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Torsors over Simplicial Schemes

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Abstract

Let X be a simplicial object in a small Grothendieck site \mathcal{C} , and let G be a sheaf of groups on \mathcal{C} . We define a notion of G -torsor over X , generalizing a definition of Gillet, and prove that there is a bijection $H^1(X, G) \cong [X, BG]$, between the set of isomorphism classes of G -torsors over X , and the set of maps in the homotopy category of simplicial presheaves on \mathcal{C} with respect to the local weak equivalences. We prove basic results about this invariant, including an exact sequence in non-abelian cohomology associated to a central extension of sheaves of groups, as well as a characterization of the sheaf cohomology group $H^2(BG, A)$, with coefficients in a sheaf of abelian groups A , in terms of central extensions of G by A .

It is well-known that, if k is a perfect field, the motivic cohomology of the classifying space $B\mathbf{GL}_n$ is a polynomial algebra over the motivic cohomology of k ; we give a proof that takes advantage of this theory of torsors over simplicial schemes.

Finally, using the work of Vistoli, we prove that, working over the complex numbers, the map in Chow groups

$$H^{2*}(B_{\text{et}}\mathbf{PGL}_p, \mathbb{Z}(*)) \rightarrow H^{2*}(B_{\text{Nis}}\mathbf{PGL}_p, \mathbb{Z}(*))$$

is injective, when p is an odd prime.

Summary for lay audience

Algebraic groups are important objects of study in Algebraic Geometry. An important example for this thesis is the projective linear group \mathbf{PGL}_n , which is the algebraic group that encodes the automorphisms, or symmetries, of projective space of dimension $n - 1$. There are various ways to construct “classifying spaces” for algebraic groups; if one can calculate the cohomology of the classifying space of an algebraic group, then one has found universal characteristic classes for the algebraic structures whose automorphism groups are given by that group.

This thesis presents results concerning the cohomology of the classifying spaces of algebraic groups. We begin in Chapter 2 by proving some basic results in a very general setting, where the classifying space is constructed using simplicial methods from Algebraic Topology. This simplicial construction has a long history, and the results of this part of the thesis are connected to work of Jardine, and earlier work of Giraud and Breen.

The general linear group plays a central role in the theory of algebraic groups; in Chapter 3, we study the motivic cohomology of its classifying space. The results of this chapter are not new, but this presentation has not appeared in print before. And, the ideas involved illustrate ways in which the results of Chapter 2 can be applied.

Finally, in Chapter 4, we prove a theorem about the relationship between the motivic cohomology of two different notions of classifying space for the projective linear group \mathbf{PGL}_p , where p is an odd prime. These two notions of classifying space correspond to the Nisnevich and étale topologies, respectively. To the best of my knowledge, the motivic cohomology of the Nisnevich classifying space has not before been studied in the literature, except in the rare cases in which it coincides with that of the étale classifying space.

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Introduction

Let's begin with an example. Over a field k , the algebraic group \mathbf{GL}_n represents a sheaf of groups on the Zariski site Sm_k of smooth k -schemes: $\mathbf{GL}_n(S)$ is the group of invertible $n \times n$ matrices with entries in the ring of global sections of the scheme S .

A simplicial (Zariski) sheaf on Sm_k is a functor $Sm_k^{op} \rightarrow \mathbf{sSet}$ that satisfies the usual patching criterion with respect to Zariski-open covers. An example of such an object is the classifying space of the sheaf of groups \mathbf{GL}_n , which is the composite

$$Sm_k^{op} \xrightarrow{\mathbf{GL}_n} \mathbf{Grp} \xrightarrow{B} \mathbf{sSet} .$$

Here, B is the nerve, or classifying space functor. This simplicial sheaf deserves its name, because there is a bijection $H^1(S, \mathbf{GL}_n) \cong [S, B\mathbf{GL}_n]$, between the set of isomorphism classes of vector bundles over S , and the set of maps in the homotopy category of simplicial sheaves on Sm_k , formed with respect to the Zariski-local equivalences.

This is a special case of a very general result of Jardine [Jar89], but it is also a manifestation of something very concrete. A rank n vector bundle V over a scheme S can be given by the data of a Zariski-open cover $\{U_i\}$ of S , and transition functions $T_{ij} \in \mathbf{GL}_n(U_i \cap U_j)$ satisfying the cocycle condition

$$T_{jk}|_{U_i \cap U_j \cap U_k} \circ T_{ij}|_{U_i \cap U_j \cap U_k} = T_{ik}|_{U_i \cap U_j \cap U_k} .$$

The Čech nerve of the cover $\{U_i\}$ is a simplicial scheme U , with

$$U_n = \bigsqcup_{(i_0, \dots, i_n)} U_{i_0} \cap \dots \cap U_{i_n}$$

and face maps given by the obvious inclusions. The natural map $U \rightarrow S$ of simplicial schemes is a Zariski-local equivalence. Moreover, the data of the transition functions is exactly what's needed to define a map $f : U \rightarrow B\mathbf{GL}_n$. In simplicial degree zero, f_0 is the unique map to the terminal sheaf, and in degree 1, the transition functions determine maps $U_i \cap U_j \rightarrow \mathbf{GL}_n$, which define f_1 . Because of the cocycle condition, the maps $(T_{ij}, T_{jk}) : U_i \cap U_j \cap U_k \rightarrow \mathbf{GL}_n \times \mathbf{GL}_n$ define a map $f_2 : U_2 \rightarrow (B\mathbf{GL}_n)_2$ such that f_1 and f_2 commute with the simplicial structure maps. The simplicial sheaf $B\mathbf{GL}_n$ is 2-coskeletal, because the nerve of any category is 2-coskeletal, so this determines the map $f : U \rightarrow B\mathbf{GL}_n$. The diagram $S \xleftarrow{\sim} U \xrightarrow{f} B\mathbf{GL}_n$ gives the map in the homotopy category $S \rightarrow B\mathbf{GL}_n$ corresponding to the vector bundle V .

In [Gil83], Henri Gillet defines a notion of vector bundle over a simplicial scheme. Using this definition, one can define a universal rank n vector bundle over the simplicial scheme $B\mathbf{GL}_n$, so that the pullback of this universal vector bundle along the classifying map $f : U \rightarrow B\mathbf{GL}_n$ gives a vector bundle over U , which coincides with the pullback of V along the Zariski-local equivalence $U \rightarrow S$.

Now, for any sheaf of groups G on a site \mathcal{C} , the classifying space construction gives a simplicial sheaf BG , which classifies G -torsors in a sense analogous to the classification of vector bundles by $B\mathbf{GL}_n$; this is the theorem Jardine proved in [Jar89]. These classifying spaces BG are the main subject of this thesis.

Let X be a simplicial object in \mathcal{C} . In Chapter 2, we define a notion of

G -torsor over X , generalizing Gillet's definition, and prove that there is a bijection $H^1(X, G) \cong [X, BG]$, between the set of isomorphism classes of G -torsors over X , and the set of maps in the homotopy category of simplicial presheaves on \mathcal{C} , formed with respect to the local weak equivalences. And, we prove some basic results about this invariant, including an exact sequence in non-abelian cohomology associated to a central extension of sheaves of groups, as well as a characterization of the sheaf cohomology group $H^2(BG, A)$, with coefficients in a sheaf of abelian groups A , in terms of central extensions of G by A .

It is well-known that, if k is a perfect field, the motivic cohomology of the classifying space $B\mathbf{GL}_n$ of the k -group \mathbf{GL}_n is a polynomial algebra over the motivic cohomology of k ; this calculation appears in a paper of Oleg Pushin [Pus04]. In Chapter 3, we give a proof of this result that takes advantage of the motivic model structure on simplicial presheaves with transfers, which is recalled in Chapter 1, as well as the theory of torsors over simplicial schemes developed in Chapter 2. Both Pushin's argument and the argument of Chapter 3 make use of vector bundles over simplicial schemes: the key step is showing that the map $E\mathbf{GL}_n \times_{\mathbf{GL}_n} \mathbb{P}^{n-1} \rightarrow B\mathbf{GL}_n$, from the Borel construction for the action of \mathbf{GL}_n on projective space, induces a monomorphism in motivic cohomology; the Borel construction $E\mathbf{GL}_n \times_{\mathbf{GL}_n} \mathbb{P}^{n-1}$ can be seen as the projectivization of the total space of the universal vector bundle over $B\mathbf{GL}_n$. Better understanding the homotopy theory of these vector bundles was the original motivation for the work of Chapter 2. Making use of the homotopy classification of torsors established there, we extend Pushin's monomorphism in motivic cohomology to an identification of motivic homotopy types

$$\mathbb{Z}_{tr}(E\mathbf{GL}_n \times_{\mathbf{GL}_n} \mathbb{P}^{n-1}) \simeq \mathbb{Z}_{tr}(B\mathbf{GL}_n \times \mathbb{P}^{n-1}),$$

after applying the free presheaf with transfers functor \mathbb{Z}_{tr} .

In motivic homotopy theory, it is typical to give the category Sm_k the Nisnevich topology. Then, if G is a Nisnevich sheaf of groups on Sm_k , the classifying space BG constructed using the nerve functor $B : \mathbf{Grp} \rightarrow \mathbf{sSet}$ classifies the Nisnevich G -torsors. So, in this context, we'll denote this construction by $B_{\text{Nis}}G$. Following Morel and Voevodsky [MV99], if G is an étale sheaf of groups on Sm_k , let $B_{\text{et}}G$ be the Nisnevich homotopy type of an étale fibrant model of $B_{\text{Nis}}G$. The object $B_{\text{et}}G$ is called the étale classifying space of G , as it classifies étale G -torsors, in the sense that there is a bijection

$$H_{\text{et}}^1(S, G) \cong [S, B_{\text{et}}G]_{\text{Nis}}$$

for any S in Sm_k . Up to motivic weak equivalence, the étale classifying space $B_{\text{et}}G$ of a linear algebraic group G is a sequential colimit of smooth k -schemes; this is a result of Morel and Voevodsky [MV99, Proposition 4.2.6].

There is a canonical map in motivic cohomology

$$H^*(B_{\text{et}}G, \mathbb{Z}(*)) \rightarrow H^*(B_{\text{Nis}}G, \mathbb{Z}(*))$$

induced by the map $B_{\text{Nis}}G \rightarrow B_{\text{et}}G$. Using the work of Angelo Vistoli [Vis07], in Chapter 4, we prove that, working over the complex numbers, and restricting attention to the Chow groups, the map

$$H^{2*}(B_{\text{et}}\mathbf{PGL}_p, \mathbb{Z}(*)) \rightarrow H^{2*}(B_{\text{Nis}}\mathbf{PGL}_p, \mathbb{Z}(*))$$

is injective, when p is an odd prime.

Chapter 1

Background on motivic cohomology

1.1 Simplicial presheaves with transfers

Let k be a perfect field. In this section, we define the additive category Cor_k of finite correspondences, as in [MVW06], and the motivic model structure on simplicial presheaves with transfers, as developed in [RØ08] and [Jar15].

Let Sm_k be the Grothendieck site of smooth, separated k -schemes, with the Nisnevich topology.

If X and Y are schemes, with X connected, an elementary correspondence from X to Y is an irreducible, closed subset of $X \times Y$ whose associated integral subscheme is finite and surjective over X . An elementary correspondence from a non-connected scheme X to Y is an elementary correspondence from a connected component of X to Y . Let $Cor(X, Y)$ be the free abelian group generated by the set of elementary correspondences from X to Y . We'll call the elements of $Cor(X, Y)$ finite correspondences.

If $f : X \rightarrow Y$ is a map in Sm_k and X is connected, then the graph of f

defines an elementary correspondence from X to Y . If X is not connected, then the sum of the components of the graph of f is a finite correspondence. The additive category Cor_k has the same objects as Sm_k , and for any schemes X, Y , the group $Cor(X, Y)$ gives the morphisms from X to Y . The law of composition is defined via the intersection product, see [MVW06, p4]. There is a functor $\gamma : Sm_k \rightarrow Cor_k$ that is the identity on objects, and takes a morphism of schemes to its graph.

Definition 1.1.1. A presheaf with transfers is an additive functor $Cor_k^{op} \rightarrow \mathbf{Ab}$. We'll write $\mathbf{PST}(k)$ for the category whose objects are presheaves with transfers, and whose morphisms are natural transformations; we'll write $\mathbf{sPST}(k)$ for the category of simplicial presheaves with transfers.

There is a free-forgetful adjunction, defined sectionwise:

$$\mathbf{sPre}(Sm_k) \begin{array}{c} \xrightarrow{\mathbb{Z}} \\ \xleftarrow{u} \end{array} \mathbf{sPre}_{\mathbb{Z}}(Sm_k)$$

where the left-hand side is the category of simplicial presheaves (ie, functors $Sm_k^{op} \rightarrow \mathbf{sSet}$), and the right-hand side is the category of simplicial abelian presheaves (ie, contravariant functors $Sm_k^{op} \rightarrow \mathbf{sAb}$).

There is a functor $\gamma_* : \mathbf{sPST}(k) \rightarrow \mathbf{sPre}_{\mathbb{Z}}(Sm_k)$ that is defined by composition with γ , and this functor has a left adjoint

$$\gamma^* : \mathbf{sPre}_{\mathbb{Z}}(Sm_k) \rightarrow \mathbf{sPST}(k)$$

which is the left Kan extension determined by

$$\gamma^*(\mathbb{Z}(\Delta^n) \otimes \mathbb{Z}(\mathrm{hom}(-, U))) = \mathbb{Z}(\Delta^n) \otimes Cor(-, U).$$

We'll use $\mathbb{Z}_{tr} : \mathbf{sPre}(Sm_k) \rightarrow \mathbf{sPST}(k)$ to denote the composite $\gamma^* \circ \mathbb{Z}$.

1.1.1 Model structures

In order to define the motivic model structure on $\mathbf{sPST}(k)$, we begin with some important definitions, following [MV99] and [Jar15].

Definition 1.1.2. An elementary distinguished square in Sm_k is a pullback diagram

$$\begin{array}{ccc} U \times_S V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & S \end{array}$$

with j an open embedding, p an étale morphism, such that the induced morphism $p^{-1}(S - U) \rightarrow S - U$ of closed subschemes (with reduced structure) is an isomorphism.

Definition 1.1.3. A simplicial presheaf X on Sm_k has the B.G.-property if $X(\emptyset)$ is contractible, and the diagram of simplicial sets

$$\begin{array}{ccc} X(S) & \longrightarrow & X(V) \\ \downarrow & & \downarrow \\ X(U) & \longrightarrow & X(U \times_S V) \end{array}$$

is a homotopy pullback for all elementary distinguished squares in Sm_k .

Recall that the projective model structure on $\mathbf{sPre}(Sm_k)$ has the sectionwise Kan fibrations for fibrations, and the sectionwise weak equivalences for weak equivalences. The cofibrations are defined by the left lifting property, and are called projective cofibrations. Now, we'll describe a localization of the projective model structure that has the Nisnevich-local equivalences for weak equivalences; this is [Jar15, Example 7.22]. Let \mathcal{F} be the smallest set of projective cofibrations of $\mathbf{sPre}(Sm_k)$ that contains

1 All the generating trivial cofibrations for the projective model structure,

which are the maps of the form:

$$\Lambda_k^n \times \text{hom}(-, U) \rightarrow \Delta^n \times \text{hom}(-, U)$$

2 Projective cofibrant replacements of the maps

$$U \cup_{U \times_S V} V \rightarrow S$$

associated to all elementary distinguished squares

3 The map

$$\emptyset \rightarrow \text{hom}(-, \emptyset)$$

from the empty simplicial presheaf to the presheaf represented by the empty scheme

and such that \mathcal{F} satisfies the closure property: if $A \rightarrow B$ is a member of \mathcal{F} , then so are all cofibrations

$$(B \times \partial\Delta^n) \cup (A \times \Delta^n) \rightarrow B \times \Delta^n .$$

Using the methods of [Jar15, Ch 7], we can form the left Bousfield localization of the projective model structure on $\mathbf{sPre}(Sm_k)$ with respect to \mathcal{F} .

Proposition 1.1.4. *A map $f : X \rightarrow Y$ of $\mathbf{sPre}(Sm_k)$ is an \mathcal{F} -local equivalence if and only if it is a Nisnevich-local equivalence.*

Proof. First, we'll show that if Z is \mathcal{F} -fibrant, then Z has the B.G.-property.

Let $(j : U \subset S, p : V \rightarrow S)$ be the data of an elementary distinguished

square, and choose a factorization

$$\begin{array}{ccc} & & Y \\ & \nearrow i & \searrow f \\ U \times_S V & \longrightarrow & V \end{array}$$

such that i is a projective cofibration, and f is a sectionwise equivalence. To show that the diagram

$$\begin{array}{ccc} Z(S) & \longrightarrow & Z(V) \\ \downarrow & & \downarrow \\ Z(U) & \longrightarrow & Z(U \times_S V) \end{array}$$

is a homotopy pullback, it suffices to show that the induced map

$$\mathbf{hom}(S, Z) \rightarrow \mathbf{hom}(U \cup_{U \times_S V} Y, Z)$$

of simplicial sets is an equivalence, and this follows from the fact that a projective cofibrant replacement of the map $U \cup_{U \times_S V} V \rightarrow S$ is in \mathcal{F} .

As Z has the B.G.-property, it follows from a theorem of Morel-Voevodsky (see [MV99, Proposition 1.16], and [Jar15, Theorem 5.39]) that there is a sectionwise equivalence $Z \rightarrow Z'$ such that Z' has the right lifting property with respect to all monomorphisms that are Nisnevich-local equivalences.

