8-6-2019 2:30 PM

Essential dimension of parabolic bundles

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Abstract

Essential dimension of a geometric object is roughly the number of algebraically independent parameters needed to define the object. In this thesis we give upper bounds for the essential dimension of parabolic bundles over a non-singular curve X of genus $g \geq 2$ using Bornes correspondence between parabolic bundles on a curve and vector bundles on a root stack. This is a generalization of the work of Biswas, Dhillon and Hoffmann on the essential dimension of vector bundles, by following their method for curves and adapting it to root stacks. In this process, we invoke the Riemann-Roch theorem of Toen for Deligne-Mumford stacks and use it to estimate the dimension of stack of parabolic bundles on X.

Keywords: Root stacks, essential dimension, vector bundles, parabolic bundles.

Summary for lay audience

Algebraic geometry is the study of solution sets of polynomials called 'Varieties'. For example, by solving the equations $ax + by + c = 0$ and $x^2 + y^2 = r^2$, we find that a line and a circle intersect at a maximum of two points in the Cartesian plane. One can ask solutions of higher degree polynomials in several variables. One of the less understood, yet seemingly tractable problem is the study of algebraic curves, these are roughly the varieties of dimension 1. One of the key methods of studying varieties is the study of functions on them. A vector bundle is roughly an assignment of a vector space to each point in the curve. The concept of vector bundles, in some sense, encode 'generalized functions' on varieties. Parabolic vector bundles are vector bundles along with the extra data of a sequence of subspaces at finitely many points on the curve.

It is useful to study the 'moduli space of parabolic vector bundles' on a curve. In a naive sense this is just the collection of all parabolic vector bundles on the curve. But we want more than that: It would be useful to have the collection of parabolic vector bundles on a curve as a honest 'variety'. This is not possible unless we restrict the class of vector bundles to a special type. Simply put, the reason for the failure is the 'symmetries' of the vector bundles. Taking these symmetries into consideration, we can construct a 'moduli stack of vector bundles'. This is not exactly a variety but it enjoys several geometric properties of a variety. Hence one can ask, what are the 'degrees of freedom' in the moduli stack of parabolic bundles. This number turns out to be the 'essential dimension' of the stack.

The essential dimension of an object, in essence, is the number of parameters required to determine the object. In this thesis we use several techniques of modern algebraic geometry, especially of algebraic stacks to find the max-

imum number of parameters required to determine a parabolic bundle over a large class of nice algebraic curves.

To Anil

Acknowledgements

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Chapter 1

Introduction

Consider the quadratic polynomial $f(x) = x^2 + ax + b$, with a and b as independent variables. By completing the square, it can be reduced to a polynomial of the form $g(y) = y^2 - c$, for some c in terms of a and b. More generally, any polynomial $p(x) = x^n + a_1 x + \cdots + a_n$, with $a_1, \ldots, a_n$ as independent variables can be reduced further to a polynomial with fewer number of independent variables as coefficients by non-degenerate Tschirnhaus transformations. It is natural to ask, given such $p(x)$ what is the minimum number of algebraically independent parameters required to determine it up to non-degenerate Tschirnhaus transformations. J.Buhler and Z.Reichstein interpret this question in terms of 'essential dimension' of a finite group, see [BR97]. Essential dimension of an object roughly estimates the number of algebraically independent parameters required to determine it. In its full generality the essential dimension of a functor $F : \texttt{Fields}/k \to \texttt{Sets}$ is defined by Merkurjev, see Definition 2.43.

The notion of essential dimension has been studied for several algebraic and geometric objects. Examples include algebraic groups, moduli spaces of curves, abelian varieties, vector bundles and, in this instance, parabolic bundles on

a smooth curve of genus $g \geq 2$. We refer the interested reader to the survey [Mer13] by Merkurjev for a broad overview.

To illustrate the depth captured by essential dimension consider the following example: Given an algebraic group G over a field k, consider the functor $K \mapsto H^1(\mathrm{spec}(K), G)$ from $Fields/k$ to $Sets$. $H^1(\mathrm{spec}(K), G)$ is precisely the set of G-torsors over K which can be further identified with the stack BG. The essential dimension of the stack $B\mathbb{G}_m$ is 0. This follows from Hilbert's theorem 90. On a similar note, the essential dimension of $B\mu_r$ is 1. This follows from Kummer theory.

The essential dimension of the stack of vector bundles over a smooth curve of any genus is calculated by Biswas, Dhillon and Hoffmann in [BDH18]. One of the key features in their work is to estimate the transcendence degree of the 'field of moduli' $k(E)$ of a given vector bundle E on a curve X/k. To do this, they make use of the stack of nilpotent endomorphisms $\mathcal{N}il_{X,n}$ parametrizing pairs (E, φ), where E is a coherent sheaf and $\varphi : E \to E$ is a nilpotent endomorphism.

The essential dimension of projective modules of rational rank has also been computed in [BDH18]. It is used to estimate the essential dimension $\mathrm{ed}_{k(E)}(\mathcal{G})$ of the residual gerbe $\mathcal{G}$ of a vector bundle. This is achieved by showing that the category of coherent sheaves whose objects are in the gerbe are equivalent to a category of projective modules of fixed rational rank. Combining the estimates of $\mathrm{trdeg}_k(k(E))$ and $\mathrm{ed}_{k(E)}(\mathcal{G})$ they arrive at an upper bound for $\mathrm{ed}_k(E)$. We show that all of these computations carry over to the root stack $\mathfrak{X}$ associated to a curve and hence by Borne's theorem 2.24 carries over to parabolic vector bundles.

In this thesis we give upper bounds for the essential dimension of the stack of parabolic bundles over a smooth curve. Parabolic bundles were introduced

by Mehta and Seshadri in [MS80]. A parabolic vector bundle $E_{.}$ on a curve X can be roughly thought of as a vector bundle E over a curve with a partial flag attached to the fibers of E at a finite number of points on the curve. This definition is generalized to higher dimensional varieties by Maruyama, Yokagawa and Simpson [Mar16], [Yok95]. Suppose X is a scheme on which a positive integer r is invertible and D an effective cartier divisor. In [Bor07] Borne proves that the category of parabolic bundles over the pair (X, D) is tensor equivalent to the category of vector bundles on a 'stack of r-th roots', denoted by $\mathfrak{X}$.

The key idea in this thesis is to use Borne's theorem to carry out the arguments by Biswas, Dhillon and Hoffmann in [BDH18]. In this pursuit we need a Hirzebruch-Riemann-Roch theorem for the root stack $\mathfrak{X}$. For this, we invoke a Grothendieck-Riemann-Roch theorem for Deligne-Mumford stacks proven by B.Toen in his PhD thesis [Toe].

This thesis is structured as follows: In chapter 2 we establish the essential priliminaries required to establish the main result. This includes definitions and properties of parabolic coherent sheaves, root stacks associated to a curve and, essential dimension. We also prove mild generalizations of relevant results of several authors mentioned before. The main theme, for the most part, is to show that all the arguments that work for a curve also work for the associated root stack. In particular, we check that the Krull-Schmidt theorem works for coherent sheaves (Theorem 2.54) on a root stack and the stack of coherent sheaves on a root stack is a smooth algebraic stack of the expected dimension (Theorem 2.62).

In chapter 3, we go through the arguments required to prove Hirzebruch-Riemann-Roch theorem for root stacks. Here, we use the cohomology theory of Gersten. This is done for line bundles over an orbifold in Toen's thesis [Toe],

we run the argument for vector bundles on the root stack and get an answer in terms of the 'parabolic data' associated to the vector bundle, see Theorem 3.19.

In chapter 4, we prove an equivalence between the category of coherent sheaves on the root stack $\mathfrak{X}$ which are objects of a residual gerbe and the category of projective modules of rational rank over a certain endomorphism algebra. This is a straightforward generalization of section 5 in [BDH18]. This allows us to reduce the problem of finding the essential dimension of the residual gerbe to that of projective modules.

In chapter 5, we construct some twisted sheaves of certain required rank on the residual gerbe of a parabolic bundle. This allows us to prove that the index of the gerbe divides the g.c.d of rank, degree and the parabolic data. This completes the required results needed to prove the main result.

Chapter 2

Preliminaries

2.1 Parabolic bundles

Let X be a scheme, D a divisor on X and $r \geq 1$ an integer. The following definition of parabolic sheaves is due to Maruyama and Yokogawa [Mar16], [Yok95].

Definition 2.1 ([Bor07])**.** The category $\mathrm{PAR}_{\frac{1}{r}}(X, D)$ of parabolic sheaves with weights integer multiples of $\frac{1}{r}$ is defined as a category whose objects are couples $(\mathcal{E}_{\cdot}, j)$, where $\mathcal{E}_{\cdot} : (\frac{1}{r}\mathbb{Z})^{op} \to \mathrm{QCoh}(X)$ is a functor, and $j : \mathcal{E}_{\cdot} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \cong \mathcal{E}_{\cdot}[1]$ a natural isomorphism such that the following triangle commutes:

$$\mathcal{E}_{\cdot} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \xrightarrow{\quad j \quad} \mathcal{E}_{\cdot}[1] \qquad (2.1.1)$$

$$id_{\mathcal{E}_{\cdot}} \otimes i \searrow \quad \swarrow$$

$$\mathcal{E}_{\cdot}$$

where $i : \mathcal{O}(-D) \to \mathcal{O}$ is the canonical inclusion. Having such a natural isomorphism j for $\mathcal{E}_{\cdot}$ is called pseudo-periodicity.

The morphisms from $(\mathcal{E}_\cdot, j)$ to $(\mathcal{E}'_\cdot, j')$ are given by natural transformations $\alpha : \mathcal{E}_\cdot \to \mathcal{E}'_\cdot$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{E}_\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) & \xrightarrow{\ \ j\ \ } & \mathcal{E}_\cdot[1] \\
\downarrow & & \downarrow \\
\mathcal{E}'_\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) & \xrightarrow{\ \ j\ \ } & \mathcal{E}'_\cdot[1]
\end{array}
$$

Definition 2.2. The category $\operatorname{Par}_{\frac{1}{r}}(X, D)$ of parabolic coherent sheaves with rational weights of denominator r (resp., vector bundles) of finite rank is the full subcategory of $\operatorname{PAR}_{\frac{1}{r}}(X, D)$ whose objects are couples $(\mathcal{E}_\cdot, j)$ such that:

1. $\mathcal{E}_\cdot : (\frac{1}{r}\mathbb{Z})^{op} \to \operatorname{QCoh}(X)$ factors through $\mathcal{E}_\cdot : (\frac{1}{r}\mathbb{Z})^{op} \to \operatorname{Coh}(X)$ (resp., $\mathcal{E}_\cdot : (\frac{1}{r}\mathbb{Z})^{op} \to \operatorname{Vect}(X)$)

2. For $\mathcal{E}_\cdot : (\frac{1}{r}\mathbb{Z})^{op} \to \operatorname{Vect}(X)$ to be a parabolic vector bundle we have the additional assumption that for $i \leq i' < i+1$, $\operatorname{coker}(\mathcal{E}_{i'} \to \mathcal{E}_i)$ is a locally free $\mathcal{O}_D$-module.

Remark 2.3. 1. When $\mathcal{E}_\cdot$ is a parabolic vector bundle, the canonical morphisms $\mathcal{E}_{i'} \to \mathcal{E}_i$ for $i \leq i' < i+1$ are monomorphisms. This follows from the commuting triangle describing the pseudo-periodicity 2.1.1.

2. Notice that for a coherent parabolic sheaf $\mathcal{E}_\cdot$ the $\operatorname{coker}(\mathcal{E}_{i'} \to \mathcal{E}_i)$ for $i \leq i' < i+1$ are coherent $\mathcal{O}_D$-modules but not necessarily locally free.

Parabolic coherent sheaves outside the support of D are equivalent to ordinary coherent sheaves on the scheme. We make this precise in the following Lemma.

Lemma 2.4. *Let $\mathcal{E}_\cdot$ be a parabolic coherent sheaf on X and $p \in X - D$, then the stalks of all the pieces of the filtration $(\mathcal{E}_{\frac{i}{r}})_p$ for $\frac{i}{r} \in \frac{1}{r}\mathbb{Z}$ are isomorphic to $(\mathcal{E}_0)_p$.*

Proof. This follows directly from the commuting triangle 2.1.1. We show this for $r = 2$. In this case, $\mathcal{E}_.$ is given by a sequence of sheaves

$$\ldots \to \mathcal{E}_1 \xrightarrow{f_1} \mathcal{E}_{\frac{1}{2}} \xrightarrow{f_0} \mathcal{E}_0 \to \ldots \tag{2.1.2}$$

By the existence of the commuting triangle 2.1.1 we have the following commuting diagram, where j is an isomorphism

$$\tag{2.1.3}$$

Taking stalks at $p \in X - D$, since $\mathcal{O}(-D)_p = \mathcal{O}_p$, we see that $(\mathcal{E}_0 \otimes \mathcal{O}(-D))_p \to (\mathcal{E}_0)_p$ is just the identity. Hence we get

$$(f_0)_p(f_1)_p = j_p \implies (f_0)_p(f_1)_p j_p^{-1} = 1 \tag{2.1.4}$$

which gives a right inverse for $(f_0)_p$. Repeating this argument for the following triangle which exists again by 2.1.1.

$$\tag{2.1.5}$$

we get a left inverse for f_0. □

Definition 2.5 (Seshadri, [MS80])**.** Let X be a smooth algebraic curve over a field k and D be a reduced effective divisor. A parabolic bundle $E_.$ is a vector bundle E along with the following data:

1. For each $p \in |D|$, there exists a filtration by subspaces

$$E(p) \supset F_{1,p} \supset \ldots \supset F_{n_p+1,p} = 0 \tag{2.1.6}$$

2. A set of rational number $(\alpha_{i,p})_{1 \leq i \leq n_p}$, called weights, associated to each $F_{i,p}$ such that

$$0 \leq \alpha_{1,p} \leq \ldots \ldots \leq \alpha_{n_p,p} < 1$$

Let $\mathrm{SPar}_{\frac{1}{r}}(X, D)$ denote the category of parabolic vector bundles with $n_p = r$ and $\alpha_{i,p} = \frac{i}{r}$ for $1 \leq i \leq r$.

Remark 2.6. Given a parabolic bundle in the sense of Maruyama and Yokogawa, i.e., an object $E_. \in \mathrm{Par}_{\frac{1}{r}}(X, D)$, we can construct a parabolic vector bundle in the sense of Seshadri [MS80] (see the above definition) as follows: By Remark 2.3, $C_i := \mathrm{coker}(E_{\frac{i}{r}} \to E_0)$ is a locally free $\mathcal{O}_D$-module. For $p \in |D|$, the stalks $(C_i)_p$ define a partial flag of the fiber $E_0(p) := E_0 \otimes_{\mathcal{O}_X} \kappa(p)$ at p. Conversely, given a parabolic vector bundle $(E_., F_{i,p})$ in the sense of Seshadri, with weights $\{\frac{i}{r}\}_{0 \leq i \leq r}$, we may define $E_{\frac{i}{r}}^p := \ker(E \to E(p)/F_{i,p})$. The vector bundles $E_{\frac{i}{r}} := \cap_{p \in |D|} E_{\frac{i}{r}}^p$ with canonical inclusions, along with pseudo-periodicity imposed on them, define an object in $Par_{\frac{1}{r}}(X, D)$.

Theorem 2.7. *The functor*

$$\mathrm{Par}_{\frac{1}{r}}(X, D) \xrightarrow{F} \mathrm{SPar}_{\frac{1}{r}}(X, D) \tag{2.1.7}$$

$$E_. \mapsto (E_0, \mathrm{coker}(g_i), \{\frac{i}{r}\}_{1 \leq i \leq r}) \tag{2.1.8}$$

as described in the above remark is an equivalence of categories. Here g_i : $E_{\frac{i}{r}} \to E_0$ are the canonical maps coming from the filtration of $E_.$.

Proof. Essential surjectivity of F follows from Remark 2.6.

Let $\mathrm{Hom}(E_., E'_.)$ be morphisms in $\mathrm{Par}_{\frac{1}{r}}(X, D)$. Suppose $f : E_. \to E'_.$ is a morphism

$$E_1 \dots \longrightarrow E_{\frac{i}{r}} \dots \longrightarrow E_{\frac{1}{r}} \longrightarrow E_0$$

$$\downarrow f_1 \qquad\qquad \downarrow f_{\frac{i}{r}} \qquad\qquad \downarrow f_{\frac{1}{r}} \qquad\qquad \downarrow f_0$$

$$E'_1 \dots \longrightarrow E'_{\frac{i}{r}} \dots \longrightarrow E'_{\frac{1}{r}} \longrightarrow E'_0$$

whose underlying morphism of vector bundles f_0 is 0. Since $E_{\frac{i}{r}} \xrightarrow{g_i} E_0$ are monomorphisms, by naturality of f, the morphisms $f_{\frac{i}{r}}$ are restrictions of f_0. Hence $f_{\frac{i}{r}}$ are also zero morphisms. This shows that F is faithful.

Now suppose $(E_0, \operatorname{coker}(g_i)) \rightarrow (E'_0, \operatorname{coker}(g'_i))$ is a morphism in $SPar_{\frac{1}{r}}(X, D)$. Then we have the following commutative diagram whose rows are exact

$$0 \longrightarrow E_{\frac{i}{r}} \longrightarrow E_0 \xrightarrow{t_i} \operatorname{coker}(g_i) \longrightarrow 0$$

$$\downarrow \qquad\qquad \vdots\, f_{\frac{i}{r}} \qquad\quad \downarrow f_0 \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$0 \longrightarrow E'_{\frac{i}{r}} \longrightarrow E'_0 \xrightarrow{t'_i} \operatorname{coker}(g'_i) \longrightarrow 0$$

The dotted arrow exists such that the diagram commutes. This is because of the fact that $E_{\frac{i}{r}}$ and $E'_{\frac{i}{r}}$ are kernels of t_i and t'_i respectively.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.2 Deligne-Mumford stacks

Definition 2.8 ([LMB00])**.** An algebraic S-stack is a stack $\mathfrak{X}$ that satisfies the following axioms:

(i) The diagonal 1-morphism of S-stacks

$$\mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$$

is representable and quasi-compact.

(ii) There exists an S-algebraic space X and a 1-morphism of S-stacks

$$X \xrightarrow{P} \mathfrak{X}$$

(automatically representable by (i)), that is surjective and smooth.

The 1-morphism P in (ii) is called a presentation of $\mathfrak{X}$. A Deligne-Mumford S-stack is an algebraic S-stack admitting an etale presentation. From now on we abbreviate Deligne-Mumford stacks as DM-stacks.

Definition 2.9. A DM-stack $\mathfrak{X}$ over a field k is called an orbifold if it is generically an algebraic space i.e, there exists an open dense substack which is representable by an algebraic space over k.

We record a Lemma about root stacks which we will prove in the following subsection. In particular Lemma 2.19 and consequently 4.5 imply this.

Lemma 2.10. *Let X be a smooth projective curve over k.The root stack $\mathfrak{X} = X_{(\mathcal{L},s,r)}$ is an orbifold over* $\mathrm{spec}(k)$.

2.3   Root stacks

In this section we will define and review properties of root stacks associated to a scheme X over $\mathrm{spec}(k)$. All the material here can be found in [Bor07] or [Cad07].

Let $r \geq 1$ be an integer, k a field whose characteristic is coprime to r, $X \to \mathrm{spec}(k)$ a noetherian scheme, $\mathcal{L}$ an invertible sheaf over X, along with a section $s \in H^0(X, \mathcal{L})$. A section of the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ over T is equivalent to a principal $\mathbb{G}_m$-bundle $P \to X$ along with a $\mathbb{G}_m$-equivariant map $P \to \mathbb{A}^1$. Let $\theta_r : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be a morphism of stacks defined by the association $(\mathcal{M}, t) \to (\mathcal{M}^{\otimes r}, t^{\otimes r})$. The sections of $[\mathbb{A}^1/\mathbb{G}_m]$ over a given

scheme as pairs of line bundle and a section $(\mathcal{M}, t)$ is made precise by the following Lemma 2.11.

Lemma 2.11. *[Example 5.13, [Ols03]] There is a correspondence between sections of $[\mathbb{A}^1/\mathbb{G}_m]$ over a scheme S and pairs $(\mathcal{M}, t)$ where $\mathcal{M}$ is an invertible sheaf over S and t a global section of $\mathcal{M}$.*

Definition 2.12. With the notation as above the stack of r-th roots, also called a root stack, is defined as

$$X_{(\mathcal{L}, s, r)} := X \times_{[\mathbb{A}^1/\mathbb{G}_m], \theta_r} [\mathbb{A}^1/\mathbb{G}_m]$$

Remark 2.13. If $Y \xrightarrow{p} X$ is a morphism of k-schemes. Then $Y \times_X X_{(\mathcal{L}, s, r)} = Y \times_X X \times_{[\mathbb{A}^1/\mathbb{G}_m], \theta_r} [\mathbb{A}^1/\mathbb{G}_m] \cong Y_{(p^*\mathcal{L}, p^*s, r)}$. Indeed, the composition $Y \to X \to [\mathbb{A}^1/\mathbb{G}_m]$ corresponds precisely to the pulled back couple $(p^*\mathcal{L}, p^*s)$ on Y.

Remark 2.14. $X_{(\mathcal{L}, s, r)}$ as a category fibered in groupoinds over $\mathrm{spec}(k)$ can be described as follows:

1. The objects of $X_{(\mathcal{L}, s, r)}$ are the quadruples $(f, \mathcal{M}, t, \varphi)$ where $f : S \to X$ is a morphism of schemes, $\mathcal{M}$ an invertible sheaf over S, $t \in H^0(X, \mathcal{L})$, and $\varphi : \mathcal{M}^{\otimes r} \cong f^*\mathcal{L}$ an isomorphism such that $\varphi(t^{\otimes r}) = f^*s$,

2. The morphisms from $(f, \mathcal{M}, t, \varphi)$ to $(g, \mathcal{N}, u, \psi)$ (over T) are the couples (h, ρ), where $h : S \to T$ is a morphism of schemes such that $gh = f$, and $\rho : \mathcal{M} \cong h^*\mathcal{N}$ is an isomorphism of sheaves such that $\rho(t) = h^*(u)$ and the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{M}^{\otimes r} & \xrightarrow{\ \rho^{\otimes r}\ } & h^*\mathcal{N}^{\otimes r} \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle h^*\psi} \\
f^*\mathcal{L} & \xrightarrow{\ =\ } & h^*g^*\mathcal{L}
\end{array}
$$

If there exist an 'r-th root' of $\mathcal{L}$ on X, then the root stack $X_{(\mathcal{L},s,r)}$ can be realized as a global quotient. This is made precise in the following theorem due to Cadman. Having such a realization helps us prove Lemma 2.19, which further helps with local arguments on the root stack. For example the restriction of a quasi-coherent sheaf $\mathcal{F}$ on $X_{(\mathcal{L},s,r)}$ to $[\mathrm{spec}(R[T]/(T^r - s))/\mu_r]$ is equivalent to a $\mathbb{Z}/r$-graded $R[T]/(T^r - s)$-module.

