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Investigating The Design Of An Aesthetic Mathematical Experience

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Abstract

In this thesis, I share my analysis of an aesthetic mathematical experience which happened while I designed and created a pop-up math story. My aim in analysing this experience is to better understand it and to make a model for other educators who are seeking to design aesthetic mathematical experiences. My research question is: What factors come into play when an educator immerses herself in the process of designing an aesthetic mathematical experience? My research uses qualitative analysis with a focus on narrative inquiry. My focus is on my personal experience designing a mathematics curriculum experience for students, using the context of a popup story that I authored. To better understand my experience, it is appropriate to see it as a narrative that I lived and to analyze it using narrative inquiry. The analysis of my experience identified five factors that came into play: surprise/insight, constructing, immersion, 'seeing as", and audience.

Keywords

Designing an aesthetic mathematical experience; Designing mathematics experience; Pop-up mathematics story; Pop-up mathematics activity; Mathematical surprise; Pop-up mathematical surprise; 'Seeing as' in math; 2D projection of 3D;

Summary for Lay Audience

In my previous master program, I had one elective course related to teaching and learning. I was interested in this subject and I chose to study education as my second master program. Since I had a background in science and mathematics, I decided to work with the professor whose focus was on mathematics education. As a research assistant, I was observing a math activity in a preservice teacher class called '"Making 10"' which was designed by my supervisor. What differentiates this activity from the other regular ones is the mathematical surprise and insight that students experience in this activity. I became motivated to display this surprise of '"Making 10"' in the pop-up context due to the experience I had in pop-up card making. I could see a similarity between the "Making 10" activity and the pop-up structure. Unexpectedly, different mathematical surprises appeared which led me to experience more mathematical discoveries and AHA! moments. I came to believe that such experiences are unique and precious in mathematics education and we, as educators, should consider providing activities in which students experience mathematical surprise and discovery. Therefore, my supervisor and I chose to investigate the process of creating the pop-up mathematical experience/surprise as my thesis research to identify the factors which helped me in this process. The aim of this thesis is to pave the way to some extend for other educators who intend to design aesthetic mathematical experiences for students.

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To my Supervisor

Dear Dr. George Gadanidis I have been blessed to have you as my supervisor. From the beginning of my master program, your advice helped me to manage the difficulties I felt in the beginning and also made my master program as one my best periods of life. I am so grateful for all efforts in supporting my research.

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My dear father and mother, Kazem and Akram thank you for your unconditional support. My dear sister, Mozhdeh, you did your best to support for me in these 2 years. Thank you.

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Chapter 1. Introduction

1.1 Introduction

The two sentences below are an origin point for my thesis:

Maryam: *Can I design and write the pop-up mathematical story as my assignment?*

George: *Of course, you can.*

I had this conversation with my supervisor in the first semester of my master program. It was also a time when he encouraged me to start thinking about what research topic I was going to work on for my thesis. We had no idea at that time that creating the pop-up math story would become the aesthetic mathematical experience that I investigated in my thesis.

1.2 Why did I try to create an aesthetic mathematical experience?

The answer to this question connects to the reason I chose to pursue a masters with a focus on mathematics education. When I was in high school and university, understanding mathematics became hard. This dampened my enthusiasm to pursue my education in physics, which was my favorite, and the field of my bachelor's degree. Having not had an appealing experience in math is sadly the case for many students. The common attitude is that math is "difficult, cold, abstract and in many cultures, largely masculine" (Ernest, 1996, p. 802). I chose to work in math education firstly to gain a deep math understanding and to help change math experiences in school for other students, so they may not have the same hard time in their mathematics learning life as I did. Papert (1980) criticized the distinction between "humanities" and "science" (p. 38). Higginson (2008) noted that mathematics in schools does not exhibit the "beauty" and "power" of mathematics.

One reason mathematics in school does not demonstrate the beauty of math is that aesthetic aspects are missing from the math education experience. Gadanidis, Borba, Hughes, and Lacerda (2016) state: “The problem with school mathematics is not that it lacks the arts, but rather that it lacks the aesthetic that is common to mathematics, the arts, and other disciplines: the aesthetic that makes the experience of these disciplines human” (p. 227). Papert (1980) by referring to Plato who wrote on his door "Let only geometers enter"(p. 38) noted that in the past, the distinction between “humanities” and “science” (p. 38) did not exist and Plato as a philosopher made connections between philosophy, humanity and science.

Contrary to many students who dislike mathematics, mathematicians are passionate about the subject because mathematicians witness the aesthetics in math in their work. Sinclair (2008) paraphrased what Krull (1930/1987) said and stated: “We are even told that the primary goals of the mathematician are aesthetic and not epistemological” (p.1). Understanding mathematics involves finding patterns and connections between mathematics concepts. Boaler (2016) stated: “when we ask mathematicians what math is, they will say it is the study of patterns that is an aesthetic, creative, and beautiful subject.” (p. 21 and 22) This is what mathematicians do and as a result experience aesthetics and insights while students lack such experiences in their math education.

1.3 Why did I choose to create a pop-up math story?

About 9 years ago, a friend of mine gave me a birthday card in which my name popped-out. I was thrilled and surprised seeing my name as the 3-D shape appear from the flat card and became so curious as to how it was made . The card came along with a CD of instruction about pop-up making. With a faint hope, I began following the CD instruction and unexpectedly it worked out well, and I could make the pop-up card, so it became my passion. This is how my pop-up life began.

Figure 1: One of the first pop-up cards I made

At the time, I was living in Malaysia, so with the friend who gave me the birthday card, we designed and made the postcard of the Petronas Twin Towers of Malaysia and started selling that in gift shops.

Figure 2: Malaysia pop-up postcard

With this history of pop-up experience, I started my master's program.

In the first semester, I witnessed a mathematical activity, called "Making 10", in which elementary school students experience math surprise. In brief: (1) starting with the number sentence __+__=10, students role a die to get the first number and calculate the second; (2) when they plot the apparently random pair of numbers, the points line up; (3) students create their own number sentences, to make the points line up in different directions, or to curve. A quick video summary of this activity is available at https://youtu.be/S2pEZIRBwso. I sensed some similarities between that activity and the pop-up structure, and I thought I might be able to implement that activity in the pop-up context to display the same math surprise in the different context.

1.4 Why was the creation of the pop-up math story considered as an aesthetic mathematical experience?

To answer this question, I need firstly to bring some math educators' and mathematicians' ideas about aesthetic mathematical experience into my discussion and secondly, to reflect on them to describe the experience I had in creating the pop-up math story.

Gadandis (2004) explained that "big math" is the "ideas that focus on mathematical relationships and integrate concepts which draw our attention and offer the opportunity for mathematical insight" (p. 10). Gadanidis et al (2016) noted that a good math experience provides opportunities for students "to explore, question, flex their imagination, discover, discuss and share their learning" and this helps the experience become "worthy of attention, worthy of conversation" (p. 238). Students can gain an aesthetic sense and see the beauty of math by searching and discovering structures within the field (Mann, 2006).

In the process of designing the pop-up math story, I encountered challenges and I needed to explore, to think hard and to have discussions with others. This path led me to find connections between mathematical structures, gain insight and surprise and see the beauty of math. This experience offered me the opportunities to see mathematics concepts as a whole, where every component fit in their place and this afforded me the sense of discovery in several moments in this process.

1.5 Why did I choose to investigate the process of creating the aesthetic mathematical experience (pop-up math story) in my thesis?

Most negative attitudes towards mathematics are caused by not having suitable experiences in school (Liljedahl, 2005). To change students' attitudes about mathematics, they need to have aesthetics experiences in school. Gadanidis et al (2016) believed that by affording students and teachers aesthetic mathematical experiences which offer them insight and surprise, we can change students' attitudes about math and their expectations from mathematics. Teachers desire to expose their students to interesting and enriched mathematical experiences in class. Gadanidis (2004) stated "teachers [] seek to touch their students through the classroom experiences they create" (p. 13). Designing the mathematics tasks in which students gain insight, experience surprise and see the beauty of math is not easy. Gadanidis, Borba, Hughes and Lacerda (2016) by being inspired by Boyd, make connection between designing insightful and surprising math experience and solving artistic puzzles and stated:

> Boyd (2001b) notes that good storytelling involves solving *artistic* puzzles of how to create situations where the audience experiences the pleasure of surprise and insight. Solving such artistic puzzles in mathematics pedagogy in ways that afford young children opportunities to engage with, to be surprised, and to gain insights into complex ideas of mathematics is not yet common in mathematics education, and it is not easy to do (p. 237).

Educators often design superficial art-math lessons that use the attractiveness of the arts to engage students with mathematics. This has the implication that school mathematics does not contain beauty itself and that it requires other disciplines like the arts to make it look interesting (Sinclair, 2001).

In this research, I am going to explore the process of creating the aesthetic mathematical experience to better understand it from an educators' perspective, using a mathematical and a pedagogical lens to reflect analytically my own experience. My goal is to understand an aesthetics mathematics experience, and to extract determinant factors for other educators who will design such aesthetic mathematical experiences. Hopefully the factors will better equip them to solve the artistic puzzles more confidently in order to develop more appealing and interesting math experiences for students.

1.6 What am I going to do in this thesis?

In this thesis, I am going to share my analysis of an aesthetic mathematical experience which happened while I designed and created a pop-up math story. As I mentioned before, my aim in analysing this experience is to better understand it and to make a model for other educators who are seeking to design aesthetic mathematical experiences. My research question is:

What factors come into play when an educator immerses herself in the process of designing an aesthetic mathematical experience?

My research uses qualitative analysis with a focus on narrative inquiry. My focus is on my personal experience designing a mathematics curriculum experience for students, using the context of a popup story that I authored. To better understand my experience, it is appropriate to see it as a narrative that I lived and to analyze it using narrative inquiry.

1.7 Theoretical framework

The theoretical framework here is based on constructivism (Piaget, 1896-1980), socio-constructivism (Vygotsky, 1978) and constructionism (Papert, 1980, 1987; Papert & Harel, 1991). I see students developing mathematically by constructing their own conceptual understanding, individually and in interaction with other people (peers, teachers, and others) as well as with other actors (technology and other tools) (Levy, 1997).

1.8 Definition of terms

Pop-up: 3D shapes which are made by 2D planes using cutting and folding.

1.9 The content of the thesis

Chapter 1 is about the rationale and significance of the study, and the purpose and research questions of thesis. In chapter 2, I review the literature related to my theoretical framework and the themes which are related to designing and analyzing an aesthetics mathematical experience. The methodology, ethics and limitation of my research are addressed in chapter 3. In chapter 4, I delve into my experience in designing the pop-up math story and I narrate my story chronologically. In chapter 5, I analyse my narrative and extract themes and a model for other educators who will to design aesthetic mathematical tasks in different contexts and settings. Chapter 6 includes recommendations for further research.

Chapter 2. Literature Review

2.1 Introduction

In the literature review below, I elaborate on the theories of constructivism, socio-constructivism, and constructionism, and ideas associated to aesthetic mathematics experience, as they relate to my research.

2.2 Literature review

The literature review that follows relates to my research question: What factors come into play when an educator immerses him/herself in the process of designing an aesthetic mathematical experience? As I explained in chapter one, my goal is to analyse and understand an *aesthetic* mathematics experience, which I had in designing and making a pop-up mathematical story, from an educator's perspectives. The math activity, "Making 10", that inspired me to design and make the pop-up math story entails characteristics of *constructivism* (students were given opportunities to construct their own understanding) and *social constructivism* (while working and negotiating meaning in collaboration with others). Re-designing the "Making 10" activity in a pop-up context, engaged me in a *constructivist* / *socio-constructivist* / *constructionist* and a *"seeing-as"* experience (Zwicky, 2003), with a focus on the *aesthetic* and on *surprise* in a *story* setting. Below, I elaborate on these ideas of constructivism, social constructivism, constructionism, seeing-as, aesthetics, surprise and story.

2.3 Constructivism

The origin of constructivism is based on Piaget's work. There are two principles: "1. Knowledge is actively constructed by the cognizing subject, not passively received from the environment. 2. Coming to know is an adaptive process that organizes one's experiential world" (Kilpatrick, 1987, p. 3). This theory is in sharp contrast to information transmission as a teaching method where information transfers from teachers to students. In constructivist theory students are not passive learners and they are actors who construct knowledge. Gadanidis (1994) stated: "Students acquire new knowledge through an active process of assimilation and accommodation, where new as well as existing knowledge is transformed as students construct more inclusive schemas of understating" (p. 93). Kamii and Ewing (1996) noted that "learning originates from inside the child" (p.260). In constructivist theory, knowledge is not transferred to the student but constructed by the student.

The method of teaching in constructivism affords students enriched, problematic and "open-ended" (Gadanidis, 1994, p. 94) learning activities in which they can explore, ask questions, talk and share their ideas (Gadanidis, 1994). In this situation, the mathematical understanding is elicited from active learning. The focus of constructivism is "the process of understanding" and it "would involve students in a problem situation from which understanding the process [] may emerge" (ibid). Fouche (1993) stated that mathematics problems should not be closed with one singular solution and should have a potential to be tackled with different methods. Teachers are more like facilitators who let students approach math puzzles with different strategies. On the other hand, students feel free to ask questions, make comments even potentially incorrect ones, and to share, and argue their ideas (ibid).

2.3.1 Agency

In constructivist learning, student agency is one of the important factors. Papert (1993) stated: "I am convinced that the best learning takes place when the learner takes charge" (p. 136). With "freedom to make choices, to investigate, and to discover", students experience the sense of agency which is enjoyable (Gadanidis, Clements & Yiu, 2018, p. 36). Agency is important in mathematics education and the aim is to provide the learning environment in which students have control on their learning (Gadanidis, 2017).

One of the main attributes of Egan's (2010) learning-in-depth project (which will be explained in the Immersion section of this chapter) is students' agency to explore the subject, accumulate information, organize information, delve into information and construct knowledge. Although Egan (2010) did not use explicitly the word "agency" for this purpose, his explanation of the process of learning-in-depth implies the meaning of agency.

2.3.2 Cognitive conflict

Gadanidis and Gieger (2010), referring to the Piaget's equilibrium theory, explained that a constructivist view necessitates the learner to encounter the conflict which was caused by some factors which would challenge their understanding of the world. Gadanidis et al. added that learning is the outcome of solving the conflict: "Learning occurs when an individual is able to resolve the conflict by rearranging their cognitive structures in such a way that the conflict is accommodated and assimilated into the individual's cognitive structure"(p. 96).

Behr and Harel (1995) pointed out two factors which need to be taken into consideration regarding cognitive conflict and the creation of positive experiences: First the conflict should not be too difficult, and it needs to be in the domain of students' abilities. Second, the conflict should be accompanied by guidance or scaffolding (when needed) for students to construct the intended knowledge. According to Vygotsky's theory (1978) of the zone of proximal development (ZPD), students can learn a new concept or skill with help of someone who is competent or knowledgeable in the specific area of learning. By having an external aid, students eventually become able to understand the concept or do the task independently. If the task or concept is too difficult, it does not fit in the ZPD and learning does not occur. Mann (2006) stated: "The pupil's insight can only be facilitated by a challenging problem that is sufficiently demanding, as well as sufficiently accessible" (p. 253).

Zaslavsky (2005) noted the connection between meaningful learning and uncertainties and introduced three different types of uncertainty in mathematics. The first is "competing claims" which might happen through discrepancy between past experiences and new knowledge; the second is an "unknown path or questionable conclusion", which might happen by not having sufficient knowledge, skills and confidence; the third is "nonreadily verifiable outcomes" which might happen by not having a clear view of the subject or not having clear reasons for proving the result and the process (p. 305). According to Zaslavsky these uncertainties in mathematics have the potential to create positive learning opportunities.

2.4 Social constructivism

Social constructivism highlights the role of "social interaction, collaboration, conversation and questioning, and idea exchange [to] foster cognitive and socioemotional growth, cement learning, and help deepen conceptual grasp" (Bruner, 1977; Dewey, 1915; Montessori, 1912; Steiner, 1996; von Glasersfeld, 1996; Vygotsky, 1978 as cited in Mays, 2015, p. 112). Communication has two effects: on the one hand, it can cause cognitive conflict, which is one element for constructing knowledge. On the other hand, it can help to solve the conflict by considering new ideas and knowing more points of view (Gadanidis & Geiger, 2010).

Gadanidis and Geiger paraphrase Goos et al. (2000) by stating: "Learning mathematics in such a community means a learner must participate in debate about new ideas and practices, offer critique of others' ideas and defend their own propositions via explanations and justifications" (p. 96). In this case, students are not inactive audiences and teachers are not the only actors in classrooms. Students are also actors who have freedom to collaborate. Gadanidis and Hoogland see mathematics experiences as a story and state: "Stories are not top-down experiences - they are interactive experiences. [ ] mathematics as story is about potential roles, rather than prescribed roles, for students" (2003, p. 489).

Socio-culturalism highlights the role of "cultural" tools like language and "physical artifacts" like calculator and computer; this theory acknowledged that through interaction with the tools, new "modes of reasoning and argumentation" can be elicited (Gadanidis & Gieger, 2010, p. 96). Egan (2010) stated: "cultural tools can be internalized and transformed into cognitive tools" (p. 207). The cultural tools can be considered, for example, to be any kind of technology which can influence human cognition. Lévy (1997), stated that "through our interaction with things, we develop skills. Through our relation to signs and information, we acquire knowledge. Each activity, each act of communication, each human interaction implies an apprenticeship" (p. 11). Tools are more than passive identities. On the contrary, they can play a role as an actor in conveying meaning to humans. In the immersion section of this chapter, I elaborate more on this topic.

Considering social constructivism and socio-cultural theories, I write below about the role of interaction with others as audience.

2.4.1 Audience

Boal (1985) in relation to the idea of 'poetics of the oppressed,' identified the potential influence of audiences on what they are exposed to. In the theatre performances designed by Boal, audience members can become what he refers to as 'spectators', by having the right to walk on stage and request to replace one of the actors and take the play in a new direction. Boal states:

> "The *poetics of the oppressed* is essentially the poetics of liberation: the spectator no longer delegates power to the characters either to think or to act in his place. The spectator frees himself; he thinks and acts for himself" (p.135)

In Boal's view, audiences can and should influence the phenomenon they are exposed to.

Gadanidis and Geiger (2010) noted that in the field of mathematics, the concept of audience is not common, especially in terms of audience beyond the environment of the class and the school. For example, it is rare that students would like to share what they learn in schools with their families and say "Let me show you what we did in math today, it is exciting" (p. 102). It has been an issue for a long time in mathematics education that deep and important ideas of math need to be shared with "nonexperts" (Gadanidis, Clements & Yiu, 2018). Students also enjoy sharing what they learn with people out of the school like their families and friends, when that learning is worth talking about (ibid).

To create engaging math activities or stories, it is necessary to know the audiences' expectations. Gadanidis and Borba (2008) make a parallel between a favourite movie and an engaging math activity and argue that when students experience an interesting math activity, they would like to talk about that and share it with their friends and families similarly to when they watch a good movie or read a good story. When students experience the pleasure of surprise, insight and have emotional moments in mathematics learning, they are excited to transfer those feelings to others (Gadanidis, 2004).

Gadanidis, Clements and Yiu (2018) stated: "We envision that young children sharing mathematical surprises and insights with the wider community may serve to offer one path to disrupt the current mathematics and mathematics education discourse" (p. 37). Math stories that are worth sharing can also serve as an important criterion for designing mathematics experiences (Gadanidis, 2012). Gadanidis (2004) referred to McKee's statement to highlight the importance of an insightful story for audiences: "Insight is the audiences' reward for paying attention, and a beautifully designed story delivers this pleasure scene after scene" (p. 10). 'Authoring' a mathematical experience can offer the vicarious pleasure for the author when they see the pleasure of the audiences' attention and insight (Gadanidis, 2004, p. 11). By offering wonderful, surprising and emotional moments, mathematics become beautiful and worth sharing. Gadanidis (2014) noted: "Good stories and audience work together. Young children are eager to share math stories that will surprise their family and friends. An audience demands stories that offer new, wonderful and surprising ways of looking at math" (p. 40).

