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Polynomial and Rational Convexity of Submanifolds of Euclidean Complex Space

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Abstract

The goal of this dissertation is to prove two results which are essentially independent, but which do connect to each other via their direct applications to approximation theory, symplectic geometry, topology and Banach algebras. First we show that every smooth totally real compact surface in $\mathbb{C}^2$ with finitely many isolated singular points of the open Whitney umbrella type is locally polynomially convex. The second result is a characterization of the rational convexity of a general class of totally real compact immersions in $\mathbb{C}^n$.

Summary for Lay Audience

In this dissertation we prove two original results that are of great interest for their applications to the theory of approximation of continuous functions. These two results unveil deep connections to other area of mathematics, such as symplectic geometry, Banach algebras and topology. More precisely, we study some geometric properties of a class of objects of complex Euclidean space. The first result (Chapter 3) essentially establishes one of those properties (which we call Polynomial convexity) to a certain class of objects. The other one (Chapter 4) provides a characterization of a different kind of objects with respect to another type of geometric property (named Rational convexity).

Statement of Co-authorship

In Chapter 3 I present a result published in a joint work [26] with my supervisor, Prof. Rasul Shafikov. In this paper we adapt the method used in [36] to prove a more general case than the one treated there. Prof. Shafikov and I had numerous brain storming sessions, where ideas were exchanged and eventually a strategy was devised. Then, we systematically started to put the paper together. I have done all the numerous computations and both Dr. Shafikov and I verified them, more than once. In subsection 3.1.1 (The characteristic foliation) I show that the method used in [36] to compute the so called characteristic foliation applies to singularities that are more general than the one we study in the paper (the Whitney Open Umbrella). The entire content of the paper (except for the introduction, which does not appear in Chapter 3 but parts of it are included in Section 1.1) was written and typed by me. Dr. Shafikov reviewed and made suggestions and/or corrections.

Acknowledgements

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Chapter 1

Introduction and Main Results

The notion of convexity of compact sets in Euclidean complex space $\mathbb{C}^n$ plays a fundamental role in complex analysis. In particular, polynomial and rational convexity (See Section 2.2 for the definitions) are of crucial importance in the general theory of approximation of continuous functions, uncovering deep connections to topology, Banach algebras, symplectic geometry, and other areas of mathematics.

In general, it is difficult to show that a compact subset of $\mathbb{C}^n$ is polynomially or rationally convex. Therefore it is important to search for criteria that may help with this task. One active area of interest is the study of the polynomial and rational convexity of embedded or immersed smooth submanifolds of $\mathbb{C}^n$ with finitely many "singular" points. Such singularities include self-intersections, complex points and other kinds, such as open Whitney umbrellas. Substantial progress has been made in this direction for Lagrangian and totally real submanifolds, thanks to the work of Alexander [2], Bedford-Klingenberg [5], Duval-Sibony [14], Forstnerič-Stout [15], Gayet [16], Gromov [18], Shafikov-Sukhov [37, 38] and other authors. However, there are still many unanswered questions in this area, two of which we are addressing in this dissertation, as described in Sections 1.1 and 1.2.

1.1   The Open Whitney Umbrella

Our first main result [26] gives a positive answer to a conjecture of Nemirovski [29] concerning the local polynomial convexity of open Whitney umbrellas. The standard open (or unfolded) Whitney umbrella is the map

$\pi : \mathbb{R}^2_{(t,s)} \to \mathbb{R}^4_{(x,u,y,v)} \cong \mathbb{C}^2_{(z=x+iy,w=u+iv)}$ given by

$$\pi(t,s) = \left(ts, \frac{2t^3}{3}, t^2, s \right). \tag{1.1.1}$$

The map π is a smooth homeomorphism onto its image, nondegenerate except at the origin. It satisfies $\pi^*\omega_{\mathrm{st}} = 0$, where $\omega_{\mathrm{st}} = dx \wedge dy + du \wedge dv$ is the standard symplectic form on $\mathbb{C}^2$, hence $\Sigma := \pi(\mathbb{R}^2)$ is a Lagrangian embedding (see Section 2.1) in $\mathbb{C}^2$, with an isolated singular point at the origin. If $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ is a local symplectomorphism, i.e., a local diffeomorphism which preserves the standard symplectic form, which we may assume, without loss of generality, to preserve the origin, then the image $\phi(\Sigma)$ is called an *open Whitney umbrella*. The first main result is the following:

Theorem 1.1.1. [26] *Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be an arbitrary smooth symplectomorphism. Then the surface $\phi(\Sigma)$ is locally polynomially convex at the origin.*

This result was proved for a generic real-analytic ϕ in [36] and for a generic smooth ϕ in [37]. Our theorem establishes polynomial convexity in *full generality* in this context. One immediate application of our main result is the following.

Corollary 1.1.2. [26] *For Σ and ϕ as in Theorem 1.1.1, there exists $\varepsilon > 0$ sufficiently small, such that any continuous function on $\phi(\Sigma) \cap \mathbb{B}(\phi(0), \varepsilon)$ can be uniformly approximated by holomorphic polynomials.*

For the proof of Theorem 1.1.1 our approach is similar to that in [36]: one constructs an auxiliary real analytic hypersurface M that contains the standard umbrella Σ. The hypersurface M is singular at the origin, but it is smooth and strictly pseudoconvex at all other points. Then one considers the so-called *characteristic foliation* on $\phi(\Sigma) \setminus \{0\}$ with respect to $\phi(M)$. It turns out that certain topological configurations of the phase portrait of the foliation guarantee local polynomial convexity of $\phi(\Sigma)$ at the origin. Direct computations yield a system of ODEs that determines the phase portrait, however, the system is degenerate, and standard tools from dynamical systems cannot be directly applied. In [36] the authors used the theory of normal forms of Bruno [7] and a result of Dumortier [8] to determine the phase portrait of the characteristic foliation. This was generalized to the smooth case in [37]. In our approach [26] we use a result of Brunella and Miari [6] to reduce the problem of determining the phase portrait of $\phi(\Sigma)$ to that of the

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so-called *principal* part of the vector field arising from the foliation. Under certain nondegeneracy conditions on the principal part, its phase portrait is topologically equivalent to that of the original vector field. The system obtained in [36] has degenerate principal part, and therefore, the result in [6] could not be applied in that case. However, a suitable modification of the auxiliary hypersurface M, introduced in [26], gives a system with a nondegenerate principal part. Our final calculations of the phase portrait of the principal part also use Bruno's normal form theory.

The proof of the corollary uses local polynomial convexity of the umbrella established in Theorem 1.1.1 and the result of Anderson, Izzo and Wermer [3]. The proof of Cor. 1 in [36] goes through in our case without any further modifications. For a better reading experience we included the proof at the end of Section 3.1.

Our interest in open Whitney umbrellas originates in the paper of Givental [17], who showed that any compact real surface S, orientable or not, admits a so-called *Lagrangian inclusion*, a map $F : S \to \mathbb{C}^2$, which is a local Lagrangian embedding except a finite number of singularities that are either double points or Whitney umbrellas. It is well-known (see, e.g., [4] or [31]) that certain surfaces do not admit a Lagrangian inclusion F without umbrellas, and so open Whitney umbrellas appear to be intricately related to the topology of the surfaces. The study of convexity properties near Whitney umbrellas is an instrumental part in this investigation. In particular, combining Theorem 1.1.1 with the results in [37] we conclude that *any Lagrangian inclusion is locally polynomially convex at every point.*

1.2 Rationally Convex Immersions

Our second main result is the following characterization of a class of rationally convex, totally real immersions in $\mathbb{C}^n$ of compact real manifolds. We refer the reader to Section 2.1 for the definitions of totally real submanifolds of $\mathbb{C}^n$ and plurisubharmonic functions, notions that are being used in the main theorem of this section.

Theorem 1.2.1. [25] *Let S be a smooth compact manifold of dimension $m \geq 1$, with or without boundary, and let $\iota : S \to \mathbb{C}^n$, $n \geq m$, be an immersion such that $\iota(S)$ is a smooth submanifold of $\mathbb{C}^n$, except at finitely many points $p_1, p_2, \ldots p_N \in \iota(S)$, where $\iota(S)$ intersects itself finitely many*

times. Suppose $\iota(S)$ is totally real and locally polynomially convex. Then the following are equivalent:

(i) *$\iota(S)$ is rationally convex;*

(ii) *There exist contractible neighborhoods W_j of p_j in $\iota(S)$, $j = 1, \ldots, N$, such that for every neighborhood Ω of $\iota(S)$, there exist neighborhoods $U_j \subset V_j$ of p_j in $\mathbb{C}^n$, $j = 1, \ldots, N$, with $\{\overline{V_j}\}_j$ pairwise disjoint, and a smooth plurisubharmonic function $\varphi : \mathbb{C}^n \to \mathbb{R}$, satisfying the following properties:*

    (a) *$U_j \cap \iota(S) = W_j$, $j = 1, \ldots, N$;*

    (b) *$\cup_{j=1}^{N} V_j$ is compactly included in Ω;*

    (c) *$dd^c\varphi = 0$ on $\cup_{j=1}^{N} U_j$;*

    (d) *φ is strictly plurisubharmonic on $\mathbb{C}^n \setminus \cup_{j=1}^{N} \overline{V_j}$;*

    (e) *$\iota^* dd^c\varphi = 0$.*

A first characterization of the rational convexity of a smooth totally real compact submanifold $S \subset \mathbb{C}^2$ was given by Duval [10], [12], who showed that S is rationally convex if and only if S is Lagrangian with respect to some Kähler form in $\mathbb{C}^2$. Subsequently, applying a different method that makes use of Hörmander's L^2 estimates, Duval and Sibony [14] extended the result to totally real embeddings of any dimension less than or equal to n. It is thanks to these remarkable results that the intrinsic connection between rational convexity and symplectic properties of real submanifolds has been revealed. In [16] Gayet analyzed totally real immersions in $\mathbb{C}^n$ of maximal dimension, with finitely many transverse double self-intersection points, showing that being Lagrangian with respect to some Kähler form in $\mathbb{C}^n$ is a sufficient condition for such immersions to be rationally convex. A similar result was proved later by Duval and Gayet [13] for immersions of maximal dimension with certain non-transverse intersections.

Theorem 1.2.1 gives a characterization of the rational convexity of a more general class of immersions in $\mathbb{C}^n$ with finitely many self-intersection points: we do not impose any restrictions on the real dimension of such immersions and do not require the self-intersections to be transverse or double. The proof of the theorem spans two sections of Chapter 4. In Section 4.1 we prove Proposition 4.1.1 which shows that the implication $(i) \Rightarrow (ii)$ in

Theorem 1.2.1 is true. It is important to note that the proof of Proposition 4.1.1 does not require S to be polynomially convex near the singular points. Proposition 4.2.1 proved in Section 4.2 shows that the other direction of Theorem 1.2.1 holds true. In the proof we follow closely the method introduced in [14] (see also [16], [38]). The condition for S to be polynomially convex near the singular points plays a key role in the proof. Note, however, that all the submanifolds considered in [10], [12], [14], [16] are polynomially convex near every point. This is a classical result for smooth totally real embeddings and, in the case of the class of immersions considered in [16], the property follows from a result of Shafikov and Sukhov [37, Theorem 1.4] who showed that every Lagrangian immersion with finitely many transverse self-intersections is locally polynomially convex. The second example in Section 4.3 shows that in general an immersion that is not locally polynomially convex may fail to be rationally convex. It is natural to ask whether local polynomial convexity is also guaranteed for immersions that are isotropic with respect to a "degenerate" Kähler form, as described in Theorem 1.2.1. This remains an open problem

In Section 4.3, using a theorem of Weinstock [41], we show that there is a "large" family of compact, totally real immersions in $\mathbb{C}^2$ with one transverse self-intersection, which are rationally convex but are not isotropic with respect to any Kähler form on $\mathbb{C}^2$, thus Gayet's theorem [16] cannot be applied to this case. However, by Theorem 1.2.1 they are isotropic with respect to a degenerate Kähler form.

We also remark that the main result in [16] is implied by Theorem 1.2.1. Indeed, in the first step of the proof of the theorem in [16] it is shown that S being Lagrangian with respect to a Kähler form $\omega = dd^c\varphi$ implies that S is isotropic with respect to a nondegenerate closed $(1,1)$-form defined on $\mathbb{C}^n$, that vanishes on sufficiently small neighborhoods of the singular points and is positive in the complement of some slightly larger neighborhoods. This is done by composing the original potential φ with a suitable non-decreasing, convex function. We note that, prior to such composition, one can multiply φ with a suitable cutoff function, and then apply the composition, this way controlling the shape of the neighborhoods mentioned above. It follows that the hypothesis of the main result in [16] implies the hypothesis of the direction $(ii) \Rightarrow (i)$ in Theorem 1.2.1.

Chapter 2

Preliminaries

2.1 General Background

In this section we review the basic background necessary for understanding the rest of this thesis. Unless otherwise specified, by "smooth" we shall mean $\mathcal{C}^\infty$-smooth and by a neighborhood of a compact connected subset $X \subset \mathbb{C}^n$ we shall mean a connected open set containing X, having compact closure. Throughout the material $B(p, r)$ denotes the open ball in $\mathbb{C}^n$ centered at $p \in \mathbb{C}^n$ and of radius $r > 0$.

If X is a real submanifold of $\mathbb{C}^n$ and $p \in X$, we say that X is *totally real at p* if the tangent space $T_p X$ does not contain any complex lines. X is said to be *totally real* if it is totally real at every point. An immediate example of a totally real submanifold of the n-dimensional euclidean complex space is $\mathbb{R}^n \subset \mathbb{C}^n$.

Let Ω be an open subset of $\mathbb{C}^n_{(z_1,\ldots,z_n)}$, $z_j = x_j + iy_j$, $j = 1, \ldots, n$, and let $\varphi : \Omega \to \mathbb{R}$ be a real-valued smooth function. As usual, we define

$$\partial\varphi := \sum_{j=1}^n \frac{\partial\varphi}{\partial z_j} dz_j, \ \ \overline{\partial}\varphi := \sum_{j=1}^n \frac{\partial\varphi}{\partial \overline{z}_j} d\overline{z}_j.$$

It follows that the usual differential of φ is given by $d\varphi = \partial\varphi + \overline{\partial}\varphi$. We shall also need the d^c differential operator, defined when acting on φ as

$$d^c\varphi := i(\overline{\partial}\varphi - \partial\varphi),$$

or, in real coordinates,

$$d^c\varphi = \sum_{j=1}^n \left(\frac{\partial\varphi}{\partial x_j}dy_j - \frac{\partial\varphi}{\partial y_j}dx_j \right). \qquad (2.1.1)$$

Definition 2.1.1. (see for example [40]) Let $D \subset \mathbb{C}^n$ be a domain. A function $\varphi : D \to [-\infty, \infty)$ is said to be *plurisubharmonic* if it is upper semicontinuous in D and for each complex line $\lambda \subset \mathbb{C}^n$ the restriction $\varphi|_{D\cap\lambda}$ is subharmonic on $D \cap \lambda$.

If φ is smooth, Definition 2.1.1 is equivalent to the following statement ([35], [41]): φ is plurisubharmonic if the $(1,1)$-form $dd^c\varphi$ is nonnegative definite (or, using another common terminology, positive semidefinite). Also, when φ is smooth, we say that φ is *strictly plurisubharmonic* if $dd^c\varphi$ is positive definite.

A *Kähler form* on $\mathbb{C}^n$ is a nondegenerate closed form ω of bidegree $(1,1)$, which is positive definite. Recall that a nondegenerate form ω is a form with the property that if $\omega(z, w) = 0$ for all $w \in \mathbb{C}^n$ then $z = 0$. For example, the standard symplectic form on $\mathbb{R}^4_{(x,u,y,v)}$, $\omega = dx \wedge dy + du \wedge dv$, is a Kähler form, since it is clearly positive definite.

A smooth function φ is called a *potential* for ω if $\omega = dd^c\varphi$. A real m-dimensional submanifold $S \subset \mathbb{C}^n$, $m \leq n$, is said to be *isotropic* with respect to a Kähler form ω if $\omega|_S = 0$. If in the above case $m = n$ then we say that S is *Lagrangian* with respect to ω.

Let $F : D \subset \mathbb{C}^n \to \mathbb{C}^n$, $F = (u_1 + iv_1, \dots, u_n + iv_n)$, be a smooth map, where D is a domain in $\mathbb{C}^n$ and u_j, v_j are smooth real-valued maps on D. For every $p \in D$ and $j \in \{1, \dots, n\}$ let

$$[d_p^c u_j] := \left[-\frac{\partial u_j}{\partial y_1}\bigg|_p \ \frac{\partial u_j}{\partial x_1}\bigg|_p \ \cdots \ -\frac{\partial u_j}{\partial y_n}\bigg|_p \ \frac{\partial u_j}{\partial x_n}\bigg|_p \right]$$

$$[d_p^c v_j] := \left[-\frac{\partial v_j}{\partial y_1}\bigg|_p \ \frac{\partial v_j}{\partial x_1}\bigg|_p \ \cdots \ -\frac{\partial v_j}{\partial y_n}\bigg|_p \ \frac{\partial v_j}{\partial x_n}\bigg|_p \right]$$

be the $1 \times 2n$ matrices associated with the operator d^c acting on $T_p\mathbb{C}^n$ and let

$$[z] := [x_1 \ y_1 \ \cdots \ x_n \ y_n],$$

where $z = (z_1, \ldots, z_n) \in T_p\mathbb{C}^n$, $z_j = x_j + iy_j$, $j = 1, \ldots, n$. Then, for all $z \in T_p\mathbb{C}^n$ we define

$$d_p^c F(z) := [d_p^c F] \cdot [z]^T,$$

where $[d_p^c F]$ is the $2n \times 2n$ matrix with complex entries, given by

$$[d_p^c F] := \begin{bmatrix} [d_p^c u_1] \\ [d_p^c v_1] \\ \ldots \\ [d_p^c u_n] \\ [d_p^c v_n] \end{bmatrix}.$$

The next technical result is required in the proof of Theorem 1.2.1.

Lemma 2.1.2. *Let $D \subset \mathbb{C}^n$ be a domain, $F : D \to \mathbb{C}^n$, $F = (F_1, \ldots, F_n)$, a smooth map such that $F(D)$ is a domain in $\mathbb{C}^n$ and $h : F(D) \to \mathbb{R}$ a smooth function. Then*

$$d_p^c(h \circ F) = d_{F(p)}h \circ d_p^c F,$$

at any point $p \in D$.

Proof. Let $p \in D$. By the complex chain rule (see for example [23, p.6]),

$$\partial_p(h \circ F) = \partial_{F(p)}h \circ \partial_p F + \bar{\partial}_{F(p)}h \circ \bar{\partial}_p F,$$

$$\bar{\partial}_p(h \circ F) = \bar{\partial}_{F(p)}h \circ \partial_p F + \partial_{F(p)}h \circ \bar{\partial}_p F.$$

So,

$$\begin{aligned} d_p^c(h \circ F) &= i\left[\bar{\partial}_p(h \circ F) - \partial_p(h \circ F)\right] \\ &= i\left[\bar{\partial}_{F(p)}h \circ \partial_p F + \partial_{F(p)}h \circ \bar{\partial}_p F - \partial_{F(p)}h \circ \partial_p F - \bar{\partial}_{F(p)}h \circ \bar{\partial}_p F\right] \\ &= \bar{\partial}_{F(p)}h \circ [-i(\partial_p F - \bar{\partial}_p F)] + \partial_{F(p)}h \circ [i(\bar{\partial}_p F - \partial_p F)] \\ &= (\bar{\partial}_{F(p)}h + \partial_{F(p)}h) \circ d_p^c F \\ &= d_{F(p)}h \circ d_p^c F. \end{aligned}$$

$\square$

One other important tool we will make use of is the standard Euclidean *distance function* defined for a subset $M \subset \mathbb{C}^n_{(z=x_1+iy_1 \ldots z=x_n+iy_n)}$ as

$$dist(z, M) = \inf\{dist(z, p) : p \in M\}$$

for all $z \in \mathbb{C}^n$.