Now, if $f : X \rightarrow Y$ is a Nisnevich-local equivalence, we can replace f up to sectionwise equivalence with a map $f_c : X_c \rightarrow Y_c$ between projective cofibrant objects. For any \mathcal{F} -fibrant object Z , the map $f_c^* : \mathbf{hom}(Y_c, Z') \rightarrow \mathbf{hom}(X_c, Z')$ is an equivalence of simplicial sets, and therefore $f_c^* : \mathbf{hom}(Y_c, Z) \rightarrow \mathbf{hom}(X_c, Z)$ is as well. It follows that f is an \mathcal{F} -local equivalence.

Conversely, say $f : X \rightarrow Y$ is an \mathcal{F} -local equivalence. Then, by the construction of the \mathcal{F} -local model structure (see [Jar15, Theorem 7.10]) there

is a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \eta \downarrow & & \downarrow \eta' \\ L_{\mathcal{F}}X & \xrightarrow{L_{\mathcal{F}}(f)} & L_{\mathcal{F}}Y \end{array}$$

such that $L_{\mathcal{F}}X$ and $L_{\mathcal{F}}Y$ are \mathcal{F} -fibrant, the map $L_{\mathcal{F}}(f)$ is a sectionwise equivalence, and the maps η, η' are in the saturation of \mathcal{F} . As all the maps in \mathcal{F} are Nisnevich-local equivalences, it follows that f is a Nisnevich-local equivalence. \square

Let $\mathcal{F}_{\mathbb{A}^1}$ be the set obtained by adding to \mathcal{F} the 0-section $U \rightarrow \mathbb{A}^1 \times U$ of the affine line, for all smooth k -schemes U . We can localize the projective model structure on $\mathbf{sPre}(Sm_k)$ with respect to $\mathcal{F}_{\mathbb{A}^1}$, and I'll call the result the projective motivic model structure. To see that the $\mathcal{F}_{\mathbb{A}^1}$ -local equivalences coincide with the motivic weak equivalences¹ observe that a $\mathcal{F}_{\mathbb{A}^1}$ -fibrant simplicial presheaf Z has the B.G.-property, and the map $Z(\mathbb{A}^1 \times U) \rightarrow Z(U)$ is a weak equivalence for all smooth k -schemes U . It follows that a motivic fibrant replacement $Z \rightarrow Z'$ is a sectionwise equivalence, and then one can argue as in the proof of Proposition 1.1.4.

For later use, we record a lemma about motivic weak equivalences:

Lemma 1.1.5. *If $f : X \rightarrow Y$ is a map of simplicial presheaves such that each map $f_n : X_n \rightarrow Y_n$ is a motivic weak equivalence, then f is a motivic weak equivalence.*

Proof. The statement follows from the assertion that, for any map $X \rightarrow Y$ of bisimplicial presheaves such that each map $X_n \rightarrow Y_n$ of vertical simplicial presheaves is a motivic weak equivalence, then the induced map on diagonals $d(X) \rightarrow d(Y)$ is a motivic weak equivalence.

¹See [MV99], where the motivic weak equivalences are called \mathbb{A}^1 -weak equivalences, and are defined for maps between simplicial sheaves on the smooth Nisnevich site. See also [Jar15, Example 7.20].

For this, recall that the motivic model structure on $\mathbf{sPre}(Sm_k)$ has the monomorphisms for cofibrations, and is therefore a category of cofibrant objects. Then, the assertion can be proved as the analogous assertion for bisimplicial sets (see, for example, [GJ09, Proposition IV.1.7]). \square

There is also a projective model structure on $\mathbf{sPST}(k)$. The fibrations and weak equivalences are defined sectionwise, and the cofibrations are defined by a left lifting property. By [Jar15, Thm 8.48] we can localize the projective model structure on $\mathbf{sPST}(k)$ at the set of cofibrations $\mathbb{Z}_{tr}(\mathcal{F})$, and I'll call the result the projective local model structure. Similarly, we can localize the projective model structure on $\mathbf{sPST}(k)$ at $\mathbb{Z}_{tr}(\mathcal{F}_{\mathbb{A}^1})$, and I'll call the result the motivic model structure.

For later use, we record a lemma about projective cofibrations of simplicial presheaves with transfers. The Dold-Kan functors $N : \mathbf{sAb} \rightarrow Ch_+(\mathbf{Ab})$ and $\Gamma : Ch_+(\mathbf{Ab}) \rightarrow \mathbf{sAb}$ can be used to define $N : \mathbf{sPST}(k) \rightarrow Ch_+(\mathbf{PST}(k))$, where $N(X : Cor_k^{op} \rightarrow \mathbf{sAb})$ is given by $N \circ X$, and $\Gamma : Ch_+(\mathbf{PST}(k)) \rightarrow \mathbf{sPST}(k)$ is defined similarly. Then N and Γ form an equivalence of categories.

As $\mathbf{PST}(k)$ is an abelian category with enough projectives [MVW06, Theorem 2.3], $Ch_+(\mathbf{PST}(k))$ has a model structure for which the weak equivalences are the homology isomorphisms, the fibrations are the maps that are epimorphisms in positive degrees, and the cofibrations are the monomorphisms that have degree-wise projective cokernels. We can use this to check if a map of simplicial presheaves with transfers is a projective cofibration:

Lemma 1.1.6. *A map $f : A \rightarrow B$ of simplicial presheaves with transfers is a projective cofibration if and only if it is a monomorphism, and $N(\text{coker}(f))$ is projective in each degree.*

Proof. Say $p : X \rightarrow Y$ is a sectionwise trivial fibration of $\mathbf{sPST}(k)$, and we

have a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

The map $N(p)$ is a homology isomorphism which is epi in positive degrees, so if $N(f)$ is a monomorphism with $N(\text{coker}(f))$ projective in each degree, then the lift exists in the image of the above diagram under N , and thus in the original diagram. The converse is similar. \square

1.1.2 Tensor products

Let $\mathbf{sPre}_{\mathbb{Z}}(\mathcal{S}m_k)$ denote the category of simplicial abelian presheaves; it has a closed symmetric monoidal structure, with

$$(A \otimes B)(U)_n = A(U)_n \otimes B(U)_n$$

and

$$\mathbf{Hom}(A, B)(U)_n = \text{hom}(A \otimes \mathbb{Z}(U) \otimes \mathbb{Z}(\Delta^n), B)$$

for all simplicial abelian presheaves A, B , all schemes U , and all $n \geq 0$.

The category Cor_k of finite correspondences has a symmetric monoidal structure, given by cartesian product of schemes. Using this, and the degreewise tensor product on \mathbf{sAb} , Day convolution defines a closed symmetric monoidal product on $\mathbf{sPST}(k)$, which I'll denote by \otimes_{tr} . It's easy to check that the free presheaf with transfers functor $\mathbb{Z}_{tr} : \mathbf{sPre}(\mathcal{S}m_k) \rightarrow \mathbf{sPST}(k)$ is monoidal with respect to the cartesian product on $\mathbf{sPre}(\mathcal{S}m_k)$ and \otimes_{tr} .

Furthermore, the product \otimes_{tr} and the motivic model structure make $\mathbf{sPST}(k)$ into a monoidal model category [RØ08, Lemma 10].

When we define the motivic cohomology groups, we'll use the following

notation from [Jar15, Chapter 8].

Notation 1.1.7. Let $\mathbb{Z}_\bullet(S^1)$ be the cokernel of the map $\mathbb{Z}(\Delta^0) \rightarrow \mathbb{Z}(S^1)$ induced by the base point of $S^1 = \Delta^1/\partial\Delta^1$. For a simplicial abelian presheaf $A \in \mathbf{sPre}_\mathbb{Z}(Sm_k)$ and $n \in \mathbb{N}$,

$$\begin{aligned}\Omega^n(A) &= \mathbf{Hom}(\mathbb{Z}_\bullet(S^1)^{\otimes n}, A) \quad \text{and} \\ \Omega^{-n}(A) &= A \otimes \mathbb{Z}_\bullet(S^1)^{\otimes n}.\end{aligned}$$

For all simplicial abelian presheaves A and $n \in \mathbb{N}$, there are natural weak equivalences

$$\Gamma(NA[-n]) \simeq \Omega^{-n}(A) \quad \text{and} \quad \Gamma(NA[n]) \simeq \Omega^n(A).$$

where Γ and N are the Dold-Kan functors.

1.2 Motivic cohomology

The motivic cohomology groups are the Nisnevich hypercohomology groups of some complexes of Nisnevich (even étale) sheaves with transfers. These groups are representable in many derived categories, and there are many approaches to constructing these categories. This section follows [Jar15, Chapter 8].

Definition 1.2.1. If K is a simplicial presheaf with transfers, then the singular complex of K is

$$C_*(K) = d(\mathbf{Hom}_{tr}(\Delta_k^\bullet, K))$$

in $\mathbf{sPST}(k)$, where Δ_k^\bullet is the cosimplicial scheme made up of the affine spaces \mathbb{A}_k^n in the usual way, and d is the diagonal functor from bisimplicial abelian groups to simplicial abelian groups.

Notation 1.2.2. For a pointed simplicial presheaf (X, x) on Sm_k , write $\mathbb{Z}_{tr}(X, x)$ for the cokernel of the map $\mathbb{Z}_{tr}(*) \xrightarrow{x} \mathbb{Z}_{tr}(X)$.

Definition 1.2.3. For $q \in \mathbb{N}$, let $\mathbb{Z}(q)$ be the simplicial presheaf with transfers

$$\mathbb{Z}(q) = C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}, e)) \otimes \mathbb{Z}_\bullet(S^1)^{\otimes q} .$$

where \mathbb{G}_m is pointed by the identity element e .

Definition 1.2.4. For any simplicial presheaf $X \in \mathbf{sPre}(Sm_k)$, and $p \in \mathbb{Z}$, $q \in \mathbb{N}$,

$$H^p(X, \mathbb{Z}(q)) = [\mathbb{Z}(X), \Omega^{2q-p}(\gamma_*\mathbb{Z}(q))]_{\text{Nis}} .$$

For any $p \in \mathbb{Z}$, $q \in \mathbb{N}$, a Nisnevich fibrant model of $\Omega^{2q-p}(\gamma_*\mathbb{Z}(q))$ is already motivic fibrant. This follows from [MVW06, Corollary 14.9]; see also [Jar15, Example 8.49]. So, there is an isomorphism

$$[\mathbb{Z}(X), \Omega^{2q-p}(\gamma_*\mathbb{Z}(q))]_{\text{Nis}} \cong [\mathbb{Z}(X), \Omega^{2q-p}(\gamma_*\mathbb{Z}(q))]_{\mathbb{A}^1}$$

for any simplicial presheaf X .

Finally, we can construct the cup product in motivic cohomology, following [MVW06] and [Jar15]. For any $q, s \in \mathbb{N}$, there is a canonical map [MVW06, Construction 3.10]:

$$\gamma_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}) \otimes \gamma_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge s}) \rightarrow \gamma_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge(q+s)}) ,$$

which induces a map

$$\mathbb{Z}(q) \otimes \mathbb{Z}(s) \rightarrow \mathbb{Z}(q+s) .$$

For any $p, r \in \mathbb{Z}$, and for any simplicial abelian presheaves A and B , there is a

canonical map [Jar15, Remark 8.36]:

$$\Omega^{-p}(A) \otimes \Omega^{-r}(B) \rightarrow \Omega^{-p-r}(A \otimes B) .$$

These maps induce the cup product pairing

$$H^p(X, \mathbb{Z}(q)) \otimes H^r(X, \mathbb{Z}(s)) \rightarrow H^{p+q}(X, \mathbb{Z}(q+s))$$

for any simplicial presheaf X .

Chapter 2

Torsors over simplicial objects

This chapter collects together several general results about torsors and non-abelian cohomology. In Section 2.2, there is a definition of G -torsor over a simplicial object X in a Grothendieck site \mathcal{C} , for any sheaf of groups G on \mathcal{C} , as well as a homotopy classification of the resulting non-abelian cohomology invariant $H^1(X, G)$. There is an exact sequence in non-abelian cohomology associated to any central extension of sheaves of groups, and this leads us to consider the cohomology group $H^2(X, A)$, where A is a sheaf of abelian groups on \mathcal{C} . In Section 2.3, we prove that, when G is a presheaf of groupoids on \mathcal{C} , there is a canonical bijection between the cohomology group $H^2(BG, A)$ and a suitably defined set of equivalence classes of central extensions of G by A . Finally, in Section 2.4, we use the results of Section 2.2 and Section 2.3 to give a characterization of $H_{\text{Nis}}^1(BG, H)$, where H is a sheaf of groups on the Nisnevich site Sm_k of smooth schemes over a field k , and G is an algebraic group over k .

2.1 Torsors

This section gives a quick introduction to the theory of torsors, following [Jar15, Chapter 9].

Let \mathcal{C} be a small Grothendieck site. If G is a sheaf of groups on \mathcal{C} , and F is a sheaf with G -action, we say that the G -action is free if it is free in each section.

Definition 2.1.1. If G is a sheaf of groups on a site \mathcal{C} , a G -torsor is a sheaf F with a free G -action, such that the canonical map $F/G \rightarrow *$ is an isomorphism in the sheaf category.

If U is an object of a site \mathcal{C} , we'll say that a G -torsor over U is a $G|_U$ -torsor on the site \mathcal{C}/U .

There is a nice characterization of G -torsors in terms of the Borel construction: if F is a sheaf with G -action, the Borel construction $EG \times_G F$ is the nerve of a sheaf of groupoids, which I'll call $E_G F$: for an object U of \mathcal{C} , $E_G F(U)$ is the groupoid with $\text{Ob } E_G F(U) = F(U)$, and $\text{Mor } E_G F(U)(x, y) = \{g \in G(U) \mid gx = y\}$. Every sheaf of groups G acts on itself by the group operation; in this case, we'll write $EG = EG \times_G G$.

Proposition 2.1.2. *If F is a sheaf with G -action, then F is a G -torsor if and only if the canonical map $EG \times_G F \rightarrow *$ is a local weak equivalence.*

Proof. Because $EG \times_G F$ is the nerve of a sheaf of groupoids, all sheaves of π_n vanish for $n \geq 2$. We have an isomorphism of sheaves $\tilde{\pi}_0(EG \times_G F) \cong F/G$.

If the G -action on F is free, then all presheaves of fundamental groups of $EG \times_G F$ vanish, so the map $EG \times_G F \rightarrow *$ is a local weak equivalence.

If the map $EG \times_G F \rightarrow *$ is a local weak equivalence, then for any object U of \mathcal{C} , and for any $f \in F(U)$, the induced map of presheaves $\pi_1(EG \times_G$

$F|_U, f) \rightarrow *$ is a local monomorphism. So, if $g \in G(U)$ satisfies $gf = f$, there is a cover of U such that g restricts to the identity element on each member of the cover. As G is a sheaf, g is the identity. \square

A map of G -torsors $F \rightarrow F'$ is a G -equivariant map.

Proposition 2.1.3. *If G is a sheaf of groups on \mathcal{C} , then the category of G -torsors on \mathcal{C} is a groupoid.*

Proof. If $f : F \rightarrow F'$ is a map of G -torsors, there is an induced diagram

$$\begin{array}{ccc} EG \times_G F & \xrightarrow{\cong} & EG \times_G F' \\ & \searrow & \swarrow \\ & BG & \end{array}$$

The natural map from the Borel construction to BG is a local fibration, and by [Jar15, Lemma 5.20], pullback diagrams of simplicial presheaves in which one leg is a local fibration are homotopy cartesian for the injective local model structure. So, the induced map on fibres, which is $f : F \rightarrow F'$, is a local weak equivalence of sheaves, hence an isomorphism. \square

2.2 Torsors over simplicial objects

The central definition of this chapter was motivated by a definition of Henri Gillet [Gil83, Example 1.1]. According to Gillet, if X is a simplicial scheme, a vector bundle V over X consists of a vector bundle V_k over X_k for all $k \geq 0$, and for all ordinal number maps $\tau : \mathbf{m} \rightarrow \mathbf{n}$, an isomorphism $\tau^*(V_m) \rightarrow V_n$.

Furthermore, Gillet says that a vector bundle V over X is determined by the data of a vector bundle V_0 over X_0 , together with an isomorphism $\alpha : d_0^*V_0 \rightarrow d_1^*V_0$ such that $d_2^*\alpha \circ d_0^*\alpha = d_1^*\alpha$.

We will use the following definition, which was suggested to me by Rick Jardine.

Definition 2.2.1. Let \mathcal{C} be a small Grothendieck site, G a sheaf of groups on \mathcal{C} , and X a simplicial object in \mathcal{C} . A G -torsor over X consists of a G -torsor F over X_0 , together with an isomorphism $\alpha : d_1^*F \rightarrow d_0^*F$ of G -torsors over X_1 , such that $s_0^*\alpha = \text{id}_F$, and the relation

$$d_0^*\alpha \circ d_2^*\alpha = d_1^*\alpha$$

holds over X_2 . If (F, α) and (F', α') are G -torsors over X , an isomorphism $(F, \alpha) \rightarrow (F', \alpha')$ consists of an isomorphism $\varphi : F \rightarrow F'$ of G -torsors over X_0 , such that the relation

$$d_0^*\varphi \circ \alpha = \alpha' \circ d_1^*\varphi$$

holds over X_1 .

Write $i : \mathbf{0} \rightarrow \mathbf{n}$ for the ordinal number map that takes 0 to i . For those who like diagrams, the cocycle condition says that the following triangle commutes

$$\begin{array}{ccc} 0^*(F) & \xrightarrow{d_2^*\alpha} & 1^*(F) \\ & \searrow d_1^*\alpha & \swarrow d_0^*\alpha \\ & & 2^*(F) \end{array}$$

And, an isomorphism $(F, \alpha) \rightarrow (F', \alpha')$ consists of an isomorphism $\varphi : F \rightarrow F'$ such that the following square commutes

$$\begin{array}{ccc} d_1^*(F) & \xrightarrow{d_1^*\varphi} & d_1^*(F') \\ \alpha \downarrow & & \downarrow \alpha' \\ d_0^*(F) & \xrightarrow{d_0^*\varphi} & d_0^*(F') \end{array}$$

Write $H^1(X, G)$ for the set of isomorphism classes of G -torsors over a sim-

plial object X , and let $[-, -]$ denote the set of maps in the homotopy category of $\mathbf{sPre}(\mathcal{C})$, with respect to the injective local model structure.