2.5 Constructionism

Constructionism adds to constructivism the main principle of "learning-by-making" (Papert, 1991, p. 6). The idea of constructionism can be traced back to Papert's experiences with cars and their internal parts like the gearbox and the differential when he was a young child (Papert, 1980). He liked them so much and, he even imagined and played with them in his mind. He was intrigued by the differential gear system in which motion transfers from one wheel to another and he gained the sense of cause and effect in this concrete object system. He was excited that "a system could be lawful and completely comprehensible without being rigidly deterministic" (Papert, 1980, p. vi). By making connections between the abstract math concepts and the gears system, he could assimilate mathematics to his mental model and this assimilation offered two significant factors: the cognitive and the affective. He stated:

> Assimilating equations to gears certainly is a powerful way to bring old knowledge to bear on a new object. But it does more as well. I am sure that such assimilations helped to endow mathematics, for me, with a positive affective tone that can be traced back to my infantile experience with cars (p. vii).

As a child, Papert appeared to engage emotionally and cognitively with gears naturally. He loved the gears system and he enjoyed understanding how they worked. Papert found the connection between gears and "body knowledge", where children can see themselves as the wheel to understand how the wheel moves and turns. He described the gears as "a transitional object" which can be used to transmit abstract math concept into "sensory" objects.

Papert (1993, 1994) as a pioneer of constructionism, developed the programming software, Logo, which he designed to be learned with minimal prerequisite knowledge, as a way of engaging young students with the "gears" of a computer. Using Logo, students were able to write simple programs to model and to explain mathematics concepts for one another. In using Logo to move the "turtle" on the computer screen, students needed to have a deep understanding of math concepts to implement them through the programming language. Papert described Logo as a "mathland" where students are immersed in math while teaching the Logo turtle to move. Papert described this immersion as natural as learning French while living in France (Papert,1980 & Harel, 1991). The outcome of this type of learning is "deepened understanding, conceptual grasp, and higher-order cognition" which are in common with the outcome of constructivism learning theory (Mays, 2015, p. 121). The focus on immersive learning resonates with Egan's theory of "learning in depth" (Egan, 2010).

Papert's childhood experiences with gears offered him a form of immersive learning, and he designed Logo as a programming environment that may offer young children an immersive math environment. Reading about Papert's childhood experience with gears, I am reminded of my experience with popups (discussed in Chapter 1), where I naturally immersed myself in an engagement with the “gears” of pop-up construction.

2.5.1 Immersion

Egan (2010) explained how by immersion, students can go beyond superficial learning. In school, because teachers are supposed to cover mandated curriculum content including different subjects, students do not have enough of time to spend in one subject and delve into that. Students in math classes do not gain the pleasure of the process and journey of mathematical exploration and discovery. Gillard (1996) stated students are afforded “the pamphlet version of discovery with numbered steps” (P. xi).

Egan introduced a educational method, “learning in depth”, which does not interfere with the school curriculum, while students have opportunities to immerse themselves in one subject. In the first week of school, all students are randomly assigned one (nature-oriented) topic to explore in 12 years of schooling. There is no assignment, score, test and deadline and students become involved in the projects following their genuine interest of learning. The purpose is building deep knowledge for students by immersing them in one subject for long time. Egan (2010) stated that learning in depth “Provides knowledge of some topics in great breadth and depth; gives a deep understanding of the nature of knowledge; engage students’ imagination and emotions in learning…” (p. 216). It may be thought that immersion in one subject for a long time can be boring (Egan, 2010, p. 30). “Educational thinkers have argued that only by learning something in significant depth can students come to grasp how knowledge works, or its nature.” (Egan, 2011, p. 6). In so doing, students gain the confidence of knowing something thoroughly.

One of the main advantages of learning in depth is: “Students' imaginations and emotions are engaged in learning.” (Egan, 2011, p. 12) and “[it] exposes students to the pleasure of learning for its own sake (Egan, 2010, p. 205). When students are immersed with one subject, they are engaged emotionally and are motivated to persist because of the pleasure of learning.

According to Liljedahl (2009), discovery and creativity are highly associated with effort and determination. Here is the comment of Henry McKean from Liljedahl's (2009) research:

> "I don't believe that any true progress arises spontaneously. I believe it is always the result of lots of hard work, covert or overt, with the understanding that old work will sometimes come into a new focus so that you get something, if not for free, then at no extra cost. Such ''inspiration'' is the outcome of covert work and so can be surprising, but the work has to have been done, even if invisibly" (p. 59).

By immersion in one subject, we pay more attention. Gadanidis (2004) saw attention as a tool to gain beauty and insight in mathematics. He stated: "Whenever we bring consciousness to bear upon a topic, either individually or communally, we engage with it in emotional and imaginative ways. We use attention to learn and extend ourselves, to incorporate a new thing, [], we extend our understanding, we become more complex, and this feels good" (Gadanidis & Hoogland, 2003, p. 10). By seeing the aesthetic mathematical experience as the good story, Gadanidis referred to McKee's (1997) statement: "Insight is the audiences' reward for paying attention, and a beautifully designed story delivers this pleasure scene after scene (p. 237, p. 10 as cited in Gadanidis & Hoogland, 2003).

Immersion can increase the chance for mathematics discoveries. Liljedahl (2009) explains two kinds of chance which can lead to mathematical discoveries. The first, extrinsic chance, happens if mathematicians obtain the ideas which inspire them from external sources. The second type of chance is intrinsic chance which happens by mixing different ideas to create a novel idea leading to insight. The source of intrinsic chance is internal (Liljedahl, 2009).

Gadanidis and Geiger (2010) stated that when we involve ourselves thoroughly with using technology, its affordances and constraints influence how we think. "When we immerse ourselves in using a technology (and this immersion is a critical component), we naturally think *with* that technology" (p. 95). As was mentioned before, technology can take different forms and it can be the technology of the paper or the technology of the computer (Gadanidis and Geiger, 2010). Gadanidis and Geiger compared Web-based tools with physical classrooms to indicate how the affordance of the tool can change our thinking about classroom communication, what should be taught, how to afford feedback and the instructor's roles, and in general about what defines knowledge and how it should be gained in an online environment (p. 95)

As I explained in Chapter one, the technology I used for designing the aesthetic mathematical experience of this thesis is popups, which involves making 3D shapes out of a 2D surface. In chapter five, I explain how by being immersed in the pop-up context, its affordances reorganized my thinking and consequently afforded me new conceptual mathematics understanding.

Immersion provides opportunities for students to think hard to gain the knowledge which is obtained by intellectual challenges. Egan (2010) states: "Learning in depth will involve each student in intensive and extensive exploration, classification, analysis, and experiments, but it will also face them with more than purely intellectual challenges" (p. 210).

2.6 Seeing as

As mentioned in the introduction, seeing one math activity, "Making 10", as the pop-up story was the starting point to design the aesthetic mathematical experience of this thesis. 'Seeing as' – seeing the two-dimensional "Making 10" activity as a quasi-three-dimensional pop-up activity – plays an important role in creating a new mathematics experience.

2.6.1 What is 'seeing as'?

Zwicky (2003) defined "seeing as" in a metaphor context. She stated that "Metaphor is a species of understanding, a form of seeing-as; it has, we might say, flex. We see, simultaneously, similarities and dissimilarities" (p. 4, left). She explained that one thing can be considered as something else in a metaphorical way while they are not the same thing. "“x is y” is not a metaphorical claim unless “x is not y” is true. In the general case, an expression is not metaphorical unless it implies [] a claim of the form “x is y” where “x is not y” is true.” (ibid, p. 5, left). A metaphor sheds light on the existing implicit similarities and connections between two things (p. 76, left). Zwicky believed that “seeing as” “[] encapsulates the mystery of meaning” (p.1, left). Zwicky added that “Metaphor is one way of showing how patterns of meaning in the world intersect and echo one another” (p. 6. left).

To understand and create metaphors, humans are likely to face some cognitive conflicts due to the inherent-hidden implications metaphors convey. Zwicky (2003), attributed words such as “recognize”, “re-think” and “think through differently” (p. 1, left) to the “seeing as” phenomenon. The action of “recognition” is used when the there is some “problem” (ibid) and it can evoke cognitive conflict. It needs “re-organization of experience, an act of contextualization, a sensing of connexions between aspects of immediate experience and other experiences” (ibid). Metaphors entail new conceptual understandings of things which can come along with cognitive conflict.

To perceive and notice metaphors, one thing should be seen from a different lens which has the potential to afford new insights and surprises. When one thing is “recognized” (ibid) as something else, it provides wonder. “The moment of recognition happens as if by magic” (ibid). Zwicky (2003) stated that: the experience of seeing how an assemblage of parts must go together, [] and understanding a metaphor are species of the same phenomenon and they all involve insight, understood re-cognition; a gestalt shift.” (ibid). Zwicky made analogies between appropriate metaphors and jokes: “Surprise is common to good metaphors and good jokes; both turn on suddenly connexions between language-games that appear distant from one another” (p. 45, left).

In what follows, I elaborate on metaphor which is the essence of ‘seeing as’.

2.6.2 Metaphor

Metaphor is the implicit connection between two domains: the “source” and the “target” (English, 1997, p. 7). Lakoff (1994) stated that “reasoning metaphorically is []

characterized by cross-domain mappings" (p. 206). Metaphor can create new "mental models based on the relational structure shared by the source and target" (English, 1997, p. 7) which can generate new understandings of both the source and target (ibid).

Van Dormolen (1991), explained how metaphor can help students to improve their mathematical understandings. *Mapping* from known domains like familiar terminology such as "degree, power and root …" (English, 1997, p. 8) can foster students' learning of abstract mathematical concepts, make the concepts accessible for non-specialists and offers new visions. Metaphor can be useful while the mathematics content is present in a concrete way. For example, the concept of function can be seen as the vending machine, and a vector as an arrow (Matos, 1991 as cited in English, 1997).

There are many philosophers who define metaphor as *seeing something as something else*. Cooper (1986) referred to some of their notions: "it gets us to see one thing as another, to see the world as one in which a thing is a certain way" (p. 227). Davidson restates that metaphors "get us to see one thing as another" (ibid).

Pimm (1988) that metaphor is conceptual. He added that metaphor indicates the "commonalities" (p. 33) and connections between two areas which apparently do not have anything to do with one another and it highlights the features of the domains which are implicit.

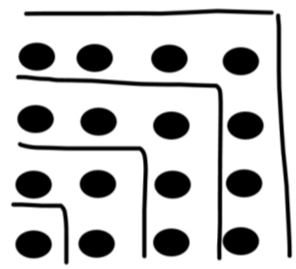
To utilize metaphor for educational purpose, the metaphor used must be appropriate otherwise it is not able to fulfill its aim. English (1997) suggested that one way to avoid metaphor misconceptions by students is to encourage them to make their own metaphors rather than serving them with our metaphors (p. 9).

Egan (1999) stated that children use metaphoric thinking easily and it indicates their ability to make and perceive different types of metaphor in language. Egan added that not only is metaphor important for language development, but also it can be applied in any intellectual task where "imagination" plays a role. Unfortunately, the current educational practice focuses least on "imaginative skills attached to metaphor, and image generation and to narrative and affective understanding" (p. 88) and focuses most on activities that need merely "logico-mathematical capacities" (p. 87).

2.6.3 Image-based reasoning and metaphor

English (1999), explained that the internal structures in image schemata can figuratively develop the mathematical abstract understanding. Because image schemata are perceived through physical experiences, the extracted metaphors are highly dependent on the structures of these experiences (p. 10). Image-based reasoning is important in mathematics education in the sense that it fosters mathematical understanding and problem solving (English, 1997, p. 10). Faraday used images and drawings to display his algebraic understating (English, 1997). Although the importance of image-based reasoning in mathematics is remarkable, it is not prevalent as much as its importance deserves.

Zwicky (2003) used gestalt shift to explain metaphor. Gestalt psychologists explained how focusing on the phenomena can cause the functional fixedness and prevent the phenomenon to serve the insight (Ashcraft, 1989). From this point of view, "in metaphor we experience a gestalt shift from one distinct intellectual and emotional complex to another in an instant of time" (Zwicky, p. 4).The proof of James Robert Brown theorem (Zwicky, 2003, p. 38, Right), is based on the picture and metaphor (Figure 3). Zwicky stated that: "Both a metaphor and a geometrical demonstration say: "look at things like this" (ibid, Left)



1+3+5+…. +(2n-1) =n^2

Figure 3: Metaphor and geometrical presentation

If this picture (Figure 4) had not been applied, the traditional inductive method would have been the only option and seeing the result as the area of the square would not have been easily discovered.

Zwicky (2003) states that "understanding a metaphor is like understanding a geometrical truth. Features of various geometrical figures or of various contexts are pulled into revealing alignment with one another by the demonstration or the metaphor" (p. 36. L). By metaphor, it is possible to see different figures merge in one figure and this can lead to artistic-mathematical discoveries.

2.7 Aesthetics

What does beauty mean in mathematics? Aristotle stated: "the supreme forms of beauty are order, symmetry, and definiteness which the mathematical sciences demonstrate in a special degree" (Aristotle, *Metaphysica,* M 3, 1078 a 36–b 2 as cited in Cellucci, 2015, p. 341). Cellucci (2015) considered various features such as "proportion, order, symmetry, definiteness, harmony, unexpectedness, inevitability, economy, simplicity, specificity, integration" (p. 341). To Mann's (2006) point of view, the beauty of math is found in the mathematical structures and he paraphrased Poincaré's idea about the mathematical sense to "see the whole" and create "harmony and relationships" (p. 253). Sinclair (2001) emphasized that the origin of aesthetics in mathematics education should be mathematical and not the aesthetics of other fields like the arts or sports which are linked to math to make it look appealing. She stated: “Such Linking also tends to undermine the aesthetic, expressive and transformative possibilities of mathematics itself” (p. 25)

Mathematicians label ‘beautiful’ a mathematical concept when “it gives understanding” (Cellucci, p. 344). Cellucci explained that math understanding means to find “the fitness of the parts to each other and to the whole” and this is an aesthetic feature of recognising the beauty in arts in the past (p. 344). Hence, the connection between understanding and beauty becomes clear. Cellucci used ‘a beautiful demonstration’ as one example to display the link between understanding and beauty (p. 346) and showed some examples to explain. In the example below, the solver should find the area of the square which is inside the bigger square. The bigger square is constituted of eight equal right triangles while the middle square (the inside square) is made of four right triangles (see Figure 4).

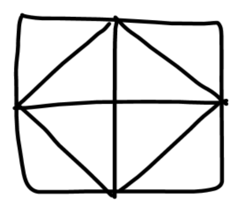
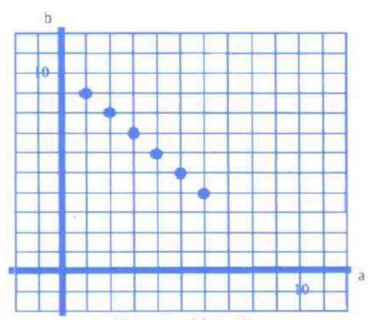


Figure 4: Understanding and beauty

By understanding that the smaller square included four similar triangles, the solver can figure out that the area of the smaller square is half the area of the given square (the bigger one).This demonstration is beautiful because it affords understanding based on the aforementioned definition of understanding and beauty (ibid). As Henderson and Taimina (2006) suggest, through such informal approaches to geometry "many levels of meaning in mathematics can be opened up in a way that most people can experience and find intellectually challenging and simulating" (p. 59).

Gadanidis and Hoogland (2003) stated: “Stories are aesthetic in nature” (p. 487). Gadanidis (2004) saw the aesthetic mathematical experience as a good story. In a good math story, students by paying attention can have opportunities to gain insight, surprise of big math ideas and see the beauty of "inner" math. "Stories [ ] attract our attention and offer us the pleasure of mathematical insight" (Gadanidis & Hoogland, 2003, p. 490). Gadanidis designed several activities and authored many math stories to indicate how simple math can lead to the big and deep math relationships which are surprising and insightful. One activity which was the inspiring activity for designing the aesthetic mathematics story of this thesis, was the ""Making 10"" activity. In this activity, students were given the equation: __+__ = 10 and a die. They were asked to find the first number by rolling a die and then calculate the second number. They repeated this until all possibilities (that is, all the numbers 1 to 6 for the first number) were exhausted. Then, they took the pairs numbers (like 1 and 9) and plotted them on a grid as ordered pairs (like (1, 9)). Once plotted, the students discovered the points aligned to form a straight line and they experienced a mathematical surprise. (http://researchideas.ca/wmt/c2.html). The reason for their surprise is that they did not expect an organized result like a straight line from randomly generated numbers by rolling a die. To illustrate, the graph (see Figure 5) displays points plotted with the constraint of __+__=10.



a+b=10

a	b
1	9
2	8
3	7
4	6
5	5
6	4

Figure 5: "Making 10" activity

In this activity, students became surprised and gained an insight of linear functions. Sinclair (2001) stated: "aesthetically-rich learning environments enable children to wonder, to notice, to imagine alternatives, [] and to experience pleasure and pride" (p.26). In the "Making 10" activity, students were asked to explore different patterns of numbers, equations, to find different shapes of graphs. "Open-ended inquiry, interest, surprise, and motivation are characteristics of an aesthetic approach" (Gadanidis and Hoogland, 2003,

p. 494; Gadanidis, 2004, p. 11). The aesthetic mathematical experience, a good math story, involves emotional moments in mathematics learning. The moment when the solvers find the solution and understand something for the first time lingers in the mind as a beautiful moment. Gadanidis (2004) referred to some teachers' comments: "The moments that I've experienced in math when the lights have gone on and I felt like 'I got it' were in fact beautiful moments, which I clearly remember even to this day." And "It's lovely. I'm lousy at it, but I love feeling my brain tumble over as it understands something for the first time." (p. 11)

Gadanidis et al. (2018) see structuring of good mathematical experiences as "a process of solving 'artistic puzzles' or 'pursuing 'artistic discovery'"(p. 37). The model of designing aesthetic experiences for young mathematicians in the research of Gadanidis, Borba, Hughes and Lacerda (2016) entails three main elements: aesthetic, mathematical and implementation elements. The aesthetic elements includes "surprise and insight", "vicarious emotional engagement" (p. 228) which is about students pleasure from witnessing other people enjoy the math experiences they shared (Gadanidis et al ,2016) and "visceral sensations" which are related to "mathematical beauty" (p. 234). Gadanidis et al intend to design math experiences in which students are exposed to complex math subjects and think hard to understand them. By using a 'low floor and high ceiling' approach, Gadanidis et al. design activities which do not require advanced math knowledge to become engaged while being rich enough to lead students to more advanced mathematical concepts. The implementation element entails two factors: cover the curriculum and flexible implementation. Gadanidis et al use the curriculum as the "content" and the complex math as the "context" to extend and enrich the curriculum content. The math experiences can be implemented based on the teachers' experiences and comfort in classes; therefore they do not interfere significantly with teacher's pedagogy and skills.

2.8 Surprise

Zaskises (2014) argued that "genuine surprise, when a result is both unexpected and unexplained, is the main catalyst for wonder in mathematics" (p. 163). Wonder can have two connotations. Wonder as a verb, which can be interpreted as “wonder why” (Sinclair and Watson, 2001), happens for a person who is in the situation where her/his curiosity is intrigued. Wonder as a noun, can be interpreted as “wonder at” (ibid), when wonder displays emotion of surprise. These two features of wonder can be interconnected and make a reciprocal connection. Curiosity of knowing the reasons can be motivation to explore which can lead to wonder at something surprising. On the other hand, wondering at something can be motivation to find the reason. From Sinclair and Watson’s (2001) point of view, when we understand underlying reasons of a surprise, how and why the surprise appears, the surprise wouldn’t stay surprising anymore.

In mathematics, surprise can emerge in the “spontaneous appreciation of beauty and elegance” (Sinclair & Watson, 2001 p. 39). To understand the underpinning reasons of the surprise in math, the questions of “why” and “how” can be raised. Having students being exposed to the surprise, which is likely to encourage them to explore to understand “why or how”, can be considered as an effective pedagogy in mathematics education. As Egan (1997) stated: “Without the initial wonder, it is hard to see how more systematic theoretical inquiry can get fruitfully under way” (p. 97). Likewise, Adhami (2007) said that by exposing students to surprising math content, they hopefully step back to “look again” (p. 34) to understand the reasons behind the math content.

