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DEPENDENCE MODELLING OF EXTREME EVENTS WITH APPLICATIONS IN FINANCE AND INSURANCE

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DEPENDENCE MODELLING OF EXTREME EVENTS WITH APPLICATIONS IN FINANCE AND INSURANCE

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Abstract

During the past few decades there has been an extensive amount of work involving the modelling of extreme events. A reasonably accurate estimate of the probabilities associated with these events contributes to a good understanding of the risk taken. Extreme Value Theory provides powerful tools to aid in investigating this risk.

In applications involving more than one random variable of interest, it is necessary to understand the extremal behaviour of the dependence structure as well as the extremal behaviour of the marginal distributions. In this thesis, the focus is on explaining the extremal dependence structure and its impact on financial and actuarial applications.

The first two contributions of this thesis are mainly focused on the extreme behaviour of two commonly used classes of multivariate distributions in finance and insurance, namely phase-type and elliptical. In the phase-type case, we examine the limiting distributions of the componentwise maxima and minima, while asymptotic results are obtained for joint threshold exceedance probabilities in the elliptical case.

The next two contributions present asymptotic results for large claims reinsurance. Specifically, we focus on ECOMOR (excédent du coût moyen relatif) and LCR (largest claims reinsurance). We provide asymptotic tail probabilities that can be used to estimate certain risk measures, such as the Value-at-Risk. Two specific models are investigated; one represents the total claims under a claims process for which each claim amount depends on the time since the previous claim and the other represents the total claims under n dependent insurance contracts.

Keywords: *Archimedean copula, Componentwise maxima, Dependence, ECOMOR and LCR reinsurance, Elliptical distribution, Extreme Value Theory, Long-tailed distribution, Marshall-Olkin distribution, Multivariate extreme value distribution, Pickands' representation, Regular variation, Tail probability, Threshold exceedances.*

Co-Authorship Statement

Each of the chapters in this thesis are my own original ideas and work, and have been accepted or submitted for publication. The works have been obtained though collaborations with my supervisor, Dr. Bruce Jones.

Chapters 2 and 3 have been published in *Insurance: Economics and Mathematics*, while the remaining two chapters have been submitted for publication. Each work is joint with Dr. Bruce Jones.

Dedicated to my Grandmother, Agafia.

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Chapter 1

Introduction

Special attention has recently been given to quantifying events with a small occurrence chance. Appropriate evaluations of the probabilities associated with these events are useful in measuring the impact of the "worst case scenario." Extreme Value Theory (EVT) provides a powerful set of tools for approximating these probabilities. Standard references on EVT with various applications including the practical aspects of estimation, as well as theoretical development are discussed in Beirlant (2004), Coles (2004), Embrechts et al. (1997), Kotz and Nadarajah (2000), McNeil et al. (2005) and Resnick (1987).

A key ingredient of this thesis is its focus on extremal dependence structures. In many financial and actuarial applications there are indications that the impact of dependence is not negligible. There is a growing body of research that addresses how to quantify this dependence. Albrecher and Teugels (2006), Bäuerle and Müller (1998), Boudreault et al. (2006), Dupuis and Jones (2006), and Tang and Vernic (2007) discuss actuarial applications for which appropriate assumptions regarding dependence are required.

Chapter 2 investigates the limiting distributions of the componentwise maxima and minima of suitably normalized independent and identically distributed (iid) multivariate phase-type random vectors. In the case of maxima, a large parametric class of multivariate extreme value distributions is obtained. The flexibility of this new class is exemplified in the bivariate setup.

In Chapter 3, we exploit a stochastic representation of bivariate elliptical distributions in order to obtain asymptotic results which are determined by the tail behaviour of the generator. Under certain specified assumptions, we present the limiting distribution of componentwise maxima, the limiting upper copula, and a bivariate version of the classical peaks over threshold result.

We consider in Chapter 4 an extension of the classical compound Poisson risk model, where the waiting time between two consecutive claims and the forthcoming claim are no longer independent. Asymptotic tail probabilities are obtained for the reinsurance amount under two reinsurance treaties, ECOMOR and LCR. These reinsurance arrangements pay amounts that are based on the upper order statistics of the claims. Simulation results are provided in order to illustrate this.

Finally, in Chapter 5, we consider a dependent portfolio of insurance contracts. Asymptotic tail probabilities of the ECOMOR and LCR reinsurance amounts are obtained under certain assumptions about the extremal dependence structure. dependence structure.

References

Albrecher, H. and Teugels, J.L. 2006. "Exponential Behavior in the Presence of Dependence in Risk Theory," Journal of Applied Probability, 43(1), 257-273.

Bäuerle, N. and Müller, A. 1998. "Modelling and Comparing Dependencies in Multivariate Risk Portfolios," *Astin Bulletin*, 28(1), 59-76.

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. 2004. *Statistics of Extremes: Theory and Applications.* Wiley, Chichester.

Boudreault, M., Cossette, H., Landriault, D. and Marceau, E. 2006. "On a Risk Model with Dependence between Interclaim Arrivals and Claim Sizes," Scandinavian Actuarial Journal, 5, 265-285.

Coles, S. 2004. *An Introduction to Statistical Modeling of Extreme Values.* Springer Series in Statistics, Springer-Verlag, London.

Dupuis, D.J. and Jones, B.L. 2006. "Multivariate Extreme Value Theory and its Usefulness in Understanding Risk,"*North American Actuarial Journal*, 10(4), 1-27.

Embrechts, P., Klüppelberg, C. and Mikosch, T. 1997. *Modelling Extremal Events for Insurance and Finance.* Springer-Verlag, Berlin.

Kotz, S. and Nadarajah, S. 2000. *Extreme Value Distributions: Theory and Applications.* Imperial College Press, London.

McNeil, A.J., Frey, R. and Embrechts, P. 2005. *Quantitative Risk Management: Concepts, Techniques and Tools.* Princeton University Press, Princeton.

Resnick, S.I. 1987. *Extreme Values, Regular Variation and Point Processes.* Springer-

Verlag, New York.

Tang, Q. and Vernic, R. 2007. "The Impact on Ruin Probabilities of the Association Structure among Financial Risks," Statistical and Probability Letters, 77(14), 1522-1525.

Chapter 2

Extreme Behaviour of Multivariate Phase-type Distributions

2.1 Introduction

Extreme value theory has received increasing attention in the actuarial literature in recent years. The severe financial implications of extreme events justify the need for such quantitative tools. Since many insurance portfolios include several (or many) dependent risks, multivariate extreme value theory is needed to properly quantify the overall risk.

The limiting distribution of the normalized componentwise maxima (minima) of a sequence of iid random vectors is a fundamental and thoroughly studied topic in the area of multivariate extreme value theory. The possible limit distributions are known as max (min) multivariate extreme value (MEV) distributions. One of the key features of these distribution functions (df) is that they cannot be specified in terms of a function involving a finite number of parameters (see Beirlant *et al.*, 2004 chapter

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8). A number of parametric families of multivariate extreme value distributions have been discussed in the literature. However, none is sufficiently broad to widely cover the entire class, and most simple families are quite restricted in their behaviour.

In this chapter, we establish the limit distribution for the normalized componentwise maxima and minima of a sequence of random vectors with multivariate phase-type (MPH) distributions. Introduced by Assaf *et al.* (1984), multivariate phase-type random vectors can be viewed as representing the times until absorption into overlapping non-empty subsets of the state space of a finite-state continuous-time Markov chain. MPH distributions have been used in reliability theory (see Assaf *et al.*, 1984), queueing theory (see Li and Xu, 2000) and ruin theory (see Cai and Li, 2005a).

The collection of limiting distributions forms a rich subclass of the max extreme value distributions. We provide some examples of bivariate phase-type distributions and explore the behaviour of the Pickands' function corresponding to the limiting distribution of componentwise maxima.

In Section 2.2, we present some preliminaries on MEV distributions and establish some of the notation that will be used throughout the chapter. This is continued in Section 2.3 where we discuss the basics of univariate phase-type distributions, including the limiting distributions of normalized maxima and minima. Section 2.4 introduces the multivariate phase-type distribution and the bivariate special case, and gives the main results of the chapter - the limiting distributions of normalized componentwise maxima and minima along with the norming constants. Some examples illustrating the flexibility of this class of distributions are provided in Section 2.5.

Conclusions are given in Section 2.6.

2.2 Preliminaries

Let $\mathbf{X}^{(1)} = (X_1^{(1)}, \ldots, X_p^{(1)})$, $\mathbf{X}^{(2)} = (X_1^{(2)}, \ldots, X_p^{(2)}), \ldots$ be a sequence of independent p-dimensional random vectors with common distribution F, and let $\mathbf{U}^{(n)}$ be a random vector with jth component

$$U_j^{(n)} = \max(X_j^{(i)},\, i = 1, \ldots n).$$

That is, $\mathbf{U}^{(n)}$ is the vector of componentwise maxima of $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n)}$. If there exist sequences of vectors of constants $\mathbf{a}^{(n)}$, $\mathbf{b}^{(n)} \in \Re^p$ and a random vector $\mathbf{U}$ with distribution G and nondegenerate marginals such that $\mathbf{a}^{(n)}\mathbf{U}^{(n)}+\mathbf{b}^{(n)}$ converges weakly to $\mathbf{U}$, then G, the limit distribution of normalized componentwise maxima, is said to be a *max extreme value distribution.* We then say that F is in the *max domain of attraction* of G with *normalizing vectors of constants* $\mathbf{a}^{(n)}$ and $\mathbf{b}^{(n)}$ and write $F \in \text{MaxDA}(G)$. Then

$$\lim_{n \to \infty} F^n(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)}) = G(\mathbf{x}), \text{ for all } \mathbf{x} \in \Re^p, \tag{2.2.1}$$

and it follows that there exist sequences of vectors of constants $\boldsymbol{\alpha}^{(n)}$, $\boldsymbol{\beta}^{(n)} \in \Re^p$ such that

$$G^n(\boldsymbol{\alpha}^{(n)}\mathbf{x} + \boldsymbol{\beta}^{(n)}) = G(\mathbf{x}), \text{ for all } \mathbf{x} \in \Re^p. \tag{2.2.2}$$

Distributions G satisfying (2.2.2) are said to be *max stable.* The class of multivariate distributions having this property is exactly the class of max extreme value distributions (see Resnick, 1987 chapter 5).

From (2.2.1), it follows that

$$\lim_{n\to\infty} n[1 - F(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)})] = -\log G(\mathbf{x}), \tag{2.2.3}$$

for all $\mathbf{x}$ such that $G(\mathbf{x}) > 0$. This relation is useful in verifying the limit distribution of the normalized componentwise maxima.

Analogous to $\mathbf{U}^{(n)}$, define $\mathbf{L}^{(n)}$ to be the vector of componentwise minima of $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n)}$. That is, $\mathbf{L}^{(n)}$ is the random vector with jth component

$$L_j^{(n)} = \min(X_j^{(i)},\ i = 1, \ldots, n).$$

If there exist sequences of vectors of constants $\mathbf{a}^{(n)}$, $\mathbf{b}^{(n)} \in \Re^p$ and a random vector $\mathbf{L}$ with distribution G and nondegenerate marginals such that $\mathbf{a}^{(n)}\mathbf{L}^{(n)} + \mathbf{b}^{(n)}$ converges weakly to $\mathbf{L}$, then G, the limit distribution of normalized componentwise minima, is said to be a *min extreme value distribution*, and F is said to be in the *min domain of attraction* of G. We write $F \in \text{MinDA}(G)$. Then

$$\lim_{n\to\infty} \bar{F}^n(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)}) = \bar{G}(\mathbf{x}), \text{ for all } \mathbf{x} \in \Re^p, \tag{2.2.4}$$

where for any distribution H of random variables $Y_1,\ldots,Y_p$,

$$\bar{H}(\mathbf{y}) = Pr(Y_1 > y_1, \ldots, Y_p > y_p)$$

is the *joint survival function* of $Y_1,\ldots,Y_p$. It follows from (2.2.4) that there exist sequences of vectors of constants $\boldsymbol{\alpha}^{(n)}$, $\boldsymbol{\beta}^{(n)} \in \Re^p$ such that

$$\bar{G}^n(\boldsymbol{\alpha}^{(n)}\mathbf{x} + \boldsymbol{\beta}^{(n)}) = \bar{G}(\mathbf{x}), \text{ for all } \mathbf{x} \in \Re^p. \tag{2.2.5}$$

Distributions G satisfying (2.2.5) are said to be *min stable*. It is easily seen that a random vector $\mathbf{L}$ is min stable if and only if $-\mathbf{L}$ is max stable. The following relation, which follows from (2.2.4), is useful in verifying the limit distribution. We have

$$\lim_{n\to\infty} n[1-\bar{F}(\mathbf{a}^{(n)}\mathbf{x}+\mathbf{b}^{(n)})] = -\log \bar{G}(\mathbf{x}), \tag{2.2.6}$$

for all $\mathbf{x}$ such that $\bar{G}(\mathbf{x}) > 0$.

A characterization of max and min domains of attraction of multivariate extreme value distributions is given by Marshall and Olkin (1983).

Necessary conditions for (2.2.1) and (2.2.4) are that each marginal F_i of F is in the (univariate) MaxDA, respectively MinDA, of the corresponding marginal G_i of G. The following classical results concerning max and min extreme value distributions in the univariate case are provided by Gnedenko (1943). In particular, if $F_i \in \text{MaxDA}(G_i)$ for some non-degenerate df G_i, then G_i belongs to the type of one of the following three df's:

$$\begin{array}{lcll} \Phi_\alpha(x) & = & \exp(-x^{-\alpha}), \quad x > 0 \ (\alpha > 0) & \Rightarrow \ G_i \text{ is of Fréchet type} \\ \Psi_\alpha(x) & = & \exp(-(-x)^{\alpha}), \ x \le 0 \ (\alpha > 0) & \Rightarrow \ G_i \text{ is of Weibull type} \\ \Lambda(x) & = & \exp(-e^{-x}), \quad x \in \Re & \Rightarrow \ G_i \text{ is of Gumbel type} \end{array} \tag{2.2.7}$$

This is the well-known Fisher-Tippett theorem. Analogously, if $F_i \in \text{MinDA}(G_i)$ for some non-degenerate df G_i, then G_i belongs to the type of one of the following three df's:

$$\begin{array}{lcll} \Phi^*_\alpha(x) & = & 1-\exp(-(-x)^{-\alpha}), \ x < 0 \ (\alpha > 0) & \Rightarrow \ G_i \text{ is of type I} \\ \Psi^*_\alpha(x) & = & 1-\exp(-x^{\alpha}), \quad x \ge 0 \ (\alpha > 0) & \Rightarrow \ G_i \text{ is of type II} \\ \Lambda^*(x) & = & 1-\exp(-e^{x}), \quad x \in \Re & \Rightarrow \ G_i \text{ is of type III} \end{array} \tag{2.2.8}$$

The dependence structure of a multivariate distribution can be characterized in terms of the *copula*. A copula is a multivariate distribution function defined on the unit cube $[0,1]^p$ with uniformly distributed marginals. According to the well-known

Sklar's Theorem (see Sklar, 1959), if G is a joint distribution function with continuous marginals $G_1, \ldots, G_p$, then there exists a unique copula, C, given by

$$C(u_1, \ldots, u_p) = G(G_1^{\leftarrow}(u_1), \ldots, G_p^{\leftarrow}(u_p)), \tag{2.2.9}$$

where $h^{\leftarrow}(u) = \inf\{x : h(x) \geq u\}$ is the generalized inverse function. The *survival copula* of a multivariate distribution is given by

$$\hat{C}(u_1, \ldots, u_p) = \bar{G}(\bar{G}_1^{\leftarrow}(u_1), \ldots \bar{G}_p^{\leftarrow}(u_p)). \tag{2.2.10}$$

A more formal definition, properties and examples of copulas are given in Nelsen (1999). The dependence structure of max extreme value distributions can be expressed in terms of the copula. In the bivariate case, it has the form

$$C(u, v) = \exp\left\{\log(uv) A\left(\frac{\log u}{\log(uv)}\right)\right\}, \tag{2.2.11}$$

where A is the unique Pickands' representation function, which is a convex function on $[0, 1]$ such that $\max(t, 1-t) \leq A(t) \leq 1$ (see Pickands, 1981). Note that, for $A(t) \equiv 1$, we have independence, and, for $A(t) = \max(t, 1-t)$, we have perfect positive dependence. For higher dimensional max extreme value distributions, representation functions for the dependence structure are given in, for example, Beirlant *et al.* (2004) and Resnick (1987). Since we focus our attention on the bivariate case, we need not discuss other representations. In the case of min extreme value distributions, the survival copula has the form (2.2.11).

2.3 Univariate Phase-Type Distributions

Let $\{Y(t),\ t \geq 0\}$ be a right-continuous, continuous-time Markov Chain (CTMC) with state space $\xi = \{\Delta, 1, \ldots, d\}$, and initial distribution $\boldsymbol{\beta} = (\alpha_0, \boldsymbol{\alpha})$. Suppose that the CTMC has infinitesimal generator

$$\mathbf{Q} = \begin{pmatrix} 0 & \mathbf{0} \\ \text{-}\mathbf{Ae} & \mathbf{A} \end{pmatrix}, \tag{2.3.1}$$

where the subgenerator $\mathbf{A} = (a_{i,j})$ is a $d \times d$ matrix, $\mathbf{0} = (0, \ldots, 0)$ is a row vector of zeroes and $\mathbf{e} = (1, \ldots, 1)'$ is a column vector of ones. Then the nonnegative random variable X of the time until absorption into state Δ is said to be *phase-type (PH)* distributed with representation $(\boldsymbol{\alpha}, \mathbf{A}, d)$. We assume that absorption into state Δ is certain, or equivalently, that the matrix $\mathbf{A}$ is nonsingular. The survival function of X, denoted by $\bar{F}$, can be expressed as follows:

$$\bar{F}(x) = \Pr(Y(x) \notin \{\Delta\}) = \boldsymbol{\alpha} e^{\mathbf{A}x}\mathbf{e}, \quad x \geq 0, \tag{2.3.2}$$

where the matrix exponential of a matrix $\mathbf{A}$ is

$$e^{\mathbf{A}} = \mathbf{I} + \sum_{i=1}^{\infty} \frac{1}{i!}\mathbf{A}^i.$$

For other properties of PH distributions, see Rolski *et al.* (1999). PH distributions have been used in reliability theory (see Neuts, 1994), queueing theory (see Asmussen, 1992) and ruin theory (see Drekic *et al.*, 2004).

All of the eigenvalues of the subgenerator $\mathbf{A}$ have negative real parts (see Rolski *et al.*, 1999). Also, the matrix $\mathbf{A}$ is of Metzler type. That is, all of its off-diagonal entries are nonnegative. Therefore, it has a real dominant eigenvalue $-\eta$ (called the

Perron eigenvalue), not necessarily unique, such that for all complex eigenvalues λ, $\text{Re}(\lambda) < -\eta$ (see MacCluer, 2000). If the matrix $\mathbf{A}$ is irreducible, then the dominant eigenvalue $-\eta$ is unique. By expressing $\mathbf{A}$ in Jordan canonical form, one can conclude that there exists a nonnegative matrix of constants $\mathbf{M}$ that satisfies:

1. if $-\eta$ is a simple eigenvalue of $\mathbf{A}$ then

$$e^{\mathbf{A}x} = e^{-\eta x}(\mathbf{M} + \boldsymbol{O}(1)), \tag{2.3.3}$$

2. if $-\eta$ has algebric multiplicity l, then there exists an integer k $(0 \leq k \leq l-1)$ such that

$$e^{\mathbf{A}x} = x^k e^{-\eta x}(\mathbf{M} + \boldsymbol{O}(1)), \tag{2.3.4}$$

where $\boldsymbol{O}(1)$ is a matrix with entries that are $o(1)$ as $x \to \infty$, and $k+1$ is the maximal order of Jordan blocks corresponding to $-\eta$, called the index of $-\eta$ (see Perko, 2001 chapter 1, or Horn and Johnson, 1985, chapter 3).

This suggests the following approach to finding the matrix $\mathbf{M}$. First, determine the eigenvalues of $\mathbf{A}$. Let $-\eta$ be the largest real eigenvalue. If $-\eta$ has algebraic multiplicity 1, then let

$$\mathbf{M} = \lim_{x\to\infty} e^{\eta x} e^{\mathbf{A}x}.$$

If $-\eta$ has algebraic multiplicity $l > 1$, then calculate the matrix

$$\lim_{x\to\infty} x^{-k} e^{\eta x} e^{\mathbf{A}x}, \tag{2.3.5}$$

for $k = 0, \ldots, l-1$, and let $\mathbf{M}$ be the matrix obtained using the largest value of k such that expression (2.3.5) does not give the zero matrix.

This approach adapts and gives a more general way of finding the matrix $\mathbf{M}$ than that of Theorem 9 from Kang and Serfozo (1999). When all of the eigenvalues are real, the Putzer algorithm (see Theorem 8.2.2., Rolski *et al.*, 1999) leads to a simpler alternative than the method described above. These results are sufficient to find the limiting distribution of the normalized maxima of a sequence of iid PH-distributed random variables. It is well-known that this distribution must be one of the three distributions in the class of generalized extreme value (GEV) distributions – the Fréchet, Weibull and Gumbel distributions given in (2.2.7). The following proposition indicates that, in this case, it is the Gumbel distribution and gives the corresponding norming constants. The norming constants for the case in which (2.3.3) holds are also given in Theorem 9 of Kang and Serfozo (1999).

