
▻ p
or
$p \wedge q$
▻ q

Conditional Introduction (→I)

$\underline{p}$
q
▻ $p \rightarrow q$

Conditional Elimination (→E)

$p \rightarrow q$
p
▻ q

Negation Introduction (¬I)

$\underline{p}$
q
$\neg q$
▻ $\neg p$

Negation Elimination (¬E)

$\underline{\neg p}$
q
$\neg q$
▻ p

Disjunction Introduction (∨I)

p
▻ $p \vee q$
or
q
▻ $p \vee q$

Disjunction Elimination (∨E)

$p \vee q$
$\underline{p}$
r
$\underline{q}$
r
▻ r

Biconditional Introduction (↔I)

$\underline{p}$
q

$\underline{q}$
p
▻ $p \leftrightarrow q$

Biconditional Elimination (↔E)

$p \leftrightarrow q$
p
▻ q
or
$p \leftrightarrow q$
q
▻ p

Universal Introduction (∀I)

p(**C**/**X**)
▻ ∀**X**p

With the conditions:
i) **C** does not occur in an open assumption.
ii) **C** does not occur in ∀**X**p.

Universal Elimination (∀E)

∀**X**p
▻ p(**C**/**X**)

Existential Elimination (∃E)

$\exists \mathbf{X}\boldsymbol{p}$

$\underline{\boldsymbol{p}(\mathbf{C}/\mathbf{X})}$

$\boldsymbol{q}$

⊳ $\boldsymbol{q}$

With the conditions:
i) **C** does not occur in an open assumption.
ii) **C** does not occur in $\exists \mathbf{X}\boldsymbol{p}$.
iii) **C** does not occur in $\boldsymbol{q}$.

Existential Introduction (∃I)

$\boldsymbol{p}(\mathbf{C}/\mathbf{X})$

⊳ $\exists \mathbf{X}\boldsymbol{p}$

!-Extant Universal Introduction (!E∀I)

$\underline{!\mathrm{E}c}$ Aux !E∀I

…

$\boldsymbol{p}(c/x)$

⊳ $!\mathrm{E}\forall x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux !E∀I.
ii) c does not occur in $!\mathrm{E}\forall x\boldsymbol{p}$.

!-Extant Universal Elimination (!E∀E)

$!\mathrm{E}\forall x\boldsymbol{p}$

$!\mathrm{E}c$

⊳ $\boldsymbol{p}(c/x)$

!-Extant Existential Elimination (!E∃E)

$!\mathrm{E}\exists x\boldsymbol{p}$

$\underline{\boldsymbol{p}(c/x) \wedge !\mathrm{E}c}$

$\boldsymbol{q}$

⊳ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in $!\mathrm{E}\exists x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

!-Extant Existential Introduction (!E∃I)

$\boldsymbol{p}(c/x)$

$!\mathrm{E}c$

⊳ $!\mathrm{E}\exists x\boldsymbol{p}$

!-Depictable Universal Introduction (!D∀I)

$\underline{!\mathrm{D}c}$ Aux !D∀I

…

$\boldsymbol{p}(c/x)$

⊳ $!\mathrm{D}\forall x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux !D∀I.
ii) c does not occur in $!\mathrm{D}\forall x\boldsymbol{p}$.

!-Depictable Universal Elimination (!D∀E)

$!\mathrm{D}\forall x\boldsymbol{p}$

$!\mathrm{D}c$

⊳ $\boldsymbol{p}(c/x)$

!-Depictable Existential Elimination (!D∃E)

$!\mathrm{D}\exists x\boldsymbol{p}$

$\underline{\boldsymbol{p}(c/x) \wedge !\mathrm{D}c}$

$\boldsymbol{q}$

⊳ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in $!\mathrm{D}\exists x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

!-Depictable Existential Introduction (!D∃I)

$\boldsymbol{p}(c/x)$

$!\mathrm{D}c$

⊳ $!\mathrm{D}\exists x\boldsymbol{p}$

!-Sentential Universal Introduction (!S∀I)

!Sc	Aux !S∀I
…	
$\boldsymbol{p}(c/x)$	

⊳ !S∀$x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux !S∀I.
ii) c does not occur in !S∀$x\boldsymbol{p}$.

!-Sentential Universal Elimination (!S∀E)

!S∀$x\boldsymbol{p}$
!Sc
⊳ $\boldsymbol{p}(c/x)$

!-Sentential Existential Introduction (!S∃I)

$\boldsymbol{p}(c/x)$
!Sc
⊳ !S∃$x\boldsymbol{p}$

!-Sentential Existential Elimination (!S∃E)

!S∃$x\boldsymbol{p}$
  $\underline{\boldsymbol{p}(c/x) \wedge !Sc}$
  $\boldsymbol{q}$
⊳ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in !S∃$x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

Extant Universal Introduction (E∀I)

Ec	Aux E∀I
…	
$\boldsymbol{p}(c/x)$	

⊳ E∀$x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux E∀I.
ii) c does not occur in E∀$x\boldsymbol{p}$.

Extant Universal Elimination (E∀E)

E∀$x\boldsymbol{p}$
Ec
⊳ $\boldsymbol{p}(c/x)$

Extant Existential Introduction (E∃I)

$\boldsymbol{p}(c/x)$
Ec
⊳ E∃$x\boldsymbol{p}$

Extant Existential Elimination (E∃E)

E∃$x\boldsymbol{p}$
  $\underline{\boldsymbol{p}(c/x) \wedge Ec}$
  $\boldsymbol{q}$
⊳ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in E∃$x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

Depictable Universal Introduction (D∀I)

Dc	Aux D∀I
…	
$\boldsymbol{p}(c/x)$	

⊳ D∀$x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux D∀I.
ii) c does not occur in D∀$x\boldsymbol{p}$.

Depictable Universal Elimination (D∀E)

D∀$x\boldsymbol{p}$
Dc
⊳ $\boldsymbol{p}(c/x)$

Depictable Existential Introduction (D∃I)

$\boldsymbol{p}(c/x)$
Dc
▻ D∃$x\boldsymbol{p}$

Depictable Existential Elimination (D∃E)

D∃$x\boldsymbol{p}$
$\underline{\boldsymbol{p}(c/x) \wedge \mathrm{D}c}$
$\boldsymbol{q}$
▻ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in D∃$x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

Sentential Universal Introduction (S∀I)

$\underline{\mathrm{S}c}$ Aux S∀I
…
$\boldsymbol{p}(c/x)$

▻ S∀$x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux S∀I.
ii) c does not occur in S∀$x\boldsymbol{p}$.

Sentential Universal Elimination (S∀E)

S∀$x\boldsymbol{p}$
Sc
▻ $\boldsymbol{p}(c/x)$

Sentential Existential Introduction (S∃I)

$\boldsymbol{p}(c/x)$
Sc
▻ S∃$x\boldsymbol{p}$

Sentential Existential Elimination (S∃E)

S∃$x\boldsymbol{p}$
$\underline{\boldsymbol{p}(c/x) \wedge \mathrm{S}c}$
$\boldsymbol{q}$
▻ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in S#∃$x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

Identity Introduction (=I)

▻ $\forall \mathbf{X}(\mathbf{X} = \mathbf{X})$

Identity Elimination (=E)

$\mathbf{C} = \mathbf{D}$

$\boldsymbol{p}$

▻ $\boldsymbol{p}(\mathbf{C}/\mathbf{D})$

Or

$\mathbf{C} = \mathbf{D}$

$\boldsymbol{p}$

▻ $\boldsymbol{p}(\mathbf{D}/\mathbf{C})$

Note that we include the rules of the extended natural deduction system (i.e De Morgan, Transposition, quantifier negation, etc.) but, since they are derivable from all of the above (and are found in any logic textbook), we do not present them here. We do include the following axioms unique to $\mathcal{JL}$ and second-order logics however.

Default Sententiality (DS)

⊳ $\forall x(!\mathrm{E}x \vee [!\mathrm{D}x \vee !\mathrm{S}x])$

(informally: all objects start, at least, at sententiality)

No Proof (NP)

⊳ $\ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$

(informally: if we can say, but only say that ***s***, then ***s*** does not obtain)

Extensionality (Ex.)

⊳ $\forall X \forall Y(X = Y \leftrightarrow \forall x(Xx \leftrightarrow Yx))$

(informally: If two predicates are identical, then any object relating to the one, relates to the other and vice-versa)

Comprehension (Comp.)

⊳ $\exists X \forall x_1, \ldots, \forall x_n(Xx_1,\ldots,x_n \leftrightarrow \boldsymbol{p}(x_1,\ldots,x_n))$

(informally: This is the axiom scheme of comprehension *i.e.* any sequence of objects that satisfy some formula ***p***, relate to some predicate *X* and vice-versa)

Lastly, definitions Ax. !E∃, Ax. !D∃, and Ax. !S∃ are axioms of $\boldsymbol{\mathcal{IL}}$.

4.8.2. Proof.

A *proof* (alternatively *derivation*) in $\boldsymbol{\mathcal{IL}}$ of ***p*** from Γ (where, $\boldsymbol{p} \in \mathrm{Sent}(\boldsymbol{\mathcal{IL}})$ and $\Gamma \subseteq \mathrm{Sent}(\boldsymbol{\mathcal{IL}})$) consists of a series:

$$
\begin{array}{ll}
1. & \Gamma \\
 & \vdots \\
\hline
m. & \boldsymbol{q}_1 \\
 & \ldots \\
n. & \boldsymbol{q}_n
\end{array}
$$

where 'Γ . . .' is a list of the sentences of Γ (where Γ is possibly empty), $\boldsymbol{p} = \boldsymbol{q}_n$, $\boldsymbol{q}_1 - \boldsymbol{q}_n$ are $\mathcal{IL}$-formulas, $\boldsymbol{q}_n$ can be derived by application of some rule of inference to formulas on lines $i < n$, and $\boldsymbol{q}_n$ falls only under the assumptions of 'Γ . . .'.

4.8.2.0. $\boldsymbol{p}$ *is provable from* Γ (denoted '$\Gamma \vdash \boldsymbol{p}$') *iff* there is a proof of $\boldsymbol{p}$ from Γ

4.8.2.1. Γ is *consistent* (in $\mathcal{IL}$) *iff* for no $\mathcal{IL}$-formula $\boldsymbol{p}$, $\Gamma \vdash \boldsymbol{p}$ and $\Gamma \vdash \neg\boldsymbol{p}$

4.8.2.2. $\varnothing \vdash \boldsymbol{p}$ is abbreviated $\vdash \boldsymbol{p}$

4.8.2.3. $\vdash \boldsymbol{p}$ indicates that $\boldsymbol{p}$ is a *theorem*

4.9. Theorems (where *x* and *y* are individual variables)

4.9.0.

4.9.0.1. $\vdash \mathrm{E}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.2. $\vdash \mathrm{D}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.3. $\vdash \mathrm{S}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.4. $\vdash !\mathrm{E}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.5. $\vdash !\mathrm{D}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

4.9.0.6. $\vdash !\mathrm{S}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

Proof: Each of theorems 4.9.0.1. - 4.9.0.6. follow from the fact that in any case of existentialization (for example, in $\mathrm{E}\exists x\boldsymbol{p}(x)$) it follows by the relevant existential elimination rule (where some *c* is an arbitrary witness for *x*, i.e. p(*c*)) that $\exists x\boldsymbol{p}(x)$ is derived by existential introduction.

4.9.1. $\vdash \forall x\mathrm{S}\exists y(x = y)$

(informally: all objects are sentential)

Proof: by DS, *x* either starts at extantiality, depictability, or sententiality. In any of those cases, sententiality is implied.

4.9.2. $\vdash \forall x(\mathrm{E}\exists y(x = y) \rightarrow \mathrm{D}\exists y(x = y) \wedge \mathrm{S}\exists y(x = y))$

(informally: if *x* is extant, then *x* is depictable and sentential)

Proof: assume *x* is extant. By 4.9.1. *x* is sentential. If *x* is not depictable then (by 4.3.1.) *x* cannot start at extantiality. Further, because *x* is extant (by 4.3.3.), *x* cannot start at sententiality either. Since *x* neither starts at extantiality nor starts at sententiality, by DS, *x* must start at depictability and (by 4.3.2.) *x* is depictable. This is a contradiction, hence *x* is depictable.

4.9.3. $\vdash \forall x(\mathrm{D}\exists y(x = y) \rightarrow \mathrm{S}\exists y(x = y))$

(informally: if *x* is depictable, then *x* is sentential)

Proof: an immediate consequence of 4.9.1.

4.9.4. $\vdash \forall x[\neg(!\mathrm{E}x \wedge !\mathrm{D}x) \wedge \neg(!\mathrm{D}x \wedge !\mathrm{S}x) \wedge \neg(!\mathrm{E}x \wedge !\mathrm{S}x)]$

Informally: all objects can start in just one order.

Proof: Assume $!\mathrm{E}x \wedge !\mathrm{D}x$. By 4.3.1., $!\mathrm{E}x$ implies $\mathrm{E}\exists y(y = x)$ and by 4.3.2., $!\mathrm{D}x$ implies $\neg\mathrm{E}\exists y(y = x)$, a contradiction. The same logic applies to $!\mathrm{D}x \wedge !\mathrm{S}x$ and $!\mathrm{E}x \wedge !\mathrm{S}x$.

Chapter 5

Meta-Results for Impossible Logic: Soundness, Completeness, and Other Proofs

•

Note that in this chapter we prove important meta-results for $\mathcal{IL}$, where, because $\mathcal{IL}$ is just an extension of $\mathcal{GL}$ by a single additional operator, what is a meta-property of $\mathcal{IL}$ is a meta-property of $\mathcal{GL}$. We proceed with the following proofs in the manner standard of most texts on second-order logic. There is rarely a need to deviate from the standard proofs, as formulas featuring the new operators (of either $\mathcal{GL}$ and $\mathcal{IL}$) require the same reckoning as those of standard second order systems. For this reason, we limit certain proofs by cases or induction to just those formulas unique to $\mathcal{IL}$ (where indicated), and only sketch certain other details accessible in relevant textbooks[29] (where indicated). It is important to note here that, although the logic is second-order, the semantics feature Henkin structures, and so completeness, compactness, and the Lowenheim-Skolem results follow in the usual way[30]. Secondly, because, by definition, each structure is Henkin Faithful (i.e. each structure satisfies every comprehension axiom), soundness follows as well[31]. We start this chapter with a quick lemma and then a proof of soundness.

§5.1. Soundness

We prove a simple lemma for satisfying formulas of the abbreviated form $\boldsymbol{p} \vee \boldsymbol{q}$. This result will make proving the universal validity of DS much simpler.

Disjunction Lemma (DL). $\mathcal{M} \vDash \boldsymbol{p} \vee \boldsymbol{q} \quad \Leftrightarrow \quad \mathcal{M} \vDash \boldsymbol{p}$ or $\mathcal{M} \vDash \boldsymbol{q}$

Proof.

$\mathcal{M} \vDash \boldsymbol{p} \vee \boldsymbol{q}$	$\Leftrightarrow$	$\mathcal{M} \vDash \neg(\neg \boldsymbol{p} \wedge \neg \boldsymbol{q})$	4.2.4.
	$\Leftrightarrow$	NOT $\mathcal{M} \vDash \neg \boldsymbol{p} \wedge \neg \boldsymbol{q}$	4.7.1.3.
	$\Leftrightarrow$	NOT $\mathcal{M} \vDash \neg \boldsymbol{p}$ and $\mathcal{M} \vDash \neg \boldsymbol{q}$	4.7.1.4.
	$\Leftrightarrow$	NOT Both	4.7.1.3.

[29] See Bell, 2006 and Shapiro, 1991 for two examples.
[30] See Shapiro, 1991, Ch. 3.
[31] Again, see Shapiro, 1991, Ch. 3.

	NOT$\mathcal{M} \vDash \boldsymbol{p}$ and NOT $\mathcal{M} \vDash \boldsymbol{q}$	
$\Leftrightarrow$	NOT NOT$\mathcal{M} \vDash \boldsymbol{p}$ or NOT NOT $\mathcal{M} \vDash \boldsymbol{q}$	DeMorgans
$\Leftrightarrow$	$\mathcal{M} \vDash \boldsymbol{p}$ or $\mathcal{M} \vDash \boldsymbol{q}$	DN

□

Soundness Theorem. Given a theory $\Gamma \subseteq \text{Sent}(\mathcal{IL})$ and an $\mathcal{IL}$ formula $\boldsymbol{p}$,

$$\Gamma \vdash \boldsymbol{s} \qquad \Rightarrow \qquad \Gamma \vDash \boldsymbol{s}$$

Proof. We only provide the relevant aspects of the proof here. Since the proof follows the standard methods for establishing the fact that each inference rule preserves truth (these proofs being the same as those found in any logic textbook featuring natural deduction) we omit the proofs for $\neg$I, $\neg$E, $\wedge$I, $\wedge$E, =I, =E, $\exists$I, and $\exists$E, and focus on the inference rules and axioms unique to $\mathcal{IL}$. We proceed with a proof by induction on the number of open assumptions in an arbitrary $\mathcal{IL}$ derivation. Let $\boldsymbol{p}_n$ indicate a formula listed at the nth position in an $\mathcal{IL}$ derivation. Let Γ_n be the set of assumptions open at position n. Sentence $\boldsymbol{p}_1$ of any derivation is either an axiom of $\mathcal{IL}$ or an open assumption. If $\boldsymbol{p}_1$ is an axiom then it is universally valid (to be proven in the subsequent passage) and entailed by any theory Γ (hence $\Gamma_1 \vDash \boldsymbol{p}_1$) and if $\boldsymbol{p}_1$ is an open assumption, then $\Gamma_1 \vDash \boldsymbol{p}_1$, as $\Gamma_1 = \{\boldsymbol{p}_1\}$ and $\{\boldsymbol{p}_1\} \vDash \boldsymbol{p}_1$. We assume by inductive hypothesis (IH) that for an arbitrary position $n > 1$, $\Gamma_m \vDash \boldsymbol{p}_m$ for each position $m < n$. We now show that for each inference rule of $\mathcal{IL}$ natural deduction, if $\Gamma_n \vdash \boldsymbol{p}_n$ (by some inference rule), then $\Gamma_n \vDash \boldsymbol{p}_n$.

We start with !-Extant Existential Introduction (!E$\exists$I). If $\boldsymbol{p}_n$ is derived from an application of !E$\exists$I then $\boldsymbol{p}_n$ is of the form $!\text{E}\exists v_n \boldsymbol{q}$ and is derived as follows

h.	$\boldsymbol{q}(c/x)$	
j.	$!\text{E}c$	
…		
n.	$!\text{E}\exists v_n \boldsymbol{q}$	h, j !E$\exists$I

By IH, $\Gamma_j \vDash !\text{E}c$ and $\Gamma_h \vDash \boldsymbol{q}(c/x)$. Further, since $!\text{E}\exists v_n \boldsymbol{q}$ has access to the assumptions open at j and h,

$$\Gamma_j \cup \Gamma_h \qquad \subseteq \qquad \Gamma_n$$

hence what $\Gamma_j \cup \Gamma_h$ entails, Γ_n entails, and so, $\Gamma_n \vDash !\text{E}c$ and $\Gamma_n \vDash \boldsymbol{q}(c/x)$. It follows then that for an arbitrary $\mathcal{IL}$ structure $\mathcal{M}$ that satisfies Γ_n,

$\mathcal{M} \vDash !\mathrm{E}c$ and $\mathcal{M} \vDash \boldsymbol{q}(c/x)$

and (by 4.7.3.) we see that $c^{\mathcal{M}a,r} \in S$, $c^{\mathcal{M}a,r} \in D$, and $c^{\mathcal{M}a,r} \in E$, and (by 4.7.3. again) $\mathcal{M} \vDash !\mathrm{E}\exists v_n\boldsymbol{q}$, hence

$\Gamma_n \vDash !\mathrm{E}\exists v_n\boldsymbol{q}$

therefore

$\Gamma_n \vDash \boldsymbol{p}_n$

By the same reasoning (*mutatis mutandis*) we can show that $\Gamma_n \vDash \boldsymbol{p}_n$ when $\boldsymbol{p}_n$ is derived by application of !D∃I, !S∃I, E∃I, D∃I, S∃I. We omit these demonstrations for brevity.

We now consider the derivation of $\boldsymbol{p}_n$ by an application of !E∃E. If $\boldsymbol{p}_n$ is derived from an application of !E∃E, then the derivation is as follows

h.	$!\mathrm{E}\exists x\boldsymbol{q}$		
…			
j.		$\underline{\boldsymbol{q}(c/x) \wedge !\mathrm{E}c}$	
…			
k.		$\boldsymbol{p}_n$	
…			
n.	$\boldsymbol{p}_n$		*h*, *j*-*k* !E∃E

By the above derivation, $\boldsymbol{p}_n$ has access to the assumptions open at *h* and *k* (save for $\boldsymbol{p}(c/x) \wedge !\mathrm{E}c$), hence

$\Gamma_h \quad \subseteq \quad \Gamma_n$

and

$\Gamma_k \quad \subseteq \quad \Gamma_n \cup \{ \boldsymbol{q}(c/x) \wedge !\mathrm{E}c \}$

By IH, $\Gamma_h \vDash !\mathrm{E}\exists x\boldsymbol{q}$ and since $\Gamma_h \subseteq \Gamma_n$,

$\Gamma_n \vDash !\mathrm{E}\exists x\boldsymbol{q}$.

By IH again, $\Gamma_k \vDash \boldsymbol{p}_n$, and because $\Gamma_k \subseteq \Gamma_n \cup \{\boldsymbol{q}(c/x) \wedge !\mathrm{E}c\}$,

$\Gamma_n \cup \{\boldsymbol{q}(c/x) \wedge !\mathrm{E}c\} \vDash \boldsymbol{p}_n$.

Note though that this implies that

$\Gamma_n \vDash \boldsymbol{p}_n$

as $\Gamma_n \vDash !\mathrm{E}\exists x\boldsymbol{q}$, and so (by 4.7.3.) for any $\mathcal{IL}$ structure $\mathcal{M}$ that satisfies Γ_n, for some $t^{\mathcal{M},a,r} \in E$

$\mathcal{M} \vDash \boldsymbol{q}(t/x) \wedge !\mathrm{E}t$

And because the constant c, by the conditions on !E∃E, does not appear in Γ, $!\mathrm{E}\exists x\boldsymbol{q}$, or $\boldsymbol{p}_n$, we have it that c is arbitrary and for any term t in place of x in $\boldsymbol{q}(t/x) \wedge !\mathrm{E}t$, when $\{\boldsymbol{p}(t/x) \wedge !\mathrm{E}t\}$is conjoined with Γ,

$\Gamma \cup \{\boldsymbol{q}(t/x) \wedge !\mathrm{E}t\} \vDash \boldsymbol{p}_n$.