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Proposition 2.1.3. *Let M be a smooth totally real submanifold of $\mathbb{C}^n$ There exists a neighborhood U of M such that $\rho := dist^2(z, M)$ is smooth and strictly plurisubharmonic on U.*

Proof. The smoothness of ρ in some small neighborhood U of M is a classical result (see for example [1, Theorem 3.1]) and we shall not insist on proving this in detail. Now, let $z_0 \in U$ and let $p \in M$ be the unique point such that $\rho(z_0) = dist^2(z_0, p)$, the uniqueness of p being guaranteed if U is small enough. Denote by ρ_p the square distance function to the tangent space $T_p M$. We have $\rho(z_0) = \rho_p(z_0)$ ([1, Theorem 3.1]). For simplicity, suppose $n = 2$ and $\dim_{\mathbb{R}} M = 2$ (the general case follows similarly). Since M is totally real, we can assume that $T_p M = \mathbb{R}^2_{(x_1, x_2)}$. If we prove that ρ_p is strictly plurisubharmonic at z_0 then it follows that ρ is also strictly plurisubharmonic at z_0. Indeed, since by continuity $\rho(z)$ gets arbitrarily close to $\rho(z_0) = \rho_p(z_0)$, as z approaches z_0, the eigenvalues of $dd^c_z \rho$ approach those of $dd^c_{z_0} \rho_p$ since ρ is smooth (in particular, C^2-smooth). The eigenvalues of $dd^c_{z_0} \rho_p$ are positive, which implies that $dd^c_z \rho$ is positive definite in a small neighborhood of z_0. Finally, to see that ρ_p is strictly plurisubharmonic at z_0 (in fact everywhere in $\mathbb{C}^2$) an easy computation shows that $dd^c \rho_p = 2(\sum_{j=1}^2 dx_j \wedge dy_j)$ which is clearly positive definite. $\qquad\square$

Let $D \subset \mathbb{C}^n$ be a domain. A continuous function $\rho : D \to \mathbb{R}$ that satisfies $\rho(z) \to \infty$ as z approaches ∂D (the boundary of D) is called an *exhaustion function* for D. The domain D is said to be *pseudoconvex* if it admits an exhaustion function which is plurisubharmonic in D.

Suppose now that ∂D is C^2-smooth and that D is compactly included in a domain $\Omega \subset \mathbb{C}^n$. A function $\rho : \Omega \to \mathbb{R}$ is called *globally defining for D* if it is C^2-smooth in some neighborhood $U \subset \Omega$ of ∂D, $\nabla \rho \neq 0$ on ∂D and $D \cap U = \{\rho < 0\}$. In this case, we say that D is pseudoconvex if it admits a globally defining function ρ which is plurisubharmonic on ∂D, i.e., if $(dd^c \rho)|_{\partial D}$ is non-negative definite. If $(dd^c \rho)|_{\partial D}$ is positive definite we say that D is *strictly pseudoconvex*. In the above two cases, we also say that ∂D is pseudoconvex, or strictly pseudoconvex, respectively.

We also mention the following classical result which we shall make use of in the proof of Theorem 1.2.1: if the function φ is (strictly) plurisubharmonic in a domain $D \subset \mathbb{C}^n$ and $\psi : \varphi(D) \to \mathbb{R}$ is a smooth (strictly) increasing and (strictly) convex function then $\psi \circ \varphi$ is (strictly) plurisubharmonic on D. The result is a direct consequence of the (strict) positiveness of $dd^c \varphi$ and the chain rule (for details see for example [35, Theorem 3, page 205]).

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2.2 Polynomial and Rational Convexity

Let X be a compact subset of $\mathbb{R}^n$, $n \geq 1$. Recall that one way to define the *convex hull* of X, denoted here by $\mathcal{C}-\text{hull}\,(X)$, is as the set of all finite real linear combinations of points of X, whose coefficients are positive and add up to 1. Let $\mathcal{L}$ denote the set of all real-linear functions defined on $\mathbb{R}^n$. It is not difficult to see that the above definition is equivalent to the following one,

$$\mathcal{C}-\text{hull}\,(X) = \{x \in \mathbb{R}^n : |F(x)| \leq \sup_X |F| , F \in \mathcal{L}\}. \qquad (2.2.1)$$

We say that X is convex if $X = \mathcal{C}-\text{hull}\,(X)$. If X is a compact subset of $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $n \geq 1$, the convex hull of X is recovered by replacing $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in equation (2.2.1).

It turns out that important notions of convexity can be defined by using a similar form of equation (2.2.1), applied to new families of functions. The following example is key for the theory of functions of several complex variables. If D is a domain in $\mathbb{C}^n$ and $\mathcal{O}(D)$ is the set of all holomorphic functions defined on D, then we may define the *holomorphically convex hull* of a compact $X \subset D$ as

$$\mathcal{H}-\text{hull}\,(X) = \{z \in D : |f(z)| \leq \sup_X |f| , f \in \mathcal{O}(D)\}. \qquad (2.2.2)$$

As expected, we say that $X \subset D$ is *holomorphically convex* if $X = \mathcal{H}-\text{hull}\,(X)$. A domain $D \subset \mathbb{C}^n$ is holomorphically convex if for any subset X compactly contained in D, $\mathcal{H}-\text{hull}\,(X)$ is also compactly contained in D. As mentioned before, the notion of holomorphic convexity plays a fundamental role in the theory of several complex variables, in particular in the study of analytic continuation: it is a classical result that a domain $D \subset \mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy, i.e., if there exists a function that is holomorphic on D but it cannot be extended holomorphically outside of D. Furthermore, the Levi problem, which was solved in the early 1950's by Oka, Bremermann and Norguet [40, p.25], states that every domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is pseudoconvex.

In this dissertation we focus on two other types of convexity of compact subsets in $\mathbb{C}^n$: polynomial and rational convexity. Next, we discuss these two concepts, their importance and their deep connection with other areas of mathematics.

Definition 2.2.1. The *polynomially convex hull* of a compact subset $X \subset \mathbb{C}^n$ is defined as

$$\widehat{X} := \{z \in \mathbb{C}^n : |P(z)| \le \sup_{w \in X} |P(w)|, \text{ for all holomorphic polynomials } P\},$$

and the *rationally convex hull* of X as

$$\mathcal{R}-\text{hull}\,(X) := \{z \in \mathbb{C}^n : |R(z)| \le \sup_{w \in X} |R(w)|,$$
$$\text{for all rational functions } R \text{ holomorphic on } X\}.$$

We say that X is *polynomially convex* if $X = \widehat{X}$ and *rationally convex* if $X = \mathcal{R}-\text{hull}\,(X)$. It is immediate to see that if X is polynomially convex then it is also rationally convex. X is said to be *polynomially convex near* $p \in X$ if for every sufficiently small $\varepsilon > 0$, the compact set $X \cap \overline{B(p,\varepsilon)}$ is polynomially convex. We say that X is *locally polynomially convex* if X is polynomially convex near all of its points.

The following statement is true.

Proposition 2.2.2. *Let $X \subset \mathbb{C}^n$ be compact. Then,*

(a) *X is polynomially convex if and only if for every point $z \in \mathbb{C}^n \setminus X$ there exists a holomorphic polynomial P such that $|P(z)| > \sup_{w \in X} |P(w)|$;*

(b) *X is rationally convex if and only if for each point $z \in \mathbb{C}^n \setminus X$ there exists a holomorphic polynomial P such that $P(z) = 0$ and $P^{-1}(0) \cap X = \emptyset$.*

Proof. Point (a) is an immediate consequence of the definition of polynomial convexity, so we focus on proving point (b). We follow the proof in [39, (1.1) page 262]. Let

$$\mathcal{R}(X) = \{z \in \mathbb{C}^n : P(z) \in P(X), \text{ for all holomorphic polynomials } P\}.$$

It is clear that point (b) is true if we prove that $\mathcal{R}-\text{hull}\,(X) = \mathcal{R}(X)$:

1. ($\subseteq$). Suppose that $z_0 \notin \mathcal{R}(X)$, which means that $P(z_0) \notin P(X)$ for some holomorphic polynomial P. It follows that the rational function $R(z) = (P(z) - P(z_0))^{-1}$ is holomorphic in a neighborhood of X (or, using Stolzenberg's terminology [39], it is *holomorphic about X*). But R has a pole at z_0 which implies that $z_0 \notin \mathcal{R}-\text{hull}\,(X)$. This proves the first inclusion.

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2. ($\supseteq$). Assume that $z_0 \notin \mathcal{R}-\text{hull}\,(X)$. Then, there exists a rational function R holomorphic about X, such that

$$|R(z_0)| > \sup_X |R|. \qquad (2.2.3)$$

In fact, we may assume that $R(z_0) = 1$. Indeed, write $R(z) = \dfrac{P(z)}{Q(z)}$, where P, Q are (coprime) holomorphic polynomials, such that Q has no zeroes on X and define $\widetilde{R}(z) = \frac{Q(z_0)}{P(z_0)}R(z)$ (by (2.2.3), $P(z_0) \neq 0$). Recycling the notation and putting $R := \widetilde{R}$, we have

$$1 = R(z_0) > \sup_X |R|. \qquad (2.2.4)$$

Write again (the new) rational function R satisfying (2.2.4) as $R(z) = \dfrac{P(z)}{Q(z)}$, where once again P, Q are coprime polynomials. Define $H(z) = P(z) - Q(z)$ which satisfies $H(z_0) = 0$. However $0 \notin H(X)$. Indeed, assuming otherwise there would be a point $z \in X$ such that $H(z) = P(z) - Q(z) = 0$, i.e. $R(z) = 1 > \sup_X |R|$ by (2.2.4) so $z \notin \mathcal{R}-\text{hull}\,(X)$ which is a contradiction, since $z \in X \subset \mathcal{R}-\text{hull}\,(X)$.

$\square$

It is easy to see that the polynomially convex hull $\widehat{X}$ of a compact $X \subset \mathbb{C}^n$ is also compact. Indeed, we can write $\widehat{X} = \cap_P X_P$, where the intersection is taken over all holomorphic polynomials P and, for a fixed such P, $X_P = \{z \in \mathbb{C}^n : |P(z)| \leq \sup_{w \in X} |P(w)|\}$. X_P is closed as the preimage of the continuous function $|P|$ of the closed set $(-\infty, \sup_{w \in X} |P(w)|] \subset \mathbb{R}$. Therefore, $\widehat{X}$ is closed as the intersection of closed sets in $\mathbb{C}^n$. It remains to show that $\widehat{X}$ is bounded. Let $B \subset \mathbb{C}^n$ be a closed ball that contains X compactly. Such ball exists since X is compact, hence bounded. As mentioned in Example 2.2.3 (a) below, B is polynomially convex, i.e., $B = \widehat{B}$. It is indeed immediate to see from the definition of the polynomial hull that $X \subset B$ implies $\widehat{X} \subset \widehat{B} = B$, so $\widehat{X}$ is bounded, hence compact.

The fact that $\mathcal{R}-\text{hull}\,(X)$ is compact follows from the following observation ([40, page 2]):

$$\mathcal{R}-\text{hull}\,(X) = \cap_P\{z \in \mathbb{C}^n : |P(z)| \geq \inf_X |P|\},$$

the intersection being taken over all holomorphic polynomials on $\mathbb{C}^n$.

The simplest examples of polynomially and rationally convex compact sets occur when $n = 1$. It can be easily seen that every compact $X \subset \mathbb{C}$ is rationally convex. Indeed, for $z_0 \in \mathbb{C} \setminus X$ choose the polynomial $P(z) = z - z_0$ whose zero locus clearly contains z_0 but it misses X. Thus, X is rationally convex by Proposition 2.2.2 (b).

It can be proved that a compact $X \subset \mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \setminus X$ is connected. To prove one direction, let X be polynomially convex and suppose that $\mathbb{C} \setminus X$ is disconnected. Then $\mathbb{C} \setminus X$ has a bounded component, say D. By the Maximum Principle, for all $z \in D$ and all holomorphic polynomials P, we have $|P(z)| \leq \sup_{w \in X} |P(w)|$ which by Proposition 2.2.2 (a) is in contradiction with the fact that X is polynomially convex. The other direction is a direct consequence of Runge's theorem (see [40, page 2] for details).

If $n > 1$ we do not have such a simple classification of polynomial or rationally convex subsets of $\mathbb{C}^n$. In fact it is in general very difficult to verify that compacts in $\mathbb{C}^n$ are polynomially or rationally convex. There are, however, some simple examples of polynomially convex (hence, rationally convex) compact subsets of $\mathbb{C}^n$, two of which we list here without proof (see [40] for details),

Example 2.2.3.

(a) Every compact convex subset of $\mathbb{C}^n$ is polynomially convex. In particular closed balls and polydisks in $\mathbb{C}^n$ are polynomially convex;

(b) Every compact subset of $\mathbb{R}^n$ is a polynomially convex subset of $\mathbb{C}^n$;

For a compact $X \subset \mathbb{C}^n$ we can define the hull with respect to the family of plurisubharmonic functions [40, page 24],

$$\text{psh}-\text{hull}\,(X) = \cap_u \{z \in \mathbb{C}^n : u(z) \leq \sup_X u\},$$

where the intersection is taken over all plurisubharmonic functions (Definition 2.1.1) on $\mathbb{C}^n$. It is a classical result that $\widehat{X} = \text{psh}-\text{hull}\,(X)$ [40, Theorem 1.3.11].

Recall that if X is a compact subset of $\mathbb{C}$, by Runge's theorem, every function holomorphic on a neighborhood of X can be approximated uniformly by rational functions with poles off X. If $\mathbb{C} \setminus X$ is connected then

every such holomorphic function can be approximated by polynomials. The generalization of Runge's theorem to complex dimensions higher than 1 is given by the Oka-Weil Theorem: *if X is polynomially (rationally) convex then every function holomorphic in a neighborhood of X is the uniform limit of holomorphic polynomials (rational functions with poles off X).* Note that in $\mathbb{C}$ every compact is rationally convex and every compact with connected complement is polynomially convex, thus the Oka-Weil Theorem is a true generalization of Runge's Theorem.

Remark 2.2.4. If $X \subset \mathbb{C}^n$ is compact then $\widehat{X} = \mathcal{H}-\mathrm{hull}\,(X)$, where the holomorphic hull is taken with respect to all functions holomorphic on $\mathbb{C}^n$. Indeed, the inclusion $\mathcal{H}-\mathrm{hull}\,(X) \subset \widehat{X}$ follows immediately from the definitions of the two respective hulls. The converse inclusion is a direct consequence of the Oka-Weil Theorem. It follows that the interior of a polynomially convex set is holomorphically convex hence, by the Levi problem, it is pseudoconvex. In fact, if X is such a polynomially convex compact subset of $\mathbb{C}^n$, with interior, and $K \Subset X$ is also compact, then $\widehat{K} \Subset X$. But, by the above observation, $\widehat{K}$ is also holomorphically convex, which proves that the interior of X is holomorphically convex, hence pseudoconvex.

There is also a natural connection between polynomial convexity, rational convexity and uniform algebras. For a compact $X \subset \mathbb{C}^n$ let $\mathcal{C}(X)$ be the uniform algebra of complex-valued functions continuous on X. Denote by $\mathcal{P}(X)$ (respectively, $\mathcal{R}(X)$) the subalgebras of continuous complex-valued functions on X that are uniform limits of holomorphic polynomials (respectively, rational functions with no poles in X). It is a classical result that if $\mathcal{P}(X) = \mathcal{C}(X)$, then X is polynomially convex. Similarly, if $\mathcal{R}(X) = \mathcal{C}(X)$, then X is rationally convex. Therefore, these two types of convexity are necessary conditions for a continuous function to be approximated by polynomials and rational functions, respectively.

With respect to set operations with polynomially (rationally) convex sets, it is true that an arbitrary intersection of polynomially (rationally) convex compact subsets of $\mathbb{C}^n$ is also polynomially (rationally) convex. However, the union of polynomially convex sets is not necessarily polynomially convex. Kallin proved in [22] that the disjoint union of three closed balls is polynomially convex, but the union of three closed polydisks need not be polynomially convex. More examples of non-polynomially convex unions of polynomially convex compacts can be found in [24], [27], [28], [32]. In [30] Ne-

mirovski proved that any finite union of closed balls, which as we mentioned are individually polynomially convex, hence rationally convex, is rationally convex.

2.2.1 Polynomial Convexity of the Union of Two Totally Real Subspaces

We include here some results of Weinstock [41] which we use in Section 4.3 to construct two examples relevant for Theorem 1.2.1. Let A be a real $n \times n$ matrix such that $i \in \mathbb{C}$ is not an eigenvalue of A and $A + i$ is invertible. Define $M(A) = (A + i)\mathbb{R}^n = \{(A + i)v^T : v \in \mathbb{R}^n\}$. We say that $z \in \mathbb{C}$ is *purely imaginary* if the real part of z is zero. The first result in [41] that we will be making use of is the following,

Theorem 2.2.5 (Weinstock). *Each compact subset of $M(A) \cup \mathbb{R}^n$ is polynomially convex if and only if A does not have any purely imaginary eigenvalues of modulus greater than* 1.

Let $R > 0$ and $D = \{\zeta \in \mathbb{C} : |\zeta| \leq R\}$. An *analytic disc* in $\mathbb{C}^n$ is the image of D via an injective map $F : D \to \mathbb{C}^n$ which is continuous in D and holomorphic in the interior of D. If X is a subset of $\mathbb{C}^n$ we say that the analytic disc $F(D)$ is *attached to X* if $\partial F(D) \subset X$ and $F(D) \not\subset X$, where $\partial F(D)$ is the boundary of $F(D)$. Similarly, let $0 < r < R$ and $\Omega = \{\zeta \in \mathbb{C} : r \leq |\zeta| \leq R\}$ be an annulus in $\mathbb{C}$. An *analytic annulus* in $\mathbb{C}^n$ is the image of an injective continuous map $F : \Omega \to \mathbb{C}^n$, such that F is holomorphic in the interior of Ω. We say that the analytic annulus $F(\Omega)$ is *attached* to some set $X \subset \mathbb{C}^n$, if $\partial F(\Omega) \subset X$ and $F(\Omega) \not\subset X$.

Remark 2.2.6. Note that the above two definitions (analytic disk, analytic annulus) can be adapted to any bounded domain in $\mathbb{C}$ with smooth boundaries.

Of course, not every subset of $\mathbb{C}^n$ can have an analytic annulus attached to it. One interesting such situation is when X is a compact polynomially convex subset of $\mathbb{C}^n$. Indeed, any such attached annulus would have to be included in the polynomially convex hull of X by the Maximum Principle, which would be in contradiction to X being polynomially convex. The second result in [41] which is of interest to us is the following,

Theorem 2.2.7 (Weinstock). *Let A be an $n \times n$ matrix with real entries whose characteristic polynomial has the form $P(\zeta) = (\zeta^2 + t^2)Q(\zeta)$, where $t > 1$ and Q is a polynomial of degree $n - 2$ with no purely imaginary roots of modulus greater than 1. Then there exists a one-parameter family of analytic annuli $F_s \subset \mathbb{C}^n$, $s > 0$, that are attached to $M(A) \cup \mathbb{R}^n$.*

In his proof, Weinstock also shows that as s approaches 0 the boundary of F_s collapses to the origin. We refer the reader to [41] for the details of the proof.