Surprise in mathematics is not confined only in theories and there is potential to be generated in other mathematical situations (Zaskis and Zaskis, 2014, 140). Zaskis focused on four types of mathematical situations which can induce surprise for students: Perceived “magic”, counterintuitive results, variation on a known result or procedure and paradoxes.

Experiencing surprise in mathematics is appealing and “contagious” (Zaskis and Zaskis, 2014, p. 165). Exposing students to surprising mathematics tasks can offer new visions and insights of mathematics contents and motive them to seek more and make the variation of the situation (Sinclair, 2006).

Watson and Mason (2007), developed and collected some activities which have the potential to create discussions about how to design surprising mathematical tasks with educational purposes. In their points of view, surprise is the “positive emotion” and mathematics is full of “philosophical and cognitive” ones; however, they admitted that it is not easy to create experiences in which the learners can witness the mathematical surprises (p. 4).

Gadanidis and Borba (2008) make a parallel between a favourite movie and an engaging math activity. One of the pleasures which movie makers provide for the audiences is the joy of surprise. This pleasure is defined by Boorstin (1990) as the voyeur’s pleasure which is “the simple joy of seeing the new and the wonderful” (p. 12). The audiences like to be exposed to what they have never experienced and become surprised in the sensible way.

Gadanidis, Hughes and Borba (2008) suggest that one of the main criteria that makes math activities worth sharing is how surprising activities are. They reported students’ feelings of surprise and joy engaging with math activities such as the patterns and relationships in Figure 5 where students could see how a combination of odd numbers can make a square. One of the purposes of this research was to indicate that if math classes provide opportunities for students to gain the sense of surprise in complex math contents, they become the math performers of experiences and emotional moments worth sharing. As a result, students are likely to share their feelings and corresponding math stories with their families and friends.

In the research conducted by Gadanidis, Hughes and Cordy (2011), they studied the effects of the math surprising experiences, which were integrated with arts and technology, on gifted students’ attitudes on math. By having students involved in some complex math concepts, students experienced surprise of how it can be possible to have more than one answer for one mathematical question (p. 415). The specific experiences displayed “the creative and imaginative side of mathematics” (p.423) in which students had opportunities to explore and understand the complex math concepts embedded with surprise and insight.

Surprise and insight are related to the AHA! experience.

2.8.1 The AHA! experience

The experience of an AHA! moment is when the solution emerges with "characteristics of brevity, suddenness, and immediate certainty" (Poincaré, 1952,as cited in Liljedahl, 2004, p.8). The AHA! experience can happen for both mathematicians and students. In other words, when a mathematician invents a new math concept or a student finds the math solution by sudden surprise and insight, both experience the AHA! moment since they both solve the problems which are problematic compared to their level of knowledge (Liljedahl, 2004). Fox (2008) called the AHA! moment a 'lightbulb' moment and he stated: "no lightbulb moment is minor to the person who had it" and it is a "profound accomplishment for the person who has it" (p. 561).

Liljedahl (2006) introduced the theory of chain discovery to address the issue of students being reluctant to work on the same math activity in different sessions and to make progress gradually to find solutions. Based on his idea, the reason originates from students' beliefs, attitudes and emotions about math and their ability to do math. Liljedahl (2006) introduced the chain of discovery as one type of problem solving (p. 142). By making discoveries in mathematics, students believe in their abilities and they are more likely to have the attitude of "I can do this" (p. 149). A chain of discovery happens when a student trying to solve a math puzzle, found solutions consistently and made discoveries consecutively (p. 142). Liljedhal (2006) stated that: "the environment for [discoveries] can be orchestrated, but the experience itself cannot" (p. 150). We as educators can provide the environment with an appropriate scaffolding for students to make discoveries; but there is not any guarantee for that. However, in Liljedahl's point of view "interaction, time and a rich task" (p. 150) are important factors for *occasioning* discoveries.

Gadanidis, G., Gadanidis, M.J. and Huang, (2005) drew an analogy between humor and the AHA! experience and acknowledged that there are similarities between them. Both are the result of creative thinking, seeing our environment or math problems with new perspectives which offer insight and surprise. Humor in the mathematical context leads students to think deeply to grasp the point of the humor and consequently to understand the covert complex math in the humor which is usually eye-opening and surprising.

2.8.2 Low floor high ceiling

Mathematics experiences designed with a low floor (easily accessible) and a high ceiling (extendable to greater complexity) approach have the potential for mathematical discoveries and gaining mathematical surprise and insight, if learners have an opportunity to immerse themselves with the experiences.

Ginsburg (2002) contended that "mathematics is big, but little children are bigger than you might think" (p. 12) and the curriculum should be "challenging" enough to offer the pleasure of math understandings. In the traditional way of mathematics teaching, children are not exposed to the complex and interesting math concepts and their cognitive abilities are underestimated (ibid). Children are only exposed to "easy math" and they do not have chance to gain the joy of understating deep math. There are two main reasons: First is the assumption about children's ability and the second is the poor design of curriculum content. The first reason is an implication of Piaget's stages of cognitive development theory where students should be old enough to be able to abstract and comprehend abstract concepts (Gadanidis, 2012). Papert (1980) corresponded this inability to the "poorly designed learning culture rather than existing in children' minds" (p. 37) and Egan (2002) noted children's natural ability to abstract when they learn language, as it is based on the ability to abstract. By refuting the idea of children's inability in abstract concepts comprehending, Papert's (1980) idea of "low floor, high ceiling" crosses the borders of grade-specific math contents (ibid, p.36). The second reason is the fragmented mathematical curriculum. Gadanidis, Clement and Yiu (2018) noted that mathematical structure is important in mathematics and for mathematicians and in the current fragmented mathematical curriculum the mathematical structure is often lost.

2.9 Math-storytelling

Developing knowledge is an outcome of human endeavors to fulfill hopes, desires and aims and it is not detached from human emotion. However, what is demonstrated in educational resources and mandated curriculum content does not indicate a sense of humanity which is the origin of knowledge development (Zazkis & Liljedahl, 2009).

Gadanidis (2012) highlighted the human aspect of story and its impact on mathematics education. He stated: "Story is not a frill that we can set aside just because we have developed a cultural pattern of ignoring it in mathematics. Story is a biological necessity. Story makes us human and adds humanity to mathematics"(p. 26). Zazkis and Liljedahl (2009) noted that in the context of story, the author is able to provide situations to reflect the origin of knowledge and make it natural and human. For example, in the story of which the plot has a hero, readers vicariously experience what the hero experiences, and feel the hero's fear, happiness and hopes.

Students can experience joy and insight within the interconnected context of story. Gadanidis and Hoogland (2003) drew attention to "multimodal" features of story which can be used as an educational tool for mathematics education. They noted that in the story context, mathematics puzzles can be presented in various ways and this context has a potential to add "emotional" effects "through image and metaphor" (p. 489) and engage the reader affectively. In such an appealing environment, the mathematical concepts can be conveyed in the most natural way.

Story can be used to connect the educational content to societal contexts and create settings for students to vicariously experience social interactions where understanding is developed socio-constructively. In these social interactions, there are opportunities to embed cultural features. This matters more in the cosmopolitan countries in which different ethnicities, cultures, and races exist. In these countries, typically the teachers and students are not in the same ethnic group and this fact might hinder learning opportunities especially for students who are in minority (Gay, 2013).

One strategy that Gay (2013) suggested for culturally responsive teaching is "using multiethnic and multicultural examples to illuminate general principles and concepts" (p. 68). This strategy can be implemented in the story format appropriately if for example the content includes humor or wonder, which are common among people regardless of their ethnicity and background. In general, the imagery, common conflicts and challenges in stories can trigger common emotions and unify students with different cultures. This unifying feeling provides valuable situations in which all types of students feel comfortable and confident (Gay, 2013).

Story can be helpful for improving students' problem solving and analytical abilities. In the story plot with a hero, the hero plays significant roles and s/he needs to overcome obstacles to obtain the goal of the plot. This process demands solving problems. Zazkis and Liljedahl (2009) argued that "[using stories in class] can assist in understanding difficult concepts and ideas, assist in solving problems"(p. 4). Students as readers of this type of story assume themselves as problem actors and they become the owners of the problem. They engage completely while accompanying the hero to overcome the challenge and they are responsible for the solutions they imagine as the hero of the story (ibid).

In general, there are remarkable benefits of using stories for the educational aim. Zaskis and Liljedahl (2009) stated that, story inherently is fun and entertaining and can spark and keep students' attention. The content of the story is internalized, and it is hard to forget. Story can make connection between different disciplines and can be used as the "cross-curriculum teaching tool" (p. 25). It increases students' motivation to learn and improve their imagination and creativity and problem-solving abilities.

Chapter 3. Methodology

3.1 Introduction

As was explained in chapter one, the main question of my research is: What factors come into play when an educator immerses him/herself in the process of designing an aesthetic mathematical experience? The aim of my research is to analyse the journey I went through to design an aesthetic mathematical experience. This journey is told as a narrative of the details and complexities of the process to extract the themes which played roles to create the pop-up mathematical story. The research method that can fulfill this demand and explain the human-centered experiences thoroughly is a narrative inquiry.

In this chapter, I explain narrative inquiry, the advantages it offers to the researcher and its differences from other conventional research methods. The aspects of narrative inquiry which are important in my own research are defined. At the end, I explain about methods I use, the ethics and limitations of this research.

3.2 Narrative inquiry

There is a close bond between human experience and narrative, as humans find the meaning of their lives through the lens of a narrative. Webster and Mertova (2007) referred to this common notion, which is also supported by the work of Bruner (1994), Clandinin and Connelly (2000), Sarbin (1986) and Elbaz (1991):

> "life as led is inseparable from a life as told…life is not 'how it was' but how it is interpreted and reinterpreted, told and retold" (p. 2).

Given an intimate connection between narrative and human experience, it is justified that human research should not be confined only to the empirical methods whose results pertain to numbers. As Webster and Mertova noted, empirical methods are not comprehensive for reflecting all aspects of human experience and narrative inquiry is one research method that is capable to supplement the deficiencies of the quantitative research method. It seems that the phrase *narrative inquiry* is first used by researchers Connelly and Clandinin (1990) to elaborate on teacher education through "personal storytelling". In their point of view, knowledge in education is obtained through the content of the educational stories and narrative inquiry is the tool to analyse the education stories which are told, heard and read by teachers.

There are two main reasons that explain why a narrative inquiry becomes important in the research area. First as mentioned before, it supplements the deficiencies of a more "conventional" research method. While quantitative research is the robust method in some disciplines, it does not have the capacity to tackle "the issues of complexity and cultural and human centredness in research" (p. 3, Webster & Mertova, 2007) and usually may "overlook issues[] considered significant by the participants in the research" (ibid). On the contrary, a narrative inherently sheds light on the complexities related to human issues like "characters, relationship and setting" (p. 4). Narrative has a capacity to present the "complexity" and "richness" of the human-centred phenomena with the broad and "holistic" view (p. 2) which is not accessible in other methods.

Changing philosophical view, from modernism to postmodernism, is the second reason that led narrative to become a research method (Webster & Mertova, 2007). Webster and Mertova (2007) by referring to Merrill and Reevs explained the differences between modernist and postmodernist perspectives. Based on Merrill modernism contends that there is one "ultimate truth" and knowledge is "objective" (p. 6). In contrast, in the postmodernist philosophical view, Reeves argued that knowledge is pertained to the individual and related to the experience and culture and it acknowledges that there is not a singular truth and knowledge is "subjective" (ibid).

It is important to know the differences between a narrative inquiry and the conventional research methods for the researcher to choose and use a proper research method. In general, the aim of the quantitative research is finding outcomes and it does not concern about the "impact of experience" while the aim of the narrative inquiry is delving into experience and its influence (Webster & Mertova, 2007, p. 5). It does not reveal the reality absolutely, but the result has "the appearance of truth and reality" or "verisimilitude" (p. 4). Reliability in quantitative methods is related to how accurate and consistent the instruments measure for instance variables, while the reliability of the narrative inquiry comes from the "trustworthiness" of the participants' statements and interviewer recordings (p. 5).

3.2.1 Narrative inquiry in educational research

Researchers use narrative as the "rich framework" to explore how humans make sense of the world through the lens of story. In Grumet's (1976) point of view, narrative is important on educational research in two ways. First, in order to make the narrative of our experience, we need to reflect and describe it by seeing the whole story including the aspects that can be a legitimate resource to make an improvement in curriculum (as cited in Webster & Mertova, 2007). According to Grumet, Webster and Mertova (2007): "It is only in the freshness and immediacy of our narratives of lived experience that curriculum can be reconceptualised, since the narrative reclaims entire areas of experience" (p. 9). Second, the narrative sheds light on the bias and interest in the experience which offers new understandings and afford insight for the future changes in the society (ibid).

3.3 Research trustworthiness in narrative inquiry

As was explained before, the focus of empirical research is on the scientific process, fact, repeatable experiments and generalizing the result. In narrative inquiry the aim of the research is human understanding of "complex and human-centered" issues. As it is obvious, the core objectives of quantitative research and narrative inquiry are different. As a result, the definition of the validity and reliability are not expected to be same (Webster & Mertova, 2007, p. 89).

Webster and Mertova (2007) noted that validity can be determined by "those who read it and they should be the ones to decide on whether and account is 'believable'" (p. 92). The reliability can be evaluated "by the accuracy and accessibility of the data, so that any reader can get hold of the relevant text or transcript" (p. 93).

Due to the different framework of the narrative inquiry we need to define new aspects, such as "access, honesty, verisimilitude, authenticity, familiarity, transferability and economy" (Huberman, 1995 as cited in Webster and Mertova, p. 94). In the following, I define these aspects and relate them to my research.

3.3.1 Access

There are two types of access: Access to the participants' background and the process of constructing knowledge in the inquiry and access to the data which can be notes or transcripts (Webster & Mertova, 2007).

To explain the access to participants' background and the process of knowledge construction, Webster and Mertova used the model of Connelly and Clandinin, which entails: "Negotiation, structures (time, place and events), tools, conclusions and risks" (p.94).

Negotiation is about how a researcher organizes the study to prepare the situation and people who are going to be involved. To understand the complexity of human-centred events, it is important to describe the narrative in terms of time, place where the events of the narrative occurred. In narrative inquiry, there are different tools like "observation, surveys, documentation (including letters, curricular and policies), interviews and transcripts" and in some research "some prototype materials or resources" are used to collect data (p. 96).

In my research, due to investigating my own narrative, I did not need to do any negotiation with participants. The narrative of my research began from the first semester of my master program (September 2017) when I chose to design the mathematical pop-up story as my assignment. The end point of my narrative was when the last mathematical discoveries happened. The place of designing and making the story, transferring the experience into narrative/words and analysing the narrative has been mostly in my home located in London Ontario.

My data sources are the notes taken in the process of creating the story, my memory to recall the details of the process, transcripts of interviews with my supervisor, conversations, my assignments related to the pop-up story, the notes of my discussions with my supervisor, sister and a friend of mine, and different versions of the story (the concrete paper story and the one with voice in the Voice thread software), the pictures of the pop-up story and the ""Making 10"" activity, the inspiring activity of making the pop-up story.

3.3.2 Honesty

In narrative inquiry a human plays the main role to collected data (Webster &Mertova, 2007). Guba and Lincoln (1981) pay attention to human characteristics to demonstrate the trustworthiness of human as an instrument (as cited in ibid). In the table below, Webster and Mertova (2007) paraphrased them:

Table 1: Different characteristics of narrative inquiry

characteristics	Explanation: Human are
"Responsiveness" (p. 97)	able to "sense [] all personal and environmental cues" (p.97)
"Adaptability" (p. 97)	"[able to] collect information about multiple factors simultaneously at multiple levels " (p. 97).
"Holistic emphasize" (p. 97)	able to understand the "complexity" of human issues and "its surrounding [] entirely" (p. 97)
"Knowledge base expansion" (p. 98)	able to work at the same time on the "propositional and tacit knowledge" (p. 98).
"Procedural immediacy" (p. 98)	"able to process data [] and generate hypotheses on the spot" (p. 98)
"Opportunities for clarification and summarisation" (p. 98)	able to summarise data on the spot and use it to "clarification and correction" (p. 98)

"Opportunity to explore atypical or idiosyncratic responses" (p. 98)	able to delve into responses to test their validity and gain a deep understanding

These features show how a human as an instrument can be as trustworthy as other instruments.

The starting point of this research was my imagining of two different activities (the "“Making 10”" activity and its pop-up representation, which will be explained in the next chapter, simultaneously identifying the similarities and differences between them. In all mathematical discoveries that happened during the process of designing the story, I used a holistic view to see the experience entirely. As an analyser, I could collect data and write it as narrative, categorize data and extract themes (the detail of all these processes are written in the following chapters). These abilities can legitimate me as a valid instrument of this narrative inquiry.

3.3.3 Verisimilitude or truthfulness

Verisimilitude can be demonstrated by three aspects (Webster & Mertova, 2007). First, if the narrative resonates with the reader's past experience or might widen the reader's horizon in that particular issue, the narrative seems truthful. Second, the narrative and the conclusions should be plausible and believable. The conclusion should not be influenced by any bias or researcher tendency. The third aspect of verisimilitude can be indicated by the "critical" events, (or turning points) and the other less critical events which have indications to prove the truthfulness of critical events and the whole narrative (Webster & Mertova, 2007).

My supervisor, George Gadanidis, was the person who witnessed the process I went trough in designing the pop-up story. After every discovery, I could not wait to share with him and have his feedbacks and comments. George is the author of the surprising *"Making 10"* mathematical story and the classroom activity that inspired me to design the pop-up story. According to him, the pop-up story could broaden his view of the ““Making 10”” activity and he could see it in 3-dimensions.

I used the ‘turning points’ to tell my narrative, with my supervisor as witness, and the themes extracted in chapter five are logical and plausible.

3.3.4 Familiarity

One of the important features of the narrative is how it makes the “familiar strange” (Webster & Mertova, p. 100). In other words, there is a tendency for humans to not consider familiar concepts in the conscious level of mind. By making familiar things strange, new understanding and insight can emerge. Webster and Mertova (2007) noted that the critical events in the narrative have the potential to present “unforeseen” things and makes the “familiar strange” (p. 100).

The narrative of my research shed light on the structure of pop-up and its connection with mathematical concepts. Likely, it makes the familiar mathematics concepts strange by seeing them through the 3-D pop-up lens.

3.3.5 Transferability

Transferability of the narrative inquiry indicates how possible it is to apply the result of the narrative inquiry in a different setting. If the narrative entails all the detail, critical events, another researcher, by reading the narrative, becomes able to use the context and results in different conditions (Webster & Mertova, 2007).

The aim of my research is analysing my narrative to identify factors which played roles in designing the aesthetics mathematical experience for other educators and teachers who are interested in designing the aesthetic mathematical experience. In doing so, I used ‘turning point’ to make it easy for the readers to track the process and as a result it becomes more convenient to use the narrative and outcomes in different contexts.

3.3.6 Economy

The economy aspect can help the researcher and also the reader to use the data in an efficient way. In the research, usually there are large amount of data which can be challenging to deal with. There are some strategies like identifying the critical events that can be helpful to categorize data and make the content and the outcome of the inquiry transferable (ibid). To use and track the data in the narrative efficiently for the readers, I used ‘turning points’ to classify the narrative into separate stages.

3.4 Method

To analyse the data, first I read all my data sources attentively to refresh my memory of my experience. As I mentioned before, the data I have access to are the notes taken in the process of creating the story, my memory to recall the details of the process, transcripts of interviews and conversations with my supervisor, my assignments related to the pop-up story, the notes of my discussions with my supervisor, notes on my insights from my discussions with my sister and a friend , pictures of the pop-up story and the "“Making 10”" activity, which was the inspiring activity of making the pop-up story, different versions of the story (the concrete paper story and the one with voice in the Voice thread software. Then I used three lenses to classify the data in my narrative in different stages (in chapter 4): turning points, critical events, and moments of surprise and insight.