Proposition 2.3.1. *Let X be a PH distributed random variable with representation $(\boldsymbol{\alpha}, \boldsymbol{A}, d)$. Then its distribution is in the MaxDA(Λ). If (2.3.3) holds, then the norming constants are*

$$a_n = \frac{1}{\eta}, \quad b_n = \frac{\log nc}{\eta}, \tag{2.3.6}$$

and if (2.3.4) holds, then the norming constants are

$$a_n = \frac{1}{\eta}, \quad b_n = \frac{\log nc + k\ \log\log n - k\ \log\eta}{\eta}, \tag{2.3.7}$$

where $c = \boldsymbol{\alpha M e}$ is assumed to be positive.

Proof. Since (2.3.3) is the special case of (2.3.4) with $k = 0$, it is sufficient to check that the convergence criterion in (2.2.3) is satisfied using the norming constants in

(2.3.7). From (2.3.2) and (2.3.4), we have

$$\begin{aligned} n\bar{F}(a_n x + b_n) &= n\boldsymbol{\alpha}[\mathbf{M} + \boldsymbol{O}(1)]\mathbf{e}(a_n x + b_n)^k e^{-\eta(a_n x + b_n)} \\ &= n[c + o(1)]\left(\frac{\log n + o(\log n)}{\eta}\right)^k e^{-x - \log nc - k\log\log n + k\log\eta} \\ &= (1 + o(1))e^{-x}\left(\frac{\log n + o(\log n)}{\log n}\right)^k \to e^{-x}, \quad n \to \infty, \qquad (2.3.8) \end{aligned}$$

which together with (2.2.7) completes the proof. ■

The limiting distribution of the normalized minima of a sequence of iid PH-distributed random variables along with the norming constants is given by Proposition 2.3.2. Here we require that $\alpha_0 = 0$. We first provide a lemma which will be used in proving the proposition.

Lemma 2.3.1. *If the random variable X is PH distributed with representation $(\boldsymbol{\alpha}, \boldsymbol{A}, d)$, then m is the minimum number of transitions needed for the underlying CTMC to be absorbed if and only if*

$$-\boldsymbol{\alpha}\boldsymbol{A}^m\boldsymbol{e} > 0, \quad \textit{and when } m \geq 2, \ -\boldsymbol{\alpha}\boldsymbol{A}^\ell\boldsymbol{e} = 0, \ \ell = 1, \ldots, m-1. \qquad (2.3.9)$$

Proof. Let $I_0 = \{i|\ \alpha_i > 0\}$ be a subset of the state space ξ and $\mathbf{a}_\Delta = -\mathbf{A}\mathbf{e}$, with ith component $a_{i,\Delta}$, be the exit rate vector from the CTMC. Then

$$-\boldsymbol{\alpha}\mathbf{A}\mathbf{e} = \sum_{i \in I_0} \alpha_i\ a_{i,\Delta}. \qquad (2.3.10)$$

If $m = 1$, then the right-hand side of (2.3.10) is positive since there exists at least one transient state with positive probability of being the initial state for which direct absorption is possible.

When $m \geq 2$, then, for $i \in I_0$, $i_1, i_2, \ldots \in \xi \setminus \{\Delta\}$, and all $\ell = 1, \ldots, m-1$,

$$a_{i,i_1} a_{i_1,i_2} \cdots a_{i_{\ell-1},\Delta} = 0. \qquad (2.3.11)$$

Also,

$$a_{i,i_1}a_{i_1,i_2}\cdots a_{i_{m-1},\Delta} > 0 \tag{2.3.12}$$

for some $\{i_1,\ldots,i_{m-1}\}$ since absorption is possible on the mth transition. Furthermore, whenever the left-hand side of (2.3.12) is not positive, it must be 0. We see this by noting that the product can be negative only if an odd number of terms are negative. However, from (2.3.11), the product of the remaining terms must be 0. Now

$$-\boldsymbol{\alpha}\mathbf{A}^{\ell}\mathbf{e} = \sum_{i\in I_0}\alpha_i \sum_{i_1,\ldots,i_{\ell-1}} a_{i,i_1}a_{i_1,i_2}\cdots a_{i_{\ell-1},\Delta}, \text{ for } \ell \geq 2. \tag{2.3.13}$$

For $\ell < m$, each term on the right-hand side of (2.3.13) vanishes due to (2.3.11), and for $\ell = m$, the right-hand side of (2.3.13) must be positive due to (2.3.12). This completes the proof of necessity. The sufficiency part of the proof follows from the same arguments. ■

Proposition 2.3.2. *Let X be a PH distributed random variable with representation $(\boldsymbol{\alpha}, \boldsymbol{A}, d)$. Then its distribution is in the $MinDA(\Psi_m^*)$ with norming constants*

$$a_n = \left(\frac{m!}{nc}\right)^{\frac{1}{m}}, \quad b_n = 0, \tag{2.3.14}$$

where the constant $c = -\boldsymbol{\alpha}\boldsymbol{A}^m\boldsymbol{e}$, and m is the minimum number of transitions needed for the CTMC to be absorbed.

Proof. Let F be the distribution function of X. It is sufficient to check the convergence criterion (2.2.6). Using Lemma 2.3.1 and the fact that $e^{\mathbf{A}x} = \mathbf{I} + \sum_{i=1}^{m}\frac{\mathbf{A}^i x^i}{i!} +$

$\boldsymbol{O}(x^m)$ as $x \downarrow 0$ we have

$$\begin{aligned} n[1-\bar{F}(a_n x + b_n)] &= n\left[1-\boldsymbol{\alpha}\left\{\mathbf{I}+\sum_{i=1}^{m}\frac{\mathbf{A}^i x^i}{i!}\left(\frac{m!}{nc}\right)^{\frac{i}{m}}+\boldsymbol{O}(n^{-1})\right\}\mathrm{e}\right] \\ &\longrightarrow x^m, \text{ as } n \to \infty, \end{aligned} \tag{2.3.15}$$

where $\mathbf{I}$ is the identity matrix. Thus, F is in the *type II* class (see (2.2.8). ■

2.4 Multivariate Phase-Type Distributions

Let $\{Y(t),\ t \geq 0\}$ be a continuous-time Markov Chain (CTMC) with finite state space $\xi = \{\Delta, 1, \ldots, d\}$ and infinitesimal generator $\mathbf{Q}$ defined as in (2.3.1). Let ξ_i, $i = 1, \ldots, p$, be nonempty stochastically closed subsets of the state space ξ such that $\bigcap_{i=1}^{p} \xi_i$ is a proper subset of ξ. A subset of the state space is said to be stochastically closed if, once the process $\{Y(t),\ t \geq 0\}$ enters the subset, it never leaves. We assume that absorption into $\bigcap_{i=1}^{p} \xi_i$ is certain. Since we are interested in the process only until it is absorbed into $\bigcap_{i=1}^{p} \xi_i$, we may assume that $\bigcap_{i=1}^{p} \xi_i$ can be viewed as one state denoted by Δ. We may write $\xi = \left(\bigcup_{i=1}^{p} \xi_i\right) \bigcup \xi_0$ for some subset $\xi_0 \subset \xi$ with $\xi_0 \bigcap \xi_i = \emptyset$ for $i = 1, \ldots, p$. Let $\boldsymbol{\beta} = (0, \boldsymbol{\alpha})$ be the initial distribution, with each component representing the probability that the process starts in a particular state in ξ.

We define $X_i = \inf\{t \geq 0 : Y(t) \in \xi_i\}$ for $i = 1, \ldots, p$. For simplicity, we may assume that $\Pr\{X_1 > 0, \ldots, X_p > 0\} = 1$, which means that the CTMC starts within ξ_0. The joint distribution of $(X_1, \ldots, X_p)$ is called a *multivariate phase-type* (MPH) distribution with representation $(\boldsymbol{\alpha}, \mathbf{A}, \xi, \xi_1, \ldots, \xi_p)$, and $(X_1, \ldots, X_p)$ is called a *phase-type* random vector (see Assaf *et al.*, 1984). Thus, a MPH distribution is a joint distribution of first passage times to various subsets of the state space ξ of a CTMC.

Examples of MPH distributions include, among many others, the multivariate exponential distributions of Marshall and Olkin (1967). The set of p-dimensional MPH distributions is dense in the set of all distributions on $[0,\infty)^p$. For further details and for discussions of the closure properties of these distributions, see Assaf *et al.* (1984) or Cai and Li (2005a). Some results on order statistics of MPH random vectors are given in Cai and Li (2005b).

Let $\bar{F}$ denote the joint survival function of a MPH distribution. Then by Assaf *et al.* (1984) we have for $0 \leq x_p \leq \cdots \leq x_1$ that

$$\bar{F}(x_1,\ldots,x_p) = \boldsymbol{\alpha}\, e^{\mathbf{A}x_p}\, \mathbf{g}_p\, e^{\mathbf{A}(x_{p-1}-x_p)}\, \mathbf{g}_{p-1} \cdots e^{\mathbf{A}(x_1-x_2)}\, \mathbf{g}_1 \mathbf{e}, \tag{2.4.1}$$

where, for $k = 1,\ldots,p$, $\mathbf{g}_k$ is a $d \times d$ diagonal matrix whose ith diagonal entry, for $i = 1,\ldots,d$, equals 1 if $i \in \xi \setminus \xi_k$ and 0 otherwise. For $p = 2$, we can interpret equation (2.4.1) as the probability that the underlying Markov chain remains in the subset $\xi \setminus \{\xi_1 \bigcup \xi_2\}$ until time x_2 and remains in the subset $\xi \setminus \xi_1$ between time x_2 and time x_1.

The random variable X_i represents the first passage time of the CTMC into ξ_i. This implies that X_i is univariate PH distributed with representation $(\boldsymbol{\alpha}_{\xi\setminus\xi_i}, \mathbf{A}_{\xi\setminus\xi_i}, d+1-|\xi_i|)$, where $\boldsymbol{\alpha}_{\xi\setminus\xi_i}$ and $\mathbf{A}_{\xi\setminus\xi_i}$ are the probability entry distribution and subgenerator matrix restricted to the state space $\xi \setminus \xi_i$. As in Section 2, η_i, k_i, and $\mathbf{M}_i$ are defined for the i-th marginal of the MPH random vector. The matrix $\mathbf{M}_i$ is extended to have dimension d, by padding it with zeroes. In order to avoid an abuse of notation, this padded matrix is denoted by $\mathbf{M}_i$.

In the bivariate case, the subgenerator has the special form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{B}_1 & \mathbf{B}_2 \\ 0 & \mathbf{A}_1 & 0 \\ 0 & 0 & \mathbf{A}_2 \end{pmatrix}, \tag{2.4.2}$$

where, for $i = 0, 1, 2$, $\mathbf{A}_i$ represents the subgenerator for states in $\xi_i \setminus \{\Delta\}$, and for $i = 1, 2$, $\mathbf{B}_i$ represents the matrix of transition intensities from states in ξ_0 to states in $\xi_i \setminus \{\Delta\}$.

The following theorem establishes the limiting distribution for bivariate PH distributions. The extension to higher dimensions will be outlined later.

Theorem 2.4.1. *Let F be the distribution function of a bivariate PH distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{A}, \xi, \xi_1, \xi_2)$. Then there exist sequences of constants $\boldsymbol{a}^{(n)}$, $\boldsymbol{b}^{(n)}$ $\in \Re^2$ such that (2.2.1) holds with G given by*

$$G(x_1, x_2) = \begin{cases} e^{-e^{-x_1}} e^{-e^{-x_2}} \exp\left\{ \frac{e^{-x_1}}{c_1} \boldsymbol{\alpha} \boldsymbol{M}_1 e^{A\left(x_2 + \log c_2 - x_1 - \log c_1\right)\eta^{-1}} \boldsymbol{g}_2 \boldsymbol{e} \right\}, \\ \qquad\qquad \text{if } x_1 + \log c_1 \le x_2 + \log c_2 \\ e^{-e^{-x_1}} e^{-e^{-x_2}} \exp\left\{ \frac{e^{-x_2}}{c_2} \boldsymbol{\alpha} \boldsymbol{M}_2 e^{A\left(x_1 + \log c_1 - x_2 - \log c_2\right)\eta^{-1}} \boldsymbol{g}_1 \boldsymbol{e} \right\}, \\ \qquad\qquad \text{if } x_2 + \log c_2 \le x_1 + \log c_1 \end{cases} \tag{2.4.3}$$

whenever $\eta_1 = \eta_2 = \eta$ and $k_1 = k_2 = k$, where $c_i = \boldsymbol{\alpha} \boldsymbol{M}_i \boldsymbol{e}$ is assumed to be positive for $i = 1, 2$. For any other case we have independence, and $G(x_1, x_2) = \exp\left(-e^{-x_1}\right) \exp\left(-e^{-x_2}\right)$.

Proof. Since X_1 and X_2 are PH distributed, both are in the MaxDA(Λ) with respective normalizing constants $a_{n,1}$, $b_{n,1}$, and $a_{n,2}$, $b_{n,2}$. From basic probability we have

$$\begin{aligned} & n[1 - \Pr(X_1 \le a_{n,1}x_1 + b_{n,1}, X_2 \le a_{n,2}x_2 + b_{n,2})] \\ &= n \Pr(X_1 > a_{n,1}x_1 + b_{n,1}) + n \Pr(X_2 > a_{n,2}x_2 + b_{n,2}) \\ &\qquad - n \Pr(X_1 > a_{n,1}x_1 + b_{n,1}, X_2 > a_{n,2}x_2 + b_{n,2}). \end{aligned} \tag{2.4.4}$$

From Proposition 2.3.1, the first two terms on the right hand side of (2.4.4) have limits e^{-x_1} and e^{-x_2}, respectively. If $\eta_1 = \eta_2 = \eta$ and $k_1 = k_2 = k$, for x_1 and x_2 such that $x_1 + \log c_1 \le x_2 + \log c_2$, from (2.3.3 or 2.3.4) and (2.4.1) we obtain

$$
\begin{aligned}
& n \Pr(X_1 > a_{n,1}x_1 + b_{n,1}, X_2 > a_{n,2}x_2 + b_{n,2}) \\
= \; & \frac{e^{-x_1}}{c_1}\left(\frac{\log n + o(\log n)}{\log n}\right)^k \boldsymbol{\alpha}(\mathbf{M}_1 + \boldsymbol{O}(1)) e^{\mathbf{A}\left(x_2+\log c_2 - x_1 - \log c_1\right)\eta^{-1}} \mathbf{g}_2 \mathbf{e} \\
\to \; & \frac{e^{-x_1}}{c_1} \boldsymbol{\alpha}\mathbf{M}_1 e^{\mathbf{A}\left(x_2+\log c_2 - x_1 - \log c_1\right)\eta^{-1}} \mathbf{g}_2 \mathbf{e}, \quad \text{as } n \to \infty,
\end{aligned}
$$

which completes the proof for this case.

If $\eta_1 > \eta_2$, then for n sufficiently large, $a_{n,1}x_1 + b_{n,1} < a_{n,2}x_2 + b_{n,2}$, and

$$
\begin{aligned}
a_{n,2}x_2 + b_{n,2} - a_{n,1}x_1 - b_{n,1} &= \left(\frac{1}{\eta_2} - \frac{1}{\eta_1}\right)\log n + o(\log n) \\
&\to \infty, \text{ as } n \to \infty.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& n \Pr(X_1 > a_{n,1}x_1 + b_{n,1}, X_2 > a_{n,2}x_2 + b_{n,2}) \\
= \; & \frac{e^{-x_1}}{c_1} \boldsymbol{\alpha}(\mathbf{M}_1 + \boldsymbol{O}(1)) \boldsymbol{O}(1) \mathbf{e} \\
\to \; & 0, \text{ as } n \to \infty.
\end{aligned}
$$

This implies that we have independence in the limit. In a similar way, the remaining cases yield the same result. ■

Starting with (2.2.11), simple algebraic computations show that, if F is a bivariate PH distribution function, then (2.2.1) holds, where G is a BEV distribution with Gumbel marginals and dependence structure given by the Pickands' representation function

$$
A(t) = \begin{cases} 1 - \frac{1-t}{c_2} \boldsymbol{\alpha}\mathbf{M}_2 e^{\mathbf{A}\frac{1}{\eta}\log\frac{c_1}{c_2}\frac{1-t}{t}} \mathbf{g}_1 \mathbf{e}, & \text{if } 0 \le t \le \frac{c_1}{c_1+c_2} \\ 1 - \frac{t}{c_1} \boldsymbol{\alpha}\mathbf{M}_1 e^{\mathbf{A}\frac{1}{\eta}\log\frac{c_2}{c_1}\frac{t}{1-t}} \mathbf{g}_1 \mathbf{e}, & \text{if } \frac{c_1}{c_1+c_2} \le t \le 1 \end{cases}. \tag{2.4.5}
$$

A number of other characterizations of MEV distributions have been proposed (see, for example, Balkema and Resnick, 1977, de Haan and Resnick, 1977). For further discussion of the different representations, see de Haan and de Ronde (1998) or Beirlant *et al.* (2004). For ease of presentation of our examples in Section 2.5, we consider only Pickands' representation.

By using the same logic as in Theorem 2.4.1 and the identity

$$\begin{aligned}
&\Pr\big(\bigcap_{i=1}^{p}\{X_i \le x_i\}\big) \\
&= 1-\sum_{i=1}^{p}\Pr(X_i > x_i)+\sum_{i<j}\Pr(X_i > x_i, X_j > x_j)-\ldots(-1)^p\Pr\big(\bigcap_{i=1}^{p}\{X_i > x_i\}\big),
\end{aligned}$$

we can obtain the limit distribution for higher dimensional MPH distributions. In order to take more advantage of the structure of $\mathbf{A}$, it is convenient to rearrange the state space. As in Cai and Li (2005a), ξ is partitioned as follows:

$$\begin{aligned}
\Gamma^{p}_{\emptyset} &= \xi_0, \\
\Gamma^{p-1}_{\{i\}} &= \xi_i \setminus \big(\bigcup_{k\neq i}(\xi_i\cap\xi_k)\big),\ i=1,\ldots,p \\
\Gamma^{p-2}_{\{i,j\}} &= (\xi_i\cap\xi_j)\setminus\big(\bigcup_{k\neq i,k\neq j}(\xi_i\cap\xi_j\cap\xi_k)\big),\ i,j=1,\ldots,p,\ i\neq j \\
&\ \vdots \\
\Gamma^{p-|\mathcal{D}|}_{\mathcal{D}} &= \big(\bigcap_{i\in\mathcal{D}}\xi_i\big)\setminus\big(\bigcup_{k\notin\mathcal{D}}\big(\bigcap_{i\in\mathcal{D}}\xi_i\cap\xi_k\big)\big),\ \mathcal{D}\subset\{1,2,\ldots,p\} \\
&\ \vdots \\
\Gamma^{0}_{\xi\setminus\{\Delta\}} &= \{\Delta\},
\end{aligned}$$

where $|\cdot|$ denotes set cardinality. Notice that, by partitioning the state space in this fashion and reordering the states so that $i<j$ whenever $i\in\Gamma^{p-|\mathcal{D}_1|}_{\mathcal{D}_1}$, $j\in\Gamma^{p-|\mathcal{D}_2|}_{\mathcal{D}_2}$ and

$|\mathcal{D}_1| < |\mathcal{D}_2|$, the subgenerator $\mathbf{A}$ becomes a block upper triangular matrix. Therefore, its eigenvalue set coincides with the union of eigenvalue sets of diagonal blocks, which simplifies the problem of finding the eigenvalues for high cardinality state spaces.

The following theorem provides the analogous limit distribution for normalized componentwise minima of bivariate PH distributed random vectors.

Theorem 2.4.2. *Let F be the distribution function of a bivariate PH distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{A}, \xi, \xi_1, \xi_2)$. Then there exist sequences of constants $\boldsymbol{a}^{(n)}$, $\boldsymbol{b}^{(n)}$ $\in \Re^2$ such that (2.2.4) holds with $\bar{G}$ given by*

$$\bar{G}(x_1, x_2) = \exp\left\{-x_1^m - x_2^m + c\min\left(\frac{x_1^m}{c_1}, \frac{x_2^m}{c_2}\right)\right\}, \tag{2.4.6}$$

where $c_i = -\boldsymbol{\alpha}\boldsymbol{A}^{m_i}\boldsymbol{g}_i\boldsymbol{e}$, $i = 1, 2$, and $c = -\boldsymbol{\alpha}\boldsymbol{A}^m\boldsymbol{e}$, provided that $m_1 = m_2 = m$, where m_i is the minimum number of transitions required in order to enter ξ_i. Otherwise, we are in the independence case and $\bar{G}(x_1, x_2) = \exp(-x^{m_1} - x^{m_2})$.