It follows then, that because $\mathcal{M} \vDash \Gamma$ and $\mathcal{M} \vDash \boldsymbol{q}(t/x) \wedge !\mathrm{E}t$,

$\mathcal{M} \vDash \boldsymbol{p}_n$

hence

$\Gamma_n \vDash \boldsymbol{p}_n$

By the same reasoning (*mutatis mutandis*) we can show that $\Gamma_n \vDash \boldsymbol{p}_n$ when $\boldsymbol{p}_n$ is derived by application of !D∃E, !S∃E, E∃E, D∃E, S∃E. Again, we omit these proofs for brevity. We now move on to the axioms of $\mathcal{IL}$. First, we consider the definitions,

Ax. !E∃	$\forall y(!\mathrm{E}\exists x(x = y)$	$\leftrightarrow$	$[(\mathrm{E}\exists x(x = y) \wedge \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$
Ax. !D∃	$\forall y(!\mathrm{D}\exists x(x = y)$	$\leftrightarrow$	$[(\neg\mathrm{E}\exists x(x = y) \wedge \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$
Ax. !S∃	$\forall y(!\exists\mathrm{S}x(x = y)$	$\leftrightarrow$	$[(\neg\mathrm{E}\exists x(x = y) \wedge \neg\mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$

Let $\boldsymbol{p}_n$ = Ax. !E∃ Because Ax. !E∃ is an axiom, it can be derived from any theory Γ, hence $\Gamma_n \vdash$ Ax. !E∃ To show that Ax. !E∃ is universally valid is to show that it is entailed by any Γ, hence $\Gamma_n \vDash$ Ax. !E∃. Let $\mathcal{M}$ be an arbitrary structure of $\mathcal{IL}$ and let d be an arbitrary element of S in $\mathcal{M}$. Then

$\mathcal{M} \vDash_{[m\mid d]a,\, r} !\mathrm{E}\exists v_n(v_n = v_m)$	$\Leftrightarrow$	for some $b \in E$, $\mathcal{M} \vDash_{[n\mid b][m\mid d]a,\, r} v_n = v_m$	4.7.3.
	$\Leftrightarrow$	for some $b \in E$, $\mathcal{M} \vDash_{[n\mid b][m\mid d]a,\, r} v_n = v_m$ and for some $b \in D$, $\mathcal{M} \vDash_{[n\mid b][m\mid d]a,\, r} v_n = v_m$ and for some $b \in S$, $\mathcal{M} \vDash_{[n\mid b][m\mid d]a,\, r} v_n = v_m$	D7
	$\Leftrightarrow$	$\mathcal{M} \vDash_{[m\mid d]a,\, r} \mathrm{E}\exists v_n(v_n = v_m)$ and $\mathcal{M} \vDash_{[m\mid d]a,\, r} \mathrm{D}\exists v_n(v_n = v_m)$ and $\mathcal{M} \vDash_{[m\mid d]a,\, r} \mathrm{S}\exists v_n(v_n = v_m)$	4.7.2.
	$\Leftrightarrow$	$\mathcal{M} \vDash_{[m\mid d]a,\, r} \mathrm{E}\exists v_n(v_n = v_m) \wedge \mathrm{D}\exists v_n(v_n = v_m)$	4.7.1.4.

and

$\mathcal{M} \vDash_{[m|d]a,\, r} \mathrm{S}\exists v_n(v_n = v_m)$

$\Leftrightarrow$ $\mathcal{M} \vDash_{[m|d]a,\, r} (\mathrm{E}\exists v_n(v_n = v_m) \wedge \mathrm{D}\exists v_n(v_n = v_m)) \wedge \mathrm{S}\exists v_n(v_n = v_m)$ 4.7.1.4.

Thus

$$\mathcal{M} \vDash_{[m|d]a,\, r} !\mathrm{E}\exists v_n(v_n = v_m) \leftrightarrow [(\mathrm{E}\exists v_n(v_n = v_m) \wedge \mathrm{D}\exists v_n(v_n = v_m)) \wedge \mathrm{S}\exists v_n(v_m = v_m)]$$

Where, because d is arbitrary,

$$\mathcal{M} \vDash_{a,\, r} \forall v_m(!\mathrm{E}\exists v_n(v_n = v_m) \leftrightarrow [(\mathrm{E}\exists v_n(v_n = v_m) \wedge \mathrm{D}\exists v_n(v_n = v_m)) \wedge \mathrm{S}\exists v_n(v_m = v_m)])$$

And lastly, because $\mathcal{M}$ is arbitrary, we have it that,

$$\vDash \forall v_m(!\mathrm{E}\exists v_n(v_n = v_m) \leftrightarrow [(\mathrm{E}\exists v_n(v_n = v_m) \wedge \mathrm{D}\exists v_n(v_n = v_m)) \wedge \mathrm{S}\exists v_n(v_m = v_m)])$$

Hence Ax. !E∃ is entailed by any theory Γ, so

$$\Gamma_n \vDash (!\mathrm{E}\exists v_n(v_n = v_m) \leftrightarrow [(\mathrm{E}\exists v_n(v_n = v_m) \wedge \mathrm{D}\exists v_n(v_n = v_m)) \wedge \mathrm{S}\exists v_n(v_m = v_m)])$$

therefore

$$\Gamma_n \vDash \boldsymbol{p}_n$$

Ax. !D∃ and Ax. !S∃ can be shown universally valid according to the same proofs, hence entailed by any theory Γ, and so these proofs will, once again, be omitted for brevity. Next consider

$\rhd \forall x((!\mathrm{E}x \vee !\mathrm{D}x) \vee !\mathrm{S}x)$ DS

First, note that DS abbreviates

$$\neg\exists x\neg\neg(\neg(\neg!\mathrm{E}\exists y(y = x) \wedge \neg!\mathrm{D}\exists y(y = x)) \wedge \neg!\mathrm{S}\exists y(y = x))$$

But, we will use the disjunction lemma to prove universally valid the disjunctive equivalent,

$$\forall x((!\mathrm{E}\exists y(y = x) \vee !\mathrm{D}\exists y(y = x)) \vee !\mathrm{S}\exists y(y = x))$$

Consider an arbitrary structure $\mathcal{M}$. By definition, the domain S of $\mathcal{M}$ is non-empty. Let $b \in S$ be an arbitrary element of $\mathcal{M}$. By D7, either

$b \in S$ and $b \in D$ and $b \in E$ Case 1

or

$b \in S$ and $b \in D$ and $b \notin E$ Case 2 or

$b \in S$ and $b \notin D$ and $b \notin E$ Case 3

To prove that DS is universally valid we show that in all of cases 1-3,

$\forall x((!\mathrm{E}\exists y(y = x) \vee !\mathrm{D}\exists y(y = x)) \vee !\mathrm{S}\exists y(y = x))$

is satisfied. Case 1.

$b \in S$ *and* $b \in D$ and $b \in E$	$\Rightarrow$	$v_m{}^{(\mathcal{M},\, [m\mid b][n\mid b]\boldsymbol{a},\, \boldsymbol{r})} = v_n{}^{(\mathcal{M},\, [m\mid b][n\mid b]\boldsymbol{a},\, \boldsymbol{r})}$	4.6.2/3.
	$\Rightarrow$	for some $b \in E$, $\mathcal{M} \vDash_{[m\mid b][n\mid b]a,\, r} v_m = v_n$	4.7.1.1.
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{E}\exists v_m(v_m = v_n)$	4.7.3.
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{E}\exists v_m(v_m = v_n)$ or $\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{ED}v_m(v_m = v_n)$	DL
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)$	DL
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)$ or $\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{S}\exists v_m(v_m = v_n)$	DL
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} (!\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)) \vee !\mathrm{S}\exists v_m(v_m = v_n)$	DL

Case 2.

$b \in S$ *and* $b \in D$ and $b \notin E$	$\Rightarrow$	$v_m{}^{(\mathcal{M},\, [m\mid b][n\mid b]\boldsymbol{a},\, \boldsymbol{r})} = v_n{}^{(\mathcal{M},\, [m\mid b][n\mid b]\boldsymbol{a},\, \boldsymbol{r})}$	4.6.2/3.
	$\Rightarrow$	for some $b \in D$ (where $d \notin E$) $\mathcal{M} \vDash_{[m\mid b][n\mid b]a,\, r} v_m = v_n$	4.7.1.1.
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{D}\exists v_m(v_m = v_n)$	4.7.3.
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{E}\exists v_m(v_m = v_n)$ or $\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{ED}v_m(v_m = v_n)$	DL
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)$	DL
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)$ or $\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{S}\exists v_m(v_m = v_n)$	DL
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} (!\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)) \vee !\mathrm{S}\exists v_m(v_m = v_n)$	DL

Case 3.

$b \in S$ *and* $b \notin D$ and $b \notin E$	$\Rightarrow$	$v_m{}^{(\mathcal{M},\, [m\mid b][n\mid b]\boldsymbol{a},\, \boldsymbol{r})} = v_n{}^{(\mathcal{M},\, [m\mid b][n\mid b]\boldsymbol{a},\, \boldsymbol{r})}$	4.6.2/3.
	$\Rightarrow$	for some $b \in S$ (where $b \notin D$ and $b \notin E$) $\mathcal{M} \vDash_{[m\mid b][n\mid b]a,\, r} v_m = v_n$	4.7.1.1.
	$\Rightarrow$	$\mathcal{M} \vDash_{[n\mid b]a,\, r} !\mathrm{S}\exists v_m(v_m = v_n)$	4.7.3.

$\Rightarrow \quad \mathcal{M} \vDash_{[n|b]a,\, r} !\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)$ DL

or

$\mathcal{M} \vDash_{[n|b]a,\, r} !\mathrm{S}\exists v_m(v_m = v_n)$

$\Rightarrow \quad \mathcal{M} \vDash_{[n|b]a,\, r} (!\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)) \vee !\mathrm{S}\exists v_m(v_m = v_n)$ DL

Thus in all cases, DS is satisfied by $\mathcal{M}$. Because b is arbitrary

$$\mathcal{M} \vDash_{a,\, r} \forall v_n((!\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)) \vee !\mathrm{S}\exists v_m(v_m = v_n))$$

Further, Because $\mathcal{M}$ is arbitrary, we have it that,

$$\vDash \forall v_n((!\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)) \vee !\mathrm{S}\exists v_m(v_m = v_n))$$

Hence DS is entailed by any theory Γ, so

$$\Gamma_n \vDash \forall v_n((!\mathrm{E}\exists v_m(v_m = v_n) \vee !\mathrm{ED}v_m(v_m = v_n)) \vee !\mathrm{S}\exists v_m(v_m = v_n))$$

therefore

$$\Gamma_n \vDash \boldsymbol{p}_n$$

Finally, we move to

$\rhd \ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$ NP

Again, consider an arbitrary structure $\mathcal{M}$ and let $\boldsymbol{s} \in \mathrm{Sent}(\mathcal{IL})$ of $\mathcal{M}$. Assume $\mathcal{M}$ fails to satisfy $\ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$ then,

NOT $\mathcal{M} \vDash_{a,\, r} \ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$	$\Leftrightarrow$	NOT $\mathcal{M} \vDash_{a,\, r} \neg(\neg\neg \ulcorner \boldsymbol{s} \urcorner \wedge \neg\neg \boldsymbol{s})$	4.2.4.
	$\Leftrightarrow$	NOT NOT $\mathcal{M} \vDash_{a,\, r} \neg\neg \ulcorner \boldsymbol{s} \urcorner \wedge \neg\neg \boldsymbol{s}$	4.7.1.3.
	$\Leftrightarrow$	$\mathcal{M} \vDash_{a,\, r} \neg\neg \ulcorner \boldsymbol{s} \urcorner$ and $\mathcal{M} \vDash_{a,\, r} \neg\neg \boldsymbol{s}$	DN, 4.7.1.4.
	$\Leftrightarrow$	NOT NOT $\mathcal{M} \vDash_{a,\, r} \ulcorner \boldsymbol{s} \urcorner$ and NOT NOT $\mathcal{M} \vDash_{a,\, r} \boldsymbol{s}$	4.7.1.3.
	$\Leftrightarrow$	$\mathcal{M} \vDash_{a,\, r} \ulcorner \boldsymbol{s} \urcorner$ and $\mathcal{M} \vDash_{a,\, r} \boldsymbol{s}$	DN

Since, for $\mathcal{M}$ to fail to satisfy $\ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$ it would need to be the case that both $\mathcal{M} \vDash_{a,\, r} \ulcorner \boldsymbol{s} \urcorner$ and $\mathcal{M} \vDash_{a,\, r} \boldsymbol{s}$. By 4.7.1., because $\mathcal{M} \vDash_{a,\, r} \ulcorner \boldsymbol{s} \urcorner$ and $\boldsymbol{s}$ has no free variables, $\boldsymbol{s} \in R\ulcorner\,\urcorner$. Our constraints on $R\ulcorner\,\urcorner$ (in §4.5.) guarantee that NOT $\mathcal{M} \vDash_{a,\, r} \boldsymbol{s}$. This is contradictory, so it follows that it is not possible for $\mathcal{M}$ to fail to satisfy $\ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$. Because these constraints apply to any $\mathcal{IL}$ structure $\mathcal{M}$,

$$\vDash \ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$$

Hence NP is entailed by any theory Γ, so

$$\Gamma_n \vDash \ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$$

therefore

$$\Gamma_n \vDash \boldsymbol{p}_n$$

As a final note, because each $\mathcal{IL}$ structure $\mathcal{M}$ is Henkin faithful, each instance of the comprehension schema is satisfied by any $\mathcal{M}$, hence Γ_n entails each comprehension axiom. And so, we see that each derivation rule unique to $\mathcal{IL}$ is truth preserving. And, because we have assumed that the standard derivation rules of second order logic are truth preserving, we may conclude that *each* derivation rule of $\mathcal{IL}$ is truth preserving. From this it follows that any sentence $\boldsymbol{s}$ in a derivation, because $\boldsymbol{s}$ follows from open assumptions in some Γ (or is an axiom), is entailed by Γ therefore, and we have it that

$$\Gamma \vdash \boldsymbol{s} \quad \Rightarrow \quad \Gamma \vDash \boldsymbol{s} \qquad \square$$

Corollary of Soundness. If $\Gamma \subseteq \text{Sent}(\mathcal{IL})$ has a model, then Γ is consistent.

Proof: Assume an arbitrary Γ has a model $\mathcal{M}$. Assume further that Γ is inconsistent. Then for some $\boldsymbol{s} \in \text{Sent}(\mathcal{IL})$, $\Gamma \vdash \boldsymbol{s}$ and $\Gamma \vdash \neg\boldsymbol{s}$. By soundness $\Gamma \vDash \boldsymbol{s}$ and $\Gamma \vDash \neg\boldsymbol{s}$. However, $\mathcal{M}$ is a model of Γ, but by definition cannot be a model of both $\boldsymbol{s}$ and $\neg\boldsymbol{s}$. $\mathcal{M}$ is not a model of Γ after all then, and we have a contradiction. Therefore, Γ must be consistent. □

§5.2. Completeness, Compactness, and Lowenheim-Skolem Results

Before proving completeness, compactness, and the Loweheim-Skolem results, some important lemmas and theorems will have to be proven. We start with a handful of results that indicate features of the deduction system and semantics that are of general use to model theorists. Their proofs have either already been carried out above or are fairly simple, (once again) found in any logic text, and, for these reasons, will be omitted here. We then move on to the lemmas and theorems unique to completeness proofs in second order logic (with Henkin semantics restricted to faithful Henkin models).

Let us start with a presentation of some concepts and definitions essential for the following proofs. Here they are in no particular order:

Let '$\bigwedge^n_{i=1} \boldsymbol{p}_i$' abbreviate formulas of the form '$\boldsymbol{p}_1 \wedge, \ldots, \wedge \boldsymbol{p}_n$'

$\mathcal{IL}_\Gamma$ is the subset of the vocabulary of $\mathcal{IL}$ where the individual constants and predicate constants of $\mathcal{IL}_\Gamma$ are all and only those among the sentences of Γ.

Consistency. Given $\Gamma \subseteq Sent(\mathcal{IL})$

Γ is *consistent* (in $\mathcal{IL}$) *iff* for no $\mathcal{IL}$-formula $\boldsymbol{p}$, $\Gamma \vdash \boldsymbol{p}$ and $\Gamma \vdash \neg\boldsymbol{p}$

Completeness. Given $\Gamma \subseteq Sent(\mathcal{IL})$

Γ is *complete* *iff* for any $\boldsymbol{s} \in Sent(\mathcal{IL})$, $\Gamma \vdash \boldsymbol{s}$ or $\Gamma \vdash \neg\boldsymbol{s}$

Extensions. Given $\mathcal{IL}$ of type (λ, *J*),

i) $\mathcal{IL}^*$ is an *extension of* $\mathcal{IL}$ *iff* $\mathcal{IL}^* = \{\mathcal{IL} \cup \{\{P_j : i \in I^*\}\} \cup \{\{c_j : j \in J^*\}\}\}$

where $I^* \cap I = \varnothing$ and $J^* \cap J = \varnothing$

ii) $\Gamma^* \subseteq Sent(\mathcal{IL}^*)$ is an $\mathcal{IL}$-*saturated extension of* Γ in $\mathcal{IL}^*$ when $\Gamma \subseteq \Gamma^*$ and, for any $\mathcal{IL}$-formula $\boldsymbol{p}$ (with at most one free variable x), there are constant symbols $c_1, c_2, c_3, c_4, c_5, c_6,$ and c_7 of $\mathcal{IL}^*$ such that,

$\Gamma^* \vdash \exists x\boldsymbol{p}(x) \rightarrow \boldsymbol{p}(c_1)$
$\Gamma^* \vdash S\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_2) \wedge Sc_2)$
$\Gamma^* \vdash D\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_3) \wedge Dc_3)$
$\Gamma^* \vdash E\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_4) \wedge Ec_4)$
$\Gamma^* \vdash !S\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_5) \wedge !Sc_5)$
$\Gamma^* \vdash !D\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_6) \wedge !Dc_6)$
$\Gamma^* \vdash !E\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_7) \wedge !Ec_7)$

And for any $\mathcal{IL}$-formula $\boldsymbol{p}$ (with at most one free variable X), there is a constant symbol P of $\mathcal{IL}^*$ such that,

$\Gamma^* \vdash \exists x\boldsymbol{p}(X) \rightarrow \boldsymbol{p}(P)$

iii) $\Gamma \subseteq Sent(\mathcal{IL})$ is *saturated* when, for any $\mathcal{IL}$-formula $\boldsymbol{p}$ (with at most one free variable x), there are constant symbols $c_1, c_2, c_3, c_4, c_5, c_6,$ and c_7 of $\mathcal{IL}$ such that,

$\Gamma \vdash \exists x\boldsymbol{p}(x) \rightarrow \boldsymbol{p}(c_1)$
$\Gamma \vdash S\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_2) \wedge Sc_2)$
$\Gamma \vdash D\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_3) \wedge Dc_3)$
$\Gamma \vdash E\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_4) \wedge Ec_4)$
$\Gamma \vdash !S\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_5) \wedge !Sc_5)$
$\Gamma \vdash !D\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_6) \wedge !Dc_6)$
$\Gamma \vdash !E\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_7) \wedge !Ec_7)$

And for any $\mathcal{IL}$-formula $\boldsymbol{p}$ (with at most one free variable X), there is a constant symbol P of $\mathcal{IL}$ such that,

$\Gamma \vdash \exists x \boldsymbol{p}(X) \to \boldsymbol{p}(P)$

Expansions and Reductions.

$\mathcal{IL}^*$-*structure* (denoted $\mathcal{M}^*$) := $(E, D, S, \mathrm{Sent}(\mathcal{IL}^*), \mathcal{C}^*, \mathcal{P}^*, \mathcal{V}^*, \{R_i : i \in I \cup I^*\}, R\ulcorner\urcorner, \{e_j : j \in J \cup J^*\}, \mathcal{R})$

$\mathcal{M}$ is called an $\mathcal{IL}$–*reduction* of $\mathcal{M}^*$ (denoted '$\mathcal{M}^*|\mathcal{IL}$')

and

$\mathcal{M}^*$ is called an $\mathcal{IL}$ *-*expansion* of $\mathcal{M}$

And, here are the theorems and lemmas essential to the proof of completeness (but taken for granted):

Cardinality Lemma 1. $|\mathcal{IL}| = |\mathrm{Form}(\mathcal{IL})|$ □

Cardinality Lemma 2. $|\mathcal{IL}_\Gamma| = \max(\aleph_0, \Gamma)$ □

Expansion Lemma. Let $\Gamma \subseteq \mathrm{Sent}(\mathcal{IL})$, let $\mathcal{IL}^*$ be any extension of $\mathcal{IL}$, let $\mathcal{M}$ be any $\mathcal{IL}$-structure, and let $\mathcal{M}^*$ be any $\mathcal{IL}^*$-expansion of $\mathcal{M}$. Then

$$\mathcal{M} \vDash \Gamma \;\leftrightarrow\; \mathcal{M}^* \vDash \Gamma \qquad \Box$$

Constants Lemma.

$$\begin{array}{lcl} \mathcal{M} \vDash \boldsymbol{p}(c_0, \ldots, c_n) & \leftrightarrow & \mathcal{M} \vDash \boldsymbol{p}[c_0^{\mathcal{M}}, \ldots, c_n^{\mathcal{M}}] \\ \mathcal{M} \vDash \boldsymbol{p}(P_0, \ldots, P_n) & \leftrightarrow & \mathcal{M} \vDash \boldsymbol{p}[P_0^{\mathcal{M}}, \ldots, P_n^{\mathcal{M}}] \end{array} \qquad \Box$$

Quantifier lemma. If x does not occur free in $\boldsymbol{p}$, then

$$\begin{array}{lcl} \Gamma \vdash \exists x(\boldsymbol{p} \wedge \boldsymbol{q}) & \leftrightarrow & (\boldsymbol{p} \wedge \exists x \boldsymbol{q}) \\ \Gamma \vdash \exists x(\boldsymbol{p} \to \boldsymbol{q}) & \leftrightarrow & (\boldsymbol{p} \to \exists x \boldsymbol{q}) \end{array} \qquad \Box$$

Deduction theorem. If $\boldsymbol{s} \in \mathit{Sent}(\mathcal{IL})$, then for any formula $\boldsymbol{p}$,

$$\Gamma \cup \{\boldsymbol{s}\} \vdash \boldsymbol{p} \quad \leftrightarrow \quad \Gamma \vdash \boldsymbol{s} \to \boldsymbol{p} \qquad \Box$$

Finiteness theorem. Where Γ' is finite and $\Gamma' \subseteq \Gamma$,

$$\Gamma \vdash \boldsymbol{p} \quad \to \quad \Gamma' \vdash \boldsymbol{p} \qquad \Box$$

Soundness theorem (as proven above).

$$\Gamma \vdash \boldsymbol{p} \quad \to \quad \Gamma \vDash \boldsymbol{p} \qquad \Box$$

Consistency lemma.

(i) Γ is consistent $\leftrightarrow$ $\Gamma \nvdash \boldsymbol{p}$ for some $\boldsymbol{\mathcal{IL}}$-formula $\boldsymbol{p}$

(ii) Γ is consistent $\leftrightarrow$ for every finite $\Gamma' \subseteq \Gamma$, $\Gamma' \nvdash \boldsymbol{p}$ for some $\boldsymbol{\mathcal{IL}}$-formula $\boldsymbol{p}$

(iii) $\boldsymbol{s} \in Sent(\boldsymbol{\mathcal{IL}})$ $\rightarrow$ $\Gamma \cup \{\boldsymbol{s}\}$ is consistent $\leftrightarrow \Gamma \nvdash \neg\boldsymbol{s}$ □

Generalization Lemma. If $\boldsymbol{p}(v_0, ..., v_n) \in$ Form($\boldsymbol{\mathcal{IL}}$), then

$$\Gamma \vdash \boldsymbol{p} \quad \rightarrow \quad \Gamma \vdash \forall v_0 \ldots \forall v_n \boldsymbol{p}$$ □

With the preliminaries out of the way, we may work our way to completeness.

Lemma 1. Given $\Gamma \subseteq Sent(\boldsymbol{\mathcal{IL}})$, where Γ is consistent,

There is a consistent $\boldsymbol{\mathcal{IL}}$-saturated extension Γ^* (in an extension $\boldsymbol{\mathcal{IL}}^*$ of $\boldsymbol{\mathcal{IL}}$) where $|\boldsymbol{\mathcal{IL}}^*| = |\boldsymbol{\mathcal{IL}}|$

Proof.