<u>Turning points.</u> One lens I used was that of “turning points”, where the narrative changes direction. Bruner (1997) highlighted the importance of “turning point” in auto-biographies and stated: “[] the self, although seemingly continuous, strikes one as curiously unstable when considered over extended lengths of time. Autobiographies, [], are typically marked by accounts of turning points featuring presumably profound changes in selfhood (p.146)”. Bruner added “Told self-narratives are more typically purpose-built for the occasion. And most lives in the process of being told are, as already noted, rather notable for their uncertainties, with their turning points, their ziggings and zaggings, their isolated episodes and events, their undigested details” (p. 155). The turning points in my narrative illustrate points where my narrative changed course.

Critical events. The second lens is what Webster and Mertova (2007) call 'critical events', which highlight the important events in narrative and their impacts on the outcome of the narrative inquiry. To Webster and Mertova's point of view "Narrative is an event-driven tool of research. The identification of key events, and the details surrounding these, are recognised forces in adequately describing the matter under research" (p. 71). Critical events can be identified by three characteristics: "time, challenge and change" (p.74). The important events which are automatically "distilled" over time, play important roles in narrative inquiry to analyse the narrative (ibid, p. 72). The critical events usually create challenges for the storyteller and lead him/her to change their points of view and understanding (ibid, p. 74).

Webster and Mertova (1998) introduced some questions which can be helpful to identify the important parts of the narrative in the context of investigation.

"1. Think of one memory you have of <context of investigation>. []

2. Thinking back to <context of investigation>, what do you remember or recall?

3. If there was one main memory of < context of investigation>. It would be …

4. Within the <context of investigation>, do you remember a particularly stressful period?

5. How would you say has it influenced you?

6. What role did others play in this event (critical others)?

7. If there was one thing you would say about that event it would be…

8. How would you describe or tell of the changing influence and long-lasting effects?" (p. 86)

As I mentioned before, part of my data is what I remember from the process and these questions helped me to identify the critical events. In retrospect, the larger the time gap between the time of events and the time of telling the story, the more memorable and critical the events are and the trivial details are not recalled (Webster & Mertova, 2007).

Furthermore, looking back to the important events offer me the opportunities to analyse the events profoundly. Webster and Mertova (2007) paraphrased Strauss (1990) and stated: "Past activities are viewed in a new light through reassessment and selective recollection" (p. 77)

Moments of surprise and insight. As I noted before, my first intention of designing the pop-up math story was demonstrating the math surprise which was in another math activity. As I discussed in chapter 1 and 2, one of the main properties of aesthetic mathematical experience is how insightful the experience is. Zwicky (2003) make connection between insight and "seeing as", to emphasise how by changing our view, we can see something as something else. Moments of conceptual surprise are often associated with conceptual insights.

These three lenses are my tools to identify the important stages in my narrative. Although they are related, each lens highlights different nuances of important points in a narrative. I use them as separate lenses due to their importance and relevance to the context of my research.

3.5 Ethical issues

Narrative inquiry, which is considered a qualitative research, inherently deals with "observation and interaction" with people to collect data. In my research, because I used my own narrative, and focused on my experiences, I did not have to apply for ethics approval, permission to carry out the research and consent to participate in the study.

3.6 Limitation

I do not write about my background and identity which can affect the result of this narrative inquiry. Making discussion about my background and identity would be beyond of the scope of this research.

Chapter 4. Stories of Experience

4.1 Introduction

In this chapter, I will share the key stages of my journey in designing an aesthetic mathematical experience. Each stage was identified using the lenses described in Chapter 3: turning points, critical events, and insight and surprise. By sharing the stages chronologically, I will identify the pattern of my experiences while designing an aesthetic mathematical experience. Then, in Chapter 5, I will analyze these stages to identify the key factors that influenced my experience as a whole.

4.2 Stages

Stage 1: Noticing a connection between a math concept and popups

The first time I noticed the connection between a mathematical concept and popups was in August 2017. Before my first semester in the master program started, I read the book *More Joy of mathematics* by Pappas (1991). I had plenty of time to delve into concepts until my curiosity was quenched. The book explains interesting math concepts and the content drew my attention and I felt enthusiastic when I read it. The book is not curriculum-intended, however.

One subject that thrilled me was the "hyperspace example" where Pappas (1991) demonstrates that by folding a sheet in which two points are located, the distance between them changes. Mathematically, folding means adding one dimension to the sheet plane which is quasi 2-D to have the 3-D shape. (see Figure 6).

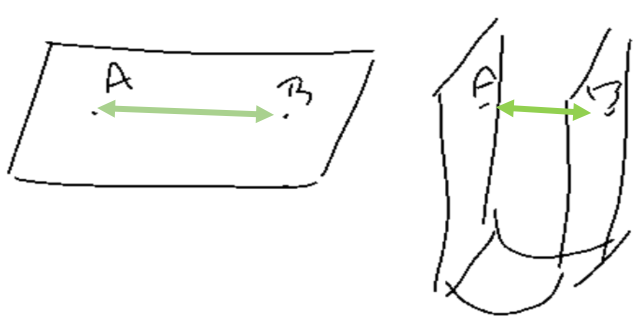


Figure 6: My first connection between popups and mathematics

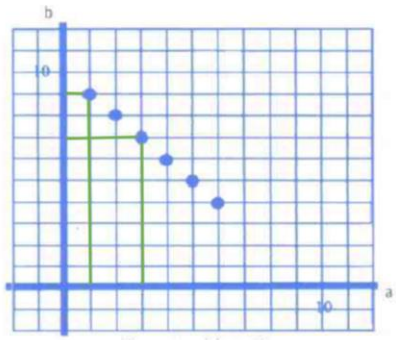
This example reminded me of my work with popups. Both popups and the hyperspace example are the result of folding the 2-D sheet. A fascinating point about popups is that they create 3-D shapes within 2-D surfaces. That is, by mere proper folding and cutting and not by adding something extra, the 3-D shapes can be created. In other words, there is an innate potential in the sheet which can be brought to life by cutting and folding. For example, you can make a pop-up castle out of the sheet. It means the castle has been always in the sheet, but hidden and not visible in any way. The sheet needs to be cut and folded properly to extract the castle out of the surface. Although the essence of popups is a sheet, there are not any traces of the popups in sheets before folding and cutting.

Stage 2: Identifying a rich math task connected to popups

2.1. What was the inspirational math activity?

In the first semester of the master program, I had to do a math project as a main assignment of a course. I could choose any math-oriented project based on my interest. Since I had a passion in writing stories and had experiences in making popups, designing and writing a pop-up mathematical story was a suitable choice. Due to the deadline of submitting the assignment, I had limited time and I was not confident about the math knowledge needed to create a good pop-up mathematical story; I did not know where I would end up. The grade of that course was all depended on this assignment. I made my decision and took the risk.

At the time, I was also an observer in a math preservice-teacher class in which I witnessed a surprising math activity: the "“Making 10”" activity. In this activity, as I explained before students should find the pair numbers for the equation _ + _ = 10 by rolling a die and calculation. Then by plotting the pairs on the grid, they found an organised pattern which was surprising to them (see Figure 7).



a+b=10

a	b
1	9
2	8
3	7
4	6
5	5
6	4

Figure 7: The inspirational math activity

2.2. Identify similarities between the ““Making 10”” activity and popups

Plotting of the number pairs in the “““Making 10””” activity as ordered pairs (a, b) led me to think to make pop-up steps with the same math constraint, where the pairs of numbers a and b were the vertical and horizontal components of steps, where a + b = 7, and where 7 was the total length of foldable planks needed to make each step (see Figure 8)

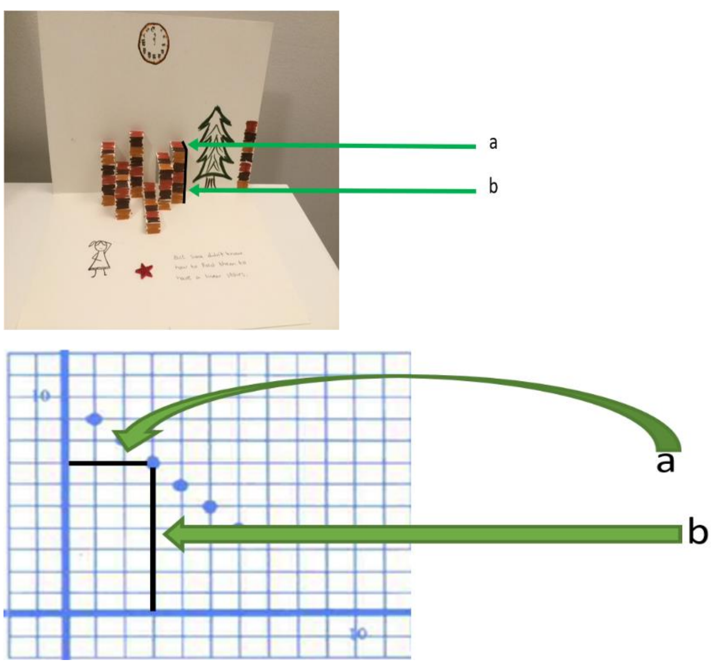


Figure 8: Seeing the points on the grid as the pop-up steps

2.3 Determining the function of the ““Making 10”” activity in the story context

Students in the preservice class also investigated the ““Making 10”” activity using the Scratch coding platform. The students chose various scenes and characters to create story-like contexts for their coding (see Figure 9). This inspired me to define a meaningful story infused by the pop-up project in which the functionality and surprise of the number pairs lining up could be demonstrated. In other words, I thought the hero of the story could use the lining up property of the pop-up steps based on the math constraint to accomplish a project such as reaching a top point of Christmas tree. The colorful features of the Scratch platform helped me imagine an atmosphere of a story in which the surprise of the ““Making 10”” activity can be displayed.

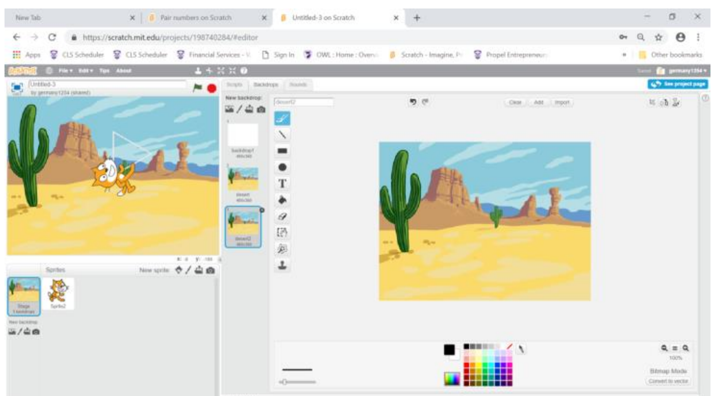


Figure 9: Scratch platform

Stage 3: The prior experiences in making the pop-up Christmas tree

I have already had the experience of making a pop-up Christmas tree, so it came to my mind that reaching the top of the tree can be defined as a pop-up project in the story. Also, I thought by making the pop-up steps based on the math constraint of the ““Making 10”” activity, the steps would line up and make a staircase (see Figure 3). Thus, I found a connection between making a pop-up “staircase” and reaching the top of the Christmas tree. However, in designing the story I used a 2-D Christmas tree and I did not make it 3-

D. I only made the steps in the pop-up format. The reason was that I wanted to not distract readers with other popups and my purpose was focusing on the pop-up stairs and the "expected surprise" resulted from applying the math constraint.

Stage 4: Making the ""Making 10"" activity in the pop-up format

In this stage, I will write about the practical processes of how I implemented my design into practice. In other words, how I made the pop-up steps and wrote the story.

4.1: Providing the supply

I provided these items: Cardboard, glue, tape, scissors and cutter.

4.2: Measuring, calculating and making

4.2.1: Determining the size of pages

4.2.2: Determining the location of Christmas tree and the pop-up steps

4.2.3: Determining the length of steps, the tree and the height of Sara, the hero of the story.

4.2.4: Making the popup steps based on the math constraint of the ""Making 10"" activity.

4.3: Designing the hero of the story, Sara.

4.4: Painting and coloring

When I made the pop-up steps, I understood that the steps did not align, and I failed to display the "expected surprise" in the pop-up context. I felt confused because I applied the same math constraint of the ""Making 10"" activity. In the activity, all number pairs aligned to form a line, but the pop-up steps did not. This discrepancy impacted on my preliminary plot and I was forced to think on the new one which seems meaningful according to the new situation. After all, I had to submit the pop-up story as my main assignment, and I could not leave it half finished because of not obtaining the "expected surprise".

Stage 5: Writing the plot

5.1. Plot

Sara, the hero of the story, wants to decorate a Christmas tree but she is not tall enough to reach the top of the tree; so, she needs to make staircase out of foldable planks relying on the wall to reach the top of the tree and to place a tree topper. (By "Foldable planks", I refer to narrow and flat pieces of wood which can be folded from different sections. "Steps" are made when the foldable planks are folded. The "staircase" made when the steps line up.)

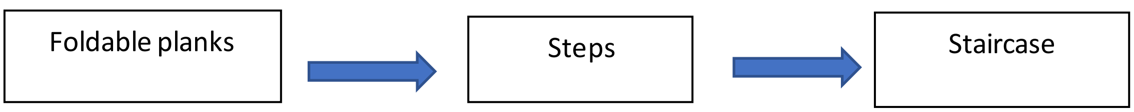


Figure 10: The sequence of these three elements: foldable planks, steps and staircase

In the following the story with illustration and popups will be displayed.

Figure 11: Page 1

This year Sara wanted to put a tree topper on top of the Christmas tree by herself, But the tree was too high. " How can I reach that point?" She whispered.

"Oh, there are many foldable planks that my grandfather made for one of his woodworking projects. Each plank has seven foldable sections. I can fold them to make "L" shaped stairs, then I can go up". That was her solution.

Figure 12: Page 2

But Sara didn't know what folding order she needed to rely on to create the stairs.

Figure 13: Page 3

Then Sara noticed the clock on the wall and thought: I can use the clock to choose a folding order, as the numbers on the clock increased in size just as the staircase would increase in height," I will follow what it shows", she said hopefully.

The clock displayed "1", so Sara folded the plank at the first segment, to create a step that was 1 section high and 6 sections wide.

Figure 14: Page 4

When the clock displayed "2", Sara folded the next plank at the second segment, 2 sections high and 5 sections wide.

"What a wise clock!" Sara was happy with its advice.

Figure 15: Page 5

When the clock displayed "3", Sara folded the next plank at the third segment, 3 sections high and 4 sections wide.

"I've never noticed how brilliant this clock is," she said when she saw the stairs she had made.

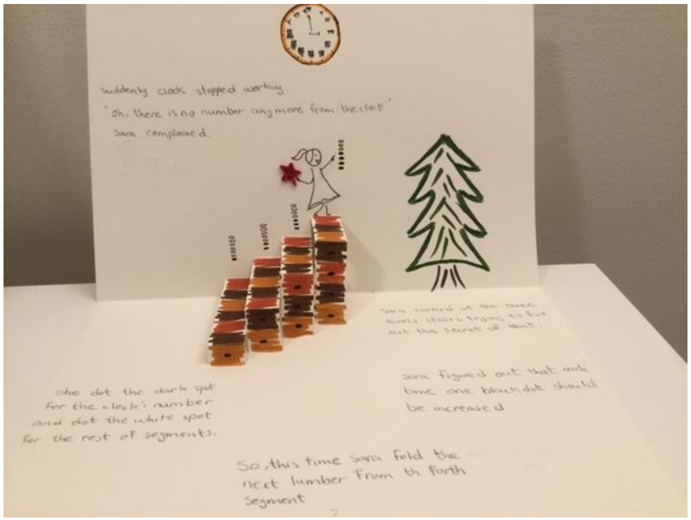


Figure 16: Page 6

Suddenly the clock stopped working.

"Oh clock," she said, don't you want to help me anymore?" Sara complained.

She looked at the first three levels of her stairs, trying to discover the secret of the clock's order.

She marked 7 circles above each stair she had made and filled some of them to record the clock's order number.

Sara figured out that each time one more circle was filled.

Figure 17: Page 7

So, for the next level of stairs, Sara continued this pattern for the remaining steps.

Figure 18: Page 8

Sara reached the top of the Christmas tree after six steps. She put the star at the top of the tree, proudly.

Figure 19: Page 9

Then Sara wanted to go down.

Sara wondered: "Can I create a new staircase to go down, instead of using the one I just made?"

"Adding black dots lead me to go up, removing black dots will lead me to go down," she thought to herself. "I want to try a new pattern for going down," she said excitingly. "After the first step, I will fold at every second crease."

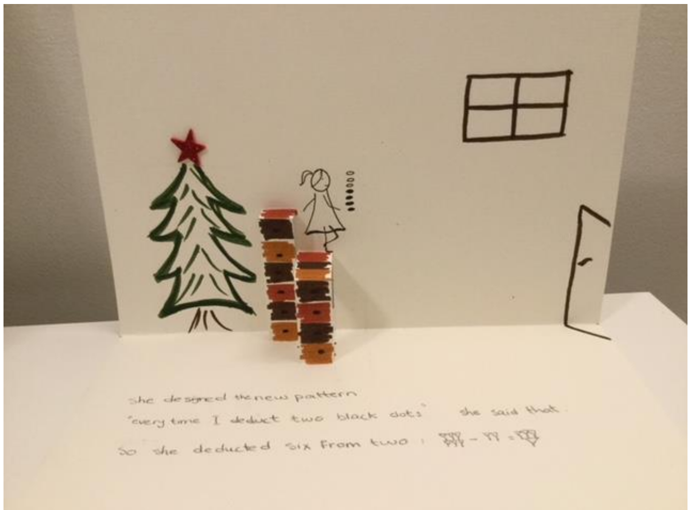


Figure 20: Page 10

Sara designed the new pattern.

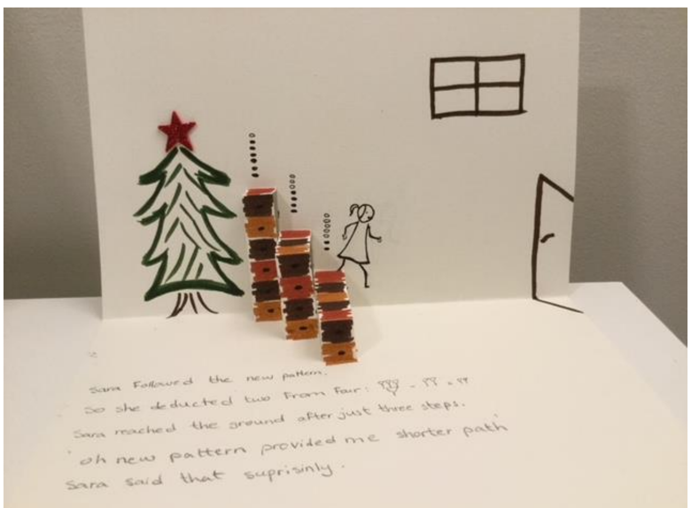


Figure 21: Page 11

"Oh, new pattern gave me the shorter path," she said surprised.

Sara reached the ground after just three steps.

Figure 22; Page 12

Sara looked at the two patterns of stairs she had made to go up and down.

"Why did I end up closer to the tree? Why did I reach the ground only after three steps?" She pondered these questions.

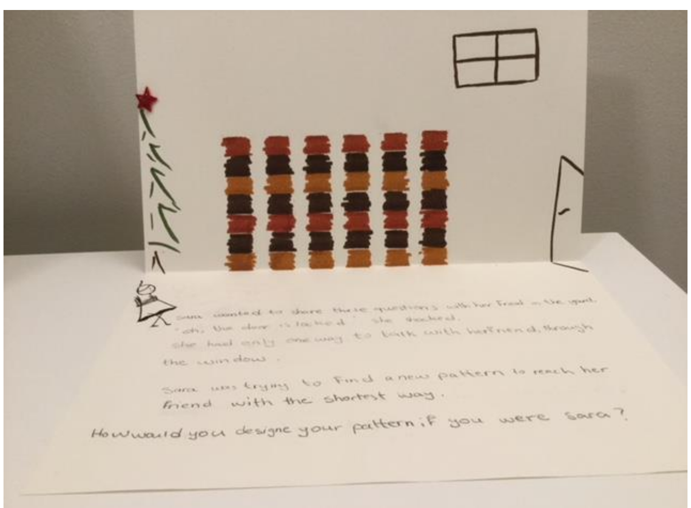


Figure 23: Page 13

Sara wanted to share the story of the Christmas tree achievement and the questions with her friend in the yard.