Theorem 2.2.2. *If G is a sheaf of groups on \mathcal{C} , and X is a simplicial object in \mathcal{C} , then there is a bijection $H^1(X, G) \cong [X, BG]$.*

Before we prove the theorem, we'll need some terminology.

Definition 2.2.3. Let \mathcal{C} be a small Grothendieck site, H a presheaf of groupoids on \mathcal{C} , and X a simplicial object in \mathcal{C} . We'll say that an H -mapping datum over X consists of an object F of $H(X_0)$, together with a morphism $\alpha : d_1^*F \rightarrow d_0^*F$ in $H(X_1)$, such that $s_0^*\alpha = \text{id}_F$, and the relation

$$d_0^*\alpha \circ d_2^*\alpha = d_1^*\alpha$$

holds in $H(X_2)$. If (F, α) and (F', α') are H -mapping data over X , an isomorphism $(F, \alpha) \rightarrow (F', \alpha')$ consists of an isomorphism $\varphi : F \rightarrow F'$ in $H(X_0)$, such that the relation

$$d_0^*\varphi \circ \alpha = \alpha' \circ d_1^*\varphi$$

holds in $H(X_1)$.

Let $\text{MD}(X, H)$ denote the set of isomorphism classes of H -mapping data over X , and let $\pi(-, -)$ denote the set of simplicial homotopy classes of maps.

Lemma 2.2.4. *If H is a presheaf of groupoids on \mathcal{C} , and X is a simplicial object in \mathcal{C} , there is a bijection*

$$\text{MD}(X, H) \cong \pi(X, BH).$$

Proof. Say $f : X \rightarrow BH$ is a map. Then $f_0 : X_0 \rightarrow \text{Ob}(H)$ corresponds to an element $F \in \text{Ob}(H)(X_0)$. The map $f_1 : X_1 \rightarrow \text{Mor}(H)$ corresponds to

$\alpha \in \text{Mor}(H)(X_1)$, and $f_2 : X_2 \rightarrow \text{Mor}(H) \times_{s,t} \text{Mor}(H)$ corresponds to a pair $(\beta, \gamma) \in \text{Mor}(H)(X_2) \times_{s,t} \text{Mor}(H)(X_2)$ (here, s and t are source and target maps).

We have $d_i \alpha = d_i^* F$ for $i = 0, 1$, and the restriction $s_0^* \alpha$ is the identity on F . Furthermore,

$$\begin{aligned}\gamma &= d_0(\beta, \gamma) = d_0^* \alpha \\ \gamma \circ \beta &= d_1(\beta, \gamma) = d_1^* \alpha \\ \beta &= d_2(\beta, \gamma) = d_2^* \alpha\end{aligned}$$

so we have $d_0^* \alpha \circ d_2^* \alpha = d_1^* \alpha$.

A simplicial homotopy $h : X \times \Delta^1 \rightarrow BH$ between $f, f' : X \rightarrow BH$ can be seen as a map $X \rightarrow B(H^{\mathbf{1}})$, where $H^{\mathbf{1}}$ is the presheaf of groupoids with $H^{\mathbf{1}}(U) = \text{Fun}(\mathbf{1}, H(U))$ for all objects U of \mathcal{C} . By adjointness, h corresponds to a map $\tilde{h} : \pi(X \times \Delta^1) \rightarrow H$ from the fundamental groupoid of $X \times \Delta^1$. As the fundamental groupoid functor respects products, and $\pi(\Delta^1)$ is the free groupoid on the ordinal number $\mathbf{1}$, such a map is equivalent to a map $\pi(X) \rightarrow H^{\mathbf{1}}$.

If $f, f' : X \rightarrow BH$ are maps, (F, α) is the H -mapping datum over X that corresponds to f , and (F', α') corresponds to f' , and $h : X \rightarrow B(H^{\mathbf{1}})$ is a homotopy from f to f' , then h_0 corresponds to a map $\varphi : F \rightarrow F'$ in $H(X_0)$, which defines an isomorphism $(F, \alpha) \rightarrow (F', \alpha')$.

Conversely, say (F, α) is an H -mapping datum over X . The object $F \in \text{Ob}(H)(X_0)$ corresponds to a map $f_0 : X_0 \rightarrow \text{Ob}(H)$, the morphism $\alpha \in \text{Mor}(H)(X_1)$ corresponds to a map $f_1 : X_1 \rightarrow \text{Mor}(H)$, and the pair $(d_2^* \alpha, d_0^* \alpha)$ corresponds to a map $f_2 : X_2 \rightarrow \text{Mor}(H) \times_{s,t} \text{Mor}(H)$.

The maps f_0, f_1 , and f_2 commute with the simplicial structure maps by

the assumptions on (F, α) . As BH is 2-coskeletal, because the nerve of any category is 2-coskeletal, this suffices to define a map $f : X \rightarrow BH$.

Say $\varphi : (F, \alpha) \rightarrow (F', \alpha')$ is an isomorphism of H -mapping data over X , and let $f' : X \rightarrow BH$ be the map defined from (F', α') . There is a homotopy $h : X \rightarrow B(H^1)$ from f to f' , where $h_0 \in \text{Ob}(H^1)(X_0)$ corresponds to the isomorphism $\varphi : F \rightarrow F'$, $h_1 \in \text{Mor}(H^1)(X_1)$ corresponds to the commutative square $d_0^* \varphi \circ \alpha = \alpha' \circ d_1^* \varphi$, and $h_2 \in \text{Mor}(H^1)(X_2) \times_{s,t} \text{Mor}(H^1)(X_2)$ corresponds to the commutative squares $1^* \varphi \circ d_2^* \alpha = d_2^* \alpha' \circ 0^* \varphi$ and $2^* \varphi \circ d_0^* \alpha = d_0^* \alpha' \circ 1^* \varphi$. \square

Lemma 2.2.5. *If $g : H \rightarrow H'$ is a sectionwise equivalence of presheaves of groupoids, and X is a simplicial object in \mathcal{C} , then the induced function*

$$g_* : \text{MD}(X, H) \rightarrow \text{MD}(X, H')$$

is a bijection.

Proof. If (F, α) is an H -mapping datum over X , then $(g_{X_0}(F), g_{X_1}(\alpha))$ is an H' -mapping datum over X , and this assignment induces the function g_* .

For any object U of \mathcal{C} , the map $g_U : H(U) \rightarrow H'(U)$ is an equivalence of groupoids by assumption, so there is a map $f_U : H'(U) \rightarrow H(U)$ with f_U left adjoint to g_U . If $\phi : U \rightarrow V$ is a map in \mathcal{C} , there is a natural isomorphism $f_U \circ \phi^* \cong \phi^* \circ f_V$, given by the following diagram

$$\begin{array}{ccccc}
 H'(V) & \xrightarrow{f_V} & H(V) & \xrightarrow{\phi^*} & H(U) \\
 \searrow \eta & & \downarrow g_V & & \downarrow g_U \\
 & & H'(V) & \xrightarrow{\phi^*} & H'(U) \\
 & & & & \downarrow f_U \\
 & & & & H(U)
 \end{array}
 \quad (2.1)$$

where η and ϵ are unit and counit isomorphisms, respectively. Given composable maps $U \xrightarrow{\phi} V \xrightarrow{\psi} W$ in \mathcal{C} , the composition of the natural isomorphisms

associated to ϕ and ψ is given by the following diagram

$$\begin{array}{ccccccc}
 H'(W) & \xrightarrow{f_W} & H(W) & \xrightarrow{\psi^*} & H(V) & & \\
 \eta \nearrow & & \downarrow g_W & & \downarrow g_V & \epsilon \nearrow & \\
 & & H'(W) & \xrightarrow{\psi^*} & H'(V) & \xrightarrow{f_V} & H(V) & \xrightarrow{\phi^*} & H(U) & & \\
 & & & & \eta \nearrow & & \downarrow g_V & & \downarrow g_U & \epsilon \nearrow & \\
 & & & & & & H'(V) & \xrightarrow{\phi^*} & H'(U) & \xrightarrow{f_U} & H(U)
 \end{array}$$

which is equal to the natural isomorphism associated to the composite $\psi \circ \phi$ by the triangle identity. In other words, the maps f_U and the natural isomorphisms of 2.1 define a pseudo-natural transformation $f : H' \rightarrow H$.

Say (F, α) is an H' -mapping datum over X . Let $\bar{F} = f_{X_0}(F)$, and let $\bar{\alpha} : d_1^* \bar{F} \rightarrow d_0^* \bar{F}$ be the composite

$$d_1^* f_{X_0}(F) \cong f_{X_1} d_1^*(F) \xrightarrow{f_{X_1}(\alpha)} f_{X_1} d_0^*(F) \cong d_0^* f_{X_0}(F).$$

Then, using the pseudo-naturality of f , $(\bar{F}, \bar{\alpha})$ is an H -mapping datum over X , and this assignment induces a function $f_* : \text{MD}(X, H') \rightarrow \text{MD}(X, H)$.

The function f_* is inverse to g_* . We'll show that $g_* \circ f_*$ is the identity on $\text{MD}(X, H')$; the other direction is similar. Let (F, α) be an H' -mapping datum over X , and define $(\bar{F}, \bar{\alpha})$ as before. The unit $\eta_0 : F \rightarrow g_{X_0} f_{X_0}(F)$ defines an isomorphism $(F, \alpha) \rightarrow (g_{X_0} f_{X_0}(F), g_{X_1}(\bar{\alpha}))$: we have a commutative diagram

$$\begin{array}{ccccccc}
 d_1^* g_{X_0} f_{X_0}(F) & \longrightarrow & g_{X_1} f_{X_1} d_1^*(F) & \xrightarrow{g(f(\alpha))} & g_{X_1} f_{X_1} d_0^*(F) & \longrightarrow & d_0^* g_{X_0} f_{X_0}(F) \\
 & & \eta_1 \uparrow & & \uparrow \eta_1 & & \\
 & & d_1^*(F) & \xrightarrow{\alpha} & d_0^*(F) & & \\
 & \nwarrow d_1^* \eta_0 & & & & \nearrow d_0^* \eta_0 &
 \end{array}$$

where η_0 is the unit of the adjunction $f_{X_0} \dashv g_{X_0}$, and η_1 is the unit of $f_{X_1} \dashv g_{X_1}$. The long way around the right-hand triangle can be represented by the

following diagram

$$\begin{array}{ccccc}
 H'(X_0) & \xrightarrow{f_{X_0}} & H(X_0) & \xrightarrow{d_0^*} & H(X_1) \\
 \searrow \eta_0 & & \downarrow g_{X_0} & & \downarrow g_{X_1} \\
 & & H'(X_0) & \xrightarrow{d_0^*} & H'(X_1) \\
 & & \searrow \eta_1 & & \downarrow g_{X_1} \\
 & & & & H'(X_1)
 \end{array}$$

ϵ_1 (arrow from $H(X_1)$ to $H'(X_1)$)
 f_{X_1} (arrow from $H'(X_1)$ to $H(X_1)$)

This is equal to the short way around the right-hand triangle by the triangle identity. The left-hand triangle commutes by a similar argument. \square

Proof of Theorem 2.2.2. There is a model structure on the category of presheaves of groupoids on a small site \mathcal{C} such that a map $K \rightarrow K'$ is a weak equivalence (resp. fibration) if and only if $BK \rightarrow BK'$ is a weak equivalence (resp. fibration) with respect to the injective local model structure on $\mathbf{sPre}(\mathcal{C})$. This model structure first appeared in [Hol08]; see also [Jar15, Proposition 9.19]. For any sheaf of groups G , let $G - \mathbf{Tors}$ be the presheaf of groupoids on \mathcal{C} with $G - \mathbf{Tors}(U)$ the groupoid of $G|_U$ -torsors on \mathcal{C}/U for all objects U of \mathcal{C} . The canonical map $G \rightarrow G - \mathbf{Tors}$ is a local weak equivalence, and $G - \mathbf{Tors}$ satisfies descent, in the sense that any fibrant model $G - \mathbf{Tors} \rightarrow (G - \mathbf{Tors})^\wedge$ is a sectionwise equivalence [Jar15, Corollary 9.27].

If X is a simplicial object in \mathcal{C} , then

$$H^1(X, G) \cong \mathrm{MD}(X, G - \mathbf{Tors}),$$

by definition. As $G - \mathbf{Tors}$ satisfies descent, Lemma 2.2.5 implies that

$$\mathrm{MD}(X, G - \mathbf{Tors}) \cong \mathrm{MD}(X, (G - \mathbf{Tors})^\wedge)$$

for any fibrant model $G - \mathbf{Tors} \rightarrow (G - \mathbf{Tors})^\wedge$. By Lemma 2.2.4,

$$\mathrm{MD}(X, (G - \mathbf{Tors})^\wedge) \cong \pi(X, B(G - \mathbf{Tors})^\wedge).$$

As X is cofibrant and $B(G - \mathbf{Tors})^\wedge$ is fibrant with respect to the injective local model structure on $\mathbf{sPre}(\mathcal{C})$, we have

$$\pi(X, B(G - \mathbf{Tors})^\wedge) \cong [X, B(G - \mathbf{Tors})^\wedge] \cong [X, BG].$$

□

2.3 Cohomology and central extensions

The results of this section require significant parts of the theory of non-abelian cohomology developed by Jardine [Jar15, Chapter 9], generalizing work of Giraud [Gir71] and Breen [Bre78]. We begin by introducing the necessary parts of this theory.

2.3.1 Presheaves of groupoids enriched in simplicial sets

The approach to non-abelian cohomology described in [Jar15, Chapter 9] makes use of the Eilenberg-Mac Lane \overline{W} functor. We will not need the details of the construction, which can be found in [Jar15, Section 9.3] or [Ste12], but we will use a few facts, which we now summarize.

Write $s_0\mathbf{Gpd}$ for the category of groupoids enriched in simplicial sets; then \overline{W} is a functor $s_0\mathbf{Gpd} \rightarrow \mathbf{sSet}$. If G is a groupoid enriched in simplicial sets, [Jar15, Corollary 9.39] says that there is a natural weak equivalence $d(BG) \rightarrow \overline{W}(G)$, where d is the diagonal functor, and BG is the bisimplicial set given by viewing G as a simplicial groupoid, and applying the nerve functor B

pointwise.

The nerve also defines a functor $B : 2\text{-Gpd} \rightarrow s_0\text{Gpd}$. If H is a 2-groupoid, then BH is the groupoid enriched in simplicial sets with

$$\text{Ob}(BH) = \text{Ob}(H) \text{ and } \text{Mor}(BH) = B(\text{Mor}(H)) .$$

It is an abuse of notation, but we will write $\overline{W}(H)$ for the simplicial set given by first applying B to the 2-groupoid H , and then applying \overline{W} .

These functors extend to the presheaf level by applying them sectionwise. There is a model structure on the category of presheaves of 2-groupoids on \mathcal{C} such that a map $A \rightarrow B$ is a weak equivalence if and only if $\overline{W}A \rightarrow \overline{W}B$ is a local weak equivalence of simplicial presheaves; this is [Jar15, Theorem 9.57].

The Eilenberg-Mac Lane functor \overline{W} is part of a Quillen equivalence between simplicial presheaves and presheaves of groupoids enriched in simplicial sets [Jar15, Theorem 9.50], and it follows from this that

$$[BG, \overline{W}(H)] \cong [G, H]_{2\text{-Gpd}}$$

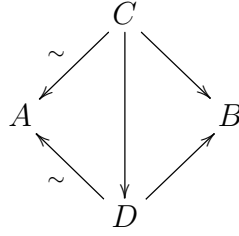
for any presheaf of groupoids G and for any presheaf of 2-groupoids H .

2.3.2 Cocycle categories

For objects A, B of a model category \mathcal{M} , write $h(A, B)$ for the category whose objects are diagrams

$$A \xleftarrow{\sim} C \longrightarrow B$$

and whose morphisms are commutative diagrams



This is the category of cocycles from A to B . The important point is that, if the model category \mathcal{M} is right proper, and if its class of weak equivalences are closed under finite products, in the sense that if $f : A \rightarrow B$ is an equivalence, then so is any map $f \times 1 : A \times C \rightarrow B \times C$, then there is a canonical bijection

$$\pi_0 h(A, B) \cong [A, B]$$

between the set of path components of $h(A, B)$ and the set of maps in $\text{Ho}(\mathcal{M})$ for any objects A, B of \mathcal{M} . This is [Jar15, Theorem 6.5].

2.3.3 Cohomology and extensions

Let A be a sheaf of abelian groups on a site \mathcal{C} . There is an associated presheaf of 2-groups $\mathbf{Iso}(A)$ defined as follows: for U an object of \mathcal{C} , the 1-morphisms of $\mathbf{Iso}(A)(U)$ are the isomorphisms

$$\alpha : A|_U \rightarrow A|_U$$

of sheaves of groups on \mathcal{C}/U , and the 2-morphisms are given by

$$\mathbf{Iso}(A)(U)(\alpha, \alpha) = A(U) \text{ for all 1-morphisms } \alpha ;$$

$$\mathbf{Iso}(A)(U)(\alpha, \beta) = \emptyset \text{ for all } \alpha \neq \beta .$$

Define a sheaf of 2-groups $A[2]$ on \mathcal{C} such that $A[2](U)$ has only the identity 1-morphism, and has $A(U)$ for 2-morphisms.