2.3 A Brief Review of Some Basic Notions in the Qualitative Theory of Differential Equations

The following are well known notions and results used in the theory of dynamic systems. We follow closely the material and notations in [9] and we shall restrict the presentation only to the concepts necessary to prove the main result of Chapter 3. Let us consider a system of differential equations in the real plane

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases} \tag{2.3.1}$$

where P, Q are polynomials with complex coefficients and of real variables $(x, y) \in \mathbb{R}^2$. Most quantitative methods fail to solve system (2.3.1) because they involve finding an explicit analytic solution which in most cases is impossible. A qualitative approach enables us to understand the geometry of such global solutions in the plane and in many cases provides important information that can be further used in the analysis at hand. We associate to system (2.3.1) the following planar vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y},$$

thus, (2.3.1) becomes

$$\dot{z} = X(z), \tag{2.3.2}$$

where $z = (x, y)$ and $\dot{z} = (\dot{x}, \dot{y})$. The integral curves of X are solutions of (2.3.2). Hence, we obtain a foliation of the real plane whose topology may provide useful information about such solutions.

A point $z \in \mathbb{R}^2$ is called *regular* if $X(z) \neq 0$ and *singular* if $X(z) = 0$. Note that in the case of a singular point z, the constant map $\varphi(t) = z, \forall t \in (-\infty, \infty)$, is a solution of (2.3.2): $\dot{\varphi} = 0 = X(z) = X(\varphi(t))$. If $I \subset \mathbb{R}^2$ is an interval containing 0 and $\varphi : I \to \mathbb{R}^2$ is a solution of (2.3.2) with $\varphi(0) = z_0 \in \mathbb{R}^2$, we say that φ is *maximal* if for every solution $\psi : J \to \mathbb{R}^2$ such that $I \subset J$ and $\varphi = \psi|_I$ we have $I = J$ and therefore $\varphi = \psi$. We say that φ is *regular* if $\dot{\varphi} \neq 0$ at every point in t. The image $\gamma_\varphi = \varphi(I)$ of a maximal solution φ which, in case φ is regular, is endowed with the orientation induced by φ, is called the *orbit* (or *trajectory*) of the maximal solution. We also say that γ_φ is the orbit of the associated vector field X. The following result holds (see [9] for the proof):

Theorem 2.3.1. *Let $\varphi : I \to \mathbb{R}^2$ be a maximal solution of the system (2.3.1). Then exactly one of the following three statements is true:*

1. *φ is a bijection onto its image;*

2. *$I = \mathbb{R}$ and φ is constant (hence the orbit γ_φ is a point);*

3. *$I = \mathbb{R}$ and there exists $\tau > 0$ such that $\varphi(t + \tau) = \varphi(t), \forall t \in \mathbb{R}$ and $\varphi(t) \neq \varphi(s)$ if $|t - s| < \tau$. In this case we say that φ is periodic of minimal period τ.*

The *phase portrait* of the vector field X is the set of orbits of X. It includes points (constant orbits) and regular orbits oriented according to the orientation inherited from the regular maximal solutions defining them.

Definition 2.3.2. Let X_1, X_2 be two planar vector fields defined on the open subsets Ω_1 and Ω_2 of $\mathbb{R}^2$, respectively. We say that X_1 is *topologically equivalent to X_2* if there exists a homeomorphism $h : \Omega_1 \to \Omega_2$ sending orbits of X_1 to orbits of X_2. More precisely, if γ_1 is the orbit of X_1 passing through $p \in \Omega_1$, then $h(\gamma_1)$ is the orbit of X_2 passing through $h(p)$. In this case we say that X_1 and X_2 belong to the same topological class of vector fields.

Let $z \in \mathbb{R}^2$ be a singular point of the vector field X. The *linear part* of X at z is given by

$$
DX(p) = \begin{pmatrix} \dfrac{\partial P}{\partial x}(p) & \dfrac{\partial P}{\partial y}(p) \\[2mm] \dfrac{\partial Q}{\partial x}(p) & \dfrac{\partial Q}{\partial y}(p) \end{pmatrix}.
$$

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We have the following classification of singular points:

Definition 2.3.3. (a) the point p is *nondegenerate* if no eigenvalue of $DX(p)$ is equal to 0;

(b) p is said to be a *nonelementary* singular point if both eigenvalues vanish; otherwise, it is said to be an *elementary* singular point;

(c) p is said to be a *saddle* if both eigenvalues of $DX(p)$ are real, nonzero and of opposite signs;

(d) p is called a *center* if there exists an open neighborhood that, in addition to the singular point, it consists of periodic orbits;

Please note that the above is by no means a complete classification of the types of singular points of planar vector fields. We just listed the ones that are of interest in our work presented in this thesis.

2.4   Bruno's Method of Normal Forms

In this section we describe the method of normal forms and sector decomposition introduced by Alexander D. Bruno [7] to describe the phase portrait of a planar system of ODE's with a singular point at the origin. As before, we shall focus on the concepts and results that are relevant for the proof of the main result of Chapter 3. We will make extensive use of the material in Bruno's textbook [7] and in the synthesis presented in [36, Section 5].

Consider the following system of two ordinary differential equations in $\mathbb{R}^2$

$$
\begin{aligned}
dx_1/dt = \dot{x}_1 = \varphi_1(x_1, x_2), \\
dx_2/dt = \dot{x}_2 = \varphi_2(x_1, x_2)
\end{aligned}
\tag{2.4.1}
$$

where φ_1, φ_2 are real analytic. Suppose that the origin is a regular point for (2.4.1). Then, the following is true [7, Theorem 1, page 98]

Theorem 2.4.1. *There exists an invertible, real analytic change of coordinates in a neighborhood of the origin,*

$$
x_i = \xi_i(y_1, y_2), \ \ \xi_i(0,0) = 0, \ \ i = 1, 2,
$$

under which system (2.4.1) becomes

$$
\dot{y}_1 = 1, \ \ \dot{y}_2 = 0.
\tag{2.4.2}
$$

Remark 2.4.2. In his textbook [7], Bruno calls the regular points of (2.4.1) *simple.* Theorem 2.4.1 states that, in a sufficiently small neighborhood of the origin, the solutions of system (2.4.1) are topologically equivalent (Definition 2.3.2) to those of (2.4.2) which is a simpler system. Citing Bruno, "(the theorem) says that the solutions of the system in a neighborhood of a simple point have simple structure".

When the origin is a singular point, i.e., when $\varphi_1(0,0) = \varphi_2(0,0) = 0$, the analysis of the phase portrait of (2.4.1) becomes more complicated. For clarity, throughout this dissertation we deal with only isolated singular points so, in this case, the origin is such a point. In the proof of Theorem 1.1.1 we analyse the phase portrait of a system of ordinary differential equations with a non-elementary singular point (Definition 2.3.3) at the origin. By applying suitable changes of coordinates, Bruno's method allows for the transformation of the original ODE system to one whose singular point is elementary and whose phase portraits are topologically related in a way that we shall describe in the remainder of this section. By using Bruno's normal forms, which we discuss below, it is somewhat easier to determine the phase portrait of an isolated elementary singular point.

2.4.1   Normal Forms of Elementary Singular Points

Consider the following ODE system

$$\dot{x}_i = \lambda_i x_i + \sigma_i x_{i-1} + \varphi_i(X), \ \ i = 1, 2, \tag{2.4.3}$$

where x_i are smooth functions of a real variable, λ_i, σ_i are real with $\sigma_1 = 0$ and φ_i are real analytic in $X = (x_1, x_2)$, such that their power series expansion at the origin do not contain constant or linear terms. We make the assumption that at least one of the eigenvalues λ_i is non-zero, i.e., $|\lambda_1| + |\lambda_2| \neq 0$, which makes the origin an elementary singular point of (2.4.3). The goal is to transform system (2.4.3) into the simplest possible form

$$\dot{y}_i = \lambda_i y_i + \sigma_i y_{i-1} + \psi_i(X), \ \ i = 1, 2, \tag{2.4.4}$$

by using a local invertible coordinate transformation

$$x_i = y_i + \xi_i(Y), \ \ i = 1, 2, \tag{2.4.5}$$

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where $Y = (y_1, y_2)$ and the power series expansions of ξ_i do not contain constant or linear terms. In general, such change of coordinates are not necessarily real analytic, which means that ξ_i can be divergent.

In what follows, for every $X = (x_1, x_2) \in \mathbb{R}^2$ and $Q = (q_1, q_2) \in \mathbb{Z}^2$ we shall use the notations $X^Q = x_1^{q_1} x_2^{q_2}$ and $|Q| = |q_1| + |q_2|$. With these notations, the expansion of ξ_i can be written as

$$\xi_i(Y) = \sum_{|Q|>1} h_{iQ} Y^Q, \ i = 1, 2.$$

It is helpful to write system (2.4.4) in the following form

$$\dot{y}_i = y_i \sum_{Q \in N_i} g_{iQ} Y^Q, \ i = 1, 2, \tag{2.4.6}$$

where

$$N_1 = \{Q = (q_1, q_2) \in \mathbb{Z}^2 : q_1 \geq -1, q_2 \geq 0, q_1 + q_2 \geq 0\},$$
$$N_2 = \{Q = (q_1, q_2) \in \mathbb{Z}^2 : q_1 \geq 0, q_2 \geq -1, q_1 + q_2 \geq 0\}.$$

Let $\Lambda = (\lambda_1, \lambda_2)$ where λ_1, λ_2 are the eigenvalues of (2.4.4) and denote by $\langle \cdot, \cdot \rangle$ the dot product in $\mathbb{R}^2$. The following statement is true ([7, Theorem 2, page 105])

Theorem 2.4.3 (The Principal Theorem on the Normal Form). *For every system of the form (2.4.3) there exists a change of coordinates (2.4.5) which transforms it into a system (2.4.4) for which $g_{iQ} = 0$ whenever $\langle Q, \Lambda \rangle = q_1 \lambda_1 + q_2 \lambda_2 \neq 0$.*

This means that the only non-zero terms in system (2.4.4) are the terms of the form $y_i g_{iQ} Y^Q$ for which $\langle Q, \Lambda \rangle = 0$. Such terms are called *resonant.*

Definition 2.4.4. A planar ODE system for which all terms are resonant is called a *normal form*. A change of coordinates that transforms a given system into a normal form is called a *normalizing transformation.*

As Bruno states it, Theorem 2.4.1 guarantees that "every formal system (2.4.3) can be put into a normal form by applying a normalizing transformation".

Let us consider now the following system of two differential equations

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{Q \in \mathbf{V}} f_{iQ} X^Q = \lambda_i x_i + x_i f_i, \quad i = 1, 2, \qquad (2.4.7)$$

where $\Lambda = (\lambda_1, \lambda_2) \neq 0$ and the set $\mathbf{V} \subset \mathbb{Z}^2$ is to be specified. In the hypothesis of the Principal Normal Form Theorem, $\varphi_i(X)$ are power series in nonnegative powers of variables and the corresponding $\mathbf{V}$ is almost completely contained in the first quadrant of the plane.

Let R^* and R_* be two vectors in $\mathbb{R}^2$ contained in the second and the fourth quadrant respectively and denote by $\mathbf{V}$ the sector bounded by R^* and R_* such that $\mathbf{V}$ contains the first quadrant. We choose R^* and R_* in such way that the sector $\mathbf{V}$ is the convex cone generated by R^* and R_*, i.e., it consists of the vectors $\alpha_1 R^* + \alpha_2 R_*$ with $\alpha_j \geq 0$.

Denote by $\mathcal{V}(X)$ the space of power series $\sum_Q f_Q X^Q$, where $Q \in \mathbf{V}$. Since in our situation such a series can have an infinite number of terms with negative exponents (even after multiplication by x_i), the notion of its convergence requires clarification. Consider first a numerical series

$$\sum_{Q \in \mathbb{Z}^2} a_Q, \qquad (2.4.8)$$

where the indices Q run through $\mathbb{Z}^2$. Let (Ω_n) be an increasing exhausting sequence of bounded domains in $\mathbb{R}^2$. Set

$$S_n = \sum_{Q \in \Omega_n} a_Q$$

(the partial sums). If the sequence (S_n) admits the limit S and this limit is independent of the choice of the sequence (Ω_n), then we say that series (2.4.8) converges to the sum S. It is well-known that if for some sequence (Ω_n) the sequence of the partial sums of the series

$$\sum_{Q \in \mathbb{Z}^2} |a_Q| \qquad (2.4.9)$$

converges, then series (2.4.8) and (2.4.9) converge. In this case we say that series (2.4.8) converges absolutely.

Under the above assumptions on R^* and R_* a series of class $\mathcal{V}(X)$ is called convergent if it converges absolutely in the set

$$\mathcal{U}_{\mathbf{V}}(\varepsilon) = \left\{ X : |X|^{R_*} \leq \varepsilon, |X|^{R^*} \leq \varepsilon, |x_1| \leq \varepsilon, |x_2| \leq \varepsilon \right\}, \qquad (2.4.10)$$

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for some $\varepsilon > 0$. As explained in detail in [7], this subset of the real plane is a natural domain of convergence for such a series. As an example we notice that when the sector $\mathbf{V}$ is defined by the vectors $R_* = (1,0)$ and $R^* = (0,1)$, i.e., when it coincides with the first quadrant, then the class $\mathcal{V}(X)$ coincides with the class of usual power series with nonnegative exponents and the set $\mathcal{U}_\mathbf{V}(\varepsilon)$ coincides with the bidisc of radius ε.

Let $\mathbf{V}$ be a sector which determines system (2.4.7). We consider changes of variables of the form

$$x_i = y_i + y_i h_i(Y), \quad i = 1, 2, \tag{2.4.11}$$

where $h_i \in \mathcal{V}(Y)$, i.e., $h_i(Y) = \sum_{Q \in \mathbf{V}} h_{iQ} Y^Q$. In the new coordinates the system takes the form

$$y_i = \lambda_i y_i + y_i g_i(Y), \quad i = 1, 2. \tag{2.4.12}$$

Theorem 2.4.5 (Second Normal Form). *Suppose that $\mathbf{V}$ is a sector as described above. Then system (2.4.7) can be transformed by a formal change of variables (2.4.11) into a normal form (2.4.12) with $g_i \in \mathcal{V}(Y)$. The coefficients of g_i satisfy $g_{iQ} = 0$ if $\langle Q, \Lambda \rangle \neq 0$.*

The normalizing change of coordinates in the above theorem in general is not convergent, even if system (2.4.7) is analytic. However, such a change of coordinates is always convergent or C^∞-smooth in $\mathcal{U}_\mathbf{V}(\varepsilon)$. For this reason the behaviour of the integral curves of systems (2.4.7) and (2.4.12) coincide in the sector given by (2.4.10) for sufficiently small $\varepsilon > 0$.

2.4.2 The Newton Diagram

Let $\mathcal{X}$ be a real analytic vector field on $\mathbb{R}^2_{(x_1, x_2)}$. Its power series expansion at 0 can be written as

$$\mathcal{X}(x) = \sum_{j=1,2} \sum_Q f_{jQ} \, x^Q x_j \, \frac{\partial}{\partial x_j}, \tag{2.4.13}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $Q = (q_1, q_2) \in \mathbb{Z}^2, q_j \geq -1$ and $x^Q = x_1^{q_1} x_2^{q_2}$. We also assume that $f_{1(i,-1)} = f_{2(-1,i)} = 0$ for all $i \in \mathbb{N} \cup \{-1\}$ (here $\mathbb{N} = \{0, 1, 2, \dots\}$). We call the subset of $\mathbb{R}^2$ defined by

$$\mathbf{D} = \{Q \in \mathbb{Z}^2 : |f_{1Q}| + |f_{2Q}| \neq 0\} \tag{2.4.14}$$

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the *support* of the vector field $\mathcal{X}$. The *Newton polygon* of $\mathcal{X}$ is defined as the convex hull Γ of the set

$$\bigcup_{Q \in \mathbf{D}} \{Q + P : P \in \mathbb{R}_+^2\},$$

where $\mathbb{R}_+ = [0, +\infty)$. It coincides with the intersection of all support half spaces of $\mathbf{D}$ (see [7], [36, Section 5]). The boundary of Γ consists of edges, which we denote by $\Gamma_j^{(1)}$, and vertices, which we denote by $\Gamma_j^{(0)}$, where j is some enumeration and the upper index denotes the dimension of the object. The union of the compact edges of Γ, which we denote by $\hat{\Gamma}$, is called the *Newton diagram* of $\mathcal{X}$.

Example 2.4.6. If $\mathbf{D}$ consists of the points $(0,2), (0,1), (1,0), (2,0)$ the Newton diagram is formed by the two vertices, $\Gamma_1^{(0)} = (0,1), \Gamma_2^{(0)} = (1,0)$ and the edge connecting them $\Gamma_1^{(1)}$.

2.4.3   Nonelementary Singular Points

We consider now the system

$$\dot{x}_i = \varphi_i(x_1, x_2), \ \varphi_i(0,0) = 0, \ i = 1, 2, \tag{2.4.15}$$

where φ_i are real analytic and the origin is an isolated nonelementary singular point. The vector field defined by (2.4.15) can be written as

$$\mathcal{X}(x_1, x_2) = \sum_{i=1,2} \sum_Q f_{iQ} \, (x_1, x_2)^Q x_i \, \frac{\partial}{\partial x_i}, \tag{2.4.16}$$

where $\varphi_i(x_1, x_2) = x_i f_i(x_1, x_2)$ and

$$f_i(x_1, x_2) = \sum_Q f_{iQ} \, (x_1, x_2)^Q. \tag{2.4.17}$$

As per Bruno's method, for each element $\Gamma_j^{(d)}$ of the Newton diagram $\hat{\Gamma}$ associated with (2.4.16), there is a corresponding sector $\mathcal{U}_j^d$ in the phase space $\mathbb{R}_{(x_1,x_2)}^2$, so that together they form a full neighbourhood of the origin (here the boundaries of the sectors are not necessarily integral curves). In each $\mathcal{U}_j^d$ one brings the system to a normal form by using power transformations

(quasihomogeneous blow-ups) which reduces the problem to the study of elementary singularities of the transformed system. This allows one to determine the behaviour of the orbits in each sector. After that the results in each sector are glued together to obtain the overall phase portrait of the system near the origin. We distinguish between two cases: that of a vertex and that of an edge.

The Case of a Vertex. We define a *unit* vector $R \in \mathbb{R}^2$ to be a vector whose coordinates are coprime integers. Let $Q = \Gamma_j^{(0)}$ be a vertex of $\hat{\Gamma}$. Consider the unit vectors $R_{j-1} = (r_{1,j-1}, r_{2,j-1})$ and $R_j = (r_{1,j}, r_{2,j})$ directional to $\Gamma_{j-1}^{(1)}$ and $\Gamma_j^{(1)}$ respectively, assuming that $r_{2,j-1} > 0$ and $r_{2,j} > 0$, so that the vectors are determined uniquely. Set $R_* = -R_{j-1}$ and $R^* = R_j$. If Q is a boundary point of $\hat{\Gamma}$, we set $R_* = (1,0)$ if Q is the right boundary point of $\hat{\Gamma}$, denoted as Q_*, and we set $R^* = (0,1)$ if Q is the left boundary point of $\hat{\Gamma}$, which we denote as Q^*. Bruno's method associates to Q a set defined by

$$
\begin{aligned}
\mathcal{U}_j^{(0)}(\varepsilon) = \{(x_1, x_2) \in \mathbb{R}^2 : & (|x_1|, |x_2|)^{R^*} \leq \varepsilon, \\
& (|x_1|, |x_2|)^{R_*} \leq \varepsilon, |x_1| \leq \varepsilon, |x_2| \leq \varepsilon\},
\end{aligned}
\tag{2.4.18}
$$

for some $\varepsilon > 0$. Applying the change of time coordinate from the old time τ to τ_1 which satisfies $d\tau_1 = (t,s)^Q d\tau$, the system (2.4.15) transforms into one of the form (2.4.7). The resulting system satisfies the assumptions of the Principal or the Second Normal Form Theorem. The behaviour of the integral curves of the normal form and the original system coincides in $\mathcal{U}_j^{(0)}(\varepsilon)$ for ε sufficiently small. See [7, page 138] for a detailed discussion and justification of these facts.