"The door is locked" she sighed.

"But I can talk to my friend through the window. Yes, I can create a new pattern again to reach the window" she said excitingly and started to think.

How would you design the pattern if you were Sara?

The end

MKT

5.2 How was the plot developed without the intended surprise of the ““Making 10”” activity?

My initial main objective of designing this story was that Sara would take advantage of the math constraint of the ““Making 10”” activity, a + b = constant, to make a “staircase” out of the foldable planks to reach the top of the Christmas tree. There were supposed to be four explicit elements as Sara, the tree topper (red star), the Christmas tree, and the foldable planks and one implicit element as the math constraint playing roles in the story. When I started to write the plot, I figured out although I had applied the math constraint in making the steps, they did not line up and a staircase did not show up, when I picked the first number randomly (as was done in the “Making 10” activity). At this point I realized something was missing and there was somehow a flaw in my design, but I could not understand what it really was. I resorted to adding another element, the magical clock, to make the steps aligned and help Sara in making a staircase. Although I applied the constraint of the ““Making 10”” activity to make the pop-up steps, I could not achieve the expected surprise when I picked the first numbers randomly. I designed the story by linking the making of the pop-up staircase to the clock, which was the external hint; so, the story did not involve the math surprise of the “Making 10” activity.

As it was said in the plot, in the middle of the story the clock did not work and Sara needed to figure out the rest of folding order by herself. On the way coming down, she used another pattern and reached the ground faster. This story became more about steepness or slope. However, it was not what I intended to offer in the story. The plot of the story gradually deviated from my original goal which was displaying the surprise and functionality of the math constraint in the ""Making 10"" activity.

Stage 6: First discovery resulted from writing an essay

After writing the story, I had to write an essay to analyze the story from the pedagogical lens. It became an opportunity to rethink about the story and solve the missing point. Below, I will explain how by delving into the story, I discovered a new way to see the pop-up steps so that the surprise of "they line up!" could be experienced. This led me to experiencing my first aesthetic-mathematical discovery, where I experienced an illumination and an "AHA" moment.

6.1: Solving the problem of the pop-up steps

In the process of writing the essay of the story, I came to understand that something was missing. I understood that despite applying the math constraint, a+b=c in making the pop-up steps, and picking the value of "a" randomly, the steps did not line up and a staircase did not materialize. I had already felt this while writing the plot, but it was much clearer now. I thought if I had not used the math constraint to make the pop-up steps, (it means the total length of the stapes had not been the same) It would have also looked random. So I asked myself: Where is the effect of the math constraint? This time I encountered the discrepancy vividly. I did not ignore it and struggled to understand the reasons that caused this mismatching.

6.2: Talking with my sister

I decided to share the story with my sister and receive her assistance. I explained what was happening to my sister from the beginning of the story. As she is a civil engineer with a math background, she welcomed the opportunity and listened to me attentively. The mathematics subject seemed simple to get her involved easily. However, we could not solve the discrepancy between the pop-up steps and the ""Making 10"" activity.

6.3: Writing the relevant math of the ""Making 10"" activity

I wrote all possible relevant math equations I could think of, to clarify the underlying mathematics of the pop-up steps and the "“Making 10”" activity.

6.4: Making the random pop-up steps look similar to the random pair numbers in the "“Making 10”" activity.

I positioned the random pop-up steps next to the grid presenting random number pairs lining up in the "“Making 10”" activity to compare visually (see Figure 8). Although, both demonstrated the same math constraint, a + b = c, the pop-up steps did not line up while the points lined up. I was frustrated and confused.

I decided to make the steps look similar to the random number pairs that were plotted on the grid (see Figure 24). I put one point in the joint of each pop-up steps. I labeled each step with one point. For example, for the first step (from left side) I corresponded (2,5) and for the second one (4,3). These pair numbers were developed: (2,5), (4.3), (1,6), (6,1), (3,4), (5,2). I added numbers of each pair; expectedly, I obtained the same number for pairs: 7.

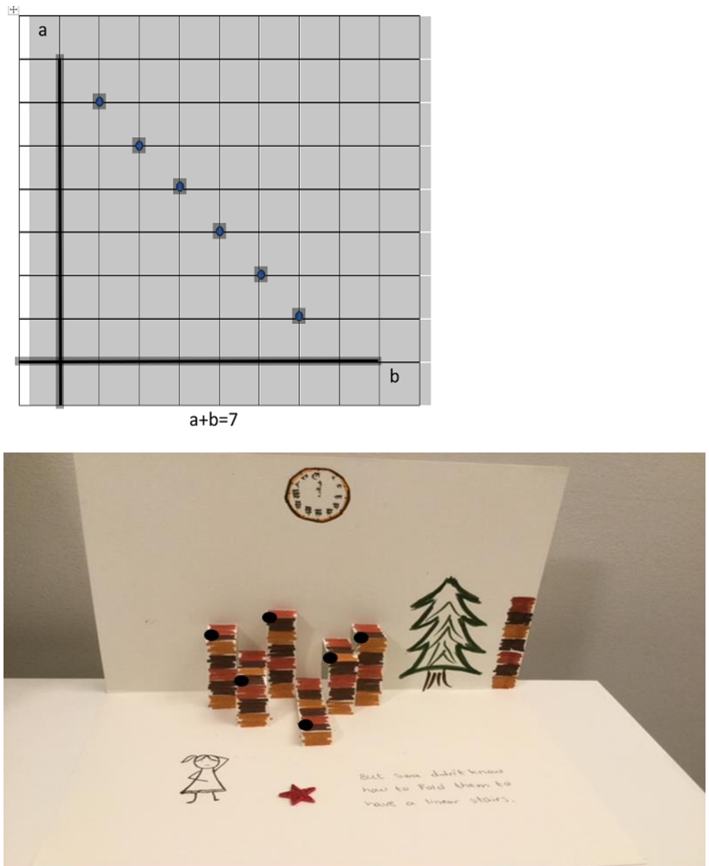


Figure 24: The pop-up steps look like the points

First discovery

Stage 7: Identifying an invisible plane where the reflection of pop-up steps line up

I took another look at the randomly created pop-up steps. After labeling each pop-up steps with a point, to indicate the points plotted in the ““Making 10” activity”, I saw each step as one point. So, I saw six points floating in the air. Without a specific and clear reason, I suddenly turned my eyes to the left side gently and realised that these points make a line in an invisible plane which was perpendicular to the two planes in which the pop-up steps were located (this can be called the third plane). It was an incredibly profound and emotional moment for me. I jumped out of the room and told my sister: they line up from the side view. It was such an insightful and unique moment. Everything started to make sense. I experienced an inner joy of solving the math problem and sensing an illumination. I have never had this feeling in my school and university time. It was like that moment only belonged to me and I discovered that by myself. To make it visible, I positioned one paper to display how the points corresponded to the pop-up steps aligned to form a line (see Figure 25). It seemed that through some traces (signs) in the planes where the pop-up steps exist, the third plane, which was implicit and did not exist apparently, was discovered and became explicit. It was like a connection between the dimensions that are visible and the dimension that is not visible. By seeing the 2-D reflection of the 3-D pop-up steps, I experienced the AHA! moment and surprise of "they line up!". However, although the type of surprises students had experienced in the ""Making 10"" activity was the same as I had in my story, the paths reaching to the surprises were different. Students followed the procedures in the ""Making 10"" activity and were led to the surprise and they did not experience the AHA! Moment. On the contrary, in my experience it was not a straightforward way. I felt confused and frustrated and struggled to experience the surprise. I thought hard to find the solution.

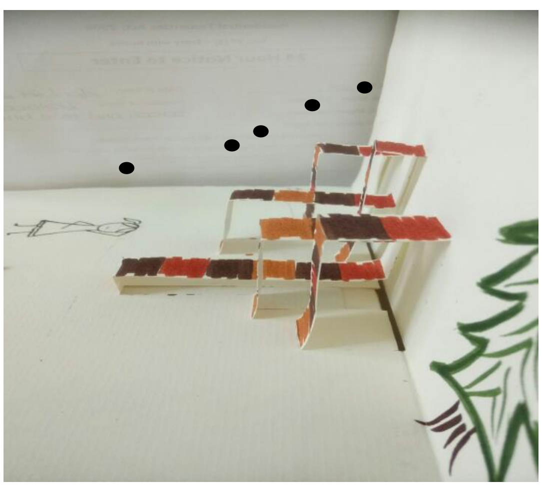


Figure 25: Seeing the pop-up steps from the side

7.1: The idea of shadow resulted from the conversation with my sister

I was excited to share what I had discovered, that "they line up", with everyone. Expectedly, the first person was my sister who had somehow involved in the progress of my story and witnessed the struggles I had gone through to figure out the discrepancy between the pop-up steps and the "“Making 10” activity". As I mentioned before, I jumped out of my room and told my sister excitingly: They line up from the side view. Then we discussed about how I could lead the reader to see what I was seeing from the side. In that discussion, I found out that the shadow of the pop-up steps is exactly the 2-D projection of the 3-D steps which line up on the side plane. I was exhilarated to try out the idea of shadow. I lit up the candle and positioned it in different places to find the best one to see the shadow of the pop-up steps properly. I saw the shadow lining up on the side wall and became delighted (see Figure 26). Due to the divergent light emitted from the candle, the line of shadow was not very precise though. It was also possible to substitute the candle for example by a flashlight in the far distance which generates parallel light rays.

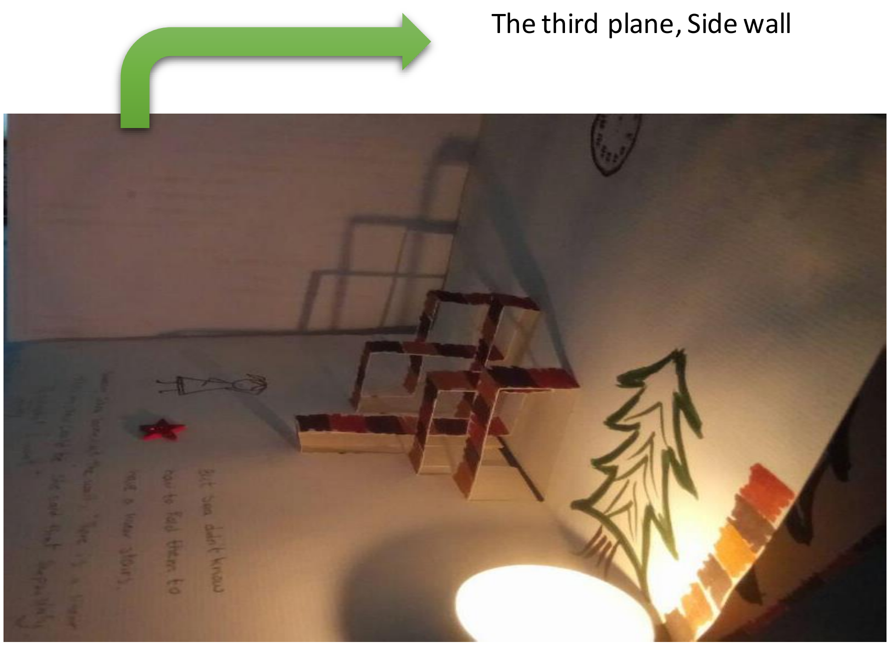


Figure 26: Seeing the pop-up steps as a shadow on the side wall

In Figures 27, it can be seen that the 2-D projections of the random pop-up steps and the organized ones are the same and even though they are different orderly in 3 dimensions, both have the same 2-D projection and make the similar linear patterns (the subtle difference between the two shadows is caused by not being the second pop-up step in the left picture and if it existed, both shadows would be identical).

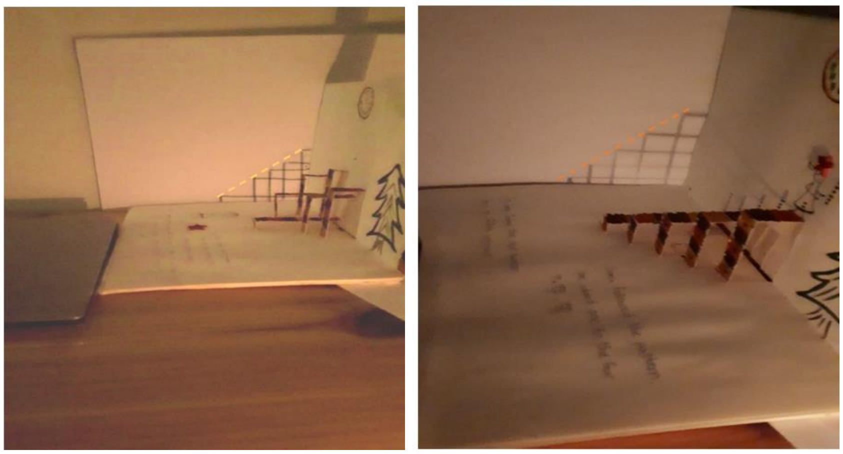


Figure 27: The same shadows for different sequence of the pop-up steps

7.2 Verification

After I found the diagonal pattern of the shadow of pop-up steps on the third plane, I delved into it to understand the reasons underpinning the phenomena. Mathematically I had initially added one dimension to the ““Making 10”” activity by applying pop-up context and in opposite through discovering the projection of the pop-up steps on the third plane, I withdrew the dimension again and saw the steps 2-D to obtain the same original surprise (see Figure 28).

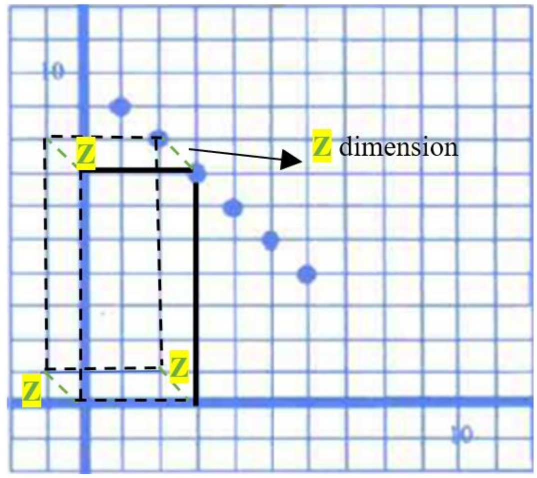


Figure 28: The z dimension is the difference between the points on the grid and the pop-up steps

7.3 Extension of the first discovery: What if x or y-coordinates become 0?

As described in the previous stage, when z-coordinate became 0, the “line” pattern appears on the X-Y plane and it can be seen from the side. Then this question was raised for me: What patterns of the steps will appear if x or y-coordinates become 0?

I understood as it is shown in Figure 29, if x-coordinate becomes 0, six rectangles with y length and z width are seen on the Y-Z plane. I saw these rectangles from the front view of the steps.

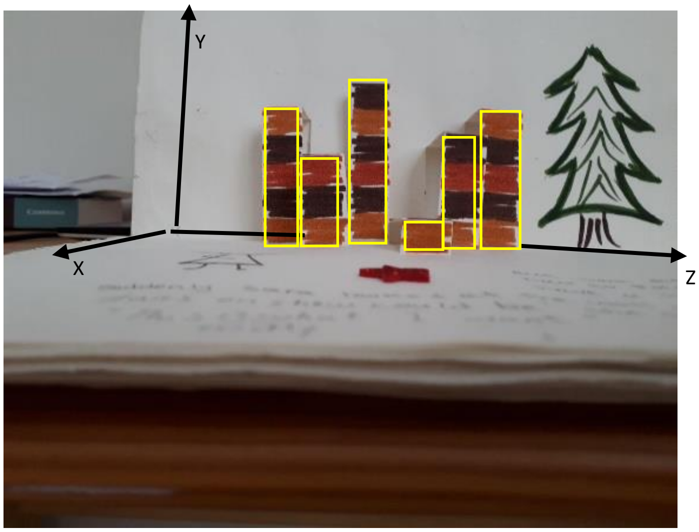


Figure 29: If x=0

If the y-coordinate becomes 0, six rectangles with x length and z width are seen on the Z-X plane. I view these rectangles from up view of the steps (see Figure 30).

Figure 30: If y = 0

In general, I figured out by making x, y, and z-coordinates equal to zero, the 2-D projections of the 3-D steps are seen on the three planes. Only one projection results in a straight line, as we have x+y=7, or x+y have a constant sum, only when z = 0. For the other two cases, the sums of x+z and y+z do not have a constant sum.

Second discovery

Stage 8: Seeing the pop-up steps as rectangles with specific property

In this stage, I will explain how I discovered that the pop-up steps formed rectangles with the same perimeter and different areas.

8.1: Reading about the relationship between area and perimeter of rectangles

Reading about a surprising math activity, led me to the second discovery in the pop-up story. While doing the course “Mathematics for Teachers”, I needed to look at the website: http://researchideas.ca/wmt/c3.html, which has a activity involving perimeter and area. Keeping that activity in my mind, I came back to my story to look at the pop-up steps attentively and noticed that there were rectangles formed by the steps. I had never noticed that there were some rectangles lying among the pop-up steps. When I looked at the sidewall, I could see the shadow of rectangles clearly. I figured out their perimeters are the same and their areas are different (see Figure 31). I experienced again a feeling of discovery. I felt that there were mysteries hidden in the pop-up steps, and it was my responsibilitys to reveal each of them one by one.

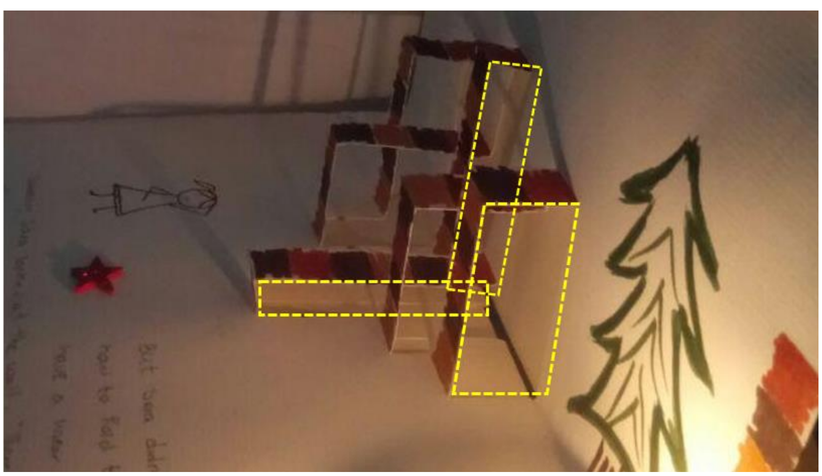


Figure 31: The pop-up steps have the same perimeters

8.2: Verification

I understood that all foldable planks have the same length, which is 7, and as a result the pop-up steps made by the planks have the constraint of $a + b = 7$. Therefore, the rectangles whose sides are made by the length of planks are expected to have the same parameters: $a + b + a + b = 14$. The shadow of the rectangles can be seen on the sidewall (see Figure 27). As I noticed more, it became clear that the rectangles marked with the same color bore more similarities and they had the same areas too (see Figure 32).

8.3: Sharing with my supervisor

In having conversation with my supervisor about that all rectangles made by the pop-up steps had the same perimeter and different areas, I realised that it is possible to have the maximal area while all perimeters are the same. This understanding led me to think about the math idea of optimization.

Figure 32: The pair of rectangles have the same area

8.4 Extension of the second discovery

Noticing the rectangles in the pop-up steps created an opportunity for me to explore more mathematics relationships when something is constant (the perimeter) while two other measurements vary (the length and the width). To clarify, I drew the table below showing some of the relationships and patterns that may be investigated.

Table 2: The vertical/horizontal sides, perimeter and area of all rectangles made by the pop-up steps

Rectangles	Vertical side	Horizental side	Perimeter	Area
1	1	6	14	6
2	2	5	14	10
3	3	4	14	12
4	4	3	14	12
5	5	2	14	10
6	6	1	14	6

Then I plot the area of each of these rectangles, against the length or the width, and I could get the parabola shown below.