Theorem 2.15 (3.4, [Bor07]). *If there exists a line bundle $\mathcal{N}$ on X and an isomorphism $\psi : \mathcal{N}^{\otimes r} \cong \mathcal{L}$ then there exists a k-scheme Y such that $[Y/\mu_r] \cong X_{(\mathcal{L},s,r)}$. Moreover, $Y = \mathrm{spec}(\frac{Sym(\mathcal{N}^{\vee})}{(\mathcal{N}^{\vee})^{\otimes}\cong\mathcal{L}^{\vee})})$ is the relative spec of the symmetric algebra $Sym(\mathcal{N}^{\vee})$ whose relations are given via $\psi^{\vee}$.*

Example 2.16. Let $X = \mathbb{A}^1$, $\mathcal{L} = \mathcal{O}_X$ and $s = x$. Then $X_{(\mathcal{L},s,r)} = [\mathbb{A}^1/\mu_r]$, where $\mu_r \subset k^*$ is the groups of r-th roots of unity. Indeed, for a given k-scheme T, the sections $[\mathbb{A}^1/\mu_r](T)$ are given by principal μ_r-bundles $P \to T$ along with μ_r-equivariant morphisms $P \to \mathbb{A}^1$. If $g_{ij} : U_{ij} \to \mu_r$ are the transition functions of P, then by composing g_{ij} with the inclusion $\mu_r \subset \mathbb{G}_m$, the principal μ_r-bundle $P \to T$ can be interpreted as a line bundle $\mathcal{M}$ over T whose r-th tensor power is trivial. Moreover, the morphism $P \to \mathbb{A}^1$ corresponds to a section t of this line bundle $\mathcal{M}$. These tuples $(\mathcal{M}, t)$ are precisely the sections of $X_{(\mathcal{L},s,r)}(T)$.

Definition 2.17. Given a root stack $X_{(\mathcal{L},s,r)}$ as described above there is a canonical 1-morphism of stacks

$$\pi : X_{(\mathcal{L},s,r)} \to X \tag{2.3.1}$$

$$(f, \mathcal{M}, t, \varphi) \mapsto f \tag{2.3.2}$$

Lemma 2.18 ([Bor07]). *The morphism $\pi : X_{(\mathcal{L},s,r)} \to X$ is proper.*

Lemma 2.19. *[Corollary 3.6, [Bor07]] For each $x \in X$ there exists an affine open $U = \mathrm{spec}(R) \overset{i}{\hookrightarrow} X$ and an $s \in R$ such that the following square is cartesian*

$$
\begin{array}{ccc}
[\mathrm{spec}(R[T]/(T^r - s))/\mu_r] & \longrightarrow & \mathrm{spec}(R) \\
\big\downarrow & & \big\downarrow i \\
X_{(\mathcal{L},s,r)} & \overset{\pi}{\longrightarrow} & X
\end{array}
$$

Proof. Choose $U = \mathrm{spec}(R) \subset X$ which trivializes $\mathcal{L}$. We have an R-module isomorphism given by

$$R^{\otimes r} \overset{\psi}{\to} R \tag{2.3.3}$$

$$m_1 \otimes \ldots \otimes m_r \mapsto m_1 \ldots m_r \tag{2.3.4}$$

This gives a line bundle $\mathcal{N}$ on U which is an r-th root of $\mathcal{L}|_U$. Now the Lemma follows by Theorem 2.15.

$\square$

Theorem 2.20 (Theorem 2.3.3, [Cad07])**.** *Let $X \to \mathrm{spec}(k)$ be a scheme over which r is invertible. Then $\mathfrak{X} = X_{(\mathcal{L},s,r)}$ is a Deligne-Mumford stack over $\mathrm{spec}(k)$.*

2.4   Sheaves on root stacks

We do not introduce the generalities of sheaves on stacks. These can be found in for example Chapter 12 and 13 of [LMB00] or [Sta18, Tag 06TF] However, we will describe explicitly the necessary definitions in the particular case of a root stack.

The category $X_{(\mathcal{L},s,r)}$ inherits a topology from $(Sch/k)_{fppf}$. Explicitly, we say that the family $\{(U_i \to X, \mathcal{M}_i, t_i, \varphi_i)\}_{i \in I} \xrightarrow{f_i} \{(U \to X, \mathcal{M}, t, \varphi)\}$ is a covering family if $\{U_i \to U\}_{i \in I}$ is a covering family in $(Sch/k)_{fppf}$. Since $X_{(\mathcal{L},s,r)}$ is a category fibered in groupoids, every morphism in $X_{(\mathcal{L},s,r)}$ is strongly cartesian and hence by Lemma 10.1 in [Sta18, Tag 06NU] these covering families give $X_{(\mathcal{L},s,r)}$ the structure of a site. By a sheaf on $X_{(\mathcal{L},s,r)}$ we mean the sheaf on this site. The structure sheaf $\mathcal{O}_{X_{(\mathcal{L},s,r)}}$ is given by a sheaf of rings $x := (U \to X, \mathcal{M}, t, \varphi) \mapsto \Gamma(U, \mathcal{O}_U)$. By an $\mathcal{O}_{X_{(\mathcal{L},s,r)}}$-module $\mathcal{F}$ we mean a sheaf on $X_{(\mathcal{L},s,r)}$ such that $\mathcal{F}(x)$ is an $\mathcal{O}_{X_{(\mathcal{L},s,r)}}(x)$-module for each $x \in X_{(\mathcal{L},s,r)}$. Note that $\mathcal{O}_{X_{(\mathcal{L},s,r)}}(x) = \Gamma(U, \mathcal{O}_U)$. We say such $\mathcal{F}$ is coherent (resp., locally free) $\mathcal{O}_{X_{(\mathcal{L},s,r)}}$-module if each $\mathcal{F}(x)$ is a coherent (resp., locally free) $\Gamma(U, \mathcal{O}_U)$-module. We also say $\mathcal{F}$ is a vector bundle over $X_{(\mathcal{L},s,r)}$ if $\mathcal{F}$ is a locally free $\mathcal{O}_{\mathfrak{X}}$-module.

2.4.1 Pushforward

The structure sheaf described above 2.4 makes the pair $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ a ringed site. We denote the category of sheaves of modules on the ringed site $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ by $\mathrm{Mod}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Consider the structure morphism $p : \mathfrak{X} \to \mathrm{spec}(k)$. There is a canonical pushforward morphism $p_* : \mathrm{Mod}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to \mathrm{Mod}(\mathrm{spec}(k), \mathcal{O}_{\mathrm{spec}(k)})$ defined in Section 12.5 of [LMB00]. Using this, one can define the global section functor in the usual way.

Theorem 2.21. *[Sta18, Tag 01DU] The category of sheaves of modules* $\mathrm{Mod}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ *on the ringed site* $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ *has enough injectives.*

Definition 2.22. Let $\mathcal{F}$ be an $\mathcal{O}_{\mathfrak{X}}$-module. The sheaf cohomology of $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ with respect to $\mathcal{F}$ is given by

$$H^i(\mathfrak{X}, \mathcal{F}) := R^i\Gamma(\mathfrak{X}, \mathcal{F}) \qquad (2.4.1)$$

Definition 2.23. The invertible sheaf $\mathcal{N}$ on $X_{(\mathcal{L},s,r)}$ as $(U \to X, \mathcal{M}, t, \varphi) \mapsto \mathcal{M}$.

This following theorem and its generalization to coherent sheaves over $X_{(\mathcal{L},s,r)}$ is the one of the most important ideas used to prove the main result of this thesis.

Theorem 2.24 (N.Borne [Bor07])**.** *Let r be a natural number, $X \to \mathrm{spec}(\mathbb{Z}[r^{-1}])$ a noetherian scheme, D an effective cartier divisor, $\mathcal{L} = \mathcal{O}_X(D)$, s the canonical section, then the functor*

$$\mathrm{Vect}(X_{(\mathcal{L},s,r)}) \xrightarrow{F} \mathrm{Par}_{\frac{1}{r}}(X, D) \qquad (2.4.2)$$

given by $F(\mathcal{F}) : (\frac{1}{r}\mathbb{Z})^{op} \longrightarrow Vect(X)$, $\frac{l}{r} \mapsto \pi_*(\mathcal{N}^{\otimes(-l)} \otimes_{\mathcal{O}_{X_{(\mathcal{L},s,r)}}} \mathcal{F})$ is an equivalence of tensorial categories.

Remark 2.25. The equivalence stated above extends to coherent sheaves on $X_{(\mathcal{L},s,r)}$ and parabolic coherent sheaves. See [BV12].

$$\mathrm{Coh}(X_{(\mathcal{L},s,r)}) \xrightarrow{F} \mathrm{CohPar}_{\frac{1}{r}}(X, D) \qquad (2.4.3)$$

2.4.2 Euler Characterisitic

Definition 2.26. Let $K \supset k$ be a field and $\mathcal{E}_1, \mathcal{E}_2$ be coherent parabolic sheaves over X_K or equivalently coherent sheaves on a root stack $\mathfrak{X}_K$. We define an Euler characteristic as follows

$$\chi(\mathcal{E}_2, \mathcal{E}_1) := \dim_K \mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_1) - \dim_K \mathrm{Ext}^1(\mathcal{E}_2, \mathcal{E}_1)$$

Lemma 2.27. *Given short exact sequences* $0 \to \mathcal{E}_1' \to \mathcal{E}_1 \to \mathcal{E}_1'' \to 0$ *(resp.,* $0 \to \mathcal{E}_2' \to \mathcal{E}_2 \to \mathcal{E}_2'' \to 0$*) of coherent parabolic sheaves, the following formula(e) hold:*

$$\chi(\mathcal{E}_2, \mathcal{E}_1) = \chi(\mathcal{E}_2', \mathcal{E}_1) + \chi(\mathcal{E}_2'', \mathcal{E}_1) \ (resp., \ \chi(\mathcal{E}_2, \mathcal{E}_1) = \chi(\mathcal{E}_2', \mathcal{E}_1) + \chi(\mathcal{E}_2'', \mathcal{E}_1))$$

Proof. We may apply the functor $\mathrm{Hom}(\mathcal{E}_2, -)$ (resp., $\mathrm{Hom}(-, \mathcal{E}_1)$) and write a long exact sequence for Ext in both cases. We know that $\mathrm{Ext}^i(-, -)$ of coherent sheaves over $\mathfrak{X}_K$ vanish for $i \geq 2$ due to Proposition-4.26 and Remark 4.27. Now the Lemma easily follows from the long exact sequence(s). □

Lemma 2.28. *If* $\mathcal{E}_2$ *and* $\mathcal{E}_1$ *are vector bundles over* $\mathfrak{X}$ *then* $\chi(\mathcal{E}_2, \mathcal{E}_1) = \chi(\mathfrak{X}, \mathcal{H}om(\mathcal{E}_2, \mathcal{E}_1))$.

Proof. We know from Lemma 4.8 that $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{E}_2, \mathcal{E}_1) = \mathcal{E}_1 \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}_2^{\vee}$ when $\mathcal{E}_2$ and $\mathcal{E}_1$ are vector bundles. So $\mathrm{Ext}^1_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{E}_2, \mathcal{E}_1) = \mathrm{Ext}^1(\mathcal{O}_{\mathfrak{X}}, \mathcal{E}_1 \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}_2^{\vee}) = \mathrm{Ext}^1(\mathcal{O}_{\mathfrak{X}}, \mathcal{H}om(\mathcal{E}_2, \mathcal{E}_1)) = H^1(\mathfrak{X}, \mathcal{H}om(\mathcal{E}_2, \mathcal{E}_1))$. □

Lemma 2.29. *If a coherent sheaf* $\mathcal{E}$ *on* $\mathfrak{X}$ *is torsion free then it is locally free.*

Proof. The question is local. Consider the open substack $U' = [spec(R')/\mu_r] \subset \mathfrak{X}$, where $R' = R[T]/(T^r - s)$. The restriction of $\mathcal{E}$ to U' is given by a finitely generated graded torsion free R'-module M. We may assume R to be local, hence a DVR. This makes R' a DVR with $\bar{T} \in R'$ as a uniformizing parameter. By fundamental theorem of graded modules over a graded PID M is a free graded R'-module. □

Lemma 2.30. *Given a parabolic coherent sheaf* $\mathcal{E}$ *on* X *there exists a decomposition* $\mathcal{E} = \mathcal{F} \oplus \mathcal{T}$, *where* $\mathcal{F}$ *is torsion free (hence locally free) and* $\mathcal{T}$ *is supported at only finitely many points.*

Proof. This is known for coherent sheaves over curves (Lemma 5.2.2 of [LP97], Lectures on vector bundles). We use the same idea for parabolic coherent sheaves. Consider the canonical morphism $\varphi : \mathcal{E} \to \mathcal{E}^{\vee\vee}$. The image $\mathcal{F}$ of φ is given by $\mathcal{F}_{\frac{l}{r}} = im(\varphi)_{\frac{l}{r}} = im(\varphi_{\frac{l}{r}}) \subset \mathcal{E}^{\vee\vee}_{\frac{l}{r}}$, for $\frac{l}{r} \in (\frac{1}{r}\mathbb{Z})^{op}$, the morphisms $\varphi_{\frac{l'}{r}} \to \varphi_{\frac{l}{r}}$ for $l+r > l' \geq l$ and the pseudo-isomorphisms are naturally induced. $\mathcal{F}$ is clearly torsion free. Let $\mathcal{T} := \ker(\varphi)$. Now we get the short exact sequence

$$0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{F} \to 0 \tag{2.4.4}$$

We claim that this short exact sequence splits. Consider the $\frac{l}{r}$-th piece of the filtration

$$0 \to \mathcal{T}_{\frac{l}{r}} \to \mathcal{E}_{\frac{l}{r}} \to \mathcal{F}_{\frac{l}{r}} \to 0 \tag{2.4.5}$$

Applying $\Gamma\mathcal{H}om(\mathcal{F}, -)$ and writing the long exact sequence for cohomology, we get the exact sequence

$$H^0(\mathcal{H}om(\mathcal{F}_{\frac{l}{r}}, \mathcal{E}_{\frac{l}{r}})) \to H^0(\mathcal{H}om(\mathcal{F}_{\frac{l}{r}}, \mathcal{F}_{\frac{l}{r}})) \to H^1(\mathcal{H}om(\mathcal{F}_{\frac{l}{r}}, \mathcal{T}_{\frac{l}{r}}))$$

The last term vanishes because $\mathcal{H}om(\mathcal{F}_{\frac{l}{r}}, \mathcal{T}_{\frac{l}{r}})$ is supported only at finitely many points. Taking inverse image of $id \in \mathrm{Hom}(\mathcal{F}_{\frac{l}{r}}, \mathcal{F}_{\frac{l}{r}}) = H^0(\mathcal{H}om(\mathcal{F}_{\frac{l}{r}}, \mathcal{F}_{\frac{l}{r}}))$ we get a section $s_{\frac{l}{r}} : \mathcal{F}_{\frac{l}{r}} \to \mathcal{E}_{\frac{l}{r}}$ of the short exact sequence 2.4.5. It can be easily verified that these sections are compatible with the morphisms $\varphi_{\frac{l'}{r}} \to \varphi_{\frac{l}{r}}$ for $l + r > l' \geq l$ giving a section of 2.4.4. $\qquad\square$

Proposition 2.31. *Let R be a DVR, π a uniformizing parameter and $K = R/\pi$ its residue field. Then*

(i.) *If $m, n \geq 1$ then $\mathrm{Hom}_R(R/\pi^n, R/\pi^m) \cong \mathrm{Ext}^1_R(R/\pi^n, R/\pi^m) = R/\pi^k$, where $k = min\{m, n\}$.*

(ii.) *Let $r \geq 1$ be a natural number. The ring $R' = R[t]/(t^r - \pi)$ is also a DVR with t as a uniformizing parameter, we have $\mathrm{Hom}_{R'}(R'/t^m, R'/t^n)^{\mu_r} = \mathrm{Ext}^1_{R'}(R'/t^m, R'/t^n)^{\mu_r} = R/\pi^k$ as R-modules for some $k \in \mathbb{N}$. Here the morphisms and extensions are of graded R'-modules.*

Proof. Consider the short exact sequence of graded R'-modules

$$0 \to R' \xrightarrow{t^m} R' \to R'/t^m \to 0 \tag{2.4.6}$$

Applying the functor $\mathrm{Hom}_{R'}(-, R'/t^n)$ to the above short exact gives a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{R'}(R'/t^m, R'/t^n) \longrightarrow \mathrm{Hom}_{R'}(R', R'/t^n) \rightharpoondown$$

$$\hookrightarrow \mathrm{Hom}_{R'}(R', R'/t^n) \longrightarrow \mathrm{Ext}^1_{R'}(R'/t^m, R'/t^n) \longrightarrow \mathrm{Ext}^1_{R'}(R', R'/t^n) \longrightarrow \ldots$$

Since R' is a free R'-module, $\mathrm{Ext}^1_{R'}(R', R'/t^n) = 0$ and we have $\mathrm{Hom}_{R'}(R', R'/t^n) = R'/t^n$ and $\mathrm{Hom}_{R'}(R', R'/t^m) = R'/t^m$. We get, $\mathrm{Ext}^1_{R'}(R'/t^m, R'/t^n) = \mathrm{coker}(R'/t^n \xrightarrow{\times t^m} R'/t^n) = R'/t^k$, where $k = min\{m, n\}$. Taking μ_r-invariants, we get $(R'/t^k)^{\mu_r} = R/\pi^k$. $\qquad\square$

Proposition 2.32. *Suppose $\mathcal{F}$ is locally free and $\mathcal{T}$ is a torsion sheaf on $\mathfrak{X}$ then $-\chi(\mathcal{T}, \mathcal{F}) = \chi(\mathcal{F}, \mathcal{T}) = rk(\mathcal{F}) \deg(\mathcal{T})$*

Proof. $\mathcal{H}om(\mathcal{F}, \mathcal{T})$ is supported only at finitely many points, locally at the level of stalks it is easy to see that $\dim_K \mathrm{Hom}(\mathcal{F}, \mathcal{T}) = rk(\mathcal{F}) \deg(\mathcal{T})$, for example for a DVR R with uniformizing parameter π and residue field K, $\dim_K \mathrm{Hom}_R(R^n, R/\pi^m) = n dim_K(R/\pi^m)$. Since $\mathcal{F}$ is locally free and $\mathcal{T}$ is finitely supported we again have $\mathrm{Ext}^1(\mathcal{F}, \mathcal{T}) = 0$. Furthermore, $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ since $\mathcal{F}$ is torsion free and $\dim_K \mathrm{Ext}^1(\mathcal{T}, \mathcal{F}) = rk(\mathcal{F}) \deg(\mathcal{T})$ since $\mathrm{Ext}^1_R(R/\pi^m, R^n) = (R/\pi^m)^n \implies \dim_K \mathrm{Ext}^1_R(R/\pi^m, R^n) = n dim_K(R/\pi^m)$. $\qquad\square$

Corollary 2.33. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are torsion sheaves on $\mathfrak{X}$, i.e. coherent sheaves supported at finitely many points then $\chi(\mathcal{T}_2, \mathcal{T}_1) = 0$.*

Proof. We may think of these sheaves $\mathcal{T}_1$ and $\mathcal{T}_2$ as sheaves supported inside the open substack $U' = [(\mathrm{spec}(R')/\mu_r] \subset \mathfrak{X}$, where $R' = R[T]/(T^r - s)$. We may localize R' making it a DVR in this case. It is enough to show that $\dim_K \mathrm{Hom}_{R'}(M_1, M_2)^{\mu_r} = \dim_K \mathrm{Ext}^1_{R'}(M_1, M_2)^{\mu_r}$ for finitely generated modules over R'. This follows from 2.31. $\qquad\qquad\square$

The following theorem is a consequence of the previous computations and the Riemann-Roch theorem we prove in Chapter 4.

Theorem 2.34. *Let $\mathcal{E}$ be coherent sheaves on $\mathfrak{X}_K$ of generic rank r and degree d respectively. If g is the genus of X, then*

$$\chi(\mathcal{E}, \mathcal{E}) = (1 - g)r^2 - \Sigma_{p \in |D|} \dim_k \mathrm{Flag}_{\mathcal{K}}(p) \qquad (2.4.7)$$

Proof. Using Lemma 2.30 we may write $\mathcal{E} = \mathcal{F} \oplus \mathcal{T}$. By Lemma 2.27 we have that

$$\chi(\mathcal{E}, \mathcal{E}) = \chi(\mathcal{F}, \mathcal{F}) + \chi(\mathcal{F}, \mathcal{T}) + \chi(\mathcal{T}, \mathcal{F}) + \chi(\mathcal{T}, \mathcal{T}) \qquad (2.4.8)$$

Now the theorem follows from 4.9, 2.33 and 2.32.

$$\square$$

2.5   Endomorphism algebras of coherent sheaves

This section deals with the theory of endomorphism algebras of coherent sheaves as done in Section 4 and Section 2 of [BDH18] for the root stack $\mathfrak{X}$. This theory applies almost verbatim for the root stack because of the fact

that an algebra of endomorphisms of coherent sheaf on $\mathfrak{X}$ is a finite dimensional algebra over the base field as argued below.

Let X be a smooth projective curve over an algebraically closed field k whose charactaristic is coprime to a fixed integer $r \geq 1$. $\mathfrak{X} = X_{(\mathcal{L},s,r)}$ be the root stack and $K \supset k$ be a field. Let E be a coherent sheaf over the root stack $\mathfrak{X}_K$.