Let $F^i \subseteq$ Form($\boldsymbol{\mathcal{IL}}$) where $\boldsymbol{p} \in F^i$ *iff* $\boldsymbol{p}$ has at most one free variable x and let $F^P \subseteq$ Form($\boldsymbol{\mathcal{IL}}$) where $\boldsymbol{p} \in F^P$ *iff* $\boldsymbol{p}$ has at most one free variable X. For each $\boldsymbol{p} \in F^i$, define constants $c_{\boldsymbol{p}}$, $c_{\mathrm{E}\boldsymbol{p}}$, $c_{\mathrm{D}\boldsymbol{p}}$, $c_{\mathrm{S}\boldsymbol{p}}$, $c_{!\mathrm{E}\boldsymbol{p}}$, $c_{!\mathrm{D}\boldsymbol{p}}$, and $c_{!\mathrm{S}\boldsymbol{p}}$ and for each $\boldsymbol{p} \in F^P$, define a predicate $P_{\boldsymbol{p}}$. We now have an extension $\boldsymbol{\mathcal{IL}}^*$ of $\boldsymbol{\mathcal{IL}}$ where, because I^* and J^* are denumerable, that makes $I^* \cup J^*$ denumerable, and therefore $|\boldsymbol{\mathcal{IL}}^*| = |\boldsymbol{\mathcal{IL}}|$. Now let

$$\begin{aligned}\Gamma' := \{&\exists x\boldsymbol{p}(x) \rightarrow \boldsymbol{p}(c_{\boldsymbol{p}}): \boldsymbol{p} \in F^i\} \cup \{\mathrm{E}\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_{\mathrm{E}\boldsymbol{p}}) \wedge \mathrm{E}c_{\mathrm{E}\boldsymbol{p}}): \boldsymbol{p} \in F^i\} \cup \\ &\{\mathrm{D}\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_{\mathrm{D}\boldsymbol{p}}) \wedge \mathrm{D}c_{\mathrm{D}\boldsymbol{p}}): \boldsymbol{p} \in F^i\} \cup \{\mathrm{S}\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_{\mathrm{S}\boldsymbol{p}}) \wedge \mathrm{S}c_{\mathrm{S}\boldsymbol{p}}): \boldsymbol{p} \in F^i\} \cup \\ &\{!\mathrm{E}\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_{!\mathrm{E}\boldsymbol{p}}) \wedge !\mathrm{E}c_{!\mathrm{E}\boldsymbol{p}}): \boldsymbol{p} \in F^i\} \cup \{!\mathrm{D}\exists x\boldsymbol{p}(x) \rightarrow (\boldsymbol{p}(c_{!\mathrm{D}\boldsymbol{p}}) \wedge !\mathrm{D}c_{!\mathrm{D}\boldsymbol{p}}): \boldsymbol{p} \in F^i\} \cup \\ &\{!\mathrm{S}\exists x\boldsymbol{p}(x) \rightarrow \boldsymbol{p}(c_{!\mathrm{S}\boldsymbol{p}}) \wedge !\mathrm{S}c_{!\mathrm{S}\boldsymbol{p}}): \boldsymbol{p} \in F^i\} \cup \{\exists X\boldsymbol{p}(X) \rightarrow \boldsymbol{p}(P_{\boldsymbol{p}}): \boldsymbol{p} \in F^P\}\end{aligned}$$

$$\Gamma^* := \Gamma \cup \Gamma'$$

By definition, Γ^* is an *$\boldsymbol{\mathcal{IL}}$-saturated extension* of Γ in $\boldsymbol{\mathcal{IL}}^*$. To prove consistency of Γ^*, assume the opposite for *reductio*. By the consistency lemma (ii), there is a finite subset $\Gamma'' \subseteq \Gamma'$ where

$$\Gamma \cup \Gamma''$$

is inconsistent. By the consistency lemma (iii), for each $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n \in \Gamma''$,

$$\Gamma \vdash \neg\bigwedge\nolimits^n_{i=1} \boldsymbol{p}_i$$

By the definition above, we see that each $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n \in \Gamma''$ is of just one of the forms

Case 1.	$\exists x\boldsymbol{p}(x)$	$\rightarrow$	$\boldsymbol{p}(c_{\boldsymbol{p}})$,
Case 2.	$\mathrm{E}\exists x\boldsymbol{p}(x)$	$\rightarrow$	$\boldsymbol{p}(c_{\mathrm{E}\boldsymbol{p}}) \wedge \mathrm{E}c_{\mathrm{E}\boldsymbol{p}}$,
Case 3.	$\mathrm{D}\exists x\boldsymbol{p}(x)$	$\rightarrow$	$\boldsymbol{p}(c_{\mathrm{D}\boldsymbol{p}}) \wedge \mathrm{D}c_{\mathrm{D}\boldsymbol{p}}$,

Case 4.	$S\exists x\boldsymbol{p}(x)$	$\rightarrow$	$\boldsymbol{p}(c_{S\boldsymbol{p}}) \wedge Sc_{S\boldsymbol{p}}$,
Case 5.	$!E\exists x\boldsymbol{p}(x)$	$\rightarrow$	$\boldsymbol{p}(c_{!E\boldsymbol{p}}) \wedge !Ec_{!E\mathbf{p}}$,
Case 6.	$!D\exists x\boldsymbol{p}(x)$	$\rightarrow$	$\boldsymbol{p}(c_{!D\boldsymbol{p}}) \wedge !Dc_{!D\boldsymbol{p}}$,
Case 7.	$!S\exists x\boldsymbol{p}(x)$	$\rightarrow$	$\boldsymbol{p}(c_{!S\boldsymbol{p}}) \wedge !Sc_{!S\boldsymbol{p}}$,

or

Case 8.	$\exists X\boldsymbol{p}(X)$	$\rightarrow$	$\boldsymbol{p}(P_{\boldsymbol{p}})$

For each $i \in \{1, \ldots, 8\}$, we show that if Γ proves the negation of any case i, Γ proves a contradiction. From this it follows that each conjunct of $\bigwedge^n_{i=1}\boldsymbol{p}_i$ is provable by Γ, hence

$$\Gamma \vdash \bigwedge^n_{i=1}\boldsymbol{p}_i$$

and we see that Γ is inconsistent, which is contradictory. We limit the demonstration to Case 7 as each other case's proof is virtually the same as Case 7's. Assume Γ proves a negated formula of the form Case 7. That is,

$$\Gamma \vdash \neg[!S\exists x\boldsymbol{p}(x) \quad \rightarrow \quad \boldsymbol{p}(c_{!S\boldsymbol{p}}) \wedge !Sc_{!S\boldsymbol{p}}] \qquad *$$

Note that the formula to the right of the turnstile in * is neither an axiom nor a member of Γ (treat '*' as indicating this formula where there is no ambiguity). That * is not an axiom is obvious, and if * were a member of Γ, then because $\Gamma \subseteq \text{Sent}(\boldsymbol{\mathcal{J}\mathcal{L}})$, $c_{!S\boldsymbol{p}} \in \boldsymbol{\mathcal{J}\mathcal{L}}$, hence $!S\boldsymbol{p} \in J$, and it follows that *J and J** are not disjoint. This outcome contradicts a necessary constraint of Extensions (i), hence $\boldsymbol{\mathcal{J}\mathcal{L}}^*$ would not be an extension (by definition). Since this is contradictory to our assumption, $* \notin \Gamma$. Because * is a derivation of Γ, by definition of *proof*, * falls only under the assumptions of Γ. So, any sub-derivations preceding * are discharged and their assumptions closed. $c_{!S\boldsymbol{p}}$ doesn't feature in Γ, hence any open assumptions, and so (if we select a variable y not free in $\boldsymbol{p}$) an application of $\forall$I in our proof gets us

$$\Gamma \vdash \forall y\neg[!S\exists x\boldsymbol{p}(x) \quad \rightarrow \quad (\boldsymbol{p}(y) \wedge !Sy)]$$

which by equivalence implies

$$\Gamma \vdash \neg\exists y[!S\exists x\boldsymbol{p}(x) \quad \rightarrow \quad (\boldsymbol{p}(y) \wedge !Sy)]$$

Where, because y is not free in $\boldsymbol{p}$, the quantifier lemma gets us

$$\Gamma \vdash \neg[!\mathrm{S}\exists x\boldsymbol{p}(x) \quad \rightarrow \quad \exists y(\boldsymbol{p}(y) \wedge !\mathrm{S}y)] \qquad \neg^{**}$$

However, by !S∃E, R, and ∃I in any derivation we can show

$$\vdash !\mathrm{S}\exists x\boldsymbol{p}(x) \quad \rightarrow \quad \exists y(\boldsymbol{p}(y) \wedge !\mathrm{S}y)$$

So, clearly

$$\Gamma \vdash !\mathrm{S}\exists x\boldsymbol{p}(x) \quad \rightarrow \quad \exists y(\boldsymbol{p}(y) \wedge !\mathrm{S}y)$$

Since, by the same reasoning, we can show for each $i \in \{1, \ldots, 6, 8\}$, $\Gamma \vdash **_i$ for case i, it follows that for each $\boldsymbol{p}_i \in \Gamma''$,

$$\text{if} \quad \Gamma \vdash \neg\boldsymbol{p}_i, \quad \text{then} \quad \Gamma \vdash \neg **_i \quad \text{and} \quad \Gamma \vdash **_i$$

As stated above, this implies Γ is inconsistent, which is contrary to our assumption. Γ* is consistent therefore. □

Lemma 2. Given a consistent $\Gamma \subseteq Sent(\boldsymbol{\mathcal{IL}})$,

there exists a *complete*, *consistent*, $\Gamma' \subseteq Sent(\boldsymbol{\mathcal{IL}})$ where $\Gamma \subseteq \Gamma'$.

Proof. Let

$$C \quad := \quad \{\Gamma' : \Gamma' \subseteq Sent(\boldsymbol{\mathcal{IL}}) \text{ and } \Gamma' \text{ is consistent and } \Gamma \subseteq \Gamma' \text{ or } \Gamma' \subseteq \Gamma\}$$

Order C by inclusion. Since the union of any two chains $a, b \in C$ is itself consistent (as either $a \subseteq b$ or $b \subseteq a$, so either $a \cup b = b$ or $a \cup b = a$ where both a and b are consistent) for any two chains $a, b \subseteq C$, $a \cup b \subseteq C$. Since C is closed under the union of chains, by Zorn's lemma, C has a maximal element Γ'. Clearly $\Gamma \subseteq \Gamma'$. Now consider some $\boldsymbol{s} \in Sent(\boldsymbol{\mathcal{IL}})$. If $\Gamma' \nvdash \boldsymbol{s}$, then by the consistency lemma (iii), $\Gamma' \cup \{\neg\boldsymbol{s}\}$ is consistent. Because Γ' is maximal consistent, $\neg\boldsymbol{s} \in \Gamma'$ or else $\Gamma' \cup \{\neg\boldsymbol{s}\}$ is itself a consistent set, where $\Gamma' \subset \Gamma' \cup \{\neg\boldsymbol{s}\}$, hence Γ' is not maximal. This leads to contradiction, so $\neg\boldsymbol{s} \in \Gamma'$, hence $\Gamma' \vdash \neg\boldsymbol{s}$ and Γ' is complete.

Theorem 1. Given a consistent $\Gamma \subseteq Sent(\boldsymbol{\mathcal{IL}})$,

there exists an *extension* $\boldsymbol{\mathcal{IL}}^+$ of $\boldsymbol{\mathcal{IL}}$ such that $|\boldsymbol{\mathcal{IL}}^+| = |\boldsymbol{\mathcal{IL}}|$ and a *complete saturated consistent* set $\Gamma^+ \subseteq Sent(\boldsymbol{\mathcal{IL}}^+)$ such that $\Gamma \subseteq \Gamma^+$

Proof. We omit the proof here as it is fairly clear how, by application of Lemma 1 to Γ we get a consistent $\mathcal{IL}$-saturated extension Γ^* (of Γ) and by an application of Lemma 2 to Γ^* we get a complete consistent saturated Γ^+. ☐

We now take a brief excursion to define a unique kind of $\mathcal{IL}$ structure, what we call a canonical structure. We define the canonical structure thus

1.1 For an arbitrary, but consistent $\Gamma \subseteq \text{Sent}(\mathcal{IL})$ define the following:

$\mathcal{S} := \{c \in \mathcal{C} : \Gamma \vdash \text{S}c\}$

$\mathcal{D} := \{c \in \mathcal{C} : \Gamma \vdash \text{D}c\}$

$\mathcal{E} := \{c \in \mathcal{C} : \Gamma \vdash \text{E}c\}$

1.2. Then, for each $c \in \mathcal{C}$, define:

$c^{=} := \{d \in \mathcal{C} : \Gamma \vdash c = d\} \cup \{\Gamma\}$

*Note that we define each $c^{=}$ as the union of the equivalence class of constants d where $\Gamma \vdash c = d$ and the singleton containing Γ. We do this in order to ensure each $c^{=}$ is unique to the theory that defines it. This (as will be made clearer in an upcoming proof) is necessary in order to deal with a potential conceptual issue that arises when treating certain classes $c^{=}$ as impossible.

1.3. Now, define:

$S^{=} := \{c^{=} : c \in \mathcal{S}\}$

$D^{=} := \{c^{=} : c \in \mathcal{D}\}$

$E^{=} := \{c^{=} : c \in \mathcal{E}\}$

1.4. Now define:

$R_i := \{(c^{=}_1, \ldots, c^{=}_{\lambda(i)}) : \Gamma \vdash P_i c_1, \ldots, c_{\lambda(i)}\}$

1.5. And let:

$\mathcal{R}$ be the set of each R_i of $\{R_i : i \in I\}$

1.6. Finally, let our structure be:

$\mathcal{M}_\Gamma := (E^=, D^=, S^=, \mathrm{Sent}(\mathcal{IL}), \mathcal{C}, \mathcal{P}, \mathcal{V}, \{R_i : i \in I\}, R\ulcorner\urcorner, \{c^=_j : j \in J\}, \mathcal{R})$

With the following constraints:

$\Gamma \vdash P = Q \quad \Leftrightarrow \quad P^{(\mathcal{M}\Gamma,\, a,\, r)} = Q^{(\mathcal{M}\Gamma,\, a,\, r)}$

$\Gamma \vdash \lceil \boldsymbol{p} \rceil \quad \Leftrightarrow \quad$ $\boldsymbol{p}$ contains free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and for some $\mathbf{C}_1 \in \mathcal{C} \cup \mathcal{P}, \ldots, \mathbf{C}_n \in \mathcal{C} \cup \mathcal{P}$,
$\mathbf{C}_1^{(\mathcal{M}\Gamma,\, a,\, r)} = \mathbf{X}_1^{(\mathcal{M}\Gamma,\, a,\, r)}$ …, and
$\mathbf{C}_n^{(\mathcal{M}\Gamma,\, a,\, r)} = \mathbf{X}_n^{(\mathcal{M}\Gamma,\, a,\, r)}$

and

$\boldsymbol{p}(\mathbf{X}_1, \ldots, \mathbf{X}_n / \mathbf{C}_1, \ldots, \mathbf{C}_n) \in R\ulcorner\urcorner$

or

$\boldsymbol{p}$ does not contain free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and $\boldsymbol{p} \in R\ulcorner\urcorner$

Subsets Lemma (SL). For any canonical structure $\mathcal{M}_\Gamma$, $E^= \subseteq D^=$, $D^= \subseteq S^=$, and $E^= \subseteq S^=$.

Proof.	$c^= \in E^=$	$\Rightarrow$	$\Gamma \vdash \mathrm{E}c$	1.1-1.3
		$\Rightarrow$	$\Gamma \vdash \mathrm{D}c$	4.10.2, $\wedge$E
		$\Rightarrow$	$c^= \in D^=$	1.1-1.3
		$\Rightarrow$	$\Gamma \vdash \mathrm{D}c$	1.1-1.3
		$\Rightarrow$	$\Gamma \vdash \mathrm{S}c$	4.10.1
		$\Rightarrow$	$c^= \in S^=$	1.1-1.3 □

We now continue with our proof of completeness.

Theorem 2. For a *consistent*, *saturated*, *complete* $\Gamma \subseteq \mathrm{Sent}(\mathcal{IL})$, $\mathcal{M}_\Gamma \vDash \Gamma$.

Proof: By induction on the degree of sentences $\boldsymbol{s}$, we show:

$\Gamma \vdash \boldsymbol{s} \quad \Leftrightarrow \quad \mathcal{M}_\Gamma \vDash \boldsymbol{s} \qquad$ *EQ*

We start with atomic formulas and formulas of the abbreviated forms Ec, Dc, Sc, !Ec, !Dc, and !Sc.

For $\boldsymbol{s}$ of the form ‘$c = d$’

$$
\begin{array}{llll}
\Gamma \vdash \boldsymbol{s} & \Rightarrow & \Gamma \vdash c = d & \\
 & \Rightarrow & d \in c^{=} & \text{by 1.2.} \\
 & \Rightarrow & \Gamma \vdash d = d & \text{by =I} \\
 & \Rightarrow & d \in d^{=} & \text{by 1.2.} \\
 & \Rightarrow & \Gamma \vdash d = d & \text{by 1.2.} \\
 & \Rightarrow & \Gamma \vdash c = d & \text{by =E} \\
 & \Rightarrow & d \in c^{=} & \\
 & \Rightarrow & d \in c^{=} \Leftrightarrow d \in d^{=} & \\
 & \Rightarrow & x \in c^{=} \Leftrightarrow x \in d^{=} & \text{on 1.2., where } x = \Gamma \text{ or } x = d \\
 & \Rightarrow & c^{=} = d^{=} & \text{by extensionality} \\
 & \Rightarrow & \mathcal{M}_{\Gamma} \vDash c = d & \text{by 4.7.1.1.} \\
 & \Rightarrow & \mathcal{M}_{\Gamma} \vDash \boldsymbol{s} &
\end{array}
$$

and

$$
\begin{array}{lllll}
 & \mathcal{M}_{\Gamma} \vDash \boldsymbol{s} & \Rightarrow & \mathcal{M}_{\Gamma} \vDash c = d & \\
 & & \Rightarrow & c^{=} = d^{=} & \text{by 4.7.1.1.} \\
* & & \Rightarrow & d \in c^{=} \Leftrightarrow d \in d^{=} & \text{by extensionality} \\
 & & \Rightarrow & \Gamma \vdash d = d & \text{by identity axioms} \\
 & & \Rightarrow & d \in d^{=} & \text{by 1.2.} \\
 & & \Rightarrow & d \in c^{=} & \text{by *} \\
 & & \Rightarrow & \Gamma \vdash c = d & \text{by 1.2.} \\
 & & \Rightarrow & \Gamma \vdash \boldsymbol{s} &
\end{array}
$$

For $\boldsymbol{s}$ of the form ‘$P = Q$’

$$
\begin{array}{llll}
\Gamma \vdash \boldsymbol{s} & \Leftrightarrow & \Gamma \vdash P = Q & \\
 & \Leftrightarrow & P^{(\mathcal{M}_{\Gamma}, \boldsymbol{a}, \boldsymbol{r})} = Q^{(\mathcal{M}_{\Gamma}, \boldsymbol{a}, \boldsymbol{r})} & \text{by 1.6.} \\
 & \Leftrightarrow & \mathcal{M}_{\Gamma} \vDash P = Q & \text{by 4.7.1.1.} \\
 & \Leftrightarrow & \mathcal{M}_{\Gamma} \vDash \boldsymbol{s} &
\end{array}
$$

For $\boldsymbol{s}$ of the form ‘Sc’

$$
\begin{array}{llll}
\Gamma \vdash \boldsymbol{s} & \Leftrightarrow & \Gamma \vdash \mathrm{S}c & \\
 & \Leftrightarrow & c^{=} \in S^{=} & \text{by 1.1. - 1.3.} \\
 & \Leftrightarrow & \mathcal{M}_{\Gamma} \vDash \mathrm{S}c & \text{by 4.7.2.} \\
 & \Leftrightarrow & \mathcal{M}_{\Gamma} \vDash \boldsymbol{s} &
\end{array}
$$

Similar proofs can be carried out for formulas of the form Dc and Ec, so we omit them here.

For $\boldsymbol{s}$ of the form ‘!Sc’

$\Gamma \vdash \boldsymbol{s}$	$\Leftrightarrow$	$\Gamma \vdash !\mathrm{S}c$	
	$\Leftrightarrow$	$\Gamma \vdash \neg\mathrm{E}c$ and $\Gamma \vdash \neg\mathrm{D}c$ and $\Gamma \vdash \mathrm{S}c$	by Ax. !S∃
	$\Leftrightarrow$	$\Gamma \nvdash \mathrm{E}c$ and $\Gamma \nvdash \mathrm{D}c$ and $\Gamma \vdash \mathrm{S}c$	by completeness/consistency
	$\Leftrightarrow$	$c^{=} \notin E^{=}$ and $c^{=} \notin D^{=}$ and $c^{=} \in E^{=}$	by 1.1. - 1.3.
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vDash !\mathrm{S}c$	by 4.7.3.
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vDash \boldsymbol{s}$	

Similar proofs can be carried out for formulas of the form !Dc and !Ec, so we omit them here.

For $\boldsymbol{s}$ of the form '$P_i\, c_1, \ldots, c_{\lambda(i)}$'

$\Gamma \vdash \boldsymbol{s}$	$\Leftrightarrow$	$\Gamma \vdash P_i\, c_1, \ldots, c_{\lambda(i)}$	
	$\Leftrightarrow$	$(c^{=}_1, \ldots, c^{=}_{\lambda(i)}) \in R_i$	by 1.4.
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vDash P_i\, c_1, \ldots, c_{\lambda(i)}$	by 4.7.1.2.
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vDash \boldsymbol{s}$	

We now move on to molecular formulas (save for formulas of the forms Ec, Dc, Sc, !Ec, !Dc, and !Sc), where we assume, by inductive hypothesis, that for $\boldsymbol{s}$ of degree $n > 0$, EQ holds for sentences of degree $m < n$ and, as was proven above, EQ holds for formulas of the forms Ec, Dc, Sc, !Ec, !Dc, and !Sc.

