8-6-2019 1:30 PM

Ricci Curvature of Noncommutative Three Tori, Entropy, and Second Quantization

Rui Dong, *The University of Western Ontario*

Supervisor: Khalkhali, Masoud, *The University of Western Ontario*

A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics

Abstract

In noncommutative geometry, the metric information of a noncommutative space is encoded in the data of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where D plays the role of the Dirac operator acting on the Hilbert space of spinors. Ideas of spectral geometry can then be used to define suitable notions such as volume, scalar curvature, and Ricci curvature. In particular, one can construct the Ricci curvature from the asymptotic expansion of the heat trace $\mathrm{Tr}(e^{-tD^2})$. In Chapter 2, we will compute the Ricci curvature of a curved noncommutative three torus. The computation is done for both conformal and a non-conformal perturbation of the flat metric. By applying Connes' pseudodifferential calculus for the noncommutative tori, we explicitly compute the second density of the heat trace expansion for the perturbed Laplacians on both functions and $1-$forms. On the other hand, in noncommutative geometry one also wants to get a good notion of an action functional which depends only on the spectrum of D, called spectral action functional. It is known that such a functional can be expressed as Tr(f(D)) for some function f. In chapter 3, we show that the von Neumann entropy, average energy, and negative free energy of the Gibbs state of the second quantized Dirac operator $d\Gamma D$ has a spectral action functional interpretation of the original Dirac operator D. To be able to carry on the computations, we have to incorporate the chemical potential μ. All those spectral action coefficients can be given in terms of the modified Bessel functions.

Co-Authorship

This thesis incorporates materials that are results of joint research, as follows:

- Chapter 2 is based on the paper:

  The Ricci Curvature for Noncommutative Three Tori, arXiv:1808.02977.

  This paper has been submitted to *Canadian Journal of Mathematics* on November 15th, 2018. This is the outcome of a joint research undertaken in collaboration with Dr. Asghar Ghorbanpour under the supervision of Professor Masoud Khalkhali. Rui Dong contributed equally as others to this paper. He also did the programming in Mathematica and typing the paper.

- Chapter 3 is based on the paper:

  Second Quantization and the Spectral Action, arXiv:1903.09624.

  This paper has been submitted to *Communications in Mathematical Physics* on May 10th, 2019. This is the outcome of a joint work with Professor Masoud Khalkhali. Rui Dong thought of applying the modified Bessel functions to calculate the spectral action coefficients and the idea of obtaining spectral actions interpreted of energy, after the entropy. The typing work of this paper is also mainly finished by Rui Dong, whose contribution was equal for the whole work of this paper.

Keywords: Noncommutative Geometry, Spectral Triples, Second Quantization, Spectral Geometry, Differential Geometry, Modified Bessel Functions, Chemical Potential, Entropy, Ricci Curvature, Scalar Curvature

Summary for lay audience

In mathematics one can describe the topological properties of a compact Hausdorff space M via the algebra $C(M)$ of all continuous complex-valued functions over M, which is a commutative C^*-algebra. By analogy, a noncommutative C^*-algebra encodes all the topological information of a noncommutative space.

In Section 1.1 and 1.2, we shall briefly review the definition of a spectral triple which encodes the geometric information of a noncommutative space. Then in Chapter 2, we shall recall the definition of the noncommutative three tori. Then we will compute the Ricci density of curved noncommutative three tori under the conformally flat metric and a specific non-conformal metric by analyzing the spectral properties of the Laplace operators.

In Section 1.3 and 1.4, we shall give a brief introduction to the spectral action principle and the second quantization. A spectral action functional is an additive functional with respect to direct sum of the spectral triples and the second quantization is one method to describe a multi-particle system in the quantum statistical mechanics. We will show, in Chapter 3, that the entropy and energy of the Gibbs state of the second quantized Dirac operator can be interpreted as spectral action functionals of the original Dirac operator. Moreover, we will explicitly compute all the spectral action coefficients for the above quantities in both Bosonic and Fermionic cases.

Acknowledgements

I would like to thank my supervisor Professor Masoud Khalkhali, for all his advice and encouragement. I am also feeling deeply grateful for our weekly NCG seminars during the past four years, which opened up my eyes to a bunch of fantastic different directions in mathematics and physics. Besides, I feel thankful to Dr. Asghar Ghorbanpour for his insightful comments in our joint research.

I also would like to thank my examiners, Professor Piotr M. Hajac, Professor Ján Mináč, Professor Rasul Shafikov, and Professor Sree Ram Valluri for carefully reading my thesis and their valuable suggestions.

Last but not least, I would like to express my thanks to my parents for their support and encouragement throughout my life.

Contents

List of Figures

List of Tables

List of Appendices

Chapter 1

Introduction to noncommutative geometry

In this chapter, we will briefly review the pseudodifferential operators and elliptic operators first, then we will introduce some basic concepts in noncommutative geometry, such as spectral triples and spectral action. Finally, we will give a short introduction about entropy and energy in the quantum statistical mechanics. This chapter contains introductory material needed to understand my two joint papers [5, 6].

1.1 Pseudodifferential operators and elliptic operators

We first briefly review the theory of pseudodifferential operators on $\mathbb{R}^m$, then we shall review the theory of pseudodifferential operators on an $m-$dimensional oriented closed manifold M following Gilkey's book [7].

To simplify the notations, we let dx, dy, and $d\xi$ denote the Lebesgue measure on $\mathbb{R}^m$ with an additional normalizing factor of $(2\pi)^{-m/2}$. Following this notation, we define the convolution product of two Schwartz class functions $f, g \in \mathcal{S}(\mathbb{R}^m)$ by

$$(f * g)(x) = \int_{\mathbb{R}^m} f(x-y)g(y)dy = \int_{\mathbb{R}^m} f(y)g(x-y)dy,$$

and the Fourier transform of f by

$$\hat{f}(\xi) = \int_{\mathbb{R}^m} e^{-ix\cdot\xi} f(x)dx.$$

We shall also use the following notation

$$d_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}, \qquad D_x^\alpha = (-i)^{|\alpha|} d_x^\alpha.$$

The following properties of Fourier transform will be used:

$$D_\xi^\alpha \hat{f}(\xi) = (-1)^{|\alpha|}\left(\widehat{x^\alpha f}\right)(\xi), \qquad \xi^\alpha \hat{f}(\xi) = \left(\widehat{D_x^\alpha f}\right)(\xi).$$

1.1.1 Pseudodifferential operators

Definition 1.1.1 *Let $U \subset \mathbb{R}^m$ be an open subset. We say $p(x,\xi)$ is a symbol of order d, and we denote it by $p \in S^d(U)$, if*

(1) the function $p(x,\xi)$ is smooth in $(x,\xi) \in U \times \mathbb{R}^m$ with compact x support,

(2) for all multi-indices (α,β), there are constants $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{d-|\beta|}.$$

We also denote by $S^{-\infty}(U) = \bigcap_d S^d(U)$, and when U is the whole Euclidean space $\mathbb{R}^m$, we may simply write the set of all symbols of order d as S^d. For a given symbol p, we define the associated operator $P(x,D) : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^m)$ by

$$P(x,D)(f)(x) = \int_{\mathbb{R}^m} e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi} p(x,\xi)f(y)dyd\xi.$$

By definition $P(x,D)$ is a linear map from $\mathcal{S}(\mathbb{R}^m)$ to $\mathcal{S}(\mathbb{R}^m)$. We can extend $P(x,D)$ from $\mathcal{S}(\mathbb{R}^m)$ to the Sobolev space H_s. We refer to Gilkey's book [7] for more details in this regard.

Lemma 1.1.2 *Let $p \in S^d$. Then $|Pf|_{s-d} \leq C|f|_s$ for all $f \in \mathcal{S}(\mathbb{R}^m)$. Thus P can be extended to a continuous map $P : H_s \to H_{s-d}$ for all s.*

The class of pseudodifferential operators is closed under the composition and adjoint. Moreover, if P and Q are two pseudodifferential operators whose symbols are $p(x,\xi)$ and $q(x,\xi)$ respectively, then we have the following asymptotic expansions of the symbols of adjoint and composition

$$\sigma(P^*) \sim \sum_\alpha d_\xi^\alpha D_x^\alpha p^* / \alpha!,$$
$$\sigma(PQ) \sim \sum_\beta d_\xi^\beta p(x,\xi) D_x^\beta q(x,\xi)/\beta!.$$

Here the relation $\sim$ means that the difference of two symbols is infinitely smoothing, namely,

$$p \sim p' \Leftrightarrow p - p' \in S^{-\infty} = \bigcap_d S^d.$$

Let $p(x,\xi) \in S^d$ have x support in $U \subset \mathbb{R}^m$, and we denote by $C_c^\infty(U)$ the set of all smooth functions with support in U. We restrict domain of P to be $C_c^\infty(U)$. Then the range of P is $C_c^\infty(U)$ as well, and thus the operator $P : C_c^\infty(U) \to C_c^\infty(U)$ is well-defined. We denote the set of all such operators by $\Psi^d(U)$.

In general, let $p(x,\xi)$ be a matrix-valued symbol, i.e., all the components of $p(x,\xi)$ belong to S^d. The corresponding operator P is then given by a matrix of pseudodifferential operators. Thus P is a map whose domain and range are vector-valued functions with compact support in U.

Now we can extend the theory of pseudodifferential operators on an oriented smooth closed Riemannian manifold M. We shall consider scalar functions first.

Definition 1.1.3 *Let $C^\infty(M)$ be the space of smooth functions on M, and let $P : C^\infty(M) \to C^\infty(M)$ be a linear operator. We say P is a pseudodifferential operator of order d if for any open chart U on M, and for any $\phi, \psi \in C_c^\infty(U)$, the localized operator $\phi P \psi$ is a pseudodifferential operator with order d on U.*

The pseudodifferential operators acting on a vector bundle V over M is defined as below.

Definition 1.1.4 *Let M be an oriented closed manifold, and $V \to M$ be a vector bundle. Let $C^\infty(V)$ denote the space of smooth sections of V. We say a linear operator $P : C^\infty(V) \to C^\infty(V)$ is a pseudodifferential operator of order d if for any open chart U on M which is a local trivilization for V and for any $\phi, \psi \in C_c^\infty(U)$, the localized operator $\phi P \psi : C_c^\infty(U, \mathbb{C}^n) \to C_c^\infty(U, \mathbb{C}^n)$ is a pseudodifferential operator of order d on U acting on $C_c^\infty(U, \mathbb{C}^n)$.*

If the vector bundle V is equipped with a Hermitian product $(\cdot,\cdot)$, then we can define an inner product over $C^\infty(V)$ by

$$\langle \xi, \eta \rangle := \int_M (\xi(x), \eta(x)) dx, \quad \xi, \eta \in C^\infty(V),$$

and complete it to a Hilbert space, which we denote by $L^2(M, V)$. We can then define the adjoint operator P^* with respect to this inner product. For this adjoint operator P^*, we have

$$\sigma_L(P^*) = \sigma_L(P)^*.$$

The order d of a pseudodifferential operator P puts some restrictions on P. In fact, we have the following theorem. One can check [7] for a proof.

Theorem 1.1.5 *Let M be an oriented closed smooth Riemannian manifold with dimension m, and $V \to M$ be a Hermitian vector bundle. For any $P \in \Psi^d(M, V)$, as a densely defined operator on $L^2(M, V)$, we have:*

(1) P is a bounded operator if $d \leq 0$.

(2) P is a compact operator if $d < 0$.

(3) P is a Dixmier class operator if $d \leq -m$.

(4) P is a trace class operator if $d < -m$.

Let U be an open subset of M. For a symbol $p \in S^d(U)$ of order d, we define the leading (principal) symbol p_L of p to be the class of p in the quotient spaces $S^d(U)/S^{d-1}(U)$. Following this definition, if P and Q are pseudodifferential operators of order d_1 and d_2 over U, then PQ is a pseudodifferential operator of order $d_1 + d_2$ and

$$\sigma_L(PQ) = \sigma_L(P)\sigma_L(Q).$$

The symbol $\sigma(P)$ is not globally well-defined on M since it is not invariant under the change of coordinates, while the leading symbol $\sigma_L(P)$ is invariant under the change of coordinates. Thus the leading symbol $\sigma_L(P)$ is globally well-defined.

Definition 1.1.6 *Let $p \in S^d(U)$ be a square matrix and U_1 be an open set such that $\overline{U}_1 \subset U$. We say p is elliptic on U_1 if there is an open subset U_2 with $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$ and there exists a symbol $p' \in S^{-d}(U)$ such that*

$$pp' - I \in S^{-\infty}(U_2), \qquad p'p - I \in S^{-\infty}(U_2).$$

We call an operator $P \in \Psi^d(M, V)$ elliptic if for any open chart $U \subset M$ and for any $\phi, \psi \in C_c^\infty(U)$, the localized operator $\phi P \psi$ is elliptic when $\phi\psi(x) \neq 0$.

If a pseudodifferential operator $P \in \Psi^d(M, V)$ is an elliptic operator, then there exists an operator $Q \in \Psi^{-d}(M, V)$ such that

$$PQ - I \in \Psi^{-\infty}(M, V), \qquad QP - I \in \Psi^{-\infty}(M, V).$$

We call Q a parametrix of P. According to Theorem 1.1.5, both $PQ - I$ and $QP - I$ are compact operators, thus P is invertible in the Calkin algebra and therefore P is a Fredholm operator.

1.2 Spectral geometry

Recall that for an operator $T : \mathcal{H} \to \mathcal{H}$ over a Hilbert space $\mathcal{H}$, the spectrum of T is defined by

$$\mathrm{spec}(T) = \{\lambda \in \mathbb{C} : (T - \lambda) \text{ is not invertible in } \mathcal{B}(\mathcal{H})\}.$$

According to the spectral theory if $T \in \mathcal{K}(\mathcal{H})$ is a self-adjoint compact operator, we can then find a complete orthonormal basis $\{\phi_n\}$ of $\mathcal{H}$ consisting of eigenvectors of T. The main idea of spectral geometry is to figure out how much geometric information one can extract by analyzing the spectrum of some given geometric operator such as the Laplace operator, on a Riemannian manifold M. The most well-known example in spectral geometry is Weyl's law (see e.g. [8]).

Theorem 1.2.1 (Weyl's Law) *Let M be a compact smooth Riemannian manifold of dimension m. Let $\triangle : L^2(M) \to L^2(M)$ be the Laplace operator over M, whose eigenvalues are $0 = \lambda_0 < \lambda_1 \leq \cdots$, each eigenvalue repeated according to its multiplicity. We denote by*

$$N(\lambda) := \#\{j : \lambda_j \leq \lambda\}.$$

Then we have

$$N(\lambda) \sim \frac{\omega_m}{(2\pi)^m}\mathrm{Vol}(M)\lambda^{m/2}, \ \text{as } \lambda \to \infty,$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$, that is

$$\omega_m = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2} + 1)}.$$

From Weyl's Law, one can deduce an asymptotic formula of the j−th eigenvalue of $\triangle$:

$$\lambda_j \sim \frac{\sqrt{2\pi}}{(\omega_m \mathrm{Vol}(M))^{2/m}} j^{2/m}, \quad j \to \infty.$$

In fact, we have the following more general result (see e.g. [7]).

Lemma 1.2.2 *Let $V \to M$ be a smooth Hermitian vector bundle over a closed smooth Riemannian manifold M of dimension m. Let $P : C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint pseudodifferential operator of order $d \geq 0$. If we order the eigenvalues such that $|\lambda_1| \leq |\lambda_2| \leq \cdots$, then there exists a constant C such that $|\lambda_j| \sim C j^{d/m}$.*

1.2.1 Laplace type operators

Let (M, g) be an oriented closed smooth Riemannian manifold of dimension m, and $V \to M$ be a rank n smooth vector bundle.

Definition 1.2.3 *We say a second order differential operator $P : C^\infty(V) \to C^\infty(V)$ is a Laplace type operator if $\sigma_L(P) = g^{ij}\xi_i\xi_j$. Namely, in any local chart U, P can be written as*

$$P = -g^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + A^k\frac{\partial}{\partial x_k} + B, \quad A^k, B \in C^\infty(U, End(V)).$$

For instance, let $\nabla : C^\infty(V) \to C^\infty(T^*M \otimes V)$ be a connection on the vector bundle V, that is, a $\mathbb{C}$-linear map satisfying

$$\nabla(f\phi) = df \otimes \phi + f\nabla\phi, \quad \forall f \in C^\infty(M), \quad \phi \in C^\infty(V).$$

We define $\nabla : C^\infty(T^*M \otimes V) \to C^\infty(T^*M \otimes T^*M \otimes V)$ by

$$\nabla(\omega \otimes \xi) \mapsto \nabla^{LC}\omega \otimes \xi + \omega \otimes \nabla\xi, \quad \omega \in \Omega^1(M), \xi \in C^\infty(V),$$

here ∇^{LC} is the Levi-Civita connection for the cotangent bundle T^*M. Now consider the second covariant derivative operator ∇^2 given by the following composition map

$$C^\infty(V) \xrightarrow{\nabla} C^\infty(T^*M \otimes V) \xrightarrow{\nabla} C^\infty(T^*M \otimes T^*M \otimes V).$$

Taking the trace over $T^*(M) \otimes T^*(M)$, we define a differential operator

$$P_\nabla = -\mathrm{Tr}(\nabla^2) : C^\infty(V) \to C^\infty(V).$$

In a local coordinate, we have

$$P_\nabla = -g^{ij}\nabla_{\partial_i}\nabla_{\partial_j} + g^{ij}\Gamma^k_{ij}\nabla_{\partial_k}.$$

Here Γ_{ij}^k's are the Christoffel symbols for the Levi-Civita connection, namely,

$$\nabla^{LC}\partial_j = \Gamma_{ij}^k dx^i \otimes \partial_k.$$

Hence P_∇ is a Laplace type operator. In fact, we have the following result [7].

Lemma 1.2.4 *Let $P : C^\infty(V) \to C^\infty(V)$ be a Laplace type operator. Then there exists a unique connection ∇ on V and a unique endomorphism $E \in C^\infty(End(V))$ so that*

$$P = P_\nabla - E.$$

Remark For a given closed smooth Riemannian manifold M, if we denote its Levi-Civita connection by

$$\nabla^{LC} : C^\infty(T(M)) \to C^\infty(T^*(M) \otimes T(M)),$$

we can always extend ∇^{LC} to the tensor fields:

$$\nabla^{LC} : T_q^p \to T_q^{p+1}.$$

In more details, in a local chart, the Levi-Civita connection ∇^{LC} follows the following rules:

(1) $\nabla^{LC} dx^k = -\Gamma_{ij}^k dx^i \otimes dx^j$,

(2) $\nabla^{LC}(\partial_{x^I}) = \sum \partial_{x^{i_1}} \otimes \cdots \otimes \nabla^{LC}\partial_{x^{i_k}} \otimes \cdots \otimes \partial_{x^{i_q}}$ for $\partial_{x^I} = \partial_{x^{i_1}} \otimes \cdots \otimes \partial_{x^{i_q}}$,

(3) $\nabla^{LC}(dx^J) = \sum dx^{j_1} \otimes \cdots \otimes \nabla^{LC} dx^{j_k} \otimes \cdots \otimes dx^{j_p}$ for $dx^J = dx^{j_1} \otimes \cdots \otimes dx^{j_p}$,

(4) $\nabla^{LC}(\omega \otimes \eta) = \nabla^{LC}\omega \otimes \eta + \omega \otimes \nabla^{LC}\eta$ for any tensors ω, η,

(5) $\nabla^{LC} f = df$ for $f \in T_0^0 = C^\infty(M)$,

(6) $\nabla^{LC}(\gamma) = \sum (-1)^{i-1} \gamma^1 \wedge \cdots \wedge \nabla^{LC}\gamma^i \wedge \cdots \wedge \gamma^k$ for any $k-$form $\gamma = \gamma^1 \wedge \cdots \wedge \gamma^k$.

When there is no confusion, we simply denote the Levi-Civita by ∇, and denote the adjoint of ∇ by ∇^*. Then we have the following result (see e.g. [11]).

Theorem 1.2.5 (Weitzenböck Formula) *Let $\triangle^1 : \Omega^1(M) \to \Omega^1(M)$ be the Laplacian on $1-$forms of M. We have the formula:*

$$\triangle^1 = \nabla^*\nabla + \mathrm{Ric}.$$

Here $\mathrm{Ric} \in \mathrm{End}(T^*M)$ is the Ricci operator.

1.2.2 Heat equation

Let $V \to M$ be a smooth Hermitian vector bundle over M. We take $P : C^\infty(V) \to C^\infty(V)$ to be an elliptic pseudodifferential operator with order $d > 0$. We also require P to be self-adjoint and $\sigma_L(P)$ to be positive definite for all $x \in M$. Then there exists some constant C, such that $\mathrm{spec}(P) \subset [C, \infty)$. The heat equation is the partial differential equation given by:

$$\left(\frac{\partial}{\partial t} + P\right) f(x,t) = 0, \quad f(x,0) = f(x) \in C^\infty(V).$$

Formally, it has a solution $f(x,t) = e^{-tP} f(x)$. We define the heat kernel $K(t,x,y) : V_y \to V_x$ to be:

$$K(t,x,y) = \sum_n e^{-t\lambda_n} \phi_n(x) \otimes \overline{\phi}_n(y).$$

Here ϕ_n is the eigenfunction of λ_n with norm 1. Thus

$$e^{-tP} f(x) = \int_M K(t,x,y) f(y) dy.$$

Also, e^{-tP} is a trace class operator for all $t > 0$ and we define the heat trace as

$$\mathrm{Tr}\left(e^{-tP}\right) = \sum_n e^{-t\lambda_n} = \int_M \mathrm{tr}\left(K(t,x,x)\right) dx.$$

Since $\mathrm{spec}(P) \subset [C, \infty)$, we get an integral representation for the operator e^{-tP};

$$e^{-tP} = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} (P - \lambda)^{-1} d\lambda,$$

where γ is a contour in the complex plane that goes around the $\mathrm{spec}(P)$ in the clockwise direction without touching it.

While the operator $(P - \lambda)^{-1}$ is not a pseudodifferential operator, the method given in [7] is trying to approximate $(P - \lambda)^{-1}$ by some pseudodifferential operator $R(\lambda)$ and then obtain properties of e^{-tP} via $R(\lambda)$. First, we shall generalize our definition of symbol:

Definition 1.2.6 *Let $\mathcal{R}$ be the closed region of $\mathbb{C}$ consisting of γ together with the component of $\mathbb{C} \backslash \gamma$ which does not contain the interval $[C, \infty)$. We say $q(x,\xi,\lambda) \in S^k(\lambda)(U)$ is a symbol of order k depending on the complex parameter $\lambda \in \mathcal{R}$ if*

(1) $q(x,\xi,\lambda)$ is smooth in $(x,\xi,\lambda) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{R}$, has compact $x-$support in U and it is holomorphic in λ.

(2) For all (α,β,γ) there exist constants $C_{\alpha,\beta,\gamma}$ such that

$$|D_x^\alpha D_\xi^\beta D_\lambda^\gamma q(x,\xi,\gamma)| \leq C_{\alpha,\beta,\gamma}(1+|\xi|+|\lambda|^{1/d})^{k-|\beta|-d|\gamma|}.$$

We say $q(x,\xi,\lambda)$ is homogeneous of order k in (ξ,λ) if

$$q(x,t\xi,t^d\lambda) = t^k q(x,\xi,\lambda), \qquad \forall t \geq 1. \tag{1.2.1}$$

We let $\Psi_k(\lambda)(U)$ be the set of all operators $Q(\lambda): C_c^\infty(U) \to C_c^\infty(U)$ with symbols $q(x,\xi,\lambda) \in S^k(\lambda)$ having $x-$support in U.

Lemma 1.2.7 ([7]) *Take $Q_i \in \Psi_{k_i}(\lambda)(U)$ with corresponding symbols q_i for $i=1,2$. Then $Q_1Q_2 \in \Psi_{k_1+k_2}(\lambda)(U)$ has symbol q where*

$$q \sim \sum_\alpha d_\xi^\alpha q_1 D_x^\alpha q_2/\alpha!.$$

Lemma 1.2.8 ([7]) *If $h: U \to \tilde{U}$ is a diffeomorphism between $U, \tilde{U} \subset \mathbb{R}^m$, then it induces a map $h_*: \Psi_k(\lambda)(U) \to \Psi_k(\lambda)(\tilde{U})$ and*

$$\sigma(h_*P) - p(h^{-1}x_1, (dh^{-1}(x_1))^t\xi_1, \lambda) \in S^{k-1}(\lambda)(\tilde{U}).$$

Using Lemma 1.2.8, we can extend the class $\Psi(\lambda)$ to closed manifolds using a partition of unity argument. We now wish to solve the following equation recursively:

$$\sigma(R(\lambda)(P-\lambda)) - I \sim 0. \tag{1.2.2}$$

We define $R(\lambda)$ with symbol $r_0 + r_1 + \cdots$, where $r_j \in S^{-d-j}(\lambda)$. We also define $p'_j(x,\xi,\lambda) = p_j(x,\xi)$ for $j < d$ and $p'_d(x,\xi,\lambda) = p_d(x,\xi) - \lambda$. Then $\sigma(P-\lambda) = \sum_{i=0}^{d} p'_j$. We denote $p'_j \in S^j(\lambda)$, and $(p'_d)^{-1} \in S^{-d}(\lambda)$. The equation (1.2.2) yields:

$$\sum_{\alpha,j,k} d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha! \sim I.$$

We rewrite it as

$$\sum_n \sum_{|\alpha|+j+d-k=n} d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha! \sim I,$$

where $j,k \geq 0$ and $k \leq d$. There are no terms with $n < 0$. For the term with $n = 0$, we get the requirement that $r_0 p'_d = I$. Thus $r_0 = (p_d - \lambda)^{-1}$ and by induction, for $n > 0$ we get the recursive formula:

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j D_x^\alpha p_k' / \alpha!.$$

We can also write

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j D_x^\alpha p_k / \alpha!.$$

We define $E(t) = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} R(\lambda) d\lambda$. Let $K'(t,x,y)$ be the kernel of $E(t)$, and $K(t,x,y)$ be the kernel of e^{-tP}. It is proved in [7] that $K'(t,x,y)$ is a smooth kernel and that $K'(t,x,y)$ can approximate $K(t,x,y)$ to arbitrary orders of t as $t \to 0$. Thus we can get the information of e^{-tP} by studying $E(t)$. We let

$$a_n(x,P) = \frac{1}{2\pi i} \int \int_\gamma e^{-\lambda} r_n(x,\xi,\lambda) d\lambda d\xi.$$

Suppose d is an even number. If we replace ξ by $-\xi$, we conclude from (1.2.1) that

$$a_n(x,P) = 0, \quad \text{when } n \text{ is odd.}$$

According to [7], $a_n(x,P) \in \mathrm{End}(V,V)$ is invariantly defined independent of the coordinate system and the local frame of V. Thus $a_n(x,P)$ is globally well-defined as an endomorphism of the vector bundle V. We can also get the following asymptotic expansion of the heat kernel on diagonal:

$$K(t,x,x) \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} a_n(x,P), \quad \text{as } t \to 0^+.$$

Thus we have the asymptotic expansion of the trace of the heat operator e^{-tP}:

$$\begin{aligned}
\mathrm{Tr}\left(e^{-tP}\right) &= \int_M \mathrm{tr}\,(K(t,x,x))\,dx \\
&\sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} \int_M \mathrm{tr}\,(a_n(x,P))\,dx \\
&\sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} a_n(P),
\end{aligned}$$

where $a_n(P) = \int_M \mathrm{tr}\,(a_n(x,P))\,dx$. The terms of $a_n(P)$ are spectral invariants of P, i.e., they only depend on the spectrum of P, which can be computed using local expressions obtained from the symbol of P. In Chapter 2, we will use the terms of $a_2(x, \triangle_0)$ and $a_2(x, \triangle_1)$ to compute the Ricci curvature of noncommutative torus $\mathbb{T}^3_\theta$.

1.3 Noncommutative Riemannian geometry

The geometric properties of a closed Riemannian spin manifold M can be described by the triple $(C^\infty(M), L^2(M, S), D)$, where S is a spinor bundle of M, and D is the Dirac operator associated with the Levi-Civita connection lifted to the spinor bundle. In fact, one can recover the geodesic distance on M by [3]

$$d(x, y) = \sup\{|f(x) - f(y)| : \|[D, f]\| \leq 1\}.$$

This triple is called the canonical spectral triple and was introduced by Alain Connes [3].

1.3.1 Spectral triples

Definition 1.3.1 *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of an involutive (unital) $*-$algebra $\mathcal{A}$ with a faithful representation π on a Hilbert space $\mathcal{H}$, and a self-adjoint operator D defined on a dense subspace of $\mathcal{H}$ such that $(D \pm i)^{-1}$ is a compact operator and $[D, \pi(a)]$ is a bounded operator for all $a \in \mathcal{A}$. A spectral triple is called even if there is a $\mathbb{Z}_2-$grading $\gamma : \mathcal{H} \to \mathcal{H}$, i.e., $\gamma^2 = 1$ and $\gamma^* = \gamma$, such that $\gamma\pi(a) = \pi(a)\gamma$ and $\gamma D = -D\gamma$.*

When there is no confusion, we may simply write the representation $\pi(a)$ as a for any $a \in \mathcal{A}$. Another example of a spectral triple is the Hodge-de Rham spectral triple. Let M be an oriented closed Riemannian manifold. The triple

$$(C^\infty(M), L^2(M, \Lambda T^*_{\mathbb{C}}M), d + d^*)$$

forms a spectral triple, and in this case, there is a $\mathbb{Z}_2-$grading γ defined by $\gamma : \omega \mapsto (-1)^{|\omega|}\omega$.

Definition 1.3.2 *We define a real structure on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to be an anti-linear isometry $J : \mathcal{H} \to \mathcal{H}$ such that:*

(1) $J^2 = \epsilon$,

(2) $JD = \epsilon' DJ$,

(3) $J\gamma = \epsilon''$,

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Table 1.1: KO-dimension

where the signs $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$ are given by the Table 1.1. Moreover,

(4) $[a, b^\circ] = 0$*, for all* $a, b \in \mathcal{A}$*, where* $b^0 = Jb^* J^{-1}$.

(5) $[[D, a], b^\circ] = 0$*, for all* $a, b \in \mathcal{A}$.

The condition (4) is called the order zero condition. It implies that $\mathcal{H}$ is an $\mathcal{A}-$bimodule with left and right action of $\mathcal{A}$ given by a and b°, and condition (5) is called the order one condition. It corresponds to the property of geometric Dirac operators of being first order elliptic differential operators.

Definition 1.3.3 *A spectral triple is finitely summable if $|D|^{-\alpha}$ is a trace class operator for some $\alpha > 0$.*

Definition 1.3.4 *A spectral triple is regular if for all $a \in \mathcal{A}$, a and $[D, a]$ are in the domain of δ^m for all positive integers m. Here $\delta(\cdot) = [|D|, \cdot]$.*

Definition 1.3.5 *A finitely summable spectral triple is of metric dimension m if the operator $|D|^{-s}$ is a trace class operator on the half plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > m\}$.*

Definition 1.3.6 *The dimension spectrum is the set of poles in $\mathbb{C}$ of the zeta functions $\zeta_{b,D}(s) := \mathrm{Tr}\,(b|D|^{-s})$ associated to the spectral triple, where b is an element in the algebra generated by the elements $\delta^m(a)$ and $\delta^m([D, a])$ for all $a \in \mathcal{A}$, and $m \in \mathbb{N}$.*

1.3.2 Noncommutative tori

We shall first define the two dimensional noncommutative torus, and then give the definition for higher dimensional noncommutative tori. For a real number $\theta \in [0, 1)$, the noncommutative two torus $C(\mathbb{T}^2_\theta)$ is the universal C^*-algebra generated by two unitary elements U and V subject to the commutation relation

$$VU = e^{2\pi i\theta} UV. \tag{1.3.1}$$

By universality of $C(\mathbb{T}^2_\theta)$, we mean that for any C^*-algebra $C^*(\tilde{U}, \tilde{V})$ with two unitary elements $\tilde{U}$ and $\tilde{V}$ which satisfy (1.3.1), there exists a unique homomorphism from $C(\mathbb{T}^2_\theta)$ to $C^*(\tilde{U}, \tilde{V})$ which maps U to $\tilde{U}$ and V to $\tilde{V}$. The C^*-algebra $C(\mathbb{T}^2_\theta)$ can be considered as the algebra of all continuous functions over a noncommutative torus $\mathbb{T}^2_\theta$. We can also define the dense subalgebra $C^\infty(\mathbb{T}^2_\theta) \subset C(\mathbb{T}^2_\theta)$ by

$$C^\infty(\mathbb{T}^2_\theta) = \left\{ a \in C(\mathbb{T}^2_\theta) \middle| a = \sum_{m,n} a_{m,n} U^m V^n, \ a_{m,n} \in \mathbb{C} \text{ rapid decay sequence} \right\}.$$

This dense subalgebra $C^\infty(\mathbb{T}^2_\theta)$ can be regarded as the algebra of smooth functions on the noncommutative torus $\mathbb{T}^2_\theta$. In fact, when $\theta = 0$, $C(\mathbb{T}^2_0)$ is isomorphic to the C^*-algebra of all continuous functions on torus $\mathbb{T}^2$, and $C^\infty(\mathbb{T}^2_0)$ is the $*-$algebra of all smooth functions on $\mathbb{T}^2$.

Similar to the two dimensional case, one can construct higher dimensional noncommutative tori. Let $\Theta = (\theta_{ij}) \in M_n(\mathbb{R})$ be an $n \times n$ skew-symmetric matrix. The algebra $C(\mathbb{T}^n_\Theta)$ is the universal C^*-algebra generated by n unitaries U_j for $1 \leq j \leq n$, subject to the commutation relations

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j, \quad 1 \leq j, k \leq n.$$

One can consult [9] for more details about noncommutative tori. In Chapter 2, we will compute the Ricci curvature and scalar curvature of noncommutative 3−tori equipped some specific metrics.

1.3.3 Spectral action

In noncommutative geometry, we are interested in constructing an action functional for a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$. A suitable construction of such an action functional is the spectral action that was proposed in [2]. A spectral action for $(\mathcal{A}, \mathcal{H}, D)$ is defined as

$$\mathrm{Tr}(f(D/\Lambda)),$$

where f is a non-negative even smooth function that is rapidly decreasing at infinity and Λ is a positive real number. We usually assume that there exists a function $h(x)$ such that

$$f(x) = \int_0^\infty h(t) e^{-tx^2} dt.$$

We also assume the existences of an asymptotic expansion of the heat trace of the form

$$\operatorname{Tr}\left(e^{-tD^2}\right) \sim \sum_{\alpha} a_\alpha t^\alpha, \quad t \to 0^+. \tag{1.3.2}$$

We define the spectral zeta function to be

$$\zeta_D(s) = \operatorname{Tr}\left(|D|^{-s}\right).$$

Here we regard $|D|$ as an operator over the orthogonal complement of $\ker D \subset \mathcal{H}$. According to [4], we have the following formula to calculate the residues of $\zeta_D(s)$:

Lemma 1.3.7 *Each non-zero term a_α with $\alpha < 0$ corresponds to a pole of $\zeta_D(s)$ at -2α with*

$$\operatorname{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2a_\alpha}{\Gamma(-\alpha)},$$

and $\zeta_D(s)$ is regular at 0 *with*

$$\zeta_D(0) + \dim \ker D = a_0.$$

By this lemma, we can get an asymptotic formula for the spectral action. For more details, one can check [4], [12] or [10].

Theorem 1.3.8 *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple that satisfies* (1.3.2)*. Then there is an asymptotic expansion of the spectral action*

$$\operatorname{Tr}\left(f(D/\Lambda)\right) = \sum_{\beta \in \mathrm{Sp}^+} f_\beta \Lambda^\beta \operatorname{Res}_{s=\beta}\zeta_D(s) + f(0)\zeta_D(0) + o(1), \quad \Lambda \to \infty,$$

where $f_\beta = \int_0^\infty f(x)x^{\beta-1}dx$, and the summation is taken over the positive part of the dimension spectrum.

In Chapter 3 we will construct some spectral actions via the second quantization of the Dirac operator D. Before that, we shall review some background about quantum statistical mechanics.

1.4 Basics of quantum statistical mechanics

In quantum statistical mechanics, the quantum mechanical states of n particles in the configuration space $\mathbb{R}^\nu$ are described by vectors of the Hilbert space $L^2(\mathbb{R}^{n\nu})$. If the number of particles is not fixed, the states are given by vectors of the full Fock space

$$\mathcal{F} = \bigoplus_{n\geq 0} L^2(\mathbb{R}^{n\nu}),$$

where $\psi = \{\psi^{(n)}\}_{n\geq 0} \in \mathcal{F}$ with $\psi^{(0)} \in \mathbb{C}$ and $\psi^{(n)} \in L^2(\mathbb{R}^{n\nu})$ for $n \geq 1$. The norm of ψ is given by

$$\|\psi\|^2 = |\psi^{(0)}|^2 + \sum_{n\geq 1} \int_{\mathbb{R}^{n\nu}} |\psi^{(n)}(x_1, \cdots x_n)|^2 dx_1 \cdots dx_n.$$

In microscopic physics, identical particles are indistinguishable and in mathematics this is reflected via the symmetry of the probability density under the interchange of coordinates. If the component $\psi^{(n)}$ of a state ψ is symmetric under the interchange of coordinates for all $n \in \mathbb{N}$, the particles are called bosons and they are said to satisfy Bose-Einstein statistics. We denote the set of all functions that satisfy such symmetric conditions by $\mathcal{F}_+$. On the other hand, if the component $\psi^{(n)}$ of a state ψ is anti-symmetric for all $n \in \mathbb{N}$, the particles are called fermions and we say they satisfy Fermi-Dirac statistics. We denote the set of all functions that satisfy the anti-symmetric conditions by $\mathcal{F}_-$. Both $\mathcal{F}_+$ and $\mathcal{F}_-$ are Hilbert subspaces of $\mathcal{F}$. In this chapter, we will briefly review the Fock space and second quantization. One can check [1] and [8] for more details.