Let $f : A[2] \rightarrow \mathbf{Iso}(A)$ be the map that includes $A[2]$ as a subobject, and define a retraction $g : \mathbf{Iso}(A) \rightarrow A[2]$ by

$$g(h : \alpha \rightarrow \alpha) = h : * \rightarrow * ,$$

for any 2-morphism h of $\mathbf{Iso}(A)(U)$.

Because $A[2]$ is a sheaf of 2-groups, we can identify $B(A[2])$ with the simplicial sheaf of morphisms from the unique object to itself; this simplicial sheaf is just BA . By [Jar15, Corollary 9.39], we have

$$\overline{WA}[2] \simeq d(BBA) \simeq K(A, 2) .$$

So, for any presheaf of groupoids G on \mathcal{C} , there are isomorphisms

$$\begin{aligned} H^2(BG, A) &= [BG, K(A, 2)] \\ &\cong [BG, \overline{WA}[2]] \\ &\cong [G, A[2]]_{2\text{-Gpd}} \\ &\cong \pi_0 h(G, A[2]) . \end{aligned}$$

As the map $f : A[2] \rightarrow \mathbf{Iso}(A)$ is a section, the induced map

$$f_* : \pi_0 h(G, A[2]) \rightarrow \pi_0 h(G, \mathbf{Iso}(A))$$

is a monomorphism.

In [Jar15, Theorem 9.66], Jardine constructs a bijection

$$\pi_0 h(G, \mathbf{Iso}(A)) \cong \pi_0 \mathbf{Ext}(G, A)$$

with the set of path components of a category $\mathbf{Ext}(G, A)$, which we'll now define.

If $p : G' \rightarrow G$ is a map of presheaves of groupoids, let $\mathrm{im}(p)$ be the presheaf of groupoids that has the same objects as G' , and with $\mathrm{im}(p)(x, y)$ given by the image of the function $G'(x, y) \rightarrow G(p(x), p(y))$. Say that a map $p : G' \rightarrow G$ is essentially surjective if the canonical map $\mathrm{im}(p) \rightarrow G$ is a local weak equivalence of presheaves of groupoids, in the sense that the induced map of nerves is a local weak equivalence of simplicial presheaves.

Say that a kernel of a map $p : G' \rightarrow G$ is a diagram

$$\begin{array}{ccc} K & \xrightarrow{j} & \mathrm{Aut}(G') \\ & \searrow & \swarrow \\ & \mathrm{Ob}(G') & \end{array}$$

where $\mathrm{Aut}(G')$ is the group object in the category of presheaves over the presheaf $\mathrm{Ob}(G)$ with $\mathrm{Aut}(G')(U) = \{\alpha : x \rightarrow x \in G'(U) \mid x \in \mathrm{Ob} G'(U)\}$, K is a group object in the category of presheaves over $\mathrm{Ob}(G')$, and j is a homomorphism of group objects over $\mathrm{Ob}(G')$, such that the following diagram is a pullback:

$$\begin{array}{ccc} K & \xrightarrow{j} & \mathrm{Aut}(G') \\ \downarrow & & \downarrow p_* \\ \mathrm{Ob}(G') & \xrightarrow{p_* \cdot e} & \mathrm{Aut}(G) \end{array}$$

Say that a kernel for p in A is a kernel $j : K \rightarrow \mathrm{Aut}(G')$ together with a map of presheaves $w : K \rightarrow A$ such that the induced map $K \rightarrow A \times \mathrm{Ob}(G')$ is a

homomorphism of group objects over $\text{Ob}(G')$ that induces an isomorphism of associated sheaves.

The objects of $\mathbf{Ext}(G, A)$ are triples (p, j, w) , where $p : G' \rightarrow G$ is essentially surjective, and (j, w) is a choice of kernel in A for p .

A morphism $\sigma : (p, j, w) \rightarrow (p', j', w')$ of this category is a local weak equivalence $\sigma : G' \rightarrow G''$ such that $p' \circ \sigma = p$, and $w' \circ \sigma_* = w$.

If G is a presheaf of groups, and $p : G' \rightarrow G$ is an essentially surjective map of presheaves of groupoids, then G' must be locally connected, i.e., a gerbe.

The map $K \rightarrow \text{Ob}(G')$ is an example of what Jardine calls a family of presheaves of groups $\mathcal{F} \rightarrow S$ over the presheaf S , which is a group object in the category of presheaves over S . For an element $x \in S(U)$, the fibre \mathcal{F}_x of the family \mathcal{F} over x is defined by the pullback diagram

$$\begin{array}{ccc} \mathcal{F}_x & \longrightarrow & \mathcal{F}|_U \\ \downarrow & & \downarrow \\ * & \xrightarrow{x} & S|_U \end{array}$$

If $(p : G' \rightarrow G, j, w)$ is an object of $\mathbf{Ext}(G, A)$, and $\alpha : x \rightarrow y$ is a morphism of $G'(U)$, then α defines an isomorphism $K_x \rightarrow K_y$ of presheaves of groups on \mathcal{C}/U by conjugation; via w , we get an induced automorphism of $A|_U$.

Let $\text{CenExt}(G, A) \subset \pi_0 \mathbf{Ext}(G, A)$ be the subset consisting of equivalence classes that have a representative $(p : G' \rightarrow G, j, w)$ with the following property: for all objects U of \mathcal{C} , and for all morphisms α of $G'(U)$, the automorphism of $A|_U$ given by conjugation with α is the identity.

If G happens to be a sheaf of groups, and $(p : G' \rightarrow G, j, w)$ is an object of $\mathbf{Ext}(G, A)$ such that G' is a sheaf of groups, then the kernel of p is necessarily isomorphic to A ; the extension (p, j, w) has the property of the last paragraph if and only if the kernel of p_U is contained in the centre of $G'(U)$ for all objects

U of \mathcal{C} .

Theorem 2.3.1. *For any presheaf of groupoids G on \mathcal{C} , and any sheaf of abelian groups A , there is a bijection*

$$H^2(BG, A) \cong \text{CenExt}(G, A).$$

Proof. We will show that $\text{CenExt}(G, A)$ is the image of the function

$$H^2(BG, A) \xrightarrow{\cong} \pi_0 h(G, A[2]) \xrightarrow{f_*} \pi_0 h(G, \mathbf{Iso}(A)) \xrightarrow{\cong} \pi_0 \mathbf{Ext}(G, A).$$

First, we'll recall the definition of the bijection

$$\psi : \pi_0 h(G, \mathbf{Iso}(A)) \rightarrow \pi_0 \mathbf{Ext}(G, A)$$

of [Jar15, Theorem 9.66]. Say

$$G \xleftarrow{g} Z \xrightarrow{F} \mathbf{Iso}(A)$$

is a cocycle from G to $\mathbf{Iso}(A)$. Define a presheaf of 2-groupoids $\mathbf{E}_Z F$, whose objects are the objects of Z . The 1-morphisms $x \rightarrow y$ of $\mathbf{E}_Z F(U)$ are pairs (α, f) with $\alpha : x \rightarrow y$ a 1-morphism of $Z(U)$ and $f \in A(U)$. A 2-morphism $(\alpha, f) \rightarrow (\beta, g)$ of $\mathbf{E}_Z F(U)$ is a 2-morphism $h : \alpha \rightarrow \beta$ of $Z(U)$ such that $F(h) = g^{-1}f$.

If $\alpha : x \rightarrow y$ is a 1-morphism of $Z(U)$, then $F(\alpha)$ is a 1-morphism of $\mathbf{Iso}(A)(U)$, ie an automorphism of $A|_U$; write α_* for this automorphism. The composite of $(\alpha, f) : x \rightarrow y$ and $(\beta, g) : y \rightarrow z$ in $\mathbf{E}_Z F(U)$ is $(\beta\alpha, g\beta_*(f)) : x \rightarrow z$.

Let $E_Z F = \pi_0(\mathbf{E}_Z F)$ be the presheaf of path component groupoids.

There is a map $\pi : \mathbf{E}_Z F \rightarrow Z$ which is the identity on objects, on 1-morphisms is $(\alpha, f) \mapsto \alpha$, and takes the 2-morphism $h : (\alpha, f) \rightarrow (\beta, g)$ to the underlying 2-morphism $h : \alpha \rightarrow \beta$. Write g_* for the composite

$$E_Z F \xrightarrow{\pi_*} \pi_0(Z) \xrightarrow{\sim} G.$$

Let $K(F) = A \times \text{Ob}(E_Z F)$; define $j : K(F) \rightarrow \text{Aut}(E_Z F)$ by letting $j_x : A|_U \rightarrow E_Z F_x$ be defined in sections by the rule $f \mapsto [(1_x, f)]$, for all $x \in \text{Ob}(E_Z F)(U)$.

Let $w : K(F) \rightarrow A$ be projection; then (j, w) is a kernel for g_* in A .

The rule $(g, F) \mapsto (g_*, j, w)$ defines a functor $h(G, \mathbf{Iso}(A)) \rightarrow \mathbf{Ext}(G, A)$, and ψ is the induced function on path components.

An element of $\pi_0 h(G, \mathbf{Iso}(A))$ is in the image of f_* if and only if it has a representative $(g : Z \rightarrow G, F : Z \rightarrow \mathbf{Iso}(A))$ such that F factors as $Z \rightarrow A[2] \subset \mathbf{Iso}(A)$. This is equivalent to the condition that, for all 1-morphisms α of $Z(U)$, $F(\alpha) = \alpha_*$ is the identity on $A|_U$. Say (g, F) is such a cocycle. We will show that for any 1-morphism σ of $E_Z F(U)$, the automorphism of $A|_U$ given by conjugation with σ is the identity.

Let $\sigma = [(\alpha, f)] : x \rightarrow y$ be a 1-morphism of $E_Z F(U)$. Let $V \rightarrow U$ be an object of \mathcal{C}/U , and let $x \mapsto x'$ and $y \mapsto y'$ under $Z(U) \rightarrow Z(V)$. For any $g \in A(V)$,

$$\begin{aligned} [(\alpha, f)][(1_{x'}, g)][(\alpha, f)]^{-1} &= [(\alpha, f)(1_{x'}, g)(\alpha^{-1}, f^{-1})] \\ &= [(\alpha, f)(\alpha^{-1}, gf^{-1})] \\ &= [(1_{y'}, f\alpha_*(gf^{-1}))] \\ &= [(1_{y'}, g)] \end{aligned}$$

So $g \mapsto g$ and σ induces the identity on $A|_U$.

We've proved that the image of our function $H^2(BG, A) \rightarrow \pi_0 \mathbf{Ext}(G, A)$ is contained in $\text{CenExt}(G, A)$.

To see the opposite inclusion, let's briefly recall the bijection

$$\phi : \pi_0 \mathbf{Ext}(G, A) \rightarrow \pi_0 h(G, \mathbf{Iso}(A)) ,$$

inverse to ψ . If $(p : G' \rightarrow G, j, w)$ is an object of $\mathbf{Ext}(G, A)$, there is a cocycle

$$G \xleftarrow{q} R(p) \xrightarrow{F(p)} \mathbf{Iso}(A).$$

The presheaf of 2-groupoids $R(p)$ has the same objects and 1-morphisms as G' , and there is a 2-morphism $\alpha \rightarrow \beta$ if and only if $p(\alpha) = p(\beta)$. For a morphism α of $G'(U)$, $F(p)(\alpha)$ is the automorphism of $A|_U$ defined by conjugation with α .

We have $\phi[(p, j, w)] = [(q, F(p))]$. Clearly, then, if $[(p, j, w)]$ is an element of $\text{CenExt}(G, A)$, then $\phi[(p, j, w)]$ is represented by a cocycle in the image of f_* , namely $(q, F(p))$. This completes the proof. \square

The elements of $H^2(BG, A)$ corresponding to central extensions of sheaves of groups can be interpreted as universal obstruction classes, in the following sense:

Theorem 2.3.2. *Let $A \rightarrow G' \xrightarrow{p} G$ be a central extension, where A, G' and G are sheaves of groups. For any simplicial presheaf X , there is an exact sequence of pointed sets*

$$H^1(X, G') \xrightarrow{p^*} H^1(X, G) \xrightarrow{\pi} H^2(X, A),$$

and if $F \in H^1(X, G)$, then $\pi(F) = F^*(c_p)$, where $c_p \in H^2(BG, A)$ classifies the extension $A \rightarrow G' \rightarrow G$.

Proof. We write $H^1(X, G) = [X, BG]$. If X is a simplicial object in the underlying site, this agrees with our previous definition by Theorem 2.2.2.

Because A is central in G' , there is an induced action $BA \times BG' \rightarrow BG'$ of the simplicial sheaf of abelian groups BA on the simplicial sheaf BG' . The Borel construction for this action is the bisimplicial sheaf $EBA \times_{BA} BG'$ with (p, q) -bisplices given by the q^{th} simplicial degree of the Borel construction for the action $A^{\times p} \times G'^{\times p} \rightarrow G'^{\times p}$.

The action of $A^{\times p}$ on $G'^{\times p}$ is free for all $p \geq 0$, so the map

$$EA^{\times p} \times_{A^{\times p}} G'^{\times p} \rightarrow G'^{\times p}$$

is a local weak equivalence. These maps induce a local weak equivalence from the diagonal

$$d(EBA \times_{BA} BG') \rightarrow BG .$$

Moreover, there is a sequence of bisimplicial sheaves

$$BG' \rightarrow EBA \times_{BA} BG' \rightarrow BBA,$$

and, taking diagonals, the sequence

$$BG' \rightarrow d(EBA \times_{BA} BG') \rightarrow d(BBA)$$

is a sectionwise fibre sequence, hence a local fibre sequence. In general, if $A \times X \rightarrow X$ is an action of a connected simplicial abelian group A on a connected simplicial set X , then the sequences

$$X \rightarrow A^{\times p} \times X \rightarrow A^{\times p}$$

are fibre sequences of connected simplicial sets, and so the sequence

$$X \rightarrow EA \times_A X \rightarrow BA$$

of bisimplicial sets induces a fibre sequence of simplicial sets after taking diagonals, by a theorem of Bousfield and Friedlander, which appears as [GJ09, Theorem IV.4.9]; the π_* -Kan condition in the hypotheses of the Bousfield-Friedlander theorem are satisfied by [GJ09, Lemma IV.4.2(1)].

As $d(BBA) \simeq K(A, 2)$, we have the exact sequence

$$H^1(X, G') \xrightarrow{P_*} H^1(X, G) \xrightarrow{\pi} H^2(X, A).$$

This is exactly the argument of [Jar15, Example 9.11] in our case.

Say $F \in H^1(X, G) = [X, BG]$. Then $\pi(F)$ is given by the diagram in the homotopy category

$$X \xrightarrow{F} BG \xleftarrow{\sim} d(EBA \times_{BA} BG') \longrightarrow d(BBA).$$

To finish the proof, we need to show that the cocycle

$$BG \leftarrow d(EBA \times_{BA} BG') \rightarrow d(BBA) \tag{*}$$

represents $c_p \in H^2(BG, A)$.

First, note that

$$EBA \times_{BA} BG' \cong BBR(p),$$

where $R(p)$ is the resolution 2-groupoid that appeared in the proof of 2.3.1. Using the natural weak equivalence $d(BH) \rightarrow \overline{W}(H)$, for H a presheaf of groupoids enriched in simplicial sets, we have a pointwise equivalence from (*)

to the cocycle

$$\overline{W}G \leftarrow \overline{W}(R(p)) \rightarrow \overline{W}(A[2]). \quad (**)$$

The functor \overline{W} has a left adjoint

$$\pi G : \mathbf{sPre} \rightarrow \mathbf{Pre}(2 - \mathbf{Gpd}),$$

where G is the loop groupoid functor to presheaves of groupoids enriched in simplicial sets, and π is the fundamental groupoid functor to presheaves of 2-groupoids.

If H is a groupoid enriched in simplicial sets, then $\pi(H)$ is the 2-groupoid with

$$\mathrm{Ob}(\pi(H)) = \mathrm{Ob}(H) \text{ and } \mathrm{Mor}(\pi(H)) = \pi(\mathrm{Mor}(H)).$$

This adjunction defines a Quillen equivalence between presheaves of 2-groupoids and the 2-equivalence model structure on simplicial presheaves of [Jar15, Theorem 5.49]. This follows from [Jar15, Proposition 9.59] and [Jar15, Proposition 9.61]. The 2-equivalence model structure has all monomorphisms for cofibrations, so that every object is cofibrant.

Now, for M a presheaf of 2-groupoids and \tilde{M} a fibrant model of M , we have natural weak equivalences

$$\pi G \overline{W} M \xrightarrow{\sim} \pi G \overline{W} \tilde{M} \xrightarrow{\sim} \tilde{M},$$

as the functor \overline{W} preserves weak equivalences. It follows from this that the cocycle obtained by applying πG to $(**)$ is pointwise equivalent to the cocycle

$$G \leftarrow R(p) \rightarrow A[2],$$

which represents the class in $\pi_0 h(G, A[2])$ corresponding to the extension

$$A \rightarrow G' \rightarrow G.$$

□

2.4 Examples

We are now in a position to give some examples of torsors over simplicial objects.

Example 2.4.1. Let G be a group object in a site \mathcal{C} ; then G represents a presheaf of groups on \mathcal{C} , and assume that this presheaf is a sheaf. Assume further that \mathcal{C} has all finite products, so that BG is a simplicial object in \mathcal{C} . Define a G -torsor over BG as follows. Over $(BG)_0 = *$ we take the trivial G -torsor, which is the sheaf G , with G -action given by left multiplication. The face maps $d_0, d_1 : G \rightarrow *$ are necessarily the same, and the pullback of G along d_i is the projection map $\pi_2 : G \times G \rightarrow G$, and G acts on $G \times G$ by left multiplication on the first factor. Let $\alpha : G \times G \rightarrow G \times G$ be the isomorphism of G -torsors over G given by $\alpha(h, g) = (hg, g)$; then (G, α) is a G -torsor over BG .