The Case of an Edge. Let $\Gamma_j^{(1)}$ be an edge of $\hat{\Gamma}$ and let $R = (r_1, r_2)$, $r_2 > 0$ be a unit directional vector of $\Gamma_j^{(1)}$. The corresponding set in the phase space is given by

$$
\mathcal{U}_j^1(\varepsilon) = \{(t,s) \in \mathbb{R}^2 : \varepsilon \leq (|t|, |s|)^R \leq 1/\varepsilon, \quad |t| \leq \varepsilon, |s| \leq \varepsilon\}.
\tag{2.4.19}
$$

Consider the power transformation given by $y_1 = t^{k_1} s^{k_2}, y_2 = t^{r_1} s^{r_2}$, where the integers k_1, k_2 are chosen such that the matrix

$$
A = \begin{pmatrix} k_1 & k_2 \\ r_1 & r_2 \end{pmatrix}
\tag{2.4.20}
$$

has the determinant equal to 1. In the matrix form, we can write $X = (t, s)$,

$$
Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},
$$

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$$F_Q = \left(\begin{array}{c} f_{1q} \\ f_{2q} \end{array} \right).$$

Then (2.4.15) can be given by

$$(\ln \dot{X}) = \sum_{Q \in \mathbf{D}} F_Q X^Q, \qquad (2.4.21)$$

where $X^Q = t^{q_1} s^{q_2}$. The power transformation can be expressed now as $Y = X^A$ taking (2.4.21) into

$$(\ln \dot{Y}) = \sum_{Q' \in D'} F'_{Q'} Y^{Q'},$$

with $Y = (y_1, y_2)$, $Q' = (A^T)^{-1}Q$, $D' = (A^T)^{-1}\mathbf{D}$, and $F'_{Q'} = AF_Q$. After division by the maximal power of y_1 one obtains a new system. Here the y_2-axis corresponds to $\{t = s = 0\}$ in the original coordinates, and therefore one needs to investigate the new system in a neighbourhood of the y_2-axis.

Chapter 3

Polynomial Convexity of the Open Whitney Umbrella

In this Chapter we prove Theorem 1.1.1 (see Section 1.1). The approach uses the method introduced in [**36**].

3.1 Reduction to a Dynamical System

We first review how the problem of local polynomial convexity near a Whitney umbrella can be reduced to the computation of the phase portrait of a certain dynamical system, a method that was introduced in [36]. In fact, the procedure works without modifications for a somewhat more general type of isolated singularities.

3.1.1 The characteristic foliation

Let $\tau : \mathbb{R}^2 \to \mathbb{R}^4 \cong \mathbb{C}^2$, $\tau(0) = 0$, be a homeomorphism onto its image, smooth except at the origin, and such that $S = \tau(\mathbb{R}^2)$ is a totally real surface in $\mathbb{C}^2$ with an isolated singular point at the origin. Suppose S is embedded in a real hypersurface M in $\mathbb{C}^2$. We define a field of lines determined at every $p \in S \setminus \{0\}$ by

$$L_p = T_p S \cap H_p M,$$

where $H_p M = T_p M \cap J T_p M$ is the complex tangent space of M at p and J is the standard complex structure on $\mathbb{C}^2$. The foliation defined by the integral

curves corresponding to this field is called the *characteristic foliation* of S (with respect to M).

Let us also suppose that M is defined as the zero locus of a function $\rho : \mathbb{C}^2 \to \mathbb{R}$, smooth and strictly plurisubharmonic near the origin,

$$M = M(\rho) = \{(z, w) \in \mathbb{C}^2 : \rho(z, w) = 0\}, \ \nabla\rho|_{M\setminus\{0\}} \neq 0,$$

and let

$$\Omega(\rho) = \{(z, w) \in \mathbb{C}^2 : \rho(z, w) < 0\}.$$

The *essential hull* K^{ess} of a compact set $K \subset \mathbb{C}^2$ is defined by $K^{\mathrm{ess}} = \hat{K} \setminus K$, and its *trace* K^{tr} by $K^{\mathrm{tr}} = K^{\mathrm{ess}} \cap K$. We note that

$$K^{\mathrm{ess}} \subseteq \widehat{K^{\mathrm{tr}}}. \tag{3.1.1}$$

Indeed, a local Maximum Principle due to Rossi [33, 40] states that if K is a compact set in $\mathbb{C}^n$, E is a compact subset of $\hat{K}$ and U is an open subset of $\mathbb{C}^n$ that contains E, then for all $f \in \mathcal{O}(U)$, $\|f\|_E = \|f\|_{(E\cap K)\cup\partial E}$, where the boundary of E is taken with respect to $\hat{K}$. Now, by choosing $E = K^{\mathrm{ess}}$ and $U = \mathbb{C}^2$, we obtain (3.1.1).

Since τ is continuous, the set $S = \tau(\mathbb{R}^2)$ is connected. Let $\varepsilon > 0$ be such that ρ is strictly plurisubharmonic in $B(0; \varepsilon)$. By a classical result (see, for example, [19, 40]), the polynomially convex hull of $\overline{S \cap B(0; \varepsilon)}$ agrees with its psh-hull (see Section 2.2). Hence, the polynomial hull of the set $\overline{S \cap B(0; \varepsilon)}$ is contained in $\overline{\Omega(\rho) \cap B(0; \varepsilon)}$. Let X be the connected component of $S \cap \overline{B(0; \varepsilon)}$ containing the origin. Then $X \setminus \{0\}$ is a smooth compact real surface embedded in $\partial\Omega(\rho)$. The following key proposition is essentially due to Duval [11] (see also Jöricke [21]).

Proposition 3.1.1. X^{tr} *cannot intersect a leaf of the characteristic foliation at a totally real point of X without crossing it.*

The original proof of Duval can be easily adapted to our situation. It is an application of Oka's characterization of polynomially convex subsets of $\mathbb{C}^n$. Oka's family of algebraic curves can be constructed from the leaves of the characteristic foliation, and because Ω is strictly pseudoconvex, it suffices to ensure that the family leaves Ω. See [36] for details.

The last step in reducing the problem to a dynamical system is provided by the following result. Recall that a *rectifiable arc* is the homeomorphic image of an interval under a Lipschitz continuous map.

Proposition 3.1.2. *Suppose that there exist two rectifiable arcs γ_1, γ_2 in X such that*

    (i) $\gamma_1 \cap \gamma_2 = \{0\}$;

    (ii) γ_j are smooth at all points except, possibly, at the origin;

    (iii) For any compact subset $K \subset X$ not contained in $\gamma_1 \cup \gamma_2$, there exists a leaf γ of the characteristic foliation of S such that $K \cap \gamma \neq \emptyset$ but K does not meet both sides of γ.

Then, X is polynomially convex.

Proof. It follows from Proposition 3.1.1 that $X^{\mathrm{tr}} \subseteq \gamma_1 \cup \gamma_2$ and from (3.1.1) that $X^{\mathrm{ess}} \subseteq \widehat{\gamma_1 \cup \gamma_2}$. A rectifiable arc is polynomially convex [40, Corollary 3.1.2]. Moreover, by [40, Theorem 3.1.1], if Y is a compact polynomially convex subset of $\mathbb{C}^n$ and Γ is a compact connected set of finite length, then $(\widehat{Y \cup \Gamma}) \setminus (Y \cup \Gamma)$ is either empty or it contains a complex purely one-dimensional analytic subvariety of the complement $\mathbb{C}^2 \setminus (Y \cup \Gamma)$. By taking Y and Γ to be the arcs γ_1, γ_2, it can be shown by following the same rationale as in [36, Corollary 2], that the union of the two arcs cannot bound a complex one-dimensional variety. Therefore, $\widehat{\gamma_1 \cup \gamma_2} = \gamma_1 \cup \gamma_2 \subset X$, so $X^{\mathrm{ess}} \subset X$. Since $\widehat{X} \setminus X \subseteq X^{\mathrm{ess}} \setminus X = \emptyset$, it follows that X is polynomially convex. $\square$

Before we proceed to the proof of Theorem 1.1.1 let us present the proof of the corollary stated in Section 1.1, assuming that Theorem 1.1.1 is true. The proof is exactly the same as the one in [36, page 9].

Proof of Corollary 1.1.2. Let $\phi(0) = p$. By Theorem 1.1.1 there exists $\varepsilon > 0$ such that $X = \overline{\phi(\Sigma) \cap \mathbb{B}(p, \varepsilon)}$ is polynomially convex. For sufficiently small ε we may further assume that $\phi(\Sigma) \cap \partial\mathbb{B}(p, \varepsilon)$ is a smooth curve. By the result of J. Anderson, A. Izzo, and J. Wermer [3, Thm. 1.5], if X is a polynomially convex compact subset of $\mathbb{C}^n$, and X_0 is a compact subset of X such that $X \setminus X_0$ is a totally real submanifold of $\mathbb{C}^n$, of class C^1, then continuous functions on X can be approximated by polynomials if and only if this can be done on X_0. We apply this result to $X = \overline{\phi(\Sigma) \cap \mathbb{B}(p, \varepsilon)}$ and $X_0 = \{p\} \cup (\phi(\Sigma) \cap \partial\mathbb{B}(p, \varepsilon))$. The set X_0, is polynomially convex. Indeed, if not, we obtain as in the proof of Proposition 3.1.2 that $\widehat{X_0} \setminus X_0$ contains a complex purely 1-dimensional analytic subvariety V of $\mathbb{C}^2 \setminus X_0$. But then V is contained in $\widehat{X}$, which contradicts Theorem 1.1.1. Furthermore, by for

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example [40], p. 122, continuous functions on X_0 can be approximated by polynomials. From this the corollary follows. $\square$

Our next goal is to find a suitable hypersurface containing the open Whitney umbrella, such that the properties of Proposition 3.1.2 are satisfied.

3.1.2   The characteristic foliation of the open Whitney umbrella

We identify $\mathbb{R}^4_{(x,u,y,v)}$ with $\mathbb{C}^2_{(z,w)}$ for computational purposes. If I_2 is the 2×2 identity matrix, we denote by

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$$

the matrix defining the standard complex structure on $\mathbb{C}^2$. Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a local symplectomorphism which, without loss of generality, is assumed to preserve the origin. Let the Jacobian matrix of ϕ at 0 be

$$D\phi(0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are the 2×2 block components given by the partial derivatives of ϕ. Since ϕ is symplectic, we have

$$A^t D - C^t B = I_2, \ \ A^t C = C^t A, \ \ D^t B = B^t D. \tag{3.1.2}$$

Let $\psi : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by the matrix

$$\Psi = \begin{pmatrix} D^t & -B^t \\ B^t & D^t \end{pmatrix}.$$

Since $\Psi J = J \Psi$, the map ψ is complex linear. We now show that Ψ is invertible. From (3.1.2) we get

$$D(\psi \circ \phi)(0) = \begin{pmatrix} I_2 & 0 \\ E & G \end{pmatrix}, \ \ \ E = (e_{ij}), e_{ij} \in \mathbb{C}, \tag{3.1.3}$$

where

$$G = (g_{ij}) = B^t B + D^t D. \tag{3.1.4}$$

Since $D\psi(0)$ is symplectic, $\det D\phi(0) = 1$, and so $\det G = \det \Psi$. We claim that

$$\det G = g_{11}g_{22} - g_{12}^2 > 0. \tag{3.1.5}$$

Indeed, let $B = (b_{jk})$, and $D = (d_{jk})$. A straightforward computation gives

$$\det G = (b_{11}b_{22} - b_{12}b_{21})^2 + (b_{11}d_{12} - b_{12}d_{11})^2 + (b_{11}d_{22} - b_{12}d_{21})^2$$
$$+ (b_{21}d_{12} - b_{22}d_{11})^2 + (b_{21}d_{22} - b_{22}d_{21})^2 + (d_{11}d_{22} - d_{12}d_{21})^2,$$

which is obviously nonnegative. If $\det G = 0$, then, for $j = 1, 2$, the following hold

$$(b_{j2} = 0) \Rightarrow (b_{j1} = 0), \quad (d_{j2} = 0) \Rightarrow (d_{j1} = 0).$$

On the other hand, if any two or more of $b_{12}, b_{22}, d_{12}, d_{22}$ do not equal 0, then the corresponding ratios $\dfrac{b_{11}}{b_{12}}, \dfrac{b_{21}}{b_{22}}, \dfrac{d_{11}}{d_{12}}, \dfrac{d_{21}}{d_{22}}$ are equal, e.g., if $b_{12} \neq 0$, $b_{22} \neq 0$, $d_{12} \neq 0$, and $d_{22} \neq 0$, then

$$\frac{b_{11}}{b_{12}} = \frac{b_{21}}{b_{22}} = \frac{d_{11}}{d_{12}} = \frac{d_{21}}{d_{22}} = \lambda \in \mathbb{R}.$$

It is not difficult to see that all possible combinations lead to $D\phi(0)$ either having two identically zero columns in the vertical $B|D$ block, or one column being a λ multiple of another. In both scenarios $\det D\phi(0) = 0$, which is a contradiction. It follows then, that $\det G > 0$, which proves that Ψ is nonsingular. Furthermore, (3.1.4) and (3.1.5) imply that $g_{11} > 0$, $g_{22} > 0$.

Now, let

$$\Sigma' = (\psi \circ \phi)(\Sigma),$$

which by construction is a totally real surface with an isolated singular point at the origin. We consider the following auxiliary hypersurface which contains Σ,

$$M = M(\rho) = \{(z, w) \in \mathbb{C}^2 : \rho(z, w) := x^2 - yv^2 + \frac{9}{4}u^2 - y^3 + C(xy - \frac{3}{2}uv) = 0\},$$

where $C > 0$. A direct computation shows that for any $C > 0$, the gradient $\nabla \rho$ does not vanish in some punctured neighbourhood of the origin. Now, put

$$M' = (\psi \circ \phi)(M) = M'(\rho'), \quad \rho' := \rho \circ (\psi \circ \phi)^{-1}.$$

It follows that M' is also smooth in some punctured neighbourhood of the origin. Clearly, $\varphi(\Sigma)$ is locally polynomially convex at the origin if and only if

$(\psi \circ \varphi)(\Sigma)$ is. We next show that, for some $C > 0$, M' is strictly pseudoconvex near the origin. Let (x', u', y', v') be the coordinates in the target space of $\psi \circ \phi$ and let

$$(D(\psi \circ \phi)(0))^{-1} = \begin{pmatrix} I_2 & 0 \\ E' & G' \end{pmatrix}, \quad E' = (e'_{ij}), G' = (g'_{ij}), \ e_{ij}, g_{ij} \in \mathbb{C}.$$

The formal Taylor expansion of $(\psi \circ \phi)^{-1}$ is given by

$$(\psi \circ \phi)^{-1}(x', u', y', v') = \left(x' + \sigma^1, u' + \sigma^2, e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3, \right.$$

$$\left. e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4 \right),$$

where

$$\sigma^i = \sum_{j+k+l+m \geq 2} h^i_{jklm} x'^j u'^k y'^l v'^m, \ h^i_{jklm} \in \mathbb{C}, \ i \in \{1, 2, 3, 4\}.$$

Then,

$$\begin{aligned} \rho'(x', u', y', v') = & (x' + \sigma^1)^2 \\ & - (e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3) \\ & \cdot (e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4)^2 \\ & + \frac{9}{4}(u' + \sigma^2)^2 - (e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3)^3 \\ & + C(x' + \sigma^1)(e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3) \\ & - \frac{3C}{2}(u' + \sigma^2)(e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4). \end{aligned}$$

A direct computation gives the Levi form of ρ',

$$L_{\rho'} = \begin{pmatrix} 2 + 2Ce'_{11} & Ce'_{12} - \dfrac{3}{2}Ce'_{21} + \dfrac{5i}{2}Cg'_{12} \\[2mm] Ce'_{12} - \dfrac{3}{2}Ce'_{21} - \dfrac{5i}{2}Cg'_{12} & \dfrac{9}{2} - 3Ce'_{22} \end{pmatrix}.$$

From this it is clear that for C sufficiently small the Levi form is strictly positive-definite. This implies that ρ' is strictly plurisubharmonic near the

origin, hence M' is strictly pseudoconvex in some punctured neighbourhood of the origin. Note that the constant C depends on the symplectomorphism ϕ.

We will show that $S = \Sigma'$ and $M = M'(\rho')$ satisfy the conditions of Proposition 4.2.1. For this, in Section 3.2, we compute the dynamical system describing the characteristic foliation of Σ', and in Section 3.3 we describe the method of reduction to the principal part of a vector field due to Brunella and Miari [6]. We use this in Section 3.4 to determine the phase portrait of the characteristic foliation.