The structure morphism $p : \mathfrak{X}_K \to \operatorname{spec}(K)$ factors as

$$\mathfrak{X}_K \xrightarrow{\pi} X_K \xrightarrow{p_0} \operatorname{spec}(K)$$

Since π is proper, the pushforward $\pi_* \mathcal{F}$ is coherent over the curve X_K. Hence the pushforward $p_* \mathcal{F} = (p_0)_* \pi_* \mathcal{F}$ is a finite dimensional K-vector space. Hence the K-algebra $\operatorname{End}(E) = H^0(\mathfrak{X}, \mathcal{E}nd(E)) = H^0(\operatorname{spec}(K), p_* \mathcal{E}nd(E))$ is a finite dimensional K-algebra. Let $\mathfrak{j}(E)$ denote the Jacobson radical of the K-algebra $\operatorname{End}(E)$. Since $\operatorname{End}(E)/\mathfrak{j}(E)$ is semi-simple Artin-Wedderburn's theorem [Jac89] states that there exists an isomorphism

$$\operatorname{End}(E)/\mathfrak{j}(E) \cong \Pi_i M_{n_i \times n_i}(D_i) \tag{2.5.1}$$

for some finite dimensional division algebras D_i over K and some natural numbers $n_i \geq 1$.

A coherent sheaf E over $\mathfrak{X}_K$ is said to be indecomposable if

$$E \cong E_1 \oplus E_2 \implies E_1 = 0 \ or \ E_2 = 0 \tag{2.5.2}$$

We have the following analog of Atiyah's Krull-Schmidt type result

Lemma 2.35. *If E is an indecomposable coherent sheaf over $\mathfrak{X}_K$ then* $\operatorname{End}(E)/\mathfrak{j}(E)$ *is a division ring D.*

Proof. By Lemma 2.30 E is either a locally free $\mathcal{O}_{\mathfrak{X}}$-module or torsion. If E is locally free then $\mathrm{End}(E)/\mathfrak{j}(E) \subset \mathrm{End}(E_0)/\mathfrak{j}(E_0)$ where E_0 is the underlying locally free sheaf on the curve X. If E is torsion then it is enough to prove the statement for the graded R'-module $E = R'/\pi^k$ for $R' = R[T]/(T^r - s)$, $k \geq 1$ where R is a DVR. If $f : E \to E$ is an endomorphism then either $\bar{1}$ goes to a unit or a power of π in R'/π^k. Hence the endomorphism is either a unit or nilpotent respectively. $\square$

The rest of the results of the section are from [BDH18] with occasionally more elaborate explanations. We include these here to make the exposition self complete.

Let R be a ring which is not necessarily commutative and $\mathfrak{n} \subset R$ be a nilpotent two sided ideal.

Lemma 2.36. *Every element $q \in R/\mathfrak{n}$ such that $q^2 = q$ admits a lift $p \in R$ with $p^2 = p$.*

Proof. Since $\mathfrak{n}$ is nilpotent, we have that $\mathfrak{n}^k = 0$ for some integer $k \geq 1$. By induction we may assume that $k = 2$. Let $p \in R$ be some lift of q, then $p^2 \equiv p \ mod \ \mathfrak{n}$. Since $\mathfrak{n}^2 = 0$ we have $(p^2 - p)^2 = 0$. Also, $p^2 \equiv p \ mod \ \mathfrak{n} \implies p^3 \equiv p^2 \ mod \ \mathfrak{n}$. Using both congruences we have that $p' = 3p^2 - 2p^3$ is another lift of q. Computing $(p')^2$, we get

$$(p')^2 = 4p^6 - 12p^5 + 9p^4 = (p^2 - p)^2(4p^2 - 4p - 3) - 2p^3 + 3p^2 = p'. \quad (2.5.3)$$

Hence $p' \in R$ is the required lift of $p \in R/\mathfrak{n}$ such that $(p')^2 = p'$. $\square$

Corollary 2.37. *Let N be a finitely generated projective $(R/\mathfrak{n})$- module. Then there is a finitely generated projective R-module M such that $M/M\mathfrak{n} \cong N$. The finitely generated projective R-module M is unique up to isomorphisms.*

Proof. Since N is a finitely generated projective $R/\mathfrak{n}$-module, there exists an $R/\mathfrak{n}$-module N' and an isomorphism of $R/\mathfrak{n}$-modules $\varphi : N \oplus N' \xrightarrow{\simeq} (R/\mathfrak{n})^r$ for some $r \in \mathbb{N}$. So N is the image of a matrix $q \in M_{r \times r}(R/\mathfrak{n})$ such that $q^2 = q$. Let $\mathfrak{N} \subset M_{r \times r}(R)$ be the two sided ideal of matrices with entries in $\mathfrak{n} \subset R$. Since $\mathfrak{n}$ is nilpotent, $\mathfrak{N}$ is also nilpotent. Using Lemma 2.36 and the fact that $M_{r \times r}(R/\mathfrak{n}) \subset M_{r \times r}(R)/M_{r \times r}(R)\mathfrak{N}$ we may lift the matrix q to a matrix $p \in M_{r \times r}(R)$ such that $p^2 = p$. The image of p is a finitely generated R-module M such that $M/M\mathfrak{n} \cong N$. Suppose M' is another finitely generated projective R-module such that $M'/M'\mathfrak{n} \cong N$. Since M and M' are direct summands of free R-modules, the induced isomorphism between $M/M\mathfrak{n}$ and $M'/M'\mathfrak{n}$ can be lifted to an automorphism f of $R/\mathfrak{n}$ modules. $\qquad\square$

Definition 2.38. A projective module M over a right artinian ring R has rank $r \in \mathbb{Q}_{>0}$ if the direct sum $M^{\oplus n}$ is free of rank nr for some $n \in \mathbb{N}$ such that $nr \in \mathbb{N}$.

For a right artinian ring R let $\mathfrak{j}$ be its jacabson radical. $\mathfrak{j}$ is the smallest ideal such that $R/\mathfrak{j}$ is semi-simple. According to Wedderburn's theorem it can be decomposed as

$$R/\mathfrak{j} \cong M_{n_1 \times n_1}(D_1) \times \ldots \times M_{n_s \times n_s}(D_s) \tag{2.5.4}$$

for some division rings D_i and integers n_i for $1 \leq i \leq s$.

Proposition 2.39. *(i.) Let $d_R := gcd(n_1, \ldots, n_s)$. There exists a finitely generated R-module M_R such that*

$$M_R/M_R\mathfrak{j} \cong M_{\frac{n_1}{d_R} \times n_1}(D_1) \times \ldots \times M_{\frac{n_s}{d_R} \times n_s}(D_s) \tag{2.5.5}$$

(ii.) If M is a projective R-module of rank $r \in \mathbb{Q}_{>0}$, then $r = n/d_R$ and $M \cong M_R^{\oplus n}$ for some integer $n \geq 1$.

Proof. The right $R/\mathfrak{j}$-module $N = M_{\frac{n_1}{d_R} \times n_1}(D_1) \times \ldots \times M_{\frac{n_s}{d_R} \times n_s}(D_s)$ is clearly a direct summand of $R/\mathfrak{j} \cong M_{n_1 \times n_1}(D_1) \times \ldots \times M_{n_s \times n_s}(D_s)$. Hence N is projective. Applying Corollary 2.37 proves (i).

If M is a projective R-module of rank $r \in \mathbb{Q}_{>0}$ then there exists an $n \in \mathbb{N}$ such that $M^{\oplus n} \cong R^{nr}$, such that $nr \in \mathbb{N}$. So we have $(M/M\mathfrak{j})^n \cong (R/\mathfrak{j})^{nr}$. By uniqueness of the decomposition, we conclude that

$$M/M\mathfrak{j} \cong M_{n_1 r \times n_1}(D_1) \times \ldots \times M_{n_s r \times n_s}(D_s)$$

with $n_i r \in \mathbb{N}$, for $1 \leq i \leq s$. We may write $r = n/d_R$ for some $n \geq 1$. So, from (i) we have $M/M\mathfrak{j} \cong (M_R/M_R\mathfrak{j})^{\oplus n}$. Applying Corollary 2.37 we conclude that $M \cong M_R^{\oplus n}$.

$\square$

Lemma 2.40. *Let E be a coherent sheaf over $\mathfrak{X}_K$ such that $\mathrm{End}(E)/\mathfrak{j}(E) = A_1 \oplus A_2$ as K-algebras. Then there exists a decomposition $E = E_1 \oplus E_2$ such that $A_1 \cong \mathrm{End}(E_1)/\mathfrak{j}(E_1)$ and $A_2 \cong \mathrm{End}(E_2)/\mathfrak{j}(E_2)$ as K-algebras.*

Proof. By the hypothesis, there exists an element $q \in \mathrm{End}(E)/j(E)$ corresponding to $(1,0)$ such that $q^2 = q$. By applying Lemma 2.36, there exists an element $p \in \mathrm{End}(E)$ such that $p^2 = p$. Hence E decomposes as $im(p) \oplus im(1-p)$. Let $E_1 = im(p)$ and $E_2 = im(1-p)$. An endomorphism $f \in \mathrm{End}(E_1)$ is equivalent to $E_1 \xrightarrow{p} E_1 \oplus E_2 \xrightarrow{g} E_1 \oplus E_2 \xrightarrow{p} E_1$. Hence $\mathrm{End}(E_1) = p\,\mathrm{End}(E)p$. Similarly $\mathrm{End}(E_2) = (1-p)\,\mathrm{End}(E)(1-p)$. Hence $\mathrm{End}(E')/j(E') \cong q(\mathrm{End}(E)/j(E))q \cong (1,0)(A_1 \times A_2)(1,0) \cong A_1$. $\square$

Lemma 2.41. *Let E be a coherent sheaf over $\mathfrak{X}_K$. Suppose we have a decomposition $\mathrm{End}(E)/j(E) \cong \Pi_i M_{n_i \times n_i}(D_i)$ as in 2.5.1. Then E admits a*

decomposition $E \cong \bigoplus_i E_i^{n_i}$ *into indecomposable coherent sheaves* E_i *such that* $\text{End}(E_i)/j(E_i) = D_i$.

Proof. By Lemma 2.40 we may assume that $\text{End}(E)/\mathfrak{j}(E)$ is simple, i.e., $\text{End}(E)/\mathfrak{j}(E) = M_{n \times n}(D)$ for some division ring D. Since $M_{1 \times n}(D)^{\oplus n} \cong M_{n \times n}(D)$ as $M_{n \times n}(D)$-algebras, $M_{1 \times n}(D)^{\oplus n}$ is a projective module of rank $1/n$ over $\text{End}(E)/\mathfrak{j}(E)$. By Lemma 2.37 we may lift $M_{1 \times n}(D)$ to a projective module M over $\text{End}(E)$ of rank $1/n$. The assignment $E_1 := M \otimes_{\text{End}(E)} E$ defines a coherent sheaf over $\mathfrak{X}_K$. Since $M^{\oplus n} \cong \text{End}(E)$, we have $E_1^{\oplus n} \cong E$. Therefore $\text{End}(E) = M_{n \times n}(\text{End}(E_1))$. Since $\text{End}(E)/j(E) = M_{n \times n}(D)$, we conclude that $\text{End}(E_1)/\mathfrak{j}(E_1) = D$. This also implies E_1 is indecomposable. $\qquad\square$

Lemma 2.42. *If* E *indecomposable vector bundle over the root stack* $\mathfrak{X}_K$ *then*

$$\dim_K \text{End}(E)/\mathfrak{j}(E) \le \text{rank}\,(E) \tag{2.5.6}$$

Proof. Since E is an indecomposable vector bundle, we have that $\text{End}(E)/\mathfrak{j}(E) = D$, a finite dimensional division algebra over K. Let $E_0 := \pi_* E$ denote the underlying vector bundle of E on the curve X_K. For any point $p \in X$, the fiber $(E_0)_p$ is a left module over $\text{End}(E) \subset \text{End}(E_0)$. Hence $\dim_K(D) \le \dim_K(\text{End}(E)) \le \dim_K(\text{End}(E_0)) \le \dim_K(E_0)_p = \text{rank}\,(E)$. $\quad\square$

2.6 Essential dimension

In this section we will recall the definition of essential dimension which is due to Merkurjev. We will compute the essential dimensional of projective modules of rank $r \in \mathbb{Q}_{>0}$ over a finite dimensional k-algebra A.

Let k be a field and denote by `Fields`/k the category of fields over k. Let F : `Fields`/k $\to$ `Sets` be a given functor. We say that an element $a \in F(K)$ is defined over an intermediate field $k \subset K' \subset K$ if there exists an element $a' \in F(K')$ whose image in $F(K)$ is a via the induced morphism $F(K') \to F(K)$.

Definition 2.43. (Merkurjev)

(1.) The essential dimension of an element $a \in F(K)$ is

$$\mathrm{ed}_k(a) := \inf_{K'} \ \mathrm{trdeg}_k K' \tag{2.6.1}$$

where the infimum is over all fields $k \subset K' \subset K$ over which a is defined.

(2.) The essential dimension of the functor F is

$$\mathrm{ed}_k(F) := \sup_a \mathrm{ed}_k(a) \tag{2.6.2}$$

where the supremum is taken over all fields $K \supset k$ and all elements $a \in F(K)$. We put $\mathrm{ed}_k(F) = -\infty$ if $F(K) = \emptyset$ for all K.

(3.) The essential dimension of a stack $\mathcal{M}$ over k is the essential dimension of the functor `Fields`/$k \to$ `Sets` that sends each field $K \supset k$ to the set of isomorphism classes of the groupoid $\mathcal{M}(K)$.

Definition 2.44. Let A be a finite dimensional k-algebra and $r \in \mathbb{Q}_{r>0}$. We define a functor denoted $\mathrm{Mod}_{A,r}$: `Fields` $\to$ `Sets` as follows:

For a field $K \supset k$, $\mathrm{Mod}_{A,r}(K)$ are the isomorphism classes of projective modules of rank $r \in \mathbb{Q}_{>0}$ over $A \otimes_k K$.

Giving upper bounds for the essential dimension of $\mathrm{Mod}_{A,r}$ for a finite dimensional k-algebra A can be reduced to giving upper bounds for the essential dimension of a division algebra as shown in the following proposition.

Proposition 2.45 (Proposition 3.2-3.5, [BDH18])**.** *(i.) If $\mathfrak{n} \subset A$ is a two-sided nilpotent ideal, then $\mathrm{ed}_k(\mathrm{Mod}_{A,r}) = \mathrm{ed}_k(\mathrm{Mod}_{A/\mathfrak{n}}, r)$.*

(ii.) If A is isomorphic to product of k-algebras A_i, then $\mathrm{ed}_k(\mathrm{Mod}_{A,r}) \leq \Sigma_i \mathrm{ed}_k(\mathrm{Mod}_{A_i,r})$.

(iii.) If $A \cong M_{n \times n}(B)$ for a k-algebra B, then $\mathrm{ed}_k(\mathrm{Mod}_{A,r}) = \mathrm{ed}_k(\mathrm{Mod}_{B,nr})$.

(iv.) If $r = n/d$ for co-prime integers $n, d \geq 1$, then $\mathrm{ed}_k(\mathrm{Mod}_{A,r}) = \mathrm{ed}_k(\mathrm{Mod}_{A,1/d})$.

Proposition 2.46. *(Corollary 3.7, [BDH18]) If A is a simple k-algebra, and $0 < r < 1$, then $\mathrm{ed}_k(\mathrm{Mod}_{A,r}) \leq r(1-r)\mathrm{dim}_k A$. Moreover, if $r\deg(A)$ is not an integer, then $\mathrm{ed}_k(\mathrm{Mod}_{A,r}) = -\infty$.*

Corollary 2.47. *If $A \cong M_{n \times n}(B)$ for some simple k-algebra B, then $\mathrm{ed}_k(\mathrm{Mod}_{A,r}) < nr(\mathrm{dim}_k B)$.*

Let $deg D$ denote the degree of a division algebra D and $v_p(n)$ denote the largest integer such that $p^{v_p(n)}$ divides n. We have

Theorem 2.48 (Corollary 3.8, [BDH18])**.** *Let D be a division algebra over k and let d be a positive integer which divides $\deg(D)$, then*

$$\mathrm{ed}_k(\mathrm{Mod}_{D,\frac{1}{d}}) \leq [l:k]\Sigma_{p|\deg(D)}p^{2v_p(\frac{degD}{d})}(p^{v_p(d)} - 1), \qquad (2.6.3)$$

where $l \supset k$ is the center of D.

2.7 Krull Schmidt theorem

The goal of this section is a brief recollection of the Krull-Schmidt theorem from the paper [Ati56] of Atiyah . As an application we derive the Krull-schmidt theorem for the category of coherent sheaves over the root stack $\mathfrak{X} = X_{(\mathcal{L},s,r)}$. In this section we fix an abelian category $\mathcal{C}$. In [Ati56], this is called an 'exact category'. For objects A and B in $\mathcal{C}$ we denote the abelian group of morphisms between them by $\mathrm{Hom}(A, B)$. For any morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ in $\mathcal{C}$, we denote by $\mathrm{Hom}(f, g)$ the induced morphism $\mathrm{Hom}(f, g) : \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, D)$ defined by composition on both sides

$$\mathrm{Hom}(f, g)(h) := ghf$$

Definition 2.49. 1. We say that a triple $\Sigma_n := (i_n, p_n, E_n)$ is a bi-chain in $\mathcal{C}$ if for all $n \geq 1$, E_n are objects in $\mathcal{C}$, $i_n : E_n \to E_{n-1}$ are monomorphisms and $p_n : E_{n-1} \to E_n$ is an epimorphsm.

2. We say that a bichain $\Sigma_n := (i_n, p_n, E_n)$ terminates if there exists an $N \in \mathbb{N}$ such that i_n and p_n are isomorphisms.

3. We say that an abelian category $\mathcal{C}$ satisfies the bi-chain condition if every bi-chain in $\mathcal{C}$ terminates.

Remark 2.50. Given a bichain Σ_n as above, the induced map $\mathrm{Hom}(p_n, i_n) : \mathrm{Hom}(E_n, E_n) \to \mathrm{Hom}(E_{n-1}, E_{n-1})$ is a monomorphism. By the left exactness of $\mathrm{Hom}(E_{n-1}, -)$, we get a monomorphism

$$0 \to \mathrm{Hom}(E_{n-1}, E_n) \xrightarrow{\mathrm{Hom}(E_{n-1},i_n)} \mathrm{Hom}(E_{n-1}, E_{n-1}).$$

By the left exactness of $\mathrm{Hom}(-, E_n)$ we get a monomorphism

$$0 \to \mathrm{Hom}(E_n, E_n) \xrightarrow{\mathrm{Hom}(p_n,E_n)} \mathrm{Hom}(E_{n-1}, E_n).$$

By composition of both the monomorphisms, we get the required monomorphism $\mathrm{Hom}(p_n, i_n)$.

Lemma 2.51 (Lemma 3,[Ati56])**.** *A bichain $\Sigma = (i_n, p_n, E_n)$ terminates if and only if the descending chain of abelian groups $\{\mathrm{Hom}(E_n, E_n), \mathrm{Hom}(p_n, i_n)\}_n$ terminates.*

Corollary 2.52. *Let $\mathcal{C}$ be an abelian category such that*

1. *For any given objects A, B in $\mathcal{C}$, $\mathrm{Hom}(A, B)$ is a finite dimensional vector space over a field k.*

2. *For any given morphisms f, g in $\mathcal{C}$, $\mathrm{Hom}(f, g)$ is k-linear.*

Then $\mathcal{C}$ satisfies the bi-chain condition.

Proof. This follows easily from the above Lemma and the fact that any chain of subspaces in a finite dimensional vector space terminates. $\square$

Proposition 2.53. *The category of coherent sheaves on a root stack $\mathfrak{X} \to \mathrm{spec}(k)$ denoted by $Coh_{\mathfrak{X}}$ is an abelian category satisfying the bi-chain condition.*

Proof. The structure map $\mathfrak{X} \xrightarrow{p} \mathrm{spec}(k)$ factors as

$$\mathfrak{X} \xrightarrow{\pi} X \xrightarrow{p_0} \mathrm{spec}(k) \tag{2.7.1}$$

Given a coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$, since π is proper, $\pi_*\mathcal{F}$ is coherent. Since $p_*\mathcal{F} \cong p_{0*}\pi_*\mathcal{F}$, we have

$$H^0(\mathfrak{X}, \mathcal{F}) = H^0(X, \pi_*\mathcal{F}). \tag{2.7.2}$$

Since X is projective, by Serre's theorem [Theorem 5.19, [Har77]] $H^0(X, \pi_*\mathcal{F})$ is a finite dimensional k-vector space. Now the proposition follows from Corollary 2.52. The second condition also follows in a similar way.

$\square$

Theorem 2.54 (Theorem-1, [Ati56]). *Let $\mathcal{C}$ be an abelian category in which the bi-chain condition holds. Then every $E \in \mathcal{C}$ can be decomposed as*

$$E = E_1 \oplus \ldots \oplus E_n \tag{2.7.3}$$

where E_i are indecomposable objects in $\mathcal{C}$. Such a decomposition is unique up to re-ordering in the sense that, given another such decomposition $E = E_1' \oplus \ldots \oplus E_m'$ then $m = n$ and $E_i \cong E_i'$ up to reordering of the indices i.

In the case where every object $E \in \mathcal{C}$ has a decomposition as mentioned above we say that 'Krull-Schmidt theorem' holds for $\mathcal{C}$.

Corollary 2.55. *Krull-Schmidt theorem holds in $Coh_{\mathfrak{X}}$, the category of coherent sheaves on the root stack $\mathfrak{X} \to \mathrm{spec}(k)$.*

Proof. This follows directly by combining 2.53 and 2.54.   □

2.8   Category of coherent sheaves on a root stack

The category of coherent sheaves on the the root stack $\mathfrak{X} = X_{(\mathcal{L},s,r)}$ is denoted by $Coh(\mathfrak{X})$. In this section we will prove that this is an abelian category.

The following Lemma follows directly from Proposition 2.31 in [Per03]. Let $\mu_r \subset k^*$ denote the group of r-th roots of unity. Since k is algebraically closed of characteristic coprime to r, μ_r is a cyclic subgroup of k^* of order r.