1.4.1 Definition of Fock spaces

In this section we denote by $\mathcal{H}$ the Hilbert space of one-particle configuration space. Here we consider the inner product $(\cdot,\cdot)$ of $\mathcal{H}$ to be conjugate linear in the first component and linear in the second component. We shall first recall the definition of the Fock space $\mathcal{F}(\mathcal{H})$, and the corresponding Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$ and the Bosonic Fock space $\mathcal{F}_+(\mathcal{H})$.

We denote by $\mathcal{H}^n = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ the n-fold tensor power of $\mathcal{H}$ when $n > 0$, and let $\mathcal{H}^0 = \mathbb{C}$. The Fock space $\mathcal{F}(\mathcal{H})$ is the completion of the pre-Hilbert space $\bigoplus_{n\geq 0} \mathcal{H}^n$. We define the operators $P_\pm$ on $\mathcal{H}^n$ by

$$P_+ (f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\pi\in S_n} f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)},$$

$$P_- (f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\pi\in S_n} (-1)^{|\pi|} f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)},$$

for all $f_1, ..., f_n \in \mathcal{H}$. Here S_n is the symmetric group of degree n. It is not difficult to see that both P_+ and P_- are projections, namely, $P_\pm^2 = P_\pm = (P_\pm)^*$. Thus $P_\pm$ can be extended by continuity to projection operators over the Fock space $\mathcal{F}(\mathcal{H})$. The Bosonic Fock space

$\mathcal{F}_+(\mathcal{H})$ and the Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$ are then defined as

$$\mathcal{F}_\pm(\mathcal{H}) = P_\pm(\mathcal{F}(\mathcal{H})),$$

and the corresponding n-particle subspaces $\mathcal{H}^n_\pm$ are defined as

$$\mathcal{H}^n_\pm = P_\pm \mathcal{H}^n.$$

1.4.2 Second quantization of operators

The structure of the Fock space allows us to amplify an operator on $\mathcal{H}$ to the whole Bose/Fermi Fock spaces $\mathcal{F}_\pm(\mathcal{H})$. This procedure is commonly referred to as the second quantization.

Let H be a self-adjoint operator on $\mathcal{H}$ with domain $D(\mathcal{H})$. For any $f_1, \cdots, f_n \in D(H)$, we define H_n on $\mathcal{H}^n_\pm$ by

$$H_n\left(P_\pm\left(f_1 \otimes \cdots \otimes f_n\right)\right) = \begin{cases} P_\pm\left(\displaystyle\sum_{i=1}^n f_1 \otimes f_2 \otimes \cdots \otimes Hf_i \otimes \cdots \otimes f_n\right) & n > 0, \\ 0 & n = 0. \end{cases}$$

The direct sum operator $\bigoplus_{n\geq 0} H_n$ is essentially self-adjoint according to [1], and the self-adjoint closure of this direct sum operator is called the second quantization of the operator H and it is denoted by $d\Gamma(H)$. Namely,

$$d\Gamma(H) = \overline{\bigoplus_{n\geq 0} H_n}.$$

In particular, if $H = \mathbb{1}$ is the identity operator, then we have

$$d\Gamma(\mathbb{1}) = N.$$

We call the operator N the number operator on $\mathcal{F}_\pm(\mathcal{H})$. The domain of N is

$$D(N) = \left\{\psi = \{\psi^{(n)}\}_{n\geq 0};\ \sum_{n\geq 0} n^2 \|\psi^{(n)}\|^2 < \infty\right\},$$

and for any $\psi \in D(N)$

$$N\psi = \{n\psi^{(n)}\}_{n\geq 0}.$$

The number operator N plays an important role in quantum statistics, as it counts the number of particles in the statistical system.

We now consider a unitary operator U on $\mathcal{H}$. First, we define U_n on $\mathcal{H}^n_\pm$ by

$$U_n\left(P_\pm\left(f_1 \otimes f_2 \otimes \cdots \otimes f_n\right)\right) = \begin{cases} P_\pm\left(Uf_1 \otimes Uf_2 \otimes \cdots \otimes Uf_n\right) & n > 0, \\ \mathbb{1} & n = 0, \end{cases}$$

and then extend it to the whole Fock space. We denote this extension by $\Gamma(U)$. It is called the second quantization of the unitary operator U,

$$\Gamma(U) = \bigoplus_{n \geq 0} U_n.$$

It is worth mentioning here that $\Gamma(U)$ is also a unitary operator on $\mathcal{F}_\pm(\mathcal{H})$. Also, if $U_t = e^{itH}$ is a strongly continuous one-parameter unitary group acting on $\mathcal{H}$, then

$$\Gamma(U_t) = e^{itd\Gamma(H)}$$

on the Fock spaces $\mathcal{F}_\pm(\mathcal{H})$.

If H is a self-adjoint Hamiltonian operator on the one-particle Hilbert space $\mathcal{H}$, then the dynamics of the ideal Bose gas and the ideal Fermi gas are described by the Schrödinger equation

$$i\hbar \frac{d\psi_t}{dt} = d\Gamma(H)\psi_t$$

on $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{F}_-(\mathcal{H})$ separately with the initial value $\psi_0 = \psi \in \mathcal{F}_\pm(\mathcal{H})$. We choose the units so that $\hbar = 1$. The solution of the Schrödinger equation gives us the evolution

$$\psi \in \mathcal{F}_\pm(\mathcal{H}) \mapsto \psi_t = e^{-itd\Gamma(H)}\psi = \Gamma(e^{-itH})\psi.$$

The evolution $\tau_t(A)$ of a bounded observable $A \in \mathcal{B}(\mathcal{F}_\pm(\mathcal{H}))$, on the other hand, is the conjugation by $\Gamma(e^{itH})$:

$$A \in \mathcal{B}(\mathcal{F}_\pm(\mathcal{H})) \mapsto \tau_t(A) = \Gamma(e^{itH})A\Gamma(e^{-itH}).$$

1.4.3 CAR and CCR relations

The CAR and CCR are acronyms of "canonical anti-commutation relations" and "canonical commutation relations", correspondingly. To describe the CAR and CCR relations, we shall define the annihilation operators and creation operators first.

Suppose $\mathcal{H}$ is a complex Hilbert space. For a vector $f \in \mathcal{H}$, we shall define the operators $a(f)$, and $a^*(f)$ acting on the Fock space $\mathcal{F}(\mathcal{H})$ by initially setting $a(f)\psi^{(0)} = 0$, $a^*(f)\psi^{(0)} = f$, $\forall f \in \mathcal{H}$, and

$$a(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \sqrt{n}\,(f, f_1)\, f_2 \otimes f_3 \otimes \cdots \otimes f_n,$$
$$a^*(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \sqrt{n+1}\, f \otimes f_1 \otimes f_2 \otimes f_3 \otimes \cdots \otimes f_n.$$

Here $\psi^{(0)} = 1 \in \mathbb{C}$. Extension by linearity then yields two densely defined operators on $\mathcal{F}(\mathcal{H})$. In fact, if $\psi^{(n)} \in \mathcal{H}^n$, we can see that

$$\|a(f)\psi^{(n)}\| \leq \sqrt{n}\|f\|\,\|\psi^{(n)}\|, \quad \|a^*(f)\psi^{(n)}\| \leq \sqrt{n+1}\|f\|\,\|\psi^{(n)}\|.$$

Thus $a(f)$ and $a^*(f)$ are well-defined in the domain $D(N^{1/2})$ of $N^{1/2}$, and

$$\|a(f)\psi\| \leq \|f\|\,\|(N+1)^{1/2}\psi\|, \quad \|a^*(f)\psi\| \leq \|f\|\,\|(N+1)^{1/2}\psi\|.$$

We call $a(f)$'s the annihilation operators, and $a^*(f)$'s the creation operators on the Fock space $\mathcal{F}(\mathcal{H})$. One can see that the maps $f \mapsto a(f)$ are anti-linear while the maps $f \mapsto a^*(f)$ are linear. Moreover, we have that $a^*(f)$ is the adjoint of $a(f)$; namely, one has

$$(a^*(f)\phi, \psi) = (\phi, a(f)\psi)$$

for any $\phi, \psi \in D(N^{1/2})$.

We can then define the annihilation operators $a_\pm(f)$ and the creation operators $a^*_\pm(f)$ on the Fermi/Bose Fock spaces $\mathcal{F}_\pm(\mathcal{H})$ correspondingly by

$$a_\pm(f) = P_\pm a(f) P_\pm, \qquad a^*_\pm(f) = P_\pm a^*(f) P_\pm.$$

Since both $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{F}_-(\mathcal{H})$ are invariant subspaces of the annihilation operator $a(f)$, and $a(f)^* = a^*(f)$, thus we have

$$a_\pm(f) = a(f) P_\pm, \qquad a^*_\pm(f) = P_\pm a^*(f).$$

A straightforward computation shows that on the Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$,

$$\{a_-(f), a_-(g)\} = \{a^*_-(f), a^*_-(g)\} = 0, \qquad \{a_-(f), a^*_-(g)\} = (f, g)\mathbb{1}, \tag{1.4.1}$$

and on the Bosonic Fock space $\mathcal{F}_+(\mathcal{H})$,

$$[a_+(f), a_+(g)] = [a_+^*(f), a_+^*(g)] = 0, \qquad [a_+(f), a_+^*(g)] = (f, g)\mathbb{1}. \tag{1.4.2}$$

The first relations (1.4.1) are called the canonical anti-commutation relations (CAR), and the second relations (1.4.2) are called the canonical commutation relations (CCR).

By CAR algebra we mean an algebra generated by elements that satisfy (1.4.1). In fact, we have the following proposition [1]:

Proposition 1.4.1 *Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{F}_-(\mathcal{H})$ be the Fermionic Fock space, and $a_-(f)$ and $a_-^*(g)$ the corresponding annihilation and creation operators on $\mathcal{F}_-(\mathcal{H})$.*

(1) For all $f \in \mathcal{H}$, we have

$$\|a_-(f)\| = \|f\| = \|a_-^*(f)\|.$$

Therefore both $a_-(f)$ and $a_-^*(g)$ have bounded extensions on $\mathcal{F}_-(\mathcal{H})$.

(2) Taking $\Omega = (1, 0, 0, \cdots)$, called the vacuum vector, and an orthonormal basis $\{f_\alpha\}$ of $\mathcal{H}$, then

$$\psi(f_{\alpha_1}, \cdots, f_{\alpha_n}) := a_-^*(f_{\alpha_1}) \cdots a_-^*(f_{\alpha_n})\,\Omega$$

form an orthonormal basis of $\mathcal{F}_-(\mathcal{H})$, when $\{f_{\alpha_1}, \cdots, f_{\alpha_n}\}$ runs over all the finite subsets of the orthonormal basis $\{f_\alpha\}$.

(3) The set of bounded operators $\{a_-(f), a_-^*(g); f, g \in \mathcal{H}\}$ is irreducible on $\mathcal{F}_-(\mathcal{H})$, i.e., the only closed subspaces of $\mathcal{F}_-(\mathcal{H})$ which are invariant under the action of the set $\{a_-(f), a_-^*(g); f, g \in \mathcal{H}\}$ are the trivial subspaces $\{0\}$ and $\mathcal{F}_-(\mathcal{H})$.

Definition 1.4.2 *We call the subalgebra of $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$ generated by $a_-(f)$, $a_-^*(g)$ and $\mathbb{1}$ the CAR algebra and denote it by $CAR(\mathcal{H})$.*

Although the CCR relations look very similar to the CAR relations, one can not simply mimic the way to define CAR algebras to deduce the definition of CCR algebras. The reason is that the annihilation operators $a_+(f)$ and the creation operators $a_+^*(g)$ are not bounded operators on $\mathcal{F}_+(\mathcal{H})$. In fact, we have

$$\|a(f)\psi^{(n)}\| = \sqrt{n}\|\psi^{(n)}\|\,\|f\|,$$

where $\psi^{(n)}$ is the $n-$fold tensor product of $f \in \mathcal{H}$ with ifself. Thus to define the CCR algebra, we first introduce the set of operators $\{\Phi(f), f \in \mathcal{H}\}$ by

$$\Phi(f) = \frac{a_+(f) + a_+^*(f)}{\sqrt{2}}.$$

Since the map $f \mapsto a_+(f)$ is anti-linear, and $f \mapsto a_+^*(f)$ is linear, then

$$a_+(f) = \frac{\Phi(f) + i\Phi(if)}{\sqrt{2}}, \quad a_+^*(f) = \frac{\Phi(f) - i\Phi(if)}{\sqrt{2}}.$$

Thus it suffices to examine the set of operators $\{\Phi(f), f \in \mathcal{H}\}$.

Let $F_+(\mathcal{H}) = P_+\left(\bigoplus_{n\geq 0} \mathcal{H}^n\right) \subseteq \mathcal{F}_+(\mathcal{H})$, i.e., $F_+(\mathcal{H})$ contains the sequences $\psi = \{\psi^{(n)}\}_{n\geq 0}$ which have only finitely many nonvanishing components.

Since for each $f \in \mathcal{H}$, $\Phi(f)$ is essentially self-adjoint on $F_+(\mathcal{H})$, $\Phi(f)$ has a unique self-adjoint extension. We still use $\Phi(f)$ to denote this self-adjoint extension

$$\Phi(f) = \overline{\frac{a(f) + a^*(f)}{\sqrt{2}}}.$$

We have the following proposition [1].

Proposition 1.4.3 *For any $f \in \mathcal{H}$, let*

$$\Phi(f) = \overline{\frac{a(f) + a^*(f)}{\sqrt{2}}}, \qquad W(f) = \exp\left(i\Phi(f)\right).$$

Let $CCR(\mathcal{H})$ denote the algebra generated by $\{W(f), f \in \mathcal{H}\}$. It follows that
(1) For any $f, g \in \mathcal{H}$, $W(f)D(\Phi(g)) = D(\Phi(g))$, and

$$W(f)\Phi(g)W(f)^* = \Phi(g) - \mathrm{Im}(f, g)\mathbb{1}.$$

(2) For each pair $f, g \in \mathcal{H}$

$$W(f)W(g) = e^{-i\mathrm{Im}(f,g)/2}W(f + g).$$

(3) $W(-f) = W(f)^*$.
(4) For each non-zero $f \in \mathcal{H}$, we have

$$\|W(f) - \mathbb{1}\| = 2,$$

and $W(0) = \mathbb{1}$.
(5) The set $\{W(f); f \in \mathcal{H}\}$ is irreducible on $\mathcal{F}_+(\mathcal{H})$, and $CCR(\mathcal{H})$ is a simple algebra.
(6) If $\|f_\alpha - f\| \to 0$, then

$$\|\left(W(f_\alpha) - W(f)\right)\psi\| \to 0$$

for all $\psi \in \mathcal{F}_+(\mathcal{H})$.

The operators $W(f)$ are called Weyl operators, and the algebra $CCR(\mathcal{H})$ is called the CCR algebra of $\mathcal{H}$.

1.4.4 Gibbs state

Let $\mathcal{H}$ be the one-particle Hilbert space and $H : \mathcal{H} \to \mathcal{H}$ be a positive operator such that $e^{-\beta H}$ is trace class. The Gibbs state ω_β is defined as a state over $\mathcal{B}(\mathcal{H})$ by

$$\omega_\beta(A) = \frac{\mathrm{Tr}(e^{-\beta H}A)}{\mathrm{Tr}(e^{-\beta H})}, \quad A \in \mathcal{B}(\mathcal{H}),$$

where $\beta > 0$ is the inverse temperature. Now we want to define a Gibbs state over $CAR(\mathcal{H})$ and $CCR(\mathcal{H})$. We define the modified Hamiltonian operator K_μ by

$$K_\mu = d\Gamma H - \mu N : \mathcal{F}_\pm(\mathcal{H}) \to \mathcal{F}_\pm(\mathcal{H}).$$

The Gibbs state $\omega_{\beta,\mu}$ is defined as

$$\omega_{\beta,\mu}(A) = \frac{\mathrm{Tr}(e^{-\beta K_\mu}A)}{\mathrm{Tr}(e^{-\beta K_\mu})}, \quad A \in CCR(\mathcal{H}) \quad \text{or} \quad CAR(\mathcal{H}).$$

To make sure the Gibbs state is well-defined, it is necessary to confirm that $\mathrm{Tr}(e^{-\beta K_\mu}) < \infty$. In fact, we have the following propositions from [1].

Proposition 1.4.4 *Let H be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and let $\beta \in \mathbb{R}$. The following conditions are equivalent:*

(1) $e^{-\beta H}$ is trace class on the one-particle Hilbert space $\mathcal{H}$.

(2) $e^{-\beta d\Gamma(H-\mu\mathbb{1})}$ is trace class on the Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$ for all $\mu \in \mathbb{R}$.

Proposition 1.4.5 *Let H be a self-adjoint operator on the one-particle Hilbert space $\mathcal{H}$, and let $\beta, \mu \in \mathbb{R}$. The following conditions are equivalent:*

(1) $e^{-\beta H}$ is trace class on the one-particle Hilbert space $\mathcal{H}$ and $\beta(H - \mu\mathbb{1}) > 0$,

(2) $e^{-\beta d\Gamma(H-\mu\mathbb{1})}$ is trace class on the Bosonic Fock space $\mathcal{F}_+(\mathcal{H})$.

1.4.5 Entropy and energy via second quantization

In this section we shall briefly introduce the density matrix and the definitions of von Neumann entropy, average energy and free energy.

Definition 1.4.6 *A density matrix $\rho : \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator such that $\rho \geq 0$ and* $\mathrm{Tr}(\rho) = 1$.

In particular, let $H : \mathcal{H} \to \mathcal{H}$ be a Hamiltonian operator. We denote the associated partition function by $Z = \mathrm{Tr}(e^{-\beta H})$. If $Z < \infty$, we can get a density matrix $\rho_\beta = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$.

Definition 1.4.7 *Let ρ be a density matrix over a Hilbert space $\mathcal{H}$. We define the von Neumann entropy of ρ to be* $\mathcal{S}(\rho) = -\mathrm{Tr}(\rho \log \rho)$.

According to this definition, von Neumann entropy is additive, i.e., given two density matrices ρ_1 and ρ_2,

$$\mathcal{S}(\rho_1 \otimes \rho_2) = \mathcal{S}(\rho_1) + \mathcal{S}(\rho_2).$$

There is another important quantity in quantum statistical mechanics called "average energy", which is defined by

$$\langle H \rangle_\beta = \frac{1}{Z}\mathrm{Tr}(He^{-\beta H}).$$

One can also compute the average energy by taking the derivative of $\log Z$ with respect to β:

$$\langle H \rangle_\beta = -\frac{\partial}{\partial \beta}(\log Z).$$

The quantity

$$F(\rho_\beta) := \langle H \rangle_\beta - \beta^{-1}\mathcal{S}(\rho_\beta)$$

is often called free energy. It is also given by

$$F(\rho_\beta) = -\beta^{-1}\log(Z).$$

In Chapter 3 we will show how to define the spectral actions via the second quantization.

Bibliography

[1] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 2*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997. Equilibrium states. Models in quantum statistical mechanics.

[2] A. H. Chamseddine and A. Connes. The spectral action principle. *Comm. Math. Phys.*, 186(3):731–750, 1997.

[3] A. Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.

[4] A. Connes and M. Marcolli. *Noncommutative geometry, quantum fields and motives*, volume 55 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI; Hindustan Book Agency, New Delhi, 2008.

[5] R. Dong, A. Ghorbanpour, and M. Khalkhali. The Ricci Curvature for Noncommutative Three Tori. *arXiv e-prints*, page arXiv:1808.02977, Aug 2018.

[6] R. Dong and M. Khalkhali. Second Quantization and the Spectral Action. *arXiv e-prints*, page arXiv:1903.09624, Mar 2019.

[7] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1995.

[8] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa. *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 2001.

[9] M. Khalkhali. *Basic noncommutative geometry*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, second edition, 2013.

[10] M. Marcolli. *Noncommutative cosmology*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

[11] S. Rosenberg. *The Laplacian on a Riemannian manifold*, volume 31 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. An introduction to analysis on manifolds.

[12] W. D. van Suijlekom. *Noncommutative geometry and particle physics*. Mathematical Physics Studies. Springer, Dordrecht, 2015.

Chapter 2

Ricci curvature for noncommutative three tori

2.1 Introduction

This chapter is a reproduction of my joint paper with Asghar Ghorbanpour and Masoud Khalkhali [8]. The spectral geometry and study of local spectral invariants of curved noncommutative tori has been the subject of intensive studies in recent years. In particular a Gauss-Bonnet theorem, the definition of scalar curvature, and the computations of scalar curvature for noncommutative two tori equipped with a curved metric has been achieved in [6, 10, 5, 9]. Building on these results, computing the scalar curvature in other dimensions and settings is carried out in [11, 17, 18, 14, 1, 7, 4, 15]. Beyond the scalar curvature, in [12] a definition of Ricci curvature in spectral terms is proposed and the Ricci density is computed for conformally flat metrics on noncommutative two tori.

In the present work we shall compute the Ricci curvature of noncommutative three tori for conformally flat metrics as well as non-conformal perturbations of the flat metric. Study of non-conformally flat metrics in three dimension is justified since even in the commutative case the class of conformally flat metrics on a three dimensional manifold is much smaller than the class of all metrics.

At the heart of the spectral formulation of Ricci curvature lies the Weitzenböck formula. This formula measures how far the Laplacian on 1-forms is from the Bochner Laplacian of the Levi-Civita connection on the cotangent bundle. It states [13, Lemma 4.8.13] that the difference is the Ricci tensor lifted to an endomorphism of the cotangent bundle denoted by Ric, and called the Ricci operator in [12]. More precisely, we have

$$\triangle_1 = \nabla^*\nabla + \mathrm{Ric}. \tag{2.1.1}$$

This result combined with Gilkey's formulas for the heat trace [13] reveals immediately that a linear combination of the Ricci operator and the scalar curvature is the density of the second coefficient of the heat trace of the Laplacian on 1-forms. That is

$$\mathrm{Tr}(e^{-t\triangle_1}) \sim a_0(\triangle_1)t^{-m/2} + a_2(\triangle_1)t^{1-m/2} + \cdots, \qquad t \to 0^+,$$

where

$$a_2(\triangle_1) = (4\pi)^{-m/2}\int_M \mathrm{tr}(\frac{1}{6}R - \mathrm{Ric})\mathrm{dvol}_g,$$

and R denotes the scalar curvature. These densities can be recovered by studying the localized heat trace $\mathrm{Tr}(Fe^{-t\triangle_1})$, where F is a smooth endomorphisms of the cotangent bundle. To isolate the Ricci operator, the second density of the heat trace of the Laplacian on functions $a_2(\triangle_0) = (4\pi)^{-m/2}\frac{1}{6}R$ enters the game where it is used to eliminate the scalar curvature present in $a_2(x, \triangle_1)$. Then the Ricci functional, as a functional on the algebra of sections of the endomorphism bundle of the cotangent bundle of M, is introduced as

$$\mathcal{R}ic(F) = \lim_{t\to 0^+} t^{\frac{m}{2}-1}\left(\mathrm{Tr}(\mathrm{tr}(F)e^{-t\triangle_0}) - \mathrm{Tr}(Fe^{-t\triangle_1})\right), \quad F \in C^\infty(\mathrm{End}(T^*M)).$$

If we denote the second density of the localized heat trace by $a_2(\mathrm{tr}(F), \triangle_0)$, the above formula can then be written as

$$\mathcal{R}ic(F) = a_2(\mathrm{tr}(F), \triangle_0) - a_2(F, \triangle_1), \quad F \in C^\infty(\mathrm{End}(T^*M)).$$

An equivalent version of the Ricci functional in terms of the spectral zeta function can be given by [12]

$$\mathcal{R}ic(F) = \begin{cases} \zeta(0, \mathrm{tr}(F), \triangle_0) - \zeta(0, F, \triangle_1) + \mathrm{Tr}(\mathrm{tr}(F)Q_0) - \mathrm{Tr}(FQ_1), & m = 2 \\ \\ \Gamma(\frac{m}{2} - 1)\mathrm{Res}_{s=\frac{m}{2}-1}\Big(\zeta(s, \mathrm{tr}(F), \triangle_0) - \zeta(s, F, \triangle_1)\Big), & m > 2, \end{cases}$$

where Q_j is the orthogonal projection on the kernel of $\triangle_j$.

This paper is organized as follows. In Section 2, we first recall the definition of the noncommutative Ricci curvature from [12]. To define the Ricci functional for the noncommutative three torus, it suffices to define the Laplacian on functions and on 1-forms.

We also recall the rearrangement lemma and Connes' pseudodifferential calculus in this section. The analogue of the de Rham complex for the noncommutative three torus is discussed in Section 2.2.2.

For the analogues of conformal $e^{-2h}(dx^2+dy^2+dz^2)$ and non-conformal $e^{-2h}(dx^2+dy^2)+dz^2$ families of metrics, the Laplacians are computed in later sections. In Section 2.3, applying the pseudodifferential calculus, the densities of the second terms are computed in the conformal case and the scalar curvature and Ricci density are computed for these metrics in Proposition 2.3.5 and Theorems 2.3.3 and 2.3.6. Finally in Section 2.4 we first compute the scalar curvature of the noncommutative three torus equipped with a non-conformally flat metric. We then compute the Ricci density for this class of metrics. It is interesting to note that two of the functions that appear in the expression for scalar curvature, Theorem 2.4.3, are the same as functions that appear in the scalar curvature of the two dimensional curved noncommutative tori [5, 10]. In Appendix A, we produce the steps that were used to compute the scalar curvature in the non-conformal case. In Appendix B, we give the list of functions obtained from the rearrangement lemma that are used in our computations.

2.2 Preliminaries

In this section we shall fix notations and review preliminaries required for the rest of the work. We will start with the definition of noncommutative three torus and then we construct the de Rham complex for it and discuss how one can define the Laplacians by fixing a metric on the noncommutative torus. Finally, we recall the definition of the Ricci functional from [12] for noncommutative three tori.

2.2.1 Noncommutative three tori

For a general introduction to topology and geometry of noncommutative tori the reader can consult [3]. Let $\theta = (\theta_{jk}) \in M_3(\mathbb{R})$ be a skew-symmetric matrix. The noncommutative 3-torus $C(\mathbb{T}^3_\theta)$ is the universal unital C^*-algebra generated by three unitary elements u_1, u_2, u_3 satisfying the relations:

$$u_k u_j = e^{2\pi i\theta_{jk}} u_j u_k, \quad j, k = 1, 2, 3.$$

We shall use both notations $C(\mathbb{T}^3_\theta)$ and $\mathbb{T}^3_\theta$ to refer to the noncommutative space represented by the algebra $C(\mathbb{T}^3_\theta)$. For $\theta = 0$, the C^*-algebra $C(\mathbb{T}^3_\theta)$ is isomorphic to the algebra of continuous functions on the 3-torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$.

There is an action of $\mathbb{T}^3$ on $C(\mathbb{T}^3_\theta)$, which is given by the 3−parameter group of automorphisms $\{\alpha_z\}$, such that

$$\alpha_z(u^m) = z^m u^m, \tag{2.2.1}$$

where for $m = (m_1, m_2, m_3) \in \mathbb{Z}^3$, we set $u^m = u_1^{m_1} u_2^{m_2} u_3^{m_3}$, and similarly, for $z = (z_1, z_2, z_3) \in \mathbb{T}^3$, we denoted $z_1^{m_1} z_2^{m_2} z_3^{m_3}$ by z^m. The set of all elements $a \in C(\mathbb{T}^3_\theta)$ for which the map $z \mapsto \alpha_z(a)$ is smooth, form an involutive dense subalgebra of $C(\mathbb{T}^3_\theta)$, which will be denoted by $C^\infty(\mathbb{T}^3_\theta)$. Alternatively, $C^\infty(\mathbb{T}^3_\theta)$ can be expressed as

$$C^\infty(\mathbb{T}^3_\theta) = \Big\{ \sum_{m\in\mathbb{Z}^3} a_m u^m : \{a_m\}_{m\in\mathbb{Z}^3} \text{ is rapidly decreasing}\Big\}.$$

By rapidly decreasing, we mean the Schwartz class condition that for all $k \in \mathbb{N}$,

$$\sup_{m\in\mathbb{Z}^3} (1 + |m|^2)^k |a_m|^2 < \infty.$$

There is a normalized faithful tracial state φ on $C(\mathbb{T}^3_\theta)$, determined by

$$\varphi(u^m) = 0, \quad \forall m \neq (0,0,0), \quad \text{and } \varphi(1) = 1.$$

The tracial state φ here plays the role of integration over $\mathbb{T}^3_\theta$. The algebra $C^\infty(\mathbb{T}^3_\theta)$ possesses three derivations, which are defined by the following relations:

$$\delta_j\Big(\sum_{m\in\mathbb{Z}^3} a_m u^m\Big) = \sum_{m\in\mathbb{Z}^3} m_j a_m u^m, \quad j = 1, 2, 3.$$

These derivations δ_j satisfy the relations

$$(\delta_j(a))^* = -\delta_j(a^*),$$
$$\varphi(a\delta_j(b)) + \varphi(\delta_j(a)b) = 0.$$

2.2.2 De Rham complex for noncommutative three tori

We will first construct the space of differential forms on $\mathbb{T}^3_\theta$. Let $W = \mathbb{C}^3$ and $\Lambda^\bullet W = \bigoplus_{j=0}^3 \Lambda^j W$ be the exterior algebra of W. The algebra

$$\Omega^\bullet \mathbb{T}^3_\theta := C^\infty(\mathbb{T}^3_\theta) \otimes \Lambda^\bullet W,$$

will play the role of the algebra of complex differential forms of the noncommutative 3-torus.

We define the exterior derivative on functions, $d_0 : \Omega^0\mathbb{T}^3_\theta \to \Omega^1\mathbb{T}^3_\theta$, by

$$d_0(a) = (i\delta_1(a), i\delta_2(a), i\delta_3(a)), \quad \forall a \in C^\infty(\mathbb{T}^3_\theta).$$

Correspondingly, exterior derivative on 1-forms, $d_1 : \Omega^1\mathbb{T}^3_\theta \to \Omega^2\mathbb{T}^3_\theta$, and on 2-forms $d_2 : \Omega^2\mathbb{T}^3_\theta \to \Omega^3\mathbb{T}^3_\theta$ are given by

$$\begin{aligned} d_1(a_1, a_2, a_3) &= (i\delta_1(a_2) - i\delta_2(a_1), i\delta_2(a_3) - i\delta_3(a_2), i\delta_1(a_3) - i\delta_3(a_1)), \\ d_2(b_1, b_2, b_3) &= i\delta_1(b_2) - i\delta_2(b_3) + i\delta_3(b_1). \end{aligned}$$

It is not difficult to check that $d_{j+1}d_j = 0$. We define the de Rham complex of the noncommutative 3-torus to be the following complex

$$\Omega^0\mathbb{T}^3_\theta \xrightarrow{d_0} \Omega^1\mathbb{T}^3_\theta \xrightarrow{d_1} \Omega^2\mathbb{T}^3_\theta \xrightarrow{d_2} \Omega^3\mathbb{T}^3_\theta. \tag{2.2.2}$$

In the commutative case, to define the Laplacian on forms, we need to fix a Riemannian metric first and find the adjoint of the exterior derivatives, d_j^* with respect to that metric. Then the Laplacian Δ_j on j-forms is defined as

$$\Delta_j = \begin{cases} d_{j-1}d_{j-1}^* + d_j^* d_j & j = 1, \\ d_j^* d_j & j = 0, \end{cases}$$

In the noncommutative case we can study specific forms of metrics where the effect of the metric can be implemented through a volume form. Then this helps us to define the adjoint of the exterior derivatives and similar to the classical case, one can define the Laplacian on j-forms. These metrics include conformal perturbation of a flat metric, as it is studied in [6, 9, 5, 10] for noncommutative two tori, and a new class of non-conformally flat metrics in which only two directions are perturbed by a conformal factor. The geometry of conformally flat metrics on $\mathbb{T}^3_\theta$ will be studied in section 2.3, and the geometry of non-conformally flat metrics will be studied in section 2.4.

2.2.3 The Ricci functional

In a noncommutative setting, as a general rule, spectral methods must be employed to formulate metric invariants. For example, in the noncommutative formulation of the Ricci curvature in [12], instead of a tensorial algebraic definition, the spectral properties of the Laplacians are used to define and compute what is called the Ricci density. This formulation allows us to define this quantity for the noncommutative three torus. In this section, we quickly review the definitions and motivations for this new formulation.

Suppose M is an $m-$dimensional closed oriented Riemannian manifold. Let $V \to M$ be a smooth Hermitian vector bundle over M and $P : C^\infty(V) \to C^\infty(V)$ be a positive elliptic differential operator of order d. The heat operator e^{-tP} is trace class for all positive values of t and it has a short time asymptotic expansion (cf. [13])

$$\mathrm{Tr}(e^{-tP}) \sim \sum_{n=0}^{\infty} a_n(P) t^{\frac{n-m}{d}}, \qquad t \to 0^+.$$

The coefficients $a_n(P)$ are given by an integral formula

$$a_n(P) = \int_M \mathrm{tr}(a_n(x,P))\mathrm{dvol}(x), \tag{2.2.3}$$

where $\mathrm{tr}(a_n(x,P))$ is the fibrewise trace and $\mathrm{dvol}(x) = \sqrt{\det g}\, dx^1...dx^m$ is the Riemannian volume form of M.

To recover the densities $a_n(x,P)$, one needs to study the localized heat trace $\mathrm{Tr}(Fe^{-tP})$ by a localizing factor $F \in C^\infty(\mathrm{End}(V))$. For an endomorphism $F \in C^\infty(\mathrm{End}(V))$, there is also a complete asymptotic expansion

$$\mathrm{Tr}(Fe^{-tP}) \sim \sum_{n=0}^{\infty} a_n(F,P) t^{\frac{n-m}{d}}, \qquad t \to 0^+,$$

where, this time the coefficients $a_n(F,P)$ can be written as the integral

$$a_n(F,P) = \int_M \mathrm{tr}(F(x)a_n(x,P))\mathrm{dvol}(x).$$

A method to compute these densities, which uses the pseudodifferential calculus, will be outlined in the next section, and will be used for differential operators on the noncommutative tori.

On the other hand, if P is a Laplace type operator, namely, a positive elliptic operator whose leading symbol is given by the inverse of the metric tensor, then there exists a unique connection ∇ on V and a unique endomorphism $E \in C^\infty(\mathrm{End}(V))$ such that [13]

$$P = P_\nabla - E,$$

where $P_\nabla : C^\infty(V) \to C^\infty(V)$ is the Bochner Laplacian of the connection defined as the composition of operators as follows

$$P_\nabla : C^\infty(V) \xrightarrow{\nabla} C^\infty(T^*M \otimes V) \xrightarrow{\nabla} C^\infty(T^*M \otimes T^*M \otimes V) \xrightarrow{-g\otimes 1} C^\infty(V).$$

The first two densities of the corresponding heat kernel for P are given by

$$a_0(x, P) = (4\pi)^{-m/2}\mathrm{I},$$
$$a_2(x, P) = (4\pi)^{-m/2}(\frac{1}{6}R(x) + E),$$

where $R(x)$ is the scalar curvature of M.

We apply the above general idea to Laplacians $\triangle_0$ and $\triangle_1$ on $\Omega^0(M)$ and $\Omega^1(M)$. The endomorphism E for the Laplacian on functions Δ_0 is zero, therefore, the first two densities in the heat kernel of $\triangle_0$ are given by

$$a_0(x, \triangle_0) = (4\pi)^{-m/2},$$
$$a_2(x, \triangle_0) = (4\pi)^{-m/2}(\frac{1}{6}R(x)).$$

By Weitzenböck formula (2.1.1), the endomorphism for Laplacian on 1-forms $\triangle_1$ is $-\mathrm{Ric}_x$, the Ricci operator on the cotangent bundle. Thus we have

$$a_0(x, \triangle_1) = (4\pi)^{-m/2}\mathrm{I},$$
$$a_2(x, \triangle_1) = (4\pi)^{-m/2}(\frac{1}{6}R(x) - \mathrm{Ric}_x).$$

These observations lead us to the following definition from [12].

Definition 2.2.1 *The Ricci functional $\mathcal{R}ic : C^\infty(\mathrm{End}(T^*M)) \to \mathbb{C}$ is defined as*

$$\mathcal{R}ic(F) = a_2(\mathrm{tr}(F), \triangle_0) - a_2(F, \triangle_1). \tag{2.2.4}$$

The Ricci functional can also be described in terms of the spectral zeta function [12, Proposition 2.2]:

$$\mathcal{R}ic(F) = \begin{cases} \zeta(0, \mathrm{tr}(F), \triangle_0) - \zeta(0, F, \triangle_1) + \mathrm{Tr}(\mathrm{tr}(F)Q_0) - \mathrm{Tr}(FQ_1), & m = 2 \\ \Gamma(\frac{m}{2} - 1)\mathrm{Res}_{s=\frac{m}{2}-1}\left(\zeta(s, \mathrm{tr}(F), \triangle_0) - \zeta(s, F, \triangle_1)\right), & m > 2. \end{cases} \tag{2.2.5}$$

Here $\zeta(s, F, \triangle_1)$ is the localized spectral zeta function defined by $\mathrm{Tr}(F\triangle_1^{-s})$ for $\Re(s) > m/2$, $\zeta(s, f, \triangle_0)$ is defined similarly, and Q_j is the orthogonal projection on the kernel of $\triangle_j$.