3.2 Calculation of the System

In this section we compute the relevant low order terms of the pullback to the parameterizing plane $\mathbb{R}^2_{(t,s)}$ of the dynamic system that determines the characteristic foliation of Σ'. We introduce the following notation for the components of the gradient of ρ',

$$\nabla \rho' = \left(R_x(t,s), R_u(t,s), R_y(t,s), R_v(t,s) \right),$$

and we also set

$$\sigma^i_x = \frac{\partial \sigma^i}{\partial x'}, \quad \sigma^i_u = \frac{\partial \sigma^i}{\partial u'}, \quad \sigma^i_y = \frac{\partial \sigma^i}{\partial y'}, \quad \sigma^i_v = \frac{\partial \sigma^i}{\partial v'}, \ i \in \{1, 2, 3, 4\}.$$

A straightforward computation gives the Jacobian matrix of $(\psi \circ \phi)^{-1}$ at the origin,

$$D(\psi \circ \phi)^{-1}(0) = \begin{pmatrix} I_2 & 0 \\ E' & G' \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ -G^{-1}E & G^{-1} \end{pmatrix}. \qquad (3.2.1)$$

The characteristic foliation of Σ' is determined at every $p \in \Sigma' \setminus \{0\}$ by

$$L_p \Sigma' = T_p \Sigma' \cap H_p M', \quad H_p M' = T_p M' \cap J(T_p M').$$

It follows that

$$\langle J X_p, \nabla \rho' \rangle = 0, \quad \text{for all } X_p \in L_p \Sigma', \ p \in \Sigma'.$$

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We thus obtain a smooth vector field $X \in T\Sigma'$, given by

$$X = \alpha \frac{\partial f}{\partial t} + \beta \frac{\partial f}{\partial s}, \qquad (3.2.2)$$

where $f : \mathbb{R}^2 \to \mathbb{R}^4$ is defined as

$$f = \psi \circ \phi \circ \pi,$$

and α, β are smooth functions on $\mathbb{R}^2$, satisfying $X_{p=f(t,s)} \in L_{p=f(t,s)}\Sigma'$, for $p \neq 0$. Consequently, we can choose

$$\alpha(t,s) = \langle J \frac{\partial f}{\partial s}, \nabla \rho' \rangle, \qquad \beta(t,s) = -\langle J \frac{\partial f}{\partial t}, \nabla \rho' \rangle. \qquad (3.2.3)$$

We conclude that the characteristic foliation of Σ' is defined by the following system of ODE's

$$\begin{cases} \dot{t} = \alpha(t,s) \\ \dot{s} = \beta(t,s). \end{cases} \qquad (3.2.4)$$

Writing

$$f(t,s) = (f_1(t,s), f_2(t,s), f_3(t,s), f_4(t,s)),$$

and using (3.1.3) and (1.1.1), we can express each f_i as a formal power series in (t,s):

$$f_1(t,s) = ts + f^1_{02}s^2 + f^1_{12}ts^2 + f^1_{21}t^2s + f^1_{03}s^3 + \sum_{j+k \geq 4} f^1_{jk}t^j s^k,$$

$$f_2(t,s) = \frac{2}{3}t^3 + f^2_{02}s^2 + f^2_{12}ts^2 + f^2_{21}t^2s + f^2_{03}s^3 + \sum_{j+k \geq 4} f^2_{jk}t^j s^k,$$

$$f_3(t,s) = g_{12}s + g_{11}t^2 + e_{11}ts + f^3_{02}s^2 + \frac{2e_{12}}{3}t^3 + f^3_{12}ts^2$$
$$+ f^3_{21}t^2s + f^3_{03}s^3 + \sum_{j+k \geq 4} f^3_{jk}t^j s^k, \qquad (3.2.5)$$

$$f_4(t,s) = g_{22}s + g_{12}t^2 + e_{21}ts + f^4_{02}s^2 + \frac{2e_{22}}{3}t^3 + f^4_{12}ts^2$$
$$+ f^4_{21}t^2s + f^4_{03}s^3 + \sum_{j+k \geq 4} f^4_{jk}t^j s^k,$$

From the above identities, putting $X_t = \dfrac{\partial f}{\partial t}$, $X_s = \dfrac{\partial f}{\partial s}$, we get

$$X_t = \begin{pmatrix} s + 2f_{21}^1 ts + f_{12}^1 s^2 \\[6pt] 2t^2 + 2f_{21}^2 ts + f_{12}^2 s^2 \\[6pt] 2g_{11}t + e_{11}s + 2e_{12}t^2 + 2f_{21}^3 ts + f_{12}^3 s^2 \\[6pt] 2g_{12}t + e_{21}s + 2e_{22}t^2 + 2f_{21}^4 ts + f_{12}^4 s^2 \end{pmatrix} + o(|(t,s)|^2), \qquad (3.2.6)$$

and

$$X_s = \begin{pmatrix} t + 2f_{02}^1 s + f_{21}^1 t^2 + 2f_{12}^1 ts + 3f_{03}^1 s^2 \\[6pt] 2f_{02}^2 s + f_{21}^2 t^2 + 2f_{12}^2 ts + 3f_{03}^2 s^2 \\[6pt] g_{12} + e_{11}t + 2f_{02}^3 s + f_{21}^3 t^2 + 2f_{12}^3 ts + 3f_{03}^3 s^2 \\[6pt] g_{22} + e_{21}t + 2f_{02}^4 s + f_{21}^4 t^2 + 2f_{12}^4 ts + 3f_{03}^4 s^2 \end{pmatrix} + o(|(t,s)|^2). \quad (3.2.7)$$

It follows from (3.2.3) that

$$\alpha(t,s) = -(X_s)_3 R_x - (X_s)_4 R_u + (X_s)_1 R_y + (X_s)_2 R_v = \sum_{j,k \geq 0} \alpha_{jk} t^j s^k,$$

$$\beta(t,s) = (X_t)_3 R_x + (X_t)_4 R_u - (X_t)_1 R_y - (X_t)_2 R_v = \sum_{j,k \geq 0} \beta_{jk} t^j s^k,$$

$$(3.2.8)$$

where $(X_t)_i, (X_s)_i$, $i = 1, \ldots, 4$, are the components of X_t, X_s, respectively. A direct computation gives

$$
\begin{aligned}
R_x = {}& 2A(x'+\sigma^1)(1+\sigma_x^1) - A(e'_{11}+\sigma_x^3)(e'_{21}x'+e'_{22}u'+g'_{12}y'+g'_{22}v'+\sigma^4)^2 \\
& - 2A(e'_{21}+\sigma_x^4)(e'_{21}x'+e'_{22}u'+g'_{12}y'+g'_{22}v'+\sigma^4) \\
& \cdot (e'_{11}x'+e'_{12}u'+g'_{11}y'+g'_{12}v'+\sigma^3) \\
& + \frac{9}{2}B(u'+\sigma^2)\sigma_x^2 - 3B(e'_{11}+\sigma_x^3)(e'_{11}x'+e'_{12}u'+g'_{11}y'+g'_{12}v'+\sigma^3)^2, \\
& + C(1+\sigma_x^1)(e'_{11}x'+e'_{12}u'+g'_{11}y'+g'_{12}v'+\sigma^3) \\
& + C(x'+\sigma^1)(e'_{11}+\sigma_x^3) \\
& - \frac{3}{2}C\sigma_x^2(e'_{21}x'+e'_{22}u'+g'_{12}y'+g'_{22}v'+\sigma^4) \\
& - \frac{3}{2}C(u'+\sigma^2)(e'_{21}+\sigma_x^4),
\end{aligned}
$$

$$
\begin{aligned}
R_u = {}& 2A(x'+\sigma^1)\sigma_u^1 - A(e'_{12}+\sigma_u^3)(e'_{21}x'+e'_{22}u'+g'_{12}y'+g'_{22}v'+\sigma^4)^2 \\
& - 2A(e'_{22}+\sigma_u^4)(e'_{21}x'+e'_{22}u'+g'_{12}y'+g'_{22}v'+\sigma^4) \\
& \cdot (e'_{11}x'+e'_{12}u'+g'_{11}y'+g'_{12}v'+\sigma^3) \\
& + \frac{9}{2}B(u'+\sigma^2)(1+\sigma_u^2) - 3B(e'_{12}+\sigma_u^3)(e'_{11}x'+e'_{12}u'+g'_{11}y'+g'_{12}v'+\sigma^3)^2 \\
& + C\sigma_u^1(e'_{11}x'+e'_{12}u'+g'_{11}y'+g'_{12}v'+\sigma^3) \\
& + C(x'+\sigma^1)(e'_{12}+\sigma_u^3) \\
& - \frac{3}{2}C(1+\sigma_u^2)(e'_{21}x'+e'_{22}u'+g'_{12}y'+g'_{22}v'+\sigma^4) \\
& - \frac{3}{2}C(u'+\sigma^2)(e'_{22}+\sigma_u^4),
\end{aligned}
$$

$$R_y = 2A(x' + \sigma^1)\sigma_y^1 - A(g'_{11} + \sigma_y^3)(e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4)^2$$
$$- 2A(g'_{12} + \sigma_y^4)(e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4)$$
$$\cdot (e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3)$$
$$+ \frac{9}{2}B(u' + \sigma^2)\sigma_y^2 - 3B(g'_{11} + \sigma_y^3)(e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3)^2,$$
$$+ C\sigma_y^1(e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3)$$
$$+ C(x' + \sigma^1)(g'_{11} + \sigma_y^3)$$
$$- \frac{3}{2}C\sigma_y^2(e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4)$$
$$- \frac{3}{2}C(u' + \sigma^2)(g'_{12} + \sigma_y^4),$$

$$R_v = 2A(x' + \sigma^1)\sigma_v^1 - A(g'_{12} + \sigma_v^3)(e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4)^2$$
$$- 2A(g'_{22} + \sigma_v^4)(e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4)$$
$$\cdot (e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3)$$
$$+ \frac{9}{2}B(u' + \sigma^2)\sigma_v^2 - 3B(g'_{12} + \sigma_v^3)(e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3)^2$$
$$+ C\sigma_v^1(e'_{11}x' + e'_{12}u' + g'_{11}y' + g'_{12}v' + \sigma^3)$$
$$+ C(x' + \sigma^1)(g'_{12} + \sigma_v^3)$$
$$- \frac{3}{2}C\sigma_v^2(e'_{21}x' + e'_{22}u' + g'_{12}y' + g'_{22}v' + \sigma^4)$$
$$- \frac{3}{2}C(u' + \sigma^2)(g'_{22} + \sigma_v^4).$$

By a direct inspection using (3.2.5), (3.2.6), (3.2.7) and (3.2.8), we find that the terms up to order 3 in the power expansion of $\alpha(t, s)$ are given by

$$\alpha_{01}s = C\left(-g'_{11}g_{12}^2 + \frac{1}{2}g'_{12}g_{12}g_{22} + \frac{3}{2}g'_{22}g_{22}^2\right)s,$$

$$\alpha_{20}t^2 = C\left(-g'_{11}g_{11}g_{12} - g'_{12}g_{12}^2 + \frac{3}{2}g'_{12}g_{11}g_{22} + \frac{3}{2}g'_{22}g_{12}g_{22}\right)t^2,$$

and those of $\beta(t, s)$ by

$$\beta_{11}ts = 2C\left(g'_{11}g_{11}g_{12} + g'_{12}g_{11}g_{22} - \frac{3}{2}g'_{12}g_{12}^2 - \frac{3}{2}g'_{22}g_{22}g_{12}\right)ts,$$

$$\beta_{30}t^3 = 2C\left(g'_{11}g_{11}^2 - \frac{1}{2}g'_{12}g_{12}g_{11} - \frac{3}{2}g'_{22}g_{12}^2\right)t^3.$$

Replacing the primed coefficients with their expressions from (3.2.1) we obtain

$$\alpha_{01} = (3C/2)g_{22}, \quad \alpha_{20} = -Cg_{12}, \quad \beta_{11} = -3Cg_{12}, \quad \beta_{30} = 2Cg_{11}. \quad (3.2.9)$$

Since g_{11}, g_{22} are positive, it follows that

$$\alpha_{01} > 0 \text{ and } \beta_{30} > 0. \tag{3.2.10}$$

and, for $g_{12} \neq 0$,

$$g_{12}\alpha_{20} < 0 \text{ and } g_{12}\beta_{11} < 0. \tag{3.2.11}$$

Combining all of the above, the dynamical system (3.2.4) defining the characteristic foliation of Σ' becomes

$$\begin{cases} \dot{t} = \alpha(t,s) = \frac{3C}{2}g_{22}s - Cg_{12}t^2 + o(|t|^2 + |s|), \\ \dot{s} = \beta(t,s) = -3Cg_{12}ts + 2Cg_{11}t^3 + o(|t|^3 + |ts|). \end{cases} \tag{3.2.12}$$

3.3   Reduction to the Principal Part

To prove that the system (3.2.12) defines a characteristic foliation satisfying the conditions of Proposition 4.2.1, we need to determine the topological structure of the vector field X in (3.2.2). Although the linear part of X does not vanish, its eigenvalues do vanish, making the origin a nonelementary isolated singularity of X. Therefore, we cannot apply standard results, such as the Hartman-Grobman theorem (see for example [9]). Instead, we will make use of a result by Brunella and Miari [6] which, under certain conditions, reduces the problem to determining the topological class of a truncated vector field.

Definition 3.3.1. *Let $\mathcal{X}$ be given as in (2.4.13) and let $\hat{\Gamma}$ be the associated Newton diagram.*

(i) The vector field

$$\mathcal{X}_\Delta(x) = \sum_{j=1,2} \sum_{Q \in \hat{\Gamma}} f_{jQ}\, x^Q x_j \frac{\partial}{\partial x_j}$$

is called the principal part of $\mathcal{X}$.

(ii) *Let* $\Gamma_1^{(1)}, \ldots, \Gamma_N^{(1)}$, *$N > 0$, be all the (compact) edges in the Newton diagram. The vector field*

$$\mathcal{X}_k(x) = \sum_{j=1,2} \sum_{Q \in \Gamma_k^{(1)}} f_Q \, x^Q x_j \frac{\partial}{\partial x_j}$$

is called the quasi-homogeneous component of the principal part $\mathcal{X}_\Delta(x)$ relative to $\Gamma_k^{(1)}$, for $k = 1, \ldots, N$.

In general, if a C^∞-smooth planar vector field does not have characteristic orbits (orbits approaching the singular point in positive or negative time with a well-defined slope limit), then an isolated singularity is either a centre or a focus, or briefly, a *centre-focus*. Following the terminology introduced in [6], we say that two vector fields on $\mathbb{R}^2$, $\mathcal{X}_1$ and $\mathcal{X}_2$, $\mathcal{X}_1(0) = \mathcal{X}_2(0) = 0$, are locally topologically equivalent *modulo centre-focus* if either one of the following cases apply:

(i) $\mathcal{X}_1$ and $\mathcal{X}_2$ have characteristic orbits and are topologically equivalent near the origin, or

(ii) $\mathcal{X}_1$ and $\mathcal{X}_2$ are both centre-foci.

Following Brunella and Miari, we say that a C^∞-smooth planar vector field $\mathcal{X}$, $\mathcal{X}(0) = 0$, has a *nondegenerate* principal part $\mathcal{X}_\Delta$, if none of its quasi-homogeneous components has singularities on $(\mathbb{R} \setminus \{0\})^2$. The main result of Brunella and Miari is the following:

Let $\mathcal{X}$ be a C^∞-smooth vector field on $\mathbb{R}^2$, $\mathcal{X}(0) = 0$, with nondegenerate principal part $\mathcal{X}_\Delta$, such that 0 is an isolated singularity of $\mathcal{X}_\Delta$. Then $\mathcal{X}$ is locally topologically equivalent to $\mathcal{X}_\Delta$ modulo centre-focus.

We remark that in Brunella and Miari's version, the Newton diagram differs from that of Bruno's (Section 2.4) by a translation of $(1,1)$. For the system (3.2.12), the Newton diagram consists of the two vertices $\Gamma_1^{(0)} = (0,2)$, $\Gamma_2^{(0)} = (4,0)$ and the edge $\Gamma_1^{(1)}$ connecting them, see Figure 3.1. The principal part of X is given by

$$X_\Delta(t,s) = \left(\alpha_{01}s + \alpha_{20}t^2\right) \frac{\partial}{\partial t} + \left(\beta_{11}ts + \beta_{30}t^3\right) \frac{\partial}{\partial s}. \qquad (3.3.1)$$

Notice that X_Δ also counts for the terms corresponding to the vertex $\Gamma_3^{(0)} = (2,1) \in \Gamma_1^{(1)}$. Clearly, X_Δ has only one quasi-homogeneous component, that
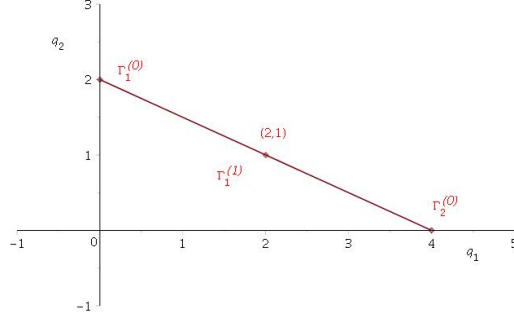
Fig. 3.1: The Newton diagram for (3.2.12) in Brunella and Miari's version.

is X_Δ itself. We claim that X_Δ is nondegenerate. Indeed, a singular point (t, s) of X_Δ would satisfy the system

$$\begin{cases} \alpha_{01} s + \alpha_{20} t^2 = 0 \\ \beta_{11} t s + \beta_{30} t^3 = 0. \end{cases} \qquad (3.3.2)$$

Note that, if $t = 0$, the only solution of (3.3.2) is the origin, hence we can assume $t \neq 0$. Thus, we obtain a linear system in s and t^2, that has nonzero solutions if and only if $\alpha_{01} \beta_{30} = \alpha_{20} \beta_{11}$. However, this is impossible, since, by (3.2.9) and (3.1.5), we have

$$\alpha_{01} \beta_{30} - \alpha_{20} \beta_{11} = 3 C^2 g_{11} g_{22} - 3 C^2 g_{12}^2 = 3 C^2 (g_{11} g_{22} - g_{12}^2) > 0. \qquad (3.3.3)$$

This proves that X_Δ is nondegenerate, with one isolated singularity at the origin. Thus, by Brunella and Miari it suffices to compute the phase portrait of X_Δ.

3.4   Final Step: the Phase Portrait

Recall that the principal part of the vector field defined by (3.2.12) is given by (3.3.1), and the corresponding ODE system is

$$\begin{cases} \dot{t} = \alpha_{01} s + \alpha_{20} t^2 = t(\alpha_{01} t^{-1} s + \alpha_{20} t) \\ \dot{s} = \beta_{11} t s + \beta_{30} t^3 = s(\beta_{11} t + \beta_{30} t^3 s^{-1}). \end{cases} \qquad (3.4.1)$$

We determine the phase portrait of X_Δ near the origin using Bruno's theory of normal forms (see Section 2.4). The Newton diagram of X_Δ

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consists of two vertices, $\Gamma_1^{(0)} = (-1, 1)$, $\Gamma_2^{(0)} = (3, -1)$, and one edge $\Gamma^{(1)}$ connecting $\Gamma_1^{(0)}$ and $\Gamma_2^{(0)}$. By Bruno's classification [7, p 138], the vertices are of Type I, so the integral curves in the sectors

$$\mathcal{U}_1^{(0)}(\varepsilon) = \{(t, s) \in \mathbb{R}^2 : (|t|, |s|)^{(1,0)} \leq \varepsilon, \ (|t|, |s|)^{(-2,1)} \leq \varepsilon\},$$
$$\mathcal{U}_2^{(0)}(\varepsilon) = \{(t, s) \in \mathbb{R}^2 : (|t|, |s|)^{(0,1)} \leq \varepsilon, \ (|t|, |s|)^{(2,-1)} \leq \varepsilon\},$$

are vertical and horizontal, respectively, in particular, they do not approach the origin.

Next, we analyze the behaviour of the orbits in the sector

$$\mathcal{U}^{(1)}(\varepsilon) = \{\{(t, s) \in \mathbb{R}^2 : \varepsilon \leq (|t|, |s|)^{(-2,1)} \leq \frac{1}{\varepsilon}, |t|, |s| \leq \varepsilon\}$$

corresponding to the edge $\Gamma^{(1)}$, whose unit directional vector is $R = (-2, 1)$. Following Bruno's method, the vector R leads to the coordinate transformation $y_1 = t, y_2 = t^{-2}s$. After the change of time parameter $d\tau_1 = y_1 d\tau$, we obtain the equivalent system

$$\begin{cases} \dot{y}_1 = y_1 \left(\alpha_{20} + \alpha_{01} y_2\right), \\ \dot{y}_2 = y_2 \left[\beta_{30} y_2^{-1} + (\beta_{11} - 2\alpha_{20}) - 2\alpha_{01} y_2\right]. \end{cases} \tag{3.4.2}$$

We are interested in the singular points along the y_2-axis, i.e., the solutions of the quadratic equation

$$-2\alpha_{01} y_2^2 + (\beta_{11} - 2\alpha_{20}) y_2 + \beta_{30} = 0, \tag{3.4.3}$$

whose discriminant is

$$\mathcal{D} = (\beta_{11} - 2\alpha_{20})^2 + 8\alpha_{01}\beta_{30}.$$

By (3.2.10), $\mathcal{D}$ is positive, hence (3.4.3) has two distinct real roots

$$y^{\pm} = \frac{\beta_{11} - 2\alpha_{20} \pm \sqrt{(\beta_{11} - 2\alpha_{20})^2 + 8\alpha_{01}\beta_{30}}}{4\alpha_{01}}. \tag{3.4.4}$$

We need to analyze the dynamics near each point $(0, y^{\pm})$, and to do so, we translate $y^{\pm}$ to the origin via he following change of coordinates

$$z_1 = y_1, \ z_2 = y_2 - y^{\pm}.$$

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As a result, the system (3.4.2) becomes

$$\begin{cases} \dot{z}_1 = z_1 \left[(\alpha_{20} + \alpha_{01} y^\pm) + \alpha_{01} z_2 \right], \\ \dot{z}_2 = z_2 \left[(\beta_{30} + \beta_{11} y^\pm - 2\alpha_{20} y^\pm - 2\alpha_{01}(y^\pm)^2) z_2^{-1} + (\beta_{11} - 2\alpha_{20} - 4\alpha_{01} y^\pm) - 2\alpha_{01} z_2 \right]. \end{cases}$$

This is a system whose linear part does not vanish, and its eigenvalues are given by

$$\lambda_1^\pm = \alpha_{20} + \alpha_{01} y^\pm, \quad \lambda_2^\pm = \beta_{11} - 2\alpha_{20} - 4\alpha_{01} y^\pm. \tag{3.4.5}$$

Lemma 3.4.1. *In the above setting, the following inequalities hold,*

$$\lambda_1^+ > 0, \quad \lambda_1^- < 0, \quad \lambda_2^+ < 0, \quad \lambda_2^- > 0.$$

Proof. Suppose first that $g_{12} = 0$. Then (3.2.9) implies that $\alpha_{20} = \beta_{11} = 0$, hence $y^\pm = \pm\sqrt{\dfrac{\beta_{30}}{2\alpha_{01}}}$. The corresponding eigenvalues become

$$\lambda_1^\pm = \pm\alpha_{01}\sqrt{\frac{\beta_{30}}{2\alpha_{01}}}, \quad \lambda_2^\pm = \mp 4\alpha_{01}\sqrt{\frac{\beta_{30}}{2\alpha_{01}}},$$

and by (3.2.10), none of them can equal zero. This proves the lemma in the case $g_{12} = 0$ so, for the rest of the proof, we assume $g_{12} \neq 0$.