Remark: If $m_1 = m_2 = m$ then the limiting distribution has the Marshall-Olkin dependence structure

$$\hat{C}(u, v) = \min(u^{1-a}v, uv^{1-b}),\ 0 < a, b < 1,$$

where $a = \dfrac{c}{c_1}$ and $b = \dfrac{c}{c_2}$.

Proof. Let $a_{n,i} = \left(\frac{m_i!}{nc_i}\right)^{\frac{1}{m_i}}$ and $b_{n,i} = 0$ be the norming constants defined as in (2.3.14). It is sufficient to verify the convergence criterion (2.2.6). Throughout the proof, we will make use of the fact that $\boldsymbol{\alpha}\mathbf{g}_i = \boldsymbol{\alpha}$ for $i = 1, 2$, since the underlying CTMC starts in ξ_0.

In the case that $m_1 = m_2 = m$, if $0 \le \frac{x_1^{m_1}}{c_1} \le \frac{x_2^{m_2}}{c_2}$, then similar to the proof of Proposition 2.3.2, we have

$$
\begin{aligned}
& n[1 - \bar{F}(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)})] \\
= {} & n\Bigg\{1 - \boldsymbol{\alpha}\left[\mathbf{I} + \sum_{i=1}^{m} \mathbf{A}^i \frac{x_1^i}{i!}\left(\frac{m!}{nc_1}\right)^{\frac{i}{m}} + O(n^{-1})\right]\mathbf{g}_1 \\
& \qquad \left[\mathbf{I} + \sum_{j=1}^{m} \mathbf{A}^j \frac{1}{j!}\left(\frac{m!}{n}\right)^{\frac{j}{m}}\left[x_2 c_2^{-\frac{1}{m}} - x_1 c_1^{-\frac{1}{m}}\right]^j + O(n^{-1})\right]\mathbf{g}_2\mathbf{e}\Bigg\} \\
= {} & n\Bigg\{\sum_{i=1}^{m}(-\boldsymbol{\alpha}\mathbf{A}^i\mathbf{g}_1\mathbf{g}_2\mathbf{e})\left(\frac{m!}{nc_1}\right)^{\frac{i}{m}}\frac{x_1^i}{i!} \\
& \qquad + \sum_{j=1}^{m}(-\boldsymbol{\alpha}\mathbf{A}^j\mathbf{g}_2\mathbf{e})\left(\frac{m!}{n}\right)^{\frac{j}{m}}\left[x_2 c_2^{-\frac{1}{m}} - x_1 c_1^{-\frac{1}{m}}\right]^j\frac{1}{j!} \\
& \qquad + \sum_{i=1}^{m-1}\sum_{j=1}^{m-i}(-\boldsymbol{\alpha}\mathbf{A}^i\mathbf{g}_1\mathbf{A}^j\mathbf{g}_2\mathbf{e})\left(\frac{m!}{n}\right)^{\frac{i+j}{m}} x_1^i c_1^{-i/m} \\
& \qquad\qquad \frac{1}{i!j!}\left[x_2 c_2^{-\frac{1}{m}} - x_1 c_1^{-\frac{1}{m}}\right]^j + O(n^{-1})\Bigg\} \\
\to {} & x_1^m + x_2^m - c\frac{x_1^m}{c_1}, \text{ as } n \to \infty,
\end{aligned}
\tag{2.4.7}
$$

where (2.4.7) follows from Lemma 2.3.1 and the fact that

$$-\boldsymbol{\alpha}\mathbf{A}^i\mathbf{g}_1\mathbf{A}^{m-i}\mathbf{g}_2\mathbf{e} = -\boldsymbol{\alpha}\mathbf{A}^m\mathbf{g}_2\mathbf{e},$$

and

$$c = c_1 + c_2 - (-\boldsymbol{\alpha}\mathbf{A}^m\mathbf{g}_1\mathbf{g}_2\mathbf{e}).$$

This completes the proof in the $m_1 = m_2 = m$ case.

In the $m_1 \neq m_2$ case we can, without loss of generality, assume that $m_1 < m_2$.

Then, for n sufficiently large $a_{n,1}x_1 < a_{n,2}x_2$, which gives the following:

$$
\begin{aligned}
& n[1 - \bar{F}(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)})] \\
= \quad & n\Bigg\{ \sum_{i=1}^{m_1}(-\boldsymbol{\alpha}\mathbf{A}^i\mathbf{g}_1\mathbf{g}_2\mathbf{e})\left(\frac{m_1!}{nc_1}\right)^{\frac{i}{m_1}} \frac{x_1^i}{i!} \\
& \quad + \sum_{j=1}^{m_2}(-\boldsymbol{\alpha}\mathbf{A}^j\mathbf{g}_2\mathbf{e})\left[x_2\left(\frac{m_2!}{nc_2}\right)^{\frac{1}{m_2}} - x_1\left(\frac{m_1!}{nc_1}\right)^{\frac{1}{m_1}}\right]^j \frac{1}{j!} \\
& \quad + \sum_{\{i,j|\frac{i}{m_1}+\frac{j}{m_2}\leq 1\}} (-\boldsymbol{\alpha}\mathbf{A}^i\mathbf{g}_1\mathbf{A}^j\mathbf{g}_2\mathbf{e})\frac{x_1^i}{i!j!}\left(\frac{m_1!}{nc_1}\right)^i \\
& \qquad \left[x_2\left(\frac{m_2!}{nc_2}\right)^{\frac{1}{m_2}} - x_1\left(\frac{m_1!}{nc_1}\right)^{\frac{1}{m_1}}\right]^j + o(n^{-1})\Bigg\} \\
\rightarrow \quad & x_1^{m_1} + x_2^{m_2}, \text{ as } n \rightarrow \infty, \qquad (2.4.8)
\end{aligned}
$$

where (2.4.8) follows from the fact that $-\boldsymbol{\alpha}\mathbf{A}^{m_1}\mathbf{g}_1\mathbf{g}_2\mathbf{e} = c_1$, from Lemma 2.3.1, its implication that

$$-\boldsymbol{\alpha}\mathbf{A}^i\mathbf{g}_1\mathbf{A}^j\mathbf{g}_2\mathbf{e} = 0 \text{ when } i < m_1 \text{ and } i + j < m_2,$$

and from the fact that

$$\left\{(i,j) : \frac{i}{m_1} + \frac{j}{m_2} \leq 1\right\} \subseteq \{(i,j) : i < m_1, i + j < m_2\}.$$

This completes the proof. ■

2.5 Examples

In this section, we present some simple examples of bivariate PH distributions. We find that, even in simple cases, we are able to achieve a wide variety of dependence structures within the BEV class. We explore this by examining the Pickands' representation function which is given by (2.4.5).

Example 1

In this example we consider a bivariate PH distribution with representation $(\boldsymbol{\alpha}, \mathbf{A}, \xi, \xi_1, \xi_2)$, where

$$\boldsymbol{\alpha} = (1,0,0), \quad \mathbf{A} = \begin{pmatrix} -a & p & q \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix}, \quad a < \min(b,c), p+q \leq a,$$

$$\xi = \{\Delta, 1, 2, 3\}, \ \ \xi_1 = \{\Delta, 2\}, \ \ \xi_2 = \{\Delta, 3\}.$$

Then one gets $\eta = a$, $k = 1$, $c_1 = 1 + \frac{q}{c-a}$, $c_2 = 1 + \frac{p}{b-a}$, and from (2.4.5), the Pickands' representation function is given by

$$A(t) = \begin{cases} 1 - t + \left(\frac{b-a}{(c-a)(b+p-a)}\right)^{1-\frac{c}{a}} q(c+q-a)^{-\frac{c}{a}} t^{\frac{c}{a}} (1-t)^{1-\frac{c}{a}}, & 0 \leq t \leq \frac{c_1}{c_1+c_2} \\ t + \left(\frac{c-a}{(b-a)(q+c-a)}\right)^{1-\frac{b}{a}} p(p+b-a)^{-\frac{b}{a}} t^{1-\frac{b}{a}} (1-t)^{\frac{b}{a}}, & \frac{c_1}{c_1+c_2} \leq t \leq 1 \end{cases}.$$

In the special case where $p = q = 0$, we have $A(t) = \max(t, 1-t)$, which corresponds to the perfect positive dependence case. In this case the underlying CTMC is certain to make a direct transition from state 1 to state Δ. Therefore, $X_1 = X_2$ with probability 1 and the componentwise maxima must also be equal.

Figure 2.1 shows the Pickands' function obtained using three different sets of parameters. In all three cases, we have assumed that $b = c$ and $p = q$. The resulting symmetry leads to symmetric $A(t)$ functions. Notice that as b and c approach a, we move closer to the independence case: $A(t) = 1$. Also, note that by choosing p and q so that $p + q = a$, the underlying CTMC cannot make a direct transition from state 1 to state Δ. Therefore, X_1 and X_2 are different with probability 1.

Figure 2.2 shows the Pickands' function for three different sets of parameters. Each of these functions is asymmetric.

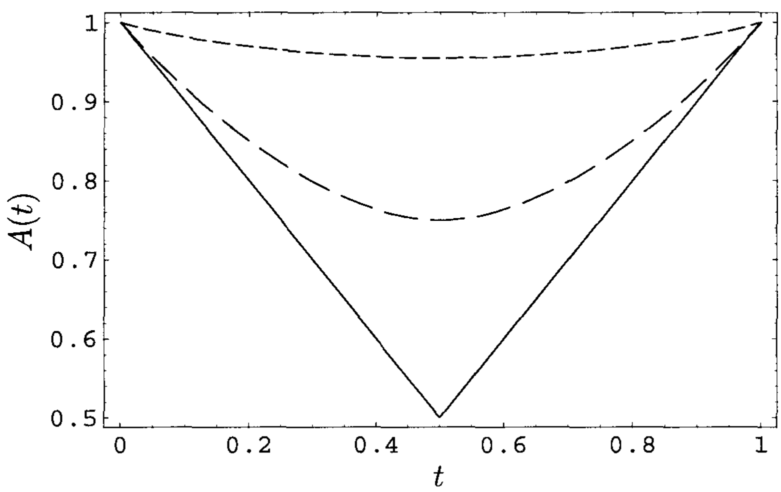

Figure 2.1: Plots of Pickands' $A(t)$ function for Example 1 with $(a, b, c, p, q) = (2, 3, 3, 0, 0) \rightarrow$ solid line, $(a, b, c, p, q) = (2, 3, 3, 1, 1) \rightarrow$ long-dashed line, $(a, b, c, p, q) = (2, 2.1, 2.1, 1, 1) \rightarrow$ short-dashed line.

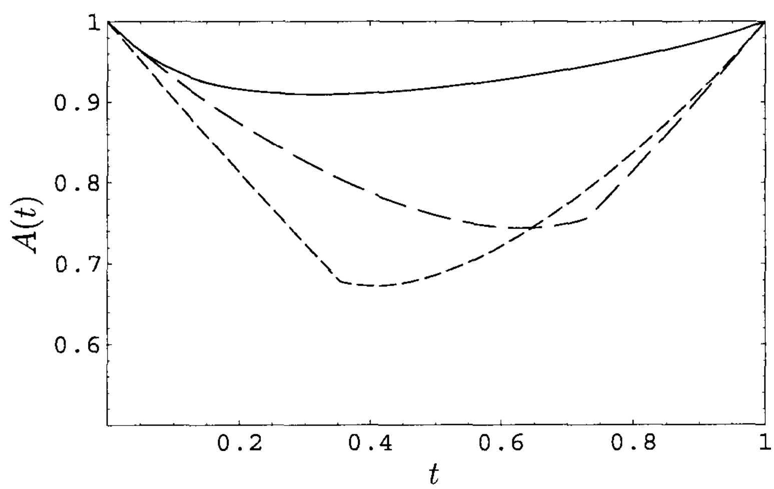

Figure 2.2: Plots of Pickands' $A(t)$ function for Example 1 with $(a, b, c, p, q) = (2, 2.1, 3, 1, 1) \rightarrow$ solid line, $(a, b, c, p, q) = (2, 3, 2.5, 0.1, 1) \rightarrow$ long-dashed line, $(a, b, c, p, q) = (2, 3, 3, 1, 0.1) \rightarrow$ short-dashed line.

Example 2

In this example, we consider the same setup as Example 1, except that we assume $a = b = c$. We also require that $a, p, q > 0$. Thus, we have

$$\mathbf{A} = \begin{pmatrix} -a & p & q \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix},$$

which implies that $\eta = a$, $k = 2$, $c_1 = q$, $c_2 = p$, and $A(t) = 1$. Notice that we have independence, even though we have satisfied all the conditions of Theorem 2.4.1 necessary for G to be given by (2.4.3).

Example 3

In this example we consider a bivariate PH distribution with representation $(\boldsymbol{\alpha}, \mathbf{A}, \xi, \xi_1, \xi_2)$, where

$$\boldsymbol{\alpha} = (p, 1-p, 0, 0),\ 0 \le p \le 1, \quad \mathbf{A} = \begin{pmatrix} -5 & 0 & 1 & 2 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix},$$

$$\xi = \{\Delta, 1, 2, 3, 4\},\ \xi_1 = \{\Delta, 3\},\ \xi_2 = \{\Delta, 4\}.$$

then $\eta = 5$, $k = 1$, $c_1 = 1 + 2p$, $c_2 = 2 - \frac{p}{2}$, which implies that

$$A(t) = \begin{cases} 1 - t + 2^{\frac{4}{5}} p (1+2p)^{-\frac{6}{5}} (4-p)^{\frac{1}{5}} t^{\frac{6}{5}} (1-t)^{-\frac{1}{5}}, & 0 \le t \le \frac{2+4p}{6+3p} \\ t + 2^{\frac{2}{5}} (2-p)(4-p)^{-\frac{7}{5}} (1+2p)^{\frac{2}{5}} t^{-\frac{2}{5}} (1-t)^{\frac{7}{5}}, & \frac{2+4p}{6+3p} \le t \le 1 \end{cases}.$$

Figure 2.3 shows the Pickands' function for three different values of the parameter p.

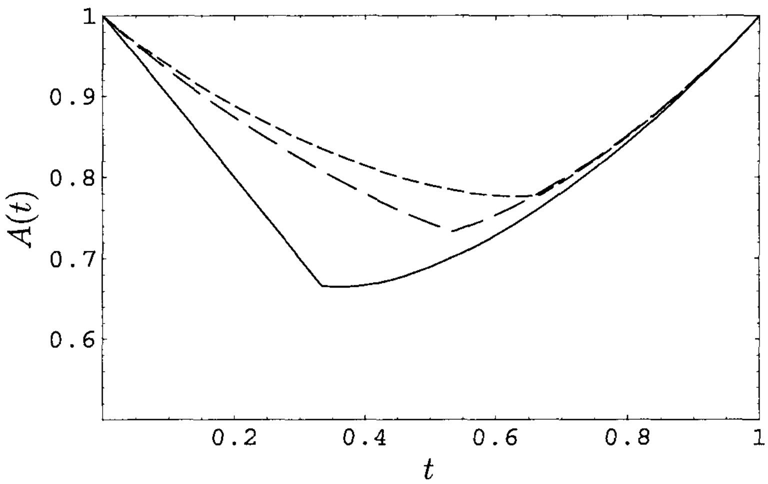

Figure 2.3: Plots of Pickands' $A(t)$ function for Example 3 with $p = 0 \rightarrow$ solid line, $p = 0.5 \rightarrow$ long-dashed line, $p = 1 \rightarrow$ short-dashed line.

2.6 Summary and Conclusions

In this chapter, we establish the set of attractors for the componentwise minima and maxima of an iid sequence of random vectors from a fairly general class of multivariate distributions known as the multivariate phase-type (MPH) distributions. The norming constants and corresponding MEV distributions are explicitly given. For the sake of simplicity, we focus on the bivariate case.

The limiting distribution of the componentwise maxima of bivariate phase-type random vectors has a complicated form. In order to investigate its behaviour, the Pickands' representation is chosen. Our examples illustrate that the limit distribution allows considerable flexibility within the BEV class. This suggests that the MPH distribution functions are well suited for statistical inference of multivariate data.

It is shown that the dependence structure of the limit distribution of componentwise minima of MPH random vectors coincides with that of the multivariate exponential (Marshall-Olkin) distributions.

References

Asmussen, S. 1992. Phase-Type Representations in Random Walk and Queueing Problems. *Annals of Probability*, 20(2), 772-789.

Assaf, D., Langberg, N.A., Savits, T.H. and Shaked, M. 1984. Multivariate Phase-Type Distributions, *Operations Research*, 32(3), 688-702.

Balkema, A.A. and Resnick, S.I. 1977. Max-Infinite Divisibility, *Journal of Applied Probability*, 14(2), 309-319.

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. 2004. *Statistics of Extremes: Theory and Applications.* Wiley, Chichester.

Cai, J. and Li, H. 2005a. Multivariate Risk Model of Phase Type, *Insurance: Mathematics and Economics*, 36(2), 137-152.

Cai, J. and Li, H. 2005b. Conditional Tail Expectations for Multivariate Phase-Type Distributions, *Journal of Applied Probability*, 42(3), 810-825.

Drekic, S., Dickson, D., Stanford, D., and Willmot, G.E. 2004. On the Distribution of the Deficit at Ruin when Claims are Phase-Type. *Scandinavian Actuarial Journal*, 2, 105-120.

de Haan, L. and Resnick, S.I. 1977, Limit Theory for Multivariate Sample Extremes, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 40, 317-337.

de Haan, L. and de Ronde, J. 1998. Sea and Wind: Multivariate Extremes at Work, *Extremes*, 1(1), 7-45.

Gnedenko, B.V. 1943. Sur la distribution limité du terme maximum d'une série aléatoaire, *Annals of Mathematics*, 44(3), 423-453.

Horn, R.A. and Johnson, C.R. 1985. *Matrix Analysis*, Cambridge University Press, Cambridge.

Kang, S. and Serfozo, R.F. 1999. Extreme Values of Phase-Type and Mixed Random Variables with Parallel-Processing Examples, *Journal of Applied Probability*, 36(1), 194-210.

Li, H. and Xu, S.H. 2000. On the Dependence Structure and Bounds of Correlated Parallel Queues and their Applications to Synchronize Stochastic Systems, *Journal of Applied Probability* 37(4), 1020-1043.

MacCluer, C.R. 2000. The Many Proofs and Applications of Perron's Theorem, *SIAM Review*, 42(3), 487-498.

Marshall, A.W., Olkin, I. 1967. A Multivariate Exponential Distribution, *Journal of the American Statistical Association*, 62, 30-44.

Marshall, A.W., Olkin, I. 1983. Domains of Attraction of Multivariate Extreme Value Distributions, *Annals of Probability*, 11(1), 168-177.

Nelsen, R. B. 1999. *An Introduction to Copulas.* Springer-Verlag, New York.

Neuts, M.F. 1994. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, Dover Publications, New York.

Perko, L. 2001. *Differential Equations and Dynamical Systems*, Springer, New York.

Pickands, J. 1981. Multivariate Extreme Value Distributions, *Bulletin of the International Statistical Institute, Proceedings of the 43rd Session, Buenos Aires*, 49, 859-878.

Resnick, S.I. 1987. *Extreme Values, Regular Variation and Point Processes.* Springer-Verlag, New York.

Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. 1999. *Stochastic Processes for Finance and Insurance*, Wiley, New York.

Sklar, A. 1959. Fonctions de répartion à n dimensions et leurs marges, Publications de l'Institut de Statistique de l'Université de Paris, 8, 229-231.

Chapter 3

Extreme Behaviour of Bivariate Elliptical Distributions

3.1 Introduction

During the past few decades there has been an extensive amount of work on the understanding of the elliptical class of distributions. The first comprehensive work was given by Fang *et al.* (1990). Primarily, these distributions allow an alternative and extension of the normal law. Elliptical distributions are easily implemented and simulated (see, for example, Breymann *et al.*, 2003; Hodgson *et al.*, 2002; Johnson, 1987; Li *et al.*, 1997; Manzotti *et al.*, 2002), and they are useful for actuarial and financial applications.

Modelling of extreme or rare events is an important and well-researched topic. When there are several random variables of interest, the dependence structure must

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be considered in investigating their extreme behaviour. This is addressed in the growing literature on multivariate extreme value theory (see, for example, Beirlant, *et al.*, 2004).

The extreme behaviour of elliptically distributed random vectors is closely related to the asymptotic property of their generator (see Berman, 1992 and Hashorva, 2005). Starting with the work of Sibuya (1960), recently many other papers have studied the extreme behaviour of elliptical random vectors, see for example Hult and Lindskog (2002), Schmidt (2002), Abdous *et al.* (2005), Demarta and McNeil (2005), and Hashorva (2005).

In this chapter, we present some results on the extreme behaviour of bivariate elliptical distributions. These results hold under certain conditions on the tail behaviour of the generator. Specifically, we give the limiting distribution of componentwise maxima of iid elliptical random vectors and find that it is exactly that obtained by Demarta and McNeil (2005) for the special case of the Student t distribution. We then present results concerning joint exceedances over a high threshold. We first provide a characterization of the limiting upper copula. We then give a bivariate version of the classical peaks over threshold result (see Balkema and de Haan, 1974, and Pickands, 1975). We close this chapter with an illustration.