Lemma 2.56. *Given an affine scheme $\mathrm{spec}(S)$ with a μ_r-action, S has the structure of a $\mathbb{Z}/r$-graded ring. Moreover, the category of $\mathbb{Z}/r$-graded S-modules and the category of μ_r-equivariant quasicoherent sheaves over $X = \mathrm{spec}(S)$ are equivalent.*

Proposition 2.57. *The category of coherent $\mathcal{O}_{\mathfrak{X}}$-modules $Coh(\mathfrak{X})$ is an abelian category.*

Proof. $Coh(\mathfrak{X})$ is a full additive subcategory of an abelian category $Sh(\mathfrak{X})$ of sheaves of $\mathcal{O}_{\mathfrak{X}}$-modules. Hence the only non-trivial statements to check are that kernels and cokernels of morphisms in $Coh(\mathfrak{X})$ are in $Coh(\mathfrak{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent sheaves. Restricitng φ to the open substack $[\operatorname{spec}(R')/\mu_r] \subset \mathfrak{X}$, where $R' = R[T]/(T^r - s)$, we see that φ is equivalent to a μ_r-equivariant morphism of finitely generated $\mathbb{Z}/r$-graded R'-modules denoted by $\varphi : M \to N$. Since R' is noetherian, $\ker(\varphi)$ and $\operatorname{coker}(\varphi)$ are finitely generated graded R'-modules. Hence the proposition follows. $\square$

We denote $Coh_{\mathfrak{X}} \to \operatorname{spec}(k)$ to be the stack of coherent sheaves on $\mathfrak{X}$ defined as follows:

1. For a k-scheme U, the objects of the groupoid $Coh_{\mathfrak{X}}(U)$ are given by coherent $\mathcal{O}_{\mathfrak{X}_U}$-modules which are flat over U.

2. The morphisms in the groupoid $Coh_{\mathfrak{X}}(U)$ are given by isomorphisms $\varphi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_{\mathfrak{X}_U}$-modules.

Consider the canonical forgetful morphism of stacks Σ, which takes a coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ to the underlying coherent sheaf $\mathcal{F}_0 := \pi_* \mathcal{F}$ on the curve X.

$$Coh_{\mathfrak{X}} \xrightarrow{\Sigma} Coh_X \qquad (2.8.1)$$

$$\mathcal{F} \mapsto \mathcal{F}_0 \qquad (2.8.2)$$

Here Coh_X is the stack of coherent sheaves on X over $\operatorname{spec}(k)$.

Let S be a k-scheme of finite type. Fix a S-valued point on Coh_X given by $E_0 : S \to Coh_X$. Note that by the Yoneda lemma, E_0 defines a coherent sheaf on $X \times_k S$. Now, consider the Cartesian square of 1-morphisms of stacks given by

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & S \\
\downarrow & & \downarrow E_0 \\
Coh_{\mathfrak{X}} & \xrightarrow{\ \Sigma\ } & Coh_X
\end{array}
$$

For an S-scheme, $S' \xrightarrow{p} S$ the objects of the groupoid $\mathcal{C}(S')$ are given by pairs $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a coherent sheaf on $\mathfrak{X}_{S'}$ and $\varphi : \mathcal{E}_0 \simeq p^* E_0$. The morphisms of $\mathcal{C}(S')$ between two objects $(\mathcal{E}, \varphi)$ and $(\mathcal{E}', \varphi')$ are isomorphims $\mathcal{E} \to \mathcal{E}'$ compatible with φ and φ' in the obvious way.

Remark 2.58.   1. We may decompose any coherent sheaf $\mathcal{E}$ on $\mathfrak{X}$ as $\mathcal{F} \oplus \mathcal{T}$, where $\mathcal{F}$ is locally free and $\mathcal{T}$ is a torsion sheaf on $\mathfrak{X}$ by Lemma 2.30. Denote by $\mathrm{Flag}(\mathcal{E})$ the canonical tuple (indexed by $|D|$) of flags attached to the locally free part $\mathcal{F}$ of $\mathcal{E}$. Hence we may associate a pair $(\mathrm{Flag}(\mathcal{E}), \mathcal{T})$ to $\mathcal{F}$.

2. A morphism $(\mathcal{E}, \varphi) \to (\mathcal{E}', \varphi')$ is an isomorphism $\mathcal{E} \xrightarrow{\eta} \mathcal{E}'$ of $\mathcal{O}_{\mathfrak{X}}$-modules compatible with φ and φ'. By Lemma 2.30 and the fact that any such isomorphism $\mathcal{F} \oplus \mathcal{T} \xrightarrow{\eta} \mathcal{F}' \oplus \mathcal{T}'$ splits as $\mathcal{F} \xrightarrow{\eta_f} \mathcal{F}'$ and $\mathcal{T} \xrightarrow{\eta_{tor}} \mathcal{T}'$ we may define $\tau(\eta)$ to be the morphism induced on flags $\mathrm{Flag}(E) \to \mathrm{Flag}(E')$ along with the morphism $\mathcal{T} \xrightarrow{\eta_{tor}} \mathcal{T}'$.

Let $\mathbb{T}$ denote the stack of coherent torsion sheaves over T_0, the torsion part of E_0. More precisely, $\mathbb{T} := Coh_{\mathfrak{X}} \times_{Coh_X, T_0} S$, via the canonical map $T_0 : \mathrm{spec}(A) \to Coh_X$

We have the following Lemma about the stack $\mathcal{C}$

Proposition 2.59. *With the above notation the 1-morphism of stacks defined by*

$$Coh_{\mathfrak{X}} \times_{Coh_X, E_0} S := \mathcal{C} \xrightarrow{\tau} \Pi_{x \in |D|} \mathrm{Flag}_{\mathcal{K}}(x) \times \mathbb{T} \qquad (2.8.3)$$

$$(\mathcal{E}, \varphi_0) \mapsto (\mathrm{Flag}(\mathcal{E}), \mathcal{T}) \qquad (2.8.4)$$

is an isomorphism.

Proof. Note that the morphism τ is a well-defined morphism of stacks by Remark 2.58. Given a tuple of flags $(F_p)_{p \in |D|}$ and the locally free sheaf F_0 on X (obtained by the decomposition $E_0 = F_0 \oplus T_0$) we can construct a locally free parabolic sheaf $\mathcal{F}$ in a unique way. Given such an $\mathcal{F}$ and a torsion sheaf $\mathcal{T}$, let $\mathcal{E}' := \mathcal{F} \oplus \mathcal{T}$. This defines a quasi-inverse functor to τ, making it an equivalence.

$\square$

Remark 2.60. Given an S-scheme S', the groupoid $\mathbb{T}(S')$ has finitely many objects up to isomorphism. Moreover, the automorphisms of each object is finite. Indeed, a coherent torsion sheaf $\mathcal{T}$ over $\mathfrak{X}_S$ with a fixed underlying coherent sheaf $\mathcal{T}_0$ over X is given by a finite number of coherent torsion sheaves over X each of which are torsion, hence finite.

Lemma 2.61. $\mathbb{T}$ *is a smooth algebraic stack of dimension* 0 *over* S.

Proof. By the previous remark $\mathbb{T}$ is given by $BG_1 \times \ldots \times BG_k$ for finite groups G_i.

$\square$

Theorem 2.62. *The stack* $Coh_{\mathfrak{X}}$ *is a smooth algebraic stack over* $\mathrm{spec}(k)$ *of dimension* $-\chi(\mathcal{E}, \mathcal{E})$ *at a point* $\mathcal{E}$.

Proof. This follows from Proposition 2.59 and Theorem 2.34.

<div align="right">□</div>

In the rest of this section we define several moduli stacks of morphisms of coherent sheaves on the root stack $\mathfrak{X}$. The goal is to relate all of these stacks to the nilpotent stack $\mathcal{N}il_{\mathfrak{X}}^n$, to help prove that it is smooth of expected dimension. This is done in Section 6 of [BDH18] for curves. Since the relation between aforementioned stacks of morphisms and the nilpotent stack is quite formal this carries over to the root stack without any changes. Such relation is shown in Theorem 4.33.

Definition 2.63.   1. We denote by $\mathcal{M}_{\mathfrak{X}}^{*\to*}$ the stack defined as follows. Given a k-scheme S, the objects of the groupoid $\mathcal{M}_{\mathfrak{X}}^{*\to*}(S)$ are morphisms $E_1 \xrightarrow{\varphi} E_2$ of coherent sheaves E_1, E_2 over $\mathfrak{X}_S$, such that the coherent sheaves
$ker(\varphi), \mathrm{im}\,(\varphi)$ and $\mathrm{coker}(\varphi)$ are flat over S. The morphisms are given by commuting squares

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ \eta_1\ } & E_1' \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
E_2 & \xrightarrow{\ \eta_2\ } & E_2'
\end{array}
$$

2. We denote by $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*} \subset \mathcal{M}_{\mathfrak{X}}^{*\to*}$ the substack defined as follows. Given a k-scheme S, the objects of the groupoid $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*}(S)$ are monomorphisms $E_1 \hookrightarrow E_2$ and morphisms are the morphisms in $\mathcal{M}_{\mathfrak{X}}^{*\to*}(S)$ i.e., $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*}(S)$ is a full subcategory of $\mathcal{M}_{\mathfrak{X}}^{*\to*}(S)$

Remark 2.64. $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*}$ is an open substack of $\mathcal{M}_{\mathfrak{X}}^{*\to*}$. Indeed, given a $\mathrm{spec}(A)$-valued point $E_1 \xrightarrow{\varphi} E_2$ in $\mathcal{M}_{\mathfrak{X}}^{*\to*}$ (i.e., an object of $\mathcal{M}_{\mathfrak{X}}^{*\to*}(\mathrm{spec}(A))$), the fiber

over it is given by the points in $\operatorname{spec}(A)$ at which φ is a monomorphism. In other words, the points S of $\operatorname{spec}(A)$ at which $\ker(\varphi) = 0$, i.e., the complement of the support of $\ker(\varphi)$. Since $\ker(\varphi)$ is coherent, S is open in $\operatorname{spec}(A)$.

Definition 2.65.   1. We denote by $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*}$ the moduli stack parametrizing the diagrams $E_1 \hookrightarrow E \hookleftarrow E_2$ of coherent sheaves over $\mathfrak{X}$. More precisely, given a k-scheme S, the objects of $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*}(S)$ are given by triples $E_1 \hookrightarrow E \hookleftarrow E_2$, where E_i and E are coherent sheaves over $\mathfrak{X}_S$ such that the coherent sheaves $\frac{E}{E_1+E_2}$, $\frac{E_1+E_2}{E_i}$ and $E_1 \cap E_2$ are flat over S. Morphisms between two objects $E_1 \hookrightarrow E \hookleftarrow E_2$ and $E_1' \hookrightarrow E' \hookleftarrow E_2'$ are given by morphisms $E \xrightarrow{\varphi} E'$ such that $\varphi(E_1) = E_1'$ and $\varphi(E_2) = E_2'$.

2. We denote by $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookrightarrow*} \subset \mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*}$ the substack defined as follows. Given a k-scheme S, the objects of the groupoid $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookrightarrow*}(S)$ are triples $E_1 \hookrightarrow E_2 \hookrightarrow E$ and morphisms are the morphisms in $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*}(S)$ i.e., $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookrightarrow*}(S)$ is a full subcategory of $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*}(S)$. The inclusion $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookrightarrow*} \subset \mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*}$ is given by imposing the condition that $E_1 \subset E_2$.

Remark 2.66. Given an object $E_1 \hookrightarrow E \hookleftarrow E_2$, $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*}(S)$, the condition that $E_1 \subset E_2$ is equivalent to the condition that $\frac{E_1}{E_1 \cap E_2} = 0$. Hence similar to the previous remark, $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookrightarrow*}$ defines an open substack of $\mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*}$.

There are canonical 1-morphisms of stacks given as follows:

1.

$$pr_{\cap} : \mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*\hookleftarrow*} \to \mathcal{M}_{\mathfrak{X}}^{*\hookrightarrow*} \tag{2.8.5}$$

$$(E_1 \hookrightarrow E \hookleftarrow E_2) \mapsto (E_1 \cap E_2 \subset E) \tag{2.8.6}$$

2.

$$pr_\rightarrow : \mathcal{M}_{\mathfrak{x}}^{* \hookrightarrow * \hookleftarrow *} \to \mathcal{M}_{\mathfrak{x}}^{* \to *} \tag{2.8.7}$$

$$(E_1 \hookrightarrow E \hookleftarrow E_2) \mapsto (E_2 \to E/E_1) \tag{2.8.8}$$

The arrow $E_2 \to E/E_1$ is given by the composition $E_2 \hookrightarrow E \twoheadrightarrow E/E_1$

Chapter 3

Riemann-Roch for Orbifolds

The ideas in this chapter are mainly from B.Toen's thesis [Toe]. We utilize this theory to derive the Hirzebruch-Riemann-Roch formula for orbifolds of dimension one, in particular for root stacks. Toen already does this for a line bundle (Corollary 3.41, [Toe]), we follow his theory to make the calculation for vector bundles.

3.1 Generalities

Given a stack $\mathfrak{X} \to Sch/S$ fibered in groupoids, we may associate a simplicial presheaf $BF_{\mathfrak{X}}$ to it. Let $\underline{K} : Sch/S \to Sp$ be a presheaf in spectra given by

$$\underline{K}(T) := K(F_{\mathfrak{X}}(T))$$

where $F_{\mathfrak{X}}(T)$ is the exact category $\mathrm{Hom}_{Cart}(T, \mathfrak{X})$ of Cartesian functors $T \to \mathfrak{X}$ over Sch/S. Here the scheme T is identified with the stack associated to it.

Let $\mathcal{E}$ be an exact category cofibered over Sch/S. We only need the examples of $Vect(\mathfrak{X})$ and $Coh(\mathfrak{X})$, the category of vector bundles and the category of coherent sheaves over the stack $\mathfrak{X}$ respectively.

Definition 3.1. Given a simplicial presheaf $BF_{\mathfrak{X}} : Sch/S \to sSet$ (associated to a stack $\mathfrak{X}$ as above) and the presheaf in spectra $\underline{K} : Sch/S \to Sp$, The K-cohomology of the stack $\mathfrak{X}$ with coefficients in $\mathcal{E}$ is the spectrum given by

$$\mathbf{H}(BF_{\mathfrak{X}}, \underline{K}) := \mathbf{R}\operatorname{Hom}_{sp}(BF_{\mathfrak{X}}, \underline{K})$$

where $\mathbf{R}\operatorname{Hom}_{sp}(BF_{\mathfrak{X}}, \underline{K}) := \operatorname{Hom}_{sp}(BF_{\mathfrak{X}}, H\underline{K})$ and $\underline{K} \hookrightarrow H\underline{K}$ being the fibrant replacement. By abuse of notation, we will denote $\mathbf{H}(BF_{\mathfrak{X}}, \underline{K})$ by $\mathbf{H}(\mathfrak{X}, \underline{K})$ or simply $\underline{\mathbf{K}}(\mathfrak{X})$.

Remark 3.2. Given a simplicial set M and a spectrum L, one can define the spectrum $\operatorname{Hom}_{sp}(M, L)_{[n]} := \operatorname{Hom}_s(M, L_{[n]})$ with the obvious induced maps $\operatorname{Hom}_{sp}(M, L)_{[n]} \to \Omega \operatorname{Hom}_{sp}(M, L)_{[n+1]}$. We can extend this definition to the case where M and L are simplicial presheaves and presheaves in spectra respectively by defining them pointwise. This makes the above definitions self contained.

Let $\int_{\mathfrak{X}} \mathcal{E} := \operatorname{Hom}_{cart}(\mathfrak{X}, \mathcal{E})$ be the category of Cartesian functors $\mathfrak{X} \to \mathcal{E}$ over Sch/S. The category $\int_{\mathfrak{X}}(\mathcal{E})$ is equivalent to the category of 'pseudo-natural transformations' of the 'pseudo-functors' $T \mapsto \mathfrak{X}(T)$ and $T \mapsto \mathcal{E}(T)$.

Definition 3.3. The K-theory of a stack $\mathfrak{X}$ is defined by

$$\mathbf{K}(\mathfrak{X}) := K(\textstyle\int_{\mathfrak{X}} \mathcal{E})$$

where $K(-) : ExCat \to Sp$ is the construction of Waldhausen [Wal85].

Proposition 3.4. *(Proposition 1.6, [Toe]) Let $\mathfrak{X} \to Sch/S$ be a stack fibered in groupoids. There is a canonical natural transformation of functors*

$$can : \mathbf{K} \to \underline{\mathbf{K}} \qquad\qquad (3.1.1)$$

In the rest of this section we will list the ingredients required to construct the 'Chern character with coefficients in representations' of a vector bundle (resp., coherent sheaf) on a Deligne-Mumford stack. For details we refer to Section 2.3, 3.1 and 3.2 in [Toe].

Consider the sheaf of groups on algebraic spaces,

$$\mu_\infty^t : (Esp/S)_{et} \to Grp \tag{3.1.2}$$

which takes an S-algebraic space X to the torsion elements in $\mathcal{O}_X(X)^*$. Using this we may construct the following sheaf of $\mathbb{Q}$-algebras

$$\mathbb{Q}[\mu_\infty^t] : (Esp/S)_{et} \to \mathbb{Q} - alg \tag{3.1.3}$$
$$X \mapsto \mathbb{Q}[\mu_\infty^t(X)] \tag{3.1.4}$$

Now consider the sheaf

$$\Lambda : (Esp/S)_{et} \to \mathbb{Q} - alg \tag{3.1.5}$$
$$X \mapsto colim_m \mathbb{Q}(\mu_m(X)) \tag{3.1.6}$$

where $\mu_m : (Esp/S)_{et} \to Grp$ is the sheaf of m-th roots of unity, given by $X \mapsto \{r \in \mathcal{O}_X(X)^* : r^m = 1\}$. This sheaf is a quotient of the sheaf of $\mathbb{Q}$-algebras $\mathbb{Q}[\mu_\infty^t]$. There is a natural projection

$$\mathbb{Q}[\mu_\infty^t] \to \Lambda \tag{3.1.7}$$

Given such a sheaf Λ, consider the associated presheaf in (Eilenberg-Maclane) spectra, $\mathcal{H}_\Lambda := K(\Lambda, o)$. This has the following properties:

$$\pi_k(\mathcal{H}_\Lambda) = 0, \text{ for } k \neq 0 \tag{3.1.8}$$

$$\pi_0(\mathcal{H}_\Lambda) \cong \Lambda \tag{3.1.9}$$

Given a presheaf in spectra K, we denote $K \otimes \Lambda := K \wedge \mathcal{H}_\Lambda$.

Definition 3.5. Let $\mathfrak{X}$ be a tame Deligne-Mumford stack over $\mathrm{spec}(k)$, The K-cohomology and G-cohomology with coefficients in representations are defined by

$$\underline{\mathbf{K}}^{rep}(\mathfrak{X}) := \mathbf{H}((I_\mathfrak{X})_{et}, \underline{K} \otimes \Lambda) \tag{3.1.10}$$

$$\underline{\mathbf{G}}^{rep}(\mathfrak{X}) := \mathbf{H}((I_\mathfrak{X})_{et}, \underline{G} \otimes \Lambda) \tag{3.1.11}$$

Theorem 3.6 (Theorem 2.15, [Toe])**.** *Let $\mathfrak{X}$ be a tame Deligne-Mumford stack,*

1. *There exists a morphism of ringed spectra*

$$\varphi_\mathfrak{X} : \mathbf{K}(\mathfrak{X}) \to \underline{\mathbf{K}}^{rep}(\mathfrak{X}) \tag{3.1.12}$$

   which is functorial for inverse images.

2. *If k contains all the roots of unity, and $\mathfrak{X}$ is regular, then the induced morphism*

$$\varphi_\mathfrak{X} : \mathbf{G}(\mathfrak{X}) \otimes \Lambda(k) \otimes \mathbb{Q} \to \underline{\mathbf{G}}^{rep}(\mathfrak{X}) \tag{3.1.13}$$

   is an isomorphism.

The construction of $\varphi_\mathfrak{X}$ in the above theorem is done as follows. First, we construct a functor

$$F_\zeta : Vect(I_F) \to Vect(I_F^t) \tag{3.1.14}$$

$$V \mapsto V^{(\zeta)} \tag{3.1.15}$$

$V^{(\zeta)}$ is roughly the 'ζ-eigen subbundle' of V.

We then take the direct sum of all such functors

$$F : \mathbf{Vect}(I_{\mathfrak{x}}) \to \bigoplus_{\mu_\infty^t(I_{\mathfrak{x}})} \mathbf{Vect}(I_{\mathfrak{x}}) \tag{3.1.16}$$

$$V \mapsto \oplus_{\zeta \in \mu_\infty^t(I_{\mathfrak{x}})} V^{(\zeta)} \tag{3.1.17}$$

This induces a morphism of presheaves in spectra over the site $(I_{\mathfrak{x}})_{et}$, which we also call

$$F : \underline{K} \to K(\bigoplus_{\mu_\infty^t} \mathbf{Vect})_{\mathbb{Q}} \cong \underline{K} \otimes \mathbb{Q}[\mu_\infty^t] \tag{3.1.18}$$

The projection $\mathbb{Q}[\mu_\infty^t] \to \Lambda$ induces a morphism of spectra $\eta : \underline{K} \otimes \mathbb{Q}[\mu_\infty^t] \to \underline{K} \otimes \Lambda$. Composing η, *can* $: \mathbf{K}(I_{\mathfrak{x}}) \to \underline{\mathbf{K}}(I_{\mathfrak{x}})$ and F gives us a morphism which we also call

$$F : \mathbf{K}(I_{\mathfrak{x}}) \to \underline{\mathbf{K}}^{rep}(\mathfrak{X})$$

We denote,

$$\varphi_{\mathfrak{x}} := F \circ \pi_{\mathfrak{x}}^* : \mathbf{K}(\mathfrak{X}) \to \underline{\mathbf{K}}^{rep}(\mathfrak{X})$$

where $\pi_{\mathfrak{x}} : I_{\mathfrak{x}} \to \mathfrak{X}$ is the canonical projection and $\pi_{\mathfrak{x}}^*$ is the pullback functor.