2.2.4 Pseudodifferential calculus and local computations

In this section, we briefly recall the definition of Connes pseudodifferential calculus [2] for C^*-dynamical systems adapted to 3-dimensional noncommutative tori and outline the necessary steps to use it to compute the heat trace densities. These densities then can be used to define the Ricci density and the scalar curvature density for the noncommutative three torus.

The action (2.2.1) on $C(\mathbb{T}^3_\theta)$ defines a C^*-dynamical system $(C(\mathbb{T}^3_\theta), \mathbb{R}^3, \alpha)$. A pseudodifferential calculus can be assigned to the given C^*-dynamical system. The symbols of order d are given by smooth maps $\rho : \mathbb{R}^3 \to C^\infty(\mathbb{T}^3_\theta)$ such that

(i) For any non-negative multi-indices α, β, there exists a positive number $C_{\alpha,\beta}$ such that

$$||\delta^\alpha \partial^\beta \rho(\xi)|| \leqslant C_{\alpha,\beta}(1 + |\xi|)^{d-|\beta|}.$$

(ii) There is a smooth map $f : \mathbb{R}^3 \backslash \{0\} \to C^\infty(\mathbb{T}^3_\theta)$ such that

$$\lim_{\lambda\to\infty} \lambda^{-d} \rho(\lambda\xi_1, \lambda\xi_2, \lambda\xi_3) = f(\xi_1, \xi_2, \xi_3).$$

Here, we use the notation that for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ we have

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial\xi_2^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial\xi_3^{\alpha_3}}, \quad \delta^\alpha = \delta_1^{\alpha_1} \delta_2^{\alpha_2} \delta_3^{\alpha_3}.$$

We shall denote the set of all symbols of order d by $S^d(\mathbb{T}^3_\theta)$. The pseudodifferential operator associated to a given symbol $\rho \in S^d(\mathbb{T}^3_\theta)$ is defined by

$$P_\rho(a) = (2\pi)^{-3} \int \int e^{-iz\cdot\xi} \rho(\xi) \alpha_z(a) dz d\xi, \quad a \in C^\infty(\mathbb{T}^3_\theta).$$

The following theorem from [2] gives a formula for the symbol of the product of pseudodifferential operators.

Theorem 2.2.2 *If* $\rho_j \in S^{d_j}(\mathbb{T}^3_\theta)$, $j = 1, 2$, *there exists a* $\rho \in S^{d_1+d_2}$ *such that* $P_\rho = P_{\rho_1} P_{\rho_2}$, *and moreover,* ρ *has an asymptotic expansion given by*

$$\rho \sim \sum_\alpha \frac{1}{\alpha!} \partial^\alpha(\rho_1) \delta^\alpha(\rho_2). \tag{2.2.6}$$

Remark For our purposes, we need more general symbols which take values in $C^\infty(\mathbb{T}^3_\theta) \otimes M_n(\mathbb{C})$. The above calculus easily extends to this setting.

In the rest of this section we outline the steps through which one can find the second density of the heat trace a_2 for a positive elliptic differential operator on $\mathbb{T}_\theta^3$ using the pseudodifferential calculus. For more details we refer the readers to [13] for the commutative case and [6, 10, 5] for the noncommutative case.

Let P be a second order positive elliptic operator on $\mathbb{T}_\theta^3$ with positive principal symbol, i.e., if we write the symbol of P as the sum of the homogeneous parts $a_2(\xi) + a_1(\xi) + a_0(\xi)$, $a_2(\xi)$ is positive and it is invertible for any nonzero $\xi \in \mathbb{R}^3$. Then the parametrix $(P-\lambda)^{-1}$ for any $\lambda \in \mathbb{C}\backslash\mathbb{R}^+$ is a pseudodifferential operator of order -2 and its symbol $\sigma((P-\lambda)^{-1})$ can be written as $b_0(\xi,\lambda) + b_1(\xi,\lambda) + \cdots$, where $b_j(\xi,\lambda)$ is homogeneous of order $-2-j$ in (ξ,λ), that is it satisfies $b_j(t\xi, t^2\lambda) = t^{-2-j} b_j(\xi,\lambda)$ for all $t \geq 0$. The terms b_j can be written in terms of a_j's and b_0 using the recursive formula for symbol product (2.2.6) applied to the equality $(P-\lambda)^{-1}(P-\lambda) \sim 1$:

$$\begin{aligned}
b_0(\xi,\lambda) =&(a_2-\lambda)^{-1},\\
b_1(\xi,\lambda) =&- b_0 a_1 b_0 - \sum_{j=1}^{3} \partial_j(b_0)\delta_j(a_2)b_0,\\
b_2(\xi,\lambda) =&- b_0 a_0 b_0 - b_1 a_1 b_0\\
&- \sum_{i=1}^{3} \Big(\partial_i(b_0)\delta_i(a_1)b_0 + \partial_i(b_1)\delta_i(a_2)b_0 + \frac{1}{2}\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)b_0\Big).
\end{aligned} \tag{2.2.7}$$

Using the Cauchy integral formula and the formula for the trace in terms of the symbols of a smoothing operator, one has the asymptotic expansion of the localized heat trace $\mathrm{Tr}(Fe^{-tP})$ as follows:

$$\mathrm{Tr}(Fe^{-tP}) \sim \sum_{n=0}^{\infty} t^{\frac{n-3}{2}} \varphi\Big(\mathrm{tr}(F \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{2\pi i} \int_\gamma e^{-\lambda} b_n(\xi,\lambda) d\lambda d\xi)\Big).$$

The geometric meaning of the second density $a_2(P)$, i.e., densities for the coefficient of the term $t^{-\frac{1}{2}}$, in the classical case is discussed in section 2.2.3. In the noncommutative case, by analogy, the second density which is given by

$$a_2(P) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{2\pi i} \int_\gamma e^{-\lambda} b_2(\xi,\lambda) d\lambda d\xi, \tag{2.2.8}$$

can be used to define the Ricci and scalar curvature for the noncommutative torus when P is a carefully chosen geometric operator. By a homogeneity argument given in [14] for noncommutative three tori, we can rewrite $a_2(P)$ as

$$a_2(P) = \frac{1}{8\pi^{7/2}} \int_{\mathbb{R}^3} b_2(\xi, -1) d\xi. \tag{2.2.9}$$

To compute the integral (2.2.9) above, one needs to apply the rearrangement lemma. Here we shall use a general version from [16, Corollary 3.5].

Proposition 2.2.3 *Suppose $\mathcal{A}$ is a C^*-algebra. Let $f_0, ..., f_p : \mathbb{R}_{\geqslant 0} \to \mathbb{C}$ be smooth functions such that for each pair of positive numbers $0 < C_1 < C_2$ and each multi-index $\alpha \in \mathbb{N}^{n+1}$, the function $f(x_0, ..., x_p) := \prod_{j=0}^{p} f_j(x_j)$ satisfies*

$$\int_0^\infty \sup_{\substack{C_1 \leqslant s_j \leqslant C_2 \\ 0 \leqslant j \leqslant n}} |u^{|\alpha|} (\partial^\alpha f)(us)| du < \infty,$$

Let $A = e^a$ for some selfadjoint element $a \in \mathcal{A}$. Then for $\rho_1, \cdots, \rho_p \in \mathcal{A}$

$$\begin{aligned} &\int_0^\infty f_0(uA) \cdot b_1 \cdot f_1(uA) \cdot \cdots \cdot b_p \cdot f_p(uA) du \\ &= A^{-1} F_\gamma(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)}, \cdots, \Delta_{(1)} \cdots \Delta_{(p)})(\rho_1 \cdot \rho_2 \cdot \cdots \cdot \rho_p), \end{aligned}$$

where $\Delta_{(j)}$ is the modular operator acting on b_j by $\Delta(b) = A^{-1} b A$, and the smooth function F is given by

$$F(s_1, ..., s_p) = \int_0^\infty f_0(u) \cdot f_1(us_1) \cdot \cdots \cdot f_p(us_p) du.$$

In the following, we first compute the Laplacians $\triangle_{0,h}$ and $\triangle_{1,h}$ and show that they are anti-unitary equivalent to operators $\tilde{\triangle}_{0,h}$ and $\tilde{\triangle}_{1,h}$ which are second order positive elliptic differential operators. Hence, the above theory can be applied to find their second densities $a_2(\tilde{\triangle}_{0,h})$ and $a_2(\tilde{\triangle}_{1,h})$. Now we can define

Definition 2.2.4 *The scalar curvature functional $\mathcal{R} : C^\infty(\mathbb{T}^3_\theta) \to \mathbb{C}$ is defined as*

$$\mathcal{R}(a) := \varphi(a a_2(\triangle_{0,h})), \qquad a \in C^\infty(\mathbb{T}^3_\theta), \tag{2.2.10}$$

and $a_2(\triangle_{0,h})$ will be called the scalar curvature density or just the scalar curvature and we denote it by R.

Similar to Definition 2.2.1, we define

Definition 2.2.5 *The Ricci curvature functional $\mathcal{R}ic : C^\infty(\mathbb{T}^3_\theta) \otimes M_3(\mathbb{C}) \to \mathbb{C}$ is defined as*

$$\mathcal{R}ic(F) := \varphi(\mathrm{tr}(F) a_2(\triangle_{0,h})) - \varphi(F a_2(\triangle_{1,h})), \qquad F \in C^\infty(\mathbb{T}^3_\theta) \otimes M_3(\mathbb{C}). \tag{2.2.11}$$

The Ricci density is then defined by the equation

$$\mathcal{R}ic(F) = \varphi(\mathrm{tr}(F\mathbf{Ric})), \qquad F \in C^\infty(\mathbb{T}^3_\theta) \otimes M_3(\mathbb{C}).$$

It can be readily seen that

$$\mathbf{Ric} = R \otimes \mathrm{I}_3 - a_2(\triangle_{1,h}).$$

Using the Mellin transform, one can show that the above definition is equivalent to the equation (2.2.5).

Remark Note that we choose to drop the effect of the volume form density *vol* on the Ricci and scalar curvature densities. We have also dropped the overall multiplicative constants in our definitions above. This means that we are ignoring a factor of $\frac{1}{48\pi^{3/2}} vol$ for the scalar curvature density and a factor of $\frac{1}{8\pi^{3/2}} vol$ for the Ricci density. Moreover, we shall use operators which are anti-unitary equivalent to the Laplacians while computing the densities. It can be seen readily that if $\tilde{\triangle} = U^* \triangle U$, for some anti-unitary operator U then

$$\mathrm{Tr}(Fe^{-t\tilde{\triangle}}) = \mathrm{Tr}(FU^* e^{-t\triangle} U) = \mathrm{Tr}(UFU^* e^{-t\triangle}).$$

Similarly, the localized heat trace densities are related as above. These two points should be taken into account while we recover the classical results in the limit $\theta \to 0$ of our formulas for the noncommutative tori.

2.3 Ricci density for conformally flat metrics

In this section we first investigate how the geometry of conformally flat metrics on three torus $\mathbb{T}^3$ can be implemented on the noncommutative three tori $\mathbb{T}^3_\theta$. We then use it to define the Laplacian on functions and on 1-forms; that is we find the Laplacian of the de Rham complex (2.2.2) with respect to the induced inner products. Then using the pseudodifferential calculus we compute the second densities of heat trace asymptotic for these operators which by Definitions 2.2.4 and 2.2.5 can be used to define the scalar curvature density and the Ricci curvature density for $\mathbb{T}^3_\theta$.

In the commutative case, if $h \in C^\infty(M)$ is a real valued function, conformally changing the Riemannian metric by the function e^{-2h} will result in changing the volume form. For instance, if the dimension of a closed Riemannian manifold M is m, and we denote the conformal change of g by $\tilde{g} = e^{-2h}g$, then the new volume form $\tilde{dx}$ is $e^{-mh}dx$. As a result, the inner products on $\Omega^0(M)$, $\Omega^1(M)$, and $\Omega^2(M)$ are given by

$$\begin{aligned}
\langle f_1, f_2\rangle_{\tilde{g}} &= \int_M f_1 \bar{f}_2 e^{-mh} dx,\\
\langle \alpha_1, \alpha_2\rangle_{\tilde{g}} &= \int_M g^{-1}(\alpha_1, \bar{\alpha}_2) e^{(2-m)h} dx,\\
\langle \omega_1, \omega_2\rangle_{\tilde{g}} &= \int_M (\wedge^2 g^{-1})(\omega_1, \bar{\omega}_2) e^{(4-m)h} dx.
\end{aligned}$$

Inspired by these classical equations, we are able to study the conformal change of metrics for noncommutative three tori. Let h be a self-adjoint positive element of $C^\infty(\mathbb{T}^3_\theta)$ and let $\varphi_0(a) = \varphi(ae^{-3h})$, for any $a \in C(\mathbb{T}^3_\theta)$. Denote the Hilbert space given by the GNS construction of $C(\mathbb{T}^3_\theta)$ with respect to the positive linear functional φ_0 by $\mathcal{H}_h^{(0)}$. In other words, the inner product of $\mathcal{H}_h^{(0)}$ is given by

$$\langle a, b\rangle_{0,h} = \varphi(b^* a e^{-3h}).$$

Let $\mathcal{H}_h^{(1)}$ denote the Hilbert space completion of $\Omega^1\mathbb{T}^3_\theta$ with respect to the inner product of $\mathcal{H}_h^{(1)}$ given by

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3)\rangle_{1,h} = \varphi(\sum_{i=1}^{3} b_i^* a_i e^{-h}).$$

Similarly, let $\mathcal{H}_h^{(2)}$ denote the Hilbert space completion of $\Omega^2\mathbb{T}^3_\theta$ with respect to the inner product of $\mathcal{H}_h^{(2)}$ given by

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3)\rangle_{2,h} = \varphi(\sum_{i=1}^{3} b_i^* a_i e^{h}).$$

We identify the formal adjoint operator d_j^* of d_j acting on elements of $\Omega^{j+1}\mathbb{T}^3_\theta \subset \mathcal{H}_h^{(j+1)}$ as follows. Let us denote $e^{h/2}$ by k. Then we have

$$d_0^*(a_1, a_2, a_3) = -i \sum_{j=1}^{3} \delta_j(a_j k^{-2}) k^6,$$

and

$$d_1^*(a_1, a_2, a_3) =$$
$$\Big(i\delta_3(a_3k^2)k^2+i\delta_2(a_1k^2)k^2, i\delta_3(a_2k^2)k^2-i\delta_1(a_1k^2)k^2, -i\delta_1(a_3k^2)k^2-i\delta_2(a_2k^2)k^2\Big).$$

Now, we can define the Laplacian on 0-forms to be $\triangle_{0,h} = d_0^* d_0$, and the Laplacian on 1-forms to be $\Delta_{1,h} = d_1^* d_1 + d_0 d_0^*$. We have

$$\triangle_{0,h}(a) = d_0^* d_0(a) = \sum_{j=1}^{3} \delta_j(\delta_j(a)k^{-2})k^6,$$

On the other hand, the Laplacian on 1-forms is given by

$$\triangle_{1,h}(a_1, a_2, a_3) =$$

$$\begin{aligned}
\Big(&\Big(\delta_2(\delta_2(a_1)k^2) + \delta_3(\delta_3(a_1)k^2) - \delta_2(\delta_1(a_2)k^2) - \delta_3(\delta_1(a_3)k^2)\Big)k^2 + \textstyle\sum \delta_1(\delta_j(a_jk^{-2})k^6),\\
&\Big(\delta_1(\delta_1(a_2)k^2) - \delta_1(\delta_2(a_1)k^2) + \delta_3(\delta_3(a_2)k^2) - \delta_3(\delta_2(a_3)k^2)\Big)k^2 + \textstyle\sum \delta_2(\delta_j(a_jk^{-2})k^6),\\
&\Big(\delta_1(\delta_1(a_3)k^2) - \delta_1(\delta_3(a_1)k^2) - \delta_2(\delta_3(a_2)k^2) + \delta_2(\delta_2(a_3)k^2)\Big)k^2 + \textstyle\sum \delta_3(\delta_j(a_jk^{-2})k^6)\Big).
\end{aligned}$$

The right multiplication operator R_{k^3} satisfies the property

$$\langle R_{k^3}a, R_{k^3}b\rangle_{0,h} = \varphi_0(k^3b^*ak^3) = \varphi(k^3b^*ak^{-3}) = \varphi(b^*a) = \langle a, b\rangle_{0,0},$$

and thus extends to a unitary operator from $\mathcal{H}_0^{(0)}$ to $\mathcal{H}_h^{(0)}$, which we still denote by R_{k^3}. Let $J : C(\mathbb{T}^3_\theta) \to C(\mathbb{T}^3_\theta)$ be the adjoint map $J(a) = a^*$. Then $R_{k^3}J : \mathcal{H}_0^{(0)} \to \mathcal{H}_h^{(0)}$ is an anti-unitary operator. Thus $\triangle_{0,h}$ is anti-unitary equivalent to

$$\tilde{\Delta}_{0,h} := JR_{k^3}^* \triangle_{0,h} R_{k^3} J = k^{-3}(J\triangle_{0,h}J)k^3 = \sum_{j=1}^{3} k^3\delta_j k^{-2}\delta_j k^3.$$

It can also be seen that

$$\langle R_k(a_1, a_2, a_3), R_k(b_1, b_2, b_3)\rangle_{1,h} = \langle (a_1, a_2, a_3), (b_1, b_2, b_3)\rangle_{1,0}.$$

Hence R_k can be extended to a unitary operator from $\mathcal{H}_0^{(1)}$ to $\mathcal{H}_h^{(1)}$, which we still denote by R_k. Then we get an anti-unitary operator $R_kJ : \mathcal{H}_0^{(1)} \to \mathcal{H}_h^{(1)}$. Therefore, $\triangle_{1,h}$ is anti-unitary equivalent to

$$\tilde{\triangle}_{1,h} := JR_k^* \triangle_{1,h} R_k J = k^{-1} J \triangle_{1,h} J k.$$

Since $JR_{k^m}J = k^m$, and $J\delta_j = -\delta_j J$, for $j = 1, 2, 3$, we have

$$JR_{k^m}\delta_i R_{k^n}\delta_j J = JR_{k^m} JJ\delta_i R_{k^n}\delta_j J = k^m \delta_i k^n \delta_j.$$

Thus,

$$\begin{gathered}\tilde{\triangle}_{1,h}(a_1, a_2, a_3) = \\ \Big(k\delta_3 k^2\delta_3 k a_1 + k\delta_2 k^2 \delta_2 k a_1 - k\delta_2 k^2 \delta_1 k a_2 - k\delta_3 k^2 \delta_1 k a_3 + \sum k^{-1}\delta_1 k^6 \delta_j k^{-1} a_j, \\ -k\delta_1 k^2 \delta_2 k a_1 + k\delta_3 k^2 \delta_3 k a_2 + k\delta_1 k^2 \delta_1 k a_2 - k\delta_3 k^2 \delta_2 k a_3 + \sum k^{-1}\delta_2 k^6 \delta_j k^{-1} a_j, \\ -k\delta_1 k^2 \delta_3 k a_1 + k\delta_1 k^2 \delta_1 k a_3 - k\delta_2 k^2 \delta_3 k a_2 + k\delta_2 k^2 \delta_2 k a_3 + \sum k^{-1}\delta_3 k^6 \delta_j k^{-1} a_j\Big).\end{gathered}$$

2.3.1 Scalar curvature

The scalar curvature for conformally flat metrics on noncommutative three tori was first computed in [14]. For the sake of completeness, we shall compute it again here. As discussed in Section 2.4, we define the scalar curvature of $\mathbb{T}^3_\theta$ to be

$$R = a_2(\tilde{\triangle}_{0,h}) = \frac{1}{8\pi^{7/2}} \int_{\mathbb{R}^3} b_2(\xi, -1) d\xi. \tag{2.3.1}$$

where $b_2(\xi, -1)$ is the second term in the asymptotic expansion of the symbol of the parametrix of $\tilde{\triangle}_{0,h}$.

To compute b_2 we need first to find the symbol of the Laplacian on functions.

Lemma 2.3.1 *Let the symbol of $\tilde{\triangle}_{0,h}$ be written as the sum of its homogeneous parts, $\sigma(\tilde{\triangle}_{0,h}) = a_2 + a_1 + a_0$. Then we have*

$$\begin{aligned} a_2 &= k^4\xi_1^2 + k^4\xi_2^2 + k^4\xi_3^2, \\ a_1 &= \sum_{i=1}^{3} (2k\delta_i(k^3) + k^3\delta_i(k^{-2})k^3)\xi_i, \\ a_0 &= \sum_{i=1}^{3} \Big(k\delta_i^2(k^3) + k^3\delta_i(k^{-2})\delta_i(k^3)\Big). \end{aligned}$$

∎

To evaluate the integral in (2.3.1), for this case, we shall first move to spherical coordinates. After performing the angular integrals, we are left with sums of integrals of the form

$$\int_0^\infty b_0^{m_0}\rho_1 b_0^{m_1}\rho_2 b_0^{m_2}\cdots\rho_l b_0^{m_p}u^{(-3/2+\sum_{j=0}^{p}m_j)}du.$$

To compute these latter integrals we need to use the following version of the rearrangement lemma. Here we present it as a corrollary of Proposition 2.2.3, but a straightforward proof can be found in [14].

Corollary 2.3.2 *Let $b_0 = (1+k^4u)^{-1}$, $\rho_j \in C^\infty(\mathbb{T}^3_\theta)$, $m_j \in \mathbb{Z}$, for $j = 0, 1, ..., p$, and set the modular operator Δ be $\Delta(x) = k^{-6}xk^6$. Then*

$$\int_0^\infty b_0^{m_0}\rho_1 b_0^{m_1}\cdots\rho_p l b_0^{m_p}u^{(-\frac{3}{2}+\sum m_j)}du =$$
$$k^{(2-4\sum m_j)}F_{m_0,,...,m_p}(\Delta_{(1)},\cdots,\Delta_{(p)})(\rho_1\cdot\rho_2\cdots\rho_p),$$

where

$$F_{m_0,\cdots,m_p}(s_1,\cdots,s_p) = \int_0^\infty (1+u)^{-m_0}\prod_{j=1}^{p}\Big(u\prod_{h=1}^{j}s_h^{\frac{2}{3}}+1\Big)^{-m_j}u^{(-\frac{3}{2}+\sum m_j)}du.$$

Proof Let u be $t^{2/3}$. Then we have $b_0 = (1+(tA)^{2/3})^{-1}$ where $A = k^6 = e^{3h}$, and it is enough to consider the following functions;

$$f_0(x) := x^{-4/3+3/2\sum_{j=0}^{p}m_j}(1+x^{2/3})^{-m_0}, \quad f_j(x) := (1+x^{2/3})^{-m_j}, \quad j = 1, ..., p.$$

If we set $F_{m_0,\cdots,m_p}(s_1,...,s_p) = F(s_1, s_1s_2,\cdots,s_1\cdots s_p)$, by Proposition 2.2.3, the result is proven.

For instance

$$F_{1,1}(s_1) = \frac{\pi}{s_1^{\frac{2}{3}}+\sqrt[3]{s_1}}, \qquad F_{2,1}(s_1) = \frac{\pi\left(\sqrt[3]{s_1}+2\right)}{2\left(\sqrt[3]{s_1}+1\right)^2\sqrt[3]{s_1}},$$

$$F_{1,1,1}(s_1,s_2) = \frac{\pi\left(\sqrt[3]{s_1}\left(\sqrt[3]{s_2}+1\right)+1\right)}{\left(\sqrt[3]{s_1}+1\right)s_1\left(\sqrt[3]{s_2}+1\right)\sqrt[3]{s_2}\left(\sqrt[3]{s_1}\sqrt[3]{s_2}+1\right)},$$

$$F_{2,1,1}(s_1,s_2) = \frac{\pi\left(\left(\sqrt[3]{s_1}+2\right)\sqrt[3]{s_1}\left(\sqrt[3]{s_2}+1\right)\left(\sqrt[3]{s_1}\sqrt[3]{s_2}+2\right)+2\right)}{2\left(\sqrt[3]{s_1}+1\right)^2 s_1\left(\sqrt[3]{s_2}+1\right)\sqrt[3]{s_2}\left(\sqrt[3]{s_1}\sqrt[3]{s_2}+1\right)^2}.$$

The complete list of these functions can be found in Appendix B.

All the ρ_j's appeared in our computations are multiples of $\delta_j(k)$ or $\delta_j^2(k)$. We want to write all ρ_j's in terms of $\log k$. To perform this step, using the expansional formula applied in [5, section 6.1], we find the corresponding formula

$$k^{-1}\delta_j(k) = f(\Delta)(\delta_j(\log k)),$$
$$k^{-1}\delta_j^2(k) = f(\Delta)(\delta_j^2(\log k)) + 2g(\Delta_{(1)}, \Delta_{(2)})(\delta_j(\log k)\cdot\delta_j(\log k)),$$

where,

$$f(x) = \int_0^1 x^{s/6}ds = \frac{6(x^{1/6}-1)}{\log x},$$

$$g(x,y) = \int_0^1\int_0^s x^{s/6}y^{t/6}dtds = \frac{36(x^{1/6}((y^{1/6}-1)\log x - \log y) + \log y)}{\log x \log y(\log x + \log y)}.$$

And finally, the result is rewritten in terms of $\nabla := \log\Delta = -3[h,\cdot]$.

Theorem 2.3.3 *For the noncommutative three tori $\mathbb{T}^3_\theta$ equipped with a conformally flat metric $g = e^{-2h}\,\mathrm{I}_3$, the scalar curvature R is given by*

$$R = a_2(\tilde{\triangle}_{0,h}) = \frac{k^{-2}}{\pi^{3/2}}\Big(K(\nabla)\left(\triangle(\log k)\right) + H(\nabla_{(1)}, \nabla_{(2)})\left(\sum \delta_j(\log k)\cdot\delta_j(\log k)\right)\Big),$$

where $\triangle(x) = \sum_{j=1}^3 \delta_j^2(x)$, $k = e^{h/2}$. The one variable function K is given by

$$K(s) = \frac{1-e^{s/3}}{s(e^{s/6}+e^{s/2})}, \tag{2.3.2}$$

and the two variable function H is given by

$$H(s,t) = -\frac{3\left(\left(e^{s/3}+3\right)s\left(e^{t/3}-1\right) - \left(e^{s/3}-1\right)\left(3e^{t/3}+1\right)t\right)}{st(s+t)e^{\frac{1}{6}(s+t)}\left(e^{(s+t)/3}+1\right)}. \tag{2.3.3}$$

∎

The classical limit $\theta \to 0$, is obtained by taking the limits of $K(s)$ and $H(s,t)$ as $t, s \to 0$. We obtain

$$\lim_{s\to 0} K(s) = -\frac{1}{6}, \qquad \lim_{(s,t)\to(0,0)} H(s,t) = \frac{1}{6}.$$

This implies that the scalar curvature R approaches the limit

$$-\frac{k^{-2}}{24\pi^{3/2}}\sum(2\delta_j^2(h) - \delta_j(h)\delta_j(h)),$$

as $\theta \to 0$. It matches with the scalar curvature $-2e^{2h}\sum(-2h_{jj} + h_j^2)$ for the three torus with the metric $g = e^{-2h}(dx^2 + dy^2 + dz^2)$ up to the factor of $k^6 48\pi^{3/2}$ due to our convention (see Remark 2.2.4).

2.3.2 The Ricci density

In this section, we shall compute the Ricci density of $\mathbb{T}_\theta^3$ equipped with a conformally flat metric. To this end, we first need to find the term (2.2.9) for $\tilde{\triangle}_{1,h}$ which is anti-unitarily equivalent to the Laplacian on 1-forms. We shall follow all the computational steps listed in the previous section to compute the scalar curvature, with one difference that the symbols are matrix valued in this case and the results will be in the matrix form. We start with the symbol of $\tilde{\triangle}_{1,h}$.

Lemma 2.3.4 *If we denote the symbol of $\tilde{\triangle}_{1,h}$ by $\sigma(\tilde{\triangle}_{1,h}) = a_2 + a_1 + a_0$, then we have*

$$
\begin{aligned}
a_2 =& (k^4\xi_1^2 + k^4\xi_2^2 + k^4\xi_3^2)\mathrm{I}_3,\\
a_1 =& \begin{pmatrix} k^5\delta_1(k^{-1}) + k^{-1}\delta_1(k^5) & -k\delta_2(k^4)k^{-1} & -k\delta_3(k^4)k^{-1} \\ k^{-1}\delta_2(k^4)k & k^3\delta_1(k) + k\delta_1(k^3) & 0 \\ k^{-1}\delta_3(k^4)k & 0 & k^3\delta_1(k) + k\delta_1(k^3) \end{pmatrix}\xi_1 \\
&+ \begin{pmatrix} k^3\delta_2(k) + k\delta_2(k^3) & k^{-1}\delta_1(k^4)k & 0 \\ -k\delta_1(k^4)k^{-1} & k^5\delta_2(k^{-1}) + k^{-1}\delta_2(k^5) & -k\delta_3(k^4)k^{-1} \\ 0 & k^{-1}\delta_3(k^4)k & k^3\delta_2(k) + k\delta_2(k^3) \end{pmatrix}\xi_2 \\
&+ \begin{pmatrix} k^3\delta_3(k) + k\delta_3(k^3) & 0 & k^{-1}\delta_1(k^4)k \\ 0 & k^3\delta_3 k + k\delta_3(k^3) & k^{-1}\delta_2(k^4)k \\ -k\delta_1(k^4)k^{-1} & -k\delta_2(k^4)k^{-1} & k^5\delta_3(k^{-1}) + k^{-1}\delta_3(k^5) \end{pmatrix}\xi_3, \\
a_0 =& \sum_{1\leqslant i,j\leqslant 3}\Big(k^{-1}\delta_i(k^6\delta_j(k^{-1})) - k\delta_j(k^2\delta_i(k))\Big)E_{ij} + \sum_{j=1}^{3} k\delta_j(k^2\delta_j(k))\mathrm{I}_3.
\end{aligned}
$$

Here E_{ij}'s are the matrix units. ∎

To compute $b_2(\xi, -1)$, we use the symbol of $\tilde{\triangle}_{1,h}$ and (2.2.7). Then (2.2.8) gives the second heat trace density $a_2(\tilde{\triangle}_{1,h})$.

Proposition 2.3.5 *With notation as above, we have*

$$\pi^{3/2}k^2 a_2(\tilde{\triangle}_{1,h}) = \Big(-\frac{1}{2}K(\nabla)\,(\triangle(\log k)) + T(\nabla_{(1)},\nabla_{(2)})\,\big(\sum \delta_i(\log k)\cdot\delta_i(\log k)\big)\Big)\,\mathrm{I}_3$$
$$+\sum_{i,j=1}^{3}\Big(F(\nabla)\Big(\delta_i\delta_j(\log k)\Big) + W(\nabla_{(1)},\nabla_{(2)})\Big(\delta_i(\log k)\cdot\delta_j(\log k)\Big)$$
$$+S(\nabla_{(1)},\nabla_{(2)})\Big([\delta_j(\log k),\delta_i(\log k)]\Big)\Big)\,E_{ij},$$

where K is the function in (2.3.2)*, and the other functions are given as follow:*

$$F(s) = \frac{e^{-\frac{s}{2}}(e^s-1)}{2(1+e^{\frac{s}{3}})s},$$
$$T(s,t) = \frac{3s(1-e^{\frac{t}{3}})(e^{\frac{2s+t}{3}} - e^{\frac{s+t}{3}} - e^{\frac{2s}{3}} - 1) + 3t(1-e^{\frac{s}{3}})(e^{\frac{s+2t}{3}} + e^{\frac{s}{3}} + e^{\frac{t}{3}} - 1)}{st(s+t)e^{\frac{3s+t}{6}}(e^{\frac{(s+t)}{3}}+1)},$$
$$W(s,t) = \frac{6(e^{\frac{s+t}{3}} + e^{\frac{2(s+t)}{3}} + 1)(se^{\frac{s+t}{3}} - e^{\frac{s}{3}}(s+t) + t)}{st(s+t)e^{\frac{s+t}{2}}(e^{\frac{s+t}{3}}+1)},$$
$$S(s,t) = \frac{1}{st(s+t)e^{\frac{1}{2}(s+t)}(e^{\frac{s+t}{3}}+1)}\Big(3s(e^{\frac{t}{3}}-1)(2e^{\frac{s+t}{3}} + e^{\frac{2s+2t}{3}} - e^{\frac{2s+t}{3}} + 1)$$
$$- 3t(e^{\frac{s}{3}}-1)(2e^{\frac{s+2t}{3}} + e^{\frac{2s+3t}{3}} - e^{\frac{s+t}{3}} + e^{\frac{t}{3}})\Big).$$

∎

Using definitions 2.2.4 and 2.2.5, Theorem 2.3.5, and Proposition 2.3.3, we can compute the Ricci density of the noncommutative three tori $\mathbb{T}^3_\theta$ equipped with a conformally flat metric $g = e^{-2h}\,\mathrm{I}_3$.

Theorem 2.3.6 *The Ricci density of $\mathbb{T}^3_\theta$ equipped with the conformally flat metric $g = e^{-2h}\,\mathrm{I}_3$ is given by*

$$\mathbf{Ric} = \pi^{-\frac{3}{2}}k^{-2}\Big(\frac{3}{2}K(\nabla)\,(\triangle(\log k)) + (H-T)(\nabla_{(1)},\nabla_{(2)})\,\big(\sum\delta_\ell(\log k)\cdot\delta_\ell(\log k)\big)\Big)\,\mathrm{I}_3$$
$$-\pi^{-\frac{3}{2}}k^{-2}\sum\Big(F(\nabla)\Big(\delta_i\delta_j(\log k)\Big) + W(\nabla_{(1)},\nabla_{(2)})\Big(\delta_i(\log k)\cdot\delta_j(\log k)\Big)$$
$$+S(\nabla_{(1)},\nabla_{(2)})\Big([\delta_j(\log k),\delta_i(\log k)]\Big)\Big)E_{ij}.$$

Here $k = e^{h/2}$, and $\triangle(a) = \sum\delta_j^2(a)$ denotes the flat Laplacian. ∎

Remark To check the result with the commutative case, we need to find the following limits:

$$\lim_{s\to 0} F(s) = \frac{1}{4}, \qquad \lim_{(s,t)\to(0,0)} T(s,t) = -\frac{1}{3}, \qquad \lim_{(s,t)\to(0,0)} W(s,t) = \frac{1}{2}.$$

Since in the commutative case the commutator term $[\delta_j(\log k), \delta_i(\log k)]$ on which S acts, automatically vanishes, we find that the $(i,j)^{th}$ entry of the Ricci density for $\theta = 0$ is given by

$$-\frac{k^{-2}}{8\pi^{3/2}}\left(\delta_{ij}\Big(\sum_{\ell=1}^{3}\delta_\ell^2(h) - \delta_\ell(h)^2\Big) + \delta_i(h)\delta_j(h) + \delta_i(\delta_j(h))\right), \tag{2.3.4}$$

where the δ_{ij} denotes the Kronecker delta. On the other hand, a direct computation in the commutative case for the metric $g = e^{-2h}\,\mathrm{I}_3$ gives the $(i,j)^{th}$ component of the Ricci operator as

$$e^{2h}\left(\delta_{ij}\Big(\sum_{\ell=1}^{3} h_{\ell\ell} - {h_\ell}^2\Big) + h_i h_j + h_{ij}\right),$$

which matches with the corresponding Ricci density in (2.3.4) after taking into the account the Remark 2.2.4.

2.4 Ricci density for non-conformal perturbations

In this section we shall compute the Ricci curvature for a metric on the noncommutative three torus which is an analogue of the metric

$$e^{-2h}(dx^2 + dy^2) + dz^2, \tag{2.4.1}$$

for some $h \in C^\infty(\mathbb{T}^3)$ in the classical case. The inner products on functions, 1-forms and 2-forms for a torus equipped with this metric are given as follows:

$$\langle f_1, f_2\rangle = \int_{\mathbb{T}^3} f_1\overline{f_2}e^{-2h}dxdydz,$$

for all $f_1, f_2 \in \Omega^0(\mathbb{T}^3)$,

$$\langle \alpha, \beta\rangle = \int_{\mathbb{T}^3}\left(\alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \alpha_3\overline{\beta_3}e^{-2h}\right)dxdydz,$$

for all $\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3) \in \Omega^1(\mathbb{T}^3)$, and

$$\langle \xi, \eta\rangle = \int_{\mathbb{T}^3}\left(\xi_1\overline{\eta_1}e^{2h} + \xi_2\overline{\eta_2} + \xi_3\overline{\eta_3}\right)dxdydz,$$

for all $\xi = (\xi_1, \xi_2, \xi_3), \eta = (\eta_1, \eta_2, \eta_3) \in \Omega^2(\mathbb{T}^3)$.