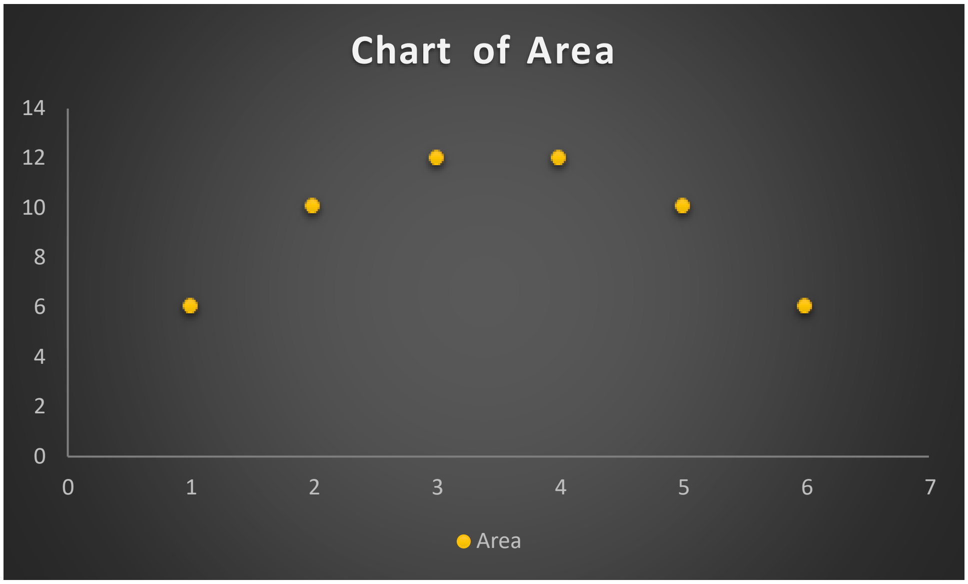


Figure 33: The parabola relation between various areas and constant perimeter

Third discovery

Stage 9: Seeing the folding of the pop-up steps as the physical sum operation

During the writing of my current thesis, I became curious to know what would happen to the steps if I folded them. Once I folded them, I saw that they made a “line” pattern. They all ended up at one row 7 units from the z-axis. I was surprised. All the steps have the different vertical sides, so I had not expected that they would line up and fall off on the same row (see Figure 34).

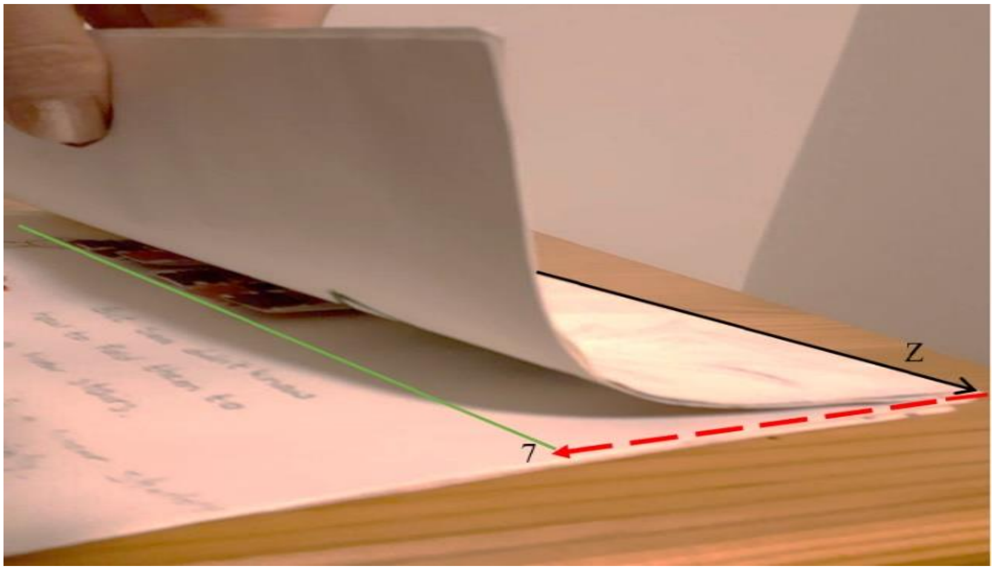


Figure 34: The pop-up steps line up by folding

9.1: Sharing with my supervisor

I shared the third discovery with my supervisor and showed him how the pop-up steps fall off at the same row by folding. In our conversation, I explained to him how the vertical and horizontal sides of the pop-up steps are added to one another. This conversation helped me to see the folding the pop-up steps as the physical sum operation.

9.2: verification

I was intrigued to see the process and find out the reasons. When I looked at the steps while folding, I noticed that the vertical side gradually folds from the place where the horizontal side ends and eventually becomes aligned with the horizontal side as it is shown in Figure 35.

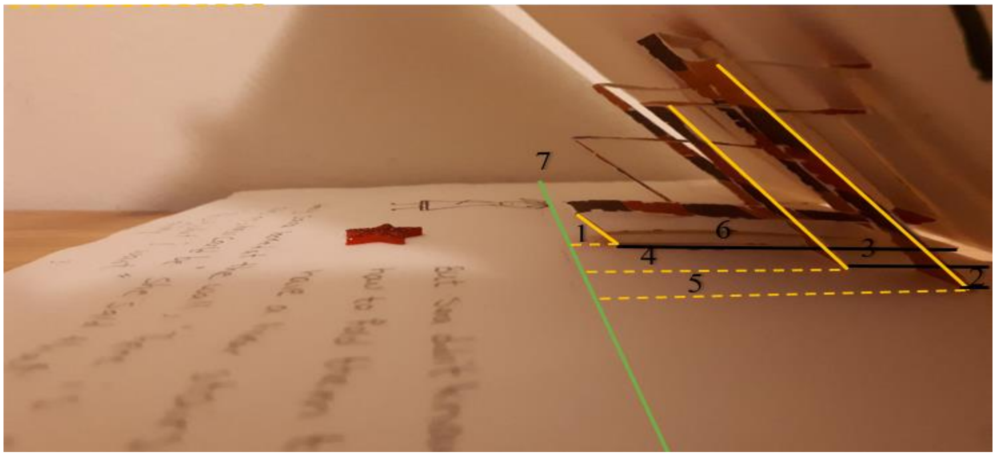


Figure 35: The process of folding the pop-up steps

On the other hand, as described earlier in this chapter, the total length of the vertical and horizontal sides of each step are equal to 7. As a result, when the steps are folded, they will end up at the row 7 units from the z-axis.

9.3: Extension of the third discovery

I was curious to see what happened if the plane is folded. I understood that the same is true in 3-D when you fold/rotate a plane. However, with the pop-up, the points behave differently in the pop-up than in typical math, because they are physically hinged. In the "Making 10" activity, the points do not fold but rather rotate as one axis folds on the other. As the points rotate, they do not all end up at the same place. I used Geogebra to see what happened if the points in "Making 10" activity rotate. As it is shown in Figure 36, six circles are formed by rotation of the points and due to the symmetry of the points around the line y=x, pairs of points end up at the same place. There were two surprising points these circles afforded:

As it is shown in Figure 38, I noticed that there is a misconception about the line. The line made by the points which is supposed to be straight, seems curvy. It looks like that the straight line is distorted, but it was not. The circles in the background cause the illusion that the line seems curvy.

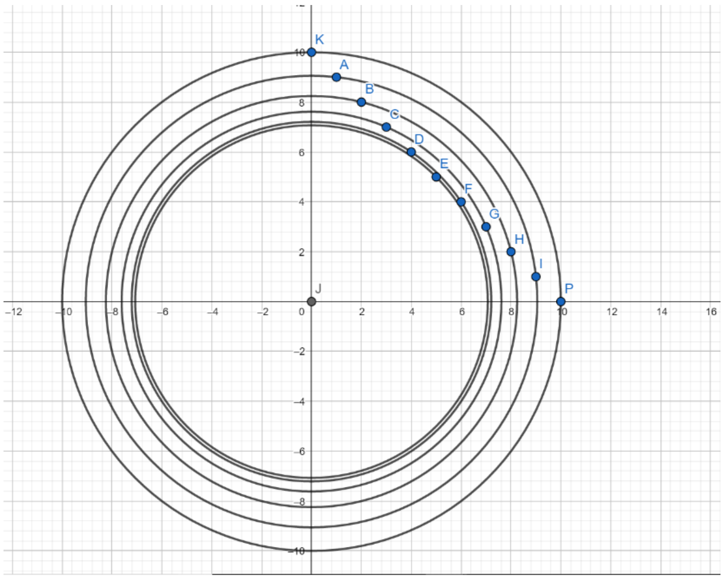


Figure 36:Circles made by the rotation of the points

This remined me of these other illusions (see Figure 37): The right line seems longer than the left one which is not correct. Due to the different sides of the arrows (outer and inner), the length of lines looks different.

Figure 37: The misconception about vision

Second, even though the distance between every two adjacent points are the same (see green lines in Figure 38), the distance between adjacent circles is not the same, with this distance decreasing from the outer circle to the inner circle.

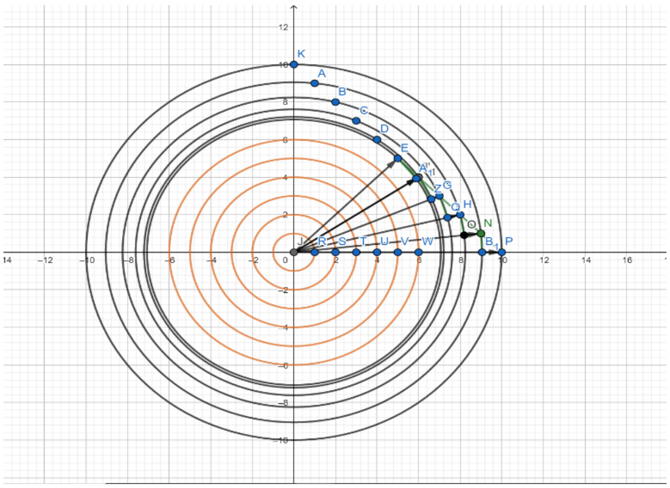


Figure 38: Different distance between adjacent circles

I figured out that when points rotate, some points have to make a bigger turn than others, that is why they end up in different places.

9.4: The shadows line up again!

After finding that folding the pages can make the steps to make a horizontal line, I put a burning candle next to the steps in the lower side and changed its positions continuously. In one position, light rays made shadows aligned to form a horizontal line. I experienced another surprise of "they line up" (Figure 41).

Figure 39: The shadow of 45 degrees light rays

9.5: Verification

I figured out that the light rays with a 45-degree degree angle can make a line of shadows. I saw that the line of the shadows appears at 7 units from the z-axis. It was surprising and reminded me of the pattern of the folded steps. In that experience, folding the steps made them end up at the height 7 from the z-axis and in the current experience, the light rays with a 45-degree angle formed the line of shadows at the height 7 from the z-axis too (see Figure 39).

I changed the position of the candle to see how the shadows would change. The shadows did not align anymore. Why 45 degrees? Because the slope of $x + y = 7$ is 1 (or 45 degrees). If the relationship is $x + 2y = 7$, this slope would change.

Mathematically, when the light ray emits at a 45-degree angle on the steps, it forms the right triangle edb by the x and y-axis (Figure 40). Due to the fact that the angle <bed and <ebd are equal to 45 degrees, the triangle is isosceles. Therefore, the ed and db sides have the same lengths. As a result, the length of the shadow, db, becomes the total length of dc and cb which is the total length of horizontal and vertical sides.

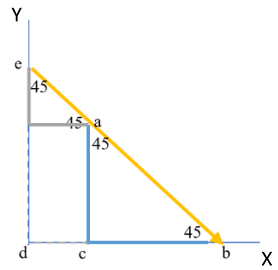


Figure 40: The underlying geometrical reason 1

As it is shown in Figure 41, I understood that the shadows of all the steps regardless of their vertical and horizontal lengths end up in the same line which is 7 units far from the z-axis. The direction of the light rays, from down to up or from up to down, does not make a difference to the pattern of shadows. In either direction, as long as the light angle is 45 degrees, the shadows of all the steps fall off on the same distance from the z-axis.

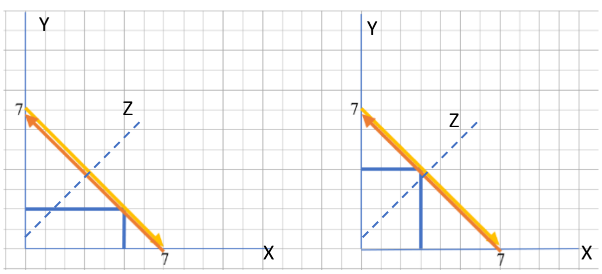
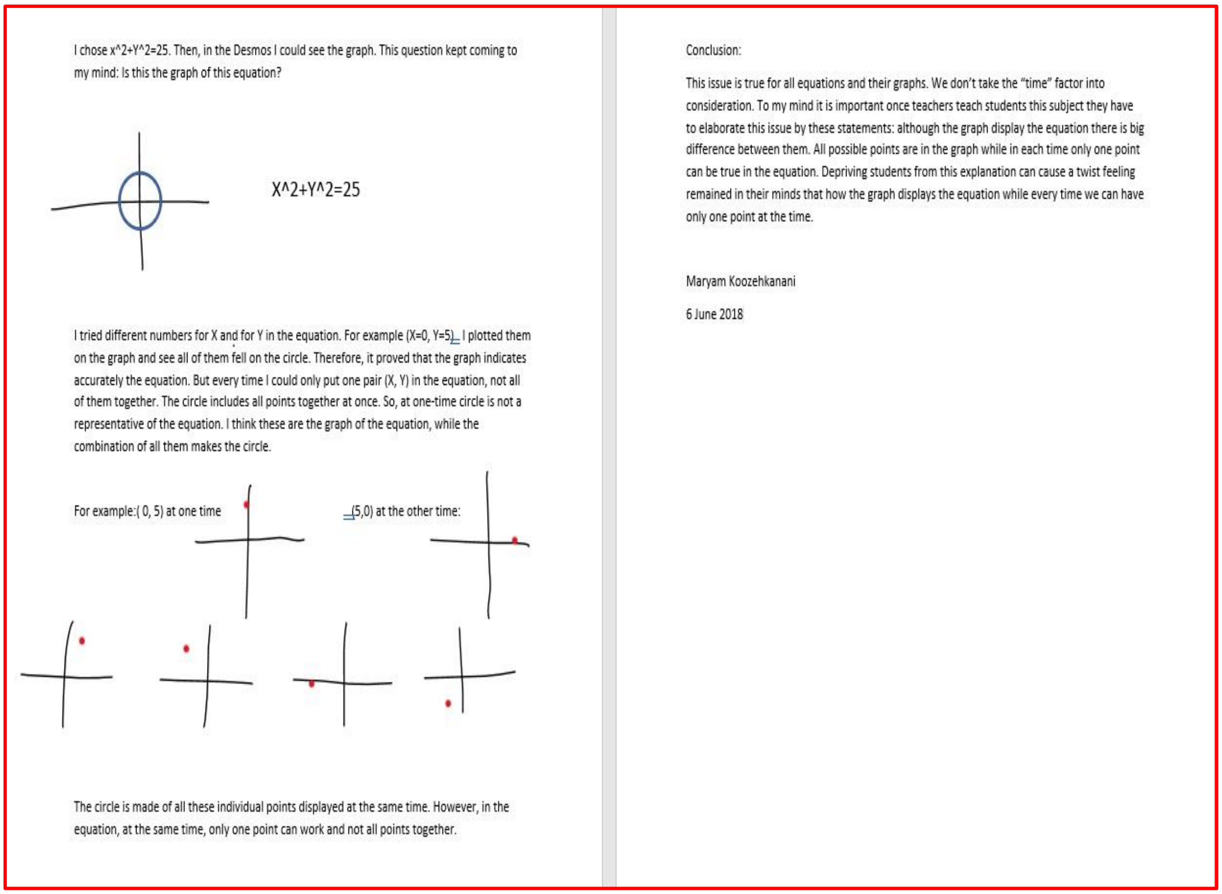


Figure 41: The underlying geometrical reason 2

Fourth discovery

Stage 10: The z axis plays the role of time

When I talked with my friend about how the 2-D projection of 3-D pop-up steps line up from the side view, we discussed more differences that these two representations (the points on the grid and the points assigned by the pop-up steps) would have. In this discussion, I realised that the z coordinate can indicate a time factor and can be seen as a time axis and the pop-up steps are then ordered in the time sequence like the chain of events. In other words, by displaying the points of the “Making 10” activity as pop-up steps, the points were streched out along the z-axis, or over time, and when the pop-up steps were seen the side view, the points are stacked and are displayed all at once, in one instant. This understanding helped me remember the comment I submitted in an online summer course about a year prior. In that comment, I drew attention that the graph of each relation may be seen as demonstrating an infinte set of points in an instant. For example, we can see the graph of the relation $X^2+Y^2=25$ as a finished product (seen all at once), or we can see it as a single point (X,Y) that travels the path of the graph over time. What follows is my comment (Figure 42):



I chose x^2+Y^2=25. Then, in the Desmos I could see the graph. This question kept coming to my mind: Is this the graph of this equation?

X^2+Y^2=25

I tried different numbers for X and for Y in the equation. For example (X=0, Y=5). I plotted them on the graph and see all of them fell on the circle. Therefore, it proved that the graph indicates accurately the equation. But every time I could only put one pair (X, Y) in the equation, not all of them together. The circle includes all points together at once. So, at one-time circle is not a representative of the equation. I think these are the graph of the equation, while the combination of all them makes the circle.

For example:(0, 5) at one time (5,0) at the other time:

The circle is made of all these individual points displayed at the same time. However, in the equation, at the same time, only one point can work and not all points together.

Conclusion:

This issue is true for all equations and their graphs. We don't take the "time" factor into consideration. To my mind it is important once teachers teach students this subject they have to elaborate this issue by these statements: although the graph display the equation there is big difference between them. All possible points are in the graph while in each time only one point can be true in the equation. Depriving students from this explanation can cause a twist feeling remained in their minds that how the graph displays the equation while every time we can have only one point at the time.

Maryam Koozehkanani

6 June 2018

Figure 42: My comment about graph and mathematical relations

I was surprised to feel that in the process of making and exploring the pop-up math story, potentially I was in the process of solving my own confusion which I was not aware of.

10. 1: Verification

The fourth discovery is a way to represent or solve my difficulty in seeing the points “line up” in the pop-up representation. It means that the pop-up steps inherently entail the element of time and due to the specific physical structure of popups, and it displays the points of the function, a+b=c, like the events happened over time which is consistent with reality, where we experience events over time.

In the last discovery, the z axis was seen as a role of time. By displaying the points of the “Making 10” activity as pop-up steps, the points were streched out along the z-axis, or over time. When the pop-up steps were seen from the side view, the points are stacked and are displayed all at once, in one instant. Basically, what a pop-up does in tranforming the “Making 10” activity into the pop-up steps is intertwining the abstract concept of time with the physical concept of space. Seeing the z axis as the axis of time creates a metaphor. I was delighted that the pop-up story became that rich and continued the discussion to find out what other kind of mathematical relation can be impelmented in the pop-up context.

Chapter 5. Analysis

5.1 Introduction

In this thesis I investigate what factors come into play when an educator immerses herself in the process of designing an aesthetic mathematical experience. In Chapter 4, I described the key stages of my narrative in designing the aesthetic mathematical experience, a pop-up mathematics story. Reflecting on my narrative as a whole, I can identify, describe and discuss five factors that were not present in my mathematics educationand in school mathematics learning more generally, which enhanced the aesthetic quality of my experience: constructing, seeing-as, surprise, immersion and audience.

5.2 Constructing

I engaged in a constructionist activity in making the pop-up math story, “Sara and Christmas Tree”. Drawing an analogy with Papert’s childhood experience with gears, I engaged with the “gears” of pop-up construction. His construction experiences began with gears. He loved gears and it was personally meaningful to him as without being taught, he was interested in seeing how the gears and differentials work. Gears became his preferrred ‘mental models’ and by assimilating math to this model, he could experience math in an affective way while he gained a new understanding of math through the gears system. He believed: “Anything is easy if you can assimilate it to your collection of models” (Papert, 1980, p. vii).