Applying π_0 to $\varphi_{\mathfrak{x}}$, we get

$$\varphi_{\mathfrak{x}} : K_0(\mathfrak{X}) \to \underline{K}_0^{rep}(\mathfrak{X})$$

Definition 3.7 ([Toe]). The cohomology and homology of a stack $\mathfrak{X}$ can be defined as follows:

$$H^p(\mathfrak{X}, q) := \pi_{dq-p}\mathbf{H}(\mathfrak{X}_{li}, \mathcal{H}^q \otimes \mathbb{Q}) \tag{3.1.19}$$

$$H_p(\mathfrak{X}, q) := \pi_{dq-p}\mathbf{H}(\mathfrak{X}_{li}, \mathcal{H}'_q \otimes \mathbb{Q}) \tag{3.1.20}$$

where $\mathcal{H}_q$ is a cohomology theory with direct images. In our case these are Eilenberg-Maclane (presheaf in) spectra $K(\mathcal{K}_q, q)$ associated to the K-theory presheaf $X \mapsto K_i(X)$ and $\mathcal{H}'_q$ is its fibrant replacement (Gersten resolution). For this cohomology theory with direct images we have d $= 1$, see Example 1.4(ii) in [Gil81].

We denote

$$H^*_{rep}(\mathfrak{X}, *) := \bigoplus_{p,q} H^p(\mathfrak{X}, q)$$

The i-th Chern classes are given by some natural transformations

$$C_i : \underline{K}_{[0]} \to \mathcal{H}^i_{[0]}$$

Here $\underline{K}$ and $\mathcal{H}^i$ are given by the presheaves in spectra associated to the exact categories of vector bundles (or coherent sheaves) over stacks and cohomology with direct images respectively. $\underline{K}_{[0]}$ and $\mathcal{H}^i_{[0]}$ are the 0-th stages of the spectra. By taking π_0 we get $C_i : K_0(\mathfrak{X}) \to H^{di}(\mathfrak{X}, i)$. Note that Toen [2] defines these for any K_m (by taking the higher homotopy functor π_m), but we only need them for $m = 0$. Since we have $d = 1$, the i-th Chern class of a vector bundle over $\mathfrak{X}$ takes values in $H^i_{rep}(\mathfrak{X}, i)$. Hence by Lemma 3.8, the pushforward of a Chern class to the point $spec(k)$ takes values in $K_0(spec(k)) = \mathbb{Z}$. Using the Chern classes $C_i : K_0(\mathfrak{X}) \to H^i(\mathfrak{X}, i)$, we may construct the Chern

character in the usual way, denoted by $Ch : K_0(\mathfrak{X}) \to H^i(\mathfrak{X}, i)$. The Chern character with coefficients in representations is given by the composition

$$Ch^{rep} : K_0(\mathfrak{X}) \xrightarrow{\varphi_{\mathfrak{X}}} \underline{K}_0^{rep}(\mathfrak{X}) \xrightarrow{Ch} H^*_{rep}(\mathfrak{X}, *)$$

3.2 Inertia stacks

This section is a quick introduction to inertia stacks, gerbes and some basic facts relating to it. All of these can be found in standard references, for example in [Sta18, Tag 050P].

Suppose $\mathfrak{X}$ is an algebraic stack over a scheme S. Let $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ denote the diagonal morphism of stacks. We denote by

$$I_{\mathfrak{X}} := \mathfrak{X} \times_{\mathfrak{X} \times_S \mathfrak{X}} \mathfrak{X} \tag{3.2.1}$$

the intertia stack. Unwinding the definition of fiber product of stacks, given a scheme $T \to S$, we have

1. The objects of $I_{\mathfrak{X}}(T)$ given by pairs (a, b, φ), where $a, b \in \mathfrak{X}(T)$ and $\varphi : \Delta(a) \to \Delta(b)$. The isomorphism φ is equivalent to a pair of isomorphisms $(a, a) \xrightarrow{(\varphi_1, \varphi_2)} (b, b)$.

2. A morphism $f : (a, b, \varphi) \to (a', b', \varphi')$ is given by pairs of isomorphisms $(f_1, f_2) : (a, b) \to (a', b')$ such that the following square commutes

$$
\begin{array}{ccc}
\Delta(a) & \xrightarrow{(f_1, f_1)} & \Delta(a') \\
\downarrow{\scriptstyle (\varphi_1, \varphi_2)} & & \downarrow{\scriptstyle (\varphi'_1, \varphi'_2)} \\
\Delta(b) & \xrightarrow[(f_2, f_2)]{} & \Delta(b')
\end{array}
$$

Let $\mathcal{A}$ denote the stack over S whose objects over a scheme T/S are pairs (a, φ) where $a \in \mathfrak{X}(T)$ and $\varphi \in Aut(a)$. The morphisms $(a, \varphi) \to (a', \varphi')$ are isomorphisms $a \xrightarrow{u} a'$ compatible with φ and φ'. We have a canonical morphism of groupoids

$$\mathcal{A}(T) \xrightarrow{\eta(T)} I_{\mathfrak{X}}(T) \tag{3.2.2}$$

$$(a, \varphi) \mapsto (a, a, (\varphi, Id)) \tag{3.2.3}$$

It has a quasi-inverse

$$I_{\mathfrak{X}}(T) \xrightarrow{\tau(T)} \mathcal{A}(T) \tag{3.2.4}$$

$$(a, b, (\varphi_1, \varphi_2)) \mapsto (a, \varphi_2^{-1}\varphi_1) \tag{3.2.5}$$

Clearly, $\tau \circ \eta = Id$. It is enough to show that $\eta \circ \tau \cong Id$, i.e, there is a natural isomorphism $(a, b, (\varphi_1, \varphi_2)) \xrightarrow{(f_1, f_2)} (a, a, (\varphi_2^{-1}\varphi_1, Id))$. This is easily seen if we put $f_1 = id$ and $f_2 = \varphi_2^{-1}$. Hence we have the following Lemma

Lemma 3.8. *The inertia stack $I_{\mathfrak{X}}$ parametrizes pairs (a, φ), where a is an object of $\mathfrak{X}$ and φ an automorphism of a.*

Definition 3.9. Let X be an algebraic space, we define a gerbe on X as a stack $\mathcal{G} \to X$ such that the following conditions are satisfied:

1. For each open set $U \to X$, there exists a covering $V \to U$ such that $\mathcal{G}(V) \neq \emptyset$.

2. Given any open set $U \to X$ and two objects $x, y \in \mathcal{G}(U)$, there exists a covering $V \to U$ and an isomorphism $x|_V \to y|_V$.

Lemma 3.10. *A stack* $p : \mathcal{G} \to X$ *over an algebraic space* X *is a gerbe if and only if the induced morphism on sheaves* $p_0 : Sh(\mathcal{G}) \to X$ *is an isomorphism.*

Proof. Given $p_0 : Sh(\mathcal{G}) \to X$ is an isomorphism, then condition (1) in Definition 3.9 follows from the fact that p_0 is an epimorphism of sheaves. Given two objects $x, y \in \mathcal{G}(U)$ over $X(U)$, they define the same object over $X(U)$. Since the sheafification morphism $\mathcal{G}^p \to Sh(\mathcal{G})$ is an epi, x and y are locally isomorphic. Here $\mathcal{G}^p$ is the presheaf given by identifying objects of $\mathcal{G}(U)$ up to isomorphism $U \mapsto \mathcal{G}(U)/_\sim$. $\square$

Definition 3.11. Let G be a sheaf of abelian groups over X. A G-gerbe $\mathcal{G} \to X$ is a gerbe along with an isomorphism of stacks

$$G \times_X \mathcal{G} \xrightarrow{\sim} I_\mathcal{G} \qquad (3.2.6)$$

Definition 3.12. Given a G-gerbe $\mathcal{G} \to X$ and a sheaf $\mathcal{F}$ on $\mathcal{G}$, the natural action

$$\mathcal{F} \times G \to \mathcal{F} \qquad (3.2.7)$$

induced via $A_\mathcal{G} \xrightarrow{\sim} I_\mathcal{G}$ is called the inertial action.

3.3 Computation

Let $\mathfrak{X}$ be a smooth orbifold of dimension one and $I_\mathfrak{X}$ its inertia stack. Let $\pi : I_\mathfrak{X} \to \mathfrak{X}$ be the canonical projection. $I_\mathfrak{X}$ splits as $\mathfrak{X} \sqcup_{i=1}^m BH_i$, where H_i are the structure groups at the ramification points. This induces a section of π.

From Section 10, Theorem 7 and Example 2 in [Bro73] we have the following Lemma.

Lemma 3.13. $H_0(\mathrm{spec}(k), q) = K_q(k)$ *and* $H_p(\mathrm{spec}(k), q) = 0$ *for* $p \neq 0$.

Proof. Since $d = 1$, we have $H_p(\mathrm{spec}(k), q) = \pi_{q-p}\mathbf{H}(\mathrm{spec}(k), \mathcal{H}_q' \otimes \mathbb{Q})$. Also, $\mathbf{H}(\mathrm{spec}(k), \mathcal{H}_q' \otimes \mathbb{Q}) = \Gamma(\mathcal{H}_q') = R\Gamma(\mathcal{H}^q)$. Hence

$$H^p(\mathrm{spec}(k), q) = \pi_{q-p}(R\Gamma(\mathcal{H}_q)) = H^{p-q}(\mathrm{spec}(k), \mathcal{H}_q) \qquad (3.3.1)$$

Since $\mathcal{H}_q = K(\mathcal{K}_q, q)$, by Example-1 in Brown [Bro73] we have

$$H^{p-q}(\mathrm{spec}(k), \mathcal{H}_q) = H^{p-q+q}(\mathrm{spec}(k), \mathcal{K}_q) = H^p(\mathrm{spec}(k), \mathcal{K}_q) \qquad (3.3.2)$$

$\mathcal{K}_q$ has a Gersten resolution $\mathcal{K}_q \to \mathcal{R}^q$ as described in 3.14. Such a resolution is a one term resolution for the point $\mathrm{spec}(k)$ whose cohomology is given by $K_q(k)$.

$\square$

Definition 3.14 (Gersten complexes). For the root stack $\mathfrak{X}$, denote by $\mathfrak{X}_{et}$ the underlying etale site inherited from $(Sch/k)_{et}$. Let $\mathcal{K}_i$ be the presheaf over $\mathfrak{X}_{et}$ given by $U \to K_i(U)$ for a k-scheme U.

Let $|U|^{(p)}$ be the set of points of U of codimension p. For each $p \geq 0$, we get a complex $\mathcal{R}^p(U)$, called the Gersten complex of U concentrated in degree $[-p, 0]$ as defined by Gersten [Gil81].

$$0 \to \bigoplus_{x \in |U|^{(0)}} K_p(k(x)) \to \bigoplus_{x \in |U|^{(1)}} K_{p-1}(k(x)) \to \ldots \to \bigoplus_{x \in |U|^{(p)}} K_0(k(x)) \to 0$$

$$(3.3.3)$$

Remark 3.15. When U is a scheme of dimension 1 the only possible codimensions of U is 0 or 1. Hence we get two possible Gersten complexes $\mathcal{R}^1(U)$ and $\mathcal{R}^0(U)$.

$$0 \to \bigoplus_{x \in |U|^{(0)}} K_1(k(x)) \to \bigoplus_{x \in |U|^{(1)}} K_0(k(x)) \to 0 \qquad (3.3.4)$$

and

$$0 \to \bigoplus_{x \in |U|^{(0)}} K_0(k(x)) \to 0 \tag{3.3.5}$$

Theorem 3.16 (Toen). *Let $\mathfrak{X}$ be a Deligne-Mumford stack, tame and proper over $\mathrm{spec}(k)$. Then*

$$\chi(\mathfrak{X}, \mathcal{F}) := \int_{\mathfrak{X}}^{rep} \tau_{\mathfrak{X}}^{rep} \tag{3.3.6}$$

In particular if $\mathfrak{X}$ is smooth and in QDM over $\mathrm{spec}(k)$, we have

$$\chi(\mathfrak{X}, \mathcal{F}) = \int_{\mathfrak{X}}^{rep} Td^{rep}(\mathfrak{X}) Ch^{rep}(\mathcal{F}) \tag{3.3.7}$$

Here QDM means Deligne-Mumford stacks over $\mathrm{spec}(k)$ whose coarse moduli space is quasi projective over $\mathrm{spec}(k)$.

Now we will compute $\chi(\mathfrak{X}, \mathcal{F})$ when $\mathfrak{X}$ is a smooth proper orbifold over $\mathrm{spec}(k)$. In this case the coarse moduli space X of $\mathfrak{X}$ is a smooth projective curve over k. In particular, it is in QDM and hence the above theorem applies. Let $\{x_i\}_{i=1}^m \subset X$ be the set of ramification points of $\pi : \mathfrak{X} \to X$. Here H_i are the structure groups at each ramification point x_i. H_i are necessarily cyclic, say $\cong \mathbb{Z}/r_i$.

We know the following facts about the inertia stack and $H_{rep}^*(\mathfrak{X}, *)$ from Toen [Toe], proof of corollary 3.41.

1. $I_{\mathfrak{X}} = \mathfrak{X} \bigsqcup_{i=1}^m BH_i$

2. $H_{rep}^0(\mathfrak{X}, 0) \cong H^0(\mathfrak{X}, 0) \bigoplus_{i=1}^m K(H_i)$

3. $H_{rep}^1(\mathfrak{X}, 1) \cong H^1(\mathfrak{X}, 1)$

So the graded ring $H^*(\mathfrak{X}, *)$ is isomorphic to $H^0(\mathfrak{X}, 0) \oplus H^1(\mathfrak{X}, 1) \bigoplus_{i=1}^m K(H_i)$.

The Chern character with coefficients in representations is given by

$$Ch^{rep} : K_*(\mathfrak{X}) \xrightarrow{\varphi_\mathfrak{X}} K_*^{rep}(\mathfrak{X}) \xrightarrow{Ch} H_{rep}^*(\mathfrak{X}, *) \qquad (3.3.8)$$

Here $K_*^{rep}(\mathfrak{X}) := K_*(I_\mathfrak{X})$. Since $I_\mathfrak{X} = \mathfrak{X} \bigsqcup_{i=1}^m BH_i$, we have

$$K_0^{rep}(\mathfrak{X}) = K_0(\mathfrak{X}) \bigoplus_{i=1}^m K_0(BH_i) \qquad (3.3.9)$$

Consequently we have,

Lemma 3.17. *The Chern character of $\mathcal{F}$ in the cohomology of representations is given by*

$$Ch^{rep}(\mathcal{F}) = Ch(\mathcal{F}) \bigoplus_{i=1}^m [V_i] \qquad (3.3.10)$$

where $Ch(\mathcal{F})$ is the Chern character in the ordinary cohomology $H^*(\mathfrak{X}, \mathcal{F})$, V_i is the H_i-representation over k induced by $\mathcal{F}$ at x_i. Since $H = \mathbb{Z}/r_i$ is diagonalizable, each V_i splits into characters. So we may write

$$V_i = \bigoplus_{d=0}^{r-1} (\chi_i^d)^{n_d} \qquad (3.3.11)$$

where $\chi_i^d : H_i \to k$ is given by $h_i^a \mapsto \zeta_i^{ad}$. When $\mathcal{F}$ is a line bundle, $V_i = \chi_i^d$ for some $0 \leq d \leq r_i - 1$, here the d coincides with the multiplicity k_i defined in Toen [Definition 3.40].

Lemma 3.18. *The Todd class in the cohomology of representations is given by*

$$Td^{rep}(\mathfrak{X}) = Td(T_\mathfrak{X}) \bigoplus_{i=1}^m Td_i$$

where $Td_i : H_i \to k$ are the characters given by $h_i^a \mapsto \frac{1}{1-\zeta_i^a}$, where ζ_i are the r_i-th roots of unity.

By the above two Lemmas, we have

$$Ch^{rep}(\mathcal{F})Td^{rep}(\mathfrak{X}) = Ch(\mathcal{F})Td(T_{\mathfrak{X}})\bigoplus_{i=1}^{m}[V_i]Td_i \qquad (3.3.12)$$

Hence we get

$$\int_{\mathfrak{X}}^{rep} Ch^{rep}(\mathcal{F})Td^{rep}(\mathfrak{X}) = \int_{\mathfrak{X}} Ch(\mathcal{F})Td(T_{\mathfrak{X}}) + \Sigma_{i=1}^{m}(\Sigma_{d=0}^{r_i-1}\int_{BH_i}([\chi_i^d])^{n_d})([Td_i]) \qquad (3.3.13)$$

The push-forward $p_{i*} : H^*(BH_i) \to H^*(\mathrm{spec}(k))$ of the class $[\chi_i^d][Td_i]$ is the average of their image as characters $\frac{1}{r_i}\Sigma_{a=0}^{r_i-1}\frac{\zeta_i^{ad}}{1-\zeta_i^a}$.

On the other hand, we have the usual formulas for the Chern character and Todd class $Ch(\mathcal{F}) = rk(\mathcal{F}) + c_1(\mathcal{F})$ and $Td(T_{\mathfrak{X}}) = 1 + \frac{1}{2}c_1(T_{\mathfrak{X}})$. Hence we get $\int_{\mathfrak{X}} Ch(\mathcal{F})Td(T_{\mathfrak{X}}) = \frac{rk(\mathcal{F})}{2}\int_{\mathfrak{X}} c_1(T_{\mathfrak{X}}) + \int_{\mathfrak{X}} c_1(\mathcal{F})$.

Hence we have,

$$\int_{\mathfrak{X}}^{rep} Ch^{rep}(\mathcal{F})Td^{rep}(\mathfrak{X}) = \frac{rk(\mathcal{F})}{2}\int_{\mathfrak{X}} c_1(T_{\mathfrak{X}}) + \int_{\mathfrak{X}} c_1(\mathcal{F}) + \Sigma_{i=1}^{m}\frac{1}{r_i}\Sigma_{d=0}^{r_i-1}\Sigma_{a=0}^{r_i-1}\frac{n_d^{(i)}\zeta_i^{ad}}{1-\zeta_i^a} \qquad (3.3.14)$$

Furthermore, we have the following formulae:

$$\frac{1}{2}\int_{\mathfrak{X}} c_1(T_{\mathfrak{X}}) = (1-g) + \Sigma_i\frac{r_i-1}{2r_i} \qquad (3.3.15)$$

$$\Sigma_{a=0}^{r_i-1}\frac{\zeta_i^{ad}}{1-\zeta_i^a} = \frac{-r_i-1}{2} - d \qquad (3.3.16)$$

$$\Sigma_{a=0}^{r_i-1}\frac{\zeta_i^{ad}+1}{1-\zeta_i^a} = -d \qquad (3.3.17)$$

Using these, we compute (a part of)the right hand side of 3.3.14.

$$\frac{1}{r_i}\Sigma_{d=0}^{r_i-1}\Sigma_{a=0}^{r_i-1}\frac{n_d^{(i)}\zeta_i^{ad}}{1-\zeta_i^a} = \frac{1}{r_i}\Sigma_{d=0}^{r_i-1}n_d^{(i)}(\frac{-r_i-1}{2}-d)$$

$$= \frac{-r_i-1}{2r_i}\Sigma_{d=0}^{r_i-1}n_d^{(i)} - \Sigma_{d=0}^{r_i-1}\frac{dn_d^{(i)}}{r_i} \qquad (3.3.18)$$

$$= \frac{-r_i-1}{2r_i}rk(\mathcal{F}) - \Sigma_{d=0}^{r_i-1}\frac{dn_d^{(i)}}{r_i}$$

Substituting equations 3.3.17 and 3.3.15 in 3.3.14, we get

$$\int_{\mathfrak{X}}^{rep} Ch^{rep}(\mathcal{F})Td^{rep}(\mathfrak{X}) = \int_{\mathfrak{X}} c_1(\mathcal{F}) + (1-g)rk(\mathcal{F}) - \Sigma_{d=0}^{r_i-1}\frac{dn_d^{(i)}}{r_i} \qquad (3.3.19)$$

We denote $\deg(\mathcal{F}) = \int_{\mathfrak{X}} c_1(\mathcal{F})$. Hence, we get

Theorem 3.19. *Let $\mathfrak{X}$ be the root stack and $\mathcal{F}$ be a vector bundle of finite rank on $\mathfrak{X}$, then*

$$\chi(\mathfrak{X},\mathcal{F}) = \deg(\mathcal{F}) + (1-g)rk(\mathcal{F}) - \Sigma_{d=0}^{r_i-1}\frac{dn_d^{(i)}}{r_i} \qquad (3.3.20)$$

Chapter 4

Transcendence degree of field of moduli

4.1 Stack of parabolic bundles, Riemann-Roch

Let X be a smooth projective curve over an algebraically closed field k. The root stack $\mathfrak{X} = X_{(\mathcal{L},s,r)}$ is a Deligne-Mumford stack over the field k whose coarse moduli space is X [Cad07]. Moreover, $\mathfrak{X}$ is a 1-dimensional smooth orbifold. We assume that $\mathfrak{X}$ is *tame*, i.e. the characteristic of the field k is co-prime to the order of inertia of every point of $\mathfrak{X}$.

Another characterization of the above mentioned root stack is:

$$X_{(\mathcal{L},s,r)} = X \times_{[\mathbb{A}^1/\mu_r],\theta_r} [\mathbb{A}^1/\mu_r] \tag{4.1.1}$$

where $\theta_r : [\mathbb{A}^1/\mu_r] \to \mathbb{A}^1/\mu_r$ is the map taking $(\mathcal{L}, s)$ to $(\mathcal{L}^{\otimes r}, s^{\otimes r})$.

For a vector bundle $\mathcal{E}$ on $\mathfrak{X}$, the Euler characteristic of $\mathfrak{X}$ is defined in [Toe] as follows:

$$\chi(\mathfrak{X}, \mathcal{E}) := \Sigma_i (-1)^i \dim_k H^i(\mathfrak{X}, \mathcal{E}) \tag{4.1.2}$$

Theorem 4.1 (Riemann-Roch). *(Toen, [Toe]) For an orbifold $\mathfrak{X}$ and a line bundle $\mathcal{L}$ on it, the Euler characteristic is given by:*

$$\chi(\mathfrak{X}, \mathcal{M}) = \int_{\mathfrak{X}} c_1(\mathcal{M}) + 1 - g - \Sigma_i \frac{k_i}{r_i} \tag{4.1.3}$$

where D is the reduced divisor of points in X over which the forgetful map $\pi : \mathfrak{X} \to X$ is ramified, $0 \leq k_i \leq r_i - 1$ are the multiplicities of $\mathcal{M}$ at $x_i \in |D|$ and r_i is the order of ramification at x_i.