Let $k = e^h$ for $h \in C^\infty(\mathbb{T}^3_\theta)$. Motivated by the classical case, we denote by $\mathcal{H}_h^{(0)}$ the Hilbert space given by the GNS construction of $C(\mathbb{T}^3_\theta)$ with respect to the positive linear functional

$$\varphi_0(a) = \varphi(ak^{-2}).$$

For 1-forms, we denote by $\mathcal{H}_h^{(1)}$ the Hilbert space, which is the completion of $\Omega^1\mathbb{T}^3_\theta$ with respect to the inner product given by

$$\langle a, b\rangle = \varphi\left(b_1^* a_1 + b_2^* a_2 + b_3^* a_3 k^{-2}\right).$$

For 2-forms, we denote by $\mathcal{H}_h^{(2)}$ the Hilbert space, which is the completion of $\Omega^2\mathbb{T}^3_\theta$ with respect to the inner product given by

$$\langle a, b\rangle = \varphi\left(b_1^* a_1 k^2 + b_2^* a_2 + b_3^* a_3\right).$$

We also need adjoints of de Rham differentials (2.2.2) with respect to the given metric. It can be shown that the adjoint of d_0 is given by

$$d_0^* : b \mapsto (-i)(\delta_1(b_1)k^2 + \delta_2(b_2)k^2 + \delta_3(b_3) - b_3 k^{-2}\delta_3(k^2)), \quad b = (b_1, b_2, b_3) \in \Omega^1\mathbb{T}^3_\theta.$$

Similarly, the adjoint of d_1 acting on an element $a = (a_1, a_2, a_3) \in \Omega^2\mathbb{T}^3_\theta$ is given by

$$d_1^* : a \mapsto \left(i\delta_2(a_1k^2) + i\delta_3(a_3), i\delta_3(a_2) - i\delta_1(a_1k^2), -i\delta_2(a_2)k^2 - i\delta_1(a_3)k^2\right).$$

To compute the spectral densities of the Laplacians for these metrics, we will follow the steps presented in section 2.2.4. By a homogeneity argument, again, the computation of contour integral can be bypassed by setting $\lambda = -1$;

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{1}{2\pi i}\int b_2(\xi, \lambda)d\lambda d\xi = \frac{1}{8\pi^{7/2}}\int_{\mathbb{R}^3} b_2(\xi, -1)d\xi.$$

Then we have integrals in ξ variable where the dependence of the integrand comes from the powers of $b_0(\xi, -1) = (1 + a_2(\xi))^{-1}$ and ξ_j. To compute these integrals, we first apply a change of variables,

$$\xi_1 = \sqrt{u(1+\eta^2)}\cos\theta, \quad \xi_2 = \sqrt{u(1+\eta^2)}\sin\theta, \quad \xi_3 = \eta, \tag{2.4.2}$$

where the domain of the new variables (u, η, θ) is given by

$$u \in [0, +\infty), \quad \eta \in (-\infty, +\infty), \quad \theta \in [0, 2\pi).$$

The Jacobian of this substitution is $\frac{1}{2}(1+\eta^2)$, and this substitution decomposes b_0 to $(1+\eta^2)^{-1}$ multiplied by a noncommutative part which depends only on u. More precisely

$$b_0(\xi, -1) = (1 + k^2\xi_1^2 + k^2\xi_2^2 + \xi_3^2)^{-1} = (1 + \eta^2 + u(1+\eta^2)k^2)^{-1} = \frac{1}{1+\eta^2} b_0(u).$$

Here we denoted $(1 + uk^2)^{-1}$ by $b_0(u)$. As a result, after applying the substitution, each term of b_2 ends up with a triple integral whose two variables (η, θ) can be separated and integrated, without involving any noncommutative terms. For instance,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \xi_2^4 \xi_3^2 b_0^3(\xi, -1)\delta_3(k^2) b_0(\xi, -1)\delta_3(k^2) b_0(\xi, -1) d\xi \\
&= \int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} \frac{u^2\eta^2(1+\eta^2)^2 \sin^4\theta}{(1+\eta^2)^5} b_0^3(u)\delta_3(k^2) b_0(u)\delta_3(k^2) b_0(u) \frac{1}{2}(1+\eta^2) d\eta d\theta du \\
&= \left(\int_{-\infty}^\infty \frac{\eta^2}{2(1+\eta^2)^2} d\eta\right)\left(\int_0^{2\pi} \sin^4\theta d\theta\right) \int_0^\infty u^2 b_0^3(u)\delta_3(k^2) b_0(u)\delta_3(k^2) b_0(u) du \\
&= \frac{3\pi^2}{16} \int_0^\infty u^2 b_0^3(u)\delta_3(k^2) b_0(u)\delta_3(k^2) b_0(u) du.
\end{aligned}$$

Applying the substitution and integrating out the η and θ variables, we end up with sums of u integrals in one of the following forms:

$$\int_0^\infty b_0(u)^{m_0} \rho_1 b_0(u)^{m_1} \rho_2 \cdots \rho_p b_0(u)^{m_p} u^{-2+\sum m_j} du,$$

or

$$\int_0^\infty b_0(u)^{m_0} \rho_1 b_0(u)^{m_1} \rho_2 \cdots \rho_p b_0(u)^{m_p} u^{-3+\sum m_j} du.$$

Here we need Proposition 2.2.3 for

$$\begin{aligned}
&f_0(x) := x^{\sum m_j - \nu}(1+x)^{-m_0}, \\
&f_j(x) := (1+x)^{-m_j}, \quad j = 1, ..., p,
\end{aligned}$$

and $a = 2h$. Here ν is equal to 2 or 3. We then get the following version of the rearrangement lemma.

Corollary 2.4.1 *Let $b_0 = (1 + uk^2)^{-1}$, $\rho_j \in C^\infty(\mathbb{T}_\theta^3)$, $m_j \in \mathbb{Z}$, for $j = 0, 1, 2, ..., p$, and*

$\Delta(x) = k^{-2}xk^2$. *Then*

$$\int_0^\infty b_0(u)^{m_0}\rho_1 b_0(u)^{m_1}\rho_2 b_0(u)^{m_2}\cdots\rho_l b_0(u)^{m_p}u^{(-\nu+\sum m_j)}du$$
$$= k^{2(-\sum_{j=0}^{p} m_j+\nu-1)}F^{[\nu]}_{m_0,m_1,\ldots,m_p}(\Delta_{(1)},\Delta_{(2)},\ldots,\Delta_{(p)})(\rho_1\cdot\rho_2\cdots\rho_p),$$

where

$$F^{[\nu]}_{m_0,m_1,\ldots,m_p}(s_1,s_2,\ldots,s_p) = \int_0^\infty (1+u)^{-m_0}\prod_{j=1}^{p}\Big(u\prod_{h=1}^{j}s_h+1\Big)^{-m_j}u^{(\sum m_j-\nu)}du.$$

$\blacksquare$

For instance,

$$\begin{aligned}
F^{[2]}_{1,1}(s_1) &= \frac{\log(s_1)}{s_1-1},\\
F^{[3]}_{2,1}(s_1) &= \frac{s_1(\log(s_1)-1)+1}{(s_1-1)^2},\\
F^{[2]}_{1,1,1}(s_1,s_2) &= \frac{(s_1s_2-1)\log(s_1)-(s_1-1)\log(s_1s_2)}{(s_1-1)s_1(s_2-1)(s_1s_2-1)},\\
F^{[3]}_{1,1,1}(s_1,s_2) &= \frac{-s_1s_2\log(s_1)+s_1s_2\log(s_1s_2)-s_2\log(s_1s_2)+\log(s_1)}{(s_1-1)(s_2-1)(s_1s_2-1)}.
\end{aligned}$$

We also need the following result from [5, Section 6.1], according to which we find the formula

$$\begin{aligned}
k^{-1}\delta_j(k) &= f(\Delta)\Big(\delta_j(\log k)\Big),\\
k^{-1}\delta_j^2(k) &= f(\Delta)(\delta_j^2(\log k)) + 2g(\Delta_{(1)},\Delta_{(2)})(\delta_j(\log k)\cdot\delta_j(\log k)),
\end{aligned}\tag{2.4.3}$$

where

$$f(x) = \int_0^1 x^{s/2}ds = \frac{2(\sqrt{x}-1)}{\log x},$$

$$g(x,y) = \int_0^1\int_0^s x^{s/2}y^{t/2}dtds = \frac{4(\sqrt{x}((\sqrt{y}-1)\log x-\log y)+\log y)}{\log x\log y(\log x+\log y)}.$$

Now we can start computing the Laplacians and their spectral densities.

2.4.1 Scalar curvature

In this section, we first find the Laplacian on functions $\triangle_{0,h}$ for the given metric and its anti-unitary equivalent differential operator $\tilde{\triangle}_{0,h}$. Then we use its symbol and its resolvent expansion to find the scalar curvature.

The Laplacian on functions $\triangle_{0,h} : C^\infty(\mathbb{T}^3_\theta) \to C^\infty(\mathbb{T}^3_\theta)$ for the metric (2.4.1), which is given by $\triangle_{0,h} = d_0^* d_0$, computes as

$$\triangle_{0,h}(a) = \delta_1^2(a)k^2 + \delta_2^2(a)k^2 + \delta_3(\delta_3(a)k^{-2})k^2.$$

We define the map $R_{0,k} : \mathcal{H}_{0,0} \to \mathcal{H}_{0,h}$ by $R_{0,k}a = ak$, for all $a \in C(\mathbb{T}^3_\theta)$. It is not hard to see that $R_{0,k}$ is an isometry from $\mathcal{H}_{0,0}$ to $\mathcal{H}_{0,h}$. That is, $\langle R_{0,k}a, R_{0,k}b\rangle_{0,h} = \langle a, b\rangle_{0,0}$. Hence, the Laplacian on functions $\triangle_{0,h}$ for the metric (2.4.1) is anti-unitary equivalent to the differential operator $(R_{0,k}J)^* \triangle_{0,h} R_{0,k} J$ on $\Omega^0\mathbb{T}^3_\theta$, which we denote by $\tilde{\triangle}_{0,h}$.

Lemma 2.4.2 *The homogeneous components of the symbol $\sigma(\tilde{\triangle}_{0,h})$ are:*

$$\begin{aligned}
a_2 &= k^2\xi_1^2 + k^2\xi_2^2 + \xi_3^2,\\
a_1 &= 2k\delta_1(k)\xi_1 + 2k\delta_2(k)\xi_2 + (k^{-1}\delta_3(k) - \delta_3(k)k^{-1})\xi_3,\\
a_0 &= k\delta_1^2(k) + k\delta_2^2(k) + k^{-1}\delta_3^2(k) - \delta_3(k)k^{-2}\delta_3(k) - k^{-1}\delta_3(k)k^{-1}\delta_3(k).
\end{aligned}$$

Proof It can be readily checked that the operator $\tilde{\triangle}_{0,h}$, on the elements of $C^\infty(\mathbb{T}^3_\theta)$, is given by

$$\begin{aligned}
\tilde{\triangle}_{0,h}(a) = {} & k^2\delta_1^2(a) + k^2\delta_2^2(a) + \delta_3^2(a)\\
&+ 2k\delta_1(k)\delta_1(a) + 2k\delta_2(k)\delta_2(a) - k^{-1}\delta_3\left(k^2\right)k^{-1}\delta_3(a) + 2k^{-1}\delta_3(k)\delta_3(a)\\
&+ k^{-1}\delta_3^2(k)a + k\delta_1^2(k)a + k\delta_2^2(k)a - k^{-1}\delta_3\left(k^2\right)k^{-2}\delta_3(k)a.
\end{aligned} \tag{2.4.4}$$

Then the symbol is given by replacing δ_j by ξ_j.

The scalar curvature of $\mathbb{T}^3_\theta$ equipped with the metric (2.4.1) is defined as in Definition 2.2.4. Similar to the conformal case it is given by (2.3.1) where b_2 is the second term in the symbol of the parametrix of $\tilde{\Delta}_{0,h}$ for this metric. The computation then shows that we have:

Theorem 2.4.3 *The scalar curvature R of the noncommutative 3-torus $\mathbb{T}^3_\theta$ equipped with the non-conformal metric* (2.4.1)*, is given by*

$$\begin{aligned}
\pi^{3/2}a_2(\tilde{\triangle}_{0,h}) = {} & K_1(\nabla)(\delta_1^2(h) + \delta_2^2(h)) + H_1(\nabla_{(1)}, \nabla_{(2)})(\delta_1(h)\cdot\delta_1(h) + \delta_2(h)\cdot\delta_2(h))\\
&+ k^{-2}K_2(\nabla)(\delta_3^2(h)) + k^{-2}H_2(\nabla_{(1)}, \nabla_{(2)})(\delta_3(h)\cdot\delta_3(h)),
\end{aligned}$$

where

$$\begin{aligned}
K_1(s) =& \frac{e^{\frac{s}{2}}(2e^s - se^s - 2 - s)}{4s(e^s-1)^2}, \\
K_2(s) =& \frac{1 - e^{2s} + 2se^s}{4se^{\frac{s}{2}}(1-e^s)^2}, \\
H_1(s,t) =& \frac{1}{e^{-\frac{s+t}{2}}(e^s-1)s(e^t-1)t(e^{s+t}-1)^2(s+t)}\Big(e^s(e^t-1)^2s^2 - e^t(e^s-1)^2t^2 \\
& \qquad - (e^s - e^t)(e^{s+t}-1)st + (1-e^s)(e^t-1)(e^{s+t}-1)(t-s)\Big), \\
H_2(s,t) =& \frac{1}{4e^{\frac{1}{2}(s+t)}(e^s-1)(e^t-1)(e^{s+t}-1)^2 st(s+t)} \times \\
& \Big((e^t-1)^2(e^{s+t} - 3e^{2s+t} - e^s - 1)s^2 \\
& + (e^s-1)^2(e^{s+2t} + e^{s+3t} - e^{2t} + 3e^t)t^2 \\
& - 2(e^s-1)(e^t-1)(e^{2(s+t)}-1)(s-t) \\
& + (e^{s+t}-1)(4e^{s+t} + e^{2s+t} - 5e^{s+2t} + e^{2s+2t} + e^s - 5e^t + 2e^{2t} + 1)st\Big).
\end{aligned}$$

∎

We can get the classical scalar curvature in the limit $\theta \to 0$, which is obtained by taking the limits of the above functions as $s, t \to 0$. We have

$$\lim_{(s,t)\to(0,0)} H_1(s,t) = 0, \qquad \lim_{(s,t)\to(0,0)} H_2(s,t) = \frac{1}{8},$$
$$\lim_{s\to 0} K_1(s) = -\frac{1}{24}, \qquad \lim_{s\to 0} K_2(s) = -\frac{1}{12}.$$

Therefore, when $\theta \to 0$, the scalar curvature approaches to

$$-\frac{1}{48\pi^{3/2}}\left(2\delta_1^2(h) + 2\delta_2^2(h) + 4e^{-2h}\delta_3^2(h) - 6e^{-2h}\delta_3(h)^2\right),$$

which is $\frac{e^{-2h}}{48\pi^{3/2}}$ multiple of the scalar curvature, $2e^{2h}(h_{11} + h_{22}) + 4h_{33} - 6(h_3)^2$, in the commutative case. This matches with our normalization of the scalar curvature density.

Remark Comparing the functions K_1 and H_1 with the corresponding functions K and H found in [5, 10] for the spectral densities of the Laplacian $k\partial^*\partial k$ reveals that

$$K_1(s) = -\frac{1}{8}K(s), \qquad H_1(s,t) = -\frac{1}{8}H(s,t). \tag{2.4.5}$$

The factor $-\frac{1}{8}$ is the result of the use of two different normalizations. In the rest of this section we shall look for a clarification of why such a relation (2.4.5) should be true.

First note that the Laplacian on functions $\tilde{\triangle}_{0,h}^{(1)}$, given in (2.4.4), is the sum of two Laplacians when we assume that $\delta_3(k) = 0$;

$$\tilde{\triangle}_{0,h} = \tilde{\triangle}_{0,h}^{(1)} \otimes 1 + 1 \otimes \tilde{\triangle}_{0,h}^{(2)},$$

where

$$\tilde{\triangle}_{0,h}^{(1)} = \sum_{i=1}^{2} k^2 \delta_i^2(a) + 2k\delta_i(k)\delta_i(a) + k\delta_i^2(k)a, \qquad \tilde{\triangle}_{0,h}^{(2)} = \delta_3^2.$$

The operator $\tilde{\triangle}_{0,h}^{(1)}$ is equal to the operator $k\partial^*\partial k$, which is anti-unitarily equivalent to the Laplacian on $C^\infty(\mathbb{T}_\theta^2)$ in [10, Section 4.1] when the complex structure is given by $\tau = i$, namely $\tau_1 = 0$, $\tau_2 = 1$. The operator $\tilde{\triangle}_{0,h}^{(2)}$ is the Laplacian of $\mathbb{T}^1$ with flat metric. Then, the local spectral invariants of $\tilde{\triangle}_{0,h}$ are related to those of $\tilde{\triangle}_{0,h}^{(1)}$ and $\tilde{\triangle}_{0,h}^{(2)}$ as we discuss next.

Let P and Q be two elliptic second order positive differential operators on $C(\mathbb{T}_\theta^d)$ and $C(\mathbb{T}_{\theta'}^{d'})$ respectively. Then $P \otimes 1 + 1 \otimes Q$ forms a positive second order elliptic differential operator on $C(\mathbb{T}_\theta^d) \otimes C(\mathbb{T}_{\theta'}^{d'})$. Moreover, for any $t > 0$ and $a \in A_\theta$ and $b \in A_{\theta,}$ we have

$$\mathrm{Tr}(a \otimes b e^{-t(P\otimes 1 + 1\otimes Q)}) = \mathrm{Tr}(ae^{-tP})\mathrm{Tr}(be^{-tQ}), \quad a \in C^\infty(\mathbb{T}_\theta^d),\ b \in C^\infty(\mathbb{T}_{\theta'}^{d'}),\ t > 0.$$

This not only gives a relations between the coefficients of asymptotic expansions as $t \to 0^+$, but also it provides a relation among the densities of these coefficients. In other words if

$$\mathrm{Tr}(ae^{-tP}) \sim \sum_{n=0}^{\infty} t^{n-\frac{d}{2}} \varphi_\theta(aa_n(P)), \qquad \mathrm{Tr}(be^{-tP}) \sim \sum_{m=0}^{\infty} t^{m-\frac{d'}{2}} \varphi_{\theta'}(ba_m(Q)),$$

where φ_θ and $\varphi_{\theta'}$ is the tracial state on $C^\infty(\mathbb{T}_\theta^d)$ and $C^\infty(\mathbb{T}_{\theta'}^{d'})$, respectively, then

$$\begin{aligned}
\mathrm{Tr}(a \otimes b e^{-t(P\otimes 1 + 1\otimes Q)}) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{m+n-\frac{d'}{2}-\frac{d}{2}} \varphi_\theta(aa_n(P))\varphi_{\theta'}(ba_m(Q)) \\
&= \sum_{l=0}^{\infty} t^{l-\frac{d'+d}{2}} \varphi_\theta \otimes \varphi_{\theta'} \left(a \otimes b\Big(\sum_{l=m+n} a_n(P) \otimes a_m(Q)\Big)\right).
\end{aligned}$$

In our case, we have

$$a_2(\tilde{\triangle}_{0,h}) = a_2(\tilde{\triangle}_{0,h}^{(1)}) \otimes a_0(\tilde{\triangle}_{0,h}^{(2)}) + a_0(\tilde{\triangle}_{0,h}^{(1)}) \otimes a_2(\tilde{\triangle}_{0,h}^{(2)}).$$

However, since $\sigma(\tilde{\triangle}^{(2)}_{0,h}) = \xi^2$, we have $a_2(\tilde{\triangle}^{(2)}_{0,h}) = 0$ and $a_0(\tilde{\triangle}^{(2)}_{0,h}) = \sqrt{\pi}$. Thus

$$a_2(\tilde{\triangle}_{0,h}) = \sqrt{\pi} a_2(\tilde{\triangle}^{(1)}_{0,h}).$$

This is the main reason why the functions of two-dimensional noncommutative two torus with conformally flat metric emerge in the formulas for the noncommutative three torus with non-conformal metric (2.4.1). On the other hand, we note that the functions K_2 and H_2 in Theorem 2.4.3 are new and do not seem to be related to functions for the noncommutative two torus.

2.4.2 Laplacian on 1-forms and the Ricci density

In this section, after finding the Laplacian on 1-forms on $\mathbb{T}^3_\theta$ equipped with the metric (2.4.1), we compute its second heat trace density. Combining with the results from the previous section, we shall then compute the Ricci density of this metric.

Recall that exterior derivative on 1-forms is given by

$$d_1(a_1, a_2, a_3) = (i\delta_1(a_2) - i\delta_2(a_1), i\delta_2(a_3) - i\delta_3(a_2), i\delta_1(a_3) - i\delta_3(a_1)),$$

and hence its formal adjoint with respect to the metric is

$$d_1^*(a_1, a_2, a_3) = \Big(i\delta_2(a_1k^2) + i\delta_3(a_3), i\delta_3(a_2) - i\delta_1(a_1k^2), -i\delta_2(a_2)k^2 - i\delta_1(a_3)k^2\Big).$$

Thus, the Laplacian on 1-forms $\triangle_{1,h}$ computes as

$$\triangle_{1,h}(a_1, a_2, a_3) = d_0 d_0^*(a_1, a_2, a_3) + d_1^* d_1(a_1, a_2, a_3) =$$

$$\begin{aligned}\Big(&\delta_1(\delta_1(a_1)k^2) + \delta_2(\delta_2(a_1)k^2) + \delta_3^2(a_1) + \delta_2(a_2)\delta_1(k^2) - \delta_1(a_2)\delta_2(k^2) - \delta_1(a_3k^{-2}\delta_3(k^2)),\\ &\delta_1(a_1)\delta_2(k^2) - \delta_2(a_1)\delta_1(k^2) + \delta_1(\delta_1(a_2)k^2) + \delta_2(\delta_2(a_2)k^2) + \delta_3^2(a_2) - \delta_2(a_3k^{-2}\delta_3(k^2)),\\ &\delta_1(a_1)\delta_3(k^2) + \delta_2(a_2)\delta_3(k^2) + \delta_1^2(a_3)k^2 + \delta_2^2(a_3)k^2 + \delta_3(\delta_3(a_3k^{-2})k^2)\Big).\end{aligned}$$

Lemma 2.4.4 *The Laplacian on 1-forms $\triangle_{1,h}$ is anti-unitary equivalent to a differential operator $\tilde{\triangle}_{1,h}$ whose symbol is the sum of the homogeneous components given by*

$$a_2 = (k^2\xi_1^2 + k^2\xi_2^2 + \xi_3^2)\,\mathrm{I}_3,$$

$$a_1 = \begin{pmatrix} \delta_1(k^2)\xi_1 + \delta_2(k^2)\xi_2 & \delta_1(k^2)\xi_2 - \delta_2(k^2)\xi_1 & -\delta_3(k^2)k^{-1}\xi_1 \\ \delta_2(k^2)\xi_1 - \delta_1(k^2)\xi_2 & \delta_1(k^2)\xi_1 + \delta_2(k^2)\xi_2 & -\delta_3(k^2)k^{-1}\xi_2 \\ k^{-1}\delta_3(k^2)\xi_1 & k^{-1}\delta_3(k^2)\xi_2 & 2k\sum_{i=1}^{2}\delta_i(k)\xi_i + [k^{-1},\delta_3(k)]\xi_3 \end{pmatrix},$$

$$a_0 = \begin{pmatrix} 0 & 0 & -\delta_1(\delta_3(k^2)k^{-1}) \\ 0 & 0 & -\delta_2(\delta_3(k^2)k^{-1}) \\ 0 & 0 & k\delta_1^2(k) + k\delta_2^2(k) + k^{-1}\delta_3(k^2\delta_3(k^{-1})) \end{pmatrix}.$$

Proof Denote by $R_{1,k} : \mathcal{H}_{1,0} \to \mathcal{H}_{1,h}$ the operator defined as

$$R_{1,k}\,(b_1, b_2, b_3) = (b_1, b_2, b_3 k)\,.$$

We notice that $R_{1,k} : \mathcal{H}_{1,0} \to \mathcal{H}_{1,h}$ is an isometry from $\mathcal{H}_{1,0}$ to $\mathcal{H}_{1,h}$. Thus $\triangle_{1,h}$ is anti-unitary equivalent to $\tilde{\triangle}_{1,h} = (R_{1,k}J)^* \triangle_{1,h} R_{1,k} J$ which is given by the formula

$$\tilde{\triangle}_{1,h}(a_1, a_2, a_3) =$$

$$\Big(\delta_1(k^2\delta_1(a_1)) + \delta_2(k^2\delta_2(a_1)) + \delta_3^2(a_1) + \delta_1(k^2)\delta_2(a_2) - \delta_2(k^2)\delta_1(a_2) - \delta_1(\delta_3(k^2)k^{-1}a_3),$$
$$\delta_2(k^2)\delta_1(a_1) - \delta_1(k^2)\delta_2(a_1) + \delta_1(k^2\delta_1(a_2)) + \delta_2(k^2\delta_2(a_2)) + \delta_3^2(a_2) - \delta_2(\delta_3(k^2)k^{-1}a_3),$$
$$k^{-1}\delta_3(k^2)\delta_1(a_1) + k^{-1}\delta_3(k^2)\delta_2(a_2) + k\delta_1^2(ka_3) + k\delta_2^2(ka_3) + k^{-1}\delta_3(k^2\delta_3(k^{-1}a_3))\Big).$$

This proves the lemma.

Then computation can be carried out to compute $a_2(\tilde{\triangle}_{1,h})$, and the final result is given in the following proposition. In this proposition to make the formulas concise, we shall use the notation

$$F^{\nabla}(\rho) := F(\nabla)(\rho), \qquad F^{\nabla}(\rho_1 \cdot \rho_2) := F(\nabla_{(1)}, \nabla_{(2)})(\rho_1 \cdot \rho_2),$$

for a given function F with one or two variables.

Proposition 2.4.5 *The second density of the heat trace for the operator $\tilde{\triangle}_{1,h}$ is given by*

$$\begin{aligned}
&\pi^{\frac{3}{2}} a_2(\tilde{\triangle}_{1,h}) = \\
&\Big(K_{22}^{\nabla}(\delta_2^2(h)) + 2W_{22}^{\nabla}(\delta_2(h)^2) + k^{-2}K_3^{\nabla}(\delta_3^2(h)) + k^{-2}H_3^{\nabla}(\delta_3(h)^2)\Big)E_{11} \\
&+ \Big(K_{11}^{\nabla}(\delta_1^2(h)) + 2W_{11}^{\nabla}(\delta_1(h)^2) + k^{-2}K_3^{\nabla}(\delta_3^2(h)) + k^{-2}H_3^{\nabla}(\delta_3(h)^2)\Big)E_{22} \\
&+ \Big(K_1^{\nabla}(\delta_1^2(h) + \delta_2^2(h)) + H_1^{\nabla}(\delta_1(h)^2 + \delta_2(h)^2) + k^{-2}H_4^{\nabla}(\delta_3(h)^2)\Big)E_{33} \\
&+ \sum k^{-c(i,j)}\Big(K_{ij}^{\nabla}(\delta_i\delta_j(h)) + S_{ij}^{\nabla}([\delta_i(h), \delta_j(h)]) + W_{ij}^{\nabla}(\{\delta_i(h), \delta_j(h)\})\Big)E_{ij}.
\end{aligned}$$

Here $[\delta_i(h), \delta_j(h)]$ and $\{\delta_i(h), \delta_j(h)\}$ denote the commutator and anti-commutator. The functions are given as the entries of the following matrices.

$$\mathbf{K} = \frac{1}{4s(e^s-1)}\begin{pmatrix} \frac{e^{2s}-2se^s-1}{e^s-1} & 0 & (s-1)e^{\frac{s}{2}} + e^{-\frac{s}{2}} \\ 0 & \frac{e^{2s}-2se^s-1}{e^s-1} & (s-1)e^{\frac{s}{2}} + e^{-\frac{s}{2}} \\ e^s - s - 1 & e^s - s - 1 & \frac{1-e^{2s}+se^{2s}+s}{e^{\frac{s}{2}}(e^s-1)} \end{pmatrix},$$

$$\mathbf{S}(s,t) = \begin{pmatrix} 0 & 1 & \frac{1}{2}e^{-\frac{s+t}{2}} \\ 1 & 0 & \frac{1}{2}e^{-\frac{s+t}{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} S_1(s,t),$$

where

$$S_1(s,t) = \frac{1}{2st} - \frac{(e^s-1)^2 e^t t + e^s s(e^t-1)^2}{2st(e^s-1)(e^t-1)(e^{s+t}-1)}.$$

Also,

$$\mathbf{W}(s,t) = \begin{pmatrix} \frac{1}{2}\cosh(\frac{s+t}{2}) & 0 & \frac{e^{-s-t}-1}{4} \\ 0 & \frac{1}{2}\cosh(\frac{s+t}{2}) & \frac{e^{-s-t}-1}{4} \\ \frac{1}{2}\sinh(\frac{s+t}{2}) & \frac{1}{2}\sinh(\frac{s+t}{2}) & \frac{W_{33}(s,t)}{H_1(s,t)} \end{pmatrix} H_1(s,t).$$

Here, H_1 is the function from Theorem 2.4.3. The function W_{33}, together with the remaining functions, are given below:

$$\begin{aligned} W_{33}(s,t) =& \frac{1}{16e^{\frac{s+t}{2}}(e^s-1)(e^t-1)(e^{s+t}-1)^2 st(s+t)} \times \\ &\Big((e^t-1)^2(1-4e^s-e^{2s}-e^{s+t}-4e^{2s+t}+e^{3s+t})s^2 \\ &+ 2(e^s+1)(e^t+1)(e^{s+t}-1)(e^s-e^t)st \\ &- (e^s-1)^2(1-4e^t-e^{2t}-e^{s+t}-4e^{2t+s}+e^{3t+s})t^2 \\ &- 4(e^s-1)(e^t-1)(e^{2(s+t)}-1)(s-t)\Big), \end{aligned}$$

$$\begin{aligned} H_3(s,t) =& \frac{1}{4e^s(e^s-1)(e^t-1)(e^{s+t}-1)^2 st(s+t)} \times \\ &\Big(e^s(e^t-1)^2(-1-3e^s+e^{s+t}-e^{2s+t})s^2 \\ &+ (e^s-1)^2(1-e^t+3e^{s+t}+e^{s+2t})t^2 \\ &- 4e^s(e^s-1)(e^t-1)(e^{s+t}-1)(s-t) \\ &+ (7e^{s+t}-7e^{2(s+t)}-e^{3(s+t)}+2e^{3s+t} \\ &+ 3e^{3s+2t}+e^{2s+3t}-3e^s-2e^{2s}-e^t+1)st\Big), \end{aligned}$$

$$K_3(s) = \frac{2 - 2e^s + se^s + s}{4s(e^s - 1)^2}, \qquad H_4(s,t) = \frac{(e^s - 1)(e^t - 1)(s + t)}{8e^{\frac{s+t}{2}}(e^{s+t} - 1)st}.$$

The power of k in the sum denoted by $c(i, j)$, counts how many of indices i, j are equal to 3. ∎

Unlike the phenomena observed for the scalar curvature in Remark 2.4.1, the functions of the heat trace densities of the Laplacian on 1-forms are not related, at least in the same way as before, to those of the Laplacian on 1-forms of the conformally flat metric. This is a consequence of a simple fact that the Laplacian on 1-forms of the product Riemannian manifolds is not the sum of the Laplacians on 1-forms of the components. In fact, if (M_1, g_1) and (M_2, g_2) are two oriented Riemannian manifolds, then the Laplacian on 1-forms on the product manifold $(M_1 \times M_2,\ g_1 \times g_2)$ is given by

$$\triangle_1 \otimes 1 + 1 \otimes \triangle_1 + \triangle_0 \otimes 1 + 1 \otimes \triangle_0 + 2d_0 \otimes d_0^* + 2d_0^* \otimes d_0,$$

where $\triangle_0$ and $\triangle_1$ are the Laplacians on functions and 1-forms for the corresponding manifolds.

Using the above proposition and Theorem 2.4.3, we obtain the Ricci density in the following theorem.

Theorem 2.4.6 *The Ricci density* **Ric** *of* $\mathbb{T}_\theta^3$ *equipped with the metric* (2.4.1) *is given by*

$$\begin{aligned}
\pi^{\frac{3}{2}}\mathbf{Ric} = &-\left(\tilde{K}_{22}^{\nabla}(\delta_2^2(h)) + 2\tilde{W}_{22}^{\nabla}(\delta_2(h)^2) + k^{-2}\tilde{K}_3^{\nabla}(\delta_3^2(h)) + k^{-2}\tilde{H}_3^{\nabla}(\delta_3(h)^2)\right)E_{11}\\
&-\left(\tilde{K}_{11}^{\nabla}(\delta_1^2(h)) + 2\tilde{W}_{11}^{\nabla}(\delta_1(h)^2) + k^{-2}\tilde{K}_3^{\nabla}(\delta_3^2(h)) + k^{-2}\tilde{H}_3^{\nabla}(\delta_3(h)^2)\right)E_{22}\\
&-k^{-2}\tilde{H}_4^{\nabla}(\delta_3(h)^2)E_{33}\\
&-\sum k^{-c(i,j)}\left(\tilde{K}_{ij}^{\nabla}(\delta_i\delta_j(h)) + S_{ij}^{\nabla}([\delta_i(h), \delta_j(h)]) + \tilde{W}_{ij}^{\nabla}(\{\delta_i(h), \delta_j(h)\})\right)E_{ij}.
\end{aligned}$$

where $\tilde{K}_{ij}$ (resp. $\tilde{W}_{ij}$ and $\tilde{H}_{ij}$) is different from K_{ij} (resp. $\tilde{W}_{ij}$ and H_{ij}) only in their diagonal entries. The new functions are given by

$$\tilde{K}_{22} = \tilde{K}_{11} = K_{11} - K_1 = \frac{-1 + e^s + se^{s/2}}{4s(1 + e^{s/2})^2}, \qquad \tilde{K}_{33} = K_{33} - K_2 = \frac{1}{4e^{s/2}},$$

$$\tilde{K}_3 = K_3 - K_2 = \frac{-1 + e^s + se^{s/2}}{4se^{s/2}(1 + e^{s/2})^2}, \qquad \tilde{W}_{11} = \tilde{W}_{22} = W_{11} - \frac{1}{2}H_1,$$

and $\tilde{H}_3 = H_3 - H_2$, $\tilde{H}_4 = H_4 - H_2$, and $\tilde{W}_{33} = W_{33}$. ∎

The classical limit of the Ricci density can be obtained by letting $s, t \to 0$. First note that in the commutative case the terms involving functions S_{ij} disappear because they act

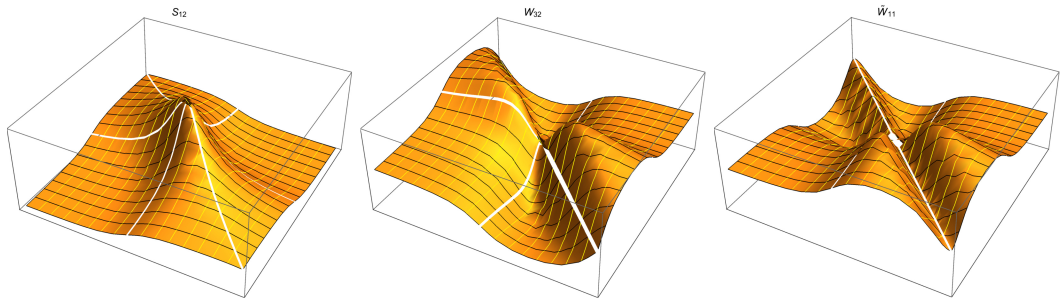


Figure 2.1: The graph of functions S_{12}, W_{32} and $\tilde{W}_{11}$.

on the commutator $[\delta_i(h), \delta_j(h)]$ which is zero. On the other hand, functions W_{ij} are anti-symmetric in their variables; $W_{ij}(s,t) = -W_{ij}(t,s)$. Hence, the terms involving them will vanish too. Moreover, since $\lim_{(s,t)\to(0,0)} H_1(s,t) = 0$, we have $\lim_{(s,t)\to(0,0)} \tilde{W}_{ij}(s,t) = 0$. The limit of the other terms are given by

$$\lim_{s\to 0} \tilde{\mathbf{K}}(s) = \begin{pmatrix} \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \end{pmatrix}, \qquad \lim_{s\to 0} \tilde{K}_3(s) = \frac{1}{8},$$

and also

$$\lim_{(s,t)\to(0,0)} \tilde{H}_3(s,t) = -\frac{1}{4}, \qquad \lim_{(s,t)\to(0,0)} \tilde{H}_4(s,t) = -\frac{1}{4}.$$

Thus when $\theta \to 0$, the Ricci density **Ric** approaches to

$$\mathbf{Ric}_0 = \frac{1}{8\pi^{\frac{3}{2}}} \times$$
$$\begin{pmatrix} e^{-2h}(2\delta_3(h)^2 - \delta_3^2(h)) - \sum_{i=1}^{2} \delta_i^2(h) & 0 & -e^{-h}\delta_1\delta_3(h) \\ 0 & e^{-2h}(2\delta_3(h)^2 - \delta_3^2(h)) - \sum_{i=1}^{2} \delta_i^2(h) & -e^{-h}\delta_2\delta_3(h) \\ -e^{-h}\delta_1\delta_3(h) & -e^{-h}\delta_2\delta_3(h) & 2e^{-2h}(\delta_3(h)^2 - \delta_3^2(h)) \end{pmatrix}$$

while the Ricci density in the classical case is given by

$$\mathbf{Ric}_{com} =$$
$$\begin{pmatrix} e^{2h}(h_{11} + h_{22}) + h_{33} - 2(h_3)^2 & 0 & h_{13} \\ 0 & e^{2h}(h_{11} + h_{22}) + h_{33} - 2(h_3)^2 & h_{23} \\ e^{2h}h_{13} & e^{2h}h_{23} & 2h_{33} - 2(h_3)^2 \end{pmatrix}.$$

The apparent discrepancy between the limit case $\mathbf{Ric}_0$ and the commutative formula $\mathbf{Ric}_{com}$ is due to our convention for the Ricci functional, and as mentioned in Remark 2.2.4 we have the relation

$$(R_{1,k}J)\mathbf{Ric}_0(R_{1,k}J)^* = R_{1,k}J\mathbf{Ric}_0 JR_{1,k^{-1}} = \frac{1}{8\pi^{\frac{3}{2}}}\mathbf{Ric}_{com}e^{-2h}.$$

Appendix A

Computations

In this section we give some details of the computation of the scalar curvature for the non-conformal metric. The full details can be found in the Mathematica file accompanying this paper.