3.2 Definitions and examples

Let $\mathbf{Z}_i = (X_i, Y_i), i = 1, 2, \ldots$ be a sequence of independent random vectors with common distribution F, and let

$$\mathbf{M}_n = (\max_{i=1,\ldots,n} X_i, \max_{i=1,\ldots,n} Y_i).$$

That is, $\mathbf{M}_n$ is the vector of componentwise maxima of $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$. If there exist sequences of vectors of constants $\mathbf{a}_n$, $\mathbf{b}_n \in \Re^2$ and a random vector $\mathbf{Z}$ with distribution G and nondegenerate marginals such that $\mathbf{a}_n\mathbf{M}_n + \mathbf{b}_n$ converges weakly to $\mathbf{Z}$, then G, the limit distribution of normalized componentwise maxima, is said to be a *bivariate extreme value distribution.* We then say that F is in the *maximum domain of attraction* of G with *normalizing vectors of constants* $\mathbf{a}_n$ and $\mathbf{b}_n$ and write $F \in \text{MDA}(G)$. It is useful to note that

$$\lim_{n\to\infty} F^n(\mathbf{a}_n\mathbf{x} + \mathbf{b}_n) = G(\mathbf{x}) \Leftrightarrow \lim_{n\to\infty} n[1 - F(\mathbf{a}_n\mathbf{x} + \mathbf{b}_n)] = -\log G(\mathbf{x}), \qquad (3.2.1)$$

for all $\mathbf{x}$ such that $G(\mathbf{x}) > 0$.

A characterization of the maximum domain of attraction of multivariate extreme value distributions is given by Marshall and Olkin (1983). Necessary conditions for (3.2.1) are that each marginal F_i of F is in the (univariate) MDA of the corresponding component G_i of G. Classical results concerning univariate maxima are given by Gnedenko (1943). In particular, if $F_i \in \text{MDA}(G_i)$ then, by the Fisher-Tippett theorem, G_i belongs to the type of the distribution

$$H_\xi(x) = \begin{cases} \exp\left\{-(1+\xi x)^{-1/\xi}\right\}, & 1+\xi x > 0, \quad \xi \neq 0 \\ \exp\{-e^{-x}\}, & -\infty < x < \infty, \quad \xi = 0 \end{cases}.$$

H_ξ is known as the *generalized extreme value distribution.* For $\alpha > 0$, $\Phi_\alpha(x) := H_{1/\alpha}(\alpha(x-1))$ is the standard Fréchet distribution, $\Psi_\alpha(x) := H_{-1/\alpha}(\alpha(x+1))$ is the standard Weibull distribution, and $\Lambda(x) := H_0(x)$ is the standard Gumbel distribution.

It is well-known (see, for example, Embrechts *et al.*, 1997) that $F_i \in MDA(H_\xi)$ if and only if there exists a positive, measurable function $a(\cdot)$ such that

$$\lim_{t \uparrow x_{F_i}} \frac{\bar{F}_i(t + xa(t))}{\bar{F}_i(t)} = \begin{cases} (1+\xi x)^{-1/\xi}, & 1+\xi x > 0, \quad \text{if } \xi \neq 0 \\ e^{-x}, & -\infty < x < \infty, \quad \text{if } \xi = 0 \end{cases}, \tag{3.2.2}$$

where x_{F_i} is the right endpoint of the support of F_i. The right-hand side of (3.2.2) is the survival function of the *generalized Pareto distribution.*

Returning to the bivariate setup, the bivariate extreme value distribution can be represented as follows:

$$G(x,y) = \exp\left\{\log\{G_1(x)G_2(y)\} A\left(\frac{\log G_1(x)}{\log\{G_1(x)G_2(y)\}}\right)\right\}, \tag{3.2.3}$$

where A is the Pickands' representation function, which is a convex function on $[0,1]$ such that $\max(t, 1-t) \leq A(t) \leq 1$ (see Pickands, 1981).

The dependence structure associated with the distribution of a random vector can be characterized in terms of a *copula.* A two-dimensional copula is a bivariate distribution function defined on $[0,1]^2$ with uniformly distributed marginals. Due to Sklar's Theorem (see Sklar, 1959), if F is a joint distribution function with continuous marginals F_1 and F_2 respectively, then there exists a unique copula, C, given by

$$C(u,v) = F(F_1^{\leftarrow}(u), F_2^{\leftarrow}(v)),$$

where $h^{\leftarrow}(u) = \inf\{x : h(x) \geq u\}$ is the generalized inverse function. Similarly, the *survival copula* is defined as the copula relative to the joint survival function and is given by

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v). \tag{3.2.4}$$

A more formal definition, properties and examples of copulas are given in Nelsen (1999). Let (U,V) be a random vector with copula C, and standard uniformly distributed marginals. The upper copula at level u is defined as follows:

$$C_u^{up}(x,y) = \Pr(U \leq F_{1,u}^{\leftarrow}(x), V \leq F_{2,u}^{\leftarrow}(y) | U > u, V > u), \tag{3.2.5}$$

where $F_{1,u}(x) = \Pr(U \leq x | U > u, V > u)$ and $F_{2,u}(y) = \Pr(V \leq y | U > u, V > u)$.

A fundamental concept in Extreme Value Theory is that of regular variation, which we now define.

Definition 3.2.1. *A positive measurable function h defined on $(0,\infty)$ and satisfying*

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\alpha, \; x > 0, \tag{3.2.6}$$

is said to be regularly varying at ∞ with index $\alpha \in \Re$, and we denote this by $h \in RV_\alpha^\infty$.

For a more thorough background on regular variation see Bingham *et al.* (1987).

We now introduce the bivariate elliptical family of distribution, using the approach of Abdous *et al.* (2005). For other properties see Fang *et al.* (1990).

Definition 3.2.2. *A bivariate elliptical random vector has the following stochastic representation:*

$$(X,Y) \stackrel{d}{=} (\mu_X, \mu_Y) + (\sigma_X RDU_1, \sigma_Y \rho RDU_1 + \sigma_Y \sqrt{1-\rho^2}\, R\, \sqrt{1-D^2}\, U_2), \tag{3.2.7}$$

where $U_1, U_2, R,$ and D are mutually independent random variables, $\mu_X, \mu_Y \in \Re$ are the respective means of X and Y, $\sigma_X, \sigma_Y > 0$ are the standard deviations, ρ is the Pearson correlation between X and Y, and $\Pr(U_i = -1) = \Pr(U_i = 1) = \frac{1}{2}$, $i = 1, 2$. *Both D and R are positive random variables and D has probability density function*

$$f_D(s) = \frac{2}{\pi\sqrt{1-s^2}}, \quad 0 < s < 1. \tag{3.2.8}$$

The random variable R is called the generator of the elliptical distributed random vector. R is the radial part of the parent spherical distribution of which the elliptical distribution is an affine transformation.

Throughout this chapter it is assumed that $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$. Therefore, the joint distribution of X and Y is symmetric, and X and Y are identically distributed. Our results can be extended to the more general setup.

The following examples give the generator pdfs for some well-known bivariate elliptical distributions. We refer to these examples later in the chapter. For more examples, see Fang *et al.* (1990), who use a more classical representation. Abdous *et al.* (2005) explain the relationship between the two representations.

Example 3.2.1. *Pearson type VII*

$$f_R(x) = \frac{2(N-1)}{m}\, x \left(1 + \frac{x^2}{m}\right)^{-N}, \quad x > 0, \; N > 1, m > 0.$$

When $m = 1$ and $N = 3/2$, we have the Cauchy distribution, and when $N = (m+2)/2$ we have the Student t distribution with m degrees of freedom.

Example 3.2.2. *Logistic*

$$f_R(x) = 4\,x\,\frac{\exp\{-x^2\}}{(1+\exp\{-x^2\})^2},\ x > 0.$$

Example 3.2.3. *Kotz*

$$f_R(x) = \frac{2s}{r^{-N/s}\Gamma(N/s)}\,x^{2N-1}\exp\{-rx^{2s}\},\ x > 0,\ N, r, s > 0.$$

When $N = 1$, $s = 1$, and $r = 1/2$, we have the normal distribution.

Finally, we shall use the notation $g(x) \sim h(x)$, as $x \to \infty$ to mean that

$$\lim_{x\to\infty}\frac{g(x)}{h(x)} = 1.$$

3.3 Main results

3.3.1 Componentwise maxima

The limiting distribution of componentwise maxima of iid elliptical random vectors is discussed in detail by Hashorva (2005). The following result shows that, in the bivariate case where the generator $R \in MDA(\Phi_\alpha)$, the limiting distribution of componentwise maxima of iid bivariate elliptical random vectors is exactly that obtained by Demarta and McNeil (2005) for the bivariate Student t distribution.

Proposition 3.3.3. *Let (X, Y) be a bivariate standardized elliptical random vector, and F its distribution function. If $R \in MDA(\Phi_\alpha)$, then $(X, Y) \in MDA(G)$, where G has Fréchet marginals, Φ_α, and the Pickands' representation is given by*

$$A(t) = t\bar{T}_{\alpha+1}\left\{\frac{[(\frac{1-t}{t})^{\frac{1}{\alpha}} - \rho]\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right\} + (1-t)\bar{T}_{\alpha+1}\left\{\frac{[(\frac{t}{1-t})^{\frac{1}{\alpha}} - \rho]\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right\}, \quad (3.3.1)$$

where $\bar{T}_\alpha$ is the survival function of a univariate Student t random variable with α degrees of freedom.

Proof. First, we show that $X \in MDA(\Phi_\alpha)$ whenever the generator $R \in MDA(\Phi_\alpha)$. The latter implies that $\bar{F}_R \in RV_{-\alpha}^{\infty}$ (see, for example, Embrechts *et al.*, 1997). Therefore, for $x > 0$, by conditioning on U_1 in (3.2.7) we get

$$\begin{aligned}\frac{\bar{F}_X(x)}{\bar{F}_R(x)} &= \frac{\Pr(RDU_1 > x)}{\bar{F}_R(x)} = \frac{1}{2}\frac{\Pr(RD > x)}{\bar{F}_R(x)} \\ &= \frac{1}{2}\int_0^1 \frac{\bar{F}_R(\frac{x}{u})}{\bar{F}_R(x)} f_D(u)\, du \to \frac{1}{2}\int_0^1 u^\alpha f_D(u)\, du \text{ as } x \to \infty, \end{aligned} \tag{3.3.2}$$

where the Dominated Convergence Theorem is used in the last step, since for x sufficiently large, the integrand is bounded by $u^{\alpha-1/2} f_D(u)$. The result can also be obtained from Lemma 2.2 of Hashorva (2005). Thus, $X \in MDA(\Phi_\alpha)$, and the normalizing constants for the maxima are given by $a_n \sim F_X^{\leftarrow}(1 - n^{-1})$ and $b_n = 0$ (see page 131 of Embrechts *et al.* 1997). It is sufficient to verify convergence criterion (3.2.1):

$$\begin{aligned}&n[1 - \Pr(X \le a_n x, Y \le a_n y)] \\ &\quad = n\Pr(X > a_n x) + n\Pr(Y > a_n y) - n\Pr(X > a_n x, Y > a_n y). \end{aligned} \tag{3.3.3}$$

Since X and $Y \in MDA(\Phi_\alpha)$, the first two terms on the right hand side of (3.3.3) have limits $x^{-\alpha}$ and $y^{-\alpha}$, respectively, and from Theorem 1 of Abdous *et al.* (2005)

we have

$$
\begin{aligned}
& n \Pr(X > a_n x, Y > a_n y) \\
= \quad & \frac{\Pr(X > a_n x, Y > a_n y)}{\Pr(X > a_n x)} n \bar{F}_X(a_n x) \\
\to \quad & x^{-\alpha} \bar{T}_{\alpha+1} \left\{ \frac{\left(\frac{y}{x} - \rho\right)\sqrt{\alpha+1}}{\sqrt{1-\rho^2}} \right\} + y^{-\alpha} \bar{T}_{\alpha+1} \left\{ \frac{\left(\frac{x}{y} - \rho\right)\sqrt{\alpha+1}}{\sqrt{1-\rho^2}} \right\}, \text{ as } n \to \infty.
\end{aligned} \tag{3.3.4}
$$

Combining (3.2.1), (3.2.3), (3.3.3) and (3.3.4) completes the proof. ■

3.3.2 Joint threshold exceedances

In financial applications, the limiting distribution of joint threshold exceedances is important in assessing the impact of extreme events affecting two or more variables of interest. For example, the losses in value of several different assets that result from a stock market crash can be viewed as dependent random variables. In analyzing the overall effect of the crash on the value of a portfolio, the dependence structure of these losses must be considered. If we are primarily interested in extreme cases, it is useful to understand the behaviour of joint exceedances over a high threshold.

When the threshold of interest for each asset is the Value at Risk (VaR), then we are interested in exceedances above high quantiles. The joint distribution of these exceedances is given by the upper copula.

The next result is motivated by the work of Breymann *et al.* (2003). There, an empirical approach was given to illustrate that the limiting upper copula of a bivariate elliptical random vector is well-fitted by the survival Clayton copula. If $R \in MDA(\Phi_\alpha)$, then under the assumption that the distribution function of the elliptical random vector is continuous with strictly increasing marginals, we can obtain an asymptotic result for the upper copula. This result is a direct implication of Theorem 2.3 of Juri and Wüthrich (2003), which states the following:

Let C be a symmetric copula such that $\hat{C}(v,v) > 0$ for all $v > 0$. Furthermore, assume that there is a strictly increasing continuous function $g : [0,\infty) \to [0,\infty)$, such that

$$\lim_{u\downarrow 0} \frac{\hat{C}(xv,v)}{\hat{C}(v,v)} = g(x), \quad x \in [0,\infty).$$

Then, there is $\theta > 0$ such that $g(x) = x^{\theta} g(1/x)$ for all $x \in (0,\infty)$. Further, for all $(x,y) \in [0,1]^2$

$$\lim_{u\uparrow 1} C_u^{up}(x,y) = x + y - 1 + G(g^{-1}(1-x), g^{-1}(1-y)),$$

where $G(x,y) := y^{\theta} g(x/y)$ for $(x,y) \in (0,1]^2$ and $G :\equiv 0$ on $[0,1]^2 \setminus (0,1]^2$.

Proposition 3.3.4. *Let (X,Y) be a standardized continuous elliptical random vector with strictly increasing margins. If $R \in MDA(\Phi_\alpha)$, then the limiting survival upper copula is given by*

$$\lim_{u\uparrow 1} \hat{C}_u^{up}(x,y) = g^{-1}(y) g\left(\frac{g^{-1}(x)}{g^{-1}(y)}\right), \tag{3.3.5}$$

where

$$g(x) = \frac{x\bar{T}_{\alpha+1}\left((x^{1/\alpha} - \rho)\frac{\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right) + \bar{T}_{\alpha+1}\left((x^{-1/\alpha} - \rho)\frac{\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right)}{2\,\bar{T}_{\alpha+1}\left\{(1-\rho)\sqrt{\frac{\alpha+1}{1-\rho^2}}\right\}}. \tag{3.3.6}$$

Proof. Letting $x > 0$, we only need to check the sufficient condition from Theorem 2.3

of Juri and Wüthrich (2003) as follows:

$$
\begin{aligned}
\frac{\hat{C}(xv, v)}{\hat{C}(v, v)} &= \frac{\Pr(X > \bar{F}_X^{\leftarrow}(xv), Y > \bar{F}_X^{\leftarrow}(v))}{\Pr(X > \bar{F}_X^{\leftarrow}(v), Y > \bar{F}_X^{\leftarrow}(v))} \\
&\sim \frac{\Pr(X > x^{-1/\alpha}\bar{F}_X^{\leftarrow}(v), Y > \bar{F}_X^{\leftarrow}(v))}{\Pr(X > \bar{F}_X^{\leftarrow}(v), Y > \bar{F}_X^{\leftarrow}(v))} \\
&\to g(x), \text{ as } v \downarrow 0,
\end{aligned}
$$

which gives the required result by applying Theorem 1 of Abdous *et al.* (2005) and the result of de Haan (1970, see page 22). ■

Proposition 3.3.4 is useful because it expresses the limiting distribution in terms of the two parameters α and ρ, which can be estimated using standard methods.

A comparison of contour plots shown in Figures 3.1, 3.2, and 3.3 indicate that, for three different α and ρ combinations, the copula in (3.3.5) is indeed similar to a Clayton copula. We have not, however, explored the relationship between the Clayton parameter and the values of α and ρ.

The main result of this chapter establishes the joint distribution of the exceedances over a high threshold when $R \in MDA(\Phi_\alpha)$ and when $R \in MDA(\Lambda)$. We first give some preliminary results.

If a distribution function $F \in MDA(\Lambda)$ has infinite support, then the auxiliary function $a(\cdot)$ that satisfies (3.2.2) is absolutely continuous with density $a'(\cdot)$ such that

$$
\lim_{t\to\infty} \frac{a(t)}{t} = 0, \ \lim_{t\to\infty} a'(t) = 0, \text{ and } \lim_{t\to\infty} \frac{a(t + xa(t))}{a(t)} = 1, \tag{3.3.7}
$$

locally uniformly in $x \in \Re$. For further details see Resnick (1987, p. 40).

The following lemma will be useful in proving the main result.

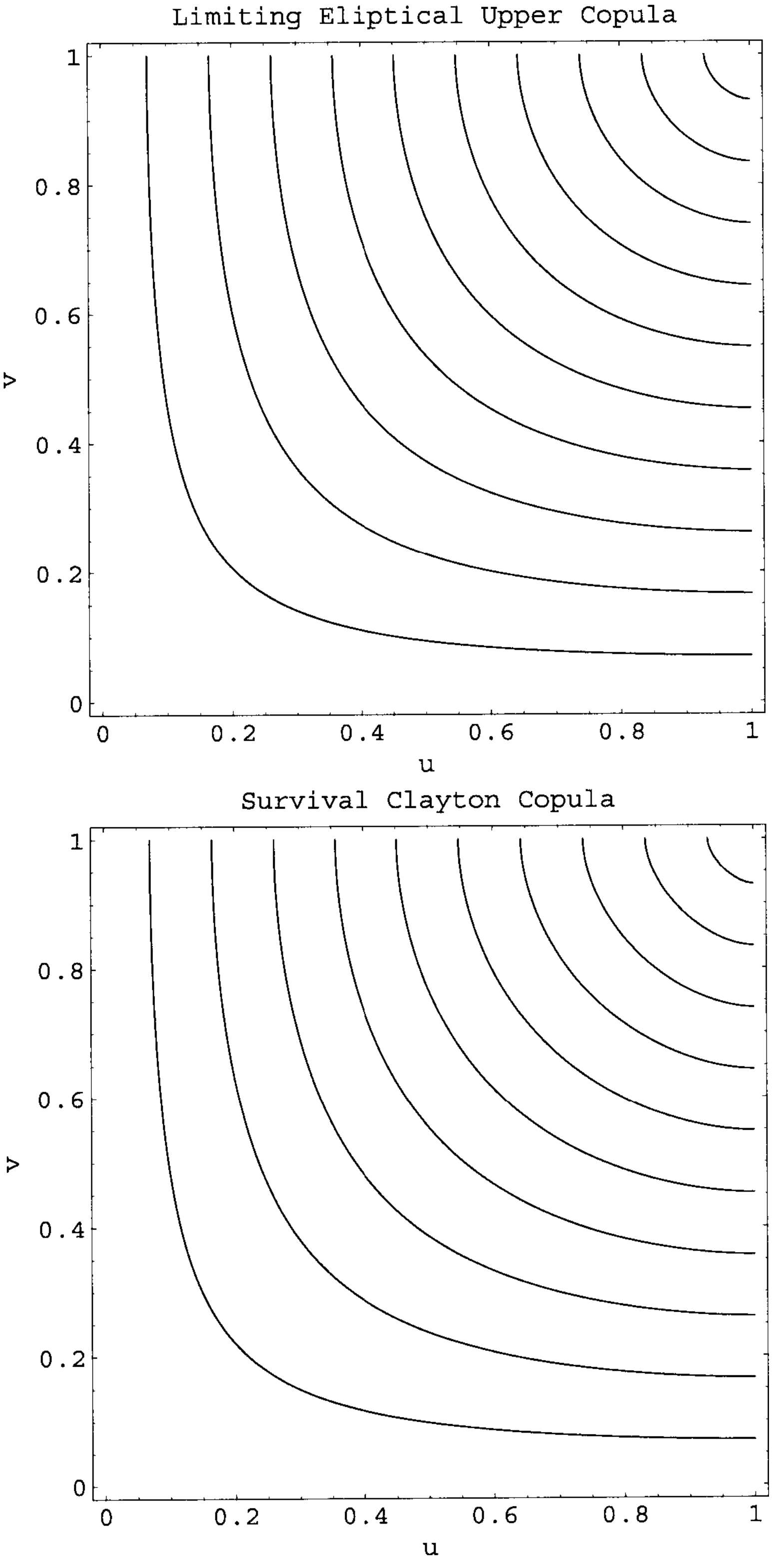

Figure 3.1: Contour plots of the limiting elliptical upper copula with $\alpha = 3$ and $\rho = 0.7$ compared with the survival Clayton copula with $\alpha = 0.9$.