Remark 4.2. Suppose $[X_i/H_i]$ is the restriction of $\mathfrak{X}$ to a Hensilian neighbourhood $\mathrm{spec}(\mathcal{O}^h_{x_i}) \to X$ then the line bundle $\mathcal{M}$ defines a 1-dimensional H_i representation over k. The multiplicities k_i are preciseley the weights of this representation. See proof of Corollary 3.39 in [Toe].

Definition 4.3. Let $\mathcal{F}$ be a vector bundle on $\mathfrak{X}$. We define the degree of $\mathcal{F}$, denoted by $\deg(\mathcal{F})$ to be

$$\deg(\mathcal{F}) = \int_{\mathfrak{X}}^{et} c_1^{et}(\mathcal{F}) \tag{4.1.4}$$

We denote the rank of $\mathcal{F}$ by $rk(\mathcal{F})$.

Lemma 4.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be vector bundles on $\mathfrak{X}$. Then,*

$$\deg(\mathcal{F} \otimes \mathcal{G}) = \deg(\mathcal{F}) rk(\mathcal{G}) + rk(\mathcal{F}) \deg(\mathcal{G}) \tag{4.1.5}$$

Proof. See Theorem 4.3 and corollary 4.6 in 4.1.3 [Bor07] □

Lemma 4.5. *Let $p : \mathrm{spec}(k(p)) \to X$ represent a point $p \in |D|$. Since k is algebraically closed, we have $\mathrm{spec}(k(p)) = \mathrm{spec}(k)$. The following square is Cartesian :*

$$[\text{spec}(\tfrac{k[T]}{T^r})/\mu_r] \xrightarrow{\quad t \quad} spec(k)$$

$$\downarrow \tilde{p} \qquad\qquad\qquad \downarrow p$$

$$\mathfrak{X} \xrightarrow{\quad \pi \quad} X$$

Proof. The line bundle $\mathcal{L}$ is trivial and the section s is 0 when restricted to spec(k). The Lemma follows readily from Example 2.4.1 in [Cad07]. $\square$

Definition 4.6. Let $\mathcal{E}$ be a vector bundle on $\mathfrak{X}$. Then $\tilde{p}^*\mathcal{E}$ is equivalent to a μ_r-equivariant module M over the ring $R' = k[T]/(T^r)$. By the proof of proposition-3 in [Bor07] there is a canonical isomorphism

$$M \cong \bigoplus_{d=0}^{r-1} (R'[d])^{n_d^{(p)}} \tag{4.1.6}$$

as graded R'-modules. Here $R'[d]$ is the d-th shift of the graded ring R'. We call the resulting vector of natural numbers $[n_0^{(p)}, \ldots n_{r-1}^{(p)}]$ as the *vector of weights* associated to $\mathcal{E}$ at x_i.

Lemma 4.7. *In the above notation, for $0 \leq l \leq r-1$, $n_l^{(p)} = \dim_k(\operatorname{coker}(\mathcal{E}_{\frac{l+1}{r}} \to \mathcal{E}_{\frac{l}{r}})_p)$. Where $\mathcal{E}_\bullet$ is the corresponding parabolic bundle of $\mathcal{F}$. Here the operation $()_p$ is taking stalk at $p \in |D|$.*

Proof. By Borne's theorem $\mathcal{E}_{\frac{l}{r}} = \pi_*(\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{N}^{\otimes -l})$. From Corollary 3.5 in Section 3.1 of [Bor07] we have the Cartesian square

$$[\text{spec}(\tfrac{R[T]}{T^r-s})/\mu_r] \xrightarrow{\quad q \quad} \text{spec}(R)$$

$$\downarrow \tilde{i} \qquad\qquad\qquad \downarrow i$$

$$\mathfrak{X} \xrightarrow{\quad \pi \quad} X$$

where i is an open immersion. Let us denote $U = \mathrm{spec}(R)$ and $R' = \frac{R[T]}{T^r - s}$ .By flat base change theorem for stacks, we have

$$\mathcal{E}_{\frac{l}{r}}|_U = i^*\pi_*(\mathcal{F} \otimes \mathcal{N}^{\otimes -l}) = q_*\tilde{i}^*(\mathcal{F} \otimes \mathcal{N}^{\otimes -l}) = q_*(T^{\tilde{l}}M). \tag{4.1.7}$$

This sheaf is equivalent to $(T^l M)^{\mu_r}$ as an R-module. Hence the map $f_l :$ $\mathcal{E}_{\frac{l+1}{r}}|_U \to \mathcal{E}_{\frac{l}{r}}|_U$ is equivalent to a map of R-modules $f_l : (T^{l+1}M)^{\mu_r} \to (T^l M)^{\mu_r}$. Suppose

$$M \cong \bigoplus_{d \in Z/r} R'[d]^{n_d} \tag{4.1.8}$$

Since $R'[l] \cong T^l R'$, we have for $0 \le l \le r - 1$,

$T^l M = (T^l R')^{n_0} \oplus (T^{l+1}R')^{n_1} \oplus \ldots \oplus (T^{l+r-1}R')^{n_{r-1}}$

Taking μ_r-invariants of each of the direct summands, for $0 \le k \le r - 1$, we get

$$(T^{l+k}R')^{\mu_r} = \begin{cases} sR & \text{if } 0 \le k \le r - l \\ s^2 R & \text{if } r - l + 1 \le k \le r - 1 \end{cases}$$

This gives us,

$(T^l M)^{\mu_r} = (sR)^{n_0} \oplus (sR)^{n_1} \oplus \ldots \oplus (sR)^{n_{r-l-1}} \oplus (sR)^{n_{r-l}} \oplus (s^2 R)^{n_{r-l+1}} \ldots \oplus (s^2 R)^{n_{r-1}}$

Hence for $1 \le l \le r - 1$, we find that $\mathrm{coker}(f_l)$ as $(\frac{sR}{s^2 R})^{n_{r-l}}$.

For $l = 0$, one can see that $\mathrm{coker}(f_0) = (\frac{R}{sR})^{n_0}$.

$\square$

Lemma 4.8. *If $\mathcal{F}$ is a vector bundle on $\mathfrak{X}$ then $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{F}) \cong \mathcal{F} \otimes \mathcal{F}^{\vee}$ as $\mathcal{O}_{\mathfrak{X}}$-modules.*

Proof. The question is local, hence can be reduced to vector bundles over the stack $[\mathrm{spec}(\frac{k[T]}{T^r})/\mu_r]$ over $\mathrm{spec}(k)$. Vector bundles over $[\mathrm{spec}(\frac{k[T]}{T^r})/\mu_r]$ are free

graded modules over $\frac{k[T]}{T^r}$. Hence the Lemma follows. See Proposition 3.12 in [Bor07]. □

Theorem 4.9. *The Euler characteristic of a vector bundle $\mathcal{E}$ is given by:*

$$\chi(\mathfrak{X}, \mathcal{E}) = \deg(\mathcal{E}) + (1-g)rk(\mathcal{E}) - \Sigma_p \Sigma_{d=0}^{r-1} \frac{dn_d^{(p)}}{r} \qquad (4.1.9)$$

Here p varies over $|D|$ and $[n_0^{(p)}, \ldots n_{r-1}^{(p)}]$ is the vector of weights at p.

Proof. See 3.19. □

Definition 4.10. Let $\mathcal{K} = [n_0, \ldots, n_{r-1}]$ be a tuple of r ordered arbitrary non-negative integers. We define $\text{Flag}_\mathcal{K}$ to be the flag variety (of type $\mathcal{K}$) of a squence of k-vector spaces $0 = V_{r-1} \subset \ldots \subset V_i \subset \ldots \subset V_0$ such that $\dim_k V_i - \dim_k V_{i+1} = n_i$.

Remark 4.11. For the rest of this thesis, $\text{Flag}_\mathcal{K}(p)$ denotes the flag variety of type $\mathcal{K} = [n_0^{(p)}, \ldots, n_{r-1}^{(p)}]$.

Theorem 4.12. *Let $\mathcal{F}$ be a vector bundle on $\mathfrak{X} = X_{\mathcal{L},s,r}$ of type $t = (r, d, \mathcal{K})$. Then the Euler characteristic of the endomorphism sheaf of $\mathcal{F}$ is given by*

$$\chi(\mathfrak{X}, \mathcal{H}om(\mathcal{F}, \mathcal{F})) = (1-g)r^2 - \Sigma_{p \in |D|} \dim_k \text{Flag}_\mathcal{K}(p) \qquad (4.1.10)$$

We may write $\chi(\mathfrak{X}, \mathcal{H}om(\mathcal{F}, \mathcal{F})) = \chi(t, t)$ because the former only depends on the type t.

Proof. By 4.9,

$$\chi(\mathfrak{X}, \mathcal{H}om(\mathcal{F}, \mathcal{F})) = \deg(\mathcal{H}om(\mathcal{F}, \mathcal{F})) + (1-g)rk(\mathcal{H}om(\mathcal{F}, \mathcal{F})) - \Sigma_{p \in |D|} \Sigma_{d=0}^{r-1} \frac{dm_d^{(p)}}{r}.$$
$$(4.1.11)$$

Here $[m_0^{(p)}, \ldots, m_{r-1}^{(p)}]$ are the vector of weights of $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ at p. By 4.4 and 4.8 we have $d(\mathcal{H}om(\mathcal{F}, \mathcal{F})) = 0$ and $r(\mathcal{H}om(\mathcal{F}, \mathcal{F})) = r(\mathcal{F})^2 = r^2$. It is enough to prove that

$$\Sigma_{d=0}^{r-1} \frac{dm_d^{(p)}}{r} = \Sigma_{p \in |D|} \dim_k \text{Flag}_{\mathcal{K}}(p) \tag{4.1.12}$$

This we establish in the next few Lemmas.

$\square$

Lemma 4.13. *Let* $\mathcal{K} = [n_0^{(p)}, \ldots, n_{r-1}^{(p)}]$ *be the vector of weights for* $\mathcal{F}$ *at* p. *Then the vector of weights for the bundle* $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ *at* p *is given by* $[m_0^{(p)}, \ldots, m_{r-1}^{(p)}]$, *where*

$$m_d^{(p)} = \Sigma_{\{i = j + d \bmod r\}} n_i n_j \tag{4.1.13}$$

Proof. In the diagram of Lemma 4.5

$$\tilde{p}^* \mathcal{H}om(\mathcal{F}, \mathcal{F}) = \bigoplus_{i,j} \text{Hom}_{R'}(R'[i]^{n_i^{(p)}}, R'[j]^{n_j^{(p)}}) = \bigoplus_{i,j \in \mathbb{Z}/r} R'[j-i]^{n_i^{(p)} n_j^{(p)}} \tag{4.1.14}$$

The Lemma easily follows.

$\square$

Lemma 4.14. $\dim_k \text{Flag}_{\mathcal{K}}(x) = n_{r-1}(n_0 + \ldots + n_{r-2}) + n_{r-2}(n_0 + \ldots + n_{r-3}) + \ldots + n_1 n_0$

Lemma 4.15. $n_{r-1}(n_0 + \ldots + n_{r-2}) + n_{r-2}(n_0 + \ldots + n_{r-3}) + \ldots + n_1 n_0 = \sum_{i \neq j} n_i n_j$, *where* i, j *varies over* $\{0, 1, \ldots, r-1\}$ *in the right hand side.*

Proof. Denote $D_r := n_{r-1}(n_0 + \ldots + n_{r-2}) + n_{r-2}(n_0 + \ldots + n_{r-3}) + \ldots + n_1 n_0$. Clearly, $D_r = n_{r-1}(n_0 + \ldots + n_{r-2}) + D_{r-1}$. Now the Lemma follows by induction on r.

$\square$

Lemma 4.16. $\sum_{d=0}^{r-1} d \sum_{i+d=j \ mod \ r} (n_i n_j) = r \sum_{i \neq j} n_i n_j$

Proof.

$$
\sum_{d=0}^{r-1} d \sum_{i+d=j \text{ in } \mathbb{Z}/r} (n_i n_j) = \Sigma_{d=0}^{r-1} d(n_i n_j) + (r-d)(n_i n_j)
$$
$$
= \Sigma_{d=0}^{r-1}(d + r - d)n_i n_j \qquad (4.1.15)
$$
$$
= r \Sigma_{i \neq j} n_i n_j
$$

$\square$

Let $\text{PBun}_X^{r,d,\mathcal{K}}$ be the stack of parabolic bundles over X of rank r, parabolic degree d and weight vectors $\mathcal{K}$. Let d_0 be the degree of the underlying vector bundle. d_0 is determined by d and $\mathcal{K}$, see 4.18

Lemma 4.17. *The forgetful functor* $\text{PBun}_X^{r,d,\mathcal{K}} \to \text{Bun}_X^{r,d_0}$ *is representable and smooth of relative dimension* $\Sigma_{x \in |D|} \dim_k \text{Flag}_{\mathcal{K}}(x)$.

Proof. Let A be a k-algebra of finite type and $\mathcal{E}_0$ be a vector bundle of rank r and degree d on $X \times_k Spec(A)$. We claim that the following commuting square is Cartesian.

$$
\begin{array}{ccc}
\prod_{x \in |D|} \text{Flag}_{\mathcal{K}}(x) & \longrightarrow & \text{spec}(A) \\
\downarrow & & \downarrow \mathcal{E}_0 \\
\text{PBun}_X^{r,d,\mathcal{K}} & \longrightarrow & \text{Bun}_X^{r,d}
\end{array}
$$

Let $\mathcal{Z}$ be the fiber product. Given a scheme S we will construct a functorial isomorphism $\psi : \mathcal{Z}(S) \to \prod_{x \in |D|} \text{Flag}_{\mathcal{K}}(x)(S)$. The objects of $\mathcal{Z}(S)$ are given by the triples $\{(\mathcal{F}, f : S \to \text{spec}(A), \varphi) : \mathcal{F}_0 \cong^\varphi \mathcal{E}_0\}$ where $\mathcal{F}$ is a parabolic vector bundle on $X \times_k S = X \times_k \text{spec}(A) \times_{\text{spec}(A)} S$ of type $t = (r, d, \mathcal{K})$. Define

$\psi(\mathcal{F}, f, \varphi) = (\sigma_x)_{x \in |D|}$. Where $\sigma_x(s)$ is the flag at x induced by $\mathcal{F}$(via φ) on $X \times \{s\} \subset X \times S$.

By the definition of parabloic structure on $\mathcal{F}_0$, ψ is a bijection. By construction one can easily check the functoriality of ψ.

$\square$

Definition 4.18. We say a vector bundle $\mathcal{F}$ on $\mathfrak{X}$ is of type $t = (r, d, \mathcal{K})$ if it has rank r, degree d and a set of weight vectors $\mathcal{K}$ at the points in $|D|$. We denote the degree of the underlying vector bundle on X by d_0. The two degrees are related by the following formula[A.6, [Bor07]].

$$d = d_0 + \Sigma_{p \in |D|} \Sigma_{k=1}^r \frac{k n_k^{(p)}}{r} \tag{4.1.16}$$

Here $[n_0^{(p)}, \ldots, n_{r-1}^{(p)}]$ are the weight vectors of $\mathcal{F}$ at $p \in |D|$.

Definition 4.19. Let A be a finitely generated k-algebra. Given coherent sheaves F_1 and F_2 over $\mathfrak{X} \times_k \mathrm{spec}(A)$, we define the stack of extensions of F_2 by F_1 denoted $\mathcal{E}xt_{\mathfrak{X}}(F_2, F_1)$ as follows. For an A-scheme $p : S \to \mathrm{spec}(A)$ the groupoid $\mathcal{E}xt_{\mathfrak{X}}(F_2, F_1)(S)$ is given by

- The objects are short exact sequences $0 \to p^*F_1 \to F \to p^*F_2 \to 0$ of coherent sheaves on $\mathfrak{X} \times_k S$.

- The morphisms are $\mathcal{O}_{\mathfrak{X} \times_k S}$-module homomorphisms $\mathcal{F} \to \mathcal{F}'$ compatible with the identity on p^*F_i.

Definition 4.20. We denote the stack parametrizing extensions of vector bundles on $\mathfrak{X}$ of type t_1 and t_2 by $\mathcal{E}xt_{\mathfrak{X}}(t_2, t_1)$. Note the difference between $\mathcal{E}xt_{\mathfrak{X}}(t_2, t_1)$ and $\mathcal{E}xt_{\mathfrak{X}}(F_2, F_1)$. The latter parametrizes extensions of fixed sheaves F_1 and F_2 while the former only asks for extensions of sheaves of fixed types t_1 and t_2.

Lemma 4.21. *Given a parabolic coherent sheaf F on (X, D) of weights $\{0, \frac{1}{r} \ldots, \frac{r-1}{r}\}$, there exists an inclusion of parabolic vector bundles $i : E_1 \to E_2$ on (X, D) of same weights whose cokernel is F.*

Proof. By prop 2.1.10 in [HL10], there exist vector bundles E_1, E_2 and a short exact sequence $0 \to E_1 \xrightarrow{i} E_2 \xrightarrow{q} F \to 0$ of coherent sheaves. Say $F' \hookrightarrow F$ is an inclusion of vector bundles. We can define, for an open set $U \subset X$, $E_2'(U) = q^{-1}(F'(U))$. Clearly E_2' defines a coherent $\mathcal{O}_X$-module. Since X is a smooth curve, the stalks O_x are discrete valuation rings. In particular they are PIDs. Hence the submodule $(E_2')_x \subset (E_2)_x$ is a free $\mathcal{O}_x$-module, making E_2' a locally free $\mathcal{O}_X$-module. Note that, by definition E_1 can be included in E_2'. This defines an inclusion $i' : E_1 \to E_2'$ whose cokernel is F'.

The above construction allows us to define a parabolic structure on E_2 given a parabolic structure on F. Hence we have obtained parabolic sheaves E_2, F and the trivial parabolic structure on E_1 makes $0 \to E_1 \xrightarrow{i} E_2 \xrightarrow{q} F \to 0$ a short exact sequence of parabolic sheaves. $\qquad\square$

We will prove the following theorem in the next section as Theorem 4.29 but we record it here to be able to prove the subsequent Theorem 4.23.

Theorem 4.22. $\mathcal{E}xt_{\mathfrak{X}}(F_2, F_1)$ *is algebraic, of finite type, smooth of relative dimension $-\chi(F_1, F_2)$ over A with all its fibers geometrically irreducible.*

Theorem 4.23. *Let $\mathfrak{X} = X_{(\mathcal{L}, s, r)}$ be the root stack and $t = (r, d, \mathcal{K})$. Then the stack of parabolic bundles $\mathrm{PBun}_{X,t}$ is isomorphic to $\mathrm{Bun}_{\mathfrak{X},t}$. Furthermore, the following assertions hold:*

  1 The stack $\mathrm{Bun}_{\mathfrak{X},t}$ is smooth of dimension $-\chi(t, t)$.

  2 The natural morphism of stacks $\mathcal{E}xt_{\mathfrak{X}}(t_2, t_1) \to \mathrm{Bun}_{\mathfrak{X}, t_1} \times \mathrm{Bun}_{\mathfrak{X}, t_2}$ is smooth with connected fibers of relative dimension $-\chi(t_2, t_1)$.

   3 $\mathcal{E}xt_{\mathfrak{X}}(t_2, t_1)$ is smooth of relative dimension $-\chi(t_2, t_1) - \chi(t_1, t_1) - \chi(t_2, t_2)$

Proof. (1) follows from 4.17, Theorem 4.9 and the fact that $\text{Bun}_{X,t}$ is smooth over $\text{spec}(k)$ of dimension $(g-1)r^2$, see Appendix in [Hof10]. The fiber of $\mathcal{E}xt_{\mathfrak{X}}(t_2, t_1) \rightarrow \text{Bun}_{\mathfrak{X},t_1} \times \text{Bun}_{\mathfrak{X},t_2}$ at an K-valued point $(F_1, F_2) : \text{spec}(K) \rightarrow \text{Bun}_{\mathfrak{X},t_1} \times \text{Bun}_{\mathfrak{X},t_2}$ is given by $\mathcal{E}xt_{\mathfrak{X}}(F_2, F_1)$. Now (2) follows from Theorem 4.29 and [Hof10]. (3) follows from (1) and (2). $\qquad\qquad\square$

4.2 Nilpotent stacks

Let $\mathcal{N}il_{n,\mathfrak{X}}$ denote the stack of nilpotent coherent sheaves on $\mathfrak{X}$ defined as follows. For a k-scheme S,

- Objects of $\mathcal{N}il_{n,\mathfrak{X}}(S)$ are pairs $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a coherent sheaf on $\mathfrak{X}$ and φ is an endomorphism of $\mathcal{E}$ such that $\varphi^n = 0$

- A morphism between $(\mathcal{E}, \varphi)$ and $(\mathcal{F}, \psi)$ is an isomorphism of sheaves $\alpha : \mathcal{E} \rightarrow \mathcal{F}$ with $\alpha\varphi = \psi\alpha$

   Our goal in this section is to prove that $\mathcal{N}il_{n,\mathfrak{X}}$ is a smooth stack and find its dimension at a given point K-valued point (E, φ) for a field $K \supset k$. For this we imitate the ideas in [BDH18] and adopt them for the root stack $\mathfrak{X}$.

Remark 4.24. Let G be a finite group acting on a ring R', assume that $|G|$ is invertible in R'. Let M and N be R'-modules on which G acts, compatible with the action on R'. If $M \xrightarrow{f} N$ be a G-equivariant surjective morphism of R'-modules then $M^G \xrightarrow{f^0} N^G$ is a surjective morphism of $R = (R')^G$-modules. Indeed, the averaging morphisms $M \rightarrow M^G$ sending m to $\frac{1}{|G|}\Sigma g.m$ and similarly $N \rightarrow N^G$ are surjective, making f^0 surjective. Averaging morphisms are well defined because $|G|$ is invertible.