For $\boldsymbol{s}$ of the form '$\neg\boldsymbol{p}$'

$\Gamma \vdash \boldsymbol{s}$	$\Leftrightarrow$	$\Gamma \vdash \neg\boldsymbol{p}$	
	$\Leftrightarrow$	$\Gamma \nvdash \boldsymbol{p}$	by consistency
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \nvDash \boldsymbol{p}$	by EQ
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vDash \neg\boldsymbol{p}$	by 4.7.1.3.
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vDash \boldsymbol{s}$	

For $\boldsymbol{s}$ of the form '$\boldsymbol{p} \wedge \boldsymbol{q}$'

$\Gamma \vdash \boldsymbol{s}$	$\Leftrightarrow$	$\Gamma \vdash \boldsymbol{p} \wedge \boldsymbol{q}$	
	$\Leftrightarrow$	$\Gamma \vdash \boldsymbol{p}$ and $\Gamma \vdash \boldsymbol{q}$	
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vdash \boldsymbol{p}$ and $\mathcal{M}_{\Gamma} \vdash \boldsymbol{q}$	by EQ
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vDash \boldsymbol{p} \wedge \boldsymbol{q}$	by 4.7.1.4.
	$\Leftrightarrow$	$\mathcal{M}_{\Gamma} \vDash \boldsymbol{s}$	

For $\boldsymbol{s}$ of the form '$\ulcorner\boldsymbol{p}\urcorner$'

$\Gamma \vdash \boldsymbol{s}$	$\Leftrightarrow$	$\Gamma \vdash \ulcorner\boldsymbol{p}\urcorner$
	$\Leftrightarrow$	$\boldsymbol{p}$ contains free variables $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ and for some $\mathbf{C}_1 \in \mathcal{C} \cup \mathcal{P}, \ldots, \mathbf{C}_n \in \mathcal{C} \cup \mathcal{P}$, $\mathbf{C}_1^{(\mathcal{M}\Gamma,\, \boldsymbol{a},\, \boldsymbol{r})} = \mathbf{Z}_1^{(\mathcal{M}\Gamma,\, \boldsymbol{a},\, \boldsymbol{r})}, \ldots,$ and $\mathbf{C}_n^{(\mathcal{M}\Gamma,\, \boldsymbol{a},\, \boldsymbol{r})} = \mathbf{Z}_n^{(\mathcal{M}\Gamma,\, \boldsymbol{a},\, \boldsymbol{r})}$

and

$\boldsymbol{p}(\mathbf{Z}_1, \ldots, \mathbf{Z}_n / \mathbf{C}_1, \ldots, \mathbf{C}_n) \in R\ulcorner\urcorner$

or

$\boldsymbol{p}$ does not contain free variables $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ and $\boldsymbol{p} \in R\ulcorner\urcorner$

			by 1.6.
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \lceil \boldsymbol{p} \rceil$	by 4.7.1
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \boldsymbol{s}$	

For $\boldsymbol{s}$ of the form '$!S\exists x\boldsymbol{p}$'

$\Gamma \vdash \boldsymbol{s}$	$\Leftrightarrow$	$\Gamma \vdash !S\exists x\boldsymbol{p}$	$!S\exists I$
	$\Leftrightarrow$	$\Gamma \vdash \boldsymbol{p}(c)$ and $\Gamma \vdash !S(c)$	by saturation
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \boldsymbol{p}(c)$ and $\mathcal{M}_\Gamma \vDash !S(c)$	by *EQ*
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \boldsymbol{p}[c^=]$ for some $c^= \in S$ (where $c^= \notin D$ and $c^= \notin E$)	by constants lemma
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash !S\exists x\boldsymbol{p}$	by 4.7.3.
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \boldsymbol{s}$	

Note that the same proofs can be carried out for sentences of the form '$!D\exists x\boldsymbol{p}$', '$!E\exists x\boldsymbol{p}$', '$S\exists x\boldsymbol{p}$', '$D\exists x\boldsymbol{p}$', '$E\exists x\boldsymbol{p}$', and '$\exists x\boldsymbol{p}$', so they will be omitted here.

For $\boldsymbol{s}$ of the form '$\exists X\boldsymbol{p}$'

$\Gamma \vdash \boldsymbol{s}$	$\Leftrightarrow$	$\Gamma \vdash \exists X\boldsymbol{p}$	
	$\Leftrightarrow$	$\Gamma \vdash \boldsymbol{p}(P)$	by saturation
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \boldsymbol{p}(P)$	by *EQ*
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \boldsymbol{p}[P^{\mathcal{M}ar}]$ for some $P^{\mathcal{M}ar} \in \mathcal{R}$	by1.5./1.4. & constants lemma
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \exists X\boldsymbol{p}$	by 4.7.3.
	$\Leftrightarrow$	$\mathcal{M}_\Gamma \vDash \boldsymbol{s}$	

Of course, that $\mathcal{M}_\Gamma$ may satisfy sentences in Γ that involve abbreviations, follows from the fact that $\mathcal{M}_\Gamma$ satisfies all sentences of Γ that the abbreviations abbreviate. □

Corollary 1: Structures $\mathcal{M}_\Gamma$ defined on complete, consistent, saturated theories Γ are Henkin faithful.

Proof: Let Γ be a complete, consistent, saturated theory. Each instance of the comprehension axiom is provable by Γ (as it's an axiom), hence

$$\Gamma \vdash \exists X \forall x_1, \ldots, \forall x_n(Xx_1,\ldots,x_n \leftrightarrow \boldsymbol{p}(x_1,\ldots,x_n)) \quad \Leftrightarrow \quad \boldsymbol{\mathcal{M}}_\Gamma \vDash \exists X \forall x_1, \ldots, \forall x_n(Xx_1,\ldots,x_n \leftrightarrow \boldsymbol{p}(x_1,\ldots,x_n)) \qquad \text{th. 2}$$

□

Note that a canonical structure $\boldsymbol{\mathcal{M}}_\Gamma$ will satisfy any complete, saturated, and consistent theory Γ, but appears to deviate significantly from the initial constraints we put on $\boldsymbol{\mathcal{IL}}$ structures (*i.e.* the domains seem to be no longer populated by object functions). Further, there is the metaphysical question of how exactly a set of constants can be extant. And lastly, there is the question of how an equivalence class $c^{=}$ can be impossible (which it would need be if, for some $c \in \boldsymbol{\mathcal{C}}$ and some $P \in \boldsymbol{\mathcal{P}}$, '$\ulcorner Pc \wedge \neg Pc \urcorner \wedge !Sc$' $\in \Gamma$). To address the second concern first, there is not necessarily an inconsistency in terms of the metaphysical properties of sets and vocabulary symbols. Theories have been proffered for representing both sets and symbols as concrete (*cf.* Charles Chihara's work on type theory without abstract sets in Chihara 1990, Hartry Field's work on foundations of math with space-time points instead of abstract entities in Field 1980, and the various programs in term formalism in Shapiro 2000). These programs aren't without objections, but they are fruitful nonetheless. For now, we will table discussions on the metaphysical nuance of such theories, as they do provide us with a means for treating (a possibly infinite number of) symbols and sets as concrete objects - where our logic works just fine regardless of these metaphysical assumptions.

Next, we comment briefly on the question of the unorthodox domain. The canonical structure contains just sets of constants (and a theory) in each domain, which do not seem to be object functions. Remember however, object functions are not abstract mathematical entities, but actual objects that serve the function of making a set of descriptions accurate of themselves (we simply represent them as mathematical functions). The sets of equivalence classes in our domains *are* objects that serve the informal function of making certain descriptions accurate of them (like, *is the union of an equivalence class and a singelton*), just as all objects serve this function in and of themselves. The equivalence classes are able to be described, and accurately. Nothing, therefore, precludes them from populating our set of objects *U*.

Lastly we comment on the latter, impossibility, question. Let's assume that $c^{=}$ is a denotation that is impossible on Γ. Consider that $c^{=}$ is a construction of $\mathcal{M}_{\Gamma}$ (itself a construction) and so $c^{=}$ gets its properties by virtue of both the construction of $\mathcal{M}_{\Gamma}$ and Γ. Note then that $c^{=}$, by virtue of Γ, is described accurately as *the denotation of a constant that relates to predicate P* and *not the denotation of a constant that relates to predicate P*. This satisfies the condition that there is some $d \in D$ where $^{+}d, {}^{-}d \in OB(c^{=})$, hence $OB(c^{=})$ is impossible on this construction. Further, $c^{=}$ is exclusively sentential. This is the case as we can't depict the relevant contradictory properties in $c^{=}$ (where presumably, there is some way of representing, imagistically, non-contradictory classes of this type) or reify such a contradictory entity according to a term formalism (the same token would literally have to be a '$P[c^{=}]$' and a '$\neg P[c^{=}]$' at the same point in physical space). Lastly, note that $c^{=}$, being a construction defined from $\mathcal{M}_{\Gamma}$ and Γ, is unique relative to $\mathcal{M}_{\Gamma}$ and Γ. That is,

Relativity Theorem. for any classes $c^{=}$ and $c^{=\prime}$ defined on structures $\mathcal{M}_{\Gamma}$ and $\mathcal{M}_{\Gamma}'$ respectively (where $\mathcal{M}_{\Gamma} \neq \mathcal{M}_{\Gamma'}$) and theories Γ and Γ′ respectively (where $\Gamma \neq \Gamma'$), $c^{=} \neq c^{=\prime}$.

Proof: by 1.2., $\Gamma \in c^{=}$. Since $\Gamma \neq \Gamma'$, $\Gamma \notin \{\Gamma'\}$, hence $\Gamma \notin c^{=\prime}$, and we have it that $c^{=} \neq c^{=\prime}$. □

We insist on this feature as without it there is the possibility that for two structures $\mathcal{M}_{\Gamma}'$ and $\mathcal{M}_{\Gamma}$ and equivalence classes $c^{=}$ and $c^{=\prime}$, $c^{=} = c^{=\prime}$. There is the further possibility that $c^{=}$ be contradictory in $\mathcal{M}_{\Gamma}'$ but not in $\mathcal{M}_{\Gamma}$. If this were the case, then $OB(c^{=})$ would be accurately described as *the denotation of a constant that relates to a predicate P and doesn't relate to a predicate P* and *not the denotation of a constant that relates to a predicate P and doesn't relate to a predicate P*, leaving $c^{=}$ defined on *every* Γ′ as exclusively sentential (if it was defined that way on *any* Γ). Theorem 1.2.1. denies this possibility as no two equivalence classes defined on the same constant can be identical on different theories.

Godel-Henkin Model Existence Theorem. Any consistent set $\Gamma \subseteq \mathrm{Sent}(\mathcal{IL})$ has a model of cardinality at most, $\max(\aleph_0, |\Gamma|)$

Proof: Let $\kappa = \max(\aleph_0, |\Gamma|)$. $\kappa = |\mathcal{JL}_\Gamma|$ by cardinality lemma 2. By Theorem 1 we can extend Γ to a complete consistent saturated theory T in a simple extension $\mathcal{JL}'$ of $\mathcal{JL}_\Gamma$ where $|\mathcal{JL}'| = |\mathcal{JL}_\Gamma| = \kappa$. By Theorem 2, the canonical structure $\mathcal{M}_T$ is a model of T and of Γ therefore. By the expansion theorem, the $\mathcal{JL}_\Gamma$-reduction $\mathcal{M}'$ of $\mathcal{M}_T$ is a model of Γ, and any $\mathcal{JL}$ -expansion $\mathcal{M}$ of $\mathcal{M}'$ is also. Lastly, if $\mathcal{C}'$ is the set of constant symbols of $\mathcal{JL}'$, then $|\mathcal{M}| = |\mathcal{M}_T| \leq |\mathcal{C}'| \leq |\mathcal{JL}'| = |\mathcal{JL}_\Gamma| = \kappa$.

□

Completeness. Given a theory $\Gamma \subseteq \mathrm{Sent}(\mathcal{JL})$ and an $\mathcal{JL}$ sentence $\boldsymbol{s}$,

$\Gamma \vDash \boldsymbol{s} \qquad \Rightarrow \qquad \Gamma \vdash \boldsymbol{s}$

Proof: If $\Gamma \nvdash \boldsymbol{s}$, then, by the consistency lemma (iii), $\Gamma \cup \{\neg \boldsymbol{s}\}$ is consistent and so, by the model existence theorem, has a model $\mathcal{M}$. $\mathcal{M} \vDash \neg \boldsymbol{s}$ therefore, where it follows that $\mathcal{M} \nvDash \boldsymbol{s}$. Since $\mathcal{M}$ is a model of Γ but not of $\boldsymbol{s}$, it follows that $\Gamma \nvDash \boldsymbol{s}$. So, $\Gamma \nvdash \boldsymbol{s} \Rightarrow \Gamma \nvDash \boldsymbol{s}$ and by the converse, $\Gamma \vDash \boldsymbol{s} \Rightarrow \Gamma \vdash \boldsymbol{s}$.

□

Compactness. For any $\Gamma \subseteq \mathrm{Sent}(\mathcal{JL})$, Γ has a model *iff* every finite subset of Γ has a model.

Proof. The proof from left to right is obvious. Conversely, if every finite subset of Γ has a model, then by corollary of soundness every finite subset of Γ is consistent and so Γ itself is consistent by the consistency lemma. Therefore Γ has a model by the model existence theorem. □

Löwenheim-Skolem Theorem. If a set Γ of second-order sentences has an infinite faithful Henkin model, it has a faithful Henkin model of any cardinality $\kappa \geq \max(\aleph_0, |\Gamma|)$.

Proof. Let $\mathcal{JL}_\Gamma'$ be the simple extension of $\mathcal{JL}_\Gamma$ obtained by adding a set $\{d_j: j \in J\}$ of new constant symbols, where $|J| = \kappa$. Define an extension Γ' as follows:

$$\Gamma' = \Gamma \cup \{\neg(d_j = d_k): j, k \in J \wedge j \neq k\}.$$

Now, let Γ_0 be any finite subset of Γ'. Clearly, only finitely many sentences of the form $\neg(d_j = d_k)$ occur in Γ_0. Let $d_{j1}, \ldots, d_{jn}$ be a list of all constant symbols occurring in $\neg(d_j = d_k)$ sentences in Γ_0. For an infinite $\mathcal{IL}_\Gamma$-model $\mathcal{M}$ of Γ, choose n distinct elements $a_1,\ldots, a_n$ from its domain S. Define $\mathcal{M}'$ to be the $\mathcal{IL}_{\Gamma'}$-expansion of $\mathcal{M}$ in which the interpretation of d_{jp} is a_p for $p = 1,\ldots, n$ and let the interpretation of d_j (such that $j \notin \{j_1, \ldots, j_n\}$) be an arbitrary element of $\mathcal{M}$. It follows that $\mathcal{M}'$ is a model of Γ_0 and so, every finite subset of Γ' has a model. Every finite subset of Γ' is consistent then, which means Γ' is consistent by the consistency theorem. Now, $|\Gamma'| = \kappa$ and so the model existence theorem implies that Γ' has a model of cardinality $\leq \kappa$. Since the interpretations of the d_j in any model of Γ' are distinct elements, these models must have a cardinality $\geq \kappa$. Γ' has a model of cardinality κ therefore and its $\mathcal{IL}_\Gamma$-reduction is a model of Γ of cardinality κ[32]. □

And, that concludes the section on meta-results. Obviously, working with a deduction system and semantics that are both sound and complete is desirable of any logic, and so any justification for including their proofs is redundant. But, second order logics with standard semantics are capable of modelling arithmetic, hence, on Godel's incompleteness theorem, are incomplete. We choose a Henkin semantics then, in order that we avoid this problem (where our structures' *faithfulness* allow for soundness). Compactness holds on this semantics as any Γ featuring a sentence equivalent to

$$\neg\exists X[\forall x\forall y\forall z(Xxy \wedge Xyz \rightarrow Xxz) \wedge \forall x(\neg Xxx \,\&\, \exists yXxy)] \qquad \text{FIN}$$

(that posits the second-order property of a finite domain) and sentences (for each $n \in \omega$) that posit the existence of at least n elements, i.e.

$$\Gamma = \{\text{FIN},\ \exists x(x = x), \exists x\exists y(x \neq y), \exists x\exists y\exists z(x \neq y \wedge x \neq z \wedge y \neq z), \ldots\} \qquad ***$$

[32] See Bell, 2006 (and course notes) for the more detailed presentation of this form of the Löwenheim-Skolem proof.

is satisfiable. On standard second-order semantics, Γ would not be satisfiable, as it essentially posits both a finite domain (since an infinite domain *would* have some subset of $S \times S$ that is assignable to some X, that satisfies FIN) and an infinite domain. *** is not satisfiable on standard second-order semantics then, but each of the finite subsets of *** are satisfiable, hence compactness fails. On our Henkin semantics however, infinite structures exist where

$$X^{(\mathcal{M}^{\Gamma}, \boldsymbol{a}, \boldsymbol{r})} \notin \boldsymbol{\mathcal{R}} \text{ (on any assignment } \boldsymbol{r}\text{)}$$

that do satisfy ***, hence there are both models for each finite subset of Γ and Γ itself.

Since it can be shown that the Lowenheim-Skolem results obtain with Henkin semantics (as does the compactness theorem), by Lindstrom's theorem, our logic is (as many suspect second order logics with Henkin semantics to be) just a two-sorted first order logic[33]. Although our logic is capable of representing theories of arithmetic (and therefore generating a Gödel sentence G), our logic also features Henkin structures where G is not entailed, and so, Gödel's incompleteness theorem does not apply. By the Lowenheim-Skolem theorem, however, categoricity is not possible. But, since our aim is to found philosophical theories of objects (and not mathematical theories), where the possibility of a lack of a standard model is not of any conceivable consequence, a lack of categoricity is not a problem either.

[33] Shapiro, 1991, pg. 14.

Chapter 6

Building on The Groundwork

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In this chapter we will interpret the informal principles and definitions required of various theories of objects and formally axiomatize a Meinongian theory as a proof of concept. An extension for (and, in part, the axiomatization of) a theory of impossible and vague objects has already been achieved with $\mathcal{IL}$, as vague objects are just a subclass of impossible objects and impossible objects are defined in the $\mathcal{IL}$ system. Capturing the essential properties of abstract and fictional objects is not possible with $\mathcal{IL}$ however as, traditionally, abstract objects have been taken to be *both* non-concrete and causally inefficacious (where causal relations cannot be captured with just the language of $\mathcal{IL}$) and fictional objects are not just non-concrete currently, they have *never* been concrete. Plato is currently non-extant therefore non-concrete, but intuitively Plato isn't fictional. At some point in time (in the past) Plato *was* extant. On the other hand, someone like Sherlock Holmes was never extant and is therefore fictional.

§6.1. Extending $\mathcal{IL}$ to a Temporal Logic ($\mathcal{ILT}$)

The reader might have noticed that in determining Plato and Sherlock Holmes non-fictional and fictional (respectively), the expressions 'currently', 'have never' and 'in the past' were used. This suggests that temporality plays a role in defining objects of this kind. Further, I suggest that with added temporal concepts a notion of causality similar to the one posited by John Stuart Mill can be captured in $\mathcal{IL}$, hence abstract objects can be represented formally. Since (despite the fact that we can represent impossible and vague objects with just $\mathcal{IL}$) we must add certain temporal concepts in order to represent abstract and fictional objects, we will extend $\mathcal{IL}$ to a temporal logic of Arthur Prior's (where **'$\mathcal{ILT}$'** indicates the language of the temporal extension), and then proceed to the axiomatization of our theory of objects. We present the following $\mathcal{ILT}$ extension as we presented the $\mathcal{GL}$ extension in chapter four (i.e. we simply add the additional items to the relevant sections). The following items should be read as though they proceed immediately after the '$\mathcal{ILT}$ extensions' in chapter four. As with chapter four, we provide an appendix featuring the complete presentation of the system. We start with the extension of our $\mathcal{ILT}$ vocabulary.

4.1. Vocabulary:

4.1.3. $\mathcal{ILT}$ Extension

P	(this symbol is read *it has at some time been the case that*)
F	(this symbol is read *it will at some time be the case that*)

4.2. Formulas

($\mathcal{ILT}$ Extension of 6.2.2.)

(b.4). If $\boldsymbol{p}$ is a formula, then P$\boldsymbol{p}$ and F$\boldsymbol{p}$ are formulas.

4.2.3.T. Form($\mathcal{ILT}$) := $\{\boldsymbol{p} : \boldsymbol{p}$ is a formula of $\mathcal{ILT}\}$.

(Abbreviations)

4.2.4. if $\boldsymbol{p}$ and $\boldsymbol{q}$ are formulas, then

…

H$\boldsymbol{p}$	abbreviates	$\neg$P$\neg\boldsymbol{p}$	'it has always been the case that'
G$\boldsymbol{p}$	abbreviates	$\neg$F$\neg\boldsymbol{p}$	'it will always be the case that'
A$\boldsymbol{p}$	abbreviates	H$\boldsymbol{p} \wedge \boldsymbol{p} \wedge$ G$\boldsymbol{p}$	'it is always the case that'

*as a matter of convention, treat each formula as though it begins with a tacit 'it is currently the case'.

4.5. Structure

$\mathcal{M}^{\mathrm{T}} := (E, D, S, T, \prec, t, \mathrm{Sent}(\mathcal{ILT}), \mathcal{C}, \mathcal{P}, \mathcal{V}, \mathcal{T}, R\ulcorner\urcorner)$

$\mathcal{M}^{\mathrm{T}}$ denotes the structure, i.e. a quattuordecuple. Like structures of $\mathcal{IL}$, $\mathcal{M}^{\mathrm{T}}$ consists of a nonempty domain S, which is a set with a well-defined subset D (possibly empty) having a well-defined subset E (possibly empty). We will define E, D, and S the same as we did for $\mathcal{IL}$, but with the following addition.

Let d^t abbreviate those $d \in D$ that are read, 'd at time t'

T is a set of points in time and $\prec$ is a relation on T (called a precedence relation) whose tuples (t, t') are read 't precedes t'' (where $(t, t') \in \prec$ is abbreviated $t \prec t'$). We assume certain constraints on $\prec$ here, where these constraints are defeasible and variations may be determined according to the logician's requirements. For now, since we are modelling informal theories of objects according to ordinary language discussions of them, we will treat time in the standard linear and discrete manner. That is, $\prec$ is irreflexive, anti-symmetrical, transitive, and non-dense. Formally *non-density* is represented thus, $\neg\forall t\forall t''[t \prec t'' \rightarrow \exists t'(t \prec t' \wedge t' \prec t'')]$. t is the point in time that all $t' \in T$ are relative to in $\mathcal{M}^{\mathrm{T}}$ (abbreviate 'a structure at time t' as '$\mathcal{M}^{\mathrm{T}}{}_{(t)}$'), that is, when $\mathcal{M}^{\mathrm{T}}{}_{(t)}$ satisfies some formula $\boldsymbol{p}$, $\boldsymbol{p}$ is said to be satisfied by $\mathcal{M}^{\mathrm{T}}$ at time t. Where no t is present in the structure, assume t = now. Sent($\mathcal{ILT}$), $\mathcal{C}$, $\mathcal{P}$, $\mathcal{V}$ are defined as they are in $\mathcal{IL}$. $\mathcal{T}$ is the set of temporally contingent elements of $\mathcal{M}^{\mathrm{T}}{}_{(t)}$, that is, those elements that vary as t does.

$$\mathcal{T} = \{\{S_t\}_{t \in T}, \{D_t\}_{t \in T}, \{E_t\}_{t \in T}, \{\{R_i : i \in I\}_t : t \in T\}, \{\{e_j : j \in J\}_t : t \in T\}, \{\mathcal{R}_t : t \in T\}.$$

We define the elements of $\mathcal{T}$ thus.

$$\{S_t\}_{t \in T} := \{(t, \mathrm{OB}_o) : t \in T \wedge \mathrm{OB}_o \in S \wedge \forall t'\ t \prec t', \text{'came to exist at time } t'\text{'} \notin \mathrm{OB}(o)\}$$

where for each $t \in T$,

$$S_t \quad := \quad \{\mathrm{OB}_o : (t, \mathrm{OB}_o) \in \{S_t\}_{t \in T}\}$$

and

$$\{D_t\}_{t \in T} := \{(t, \mathrm{OB}_o) : t \in T \wedge \mathrm{OB}_o \in S_t \wedge [\text{'concrete at time } t\text{'} \in OB(o) \vee \text{'non-concrete at time } t\text{'} \in OB(o)] \wedge \text{for any } d \in D, {}^{-}d^{t}, {}^{+}d^{t} \notin OB(o)\}$$

where for each $t \in T$,

$$D_t \quad := \quad \{\mathrm{OB}_o : (t, \mathrm{OB}_o) \in \{D_t\}_{t \in T}\}$$

and

$$\{E_t\}_{t \in T} := \{(t, \mathrm{OB}_o) : t \in T \wedge \mathrm{OB}_o \in D_t \wedge \text{'concrete at time } t\text{'} \in OB(o) \wedge \text{'non-concrete at time } t\text{'} \notin OB(o) \wedge \text{for any } d \in D, {}^{-}d^{t}, {}^{+}d^{t} \notin OB(o)\}$$

where for each $t \in T$,

$$E_t \quad := \quad \{\mathrm{OB}_o : (t, \mathrm{OB}_o) \in \{E_t\}_{t \in T}\}$$

and define $\{\{R_i^t : i \in I\}_t : t \in T\}$ to be a family of families of relations on S with the constraint,

for each $i \in I$ and $t \in T$,

$$R_i^t \in \{R_i^t : i \in I\}_t \quad \Rightarrow \quad R_i^t \subseteq S_t^{\lambda(i)}$$

where we add a final constraint that, for any $OB_o \in S_t$ and any $d \in D$

$^{+}d, {}^{-}d \in OB(o) \quad \Rightarrow \quad R_d = R_i^t$ (for some $i \in I$ where $\lambda(i) = 1$) and $OB(o) \notin R_i^t$

next, define $\{\{e_j^t : j \in J\}_t : t \in T\}$ to be a family of families of designated elements of S with the constraint,

for each $j \in J$ and $t \in T$

$e_j^t \in \{e_j^t : j \in J\}_t \quad \Rightarrow \quad e_j^t \in S_t$

and (with $\mathbf{R}_t$ abbreviating $\{R_i : i \in I\}_t$)