If we take $G = \mathbf{GL}_n$, this construction corresponds to the “universal vector bundle” over $B\mathbf{GL}_n$ of [Pus04], which is the Borel construction $E\mathbf{GL}_n \times_{\mathbf{GL}_n} \mathbb{A}^n$.

Example 2.4.2. Let S be a scheme, and let $\{U_i \rightarrow S\}_{i=1}^m$ be an étale cover. There is a simplicial scheme U with

$$U_n = \bigsqcup_{(i_0, \dots, i_n)} U_{i_0} \times_S \cdots \times_S U_{i_n}$$

and a canonical map of simplicial schemes $f : U \rightarrow S$, which is an étale-local equivalence; see [Jar15, Example 4.17]. By Theorem 2.2.2, the induced map

$$f^* : H_{\text{et}}^1(S, G) \rightarrow H_{\text{et}}^1(U, G)$$

is an isomorphism, for any étale sheaf of groups G .

Further examples are given by the classifying spaces of algebraic groups. In the following, an inner automorphism of a sheaf of groups is an automorphism given by conjugation with a global section.

Proposition 2.4.3. *Let k be a field, G a k -group, and H a Nisnevich sheaf of groups on Sm_k . Then,*

$$H_{\text{Nis}}^1(BG, H) \cong \text{hom}(G, H) / \text{inner automorphisms of } H.$$

Proof. We have

$$H_{\text{Nis}}^1(BG, H) \cong [BG, BH] \cong \pi(BG, B(H - \mathbf{Tors}));$$

see the proof of Theorem 2.2.2.

We'll begin by defining a function

$$\alpha : \pi(BG, B(H - \mathbf{Tors})) \rightarrow \text{hom}(G, H) / \text{inner automorphisms of } H.$$

Given a class in $\pi(BG, B(H - \mathbf{Tors}))$, choose a representative $\phi : BG \rightarrow B(H - \mathbf{Tors})$. Write T for the H -torsor over $\text{Spec } k$ corresponding to $\phi_0 : \text{Spec } k \rightarrow \text{Ob}(H - \mathbf{Tors})$. Any Nisnevich torsor over the spectrum of a field is isomorphic to the trivial torsor, so choose an isomorphism $\tau_k : T \rightarrow H$, where H is the trivial H -torsor over $\text{Spec } k$. For an object U of Sm_k , write T_U, H_U

for the restriction of T, H to U ; the choice of τ_k determines an isomorphism of H -torsors over U , $\tau_U : T_U \rightarrow H_U$, for every object U of Sm_k . Furthermore, the choice of τ_k allows us to define a map $\psi : BG \rightarrow B(H - \mathbf{Tors})$, where $\psi_0 : \text{Spec } k \rightarrow \text{Ob}(H - \mathbf{Tors})$ corresponds to the trivial H -torsor H , and ψ_1 is defined by the rule $\psi_1(g) = \tau_U \phi_1(g) \tau_U^{-1}$ for every $g \in G(U)$. By construction, $\tau : \phi \Rightarrow \psi$ defines a simplicial homotopy. In simplicial degree 1, we have $\psi_1 : G \rightarrow \text{Aut}(H) = H$; define

$$\alpha([\phi]) = [\psi_1].$$

To see this is well-defined, let $\phi' : BG \rightarrow B(H - \mathbf{Tors})$ be a map with $\sigma : \phi \Rightarrow \phi'$ a simplicial homotopy. Write T' for the H -torsor over $\text{Spec } k$ corresponding to ϕ'_0 , and choose an isomorphism $\tau'_k : T' \rightarrow H$, which determines a simplicial homotopy $\tau' : \phi' \Rightarrow \psi'$. Let $\mu : \psi \Rightarrow \psi'$ be the composition $\tau' \circ \sigma \circ \tau^{-1}$. The homotopy μ gives an isomorphism $H \rightarrow H$ of H -torsors over $\text{Spec } k$, i.e., an element $h \in H(k)$, and we have $\psi'_1 = h\psi_1 h^{-1}$, so that α is well-defined.

The function α is a bijection, as it has inverse

$$\beta : \text{hom}(G, H) / \text{inner automorphisms of } H \rightarrow \pi(BG, B(H - \mathbf{Tors})),$$

defined as follows: given a class in the domain, choose a representative $f : G \rightarrow H$, and let

$$\beta([f]) = [BG \xrightarrow{B(f)} BH \rightarrow B(H - \mathbf{Tors})].$$

If $h \in H(k)$ and $f' = hfh^{-1}$, then h defines a simplicial homotopy $B(f) \Rightarrow B(f')$, so β is well-defined. \square

Example 2.4.4. The motivating example for Theorem 2.3.2 is the Brauer

group. On the étale site $(Sch_S)_{\text{ét}}$ over a scheme S , the central extension of sheaves of groups

$$\mathbb{G}_m \rightarrow \mathbf{GL}_n \xrightarrow{p} \mathbf{PGL}_n$$

corresponds to a class $c_p \in H_{\text{ét}}^2(B\mathbf{PGL}_n, \mathbb{G}_m)$, and a \mathbf{PGL}_n -torsor F over S lifts to a \mathbf{GL}_n -torsor if and only if the class $F^*(c_p)$ vanishes in $H_{\text{ét}}^2(S, \mathbb{G}_m)$.

Example 2.4.5. If S is a smooth scheme over a field, then the Nisnevich cohomology group $H_{\text{Nis}}^2(S, \mathbb{G}_m)$ is zero, and so all Nisnevich \mathbf{PGL}_n -torsors over S lift to \mathbf{GL}_n -torsors. The situation is more interesting over a simplicial scheme.

By Proposition 2.4.3, for any k -group G , a Nisnevich \mathbf{PGL}_n -torsor over BG is given by a homomorphism $f : G \rightarrow \mathbf{PGL}_n$, and f lifts to a \mathbf{GL}_n -torsor over BG if and only if f factors through the canonical map $p : \mathbf{GL}_n \rightarrow \mathbf{PGL}_n$ up to simplicial homotopy, and hence on the nose.

By Theorem 2.3.1 and Theorem 2.3.2, the central extension $\mathbb{G}_m \rightarrow \mathbf{GL}_n \rightarrow \mathbf{PGL}_n$ corresponds to an element c_p of the group $H_{\text{Nis}}^2(B\mathbf{PGL}_n, \mathbb{G}_m)$, and a homomorphism $f : G \rightarrow \mathbf{PGL}_n$ factors through \mathbf{GL}_n if and only if $f^*(c_p)$ vanishes in $H_{\text{Nis}}^2(BG, \mathbb{G}_m)$. This is a statement about motivic cohomology, because of the isomorphism

$$H_{\text{Nis}}^2(X, \mathbb{G}_m) \cong H^3(X, \mathbb{Z}(1)),$$

for any simplicial presheaf X .

Chapter 3

The motivic cohomology of $B\mathbf{G}\mathbf{l}_n$

This chapter presents calculations of the motivic cohomology of $B\mathbf{G}\mathbf{l}_n$ and $B\mathbf{S}\mathbf{l}_n$. These results are well-known; a calculation of the motivic cohomology of $B\mathbf{G}\mathbf{l}_n$ appears in a paper of Pushin [Pus04] (in which the author studies the relationship between higher Chern classes and reduced power operations in motivic cohomology), and the result about $\mathbf{S}\mathbf{l}_n$ follows easily from this. The argument has a long history, going back to the work of Grothendieck [Gro58] and Gillet [Gil81]. The proofs in this chapter take advantage of the motivic model structure on simplicial presheaves with transfers, as well as the approach to torsors over simplicial schemes developed in the previous chapter. While the fundamental ideas are the same as those underlying Pushin's argument, these ideas appear in a different form in this setting. The key step in the calculation of the motivic cohomology of $B\mathbf{G}\mathbf{l}_n$ is a projective bundle theorem, which appears here as an identification of the motivic homotopy type of

$$\mathbb{Z}_{tr}(E\mathbf{G}\mathbf{l}_n \times_{\mathbf{G}\mathbf{l}_n} \mathbb{P}^{n-1}),$$

the free simplicial presheaf with transfers on the Borel construction for the action of \mathbf{GL}_n on \mathbb{P}^{n-1} , with $\mathbb{Z}_{tr}(B\mathbf{GL}_n \times \mathbb{P}^{n-1})$.

3.1 The general linear group

Let k be a perfect field. In this section, we will prove the following

Theorem 3.1.1. *The motivic cohomology of $B\mathbf{GL}_n$ is a polynomial algebra over the cohomology of the base field*

$$H^*(B\mathbf{GL}_n, \mathbb{Z}(*)) \cong H^*(k, \mathbb{Z}(*))[c_1, \dots, c_n]$$

with $c_i \in H^{2i}(B\mathbf{GL}_n, \mathbb{Z}(i))$.

The proof is by induction on n ; we'll begin by using the identification of motivic homotopy types $\mathbb{P}^\infty \simeq B\mathbb{G}_m$ to prove the base case.

If X is a scheme, we have $H^2(X, \mathbb{Z}(1)) \cong \text{Pic}(X)$. Via this isomorphism, the line bundle $\mathcal{O}(1)$ defines an element $c \in H^2(\mathbb{P}^n, \mathbb{Z}(1))$.

Proposition 3.1.2. *The motivic cohomology of projective space is given by*

$$H^*(\mathbb{P}^n, \mathbb{Z}(*)) \cong H^*(k, \mathbb{Z}(*))[c] / (c^{n+1}).$$

For this, see e.g. [MVW06, Corollary 15.5], which says that there is an identification of motivic homotopy types $\mathbb{Z}_{tr}(\mathbb{P}^n) \simeq \bigoplus_{i=0}^n \mathbb{Z}(i)$.

Proposition 3.1.3. *The motivic cohomology of infinite-dimensional projective space is given by*

$$H^*(\mathbb{P}^\infty, \mathbb{Z}(*)) \cong H^*(k, \mathbb{Z}(*))[c].$$

Proof. Let $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, and take a Nisnevich fibrant replacement

$\Omega^{2q-p}(\gamma_*\mathbb{Z}(q)) \rightarrow F(p, q)$. Then, for any simplicial presheaf X on Sm_k ,

$$H^p(X, \mathbb{Z}(q)) \cong \pi_0 \mathbf{hom}(\mathbb{Z}(X), F(p, q)) ,$$

where \mathbf{hom} denotes the mapping space in $\mathbf{sPre}_{\mathbb{Z}}(Sm_k)$. There is a tower of fibrations

$$\mathbf{hom}(\mathbb{Z}(\mathbb{P}^0), F(p, q)) \leftarrow \mathbf{hom}(\mathbb{Z}(\mathbb{P}^1), F(p, q)) \leftarrow \dots$$

Write $P_n = \mathbf{hom}(\mathbb{Z}(\mathbb{P}^n), F(p, q))$, and consider the Milnor exact sequence (see, e.g., [GJ09, Proposition VI.2.15])

$$* \rightarrow \lim^1 \pi_1 P_n \rightarrow \pi_0(\varprojlim_n P_n) \rightarrow \varprojlim_n \pi_0 P_n \rightarrow * .$$

Note that

$$\begin{aligned} \pi_1 \mathbf{hom}(\mathbb{Z}(\mathbb{P}^n), F(p, q)) &\cong \pi_0 \mathbf{hom}(\mathbb{Z}_{\bullet}(S^1), \mathbf{hom}(\mathbb{Z}(\mathbb{P}^n), F(p, q))) \\ &\cong \pi_0 \mathbf{hom}(\mathbb{Z}_{\bullet}(S^1) \otimes \mathbb{Z}(\mathbb{P}^n), F(p, q)) \\ &\cong \pi_0 \mathbf{hom}(\mathbb{Z}(\mathbb{P}^n), \Omega F(p, q)) , \end{aligned}$$

and therefore

$$\pi_1 \mathbf{hom}(\mathbb{Z}(\mathbb{P}^n), F(p, q)) \cong H^{p-1}(\mathbb{P}^n, \mathbb{Z}(q)) .$$

The tower of abelian groups

$$H^{p-1}(\mathbb{P}^0, \mathbb{Z}(q)) \leftarrow H^{p-1}(\mathbb{P}^1, \mathbb{Z}(q)) \leftarrow \dots$$

satisfies the Mittag-Leffler condition, so $\lim^1 \pi_1 P_n = 0$. As

$$\varprojlim_n \mathbf{hom}(\mathbb{Z}(\mathbb{P}^n), F(p, q)) \cong \mathbf{hom}(\mathbb{Z}(\mathbb{P}^\infty), F(p, q))$$

the \lim^1 exact sequence implies that

$$H^p(\mathbb{P}^\infty, \mathbb{Z}(q)) \cong \varprojlim_n H^p(\mathbb{P}^n, \mathbb{Z}(q)).$$

□

Let's recall the Borel construction. If G is a sheaf of groups on a site \mathcal{C} , and F is a sheaf with G -action, the Borel construction $EG \times_G F$ is the nerve of a sheaf of groupoids, which I'll call $E_G F$: for an object U of \mathcal{C} , $E_G F(U)$ is the groupoid with $\text{Ob } E_G F(U) = F(U)$, and $\text{Mor } E_G F(U)(x, y) = \{g \in G(U) \mid gx = y\}$. Every sheaf of groups G acts on itself by the group operation; in this case, we'll write $EG = EG \times_G G$.

Now we can finish the case $n = 1$ of Theorem 3.1.1.

Proposition 3.1.4. *There is a zigzag of motivic weak equivalences, giving $\mathbb{P}^\infty \simeq B\mathbb{G}_m$.*

Proof. The inclusion $i : \mathbb{A}^n - \{0\} \subset \mathbb{A}^{n+1} - \{0\}$ defined by

$$(s_1, \dots, s_n) \mapsto (s_1, \dots, s_n, 0)$$

contracts onto the north pole $(0, \dots, 0, 1)$ by an algebraic homotopy

$$(\mathbb{A}^n - \{0\}) \times \mathbb{A}^1 \rightarrow \mathbb{A}^{n+1} - \{0\}$$

which is defined by

$$((s_1, \dots, s_n), t) \mapsto (ts_1, \dots, ts_n, 1 - t).$$

Let $F(X)$ denote the fibrant model of a simplicial presheaf X with respect to the motivic model structure on $\mathbf{sPre}(Sm_k)$. As all objects are cofibrant for this model structure, the above algebraic homotopy gives a homotopy

$$F(\mathbb{A}^n - \{0\}) \times \Delta^1 \rightarrow F(\mathbb{A}^{n+1} - \{0\})$$

from the map $i_* : F(\mathbb{A}^n - \{0\}) \rightarrow F(\mathbb{A}^{n+1} - \{0\})$ to the constant map onto the image of the north pole $(0, \dots, 0, 1)$ in $F(\mathbb{A}^{n+1} - \{0\})$. So, the colimit $\varinjlim_n F(\mathbb{A}^n - \{0\})$ is sectionwise contractible, and, by motivic descent, the colimit $\mathbb{A}^\infty - \{0\} = \varinjlim_n (\mathbb{A}^n - \{0\})$ is motivically contractible.

Now, \mathbb{G}_m acts on $\mathbb{A}^n - \{0\}$ without fixed points, so the map

$$E\mathbb{G}_m \times_{\mathbb{G}_m} (\mathbb{A}^n - \{0\}) \rightarrow (\mathbb{A}^n - \{0\})/\mathbb{G}_m$$

is a local weak equivalence. The map from the Nisnevich sheaf quotient $(\mathbb{A}^n - \{0\})/\mathbb{G}_m$ to \mathbb{P}^{n-1} induced by the canonical map $\mathbb{A}^n \rightarrow \mathbb{P}^{n-1}$ is easily seen to be an isomorphism, as it induces an isomorphism at every Hensel local ring. So, the induced map on the colimits

$$E\mathbb{G}_m \times_{\mathbb{G}_m} (\mathbb{A}^\infty - \{0\}) \rightarrow \mathbb{P}^\infty$$

is a local weak equivalence.

The canonical map $\pi : E\mathbb{G}_m \times_{\mathbb{G}_m} (\mathbb{A}^\infty - \{0\}) \rightarrow B\mathbb{G}_m$ is the projection $\mathbb{G}_m^{\times n} \times (\mathbb{A}^\infty - \{0\}) \rightarrow \mathbb{G}_m^{\times n}$ in simplicial degree n , and π is therefore a motivic

weak equivalence by contractibility of $\mathbb{A}^\infty - \{0\}$ and Lemma 1.1.5. \square

Fix $n \geq 2$, and let $\mathbb{P}E$ denote the Borel construction for the action of \mathbf{GL}_n on \mathbb{P}^{n-1} . So, $\mathbb{P}E$ is a simplicial scheme, with

$$\mathbb{P}E_k = \mathbf{GL}_n^{\times k} \times \mathbb{P}^{n-1},$$

and with $d_0 : \mathbf{GL}_n \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$ the action, and $d_1 : \mathbf{GL}_n \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$ projection.

There is a canonical map $\pi : \mathbb{P}E \rightarrow B\mathbf{GL}_n$, which is projection in each degree.