We next observe that, by substituting (3.4.4) in (3.4.5), we obtain

$$\lambda_2^\pm = \mp\sqrt{(\beta_{11} - 2\alpha_{20})^2 + 8\alpha_{01}\beta_{30}},$$

which, by (3.2.10), cannot be zero, hence the last two inequalities of the lemma follow.

Suppose now that

$$\lambda_1^+ = \alpha_{20} + \alpha_{01} y^+ \leq 0.$$

Then, by substituting the expression (3.4.4) for y^+, we get

$$\beta_{11} + 2\alpha_{20} + \sqrt{(\beta_{11} - 2\alpha_{20})^2 + 8\alpha_{01}\beta_{30}} \leq 0. \tag{3.4.6}$$

By (3.2.11), if $g_{12} < 0$ then $\beta_{11} + 2\alpha_{20} > 0$, hence (3.4.6) cannot be true. If $g_{12} > 0$ then $\beta_{11} + 2\alpha_{20} < 0$, so (3.4.6) leads to

$$(\beta_{11} - 2\alpha_{20})^2 + 8\alpha_{01}\beta_{30} < (\beta_{11} + 2\alpha_{20})^2,$$

which, after simplifications, becomes

$$\alpha_{01}\beta_{30} - \alpha_{20}\beta_{11} < 0,$$

hence contradicting (3.3.3). Thus, in both cases, we conclude that $\lambda_1^+ > 0$.

Substituting y^- in the first equation of (3.4.5) with its expression (3.4.4), we get

$$\lambda_1^- = \frac{1}{4}\left(2\alpha_{20} + \beta_{11} - \sqrt{(\beta_{11} - 2\alpha_{20})^2 + 8\alpha_{01}\beta_{30}}\right). \qquad (3.4.7)$$

If $g_{12} > 0$ then $2\alpha_{20} + \beta_{11} < 0$, hence by (3.4.7), $\lambda_1^- < 0$. If $g_{12} < 0$, then $2\alpha_{20} + \beta_{11} > 0$. In this case, suppose $\lambda_1^- \geq 0$. By (3.4.7), it follows that

$$2\alpha_{20} + \beta_{11} \geq \sqrt{(\beta_{11} - 2\alpha_{20})^2 + 8\alpha_{01}\beta_{30}},$$

and since $2\alpha_{20} + \beta_{11} > 0$,

$$(2\alpha_{20} + \beta_{11})^2 \geq (\beta_{11} - 2\alpha_{20})^2 + 8\alpha_{01}\beta_{30},$$

which leads to

$$8(\alpha_{01}\beta_{30} - \alpha_{20}\beta_{11}) \leq 0.$$

Again, this contradicts (3.3.3), and it follows that $\lambda_1^- < 0$, which proves the lemma. $\qquad \square$

By Lemma 3.4.1, for both y^+ and y^-, the corresponding eigenvalues are of opposite signs, hence the phase portrait of system (3.4.2) is a saddle at the origin. It follows that, in (y_1, y_2)-coordinates, the y_2-axis and the lines $\{y_2 = y^+\}$, $\{y_2 = y^-\}$ are integral curves. Let $L_1 = \{(y_1, y^+) : y_1 > 0\}$, $L_2 = \{(y_1, y^+) : y_1 < 0\}$, $L_3 = \{(y_1, y^-) : y_1 > 0\}$, $L_4 = \{(y_1, y^-) : y_1 < 0\}$, $L_5 = \{(0, y_2) : y_2 > y^+\}$, $L_6 = \{(0, y_2) : y_2 < y^-\}$ and $I = \{(0, y_2) : \min\{y^-, y^+\} < y_2 < \max\{y^-, y^+\}\}$. In the strip $\{(y_1, y_2) : y_1 \in \mathbb{R}, \min\{y^-, y^+\} < y_2 < \max\{y^-, y^+\}\}$ of $\mathbb{R}^2_{(y_1, y_2)}$, the integral curves are asymptotic to L_1 and L_3 or to L_2 and L_4, and do not touch I. The rest of the orbits are asymptotic to L_2, L_5 or to L_5, L_1 or to L_6, L_4 or, finally, to L_6 and L_3. This means that in the original system there are two integral curves $s = y^\pm t^4$ entering the origin while the other integral curves are in the complement of these two curves. Lastly, we observe that for a sufficiently small $\varepsilon > 0$, the curves $s = y^\pm t^2$ enter $\mathcal{U}^{(1)}(\varepsilon)$, which completes the analysis for the edge $\Gamma^{(1)}$ of the Newton diagram.

Gluing the orbits in all three sectors corresponding to $(\Gamma_1^{(0)}, \Gamma_2^{(0)}, \Gamma^{(1)})$, we see that the phase portrait near the origin of the system (3.2.12) is a saddle. By letting γ_1 and γ_2 be the curves $s = y^{\pm} t^2$, we conclude that any small enough compact K which is not contained in $\gamma_1 \cup \gamma_2$ will meet one of the orbits of the characteristic foliation at exactly one point, which shows that the conditions of Proposition 4.2.1 are met. This completes the proof.

Chapter 4

A Characterization of Rationally Convex Immersions

We now turn our attention to the second main result of this dissertation, Theorem 1.2.1, Section 1.2. Recall that the main object of study is an immersion $\iota : S \to \mathbb{C}^n$ of a smooth manifold S of real dimension $m \leq n$ into $\mathbb{C}^n$ such that $\iota(S)$ is smooth except at finitely many points where it self-intersects finitely many times. In addition, $\iota(S)$ is compact and totally real. For the rest of the chapter, we shall commit a mild abuse of notation and keep the notation S for the image in $\mathbb{C}^n$ of the given manifold S via the immersion.

4.1 The Necessary Condition for the Rational Convexity of S

In this section we prove that (i) implies (ii) in Theorem 1.2.1. As we already mentioned in the introduction, in this case we do not require the totally real immersion to be locally polynomially convex. In fact, we shall prove the following more general result, where the rational convexity of S implies the existence of a family of degenerate Kähler forms with respect to which it is isotropic.

Proposition 4.1.1. *Let S be the immersion defined in Theorem 1.2.1 without assuming that it is locally polynomially convex. If S is rationally convex then for every sufficiently small $\varepsilon > 0$ there exist contractible neighborhoods W_ε^j of p_j in S, $j = 1, \ldots, N$, such that for every neighborhood Ω of S there*

exist neighborhoods $U_\varepsilon^j \subset V_\varepsilon^j \Subset B(p_j, \varepsilon) \cap \Omega$ *of* p_j, $j = 1, \ldots, N$ *and a smooth plurisubharmonic function* $\varphi_\varepsilon : \mathbb{C}^n \to \mathbb{R}$ *such that* $U_\varepsilon^j \cap S = W_\varepsilon^j$ *for all* j, $dd^c\varphi_\varepsilon = 0$ *on* $\cup_{j=1}^N \overline{U_\varepsilon^j}$, φ_ε *is strictly plurisubharmonic on* $\mathbb{C}^n \setminus \bigcup_{j=1}^N \overline{V_\varepsilon^j}$ *and* $\iota^* dd^c\varphi_\varepsilon = 0$.

A consequence of the above proposition is the following corollary which may be useful in applications. Its proof is given at the end of this section.

Corollary 4.1.2. *If* S *is rationally convex then for all integers* $k \geq 2$ *there exists a* C^k-*smooth plurisubharmonic function* $\varphi_0 : \mathbb{C}^n \to \mathbb{R}$ *which is strictly plurisubharmonic on* $\mathbb{C}^n \setminus \{p_1, \ldots, p_N\}$ *and such that* $\iota^* dd^c\varphi_0 = 0$.

Fix $j \in \{1, \ldots, N\}$ and suppose that p_j is a point where S self-intersects l times. For a sufficiently small $r > 0$, the set $S \cap B(p_j, r)$ is the union of l compact smooth submanifolds with boundary, $S_1, \ldots, S_l \subset S$, such that $S_k \cap S_m = \{p_j\}$, $k \neq m$. We say that $S_1, \ldots, S_l$ are *smooth components of* S *at* p_j. The proof of Proposition 4.1.1 relies on the construction of a suitable function defined on a neighborhood of each smooth component of S at each singular point and on the patching of all such functions into one that has the required properties (as per Lemma 4.1.5). The following two lemmas (4.1.3 and 4.1.5) will be applied separately to each smooth component of S at each singular point. More generally, we prove the lemmas for a smooth, totally real submanifold M of $\mathbb{C}^n$, with or without boundary.

Lemma 4.1.3. *For every point* $p \in M$ *there exists a smooth function* $\tilde{f} : \mathbb{C}^n \to \mathbb{R}$, *with compact support*, $p \in supp(\tilde{f})$, *such that* $\tilde{f}$ *has a local minimum at* p *and satisfies* $\iota^* d^c\tilde{f} = 0$, *where* $\iota : M \to \mathbb{C}^n$ *is the inclusion map.*

Remark 4.1.4. Note that the lemma does not require p to be a *strict* local minimum point for f.

Proof of Lemma 4.1.3. Suppose first that $\dim_\mathbb{R} M = n$. For each $q \in M$, there exists a global complex-affine change of coordinates, **which depends on** q,

$$\Phi : \mathbb{C}^n_{z=(z_1, \ldots, z_n)} \to \mathbb{C}^n_{w=(w_1, \ldots, w_n)},$$

where $z_j = x_j + iy_j, w_j = u_j + iv_j$, $x_j, y_j, u_j, v_j \in \mathbb{R}, \forall j = 1 \ldots n$, such that $\Phi(q) = 0$ and $T_0 M' = \mathbb{R}^n_{u:=(u_1, \ldots, u_n)}$, with $M' = \Phi(M)$. Suppose that $\Phi(z) = A(z-q)$, where A is a complex $n \times n$ invertible matrix (as a complex-affine map, we can always represent Φ like this). Let $J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ be the

matrix that gives the standard complex structure of $\mathbb{C}^n$, corresponding to multiplication by i. Denote

$$JT_qM := \{i(z - q) + q \,:\, z \in T_qM\}.$$

Claim A. $\Phi(JT_qM) = \mathbb{R}_v^n \subset \mathbb{C}_w^n$, for all $q \in M$.

Proof of Claim A. Let $w \in JT_qM$, hence there exists $z \in T_qM$ such that $w = i(z - q) + q$. Then,

$$
\begin{aligned}
\Phi(w) &= \Phi(i(z - q) + q) \\
&= A\left[i(z - q) + q - q\right] \\
&= iA(z - q) = i\Phi(z) \in JT_0M' = \mathbb{R}_v^n.
\end{aligned}
$$

The converse inclusion follows similarly. $\qquad\square$

Let $p \in M$ be arbitrarily fixed. Since M is totally real, there exists a (small enough) neighborhood U of p such that

$$\mathcal{F}_U := \{JT_qM \cap U \,|\, q \in M \cap U\}$$

is a foliation of U.

Let V be a neighborhood of p in M such that $V \Subset M \cap U$ and let $f : M \to \mathbb{R}$ be a smooth nonnegative function such that $supp\, f = \overline{V}$, with a strict local minimum, equal to 0, at p. Define $\widetilde{f} : U \to \mathbb{R}$ as

$$\widetilde{f}(z) = f(q), \text{ for each } z \in U \cap JT_qM, q \in M \cap U,$$

which is well defined, since $\mathcal{F}_U$ is a foliation of U and, shrinking U if necessary, for each $q \in M \cap U$ we have $M \cap U \cap JT_qM = \{q\}$. Multiply $\widetilde{f}$ with a suitable smooth cut-off function to obtain a new $\widetilde{f}$ (maintaining the same notation) which is defined on the entire $\mathbb{C}^n$ and satisfies

$$
\widetilde{f}(z) =
\begin{cases}
f(q), & \forall z \in JT_qM \cap U', q \in V, \\
0, & \forall z \in \mathbb{C}^n \setminus U'',
\end{cases}
$$

where $U' \subset U'' \Subset U$ and $U' \cap M = V$. Note that $\widetilde{f}$ is nonnegative, smooth, with compact support and with 0 as a non-strict local minimum value at p.

For every $q \in V$, define

$$h := \widetilde{f} \circ \Phi^{-1},$$

where, again, Φ (and therefore h) depends on the choice of q. Then, h is a smooth real-valued function satisfying

$$h(w) = h(0) = f(q), \ \forall w \in W \cap J\mathbb{R}^n_u = W \cap \mathbb{R}^n_{v=(v_1,\dots,v_n)},$$

for some neighborhood W of the origin $0 \in \mathbb{C}^n_w$. Hence, h is constant in $W \cap \mathbb{R}^n_v$, which means that

$$\frac{\partial h}{\partial v_j}(0) = 0, \tag{4.1.1}$$

for all $j = 1 \dots n$. By (2.1.1) we have

$$d^c = \sum_{j=1}^n \left(\frac{\partial}{\partial u_j} dv_j - \frac{\partial}{\partial v_j} du_j \right),$$

and by (4.1.1) and the fact that $dv_j = 0$ on $\mathbb{R}^n_u$ we get

$$j^* d^c_0 h = 0, \tag{4.1.2}$$

where $j : \mathbb{R}^n_u \to \mathbb{C}^n$ is the inclusion map.

Claim B. If $j^* d^c_0 h = 0$ then $\iota^* d^c_q \widetilde{f} = 0$.

Proof of Claim B. By Lemma 2.1.2,

$$d^c_0 h = d^c_0(\widetilde{f} \circ \Phi^{-1}) = d_{\Phi^{-1}(0)} \widetilde{f} \circ d^c_0 \Phi^{-1} = d_q \widetilde{f} \circ d^c_0 \Phi^{-1}. \tag{4.1.3}$$

Let ν be a tangent vector in $T_q M$. Then, $d^c_q \widetilde{f}(\nu) = i\bar\partial_q \widetilde{f}(\nu) - i\partial_q \widetilde{f}(\nu) = \bar\partial_q \widetilde{f}(-i\nu) + \partial_q \widetilde{f}(-i\nu) = d_q \widetilde{f}(-i\nu)$, where we used the complex anti-linearity of $\bar\partial$ and the complex linearity of ∂. Because Φ is (bi)holomorphic, we have $d^c_0 \Phi^{-1} = -id_0 \Phi^{-1}$. Since $d_0 \Phi^{-1}$ is a vector space isomorphism, there exists $\xi \in T_0 M' = \mathbb{R}^n_u$ such that $\nu = d_0 \Phi^{-1}(\xi)$, hence $-i\nu = -id_0 \Phi^{-1}(\xi) = d^c_0 \Phi^{-1}(\xi)$. Since ν was arbitrarily fixed in $T_q M$ and by (4.1.3), the claim follows. $\square$

By Claim B and by (4.1.2) it follows that $\iota^* d^c_q \widetilde{f} = 0$ and, since q was arbitrarily fixed in V and by the fact that on $\mathbb{C}^n \setminus U''$ we have $\widetilde{f} \equiv 0$, we conclude that

$$\iota^* d^c \widetilde{f} = 0.$$

If $\dim_{\mathbb{R}} M < n$ and $p \in M$, there exists U, a neighborhood of p in $\mathbb{C}^n$, such that $M \cap U$ is included in a compact, totally real submanifold $\widetilde{M}$ of

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$\mathbb{C}^n$ of real dimension n. By what we proved already, there exists a smooth function $\widetilde{f} : \widetilde{M} \to \mathbb{R}$, with compact support, which can be chosen such that $supp(f) \subset U$ and with a local minimum at p, such that $\widetilde{\iota}^* d^c \widetilde{f} = 0$, where $\widetilde{\iota} : \widetilde{M} \to \mathbb{C}^n$ is the inclusion map. Then, the restriction $\widetilde{f}\big|_M$ satisfies the same properties for M. This completes the proof of Lemma 4.1.3. $\qquad\square$

Lemma 4.1.5. *For every point $p \in M$ and every sufficiently small $\varepsilon > 0$, there exist a neighborhood Ω_ε of M, a smooth nonnegative function $g_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ and neighborhoods $V_\varepsilon, W_p \subset \Omega_\varepsilon$ of p such that*

(i) $V_\varepsilon \Subset B(p, \varepsilon) \Subset W_p$;

(ii) $g_\varepsilon \equiv 0$ *in* V_ε;

(iii) g_ε *is plurisubharmonic in* Ω_ε *and strictly plurisubharmonic in* $\Omega_\varepsilon \setminus \overline{V}_\varepsilon$;

(iv) $g_\varepsilon = C_\varepsilon \cdot dist^2(\cdot, M)$ *in* $\Omega_\varepsilon \setminus W_p$, *for some constant* $C_\varepsilon > 0$;

(v) $\iota^* d^c g_\varepsilon = 0$.

Remark 4.1.6. The neighborhood W_p depends only on p, not on ε, therefore the notation.

Proof of Lemma 4.1.5. Without loss of generality, suppose $p = 0 \in \mathbb{C}^n$. By Lemma 4.1.3, there exists a smooth function $\widetilde{f}$ defined on $\mathbb{C}^n$, such that $\widetilde{f}$ has compact support near the origin, $\iota^* d^c \widetilde{f} = 0$ and $\widetilde{f}$ has a local minimum at the origin. In fact, by construction, $\widetilde{f}$ is nonnegative, with a (non-strict) local minimum equal to 0 attained on $JT_0 M$. Let $W_p := (\text{supp }\widetilde{f})^\circ$.

Suppose $\widetilde{\Omega}$ is a neighborhood of M on which $dist^2(\cdot, M)\big|_{\widetilde{\Omega}}$ is smooth and strictly plurisubharmonic. We can also assume that $W_p \Subset \widetilde{\Omega}$. Then, for a sufficiently small $C > 0$, the function defined on $\widetilde{\Omega}$ as

$$\widetilde{g} = dist^2(\cdot, M) + C\widetilde{f},$$

satisfies,

(a) $\widetilde{g}$ is strictly plurisubharmonic in $\widetilde{\Omega}$;

(b) $\widetilde{g}$ is nonnegative and it has a strict local minimum at the origin which is equal to 0;

(c) $\widetilde{g} = dist^2(\cdot, M)$ on $\widetilde{\Omega} \setminus W_p$;

(d) $\iota^* d^c \widetilde{g} = 0$, by Lemma 4.1.3 and by the fact that $\iota^* d^c [dist^2(\cdot, M)] = 0$.