$\{\boldsymbol{\mathcal{R}}_t : t \in T\}$ is a family of sets of relations on S where for any $t \in T$ and relation R on S,

$R \in \boldsymbol{\mathcal{R}}_t \quad \Rightarrow \quad R \in \cup\mathbf{P}(S_t^n)$

and

$\mathbf{R}_t \subseteq \boldsymbol{\mathcal{R}}_t$

Lastly, $R\ulcorner\urcorner$ is defined the same as in $\boldsymbol{\mathcal{IL}}$ save for the fact that we change its constraint to

for any $OB_o \in S_t$ and any $d \in D$,

if $d \neq$ ‘concrete’, ‘non-concrete’, ‘impossible’, ‘abstract’, ‘fictional’, or ‘vague’, then

$^{+}d, {}^{-}d \in OB(o) \quad \Rightarrow \quad OB_o = e_j$ and $R_d = R_i \quad \Rightarrow \quad$ ‘$P_ic_j \wedge \neg P_ic_j$’ $\in R\ulcorner\urcorner$

or

if $d =$ ‘concrete’, ‘non-concrete’, ‘impossible’, ‘abstract’, ‘fictional’, or ‘vague’, then (where $\boldsymbol{p}$ captures the relevant mode of existence)

$^{+}d, {}^{-}d \in OB(o) \quad \Rightarrow \quad OB_o = e_j \quad \Rightarrow \quad$ ‘$\boldsymbol{p}(c_j) \wedge \neg\boldsymbol{p}(c_j)$’ $\in R\ulcorner\urcorner$

4.6. Interpretation

4.6.1. (Variable Assignment) Given the structure, $\boldsymbol{\mathcal{M}}^{\mathrm{T}}$ of type (λ, J),

A-sequence := a countable sequence of elements of S_t (denoted, ‘$\boldsymbol{a}^t = (a_0, a_1, \ldots)$’)

R-sequence := a countable sequence of elements of $\boldsymbol{\mathcal{R}}_t$ (denoted, ‘$\boldsymbol{r}^t = (R_0, R_1, \ldots)$’) with the following constraint:

For each n, the nth R in $\boldsymbol{r}$ is of degree $v(n)$

4.6.2. (Interpreting the Symbols) Given $\boldsymbol{\mathcal{M}}$, $\boldsymbol{a}^t$, $\boldsymbol{r}^t$, $\boldsymbol{t}$ (where we read ‘$\mathbf{V}^{(\boldsymbol{\mathcal{M}}, \boldsymbol{at}, \boldsymbol{rt}, \boldsymbol{t})}$’ as the element of the domain of $\boldsymbol{\mathcal{M}}$ that $\mathbf{V}$ is interpreted-by/names/is-assigned-from-$\boldsymbol{a}^t$-or-$\boldsymbol{r}^t$ at time t),

Interpretation of $\boldsymbol{\mathcal{IL}}$ in $(\boldsymbol{\mathcal{M}}, \boldsymbol{a}^t, \boldsymbol{r}^t, \boldsymbol{t})$:=

i) $P_i^{(\boldsymbol{\mathcal{M}}, \boldsymbol{at}, \boldsymbol{rt}, \boldsymbol{t})} = R_i^t$

ii) $V_n^{(\boldsymbol{\mathcal{M}}, \boldsymbol{at}, \boldsymbol{rt}, \boldsymbol{t})} = R_n$

iii) $c_j^{(\boldsymbol{\mathcal{M}}, \boldsymbol{at}, \boldsymbol{rt}, \boldsymbol{t})} = e_i^t$

iv) $v_n^{(\boldsymbol{\mathcal{M}}, \boldsymbol{at}, \boldsymbol{rt}, \boldsymbol{t})} = a_n$

4.6.3. (Variant Assignment)

For $n \in \omega$, $b \in S_t$,

$[n|b]\boldsymbol{a}^t := (a_0, a_1, ..., a_{n-1}, b, a_{n+1}, ...)$

For $n \in \omega$, $S \in \mathcal{R}_t$ (where S is of degree $v(n)$)

$[n|S]\boldsymbol{r}^t := (R_0, R_1, ..., R_{n-1}, S, R_{n+1}, ...)$

4.7. Satisfaction

4.7.1. For $\boldsymbol{p} \in Form(\mathcal{ILT})$,

$\boldsymbol{a}^t$, $\boldsymbol{r}^t$ *satisfy* $\boldsymbol{p}$ *in* $\mathcal{M}^{\mathrm{T}}$ at time t (denoted, '$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at}, \boldsymbol{rt}} \boldsymbol{p}$') :=

…

4.7.9. $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at}, \boldsymbol{rt}} \mathrm{P}\boldsymbol{p}$ $\Leftrightarrow$ for some t' where $t' \prec t$, $\mathcal{M}^{\mathrm{T}}_{(t')} \vDash_{\boldsymbol{at'}, \boldsymbol{rt'}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at}, \boldsymbol{rt}} \mathrm{F}\boldsymbol{p}$ $\Leftrightarrow$ for some t' where $t \prec t'$, $\mathcal{M}^{\mathrm{T}}_{(t')} \vDash_{\boldsymbol{at'}, \boldsymbol{rt'}} \boldsymbol{p}$

(where two proofs similar to the one generating satisfaction rules for universalized statements gets us)

4.7.10. $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at}, \boldsymbol{rt}} \mathrm{H}\boldsymbol{p}$ $\Leftrightarrow$ for all t' where $t' \prec t$, $\mathcal{M}^{\mathrm{T}}_{(t')} \vDash_{\boldsymbol{at'}, \boldsymbol{rt'}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at}, \boldsymbol{rt}} \mathrm{G}\boldsymbol{p}$ $\Leftrightarrow$ for all t' where $t \prec t'$, $\mathcal{M}^{\mathrm{T}}_{(t')} \vDash_{\boldsymbol{at'}, \boldsymbol{rt'}} \boldsymbol{p}$

4.8. Deduction.

Note that here we only provide the bare minimum set of temporal axioms and inference rules required for deducing sentences entailed by theories Γ on our above temporal semantics and that are universal validities on our above temporal semantics. We do this in order that we not put any undue ontological constraints on the system. Which modal system the logician chooses is up to them, hence, further axioms/inference rules may always be added, or current axioms/inference rules removed.

4.8.3. Inference Rules

(where 'temporal scope' denotes the scopes of assumptions including the scope of primary assumptions, Aux FE, Aux PE, Aux GE, Aux HE)

Will At Some Point Elimination (FE)

	Fp		
		$\underline{p}$	Aux FE
		q_1	
		…	
		q_n	
⊳	Fq_n		

Has At Some Point Elimination (PE)

	Pp		
		$\underline{p}$	Aux PE
		q_1	
		…	
		q_n	
⊳	Pq_n		

Will Always Elimination (GE)

	Gp		
		$\underline{p}$	Aux GE
		q_1	
		…	
		q_n	
⊳	Gq_n		

Has Always Elimination (HE)

	Hp		
		$\underline{p}$	Aux HE
		q_1	
		…	
		q_n	
⊳	Hq_n		

Has Always Introduction (HI)

	p
⊳	Hp

Where p is a theorem.

Will Always Introduction (GI)

	p
⊳	Gp

Where p is a theorem.

Has Will Always Elimination (PGE)

	PGp
⊳	p

Where PGp and p appear in the same temporal scope.

Will Has Always Elimination (FHE)

	FHp
⊳	p

Where FHp and p appear in the same temporal scope.

Past Definition (PHD)

⊳ $\neg\mathrm{H}\neg p \leftrightarrow \mathrm{P}p$

and

⊳ $\neg\mathrm{P}\neg p \leftrightarrow \mathrm{H}p$

Future Definition (FGD)

⊳ $\neg\mathrm{G}\neg p \leftrightarrow \mathrm{F}p$

and

⊳ $\neg\mathrm{F}\neg p \leftrightarrow \mathrm{G}p$

And that concludes the $\mathcal{ILT}$ extension of $\mathcal{IL}$. However, before moving on to our discussion of representation of objects, it is worth taking a moment to discuss an interesting result of $\mathcal{ILT}$.

It is well known that a defect of many first-order and higher-order temporal logics is an inability to represent concepts involving ancestry and statements positing existence [34]. The problem arises from a dilemma

[34] Rescher, 1971 – Chapters 13 and 20.

temporal logicians face when defining the domains of their structures. Domains of $\mathcal{LT}$ structures must either remain fixed across all $t \in T$ or contain only those elements extant at each $t \in T$. If the former is the case, then a true statement like 'the present king of France does not exist' is currently false, and if the latter is the case, then a true statement like 'Pierre Trudeau is an ancestor of Justin Trudeau' is currently false. The former statement is false as King Louis is in the domain *currently* as well as in the relation containing kings of France. King Louis exists on these structures therefore and satisfies the condition for being the king of France. Alternatively, should the domain be relative to t, the latter statement fails as Pierre Trudeau is not currently in the domain and therefore not in a tuple next to Justin in the ancestor relation.

Issues like these do not arise in $\mathcal{ILT}$ as, although the domain S in any structure $\mathcal{M}^{\mathrm{T}}{}_{(t)}$ is defined relative to those objects that exist at t, the depictable and/or sentential domains for any t contain all elements that have ever existed up until t. Further, the informal predicate 'exists' is neither treated as a status captured by the existential operator nor as a primitive formal predicate. In $\mathcal{ILT}$ 'exists' in this sense means, exists in space and time, and hence is captured by extantiality. Let 'the present king of France does not exist' be captured by

(PK) $\quad \exists x[(Kx \wedge \forall y(Ky \rightarrow y = x) \wedge \neg!\mathrm{E}x)]$

Provided that

$$x^{(\mathcal{M},\, a,\, r\, t)} \in K^{(\mathcal{M},\, a,\, r\, t)},$$

$$K^{(\mathcal{M},\, a,\, r\, t)} = \{x^{(\mathcal{M},\, a,\, r\, t)}\},$$

and

$$x^{(\mathcal{M},\, a,\, r\, t)} \notin E,$$

(PK) is satisfied by any $\mathcal{M}^{\mathrm{T}}{}_{(t)}$ meeting both the above conditions and the condition that t be a point in time after the last king of France has died. Of course, on this way of looking at 'the king of France', we assume that once any king of France is succeeded, he is no longer in the relation of kings of France and, for any king of France, unless he has a successor, he remains king even after death. If, on the other hand, we read 'the present king of France' as denoting a fictional or abstract figure, then $K^{(\mathcal{M},\, a,\, r\, t)}$ is not actually the relation containing real life kings of France, but the relation containing a fictional king of France. Either way, unless France puts in place another king as its head of state,

$$K^{(\boldsymbol{\mathcal{M}},\, \boldsymbol{a},\, \boldsymbol{r}\, \boldsymbol{t})} = \{x^{(\boldsymbol{\mathcal{M}},\, \boldsymbol{a},\, \boldsymbol{r}\, \boldsymbol{t})}\}$$

Now, let ‘Pierre Trudeau is an ancestor of Justin Trudeau’ be captured by

(A) *Apj*

Where if

$$(p^{(\boldsymbol{\mathcal{M}},\, \boldsymbol{a},\, \boldsymbol{r}\, \boldsymbol{t})}, j^{(\boldsymbol{\mathcal{M}},\, \boldsymbol{a},\, \boldsymbol{r}\, \boldsymbol{t})}) \in A^{(\boldsymbol{\mathcal{M}},\, \boldsymbol{a},\, \boldsymbol{r}\, \boldsymbol{t})}$$

then (A) is satisfied by any $\boldsymbol{\mathcal{M}}^{\mathrm{T}}{}_{(t)}$ where *t* is a point in time after the birth of Justin Trudeau. From this we see that the concept of ancestry, as well as definite descriptions, do not pose a problem for $\boldsymbol{\mathcal{ILT}}$. That aside, *aside*, we now return to our axiomatization of a theory of objects.

§6.2. Defining Impossible, Vague, Abstract, and Fictional Objects

We start with a brief discussion of the defining features of object kinds to be captured formally, and then we define each kind formally. Impossible objects in general have already been elaborated on, so we’ll focus on a subset of them that have featured in quite a lot of philosophical debate. Here, I’m speaking of vague objects. Vague objects are not to be confused with vague predicates. The latter’s vagueness is a matter of epistemic indeterminacy and the former’s is a matter of ontological indeterminacy. Vague predicates are those the extensions of which are not decidable (in an informal sense), that is, we are often not certain whether the predicate applies to a particular object or not. Take a color property like ‘reddish-orange’ for example. This could be said to be a vague property (therefore a referent of a vague predicate), as the greater preciseness of ‘reddish-orange’ leaves the problem of *whether or not the property is exemplified by the object* applicable to a much larger number of objects. To illustrate the problem better, picture four apples on a table – a granny smith apple, two gala apples, and a red delicious apple. Imagine you are asked to retrieve a particular apple, by request. If you were told to retrieve the apple that possessed a coloring, this lack of precision in description would ensure that you know without doubt which apples were candidates (as they all possess a coloring), but you wouldn’t know which one to choose. *Has a coloring* is a non-vague (but imprecise) property for this reason. If you were asked to choose the apple that was reddish-orange however, the greater precision of this description would narrow your options down to the gala apples for certain, but what if, for either of the gala apples, you couldn't decide whether it was reddish-orange or orangish-red?

This is a common issue we face when trying to determine the properties of objects. In this case, we see that reddish orange is a less decidable (but more precise) property. Properties (and the predicates that capture them), where cases of indeterminacy like this arise, are said to be vague (*i.e.* we just don't *know* if the object has the property or not). Determining a vague object on the other hand has nothing to do with what we do or do not know - what we observe or do not observe. Vagueness on this account is a property of the object *per se*. In the case of the gala apple then, it's being a vague object would mean that reddish-orange*ness*, even if there were no human beings (or any entity present) to observe the apple or describe it, would be a property that may or may not be exemplified by the apple. The apple would not be an object with and without reddish orange*ness* (hence *T-impossible*) it is something more nebulous. The gala apple (and other vague objects) are those that, at best, we can say have fuzzy identity boundaries or, are objects that cannot be said to identify or not with certain objects.

By appeal to the following argument[35], we see that it must be the case that vague objects are objects that are not determined identical to themselves, hence impossible. Consider if an object a is vague it is said to be of indeterminate identity, i.e., for some object b it is indeterminate as to whether $a = b$. But, then b has the property of being *an object x where it is indeterminate as to whether x = a* where, because $a = a$, a does not have this property. So, if b has a property that a doesn't have, by Leibnitz' principle, it is not the case that $a = b$, hence it is able to be determined whether or not $a = b$, and a is *not* of indeterminate identity. It follows then that a is not a

[35] This argument is based on Gareth Evans' formalized version in *Evans, 78*. Note that by defining vague objects from this argument, we assume only one of many possible definitions of vague objects. We may treat 'vague objects' in this work as, more accurately, *Evans-Vague Objects*. We choose Evans' formulation, as it is most common in the literature. That said, the system is amenable to other definitions of vague objects. For one example, vague objects may also be defined as real world incomplete objects (i.e. concrete objects o where there is at least one property $\boldsymbol{p}$ where it is indeterminate as to whether o has property $\boldsymbol{p}$) where we (might) represent such an object as follows: $\iota x[!Ex \wedge \ulcorner \exists X \neg (Xx \vee \neg Xx) \urcorner]$ or, *the extant object of which we say, but only say, 'it is not the case that the object has or doesn't have at least one property'*.

vague object. Note though, that (on the above argument) if it were indeterminate as to whether $a = a$, it wouldn't follow that b has a property that a doesn't have. We have it then that if a is to be truly vague it must be the case that it not be determined identical to itself. Vague objects, then, are those objects that are not able to be determined identical to some object and are not determined identical to themselves. That said, it is too weak a claim, evidently, to say that *a vague object o is an object that is not able to be determined identical to some object* as *o* is an object not even identical to itself, hence not identical to any object. Vague objects are just *I-Impossible* objects that, because all objects are said to be identical to themselves, are - by definition (and a little redundantly) - impossible.

Abstract objects are much simpler to define. They're just non-concrete objects said to be causally inefficacious. However, though abstract objects are easy to define informally (if not easy to conceptualize), they are difficult to capture in formal logics. The problem isn't non-concreteness, that concept has already been defined. The problem is a need to capture the concept of causality. There have been attempts to capture causality in formal theories. But, being theories that (if we employed them here) would require an additional extension to $\mathcal{ILT}$, these logics will not be employed (and therefore not surveyed either). Instead, the aim is to capture causality with just the machinery of $\mathcal{ILT}$. With our present system, it is possible to capture a theory of causality similar to that of John Stuart Mill's[36] (albeit with a greater emphasis on interventionism).

We will present the causal theory shortly. However, now that we have informally defined abstract objects, it would be a good time to informally define fictional objects. Intuitively, a fictional object is a non-concrete, non-abstract object that is the creation of a (at one time) extant individual. From this, it may seem a natural conclusion that a new predicate be defined into $\mathcal{ILT}$, one that captures the creation of a non-concrete, non-abstract object by an extant object. However, this is unnecessary. Should a non-concrete, non-abstract object exist that is said to have never been extant - that some extant object created it is implied. Since the object is discussed, but isn't extant or abstract, the only way the object could have come into existence is through conception by some sentient being.

[36] Mill, 2016 – Ch 5.

Fictional objects then, are just non-concrete, non-abstract objects that have never been (nor will they ever be) extant.

We may now present our formal theory of causality. Let

$\boldsymbol{p} \dashv \boldsymbol{q}$ abbreviate $\mathrm{A}[\mathrm{P}(\boldsymbol{p} \rightarrow \mathrm{F}\boldsymbol{q}) \wedge \mathrm{F}(\boldsymbol{q} \rightarrow \mathrm{P}\boldsymbol{p})] \wedge \mathrm{A}\neg[\boldsymbol{p} \leftrightarrow \boldsymbol{q}]$

Informally, we read $\boldsymbol{p} \dashv \boldsymbol{q}$ as '$\boldsymbol{p}$ causes $\boldsymbol{q}$' and the formula to the right can be read as 'it is always the case that when the circumstance(s) captured by $\boldsymbol{p}$ *have* obtained, the circumstance(s) captured by $\boldsymbol{q}$ *will* obtain, and vice-versa (but these circumstances do not occur simultaneously)'. This precludes a definition from being treated as a cause (as the necessary and sufficient conditions for $\boldsymbol{p}$ always occur simultaneously with $\boldsymbol{q}$) but still allows for any possible u (in $\boldsymbol{p}(u)$) to causally affect itself. The idea here then, is that when circumstances $C_1, \ldots, C_{n-1}$ always precede circumstance C_n (where if any of $C_1, \ldots, C_{n-1}$ do not occur, C_n will not occur) we treat $C_1, \ldots, C_{n-1}$ as the cause of C_n (*cf.* Mill, 2016 and Woodward, 2003). Further we treat any objects involved with C_1 or, …, or C_{n-1} as causally affecting any objects involved with Cn. From this, we get

Abstract(u) abbreviates $!\mathrm{D}u \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(u) \dashv \boldsymbol{q}(y))$

We can read this as 'u is abstract *iff* it's non-concrete and it's always the case that there are no extant objects y that u causally affects'. Whether or not the abstract object itself is able to be causally affected is a matter for the metaphysician to decide. However, if one desires causal inefficacy to imply neither *causing* nor *being causally affected*, the following clause can be added to the above abbreviation:

$\wedge\ \mathrm{A}\neg\exists z(\boldsymbol{r}(z) \dashv \boldsymbol{p}(u))$

Note that because the above definition is in schematic form, and because there is at least a denumerable set of formulas of $\mathcal{ILT}$, there is no effective means of proving an object abstract nor representing it as such in the language (short of relying on the abbreviation). At this point, the best we can do is prove an object non-abstract by establishing a negation of an instance of the above schema. For this reason, we introduce an *abstract* predicate 'Abstract' into the language of $\mathcal{ILT}$ as follows

4.1.3. $\mathcal{ILT}$ Extension

…

Abstract (a predicate indicating an abstract object)

where

($\boldsymbol{\mathcal{ILT}}$ Extension of 4.2.1.)

…

'*Abstract*(u)' is an atomic formula of $\boldsymbol{\mathcal{ILT}}$

Since 'Abstract' just acts as an abbreviating predicate for '$!\mathrm{D}t \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(u) \prec \boldsymbol{q}(y))$', the satisfaction conditions for the new predicate will be as follows:

4.7.11. $\boldsymbol{\mathcal{M}}^{\mathrm{T}}{}_{(t)} \vDash_{a,r} \mathrm{Abstract}(u) \iff$ for some $b \in D$ (where $b \notin E$) $\boldsymbol{\mathcal{M}}^{\mathrm{T}}{}_{(t)} \vDash_{[n|b]a,\,r} v_n = u$

and

for some $\boldsymbol{p}, \boldsymbol{q} \in \mathrm{Form}(\boldsymbol{\mathcal{ILT}})$

for all t, $\boldsymbol{\mathcal{M}}^{\mathrm{T}}{}_{(t)} \vDash_{a,r}$

$$\neg\mathrm{E}\exists y(\mathrm{A}[\mathrm{P}(\boldsymbol{p}(u) \rightarrow \mathrm{F}\boldsymbol{q}(y)) \wedge \mathrm{F}(\boldsymbol{q}(y) \rightarrow \mathrm{P}\,\boldsymbol{p}(u))] \wedge \mathrm{A}\neg[\boldsymbol{p}(u) \leftrightarrow \boldsymbol{q}(y)])$$

Where obviously

$$\forall x[\mathrm{Abstract}(x) \leftrightarrow (!\mathrm{D}x \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(x) \prec \boldsymbol{q}(y)))]$$

is universally valid and so we add it as an axiom to $\boldsymbol{\mathcal{ILT}}$. Now we may speak of abstract objects without treating the abstraction predicate as an abbreviation of a conjunction, the number of conjuncts of which, are at least countably infinite. That said, all other object kind predicates can remain as abbreviations in $\boldsymbol{\mathcal{ILT}}$**.** The definitions are as follows.