Let $\{U_1, \dots, U_n\}$ be the usual open cover of \mathbb{P}^{n-1} , for which U_i is the subset where $x_i \neq 0$. This is a trivializing cover for the line bundle $\mathcal{O}(1)$, which has transition functions $x_i/x_j \in \mathbb{G}_m(U_i \cap U_j)$. The cover $\{U_i\}$ defines a Čech hypercover $U_\bullet \rightarrow \mathbb{P}^{n-1}$, which pulls back to hypercovers $d_0^*U_\bullet \rightarrow \mathbf{GL}_n \times \mathbb{P}^{n-1}$ and $d_1^*U_\bullet \rightarrow \mathbf{GL}_n \times \mathbb{P}^{n-1}$. The pullback

$$\begin{array}{ccc} V_\bullet & \xrightarrow{p_0} & d_0^*U_\bullet \\ p_1 \downarrow & & \downarrow \\ d_1^*U_\bullet & \longrightarrow & \mathbf{GL}_n \times \mathbb{P}^{n-1} \end{array}$$

defines a hypercover $V_\bullet \rightarrow \mathbf{GL}_n \times \mathbb{P}^{n-1}$, which corresponds to the open cover $\{d_1^{-1}(U_i) \cap d_0^{-1}(U_j)\}_{i,j=1}^n$ of $\mathbf{GL}_n \times \mathbb{P}^{n-1}$. This is a trivializing cover for $d_0^*\mathcal{O}(1)$ and $d_1^*\mathcal{O}(1)$, and the elements

$$p_1^*d_1^*(x_i)/p_0^*d_0^*(x_j) \in \mathbb{G}_m(d_1^{-1}(U_i) \cap d_0^{-1}(U_j))$$

define an isomorphism $\alpha : d_1^*\mathcal{O}(1) \rightarrow d_0^*\mathcal{O}(1)$. The pair $(\mathcal{O}(1), \alpha)$ defines a \mathbb{G}_m -torsor over $\mathbb{P}E$.

By Theorem 2.2.2, $(\mathcal{O}(1), \alpha)$ corresponds to an element $\xi \in [\mathbb{P}E, B\mathbf{G}_m]$. We have

$$[\mathbb{P}E, B\mathbf{G}_m] \cong [\mathbb{Z}_{tr}(\mathbb{P}E), \mathbb{Z}(1)]_{\mathbb{A}^1};$$

using the cup product on motivic cohomology, we get classes

$$\xi^j \in [\mathbb{Z}_{tr}(\mathbb{P}E), \mathbb{Z}(j)]_{\mathbb{A}^1} \quad \text{for } 0 \leq j \leq n-1.$$

Let I^j be a motivic fibrant and cofibrant model of $\mathbb{Z}(j)$ in $\mathbf{sPST}(k)$, and let $f^j : \mathbb{Z}_{tr}(\mathbb{P}E) \rightarrow I^j$ represent ξ^j .

We'll use ψ to denote the composition

$$\mathbb{Z}_{tr}(\mathbb{P}E) \longrightarrow \mathbb{Z}_{tr}(\mathbb{P}E) \otimes_{tr} \mathbb{Z}_{tr}(\mathbb{P}E) \xrightarrow{\pi \otimes (\oplus f^j)} \mathbb{Z}_{tr}(B\mathbf{GL}_n) \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j$$

where the first map is induced by the diagonal $\mathbb{P}E \rightarrow \mathbb{P}E \times \mathbb{P}E$.

The following projective bundle theorem is the key step in the proof of Theorem 3.1.1.

Theorem 3.1.5. *The map*

$$\psi : \mathbb{Z}_{tr}(\mathbb{P}E) \rightarrow \mathbb{Z}_{tr}(B\mathbf{GL}_n) \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j$$

is a motivic weak equivalence.

Proof. If X is a simplicial presheaf with transfers, then there is a short exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{\infty} \mathrm{sk}_i X \longrightarrow \bigoplus_{i=0}^{\infty} \mathrm{sk}_i X \longrightarrow X \longrightarrow 0 \quad (3.1)$$

Just in this proof, we'll write $E = \mathbb{Z}_{tr}(\mathbb{P}E)$ and $B = \mathbb{Z}_{tr}(B\mathbf{GL}_n)$. Form the

diagram

$$\begin{array}{ccccc}
\bigoplus_{i=0}^{\infty} \mathrm{sk}_i E & \xrightarrow{j} & \bigoplus_{i=0}^{\infty} \mathrm{sk}_i E & \longrightarrow & E \\
\downarrow d & & \downarrow d & & \downarrow \\
\bigoplus_{i=0}^{\infty} \mathrm{sk}_i E \otimes_{tr} E & \longrightarrow & \bigoplus_{i=0}^{\infty} \mathrm{sk}_i E \otimes_{tr} E & \longrightarrow & E \otimes_{tr} E \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{i=0}^{\infty} \mathrm{sk}_i B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j & \xrightarrow{j'} & \bigoplus_{i=0}^{\infty} \mathrm{sk}_i B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j & \longrightarrow & B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j
\end{array}$$

Here, the middle row is obtained by tensoring the top row with E , and the bottom row is obtained by letting $X = B$ in 3.1, and tensoring with $\bigoplus_{j=0}^{n-1} I^j$. The right vertical composition is our map ψ , and $d = \bigoplus_{i=0}^{\infty} d_i$, where d_i is \mathbb{Z}_{tr} applied to the canonical map $\mathrm{sk}_i \mathbb{P}E \rightarrow \mathrm{sk}_i \mathbb{P}E \times \mathbb{P}E$. The map labelled j is a cofibration by Lemma 1.1.6. The map j' is a cofibration, as $\bigoplus_{j=0}^{n-1} I^j$ is cofibrant. It follows that both the top and bottom rows of the diagram are cofibre sequences, so if we can show that the composition

$$\bigoplus_{i=0}^{\infty} \mathrm{sk}_i E \rightarrow \bigoplus_{i=0}^{\infty} \mathrm{sk}_i B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j$$

is a motivic weak equivalence, it will follow that ψ is as well.

For any simplicial scheme S , $\mathrm{sk}_i(\mathbb{Z}_{tr}(S))$ is cofibrant, as $N(\mathrm{sk}_i(\mathbb{Z}_{tr}(S)))$ is projective in each degree. So, it suffices to show that each component

$$\sigma_i : \mathrm{sk}_i E \rightarrow \mathrm{sk}_i B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j$$

is a motivic weak equivalence.

We'll prove this by induction on i . First, we need some notation. For $k \geq 0$, let $v : \Delta^0 \rightarrow \Delta^k$ be the vertex corresponding to the functor $\mathbf{0} \rightarrow \mathbf{k}$ that

takes 0 to k ; this gives us a map

$$\beta_k : \mathbb{P}E_k \cong \mathbb{P}E_k \times \Delta^0 \rightarrow \mathbb{P}E_k \times \Delta^k .$$

Let $\alpha_k : \mathbb{P}E_k \times \Delta^k \rightarrow \mathbb{P}E$ be the canonical map, and let

$$\xi_k^j = (\alpha_k \circ \beta_k)^*(\xi^j) \in [\mathbb{Z}_{tr}(\mathbb{P}E_k), \mathbb{Z}(j)]_{\mathbb{A}^1} \quad \text{for } 0 \leq j \leq n-1 .$$

Let f_k^j be the composite

$$f^j \circ \mathbb{Z}_{tr}(\alpha_k \circ \beta_k) : \mathbb{Z}_{tr}(\mathbb{P}E_k) \rightarrow I^j .$$

Then, f_k^j represents ξ_k^j . Note that $f_0 : \mathbb{Z}_{tr}(\mathbb{P}^{n-1}) \rightarrow I$ represents the class in $[\mathbb{Z}_{tr}(\mathbb{P}^{n-1}), \mathbb{Z}(1)]_{\mathbb{A}^1}$ corresponding to $\mathcal{O}(1)$. So, σ_0 is the map

$$\bigoplus_{j=0}^{n-1} f_0^j : \mathbb{Z}_{tr}(\mathbb{P}^{n-1}) \rightarrow \bigoplus_{j=0}^{n-1} I^j$$

which is a motivic weak equivalence by [MVW06, Theorem 15.12].

Now, assume that for some $i > 0$, each σ_m is a motivic weak equivalence for $m < i$. Write $\text{sk}_i E$ as the following pushout:

$$\begin{array}{ccc} P_E & \xrightarrow{a_3} & \text{sk}_{i-1} E \\ a_4 \downarrow & & \downarrow a_6 \\ E_i \otimes \Delta^i & \xrightarrow{a_5} & \text{sk}_i E \end{array}$$

where P_E is given by the pushout:

$$\begin{array}{ccc} L_i E \otimes \partial \Delta^i & \xrightarrow{a_1} & L_i E \otimes \Delta^i \\ a_2 \downarrow & & \downarrow \\ E_i \otimes \partial \Delta^i & \longrightarrow & P_E \end{array}$$

Both of these diagrams are homotopy pushouts, as all objects appearing in them are cofibrant, and the maps labelled a_1 and a_4 are easily seen to be cofibrations.

Writing $\mathrm{sk}_i B$ as a pushout, as above, and tensoring with $\bigoplus_{j=0}^{n-1} I^j$, we get the following pushout

$$\begin{array}{ccc} P_B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j & \longrightarrow & \mathrm{sk}_{i-1} B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j \\ \downarrow & & \downarrow b_6 \\ (B_i \otimes \Delta^i) \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j & \xrightarrow{b_5} & \mathrm{sk}_i B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j \end{array}$$

which is a homotopy pushout as before. We'll show that σ_i is a weak equivalence by giving a natural transformation from the diagram defining $\mathrm{sk}_i E$ to this one, which is a weak equivalence in each component, and which induces σ_i on the pushouts.

Note that $b_6 \circ \sigma_{i-1} = \sigma_i \circ a_6$, and σ_{i-1} is a weak equivalence by the inductive hypothesis.

There is a diagonal map $E_i \rightarrow E_i \otimes_{tr} E_i$, which is \mathbb{Z}_{tr} applied to the diagonal map $\mathbb{P}E_i \rightarrow \mathbb{P}E_i \times \mathbb{P}E_i$, and from this and the diagonal map on Δ^i we get a map

$$E_i \otimes \Delta^i \rightarrow (E_i \otimes_{tr} E_i) \otimes \Delta^i \otimes \Delta^i.$$

Then,

$$(E_i \otimes_{tr} E_i) \otimes \Delta^i \otimes \Delta^i \cong (E_i \otimes \Delta^i) \otimes_{tr} (E_i \otimes \Delta^i),$$

and there is a map

$$(\pi_i \otimes \Delta^i) \otimes_{tr} \left(\bigoplus_{j=0}^{n-1} f^j \circ \alpha_i \right) : (E_i \otimes \Delta^i) \otimes_{tr} (E_i \otimes \Delta^i) \rightarrow (B_i \otimes \Delta^i) \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j.$$

Let

$$\tau_i : E_i \otimes \Delta^i \rightarrow (B_i \otimes \Delta^i) \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j$$

denote the composition of these maps; then, $\sigma_i \circ a_5 = b_5 \circ \tau_i$.

Consider the map $1 \times v : \Delta^i \times \Delta^0 \rightarrow \Delta^i \times \Delta^i$, where v is still our chosen vertex. There is a homotopy $h : \Delta^i \times \Delta^1 \rightarrow \Delta^i \times \Delta^i$ between $1 \times v$ and the diagonal map; the homotopy is defined by an obvious natural transformation between the functors $\mathbf{i} \rightarrow \mathbf{i} \times \mathbf{i}$ that correspond to these two maps. This gives a homotopy between τ_i and the following map:

$$E_i \otimes \Delta^i \xrightarrow{\delta} (E_i \otimes \Delta^i) \otimes_{tr} E_i \xrightarrow{\epsilon} (B_i \otimes \Delta^i) \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j \quad (3.2)$$

where

$$\delta = \mathbb{Z}_{tr} \left(\mathbb{P}E_i \times \Delta^i \xrightarrow{1 \times pr} (\mathbb{P}E_i \times \Delta^i) \times \mathbb{P}E_i \right)$$

and $\epsilon = (\pi_i \otimes \Delta^i) \otimes_{tr} \bigoplus_{j=0}^{n-1} f_i^j$. Again by [MVW06, Theorem 15.12], the map

$$E_i \longrightarrow E_i \otimes_{tr} E_i \xrightarrow{\pi_i \otimes_{tr} \bigoplus f_i^j} B_i \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j$$

is a motivic weak equivalence, and so the map 3.2 is as well.

The maps σ_{i-1} and τ_i restrict to the same map $\nu_i : P_E \rightarrow P_B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j$. To see that this map is a motivic weak equivalence, note that it is induced by a natural transformation from the pushout diagram defining P_E to the one defining $P_B \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j$, which is a weak equivalence in each component.

For this, note that the homotopy h restricts to the boundary of Δ^i as:

$$\begin{array}{ccc} \Delta^i \times \Delta^1 & \xrightarrow{h} & \Delta^i \times \Delta^i \\ \uparrow \lrcorner & & \uparrow \lrcorner \\ \partial \Delta^i \times \Delta^1 & \xrightarrow{\bar{h}} & \partial \Delta^i \times \Delta^i \end{array}$$

and \tilde{h} is a homotopy from the “diagonal” map

$$\partial\Delta^i \rightarrow \partial\Delta^i \times \partial\Delta^i \hookrightarrow \partial\Delta^i \times \Delta^i$$

to $1 \times v : \partial\Delta^i \rightarrow \partial\Delta^i \times \Delta^i$. Using this, one can check that the components of this natural transformation are equivalences, finishing the proof. \square

Corollary 3.1.6. *The induced map on motivic cohomology*

$$\pi^* : H^*(B\mathbf{GL}_n, \mathbb{Z}(*)) \rightarrow H^*(\mathbb{P}E, \mathbb{Z}(*))$$

is a monomorphism.

Proof. As $\mathbb{Z}(0)$ is both motivic fibrant and cofibrant, we can take $I^0 = \mathbb{Z}(0)$.

Let $pr : \bigoplus_{j=0}^{n-1} I^j \rightarrow \mathbb{Z}(0)$ be projection, and define

$$\phi : \mathbb{Z}_{tr}(B\mathbf{GL}_n) \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j \rightarrow \mathbb{Z}_{tr}(B\mathbf{GL}_n) \otimes_{tr} \mathbb{Z}(0) \cong \mathbb{Z}_{tr}(B\mathbf{GL}_n)$$

as $\phi = 1 \otimes_{tr} pr$. Because the projection pr has a section, the map ϕ has a section. The following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}_{tr}(\mathbb{P}E) & \xrightarrow{\psi} & \mathbb{Z}_{tr}(B\mathbf{GL}_n) \otimes_{tr} \bigoplus_{j=0}^{n-1} I^j \\ & \searrow \pi & \swarrow \phi \\ & & \mathbb{Z}_{tr}(B\mathbf{GL}_n) \end{array}$$

so the claim follows by the previous lemma. \square

Using the same argument one uses to calculate the motivic cohomology of projective space, Theorem 3.1.5 implies that the map

$$\pi^* \oplus \xi \cdot \pi^* \oplus \cdots \oplus \xi^{n-1} \cdot \pi^* : \bigoplus_{i=0}^{n-1} H^{2(n-i)}(B\mathbf{GL}_n, \mathbb{Z}(n-i)) \rightarrow H^{2n}(\mathbb{P}E, \mathbb{Z}(n))$$

is an isomorphism, and therefore that there are unique elements $c_i \in H^{2i}(B\mathbf{GL}_n, \mathbb{Z}(i))$ such that

$$\xi^n + c_1 \cdot \xi^{n-1} + \cdots + c_n = 0 .$$

If F is a \mathbf{GL}_n -torsor over a simplicial scheme X , corresponding to a map in the Nisnevich homotopy category $f : X \rightarrow B\mathbf{GL}_n$, then there is an induced map

$$f^* : H^*(B\mathbf{GL}_n, \mathbb{Z}(*)) \rightarrow H^*(X, \mathbb{Z}(*)) .$$

Define the Chern classes of F to be the elements

$$c_i(F) = f^*(c_i) \in H^{2i}(X, \mathbb{Z}(i)) .$$

Proposition 3.1.7. *The motivic cohomology ring of $B\mathbb{G}_m^{\times n}$ is the polynomial algebra*

$$H^*(B\mathbb{G}_m^{\times n}, \mathbb{Z}(*)) \cong H^*(k, \mathbb{Z}(*))[p_1^*(c), \dots, p_n^*(c)]$$

where $p_i : B\mathbb{G}_m^{\times n} \rightarrow B\mathbb{G}_m$ is the projection map, and $c \in H^2(B\mathbb{G}_m, \mathbb{Z}(1))$ is the generator from Proposition 3.1.3.