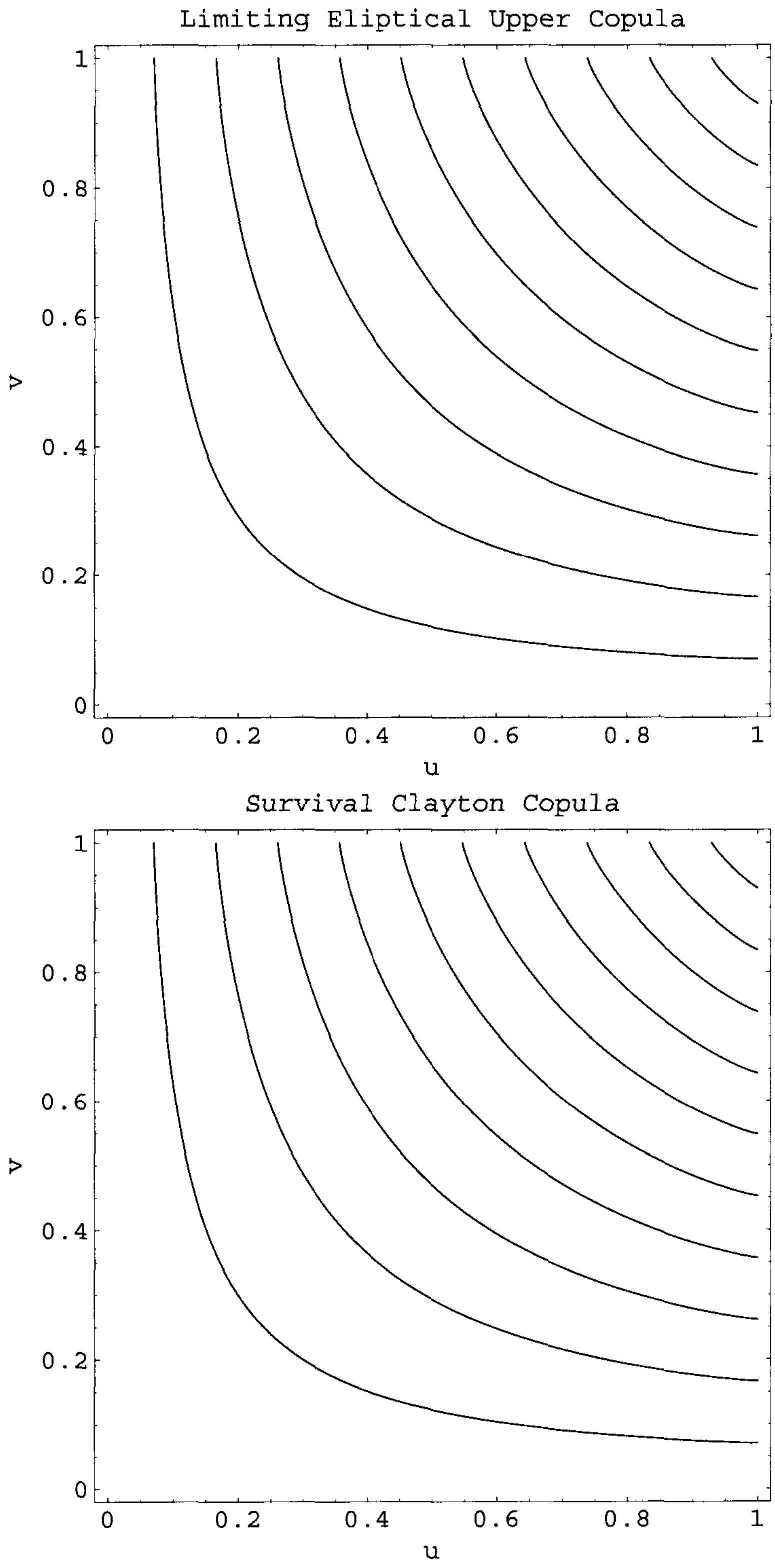

Figure 3.2: Contour plots of the limiting elliptical upper copula with $\alpha = 4$ and $\rho = -0.6$ compared with the survival Clayton copula with $\alpha = 0.25$.

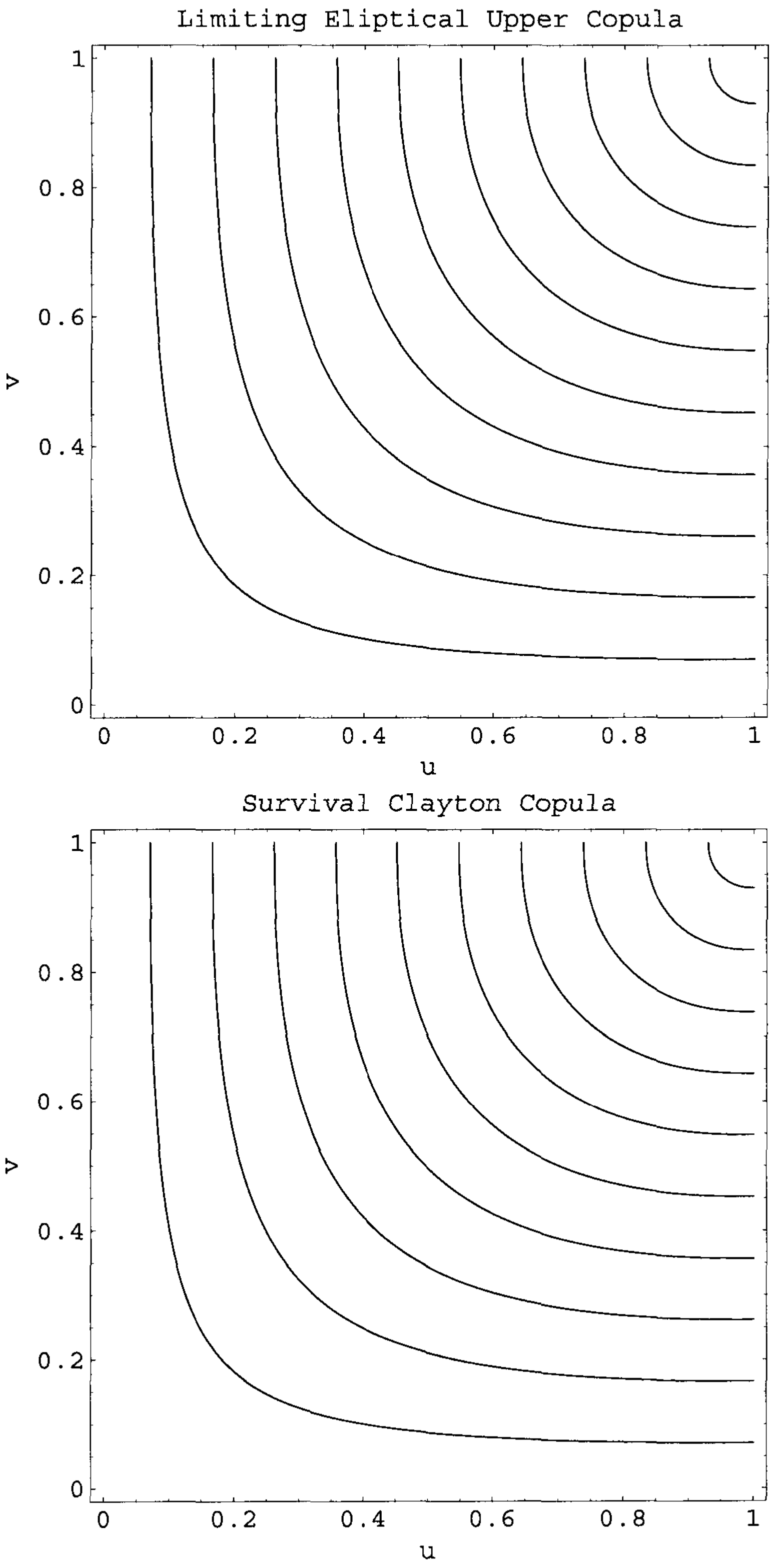

Figure 3.3: Contour plots of the limiting elliptical upper copula with $\alpha = 5$ and $\rho = 0.9$ compared with the survival Clayton copula with $\alpha = 1.5$.

Lemma 3.3.1. *If $F \in MDA(\Lambda)$ with $x_F = \infty$ and auxiliary function $a(\cdot)$, then, provided that $h(t) = o(a(t))$, the following holds for any x:*

$$\lim_{t\to\infty} \frac{\bar{F}(t + xa(t) + h(t))}{\bar{F}(t)} = \exp\{-x\}. \tag{3.3.8}$$

Proof. Let $h(t) = o(a(t))$. Then it is sufficient to verify that $\bar{F}(t + h(t)) \sim \bar{F}(t)$. Using a representation of Von Mises functions (see Resnick, 1987, p. 40) we need only prove that

$$\lim_{t\to\infty} \int_t^{t+h(t)} \frac{1}{a(u)}\, du = 0. \tag{3.3.9}$$

Let $\varepsilon, \delta > 0$, then since $a(\cdot)$ is positive, for t sufficiently large we get

$$\int_t^{t+h(t)} \frac{1}{a(u)}\, du \le \int_t^{t+a(t)\varepsilon} \frac{1}{a(u)}\, du = \int_0^{\varepsilon} \frac{a(t)}{a(t + za(t))}\, dz < (1+\delta)\varepsilon,$$

where the last inequality is implied by (3.3.7), which completes the proof. ■

Theorem 3.3.1. *Let (X, Y) be a bivariate standard elliptical random vector with $-1<\rho<1$.*

(a) Let $R \in MDA(\Phi_\alpha)$. Then whenever $x, y > 0$,

$$\begin{aligned} &\lim_{t\to\infty} \Pr(X > t + xa(t), Y > t + ya(t) | X > t, Y > t) \qquad (3.3.10)\\ &= \frac{\left(1+\frac{x}{\alpha}\right)^{-\alpha} \bar{T}_{\alpha+1}\left\{\left(\frac{\alpha+y}{\alpha+x} - \rho\right)\sqrt{\frac{\alpha+1}{1-\rho^2}}\right\} + \left(1+\frac{y}{\alpha}\right)^{-\alpha} \bar{T}_{\alpha+1}\left\{\left(\frac{\alpha+x}{\alpha+y} - \rho\right)\sqrt{\frac{\alpha+1}{1-\rho^2}}\right\}}{2\, \bar{T}_{\alpha+1}\left\{(1-\rho)\sqrt{\frac{\alpha+1}{1-\rho^2}}\right\}}, \end{aligned}$$

where $a(\cdot)$ is defined by (3.2.2).

(b) Let $R \in MDA(\Lambda)$ with auxiliary function $a(\cdot)$ and infinite right endpoint. If $a \in RV_\alpha^\infty$, $\alpha \le 1$, then whenever $x, y > 0$,

$$\begin{aligned} &\lim_{t\to\infty} \Pr(X > t + xa(t), Y > t + ya(t) | X > t, Y > t)\\ &= \exp\left\{-\frac{x+y}{2} K^{\alpha-1}(\rho)\right\}, \qquad (3.3.11) \end{aligned}$$

where $K(\rho) = \sqrt{(\rho+1)/2}$.

If $\rho = 1$, *then*

$$\lim_{t\to\infty} \Pr(X > t + xa(t), Y > t + xa(t) | X > t, Y > t) = \exp\{-\max(x, y)\}.$$

Remarks:

1. When $\rho = -1$, there exists a $t_0 > 0$ such that for all $t > t_0$, $\Pr(X > t, Y > t) = 0$. Since it does not make sense to condition on the event $\{X > t, Y > t\}$ in this case, an equivalent result cannot be obtained.

2. In the Gaussian case, $\alpha = -1$ and (3.3.11) coincides with the result of Juri and Wüthrich (2003).

Proof. (a) If $R \in MDA(\Phi_\alpha)$, then $a(t) \sim \frac{t}{\alpha}$ (see p. 159 Embrechts *et al.* 1997). Then the proof of (a) follows from Theorem 1 of Abdous *et al.* (2005).

(b) Let $x, y \geq 0$, and we assume that $\rho \in [0, 1)$ (the $\rho \in (-1, 0)$ case follows the same reasoning). We now prove that when $t \to \infty$ the following holds:

$$\frac{\Pr(X > t{+}xa(t), Y > t{+}xa(t))}{\bar{F}_R\left(\frac{t}{K(\rho)}\right)} \sim 2\frac{a(t)}{t}\frac{K^{2-\alpha}(\rho)}{\sqrt{1-K^2(\rho)}} \times \exp\left\{-K^{\alpha-1}(\rho)\frac{x+y}{2}\right\}. \tag{3.3.12}$$

By conditioning on U_1, U_2 and D, from Definition 3.2.2, for t sufficiently large, we obtain

$$\begin{aligned}\Pr(X > t + xa(t), Y > t + xa(t)) \\ = \frac{1}{2\pi}\Bigg[\int_0^1 \bar{F}_R\left(\max\left\{\frac{t+a(t)x}{u}, \frac{t+a(t)y}{f(u,\rho)}\right\}\right)\frac{1}{\sqrt{1-u^2}}\,du \\ + \int_{\sqrt{1-\rho^2}}^1 \bar{F}_R\left(\frac{t+a(t)x}{g(u,\rho)}\right)\frac{1}{\sqrt{1-u^2}}\,du\Bigg],\end{aligned} \tag{3.3.13}$$

where $f(u,\rho) = \rho u + \sqrt{1-\rho^2}\sqrt{1-u^2}$ and $g(u,\rho) = \rho u - \sqrt{1-\rho^2}\sqrt{1-u^2}$. Note that we have used the fact that $g(u,\rho) < 0$ when $u < \sqrt{1-\rho^2}$. Some simple algebraic computations allow one to express (3.3.13) as

$$\begin{aligned} &\Pr(X > t + xa(t), Y > t + xa(t)) \\ &\quad = \frac{1}{2\pi}\{I_1(t,x,y,\rho) + I_2(t,x,y,\rho) + I_3(t,x,y,\rho)\}, \end{aligned} \tag{3.3.14}$$

where the three integrals I_1, I_2, and I_3 are

$$I_1(t,x,y,\rho) = \int_0^{u(t,x,y,\rho)} \bar{F}_R\left(\frac{t+a(t)x}{u}\right)\frac{1}{\sqrt{1-u^2}}\,du, \tag{3.3.15}$$

$$I_2(t,x,y,\rho) = \int_{u(t,x,y,\rho)}^{1} \bar{F}_R\left(\frac{t+a(t)y}{f(u,\rho)}\right)\frac{1}{\sqrt{1-u^2}}\,du, \tag{3.3.16}$$

$$I_3(t,x,y,\rho) = \int_{\sqrt{1-\rho^2}}^{1} \bar{F}_R\left(\frac{t+a(t)y}{g(u,\rho)}\right)\frac{1}{\sqrt{1-u^2}}\,du, \tag{3.3.17}$$

and

$$u(t,x,y,\rho) = \left(\frac{\left(\frac{t+a(t)y}{t+a(t)x}\right)^2 - 2\rho\left(\frac{t+a(t)y}{t+a(t)x}\right) + 1}{1-\rho^2}\right)^{-1/2}. \tag{3.3.18}$$

We now have to determine the rates of convergence for each of the three integrals defined in (3.3.15), (3.3.16), and (3.3.17). First, we establish that

$$\begin{aligned} I_1(t,x,y,\rho) \sim &\frac{K^{2-\alpha}(\rho)}{\sqrt{1-K^2(\rho)\frac{a(t)}{t}}}\bar{F}_R\left(\frac{t}{K(\rho)}\right) \\ &\times \exp\left\{-K^{\alpha-1}(\rho)\frac{x+y}{2}\right\}, \text{ as } t\to\infty. \end{aligned} \tag{3.3.19}$$

The change of variable $u(t,x,y,\rho)/u = 1 + za(t)/t$ in (3.3.15) gives

$$\begin{aligned} I_1(t,x,y,\rho) = &\frac{a(t)}{t}u(t,x,y,\rho)\times \\ &\int_0^\infty \bar{F}_R\left(\frac{t+(x+z)a(t)+xza^2(t)/t}{u(t,x,y,\rho)}\right)\frac{(1+za(t)/t)^{-2}}{\sqrt{1-\left(\frac{u(t,x,y,\rho)}{1+za(t)/t}\right)^2}}\,dz. \end{aligned} \tag{3.3.20}$$

Using Lemma 3.3.1 and the fact that $a(\cdot) \in RV_\alpha^\infty$, straightforward computations yield that

$$\frac{\bar{F}_R\left(\frac{t+(x+z)a(t)+xza^2(t)/t}{u(t,x,y,\rho)}\right)}{\bar{F}_R(t/K(\rho))} \sim \exp\left\{-K^{\alpha-1}(\rho)\left(z+\frac{x+y}{2}\right)\right\}, \text{ as } t \to \infty. \quad (3.3.21)$$

Since $e^{-z} < 1/z(z+1)$ for $z \geq 2$ the integral in (3.3.20) is bounded, and the Dominated Convergence Theorem together with (3.3.7), (3.3.18), and (3.3.21) leads to (3.3.19).

In a similar manner asymptotic equivalences for I_2 and I_3 can be found. The one-to-one mapping $u \mapsto f(z,\rho)$ reduces (3.3.16) to

$$I_2(t,x,y,\rho) = \int_\rho^{z(t,x,y,\rho)} \bar{F}_R\left(\frac{t+a(t)y}{z}\right)\frac{1}{\sqrt{1-z^2}}\,dz, \quad (3.3.22)$$

where

$$z(t,x,y,\rho) = f(u(t,x,y,\rho),\rho). \quad (3.3.23)$$

The change of variable $z-\rho = (z(t,x,y,\rho)-\rho)/\left(1+\varsigma a(t)/t\right)$ in (3.3.22) yields

$$\begin{aligned} I_2(t,x,y,\rho) &= (z(t,x,y,\rho)-\rho)\frac{a(t)}{t}\times \\ &\int_0^\infty \bar{F}_R\left(\frac{t+(y+\varsigma)a(t)+y\varsigma a^2(t)/t}{z(t,x,y,\rho)+\rho\varsigma a(t)/t}\right)\frac{\left(1+\varsigma a(t)/t\right)^{-2}}{\sqrt{1-\left(\frac{z(t,x,y,\rho)+\rho\varsigma\frac{a(t)}{t}}{1+\varsigma a(t)/t}\right)^2}}\,d\varsigma, \quad (3.3.24)\end{aligned}$$

and straightforward computations together with Lemma 3.3.1 and the Dominated Convergence Theorem give

$$\begin{aligned} I_2(t,x,y,\rho) &\sim \frac{K^{2-\alpha}(\rho)}{\sqrt{1-K^2(\rho)\frac{a(t)}{t}}}\bar{F}_R\left(\frac{t}{K(\rho)}\right) \\ &\times\ \exp\left\{-K^{\alpha-1}(\rho)\frac{x+y}{2}\right\}, \text{ as } t \to \infty. \quad (3.3.25)\end{aligned}$$

The change of variable $z = g(u, \rho)$ in (3.3.17) yields

$$I_3(t, x, y, \rho) = \int_0^{\rho} \bar{F}_R\left(\frac{t + a(t)y}{z}\right) \frac{1}{\sqrt{1 - z^2}}\, dz. \tag{3.3.26}$$

In a similar way as for the previous two integrals, the rate of convergence for I_3 can be found when $\rho > 0$:

$$I_3(t, x, y, \rho) \sim \frac{\rho^{2-\alpha}}{\sqrt{1 - \rho^2}} \frac{a(t)}{t} \bar{F}_R\left(\frac{t}{\rho}\right) \exp\left\{-\rho^{\alpha-1} y\right\}, \text{ as } t \to \infty, \tag{3.3.27}$$

and by (3.3.17) $I_3 \equiv 0$ when $\rho = 0$. Moreover, when $\rho \geq 0$ it follows that $\rho < K(\rho)$, and since $\bar{F}_R$ is rapidly varying (see, for example, Embrechts, *et al.* 1997, p. 140) and using (3.3.27) we get

$$I_3(t, x, y, \rho) = o\left(\bar{F}_R\left(\frac{t}{K(\rho)}\right) \frac{a(t)}{t}\right). \tag{3.3.28}$$

Combining (3.3.14), (3.3.19), (3.3.25) and (3.3.28) gives (3.3.12) and (3.3.11), which completes the proof. ■

The Pearson type VII generator given in Example 3.2.1 is in the maximum domain of attraction of the Fréchet distribution with $\alpha = 2(N - 1)$, and the generators given in Examples 3.2.2 and 3.2.3 are in the maximum domain of attraction of the Gumbel distribution. The auxiliary functions $a(\cdot)$ are regularly varying with indices -1 and $1 - 2s$ for the Logistic and Kotz cases, respectively.

3.4 Illustration

In this section, we explore the sensitivity of the probabilities obtained from the limit distribution given by (3.3.10) in Theorem 3.3.1 to the values of α and ρ, and we illustrate how the theorem can be used in analyzing the joint distribution of returns on two stocks in the presence of an extreme event such as a market crash.