The computation starts from the formula for b_2 given by (2.2.7). We first plug in formula b_1 and write $b_2(\xi,\lambda)$ in terms of b_1 and the homogeneous parts of the symbol a_2, a_1 and a_0:

$$
\begin{aligned}
b_2(\xi,\lambda) = &- b_0a_0b_0 - b_1a_1b_0 - \partial_1(b_0)\delta_1(a_1)b_0 - \partial_2(b_0)\delta_2(a_1)b_0 - \partial_3b_0\delta_3(a_1)b_0 \\
&- \partial_1(b_1)\delta_1(a_2)b_0 - \partial_2(b_1)\delta_2(a_2)b_0 - \partial_3(b_1)\delta_3(a_2)b_0 \\
&- \frac{1}{2}\partial_1^2(b_0)\delta^2(a_2)b_0 - \frac{1}{2}\partial_2^2(b_0)\delta_2^2(a_2)b_0 - \frac{1}{2}\partial_3^2(b_0)\delta_3^2(a_2)b_0 \\
&- \partial_2\partial_3(b_0)\delta_3\delta_2(a_2)b_0 - \partial_1\partial_2(b_0)\delta_2\delta_1(a_2)b_0 - \partial_1\partial_3(b_0)\delta_1\delta_3(a_2)b_0.
\end{aligned}
$$

The next step is to plug a_j's from Lemma 2.3.1 into the above formula. Note that the derivatives of b_0 can be written as

$$
\partial_1(b_0) = -2\xi_1k^2b_0^2, \quad \partial_2(b_0) = -2\xi_2k^2b_0^2, \quad \partial_3(b_0) = -2\xi_3b_0^2.
$$

The complete outcome is long and involves 465 terms. Here we only display the result for a sample term $\partial_3(b_0)\delta_3(a_2)b_0$ below.

$$
\begin{aligned}
&\partial_3(b_0)\delta_3(a_2)b_0 \\
&= -4\xi_1^5\xi_3k^2b_0^2\delta_1(k^2)b_0^2\delta_3(k^2)b_0 - 8\xi_1^5\xi_3k^2b_0^3\delta_1(k^2)b_0\delta_3(k^2)b_0 \\
&\quad -4\xi_1^4\xi_3^2b_0^2\delta_3(k^2)b_0^2\delta_3(k^2)b_0 - 8\xi_1^4\xi_3^2b_0^3\delta_3(k^2)b_0\delta_3(k^2)b_0 \\
&\quad +2\xi_1^4b_0^2\delta_3(k^2)b_0\delta_3(k^2)b_0 - 4\xi_1^4\xi_2\xi_3k^2b_0^2\delta_2(k^2)b_0^2\delta_3(k^2)b_0
\end{aligned}
$$

$$-8\xi_1^4\xi_2\xi_3k^2b_0^3\delta_2(k^2)b_0\delta_3(k^2)b_0 - 8\xi_1^3\xi_2^2\xi_3k^2b_0^2\delta_1(k^2)b_0^2\delta_3(k^2)b_0$$
$$+4\xi_1^3\xi_3b_0k\delta_1(k)b_0^2\delta_3(k^2)b_0 - 16\xi_1^3\xi_2^2\xi_3k^2b_0^3\delta_1(k^2)b_0\delta_3(k^2)b_0$$
$$+4\xi_1^3\xi_3b_0^2k\delta_1(k)b_0\delta_3(k^2)b_0 + 4\xi_1^2\xi_2^2b_0^2\delta_3(k^2)b_0\delta_3(k^2)b_0$$
$$-8\xi_1^2\xi_2^2\xi_3^2b_0^2\delta_3(k^2)b_0^2\delta_3(k^2)b_0 - 16\xi_1^2\xi_2^2\xi_3^2b_0^3\delta_3(k^2)b_0\delta_3(k^2)b_0$$
$$+2\xi_1^2\xi_3^2b_0k^{-1}\delta_3(k)b_0^2\delta_3(k^2)b_0 - 2\xi_1^2\xi_3^2b_0\delta_3(k)k^{-1}b_0^2\delta_3(k^2)b_0$$
$$+2\xi_1^2\xi_3^2b_0^2k^{-1}\delta_3(k)b_0\delta_3(k^2)b_0 - 2\xi_1^2\xi_3^2b_0^2\delta_3(k)k^{-1}b_0\delta_3(k^2)b_0$$
$$-\xi_1^2b_0k^{-1}\delta_3(k)b_0\delta_3(k^2)b_0 + \xi_1^2b_0\delta_3(k)k^{-1}b_0\delta_3(k^2)b_0$$
$$-8\xi_1^2\xi_2^3\xi_3k^2b_0^2\delta_2(k^2)b_0^2\delta_3(k^2)b_0 - 16\xi_1^2\xi_2^3\xi_3k^2b_0^3\delta_2(k^2)b_0\delta_3(k^2)b_0$$
$$+4\xi_1^2\xi_2\xi_3b_0k\delta_2(k)b_0^2\delta_3(k^2)b_0 + 4\xi_1^2\xi_2\xi_3b_0^2k\delta_2(k)b_0\delta_3(k^2)b_0$$
$$-4\xi_1\xi_2^4\xi_3k^2b_0^2\delta_1(k^2)b_0^2\delta_3(k^2)b_0 - 8\xi_1\xi_2^4\xi_3k^2b_0^3\delta_1(k^2)b_0\delta_3(k^2)b_0$$
$$+4\xi_1\xi_2^2\xi_3b_0k\delta_1(k)b_0^2\delta_3(k^2)b_0 + 4\xi_1\xi_2^2\xi_3b_0^2k\delta_1(k)b_0\delta_3(k^2)b_0$$
$$+2\xi_2^4b_0^2\delta_3(k^2)b_0\delta_3(k^2)b_0 - \xi_2^2b_0k^{-1}\delta_3(k)b_0\delta_3(k^2)b_0$$
$$+\xi_2^2b_0\delta_3(k)k^{-1}b_0\delta_3(k^2)b_0 - 4\xi_2^4\xi_3^2b_0^2\delta_3(k^2)b_0^2\delta_3(k^2)b_0$$
$$-8\xi_2^4\xi_3^2b_0^3\delta_3(k^2)b_0\delta_3(k^2)b_0 + 2\xi_2^2\xi_3^2b_0k^{-1}\delta_3(k)b_0^2\delta_3(k^2)b_0$$
$$-2\xi_2^2\xi_3^2b_0\delta_3(k)k^{-1}b_0^2\delta_3(k^2)b_0 + 2\xi_2^2\xi_3^2b_0^2k^{-1}\delta_3(k)b_0\delta_3(k^2)b_0$$
$$-2\xi_2^2\xi_3^2b_0^2\delta_3(k)k^{-1}b_0\delta_3(k^2)b_0 - 4\xi_2^5\xi_3k^2b_0^2\delta_2(k^2)b_0^2\delta_3(k^2)b_0$$
$$-8\xi_2^5\xi_3k^2b_0^3\delta_2(k^2)b_0\delta_3(k^2)b_0 + 4\xi_2^3\xi_3b_0k\delta_2(k)b_0^2\delta_3(k^2)b_0$$
$$+4\xi_2^3\xi_3b_0^2k\delta_2(k)b_0\delta_3(k^2)b_0.$$

Then we apply the substitution given in (2.4.2) and integrate with respect to η and θ. The result then is

$$\frac{1}{\pi^2}\int_{-\infty}^{+\infty}\int_0^{2\pi} b_2(u,\eta,\theta,-1)\frac{1+\eta^2}{2}d\theta d\eta =$$
$$2u^3k^2b_0^2\delta_1(k)k^3b_0^2k\delta_1(k)b_0 + 2u^3k^2b_0^2\delta_1(k)k^3b_0^2\delta_1(k)kb_0 + 2u^3k^2b_0^2\delta_2(k)k^3b_0^2k\delta_2(k)b_0$$
$$+2u^3k^2b_0^2\delta_2(k)k^3b_0^2\delta_2(k)kb_0 + 4u^3k^4b_0^3k\delta_1(k)b_0k\delta_1(k)b_0 + 4u^3k^4b_0^3k\delta_1(k)b_0\delta_1(k)kb_0$$
$$+4u^3k^4b_0^3k\delta_2(k)b_0k\delta_2(k)b_0 + 4u^3k^4b_0^3k\delta_2(k)b_0\delta_2(k)kb_0 + 4u^3k^4b_0^3\delta_1(k)kb_0k\delta_1(k)b_0$$
$$+4u^3k^4b_0^3\delta_1(k)kb_0\delta_1(k)kb_0 + 4u^3k^4b_0^3\delta_2(k)kb_0k\delta_2(k)b_0 + 4u^3k^4b_0^3\delta_2(k)kb_0\delta_2(k)kb_0$$
$$+2u^3k^2b_0^2k\delta_1(k)k^2b_0^2k\delta_1(k)b_0 + 2u^3k^2b_0^2k\delta_1(k)k^2b_0^2\delta_1(k)kb_0 - 2u^2k^4b_0^3k\delta_1\left(\delta_1(k)\right)b_0$$
$$+2u^3k^2b_0^2k\delta_2(k)k^2b_0^2\delta_2(k)kb_0 + 2u^3k^2b_0^2k\delta_2(k)k^2b_0^2k\delta_2(k)b_0 - 2u^2k^4b_0^3k\delta_2\left(\delta_2(k)\right)b_0$$
$$-4u^2k^4b_0^3\delta_1(k)\delta_1(k)b_0 - 2u^2k^4b_0^3\delta_1\left(\delta_1(k)\right)kb_0 - 4u^2k^4b_0^3\delta_2(k)\delta_2(k)b_0$$
$$-2u^2k^4b_0^3\delta_2\left(\delta_2(k)\right)kb_0 - 2u^2b_0^2k\delta_3(k)b_0k\delta_3(k)b_0 - 2u^2b_0^2k\delta_3(k)b_0\delta_3(k)kb_0$$

$$
\begin{gathered}
+2u^2b_0^2k\delta_3(k)b_0^2k\delta_3(k)b_0+2u^2b_0^2k\delta_3(k)b_0^2\delta_3(k)kb_0-2u^2b_0^2\delta_3(k)kb_0k\delta_3(k)b_0\\
-2u^2b_0^2\delta_3(k)kb_0\delta_3(k)kb_0+2u^2b_0^2\delta_3(k)kb_0^2k\delta_3(k)b_0+2u^2b_0^2\delta_3(k)kb_0^2\delta_3(k)kb_0\\
+4u^2b_0^3k\delta_3(k)b_0k\delta_3(k)b_0+4u^2b_0^3k\delta_3(k)b_0\delta_3(k)kb_0+4u^2b_0^3\delta_3(k)kb_0k\delta_3(k)b_0\\
+4u^2b_0^3\delta_3(k)kb_0\delta_3(k)kb_0-8u^2k^2b_0^2k\delta_1(k)b_0k\delta_1(k)b_0-6u^2k^2b_0^2k\delta_1(k)b_0\delta_1(k)kb_0\\
-8u^2k^2b_0^2k\delta_2(k)b_0k\delta_2(k)b_0-6u^2k^2b_0^2k\delta_2(k)b_0\delta_2(k)kb_0-6u^2k^2b_0^2\delta_1(k)kb_0k\delta_1(k)b_0\\
-4u^2k^2b_0^2\delta_1(k)kb_0\delta_1(k)kb_0-6u^2k^2b_0^2\delta_2(k)kb_0k\delta_2(k)b_0-4u^2k^2b_0^2\delta_2(k)kb_0\delta_2(k)kb_0\\
-2u^2b_0k\delta_1(k)k^2b_0^2k\delta_1(k)b_0-2u^2b_0k\delta_1(k)k^2b_0^2\delta_1(k)kb_0-2u^2b_0k\delta_2(k)k^2b_0^2k\delta_2(k)b_0\\
-2u^2b_0k\delta_2(k)k^2b_0^2\delta_2(k)kb_0+ub_0^2k\delta_3\left(\delta_3(k)\right)b_0+2ub_0^2\delta_3(k)\delta_3(k)b_0\\
+ub_0^2\delta_3\left(\delta_3(k)\right)kb_0-2ub_0^3k\delta_3\left(\delta_3(k)\right)b_0-4ub_0^3\delta_3(k)\delta_3(k)b_0\\
-2ub_0^3\delta_3\left(\delta_3(k)\right)kb_0+3uk^2b_0^2k\delta_1\left(\delta_1(k)\right)b_0+3uk^2b_0^2k\delta_2\left(\delta_2(k)\right)b_0\\
+4uk^2b_0^2\delta_1(k)\delta_1(k)b_0+uk^2b_0^2\delta_1\left(\delta_1(k)\right)kb_0+4uk^2b_0^2\delta_2(k)\delta_2(k)b_0\\
+uk^2b_0^2\delta_2\left(\delta_2(k)\right)kb_0+ub_0k^{-1}\delta_3(k)b_0k\delta_3(k)b_0+ub_0k^{-1}\delta_3(k)b_0\delta_3(k)kb_0\\
-ub_0k^{-1}\delta_3(k)b_0^2k\delta_3(k)b_0-ub_0k^{-1}\delta_3(k)b_0^2\delta_3(k)kb_0+4ub_0k\delta_1(k)b_0k\delta_1(k)b_0\\
+2ub_0k\delta_1(k)b_0\delta_1(k)kb_0+4ub_0k\delta_2(k)b_0k\delta_2(k)b_0+2ub_0k\delta_2(k)b_0\delta_2(k)kb_0\\
-ub_0\delta_3(k)k^{-1}b_0k\delta_3(k)b_0-ub_0\delta_3(k)k^{-1}b_0\delta_3(k)kb_0+ub_0\delta_3(k)k^{-1}b_0^2k\delta_3(k)b_0\\
+ub_0\delta_3(k)k^{-1}b_0^2\delta_3(k)kb_0-ub_0^2k^{-1}\delta_3(k)b_0k\delta_3(k)b_0-ub_0^2k^{-1}\delta_3(k)b_0\delta_3(k)kb_0\\
-ub_0^2k\delta_3(k)b_0k^{-1}\delta_3(k)b_0+ub_0^2k\delta_3(k)b_0\delta_3(k)k^{-1}b_0+ub_0^2\delta_3(k)k^{-1}b_0k\delta_3(k)b_0\\
+ub_0^2\delta_3(k)k^{-1}b_0\delta_3(k)kb_0-ub_0^2\delta_3(k)kb_0k^{-1}\delta_3(k)b_0+ub_0^2\delta_3(k)kb_0\delta_3(k)k^{-1}b_0\\
-b_0k^{-1}\delta_3\left(\delta_3(k)\right)b_0-b_0k\delta_1\left(\delta_1(k)\right)b_0-b_0k\delta_2\left(\delta_2(k)\right)b_0+b_0^2k^{-1}\delta_3\left(\delta_3(k)\right)b_0\\
-b_0^2\delta_3\left(\delta_3(k)\right)k^{-1}b_0+b_0\delta_3(k)k^{-2}\delta_3(k)b_0+b_0k^{-1}\delta_3(k)k^{-1}\delta_3(k)b_0\\
-b_0^2k^{-1}\delta_3(k)k^{-1}\delta_3(k)b_0+b_0^2\delta_3(k)k^{-1}\delta_3(k)k^{-1}b_0+\frac{1}{2}b_0k^{-1}\delta_3(k)b_0k^{-1}\delta_3(k)b_0\\
-\frac{1}{2}b_0k^{-1}\delta_3(k)b_0\delta_3(k)k^{-1}b_0-\frac{1}{2}b_0\delta_3(k)k^{-1}b_0k^{-1}\delta_3(k)b_0\\
+\frac{1}{2}b_0\delta_3(k)k^{-1}b_0\delta_3(k)k^{-1}b_0.
\end{gathered}
$$

To perform u integration, we apply Corollary 2.4.1 where the functions $F^{[\nu]}_{m_0,\cdots,m_p}$ show up in the result. The ρ terms appearing in the outcome expression include $\delta_j(k)$ and $\delta_j^2(k)$ multiplied by a power of k. We use the following identities to bring all these ρ's into the form $k^{-1}\delta_j$ or $k^{-1}\delta_j^2(k)$.

$$
F(\Delta)(\rho_1\rho_2)=F(\Delta_{(1)}\Delta_{(2)})(\rho_1\cdot\rho_2),\qquad F(\Delta)(k^m\rho k^n)=k^{m+n}\Delta^{\frac{n}{2}}F(\Delta)(\rho),
$$

$$F(\Delta_{(1)}, \Delta_{(2)})(k^l \rho_1 \cdot k^m \rho_2 k^n) = k^{l+m+n} \Delta_{(1)}^{\frac{m+n}{2}} \Delta_{(2)}^{\frac{n}{2}} F(\Delta_{(1)} \Delta_{(2)})(\rho_1 \cdot \rho_2).$$

These identities are consequences of the fact that Δ is a C^*-algebra automorphism which commutes with k and also $xk = k\Delta^{\frac{1}{2}}(x)$. Applying the aforementioned identities, the integral of b_2, up to the total factor π^2, is equal to

$$\begin{aligned}
&\Big((3+\Delta^{\frac{1}{2}})F^{[2]}_{2,1}(\Delta)\Big(k^{-1}\delta_1^2(k)\Big) - F^{[2]}_{1,1}(\Delta)\Big(k^{-1}\delta_1^2(k)\Big) - 2(1+\Delta^{\frac{1}{2}})F^{[2]}_{3,1}(\Delta)\Big(k^{-1}\delta_1^2(k)\Big)\Big)\\
&+\Big(2\Delta_{(1)}(\Delta_{(2)}^{\frac{1}{2}}+2)F^{[2]}_{1,1,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)\Big)\\
&-2\Delta_{(1)}^2(\Delta_{(2)}^{\frac{1}{2}}+1)F^{[2]}_{1,2,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)\Big)\\
&-2\Delta_{(1)}(3\Delta_{(2)}^{\frac{1}{2}}+2\Delta_{(1)}^{\frac{1}{2}}\Delta_{(2)}^{\frac{1}{2}}+3\Delta_{(1)}^{\frac{1}{2}}+4)F^{[2]}_{2,1,1}\Big(\Delta_{(1)},\Delta_{(2)})(k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)\Big)\\
&+2\Delta_{(1)}^2(1+\Delta_{(1)}^{\frac{1}{2}})(1+\Delta_{(2)}^{\frac{1}{2}})F^{[2]}_{2,2,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)\Big)\\
&+4\Delta_{(1)}(1+\Delta_{(1)}^{\frac{1}{2}})(1+\Delta_{(2)}^{\frac{1}{2}})F^{[2]}_{3,1,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)\Big)\\
&+4\Delta_{(1)}^{\frac{1}{2}}F^{[2]}_{2,0,1}\Big(\Delta_{(1)},\Delta_{(2)})(k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)\Big)\\
&-4\Delta_{(1)}^{\frac{1}{2}}F^{[2]}_{3,0,1}\Big(\Delta_{(1)},\Delta_{(2)})(k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)\Big)\Big)\\
&+\Big((3+\Delta^{\frac{1}{2}})F^{[2]}_{2,1}(\Delta)\Big(k^{-1}\delta_2^2(k)\Big) - F^{[2]}_{1,1}(\Delta)\Big(k^{-1}\delta_2^2(k)\Big)\\
&-2(1+\Delta^{\frac{1}{2}})F^{[2]}_{3,1}(\Delta)\Big(k^{-1}\delta_2^2(k)\Big)\Big)\\
&+\Big(2\Delta_{(1)}(\Delta_{(2)}^{\frac{1}{2}}+2)F^{[2]}_{1,1,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_2(k)\cdot k^{-1}\delta_2(k)\Big)\\
&-2\Delta_{(1)}^2(\Delta_{(2)}^{\frac{1}{2}}+1)F^{[2]}_{1,2,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_2(k)\cdot k^{-1}\delta_2(k)\Big)\\
&-2\Delta_{(1)}(3\Delta_{(2)}^{\frac{1}{2}}+2\Delta_{(1)}^{\frac{1}{2}}\Delta_{(2)}^{\frac{1}{2}}+3\Delta_{(1)}^{\frac{1}{2}}+4)F^{[2]}_{2,1,1}\Big(\Delta_{(1)},\Delta_{(2)})(k^{-1}\delta_2(k)\cdot k^{-1}\delta_2(k)\Big)\\
&+2\Delta_{(1)}^2(1+\Delta_{(1)}^{\frac{1}{2}})(1+\Delta_{(2)}^{\frac{1}{2}})F^{[2]}_{2,2,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_2(k)\cdot k^{-1}\delta_2(k)\Big)\\
&+4\Delta_{(1)}(1+\Delta_{(1)}^{\frac{1}{2}})(1+\Delta_{(2)}^{\frac{1}{2}})F^{[2]}_{3,1,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_2(k)\cdot k^{-1}\delta_2(k)\Big)\\
&+4\Delta_{(1)}^{\frac{1}{2}}F^{[2]}_{2,0,1}\Big(\Delta_{(1)},\Delta_{(2)})(k^{-1}\delta_2(k)\cdot k^{-1}\delta_2(k)\Big)\\
&-4\Delta_{(1)}^{\frac{1}{2}}F^{[2]}_{3,0,1}\Big(\Delta_{(1)},\Delta_{(2)})(k^{-1}\delta_2(k)\cdot k^{-1}\delta_2(k)\Big)\Big)\\
&+k^{-2}\Big(-F^{[2]}_{1,1}(\Delta)\Big(k^{-1}\delta_3^2(k)\Big)+(1+\Delta^{\frac{1}{2}})F^{[2]}_{2,1}(\Delta)\Big(k^{-1}\delta_3^2(k)\Big)\\
&\qquad -2(1+\Delta^{\frac{1}{2}})F^{[3]}_{3,1}(\Delta)\Big(k^{-1}\delta_3^2(k)\Big)+(1-\Delta^{-\frac{1}{2}})F^{[3]}_{2,1}(\Delta)\Big(k^{-1}\delta_3^2(k)\Big)\Big)\\
&+k^{-2}\Big((\Delta_{(1)}-\Delta_{(1)}^{\frac{1}{2}})(\Delta_{(2)}^{\frac{1}{2}}+1)F^{[2]}_{1,1,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\Big)\\
&\qquad -2\Delta_{(1)}(\Delta_{(1)}^{\frac{1}{2}}+1)(\Delta_{(2)}^{\frac{1}{2}}+1)F^{[2]}_{2,1,1}(\Delta_{(1)},\Delta_{(2)})\Big(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\Delta_{(1)}^{\frac{1}{2}} F^{[2]}_{2,0,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\\
&+ (1+\Delta_1^{-\frac{1}{2}}) F^{[2]}_{1,0,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\\
&+ (\Delta_{(1)}^{\frac{1}{2}} - \Delta_{(1)})(\Delta_{(2)}^{\frac{1}{2}}+1) F^{[3]}_{1,2,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\\
&- (\Delta_{(1)}^{\frac{1}{2}} - \Delta_{(2)}^{-\frac{1}{2}})(1+\Delta_{(1)}^{\frac{1}{2}} - \Delta_{(2)}^{\frac{1}{2}} + \Delta_{(1)}^{\frac{1}{2}}\Delta_{(2)}^{\frac{1}{2}})\\
&\qquad\qquad\qquad\qquad F^{[3]}_{2,1,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\\
&+ 2\Delta_{(1)}(\Delta_{(1)}^{\frac{1}{2}}+1)(\Delta_{(2)}^{\frac{1}{2}}+1) F^{[3]}_{2,2,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\\
&+ 4\Delta_{(1)}(\Delta_{(1)}^{\frac{1}{2}}+1)(\Delta_{(2)}^{\frac{1}{2}}+1) F^{[3]}_{3,1,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\\
&+ \frac{1}{2}(\Delta_{(1)}^{-\frac{1}{2}}-1)(\Delta_2^{-\frac{1}{2}}-1) F^{[3]}_{1,1,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\\
&- 4\Delta_{(1)}^{\frac{1}{2}} F^{[3]}_{3,0,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\\
&- (1-\Delta_1^{-\frac{1}{2}}\Delta_2^{-\frac{1}{2}}) F^{[3]}_{2,0,1}(\Delta_{(1)}, \Delta_{(2)}) \left(k^{-1}\delta_3(k)\cdot k^{-1}\delta_3(k)\right)\Big).
\end{aligned}
$$

In the above formula, we grouped the terms with the same sequence of ρ_j's together. The terms which has $k^{-1}\delta_1^2(k)$ have exactly the exactly the same functions as the term $k^{-1}\delta_2^2(k)$, and it reads

$$
\pi^2\Big((3+\Delta^{\frac{1}{2}})F^{[2]}_{2,1}(\Delta) - F^{[2]}_{1,1}(\Delta) - 2(1+\Delta^{\frac{1}{2}})F^{[2]}_{3,1}(\Delta)\Big).
$$

If we substitute the functions $F^{[\nu]}_{m_0,m_1}$ in the above expression, we get:

$$
\psi_1(s_1) := -\frac{\pi^2\sqrt{s_1}(s_1\log(s_1)+\log(s_1)-2s_1+2)}{(\sqrt{s_1}-1)^3(\sqrt{s_1}+1)^2}.
$$

The function for $k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)$ is the same as the function for $k^{-1}\delta_1(k)\cdot k^{-1}\delta_1(k)$ and it is given by

$$
\begin{aligned}
&\phi_1(s_1,s_2) = \frac{2\pi^2\sqrt{s_1}\sqrt{s_2}}{(\sqrt{s_1}-1)(s_1-1)(\sqrt{s_2}-1)(s_2-1)(\sqrt{s_1 s_2}-1)(s_1 s_2-1)^2}\times\\
&\Big(s_1^{3/2}(s_2^{5/2} - s_2^{1/2} + 2s_2^{3/2}\log(s_2) + s_2^2(\log(s_1 s_2)-2) - 2s_2\log(s_1 s_2) + \log(s_1 s_2) + 2)\\
&\quad + s_2 s_1^{5/2}(s_2^{3/2}(\log(s_1)-1) - (s_2-1)(\log(s_1 s_2)-2) - \sqrt{s_2}(\log(s_1 s_2)-1))\\
&\quad + s_2 s_1^2(s_2(\log(s_2)-1)+1) - s_1(s_2^2(\log(s_1 s_2)-1) - 2s_2\log(s_1) + \log(s_1 s_2)+1)\\
&\quad - \sqrt{s_1}(s_2^{3/2}(\log(s_1 s_2)+1) - s_2(\log(s_1 s_2)+2) + \log(s_1 s_2) - \sqrt{s_2}(\log(s_1)+1)+2)\\
&\quad + 1 - s_2 + \log(s_2)\Big).
\end{aligned}
$$

Also, the functions for $k^{-1}\delta_3(k)k^{-1}\delta_3(k)$ and $k^{-1}\delta_3^2(k)$ are given by

$$\psi_2(s_1) = \frac{2\pi^2\left(-s_1^2 + 2s_1\log(s_1) + 1\right)}{\sqrt{s_1}\,(s_1-1)^2\log(s_1)},$$

and

$$\begin{aligned}\phi_2(s_1,s_2) = \;& \frac{2\pi^2}{\sqrt{s_1s_2}(s_1-1)(s_2-1)(s_1s_2-1)^2\log(s_1)\log(s_2)\log(s_1s_2)}\times\\ &\Big((s_1^2s_2^2 - 5s_1s_2^2 + s_1^2s_2 + 2s_2^2 + 4s_1s_2 - 5s_2 + s_1 + 1)(s_1s_2-1)\log(s_1)\log(s_2)\\ &+ (s_1-1)^2(s_2^3s_1 + s_1s_2^2 - s_2^2 + 3s_2)\log(s_2)^2 - (s_2-1)^2(3s_1^2s_2 - s_2s_1 + s_1 + 1)\log(s_1)^2\\ &+ 2(s_2-1)(s_1-1)(s_1^2s_2^2-1)\log(\frac{s_2}{s_1})\Big).\end{aligned}$$

Finally, we would like to express the result in term of $\log k$ and $\nabla := \log\Delta = [-2h,\cdot]$. To do so we first need to use the formula (2.4.3), then replace Δ with e^{∇}. For example, the term involving $\delta_1^2(\log(k))$ comes from $\psi_1(\Delta)(k^{-1}\delta_1^2(k))$ and it is given by

$$\begin{aligned}\psi_1(\Delta)f(\Delta) &= -\frac{\pi^2\sqrt{\Delta}(s_1\log(\Delta) + \log(\Delta) - 2\Delta + 2)}{(\sqrt{\Delta}-1)^3(\sqrt{\Delta}+1)^2}\frac{2(\sqrt{\Delta}-1)}{\log(\Delta)}\\ &= 2\pi^2\frac{e^{\frac{\nabla}{2}}(2e^{\nabla} - \nabla e^{\nabla} - 2 - \nabla)}{\nabla(e^{\nabla}-1)^2}\nabla.\end{aligned}$$

Multiplying the overall factor $(4\pi)^{-\frac{3}{2}}$ and factoring out the powers of π, we get the function $K_1(s)$ given in Theorem 2.3.3. Similarly, other function are obtained as

$$\begin{aligned}K_2(s) &= \frac{1}{8\pi^2}\psi_2(e^s)f(s)\\ H_1(s,t) &= \frac{1}{8\pi^2}\left(\phi_1(e^s,e^t)f(e^s)f(e^t) + 2\psi_1(e^se^t)g(e^s,e^t)\right),\\ H_2(s,t) &= \frac{1}{8\pi^2}\left(\phi_2(e^s,e^t)f(e^s)f(e^t) + 2\psi_2(e^se^t)g(e^s,e^t)\right).\end{aligned}$$

Appendix B

Functions from the rearrangement lemma

In this appendix we list all the functions obtained from the rearrangement lemmas 2.3.2 and 2.4.1 which are required in the computations. First we have the functions from the conformally flat case in section 2.3:

$$F_{1,1}(s_1) = \pi / \left(s_1^{2/3} + \sqrt[3]{s_1} \right),$$

$$F_{2,1}(s_1) = \pi \left(\sqrt[3]{s_1} + 2 \right) / \left(2 \left(\sqrt[3]{s_1} + 1 \right)^2 \sqrt[3]{s_1} \right),$$

$$F_{3,1}(s_1) = \pi \left(3 s_1^{2/3} + 9 \sqrt[3]{s_1} + 8 \right) / \left(8 \left(\sqrt[3]{s_1} + 1 \right)^3 \sqrt[3]{s_1} \right),$$

$$F_{1,1,1}(s_1, s_2) = \frac{\pi \left(\sqrt[3]{s_1} \left(\sqrt[3]{s_2} + 1 \right) + 1 \right)}{\left(\sqrt[3]{s_1} + 1 \right) s_1 \left(\sqrt[3]{s_2} + 1 \right) \sqrt[3]{s_2} \left(\sqrt[3]{s_1} \sqrt[3]{s_2} + 1 \right)},$$

$$F_{1,2,1}(s_1, s_2) = \frac{\pi \left(2 s_1^{2/3} \left(\sqrt[3]{s_2} + 1 \right)^2 + \sqrt[3]{s_1} \left(\sqrt[3]{s_2} + 2 \right)^2 + \sqrt[3]{s_2} + 2 \right)}{2 \left(\sqrt[3]{s_1} + 1 \right)^2 s_1^{5/3} \left(\sqrt[3]{s_2} + 1 \right)^2 \sqrt[3]{s_2} \left(\sqrt[3]{s_1} \sqrt[3]{s_2} + 1 \right)},$$

$$F_{2,1,1}(s_1, s_2) = \frac{\pi \left(\left(\sqrt[3]{s_1} + 2 \right) \sqrt[3]{s_1} \left(\sqrt[3]{s_2} + 1 \right) \left(\sqrt[3]{s_1} \sqrt[3]{s_2} + 2 \right) + 2 \right)}{2 \left(\sqrt[3]{s_1} + 1 \right)^2 s_1 \left(\sqrt[3]{s_2} + 1 \right) \sqrt[3]{s_2} \left(\sqrt[3]{s_1} \sqrt[3]{s_2} + 1 \right)^2},$$

$$\begin{aligned} F_{2,2,1}(s_1, s_2) = {} & \frac{\pi}{2 \left(\sqrt[3]{s_1} + 1 \right)^3 s_1^{5/3} (\sqrt[3]{s_2} + 1)^2 \sqrt[3]{s_2} (\sqrt[3]{s_1} \sqrt[3]{s_2} + 1)^2} \times \\ & \Big((\sqrt[3]{s_2} + 1)^2 (s_1^{4/3} \sqrt[3]{s_2} + s_1^{2/3} (\sqrt[3]{s_2} + 6) + s_1 (3 \sqrt[3]{s_2} + 2)) \\ & \qquad + \sqrt[3]{s_1} (2 s_2^{2/3} + 7 \sqrt[3]{s_2} + 6) + \sqrt[3]{s_2} + 2 \Big), \end{aligned}$$

$$
\begin{aligned}
F_{3,1,1}(s_1,s_2) = {} & \frac{\pi}{8s_1\sqrt[3]{s_2}(\sqrt[3]{s_1}+1)^3(\sqrt[3]{s_2}+1)(\sqrt[3]{s_1s_2}+1)^3}\times \\
& \Big((24\sqrt[3]{s_1}+3s_2^{2/3}s_1^{5/3}+27\sqrt[3]{s_2}s_1+8s_2^{2/3}s_1+8s_1)\left(\sqrt[3]{s_2}+1\right) \\
& \qquad + (9s_1^{4/3}\sqrt[3]{s_2}+24s_1^{2/3})\left(\sqrt[3]{s_2}+1\right)^2+8\Big).
\end{aligned}
$$

The list of functions required in the computations for the non-conformal metric is the following:

$$
\begin{aligned}
F^{[2]}_{1,1}(s_1) &= \log(s_1)/(s_1-1), \\
F^{[2]}_{2,1}(s_1) &= (s_1-\log(s_1)-1)/(s_1-1)^2, \\
F^{[2]}_{3,1}(s_1) &= ((s_1-4)s_1+2\log(s_1)+3)/(2(s_1-1)^3), \\
F^{[3]}_{3,1}(s_1) &= (s_1^2-2s_1\log(s_1)-1)/(2(s_1-1)^3), \\
F^{[3]}_{2,1}(s_1) &= (s_1(\log(s_1)-1)+1)/(s_1-1)^2, \\
F^{[2]}_{1,0,1}(s_1,s_2) &= \log(s_1s_2)/(s_1s_2-1), \\
F^{[2]}_{1,1,1}(s_1,s_2) &= \frac{((s_1s_2-1)\log(s_1)-(s_1-1)\log(s_1s_2))}{(s_1-1)s_1(s_2-1)(s_1s_2-1)}, \\
F^{[2]}_{2,0,1}(s_1,s_2) &= (s_1s_2-\log(s_1s_2)-1)/(s_1s_2-1)^2, \\
F^{[2]}_{1,2,1}(s_1,s_2) &= \frac{1}{(s_1-1)^2s_1^2(s_2-1)^2(s_1s_2-1)}\Big((s_1-1)^2\log(s_1s_2) \\
&\quad + (s_1s_2-1)(s_1(-s_2)+(s_1(s_2-2)+1)\log(s_1)+s_1+s_2-1)\Big), \\
F^{[2]}_{2,1,1}(s_1,s_2) &= \frac{1}{(s_1-1)^2s_1(s_2-1)(s_1s_2-1)^2}\Big((s_1-1)^2\log(s_1s_2) \\
&\quad + (s_1s_2-1)((s_1-1)s_1(s_2-1)+(1-s_1s_2)\log(s_1))\Big), \\
F^{[2]}_{2,2,1}(s_1,s_2) &= \frac{1}{(s_1-1)^3s_1^2(s_2-1)^2(s_1s_2-1)^2}\Big(-(s_1-1)^3\log(s_1s_2) \\
&\quad + (s_1-1)(s_2-1)(s_1s_2-1)(s_1(s_1s_2-s_1+s_2)-1) \\
&\quad - (s_1s_2-1)^2(s_1(2s_2-3)+1)\log(s_1)\Big) \\
F^{[2]}_{3,0,1}(s_1,s_2) &= \frac{(s_1s_2-3)(s_1s_2-1)+2\log(s_1s_2)}{(2(s_1s_2-1)^3)}, \\
F^{[2]}_{3,1,1}(s_1,s_2) &= \frac{1}{2(s_1-1)^3s_1(s_2-1)(s_1s_2-1)^3}\times \\
&\quad \Big(2(s_1s_2-1)^3\log(s_1)-2(s_1-1)^3\log(s_1s_2) \\
&\quad + s_1(s_1-1)(s_2-1)(s_1s_2-1)((s_1-3)s_1s_2-3s_1+5)\Big),
\end{aligned}
$$

$$
\begin{aligned}
F^{[3]}_{1,1,1}(s_1,s_2) &= \frac{(-s_1 s_2+1)\log(s_1)+(s_1-1)s_2\log(s_1 s_2)}{(s_1-1)(s_2-1)(s_1 s_2-1)},\\
F^{[3]}_{2,0,1}(s_1,s_2) &= \Big(-s_1 s_2 + s_1 s_2\log(s_1 s_2)+1\Big)/(s_1 s_2-1)^2,\\
F^{[3]}_{3,0,1}(s_1,s_2) &= \Big(s_1^2 s_2^2 - 2 s_1 s_2\log(s_1 s_2)-1\Big)/(2(s_1 s_2-1)^3),\\
F^{[3]}_{1,2,1}(s_1,s_2) &= \frac{1}{(s_1-1)^2 s_1 (s_2-1)^2(s_1 s_2-1)}\Big((s_1 s_2-1)((s_1-1)(s_2-1)\\
&\quad + (s_1-s_2)\log(s_1)) - (s_1-1)^2 s_2\log(s_1 s_2)\Big),\\
F^{[3]}_{2,1,1}(s_1,s_2) &= \frac{1}{(s_1-1)^2(s_2-1)(s_1 s_2-1)^2}\Big((s_1 s_2-1)^2\log(s_1)\\
&\quad - (s_1-1)((s_2-1)(s_1 s_2-1)+(s_1-1)s_2\log(s_1 s_2))\Big),\\
F^{[3]}_{2,2,1}(s_1,s_2) &= \frac{1}{(s_1-1)^3 s_1 (s_2-1)^2(s_1 s_2-1)^2}\Big(s_1^2 + s_2^3 s_1^3(\log(s_1)-2)\\
&\quad + s_2^2 s_1^3(3-2\log(s_1)) + s_2 s_1^3(\log(s_1 s_2)-1) + s_2^3 s_1^2(\log(s_1)+2)\\
&\quad - s_2^2 s_1^2(2\log(s_1)) + s_2 s_1^2(4\log(s_1)-3\log(s_1 s_2)-3) - 2 s_1\log(s_1)\\
&\quad + s_2^2 s_1(-2\log(s_1)-3) + s_2 s_1(\log(s_1)+3\log(s_1 s_2)+3)\\
&\quad + s_2(\log(s_1)-\log(s_1 s_2)+1)-1\Big)\\
F^{[3]}_{3,1,1}(s_1,s_2) &= \frac{1}{2(s_1-1)^3(s_2-1)(s_1 s_2-1)^3}\times\\
&\quad \Big(2(s_1-1)^3 s_2\log(s_1 s_2) - 2(s_1 s_2-1)^3\log(s_1)\\
&\quad + (s_1-1)(s_2-1)(s_1 s_2-1)((s_1+1)s_1 s_2 + s_1 - 3)\Big).
\end{aligned}
$$

Bibliography

[1] T. A. Bhuyain and M. Marcolli. The Ricci flow on noncommutative two-tori. *Lett. Math. Phys.*, 101(2):173–194, 2012.