Proof. We have $B\mathbb{G}_m^{\times n} \simeq (\mathbb{P}^\infty)^{\times n}$, which is the colimit of

$$(\mathbb{P}^1)^{\times n} \rightarrow (\mathbb{P}^2)^{\times n} \rightarrow \dots .$$

As $\mathbb{Z}_{tr}(\mathbb{P}^m) \simeq \bigoplus_{i=0}^m \mathbb{Z}(i)$ [MVW06, Corollary 15.5], we have

$$H^*((\mathbb{P}^m)^{\times n}, \mathbb{Z}(*)) \cong H^*(k, \mathbb{Z}(*))[p_1^*(c), \dots, p_n^*(c)] / (p_1^*(c)^{m+1}, \dots, p_n^*(c)^{m+1}),$$

where $c \in H^2(\mathbb{P}^m, \mathbb{Z}(1))$, and an argument analogous to the proof of Proposition 3.1.3 gives the result. \square

Let $P_{1,n-1} \subset \mathbf{GL}_n$ be the subgroup-scheme consisting of matrices $(a_{i,j})$ with $a_{2,1} = a_{3,1} = \dots = a_{n,1} = 0$. These are the matrices that fix the line spanned by $e_1 = [1 \ 0 \ \dots \ 0]^T$. Consider the diagram of inclusions

$$\mathbb{G}_m \times \mathbf{GL}_{n-1} \hookrightarrow P_{1,n-1} \hookrightarrow \mathbf{GL}_n .$$

The nerve functor preserves products, so there is an induced diagram

$$B\mathbb{G}_m \times B\mathbf{GL}_{n-1} \hookrightarrow BP_{1,n-1} \hookrightarrow B\mathbf{GL}_n .$$

The inclusion $P_{1,n-1} \hookrightarrow \mathbf{GL}_n$ induces a map $EP_{1,n-1} \rightarrow E\mathbf{GL}_n$ between contractible free $P_{1,n-1}$ -spaces, which induces a local weak equivalence

$$BP_{1,n-1} \cong EP_{1,n-1} / P_{1,n-1} \rightarrow E\mathbf{GL}_n / P_{1,n-1} .$$

There is a map $\mathbf{GL}_n \rightarrow \mathbb{P}^{n-1}$ with $M \mapsto Me_1$, and this induces a \mathbf{GL}_n -equivariant isomorphism between the Nisnevich sheaf quotient $\mathbf{GL}_n / P_{1,n-1}$ and \mathbb{P}^{n-1} , as can be checked by evaluating at the Hensel local rings. From this, we have an isomorphism

$$E\mathbf{GL}_n / P_{1,n-1} \cong E\mathbf{GL}_n \times_{\mathbf{GL}_n} \mathbb{P}^{n-1} = \mathbb{P}E .$$

As schemes, $P_{1,n-1} \cong \mathbb{G}_m \times \mathbf{GL}_{n-1} \times \mathbb{A}^{n-1}$, and so the inclusion $\mathbb{G}_m \times \mathbf{GL}_{n-1} \hookrightarrow P_{1,n-1}$ is a motivic weak equivalence. By Lemma 1.1.5, the induced map $B\mathbb{G}_m \times B\mathbf{GL}_{n-1} \rightarrow BP_{1,n-1}$ is a motivic weak equivalence. The inclusion

$B\mathbb{G}_m \times B\mathbf{GL}_{n-1} \hookrightarrow B\mathbf{GL}_n$ factors as

$$\begin{array}{ccc} B\mathbb{G}_m \times B\mathbf{GL}_{n-1} & \hookrightarrow & B\mathbf{GL}_n \\ \simeq \downarrow & & \uparrow \pi \\ BP_{1,n-1} & \xrightarrow{\simeq} & \mathbb{P}E \end{array}$$

So, the induced map on motivic cohomology

$$H^*(B\mathbf{GL}_n, \mathbb{Z}(*)) \rightarrow H^*(B\mathbb{G}_m \times B\mathbf{GL}_{n-1}, \mathbb{Z}(*))$$

is a monomorphism.

Lemma 3.1.8. *The map induced by inclusion $B\mathbb{G}_m^{\times n} \rightarrow B\mathbf{GL}_n$ induces a monomorphism on motivic cohomology.*

Proof. By induction, we may assume $B\mathbb{G}_m^{\times n-1} \rightarrow B\mathbf{GL}_{n-1}$ induces a monomorphism on cohomology. As the inclusion $B\mathbb{G}_m \times B\mathbf{GL}_{n-1} \rightarrow B\mathbf{GL}_n$ induces a monomorphism on cohomology, it suffices to show that the inclusion $i : B\mathbb{G}_m^{\times n} \rightarrow B\mathbb{G}_m \times B\mathbf{GL}_{n-1}$ induces a monomorphism on cohomology.

We have $B\mathbb{G}_m \times B\mathbf{GL}_{n-1} \simeq \mathbb{P}^\infty \times B\mathbf{GL}_{n-1}$, which is the colimit of the diagram

$$\mathbb{P}^1 \times B\mathbf{GL}_{n-1} \rightarrow \mathbb{P}^2 \times B\mathbf{GL}_{n-1} \rightarrow \dots$$

With $c \in H^2(\mathbb{P}^m, \mathbb{Z}(1))$ as before, and p_1 the projection map, we have

$$H^*(\mathbb{P}^m \times B\mathbf{GL}_{n-1}, \mathbb{Z}(*)) \cong H^*(B\mathbf{GL}_{n-1}, \mathbb{Z}(*))[p_1^*(c)] / (p_1^*(c)^{m+1}),$$

using again [MVW06, Corollary 15.5]. An argument analogous to the proof of Proposition 3.1.3 shows that

$$H^*(\mathbb{P}^\infty \times B\mathbf{GL}_{n-1}, \mathbb{Z}(*)) \cong H^*(B\mathbf{GL}_{n-1}, \mathbb{Z}(*))[p_1^*(c)]$$

where now $c \in H^2(\mathbb{P}^\infty, \mathbb{Z}(1))$. As $i^*(p_1^*(c)) = p_1^*(c)$, where the right-hand side is the generator from Proposition 3.1.7, the inductive hypothesis finishes the proof. \square

We need a couple of lemmas before we can finish the proof of Theorem 3.1.1.

Lemma 3.1.9. *Let G be a presheaf of groups on a site \mathcal{C} , and let g be a global section of G . The map $BG \rightarrow BG$ induced by the inner automorphism of G defined by g is homotopic to the identity.*

Proof. Define a homotopy $BG \times \Delta^1 \rightarrow BG$ as the nerve of a functor $H : G \times \mathbf{1} \rightarrow G$. For any object U of \mathcal{C} , the category $G(U) \times \mathbf{1}$ has objects $\{0, 1\}$, and $\text{hom}(0, 0) = \text{hom}(0, 1) = \text{hom}(1, 1) = G(U)$. For $f \in \text{hom}(0, 0)$, let $H(f) = f$, for $f \in \text{hom}(0, 1)$, let $H(f) = gf$, and for $f \in \text{hom}(1, 1)$, let $H(f) = fgf^{-1}$. \square

Lemma 3.1.10. *If X is a simplicial scheme, $U, V \subset X$ are maps of simplicial schemes such that $U_n, V_n \subset X_n$ is a Zariski cover for all $n \geq 0$, and we have $\alpha \in H^p(X, \mathbb{Z}(q))$ and $\beta \in H^r(X, \mathbb{Z}(s))$ such that α vanishes when restricted to U , and β vanishes when restricted to V , then the cup product $\alpha \cdot \beta$ is zero.*

Proof. Let F be a Nisnevich fibrant model of $\Omega^{2q-p}(\gamma_*\mathbb{Z}(q))$. As $i : \mathbb{Z}(U) \rightarrow \mathbb{Z}(X)$ is a cofibration, the sequence

$$\mathbf{hom}(\mathbb{Z}(X)/\mathbb{Z}(U), F) \rightarrow \mathbf{hom}(\mathbb{Z}(X), F) \xrightarrow{i^*} \mathbf{hom}(\mathbb{Z}(U), F)$$

is a fibre sequence of simplicial sets. It follows that we can choose a representative $f : \mathbb{Z}(X) \rightarrow F$ of α such that $i^*(f) = 0$, and therefore f factors through the presheaf-theoretic quotient $\mathbb{Z}(X)/\mathbb{Z}(U)$. Similarly, there is a representative of β that factors through $\mathbb{Z}(X)/\mathbb{Z}(V)$. But, the condition that U and V

give a Zariski cover of X in every simplicial degree implies that the map

$$\mathbb{Z}(X) \xrightarrow{\Delta} \mathbb{Z}(X) \otimes \mathbb{Z}(X) \rightarrow (\mathbb{Z}(X)/\mathbb{Z}(U)) \otimes (\mathbb{Z}(X)/\mathbb{Z}(V))$$

induces the zero map at every local ring, so that this map is zero in the homotopy category of $\mathbf{sPre}_{\mathbb{Z}}(Sm_k)$ with respect to the Zariski topology, and hence zero in the homotopy category with respect to the Nisnevich topology. \square

Let $\sigma_i = \sigma_i(p_1^*(c), \dots, p_n^*(c)) \in H^{2i}(B\mathbb{G}_m^{\times n}, \mathbb{Z}(i))$ be the elementary symmetric polynomials. The following result finishes the proof of Theorem 3.1.1.

Proposition 3.1.11. *The image of $H^*(B\mathbf{GL}_n, \mathbb{Z}(*))$ in $H^*(B\mathbb{G}_m^{\times n}, \mathbb{Z}(*))$ is $H^*(k, \mathbb{Z}(*))[\sigma_1, \dots, \sigma_n]$.*

Proof. The $H^*(k, \mathbb{Z}(*))$ -algebra generated by the σ_i is the subalgebra of $H^*(B\mathbb{G}_m^{\times n}, \mathbb{Z}(*))$ invariant under the action of the symmetric group S_n on $B\mathbb{G}_m^{\times n}$, where S_n permutes the factors. This action is the restriction to $B\mathbb{G}_m^{\times n}$ of the S_n action on $B\mathbf{GL}_n$ given by the inner automorphisms defined by permutation matrices. By Lemma 3.1.9, the image of $H^*(B\mathbf{GL}_n, \mathbb{Z}(*))$ is contained in the subalgebra generated by the σ_i . The image is equal to this subalgebra, as the Chern class $c_i \in H^{2i}(B\mathbf{GL}_n, \mathbb{Z}(i))$ restricts to σ_i , as we'll now show.

For $n \geq 1$, let $E^n = E\mathbf{GL}_n \times_{\mathbf{GL}_n} \mathbb{A}^n$ be the Borel construction for the usual action of \mathbf{GL}_n on \mathbb{A}^n , and let $D^n = E\mathbb{G}_m^{\times n} \times_{\mathbb{G}_m^{\times n}} \mathbb{A}^n$ be the restriction to the maximal torus. Note that $D^n = p_1^*(E^1) \oplus \dots \oplus p_n^*(E^1)$. Let $\xi_T \in H^2(\mathbb{P}(D^n), \mathbb{Z}(1))$ be the restriction of $\xi_n \in H^2(\mathbb{P}(E^n), \mathbb{Z}(1))$, and let $c'_i \in H^{2i}(B\mathbb{G}_m^{\times n}, \mathbb{Z}(i))$ be the restriction of c_i . The elements c'_i are characterized by the equation

$$\xi_T^n + c'_1 \cdot \xi_T^{n-1} + \dots + c'_n = 0 .$$

So, it suffices to show that

$$\prod_{i=1}^n (\xi_T + p_i^*(c)) = 0 .$$

Let $(\mathcal{O}(1), \alpha_n)$ be the \mathbb{G}_m -torsor on $\mathbb{P}(E^n)$ that corresponds to ξ_n . The restriction of $(\mathcal{O}(1), \alpha_n)$ to $\mathbb{P}(p_i^*(E^1))$ is isomorphic to the pullback of $(\mathcal{O}(1), \alpha_1)$ along the map $\mathbb{P}(p_i^*(E^1)) \rightarrow \mathbb{P}(E^1)$. So, the following diagram commutes in the Nisnevich homotopy category

$$\begin{array}{ccccc} \mathbb{P}(p_i^*(E^1)) & \longrightarrow & \mathbb{P}(D^n) & \longrightarrow & \mathbb{P}(E^n) \\ \downarrow & & & & \downarrow \xi_n \\ \mathbb{P}(E^1) & \xrightarrow{\xi_1} & & \longrightarrow & B\mathbb{G}_m \end{array}$$

From this it follows that ξ_T restricts to $p_i^*(\xi_1)$ on $\mathbb{P}(p_i^*(E^1))$, and so $\xi_T + p_i^*(c)$ vanishes when restricted to $\mathbb{P}(p_i^*(E^1))$, as $\xi_1 + c = 0$.

For each $1 \leq i \leq n$, let

$$C_i = p_1^*(E^1) \oplus \cdots \oplus \widehat{p_i^*(E^1)} \oplus \cdots \oplus p_n^*(E^1) .$$

The obvious map $\mathbb{P}(D^n) - \mathbb{P}(C_i) \rightarrow \mathbb{P}(p_i^*(E^1))$ is a motivic weak equivalence, as it is a rank $n - 1$ vector bundle in each degree. The inclusion $\mathbb{P}(p_i^*(E^1)) \rightarrow \mathbb{P}(D^n) - \mathbb{P}(C_i)$ is a section of the projection, therefore a motivic weak equivalence as well. So, $\xi_T + p_i^*(c)$ vanishes when restricted to $\mathbb{P}(D^n) - \mathbb{P}(C_i)$.

Now, the result follows from Lemma 3.1.10 □

3.2 The special linear group

In this section, we prove the following

Theorem 3.2.1. *Let k be a perfect field. Then the motivic cohomology of $B\mathbf{SL}_n$ is the polynomial algebra*

$$H^*(B\mathbf{SL}_n, \mathbb{Z}(*)) \cong H^*(k, \mathbb{Z}(*))[c_2, \dots, c_n],$$

where $c_i \in H^{2i}(B\mathbf{SL}_n, \mathbb{Z}(i))$.

Let T_n denote the subgroup of diagonal matrices in \mathbf{SL}_n ; there is an isomorphism $\mathbb{G}_m^{\times n-1} \rightarrow T_n$ defined by

$$(a_1, \dots, a_{n-1}) \mapsto \text{diag}(a_1, \dots, a_{n-1}, a_1^{-1} \dots a_{n-1}^{-1}).$$

The composite

$$\mathbb{G}_m^{\times n-1} \xrightarrow{\cong} T_n \hookrightarrow \mathbb{G}_m^{\times n} \xrightarrow{p_n} \mathbb{G}_m$$

is equal to the composite

$$\mathbb{G}_m^{\times n-1} \xrightarrow{\det} \mathbb{G}_m \xrightarrow{(-)^{-1}} \mathbb{G}_m.$$

The inversion map $(-)^{-1}$ induces the map on $H^*(B\mathbb{G}_m, \mathbb{Z}(*))$ defined by $c \mapsto -c$, and the determinant induces the map $H^*(B\mathbb{G}_m, \mathbb{Z}(*)) \rightarrow H^*(B\mathbb{G}_m^{\times n-1}, \mathbb{Z}(*))$ defined by $c \mapsto p_1^*(c) + \dots + p_{n-1}^*(c)$.

Let ϕ be the composition $\mathbb{G}_m^{\times n-1} \cong T_n \hookrightarrow \mathbb{G}_m^{\times n}$; then

$$\phi^*(p_n^*(c)) = -(p_1^*(c) + \dots + p_{n-1}^*(c)).$$

It follows that the map on motivic cohomology induced by $BT_n \hookrightarrow B\mathbb{G}_m^{\times n}$ is

the quotient map

$$H^*(k, \mathbb{Z}(*))[p_1^*(c), \dots, p_n^*(c)] \rightarrow H^*(k, \mathbb{Z}(*))[p_1^*(c), \dots, p_n^*(c)] / (p_1^*(c) + \dots + p_n^*(c)) .$$

Consider the diagram of inclusions

$$\begin{array}{ccc} T_n & \longrightarrow & \mathbf{S}\mathbf{l}_n \\ \downarrow & & \downarrow \\ \mathbb{G}_m^{\times n} & \longrightarrow & \mathbf{G}\mathbf{l}_n \end{array}$$

The image of $H^*(B\mathbf{G}\mathbf{l}_n, \mathbb{Z}(*))$ in $H^*(BT_n, \mathbb{Z}(*))$ is $H^*(k, \mathbb{Z}(*))[\sigma_2, \dots, \sigma_n]$.

The image of $H^*(B\mathbf{S}\mathbf{l}_n, \mathbb{Z}(*))$ is contained in the symmetric part of $H^*(BT_n, \mathbb{Z}(*))$,

as the restriction of the S_n -action on $B\mathbb{G}_m^{\times n}$ to BT_n is also the restriction of

an S_n -action on $B\mathbf{S}\mathbf{l}_n$ given by inner automorphisms: the transposition that

exchanges the i^{th} and $(i+1)^{\text{th}}$ factor is defined by the block diagonal matrix

with

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in columns i and $i+1$. So, $H^*(k, \mathbb{Z}(*))[\sigma_2, \dots, \sigma_n]$ is the image of $H^*(B\mathbf{S}\mathbf{l}_n, \mathbb{Z}(*))$.

Lemma 3.2.2. *The inclusion $T_n \hookrightarrow \mathbf{S}\mathbf{l}_n$ induces a monomorphism*

$$H^*(B\mathbf{S}\mathbf{l}_n, \mathbb{Z}(*)) \rightarrow H^*(BT_n, \mathbb{Z}(*))$$

Proof. This is analogous to the proof of the corresponding result for $\mathbf{G}\mathbf{l}_n$. Let

$$SP_{1,n-1} = \{(m_{ij}) \in \mathbf{S}\mathbf{l}_n \mid m_{2,0} = \dots = m_{n,0} = 0\} .$$

As schemes, $SP_{1,n-1} \cong \mathbf{G}\mathbf{l}_{n-1} \times \mathbb{A}^{n-1}$. The projection map $SP_{1,n-1} \rightarrow \mathbf{G}\mathbf{l}_{n-1}$

has a section, which takes a matrix $N = (n_{ij})$ to the block diagonal matrix

(a_{ij}) with $a_{1,1} = \det(N)^{-1}$ and $a_{i,j} = n_{i-1,j-1}$ for all $i, j > 1$. It follows that the induced map $B\mathbf{GL}_{n-1} \rightarrow BSP_{1,n-1}$ is a motivic weak equivalence.

The inclusion $SP_{1,n-1} \hookrightarrow \mathbf{S}\mathbf{l}_n$ induces a map $BSP_{1,n-1} \rightarrow E\mathbf{S}\mathbf{l}_n/SP_{1,n-1}$, which is a local weak equivalence. Furthermore, $E\mathbf{S}\mathbf{l}_n/SP_{1,n-1} \cong E\mathbf{S}\mathbf{l}_n \times_{\mathbf{s}\mathbf{l}_n} \mathbb{P}^{n-1}$.

Arguing exactly as in the proof of Theorem 3.1.5, the map $E\mathbf{S}\mathbf{l}_n \times_{\mathbf{s}\mathbf{l}_n} \mathbb{P}^{n-1} \rightarrow B\mathbf{S}\mathbf{l}_n$ induces a monomorphism on motivic cohomology, and so the map $BSP_{1,n-1} \rightarrow B\mathbf{S}\mathbf{l}_n$ does as well.