Table 3.1: Probabilities from equation (3.3.10) with $x = 1$ and $y = 3$ for various α and ρ

$\alpha\backslash\rho$	-0.9	-0.7	-0.2	0	0.1	0.5	0.8
3	0.2155	0.2145	0.2110	0.2089	0.2076	0.1994	0.1835
4	0.1971	0.1962	0.1929	0.1910	0.1898	0.1820	0.1664
5	0.1856	0.1847	0.1817	0.1799	0.1788	0.1714	0.1563

Table 3.1 shows joint probabilities obtained from equation (3.3.10) with $x = 1$ and $y = 3$ for several values of α and ρ. We observe that relatively small changes in α (relative to the range of possible values, $\alpha > 0$) lead to similar changes in the probabilities as relatively large changes in ρ $(-1 < \rho < 1)$. Therefore, these probabilities are sensitive to the value of α, while the value of ρ does not have an large impact.

Table 3.2: Approximate Values of $\Pr(X > 0.25+x, Y > 0.25+y \mid X > 0.25, Y > 0.25)$

x	y	Probability
0.1	0.1	0.2603
0.1	0.2	0.1456
0.1	0.3	0.0826
0.2	0.2	0.0953
0.2	0.3	0.0606
0.3	0.3	0.0427

We now illustrate the used of Theorem 3.3.1 in analyzing the conditional joint distribution of returns on two stocks when both are subject to large losses. Let X represent the negative daily log return for a given stock, and let Y represent the negative daily log return for another stock. Assume that (X, Y) is elliptically distributed with mean vector $(0,0)$, standard deviation vector $(0.01, 0.01)$, $\alpha = 4$ and $\rho = 0.5$. These parameters were chosen arbitrarily, but are intended to be plausible. We are interested in the conditional distribution of (X, Y) given that a significant loss has occurred on both stocks (perhaps due to a market crash). Specifically, we condition on the event that the negative log return on both stocks exceeds 0.25. That is, both stocks have decreased in value by at least (approximately) 22 percent. Table 3.2 shows several probabilities obtained from the conditional distribution of interest using the result of Theorem 3.3.1 (a). Calculations such as this allow one to correctly capture the impact of the dependence structure when analyzing the severity investment losses under extreme market conditions.

References

Abdous, B., Fougères, A.L. and Ghoudi, K. 2005. "Extreme Behaviour for Bivariate Elliptical Distributions," *The Canadian Journal of Statistics*, 33(3), 317-334.

Balkema, A.A. and de Haan, L. 1974. "The Residual Life Time at Great Age," *Annals of Probability*, 2(5), 792-804.

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. 2004. *Statistics of Extremes: Theory and Applications.* Wiley, Chichester.

Berman, M.S. 1992. *Sojourns and Extremes of Stochastic Processes.* Wadsworth & Brooks/Cole, Belmont, California.

Bingham, N.H., Goldie, C.M., and Teugels, J.L. 1987. *Regular Variation.* Cambridge

University Press, Cambridge.

Breymann, W., Dias, A. and Embrechts, P. 2003. "Dependence Structures for Multivariate High-frequency Data in Finance," *Quantitative Finance*, 3(1), 1-14.

Demarta, S. and McNeil, A.J. 2005. "The t Copula and Related Copulas," *International Statistical Review*, 73(1), 111-129.

Embrechts, P., Klüppelberg, C. and Mikosch, T. 1997. *Modelling Extremal Events for Insurance and Finance.* Springer-Verlag, Berlin.

Fang, K.T., Kotz, S. and Ng, K.W. 1990. *Symmetric Multivariate and Related Distributions.* Monographs on Statistics and Applied Probability, vol. 36, Chapman and Hall, London.

Gnedenko, B.V. 1943. "Sur la distribution limité du terme maximum d'une série aléatoaire," *Annals of Mathematics*, 44(3), 423-453.

de Haan, L. 1970. *On Regular Variation and its Application to the Weak Convergence of Sample Extremes.* Mathematisch Centrum Tract 32, Amsterdam.

Hashorva, E. 2005. "Extremes of Asymptotically Spherical and Elliptical Random Vectors," *Insurance: Mathematics and Economics*, 36(3), 285-302.

Hodgson, D.J., Linton, O. and Vorkink, K. 2002. "Testing the Capital Asset Pricing Model Efficiently under Elliptical Symmetry: A Parametric Approach," *Journal of Applied Econometrics*, 17(6), 617-639.

Hult, H. and Lindskog, F. 2002, "Multivariate Extremes, Aggregation and Dependence in Elliptical Distributions," *Advances in Applied Probability*, 34(3), 587-608.

Johnson, M.E. 1987. *Multivariate Statistical Simulation.* Wiley, New York.

Juri, A. and Wüthrich, M.V. 2003. "Tail Dependence from a Distributional Point of

View," *Extremes*, 6(3), 213-246.

Li, R.Z., Fang, K.T. and Zhu, L.X. 1997. "Some Q-Q Probability Plots to Test Spherical and Elliptical Symmetry," *Journal of Computational and Graphical Statistics*, 6(4), 435-450.

Manzotti, A., Pérez, F.J. and Quiroz, A.J. 2002. "A Statistic for Testing the Null Hypothesis of Elliptical Symmetry," *Journal of Multivariate Analysis*, 81(2), 274-285.

Marshall, A.W., Olkin, I. 1983. "Domains of Attraction of Multivariate Extreme Value Distributions," *Annals of Probability*, 11(1), 168-177.

Nelsen, R. B. 1999. *An Introduction to Copulas.* Springer-Verlag, New York.

Pickands, J. 1975. "Statistical Inference using Extreme Order Statistics," *Annals of Statistics*, 3(1), 119-131.

Pickands, J. 1981. "Multivariate Extreme Value Distributions," *Bulletin of the International Statistical Institute, Proceedings of the 43rd Session, Buenos Aires*, 49, 859-878.

Resnick, S.I. 1987. *Extreme Values, Regular Variation and Point Processes.* Springer-Verlag, New York.

Schmidt, R. 2002. "Tail Dependence for Elliptical Contoured Distributions," *Mathematical Methods of Operations Research*, 55(2), 301-327.

Sibuya, M. 1960. "Bivariate Extreme Statistics," *Annals of the Institute of Statistical Mathematics*, 11, 195-210.

Sklar, A. 1959. "Fonctions de répartion à n dimensions et leurs marges," Publications de l'Institut de Statistique de l'Université de Paris, 8, 229-231.

Chapter 4

Dependence and the Asymptotic Behavior of Large Claims Reinsurance

4.1 Introduction

Insurance companies often seek reinsurance to protect themselves against catastrophic losses. Such reinsurance comes in many forms. Excess of loss and stop loss coverages are common, and the risks associated with these coverages have been thoroughly studied in the literature. Two lesser-known reinsurances are ECOMOR (excédent du coût moyen relatif) and LCR (largest claims reinsurance). This may be due to their mathematical complexity. Under ECOMOR, the reinsurer pays the sum of the exceedances of the l largest claims over the $l+1$st largest claim. Under LCR, the reinsurer pays the sum of the l largest claims. These forms of reinsurance were introduced to actuaries by Thépaut (1950) and Ammeter (1964), respectively.

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The purpose of this chapter is to establish the asymptotic tail probabilities of the reinsurance amount under ECOMOR and LCR. This problem is considered by Ladoucette and Teugels (2006a and b) under the assumption that the claim amounts are iid and independent of the claim arrival process. Kremer (1998) provides an upper bound for the reinsurance premium when the claim amounts are not necessarily independent. In this chapter, we consider a different dependence assumption. That is, we assume that the interarrival time and the forthcoming claim size are dependent. In the context of ruin theory, similar risk models are discussed by Albrecher and Boxma (2004), Albrecher and Teugels (2006) and Boudreault *et al.* (2006).

We consider a risk process for which the claim sizes $X_i, i = 1, 2, \ldots$ are assumed to be positive iid rvs with common distribution function F. Moreover, the claim arrival process $\{N(u), u \geq 0\}$ is assumed to be a homogeneous Poisson process with intensity $\lambda > 0$. Let $X_{N(t),1} \geq X_{N(t),2}, \ldots$ be the order statistics corresponding to the claim sizes occurring on the time horizon of interest, $[0, t]$. Then the reinsurance amounts under ECOMOR and LCR are given by

$$E_l(t) = \sum_{i=1}^{l} (X_{N(t),i} - X_{N(t),l+1}) I_{\{N(t)>l\}}, \tag{4.1.1}$$

and

$$L_l(t) = \sum_{i=1}^{l} X_{N(t),i} I_{\{N(t)\geq l\}}. \tag{4.1.2}$$

As stated above, our primary objective is to obtain asymptotic tail probabilities for the reinsurance amount under ECOMOR and LCR reinsurance treaties. These results can be used in analyzing risk measures associated with these contracts.

4.2 Preliminaries

4.2.1 Definitions

The dependence structure associated with the distribution of a random vector can be characterized in terms of a *copula*. A two-dimensional copula is a bivariate distribution function defined on $[0,1]^2$ with uniformly distributed marginals. Due to Sklar's Theorem (see Sklar, 1959), if F is a joint distribution function with continuous marginals F_1 and F_2 respectively, then there exists a unique copula, C, given by

$$C(u,v) = F(F_1^{\leftarrow}(u), F_2^{\leftarrow}(v)),$$

where $h^{\leftarrow}(u) = \inf\{x : h(x) \geq u\}$. Similarly, the *survival copula* is defined as the copula relative to the joint survival function and is given by

$$\widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v).$$

A more formal definition and properties of copulas are given in Nelsen (1999).

There are many characterizations of heavy-tailed distributions, but one of the largest families is the class $\mathcal{L}$ of long-tailed distributions. By definition, a df $F = 1-\bar{F}$ belongs to $\mathcal{L}$ if

$$\lim_{t\to\infty} \frac{\bar{F}(t+x)}{\bar{F}(t)} = 1, \text{ for all } x \in \Re.$$

Note that, the convergence is uniform on compact subsets of $\Re$. For more details on heavy-tailed distributions, we refer the reader to Bingham *et al.* (1987) and Embrechts *et al.* (1997).

The long-tailed distributions form a large class that included distributions with regularly varying tails as well as the more general subexponential class. An example of a long-tailed distribution is the lognormal distribution, which is also a subexponential distribution, but does not have a regularly varying tail. The Pareto distribution is an example which does have a regularly varying tail.

The long-tailed distributions are a special case of the more general class $\mathcal{S}(a)$ for which

$$\lim_{t \to \infty} \frac{\bar{F}(t+x)}{\bar{F}(t)} = e^{-ax}, \text{ for all } x \in \Re, a \geq 0.$$

In addition to the long-tailed distributions, $\mathcal{S}(a)$ includes light-tailed distributions such as the exponential, Erlang, and more general phase-type distributions. While we focus on the long-tailed distributions in this chapter, we briefly discuss an extension of our results in the light-tailed case.

An important concept that is crucial to establishing the main results of this chapter is vague convergence. Let $\{\mu_n, n \geq 1\}$ be a sequence of measures on a locally compact space $\mathbb{E}$ with countable base. Then μ_n *converges vaguely* to some measure μ (written $\mu_n \xrightarrow{v} \mu$) if for all bounded continuous functions f with compact support we have

$$\lim_{n \to \infty} \int_{\mathbb{E}} f \, d\mu_n = \int_{\mathbb{E}} f \, d\mu.$$

A thorough background on vague convergence is given by Kallenberg (1983) and Resnick (1987).

4.2.2 Assumptions and Examples

Let W_i be the time between the $(i-1)^{st}$ and i^{th} claims. This model relaxes the usual assumption of independence between W_i and X_i. This may be necessary, for example, in modelling earthquake insurance, where the intensity of the earthquake depends on the time since the last occurrence. The following assumptions for the underlying dependence structure are sufficient to establish our main results.

Assumption 4.2.1. *The random vectors (X_i, W_i), $i = 1, \ldots, N(t)$, are mutually independent and identically distributed, and the generic pair (X_1, W_1) has absolutely continuous copula C with corresponding survival copula $\widehat{C}$.*

Assumption 4.2.2. *There exists a $v_0 \in (0,1)$ and a continuous bounded function g such that*

$$\lim_{u \downarrow 0} \frac{\widehat{c}_2(u,v)}{u} = g(v), \text{ for all } v \in [v_0, 1],$$

where $\widehat{c}_2(u,v) := \partial_v \widehat{C}(u,v)$.

Below are some examples of copulas given in Nelsen (1999) which satisfy Assumptions 4.2.1 and 4.2.2.

Example 4.2.1. *Independence*

$$C(u,v) = uv,$$

with $g(v) = 1$.

Example 4.2.2. *Ali-Mikhail-Haq*

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}, \ \theta \in [-1,1],$$

with $g(v) = 1 + \theta(1 - 2v)$.

Example 4.2.3. *Clayton*

$$C(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta},\ \theta \in (0, \infty),$$

with $g(v) = (1+\theta)(1-v)^{\theta}$.

Example 4.2.4. *Farlie-Gumbel-Morgenstern*

$$C(u,v) = uv + \theta uv(1-u)(1-v),\ \theta \in [-1, 1],$$

with $g(v) = 1 + \theta(1 - 2v)$.

Example 4.2.5. *Frank*

$$C(u,v) = -\frac{1}{\theta} \ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right),\ \theta \in \Re \setminus \{0\},$$

with $g(v) = \theta e^{\theta(1-v)}/(e^{\theta} - 1)$.

Example 4.2.6. *Plackett*

$$C(u,v) = \frac{1 + (\theta-1)(u+v) - \sqrt{(1 + (\theta-1)(u+v))^2 - 4uv\theta(\theta-1)}}{2(\theta-1)},\ \theta \in \Re_+ \setminus \{1\},$$

with $g(v) = \theta/(1 + (\theta-1)v)^2$.

Note that, while all six of the above examples involve a symmetric copula, this is not necessary. In particular, Assumptions 4.2.1 and 4.2.2 are satisfied by the asymmetric copula,

$$C_{k,l}(u,v) = u^{1-k} v^{1-l} C(u^k, v^l),$$

for many of the well-known absolutely continuous symmetric copulas C given in Nelsen (1999) and $0 < k, l < 1$. This construction of an asymmetric copula was proposed by Khoudraji (1995).

We also note that Assumptions 4.2.1 and 4.2.2 imply the existence of the limit

$$\lim_{x \to \infty} \Pr(W_1 \leq w \mid_{X_1 > x}).$$

This is a special case of the characterization of random vectors with one extreme component given by Heffernan and Resnick (2007).

Two examples of copulas that do not satisfy Assumption 4.2.2 are the Gaussian copula with $\rho \neq 0$ (see Abdous *et al.*, 2005) and the Gumbel copula.

4.3 Main results

4.3.1 Order statistics

In the first part of this section, we derive the asymptotic behavior of the lth largest order statistic $X_{N(t),l}$. Recall that the joint pdf of the interarrival times conditioned on the number of claims by time t is

$$f_{W_1,\ldots,W_n|N(t)=n}(w_1, \ldots, w_n) = \frac{n!}{t^n}, \text{ on } D_n = \left\{ \mathbf{w} : 0 < \sum_{j=1}^{i} w_j < t, i = 1, \ldots, n \right\}$$

(see, for example, Embrechts *et al.*, 1997, p. 187), and the marginals are identically distributed with common density

$$f_{W|N(t)=n}(w) = \frac{n(t-w)^{n-1}}{t^n}, \ 0 < w < t.$$

Proposition 4.3.5. *If Assumptions 4.2.1 and 4.2.2 are satisfied with $v_0 = e^{-\lambda t}$, then for any integer $l \geq 1$ we have*

$$\Pr(X_{N(t),l} > s) \sim [\Pr(X_1 > s)]^l \, K(l) \text{ as } s \to \infty,$$

where

$$K(l) = \int_0^t \int_0^{t-\omega_1} \cdots \int_0^{t-\sum_{i=1}^{l-1}\omega_i} h(t - \sum_{i=1}^{l} \omega_i, l) \prod_{i=1}^{l} g(e^{-\lambda\omega_i}) \, d\boldsymbol{w}$$

and

$$h(x, l) = e^{-\lambda t} \lambda^l \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} \binom{n+l}{l}.$$

Proof. For simplicity, we focus on the case in which $l = 1$. Extensions to $l > 1$ follow the same logic. We have

$$\begin{aligned} \Pr(X_{N(t),1} > s) &= \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \Pr\left(X_{N(t),1} > s \,|_{N(t)=n}\right) \qquad (4.3.1) \\ &= \sum_{n=1}^{\infty} e^{-\lambda t} \lambda^n \int_{D_n} \Pr\left(X_{N(t),1} > s \,|_{\boldsymbol{W}=\boldsymbol{w},\, N(t)=n}\right) d\boldsymbol{w} \\ &= \sum_{n=1}^{\infty} e^{-\lambda t} \lambda^n \int_{D_n} \left\{ 1 - \prod_{i=1}^{n} [1 - \Pr(X_1 > s \,|_{W_1=w_i})] \right\} d\boldsymbol{w}. \end{aligned}$$

Now,

$$\begin{aligned} &\sum_{n=1}^{\infty} e^{-\lambda t} \lambda^n \int_{D_n} \left\{ \sum_{i=1}^{n} \frac{\Pr(X_1 > s \,|_{W_1=w_i})}{\Pr(X_1 > s)} \right\} d\boldsymbol{w} \\ &= \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} n^2 \int_0^t \frac{\Pr(X_1 > s \,|_{W_1=w})}{\Pr(X_1 > s)} \times \frac{(t-w)^{n-1}}{t^n} \, dw. \qquad (4.3.2) \end{aligned}$$

And since the inequality

$$n \int_0^t \frac{\Pr(X_1 > s \,|_{W_1=w})}{\Pr(X_1 > s)} \times \frac{(t-w)^{n-1}}{t^n} \, dw < e^{\lambda t} \frac{n}{\lambda} \int_0^t \frac{(t-w)^{n-1}}{t^n} \, dw = e^{\lambda t}/\lambda$$

holds for any $s > 0$, we can apply the Dominated Convergence Theorem to show that (4.3.2) is asymptotically equivalent to

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-\lambda t} \frac{\lambda^n}{(n-1)!} n \int_0^t g(e^{-\lambda w})(t-w)^{n-1}\, dw \\ = e^{-\lambda t} \lambda \int_0^t g(e^{-\lambda w}) \sum_{n=0}^{\infty} \frac{(n+1)}{n!} [\lambda(t-w)]^n\, dw.\end{aligned}$$

Note that we used the fact that $\Pr(X_1 > s\, |_{W_1=w}) \sim \Pr(X_1 > s) g(e^{-\lambda w})$, which is a straightforward implication of Assumption 4.2.2. The interchange of the summation and integral is due to Pratt's Lemma (see Pratt, 1960). In a similar manner, the remaining terms of (4.3.1) can be shown to be $o(\Pr(X_1 > s))$. The proof for the case $l = 1$ is now complete. ∎

Some examples with a simple closed form for the asymptotic constant $K(1)$ from Proposition 4.3.5 are now given. In Example 4.2.1, the explicit form of the asymptotic constant is $K(l) = (\lambda t)^l / l!$, which is the l^{th} factorial moment of the counting process. That is,

$$K(l) = \mathbb{E}\left\{ \frac{N(t)(N(t)-1)\ldots(N(t)-l+1)}{l!} \right\}.$$

Examples 4.2.2 and 4.2.4 imply that $K(1) = \lambda t - (1 - e^{-2\lambda t})\theta/2$. In the case of Example 4.2.6, we have

$$K(1) = 1 - \frac{\theta}{\theta - 1 + e^{\lambda t}} - \frac{\lambda t + \theta \ln(\theta) - \theta \ln(\theta - 1 + e^{\lambda t})}{\theta - 1}.$$

For other cases, including $l > 1$, if a closed form is obtainable it is long and complicated.

4.3.2 ECOMOR and LCR reinsurance

This section contains the main results of this chapter. More specifically, the asymptotic tail probability results for the ECOMOR and LCR reinsurances on finite horizon $[0, t]$ are obtained. Recall that we allow dependence between claim amount and inter-arrival time and the number of claims process is assumed to be homogeneous Poisson. These results are motivated by the work of Ladoucette and Teugels (2006a) which assumes that the claim and number of claims processes are independent; the counting process is assumed to be a mixed Poisson process. They provide explicit results for the ECOMOR reinsurance when the horizon is finite. Specifically,

$$\Pr(E_l(t) > s) \sim \Pr(X_{N(t),1} > s) \text{ as } s \to \infty,$$

for any $l \geq 1$, provided that $X_1 \in \mathcal{L}$. We conclude that the same results follow under our assumptions for both reinsurances. This implies that the tail of the reinsurer's total claims is (asymptotically) the same regardless of how many upper order statistics are reinsured.

Theorem 4.3.1. *If Assumptions 4.2.1 and 4.2.2 are satisfied with $v_0 = e^{-\lambda t}$, and $X_1 \in \mathcal{L}$, then for any integer $l \geq 1$ we have*

$$\Pr(E_l(t) > s) \sim \Pr(L_l(t) > s) \sim \Pr(X_{N(t),1} > s) \text{ as } s \to \infty.$$

Remark: Using the logic of the following proof, one can conclude that the same asymptotic results hold in the case of independence and a mixed Poisson counting process. This provides an alternative proof for the tail probability of $E_l(t)$ to that given in Ladoucette and Teugels (2006a).