Let $0 < \varepsilon' < \varepsilon$ be sufficiently small, by which we mean that $B(0, \varepsilon)$ is included in a neighborhood of the origin $U \Subset W_p$ on which $\widetilde{g}$ has a strict minimum at the origin, and let $a_\varepsilon := \max\{\widetilde{g}(z) : z \in \overline{B(0, \varepsilon')}\}$. By making ε' even smaller if necessary, there exists a neighborhood $0 \in V_\varepsilon \subset B(0, \varepsilon)$ such that $\widetilde{g}(z) > a_\varepsilon$ for all $z \in U \setminus \overline{V}_\varepsilon$ and $\widetilde{g}(z) \le a_\varepsilon$ for all $z \in \overline{V}_\varepsilon$.

Define $\sigma_\varepsilon : \mathbb{R}_{\ge 0} \to \mathbb{R}$ to be a nonnegative smooth, convex, non-decreasing function such that $\sigma_\varepsilon(t) = 0$ for all $t \in [0, a_\varepsilon]$. It follows that the function

$$\widetilde{g}_\varepsilon := \sigma_\varepsilon \circ \widetilde{g}$$

is identically zero in $\overline{V}_\varepsilon$. Since $dist^2(\cdot, M) = 0$ on M and $f = 0$ on the complement of $W_p = (\operatorname{supp} f)^\circ$, the function $\widetilde{g}$ vanishes on $M \setminus W_p$, hence $\widetilde{g}_\varepsilon$ vanishes on a neighborhood Ω'_ε of $M \setminus W_p$. Note that, for sufficiently small $\varepsilon > 0$ (hence, for sufficiently small a_ε), we can ensure that $\Omega'_\varepsilon \cap V_\varepsilon = \emptyset$. Since $\widetilde{g}$ is strictly plurisubharmonic in $\widetilde{\Omega}$, $\widetilde{g}_\varepsilon$ is plurisubharmonic in $\widetilde{\Omega}$ and strictly plurisubharmonic in $\widetilde{\Omega} \setminus (\overline{\Omega'}_\varepsilon \cup \overline{V}_\varepsilon)$. Moreover, since $\iota^* d^c \widetilde{g} = 0$ and

$$dd^c \widetilde{g}_\varepsilon = dd^c(\sigma_\varepsilon \circ \widetilde{g}) = \sigma''_\varepsilon d\widetilde{g} \wedge d^c \widetilde{g} + \sigma'_\varepsilon dd^c \widetilde{g},$$

it follows that $\iota^* dd^c \widetilde{g}_\varepsilon = 0$.

The set $\Omega_\varepsilon := \Omega'_\varepsilon \cup W_p$ is a neighborhood of M. Let $\mathcal{X} : \mathbb{C}^n \to \mathbb{R}$ be a smooth cut-off function, which is identically 0 on a neighborhood of the origin Z and equal to 1 on the complement of a larger neighborhood Z', where $V_\varepsilon \subset Z \Subset Z' \Subset W_p$. We can also ensure that $Z' \cap \Omega'_\varepsilon = \emptyset$ (because $\Omega'_\varepsilon \cap V_\varepsilon = \emptyset$). For a sufficiently small constant $C_\varepsilon > 0$, the function defined on Ω_ε as

$$g_\varepsilon := \widetilde{g}_\varepsilon + C_\varepsilon(\mathcal{X} \cdot dist^2(\cdot, M))$$

has the required properties. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let S be the immersion considered at the beginning of this section, with finitely many double self-intersections, $p_1, \dots, p_N \in S$.

Lemma 4.1.7. *For any sufficiently small $\varepsilon > 0$ there exist a neighborhood Ω_ε of S, neighborhoods $p_j \in V_\varepsilon^j \subset \Omega_\varepsilon$, with $\operatorname{diam} V_\varepsilon^j < \varepsilon$ for all $j = 1 \dots N$, and a smooth function $\rho_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ such that*

(i) $\rho_\varepsilon \equiv 0$ in V_ε^j, $j = 1 \ldots N$;

(ii) ρ_ε is plurisubharmonic in Ω_ε and strictly plurisubharmonic in $\Omega_\varepsilon \setminus \cup_{j=1}^N \overline{V_\varepsilon^j}$;

(iii) $\iota^* dd^c \rho_\varepsilon = 0$.

Proof. Without any loss of generality we can suppose that S has only one (double) self-intersection at the origin. The construction can be easily extended to the general case.

Let S_1, S_2 be two smooth components of S at the origin and let $\varepsilon > 0$ be sufficiently small. By Lemma 4.1.5, it follows that, for $j = 1, 2$, there exist Ω_ε^j, a neighborhood of S_j, a smooth nonnegative function $g_\varepsilon^j : \Omega_\varepsilon^j \to \mathbb{R}$ and neighborhoods $0 \in V_\varepsilon^j \Subset B(0, \varepsilon) \Subset W^j \subset \Omega_\varepsilon^j$ such that

(i) g_ε^j is plurisubharmonic in Ω_ε^j and strictly plurisubharmonic in $\Omega_\varepsilon^j \setminus \overline{V}_\varepsilon^j$;

(ii) $g_\varepsilon^j = 0$ in V_ε^j;

(iii) $g_\varepsilon^j = C_\varepsilon \cdot dist^2(\cdot, S_j)$ in $\Omega_\varepsilon^j \setminus W^j$, where $C_\varepsilon := \min\{C_\varepsilon^1, C_\varepsilon^2\}$ and the constants C_ε^j are given by Lemma 4.1.5;

(iv) $\iota^* dd^c g_\varepsilon^j = 0$.

Let $V_\varepsilon := V_\varepsilon^1 \cap V_\varepsilon^2$. By construction, $V_\varepsilon \Subset B(0, \varepsilon) \Subset \Omega_\varepsilon^j, j = 1, 2$. Make the neighborhoods $\Omega_\varepsilon^1, \Omega_\varepsilon^2$ narrow enough so that $(\Omega_\varepsilon^1 \cap \Omega_\varepsilon^2) \setminus V_\varepsilon = \emptyset$. Let $\Omega_\varepsilon := \Omega_\varepsilon^1 \cup \Omega_\varepsilon^2 \cup V_\varepsilon$, and define $\rho_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ as

$$\rho_\varepsilon(z) = \begin{cases} 0, & z \in \overline{V}_\varepsilon, \\ g_\varepsilon^1(z), & z \in \Omega_\varepsilon^1 \setminus \overline{V}_\varepsilon, \\ g_\varepsilon^2(z), & z \in \Omega_\varepsilon^2 \setminus \overline{V}_\varepsilon, \end{cases}$$

which satisfies

$$\rho_\varepsilon(z) = \begin{cases} 0, & z \in \overline{V}_\varepsilon, \\ C_\varepsilon \cdot dist^2(\cdot, S), & z \in \Omega_\varepsilon \setminus W_p, \end{cases}$$

where $W_p = W_p^1 \cup W_p^2$. Lastly, extend Ω_ε to a full neighborhood of S and ρ_ε accordingly to obtain the required function. $\qquad \square$

In the following, we set $S_\delta := \{z \in \mathbb{C}^n : dist(z, S) \leq \delta\}$, where $\delta > 0$.

Lemma 4.1.8. *For every neighborhood Ω of S, there exists $\delta_0 > 0$, which depends on Ω, such that $\mathcal{R}-\mathrm{hull}\,(S_\delta) \Subset \Omega$ for all $0 < \delta \leq \delta_0$.*

Proof. Let Ω be a neighborhood of S. It suffices to show that there exists $\nu_0 \in \mathbb{Z}^+$ such that $\mathcal{R}-\mathrm{hull}\,(S_{1/\nu}) \Subset \Omega$ for all integers $\nu \geq \nu_0$. In fact, because $S_{1/(\nu+1)} \subset S_{1/\nu}$ for all ν, it is enough to prove that there exists $\nu_0 \in \mathbb{Z}^+$ such that $\mathcal{R}-\mathrm{hull}\,(S_{1/\nu_0}) \Subset \Omega$. Assuming the contrary, we obtain a sequence $\{z_\nu \in \mathbb{C}^n : \nu \in \mathbb{Z}^+\}$ such that $z_\nu \in \mathcal{R}-\mathrm{hull}\,(S_{1/\nu})$ and $z_\nu \notin \Omega$ for all $\nu \in \mathbb{Z}$. Since $\mathcal{R}-\mathrm{hull}\,(S_1)$ is compact we may assume that this sequence converges to some $z \in \mathbb{C}^n \setminus \Omega$, which means that $z \notin S$, since $S \subset \Omega$. Recall that for any compact $X \subset \mathbb{C}^n$ we have (see for example [39, Proposition 1.1])

$$\mathcal{R}-\mathrm{hull}\,(X) = \{z \in \mathbb{C}^n : f(z) \in f(X), \text{ for all holomorphic polynomials } f\}.$$

Since $z \notin S$ and S is rationally convex, there exists a holomorphic polynomial P such that $P(z) \notin P(S)$. By a continuity/compactness type of argument one can easily show that $P(S) = \cap_{\nu=1}^{\infty} P(S_{1/\nu})$. Since $P(z) \notin P(S)$ and $P(S)$ is compact, there exists $r > 0$ such that $B(P(z),r) \cap P(S) = \emptyset$ so, for all but finitely many elements of the sequence, we have $P(z_\nu) \in B(P(z),r)$. On the other hand, since $P(S) = \cap_{\nu=1}^{\infty} P(S_{1/\nu})$, there exists a neighborhood U of $P(S)$ such that $U \cap B(P(z),r) = \emptyset$ and all but finitely many elements $P(z_\nu)$ belong to U, which leads to a contradiction. $\qquad\square$

Proof of Proposition 4.1.1. For $\varepsilon > 0$ sufficiently small, let Ω_ε be the neighborhood of S, $\widetilde{V}_\varepsilon^j \subset \Omega_\varepsilon$, diam $\widetilde{V}_\varepsilon^j < \varepsilon$, the neighborhoods of the self-intersection points p_j, $j = 1, \ldots, N$, and ρ_ε the function given by Lemma 4.1.7. Let Ω be a neighborhood of S such that $\Omega \Subset \Omega_\varepsilon$. By Lemma 4.1.8 there exists $\delta > 0$ such that $\mathcal{R}-\mathrm{hull}\,(S_\delta) \Subset \Omega$. By Duval-Sibony [14, Theorem 2.1] there exists a smooth plurisubharmonic function $\psi_\delta : \mathbb{C}^n \to \mathbb{R}$ which is strictly plurisubharmonic on $\mathbb{C}^n \setminus \mathcal{R}-\mathrm{hull}\,(S_\delta)$ and satisfies $dd^c\psi_\delta|_{\mathcal{R}-\mathrm{hull}(S_\delta)} = 0$. Let $W_\varepsilon^j := \widetilde{V}_\varepsilon^j \cap S$, which for sufficiently small $\varepsilon > 0$ is contractible, $U_\varepsilon^j := \widetilde{V}_\varepsilon^j \cap S_\delta$, $V_\varepsilon^j := \widetilde{V}_\varepsilon^j \cap \Omega$ and define

$$\varphi_\varepsilon(z) := \psi_\delta + C\mathcal{X}(z)\rho_\varepsilon(z),$$

where C is a positive constant and $\mathcal{X}$ is a smooth cutoff function equal to 1 in a neighborhood of S which contains Ω and equal to 0 in the complement of a slightly larger neighborhood of S, both neighborhoods being compactly included in Ω_ε. Then, for a sufficiently small $C > 0$, φ_ε is strictly plurisubharmonic in $\mathbb{C}^n \setminus \bigcup_{j=1}^{k} \overline{V_\varepsilon^j}$ and the rest of the properties stated in Proposition 4.1.1 are satisfied. $\qquad\square$

Proof of Corollary 4.1.2. Let $\varphi_j := \varphi_{\varepsilon_j} : \mathbb{C}^n_{z=(z_1,\ldots,z_n)} \to \mathbb{R}^n$, $z_\mu = x_\mu + ix_{\mu+n}$, $\mu = 1, \ldots, n$, where $\{\varepsilon_j\}_{j \in \mathbb{Z}^+}$ is a decreasing sequence of (sufficiently small) positive numbers converging to 0 and φ_{ε_j} are the functions given by Proposition 4.1.1. Let $B \subset \mathbb{C}^n$ be a closed ball such that $\cup_j \Omega_{\varepsilon_j} \Subset B$, where Ω_{ε_j} are the neighborhoods of S given by Lemma 4.1.7. Then, there exist positive reals, $\alpha_j > 0$, $j \in \mathbb{Z}^+$, such that

$$\varphi_0 := \sum_j \alpha_j \varphi_j < \infty, \quad \sum_j \alpha_j \frac{\partial^l \varphi_j}{\partial x_{\mu_1} \ldots \partial x_{\mu_l}} < \infty$$

in B, for all $1 \leq l \leq k$ and $\mu_1, \ldots, \mu_l \in \{1, \ldots, 2n\}$ where the convergence of the series is uniform. By making use of a classical result in real single variable calculus, see for example [34, Theorem 7.17], it is straightforward to show that φ_0 is C^k-smooth in B and that

$$\frac{\partial^l \varphi_0}{\partial x_{\mu_1} \ldots \partial x_{\mu_l}} = \sum_j \alpha_j \frac{\partial^l \varphi_j}{\partial x_{\mu_1} \ldots \partial x_{\mu_l}}$$

for all $1 \leq l \leq k$ and $\mu_1, \ldots, \mu_l \in \{1, \ldots, 2n\}$. In particular, we have

$$dd^c \varphi_0 = \sum_j \alpha_j dd^c \varphi_j. \tag{4.1.4}$$

In $\mathbb{C}^n \setminus B$, we have $\varphi_0 = C\psi$, where $C = \sum_j \alpha_j < \infty$, hence φ_0 is C^∞-smooth there and clearly satisfies (4.1.4). It follows that φ_0 satisfies the required properties. $\qquad\square$

4.2 The Sufficient Condition for S to be Rationally Convex

Let S be the immersion defined in Theorem 1.2.1. In this section we prove the converse statement $(ii) \Rightarrow (i)$ in Theorem 1.2.1 which translates into the following result.

Proposition 4.2.1. *Suppose that S is locally polynomially convex and that there exist contractible neighborhoods W_j of p_j in S such that for any neighborhood Ω of S, there exist a smooth plurisubharmonic function $\varphi : \mathbb{C}^n \to \mathbb{R}$*

and neighborhoods $U_j \subset V_j$ of p_j, $j = 1, \ldots, N$, where $\{V_j\}_j$ are pairwise disjoint, such that $U_j \cap S = W_j$, $\cup_{j=1}^{N} V_j \Subset \Omega$, $dd^c\varphi = 0$ on $\cup_{j=1}^{N} \overline{U}_j$, φ is strictly plurisubharmonic on $\mathbb{C}^n \setminus \bigcup_{j=1}^{N} \overline{V}_j$ and $\iota^* dd^c\varphi = 0$. Then S is rationally convex.

In the proof we make use of the following technical lemma proved in [16] (see also [38]). In [16], the lemma is proved for totally real immersions of maximal dimension. We remark that the lemma is true in the more general case that we consider in this material, the proof being practically the same.

Lemma 4.2.2. *Let ϕ be a smooth plurisubharmonic function on $\mathbb{C}^n$ and h a smooth complex-valued function on $\mathbb{C}^n$ satisfying the following properties*

(1) $|h| \leq e^\varphi$ and $X := \{|h| = e^\varphi\}$ is compact;

(2) $\overline{\partial}h = O(dist(\cdot, S)^{\frac{3n+5}{2}})$;

(3) $|h| = e^\varphi$ with order at least 1 on S;

(4) For any point $p \in X$ at least one of the two following conditions holds: (i) h is holomorphic in a neighborhood of p, or (ii) p is a smooth point of S and φ is strictly plurisubharmonic at p.

Then X is rationally convex.

Remark 4.2.3. As it was already mentioned in [38], it can be seen from the proof of the lemma in [16] that it is enough to assume that φ is only continuous and not necessarily smooth at points where h is holomorphic. In fact, the Lemma is also valid if φ is only continuous at points in the complement of a neighborhood of S.

To prove Proposition 4.2.1 we will follow closely the method used in [14] (see also [16], [38]). In Step 1 below we construct the function h from the given plurisubharmonic function, φ. The resulting pair of functions, (h, φ), does not entirely comply with the requirements of Lemma 4.2.2, because some of the points of S do not satisfy condition (4) of the lemma. That is why, in the second and last step of the proof, we further perturb φ such that the modified function is strictly plurisubharmonic at all points of S where h is not holomorphic. In this last step we take full advantage of the polynomial convexity of S near the singular points.

Step 1: construction of the function h. The conditions $(2), (3)$ in Lemma 4.2.2 that h and φ must satisfy translate into the condition $\iota^*(d^c\varphi - d(\arg h)) = 0$, or in other words, the closed 1-form $\iota^* d^c\varphi$ has to have integer periods (see [16], [38]). We can meet this condition by perturbing φ accordingly.

Let Σ be an open subset of $\mathbb{C}^n$ such that S is a deformation retract of Σ. Consequently, $H_1(\Sigma, \mathbb{Z}) \cong H_1(S, \mathbb{Z})$. Let $\gamma_1, \ldots, \gamma_l$ be a basis for $H_1(\Sigma, \mathbb{Z})$ which we may consider to be supported on $S \setminus \cup_{j=1}^N \overline{V}_j$. By the de Rham theorem, there exist closed 1-forms $\beta_1, \ldots, \beta_l$ on Σ, with compact support, such that $\int_{\gamma_j} \beta_k = \delta_{jk}$. In fact, we can choose β_k such that they vanish on each U_j, $j = 1, \ldots, N$.