Concrete(u)	abbreviates	$!\mathrm{E}u$
Non-concrete(u)	abbreviates	$!\mathrm{D}u$
Abstract(u)	abbreviates	$!\mathrm{D}u \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(u) \prec \boldsymbol{q}(y))$
Fictional(u)	abbreviates	$!\mathrm{D}u \wedge (\neg \mathit{Abstract}(u) \wedge \mathrm{A}\neg!\mathrm{E}u)$
T-impossible(u)	abbreviates	${}^{s}Tu$
Vague(u)	abbreviates	${}^{s}Iu$

$Impossible(u)$ abbreviates $(\exists X)X\text{-}impossible(u) \vee Vague(u)$

Where the as yet accounted for abbreviations are

$^{s}Abstract(u)$ abbreviates $\ulcorner(!\mathrm{D}u \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(u) \dashv \boldsymbol{q}(y))) \wedge \neg(!\mathrm{D}u \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(u) \dashv \boldsymbol{q}(y)))\urcorner \wedge !\mathrm{S}u$

$^{s}Fictional(u)$ abbreviates $\ulcorner(!\mathrm{D}u \wedge (\neg Abstract(u) \wedge \mathrm{A}\neg!\mathrm{E}u)) \wedge \neg(!\mathrm{D}u \wedge (\neg Abstract(u) \wedge \mathrm{A}\neg!\mathrm{E}u))\urcorner \wedge !\mathrm{S}u$

$^{s}Vague(u)$ abbreviates $\ulcorner {}^{s}Iu \wedge \neg {}^{s}Iu\urcorner \wedge !\mathrm{S}u$

To comment briefly on the soundness and completeness of $\mathcal{ILT}$ – we will not prove these properties here. However, they may be assumed provable as we need only test our abstractness axiom for universal validity (which is clearly the case) as a necessary condition for soundness and account for atomic formulas of the form '*Abstract*(*t*)' in our definitions of canonical structures as a necessary condition for completeness. From here we add the required universal validities as axioms (if they cannot be proven) and carry out the same proofs that already exist for showing soundness and completeness for Priorian tense logics for $\mathcal{ILT}$[37].

And that's it. That is, that is all that can be added to the system without putting any unwarranted constraints on the metaphysician. Of course, even the definitions of the above kinds of objects need not be assumed in a logic of objects (Zalta, for instance, treats all non-concrete objects as abstract, drawing no further distinctions[38]). And, there are likely many other object kinds that can be defined from the machinery of $\mathcal{ILT}$. But, it was the aim of this project to provide a system that could capture all of the desired delineation of object kinds, so it is at least necessary to show that $\mathcal{ILT}$ can translate the informal definitions of object kinds that feature perennially in philosophical discussion.

§6.3. Proof of Concept: A Meinongian Logic From $\mathcal{ILT}$

That said, even though this is not a project in metaphysics, for the sake of a proof of concept, we provide an example of a theory of objects that may be captured by $\mathcal{ILT}$ (where the definitions and principles involved need not be assumed a facet of our system), and then address a few questions and concerns. We start by

[37] For one such proof see Rescher, 1971 pg. 241.

[38] Zalta, 1983

representing a Meinongian theory of objects (or at least the cornerstones of a Meinongian theory). Although $\boldsymbol{\mathcal{ILT}}$ is non-Meinongian (i.e. it allows for the capturing of theories of objects that do not satisfy the *naïve object theor*y principle) Meinongian theories of objects can still be captured by $\boldsymbol{\mathcal{ILT}}$. Essential to a formal Meinongian theory is a formalization of the NOT principle, that is, *for any possible combination of properties there exists an object that exemplifies all and only those properties*. As well, the concepts of existence (i.e. material and temporal being), subsistence (i.e. non-temporal being), and outside-being (i.e. being an object that *might* neither exist nor subsist) need to be defined. Luckily, the three concepts track concreteness, abstractness, and sententiality respectively and so we need only apply the definitions that $\boldsymbol{\mathcal{ILT}}$ already provides. As for the NOT principle, something like the following axiom and schema pair (call them 'NOT-Simpliciter' and 'NOT-Schema' respectively) should do the trick,

$$\forall Y \exists x(Yx \wedge \forall Z[Zx \rightarrow (Z = Y)]) \qquad \text{NOT-Simpliciter}$$

*where Y is of degree 1.

NOT-Simpliciter states that 'for any one-place predicate, there exists some x that relates to that (and only that) predicate (or, for any property, there is an object that exemplifies that property simpliciter). Next, we have NOT-Schema,

$$\forall Y_1, \ldots, \forall Y_n [(\neg(Y_1 = \ldots \wedge \neg(Y_1 = Y_n) \wedge \ldots \wedge \neg(Y_{n\text{-}1} = Y_n)) \rightarrow \exists x([(Y_1x \vee {}^sY_1x) \wedge \ldots \wedge (Y_nx \vee {}^sY_nx)] \wedge \forall Z(Zx \rightarrow [Z = Y_1 \vee \ldots \vee Z = Y_n]))]$$

*where for each $i \in \{1, \ldots, n\}$, Y_i is of degree 1.

NOT-Schema states that 'for any n distinct one-place predicates, there exists some x that relates to those (and only those) predicates (in either their standard or sententializing form). Pairing the NOT principle off into an axiom and an axiom schema makes the proof system a bit more cumbersome, but the two principles capture the general idea of NOT. As well, the possibility of sententialization denies any inconsistencies that may arise should, for any $i, j \in \{1, \ldots, n\}$, $Y_ix \rightarrow \neg Y_jx$. The NOT axioms of the Meinongian theory guarantee an object for every combination of predicates. As an example, consider that the following instance (presented in a qausi-formal manner) of the NOT-Schema,

$$\neg(Square = Circular) \rightarrow \exists x[(Square(x) \vee {}^sSquare(x)) \wedge (Circular(x) \vee {}^sCircular(x) \;\wedge \forall Z(Zx \rightarrow [Z = Square \vee Z = Circular])]$$

guarantees us the existence of the square circle, as (if the theory accurately represents the fact that square things are not circular things and vice-versa), with *modus ponens*, we get

$$\exists x[(Square(x) \vee {}^{s}Square(x)) \wedge (Circular(x) \vee {}^{s}Circular(x) \ \wedge \forall Z(Zx \rightarrow [Z = Square \vee Z \ = Circular])]$$

where because the theory is assumed accurate, some logic shows that

$$\exists x[({}^{s}Square(x) \wedge {}^{s}Circular(x)) \ \wedge \forall Z(Zx \rightarrow [Z = Square \vee Z \ = Circular])]$$

which is actually an abbreviation of

$$\exists x[(\ulcorner Square(x) \wedge \neg Square(x)\urcorner \wedge !\mathrm{S}x) \wedge (\ulcorner Circular(x) \wedge \neg Circular(x)\urcorner \wedge !\mathrm{S}x)) \ \wedge \forall Z(Zx \rightarrow [Z = Square \vee Z \ = Circular])]$$

So, the sentence is consistent in and of itself.

Other objects of interest to Meinong, like the existent golden mountain can be represented too. Let's adopt the convention of abbreviating $\exists x\boldsymbol{p}(x) \wedge \forall y(\boldsymbol{p}(y) \rightarrow y = x)$ as $\exists!x\boldsymbol{p}(x)$ in order that we may represent the existent golden mountain thus

$$\exists!x([\mathrm{Golden}(x) \wedge \mathrm{Mountain}(x)] \wedge \mathrm{E}!x)$$

The existent golden mountain is straightforwardly modelled by structures that have an element in their domain that is in the *golden* and *mountain* relations (where nothing else in the domain is in both of those relations). Of course, we would be acting disingenuously if we failed to acknowledge that *exists* on Meinong's account means *concrete* on ours. For this reason, we will represent the concrete golden mountain too. The *concrete* golden mountain is a bit problematic in terms of representation as the concrete golden mountain is not concrete. However, this just makes the concrete golden mountain impossible. So, we would represent it thus

$$\exists!x([\mathrm{Golden}(x) \wedge \mathrm{Mountain}(x)] \wedge {}^{s}Concrete(\mathrm{x}))$$

which is actually an abbreviation of

$$\exists x([\mathrm{Golden}(x) \wedge \mathrm{Mountain}(x)] \wedge [\ulcorner !\mathrm{E}x \wedge \neg !\mathrm{E}x\urcorner \wedge !\mathrm{S}x])$$

which is again, a sentence that is consistent in and of itself. We see then, that the foundation for a Meinongian theory of objects (as well as the objects that are mainstays of Meinongian theory) is able to be captured in $\mathcal{ILT}$. A model of the Meinongian theory would be any $\mathcal{ILT}$ structure that satisfies NOT-Simpliciter and each instance of the NOT-Schema. However, what proceeds from the basic principles and definitions of the theory is a matter for the Meinongian scholar to determine. Since we are not Meinongian scholars, we end our presentation of the Meinongian theory of objects here.

§6.4. Tying Up Loose Ends

We will end this chapter with some commentary on other possibilities for extensions and, in general, the various directions we may take $\boldsymbol{\mathcal{ILT}}$. To start, there is an obvious question evoked, how do the impossible objects of $\boldsymbol{\mathcal{ILT}}$ compare to the impossible objects of standard modal possible world logics. It seems, should we extend $\boldsymbol{\mathcal{ILT}}$ to include the possibility and necessity operators ('◊' and '☐' respectively), we would have possible impossible objects (i.e. $\Diamond\exists x Impossible(x)$), but could we also have impossible impossible objects (i.e. $\neg\Diamond\exists x[Impossible(x) \wedge \boldsymbol{p}(x)]$ or $\exists x\neg\Diamond Impossible(x)$) too? And, isn't the former notion counterintuitive and the latter redundant?

To me, it seems like both representations are unproblematic. The former sentence is only counterintuitive when represented in natural language terms of inadequate description, and the latter sentence only seems redundant for the same reason. As for why I suspect that such representations are unproblematic - as well as potentially useful statements in theories of objects captured by modal extensions of $\boldsymbol{\mathcal{ILT}}$ - with more elaborate description we see that '$\Diamond\exists x\mathrm{Impossible}(x)$' actually says (in natural language terms) *in some possible context there exists some x where x, as described, violates the law of non-contradiction*. This statement should neither strike the reader as counter-intuitive nor contradictory *per se*. What may clear up the confusion is the informal convention, *use 'not-possible' for objects not found in any world or context* and *use 'impossible' for objects described in such a way that they violate the law of non-contradiction*.

In regard to the dual possibilities for representing impossible impossible objects, i.e.

$$\neg\Diamond\exists \mathrm{x}[\mathrm{Impossible}(x) \wedge \boldsymbol{p}(x)] \qquad \text{(O1)}$$

and

$\exists x \neg \Diamond \text{Impossible}(x)$ (O2)

with greater description we get the natural language translations *there is no context where there is an x where x, as described, violates the law of non-contradiction and has certain other defining features* and *there is some x where there is no context where x, as described, violates the law of non-contradiction*, respectively. The redundancy dissolves for both O1 and O2 when we establish, in natural language terms, what they actually mean when modelled formally. And, we want $\boldsymbol{\mathcal{ILT}}$ to have non-possible objects like x in O1. A possibility operator would be quite helpful for instance, should we want to indicate the non-possible property that is exemplified by say, the square circle, should it be said to be both *of philosophical interest* and not *of philosophical interest* simultaneously. If we let S, C, and P be *square*, *circular*, and *of philosophical interest* (respectively), it becomes clear that we would want a sentence like

$$P[\iota x({}^{s}Sx \wedge {}^{s}Cx)] \wedge \neg P[\iota x({}^{s}Sx \wedge {}^{s}Cx)]$$

to not obtain in any possible world, as it is straightforwardly contradictory. Lastly, to comment briefly on O2, objects with consistent properties are not impossible in any context and, since O2 captures that fact with the use of the possibility operator, we see again the use a modal extension of $\boldsymbol{\mathcal{ILT}}$ would serve. I conjecture then, that the only potential for problems with a modal extension of $\boldsymbol{\mathcal{ILT}}$ would have been those that arise from representing *impossible impossible objects* and *possible impossible objects*, where because such problems have been shown unfounded, consistent modal extensions are not only desirable, but realizable.

The above paragraph hints at an important feature of $\boldsymbol{\mathcal{IL}}$. The feature is as follows - say/only-say sentences $\ulcorner \boldsymbol{s} \urcorner$ cannot be derived. This isn't strictly true of course, $\ulcorner \boldsymbol{s} \urcorner$ may be derived from contradictory formulas of the form $\boldsymbol{q} \wedge \neg \boldsymbol{q}$ or $\neg(t = t)$ via negation elimination. However, if this derivation were an option, we would be working with an inconsistent theory and anything could be derived. What is meant is that there is not (nor can there be) a derivation rule for the introduction of $\ulcorner \;\; \urcorner$. Say only say ('S/OS') formulas are defining features of impossible objects (*i.e.* informally, S/OS statements 'sententialize' objects, thereby making them impossible). If $\boldsymbol{\mathcal{IL}}$ had an inference rule that allowed us to introduce the operator $\ulcorner\ \urcorner$, there would be no conceivable conditions on this rule that would preclude any contradictory formula (of the forms demonstrated above) from being cornered by $\ulcorner\ \urcorner$. The only real option is the condition '$\boldsymbol{s}$ was previously cornered by $\ulcorner\ \urcorner$'. Of course, this is a condition that can never be met, as the *no proof axiom* $\ulcorner \boldsymbol{s} \urcorner \rightarrow \neg \boldsymbol{s}$ guarantees that a contradiction arises should both $\ulcorner \boldsymbol{s} \urcorner$ and $\boldsymbol{s}$ feature in any derivation (which makes sense, since $\boldsymbol{s}$ - on this scenario - is either assumed or derived which means it's not just *said and only said*). So, with no conceivable conditions on this hypothetical inference rule, we would be able to infer for any contradictory sentence $\boldsymbol{s}$, $\ulcorner \boldsymbol{s} \urcorner$. From this fact, it follows that exclusively sentential terms u cannot possess contradictory properties as any predicate P where $Pt \wedge \neg Pt$ implies ${}^{s}Pt$. This means that formulas like

$$P[\iota x({}^{s}Sx \wedge {}^{s}Cx)] \wedge \neg P[\iota x({}^{s}Sx \wedge {}^{s}Cx)]$$

would imply an informal contradiction. ${}^{s}P[\iota x({}^{s}Sx \wedge {}^{s}Cx)]$ is able to be derived if we allow the introduction of $\ulcorner\ \urcorner$, so the predicate P is both sententializing, hence P is contradictory but consistent, but is also straightforwardly contradictory, therefore inconsistent. This result is undesirable.

Since we want $\mathcal{IL}$ to mirror ordinary language conversations about impossible objects, and impossible objects are able to be described - in ordinary conversations - in contradictory but consistent ways (according to certain properties) but also in contradictory and inconsistent ways (according to certain other properties), we do not allow for the introduction of the $\ulcorner\ \urcorner$ operator in $\mathcal{IL}$. Instead, cornered formulas are decided as a matter of assumption when formulating a theory Γ in $\mathcal{IL}$. If the logician wants a theory of the square circle (call it 'Γ_{SC}') then $\iota x({}^{s}Sx \wedge {}^{s}Cx)$ is included in Γ_{SC} at the logician's discretion, hence the relevant cornered formulas are included too. This formal convention mirrors the informal convention of introducing impossible objects into ordinary language philosophical discussions at the interlocutor's discretion (as discussed in §4.7). So, for any formulas of the form $\ulcorner \boldsymbol{s} \urcorner$ (for any theory Γ), if $\Gamma \vdash \ulcorner \boldsymbol{s} \urcorner$ then $\ulcorner \boldsymbol{s} \urcorner$ must feature in Γ itself (as a formula or subformula). In other words, $\mathcal{IL}$ doesn't prove objects permissively inconsistent; it reasons according to objects that are assumed permissively inconsistent. Again, this is in keeping with ordinary conversations too, as obviously, philosophers do not conceive of a square circle and then reason towards its square*ness* and circle*ness*, they assume an object that is simultaneously square and circular (by hypothesis) and reason from there.

Since this is a project on a logic of objects, it is only appropriate that we close this final chapter with a discussion of the objects defined in this logic. The nature of the objects of $\mathcal{GL}$ - *viz*. object functions - are decidedly nebulous. And that is exactly the point. As hinted at in §4.5, we do not explain here what realizes any type of object function, save for extant objects. This is, once again, done out of a spirit of pluralism and a need to not put any undue constraints on the metaphysician. Naturally, certain questions arise. For example, *what realizes the object function that is Homer Simpson?*, *what happens to an object function that goes from being extant to not extant, but depictable, (like Pierre Trudeau from the above example)?*, *'depictability' and 'sententiality' suggest a describer, is the existence of objects that start at depictability and sententiality contingent on the existence of an entity that first describes them?*, and *are objects that start at depictability and sententiality eternal?*.

We do not answer any of these questions here. We want it to be the case that Homer Simpson may be a Platonic particular, an object of a thought, a constitutive object (viz. an object constituted of the conception of Matt Groening, descriptions in a script, the voice acting of Dan Castellaneta, animation cells, etc.), or any other type of object. Further, we want it to be the case that formerly extant objects may exist as they were when extant, save for they exist outside of space and time, or they are transformed into objects of a different type, or they persist as ideas, etc. Lastly, we want the option of *objects that start at depictability and sententiality being contingent on the existence of an entity that first describes such objects* open to the metaphysician as well as the option that *objects that start at depictability and sententiality are eternal*. Note though, that this latter option is open in $\mathcal{GL}$ and $\mathcal{IL}$, but isn't open in $\mathcal{ILT}$ as $\mathcal{ILT}$ is defined. That said, nothing stops the logician from defining $\mathcal{ILT}$ in such a way that these constraints are removed. So, other than positing the various modes of existence that it does, $\mathcal{GL}$ and its extensions are otherwise agnostic in regard to how else objects might exist. We will leave such explication to the metaphysician.

From the discussion above, we see that we have tied up all important loose ends and so, we conclude this chapter (and project).

Appendix: Syntax, Semantics, and Deduction Theory of $\mathcal{ILT}$

0. Preliminaries:

Let λ be a function $\lambda : I \rightarrow \omega$, that maps indices in I to natural numbers in ω.

Let v be a function $v : \omega \rightarrow \omega$, that maps natural numbers to natural numbers in ω (not necessarily to themselves).

1. Vocabulary:

1.1. Vocabulary for A Standard Second-Order System

$v_0, v_1, \ldots$	*individual variables*
$V_0, V_1, \ldots$	*predicate variables of degree* $v(n)$
for each $i \in I$, a predicate symbol P_i of degree $\lambda(i)$	*predicate symbols*
for each $j \in J$ an individual constant c_j	*individual constants*
$=$	*equality symbol*
$\neg$	*logical operators*: *negation*
$\wedge$	*logical operators: conjunction*
$\exists$	*existential quantifier symbol*
(,), [,]	*punctuation symbols*

* individual variables and constants are called *individual terms,* where t, u (possibly with subscripts) denote arbitrary individual terms.

** let T, U (possibly with subscripts) denote arbitrary predicate constants and predicate variables.

1.2. $\mathcal{IL}$ Extension

E$\exists$	(this quantifier binds an *extantial* object, or a concrete object)
D$\exists$	(this quantifier binds a *depictable* object, or an object able to be represented imagistically)
S$\exists$	(this quantifier binds a *sentential* object, or a verbally representable object)
!E$\exists$	(this quantifier binds an object starting at *extantiality*, or a depictable, sentential, extant object)
!D$\exists$	(this quantifier binds an object starting at *depictability*, or a depictable, sentential, non-extant object)
!S$\exists$	(this quantifier binds an object starting at *sententiality*, or an exclusively sentential object)
$\ulcorner\ \urcorner$	(read as *it is said, and only said, that…*)

*** let **C**, **D** (possibly with subscripts) denote arbitrary individual constants and predicate constants.

**** let **X**, **Y** (possibly with subscripts) denote arbitrary individual variables and predicate variables.

**** let **V**, **W** (possibly with subscripts) denote variable and constant symbols of either kind.

1.3. $\mathcal{ILT}$ Extension

P	(this symbol is read *it has at some time been the case that*)
F	(this symbol is read *it will at some time be the case that*)
Abstract	(a predicate indicating abstract*ness*)

2. Formulas

2.1.T. *Atomic formulas* of $\boldsymbol{\mathcal{L}}$:= finite strings (of the basic symbols (i) - (iv)) either of the forms $T_i t_1,\ldots t_{\lambda(i)}$, $Abstract(t)$, $t = u$, or $T = U$

2.2. *Formulas* of $\boldsymbol{\mathcal{L}}$ (or $\boldsymbol{\mathcal{L}}$-*formulas*) := finite strings (of the basic symbols (i) - (vii)) defined in the following recursive manner:

(a) any *atomic formula* is a *formula*

(b.0) if $\boldsymbol{p}$, $\boldsymbol{q}$ are *formulas*, so also are $\neg\boldsymbol{p}$, $\boldsymbol{p} \wedge \boldsymbol{q}$, $\exists x\boldsymbol{p}$, $\exists X\boldsymbol{p}$ (where x is any variable v_i and X any variable V_i)
(b.1). If $\boldsymbol{p}$ is a formula, then $\ulcorner\boldsymbol{p}\urcorner$ is a formula.
(b.2). If $\boldsymbol{p}$ is a formula, then $\mathrm{E}\exists x\boldsymbol{p}$, $\mathrm{D}\exists x\boldsymbol{p}$, $\mathrm{S}\exists x\boldsymbol{p}$ are formulas.
(b.3). If $\boldsymbol{p}$ is a formula, then $!\mathrm{E}\exists x\boldsymbol{p}$, $!\mathrm{D}\exists x\boldsymbol{p}$, $!\mathrm{S}\exists x\boldsymbol{p}$ are formulas.
(b.4). If $\boldsymbol{p}$ is a formula, then $\mathrm{P}\boldsymbol{p}$ and $\mathrm{F}\boldsymbol{p}$ are formulas.

(c) a finite string of symbols is a formula exactly when it follows from finitely many applications of (a) and (b)

2.3.T. Form($\boldsymbol{\mathcal{ILT}}$) := $\{\boldsymbol{p} : \boldsymbol{p}$ is a formula of $\boldsymbol{\mathcal{ILT}}\}$.

A *sentence* is a formula with no free variables.

(Abbreviations)

2.4. if $\boldsymbol{p}$ and $\boldsymbol{q}$ are formulas, then

$\boldsymbol{p} \vee \boldsymbol{q}$	abbreviates	$\neg(\neg\boldsymbol{p} \wedge \neg\boldsymbol{q})$	
$\boldsymbol{p} \rightarrow \boldsymbol{q}$	abbreviates	$\neg\boldsymbol{p} \vee \boldsymbol{q}$	
$\boldsymbol{p} \leftrightarrow \boldsymbol{q}$	abbreviates	$(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\boldsymbol{q} \rightarrow \boldsymbol{p})$	
$\forall X\boldsymbol{p}$	abbreviates	$\neg\exists X\neg\boldsymbol{p}$	
$\forall x\boldsymbol{p}$	abbreviates	$\neg\exists x\neg\boldsymbol{p}$	
$\mathrm{E}\forall x\boldsymbol{p}$,	abbreviates	$\neg\mathrm{E}\exists x\neg\boldsymbol{p}$	
$\mathrm{D}\forall x\boldsymbol{p}$,	abbreviates	$\neg\mathrm{D}\exists x\neg\boldsymbol{p}$	
$\mathrm{S}\forall x\boldsymbol{p}$,	abbreviates	$\neg\mathrm{S}\exists x\neg\boldsymbol{p}$	
$!\mathrm{E}\forall x\boldsymbol{p}$,	abbreviates	$\neg!\mathrm{E}\exists x\neg\boldsymbol{p}$	
$!\mathrm{D}\forall x\boldsymbol{p}$,	abbreviates	$\neg!\mathrm{D}\exists x\neg\boldsymbol{p}$	
$!\mathrm{S}\forall x\boldsymbol{p}$,	abbreviates	$\neg!\mathrm{S}\exists x\neg\boldsymbol{p}$	
$\boldsymbol{p}([\backepsilon x\boldsymbol{q}(x)])$	abbreviates	$\exists x(\boldsymbol{q}(x) \wedge \boldsymbol{p}(x))$	
$\boldsymbol{p}([\iota x\boldsymbol{q}(x)])$	abbreviates	$\exists x[(\boldsymbol{q}(x) \wedge \forall y(\boldsymbol{q}(y) \rightarrow y = x)) \wedge \boldsymbol{p}(x)]$	
$\mathrm{E}c$	abbreviates	$\mathrm{E}\exists x(x = c)$	
$\mathrm{D}c$	abbreviates	$\mathrm{D}\exists x(x = c)$	
$\mathrm{S}c$	abbreviates	$\mathrm{S}\exists x(x = c)$	
${}^{s}Tt$	abbreviates	$\ulcorner Tt \wedge \neg Tt\urcorner \wedge\, !\mathrm{S}\exists x(x = t)$	
${}^{s}It$	abbreviates	$\ulcorner\neg t = t\urcorner \wedge\, !\mathrm{S}\exists x(x = t)$	
$\mathrm{H}\boldsymbol{p}$	abbreviates	$\neg\mathrm{P}\neg\boldsymbol{p}$	‘it has always been the case that’
$\mathrm{G}\boldsymbol{p}$	abbreviates	$\neg\mathrm{F}\neg\boldsymbol{p}$	‘it will always be the case that’
$\mathrm{A}\boldsymbol{p}$	abbreviates	$\mathrm{H}\boldsymbol{p} \wedge \boldsymbol{p} \wedge \mathrm{G}\boldsymbol{p}$	‘it is always the case that’

*as a matter of convention, treat each formula as though it begins with a tacit 'it is currently the case'.