Lemma 4.25. *Let $\mathcal{F}$ be a coherent sheaf on the root stack $\mathfrak{X}$ and $\pi : \mathfrak{X} \to X$ be the canonical map. Then, for $i \geq 1$, $R^i \pi_* \mathcal{F} = 0$.*

Proof. It is enough to prove that π_* is exact. π can be restricted to a morphism of the form $U' = [\mathrm{spec}(R')/\mu_r] \to U = \mathrm{spec}(R)$, where $R' = R[T]/(T^r - s)$. The restriction $\mathcal{F}|_{U'}$ of $\mathcal{F}$ to $U' = [Y/\mu_r]$ is equivalent to a μ_r equivariant R'-module M. The pushforward $\pi_*(\mathcal{F}|_U)$ is the sheaf associated to the R-module M^{μ_r}. So to prove that π_* is exact, it is enough to prove that the fixed point functor $()^{\mu_r}$ is exact. But this is true because of the remark 4.24 and the fact that $r = |\mu_r|$ is invertible in R. $\qquad\square$

Proposition 4.26. *Let $\mathfrak{X} = X_{(\mathcal{L}, s, r)}$ be the root stack and S be a k-scheme. We write $p = p_0 \pi$, where $\pi : \mathfrak{X}_S \to X_S$ is the canonical map to the curve and $p_0 : X_S \to S$ is the structure map of the curve. Let $\mathcal{F}$ be a coherent sheaf on $\mathfrak{X}$. Then $\mathbf{R}p_* \mathcal{F}$ can be represented by a two term complex of vector bundles in $D^b(S)$.*

Proof. Since $R^i \pi_*(\mathcal{I}) = 0$ for an injective object $\mathcal{I}$ by Proposition 2.58 in [Huy06] we may write $\mathbf{R}p_* = \mathbf{R}(p_0)_* \mathbf{R}\pi_*$. Since π is proper and $R^i \pi_* = 0$ for $i \geq 1$ by Lemma 4.25, $\mathbf{R}\pi_*$ takes a coherent sheaf to a one term complex of coherent sheaves. Since $p_0 : X_S \to S$ is a smooth curve, $\mathbf{R}p_{0*}$ of a coherent sheaf may be represented by a two term complex of vector bundles on S. The lower order terms in the Leray spectral sequence reads as $0 \to R^1(p_0)_*(\pi_* \mathcal{F}) \to R^1 p_* \mathcal{F} \to (p_0)_* R^1 \pi_* \mathcal{F} \to R^2(p_0)_*(\pi_* \mathcal{F}) \to R^2 p_* \mathcal{F} \to (p_0)_* R^2 \pi_* \mathcal{F} \to \dots.$

We have, $R^2(p_0)_*(\pi_* \mathcal{F}) = 0$ since X is a smooth curve. $(p_0)_* R^2 \pi_* \mathcal{F} = 0$ since $R^i \pi_* \mathcal{F} = 0$ for $i \geq 1$ by Lemma 4.25. Hence $R^2 p_* \mathcal{F} = 0$.

Similarly, for large i the sequence reads as

$\dots \to R^i(p_0)_*(\pi_* \mathcal{F}) \to R^i p_* \mathcal{F} \to (p_0)_* R^i \pi_* \mathcal{F} \to \dots$

By the same reasoning as for the $i = 2$ case, the $R^i p_* \mathcal{F}$ vanish for all $i \geq 2$. Hence $\mathbf{R}p_* \mathcal{F}$ can be represented by a two term complex of vector bundles in $D^b(S)$.

$\square$

Remark 4.27. Let $\mathcal{F}_1, \mathcal{F}_2$ be coherent $\mathcal{O}_{\mathfrak{X}_S}$-modules flat over S and $p : \mathfrak{X}_S \to S$ be the projection map. Since p is proper, $p_* \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$ is a coherent sheaf over S. The object $\mathbf{R}p_* \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$ in the derived category of coherent sheaves $D^b(S)$ is representable by a two term complex $E_0 \xrightarrow{\varphi} E_1$ of locally free sheaves over S. This follows from Proposition 4.26.

Remark 4.28. For any two term complex of locally free sheaves $E_0 \xrightarrow{\varphi} E_1$ as above, one can associate a Picard stack to it, given by the quotient stack $[E_1/E_0]$, where E_0 acts on E_1 via φ. Two such complexes which are quasi-isomorphic give isomorphic Picard stacks [SGA4, Exp. XVIII]. We describe the quotient stack as follows, using Lemma 0.1 in [Hei04].

If $T \to S$ is morphism of schemes, $T = \operatorname{spec}(A)$ is affine, then

- $ob([E_1/E_0](T)) = H^0(T, f^* E_1)$

- Given two objects s and t, a morphism between them in $[E_1/E_0](T)$ is given by $h \in H^0(T, f^* E_0)$ such that $\varphi(h) = t - s$.

Theorem 4.29. *With the hypothesis in 4.27, let $S = \operatorname{spec}(K)$, for a field K, the stack of extensions $\mathcal{E}xt(\mathcal{F}_1, \mathcal{F}_2)$ is isomorphic to $[E_1/E_0]$. Hence it is smooth of dimension $\dim_K \operatorname{Ext}^1_{\mathfrak{X}}(\mathcal{F}_1, \mathcal{F}_2) - \dim_K \operatorname{Hom}_{\mathfrak{X}}(\mathcal{F}_1, \mathcal{F}_2)$.*

Proof. The length 1 complex $E_0 \xrightarrow{\varphi} E_1$ obtained in 4.27 has cohomology $H^0(\varphi) = H^0(\mathbf{R}p_* \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)) = \operatorname{Ext}^0(\mathcal{F}_1, \mathcal{F}_2) = \operatorname{Hom}_{\mathfrak{X}}(\mathcal{F}_1, \mathcal{F}_2)$ and $H^1(\varphi) = H^1(\mathbf{R}p_* \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)) = \operatorname{Ext}^1_{\mathfrak{X}}(\mathcal{F}_1, \mathcal{F}_2)$. Hence we have $\dim_K[E_1/E_0] = \dim_K E_1 - \dim_K E_0 = \dim_K \operatorname{Ext}^1_{\mathfrak{X}}(\mathcal{F}_1, \mathcal{F}_2) - \dim_K \operatorname{Hom}_{\mathfrak{X}}(\mathcal{F}_1, \mathcal{F}_2)$.

□

Definition 4.30. Let

$$0 \to K \to E \to Q \to 0 \qquad (4.2.1)$$

be a short exact sequence and let $f : \tilde{K} \to K$ be a morphism of vector bundles over $\mathfrak{X}$.

Let $\mathcal{E}xt_{\mathfrak{X}}(Q, f : \tilde{K} \to K)/E$ denote the stack parametrzing the following diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K} & \longrightarrow & \tilde{E} & \longrightarrow & Q & \longrightarrow & 0 \\
& & \downarrow f & & \downarrow & & \downarrow \text{id} & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0
\end{array}
$$

(def 3.7 in [BH08])

Lemma 4.31. *The following square of 1-morphisms of stacks is Cartesian and its vertical right arrow (hence the left) is smooth.*

$$
\begin{array}{ccc}
\mathcal{E}xt_{\mathfrak{X}}(Q, f : \tilde{K} \to K)/E & \longrightarrow & \mathcal{E}xt_{\mathfrak{X}}(Q, \tilde{K}) \\
\downarrow & & \downarrow \mathbf{f}_* \\
spec(A) & \longrightarrow & \mathcal{E}xt_{\mathfrak{X}}(Q, K)
\end{array}
$$

Proof. The proof is similar to the one in Lemma 3.8 of [BH08]. We imitate it here for the case of the root stack $\mathfrak{X}$ in the place of the curve X.

We can represent the morphism $f_* : RHom(Q, K) \to RHom(Q, \tilde{K})$ in $D^b(\text{spec}(A))$ by a chain morphism

$$
\begin{array}{ccc}
E^0 & \xrightarrow{\;f^0\;} & \tilde{E}^0 \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\tilde{\delta}} \\
E^1 & \xrightarrow{\;f^1\;} & \tilde{E}^1
\end{array}
$$

of length one complexes, where $E^i, \tilde{E}^i$ are vector bundles over $\mathrm{spec}(A)$. We established the 1-isomorphisms $[E^1/E^0] \cong \mathcal{E}xt(Q, K)$ and $[\tilde{E}^1/\tilde{E}^0] \cong \mathcal{E}xt(Q, \tilde{K})$ in Lemma 4.29. Furthermore, by construction the following square commutes up to 2-isomorphism

$$
\begin{array}{ccc}
\frac{E^1}{E^0} & \xrightarrow{\;\cong\;} & \mathcal{E}xt(Q, K) \\
\downarrow{\scriptstyle(f^1, f^0)} & & \downarrow{\scriptstyle f_*} \\
\frac{\tilde{E}^1}{\tilde{E}^0} & \xrightarrow{\;\cong\;} & \mathcal{E}xt(Q, \tilde{K})
\end{array}
$$

Hence the smoothness and surjectivity of the 1-morphism (f^1, f^0) implies the same for f_*.

The diagram of Picard stacks is Cartesian

$$
\begin{array}{ccc}
(E^1 \oplus \tilde{E}^0)/E^0 & \longrightarrow & E^1/E^0 \\
\downarrow{\scriptstyle(f^1, -\tilde{\delta})} & & \downarrow{\scriptstyle(f^1, f^0)} \\
\tilde{E}^1 & \longrightarrow & \tilde{E}^1/\tilde{E}^0
\end{array}
$$

The canonical morphisms $E^1 \oplus \tilde{E}^0 \to (E^1 \oplus \tilde{E}^0)/E^0$ and $\tilde{E}^1 \to \tilde{E}^1/\tilde{E}^0$ are smooth and surjective. Now it is enough to show that the morphism $(f^1, -\tilde{\delta}) : E^1 \oplus E^0 \to \tilde{E}^1$. Smoothness follows from the fact that this is a morphism of vector bundles. Notice that by the choice of $E^i, \tilde{E}^i$, the cokernel of $(f^1, -\tilde{\delta})$ is the same as the cokernel of the k-linear map

$$f_* : \operatorname{Ext}^1_{\mathfrak{X}}(Q, K) \to \operatorname{Ext}^1_{\mathfrak{X}}(Q, \tilde{K})$$

The cokernel of f_* is 0 by using the long exact sequence for Ext and the fact that $\operatorname{Ext}^2_{\mathfrak{X}}(Q, \ker(f)) = 0$. $\qquad\square$

Lemma 4.32. $pr_{ext} : \mathcal{M}^{*\hookrightarrow*\hookrightarrow*}_{\mathfrak{X}} \to \mathcal{M}^{*\to*}_{\mathfrak{X}}$ *sending* $(E_1 \subset E_2 \subset E)$ *to the composition* $E_2 \hookrightarrow E \twoheadrightarrow \frac{E}{E_1}$ *is smooth of relative dimension* $-\chi(\frac{E}{E_2}, E_1)$.

Proof. Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of coherent sheaves over $\mathcal{O}_{\mathfrak{X}}$. We get a short exact sequence of coherent sheaves:

$$0 \to im(\varphi) \to \mathcal{G} \to \operatorname{coker}(\varphi) \to 0 \qquad (4.2.2)$$

Consider the moduli stack parametrizing the coherent sheaves E which fit into the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & E & \longrightarrow & \operatorname{coker}(\varphi) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & im(\varphi) & \longrightarrow & \mathcal{G} & \longrightarrow & \operatorname{coker}(\varphi) & \longrightarrow & 0
\end{array}
$$

This moduli stack is $\mathcal{E}xt_{\mathfrak{X}}(Q, f)/E$ that we defined above. Here the surjective morphism $\varphi : \mathcal{F} \twoheadrightarrow im(\varphi)$ is denoted by f and $Q := \operatorname{coker}(\varphi)$. On the other hand the stack of such extensions with $\varphi : \mathcal{F} \to \mathcal{G}$ factoring as $\mathcal{F} = E_2 \hookrightarrow E \twoheadrightarrow \frac{E}{E_1}$ is precisely the fiber of pr_{ext} at the point φ. So by the previous Lemma, the relative dimension of pr_{ext} at $E_1 \subset E_2 \subset E$ is

$$\dim_{(E_1 \subset E_2 \subset E)}(\mathcal{E}xt_{\mathfrak{X}}(Q, f)/E) = \dim\mathcal{E}xt_{\mathfrak{X}}(Q, \mathcal{F}) - \dim\mathcal{E}xt_{\mathfrak{X}}(Q, im(\varphi))$$

$$= -\chi(\mathrm{coker}(\varphi), \mathcal{F}) + \chi(\mathrm{coker}(\varphi), im(\varphi))$$

$$= -\chi(\frac{E}{E_2}, E_2) + \chi(\frac{E}{E_2}, \frac{E_2}{E_1})$$

$$= -\chi(\frac{E}{E_2}, E_1)$$

$\square$

Theorem 4.33. *The following commutative diagram of stacks is Cartesian*

$$
\begin{array}{ccc}
Nil_{\mathfrak{X}}^{n+1} & \xrightarrow{(E,\varphi) \mapsto (\ker(\varphi) \subset E \supset im(\varphi))} & \mathcal{M}_{\mathfrak{X}}^{* \hookrightarrow * \hookleftarrow *} \\
\downarrow{\scriptstyle pr_n} & & \downarrow{\scriptstyle pr_\to} \\
Nil_{\mathfrak{X}}^{n} & \xrightarrow{(\mathcal{F},\psi) \mapsto (\psi : \mathcal{F} \to \mathcal{F})} & \mathcal{M}_{\mathfrak{X}}^{* \to *}
\end{array}
$$

and the fiber product of the stacks

$$
\begin{array}{ccc}
\mathcal{M}_{\mathfrak{X}}^{* \to *} \times_{Coh_{\mathfrak{X}}} \mathcal{M}_{\mathfrak{X}}^{* \hookrightarrow *} & \xrightarrow{\hspace{5cm}} & \mathcal{M}_{\mathfrak{X}}^{* \hookrightarrow *} \\
\downarrow & & \downarrow{\scriptstyle pr_{sub}} \\
\mathcal{M}_{\mathfrak{X}}^{* \to *} & \xrightarrow{(\psi : \mathcal{F}' \to \mathcal{F}) \mapsto \ker \psi} & Coh_{\mathfrak{X}}
\end{array}
$$

makes the following diagram Cartesian as well:

$$
\begin{array}{ccc}
\mathcal{M}_{\mathfrak{X}}^{* \hookrightarrow * \hookleftarrow *} & \xrightarrow{(E_1 \subset E \supset E_2) \mapsto (E_1 \subset E_1 + E_2 \subset E)} & \mathcal{M}_{\mathfrak{X}}^{* \hookrightarrow * \hookrightarrow *} \\
\downarrow{\scriptstyle pr_\to \times pr_\cap} & & \downarrow{\scriptstyle pr_{ext}} \\
\mathcal{M}_{\mathfrak{X}}^{* \to *} \times_{Coh_{\mathfrak{X}}} \mathcal{M}_{\mathfrak{X}}^{* \hookrightarrow *} & \xrightarrow{(\psi : \mathcal{F}' \to \mathcal{F}, \ker \psi \subset \mathcal{F}") \mapsto (\psi + 0 : \frac{\mathcal{F}' \oplus \mathcal{F}"}{\ker \psi} \to \mathcal{F})} & \mathcal{M}_{\mathfrak{X}}^{* \to *}
\end{array}
$$

Proof. This is exactly Theorem 6.1 in [BDH18]. The proof only uses the fact that category of coherent sheaves is abelian, the same is true of parabolic coherent sheaves by Proposition 2.57. □

Lemma 4.34. *Suppose* $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ *is a short exact sequence of parabolic bundles whose parabolic data is given by* $\mathcal{K}_1, \mathcal{K}$ *and* $\mathcal{K}_2$ *respectively then for each* $x \in |D|$,

$$\dim_K \mathrm{Flag}_{\mathcal{K}_1}(x) + \dim_K \mathrm{Flag}_{\mathcal{K}_2}(x) \leq \dim_K \mathrm{Flag}_{\mathcal{K}}(x)$$

Proof. See Lemma 11.1 in [BDL12]. □

Corollary 4.35. *The stack* $\mathcal{N}il_{n,\mathfrak{X}}$ *is smooth over* k. *Its dimension at the* K-*valued point given by a coherent sheaf* E *on* $\mathfrak{X}_K$ *and* $\varphi \in \mathrm{End}(E)$ *with* $\varphi^n = 0$ *is*

$$\dim_{(E,\varphi)}(\mathcal{N}il_{\mathfrak{X},n}) = (g-1)\Sigma_{i=1}^n r_i^2 + \Sigma_{i=1}^n \Sigma_{p \in |D|} \mathrm{Flag}_{\mathcal{K}_{i,p}}(p), \qquad (4.2.3)$$

where r_i *denotes the rank of the underlying coherent sheaf over* X_K *of* $im(\varphi^{i-1})/im(\varphi^i)$ *over the generic point of* X_K

Proof. The 1-morphism of stacks $\mathcal{E}xt_{\mathfrak{X}}(t_2, t_1) \to Coh_{\mathfrak{X},t_1} \times Coh_{\mathfrak{X},t_2}$ is smooth of relative dimension $-\chi(t_2, t_1)$. This follows from Theorem 4.23.

Hence $pr_{sub} : \mathcal{M}_{\mathfrak{X}}^{* \hookrightarrow *} \to Coh_{\mathfrak{X}}$ defined by $(E_0 \subset E_1) \mapsto E_0$ is smooth of relative dimension $-\chi(\frac{E_1}{E_0}, E_0) - \chi(\frac{E_1}{E_0}, \frac{E_1}{E_0}) = -\chi(\frac{E_1}{E_0}, E_1)$ at the point $(E_0 \subset E_1)$.

By Lemma 4.31 $pr_{ext} : \mathcal{M}_{\mathfrak{X}}^{* \hookrightarrow * \hookrightarrow *} \to \mathcal{M}_{\mathfrak{X}}^{* \to *}$ defined by $(E_1 \subset E_3 \subset E) \mapsto (E_3 \hookrightarrow E \to \frac{E}{E_1})$ is smooth of relative dimension $-\chi(\frac{E}{E_1}, E_1) + \chi(\frac{E_3}{E_1}, E) = -\chi(\frac{E}{E_3}, E_1)$ at a K-valued point $(E_1 \subset E_3 \subset E)$.

pr_{ext} is smooth implies $pr_\to \times pr_\cap$ is smooth by the third Cartesian square in 4.33.

pr_{sub} is smooth imples pr_1 is smooth by the above Cartesian square.

Hence $pr_\to = pr_1 \circ (pr_\to \times pr_\cap)$ is smooth. It also follows that $\dim(pr_\to) = \dim(pr_{ext}) + \dim(pr_{sub})$.

Hence $pr_\to$ is smooth of relative dimension $-\chi(\frac{E_1}{E_1 \cap E_2}, E_1) - \chi(\frac{E}{E_1 + E_2}, E_1) = -\chi(\frac{E}{E_2}, E_1)$ at a K-valued point $(E_1 \subset E \supset E_2)$.

Therefore the morphism pr_n is smooth with relative dimension $-\chi(E/im\varphi, \ker \varphi) = -\chi(\frac{E}{im\varphi}, \frac{E}{im\varphi}) = (g-1)r_1^2 + \Sigma_{p \in |D|} \mathrm{Flag}_{\mathcal{K}_{1,p}}(p)$. at the point (E, φ). Here $\mathcal{K}_{1,p}$ is the weight vector of $\frac{E}{im\varphi}$ at p. Using induction on n we conclude that the dimension of $\mathcal{N}il_{n,\mathfrak{x}}$ at (E, φ) is $(g-1)\Sigma_{i=1}^n r_i^2 + \Sigma_{i=1}^n \Sigma_{p \in |D|} \mathrm{Flag}_{\mathcal{K}_{i,p}}(p)$ $\qquad \square$

Lemma 4.36. *Let $\mathcal{C}$ be the closure of a point by E in $Coh_{\mathfrak{x}}(K)$, then*

$$\dim_k \mathcal{C} = \mathrm{trdeg}_k(k(E)) - \dim_K \mathrm{End}(E) \qquad (4.2.4)$$

Proof. We may assume $\mathcal{C}$ to be a quotient stack $[U/H]$ for some scheme U and some algebraic group H . Let $\mathcal{G} \hookrightarrow \mathcal{C}$ be the residual gerbe. We have the following Cartesian square

$$R \longrightarrow U$$
$$\downarrow H \qquad \downarrow H$$
$$\mathcal{G} \longrightarrow \mathcal{C}$$

We have $\dim_k(U) - \dim_k(\mathcal{C}) = \dim_k H$ and $\dim_{k(\mathcal{G})} R - \dim_{k(\mathcal{G})}\mathcal{G} = \dim_k H$. Combining them with the equations $\dim_k R = \dim_k U$ (since R is an open dense subscheme of U) and $\dim_k U = \mathrm{trdeg}_k k(U) = \mathrm{trdeg}_k(k(\mathcal{G})) + \dim_{k(\mathcal{G})} U$ we get the required formula.

$\square$

Corollary 4.37. *Let E be an indecomposable vector bundle with set of weight vectors $\mathcal{K}$ over $\mathfrak{X}_K$ for an algebraically closed field $K \supset k$. If r_i denotes the generic rank of $im(\varphi^{i-1})/im(\varphi^i)$ for a general element φ of the Jacobson radical $j(E)$ in $\mathrm{End}_{\mathfrak{X}}(E)$, then*

$$\mathrm{trdeg}_k k(E) \leq 1 + (g-1)\Sigma_i r_i^2 + \Sigma_{x \in |D|}\dim_K \mathrm{Flag}_{\mathcal{K}}(x) \qquad (4.2.5)$$

Proof. Let E_0 be the underlying vector bundle of E. Since E is indecomposable, Krull-Schmidt for vector bundles implies that $\mathrm{End}_X(E_0)/j(E_0) = K$, since $\mathrm{End}_{\mathfrak{X}}(E) \subset \mathrm{End}_X(E_0)$ it follows that $\mathrm{End}_{\mathfrak{X}}(E)/j(E) = K$.