As explained in chapter one, I was thrilled by seeing the pop-up card where my name popped out and it became my passion to learn how to make it. Nobody told me to do it and nobody taught me, but it was personally meaningful. I was intrigued by seeing how 3-D shapes can be made by the 2-D sheet. I would delve into the pop-ups to understand the underlying geometrical reasons of this product that seemed magical to me. I was amazed by the structure of pop-ups and it influenced me, both cognitively and affectively. Using Papert’s terminology, pop-ups became my “collection of models”, which then became my gears.

When I witnessed the "Making 10"activity, it resonated with the basic structure of the pop-up steps. I assimilated the math of the activity to pop-ups and due to the independence I had in choosing the project as my assignment, I used this assimilation for presenting the "Making 10" activity in the pop-up context. Furthermore, the physical embodiment of the pop-up story provided me further agency to examine the story with different perspectives, which allowed me to explore abstract math ideas in a tangible and concrete embodiment. Whenever an idea came up, I could take the story and try the idea in a practical way. I could see the steps with different angles and compare it with other physical objects. If I had not constructed the physical version of the story myself, I could not delve into the pop-up steps. All discoveries happened when I had the story in my hands, during which I sometimes compared it with the "Making 10" activity and sometimes looked at the steps attentively. The physical version of the story afforded me agency to touch the story and see vividly what I imagined in a concrete way instantly, as well as providing special affordances leading to artistical-mathematical discoveries.

Constructing the "Making 10" activity (2-D) in the pop-up context (3-D) was significantly mathematical. To demonstrate the surprise of "Making 10", I needed to solve the puzzle, which led me to new mathematical ideas, surprise and insight. The new mathematics, such as the 2-D projection of 3-D, optimization, the physical representation of sum operation and conceptual understanding of seeing the z axis as the time axis, were all effects of the pop-up context and my agency to work on it.

Papert (1993) stated: "I am convinced that the best learning takes place when the learner takes charge" (p. 136). Students, by having "freedom to make choices, to investigate, and to discover", experience the sense of agency that is enjoyable (Gadanidis, Clements & Yiu, 2018, p. 36). The main attributes of Egan's in-depth learning project is students' agency to choose how to explore the subject, accumulate information, organize it, and delve into it and construct knowledge.

Papert (1980) used the term "mathland" to describe learning math through constructing with computer programming where students learn math naturally as learning French while living in France (p.6). For me the pop-up story became the "pop-up mathland". The more I investigated, the more I discovered. It was as if *I was walking through the steps to* *explore around. While the sun shining, I discovered the shadow of the steps lining up in the front mountain. Suddenly I looked at the walls made by the steps and found all have the same perimeter and different areas. Then "Boof!" the page fell down on me. When I was hit I saw all the steps line up at 7, "oh again they line up". I was overwhelmed and I wanted to get out through the z axis, somebody shouted, you are walking through time axis. I screamed and amazed of what was going on, suddenly I saw the sign "Welcome to the Pop-up Mathland".*

This story was a metaphor for what happened to me through creating the pop-up story and exploring it. I was naturally exposed to different math ideas. Gears "illustrate many powerful "advanced" mathematical ideas" (Papert, 1980, viii). Similarly, Papert found pop-ups "illustrate many powerful "advanced" mathematical ideas" (ibid).

Papert (1980) used the phrase "transitional object" to describe gears when he could see the connection between the sensory objects and abstract concepts of math. For me, pop-ups play the "transitional object" role in the discoveries that happened while designing the pop-up math story. In all discoveries, the abstract math ideas were presented in concrete ways. Pop-ups make connections between mathematical ideas and physics (light and shadows), between the abstract concept of time and the space, and intertwined the action of movement (spatial) with the sum operation (non spatial).

5.3 'Seeing as'

My seeing-as experiences:

- Seeing the points on the grid as the pop-up steps
- Seeing the pop-up steps as shadows
- Seeing the pop-up steps as rectangles
- Seeing the pop-up steps as physical sum operations
- Seeing the z axis as the time axis

My seeing-as experiences were instrumental in affording me mathematical and pedagogical surprises and insights. In this section, I note briefly each ‘seeing-as’ experience and explain it mathematically and pedagogically. I elaborate more on the first ‘seeing-as’ experience because it predisposed the rest ‘seeing-as’ experiences.

5.3.1 Seeing the points on the grid as the pop-up steps

In my attempt to design an aesthetic mathematical experience, the first seeing-as opportunity arose from my decision to use a pop-up story to represent the two-dimensional mathematical surprise of “they line up!” in the “Making 10” activity in a three dimensional version, as described in Chapter 4.

Zwicky (2003) states that “metaphor is a species of understanding, a form of seeing-as. We see simultaneously, similarities and dissimilarities” (p. 4). “Seeing-as” can illuminate resemblances between phenomena that create significant turning points in what we understand and how we understand it. Seeing-as requires the metaphorical process of making connections between seemingly disparate phenomena or situations, which are not initially obvious, and which lead to a reorganization of knowledge.

Initially, I saw the x and y components of points in the “Making 10” activity as steps that would materialize as a page of the story opened (see Figure 43). I thought that representing the points as steps would also lead to the surprise of “they line up!”.

Figure 43: The first ‘seeing as’

However, the surprise in "Making 10" was due to the points lining up even though the first coordinate was chosen randomly. When I tried representing the points as steps in a random fashion, the points they represented did not appear to line up, as shown in Figure 43. Putting myself in the position of trying to see the "Making 10" surprise as a pop-up step surprise created some mathematical and pedagogical confusion. Mathematically, what was the reason for the random steps not lining up as the random points did? Pedagogically, I wondered how I might structure the pop-up story so the reader can experience the surprise of "they line up!"

The first seeing-as experience afforded me the opportunity to think hard mathematically, to experience cognitive conflict, followed by mathematical surprise and insight, which led to further seeing-as insights:

- Seeing the pop-up steps as shadows
- Seeing the pop-up steps as rectangles
- Seeing the pop-up steps as physical sum operation
- Seeing the z axis as the time axis

5.3.2 Seeing the pop-up steps as shadows

As described in greater detail in Chapter 4, I eventually discovered a way to experience the surprise of the randomly constructed steps lining up in the pop-up, just as the "Making 10" points lined up. As shown in Figure 26, I saw the random steps line up when I viewed them from the side, like shadows on a wall.

This discovery of seeing the steps line up as shadows created a new surprise for me. The surprise, however, is quite different from the surprise experienced in the "Making 10" activity. In the "Making 10" activity, the students are led to the "They line up!" surprise by the activity sequence. That is, the surprise is predetermined, and they have no choice but to notice that the randomly plotted points line up. In the case of modelling the random points as 3-D steps, I was initially led to a state of confusion. Unlike the students, the surprise I eventually experienced was not predetermined, and I had to "think hard" to discover it. The possibly also existed that I might never have discovered it. Having challenges, illuminating the solution with surprise, and insight after a lot of efforts are all related to the "AHA!" moment I experienced in this stage.

Pedagogically, my experience is more similar to the next part of the "Making 10" activity, where students have to think hard and experience some confusion and failure as they try to design equations that would lead to graphs that point in different directions, or that may even curve.

In the extension of this part as it is described in chapter 4, by making x, y, and z-coordinates equal to zero, the 2-D projections of the 3-D steps are seen on the three planes. Only one projection results in a straight line, as we have $x+y=7$, or $x+y$ have a constant sum, only when $z = 0$. For the other two cases, the sums of $x+z$ and $y+z$ do not have a constant sum (Figure 29 & 30 in Chapter 4).

Mathematically, this experience highlights the reason why the points in __ + __ = 10 line up. They line up as all sums have the same constraint. This is why the order of the points can be random and they will still line up.

Pedagogically, this creates an opportunity to help students realize that in mathematics, constraints lead to mathematical patterns.

5.3.3 Seeing the pop-up steps as rectangles

Seeing the pop-up steps as rectangles with specific properties has mathematical and pedagogical effects. Mathematically, due to the constraint of $x+y= 7$ applied in making the steps, the total length of the horizontal and vertical parts for each step is the same. On the other hand, the perimeter of each rectangle entails two vertical and horizontal parts. As a result, the rectangles constructed by the steps have the same perimeter of 14 (see Figure 31, chapter 4). In "Making 10", it is also possible to attribute imaginary rectangles to the points by drawing the shortest lines from the points to the x-axis and the y-axis (see Figure 44), and the rectangles would also have the same perimeter.

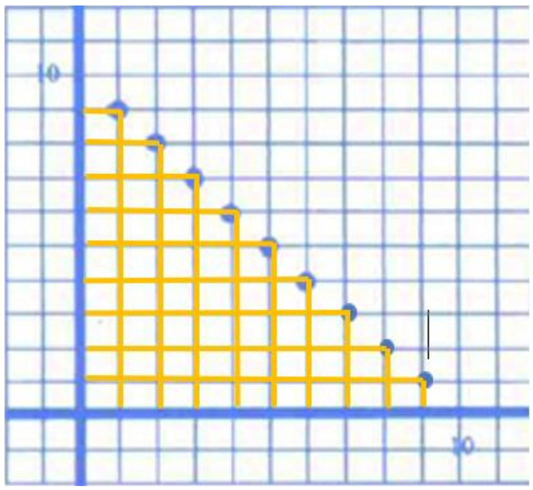


Figure 44: The virtual rectangles on the grid

Pedagogically, noticing the rectangles is more likely to happen with the pop-up steps than in “Making 10”. The steps materialize and reveal the imaginary rectangles in “Making 10” due to the 3-D context of popups. I was able to find the imaginary rectangles in “Making 10” by as a result of knowing about the rectangles formed by the steps.

Extending this idea and applying the constraint of a constant perimeter led to the idea of optimization (see Table 2 & Figure 33 in chapter 4), which has mathematical and pedagogical effects.

Mathematically, applying the constraint of a constant perimeter leads to an investigation of optimization. In this case, students have the opportunity to discover a new big math context, in which they find the maximum area of a rectangle where the perimeter is constant. Pedagogically, this can be made more tangible for students by relating it to a problem of enclosing the largast rectangular area possible with a given (say 20 metres) length of fence.

Mathematically and pedagogically, there is also the opportunity for the teacher to ask students to relate or constrain the rectangles in a new way, such as making the area constant. Would this also lead to an optimization problem? Would the optimization involve a maximum again? How would the graph be similar or different? Why?

There is also an opportunity here to revisit the role of constaints in the mathmatics so far, and in mathematics more genarally:

- First constraint: x + y = constant
- Second constraint: perimeter = constant
- Third constraint: area = constant

5.3.4 Seeing the pop-up steps as a physical sum operation

Unlike the other “line” I saw in seeing the steps as shadows, which was diagonal, in this ‘seeing-as’ experience, the line that appeared by folding the pop-up steps was horizontal, and it represents the value of the constraint. Pedagogically, this surprising discovery may lead to further attention on the constraint of the sum being 7, and its importance in creating mathematical patterns.

Mathematically, the folding pop-up steps can be the new representation of sum operations for students.

By extending this experience, seeing the pop-up steps as physical sum operation, the shadow of 45 degrees light rays of the pop-up steps made the line exactly where the steps fall off by folding (see Figure 39).

Pedagogically, the shadows played important roles in my aesthetic mathematical experience and I experienced the surprise of “they line up” twice from them. In both surprises, the shadows aligned differently and through different processes. In the second “seeing-as” experience, the shadows made a diagonal line on the side plane while in the extension of the fourth “seeing-as” experience, the shadows made a straight line on the planes of the steps. What is noteworthy is that in both cases, because all the steps have the same total length, the shadows formed the “line” patterns similar to the “Making 10” activity.

5.3.5 Seeing the z axis as time

Pedagogically, firstly by seeing the z axis as time, pop-up steps and its shadow can be considred one representation in which students witness the process of how a mathematical graph is developed. Secondly, this idea can be a starting point for more investigation. Students can be asked to find which properties the graphs should have to make it possible to be developed by the pop-up steps. In this discusssion, there is a potential for understanding the differences between types of functions and mathematical relations.

Mathematically, by seeing the z axis as time, it is possible to illustrate that the points constitute the graph of some mathematical relations by the pop-up steps over the z axis. In this case, seeing the pop-up steps from the side or their shadow displays the whole graph as the finished product. This discussion is still open and more investigation is needed to find out what kind of mathematical graphs can be developed through the pop-up steps.

As discussed in Chapter 4, my experience in trying to see the 2-D “Making 10” activity as a quasi-3D pop-up representation also led me to (1) discoveries related to the mathematics of lines in 3-D space, (2) seeing the pop-up steps as rectangles and making connections to another surprising 2-D activity related to optimization in the context of area and perimeter relationships, (3) seeing the pop-up steps as a physical sum operations, and (4) seeing the z axis as representing the dimension of time.

In summary, by making the “Making 10” activity in the pop-up context, I transferred the activity from 2-D to 3-D. Due to the inherent dynamic geometrical property of pop-ups, the 3-D world of pop-ups is not static and can be transformed to 2-D by folding. As a result, 2-D and 3-D worlds are combined and the potential of seeing things in multiple ways can be provided. Seeing the 2-D versions of 3-D steps (by different methods and perspectives) led me to make artistic, mathematical, and pedagogical discoveries. First, the 3-D pop-up steps can be projected in different planes. For example, the steps are seen as a diagonal pattern of shadows when they are viewed from the side (see Figure 27).

Similarly, the steps can be seen as a horizontal line of shadows in the planes of the steps with the 45-degree light (see Figure 39). In this conversion, the steps are not transformed themselves, but their shadows of projections are the result of different transformations. Second, the steps can be seen as rectangles whose perimeters are the same. The rectangles are visible when the steps are seen from the side (see Figure 31). Pop-ups work like a hinge that can fold the pages and turn 3-D to 2-D and vice versa. This property combines two horizontal and vertical sides of pop-ups and makes it possible to see the folding action as the sum operation. In general, all discoveries other than the last one (seeing the z axis as time) happened by seeing the 3-D steps in 2-D with different perspectives, with the underpinning reasons of all of them being the math constraint of $a + b = 7$. In the last discovery, the inherent physical properties of pop-ups and comparing two representations in “Making 10” and the pop-up steps, led to see the z axis to be seen as time.

5.4 Surprise and insight

This aesthetic mathematical experience, the pop-up math story that entails several math surprises and insights, was originally developed by my two sources of surprise: the surprise of the “Making 10” activity (as the aim) and the surprise of the pop-up structure (as the tools for reaching the aim), which was always magical to me. These two different surprises diverged in the pop-up mathematical story, Sara and Christmas Tree, and resonated with one another and created more surprises. Pop-ups added new surprises/insights to the “Making 10” activity and the “Making 10” activity added more surprises/insights to the pop-ups.

Five insightful surprises were emereged unexprctedly:

The surprise of finding the 2-D projection of 3-D pop-up steps line up from the side view

The surprise of finding that the rectangles made by the steps have specific properties

The surprise of finding the linear pattern of folded pages

The surprise of fining the linear pattern of shadows that resulted from the 45-degree light ray

The surprise of seeing the z axis as time

The first surprise was different from other surprises and it happened while I was trying to find the expected surprise of the "Making 10" activity. I was stuck, confused and frustrated. It was not the case for other surprises, which were illuminated without me trying to find the solution. However, all surprises were accompanied with mathematical insight and beauty, and offered me a new and profound understanding of the specific math ideas. In mathematics, "surprise can emerge in the "spontaneous appreciation of beauty and elegance" (Sinclair & Watson, 2001 p. 39).

Pedagogically, surprise can have a positive effect on mathematics education. To understand the underpinning reasons of surprises in math, the questions of "why" and "how" can be raised. Having students being exposed to the surprise, which is likely to encourage them to explore and understand "why or how", can be considered as an effective pedagogy in mathematics education. As Egan (1997) stated: "Without the initial wonder, it is hard to see how more systematic theoretical inquiry can get fruitfully under way" (p. 97). Likewise, Adhami (2007) said that by exposing students to surprising math content, they hopefully step back to "look again" (p. 34) to understand the reasons behind the math content.

My pop-up construction was focused on surprise and insight from the start, as I tried to replicate the surprising and insightful "Making 10" activity in a pop-up context. I doubt that I would have made the discoveries I did without this focus. I do not think my experience or my pop-up story would have been as aesthetically pleasing without this constant reminder to focus on surprise and insight.

5.5 Immersion

I was involved with the pop-up math story for a considerable time, which allowed me to immerse myself in this experience. I started writing and designing the story in the first semester of 2017. I did not finish it at that time, and it became a continuous process in my graduate work. That is, in the following semesters, I chose my essays about the story (based on my supervisor's advice) to analyse it with different perspectives. In the other course, I presented the story with audio through Voice thread software and again I wrote about it through an educator's perspective. Then, I chose to analyse the process of creating the pop-up story as my thesis subject. Therefore, I had more chances to be exposed to the story and the last two discoveries, that folded pop-up steps make the horizontal line pattern and seeing the z axis as the axis of time, happened while I had already started writing the thesis. Spending this amount of time (from the first semester until recently) on the pop-up math story provided me with opportunities to gradually gain insights and new visions about the mathematical content of the story.

Spending a lot of time with the pop-up story was appealing to me. My topic was one that I chose and one that I was passionate about. I had agency to choose the subject of designing a pop-up mathematical story as my assignment in the first semester. Dr. Gadanidis, my instructor in the course, approved this project and allowed me to choose writing and designing the pop-up math story, which was in accordance with my interest and strength. I immersed myself in what I had developed based on my passion.

If students have freedom to choose their assignment subjects, they might have more opportunities to apply their talents and strengths, and they are likely to be more involved and stay motivated to accomplish their projects. Students are more likely to immerse themselves with the projects that entail their interests. In this aspect, my experience is different from Egan's idea that topics are to be assigned to students.

As discussed in chapter 4, I experienced immersion in the following ways: (1) Immersion predisposed me to pay more attention to some ambiguities and provided me the chances to delve into them, think hard and find the solution, (2) my immersion process intertwined with persistent work, trying different methods and talking with others, (3) due to my pop-up-immersed mind, I discovered new mathematics in the story by chance and I found connections between mathematical ideas and the pop-up steps, and (4) immersing myself with pop-ups as a tool influenced my thinking and led me to new conceptual understandings of some math ideas. I exploited the affordances of the pop-up structure and it reorganised my math knowledge construction.

5.5.1 Examples of immersion with thinking hard/attention manifestations

The source of the first artistic-mathematical discovery, in which the diagonal pattern was found from the side view, was paying attention to the discrepancy between the "Making 10" activity and the pop-up steps. It was a state of confusion and I needed to think hard, concentrate and try different ways to find the expected surprise. Eventually, I found the "line" pattern where I had not expected to and became surprised. It gave me insight into the complex math of the 2-D projection of 3-D volumes. Although the initial math of the story was the linear function of "Making 10", by thinking hard I got the insight of the complex math of projection. Immersion offered me the opportunity to pay more attention. Gadanidis (2004) saw attention as a tool to gain beauty and insight in mathematics.

Being involved with the math content of the story does not need any previous math knowledge (low floor), and it has the potential to introduce the advanced math concepts of projection (high ceiling). If the story were not deep and enriched enough, thinking hard would not have been needed to make the artistic-mathematical discoveries that brought joy, surprise and insight. The pop-up story provided me the scaffolding to think hard and to gain the surprise of exposing the advanced math in a concrete way.

Immersion provides opportunities for students to think hard to gain the knowledge that is obtained by intellectual challenges and solve it by thinking hard. Egan (2010) states: "Learning in depth will involve each student in intensive and extensive exploration, classification, analysis, and experiments, but it will also face them with more than purely intellectual challenges" (p. 210).

5.5.2 Examples of immersion with persistent work manifestation

When I understood that in spite of applying the same math constraint in the "Making 10"activity in constructing the pop-up steps, I could not reach the expected surprise, I tested different methods to find the reason. In that time, I was not aware that I chose different planes from the ones in the "“Making 10”" activity and it caused me to not see the linear pattern of the pop-up steps.