3. Interdefinability

Ax. !E∃	$\forall y(!\mathrm{E}\exists x(x = y)$	$\leftrightarrow$	$[(\mathrm{E}\exists x(x = y) \wedge \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$
	$!\mathrm{E}c$ abbreviates	$!\mathrm{E}\exists x(x = c)$	
Ax. !D∃	$\forall y(!\mathrm{D}\exists x(x = y)$	$\leftrightarrow$	$[(\neg\mathrm{E}\exists x(x = y) \wedge \mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$
	$!\mathrm{D}c$ abbreviates	$!\mathrm{D}\exists x(x = c)$	
Ax. !S∃	$\forall y(!\exists \mathrm{S}x(x = y)$	$\leftrightarrow$	$[(\neg\mathrm{E}\exists x(x = y) \wedge \neg\mathrm{D}\exists x(x = y)) \wedge \mathrm{S}\exists x(x = y)])$
	$!\mathrm{S}c$ abbreviates	$!\mathrm{S}\exists x(x = c)$	

4. Concrete, Non-Concrete, Impossible (and sententializing variants)

$\mathit{Concrete}(u)$	abbreviates	$!\mathrm{E}u$
$\mathit{Non\text{-}concrete}(u)$	abbreviates	$!\mathrm{D}u$
$\mathit{Abstract}(u)$	abbreviates	$!\mathrm{D}u \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(u) \dashv \boldsymbol{q}(y))$
$\mathit{Fictional}(u)$	abbreviates	$!\mathrm{D}u \wedge (\neg\mathit{Abstract}(u) \wedge \mathrm{A}\neg!\mathrm{E}u)$
$\mathit{T\text{-}impossible}(u)$	abbreviates	${}^{s}Tu$
$\mathit{Vague}(u)$	abbreviates	${}^{s}Iu$
$\mathit{Impossible}(u)$	abbreviates	$(\exists X)X\text{-}\mathit{impossible}(u) \vee \mathit{Vague}(u)$
${}^{s}\mathit{Concrete}(u)$	abbreviates	$\lceil !\mathrm{E}u \wedge \neg!\mathrm{E}u\rceil \wedge !\mathrm{S}u$
${}^{s}\mathit{Non\text{-}concrete}(u)$	abbreviates	$\lceil !\mathrm{D}u \wedge \neg!\mathrm{D}u\rceil \wedge !\mathrm{S}u$
${}^{s}\mathit{T\text{-}Impossible}(u)$	abbreviates	$\lceil {}^{s}Tu \wedge \neg{}^{s}Tu\rceil \wedge !\mathrm{S}\exists x(x = u)$
${}^{s}\mathit{Abstract}(u)$	abbreviates	$\lceil(!\mathrm{D}u \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(u) \dashv \boldsymbol{q}(y))) \wedge \neg(!\mathrm{D}u \wedge \mathrm{A}\neg\mathrm{E}\exists y(\boldsymbol{p}(u) \dashv \boldsymbol{q}(y)))\rceil \wedge !\mathrm{S}u$
${}^{s}\mathit{Fictional}(u)$	abbreviates	$\lceil(!\mathrm{D}u \wedge (\neg\mathit{Abstract}(u) \wedge \mathrm{A}\neg!\mathrm{E}u)) \wedge \neg(!\mathrm{D}u \wedge (\neg\mathit{Abstract}(u) \wedge \mathrm{A}\neg!\mathrm{E}u))\rceil \wedge !\mathrm{S}u$
${}^{s}\mathit{Vague}(u)$	abbreviates	$\lceil {}^{s}Iu \wedge \neg{}^{s}Iu\rceil \wedge !\mathrm{S}u$
${}^{s}\mathit{Impossible}(u)$	abbreviates	$\lceil(\exists X)X\text{-}\mathit{impossible}(u) \wedge \neg(\exists X)X\text{-}\mathit{impossible}(u)\rceil \wedge !\mathrm{S}\exists x(x = u)$ $\vee$ $\lceil \mathit{Vague}(u) \wedge \neg\mathit{Vague}(u)\rceil \wedge !\mathrm{S}\exists x(x = u)$

5. Structure

5.1. $\boldsymbol{\mathcal{M}}^{\mathrm{T}} := (E, D, S, T, \prec, t, \mathrm{Sent}(\boldsymbol{\mathcal{ILT}}), \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{T}}, R\ulcorner\urcorner)$

6. Interpretation

6.1. (Variable Assignment) Given the structure, $\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)}$ of type (λ, J),

A-sequence := a countable sequence of elements of S_t (denoted, '$\boldsymbol{a^t} = (a_0, a_1, \ldots)$')

R-sequence := a countable sequence of elements of $\boldsymbol{\mathcal{R}}_t$ (denoted, '$\boldsymbol{r^t} = (R_0, R_1, \ldots)$') with the following constraint:

For each n, the nth R in $\boldsymbol{r}$ is of degree $v(n)$

6.2. (Interpreting the Symbols) Given $\boldsymbol{\mathcal{M}}$, $\boldsymbol{a^t}$, $\boldsymbol{r^t}$, $\boldsymbol{t}$ (where we read '$\mathbf{V}^{(\mathcal{M},\, at,\, rt,\, t)}$' as the element of $\boldsymbol{\mathcal{M}}$ that $\mathbf{V}$ is interpreted-by/names/is-assigned at time t),

Interpretation of $\boldsymbol{\mathcal{IL}}$ in ($\boldsymbol{\mathcal{M}}$, $\boldsymbol{a^t}$, $\boldsymbol{r^t}$, $\boldsymbol{t}$) :=

i) $P_i^{(\mathcal{M},\, at,\, rt,\, t)} = R_i^t$

ii) $V_n^{(\mathcal{M},\, at,\, rt,\, t)} = R_n$

iii) $c_j^{(\mathcal{M},\, at,\, rt,\, t)} = e_i^t$

iv) $v_n^{(\mathcal{M},\, at,\, rt,\, t)} = a_n$

6.3. (Variant Assignment)

For $n \in \omega$, $b \in S_t$,

$[n|b]\boldsymbol{a^t} := (a_0, a_1, \ldots, a_{n-1}, b, a_{n+1}, \ldots)$

For $n \in \omega$, $S \in \boldsymbol{\mathcal{R}}_t$ (where S is of degree $v(n)$)

$[n|S]\boldsymbol{r^t} := (R_0, R_1, \ldots, R_{n-1}, S, R_{n+1}, \ldots)$

7. Satisfaction

7.1. For $\boldsymbol{p} \in Form(\boldsymbol{\mathcal{ILT}})$,

$\boldsymbol{a^t}$, $\boldsymbol{r^t}$ *satisfy* $\boldsymbol{p}$ *in* $\boldsymbol{\mathcal{M}}^{\mathrm{T}}$ at time t (denoted, '$\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} \boldsymbol{p}$') :=

7.1.1. for terms t, u,

$\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} t = u \quad \Leftrightarrow \quad t^{(\mathcal{M},\, at,\, rt,\, t)} = u^{(\mathcal{M},\, at,\, rt,\, t)}$

for predicates T, U,

$\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} T = U \quad \Leftrightarrow \quad T^{(\mathcal{M},\, at,\, rt,\, t)} = U^{(\mathcal{M},\, at,\, rt,\, t)}$

7.1.2. for terms $t_1, \ldots, t_{\lambda(i)}$ and predicate T_i

$\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} T_i\, t_1, \ldots, t_{\lambda(i)} \quad \Leftrightarrow \quad (t_1^{(\mathcal{M},\, at,\, rt,\, t)}, \ldots, t_{\lambda(i)}^{(\mathcal{M},\, at,\, rt,\, t)}) \in T_i^{(\mathcal{M},\, at,\, rt,\, t)}$

7.1.3. $\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} \neg\boldsymbol{p} \quad \Leftrightarrow \quad$ *it is not the case that* $\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} \boldsymbol{p}$

7.1.4. $\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} \boldsymbol{p} \wedge \boldsymbol{q} \quad \Leftrightarrow \quad \boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} \boldsymbol{p}$ and $\boldsymbol{\mathcal{M}}^{\mathrm{T}}_{(t)} \vDash_{at,\, rt} \boldsymbol{q}$

7.1.5. $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \exists V_n\boldsymbol{p}$ $\Leftrightarrow$ for some $S \in \mathcal{R}_t$ of degree $v(n)$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, [n|S]rt}} \boldsymbol{p}$

7.1.6. $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \ulcorner \boldsymbol{p} \urcorner$ $\Leftrightarrow$ $\boldsymbol{p}$ contains free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and for some $\mathbf{C}_1 \in \mathcal{C} \cup \mathcal{P}, \ldots,$ $\mathbf{C}_n \in \mathcal{C} \cup \mathcal{P}$, $\mathbf{C}_1^{(\mathcal{M},\, a,\, r)} = \mathbf{X}_1^{(\mathcal{M},\, a,\, r)}, \ldots,$ and $\mathbf{C}_n^{(\mathcal{M},\, a,\, r)} = \mathbf{X}_n^{(\mathcal{M},\, a,\, r)}$

and

$\boldsymbol{p}(\mathbf{X}_1, \ldots, \mathbf{X}_n / \mathbf{C}_1, \ldots, \mathbf{C}_n) \in R\ulcorner\urcorner$

or

$\boldsymbol{p}$ does not contain free variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and $\boldsymbol{p} \in R\ulcorner\urcorner$

7.1.7. $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \exists v_n\boldsymbol{p}$ $\Leftrightarrow$ for some $b \in S_t$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \mathrm{E}\exists v_n\boldsymbol{p}$ $\Leftrightarrow$ E_t is non-empty and for some $b \in E_t$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \mathrm{D}\exists v_n\boldsymbol{p}$ $\Leftrightarrow$ D_t is non-empty and for some $b \in D_t$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \mathrm{S}\exists v_n\boldsymbol{p}$ $\Leftrightarrow$ for some $b \in S_t$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{[n|b]a,\, r} \boldsymbol{p}$

7.1.8. $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{E}\exists v_n\boldsymbol{p}$ $\Leftrightarrow$ E_t is non-empty and for some $b \in E_t$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{D}\exists v_n\boldsymbol{p}$ $\Leftrightarrow$ D_t is non-empty and for some non-empty $X \subseteq D_t$, $X \cap E_t = \varnothing$

and

for some $b \in D_t$ (where $b \notin E_t$), $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{S}\exists v_n\boldsymbol{p}$ $\Leftrightarrow$ for some $b \in S_t$ (where $b \notin E_t$ and $b \notin D_t$), $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

7.1.9. for a term t

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{E}t$ $\Leftrightarrow$ $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{E}\exists x(x = t)$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{D}t$ $\Leftrightarrow$ $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{D}\exists x(x = t)$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{S}t$ $\Leftrightarrow$ $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{S}\exists x(x = t)$

7.1.10. for a term t

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} {}^{s}P_i t$ $\Leftrightarrow$ $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \ulcorner P_i t \wedge \neg P_i t \urcorner$ and $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{S}t$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} {}^{s}V_i t$ $\Leftrightarrow$ $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \ulcorner V_i t \wedge \neg V_i t \urcorner$ and $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} !\mathrm{S}t$

7.1.11. $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \forall v_n\boldsymbol{p}$ $\Leftrightarrow$ for all $b \in S_t$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

7.1.12. $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \mathrm{E}\forall v_n\boldsymbol{p}$ $\Leftrightarrow$ if E_t is non-empty, then for all $b \in E_t$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{at,\, rt}} \mathrm{D}\forall v_n\boldsymbol{p}$ $\Leftrightarrow$ if D_t is non-empty, then for all $b \in D_t$, $\mathcal{M}^{\mathrm{T}}_{(t)} \vDash_{\boldsymbol{[n|b]at,\, rt}} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} \mathrm{S}\forall v_n\boldsymbol{p}$ $\Leftrightarrow$ for all $b \in S_t$, $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{[n|b]at,\,rt} \boldsymbol{p}$

7.1.13. $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} !\mathrm{E}\forall v_n\boldsymbol{p}$ $\Leftrightarrow$ if E_t is non-empty, then for all $b \in E_t$, $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{[n|b]at,\,rt} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} !\mathrm{D}\forall v_n\boldsymbol{p}$ $\Leftrightarrow$ if D_t is non-empty and for some non-empty $X \subseteq D_t$, $X \cap E_t = \varnothing$,
then for all $b \in D_t$ (where $b \notin E_t$), $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{[n|b]at,\,rt} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} !\mathrm{S}\forall v_n\boldsymbol{p}$ $\Leftrightarrow$ for all $b \in S_t$ (where $b \notin E_t$ and $b \notin D_t$), $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{[n|b]at,\,rt} \boldsymbol{p}$

7.1.14. $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} \mathrm{P}\boldsymbol{p}$ $\Leftrightarrow$ for some t' where $t' \prec t$, $\mathcal{M}^{\mathrm{T}}{}_{(t')} \vDash_{at',\,rt'} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} \mathrm{F}\boldsymbol{p}$ $\Leftrightarrow$ for some t' where $t \prec t'$, $\mathcal{M}^{\mathrm{T}}{}_{(t')} \vDash_{at',\,rt'} \boldsymbol{p}$

(where two proofs similar to the one generating satisfaction rules for universalized statements gets us)

7.1.15. $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} \mathrm{H}\boldsymbol{p}$ $\Leftrightarrow$ for all t' where $t' \prec t$, $\mathcal{M}^{\mathrm{T}}{}_{(t')} \vDash_{at',\,rt'} \boldsymbol{p}$

$\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} \mathrm{G}\boldsymbol{p}$ $\Leftrightarrow$ for all t' where $t \prec t'$, $\mathcal{M}^{\mathrm{T}}{}_{(t')} \vDash_{at',\,rt'} \boldsymbol{p}$

7.1.16. $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} \mathrm{Abstract}(u)$ $\Leftrightarrow$ for some $b \in D^t$ (where $b \notin E^t$) $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{[n|b]at,\,rt} v_n = u$

and

for all $\boldsymbol{p}, \boldsymbol{q} \in \mathrm{Form}(\mathcal{ILT})$

for all t, $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt}$

$$\neg\mathrm{E}\exists y(\mathrm{A}[\mathrm{P}(\boldsymbol{p}(u) \to \mathrm{F}\boldsymbol{q}(y)) \wedge \mathrm{F}(\boldsymbol{q}(y) \to \mathrm{P}\,\boldsymbol{p}(u))] \wedge \mathrm{A}\neg[\boldsymbol{p}(u) \leftrightarrow \boldsymbol{q}(y)])$$

7.1.17.
We say that an $\mathcal{ILT}$ formula $\boldsymbol{p}$ is *satisfiable* if for some $\mathcal{ILT}$ structure $\mathcal{M}^{\mathrm{T}}{}_{(t)}$ and variable assignments $\boldsymbol{a^t}$, $\boldsymbol{r^t}$, $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} \boldsymbol{p}$.

7.1.18.
We say that an $\mathcal{ILT}$ formula $\boldsymbol{p}$ is *valid* if for some $\mathcal{ILT}$ structure $\mathcal{M}^{\mathrm{T}}{}_{(t)}$ and all variable assignments $\boldsymbol{a^t}$, $\boldsymbol{r^t}$, $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash_{at,\,rt} \boldsymbol{p}$.

7.1.19.
We say that an $\mathcal{ILT}$ formula $\boldsymbol{p}$ is *universally valid* (‘$\varnothing \vDash \boldsymbol{s}$’ or ‘$\vDash \boldsymbol{s}$’) if for all $\mathcal{ILT}$ structures $\mathcal{M}^{\mathrm{T}}{}_{(t)}$, $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash \boldsymbol{p}$.

7.1.20.
For any $\Gamma \subseteq \mathrm{Sent}(\mathcal{ILT})$ and any $\mathcal{IL}$ structures $\mathcal{M}^{\mathrm{T}}{}_{(t)}$, $\mathcal{M}^{\mathrm{T}}{}_{(t)}$ is a *model* of Γ (‘$\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash \Gamma$’) if, for each $\boldsymbol{s} \in \Gamma$, $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash \boldsymbol{s}$.

7.1.21.
For any $\Gamma \subseteq \mathrm{Sent}(\mathcal{ILT})$ and any $\boldsymbol{s} \in \mathrm{Sent}(\mathcal{IL})$, we say that Γ *entails* $\boldsymbol{s}$ (‘$\Gamma \vDash \boldsymbol{s}$’) if, for all $\mathcal{ILT}$ structures $\mathcal{M}^{\mathrm{T}}{}_{(t)}$,

if $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash \Gamma$, then $\mathcal{M}^{\mathrm{T}}{}_{(t)} \vDash \boldsymbol{s}$

8. Natural Deduction of $\mathcal{ILT}$.

8.1. Inference Rules (where ‘$\rhd$’ is read ‘from what preceded, infer…’)

Will At Some Point Elimination (FE)

	F$\boldsymbol{p}$		
		$\underline{\boldsymbol{p}}$	Aux FE
		$\boldsymbol{q}_1$	
		…	
		$\boldsymbol{q}_n$	
▻	F$\boldsymbol{q}_n$		

Has At Some Point Elimination (PE)

	P$\boldsymbol{p}$		
		$\underline{\boldsymbol{p}}$	Aux PE
		$\boldsymbol{q}_1$	
		…	
		$\boldsymbol{q}_n$	
▻	P$\boldsymbol{q}_n$		

Will Always Elimination (GE)

	G$\boldsymbol{p}$		
		$\underline{\boldsymbol{p}}$	Aux GE
		$\boldsymbol{q}_1$	
		…	
		$\boldsymbol{q}_n$	
▻	G$\boldsymbol{q}_n$		

Has Always Elimination (HE)

	H$\boldsymbol{p}$		
		$\underline{\boldsymbol{p}}$	Aux HE
		$\boldsymbol{q}_1$	
		…	
		$\boldsymbol{q}_n$	
▻	H$\boldsymbol{q}_n$		

Has Always Introduction (HI)

	$\boldsymbol{p}$
▻	H$\boldsymbol{p}$

Where $\boldsymbol{p}$ is a theorem.

Will Always Introduction (GI)

	$\boldsymbol{p}$
▻	G$\boldsymbol{p}$

Where $\boldsymbol{p}$ is a theorem.

Has Will Always Elimination (PGE)

	PG$\boldsymbol{p}$
▻	$\boldsymbol{p}$

Where PG$\boldsymbol{p}$ and $\boldsymbol{p}$ appear in the same temporal scope.

Will Has Always Elimination (FHE)

	FH$\boldsymbol{p}$
▻	$\boldsymbol{p}$

Where FH$\boldsymbol{p}$ and $\boldsymbol{p}$ appear in the same temporal scope.

Past Definition (PHD)

▻ $\neg\mathrm{H}\neg\boldsymbol{p} \leftrightarrow \mathrm{P}\boldsymbol{p}$

and

▻ $\neg\mathrm{P}\neg\boldsymbol{p} \leftrightarrow \mathrm{H}\boldsymbol{p}$

Future Definition (FGD)

▻ $\neg\mathrm{G}\neg\boldsymbol{p} \leftrightarrow \mathrm{F}\boldsymbol{p}$

and

▻ $\neg\mathrm{F}\neg\boldsymbol{p} \leftrightarrow \mathrm{G}\boldsymbol{p}$

Reiteration (R)

	$\boldsymbol{p}$
▻	$\boldsymbol{p}$

With the Following Conditions:

i) for any formula $\boldsymbol{p}$ outside the scope of Aux FE or Aux GE, only formulas $\boldsymbol{p}$ = P$\boldsymbol{q}$ or $\boldsymbol{p}$ = G$\boldsymbol{q}$ may be reiterated within the scope of Aux FE or Aux GE.

ii) for any formula $\boldsymbol{p}$ outside the scope of Aux PE or Aux HE, only formulas $\boldsymbol{p}$ = F$\boldsymbol{q}$ or $\boldsymbol{p}$ = H$\boldsymbol{q}$ may be reiterated within the scope of Aux FE or Aux GE.

Has Some Reiteration (PR)

$\mathrm{H}\boldsymbol{q}$
$\underline{\boldsymbol{p}}$ Aux PE
▻ $\boldsymbol{q}$

Has Always Reiteration (HR)

$\mathrm{H}\boldsymbol{q}$
$\underline{\boldsymbol{p}}$ Aux HE
▻ $\boldsymbol{q}$

Will Some Reiteration (FR)

$\mathrm{G}\boldsymbol{q}$
$\underline{\boldsymbol{p}}$ Aux FE
▻ $\boldsymbol{q}$

Will Always Reiteration (GR)

$\mathrm{G}\boldsymbol{q}$
$\underline{\boldsymbol{p}}$ Aux GE
▻ $\boldsymbol{q}$

Conjunction Introduction ($\wedge$I)

$\boldsymbol{p}$
$\boldsymbol{q}$
▻ $\boldsymbol{p} \wedge \boldsymbol{q}$

With the Following Conditions:

i) if $\boldsymbol{p}$ is outside the scope of Aux FE or Aux GE (where $\boldsymbol{p} \neq \mathrm{P}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{G}\boldsymbol{r}$), $\boldsymbol{p} \wedge \boldsymbol{q}$ cannot feature within the scope of Aux FE or Aux GE unless $\boldsymbol{p}$ features within the scope of Aux FE or Aux GE (respectively).

ii) if $\boldsymbol{q}$ is outside the scope of Aux FE or Aux GE (where $\boldsymbol{q} \neq \mathrm{P}\boldsymbol{r}$ and $\boldsymbol{q} \neq \mathrm{G}\boldsymbol{r}$), $\boldsymbol{p} \wedge \boldsymbol{q}$ cannot feature within the scope of Aux FE or Aux GE unless $\boldsymbol{q}$ features within the scope of Aux FE or Aux GE (respectively).

iii) if $\boldsymbol{p}$ is outside the scope of Aux PE or Aux HE (where $\boldsymbol{p} \neq \mathrm{F}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{H}\boldsymbol{r}$), $\boldsymbol{p} \wedge \boldsymbol{q}$ cannot feature within the scope of Aux PE or Aux HE unless $\boldsymbol{p}$ features within the scope of Aux PE or Aux HE (respectively).

iv) if $\boldsymbol{q}$ is outside the scope of Aux PE or Aux HE (where $\boldsymbol{q} \neq \mathrm{F}\boldsymbol{r}$ and $\boldsymbol{q} \neq \mathrm{H}\boldsymbol{r}$), $\boldsymbol{p} \wedge \boldsymbol{q}$ cannot feature within the scope of Aux PE or Aux HE unless $\boldsymbol{q}$ features within the scope of Aux PE or Aux HE (respectively).