[2] A. Connes. C^* algèbres et géométrie différentielle. *C. R. Acad. Sci. Paris Sér. A-B*, 290(13):A599–A604, 1980.

[3] A. Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.

[4] A. Connes and F. Fathizadeh. The term a_4 in the heat kernel expansion of noncommutative tori. *arXiv:1611.09815 [math.QA]*, 2016.

[5] A. Connes and H. Moscovici. Modular curvature for noncommutative two-tori. *J. Amer. Math. Soc.*, 27(3):639–684, 2014.

[6] A. Connes and P. Tretkoff. The Gauss-Bonnet theorem for the noncommutative two torus. In *Noncommutative geometry, arithmetic, and related topics*, pages 141–158. Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[7] L. Dabrowski and A. Sitarz. An asymmetric noncommutative torus. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 11:Paper 075, 11, 2015.

[8] R. Dong, A. Ghorbanpour, and M. Khalkhali. The Ricci Curvature for Noncommutative Three Tori. *arXiv e-prints*, page arXiv:1808.02977, Aug 2018.

[9] F. Fathizadeh and M. Khalkhali. The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure. *J. Noncommut. Geom.*, 6(3):457–480, 2012.

[10] F. Fathizadeh and M. Khalkhali. Scalar curvature for the noncommutative two torus. *J. Noncommut. Geom.*, 7(4):1145–1183, 2013.

[11] F. Fathizadeh and M. Khalkhali. Scalar curvature for noncommutative four-tori. *J. Noncommut. Geom.*, 9(2):473–503, 2015.

[12] R. Floricel, A. Ghorbanpour, and M. Khalkhali. The Ricci curvature in noncommutative geometry. *Accepted to appear in Journal of Nonccommutative Geometry, arXiv:1612.06688 [math.QA]*, Dec. 2016.

[13] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1995.

[14] M. Khalkhali, A. Moatadelro, and S. Sadeghi. A scalar curvature formula for the noncommutative 3-torus. *arXiv:1610.04740 [math.OA]*, 2016.

[15] M. Khalkhali and A. Sitarz. Gauss-Bonnet for matrix conformally rescaled Dirac. *J. Math. Phys.*, 59(6):063505, 10, 2018.

[16] M. Lesch. Divided differences in noncommutative geometry: rearrangement lemma, functional calculus and expansional formula. *J. Noncommut. Geom.*, 11(1):193–223, 2017.

[17] M. Lesch and H. Moscovici. Modular curvature and Morita equivalence. *Geom. Funct. Anal.*, 26(3):818–873, 2016.

[18] Y. Liu. Modular curvature for toric noncommutative manifolds. *arXiv:1510.04668 [math.QA]*, 2015.

Chapter 3

Second quantization and spectral action

3.1 Introduction

This chapter is a reproduction of my joint paper with Masoud Khalkhali [4]. The spectral action principle of Connes and Chamseddine was originally developed mainly to give a conceptual and geometric formulation of the standard model of particle physics [2]. The spectral action can be defined for spectral triples $(A, \mathcal{H}, D)$, even when the algebra A is not commutative. An interesting feature here is the additivity of the spectral action with respect to the direct sum of spectral triples. Conversely, one can wonder whether a given additive functional on spectral triples is obtained via an spectral action.

In a recent paper [3], Chamseddine, Connes, and van Suijlekom have shown that the von Neumann entropy of the Gibbs state naturally defined by a Fermionic second quantization of a spectral triple is in fact spectral and they find a universal function that defines the spectral action.

In this paper we show that by incorporating chemical potentials one can extend the formalism of spectral action principle to both Bosonic and Fermionic second quantization. In fact we show that the von Neumann entropy, the average energy, and the negative free energy of the thermal equilibrium state defined by the Bosonic, or Fermionic, grand partition function, with a given chemical potential, can be expressed as spectral actions. We show that all spectral action coefficients can be expressed in terms of the modified Bessel functions of the second kind. In the Fermionic case, we show that the spectral action coefficients for the von Neumann entropy, in the limit when the chemical potential μ approaches to 0, can be expressed in terms of the Riemann zeta function. This recovers the recent result of Chamseddine-Connes-van Suijlekom in [3].

It should be noted that without the use of chemical potentials, the natural spectral function for the von Neumann entropy in the Bosonic case is singular at $t = 0$, and in fact the corresponding functional is not spectral.

In searching for a suitable expression of spectral action coefficients in all six cases studied in this paper, we were naturally led to the class of modified Bessel functions of the second kind. In Section 3 some basic properties of these functions are derived. In section 2 we recall some of the main concepts and results from the theory of second quantization. Our main results are presented in Sections 4 and 5.

3.2 Second quantization basics

In this section, mainly to fix our notation and terminology, we shall recall some basic definitions and facts from the theory of second quantization in quantum statistical mechanics. We shall largely follow [1].

3.2.1 Fock space and second quantization

In this section we shall first recall the definition of the Fock space $\mathcal{F}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$, and the correspondding Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$ and the Bosonic Fock space $\mathcal{F}_+(\mathcal{H})$ [1]. Here we will regard $\mathcal{F}_\pm(\mathcal{H})$ as subspaces of $\mathcal{F}(\mathcal{H})$, although one can also treat them as the quotient spaces of $\mathcal{F}(\mathcal{H})$ instead. After that we shall recall the procedure of second quantization.

Let $\mathcal{H}$ be a complex Hilbert space. We denote by $\mathcal{H}^n = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ the n-fold tensor product of $\mathcal{H}$ with itself when $n > 0$, and let $\mathcal{H}^0 = \mathbb{C}$. The Fock space $\mathcal{F}(\mathcal{H})$ is the completion of the pre-Hilbert space $\bigoplus_{n\geq 0} \mathcal{H}^n$. Define the projection operators $P_\pm$ on $\mathcal{H}^n$ by

$$P_+ (f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\pi \in S_n} f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)},$$
$$P_- (f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\pi \in S_n} (-1)^{|\pi|} f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)},$$

for all $f_1, ..., f_n \in \mathcal{H}$. Since $P_\pm$ are bounded operators with norm 1 on $\bigoplus_{n\geq 0} \mathcal{H}^n$, they can be extended by continuity to bounded projection operators on the Fock space $\mathcal{F}(\mathcal{H})$. The Bosonic Fock space $\mathcal{F}_+(\mathcal{H})$ and the Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$ are then defined by

$$\mathcal{F}_\pm(\mathcal{H}) = P_\pm(\mathcal{F}(\mathcal{H})).$$

The corresponding n-particle subspaces $\mathcal{H}^n_\pm$ are defined by $\mathcal{H}^n_\pm = P_\pm \mathcal{H}^n$.

The structure of the Fock space allows us to amplify an operator on $\mathcal{H}$ to the whole Bose/Fermi Fock spaces $\mathcal{F}_\pm(\mathcal{H})$. This procedure is commonly referred to as second quantization.

Let H be a self-adjoint operator on $\mathcal{H}$ with domain $D(\mathcal{H})$. We define H_n on $\mathcal{H}^n_\pm$ by

$$H_n\left(P_\pm\left(f_1\otimes\cdots\otimes f_n\right)\right)=\begin{cases}P_\pm\left(\displaystyle\sum_{i=1}^n f_1\otimes f_2\otimes\cdots\otimes Hf_i\otimes\cdots\otimes f_n\right) & n>0,\\ 0 & n=0,\end{cases}$$

for all $f_i \in D(H)$. The direct sum operator $\bigoplus_{n\geq 0} H_n$ is essentially self-adjoint, and the self-adjoint closure of this direct sum operator is called the second quantization of the operator H and it is denoted by $d\Gamma(H)$. Namely,

$$d\Gamma(H)=\overline{\bigoplus_{n\geq 0} H_n}.$$

In particular, let $H = \mathbb{1}$ be the identity operator. Then we have

$$d\Gamma(\mathbb{1}) = N,$$

where N is the number operator on $\mathcal{F}_\pm(\mathcal{H})$, whose domain is defined by

$$D(N)=\left\{\psi=\{\psi^{(n)}\}_{n\geq 0};\ \sum_{n\geq 0} n^2\|\psi^{(n)}\|^2<\infty\right\},$$

and for any $\psi \in D(N)$

$$N\psi = \{n\psi^{(n)}\}_{n\geq 0}.$$

For a unitary operator U on $\mathcal{H}$, first we define U_n on $\mathcal{H}^n_\pm$ by

$$U_n\left(P_\pm\left(f_1\otimes f_2\otimes\cdots\otimes f_n\right)\right)=\begin{cases}P_\pm\left(Uf_1\otimes Uf_2\otimes\cdots\otimes Uf_n\right) & n>0,\\ \mathbb{1} & n=0,\end{cases}$$

and then extend it to the whole Fock space.

We denote this extension by $\Gamma(U)$, called the second quantization of the unitary operator U,

$$\Gamma(U)=\bigoplus_{n\geq 0} U_n.$$

It is worth noticing that here $\Gamma(U)$ is also a unitaty operator on $\mathcal{F}_\pm(\mathcal{H})$. Also, if $U_t = e^{itH}$ is a strongly continuous one-parameter unitary group acting on $\mathcal{H}$, then

$$\Gamma(U_t) = e^{itd\Gamma(H)}$$

on the Fock spaces $\mathcal{F}_\pm(\mathcal{H})$.

If H is a self-adjoint Hamiltonian operator on the one-particle Hilbert space $\mathcal{H}$, then the dynamics of the ideal Bose gas and the ideal Fermi gas are described by the Schrödinger equation

$$i\hbar\frac{d\psi_t}{dt} = d\Gamma(H)\psi_t$$

on $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{F}_-(\mathcal{H})$, separately. We choose the units so that $\hbar = 1$. The solution of the Schrödinger equation gives us the evolution

$$\psi \in \mathcal{F}_\pm(\mathcal{H}) \mapsto \psi_t = e^{-itd\Gamma(H)}\psi = \Gamma(e^{-itH})\psi,$$

and the evolution of a bounded observable $A \in \mathcal{B}(\mathcal{F}_\pm(\mathcal{H}))$ is given by conjugation as

$$A \in \mathcal{B}(\mathcal{F}_\pm(\mathcal{H})) \mapsto \tau_t(A) = \Gamma(e^{itH})A\Gamma(e^{-itH}).$$

Next we shall introduce the Gibbs grand canonical equilibrium state ω of a particle system at inverse temperature $\beta \in \mathbb{R}$, and with chemical potential $\mu \in \mathbb{R}$. Let

$$K_\mu = d\Gamma(H - \mu\mathbb{1}) = d\Gamma(H) - \mu N$$

be the modified Hamiltonian. Then ω is defined by

$$\omega(A) = \frac{\mathrm{Tr}\left(e^{-\beta K_\mu}A\right)}{\mathrm{Tr}\left(e^{-\beta K_\mu}\right)}, \qquad A \in \mathcal{B}(\mathcal{F}_\pm(\mathcal{H})).$$

Here we assume the operator $e^{-\beta K_\mu}$ is a trace-class operator.

If we have two one-particle spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and self-adjoint operators

$$H_1 : \mathcal{H}_1 \to \mathcal{H}_1, \qquad H_2 : \mathcal{H}_2 \to \mathcal{H}_2,$$

then we can form the direct sum $H_1 \oplus H_2 : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$.

Lemma 3.2.1 *We have $e^{-d\Gamma(H_1 \oplus H_2)} = e^{-d\Gamma(H_1)} \otimes e^{-d\Gamma(H_2)}$. Moreover, when $e^{-d\Gamma(H_i)}$ are trace-*

class operators for $i = 1, 2$, we can define the density operators

$$\rho = \frac{e^{-d\Gamma(H_1 \oplus H_2)}}{\mathrm{Tr}(e^{-d\Gamma(H_1 \oplus H_2)})},$$
$$\rho_i = \frac{e^{-d\Gamma(H_1)}}{\mathrm{Tr}(e^{-d\Gamma(H_1)})}, \qquad i = 1, 2,$$

and we have $\rho = \rho_1 \otimes \rho_2$.

Proof It is clear that $e^{-H_1 \oplus H_2} = e^{-H_1} \oplus e^{-H_2}$. Thus

$$e^{-d\Gamma(H_1 \oplus H_2)} = \Gamma\left(e^{-H_1} \oplus e^{-H_2}\right) = \Gamma\left(e^{-H_1}\right) \otimes \Gamma\left(e^{-H2}\right) = e^{-d\Gamma(H_1)} \otimes e^{-d\Gamma(H_2)}.$$

When the operators $e^{-d\Gamma(H_i)}$ are positive trace-class operators for $i = 1, 2$, then

$$\rho = \frac{e^{-d\Gamma(H_1 \oplus H_2)}}{\mathrm{Tr}(e^{-d\Gamma(H_1 \oplus H_2)})} = \frac{e^{-d\Gamma(H_1)} \otimes e^{-d\Gamma(H_2)}}{\mathrm{Tr}(e^{-d\Gamma(H_1)})\mathrm{Tr}(e^{-d\Gamma(H_2)})} = \rho_1 \otimes \rho_2.$$

3.2.2 CAR and CCR algebras

Both of the CAR and CCR algebras are constructed with the help of creation and annihilation operators. Because of that, we shall recall the definitions of annihilation and creation operators first.

Let $\mathcal{H}$ be a complex Hilbert space. For each $f \in \mathcal{H}$, we define the annihilation operator $a(f)$, and the creation operator $a^*(f)$ acting on the Fock space $\mathcal{F}(\mathcal{H})$ by initially setting $a(f)\psi^{(0)} = 0$, $a^*(f)\psi^{(0)} = f$, for all $f \in \mathcal{H}$, and

$$a(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \sqrt{n}\,(f, f_1)\, f_2 \otimes f_3 \otimes \cdots \otimes f_n,$$
$$a^*(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \sqrt{n+1} f \otimes f_1 \otimes f_2 \otimes f_3 \otimes \cdots \otimes f_n.$$

Here $\psi^{(0)} = 1 \in \mathbb{C}$. One can see that the maps $f \mapsto a(f)$ are anti-linear while the maps $f \mapsto a^*(f)$ are linear. Also, one can show that $a(f)$ and $a^*(f)$ have well-defined extensions to $D(N^{1/2})$, the domain of the operator $N^{1/2}$. Moreover, we have that $a^*(f)$ is the adjoint of $a(f)$; namely, for any $\phi, \psi \in D(N^{1/2})$, one has

$$(a^*(f)\phi, \psi) = (\phi, a(f)\psi).$$

We can then define the annihilation operators $a_\pm(f)$ and the creation operators $a^*_\pm(f)$ on the Fermi/Bose Fock spaces $\mathcal{F}_\pm(\mathcal{H})$ by

$$a_\pm(f) = P_\pm a(f) P_\pm, \qquad a^*_\pm(f) = P_\pm a^*(f) P_\pm.$$

Moreover, since the annihilation operator $a(f)$ keeps the subspaces $\mathcal{F}_\pm(\mathcal{H})$ invariant, we have

$$a_\pm(f) = a(f) P_\pm, \qquad a^*_\pm(f) = P_\pm a^*(f).$$

One computes straightforwardly that on the Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$,

$$\{a_-(f), a_-(g)\} = \{a^*_-(f), a^*_-(g)\} = 0, \qquad \{a_-(f), a^*_-(g)\} = (f, g)\mathbb{1},$$

and on the Bosonic Fock space $\mathcal{F}_+(\mathcal{H})$,

$$[a_+(f), a_+(g)] = [a^*_+(f), a^*_+(g)] = 0, \qquad [a_+(f), a^*_+(g)] = (f, g)\mathbb{1}.$$

The first relations are called the canonical anti-commutation relations (CAR), and the second relations are called the canonical commutation relations (CCR).

Roughly speaking, the CAR algebra is the algebra generated by the annihilation operators $a_-(f)$ and creation operators $a^*_-(f)$. In fact, we have the following proposition [1]:

Proposition 3.2.2 *Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{F}_-(\mathcal{H})$ be the Fermionic Fock space, and $a_-(f)$ and $a^*_-(g)$ the corresponding annihilation and creation operators on $\mathcal{F}_-(\mathcal{H})$.*

(1) For all $f \in \mathcal{H}$, we have

$$\|a_-(f)\| = \|f\| = \|a^*_-(f)\|.$$

Therefore both $a_-(f)$ and $a^*_-(g)$ have bounded extensions on $\mathcal{F}_-(\mathcal{H})$.

(2) Taking $\Omega = (1, 0, 0, \cdots)$, called the vacuum vector, and $\{f_\alpha\}$ an orthonormal basis of $\mathcal{H}$, then

$$\psi(f_{\alpha_1}, ..., f_{\alpha_n}) := a^*_-(f_{\alpha_1}) \cdots a^*_-(f_{\alpha_n})\,\Omega$$

is an orthonormal basis of $\mathcal{F}_-(\mathcal{H})$, when $\{f_{\alpha_1}, ..., f_{\alpha_n}\}$ runs over all the finite subsets of the orthonormal basis $\{f_\alpha\}$.

(3) The set of bounded operators $\{a_-(f), a^*_-(g); f, g \in \mathcal{H}\}$ is irreducible on $\mathcal{F}_-(\mathcal{H})$.

Definition 3.2.3 *We call the subalgebra of $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$ generated by $a_-(f)$, $a^*_-(g)$ and $\mathbb{1}$ the CAR algebra and denote it by $CAR(\mathcal{H})$.*

Although the CCR rules looks very similar to the CAR rules, however, one can not simply mimic the previous definition of CAR algebras to deduce the definition of CCR algebras. The reason is that the annihilation operators $a_+(f)$ and the creation operators $a_+^*(g)$ are not bounded operators on $\mathcal{F}_+(\mathcal{H})$.

First we introduce the set of operators $\{\Phi(f), f \in \mathcal{H}\}$ by

$$\Phi(f) = \frac{a_+(f) + a_+^*(f)}{\sqrt{2}}.$$

Since the map $f \mapsto a_+(f)$ is anti-linear, and $f \mapsto a_+^*(f)$ is linear, then

$$a_+(f) = \frac{\Phi(f) + i\Phi(if)}{\sqrt{2}}, \quad a_+^*(f) = \frac{\Phi(f) - i\Phi(if)}{\sqrt{2}}.$$

Thus it suffices to examine the set of operators $\{\Phi(f), f \in \mathcal{H}\}$.

Let $F_+(\mathcal{H}) = P_+\left(\bigoplus_{n\geq 0} \mathcal{H}^n\right) \subseteq \mathcal{F}_+(\mathcal{H})$, i.e. $F_+(\mathcal{H})$ contains the sequences $\psi = \{\psi^{(n)}\}_{n\geq 0}$ which have only a finite number of nonvanishing components.

Since for each $f \in \mathcal{H}$, $\Phi(f)$ is essentially self-adjoint on $F_+(\mathcal{H})$, $\Phi(f)$ can be extended to a self-adjoint operator, we still use $\Phi(f)$ to denote the selfadjoint operator

$$\Phi(f) = \frac{\overline{a(f) + a^*(f)}}{\sqrt{2}}.$$

We have the following proposition [1]:

Proposition 3.2.4 *For each $f \in \mathcal{H}$, let*

$$\Phi(f) = \frac{\overline{a(f) + a^*(f)}}{\sqrt{2}}, \qquad W(f) = \exp\left(i\Phi(f)\right).$$

Let $CCR(\mathcal{H})$ denote the algebra generated by $\{W(f), f \in \mathcal{H}\}$. It follows that

(1) For any $f, g \in \mathcal{H}$, $W(f)D(\Phi(g)) = D(\Phi(g))$, and

$$W(f)\Phi(g)W(f)^* = \Phi(g) - \mathrm{Im}(f, g)\mathbb{1}.$$

(2) For each pair $f, g \in \mathcal{H}$

$$W(f)W(g) = e^{-i\mathrm{Im}(f,g)/2}W(f + g).$$

(3) $W(-f) = W(f)^*$.

(4) *For each* $f \in \mathcal{H}\backslash\{0\}$

$$\|W(f) - \mathbb{1}\| = 2,$$

and $W(0) = \mathbb{1}$.

(5) *The set* $\{W(f); f \in \mathcal{H}\}$ *is irreducible on* $\mathcal{F}_+(\mathcal{H})$*, and* $CCR(\mathcal{H})$ *is a simple algebra.*

(6) *If* $\|f_\alpha - f\| \to 0$*, then*

$$\|(W(f_\alpha) - W(f))\psi\| \to 0$$

for all $\psi \in \mathcal{F}_+(\mathcal{H})$.

The operators $W(f)$ *are called Weyl operators, and the algebra* $CCR(\mathcal{H})$ *is called the CCR algabra of* $\mathcal{H}$.

3.2.3 Gibbs states

Let K_μ denote the modified Hamiltonian operator

$$K_\mu = d\Gamma(H - \mu\mathbb{1}).$$

In the Fermionic case, we can define the Gibbs state $\omega(A)$ over the CAR algebra $CAR(\mathcal{H})$ by

$$\omega(A) = \frac{\mathrm{Tr}\left(e^{-\beta K_\mu}A\right)}{\mathrm{Tr}\left(e^{-\beta K_\mu}\right)}, \quad \forall A \in CAR(\mathcal{H}).$$

Here we assume the operator $e^{-\beta K_\mu}$ is a trace-class operator on $\mathcal{F}_-(\mathcal{H})$. In fact, we have the following proposition [1]:

Proposition 3.2.5 *Let* H *be a self-adjoint operator on the Hilbert space* $\mathcal{H}$ *and let* $\beta \in \mathbb{R}$. *The following conditions are equivalent:*

(1) $e^{-\beta H}$ *is trace-class on the one-particle Hilbert space* $\mathcal{H}$.

(2) $e^{-\beta d\Gamma(H-\mu\mathbb{1})}$ *is trace-class on the Fermionic Fock space* $\mathcal{F}_-(\mathcal{H})$ *for all* $\mu \in \mathbb{R}$.

In the Bosonic case, we can define the Gibbs state $\omega(A)$ over the CCR algebra $CCR(\mathcal{H})$ by

$$\omega(A) = \frac{\mathrm{Tr}\left(e^{-\beta K_\mu}A\right)}{\mathrm{Tr}\left(e^{-\beta K_\mu}\right)}, \quad \forall A \in CCR(\mathcal{H}).$$

Similarly as in the case of Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$, it is implicitly assumed that the operator $e^{-\beta K_\mu}$ is trace-class on $\mathcal{F}_+(\mathcal{H})$, in fact, we have the fololowing proposition [1]:

Proposition 3.2.6 *Let H be a self-adjoint operator on the one-particle Hilbert space $\mathcal{H}$, let $\beta, \mu \in \mathbb{R}$. The following conditions are equivalent:*

(1) $e^{-\beta H}$ is trace-class on the one-particle Hilbert space $\mathcal{H}$ and $\beta(H - \mu\mathbb{1}) > 0$,

(2) $e^{-\beta d\Gamma(H-\mu\mathbb{1})}$ is trace-class on the Bosonic Fock space $\mathcal{F}_+(\mathcal{H})$.

3.2.4 Entropy and energy

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. We can construct the Bosonic and Fermionic Fock spaces $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{F}_-(\mathcal{H})$, respectively. Let $D_\mu = \sqrt{D^2 - \mu\mathbb{1}}$. Suppose the operator $e^{-d\Gamma D_\mu}$ is a trace-class operator on $\mathcal{F}_+(\mathcal{H})$, or on $\mathcal{F}_-(\mathcal{H})$. Then we can define the density matrix

$$\rho = \frac{e^{-d\Gamma D_\mu}}{\mathrm{Tr}\left(e^{-d\Gamma D_\mu}\right)}.$$

In this section, we will show that when the operator e^{-D_μ} is trace class on $\mathcal{H}$, the von Neumann entropy, the average energy, as well as the negative free energy of ρ can be expressed as spectral actions for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

First let us briefly recall the von Neumann entropy and the energy. Consider a density matrix ρ on a Hilbert space $\mathcal{H}$, i.e. ρ is a positive trace-class operator with $\mathrm{Tr}(\rho) = 1$. Its von Neumann entropy is defined to be

$$\mathcal{S}(\rho) := -\mathrm{Tr}(\rho \log \rho).$$

Consider an observable, that is a self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$, and let $\rho = \frac{1}{Z}\exp(-\beta H)$ be a thermal density matrix, at some inverse temperature β. Here $Z = \mathrm{Tr}(\exp(-\beta H))$ is the canonical partition function. Then the average energy $\langle H \rangle = \mathrm{Tr}(\rho H)$ is given by

$$\langle H \rangle = -\frac{\partial}{\partial \beta}(\log Z), \tag{3.2.1}$$

and the free energy $F(\rho)$ is defined by

$$F(\rho) = -\frac{1}{\beta}\log Z.$$

It is easy to see that

$$-F(\rho) = \frac{1}{\beta}S(\rho) - E(\rho). \tag{3.2.2}$$

In a given spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the operator $e^{-d\Gamma D_\mu}$ is well-defined on both $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{F}_-(\mathcal{H})$.

According to the proposition 3.2.5, the operator $e^{-d\Gamma D_\mu}$ is trace-class on $\mathcal{F}_-(\mathcal{H})$ if and only if the operator e^{-D_μ} is trace-class on $\mathcal{H}$. Thus suppose e^{-D_μ} is trace-class on $\mathcal{H}$. Then we can define a density matrix

$$\rho(d\Gamma D_\mu) = \frac{e^{-d\Gamma D_\mu}}{\mathrm{Tr}(e^{-d\Gamma D_\mu})}$$

on $\mathcal{F}_-(\mathcal{H})$. The map $D \mapsto \mathcal{S}(\rho(d\Gamma D_\mu))$ gives rise to a spectral action, and this spectal action is an additive functional on spectral triples. In fact, suppose $D = S \oplus T$ is an orthogonal decomposition, then

$$\sqrt{D^2 - \mu\mathbb{1}} = \sqrt{S^2 - \mu\mathbb{1}} \oplus \sqrt{T^2 - \mu\mathbb{1}},$$

which we denote as $D_\mu = S_\mu \oplus T_\mu$. According to Lemma 3.2.1,

$$\rho(d\Gamma D_\mu) = \rho(d\Gamma S_\mu) \otimes \rho(d\Gamma T_\mu),$$

and since we have the entropy

$$\mathcal{S}(\rho(d\Gamma S_\mu) \otimes \rho(d\Gamma T_\mu)) = \mathcal{S}(\rho(d\Gamma S_\mu)) + \mathcal{S}(\rho(d\Gamma T_\mu)),$$

thus the map $D \mapsto \mathcal{S}(\rho(d\Gamma D_\mu))$ gives rise to a well-defined spectral action.

Now for a given chemical potential μ, the map $D \mapsto \langle d\Gamma D_\mu \rangle$ gives us a spectral action as well. According to Lemma 3.2.1, this action is additive. For simplicity, we take the inverse temperature $\beta = 1$ here.

3.3 Modified Bessel functions of the second kind

The modified Bessel functions $\{I_\nu(z), K_\nu(z)\}$ are the solutions of the modified Bessel's equation

$$z^2 y'' + zy' - (z^2 + \nu^2)y = 0,$$

where

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{2n}}{\Gamma(n+\nu+1)n!},$$

and

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}, \quad -\pi < \arg z < \pi. \tag{3.3.1}$$

The right-hand side of (3.3.1) should be determined by taking the limit when ν is an integer. The function $I_\nu(z)$ is called the modified Bessel function of the first kind, and $K_\nu(z)$ the modified Bessel function of the second kind.

We shall introduce some basic properties of the modified Bessel function of the second kind. For more detail, one can check the references [8, 7, 5].

Lemma 3.3.1 *When $\alpha \in \mathbb{R}$, one has the formula*

$$K_\alpha(z) = K_{-\alpha}(z).$$

Lemma 3.3.2 *We have the formula*

$$\frac{d}{dz}K_0(z) = -K_1(z).$$

Lemma 3.3.3 *For $\alpha > 0$, when $z \to 0^+$, one has the asymptotics*

$$K_\alpha(z) \sim \begin{cases} -\log(\frac{z}{2}) - \gamma & \alpha = 0, \\ \frac{\Gamma(\alpha)}{2}\left(\frac{2}{z}\right)^\alpha & \alpha > 0, \end{cases}$$

where γ is Euler's constant. When $z \nearrow \infty$, one has

$$K_\alpha(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}.$$

Lemma 3.3.4 *One has the integral representation formula of the function $K_\nu(z)$:*

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zx}(x^2-1)^{\nu-1/2}dx \qquad \textit{for} \quad \nu > -\frac{1}{2}.$$

Lemma 3.3.5 *Let $K_\nu(z)$ be the modified Bessel functions of the second kind. Then one has [5, 8.486]*

$$zK_{\nu-1}(z) - zK_{\nu+1}(z) = -2\nu K_\nu(z), \tag{3.3.2}$$

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2\frac{\partial}{\partial z}K_\nu(z), \tag{3.3.3}$$

$$z\frac{\partial}{\partial z}K_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z), \tag{3.3.4}$$

$$z\frac{\partial}{\partial z}K_\nu(z) - \nu K_\nu(z) = -zK_{\nu+1}(z). \tag{3.3.5}$$

Lemma 3.3.6 *When $\nu > -\frac{1}{2}$, $a > 0$, and $x > 0$, we have the integral formula [5, 8.432]*

$$x^\nu K_\nu(ax) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty \frac{\cos xt}{(t^2+a^2)^{\nu+\frac{1}{2}}}dt.$$

Using Lemma 3.3.6, we obtain the following Lemma:

Lemma 3.3.7 *When $\nu > -\frac{1}{2}$, $a > 0$, and $x \in \mathbb{R}\backslash\{0\}$, one has*

$$|x|^\nu K_\nu\left(a|x|\right) = \frac{\pi\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}{\Gamma\left(\frac{1}{2}\right)}\widehat{\psi}_{\nu,a}(x), \tag{3.3.6}$$

and

$$e^{i\pi x}|x|^\nu K_\nu\left(a|x|\right) = \frac{\pi\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}{\Gamma\left(\frac{1}{2}\right)}\widehat{\phi}_{\nu,a}(x), \tag{3.3.7}$$

where

$$\psi_{\nu,a}(t) = \frac{1}{\left((2\pi t)^2+a^2\right)^{\nu+\frac{1}{2}}}, \quad \phi_{\nu,a}(t) = \psi_{\nu,a}\left(t+\frac{1}{2}\right),$$

and $\widehat{\psi}_{\nu,a}$, $\widehat{\phi}_{\nu,a}$ denote the corresponding Fourier transforms of $\psi_{\nu,a}$ and $\phi_{\nu,a}$. Namely,

$$\widehat{\psi}_{\nu,a}(x) = \int_{-\infty}^\infty \psi_{\nu,a}(t)e^{-2\pi ixt}dt, \quad \widehat{\phi}_{\nu,a}(x) = \int_{-\infty}^\infty \phi_{\nu,a}(t)e^{-2\pi ixt}dt.$$

Proof According to Lemma 3.3.6, one has

$$|x|^\nu K_\nu(a|x|) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}{2\Gamma\left(\frac{1}{2}\right)}\int_{-\infty}^\infty \frac{1}{\left(t^2+a^2\right)^{\nu+\frac{1}{2}}}e^{-ixt}dt,$$

and then changing the variable $t \mapsto 2\pi t$, one can get formulae (3.3.6) and (3.3.7).

From Lemma 3.3.7, one can easily deduce the following lemma:

Lemma 3.3.8 *When $\nu \geq -\frac{1}{2}$, $a > 0$, and $x \in \mathbb{R}\backslash\{0\}$, one has*

$$|x|^{\nu+2} K_\nu\left(a|x|\right) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}{-4\pi\Gamma\left(\frac{1}{2}\right)}\widehat{\psi''}_{\nu,a}(x), \tag{3.3.8}$$

and

$$e^{i\pi x}|x|^{\nu+2}K_\nu\left(a|x|\right) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}{-4\pi\Gamma\left(\frac{1}{2}\right)}\widehat{\phi''}_{\nu,a}(x). \tag{3.3.9}$$

3.3.1 Poisson summation and asymptotic expansions

To continue, we need the following version of the Poisson's summation formula:

Lemma 3.3.9 (Poisson's summation formula [6]) *If a function $f(x)$ is integrable, tends to zero at infinity, and $xf'(x) \in L^p(0,\infty), (1 < p \leq 2)$, then*

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} f(n) - \int_0^N f(t)dt\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} g(n) - \int_0^N g(x)dx\right),$$

where

$$g(x) = 2\int_0^\infty \cos(2\pi xt)f(t)dt.$$

By this lemma we can deduce the following asymptotic expansion formulae [6]:

Lemma 3.3.10 *When $a \to 0^+$, we have the following asymptotic expansions*

$$\sum_{n=1}^{\infty}(-1)^{n+1}K_0(an) \sim \frac{\log\pi-\gamma}{2} - \frac{\log a}{2}, \tag{3.3.10}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}a\,n\,K_1(an) \sim \frac{1}{2}, \tag{3.3.11}$$

$$\sum_{n=1}^{\infty}K_0(an) \sim \frac{\gamma-\log(4\pi)}{2} + \frac{\log a}{2} + \frac{\pi}{2a}, \tag{3.3.12}$$

$$\sum_{n=1}^{\infty}a\,n\,K_1(an) \sim -\frac{1}{2} + \frac{\pi}{2a}, \tag{3.3.13}$$

where γ is Euler's constant.

Proof Let us consider the formula (3.3.12) first. Let

$$f_0(t) = \frac{\pi}{((2\pi t)^2 + a^2)^{1/2}}, \qquad g_0(x) = K_0(ax), \qquad a > 0, \quad x > 0.$$

By the equation (3.3.6), we have

$$g_0(x) = 2\int_0^\infty \cos(2\pi xt) f_0(t)dt.$$

And

$$\begin{aligned}
&\lim_{N\to\infty}\left(\sum_{n=1}^{N} f_0(n) - \int_0^N f_0(t)dt\right)\\
&= \sum_{n=1}^{\infty}\left(\frac{\pi}{((2\pi n)^2+a^2)^{1/2}} - \frac{1}{2n}\right) + \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2n} - \frac{\log N}{2}\right)\\
&+ \lim_{N\to\infty}\left(\frac{\log N}{2} - \int_0^N \frac{\pi}{((2\pi t)^2+a^2)^{1/2}}dt\right)\\
&= \sum_{n=1}^{\infty}\left(\frac{\pi}{((2\pi n)^2+a^2)^{1/2}} - \frac{1}{2n}\right) + \frac{\gamma}{2} + \lim_{N\to\infty}\left(\frac{\log N}{2} - \frac{1}{2}\log\left(\frac{2\pi N}{a} + \sqrt{1+\left(\frac{2\pi N}{a}\right)^2}\right)\right)\\
&= \sum_{n=1}^{\infty}\left(\frac{\pi}{((2\pi n)^2+a^2)^{1/2}} - \frac{1}{2n}\right) + \frac{\gamma}{2} - \frac{1}{2}\log\frac{4\pi}{a},
\end{aligned} \tag{3.3.14}$$

and

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} g_0(n) - \int_0^N g_0(x)dx\right) = \sum_{n=1}^{\infty} K_0(an) - \frac{\pi}{2a}, \tag{3.3.15}$$

applying Lemma 3.3.9 to (3.3.14) and (3.3.15),

$$\sum_{n=1}^{\infty}\left(\frac{\pi}{((2\pi n)^2+a^2)^{1/2}} - \frac{1}{2n}\right) + \frac{\gamma}{2} - \frac{1}{2}\log\frac{4\pi}{a} = \sum_{n=1}^{\infty} K_0(an) - \frac{\pi}{2a}.$$

Thus we get the asymptotic formula (3.3.12).