I tried different ways to find the cause of the discrepancy between "“Making 10” and the pop-up steps. First, I compared the pop-up steps with the “Making 10” activity visually. To do so, I located the steps next to the plot of “Making 10”. Then I had conversations with my sister about the underlying mathematics of pop-up steps and “Making 10”. Then, I wrote what we had discussed. In the next step, I tried to make the pop-up steps look similar to the points in "“Making 10”by labelling each step with one point. In the illumination moment, I suddenly noticed an invisible plane (from the side view) that was perpendicular to the two planes where the steps are located. By immersing myself in the activity and by persisting over a long period of time, I noticed the projection of the pop-up steps lining up on the plane and understood why the steps do not line up themselves.

According to Liljedahl (2009), discovery and creativity are highly associated with effort and determination. Here is the comment of Henry McKean in the Liljedahl’s (2009) research:

> I don’t believe that any true progress arises spontaneously. I believe it is always the result of lots of hard work, covert or overt, with the understanding that old work will sometimes come into a new focus so that you get something, if not for free, then at no extra cost. Such ‘‘inspiration’’ is the outcome of covert work and so can be surprising, but the work has to have been done, even if invisibly (p. 59).

5.5.3 Immersion examples with serendipity manifestation

I could find the second discovery about the rectangles of the pop-up steps by the external chance, when I came across the math activity about the relationship between areas and perimeters of rectangles. It triggered my mind and drew my attention back to the pop-up steps with a new lens. I found out that the rectangles formed by the pop-up steps have the same perimeter and different areas. Due to my mind being immersed with the pop-up math story, when I came across the different activity, I found the connection between them. Liljedahl (2009) stated that the extrinsic chance happens if mathematicians obtain an idea that inspires them from external sources.

The third discovery, that pop-up steps make the horizontal linear pattern when they are folded, was made by intrinsic chance when I was thinking about the impact of folding on the pop-up cards in general. Initially, the pop-up steps were not on my mind at all but suddenly I thought about what would happen to the steps if they were folded. Then I tried it out and saw that the steps aligned surprisingly. This discovery, which happened by thinking about the effect of folding on pop-ups, can be connected to the second type of chance, intrinsic chance. Intrinsic chance can occur by mixing different ideas properly and making new and interesting ones or ideas coming to mind that lead to insight. The source of intrinsic chance is internal (Liljedahl, 2009).

The two artistic-mathematical discoveries happened by chance because my mind was immersed with the pop-up-math story; therefore, when I came across different ideas by extrinsic or intrinsic sources, my mind made the connection between them and the pop-up steps. While the mind is immersed in one subject, it would be more potential to find other ideas which can be possibly relevant.

5.5.4 Immersion with tools

By seeing the “Making 10” activity as the pop-up steps, I exploited the affordances of the pop-up structure: the 3-D and 2-D convertible feature, movement (folding and opening), creating 3-D shapes and displaying the points like the events over time. These affordances created new mathematical ideas for me: the relations between 2-D and 3D plotting, relation between area and perimeter of rectangles made by the pop-up steps, a physical presentation of the sum operation and seeing the z axis as the time. All this math content emerged in surprising ways and experiencing it in a pop-up context provided me with new math insights and perspectives. Considering pop-ups as one type of tool, it is worth mentioning Lévy’s idea of technology. According to Lévy (1997), by communicating with tools, humans can learn and obtain new methods of thinking and “skills”. In other words, tools can play a role as an actor conveying meaning to humans. Gadanidis and

Geiger (2010) stated that when we involve ourselves thoroughly with using technology, its affordances and constraints influence how we think. They state that "when we immerse ourselves in using a technology (and this immersion is a critical component), we naturally think *with* that technology" (p. 95). The affordance of the pop-up structure led me to rethink the math ideas and influenced my math knowledge constructions.

All the examples I explained were the different manifestations of immersion and its positive impacts on my process of designing the aesthetic mathematical experience. Egan (2010), by introducing a new educational method of learning in depth, highlighted the importance of immersion in profound learning.

One of the criticisms of learning in depth is that it can cause boredom (Egan, 2010, p. 30). This is what I thought too. I wanted to write new stories, but my supervisor led me to come back to my first story and write about it in different assignments. As a result, I had more chances to delve into the content of the story and experience more interesting mathematics. Egan stated "boredom is a product of ignorance, not of knowledge" (p. 14). Unexpectedly, the more I immersed myself in the story, the more knowledge and joy from learning I gained.

The more time I spent with the Sara and Christmas Tree story, the more discoveries, surprises, and insightful moments I experienced. The first artistic-mathematical discovery happened when I came back to the story to write about it through an educational lens. It brought the missing point, the expected surprise of the "Making 10" activity, into the forefront and caused me to delve into the mathematical process to find the lining up pattern where I had not expected. The second artistic-mathematical discovery was revealed when I returned to the story to write the next assignment. This time I could see rectangles with the same perimeter and different areas. Unexpectedly, the third discovery, folding the pop-up steps, happened when I was writing my first section of my thesis. I thought I had already found everything surprising about the story and they are all ready to be analysed. About three months ago, during the discussion with a friend of mine, the last discovery (seeing the z axis as the time axis) emerged from our conversation. The more I immersed with the story (in different ways), the more discoveries happened. I have never had this feeling of math discovery when I was in school. Likewise, I did not have a chance to immerse in one subject. Egan (2011) describes schools: “The condition that can really engage their interest in topics is too rarely realized in classes where teachers have to keep moving across the surface level of mandated curricula just to ensure 'coverage.'” (p. 14). In school, I had never spent significant time to delve into one subject and could not experience the joy of genuine learning and knowledge. The more I spent time thinking deeply about the story, the more I wondered about mathematic concepts hidden in the story.

5.6 Audience

My sister: In sharing the first discovery with my sister, I explained how the points assigned by the pop-up steps do line up from the side view. We discussed how I could lead the reader to see what I was seeing from the side. In that discussion, I came to realize that the shadow of the pop-up steps is exactly the 2-D projection of the 3-D steps which line up on the side plane.

My supervisor: In having conversations with my supervisor about how all the rectangles made by the pop-up steps had the same perimeter and different areas, I realised that it is possible to have the maximal area while all perimeters are the same. This understanding led me to think about the math idea of optimization.

I shared the third discovery with my supervisor and showed him how the pop-up steps fall off at the same row by folding. In our conversation, I explained to him how the vertical and horizontal sides of the pop-up steps are added to one another. This conversation helped me to see the folding the pop-up steps as the physical sum operation.

My friend: When I talked with my friend about how the 2-D projection of 3-D pop-up steps line up from the side view, we discussed more differences that these two representations (the points on the grid and the points assigned by the pop-up steps) would have. In this discussion, I realised that the z coordinate can indicate the time factor, and the pop-up steps are ordered in the time sequence like the chain of events.

My general audience: Providing the pleasure of surprise, specifically the surprise of the "Making 10" activity for the audiences of my story, was my main motivation to find the expected surprise. Unexpectedly, I found new surprises, math connections and relationships, which were not part of the "Making 10" activity.

Pedagogically, considering audiences' expectations is important in designing aesthetic-mathematical experiences, which contain mathematical surprise and insight. When students become surprised in math activities, as Gadanidis and Borba (2008) stated, they would like to share the surprise with their friends and families who are not math experts. As I designed the pop-up story, I wanted to create a surprise for the audience, as surprise is part of what makes a good story. I tried to imagine if and how the audience might be surprised, and this thinking-with-audience affected my work.

My audiences (my sister, my supervisor and my friend) in the process of creating the mathematics experience were actively engaged and influenced the experience. Boal (1985), in the idea of 'poetics of the oppressed', demonstrates that audience can have authority to change the flow of the performance and effect from what they are exposed to, and this was the case for the audiences of my story. Mathematically, having an active audience to talk to can be helpful to bounce the ideas back and forth. In sharing ideas and brainstorming, there is potential to gain a deeper mathimatical understanding and become aware of different perspectives.

My sister, supervisor and a friend of mine were my audiences who were involved chronologically in the process of designing the pop-up math story and became the active audience who played roles in enriching all the mathematical discoveries I experienced. The conversation with my sister led to the idea of shadow and helped me to see the phenomena by a different perspective, therefore enriching my experience. The fruits of sharing the second discovery with my supervisor was the idea of optimization, which extended my findings to the bigger math ideas and the fruit of sharing the third discovery with my supervisor was seeing the folding action (of the pop-up steps) as a physical sum operation (as described in chapter 4), leading me to see it in a more functional way. The talking with my friends opened my eyes to the mathematical equations, and their graphs developed and showed me that pop-ups can be used to graph the mathematical equation and led me to think that which kinds of mathematical equations can be graphed in the pop-up context.

5.7 Conclusion

My research question was:

What factors come into play when an educator immerses herself in the process of designing an aesthetic mathematical experience?

The result of investigation in this research identified five factors that enhanced the aesthetic nature of my experience and of my pop-up story: surprise/insight, constructing, immersion, 'seeing-as', and audience.

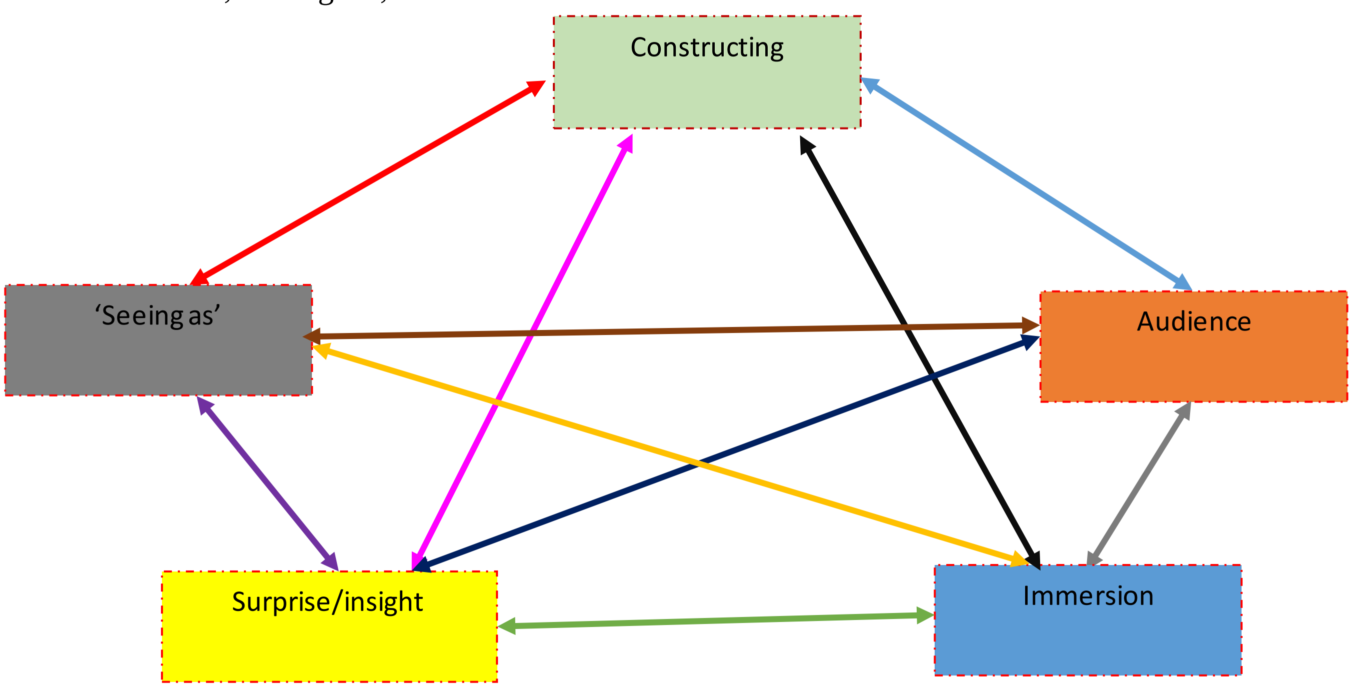


Figure 45: The model of designing of aesthetic mathematical experience of this research

As it is displayed in the Figure 45, these factors are interconnected and can influence one another.

In the following, I explain each connection of the pair of factors, which are indicated by the colorful two-sided arrows.

‘Seeing-as’/Constructing (The red arrow)

Seeing the 2-D (the points on the grid) in 3-D (the pop-up steps) influenced the construction of the story.

Constructing the physical embodiment of the mathematical experience influenced me to experience ‘seeing as’.

Constructing/Audience (The blue arrow)

The expectation of audience affected my construction of the mathematical experience.

Constructing the physical embodiment of the mathematical experience entertained my audiences and motivated them to participate actively.

Audience/Immersion (The grey arrow)

Considering the audiences’ expectations and their engagement in the process had a positive impact on the immersion factor.

Talking with others (audiences) and sharing ideas are part of my immersion experience.

Immersion/Surprise and insight (The green arrow)

Immersion provided the opportunities to find mathematical surprise and insight.

Each surprise encouraged me to immerse myself more with the pop-up story to discover additional surprises.

Surprise/‘Seeing-as’ (The purple arrow)

‘Seeing-as’ led me to find surprises.

Surprises in the second and third discoveries led me to see the pop-up steps as rectangles and physical sum operations.

Constructing/Surprise (The pink arrow)

Constructing the “Making 10” activity in the pop-up context created new and unexpected surprises.

Presenting the surprise of the “Making 10” activity was the reason to construct the pop-up story.

Constructing/Immersion (The black arrow)

Constructing with the ‘gears’ (Papert, 1980) of popups was my passion, therefore I wanted to immerse myself.

Prior immersion with the pop-up structure predisposed me to see the connection between the “Making 10” activity and the pop-up structure and led me to construct the pop-up math story.

Immersion/‘Seeing-as’ (The orange arrow)

Prior immersion with the pop-up structure predisposed me to see the points on the grid as the pop-up steps.

Seeing the points on the grid as the pop-up steps provided an entertaining mathematical experience in which I would like to be immersed.

‘Seeing-as’/Audience (The brown arrow)

See the points on the grid as the pop-up step provided an entertaining mathematical experience that was engaging to the audience and encouraged them to get involved.

The audiences’ ideas enriched my ‘seeing-as’ experience and led to more ‘seeing-as’ experiences.

Audience/Surprise and insight (The navy arrow)

> The audiences' ideas enriched the mathematical discoveries and led me to find new mathematical surprises and insights.
>
> Sharing the mathematical surprise and insight I found with my audiences, encouraged them to participate actively in my discussions. Furthermore, considering surprise as an appealing element for audience motivated me to find the expected surprise of the "Making 10" activity.

This model can be compared with the model for mathematics education reform developed by Gadanidis, Borba, Hughes and Lacerda (2016) in designing aesthetic experiences for young mathematicians. The aesthetic elements of the model, which include "1. Surprise and insight, 2. Vicarious emotional engagement, 3. Visceral sensations" (p. 226, ibid) are embedded in the pop-up math story. Surprise and insight are the main elements in all mathematical discoveries and when I shared them with my friends, family and professional audiences, I could see how they engaged and were surprised and excited, and I enjoyed this vicariously. In addition, I witnessed the mathematical beauty attributed to the visceral sensation (Gadanidis et al., 2016). The mathematical elements of the model include "thinking hard" and "low floor, high ceiling" (p.226), which are both in the immersion element of the model of this chapter. Due to the "low floor, high ceiling" property of the math in the pop-up story, I could find advanced math relationships. The implementation element includes "covering the curriculum" and "flexible implementation" (p. 226), which are consistent with the math ideas of the pop-up story. Finding the missing numbers, linear function, 2-D projection of 3-D shapes (which can be connected to matrices), and area and perimeter and their relations are compatible with curriculum and can be implemented in different ways based on the teachers' experiences and knowledge.

In general, the elements of the Gadanidis et al. model for mathematics education reform explain the properties of the aesthetics experience. My thesis model adds to this work, by thoroughly studying a teacher's experience of creating opportunities for math surprises and insight. Its main contribution is the in-depth study of the experience and identification of the actions that educators need to take to when designing an aesthetic-mathematical experience. The factors, such as immersion, constructing, and 'seeing as', in my thesis model focus on what actions in educators' experiences can equip them to successfully create a surprising, insightful mathematical experience.

Chapter 6. Implications and future investigations

6.1 Introduction

In this research, I investigated the process of how I created the aesthetic-mathematical experience as a pop-up math story, Sara and Christmas Tree. The specific affordances of pop-ups showed new understandings of some math knowledge and offered the intellectual intersections between domains such as: linear function, projection and light/shadow (physics); optimization and linear function (through 3-D geometrical shapes); sum operation, linear function, spatial movement and light/shadow; and time and space. These features make the experience profound, rich and deep. The aim of this research was to help other educators create aesthetic math experiences, like the pop-up math story, that are worth sharing and offer math surprise and insight. In this study, I tried to identify factors that determined the creation of mathematics experiences. In doing so, I analysed the path I went through and used narrative inquiry focusing on turning points and critical events to identify the actors that played significant roles and influenced me to create the new mathematics experience. Based on the literature review and data analysis, five factors have been drawn: Surprise and insight, constructing, immersion, seeing-as and audience. Due to the limitation of this thesis, it needs to be acknowledged that the conclusion of this research is not definitive and further research is needed to validate the outcome of this study.

6.2 Implications

1. This research began with creating the pop-up math story. Although I had experience in pop-up making, I did not know that the structure of the pop-ups can be linked to mathematical ideas. In this process, I was exposed to different math concepts in profound and tangible ways. When I shared them as the poster in two conferences, it grabbed the audience's attention and they experienced math surprise and insight in pop-ups, which was a new representation that they had not seen before. They discussed it with one another, interpreted it in different ways and even tried to make it on their own.

From my point of view, what happened mathematically in designing the pop-up story, Sara and Christmas Tree, was beyond seeing it as a singular experience that occurred once. In other words, the mathematical ideas that emerged in the pop-up experience have been deep and intellectual enough to consider the pop-up context as the new representation or manipulative in mathematics education. The inherent pop-up structures can be linked to different math ideas and pop-ups can allow new understandings of old math knowledge. Although this claim requires further research, the quantity and quality of the math conceptual understandings, surprises and insights illuminated in this singular experience make the pop-ups a legitimate consideration as a new strand that is worth trying to design new mathematics experiences in the pop-up context.

2. Offering new hands-on activities in the contemporary lifestyle where children are immersed with technology and digital-based activities can show new understandings and visions about math ideas. To clarify, I should acknowledge that the illumination moments I experienced in the pop-up math story, which was developed in the concrete and physical version, were less likely to happen in the digital-based activity, such as on the computer. Papert (1980) gave the label “transitional objects” to the gears, which illustrated the potential to transfer abstract math ideas through “sensory” objects. As I mentioned before, pop-ups, by its specific affordances, introduced the intersection between different math ideas, the abstract math ideas and physics, and math ideas and tangible objects, and played the role as the “transitional objects” like the real meaning of the term. Both ‘gears’ and pop-ups were hands-on activities that were capable of intertwining concepts in different domains. From my point of view, hands-on activities have the significant potential to converge ideas from different domains and offer insight and deep understandings due to their physical property.

3. Presenting the math discoveries occurred in designing, the Sara and Christmas Tree story can be an engaging math performance for students in different grades. The mathematical content in the story is quite broad, covers the curriculum and, due to the low floor/high ceiling approach, students in elementary and high school can get involved differently. In the related activities they would have opportunities to explore potential

mathematical discoveries and find the connection between different math ideas in the pop-up context.

6.3 Recommendations for future investigation

Practical

1. Based on the discoveries that occurred in the story, related hands-on activities and lesson plans can be designed. Students can make the pop-up steps by themselves and they can explore and find different patterns. Students can be encouraged to do related outdoor activities, like taking photos of different objects to see the shadow and find new patterns.

2. These activities and the story can be tried in different classes and, based on how students and teachers engage and interact, the quality of activities can be modified, improved and evolved.

3. The related computer software can be linked to the pop-up story and the related hands-on activity. Students can see different presentations of math concepts and see the advantage of the affordances of both hands-on and digital-based activities.

Research

1. As it is mentioned in the limitation of the research, some issues have not been investigated in this study, which could have impacted the conclusion. The issues like my identity and background, future perspective, culture, the environment and the situation I was living in while designing the mathematics experience was not tackled in this study. In future research, these factors can be taken into consideration.

2. Further investigation is recommended to find the mathematics ideas that can be implemented in the pop-up context to design new pop-up aesthetic-mathematical experiences.

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