Proof. For simplicity we give the proof for $E_1(t)$ and $L_2(t)$. In light of the proof of Proposition 4.3.5 and the fact that $X_1 \in \mathcal{L}$, it is easy to obtain that

$$\lim_{s\to\infty} \frac{\Pr(X_{N(t),1} > s + x, X_{N(t),2} \le y)}{\Pr(X_1 > s)} = H(y)$$

for any real x and nonnegative y, where

$$H(y) := e^{-\lambda t} \sum_{n=1}^{\infty} \lambda^n \sum_{i=1}^{n} \int_{D_n} g(e^{-\lambda w_i}) \prod_{j \ne i} \Pr(X_1 \le y \,|_{W_1 = w_j})\, d\mathbf{w}. \tag{4.3.3}$$

This implies that

$$\mu_s(\cdot) := \frac{\Pr\left((X_{N(t),1} - s, X_{N(t),2}) \in \cdot\right)}{\Pr(X_1 > s)} \xrightarrow{v} \mu(\cdot), \tag{4.3.4}$$

where $\mu\left((x,\infty] \times [0,y]\right) = H(y)$. Note that $H(y) \le K(1)$ holds for any positive y and $H(y) \to K(1)$ when $y \to \infty$. Thus, $\mu\left(\{(x,y) : x \ge y \ge 0\}\right) = K(1)$, which together with (4.3.4) and Proposition 4.3.5 completes the proof for the ECOMOR case.

The LCR case is slightly different. For any $M > 0$,

$$\mu\left(\{(x,y) : x + y \ge 0, x > -M\}\right) = K(1)$$

. Now, since $\mu((a,b] \times [0,y]) = 0$,

$$\lim_{M\to\infty} \mu\left(\{(x,y) : x + y > 0, x \le -M, y \ge 0\}\right) = 0, \tag{4.3.5}$$

which completes the proof for the LCR case. ■

A straightforward generalization to the case in which $X_1 \in \mathcal{S}(a)$ is obtained for ECOMOR reinsurance. In this case, μ in (4.3.4) is given by

$$\mu((x,\infty] \times [0,y]) = e^{-ax} H(y),$$

where the function H is defined in (4.3.3). This modest extension, which we can provide only for $E_1(t)$, is stated as Proposition 4.3.6.

Proposition 4.3.6. *If Assumptions 4.2.1 and 4.2.2 are satisfied with $v_0 = e^{-\lambda t}$, and $X_1 \in \mathcal{S}(a)$, $a > 0$, then*

$$\Pr(E_1(t) > s) \sim \Pr(X_1 > s) \int_0^\infty a e^{-az} H(z)\, dz \text{ as } s \to \infty.$$

As a special case, under independence we have $H(z) = \lambda t \exp\{-\lambda t \Pr(X_1 > z)\}$.

4.3.3 Another Dependence Model

Boudreault *et al.* (2006) consider a risk process for which each claim amount is dependent on the waiting time until the claim as follows:

$$\Pr(X_1 > x \,|_{W_1 = w}) = e^{-\beta w} \bar{F}_1(x) + (1 - e^{-\beta w}) \bar{F}_2(x),$$

where $F_1 = 1 - \bar{F}_1$ and $F_2 = 1 - \bar{F}_2$ are distribution functions of positive random variables such that F_2 has a heavier tail than F_1. It follows that

$$\frac{\Pr(X_1 > x \,|_{W_1 = w})}{\Pr(X_1 > x)} \sim \frac{\lambda + \beta}{\beta}(1 - e^{-\beta w}), \;\; x \to \infty. \tag{4.3.6}$$

Therefore, Proposition 4.3.5 holds with $g(e^{-\lambda w})$ replaced by the right hand side of (4.3.6), and Theorem 4.3.1 holds. This illustrates that, even when we cannot explicitly characterize the dependence structure of W_1 and X_1 via the copula, we can still obtain the asymptotic results as long as the limit of $\Pr(X_1 > x \,|_{W_1 = w}) / \Pr(X_1 > x)$ exists.

4.4 Simulation Study

To explore the results given in Proposition 4.3.5 and Theorem 4.3.1, a simulation study was performed using the software R (see R Development Core Team, 2007). It was assumed that claim amounts have a Pareto distribution with distribution function

$$F_{X_1}(x) = 1 - (1+x)^{-\alpha}, \ x \geq 0$$

with α equal to 1 and 2. The dependence of the claim amount and the waiting time until the claim is given by the Ali-Mikhail-Haq copula given in Example 4.2.2 with values of θ equal to -0.9, 0.1 and 0.9.

Each analysis consists of 1,000,000 simulations of the risk process with $\lambda = 1$ and time horizon $t = 50$. For each simulation, the values of $X_{N(50),1}$, $L_2(50)$ and $E_1(50)$ were calculated. Probabilities associated with these three random variables were then estimated from the empirical distributions arising from the simulated samples of size 1,000,000. Probabilities associated with the random variable X_1, were estimated from the empirical distribution of all simulated claim amounts. These estimates were used to obtain the ratios presented in Tables 4.1, 4.2, 4.3 and 4.4.

For the ratios in Tables 4.1 and 4.2, the speed of convergence increases with θ, the strength of dependence. For $\alpha = 2$ the ratios converge quickly to 1.

Table 4.1: Estimated probability ratios, $\Pr(X_{N(50),1} > s)/K(1)\Pr(X_1 > s)$, when $\alpha = 1$ and $\theta \in \{-0.9, 0.1, 0.9\}$.

s	-0.9	0.1	0.9
500	0.9413	0.9533	0.9623
1000	0.9654	0.9772	0.9852
2000	0.9782	0.9884	0.9978
2500	0.9806	0.9908	0.9997
4000	0.9843	0.9942	1

The probabilities involving $L_2(50)$ and $E_1(50)$ are compared with those involving $X_{N(50),1}$ in Tables 4.3 and 4.4 for $\theta \in \{-0.9, 0.1, 0.9\}$ and $\alpha = 1$, $\alpha = 2$, respectively.

Table 4.2: Estimated probability ratios, $\Pr(X_{N(50),1} > s)/K(1)\Pr(X_1 > s)$, when $\alpha = 2$ and $\theta \in \{-0.9, 0.1, 0.9\}$.

s	-0.9	0.1	0.9
50	0.9815	0.9924	0.9999
150	0.9907	0.9997	1.0074
250	0.9912	0.9998	1.0088
500	0.9912	1.0010	1.0088
1000	0.9912	1.0010	1.0088

For both cases, there does not appear to be an effect from θ, indicating that unlike the maximum, LCR and ECOMOR are not affected by the strength of dependence. In addition, when $\alpha = 2$, the rate of convergence is faster than when $\alpha = 1$.

Table 4.3: Estimated probability ratios, $\Pr(L_2(50) > s)/\Pr(X_{N(50),1} > s)$ and $\Pr(E_1(50) > s)/\Pr(X_{N(50),1} > s)$, when $\alpha = 1$ and $\theta \in \{-0.9, 0.1, 0.9\}$.

	LCR			ECOMOR		
$s\backslash\theta$	-0.9	0.1	0.9	-0.9	0.1	0.9
500	1.2169	1.2155	1.2165	0.8394	0.8401	0.8395
1000	1.1456	1.1443	1.1422	0.8931	0.8928	0.8944
2000	1.0906	1.0853	1.0876	0.9334	0.9590	0.9332
2500	1.0740	1.0750	1.0755	0.9389	0.9402	0.9461
4000	1.0535	1.0509	1.0533	0.9563	0.9613	0.9594

Table 4.4: Estimated probability ratios, $\Pr(L_2(50) > s)/\Pr(X_{N(50),1} > s)$ and $\Pr(E_1(50) > s)/\Pr(X_{N(50),1} > s)$, when $\alpha = 2$ and $\theta \in \{-0.9, 0.1, 0.9\}$.

	LCR			ECOMOR		
$s\backslash\theta$	−0.9	0.1	0.9	−0.9	0.1	0.9
50	1.5466	1.5630	1.5508	0.7270	0.7268	0.7269
150	1.1964	1.1744	1.1797	0.8787	0.8873	0.8820
250	1.1023	1.0966	1.1006	0.9223	0.9136	0.9301
500	1.0506	1.0598	1.0276	0.9545	0.9620	0.9585
1000	1	1.0189	1.0204	0.9821	0.9783	1

References

Abdous, B., Fougères, A.L. and Ghoudi, K. 2005. "Extreme Behaviour for Bivariate Elliptical Distributions," *The Canadian Journal of Statistics*, 33(3), 317-334.

Albrecher, H. and Boxma, O.J. 2004. "A Ruin Model with Dependence between Claim Sizes and Claim Intervals," *Insurance: Mathematics and Economics*, 35(1), 245-254.

Albrecher, H. and Teugels, J.L. 2006. "Exponential Behavior in the Presence of Dependence in Risk Theory," *Journal of Applied Probability*, 43(1), 257-273.

Ammeter, H. 1964. "The Rating of Largest Claim Reinsurance Covers," *Quarterly Letter from the Algemeine Reinsurance Companies Jubilee*, Number 2, 5-17.

Bingham, N.H., Goldie, C.M., and Teugels, J.L. 1987. *Regular Variation.* Cambridge University Press, Cambridge.

Boudreault, M., Cossette, H., Landriault, D. and Marceau, E. 2006. "On a Risk Model with Dependence between Interclaim Arrivals and Claim Sizes," *Scandinavian Actuarial Journal*, 5, 265-285.

Embrechts, P., Klüppelberg, C. and Mikosch, T. 1997. *Modelling Extremal Events for Insurance and Finance.* Springer-Verlag, Berlin.

Heffernan, J.E. and Resnick, S.I. 2007. "Limit laws for random vectors with an extreme component," *Annals of Applied Probability*, 17(2), 537-571.

Kallenberg, O. 1983. *Random Measures*, 3rd edition Akademie-Verlag, Berlin.

Kremer, E. 1998. "Largest Claims Reinsurance Premiums under Possible Claims Dependence," *ASTIN Bulletin*, 28(2), 257-267.

Khoudraji, A. 1995. "Contributions à l'étude des copules et à la modélasion des valeurs extrêmes bivariées," Ph.D. thesis, Université Laval, Québec, Canada.

Ladoucette, S.A. and Teugels, J.L. 2006a. "Reinsurance of Large Claims," *Journal of Computational and Applied Mathematics*, 186(1), 163-190.

Ladoucette, S.A. and Teugels, J.L. 2006b. "Analysis of Risk Measures for Reinsurance Layers," *Insurance: Mathematics and Economics*, 38(3), 360-369.

Nelsen, R. B. 1999. *An Introduction to Copulas.* Springer-Verlag, New York.

Pratt, J.W. 1960. "On Interchanging limits and integrals," *Annals of Mathematical Statistics*, 31(1), 74-77.

R Development Core Team (2007). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0, URL `http://www.R-project.org`.

Resnick, S.I. 1987. *Extreme Values, Regular Variation and Point Processes.* Springer-

Verlag, New York.

Sklar, A. 1959. "Fonctions de répartion à n dimensions et leurs marges," Publications de l'Institut de Statistique de l'Université de Paris, 8, 229-231.

Thépaut, A. 1950. "Une nouvelle forme de réassurance: le traité d'excédent de coût moyen relatif (ECOMOR)," *Bulletin Trimestriel de l'Institut des Actuaires Français*, 49, 273-343.

Chapter 5

Asymptotic Tail Probabilities for Large Claims Reinsurance of a Portfolio of Dependent Risks

5.1 Introduction

Insurance companies often use reinsurance as a mechanism for sharing risk, particularly when there is the possibility of catastrophic losses. Two appealing reinsurances are ECOMOR (excédent du coût moyen relatif) and LCR (largest claims reinsurance). Under ECOMOR, the reinsurer pays the sum of the exceedances of the l largest claims over the $l+1$st largest claim. Under LCR, the reinsurer pays the sum of the l largest claims. ECOMOR and LCR treaties were proposed by Thépaut (1950) and Ammeter (1964), respectively.

We consider a portfolio of n similar insurance contracts. The associated loss random variables $X_i, i = 1, \ldots, n$ are assumed to be dependent and identically dis-

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tributed with common df $F = 1 - \bar{F}$ and dependence structure given by a suitable copula. Let $X_{1,n} \geq \ldots \geq X_{n,n}$ be the corresponding upper order statistics. Then the reinsurance amounts under ECOMOR and LCR are given by

$$E_l = \sum_{i=1}^{l} (X_{i,n} - X_{l+1,n}),$$

and

$$L_l = \sum_{i=1}^{l} X_{i,n}.$$

The purpose of this chapter is to establish the asymptotic tail probabilities of the reinsurance amount under ECOMOR and LCR for a portfolio of dependent insurance contracts. This may be quite useful for risk management purposes, as it allows one to determine high quantiles of the reinsurance amount and therefore enables one to obtain capital amounts that will be adequate with high probability. This can also be done by performing a simulation study. However, to estimate high quantiles, a very large number of simulations are required, and since multivariate outcomes must be generated, the computations may be very time consuming.

5.2 Preliminaries

Let Y_i, $i = 1, 2, \ldots$ be a sequence of independent random variables with common distribution F, and let M_n be the maximum of $Y_1, \ldots, Y_n$. If there exist constants a_n, b_n and a random variable Z with nondegenerate df G such that $a_n M_n + b_n$ converges weakly to Z, then F is in the *maximum domain of attraction* of G and we write $F \in \mathrm{MDA}(G)$. Moreover, by the Fisher-Tippett theorem (see, for example, Embrechts *et al.*, 1997), G belongs to the type of the distribution

$$H_\xi(x) = \begin{cases} \exp\left\{-(1+\xi x)^{-1/\xi}\right\}, & 1+\xi x > 0, \quad \xi \neq 0 \\ \exp\{-e^{-x}\}, & -\infty < x < \infty, \quad \xi = 0 \end{cases}.$$

H_ξ is known as the *generalized extreme value distribution.* For $\alpha > 0$, $\Phi_\alpha(x) := H_{1/\alpha}(\alpha(x-1))$ is the standard Fréchet distribution, $\Psi_\alpha(x) := H_{-1/\alpha}(\alpha(x+1))$ is the standard Weibull distribution, and $\Lambda(x) := H_0(x)$ is the standard Gumbel distribution.

The dependence structure associated with the distribution of a random vector can be characterized in terms of a *copula.* An n-dimensional copula is a multivariate df defined on $[0,1]^n$ with uniformly distributed marginals. Due to Sklar's Theorem (see Sklar, 1959), if $X_1, \ldots, X_n$ has a joint distribution function with continuous marginals, then there exists a unique copula, C, such that

$$\Pr(X_1 \leq x_1, \ldots, X_n \leq x_n) = C\big(\Pr(X_1 \leq x_1), \ldots, \Pr(X_n \leq x_n)\big).$$

Similarly, the *survival copula*, $\widehat{C}$, is defined as the copula relative to the joint survival function and satisfies

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) = \widehat{C}\big(\Pr(X_1 > x_1), \ldots, \Pr(X_n > x_n)\big).$$

A well-known class of copulas is the Archimedean class. By definition, an *Archimedean copula* C is given by

$$C(u_1, \ldots, u_n) = \varphi^{-1}\left(\sum_{i=1}^{n} \varphi(u_i)\right),$$

where $\varphi : [0,1] \mapsto [0,\infty)$ is its generator. Some regularity conditions are necessary to ensure that C is a valid copula (see Kimberling, 1974 and Nelsen, 1999, chapter 4).

An important concept that is crucial to establishing the main results of this chapter is vague convergence. Let $\{\mu_n, n \geq 1\}$ be a sequence of measures on a locally compact space $\mathbb{E}$ with countable base. Then μ_n *converges vaguely* to some measure μ (written $\mu_n \xrightarrow{v} \mu$) if for all bounded continuous functions f with compact support we have

$$\lim_{n\to\infty} \int_{\mathbb{E}} f \, d\mu_n = \int_{\mathbb{E}} f \, d\mu.$$

A thorough background on vague convergence is given by Kallenberg (1983) and Resnick (1987).

5.3 Main Results

Throughout this chapter it is assumed that the common df $F = 1 - \bar{F}$ has positive support and infinite right endpoint. For ease of exposition, we first assume that the survival copula, which describes the dependence among portfolio risks, is a member of the Archimedean class. This setup is used by Wüthrich (2003) and Alink *et al.* (2004 and 2005) in order to characterize the asymptotic tail behavior for a sum of dependent random variables. A similar problem is discussed by Albrecher *et al.* (2006), Barbe *et al.* (2006) and Kortschak and Albrecher (2007), when a more general dependence structure is assumed. Since the ECOMOR and LCR reinsurances are linear combinations of the order statistics, studying the asymptotic tail probability for the losses associated with these reinsurance treaties is closely related to the aforementioned problem.

We make the additional assumption that the generator φ of the survival copula is regularly varying at 0 with index $-\alpha$ ($\varphi \in RV^0_{-\alpha}$). That is,

$$\lim_{t \uparrow 0} \frac{\varphi(tx)}{\varphi(t)} = x^{-\alpha},$$

for any positive x. For more details on regular variation, we refer the reader to Bingham *et al.* (1987).

The Clayton copula is an example of an Archimedean copula with generator, $\varphi(u) = u^{-\alpha} - 1$, which satisfies the property $\varphi \in RV^0_{-\alpha}$. This copula has the form

$$C(u_1, \ldots, u_n) = \left(1 - n + \sum_{i=1}^{n} u_i^{-\alpha}\right)^{-1/\alpha},$$

where $\alpha > 0$.

Our assumption that the individual loss df F has infinite right endpoint implies that only $F \in MDA(\Phi_\beta)$ or $F \in MDA(\Lambda)$ may hold. We consider these two cases in turn.

5.3.1 Results for F in MDA of Fréchet

If $F \in MDA(\Phi_\beta)$ and $\varphi \in RV^0_{-\alpha}$, then for any positive $x_1, \ldots, x_l$ with $1 \leq l \leq n$,

$$\lim_{t \to \infty} \frac{\Pr(X_1 > tx_1, \ldots, X_l > tx_l)}{\bar{F}(t)} = \left(\sum_{i=1}^{l} x_i^{\alpha\beta}\right)^{-1/\alpha}, \tag{5.3.1}$$

provided that $0 < \alpha < \infty$ (see Alink *et al.* 2004).

Now, as a result of our assumptions, the random variables $X_1, \ldots, X_n$ are exchangeable. Therefore,

$$\begin{aligned} &\Pr(X_{1,n} > tx_1, \ldots,\ X_{l,n} > tx_l) \qquad (5.3.2) \\ &= \sum_{(k_1,\ldots,k_l) \in A_l} \frac{n!}{k_1! \cdots k_l!\ (n - k_1 - \cdots - k_l)!} \Pr\Big(\{X_1, \ldots, X_{k_1} > tx_1\}, \\ &\qquad \{tx_2 < X_{k_1+1}, \ldots, X_{k_1+k_2} \leq tx_1\}, \cdots, \{X_{k_1+\ldots+k_l+1}, \ldots, X_n \leq tx_l\}\Big), \end{aligned}$$

for any $x_1 > \ldots > x_l$, where $A_l = \{(k_1, \ldots, k_l) : i \le k_1 + \ldots + k_i \le n, i = 1, \ldots, l\}$. Each term on the right-hand side of (5.3.2) can be expressed as a linear combination of joint survival probabilities. This fact combined with (5.3.1) allows us to conclude that there exists a positive function f_l such that

$$\Pr(X_{1,n} > tx_1, \ldots, X_{l,n} > tx_l) \sim \bar{F}(t) f_l(x_1, \ldots x_l), \ t \to \infty. \tag{5.3.3}$$

Under more general assumptions for which the exchangeability property does not hold, a similar but even more cumbersome relationship to that in (5.3.2) can be obtained.

Now, relation (5.3.3) implies that

$$\frac{\Pr\left((X_{1,n}/t, \ldots, X_{l,n}/t) \in \cdot\right)}{\Pr(X_1 > t)} \xrightarrow{v} \mu_l(\cdot),$$

holds on $[0, \infty]^l \setminus \{\mathbf{0}\}$ where the measure μ_l is given by

$$\mu_l\big((x_1, \infty] \times \cdots \times (x_l, \infty]\big) := f_l(x_1, \ldots, x_l). \tag{5.3.4}$$

We now have the essential development for the main results of this subsection, which are stated in the following theorem.

Theorem 5.3.1. *Let $(X_1, \ldots, X_n)$ be a positive random vector with an Archimedean survival copula for which the generator satisfies $\varphi \in RV^0_{-\alpha}$ with $\alpha \in (0, \infty)$. In addition, the marginals are identically distributed with df $F \in MDA(\Phi_\beta)$. For $l = 1, \ldots, n-1$, the asymptotic tail probability for E_l, the reinsurance amount under an ECOMOR treaty, is given by*

$$\Pr(E_l > t) \sim C_{EF}(l, \alpha, \beta)\ \bar{F}(t) \text{ as } t \to \infty,$$

where

$$C_{EF}(l, \alpha, \beta) = \mu_{l+1}\left(\boldsymbol{x} : \sum_{i=1}^{l} x_i - l x_{l+1} \ge 1, x_1 \ge \cdots \ge x_{l+1} \ge 0\right),$$

with μ_l defined by (5.3.4).