If we replace a by $2a$ in formula (3.3.12), we get

$$\sum_{n=1}^{\infty} K_0(2an) \sim \frac{\gamma - \log(4\pi)}{2} + \frac{\log(2a)}{2} + \frac{\pi}{4a}, \qquad a\to 0^+,$$

and

$$\sum_{n=1}^{\infty}(-1)^{n+1}K_0(an) = \sum_{n=1}^{\infty} K_0(an) - 2\sum_{n=1}^{\infty} K_0(2an)$$

$$\begin{aligned}&\sim \left(\frac{\gamma-\log(4\pi)}{2}+\frac{\log a}{2}+\frac{\pi}{2a}\right)-2\left(\frac{\gamma-\log(4\pi)}{2}+\frac{\log(2a)}{2}+\frac{\pi}{4a}\right)\\&=\frac{\log\pi-\gamma}{2}-\frac{\log a}{2}.\end{aligned}$$

Thus we proved (3.3.10).

To prove (3.3.13), let

$$f_1(t)=\frac{a\pi}{((2\pi t)^2+a^2)^{3/2}},\qquad g_1(x)=x\,K_1(ax),\quad a>0,\quad x>0.$$

We have

$$g_1(x)=2\int_0^\infty \cos(2\pi xt)f_1(t)dt. \tag{3.3.16}$$

Since

$$\lim_{N\to\infty}\left(\sum_{n=1}^N f_1(n)-\int_0^N f_1(t)dt\right)=\sum_{n=1}^\infty\left(\frac{a\pi}{((2\pi n)^2+a^2)^{3/2}}\right)-\frac{1}{2a}, \tag{3.3.17}$$

and

$$\lim_{N\to\infty}\left(\sum_{n=1}^N g_1(n)-\int_0^N g_1(x)dx\right)=\sum_{n=1}^\infty n\,K_1(an)-\frac{\pi}{2a^2}, \tag{3.3.18}$$

Applying Lemma 3.3.9 to (3.3.17) and (3.3.18) again, we have

$$\sum_{n=1}^\infty\left(\frac{a\pi}{((2\pi n)^2+a^2)^{3/2}}\right)-\frac{1}{2a}=\sum_{n=1}^\infty n\,K_1(an)-\frac{\pi}{2a^2},$$

multiplying by a on both sides, we get the asymptotic formula (3.3.13).

Finally, since

$$\begin{aligned}\sum_{n=1}^\infty(-1)^{n+1}a\,n\,K_1(an)&=\sum_{n=1}^\infty a\,n\,K_1(an)-\sum_{n=1}^\infty 2\,a\,n\,K_1(2an)\\&\sim -\frac{1}{2}+\frac{\pi}{2a}-2\left(-\frac{1}{2}+\frac{\pi}{4a}\right)\\&=\frac{1}{2}.\end{aligned}$$

Thus we proved (3.3.11).

Remark This is consistent with the formulae given in [5, 8.526], where if we take $t=0$, when $x\to 0^+$, we can get the formulae (3.3.10) and (3.3.12) then.

3.4 The case of Fermionic Fock space

When the one-particle Hilbert space is $\mathcal{H} = \mathbb{C}$, then the Fermionic Fock space is $\mathcal{F}_{-}(\mathcal{H}) = \mathbb{C} \bigoplus \mathcal{H}$. Suppose we have a Dirac operator $D : \mathcal{H} \to \mathcal{H}$. Thus $D = D^*$ and $\sigma(D) = x \in \mathbb{R}$. Consider the Hamiltonian operator given by $H = D^2$. Clearly the spectrum of H is $\sigma(H) = x^2$. Denote the chemical potential by μ, where $\mu \le 0$, and let $D_\mu = \sqrt{H - \mu \mathbb{1}}$. Then the spectrum of $d\Gamma D_\mu$ is $\sigma(d\Gamma D_\mu) = \{0, \sqrt{x^2 - \mu}\}$, and we get a density matrix

$$\rho = \frac{e^{-d\Gamma D_\mu}}{\operatorname{Tr}\left(e^{-d\Gamma D_\mu}\right)}.$$

3.4.1 The von Neumann entropy in the Fermionic second quantization

In the Fermionic Fock space, the von Neumann entropy of ρ is given by

$$\mathcal{S}(\rho) = -\operatorname{Tr}(\rho \log \rho) = \frac{\sqrt{x^2 - \mu}}{\mathrm{e}^{\sqrt{x^2-\mu}} + 1} + \log\left(1 + \mathrm{e}^{-\sqrt{x^2-\mu}}\right). \tag{3.4.1}$$

It is worth noticing that we can still define D_μ for a general spectral triple, and when $\mu < 0$, the difference between D and D_μ is

$$D - D_\mu = \frac{\mu}{D + D_\mu},$$

which is a compact operator. Thus D_μ here plays the role of a fluctuation of D, even though there is no $*-$algebra here.

Let

$$h_\mu(x) = \mathcal{S}(\rho) = \frac{\sqrt{x^2 - \mu}}{e^{\sqrt{x^2-\mu}} + 1} + \log\left(1 + e^{-\sqrt{x^2-\mu}}\right).$$

Notice that when $\mu = 0$, we get the same function $h(x)$ as in [3]. The derivative of $h_\mu(x)$ is

$$h'_\mu(x) = -\frac{x}{4\cosh^2\left(\frac{\sqrt{x^2-u}}{2}\right)}.$$

According to [3],

$$h(\sqrt{x}) = \log(2) + \sum_{n=1}^{\infty} (-1)^n \frac{1 - 2^{-2n}}{n} \pi^{-n} \xi(2n) \frac{x^n}{n!}.$$

Thus we get the expansion of $h_\mu(x)$:

$$h_\mu(x) = \log(2) + \sum_{n=1}^{\infty} (-1)^n \frac{1 - 2^{-2n}}{n} \pi^{-n} \xi(2n) \frac{(x^2 - \mu)^n}{n!}.$$

Also, according to proposition 4.4 in [3],

$$h(x) = \int_0^\infty e^{-tx^2} \tilde{g}(t) dt,$$

where

$$\tilde{g}(t) = \frac{1}{2t} \sum_{n \in \mathbb{Z}} \left(2\pi^2 (2n+1)^2 t - 1\right) e^{-\pi^2 (2n+1)^2 t}. \tag{3.4.2}$$

Thus

$$h_\mu(x) = \int_0^\infty e^{-t(x^2-\mu)} \tilde{g}(t) dt = \int_0^\infty e^{-tx^2} \tilde{g}_\mu(t) dt, \tag{3.4.3}$$

where $\tilde{g}_\mu(t) := e^{\mu t} \tilde{g}(t)$.

Now we want to compute the moments of the function $h_\mu(x)$, that is the integral

$$\int_0^\infty h_\mu(x) x^\nu dx.$$

To this end, one can first compute the two integrals

$$\int_0^\infty \log\left(1 + e^{-\sqrt{x^2-\mu}}\right) x^\nu dx \quad \text{and} \quad \int_0^\infty \frac{\sqrt{x^2-\mu}}{e^{\sqrt{x^2-\mu}} + 1} x^\nu dx,$$

separately, and then sum them up.

Lemma 3.4.1 *We have the integral formula:*

$$\int_1^\infty e^{-zx} (x^2 - 1)^{\nu - \frac{1}{2}} x dx = \frac{2^\nu}{\sqrt{\pi}} \Gamma\left(\nu + \frac{1}{2}\right) z^{-\nu} K_{\nu+1}(z). \tag{3.4.4}$$

Proof According to Lemma 3.3.4, one has the integral formula:

$$\int_1^\infty e^{-zx} (x^2 - 1)^{\nu - \frac{1}{2}} x dx = -\frac{2^\nu}{\sqrt{\pi}} \Gamma\left(\nu + \frac{1}{2}\right) z^{-\nu} \left(\frac{\partial}{\partial z} K_\nu(z) - \nu K_\nu(z) z^{-1}\right),$$

and using (3.3.4) and (3.3.5)

$$\frac{\partial}{\partial z}K_\nu(z) = -\frac{1}{2}\left(K_{\nu-1}(z) + K_{\nu+1}(z)\right),$$

$$\frac{\nu K_\nu(z)}{z} = -\frac{1}{2}\left(K_{\nu-1}(z) - K_{\nu+1}(z)\right).$$

From which we get the formula (3.4.4).

Lemma 3.4.2 *We have the following integral formula:*

$$\int_1^\infty e^{-zx}(x^2-1)^{\nu-\frac{1}{2}}x^2dx = \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\nu+\frac{1}{2}\right)z^{-\nu-1}(zK_\nu(z) + (1+2\nu)K_{\nu+1}(z)). \tag{3.4.5}$$

Proof Taking the derivative with respect to z on both sides of the formula (3.4.4), one has

$$\int_1^\infty e^{-zx}(x^2-1)^{\nu-\frac{1}{2}}x^2dx = \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\nu+\frac{1}{2}\right)\left(\nu z^{-\nu-1}K_{\nu+1}(z) - z^{-\nu}\frac{\partial}{\partial z}K_{\nu+1}(z)\right). \tag{3.4.6}$$

Using (3.3.4), one has

$$\frac{\partial}{\partial z}K_{\nu+1}(z) = -K_\nu(z) - \frac{\nu+1}{z}K_{\nu+1}(z). \tag{3.4.7}$$

Now substituting (3.4.7) into (3.4.6), finally we get the desired formula (3.4.5).

Lemma 3.4.3 *When $\nu > -1$, one has*

$$\int_0^\infty \log\left(1+e^{-\sqrt{x^2-\mu}}\right)x^\nu dx = (-\mu)^{\frac{\nu+2}{4}}\,2^{\frac{\nu}{2}}\,\frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{\nu+1}{2}\right)\sum_{n=1}^\infty(-1)^{n+1}n^{-\frac{\nu}{2}-1}K_{\frac{\nu}{2}+1}\left(n\sqrt{-\mu}\right). \tag{3.4.8}$$

Proof Notice that

$$\log\left(1+e^{-\sqrt{x^2-\mu}}\right) = \sum_{n=1}^\infty(-1)^{n+1}\frac{1}{n}\,e^{-n\sqrt{x^2-\mu}},$$

and hence

$$\int_0^\infty \log\left(1+e^{-\sqrt{x^2-\mu}}\right)x^\nu dx = \sum_{n=1}^\infty(-1)^{n+1}\frac{1}{n}\int_0^\infty e^{-n\sqrt{x^2-\mu}}x^\nu dx. \tag{3.4.9}$$

Consider the integral

$$\int_0^\infty e^{-n\sqrt{x^2-\mu}}x^\nu dx,$$

and substitute x by $y = \sqrt{x^2 - \mu}$ to get:

$$\int_0^\infty e^{-n\sqrt{x^2-\mu}} x^\nu dx = \int_{\sqrt{-\mu}}^\infty e^{-ny}(y^2+\mu)^{\frac{\nu-1}{2}} y dy.$$

Substitute $z = \frac{y}{\sqrt{-\mu}}$, to obtain

$$\int_0^\infty e^{-n\sqrt{x^2-\mu}} x^\nu dx = (-\mu)^{\frac{\nu+1}{2}} \int_1^\infty e^{-n\sqrt{-\mu}z}(z^2-1)^{\frac{\nu-1}{2}} z dz.$$

Thus using Lemma 3.4.1, one has

$$\int_0^\infty e^{-n\sqrt{x^2-\mu}} x^\nu dx = \frac{2^{\frac{\nu}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{\nu+1}{2}\right) \left(n\sqrt{-\mu}\right)^{-\frac{\nu}{2}} K_{\frac{\nu}{2}+1}\left(n\sqrt{-\mu}\right). \tag{3.4.10}$$

Using (3.4.10) and (3.4.9), then finally we get formula (3.4.8).

Lemma 3.4.4 *When $\nu > -1$,*

$$\begin{aligned}&\int_0^\infty \frac{\sqrt{x^2-\mu}}{e^{\sqrt{x^2-\mu}}+1} x^\nu dx \\ &= (-\mu)^{\frac{\nu+2}{4}} 2^{\frac{\nu}{2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\nu+1}{2}\right) \sum_{n=1}^\infty (-1)^{n+1} \left(\sqrt{-\mu}\, n^{-\frac{\nu}{2}} K_{\frac{\nu}{2}}\left(n\sqrt{-\mu}\right) + (1+\nu) n^{-\frac{\nu}{2}-1} K_{\frac{\nu}{2}+1}\left(n\sqrt{-\mu}\right)\right).\end{aligned} \tag{3.4.11}$$

Proof Since

$$\frac{y}{1+e^y} = (-y)\left(\log\left(1+e^{-y}\right)\right)',$$

for $y > 0$ we obtain

$$\frac{y}{1+e^y} = \sum_{n=1}^\infty (-1)^{n+1} y e^{-ny}.$$

Let $y = \sqrt{x^2-\mu}$. Then

$$\begin{aligned}\int_0^\infty \frac{\sqrt{x^2-\mu}}{e^{\sqrt{x^2-\mu}}+1} x^\nu dx &= \int_{\sqrt{-\mu}}^\infty \frac{y}{e^y+1}(y^2+\mu)^{\frac{\nu-1}{2}} y dy \\ &= \sum_{n=1}^\infty (-1)^{n+1} \int_{\sqrt{-\mu}}^\infty e^{-ny}(y^2+\mu)^{\frac{\nu-1}{2}} y^2 dy.\end{aligned}$$

Changing the variable again, let $z = \frac{y}{\sqrt{-\mu}}$. We obtain the formula:

$$\int_0^\infty \frac{\sqrt{x^2-\mu}}{e^{\sqrt{x^2-\mu}}+1} x^\nu dx = (-\mu)^{\frac{\nu+1}{2}} \sum_{n=1}^\infty (-1)^{n+1} \int_1^\infty e^{-n\sqrt{-\mu}z} \left(z^2-1\right)^{\frac{\nu-1}{2}} z^2 dz.$$

Now applying Lemma 3.4.2 to this equation, one gets the integral formula (3.4.11).

Lemma 3.4.5 *For $\nu > -1$, one has*

$$\int_0^\infty h_\mu(x) x^\nu dx = (-\mu)^{\frac{\nu}{4}+1} 2^{\frac{\nu}{2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\nu+1}{2}\right) \sum_{n=1}^\infty (-1)^{n+1} \left(n^{-\frac{\nu}{2}} K_{\frac{\nu}{2}+2}\left(n\sqrt{-\mu}\right)\right). \tag{3.4.12}$$

And

$$\int_0^\infty h_\mu(x) x^\nu dx \sim \frac{1-2^{-\nu-1}}{\nu+1} \Gamma(\nu+3)\zeta(\nu+2)$$

as $\mu \to 0^-$.

Proof Using propositions 3.4.3 and 3.4.4, we have

$$\int_0^\infty h_\mu(x) x^\nu dx$$
$$= (-\mu)^{\frac{\nu}{4}+\frac{1}{2}} 2^{\frac{\nu}{2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right) \sum_{n=1}^\infty (-1)^{n+1} \left(\sqrt{-\mu} K_{\frac{\nu}{2}}\left(n\sqrt{-\mu}\right) n^{-\frac{\nu}{2}} + (2+\nu) K_{\frac{\nu}{2}+1}\left(n\sqrt{-\mu}\right) n^{-\frac{\nu}{2}-1}\right).$$

By applying (3.3.2) to this equation, we get the integral formula (3.4.12).

For the second statement, we use the asymptotics

$$K_\alpha(z) \sim \begin{cases} -\log(\frac{z}{2}) - \gamma & \alpha = 0, \\ \frac{\Gamma(\alpha)}{2}\left(\frac{2}{z}\right)^\alpha & \alpha > 0. \end{cases}$$

Thus for $\nu \geq 0$,

$$\int_0^\infty h_\mu(x) x^\nu dx \sim 2^{\nu+1} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{\nu}{2}+2\right) \sum_{n=1}^\infty (-1)^{n+1} n^{-\nu-2} \tag{3.4.13}$$

as $\mu \to 0^-$. By using

$$\sum_{n=1}^\infty (-1)^{n+1} n^{-\nu-2} = \left(1-2^{-\nu-1}\right)\zeta(\nu+2), \tag{3.4.14}$$

and the Legendre duplication formula for the gamma function

$$\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu+2}{2}\right) = 2^{-\nu}\sqrt{\pi}\,\Gamma(\nu+1)\,, \tag{3.4.15}$$

in (3.4.13), we get

$$\begin{aligned}\int_0^\infty h_\mu(x)x^\nu dx &\sim (\nu+2)\Gamma(\nu+1)\,(1-2^{-\nu-1})\zeta(\nu+2)\\ &= \frac{1-2^{-\nu-1}}{\nu+1}\Gamma(\nu+3)\zeta(\nu+2),\end{aligned}$$

which is the same as [3, Lemma 4.5].

We denote the $a-$th order spectral action coefficient of $h_\mu(\sqrt{x})$ by $\gamma_\mu(a)$; namely,

$$\gamma_\mu(a) = \int_0^\infty t^a \tilde{g}_\mu(t)dt = \int_0^\infty t^a e^{\mu t}\tilde{g}(t)dt. \tag{3.4.16}$$

It is clear that for a fixed chemical potential $\mu < 0$, the equation (3.4.16) is an entire function with respect to $a \in \mathbb{C}$. According to the Lemma 3.4.5, we can deduce that when the order $a < 0$, the coefficient of t^a in the heat expansion is

$$\gamma_\mu(a) = \frac{1}{\Gamma(-a)}\int_0^\infty h_\mu(x^{\frac{1}{2}})x^{-a-1}dx \tag{3.4.17}$$

$$= \frac{2}{\Gamma(-a)}\int_0^\infty h_\mu(x)x^{-2a-1}dx \tag{3.4.18}$$

$$= \frac{1}{\sqrt{\pi}}2^{-a+\frac{1}{2}}(-\mu)^{-\frac{a}{2}+\frac{3}{4}}\sum_{n=1}^\infty(-1)^{n+1}\left(n^{a+\frac{1}{2}}K_{-a+\frac{3}{2}}\left(n\sqrt{-\mu}\right)\right), \quad a<0. \tag{3.4.19}$$

Now we show that for any fixed chemical potential $\mu < 0$, the function (3.4.19) is an entire function with respect to $a \in \mathbb{C}$, so that the function (3.4.19) can give rise to spectral action coefficients for any order a.

Proposition 3.4.6 *For any fixed chemical potential $\mu < 0$, the function* (3.4.19) *is an entire function in $a \in \mathbb{C}$. Hence we have the formula*

$$\gamma_\mu(a) = \frac{1}{\sqrt{\pi}}2^{-a+\frac{1}{2}}(-\mu)^{-\frac{a}{2}+\frac{3}{4}}\sum_{n=1}^\infty(-1)^{n+1}\left(n^{a+\frac{1}{2}}K_{-a+\frac{3}{2}}\left(n\sqrt{-\mu}\right)\right) \tag{3.4.20}$$

for all a.

Proof We only need to show that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(n^{a+\frac{1}{2}}K_{-a+\frac{3}{2}}\left(n\sqrt{-\mu}\right)\right) \tag{3.4.21}$$

is an entire function in $a \in \mathbb{C}$. In fact, using the integral expression for the Bessel function $K_\nu(z)$ [5, 8.432], we have

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt, \qquad |\arg z| < \frac{\pi}{2} \text{ or } \mathrm{Re}(z) = 0 \text{ and } \nu = 0.$$

We see that for a fixed $z > 0$ the function $K_\nu(z)$ is an entire function with respect to $\nu \in \mathbb{C}$. Now we need to show that equation (3.4.21) is locally uniformly convergent. In fact, for $|\nu| \le R$,

$$|K_\nu(z)| \le \int_0^\infty e^{-z\cosh t}\cosh(Rt)dt = K_R(z).$$

For $|-a+\frac{3}{2}| \le R$, where $R < \infty$, we have

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}\left(n^{a+\frac{1}{2}}K_{-a+\frac{3}{2}}\left(n\sqrt{-\mu}\right)\right)\right| \le \sum_{n=1}^{\infty} n^{R+2}K_R\left(n\sqrt{-\mu}\right).$$

Since we have the asymptotic expansion

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z} \qquad z \to \infty,$$

it follows that the series

$$\sum_{n=1}^{\infty} n^{R+2}K_R\left(n\sqrt{-\mu}\right)$$

is convergent. Therefore the series (3.4.21) is locally uniformly convergent, and the function (3.4.20) is an entire function. Now according to (3.4.16), $\gamma_\mu(a)$ is an entire function, hence the function (3.4.20) gives the spectral action coefficients for all a.

Interestingly, we can express the spectral action coefficients $\gamma_\mu(a)$ in a more concise way via the Poisson summation formula.

Proposition 3.4.7 *For any fixed chemical potential $\mu < 0$, we have the expression for $\gamma_\mu(a)$:*

$$\gamma_\mu(a) = \frac{\Gamma(a)}{2}\sum_{n=-\infty}^{\infty}\frac{(2a-1)(2n+1)^2\pi^2+\mu}{((2n+1)^2\pi^2-\mu)^{a+1}}. \tag{3.4.22}$$

.

Proof Using Lemma 3.3.8, and using the Poisson summation formula, when $\nu \geq -\frac{1}{2}$, $a > 0$, we have

$$\sum_{n=1}^{\infty}(-1)^n |n|^{\nu+2} K_\nu(a|n|) = \frac{1}{2}\frac{\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}{-4\pi\Gamma\left(\frac{1}{2}\right)}\sum_{n=-\infty}^{\infty}\phi''_{\nu,a}(n), \tag{3.4.23}$$

where $\phi_{\nu,a}(x) = \frac{1}{((2x+1)^2\pi^2+a^2)^{\nu+\frac{1}{2}}}$. Since we have the equation

$$K_{-a+\frac{3}{2}}\left(n\sqrt{-\mu}\right) = K_{a-\frac{3}{2}}\left(n\sqrt{-\mu}\right),$$

applying the formula (3.4.23) to proposition 3.4.6 we then get the equatoin (3.4.22) when $a \geq \frac{3}{2}$. Now, in proposition 3.4.6 we saw that $\gamma_\mu(a)$ is an entire function. It follows that the function (3.4.22) has an analytic extension to the whole complex plane $\mathbb{C}$, and therefore equation (3.4.22) is true for all $a \in \mathbb{C}$.

Remark The second expression of $\gamma_\mu(a)$ is in the sense of analytic continuation. Thus for example we have,

$$\gamma_\mu\left(\frac{1}{2}\right) = \lim_{a\to 1/2^+}\frac{\Gamma(a)}{2}\sum_{n=-\infty}^{\infty}\frac{(2a-1)(2n+1)^2\pi^2+\mu}{((2n+1)^2\pi^2-\mu)^{a+1}},$$

and

$$\lim_{\mu\to 0^-}\gamma_\mu\left(\frac{1}{2}\right) = \lim_{a\to 1/2^+}\frac{\Gamma(a)}{2}\sum_{n=-\infty}^{\infty}\frac{(2a-1)}{((2n+1)^2\pi^2)^a} = \frac{1}{2\sqrt{\pi}}.$$

Next we prove that when the chemical potential $\mu \to 0^-$, we can get the same coefficients given in [3]. We follow the same notation as in [3], and denote

$$\gamma(a) = \frac{1-2^{-2a}}{a}\pi^{-a}\xi(2a),$$

where $\xi(z)$ is the Riemann ξ−function.

Theorem 3.4.8 *For all $a \in \{\frac{n}{2} : n \in \mathbb{Z}\}$, when the chemical potential μ approaches to 0, we have*

$$\lim_{\mu\to 0^-}\gamma_\mu(a) = \gamma(a).$$

Proof Since the spectral action coefficients are given by

$$\gamma_\mu(a) = \int_0^\infty \tilde{g}_\mu(t)t^a dt = \int_0^\infty e^{\mu t}\tilde{g}(t)t^a dt,$$

where $g(t)$ is given by equation (3.4.2), we obtain

$$\lim_{\mu\to 0^-} \gamma_\mu(a) = \int_0^\infty \tilde{g}(t)t^a dt = \gamma(a).$$

Summarizing the above computations, we get the following proposition:

Proposition 3.4.9 *(1) For a given chemical potential $\mu < 0$, the coefficient of t^a in the heat expansion is given by $\gamma_\mu(a)$, where*

$$\gamma_\mu(a) = \int_0^\infty t^a \tilde{g}_\mu(t)dt,$$

and we have the following two explicit expressions of $\gamma_\mu(a)$:

$$\gamma_\mu(a) = \frac{1}{\sqrt{\pi}} 2^{-a+\frac{1}{2}}(-\mu)^{-\frac{a}{2}+\frac{3}{4}} \sum_{n=1}^{\infty}(-1)^{n+1}\left(n^{a+\frac{1}{2}}K_{-a+\frac{3}{2}}\left(n\sqrt{-\mu}\right)\right),$$

and

$$\gamma_\mu(a) = \frac{\Gamma(a)}{2}\sum_{n=-\infty}^{\infty}\frac{(2a-1)(2n+1)^2\pi^2+\mu}{((2n+1)^2\pi^2-\mu)^{a+1}}.$$

Moveover, $\gamma_\mu(a)$ is an entire function in $a \in \mathbb{C}$.
(2) For any given order a, when the chemical potential μ approaches to 0, $\gamma_\mu(a)$ converges to $\gamma(a)$, namely,

$$\lim_{\mu\to 0^-}\gamma_\mu(a) = \gamma(a), \quad a \in \left\{\frac{n}{2} : n \in \mathbb{Z}\right\},$$

where $\gamma(a) = \frac{1-2^{-2a}}{a}\pi^{-a}\xi(2a)$.

3.4.2 The average energy in the Fermionic second quantization

Now we shall consider the average energy when the one-particle Hilbert space is $\mathcal{H} = \mathbb{C}$. We denote by $Z = \mathrm{Tr}(e^{-\beta d\Gamma D_\mu})$ the partition function. Then

$$Z = 1 + e^{-\beta\sqrt{x^2-\mu}}.$$

According to (3.2.1),

$$\langle d\Gamma D_\mu \rangle = -\frac{\partial}{\partial \beta}(\log Z)\Big|_{\beta=1} = \frac{\sqrt{x^2-\mu}}{1+e^{\sqrt{x^2-\mu}}}.$$

Interestingly, this is just the first part on the right-hand side of our von Neumann entropy formula (3.4.1).

We denote this function by $u_\mu(x)$,

$$u_\mu(x) = \frac{\sqrt{x^2-\mu}}{1+e^{\sqrt{x^2-\mu}}}.$$

Now let us consider the function $u_0(x)$ first. Since we have the expansion

$$u_0(x) = \frac{x}{1+e^x} = \sum_{n=1}^{\infty}(-1)^{n+1} x e^{-nx}, \tag{3.4.24}$$

and (cf. e.g. [7])

$$\sqrt{x}e^{-n\sqrt{x}} = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-5/2}\left(\frac{n^2}{4}-\frac{t}{2}\right)e^{-\frac{n^2}{4t}}e^{-tx}dt,$$

we obtain

$$u_\mu(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-5/2}\left(\frac{n^2}{4}-\frac{t}{2}\right)e^{-\frac{n^2}{4t}}e^{-t(x^2-\mu)}dt.$$

When $\mu < 0$, using the Fubini theorem, we can exchange the infinite sum and the integral, so that

$$u_\mu(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-5/2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n^2}{4}-\frac{t}{2}\right)e^{-\frac{n^2}{4t}}e^{-t(x^2-\mu)}dt, \qquad x \geq 0.$$

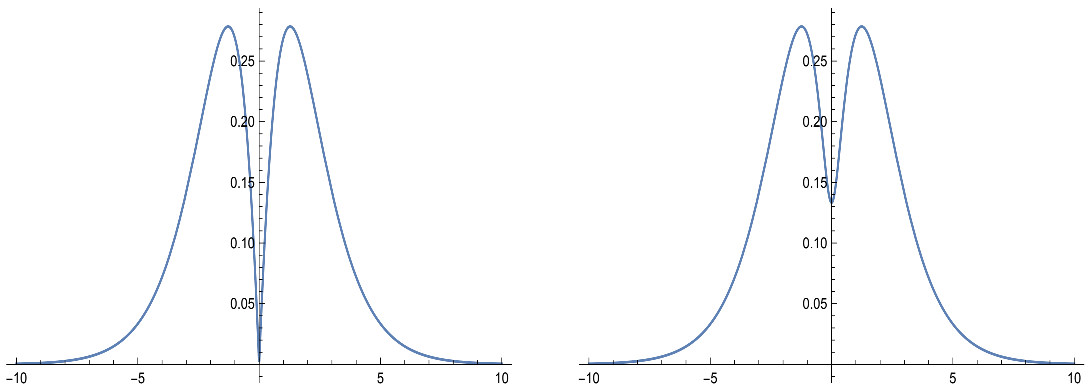
Let

$$r_\mu(t) = \frac{1}{\sqrt{\pi}}t^{-5/2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n^2}{4}-\frac{t}{2}\right)e^{-\frac{n^2}{4t}}e^{\mu t}.$$

Then we obtain the following expression for the Laplace transform of r_μ:

$$u_\mu(\sqrt{x}) = \int_0^\infty r_\mu(t)e^{-tx}dt, \qquad \mu < 0, \quad x \geq 0.$$

Therefore, the function $u_\mu(\sqrt{x})$ is a well-defined spectral action function. Notice that here we can not take the chemical potential $\mu = 0$, since the function $u_0(x)$ is singular at $x = 0$.



(a) The image of $u_0(x)$ (b) The image of $u_{-0.1}(x)$

Figure 3.1: The image of $u_0(x)$ and $u_{-0.1}(x)$

When $a < 0$, the spectral action coefficient of t^a is given by

$$\omega_\mu(a) = \frac{1}{\Gamma(-a)} \int_0^\infty u_\mu(\sqrt{x})x^{-a-1}dx = \frac{2}{\Gamma(-a)} \int_0^\infty u_\mu(x)x^{-2a-1}dx.$$

Using Lemma 3.4.4, we can express $\omega_\mu(a)$ as follows:

Proposition 3.4.10 *For any fixed chemical potential $\mu < 0$, the function $\omega_\mu(a)$ is given by*

$$\omega_\mu(a) = \frac{(2\sqrt{-\mu})^{-a+\frac{1}{2}}}{\sqrt{\pi}} \sum_{n=1}^{\infty}(-1)^n \left(2a\, n^{a-\frac{1}{2}} K_{-a+\frac{1}{2}}\left(n\sqrt{-\mu}\right) - n^{a+\frac{1}{2}} \sqrt{-\mu} K_{-a-\frac{1}{2}}\left(n\sqrt{-\mu}\right)\right), \tag{3.4.25}$$

and moreover, it can be extended to an entire function in a.

Proof Taking any $\mu < 0$, and using the same argument as in the proof of proposition 3.4.6, we can show that $\omega_\mu(a)$ can be extended to an entire function as well.

Now we want to find a more explicit expression for $\omega_\mu(a)$ using the Poisson summation formula.

Proposition 3.4.11 *For any fixed chemical potential $\mu < 0$, we can express $\omega_\mu(a)$ as*

$$\omega_\mu(a) = \Gamma(a+1) \sum_{n=-\infty}^{\infty} \frac{(2n+1)^2\pi^2}{((2n+1)^2\pi^2 - \mu)^{a+1}} - \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}}\Gamma\left(a - \frac{1}{2}\right). \tag{3.4.26}$$

Proof Using (3.3.7) and applying Poisson's summation formula, we obtain, for any $\nu > 0$ and $z > 0$,

$$\sum_{n=1}^{\infty}(-1)^n n^\nu K_\nu(zn) = \frac{\sqrt{\pi}}{2}\Gamma\left(\nu+\frac{1}{2}\right)(2z)^\nu \sum_{n=-\infty}^{\infty}\frac{1}{((2\pi n+\pi)^2+z^2)^{\nu+\frac{1}{2}}} - \frac{\Gamma(\nu)}{4}\left(\frac{2}{z}\right)^\nu. \quad (3.4.27)$$

When $a > \frac{1}{2}$, we can combine the above equation with (3.4.25), and after simplification, we can deduce the equation (3.4.26). Now since $\omega_\mu(a)$ is an entire function, we conclude that (3.4.26) is valid in the whole complex plane.

Now we want to see how the spectral action coefficients $\omega_\mu(a)$ behave when $\mu \to 0^-$.

Proposition 3.4.12 *When the order $a \le 0$, we have the limit*

$$\lim_{\mu\to 0^-}\omega_\mu(a) = \frac{2-2^{1-2a}}{2a-1}\pi^{-a}\xi(2a).$$

When $a = \frac{1}{2}$, we have the asymptotic formula

$$\omega_\mu(a) \sim \frac{1-\log\pi+\gamma}{2\sqrt{\pi}} + \frac{\log\sqrt{-\mu}}{2\sqrt{\pi}}, \qquad \mu \to 0^-.$$

When $a > \frac{1}{2}$, we have the asymptotic approximation

$$\omega_\mu(a) \sim \frac{2-2^{1-2a}}{2a-1}\pi^{-a}\xi(2a) - \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}}\Gamma\left(a-\frac{1}{2}\right), \qquad \mu \to 0^-. \quad (3.4.28)$$

Proof For $a < 0$, we have

$$\lim_{\mu\to 0^-}\omega_\mu(a) = \lim_{\mu\to 0^-}\frac{2}{\Gamma(-a)}\int_0^\infty u_\mu(x)x^{-2a-1}dx = \frac{2}{\Gamma(-a)}\int_0^\infty \frac{x^{-2a}}{1+e^x}dx.$$

Applying (3.4.24), we get

$$\int_0^\infty \frac{x^{-2a}}{1+e^x}dx = \frac{(2-2^{1+2a})\Gamma(1-2a)\zeta(1-2a)}{\Gamma(-a)} = \frac{2-2^{1-2a}}{2a-1}\pi^{-a}\xi(2a).$$

When $a = 0$, since

$$\omega_\mu(0) = u_\mu(0),$$

we deduce that

$$\lim_{\mu\to 0^-}\omega_\mu(0) = \lim_{\mu\to 0^-}u_\mu(0) = 0.$$

When $a = \frac{1}{2}$, using Lemma 3.3.10, we see that

$$\omega_\mu(a) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^{n+1} \left(n\sqrt{-\mu}\, K_1(n\sqrt{-\mu}) - K_0(n\sqrt{-\mu}) \right)$$
$$\sim \frac{1 - \log \pi + \gamma}{2\sqrt{\pi}} + \frac{\log \sqrt{-\mu}}{2\sqrt{\pi}}, \qquad \mu \to 0^-.$$

When $a > \frac{1}{2}$, using proposition 3.4.11, we have the limit

$$\begin{aligned}\lim_{\mu \to 0^-} \left(\omega_\mu(a) + \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}} \Gamma\left(a - \frac{1}{2}\right) \right) &= \Gamma(a+1) \sum_{n=-\infty}^{\infty} \frac{1}{((2n+1)^2\pi^2)^a} \\ &= \Gamma(a+1) \left(2 - 2^{1-2a}\right) \pi^{-2a} \zeta(2a) \\ &= \frac{2 - 2^{1-2a}}{2a - 1} \pi^{-a} \xi(2a).\end{aligned}$$

From which (3.4.28) follows.

In particular, using proposition 3.4.12, we get the expansion of $u_0(\sqrt{x})$ as follows:

$$u_0(\sqrt{x}) = \frac{\sqrt{x}}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{2 - 2^{1-2n}}{(2n-1)n!} \pi^{-n} \xi(2n) x^n.$$

3.4.3 The negative free energy in the Fermionic Fock space

Since the free energy is the difference between average energy and von Neumann entropy, in the case of Fermionic second quantization it is natural to define the spectral action function with respect to the negative free energy to be

$$v_\mu(x) = h_\mu(x) - u_\mu(x) = \log\left(1 + e^{-\sqrt{x^2-\mu}}\right). \tag{3.4.29}$$

Proposition 3.4.13 *When chemical potential $\mu < 0$, we have the following equation:*

$$v_\mu(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \sum_{n=1}^{\infty} (-1)^{n+1}\, t^{-\frac{3}{2}} e^{-\frac{n^2}{4t} + t\mu} e^{-tx^2} dt.$$

Proof Since

$$\log(1 + e^{-x}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-nx}}{n},$$

and

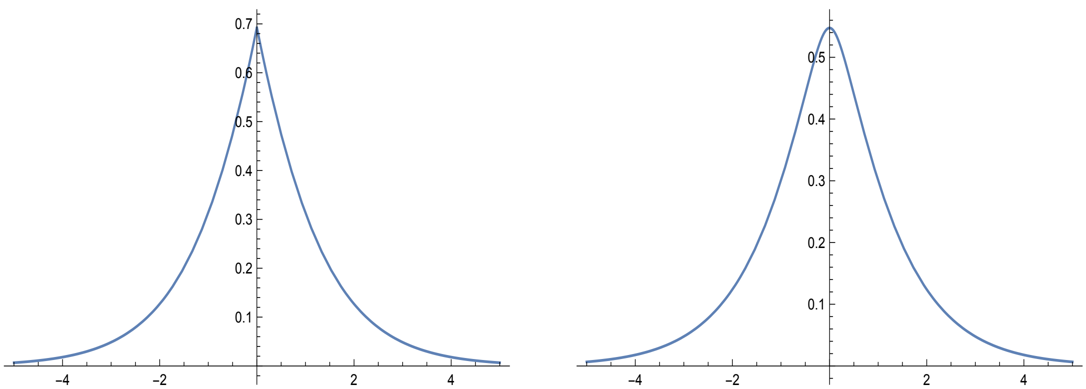
$$e^{-n\sqrt{x}} = \frac{n}{2\sqrt{\pi}} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{n^2}{4t}} e^{-tx} dt,$$

we have

$$v_\mu(x) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^\infty \int_0^\infty (-1)^{n+1} t^{-\frac{3}{2}} e^{-\frac{n^2}{4t}+t\mu} e^{-tx^2} dt.$$

Now since $\mu < 0$, we can apply the Fubini theorem to get the equation (3.4.29).

Therefore the function $v_\mu(\sqrt{x})$ is a well-defined spectral action function when $\mu < 0$, while when $\mu = 0$, $v_0(\sqrt{x})$ is not a well-defined spectral action function since it is singular at $x = 0$.