We show next that there exist smooth functions $\psi_k, k = 1, \ldots, l$, with compact support in $\mathbb{C}^n$, such that $\iota^* d^c\psi_k = \iota^*\beta_k$ and $\psi_k|_{U_j} \equiv 0$, $j = 1, \ldots, N$. Suppose first that $\dim_{\mathbb{R}} S = n$ and let $\widetilde{S} := S \setminus \cup_{j=1}^N U_j$, which is a compact smooth submanifold of $\mathbb{C}^n$ with boundary. Fix $k \in \{1, \ldots l\}$ and, to simplify the notations, denote $\psi := \psi_k, \beta := \beta_k$. Let $p \in \widetilde{S}$ be arbitrarily fixed. In fact, without losing any generality, suppose $p = 0$ and $T_0\widetilde{S} = \mathbb{R}^n$. Then there exists a neighborhood of the origin W in $\mathbb{C}^n$ and smooth real-valued functions $r_1, \ldots, r_n$ defined on $\widetilde{W} := W \cap \widetilde{S}$, such that $y_j = r_j(\mathrm{x})$ for all $z = \mathrm{x} + i\mathrm{y} \in \widetilde{W}$, $\mathrm{x} = (x_1, \ldots, x_2), \mathrm{y} = (y_1, \ldots, y_n)$, and $\frac{\partial r_j}{\partial x_k}(0) = 0$ for all $j, k \in \{1, \ldots, n\}$. We would like to find smooth functions $\alpha_j : W \to \mathbb{R}$, $j = 1, \ldots, n$, so that if we define

$$\psi_0(z) := \sum_{j=1}^n \alpha_j(z)\left[r_j(\mathrm{x}) - y_j\right], \tag{4.2.1}$$

then $\iota^* d^c\psi_0 = \iota^*\beta$ on some neighborhood of the origin included in $\widetilde{W}$. A direct calculation gives

$$\iota^* d^c\psi_0 = \sum_{j=1}^n \left(\widetilde{\alpha}_j + \sum_{k=1}^n \widetilde{\alpha}_k P_{jk}\right) dx_j,$$

where $\widetilde{\alpha}_j$ is the pullback of α_j on $\widetilde{W}$, given by $\widetilde{\alpha}_j(\mathrm{x}) = \alpha_j(x_1, r_1(\mathrm{x}), \ldots, x_n, r_n(\mathrm{x}))$, and $P_{jk} = \sum_{l=1}^n \frac{\partial r_k}{\partial x_j}\frac{\partial r_j}{\partial x_l}$, $j, k = 1, \ldots, n$. For every $z \in \widetilde{W}$ let $A(z) = [a_{jk}(z)]_{1 \le j,k \le n}$ be the $n \times n$ matrix whose components are given by

$$a_{jj}(z) = 1 + P_{jj}(z),$$

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$$a_{jk}(z) = P_{jk}(z), j \neq k.$$

We note that $A(0) = I_n$, hence A is invertible in a neighborhood $\widetilde{Z} \subset \widetilde{W}$ of the origin. Applying the same construction to each point $p \in \widetilde{S}$, we obtain neighborhoods $\widetilde{Z}_p$ in $\widetilde{S}$ on which the corresponding matrices are non-singular. By the compactness of $\widetilde{S}$, assume that the cover given by such neighborhoods is finite, $\{\widetilde{Z}_\nu\}_{1 \leq \nu \leq s}$, $s \in \mathbb{Z}^+$ and denote by A_ν the corresponding matrices. Let $\{\rho_\nu\}_{1 \leq \nu \leq s}$ be a smooth partition of unity subordinate to this cover. In each $\widetilde{Z}_\nu$, write the pullback of β in local coordinates, $\iota^*\beta = \sum_{j=1}^n \beta_j dx_j$ and let $\beta_\nu^j := \rho_\nu \beta_j$, so

$$\iota^*\beta = \sum_{j=1}^n \left(\sum_{\nu=1}^s \beta_\nu^j \right) dx_j.$$

Then for all $1 \leq \mu \leq s$, the linear system in $\widetilde{\alpha}_1^\mu(z), \ldots, \widetilde{\alpha}_n^\mu(z)$, $z \in \widetilde{Z}_\mu$,

$$\widetilde{\alpha}_j^\mu(z) + \sum_{k=1}^n \widetilde{\alpha}_k^\mu(z) P_{jk}^\mu(z) = \sum_{\nu=1}^s \beta_\nu^j(z), \quad j = 1, \ldots, n. \tag{4.2.2}$$

has smooth solutions in $\widetilde{Z}_\mu$, since the system matrix A_μ is invertible there. By defining $\alpha_j^\mu(z) := \widetilde{\alpha}_j^\mu(\mathrm{x})$ for all $z = \mathrm{x} + i\mathrm{y}$ in a neighborhood Z_μ in $\mathbb{C}^n$ such that $\widetilde{Z}_\mu = Z_\mu \cap \widetilde{S}$, $Z_\mu \subset W_\mu$, and by applying formula (4.2.1) accordingly, we obtain a smooth function ψ_μ defined on Z_μ which satisfies $\iota^*d^c\psi_\mu = \iota^*\beta$. Let $\Omega := \cup_{\mu=1}^s Z_\mu$, hence $\widetilde{S} \subset \Omega$. Since β has compact support, ψ_μ also has compact support, so we can extend it smoothly to $\mathbb{C}^n$ by letting it be equal to zero everywhere else. Then the smooth function given by $\widetilde{\psi} := \sum_{\mu=1}^s \psi_\mu$ satisfies the required properties. If $\dim_\mathbb{R} S < n$, we locally embed $\widetilde{S}$ in a compact, totally real submanifold $\widetilde{M} \subset \mathbb{C}^n$ of real dimension n, apply the above construction to obtain ψ for $\widetilde{M} \cup (\cup_{j=1}^n U_j)$ and then locally restrict ψ to $\widetilde{S} \cup (\cup_{j=1}^n U_j)$.

Let $\lambda := (\lambda_1, \ldots, \lambda_l)$ be an l-tuple of rational numbers and define $\varphi_\lambda := \varphi + \lambda_1 \psi_1 + \cdots + \lambda_l \psi_l$. For sufficiently small λ (i.e., each λ_k is sufficiently small) φ_λ is strictly plurisubharmonic in $\mathbb{C}^n \setminus \cup_{j=1}^N \overline{V}_j$ and pluriharmonic on each U_j. Further, we can find $M \in \mathbb{Z}$ by adjusting λ accordingly, such that $\int_{\gamma_k} \iota^*d^c\varphi_\lambda \in 2\pi\mathbb{Z}/M$, $k = 1, \ldots, l$. Let us recycle the notation of the initial function and define $\varphi := M\varphi_\lambda$. The form $\iota^*d^c\varphi$ is closed and it has periods which are integer multiples of 2π.

Fix $p \in S$ and define $\mu : S \cup (\cup_{j=1}^{N} \overline{U}_j) \to \mathbb{R}/2\pi\mathbb{Z}$ given by $\mu(z) := \int_{p}^{z} d^c\varphi$ which clearly satisfies $\iota^* d^c\varphi = d\mu$. Fix $j \in \{1, \ldots, N\}$ and $z \in U_j$. Let $\gamma : [0,1] \to \mathbb{C}^n$ be a rectifiable path such that $\gamma(0) = p, \gamma(1) = z$. Then, the function $\mu_z(w) := \int_{\gamma} d^c\varphi + \int_{z}^{w} d^c\varphi$, $w \in U_j$, does not depend on the path from z to w, since $d^c\varphi$ is closed in U_j. An easy computation shows that, in a neighborhood of z, φ and μ_z satisfy the Cauchy-Riemann equations, hence the function $\varphi + i\mu_z$ is holomorphic in U_j. Because $\mu : S \cup (\cup_{j=1}^{N} \overline{U}_j) \to \mathbb{R}/2\pi\mathbb{Z}$ and $\operatorname{Arg} e^{\varphi + i\mu} = \mu$, the function $h_1 : S \cup (\cup_{j=1}^{N} U_j) \to \mathbb{C}^n$ given as $h_1 = e^{\varphi + i\mu}$ is well defined and holomorphic in U_j, $j = 1, \ldots, N$. By a classical result in [20], there exists a smooth function $h : \mathbb{C}^n \to \mathbb{C}$ such that

$$h|_S = h_1|_S,$$

$$\overline{\partial} h = O(dist(\cdot, S)^r), \text{ for all } r \in \mathbb{Z}^+$$

and h is holomorphic in neighborhoods Z_j of p_j, $Z_j \Subset U_j$, where the boundary of Z_j can be chosen to be arbitrarily "close" to that of U_j, i.e., for any given small tubular neighborhood V of ∂U_j, we can find such Z_j with ∂Z_j being included in V. This completes Step 1 of the proof.

Step 2: further perturbation of φ. For each singular point $p_j \in S$ there exists a region, $S \cap (\overline{V}_j \setminus Z_j)$, on which we cannot guarantee φ to be strictly plurisubharmonic, nor do we know whether h is holomorphic, therefore condition (4) of Lemma 4.2.2 may not be satisfied there. We address this issue by taking advantage of the local polynomial convexity of S. We make use of the following result (Lemma 4.2.4), which is essentially due to Forstnerič and Stout [15] who stated it for totally real discs with finitely many hyperbolic points in $\mathbb{C}^2$. Subsequently Shafikov and Sukhov stated it for Lagrangian inclusions in $\mathbb{C}^2$ [38, Lemma 3.3] and for a different setting in $\mathbb{C}^n$ [37, Lemma 5.3]. Again, S denotes the immersion defined above with $p_1, \ldots, p_N$ being its self-intersection points.

Lemma 4.2.4. *If S is locally polynomially convex then, for all sufficiently small $\delta > 0$, there exists a neighborhood Ω of S in $\mathbb{C}^n$ and a continuous non-negative plurisubharmonic function $\rho : \Omega \to \mathbb{R}$ such that $S = \{\rho = 0\}$ and $\rho = dist(\cdot, S)^2$ on $\Omega \setminus \cup_{j=1}^{N} B(p_j, \delta)$; in particular, ρ is smooth and strictly plurisubharmonic there.*

For a proof of the lemma we refer the reader to [37]. The proof is essentially identical and it applies verbatim without requiring that the self-intersections be transverse or double. In fact, the main ingredient allowing

the proof to flow through is the local polynomial convexity and not the type of singularities of S.

To finalize Step 2 in the proof of Proposition 4.2.1, let $\delta > 0$ be small enough such that $B(p_j, \delta) \cap S \Subset W_j = U_j \cap S$, for all j. Let Ω be the neighborhood of S and $\rho : \Omega \to \mathbb{R}$ the function given by Lemma 4.2.4 corresponding to δ. By hypothesis, we can assume that $V_j \Subset \Omega$ for all $j \in \{1, \ldots, N\}$. By the construction in Step 1, the neighborhoods $Z_j \Subset U_j$ can be chosen so that ∂Z_j is arbitrarily close to ∂U_j, as defined above, thus we may assume that $B(p_j, \delta) \cap S \Subset Z_j \cap S$. We extend ρ smoothly on $\mathbb{C}^n \setminus \Omega$ to a function which is still denoted by ρ, by multiplying it by a suitable cut-off function. Then, the function

$$\widetilde{\varphi} := \varphi + C \cdot \rho$$

is plurisubharmonic for a sufficiently small $C > 0$. Also, since $C > 0$ and ρ is strictly plurisubharmonic on $\Omega \setminus \cup_{j=1}^N B(p_j, \delta)$, it follows that $\widetilde{\varphi}$ is strictly plurisubharmonic on S, everywhere h is not holomorphic. Removing the tilde and denoting $\widetilde{\varphi}$ by φ, it follows that the pair (h, φ) satisfies the conditions of Lemma 4.2.2 which shows that the set $X = \{|h| = e^\varphi\}$ is rationally convex. Clearly, $S \subset X$ and, by multiplying h with a suitable cut-off function, we can construct for any neighborhood Ω of S such a set X which is included in Ω. This proves that S is rationally convex.

Remark 4.2.5. In [16], the final perturbation of φ is done without the requirement of S to be locally polynomially convex. We were not able to adapt this technique to our more general case.

4.3 Examples

4.3.1 Rationally Convex Immersions in $\mathbb{C}^2$ that are not Isotropic with respect to any Kähler Form

By Theorem 1.2.1, $(i) \Rightarrow (ii)$, the rational convexity of S implies the existence of a nonnegative closed form of bidegree $(1, 1)$, $\omega := dd^c\varphi$, defined on $\mathbb{C}^n$, which is positive outside some pairwise disjoint neighborhoods V_j of p_j, $j = 1, \ldots, N$, it is identically zero on some smaller neighborhoods $p_j \in U_j \subset V_j$ and S is isotropic with respect to ω. It is natural to ask whether the same assumption of rational convexity for S leads to the existence of a genuine

Kähler form with respect to which S is isotropic, similar to the result in [14]. We show in the following that there exist compact immersions in $\mathbb{C}^2$ with one transverse self-intersection which are rationally convex but which are not Lagrangian with respect to any Kähler form and, in fact, that this class of immersions is not just an isolated case.

Denote by $\mathcal{W}$ the set of all 2×2 matrices with real entries such that for all $A \in \mathcal{W}$ the following properties are satisfied:

(a) $(A + i)$ is invertible;

(b) i is not an eigenvalue of A;

(c) A has no purely imaginary eigenvalue of modulus greater than 1.

It is straightforward to show that $\mathcal{W}$ is an open subset of the space of 2×2 matrices with real entries, $\mathcal{M}_{2 \times 2}(\mathbb{R})$. Let $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathcal{W}$ and define the following 2-dimensional subspace of $\mathbb{C}^2$

$$M(A) := (A + i)\mathbb{R}^2.$$

It can easily be verified that condition (b) above is equivalent to $M(A)$ being totally real. By Theorem 2.2.5 (Weinstock) it follows that every compact subset of $M(A) \cup \mathbb{R}^2$ is polynomially convex, hence rationally convex. For some $r > 0$ let $S_1(A) := \mathbb{R}^2 \cap \overline{B(0,r)}$, $S_2(A) := M(A) \cap \overline{B(0,r)}$ and

$$S(A) := S_1(A) \cup S_2(A).$$

Then $S(A)$ is a totally real, compact, rationally convex surface in $\mathbb{C}^2$, smooth everywhere except at the origin where it has a double self-intersection. Let $\iota_1, \iota_2 : \mathbb{R}^2_{(t,s)} \to \mathbb{C}^2_{(z_1,z_2)}$ be the maps given as

$$\iota_1(t, s) = (t, s),$$
$$\iota_2(t, s) = (xt + yt + it, zt + ws + is),$$

which satisfy $\iota_1(\mathbb{R}^2) = \mathbb{R}^2 = T_0 S_1(A)$ and $\iota_2(\mathbb{R}^2) = M(A) = T_0 S_2(A)$. Suppose that $S(A)$ is Lagrangian with respect to some Kähler form ω. This implies in particular that the restriction of $\omega(0)$ to the two tangent spaces of $S(A)$ at the origin vanishes or, equivalently, $\iota_1^* \omega = 0$ and $\iota_2^* \omega = 0$, where ι_1^*, ι_2^* denote the respective pullbacks. Since ω is Kähler, we can write

$$\omega = h_1 dz_1 \wedge d\bar{z}_1 + \bar{h} dz_1 \wedge d\bar{z}_2 + h dz_2 \wedge d\bar{z}_1 + h_2 dz_2 \wedge d\bar{z}_2,$$

where at each point $p \in \mathbb{C}^2$, $H_\omega(p) := \begin{bmatrix} h_1(p) & \overline{h}(p) \\ h(p) & h_2(p) \end{bmatrix}$ is a positive definite Hermitian matrix. The rest of our analysis takes place at the origin, and for simplicity, we shall use the notations $H_\omega := H_\omega(0), h_1 := h_1(0)$, etc. Direct calculations give

$$\iota_1^* \omega = (\overline{h} - h)dt \wedge ds,$$

and

$$\iota_2^* \omega = \{(\overline{h} - h)(xw - yz + 1) + i[2h_1 y - 2h_2 z - (\overline{h} + h)(x - w)]\}dt \wedge ds. \quad (4.3.1)$$

The condition $\iota_1^* \omega = 0$ implies $h = \overline{h} \in \mathbb{R}$ so H_ω has only real entries. Then (4.3.1) becomes

$$\iota_2^* \omega = 2i(-hx + hw + h_1 y - h_2 z)dt \wedge ds.$$

and $\iota_2^* \omega = 0$ gives

$$hx - hw - h_1 y + h_2 z = 0. \quad (4.3.2)$$

We just showed that, if $A \in \mathcal{W}$ and ω is a Kähler form in $\mathbb{C}^2$ such that $S(A)$ is Lagrangian with respect to ω then, at the origin, the Hermitian matrix H_ω associated with ω is in fact a positive definite symmetric matrix with real entries and equation (4.3.2) has to be satisfied.

Now, let $\widetilde{A} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. It is easy to see that $\widetilde{A} \in \mathcal{W}$, hence $S(\widetilde{A})$ is rationally convex. Suppose that $S(\widetilde{A})$ is Lagrangian with respect to some Kähler form ω in $\mathbb{C}^2$. Then, $\widetilde{A}$ has to satisfy equation (4.3.2). Substituting the entries of $\widetilde{A}$ in (4.3.2) we obtain $h = -(h_1 + h_2)$. Recall that $H_\omega = \begin{bmatrix} h_1 & h \\ h & h_2 \end{bmatrix}$ is positive definite which, by Sylvester's criterion, is equivalent to H_ω satisfying $h_1 > 0$ and $\det H_\omega > 0$, which also implies that $h_2 > 0$. However, since $h = -(h_1 + h_2)$, we have $\det H_\omega = h_1 h_2 - h^2 = -h_1^2 - h_2^2 - h_1 h_2 < 0$ which is a contradiction. It follows that $S(\widetilde{A})$ is not Lagrangian with respect to any Kähler form ω. Note that by Theorem 1.2.1 there exist a family of degenerate Kähler forms with respect to which $S(\widetilde{A})$ is indeed isotropic. Furthermore, the following holds.

Proposition 4.3.1. *The set of matrices $A \in \mathcal{W}$ such that $S(A)$ is not Lagrangian with respect to any Kähler form defined on $\mathbb{C}^2$ contains a nonempty subset which is open in $\mathcal{M}_{2 \times 2}(\mathbb{R})$.*

Proof. Define the set

$$\widetilde{\mathcal{W}} := \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathcal{W} : x - w \neq 0 \right\}.$$

Since $\mathcal{W}$ is open in $\mathcal{M}_{2\times2}(\mathbb{R})$ it is immediate that $\widetilde{\mathcal{W}}$ is also open in $\mathcal{M}_{2\times2}(\mathbb{R})$. $\widetilde{A} \in \widetilde{\mathcal{W}}$, where $\widetilde{A}$ is the matrix we defined in the example above, hence $\widetilde{\mathcal{W}} \neq \emptyset$. Define

$$F : \widetilde{\mathcal{W}} \ni \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto x^2 + w^2 - 2xw + 2yz \in \mathbb{R},$$

which is a continuous function. Therefore, $F^{-1}((-\infty, 0))$ is open in $\mathcal{M}_{2\times2}(\mathbb{R})$ and it is nonempty because $\widetilde{A} \in F^{-1}((-\infty, 0))$. Let $A := \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in F^{-1}((-\infty, 0))$ and suppose that $S(A)$ is Lagrangian with respect to some Kähler form ω. Then, (4.3.2) is satisfied: $h(x - w) - h_1 y + h_2 z = 0$ and, since $x - w \neq 0$,

$$h = \frac{h_1 y - h_2 z}{x - w}.$$

A direct computation gives

$$\det H_\omega = h_1 h_2 - h^2 = \frac{h_1 h_2(x^2 + w^2 - 2xw + 2yz) - h_1^2 y^2 - h_2^2 z^2}{(x - w)^2}$$

$$= \frac{h_1 h_2 F(A) - h_1^2 y^2 - h_2^2 z^2}{(x - w)^2}.$$

But $h_1 h_2 > 0, h_1^2 y^2 \geq 0, h_2^2 z^2 \geq 0$ and $F(A) < 0$ by construction, hence $\det H_\omega < 0$ which is a contradiction. It follows that for every element A of the nonempty open set $F^{-1}((-\infty, 0))$, $S(A)$ cannot be Lagrangian with respect to any Kähler form on $\mathbb{C}^2$, which ends the proof. $\square$

4.3.2 An Immersion in $\mathbb{C}^2$ which is not Locally Polynomially Convex

Consider the matrix

$$A = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix},$$

where $t > 1$. Maintaining the notations from the previous subsection, let $S := M(A) \cup \mathbb{R}^2 \subset \mathbb{C}^2$. By a result of Weinstock [41, Theorem 4], there exists

a continuous one-parameter family of analytic annuli in $\mathbb{C}^2$ whose boundaries lie in S and that converges to the origin as the parameter approaches 0. Let $K := S \cap \overline{B(0, \delta)}$ for some $\delta > 0$. Clearly K is not polynomially convex since, every annulus attached to K is included in $\widehat{K}$ by the Maximum Principle, hence S is not locally polynomially convex at the origin. If p is a point in the interior of one of the annuli attached to K and V is an algebraic variety passing through p, then it is known that V should intersect all the members of the family. This implies that V would intersect either the boundaries of the annuli or it would pass through the origin. Either way, V would intersect K which proves that $p \in \mathcal{R}-\text{hull}(K)$, hence K is not rationally convex.

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