For $l = 1, \dots, n$, the asymptotic tail probability for L_l, the reinsurance amount under an LCR treaty, is given by

$$\Pr(L_l > t) \sim C_{LF}(l, \alpha, \beta)\ \bar{F}(t) \text{ as } t \to \infty,$$

where

$$C_{LF}(l, \alpha, \beta) = \mu_l \left(\boldsymbol{x} : \sum_{i=1}^{l} x_i \geq 1, x_1 \geq \cdots \geq x_l \geq 0 \right).$$

It should be noted that in order to obtain these results, we used the fact that each measure μ_l contributes zero mass to $\bigcup_{i=1}^{l} \{x_i = \infty\}$.

5.3.2 Result for F in MDA of Gumbel

As in the Fréchet case, the first step is to establish the joint tail extreme behavior. It is well-known (see, for example, Embrechts *et al.*, 1997) that if $F \in MDA(\Lambda)$, then there exists a positive, measurable function $a(\cdot)$ such that

$$\lim_{t \to \infty} \frac{\bar{F}(t + xa(t))}{\bar{F}(t)} = e^{-x}, \tag{5.3.5}$$

for any real x. Once again, we assume that $\varphi \in RV^0_{-\alpha}$, which gives that

$$\lim_{t \to \infty} \frac{\Pr(X_1 > t + x_1 a(t), \dots, X_l > t + x_l a(t))}{\bar{F}(t)} = \left(\sum_{i=1}^{l} e^{\alpha x_i} \right)^{-1/\alpha}, \tag{5.3.6}$$

for any real $x_1, \dots x_l$ with $1 \leq l \leq n$ (see Alink *et al.* 2004).

In the same manner as the previous subsection, we have

$$\Pr(X_{1,n} > t + x_1 a(t), \dots, X_{l,n} > t + x_l a(t)) \sim \bar{F}(t)\ g_l(x_1, \dots x_l), \tag{5.3.7}$$

where g_l is a positive function.

Now, relation (5.3.7) implies that

$$\frac{\Pr\Big(\big((X_{1,n}-t)/a(t),\ldots,(X_{l,n}-t)/a(t)\big)\in\cdot\Big)}{\Pr(X_1>t)}\xrightarrow{v}\nu_l(\cdot),$$

holds on $(-\infty,\infty]^l$ where the measure ν_l is given by

$$\nu_l\big((x_1,\infty]\times\cdots\times(x_l,\infty]\big):=g_l(x_1,\ldots,x_l). \tag{5.3.8}$$

Now, we are able to give the main result from this subsection, which is only for the LCR reinsurance. This is stated as Theorem 5.3.2.

Theorem 5.3.2. *Let $(X_1,\ldots,X_n)$ be a positive random vector with an Archimedean survival copula for which the generator satisfies $\varphi\in RV^0_{-\alpha}$ with $\alpha\in(0,\infty)$. In addition, the marginals are identically distributed with df $F\in MDA(\Lambda)$. For $l=1,\ldots,n$, we have*

$$\Pr(L_l>lt)\sim C_{LG}(l,\alpha,\beta)\,\bar{F}(t)\text{ as }t\to\infty,$$

where

$$C_{LG}(l,\alpha)=\nu_l\left(\boldsymbol{x}:\sum_{i=1}^{l}x_i\geq 0,x_1\geq\cdots\geq x_l\right),$$

with ν_l defined by (5.3.8).

Two more remarks are useful in understanding Theorem 5.3.2. First, note that each measure ν_l contributes zero mass to $\bigcup_{i=1}^{l}\{x_i=\infty\}$. Second, ν_l has no mass on regions around $-\infty$. This is obvious for $l=1$, so we consider the case in which $l>1$. It is sufficient to check that

$$\lim_{M\to\infty}\nu_l\left(\mathbf{x}:\sum_{i=1}^{l}x_i\geq 0,x_1\geq\cdots\geq x_{l-1}\geq -M\geq x_l\right)=0. \tag{5.3.9}$$

In doing so, we first mention that the following clearly holds

$$\begin{aligned}\Pr(X_{1,n}>t)&=\binom{n}{1}\Pr(X_1>t)-\cdots+(-1)^{n+1}\binom{n}{n}\Pr(X_1>t,\ldots,X_n>t)\\&\sim\Delta\bar{F}(t),\text{ as }t\to\infty,\end{aligned}\tag{5.3.10}$$

where the last step is due to (5.3.6) and Δ is a positive constant. Combining (5.3.5) and (5.3.10), we have

$$\begin{aligned}\nu_l & \left(\mathbf{x} : \sum_{i=1}^{l} x_i \geq 0, x_1 \geq \cdots \geq x_{l-1} \geq -M \geq x_l\right) \\ & \leq \lim_{t\to\infty} \frac{\Pr\left(X_{1,n} > t + a(t)\frac{M}{l-1}\right)}{\bar{F}(t)} = \Delta e^{-M/(l-1)},\end{aligned}$$

which leads to (5.3.9).

5.3.3 Examples

In this subsection, examples for the limiting constants from Theorems 5.3.1 and 5.3.2 are given. In order to avoid long computations, a portfolio consisting of $n = 3$ insurance contracts is considered. First, the Fréchet case is explored. From (5.3.2), we have

$$\begin{aligned}\Pr(X_{1,3} > tx_1, X_{2,3} > tx_2) &= \Pr(X_1, X_2, X_3 > tx_1) + 3\Pr(X_1, X_2 > tx_1, X_3 \leq tx_2) \\ &\quad + 3\Pr(X_1, X_2 > tx_1, tx_2 < X_3 \leq tx_1) \\ &\quad + 3\Pr(X_1 > tx_1, tx_2 < X_2, X_3 \leq tx_1) \\ &\quad + 6\Pr(X_1 > tx_1, tx_2 < X_2 \leq tx_1, X_3 \leq tx_2),\end{aligned}$$

for any $x_1 > x_2 > 0$. Otherwise,

$$\Pr(X_{1,3} > tx_1, X_{2,3} > tx_2) = \Pr(X_1, X_2, X_3 > tx_2) + 3\Pr(X_1, X_2 > tx_2, X_3 \leq tx_2).$$

Straightforward computations together with (5.3.1) yield the following

$$f_2(x_1, x_2) = \begin{cases} (3^{-1/\alpha} - 3\cdot 2^{-1/\alpha})x_1^{-\beta} + 6(x_1^{\alpha\beta} + x_2^{\alpha\beta})^{-1/\alpha} & \\ \qquad -3(x_1^{\alpha\beta} + 2x_2^{\alpha\beta})^{-1/\alpha}, & 0 < x_2 < x_1 \\ (3\cdot 2^{-1/\alpha} - 2\cdot 3^{-1/\alpha})x_2^{-\beta}, & 0 < x_1 \leq x_2 \end{cases} . \quad (5.3.11)$$

In a similar manner, if $F \in MDA(\Lambda)$ then (5.3.6) yields

$$g_2(x_1, x_2) = \begin{cases} (3^{-1/\alpha} - 3 \cdot 2^{-1/\alpha})e^{-x_1} + 6(e^{\alpha x_1} + e^{\alpha x_2})^{-1/\alpha} & \\ \qquad -3(e^{\alpha x_1} + 2e^{\alpha x_2})^{-1/\alpha}, & 0 < x_2 < x_1 \\ (3 \cdot 2^{-1/\alpha} - 2 \cdot 3^{-1/\alpha})e^{-x_2}, & 0 < x_1 \leq x_2 \end{cases} . \quad (5.3.12)$$

The measure $\mu_2\big((x_1, \infty] \times (x_2, \infty]\big) := f_2(x_1, x_2)$, and it follows from Theorem 5.3.1 that the respective constants for ECOMOR and LCR are

$$\begin{aligned} & C_{EF}(1, \alpha, \beta) \\ &= \mu_2\big((x_1, x_2) : x_1 - x_2 \geq 1, 0 \leq x_2 \leq x_1\big) \\ &= 6\beta \int_0^\infty t^{\alpha\beta - 1} \left\{ \left[t^{\alpha\beta} + (1+t)^{\alpha\beta}\right]^{-1-1/\alpha} - \left[2t^{\alpha\beta} + (1+t)^{\alpha\beta}\right]^{-1-1/\alpha} \right\} dt \end{aligned}$$

and

$$\begin{aligned} & C_{LF}(2, \alpha, \beta) \\ &= \mu_2\big((x_1, x_2) : x_1 + x_2 \geq 1, 0 \leq x_2 \leq x_1\big) \\ &= \mu_2\big((x_1, x_2) : x_1 = x_2 \geq 1/2\big) + \mu_2\big((x_1, x_2) : x_1 + x_2 \geq 1, 0 \leq x_2 < x_1\big) \\ &= f_2(1/2, 1/2) \\ &\quad + 6(1+\alpha)\beta^2 \int_{1/2}^1 \int_{1-s}^s (st)^{\alpha\beta - 1} \left[\left(s^{\alpha\beta} + t^{\alpha\beta}\right)^{-2-1/\alpha} - \left(s^{\alpha\beta} + 2t^{\alpha\beta}\right)^{-2-1/\alpha} \right] dt\, ds \\ &\quad + 6(1+\alpha)\beta^2 \int_1^\infty \int_0^s (st)^{\alpha\beta - 1} \left[\left(s^{\alpha\beta} + t^{\alpha\beta}\right)^{-2-1/\alpha} - \left(s^{\alpha\beta} + 2t^{\alpha\beta}\right)^{-2-1/\alpha} \right] dt\, ds \\ &= 3 + 3 \cdot 2^{-1/\alpha}(2^\beta - 1) + 3^{-1/\alpha}(1 - 2^{\beta+1}) \\ &\quad + 6(1+\alpha)\beta^2 \int_{1/2}^1 \int_{1-s}^s (st)^{\alpha\beta - 1} \left[\left(s^{\alpha\beta} + t^{\alpha\beta}\right)^{-2-1/\alpha} - \left(s^{\alpha\beta} + 2t^{\alpha\beta}\right)^{-2-1/\alpha} \right] dt\, ds. \end{aligned}$$

The measure $\nu_2\big((x_1, \infty] \times (x_2, \infty]\big) := g_2(x_1, x_2)$ and from Theorem 5.3.2 the limiting constant for LCR is

$$
\begin{aligned}
& C_{LG}(2,\alpha) \\
&= \nu_2\big((x_1,x_2) : x_1 + x_2 \geq 0, x_1 \geq x_2\big) \\
&= \nu_2\big((x_1,x_2) : x_1 = x_2 \geq 0\big) + \nu_2\big((x_1,x_2) : x_1 + x_2 \geq 0, x_1 > x_2\big) \\
&= 3 \cdot 2^{-1/\alpha} - 2 \cdot 3^{-1/\alpha} \\
&\quad +6(1+\alpha) \int_0^\infty \int_{-s}^{s} e^{\alpha(s+t)} \left[\big(e^{\alpha s} + e^{\alpha t}\big)^{-2-1/\alpha} - \big(e^{\alpha s} + 2e^{\alpha t}\big)^{-2-1/\alpha} \right] dt\, ds.
\end{aligned}
$$

Numcrical exemplifications of our main results are now considered for the LCR treaty. It is assumed that each marginal is a two-parameter Pareto distribution with df

$$
F_{Pareto}(x;\beta,\gamma) = 1 - \left(1 + \gamma \frac{x}{\beta}\right)^{-\beta}, \ x \geq 0
$$

in order to illustrate Theorem 5.3.1 and exponentially distributed for Theorem 5.3.2. In both cases, the expected value is set to 10,000, which implies that the Pareto parameters should satisfy $\gamma = \beta/((\beta-1) \times 10,000)$. We performed the calculations for $\beta = 2,3,4,5$. For both the Pareto and exponential cases we considered $\alpha = 2,3,5,7,9,10$. The following tables show the values of the asymptotic constants and the resulting quantiles at level 0.999.

Tables 5.1 and 5.2 show that, as α increases, the asymptotic constants $C_{LF}(2,\alpha,\beta)$ decrease. This makes the corresponding quantile decrease, which is expected since an increasing value of α results in a stronger dependence between the insurance contracts. Changing the value of α does not have a significant impact on the quantiles, but the sensitivity to β is quite apparent. This indicates that poor quantification of the tail

Table 5.1: Asymptotic constants, $C_{LF}(2, \alpha, \beta)$

α	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$
2	8.6293	17.2031	34.3509	68.6358
3	8.5542	17.0840	34.1435	68.2577
5	8.4062	16.8037	33.5987	67.1870
7	8.3146	16.6248	33.2452	66.4851
9	8.2557	16.5087	33.0147	66.0263
10	8.2336	16.4651	32.9280	65.8535

Table 5.2: Quantile estimates of L_2 at 0.999 level

α	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$
2	$918,940$	$496,296$	$378,419$	$330,997$
3	$914,891$	$495,102$	$377,801$	$330,587$
5	$906,852$	$492,269$	$376,164$	$329,417$
7	$901,844$	$490,445$	$375,092$	$328,642$
9	$898,606$	$489,254$	$374,388$	$328,132$
10	$897,393$	$488,805$	$374,122$	$327,939$

index β may yield incorrect results. A heavier tail, which corresponds to a lower value of β, results in larger quantiles.

The asymptotic constant $C_{LG}(2, \alpha)$ and quantile from Table 5.3 exhibit the same behaviour as in Tables 5.1 and 5.2, regarding changes in the strength of dependence. As anticipated, the quantiles for the exponential case are smaller than the corresponding Pareto quantiles, due to the light-tail extreme behaviour of the exponential distribution.

Table 5.3: Asymptotic constants, $C_{LG}(2,\alpha)$ and quantile estimates of L_2 at level 0.999

α	$C_{LG}(2,\alpha)$	Quantile
2	2.1367	153,340
3	2.1294	153,272
5	2.0983	152,978
7	2.0770	152,774
9	2.0630	152,638
10	2.0576	152,586

5.4 Other Dependence Structures

In the previous section it was assumed that the survival copula is Archimedean, and some regularity conditions were imposed. The main purpose of this section is to extend those results.

5.4.1 Archimedean Copula

A natural question is how do the asymptotic results differ when the copula itself (rather than the survival copula) is assumed to be Archimedean? This can be done, but we give up some simplicity. In this case, we assume that the generator φ is regularly varying at 1. By definition, this means that for any positive x the following holds

$$\lim_{t\downarrow 0} \frac{\varphi(1-tx)}{\varphi(1-t)} = x^{\alpha},$$

and we write $\varphi \in RV^1_{\alpha}$. Furthermore, the index satisfies the condition that $\alpha \geq 1$ (see Juri and Wütrich, 2003). The Gumbel copula is an example of such a copula with regularly varying generator $\varphi(u) = (-\ln u)^{\alpha}$, which satisfies the latter property

($\varphi \in RV_\alpha^1$).

$$C(u_1, \ldots, u_n) = \exp\left(-\left[\sum_{i=1}^{n}(-\ln u_i)^\alpha\right]^{1/\alpha}\right),$$

where $\alpha \geq 1$.

Upon defining the joint tail extreme behavior, the same steps as in the case of the survival Archimedean copula are followed, where (5.3.1) and (5.3.6) are replaced respectively by

$$\lim_{t\to\infty}\frac{\Pr(X_1 > tx_1, X_2 > tx_2)}{\bar{F}(t)} = x_1^{-\beta} + x_2^{-\beta} - \left(x_1^{-\alpha\beta} + x_2^{-\alpha\beta}\right)^{1/\alpha}, \quad x_1, x_2 > 0,$$

in the Fréchet case, and

$$\begin{aligned}\lim_{t\to\infty}&\frac{\Pr(X_1 > t + x_1 a(t), X_2 > t + x_2 a(t))}{\bar{F}(t)}\\ &= e^{-x_1} + e^{-x_2} - \left(e^{-\alpha x_1} + e^{-\alpha x_2}\right)^{1/\alpha}, \quad -\infty < x_1, x_2 < \infty,\end{aligned}$$

in the Gumbel case (see Juri and Wütrich, 2003) provided that $1 < \alpha < \infty$. For simplicity, the bivariate case has been considered, but the result can be extended to the multivariate case, which is more cumbersome.

5.4.2 Extension

All previous cases were done under the assumption of exchangeability, which simplifies the computations since we deal with order statistics. We recognize that this assumption may be questionable, but extensions can be made when it does not hold, though they are tedious.

Earlier we mentioned that the joint tail extreme behaviour is essential to characterize the tail probability for the ECOMOR and LCR reinsurances. In the case that the exchangeability property fails to hold we can still make the same characterization, provided that for any set $I \subseteq \{1, \ldots, n\}$ the following exist

$$\lim_{t\to\infty} \frac{\Pr(X_i > tx_i, i \in I)}{V(t)}, \; x_i > 0,$$

in the Fréchet case, and

$$\lim_{t\to\infty} \frac{\Pr(X_i > t + a(t)x_i, i \in I)}{V(t)}, \; -\infty < x_i < \infty,$$

for Gumbel, where $V(\cdot)$ is a positive-valued function.

5.5 Conclusions

In this chapter, we provide a procedure to understand the tail behavior of the ECOMOR and LCR reinsurances for a portfolio of dependent insurance contracts. First, a specific dependence structure is considered. Namely, the survival copula is assumed to be Archimedean. This choice of dependence structure aids in giving closed form results, while the exchangeability between random variables simplifies the analysis. Finally, we note that our main results can be extended, provided that we control the limiting joint tail probabilities.

References

Albrecher, H. , Asmussen, S. and Kortschak, D. 2006. "Tail Asymptotics for the Sum of Two Heavy-Tailed Dependent Risks," *Extremes*, 9 (2), 107-130.

Alink, S. , Löwe, M. and Wüthrich, M.V. 2004. "Diversification of Aggregate Dependent Risks," *Insurance: Mathematics and Economics*, 35(1), 77-95.

Alink, S. , Löwe, M. and Wüthrich, M.V. 2005. "Analysis of the Expected Shortfall of Aggregate Dependent Risks," *ASTIN Bulletin*, 35(1), 25-43.

Ammeter, H. 1964. "The Rating of Largest Claim Reinsurance Covers," *Quarterly Letter from the Algemeine Reinsurance Companies Jubilee*, Number 2, 5-17.

Barbe, P., Fougères, A.-L. and Genest, C. 2006. "On the Tail Behavior of Sums of Dependent Risks," *ASTIN Bulletin*, 36(2), 361 - 373.

Bingham, N.H., Goldie, C.M., and Teugels, J.L. 1987. *Regular Variation.* Cambridge University Press, Cambridge.

Embrechts, P., Klüppelberg, C. and Mikosch, T. 1997. *Modelling Extremal Events for Insurance and Finance.* Springer-Verlag, Berlin.

Juri, A. and Wütrich, M. V., 2003, "Tail dependence from a distributional point of view", *Extremes*, 6(3), 213-246.

Kallenberg, O. 1983. *Random Measures*, 3rd edition Akademie-Verlag, Berlin.

Kimberling, C. H. 1974."A Probabilistic Interpretation of Complete Monotonicity," *Aequationes Mathematica*, 10, 152-164.

Kortschak, D. and Albrecher, H. 2007. "Asymptotic Results for the Sum of Dependent Non-identically Distributed Random Variables," RICAM Report 2007-04.

Nelsen, R. B. 1999. *An Introduction to Copulas.* Springer-Verlag, New York.

Resnick, S.I. 1987. *Extreme Values, Regular Variation and Point Processes.* Springer-Verlag, New York.

Sklar, A. 1959. "Fonctions de répartion à n dimensions et leurs marges," Publications de l'Institut de Statistique de l'Université de Paris, 8, 229-231.

Thépaut, A. 1950. "Une nouvelle forme de réassurance: le traité d'excédent de coût moyen relatif (ECOMOR)," *Bulletin Trimestriel de l'Institut des Actuaries Français*, 49, 273-343.

Wüthrich, M.V. 2003. "Asymptotic Value-at-Risk Estimates for Sums of Dependent Random Variables," *ASTIN Bulletin*, 33(1), 75-92.

Chapter 6

Future Research

Some extensions of the present work might be considered for future research. In the last two chapters of this thesis, asymptotic results for large claims reinsurance are studied in two specific models. A related practical problem is to actually price these insurance products. In other words, the pure premiums for ECOMOR and LCR need to be quantified. We have already partially addressed this problem by providing asymptotic tail probabilities. Exact results will be difficult to obtain, but it may be feasible to find upper and/or lower bounds.

Another idea for future research focuses on the diversification effect of a portfolio with dependent financial risks. Under the assumption that risks are diversified, it is generally accepted that the subadditivity of the most popular risk measure, Value-at-Risk, holds. In order to determine the minimum capital charge required by regulators, a lower bound for the overall risk must be obtained. More specifically, the international Basel II regulations express that the minimal amount of capital for paying the possible future claims should cover at least 99% of the realizations. One way of addressing this problem is to consider the case in which the multivariate regular variation property holds. In this situation, the associated spectral measure needs to be explored to aid in deciding if the asymptotic diversification effect is satisfied.

Appendix A

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