But now we are done: the inclusion $BT_n \hookrightarrow B\mathbf{S}\mathbf{l}_n$ factors as

$$BT_n \xrightarrow{f} B\mathbf{GL}_{n-1} \xrightarrow{g} BSP_{1,n-1} \hookrightarrow B\mathbf{S}\mathbf{l}_n$$

where f is the map defined by

$$\text{diag}(a_1, \dots, a_{n-1}, a_1^{-1} \dots a_{n-1}^{-1}) \mapsto \text{diag}(a_1, \dots, a_{n-1})$$

and g is the map defined previously. As $T_n \cong \mathbb{G}_m^{n-1}$, we've already seen that f induces a monomorphism on cohomology. \square

Chapter 4

The Nisnevich classifying space of \mathbf{PGL}_p

In this chapter, we prove the following

Theorem 4.0.1. *Let p be an odd prime. Over the complex numbers, the canonical homomorphism in motivic cohomology*

$$H^{2*}(B_{ct}\mathbf{PGL}_p, \mathbb{Z}(*)) \rightarrow H^{2*}(B_{Nis}\mathbf{PGL}_p, \mathbb{Z}(*))$$

is injective.

The hypothesis that the base field is the complex numbers is used in the proof of Lemma 4.2.2; the hypothesis that p is an odd prime is used in the work of Vistoli [Vis07], upon which the proof of Theorem 4.0.1 relies.

4.1 Preliminaries

In this section, let k be a perfect field, and let Sm_k be the Nisnevich site of smooth, separated k -schemes.

For G a presheaf of groups on Sm_k , we'll use $B_{\text{Nis}}G$ to denote the Nisnevich classifying space of G , which is the simplicial presheaf on Sm_k with

$$B_{\text{Nis}}G(U) = B(G(U)) \quad \text{for all schemes } U,$$

where $B(G(U))$ is the nerve of the group $G(U)$.

Following Morel and Voevodsky [MV99], let $B_{\text{et}}G$ be the Nisnevich homotopy type of an étale fibrant model of $B_{\text{Nis}}G$. Explicitly, choose a map

$$j : B_{\text{Nis}}G \rightarrow F_{\text{et}}(B_{\text{Nis}}G),$$

where j is an étale local equivalence, and $F_{\text{et}}(B_{\text{Nis}}G)$ is injective fibrant with respect to the étale topology. Then $F_{\text{et}}(BG)$ is a model of $B_{\text{et}}G$.

For any smooth k -scheme U , $B_{\text{et}}G$ classifies étale G -torsors, in the sense that

$$H_{\text{et}}^1(U, G) \cong [X, B_{\text{et}}G]_{\text{Nis}}$$

In [MV99, Lemma 4.1.18], Morel and Voevodsky observe the following

Proposition 4.1.1. *Let G be a presheaf of groups. The map $B_{\text{Nis}}G \rightarrow B_{\text{et}}G$ is a Nisnevich local equivalence if and only if G is an étale sheaf, and one of the following equivalent conditions holds:*

1. *for any smooth scheme S over k , one has $H_{\text{Nis}}^1(S, G) \cong H_{\text{et}}^1(S, G)$.*
2. *for any smooth scheme S over k and a point x of S , one has*

$$H_{\text{et}}^1(\text{Spec}(\mathcal{O}_{S,x}^h), G) = * .$$

And, they point out [MV99, Lemma 4.3.6] that general linear groups satisfy

these conditions; so

$$B_{\text{Nis}}\mathbf{GL}_n \rightarrow B_{\text{et}}\mathbf{GL}_n$$

is a Nisnevich local equivalence for all $n > 0$.

If k is a perfect field, we've seen that the motivic cohomology of $B\mathbf{GL}_n$ is a polynomial algebra over the cohomology of the base field:

$$H^*(B_{\text{Nis}}\mathbf{GL}_n, \mathbb{Z}(*)) \cong H^*(k, \mathbb{Z}(*))[c_1, \dots, c_n]$$

with $c_i \in H^{2i}(B_{\text{Nis}}\mathbf{GL}_n, \mathbb{Z}(i))$.

If G is a presheaf of groups on the Nisnevich site Sm_k as before, and $f : G \rightarrow \mathbf{GL}_n$ is a representation, then we can define the Chern classes of f to be

$$c_i(f) = f^*(c_i) \in H^{2i}(B_{\text{Nis}}G, \mathbb{Z}(i)) .$$

As we have an identification $B_{\text{Nis}}\mathbf{GL}_n \simeq B_{\text{et}}\mathbf{GL}_n$ for the Nisnevich topology, we have an identification of motivic homotopy types, so that we can define Chern classes in the motivic cohomology of $B_{\text{et}}G$ in the same way. Because the canonical map in the homotopy category $B_{\text{Nis}}G \rightarrow B_{\text{et}}G$ is natural in G , the homomorphism in motivic cohomology

$$H^{2*}(B_{\text{et}}G, \mathbb{Z}(*)) \rightarrow H^{2*}(B_{\text{Nis}}G, \mathbb{Z}(*))$$

takes Chern classes to Chern classes.

4.2 Proof of Theorem 4.0.1

We begin with a couple of lemmas.

Let X be a simplicial set, and let F be any field. There is an adjunction

$$\Gamma^* : \mathbf{s}(F - \mathbf{vec}) \rightleftarrows \mathbf{sPre}_F(Sm_k) : \Gamma_*$$

between simplicial F -vector spaces and presheaves of simplicial F -vector spaces, where Γ^* is the constant presheaf functor, and Γ_* is global sections.

Lemma 4.2.1. *Let F be a field, and let X be a simplicial set such that all singular homology groups $H_r(X, F)$ are finite-dimensional. Then, the motivic cohomology ring of Γ^*X can be written as the tensor product*

$$H^*(\Gamma^*X, F(*)) \cong H^*(k, F(*)) \otimes H^*(X, F),$$

where elements of $H^r(X, F)$ are seen as elements of the motivic cohomology group $H^r(\Gamma^*X, F(0))$.

Proof. The adjunction $\Gamma^* \dashv \Gamma_*$ is a Quillen adjunction for the injective local model structure on $\mathbf{sPre}_F(Sm_k)$ and the usual model structure on $\mathbf{s}(F - \mathbf{vec})$.

So, we have

$$\begin{aligned} H^p(\Gamma^*X, F(q)) &= [F(\Gamma^*X), F(q)[-p]] \\ &\cong [FX, F(q)[-p](k)]. \end{aligned}$$

For any simplicial F -vector spaces C and D , the F -vector space $[C, D]$ of maps in the homotopy category of $\mathbf{s}(F - \mathbf{vec})$ is isomorphic to $\pi(C, D)$, the space of maps from C to D in $\mathbf{s}(F - \mathbf{vec})$ modulo chain homotopy, as all simplicial F -vector spaces are both fibrant and cofibrant. The obvious map

$$\pi(C, D) \rightarrow \prod_{n \geq 0} \mathrm{hom}(H_n(C), H_n(D))$$

that sends a homotopy class $[f : C \rightarrow D]$ to the induced maps $(f_* : H_0(C) \rightarrow H_0(D), \dots)$ is an isomorphism, so that

$$[C, D] \cong \prod_{n \geq 0} \text{hom}(H_n(C), H_n(D)) .$$

For $n \geq 0$, we have

$$H_n(F(q)[-p](k)) = H^{p-n}(k, F(q)),$$

and so we have

$$H^p(\Gamma^* X, F(q)) \cong \prod_{n \geq 0} \text{hom}(H_n(X, F), H^{p-n}(k, F(q))) .$$

As $H^r(k, F(0)) \cong F$ if $r = 0$, and is zero otherwise, we have

$$H^*(\Gamma^* X, F(0)) \cong H^*(X, F) .$$

If V, W are F -vector spaces with V finite-dimensional, then the canonical map

$$V^\vee \otimes W \rightarrow \text{hom}(V, W)$$

is an isomorphism. So, by our assumptions on X , we have

$$H^p(\Gamma^* X, F(q)) \cong \prod_{n \geq 0} H^n(X, F) \otimes H^{p-n}(k, F(q)) .$$

□

Lemma 4.2.2. *Let p be an odd prime. Over the complex numbers, the homo-*

morphism in motivic cohomology

$$H^{2*}(B_{\text{et}}(\mu_p \times \mu_p), \mathbb{Z}(*)) \rightarrow H^{2*}(B_{\text{Nis}}(\mu_p \times \mu_p), \mathbb{Z}(*))$$

is injective.

Proof. As \mathbb{C} contains a primitive p^{th} root of unity, the obvious map

$$\Gamma^* BC_p \rightarrow B_{\text{Nis}} \mu_p$$

is a Nisnevich local equivalence, where C_p denotes the cyclic group with p elements. We'll begin the proof by showing that the homomorphism in motivic cohomology

$$H^{2*}(B_{\text{et}} \mu_p, \mathbb{Z}(*)) \rightarrow H^{2*}(\Gamma^* BC_p, \mathbb{Z}(*))$$

is injective.

The Chow ring of $B_{\text{et}} \mu_p$ is generated by the first Chern class t of the embedding $\mu_p \subset \mathbb{G}_m$ ([Vis07, p190]):

$$H^{2*}(B_{\text{et}} \mu_p, \mathbb{Z}(*)) \cong \mathbb{Z}[t]/(p \cdot t) .$$

Furthermore,

$$H^2(B_{\text{Nis}} \mu_p, \mathbb{Z}(1)) \cong H^1(B_{\text{Nis}} \mu_p, \mathbb{G}_m) \cong \text{hom}(\mu_p, \mathbb{G}_m) \cong \mathbb{Z}/p ,$$

using Proposition 2.4.3. Let c denote the first Chern class of $\mu_p \subset \mathbb{G}_m$ in $H^2(B_{\text{Nis}} \mu_p, \mathbb{Z}(1))$: we want to show that c has infinite multiplicative order in

$$H^{2*}(B_{\text{Nis}} \mu_p, \mathbb{Z}(*)) \cong H^{2*}(\Gamma^* BC_p, \mathbb{Z}(*)) .$$

For this we use \mathbb{Z}/p coefficients.

By Bloch-Kato (see e.g. [HW19]), we have

$$H^0(\mathbb{C}, \mathbb{Z}/p(i)) \cong H_{\text{et}}^0(\mathbb{C}, \mu_p^{\otimes i}) \cong \mathbb{Z}/p ,$$

and any generator $\tau \in H_{\text{et}}^0(\mathbb{C}, \mu_p)$ defines an isomorphism $H_{\text{et}}^0(\mathbb{C}, \mu_p^{\otimes i}) \cong H_{\text{et}}^0(\mathbb{C}, \mu_p^{\otimes i+1})$, so that τ has infinite multiplicative order in $H^*(\mathbb{C}, \mathbb{Z}/p(*))$.

By Lemma 4.2.1,

$$H^2(\Gamma^* BC_p, \mathbb{Z}/p(0)) \cong H^2(BC_p, \mathbb{Z}/p) \cong \mathbb{Z}/p ,$$

and a generator y of this group has infinite multiplicative order in $H^*(\Gamma^* BC_p, \mathbb{Z}/p(0))$.

Again by Lemma 4.2.1, $\tau \cdot y \in H^2(\Gamma^* BC_p, \mathbb{Z}/p(1))$ has infinite multiplicative order.

Up to a choice of the generator y , we have $c \mapsto \tau \cdot y$ under the map

$$H^2(\Gamma^* BC_p, \mathbb{Z}(1)) \rightarrow H^2(\Gamma^* BC_p, \mathbb{Z}/p(1)) . \quad (4.1)$$

To see this, consider the diagram

$$\begin{array}{ccc} H^2(B_{\text{et}} \mu_p, \mathbb{Z}(1)) & \longrightarrow & H^2(B_{\text{Nis}} \mu_p, \mathbb{Z}(1)) \\ \alpha \downarrow & & \downarrow \\ H^2(B_{\text{et}} \mu_p, \mathbb{Z}/p(1)) & \longrightarrow & H^2(B_{\text{Nis}} \mu_p, \mathbb{Z}/p(1)) \\ \beta \downarrow & & \downarrow \\ H_{\text{et}}^2(B_{\text{et}} \mu_p, \mathbb{Z}/p(1)) & \xrightarrow{\gamma} & H_{\text{et}}^2(B_{\text{Nis}} \mu_p, \mathbb{Z}/p(1)) \end{array}$$

As $H^2(B_{\text{et}} \mu_p, \mathbb{Z}(1))$ is p -torsion, the map labelled α is injective, by the universal coefficients sequence [MVW06, p27]; the map labelled β is injective, by the Beilinson-Lichtenbaum conjecture [Voe11, Theorem 6.17]; and the map

labelled γ between étale motivic cohomology groups is an isomorphism, as $B_{\text{Nis}} \mu_p$ and $B_{\text{et}} \mu_p$ are étale-locally equivalent. It follows that the map 4.1 does not kill c . Using Lemma 4.2.1 and that the ground field is algebraically closed, every non-zero element of $H^2(\Gamma^* BC_p, \mathbb{Z}/p(1))$ is equal to $\tau \cdot y$ for some choice of generator $y \in H^2(BC_p, \mathbb{Z}/p)$. So, as $c \mapsto \tau \cdot y$, c has infinite multiplicative order in $H^{2*}(B_{\text{Nis}} \mu_p, \mathbb{Z}(*))$.

Now we can prove the lemma. We have [Vis07, p194]:

$$H^{2*}(B_{\text{et}}(\mu_p \times \mu_p), \mathbb{Z}(*)) \cong \mathbb{Z}[s, r]/(p \cdot s, p \cdot r),$$

with $s = \pi_1^*(t)$ and $r = \pi_2^*(t)$, where $\pi_i : \mu_p \times \mu_p \rightarrow \mu_p$ are the projection homomorphisms. As the image of t in $H^2(B_{\text{Nis}} \mu_p, \mathbb{Z}/p(1))$ is $\tau \cdot y$, the image of $\pi_i^*(t)$ in $H^2(B_{\text{Nis}}(\mu_p \times \mu_p), \mathbb{Z}/p(1))$ is $\tau \cdot \pi_i^*(y)$. By Lemma 4.2.1, the composition

$$H^{2*}(B_{\text{et}}(\mu_p \times \mu_p), \mathbb{Z}(*)) \rightarrow H^{2*}(B_{\text{Nis}}(\mu_p \times \mu_p), \mathbb{Z}(*)) \rightarrow H^{2*}(B_{\text{Nis}}(\mu_p \times \mu_p), \mathbb{Z}/p(*))$$

is injective, finishing the proof. \square

The proof of Theorem 4.0.1 relies on the work of Vistoli on the Chow ring of the étale classifying space of \mathbf{PGL}_p [Vis07]. By [Voe03, Corollary 6.2], we have

$$A_G^* \cong H^{2*}(B_{\text{et}} G, \mathbb{Z}(*)),$$

where G is a linear algebraic group, and A_G^* is the Chow ring of the classifying space of G , in the sense of Totaro [Tot99].

Proof of Theorem 4.0.1. In [Vis07], Vistoli defines a subgroup $C_p \times \mu_p \subset \mathbf{PGL}_p$, as follows.

Let ω be a primitive p^{th} root of unity in \mathbb{C} , and let τ be the diagonal matrix $\text{diag}(\omega, \omega^2, \dots, \omega^{p-1}, 1)$. Then τ generates a subgroup of \mathbf{PGL}_p isomorphic to

μ_p .

Let σ be the permutation matrix corresponding to the cycle $(1\ 2\ \dots\ p) \in S_p$. Then σ generates a subgroup of \mathbf{PGL}_p isomorphic to C_p , the cyclic group of order p , viewed as a \mathbb{C} -group scheme in the usual way. In our case, $C_p \cong \mu_p$.

In \mathbf{GL}_p , we have $\tau\sigma = \omega\sigma\tau$, so σ and τ commute in \mathbf{PGL}_p , and they generate a subgroup of \mathbf{PGL}_p isomorphic to $C_p \times \mu_p$.

Just for this proof, write

$$CH^*X = H^{2*}(X, \mathbb{Z}(*)) .$$

Let T_{PGL_p} be the standard maximal torus in \mathbf{PGL}_p , consisting of classes of diagonal matrices. By work of Totaro and Vistoli, [Vis07, Proposition 9.3] and [Vis07, Proposition 9.4], the inclusions $T_{PGL_p} \subset \mathbf{PGL}_p$ and $C_p \times \mu_p \subset \mathbf{PGL}_p$ induce an injective homomorphism

$$CH^*B_{\text{et}}\mathbf{PGL}_p \rightarrow CH^*B_{\text{et}}T_{PGL_p} \times CH^*B_{\text{et}}(C_p \times \mu_p) .$$

By Lemma 4.2.2, the natural map

$$CH^*B_{\text{et}}(C_p \times \mu_p) \rightarrow CH^*B_{\text{Nis}}(C_p \times \mu_p)$$

is injective.

As group-schemes, we have

$$T_{PGL_p} \cong T_{Sl_p} \cong \mathbb{G}_m^{\times p-1} ,$$

so that $B_{\text{et}}T_{PGL_p} \simeq B_{\text{Nis}}T_{PGL_p}$.

Consider the commutative diagram

$$\begin{array}{ccc}
 CH^*B_{\text{et}}\mathbf{PGL}_p & \longrightarrow & CH^*B_{\text{Nis}}\mathbf{PGL}_p \\
 \downarrow & & \downarrow \\
 CH^*B_{\text{et}}T_{\mathbf{PGL}_p} \times CH^*B_{\text{et}}(C_p \times \mu_p) & \longrightarrow & CH^*B_{\text{Nis}}T_{\mathbf{PGL}_p} \times CH^*B_{\text{Nis}}(C_p \times \mu_p)
 \end{array}$$

The bottom route around the square is injective, and it follows that

$$CH^*B_{\text{et}}\mathbf{PGL}_p \rightarrow CH^*B_{\text{Nis}}\mathbf{PGL}_p$$

is injective.

□

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