Conjunction Elimination ($\wedge$E)

$\boldsymbol{p} \wedge \boldsymbol{q}$
▻ $\boldsymbol{p}$
or
$\boldsymbol{p} \wedge \boldsymbol{q}$
▻ $\boldsymbol{q}$

With the Following Conditions:

i) if $\boldsymbol{p} \wedge \boldsymbol{q}$ is outside the scope of Aux FE or Aux GE (and $\boldsymbol{p} \neq \mathrm{P}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{Gr}$), $\boldsymbol{p}$ cannot be detached (from $\boldsymbol{p} \wedge \boldsymbol{q}$) within the scope of Aux FE or Aux GE unless $\boldsymbol{p} \wedge \boldsymbol{q}$ features within the scope of Aux FE or Aux GE (respectively).

ii) if $\boldsymbol{p} \wedge \boldsymbol{q}$ is outside the scope of Aux FE or Aux GE (and $\boldsymbol{q} \neq \mathrm{P}\boldsymbol{r}$ and $\boldsymbol{q} \neq \mathrm{Gr}$), $\boldsymbol{q}$ cannot be detached (from $\boldsymbol{p} \wedge \boldsymbol{q}$) within the scope of Aux FE or Aux GE unless $\boldsymbol{p} \wedge \boldsymbol{q}$ features within the scope of Aux FE or Aux GE (respectively).

iii) if $\boldsymbol{p} \wedge \boldsymbol{q}$ is outside the scope of Aux PE or Aux HE (and $\boldsymbol{p} \neq \mathrm{F}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{Hr}$), $\boldsymbol{p}$ cannot be detached (from $\boldsymbol{p} \wedge \boldsymbol{q}$) within the scope of Aux PE or Aux HE unless $\boldsymbol{p} \wedge \boldsymbol{q}$ features within the scope of Aux PE or Aux HE (respectively).

iv) if $\boldsymbol{p} \wedge \boldsymbol{q}$ is outside the scope of Aux PE or Aux HE (and $\boldsymbol{q} \neq \mathrm{F}\boldsymbol{r}$ and $\boldsymbol{q} \neq \mathrm{Hr}$), $\boldsymbol{q}$ cannot be detached (from $\boldsymbol{p} \wedge \boldsymbol{q}$) within the scope of Aux PE or Aux HE unless $\boldsymbol{p} \wedge \boldsymbol{q}$ features within the scope of Aux PE or Aux HE (respectively).

Conditional Introduction ($\rightarrow$I)

$\underline{\boldsymbol{p}}$
$\boldsymbol{q}$
$\rhd$ $\boldsymbol{p} \rightarrow \boldsymbol{q}$

Negation Introduction ($\neg$I)

$\underline{\boldsymbol{p}}$
$\boldsymbol{q}$
$\neg\boldsymbol{q}$
$\rhd$ $\neg\boldsymbol{p}$

Disjunction Introduction ($\vee$I)

$\boldsymbol{p}$
$\rhd$ $\boldsymbol{p} \vee \boldsymbol{q}$
or
$\boldsymbol{q}$
$\rhd$ $\boldsymbol{p} \vee \boldsymbol{q}$

Conditional Elimination ($\rightarrow$E)

$\boldsymbol{p} \rightarrow \boldsymbol{q}$
$\boldsymbol{p}$
$\rhd$ $\boldsymbol{q}$

With the Following Condition:

i) $\boldsymbol{p} \rightarrow \boldsymbol{q}$ and $\boldsymbol{p}$ must be in the same temporal scope (where the scopes of Aux FE, Aux PE, Aux GE, and Aux HE and the scope of primary assumptions are temporal scopes).

Negation Elimination ($\neg$E)

$\underline{\neg\boldsymbol{p}}$
$\boldsymbol{q}$
$\neg\boldsymbol{q}$
$\rhd$ $\boldsymbol{p}$

Disjunction Elimination ($\vee$E)

$\boldsymbol{p} \vee \boldsymbol{q}$
$\underline{\boldsymbol{p}}$ Aux $\vee$E
$\boldsymbol{r}$
$\underline{\boldsymbol{q}}$ Aux $\vee$E
$\boldsymbol{r}$
$\rhd$ $\boldsymbol{r}$

With the Following Conditions:

i) if $\boldsymbol{p} \vee \boldsymbol{q}$ is outside the scope of Aux FE or Aux GE (and $\boldsymbol{p} \neq$ P$\boldsymbol{r}$ and $\boldsymbol{p} \neq$ Gr), $\boldsymbol{p}$ cannot be assumed (as Aux $\vee$E) within the scope of Aux FE or Aux GE unless $\boldsymbol{p} \vee \boldsymbol{q}$ features within the scope of Aux FE or Aux GE (respectively).

ii) if $\boldsymbol{p} \vee \boldsymbol{q}$ is outside the scope of Aux FE or Aux GE (and $\boldsymbol{q} \neq$ P$\boldsymbol{r}$ and $\boldsymbol{q} \neq$ Gr), $\boldsymbol{q}$ cannot be assumed (as Aux $\vee$E) within the scope of Aux FE or Aux GE unless $\boldsymbol{p} \vee \boldsymbol{q}$ features within the scope of Aux FE or Aux GE (respectively).

iii) if $\boldsymbol{p} \vee \boldsymbol{q}$ is outside the scope of Aux PE or Aux HE (and $\boldsymbol{p} \neq$ F$\boldsymbol{r}$ and $\boldsymbol{p} \neq$ Hr), $\boldsymbol{p}$ cannot be assumed (as Aux $\vee$E) within the scope of Aux PE or Aux HE unless $\boldsymbol{p} \vee \boldsymbol{q}$ features within the scope of Aux PE or Aux HE (respectively).

iv) if $\boldsymbol{p} \vee \boldsymbol{q}$ is outside the scope of Aux PE or Aux HE (and $\boldsymbol{q} \neq$ F$\boldsymbol{r}$ and $\boldsymbol{q} \neq$ Hr), $\boldsymbol{q}$ cannot be assumed (as Aux $\vee$E) within the scope of Aux PE or Aux HE unless $\boldsymbol{p} \vee \boldsymbol{q}$ features within the scope of Aux PE or Aux HE (respectively).

Biconditional Elimination (↔E)

$\boldsymbol{p \leftrightarrow q}$
$\boldsymbol{p}$
▻ $\boldsymbol{q}$
or
$\boldsymbol{p \leftrightarrow q}$
$\boldsymbol{q}$
▻ $\boldsymbol{p}$

With the Following Condition:

i) $\boldsymbol{p \leftrightarrow q}$ and $\boldsymbol{p}$ (or $\boldsymbol{q}$) must be in the same temporal scope.

Biconditional Introduction (↔I)

$\underline{\boldsymbol{p}}$
$\boldsymbol{q}$

$\underline{\boldsymbol{q}}$
$\boldsymbol{p}$
▻ $\underline{\boldsymbol{p \leftrightarrow q}}$

Universal Introduction (∀I)

$\boldsymbol{p}(\mathbf{C}/\mathbf{X})$
▻ $\forall\mathbf{X}\boldsymbol{p}$

With the conditions:

i) **C** does not occur in an open assumption.

ii) **C** does not occur in $\forall\mathbf{X}\boldsymbol{p}$.

iii) if $\boldsymbol{p}$ is outside the scope of Aux FE or Aux GE (where $\boldsymbol{p} \neq \mathrm{P}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{G}\boldsymbol{r}$), $\forall\mathbf{X}\boldsymbol{p}$ cannot feature within the scope of Aux FE or Aux GE unless $\boldsymbol{p}$ features within the scope of Aux FE or Aux GE (respectively).

iv) if $\boldsymbol{p}$ is outside the scope of Aux PE or Aux HE (where $\boldsymbol{p} \neq \mathrm{F}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{H}\boldsymbol{r}$), $\forall\mathbf{X}\boldsymbol{p}$ cannot feature within the scope of Aux PE or Aux HE unless $\boldsymbol{p}$ features within the scope of Aux PE or Aux HE (respectively).

Universal Elimination (∀E)

$\forall\mathbf{X}\boldsymbol{p}$
▻ $\boldsymbol{p}(\mathbf{C}/\mathbf{X})$

With the conditions:

i) if $\forall\mathbf{X}\boldsymbol{p}$ is outside the scope of Aux FE or Aux GE (where $\boldsymbol{p} \neq \mathrm{P}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{G}\boldsymbol{r}$), $\boldsymbol{p}(\mathbf{C}/\mathbf{X})$ cannot feature within the scope of Aux FE or Aux GE unless $\forall\mathbf{X}\boldsymbol{p}$ features within the scope of Aux FE or Aux GE (respectively).

ii) if $\forall\mathbf{X}\boldsymbol{p}$ is outside the scope of Aux PE or Aux HE (where $\boldsymbol{p} \neq \mathrm{F}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{H}\boldsymbol{r}$), $\boldsymbol{p}(\mathbf{C}/\mathbf{X})$ cannot feature within the scope of Aux PE or Aux HE unless $\forall\mathbf{X}\boldsymbol{p}$ features within the scope of Aux PE or Aux HE (respectively).

Existential Elimination (∃E)

$\exists\mathbf{X}\boldsymbol{p}$
  $\underline{\boldsymbol{p}(\mathbf{C}/\mathbf{X})}$ Aux ∃E
  $\boldsymbol{q}$
⊳ $\boldsymbol{q}$

With the conditions:

i) **C** does not occur in an open assumption.

ii) **C** does not occur in $\exists\mathbf{X}\boldsymbol{p}$.

iii) **C** does not occur in $\boldsymbol{q}$.

iv) if $\exists\mathbf{X}\boldsymbol{p}$ is outside the scope of Aux FE or Aux GE (where $\boldsymbol{p} \neq \mathrm{P}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{G}\boldsymbol{r}$), Aux ∃E cannot feature within the scope of Aux FE or Aux GE unless $\exists\mathbf{X}\boldsymbol{p}$ features within the scope of Aux FE or Aux GE (respectively).

v) if $\exists\mathbf{X}\boldsymbol{p}$ is outside the scope of Aux PE or Aux HE (where $\boldsymbol{p} \neq \mathrm{F}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{H}\boldsymbol{r}$), Aux ∃E cannot feature within the scope of Aux PE or Aux HE unless $\exists\mathbf{X}\boldsymbol{p}$ features within the scope of Aux PE or Aux HE (respectively).

Existential Introduction (∃I)

$\boldsymbol{p}(\mathbf{C}/\mathbf{X})$
⊳ $\exists\mathbf{X}\boldsymbol{p}$

i) if $\boldsymbol{p}$ is outside the scope of Aux FE or Aux GE (where $\boldsymbol{p} \neq \mathrm{P}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{G}\boldsymbol{r}$), $\exists\mathbf{X}\boldsymbol{p}$ cannot feature within the scope of Aux FE or Aux GE unless $\boldsymbol{p}$ features within the scope of Aux FE or Aux GE (respectively).

ii) if $\boldsymbol{p}$ is outside the scope of Aux PE or Aux HE (where $\boldsymbol{p} \neq \mathrm{F}\boldsymbol{r}$ and $\boldsymbol{p} \neq \mathrm{H}\boldsymbol{r}$), $\exists\mathbf{X}\boldsymbol{p}$ cannot feature within the scope of Aux PE or Aux HE unless $\boldsymbol{p}$ features within the scope of Aux PE or Aux HE (respectively).

*From here we assume that the introduction and elimination rules for each sortal quantifier feature the same conditions (*mutatis mutandis*) as the above quantifier introduction and elimination rules.

!-Extant Universal Introduction (!E∀I)

$\underline{!\mathrm{E}c}$ Aux !E∀I
…
$\boldsymbol{p}(c/x)$

⊳ $!\mathrm{E}\forall x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux !E∀I.
ii) c does not occur in $!\mathrm{E}\forall x\boldsymbol{p}$.

!-Extant Universal Elimination (!E∀E)

$!\mathrm{E}\forall x\boldsymbol{p}$
$!\mathrm{E}c$
⊳ $\boldsymbol{p}(c/x)$

!-Extant Existential Elimination (!E∃E)

$!\mathrm{E}\exists x\boldsymbol{p}$
  $\underline{\boldsymbol{p}(c/x) \wedge !\mathrm{E}c}$
  $\boldsymbol{q}$
⊳ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in $!\mathrm{E}\exists x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

!-Extant Existential Introduction (!E∃I)

$\boldsymbol{p}(c/x)$
$!\mathrm{E}c$
⊳ $!\mathrm{E}\exists x\boldsymbol{p}$

!-Depictable Universal Introduction (!D∀I)

$\underline{!Dc\quad\quad}$ Aux !D∀I
…
$\boldsymbol{p}(c/x)$

▻ !D∀$x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux !D∀I.
ii) c does not occur in !D∀$x\boldsymbol{p}$.

!-Depictable Existential Introduction (!D∃I)

$\boldsymbol{p}(c/x)$
!Dc
▻ !D∃$x\boldsymbol{p}$

!-Sentential Universal Introduction (!S∀I)

$\underline{!Sc\quad\quad}$ Aux !S∀I
…
$\boldsymbol{p}(c/x)$

▻ !S∀$x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux !S∀I.
ii) c does not occur in !S∀$x\boldsymbol{p}$.

!-Sentential Existential Introduction (!S∃I)

$\boldsymbol{p}(c/x)$
!Sc
▻ !S∃$x\boldsymbol{p}$

Extant Universal Introduction (E∀I)

$\underline{Ec\quad\quad}$ Aux E∀I
…
$\boldsymbol{p}(c/x)$

▻ E∀$x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux E∀I.
ii) c does not occur in E∀$x\boldsymbol{p}$.

!-Depictable Universal Elimination (!D∀E)

!D∀$x\boldsymbol{p}$
!Dc
▻ $\boldsymbol{p}(c/x)$

!-Depictable Existential Elimination (!D∃E)

!D∃$x\boldsymbol{p}$
$\underline{\boldsymbol{p}(c/x) \wedge !Dc}$
$\boldsymbol{q}$
▻ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in !D∃$x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

!-Sentential Universal Elimination (!S∀E)

!S∀$x\boldsymbol{p}$
!Sc
▻ $\boldsymbol{p}(c/x)$

!-Sentential Existential Elimination (!S∃E)

!S∃$x\boldsymbol{p}$
$\underline{\boldsymbol{p}(c/x) \wedge !Sc}$
$\boldsymbol{q}$
▻ $\boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in !S∃$x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

Extant Universal Elimination (E∀E)

E∀$x\boldsymbol{p}$
Ec
▻ $\boldsymbol{p}(c/x)$

Extant Existential Introduction (E∃I)

$\quad \boldsymbol{p}(c/x)$

$\quad \mathrm{E}c$

$\rhd \quad \mathrm{E}\exists x\boldsymbol{p}$

Depictable Universal Introduction (D∀I)

$\quad \underline{\mathrm{D}c\qquad}$ $\qquad$ Aux D∀I

$\quad \ldots$

$\quad \boldsymbol{p}(c/x)$

$\rhd\ \mathrm{D}\forall x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux D∀I.
ii) c does not occur in $\mathrm{D}\forall x\boldsymbol{p}$.

Depictable Existential Introduction (D∃I)

$\quad \boldsymbol{p}(c/x)$

$\quad \mathrm{D}c$

$\rhd \quad \mathrm{D}\exists x\boldsymbol{p}$

Sentential Universal Introduction (S∀I)

$\quad \underline{\mathrm{S}c\qquad}$ $\qquad$ Aux S∀I

$\quad \ldots$

$\quad \boldsymbol{p}(c/x)$

$\rhd\ \mathrm{S}\forall x\boldsymbol{p}$

With the conditions:
i) c does not occur in an open assumption outside the scope of Aux S∀I.
ii) c does not occur in $\mathrm{S}\forall x\boldsymbol{p}$.

Sentential Existential Introduction (S∃I)

$\quad \boldsymbol{p}(c/x)$

$\quad \mathrm{S}c$

$\rhd \quad \mathrm{S}\exists x\boldsymbol{p}$

Extant Existential Elimination (E∃E)

$\quad \mathrm{E}\exists x\boldsymbol{p}$

$\qquad \underline{\boldsymbol{p}(c/x) \wedge \mathrm{E}c}$

$\qquad \boldsymbol{q}$

$\rhd \quad \boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in $\mathrm{E}\exists x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

Depictable Universal Elimination (D∀E)

$\quad \mathrm{D}\forall x\boldsymbol{p}$

$\quad \mathrm{D}c$

$\rhd \quad \boldsymbol{p}(c/x)$

Depictable Existential Elimination (D∃E)

$\quad \mathrm{D}\exists x\boldsymbol{p}$

$\qquad \underline{\boldsymbol{p}(c/x) \wedge \mathrm{D}c}$

$\qquad \boldsymbol{q}$

$\rhd \quad \boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in $\mathrm{D}\exists x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

Sentential Universal Elimination (S∀E)

$\quad \mathrm{S}\forall x\boldsymbol{p}$

$\quad \mathrm{S}c$

$\rhd \quad \boldsymbol{p}(c/x)$

Sentential Existential Elimination (S∃E)

$\quad \mathrm{S}\exists x\boldsymbol{p}$

$\qquad \underline{\boldsymbol{p}(c/x) \wedge \mathrm{S}c}$

$\qquad \boldsymbol{q}$

$\rhd \quad \boldsymbol{q}$

With the conditions:
i) c does not occur in an open assumption.
ii) c does not occur in $\mathrm{S}\#\exists x\boldsymbol{p}$.
iii) c does not occur in $\boldsymbol{q}$.

Identity Introduction (=I)	
⊳	$\forall \mathbf{X}(\mathbf{X} = \mathbf{X})$

Identity Elimination (=E)	
	$\mathbf{C} = \mathbf{D}$
	$\boldsymbol{p}$
⊳	$\boldsymbol{p}(\mathbf{C}/\mathbf{D})$
Or	
	$\mathbf{C} = \mathbf{D}$
	$\boldsymbol{p}$
⊳	$\boldsymbol{p}(\mathbf{D}/\mathbf{C})$

Note that we include the rules of the extended natural deduction system (i.e De Morgan, Transposition, quantifier negation, etc.) but, since they are derivable from all of the above (and are found in any logic textbook), we do not present them here. We do include the following axioms unique to $\mathcal{ILT}$ and second-order logics however.

Default Sententiality (DS)	
⊳	$\forall x(!\mathrm{E}x \vee [!\mathrm{D}x \vee !\mathrm{S}x])$
(informally: all objects start, at least, at sententiality)	

No Proof (NP)	
⊳	$\lceil \boldsymbol{p} \rceil \rightarrow \neg \boldsymbol{p}$
(informally: if we can say, but only say that ***p***, then ***p*** does not obtain)	

Extensionality (Ex.)	
⊳	$\forall X \forall Y(X = Y \leftrightarrow \forall x(Xx \leftrightarrow Yx))$
(informally: If two predicates are identical, then any object relating to the one, relates to the other and vice-versa)	

Comprehension (Comp.)	
⊳	$\exists X \forall x_1, \ldots, \forall x_n(Xx_1, \ldots, x_n \leftrightarrow \boldsymbol{p}(x_1, \ldots, x_n))$
(informally: This is the axiom scheme of comprehension *i.e*. any sequence of objects that satisfy some formula ***p***, relate to some predicate *X* and vice-versa)	

Abstract Schema (AS)	
⊳	$\forall x[\text{Abstract}(x) \leftrightarrow (!\text{D}x \wedge \text{A}\neg\text{E}\exists y(\boldsymbol{p}(x) \prec \boldsymbol{q}(y)))]$

Lastly, definitions Ax. !E∃, Ax. !D∃, Ax. !S∃ are axioms of $\mathcal{ILT}$.

8.2. Proof.

A *proof* (alternatively *derivation*) in $\mathcal{IL}$ of $\boldsymbol{p}$ from Γ (where, $\boldsymbol{p} \in \text{Sent}(\mathcal{IL})$ and $\Gamma \subseteq \text{Sent}(\mathcal{IL})$) consists of a series:

$$
\begin{array}{ll}
1. & \Gamma \\
 & \vdots \\
m. & \boldsymbol{q}_1 \\
 & \dots \\
n. & \boldsymbol{q}_n
\end{array}
$$

where 'Γ . . .' is a list of the sentences of Γ (where Γ is possibly empty), $\boldsymbol{p} = \boldsymbol{q}_n$, $\boldsymbol{q}_1 - \boldsymbol{q}_n$ are $\mathcal{IL}$-formulas, $\boldsymbol{q}_n$ can be derived by application of some rule of inference to formulas on lines $i < n$, and $\boldsymbol{q}_n$ falls only under the assumptions of 'Γ. . .'.

8.2.0. $\boldsymbol{p}$ *is provable from* Γ (denoted '$\Gamma \vdash \boldsymbol{p}$') *iff* there is a proof of $\boldsymbol{p}$ from Γ

8.2.1. Γ is *consistent* (in $\mathcal{IL}$) *iff* for no $\mathcal{IL}$-formula $\boldsymbol{p}$, $\Gamma \vdash \boldsymbol{p}$ and $\Gamma \vdash \neg\boldsymbol{p}$

8.2.2. $\varnothing \vdash \boldsymbol{p}$	is abbreviated	$\vdash \boldsymbol{p}$
8.2.3. $\vdash \boldsymbol{p}$	indicates that	$\boldsymbol{p}$ is a *theorem*

9. Theorems (where *x* and *y* are individual variables)

9.0.

9.0.1.	$\vdash \text{E}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$
9.0.2.	$\vdash \text{D}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$
9.0.3.	$\vdash \text{S}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$
9.0.4.	$\vdash !\text{E}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$
9.0.5.	$\vdash !\text{D}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$
9.0.6.	$\vdash !\text{S}\exists x\boldsymbol{p}(x) \rightarrow \exists y\boldsymbol{p}(y)$

Proof: Each of theorems 9.0.1. - 9.0.6. follow from the fact that in any case of existentialization (for example, in $E\exists x\boldsymbol{p}(x)$) it follows by the relevant existential elimination rule (where some c is an arbitrary witness for x, i.e. $p(c)$) that $\exists x\boldsymbol{p}(x)$ is derived by existential introduction.

9.1. $\vdash \forall x S\exists y(x = y)$

(informally: all objects are sentential)

Proof: by DS, x either starts at extantiality, depictability, or sententiality. In any of those cases, sententiality is implied.

9.2. $\vdash \forall x(E\exists y(x = y) \rightarrow D\exists y(x = y) \wedge S\exists y(x = y))$

(informally: if x is extant, then x is depictable and sentential)

Proof: assume x is extant. By 10.1. x is sentential. If x is not depictable then (by Ax. !E∃) x cannot start at extantiality. Further, because x is extant (by Ax. !S∃), x cannot start at sententiality either. Since x neither starts at extantiality nor starts at sententiality, by DS, x must start at depictability and (by !D∃) x is depictable. This is a contradiction, hence x is depictable. That x is sentential follows from theorem 10.1.

9.3. $\vdash \forall x(D\exists y(x = y) \rightarrow S\exists y(x = y))$

(informally: if x is depictable, then x is sentential)

Proof: an immediate consequence of 9.1.

9.4. $\vdash \forall x[\neg(!Ex \wedge !Dx) \wedge \neg(!Dx \wedge !Sx) \wedge \neg(!Ex \wedge !Sx)]$

Informally: all objects can start in just one order.

Proof: Assume $!Ex \wedge !Dx$. By Ax. !E∃, $!Ex$ implies $E\exists y(y = x)$ and by Ax. !D∃, $!Dx$ implies $\neg E\exists y(y = x)$, a contradiction. The same logic applies to $!Dx \wedge !Sx$ and $!Ex \wedge !Sx$.

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