(a) The image of $v_0(x)$ (b) The image of $v_{-0.1}(x)$

Figure 3.2: The image of $v_0(x)$ and $v_{-0.1}(x)$

We denote by $\lambda_\mu(a)$ the spectral action coefficient of $v_\mu(x)$ of order a. For $a < 0$ we have

$$\lambda_\mu(a) = \frac{1}{\Gamma(-a)} \int_0^\infty v_\mu(\sqrt{x}) x^{-a-1} dx = \frac{2}{\Gamma(-a)} \int_0^\infty v_\mu(x) x^{-2a-1} dx. \tag{3.4.30}$$

Using an argument similar to the subsection 3.4.2, we obtain the following proposition. We omit the proof which is similar to the proof of proposition 3.4.10, 3.4.11,3.4.12.

Proposition 3.4.14 *For a given chemical potential $\mu < 0$, we can get a spectral action from the negative free energy of the Fermionic second quantization, and this spectral action function is given by the function $v_\mu(\sqrt{x})$, where*

$$v_\mu(x) = \log\left(1 + e^{-\sqrt{x^2-\mu}}\right),$$

The spectral action coefficients of $v_\mu(\sqrt{x})$ are given by the following two functions:

$$\lambda_\mu(a) = \frac{2^{-a+\frac{1}{2}}}{\sqrt{\pi}}(-\mu)^{-\frac{a}{2}+\frac{1}{4}}\sum_{n=1}^{\infty}(-1)^{n+1}n^{a-\frac{1}{2}}K_{-a+\frac{1}{2}}\left(n\sqrt{-\mu}\right),$$

and

$$\lambda_\mu(a) = -\frac{\Gamma(a)}{2}\sum_{n=-\infty}^{\infty}\frac{1}{((2n+1)^2\pi^2-\mu)^a} + \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}}\Gamma\left(a-\frac{1}{2}\right).$$

Moreover, for any fixed chemical potential $\mu < 0$, $\lambda_\mu(a)$ is an entire function. When the order $a < 0$, we have the limit

$$\lim_{\mu\to 0^-}\lambda_\mu(a) = \frac{2^{-2a}-1}{(2a-1)a}\pi^{-a}\xi(2a).$$

When $a = 0$,

$$\lim_{\mu\to 0^-}\lambda_\mu(0) = \log 2.$$

When $a = \frac{1}{2}$, we have the asymptotic expansion:

$$\lambda_\mu\left(\frac{1}{2}\right) \sim \frac{\log\pi - \gamma}{2\sqrt{\pi}} - \frac{\log\sqrt{-\mu}}{2\sqrt{\pi}}, \qquad \mu \to 0^-.$$

When $a > \frac{1}{2}$, we have the asymptotic approximation:

$$\lambda_\mu(a) \sim \frac{2^{-2a}-1}{(2a-1)a}\pi^{-a}\xi(2a) + \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}}\Gamma\left(a-\frac{1}{2}\right), \qquad \mu \to 0^-.$$

Also, we can expand $v_0(\sqrt{x})$ as

$$v_0(\sqrt{x}) = \log 2 - \frac{\sqrt{x}}{2} + \sum_{n=1}^{\infty}(-1)^n\frac{2^{-2n}-1}{n(2n-1)n!}\pi^{-n}\xi(2n)x^n.$$

3.5 The case of Bosonic Fock space

As in the case of Fermionic Fock space, we can also define the spectral actions in the case of Bosonic Fock space. Let $\mathcal{H} = \mathbb{C}$ be the 1-particle Hilbert space. Then the Bosonic Fock space is given by $\mathcal{F}_+(\mathcal{H}) = \bigoplus_{n=0}^{\infty}\mathcal{H}^n$. Let $D : \mathcal{H} \to \mathcal{H}$ be the Dirac operator on $\mathcal{H}$. We denote by $H = D^2$ the corresponding Hamiltonian operator. If the spectrum of D is

$\sigma(D) = \{x\}$, x$\in \mathbb{R}$, then obviously $\sigma(H) = \{x^2\}$. Let $\mu < 0$ be the chemical potential, and let $D_\mu = \sqrt{H - \mu \mathbb{1}}$. The spectrum of $d\Gamma D_\mu$ is $\sigma(d\Gamma D_\mu) = \{n\sqrt{x^2 - \mu} : n = 0, 1, 2, 3, \cdots\}$. Since the chemical potential $\mu < 0$, we can define a density matrix

$$\rho = \frac{e^{-d\Gamma D_\mu}}{\mathrm{Tr}\left(e^{-d\Gamma D_\mu}\right)}.$$

3.5.1 The von Neumann entropy in the Bosonic second quantization

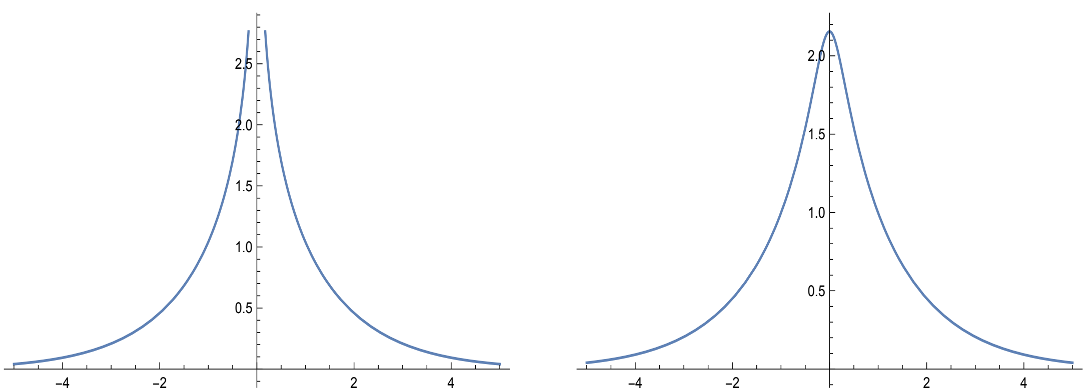
We define a function $k_\mu(x)$ by

$$k_\mu(x) := \mathcal{S}(\rho) = -\mathrm{Tr}\left(\rho \log \rho\right) = -\frac{\sqrt{x^2 - \mu}}{1 - e^{\sqrt{x^2-\mu}}} - \log\left(1 - e^{-\sqrt{x^2-\mu}}\right).$$

In the Bosonic Fock space case, we cannot take the chemical potential $\mu = 0$, since the function $k_0(x)$ is singular at $x = 0$:

Lemma 3.5.1 *The function $k_0(x)$ is an even positive function of the variable $x \in \mathbb{R}\backslash\{0\}$, and its derivative is*

$$k_0'(x) = -\frac{x}{4\sinh^2(\frac{x}{2})}.$$



(a) The image of $k_0(x)$ (b) The image of $k_{-0.1}(x)$

Figure 3.3: The image of $k_0(x)$ and $k_{-0.1}(x)$

Compare this to the function $h_0(x)$ in section 3.4.1, or the function $h(x)$ in [3], where

$$h_0'(x) = -\frac{x}{4\cosh^2\left(\frac{x}{2}\right)}.$$

Similar to $h_0(x)$, we shall prove that the function $k_0(\sqrt{x})$ is also given by the Laplace transform when $x \neq 0$. To prove this, we need the following lemma(compare this with Lemma 4.2 in [3]):

Lemma 3.5.2 *For $x > 0$,*

$$\sum_{\mathbb{Z}} \frac{(2\pi n)^2 - x}{((2\pi n)^2 + x)^2} = -\frac{1}{4\sinh^2(\frac{\sqrt{x}}{2})}.$$

Proof We use the Eisenstein series [3]

$$\sum_{\mathbb{Z}} \frac{1}{(\pi n + x)^2} = \frac{1}{\sin^2 x},$$

in conjunction with

$$\sinh x = -i\sin(ix).$$

Thus

$$\begin{aligned}\frac{1}{4\sinh^2(\frac{\sqrt{x}}{2})} &= -\frac{1}{4\sin^2(i\frac{\sqrt{x}}{2})}\\ &= -\sum_{\mathbb{Z}} \frac{1}{4(\pi n + i\frac{\sqrt{x}}{2})^2}\\ &= -\sum_{\mathbb{Z}} \frac{(2\pi n)^2 - x}{((2\pi n)^2 + x)^2}.\end{aligned}$$

Now since one has the equation

$$\int_0^\infty \left(2(2\pi n)^2 t - 1\right) e^{-(2\pi n)^2 t - tx} dt = \frac{(2\pi n)^2 - x}{\left((2\pi n)^2 + x\right)^2},$$

by the Fubini theorem we have the formula

$$-\frac{1}{4\sinh^2\frac{\sqrt{x}}{2}} = \int_0^\infty f(t)e^{-tx}dt, \quad f(t) := \sum_{n\in\mathbb{Z}} \left(2(2\pi n)^2 t - 1\right) e^{-(2\pi n)^2 t},$$

when $x > 0$.

Now we have the following lemma:

Lemma 3.5.3 *Let*

$$f(t) = \sum_{n\in\mathbb{Z}} \left(2(2\pi n)^2 t - 1\right) e^{-(2\pi n)^2 t}.$$

The function $f(t)$ is rapidly decreasing as $t \to 0^+$.

Proof Consider the theta function

$$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}.$$

Let

$$g(t) = -2t\,\theta'(t) - \theta(t).$$

We have $f(t) = g(4\pi t)$. Thus it suffices to show that $g(t)$ is rapidly decreasing as $t \to 0^+$. Now, using the Jacobi inversion formula,

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right),$$

we have

$$g(t) = -2t\left(-\frac{1}{2}t^{-3/2}\theta\left(\frac{1}{t}\right) - \frac{1}{\sqrt{t}}\theta'\left(\frac{1}{t}\right)t^{-2}\right) - \theta(t) \tag{3.5.1}$$

$$= t^{-1/2}\theta\left(\frac{1}{t}\right) + 2t^{-3/2}\theta'\left(\frac{1}{t}\right) - \theta(t) \tag{3.5.2}$$

$$= 2t^{-3/2}\theta'\left(\frac{1}{t}\right). \tag{3.5.3}$$

Since as $t \to 0^+$, $\theta'\left(\frac{1}{t}\right)$ is rapidly decreasing, $g(t)$ is rapidly decreasing, and also the function $f(t)$ is rapidly decreasing as $t \to 0^+$.

Thus we have the following proposition:

Proposition 3.5.4 *When $x > 0$, one has*

$$k_0(x) = \int_0^\infty e^{-tx^2}\tilde{f}(t)dt, \tag{3.5.4}$$

where

$$k_0(x) = -\frac{|x|}{1 - e^{|x|}} - \log\left(1 - e^{-|x|}\right),$$

and

$$\tilde{f}(t) = \frac{f(t)}{2t} = \frac{1}{2t}\sum_{n \in \mathbb{Z}}\left(2(2\pi n)^2 t - 1\right)e^{-(2\pi n)^2 t}.$$

Proof According to Lemma 3.5.3, $\tilde{f}(t)$ is rapidly decreasing as $t \to 0^+$. Thus when $x > 0$, the integral on the right hand side is well-defined. We denote the integral on the right hand side of (3.5.4) by $\tilde{k}(x)$. We have

$$\partial_x \tilde{k}(x) = -2x \int_0^\infty e^{-tx^2} t \tilde{f}(t) dt = \frac{x}{4\sinh^2 \frac{x}{2}} = -\partial_x k_0(x), \tag{3.5.5}$$

and since both $k_0(x)$ and $\tilde{k}(x)$ approach to 0 when $x \to \infty$, thus $k_0(x) = \tilde{k}(x)$.

Thus immediately we have

Proposition 3.5.5 *When the chemical potential $\mu < 0$, for all $x \in \mathbb{R}$,*

$$k_\mu(x) = \int_0^\infty e^{-tx^2} \tilde{f}_\mu(t) dt,$$

where

$$\tilde{f}_\mu(t) = e^{\mu t} \tilde{f}(t) = \frac{e^{\mu t}}{2t} \sum_{n \in \mathbb{Z}} \left(2(2\pi n)^2 t - 1\right) e^{-(2\pi n)^2 t}.$$

For the Bosonic Fock space, we can get similar results as in the Fermionic Fock space case. The main difference between them is that we get alternating sum from Fermionic second quantization, while we get just a sum in the Bosonic second quantization.

Lemma 3.5.6 *When $\nu > -1$, one has the integral formula*

$$\int_0^\infty k_\mu(x) x^\nu dx = (-\mu)^{\frac{\nu}{4}+1} 2^{\frac{\nu}{2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\nu+1}{2}\right) \sum_{n=1}^\infty n^{-\frac{\nu}{2}} K_{\frac{\nu}{2}+2}\left(n\sqrt{-\mu}\right),$$

and

$$\int_0^\infty k_\mu(x) x^\nu dx \sim \Gamma(\nu+1)(\nu+2)\zeta(\nu+2)$$

as $\mu \to 0^-$.

Proof The proof of this proposition is the same as the proof of the Lemma 3.4.5.

We denote by $\chi_\mu(a)$ the $a-$th order spectral action coefficient of $k_\mu(\sqrt{x})$, that is,

$$\chi_\mu(a) = \int_0^\infty t^a \tilde{f}_\mu(t) dt = \int_0^\infty t^a e^{\mu t} f(t) dt.$$

Similar to the proposition 3.4.6 and proposition 3.4.7, we have the following proposition:

Proposition 3.5.7 *For a fixed chemical potential $\mu < 0$, we can express the $a-$th order spectral action coefficient of $k_\mu(\sqrt{x})$ as:*

$$\chi_\mu(a) = \frac{1}{\sqrt{\pi}} 2^{-a+\frac{1}{2}} (-\mu)^{-\frac{a}{2}+\frac{3}{4}} \sum_{n=1}^{\infty} n^{a+\frac{1}{2}} K_{-a+\frac{3}{2}}\left(n\sqrt{-\mu}\right), \tag{3.5.6}$$

and

$$\chi_\mu(a) = -\frac{\Gamma(a)}{2} \sum_{n=-\infty}^{\infty} \frac{(2a-1)(2n)^2\pi^2 + \mu}{((2n)^2\pi^2 - \mu)^{a+1}}. \tag{3.5.7}$$

Moreover, the expressions (3.5.6) *and* (3.5.7) *both are entire functions of $a \in \mathbb{C}$.*

Proof The proof is similar to the proof of proposition 3.4.6 and proposition 3.4.7.

Now let us investigate the behaviour of $\chi_\mu(a)$ when $\mu \to 0^-$.

Proposition 3.5.8 *When $a < 0$, we have the limit*

$$\lim_{\mu\to 0^-} \chi_\mu(a) = -\frac{2^{-2a}}{a} \pi^{-a} \xi(2a).$$

When $a = 0$, we have the asymptotic expansion

$$\chi_\mu(0) \sim 1 - \log\sqrt{-\mu}, \qquad \mu \to 0^-. \tag{3.5.8}$$

When $a \geq \frac{1}{2}$, we have the asymptotic expansion

$$\chi_\mu(a) \sim -\frac{2^{-2a}}{a} \pi^{-a} \xi(2a) + \frac{\Gamma(a)}{2} (-\mu)^{-a}, \qquad \mu \to 0^-.$$

Moreover, we have the expansion of $k_0(x)$ as follows

$$k_0(\sqrt{x}) = 1 - \frac{\log x}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\xi(2n) x^n}{2^{2n}\pi^n n\, n!}. \tag{3.5.9}$$

Proof When $a < 0$, using the equation (3.5.6) and Lemma 3.3.3, we have

$$\begin{aligned}
\chi_\mu(a) &\sim \frac{2^{-2a+1}}{\sqrt{\pi}} \Gamma\left(-a + \frac{3}{2}\right) \zeta(-2a+1) \\
&= \left(-a + \frac{1}{2}\right) \frac{2^{-2a+1}}{\sqrt{\pi}} \Gamma\left(-a + \frac{1}{2}\right) \zeta(-2a+1) \\
&= -\frac{2^{-2a}}{a} \pi^{-a} \xi(-2a+1).
\end{aligned}$$

Now since $\xi(-2a+1) = \xi(2a)$, we have that

$$\lim_{\mu\to 0^-} \chi_\mu(a) = -\frac{2^{-2a}}{a}\pi^{-a}\xi(2a).$$

When $a = \frac{1}{2}$, using Lemma 3.3.10, we have

$$\chi_\mu\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}\sqrt{-\mu}\sum_{n=1}^{\infty} n\,K_1(n\sqrt{-\mu}) \sim -\frac{1}{2\sqrt{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{-\mu}}.$$

When $a > \frac{1}{2}$, according to proposition 3.5.7,

$$\chi_\mu(a) - \frac{\Gamma(a)}{2}(-\mu)^{-a} = -\frac{\Gamma(a)}{2}\sum_{n\neq 0}\frac{(2a-1)(2n)^2\pi^2+\mu}{((2n)^2\pi^2-\mu)^{a+1}}.$$

Taking the limit of both sides,

$$\lim_{\mu\to 0^-}\left(\chi_\mu(a) - \frac{\Gamma(a)}{2}(-\mu)^{-a}\right) = -\frac{\Gamma(a)}{2}\sum_{n\neq 0}\frac{(2a-1)}{((2n)^2\pi^2)^a} = -\frac{2^{-2a}}{a}\pi^{-a}\xi(2a),$$

we get the asymptotic expansion

$$\chi_\mu(a) \sim -\frac{2^{-2a}}{a}\pi^{-a}\xi(2a) + \frac{\Gamma(a)}{2}(-\mu)^{-a}, \qquad \mu\to 0^-.$$

Since when $a = n$ is a positive integer,

$$\chi_\mu(n) = (-1)^n\left(k_\mu(\sqrt{x})\right)^{(n)}\Big|_{x=0},$$

we have

$$\left(k_\mu(\sqrt{x})\right)^{(n)}\Big|_{x=0} \sim (-1)^n\left(-\frac{2^{-2n}}{n}\pi^{-n}\xi(2n) + \frac{(n-1)!}{2}(-\mu)^{-n}\right), \qquad \mu\to 0^-.$$

Thus we can expand $k_0(\sqrt{x})$ as

$$k_0(\sqrt{x}) = c - \frac{\log x}{2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\xi(2n)}{2^{2n}\pi^n n}\frac{x^n}{n!},$$

where c is a constant. To find the value of c, we can simply take $\lim_{x\to 0}\left(k_0(\sqrt{x}) + \frac{\log x}{2}\right)$, that is,

$$c = \lim_{x\to 0}\left(k_0(\sqrt{x}) + \frac{\log x}{2}\right) = \lim_{x\to 0}\left(-\frac{\sqrt{x}}{1-e^{\sqrt{x}}} - \log\left(1-e^{-\sqrt{x}}\right) + \frac{\log x}{2}\right) = 1.$$

Then we get the expansion of $k_0(\sqrt{x})$:

$$k_0(\sqrt{x}) = 1 - \frac{\log x}{2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\xi(2n)x^n}{2^{2n}\pi^n n\, n!}, \tag{3.5.10}$$

where $\xi(s)$ is the Riemann's ξ function.

Since when the order $a = 0$,

$$\chi_\mu(0) = k_\mu(0) = k_0(\sqrt{-\mu}),$$

according to (3.5.9) we can deduce the asymptotic approximation of $\chi_\mu(0)$:

$$\chi_\mu(0) \sim 1 - \log\sqrt{-\mu}, \qquad \mu \to 0^-. \tag{3.5.11}$$

3.5.2 The average energy in the Bosonic second quantization

As in the case of Fermionic second quantization, if we take the one-particle Hilbert space to be $\mathcal{H} = \mathbb{C}$, we get the partition function

$$Z = \mathrm{Tr}\left(e^{-\beta d\Gamma D_\mu}\right) = \frac{1}{1-e^{-\beta\sqrt{x^2-\mu}}}.$$

Then by the formula (3.2.1),

$$\langle d\Gamma D_\mu\rangle = -\frac{\partial}{\partial\beta}(\log Z)\Big|_{\beta=1} = -\frac{\sqrt{x^2-\mu}}{1-e^{\sqrt{x^2-\mu}}}.$$

We define $p_\mu(x)$ by

$$p_\mu(x) = -\frac{\sqrt{x^2-\mu}}{1-e^{\sqrt{x^2-\mu}}}.$$

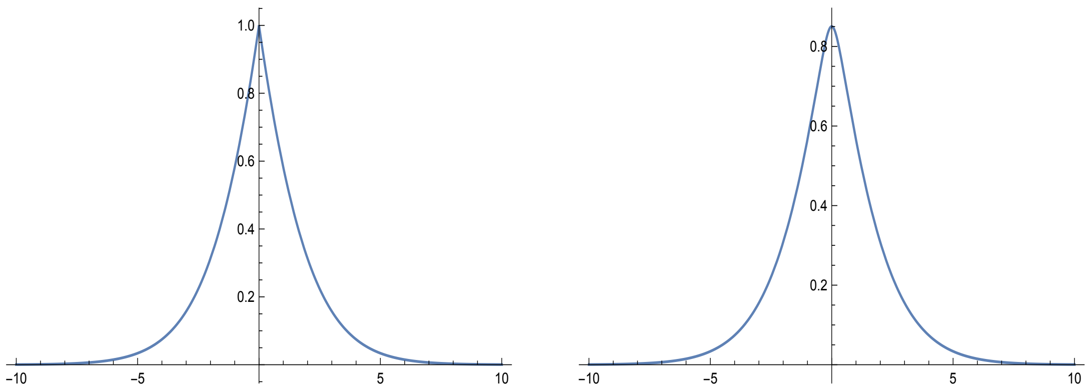
As with the discussion in section 3.4.2, with the chemical potential $\mu < 0$, the function $p_\mu(x)$ is given by the following Laplace transform:

$$p_\mu(\sqrt{x}) = \int_0^\infty s_\mu(t)e^{-tx}dt, \qquad \mu < 0, \quad x \geq 0, \tag{3.5.12}$$

where

$$s_\mu(t) = \frac{1}{\sqrt{\pi}} t^{-5/2} \sum_{n=1}^{\infty} \left(\frac{n^2}{4} - \frac{t}{2} \right) e^{-\frac{n^2}{4t}} e^{\mu t}.$$

Unlike the Fermionic second quantization, here we cannot take $\mu = 0$, as the integral on the right-hand side of the formula (3.5.12) does not converge. This is consistent with the fact that $p_0(x)$ is singular at $x = 0$.



(a) The image of $p_0(x)$ (b) The image of $p_{-0.1}(x)$

Figure 3.4: The image of $p_0(x)$ and $p_{-0.1}(x)$

When the order $a < 0$, we denote by $\alpha_\mu(a)$ the spectral action coefficient of the spectral action function $p_\mu(\sqrt{x})$, namely,

$$\alpha_\mu(a) = \frac{1}{\Gamma(-a)} \int_0^\infty p_\mu(\sqrt{x}) x^{-a-1} dx = \frac{2}{\Gamma(-a)} \int_0^\infty p_\mu(x) x^{-2a-1} dx.$$

Using the same argument as in section 3.4.2, we have

Proposition 3.5.9 *For any fixed chemical potential $\mu < 0$,*

$$\alpha_\mu(a) = \frac{2^{-a+\frac{1}{2}}}{\sqrt{\pi}} (-\mu)^{-\frac{a}{2}+\frac{1}{4}} \sum_{n=1}^{\infty} \left(n^{a+\frac{1}{2}} \sqrt{-\mu}\, K_{-a-\frac{1}{2}}\left(n\sqrt{-\mu}\right) - 2\, a\, n^{a-\frac{1}{2}} K_{-a+\frac{1}{2}}\left(n\sqrt{-\mu}\right) \right), \quad (3.5.13)$$

and it can be extended to a holomorphic function on $\mathbb{C}$. Thus this formula gives the spectral action coefficients of all orders. Moreover, we have yet another expression for $\alpha_\mu(a)$:

$$\alpha_\mu(a) = -\Gamma(a+1) \sum_{n=-\infty}^{\infty} \frac{(2n)^2\pi^2}{((2n)^2\pi^2 - \mu)^{a+1}} + \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}} \Gamma\left(a - \frac{1}{2}\right), \quad (3.5.14)$$

and it can also be extended to an entire function for any fixed chemical potential $\mu < 0$.

When the chemical potential $\mu \to 0^-$, we have the following proposition.

Proposition 3.5.10 *For a fixed order $a \leq 0$, we have*

$$\lim_{\mu \to 0^-} \alpha_\mu(a) = \frac{2^{1-2a}}{1-2a} \pi^{-a} \xi(2a).$$

For $a = \frac{1}{2}$, we have the asymptotic expansion:

$$\alpha_\mu\left(\frac{1}{2}\right) \sim \frac{-\gamma - 1 + \log(4\pi)}{2\sqrt{\pi}} - \frac{\log \sqrt{-\mu}}{2\sqrt{\pi}}, \qquad \mu \to 0^-.$$

For $a > \frac{1}{2}$, we have the asymptotic approximation

$$\alpha_\mu(a) \sim \frac{2^{1-2a}}{1-2a} \pi^{-a} \xi(2a) + \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}} \Gamma\left(a - \frac{1}{2}\right), \qquad \mu \to 0^-.$$

Moreover, we have the expansion of $p_0(\sqrt{x})$

$$p_0(\sqrt{x}) = 1 - \frac{\sqrt{x}}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{1-2n}}{1-2n} \pi^{-n} \xi(2n) \frac{x^n}{n!}.$$

3.5.3 The negative free energy in the Bosonic second quantization

Similar to the Fermionic second quantization, we define the spectral action function with respect to the negative free energy in the Bosonic second quantization to be

$$q_\mu(\sqrt{x}) = -\log\left(1 - e^{-\sqrt{x^2 - \mu}}\right).$$

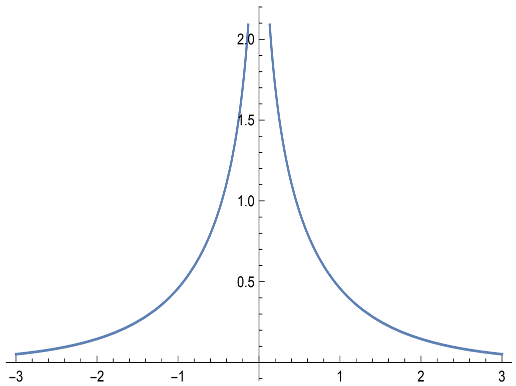
It is obvious that the chemical potential must be negative, $\mu < 0$.

We denote by $\beta_\mu(a)$ the spectral action coefficients of $q_\mu(\sqrt{x})$; namely,

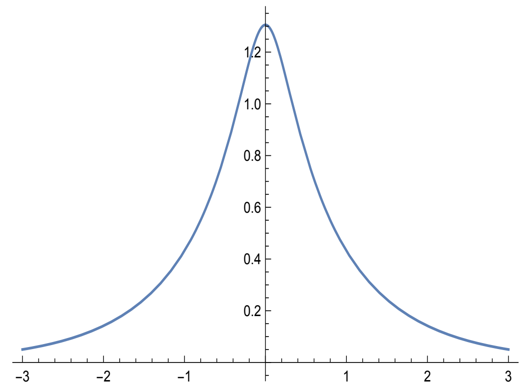
$$\beta_\mu(a) = \frac{1}{\Gamma(-a)} \int_0^\infty q_\mu(\sqrt{x}) x^{-a-1} dx = \frac{2}{\Gamma(-a)} \int_0^\infty q_\mu(x) x^{-2a-1} dx.$$

Using the same argument as before, we deduce that

Proposition 3.5.11 *For any fixed chemical potential $\mu < 0$, $\beta_\mu(a)$ is an entire function.*



(a) The image of $q_0(x)$



(b) The image of $q_{-0.1}(x)$

Figure 3.5: The image of $q_0(x)$ and $q_{-0.1}(x)$

Moreover it has the following two explicit expressions:

$$\beta_\mu(a) = \frac{2^{-a+\frac{1}{2}}}{\sqrt{\pi}}(-\mu)^{-\frac{a}{2}+\frac{1}{4}} \sum_{n=1}^{\infty} n^{a-\frac{1}{2}} K_{-a+\frac{1}{2}}\left(n\sqrt{-\mu}\right),$$

and

$$\beta_\mu(a) = \frac{\Gamma(a)}{2} \sum_{n=-\infty}^{\infty} \frac{1}{((2n)^2\pi^2 - \mu)^a} - \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}}\Gamma\left(a - \frac{1}{2}\right).$$

When $\mu \to 0^-$, the behaviour of the spectral action coefficients is as follows

Proposition 3.5.12 *For any order $a < 0$, we have*

$$\lim_{\mu \to 0^-} \beta_\mu(a) = \frac{2^{-2a}\pi^{-a}}{(2a-1)a}\xi(2a).$$

When $a = 0$,

$$\beta_\mu(0) \sim -\log\sqrt{-\mu}, \qquad \mu \to 0^-.$$

When $a = \frac{1}{2}$,

$$\beta_\mu\left(\frac{1}{2}\right) \sim \frac{\gamma - \log(4\pi)}{2\sqrt{\pi}} + \frac{\log\sqrt{-\mu}}{2\sqrt{\pi}} + \frac{\sqrt{\pi}}{2\sqrt{-\mu}}, \qquad \mu \to 0^-.$$

When $a > \frac{1}{2}$,

$$\beta_\mu(a) \sim \frac{2^{-2a}\pi^{-a}}{(2a-1)a}\xi(2a) + \frac{\Gamma(a)}{2}(-\mu)^{-a} - \frac{(-\mu)^{-a+\frac{1}{2}}}{4\sqrt{\pi}}\Gamma\left(a - \frac{1}{2}\right), \qquad \mu \to 0^-.$$

Moreover, we have the expansion of $q_0(\sqrt{x})$ as follows:

$$q_0(\sqrt{x}) = \frac{\sqrt{x}}{2} - \frac{\log x}{2} + \sum_{n=1}^{\infty}(-1)^n \frac{2^{-2n}\pi^{-n}}{(2n-1)n}\xi(2n)\frac{x^n}{n!}.$$

Appendix C

Spectral action basics

In this appendix we shall briefly recall the spectral action principle, originally formulated by Chamseddine and Connes [2]. Assume $(\mathcal{A}, \mathcal{H}, D)$ is a finitely summable regular spectral triple with simple dimension spectrum, The spectral action is defined as

$$\mathrm{Tr}(f(D/\Lambda)),$$

where $f(x)$ is a non-negative even smooth function which is rapidly decreasing at $\pm\infty$, and Λ is a positive number called mass scale, or cutoff. Note that $f(D/\Lambda)$ is a trace-class operator. We denote by $\chi(x) = f(\sqrt{x})$, and assume that $\chi(x)$ is given as a Laplace transform

$$\chi(x) = \int_0^\infty e^{-sx} g(s) ds,$$

where $g(s)$ is rapidly decreasing near 0 and ∞. We also assume that there is a heat trace expansion

$$\mathrm{Tr}\left(e^{-tD^2}\right) \sim \sum_\alpha a_\alpha t^\alpha, \qquad t \to 0^+,$$

It was proved in [2] that the spectral action has an asymptotic expansion for $\Lambda \to \infty$, namely,

$$\mathrm{Tr}(\chi(D^2/\Lambda)) \sim \sum a_\alpha \Lambda^{-\alpha} \int_0^\infty s^\alpha g(s) ds.$$

When $\alpha < 0$, by the Mellin transform,

$$s^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty e^{-sx} x^{-\alpha-1} dx.$$

Thus the spectral action coefficient is

$$\int_0^\infty s^\alpha g(s)ds = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \chi(x) x^{-\alpha-1} dx.$$

When $\alpha = 0$,

$$\int_0^\infty g(s)ds = \chi(0).$$

When $\alpha > 0$, the spectral action coefficient $\int_0^\infty s^\alpha g(s)ds$ is of order Λ^{-1}. Thus we get

$$\mathrm{Tr}(\chi(D^2/\Lambda)) \sim \sum_{\alpha<0} a_\alpha \Lambda^{-\alpha} \frac{1}{\Gamma(-\alpha)} \int_0^\infty \chi(x) x^{-\alpha-1} dx + a_0 \chi(0) + O(\Lambda^{-1}), \quad \Lambda \to \infty.$$

And when $\alpha = n$ is a positive integer, since $(\partial_x)^n (e^{-sx}) = (-1)^n s^n e^{-sx}$, we have that

$$\int_0^\infty s^n g(s)ds = (-1)^n \left(\int_0^\infty (\partial_x)^n (e^{-sx}) g(s) ds \right) \Bigg|_{x=0} = (-1)^n \chi^{(n)}(0).$$

Bibliography

[1] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 2*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997. Equilibrium states. Models in quantum statistical mechanics.

[2] A. H. Chamseddine and A. Connes. The spectral action principle. *Comm. Math. Phys.*, 186(3):731–750, 1997.

[3] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom. Entropy and the spectral action. *Communications in Mathematical Physics*, Feb 2019.

[4] R. Dong and M. Khalkhali. Second Quantization and the Spectral Action. *arXiv e-prints*, page arXiv:1903.09624, Mar 2019.

[5] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[6] A. P. Guinand. On Poisson's summation formula. *Ann. of Math. (2)*, 42:591–603, 1941.

[7] B. M. Project, H. Bateman, A. Erdélyi, and U. S. O. of Naval Research. *Tables of integral transforms: based, in part, on notes left by Harry Bateman*. Number v. 1 in Bateman manuscript project. McGraw-Hill, 1954.

[8] N. Temme. *Special functions: an introduction to the classical functions of mathematical physics*. A Wiley-Interscience Publication. Wiley, 1996.

Chapter 4

Conclusion

In noncommutative geometry, the geometric information is encoded in a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. If we take

$$\mathcal{A} = C^\infty(\mathbb{T}^3_\theta), \quad \mathcal{H} = L^2(\mathbb{T}^3_\theta, \Omega^\bullet \mathbb{T}^3_\theta), \quad D = d + d^*,$$

where d^* is determined by the metric over $\mathbb{T}^3_\theta$, then the spectral triple

$$(C^\infty(\mathbb{T}^3_\theta), L^2(\mathbb{T}^3_\theta, \Omega^\bullet \mathbb{T}^3_\theta), d + d^*)$$

encodes all the geometric information of this given noncommutative three-torus. By computing the coefficients $a_2(\triangle_0)$ and $a_2(\triangle_1)$, we can get the Ricci density under the given metric, which is given in section 2.3.2 and section 2.4.2. Moreover, we can obtain the scalar curvature as a byproduct, which is given in section 2.3.1 and section 2.4.1.

On the other hand, in the noncommutative geometry setting, we would like to get a good notion of an action functional, which can be associated to an arbitrary finitely summable spectral triple that depends only on the spectrum of D. The spectral action

$$\mathrm{Tr}(f(D/\Lambda))$$

is a suitable construction of such an action functional. Moreover, by doing the second quantization, we can obtain the second quantized operator $e^{d\Gamma\sqrt{D^2-\mu}}$, where μ is the chemical potential. When the partition function $Z = \mathrm{Tr}(e^{d\Gamma\sqrt{D^2-\mu}}) < \infty$, we can obtain a density matrix ρ. We can interpret the entropy, average energy, and negative free energy of ρ as the spectral actions of the Dirac operator D with the help of the chemical potential μ. It is worth to mention that the spectral action coefficients deduced from the Bosonic /Fermionic second quantization are very similar to each other. For instance, The spectral action coefficients of the entropy via Fermionic second quantization given in Proposition 3.4.9 can be expressed by an alternating sum of modified Bessel functions of the second kind, while the spectral action coefficients of the entropy via Bosonic second quantization given in Proposition 3.5.7 can be expressed by the sum, instead of alternating sum, of the same terms. This phenomenon reflects the difference between the anti-symmetric property of Fermionic second quantization and the symmetric property of Bosonic second quantization.

Rui Dong | Curriculum Vitae

Research Interests

- Differential Geometry
- Noncommutative Geometry
- Spectral Geometry
- Mathematical Physics
- Operator Algebras
- Functional Analysis

Education

- Ph.D., Department of Mathematics, Western University, 2015 – current.
- Master of Science, Department of Mathematics, Western University, 2014 – 2015.

Publication List

- (R.Dong, A.Ghorbanpour, M.Khalkhali)The Ricci Curvature for Noncommutative Three Tori, arXiv:1808.02977. Submitted.
- (R.Dong, M.Khalkhali)Second Quantization and the Spectral Action, arXiv:1903.09624. Submitted.

Teaching Experience

In 2014

- Completing the Teaching Assistant Training Program(TATP) in Western University.
- Teaching Assistant for the course Math 1228A, Math 1228B and Math 1229A in Fall and Winter.

In 2015

- Teaching Assistant for the course Math 1225A, Math 1228A and Math 1229A in Summer.
- Teaching Assistant for the course Math 1228A, Math 1228B and Math 1229A in Fall and Winter.

In 2016

- Teaching Assistant for the course Math 1225A, Math 1228A and Math 1229A in Summer.
- Teaching Assistant for the course Math 1500A, Math 1501B in Fall and Winter.

In 2017

- Teaching Assistant for the course Math 1225B in Summer.
- Teaching Assistant for the course Math 1225B and Math 1228B in Fall and Winter.

In 2018

- Teaching Assistant for the course Math 1228A, Math 1500A in Fall.

Selected Workshops and Seminars

In 2016

- "The local Index Formula" in the NCG Seminar in Western University on April 7th.
- "Random Non-commutative Geometries and Matrix Integrals" for the Comprehensive Exam Presentation in Western University on June 21st.

In 2017

- "The Ricci Curvature in Noncommutative Geometry" in the NCG Seminar in Western University on July 14th.

In 2018

- "Entropy and the Spectral Action " in the NCG Seminar in Western University on October 18th.

Technical and Personal skills

- **Programming Languages:** Proficient in: Python, Sage, Mathematica, LaTeX.