Let $\mathcal{C}$ be the closure of a point given by E in $Coh_{\mathfrak{X}}$. By the previous proposition

$$\dim_k \mathcal{C} = \mathrm{trdeg}_k k(E) - \dim_K \mathrm{End}_{\mathfrak{X}}(E) \qquad (4.2.6)$$

Choose $n \in \mathbb{N}$ with $(j(E))^n = 0$. Let $\mathcal{N} \subset Nil_{\mathfrak{X},n}$ be the closure of the points (E, φ) with $\varphi \in j(E)$ such that each $im(\varphi)^{i-1}/im(\varphi)^i$ has generic rank r_i.

The fiber of the forgetful 1-morphism $\mathcal{N} \to \mathcal{C}$ over the dense point $E : spec(K) \to \mathcal{C}$ contains an open dense subscheme of $j(E)$. So we have,

$$\dim_k \mathcal{N} \geq \dim_k \mathcal{C} + \dim_K j(E) = \dim_k \mathcal{C} + \dim_K \operatorname{End}_{\mathfrak{x}}(E) - 1 = \operatorname{trdeg}_k k(E) - 1$$
$$(4.2.7)$$

From the previous corollary we have,

$$\operatorname{trdeg}_k k(E) \leq 1 + (g-1)\Sigma_{i=1}^n r_i^2 + \Sigma_{x \in |D|} \dim_K \operatorname{Flag}_{\mathcal{K}}(x) \qquad (4.2.8)$$

$\square$

Remark 4.38. We may assume that $k(E) = k(E \otimes_K L)$ for any field $L \supset K$

Lemma 4.39. *Let E be a vector bundle of type $t = (r, d, \mathcal{K})$ over $\mathfrak{X}_K$ for a field $K \supset k$. If E is not simple, and the curve X has genus $g \geq 2$, then*

$$\operatorname{trdeg}_k(k(E)) \leq (g-1)(r^2 - r) + 2 + \Sigma_{x \in |D|} \dim_K \operatorname{Flag}_{\mathcal{K}}(x) \qquad (4.2.9)$$

Proof. By the above remark we may assume K is algebraically closed. Hence by Krull-Schmidt E can be written as a direct sum of indecomposable vector bundles E_i over $\mathfrak{X}_K$ of rank $r_i \geq 1$ and ramification data $\mathcal{K}_i$. The above corollary says that

$$\operatorname{trdeg}_k k(E_j) \leq 1 + (g-1)\sum_i r_{ij}^2 + \Sigma_{x \in |D|} \dim(\operatorname{Flag}_{\mathcal{K}_|}(x)) \qquad (4.2.10)$$

for some integers $r_{ij} \geq 1$ such that $\sum_i r_{ij} = r_j$.

We also have that $\Sigma_j \Sigma_{x \in |D|} \dim_K \operatorname{Flag}_{\mathcal{K}_j}(x) \leq \Sigma_{x \in |D|} \dim_K \operatorname{Flag}_{\mathcal{K}}(x)$ by 4.34.

Using

$$\operatorname{trdeg}_k k(E) \leq \sum_j \operatorname{trdeg}_k k(E_j) \qquad (4.2.11)$$

we have

$$\text{trdeg}_k k(E) \le \sum_j 1 + (g-1) \sum_{i,j} r_{ij}^2 + \Sigma_{x \in |D|} \dim_K \text{Flag}_{\mathcal{K}}(x) \qquad (4.2.12)$$

Because E is not simple, the sum $\sum_{i,j} r_{ij} = r$ has at least two summands. Hence

$$\text{trdeg}_k k(E) \le (g-1)(r^2-r) + \Sigma_{x \in |D|} \dim_K \text{Flag}_{\mathcal{K}}(x) + 2 - (g-2)(r-2) \quad (4.2.13)$$

$r \ge 2$ and $g \ge 2$ implies $(g-2)(r-2) \ge 0$.

So

$$\text{trdeg}_k k(E) \le (g-1)(r^2 - r) + \Sigma_{x \in |D|} \dim_K \text{Flag}_{\mathcal{K}}(x) + 2 \qquad (4.2.14)$$

$\square$

Chapter 5

Gerbes and central simple algebras

Let X be a non-singular algebraic curve over an algebraically closed field k, D be a reduced divisor and an integer $r \geq 1$ coprime to $char(k)$ as in the previous section. Let $\mathfrak{X} = X_{(\mathcal{L},s,r)}$ be the root stack associated to such data.

Let $K \supset k$ be a field and let $\mathcal{G}$ denote the residual gerbe of a point $E : spec(K) \to Coh_{\mathfrak{X}}$. $\mathcal{G}$ is an algebraic stack over $\operatorname{spec}(k(\mathcal{G}))$, where the field $k(\mathcal{G})$ is the field of moduli of E (see chapter 11 of [LMB00]). The field of moduli is sometimes also denoted by $k(E)$. E is equivalent to choosing a parabolic coherent sheaf $E_\bullet$ on X with rational weights $\{0, \frac{1}{r}, \ldots, \frac{r-1}{r}\}$ (Borne-Vistoli).

$\mathcal{G}$ is in fact a gerbe over $k(\mathcal{G})$ (chapter 11 of [LMB00]). By the definition of a gerbe, there exists a finitely generated $k(\mathcal{G})$-algebra B such that $\mathcal{G}(B) \neq \emptyset$. By the Hilbert's Nullstellensatz there exists a maximal ideal $\mathfrak{m} \subset B$ such that the field $l = B/\mathfrak{m}$ is a finite extension of $k(\mathcal{G})$. We denote

$$d := [l : k(\mathcal{G})] < \infty \tag{5.0.1}$$

By taking the image of an object in $\mathcal{G}(B)$ to $\mathcal{G}(l)$ we have that $\mathcal{G}(l) \neq \emptyset$. We choose a coherent sheaf $F \in \mathcal{G}(l)$. Let $\pi : \mathfrak{X}_l \to \mathfrak{X}_{k(\mathcal{G})}$ be the canonical projection, and let

$$A := \operatorname{End}_{\mathfrak{X}}(\pi_* F)$$

Definition 5.1. We say that a coherent sheaf E over $\mathfrak{X}_K$ is simple if $\operatorname{End}_{\mathfrak{X}_K}(E) = K$.

Remark 5.2. Let $\mathcal{G}$ be a residual gerbe of $Coh_{\mathfrak{X}}$ parametrizing simple coherent sheaves. Then $\mathcal{G}$ is a gerbe with band $\mathbb{G}_m$ over $k(\mathcal{G})$ and A is a central simple algebra of degree d over $k(\mathcal{G})$ as demonstrated in the next few Lemmas.

Consider the following Cartesian square

$$
\begin{array}{ccc}
\mathfrak{X}_l & \xrightarrow{\ \pi\ } & \mathfrak{X}_{k(\mathcal{G})} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\operatorname{spec}(l) & \xrightarrow{\ s\ } & \operatorname{spec}(k(\mathcal{G}))
\end{array}
$$

Suppose $\mathcal{E}$ is a coherent sheaf on $\mathfrak{X}_{k(\mathcal{G})}$, then $p_* \pi^* \mathcal{E} \cong s^* q_* \mathcal{E}$. This is because locally the diagram can be viewed as follows

$$
\begin{array}{ccc}
[Y_l/\mu_r] & \xrightarrow{\ t\ } & [Y_{k(\mathcal{G})}/\mu_r] \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\operatorname{spec}(l) & \xrightarrow{\ \pi\ } & \operatorname{spec}(k(\mathcal{G}))
\end{array}
$$

and the isomorphism can be reduced to μ_r-equivariant morphisms of coherent sheaves over schemes.

Remark 5.3. The $\pi : \mathfrak{X}_l \to \mathfrak{X}_{k(\mathcal{G})}$ used in this section is different from the canonical morphism $\pi : \mathfrak{X} \to X$

Remark 5.4. We may assumer that $l/k(\mathcal{G})$ is galois. $\pi^*\pi_*\mathcal{F} \cong \bigoplus_{\sigma \in Gal(l/k(\mathcal{G}))} \sigma^*\mathcal{F}$, where $\sigma^*\mathcal{F}$ are the Galois conjugates of $\mathcal{F}$.

Lemma 5.5. $\pi^*\pi_*\mathcal{F} \cong \mathcal{F}^{\oplus d}$ as $\mathcal{O}_{\mathfrak{X}_l}$-modules.

Proof. We have that $\pi^*\pi_*\mathcal{F} \cong \bigoplus_{\sigma \in Gal(l/k(\mathcal{G}))} \sigma^*\mathcal{F}$ by Remark-5.4. Choose a field L containing both l and K such that $\mathcal{F} \otimes_l L \cong E \otimes_K L$, i.e., $t^*\mathcal{F} \cong s^*E$ with the notation in the following square of embeddings

$$
\begin{array}{ccc}
L & \xleftarrow{\;\;s\;\;} & K \\
{\scriptstyle t}\big\uparrow & & \big\uparrow \\
l & \xleftarrow{\;\;i\;\;} & k
\end{array}
$$

Let $\sigma \in Gal(l/k)$ be an element in the galois group. We also denote the induced automorphisms $\mathfrak{X}_l \to \mathfrak{X}_l$ by σ. We have $(\sigma t)^*\mathcal{F} \cong \mathcal{F} \otimes_l L$ via the galois automorphism σ. On the other hand $(\sigma t)^*\mathcal{F} \cong t^*\sigma^*\mathcal{F} \cong \sigma^*\mathcal{F} \otimes_l L$. Hence $\sigma^*\mathcal{F} \otimes_l L \cong E \otimes_K L$. This implies $\sigma^*\mathcal{F} \in \mathcal{G}(l)$. But since E is simple, $\mathcal{G}$ is a $\mathbb{G}_m$-gerbe, so there is only one object in $\mathcal{G}(l)$ up to isomorphism. Hence $\sigma^*\mathcal{F} \cong \mathcal{F}$. The Lemma follows.

$\square$

Lemma 5.6. *Let* $\pi : \mathfrak{X}_l \to \mathfrak{X}_{k(\mathcal{G})}$ *be the morphism defined above. Suppose* $\mathcal{F}$ *is simple then the* $k(\mathcal{G})$-*algebra* $A = \mathrm{End}_{\mathfrak{X}}(\pi_*\mathcal{F})$ *is a central simple algebra.*

Proof. Since π is a flat morphism and $\mathcal{F}$ is a coherent sheaf, π^* commutes with the endomorphism sheaf $\pi^*\mathcal{E}nd(\pi_*\mathcal{F}) \cong \mathcal{E}nd(\pi^*\pi_*\mathcal{F})$. Taking global sections, we get $\mathrm{End}(\pi_*\mathcal{F}) \otimes_{k(\mathcal{G})} l \cong \mathrm{End}(\pi^*\pi_*\mathcal{F})$. By Lemma 5.5 $\pi^*\pi_*\mathcal{F} \cong \mathcal{F}^{\oplus d}$ as

$\mathcal{O}_{\mathfrak{X}_l}$-modules, we get $A \otimes_{k(\mathcal{G})} l \cong \text{End}(\mathcal{F}^{\oplus d}) \cong M_{d \times d}(\text{End}(\mathcal{F}))$ as algebras. But $\mathcal{F}$ is simple, hence $A \otimes_{k(\mathcal{G})} l \cong M_{d \times d}(l)$.

$\square$

Lemma 5.7. *With the notation in Lemma 5.6, $\mathcal{G}$ defines the same Brauer class as $A = \text{End}(\pi_* \mathcal{F})$ over $k(\mathcal{G})$.*

Proof. Since $\mathcal{F}$ is simple the residual gerbe $\mathcal{G} \to \text{spec}(k(\mathcal{G}))$ is a $\mathbb{G}_m$-gerbe. There exists a canonical locally free sheaf $\mathcal{E}$ on $\mathcal{G}$, given by $(U \xrightarrow{t} \text{spec}(k(\mathcal{G}), E') \mapsto p_* E'$. Here E' is a locally free sheaf on $\mathfrak{X}_U$ which is an object of $\mathcal{G}(U)$ and $p : \mathfrak{X}_U \to U$ is the base change of the structure map. Since p is proper, $p_* E'$ is a locally free sheaf. $\mathcal{E}$ is a $\mathcal{G}$-twisted sheaf in the sense of [Lie08]. The correspondence between $\mathcal{G}$-twisted sheaves and Azumaya algebras on $\text{spec}(k(\mathcal{G}))$ is determined by $\mathcal{E} \mapsto \mathcal{E}nd(\mathcal{E})$. So $\text{End}(p_* E')$ defines a central simple algebra over $\text{spec}(k(\mathcal{G}))$. Now it is enough to show that $\text{End}(p_* E')$ defines the same Brauer class as $\text{End}(\pi_* \mathcal{F})$ in $Br(k(\mathcal{G}))$. But $\mathcal{G}$ is a gerbe, hence there exists covers $U' \xrightarrow{q} U$ and $U' \xrightarrow{s} \text{spec}(l)$ such that $q^* p_* E' \cong s^* \pi_* \mathcal{F}$ inducing an isomorphism $\text{End}(p_* E') \otimes \mathcal{O}_{U'} \cong \text{End}(\pi_* \mathcal{F}) \otimes \mathcal{O}_{U'}$

$\square$

Theorem 5.8. *In the situation as above, consider a field $K \supset k(\mathcal{G})$. Then the following categories are equivalent:*

- *the category of coherent sheaves E over $\mathfrak{X}_K$ which are objects in $\mathcal{G}(K)$, and*

- *the category of projective modules M of rank $1/d$ over $A_K = A \otimes_{k(\mathcal{G})} K$*

Proof. We will describe mutually inverse functors between the two categories.

Suppose E is a coherent sheaf over $\mathfrak{X}_K$ which is an object in $\mathcal{G}(K)$. We define

$$M := \operatorname{Hom}_{\mathfrak{X}_K}((\pi_* F) \otimes_{k(\mathcal{G})} K, E)$$

Since $\mathcal{G}$ is a gerbe over $k(\mathcal{G})$, there exists a field extension L containing l and K such that $E \otimes_K L$ and $F \otimes_l L$ are isomorphic over $\mathfrak{X}_L$. Note that we may assume that $[L : K] < \infty$.

We have following isomorphisms since $\pi_* \mathcal{F}$ and E are coherent and L is K-flat

$$\operatorname{Hom}_{\mathfrak{X}_K}(\pi_* F \otimes_{k(\mathcal{G})} K, E) \otimes_K L \cong \operatorname{Hom}_{\mathfrak{X}_L}(\pi_* F \otimes_{k(\mathcal{G})} L, E \otimes_K L)$$
$$\cong \operatorname{Hom}_{\mathfrak{X}_L}(\pi_* F \otimes_{k(\mathcal{G})} L, F \otimes_l L) \qquad (5.0.2)$$

Hence,

$$M \otimes_K L \cong \operatorname{Hom}_{\mathfrak{X}_L}((\pi_* F) \otimes_{k(\mathcal{G})} L, F \otimes_l L)$$
$$\cong \operatorname{Hom}_{\mathfrak{X}_L}((\pi_* F) \otimes_{k(\mathcal{G})} l \otimes_l L, F \otimes_l L) \qquad (5.0.3)$$
$$\cong \operatorname{Hom}_{\mathfrak{X}_L}((\pi_* F) \otimes_{k(\mathcal{G})} l, F) \otimes_l L$$

But we have

$$\pi^* \pi_* \mathcal{F} = \pi_* \mathcal{F} \otimes_{k(\mathcal{G})} l$$

So we get,

$$M \otimes_K L = \operatorname{Hom}_{\mathfrak{X}_L}(\pi^* \pi_* F, F) \otimes_l L$$

By the adjunction $\pi^* \dashv \pi_*$ it follows that $\operatorname{Hom}_{\mathfrak{X}_l}(\pi^* \pi_* \mathcal{F}, \mathcal{F}) \cong \operatorname{Hom}_{\mathfrak{X}_{k(G)}}(\pi_* \mathcal{F}, \pi_* \mathcal{F}) = A$ as $k(\mathcal{G})$-vector spaces. So

$$\operatorname{Hom}_{\mathfrak{X}_l}(\pi^* \pi_* \mathcal{F}, \mathcal{F})^{\oplus d} \cong \operatorname{Hom}_{\mathfrak{X}_{k(G)}}(\pi_* \mathcal{F}, \pi_* \mathcal{F}) \otimes_{k(\mathcal{G})} l$$

as l-vector spaces. Hence, $(M \otimes_K L)^{\oplus d} \cong A \otimes_{k(\mathcal{G})} L$.

Therefore, the A_L-module $(M \otimes_K L)^{\oplus d}$ is free of rank one. Hence the underlying A_K-module $M^{\oplus d[L:K]}$ is free of rank $[L : K]$. Hence the A_K-module M is projective of rank $\frac{1}{d}$.

Conversely, let M be a projective A_K module of rank $\frac{1}{d}$. Then define

$$E := M \otimes_A \pi_* F \tag{5.0.4}$$

Choose an appropriate field L containing both l and $k(\mathcal{G})$, we have

$$M \otimes_K L \cong \mathrm{Hom}_{\mathfrak{X}_L}(\pi_* F \otimes_{k(\mathcal{G})} L, F \otimes_l L) \tag{5.0.5}$$

Since both are projective A_L-modules of rank $\frac{1}{d}$ by Corollary 2.2 in [BDH18].

$$\begin{aligned} E \otimes_K L &\cong (M \otimes_K L) \otimes_A (\pi_* F) \\ &\cong \mathrm{Hom}(\pi_* F \otimes_{k(\mathcal{G})} L, F \otimes_l L) \otimes_A \pi_* F \end{aligned} \tag{5.0.6}$$

Hence, by the same calculation as above using the adjunction $\pi^* \dashv \pi_*$ we have

$$\begin{aligned} (E \otimes_K L)^{\oplus d} &\cong (A_L) \otimes_A \pi_* F \\ &\cong \pi_* F \otimes_{k(\mathcal{G})} L \\ &\cong (F \otimes_l L)^{\oplus d} \end{aligned} \tag{5.0.7}$$

This implies $E \otimes_K L \cong F \otimes_l L$. Hence E is an object in $\mathcal{G}(K)$.

$\square$

Corollary 5.9. *Let $\mathcal{G}$ be the residual gerbe of a coherent sheaf $E \in Coh_{\mathfrak{X}}(K)$. Then,*

(i.) Given any field $K \supset k(\mathcal{G})$, all the objects in the groupoid $\mathcal{G}(K)$ are isomorphic.

(ii.) For the $k(\mathcal{G})$-algebra $A = \mathrm{End}(\pi_* \mathcal{F})$ and the integer d defined in 5.0.1, we have

$$\mathrm{ed}_{k(\mathcal{G})}(\mathcal{G}) = \mathrm{ed}_{k(\mathcal{G})}(\mathrm{Mod}_{A,1/d}). \tag{5.0.8}$$

Proof. By Proposition-2.39 (ii) we have that there exists only one projective module over $A \otimes_{k(\mathcal{G})} K$ of rank $1/d$ up to isomorphism. (i) follows from the equivalence in 5.8. (ii) also follows from 5.8 and 2.39. $\qquad\square$

Chapter 6

Main theorem

Let X be a smooth projective curve of genus g over an algebraically closed field k and let D be a reduced divisor on X, $r \geq 1$ be an integer which coprime to the characteristic of k, $\mathfrak{X} = X_{(\mathcal{L},s,r)}$ be the associated root stack. $Coh_{\mathfrak{X}}$ be the category of coherent sheaves on $\mathfrak{X}$. $\text{Bun}_{\mathfrak{X},t} \subset Coh_{\mathfrak{X}}$ be the open substack that parametrizes vector bundles on $\mathfrak{X}$ of type $t = (r, d, \mathcal{K})$. Here r and d denote the rank and degree, $\mathcal{K}$ denote a set of weight vectors (aka parabolic data). By [Bor07], $\text{Bun}_{\mathfrak{X},t} = \text{PBun}_{X,t}$, the stack of parabolic bundles on X of type $t = (r, d, \mathcal{K})$ and weights $\{0, \frac{1}{r} \ldots, \frac{r-1}{r}\}$.

Given a scheme U over k, a 1-morphism $\lambda : U \to \text{Bun}_{\mathfrak{X},t}$ is equivalent to a family $E_{\bullet}$ of parabolic bundles on $X \times U$ of type t.

Let $p_U : X \times U \to U$ and $\pi : X \times U \to X$ denote the canonical projections. Since p_U is a proper morphism there exists an $n >> 0$ such that $R^1(p_U)_*(E(n)) = 0$, where $E(n) := E \otimes \pi^* \mathcal{O}(n)$.

Remark 6.1 (Universal bundles). An object of the $X \times \text{Bun}_{\mathfrak{X},t}$ as a category fibered over Sch/k is a 1-morphism $(t, \lambda) : U \to X \times \text{Bun}_{\mathfrak{X},t}$. Here (t, λ) is equivalent to a morphism of schemes $t : U \to X$ and a morphism $\lambda : U \to$

$\mathrm{Bun}_{\mathfrak{X},t}$. λ is equivalent to a parabolic bundle $E_\bullet$ on $X \times U$ of type t. On the other hand, we may factorize (t, λ) as follows:

$$
\begin{array}{ccc}
 & X \times U & \\
 {\scriptstyle t\times 1}\nearrow & & \searrow {\scriptstyle 1\times\lambda} \\
 U \xrightarrow{\qquad (t,\lambda) \qquad} & & X \times \mathrm{Bun}_{\mathfrak{X}}
\end{array}
$$

For $0 \leq i \leq r$, define $\Sigma_i(t, \lambda) := (t \times 1)^* E_{\frac{i}{r}}$. These Σ_i define vector bundles on $X \times \mathrm{Bun}_{\mathfrak{X}}$ such that $E_{\frac{i}{r}} = (\lambda \times 1)^* \Sigma_i$.

Theorem 6.2. *Let $E : spec(K) \to \mathrm{Bun}_{\mathfrak{X},t}$ be a vector bundle of type t on $\mathfrak{X}_K$, i.e, a parabolic bundle $E_\bullet$ of type t on X_K. For each $0 \leq i \leq r$, there exist vector bundles $\mathcal{E}_{\frac{i}{r}}$ of rank $h^0(X, E_{\frac{i}{r}}(n))$ on some dense open substack $\mathcal{B}_n$ of $\mathrm{Bun}_{\mathfrak{X},t}$ for some natural number n.*

Proof. Given a scheme U over k, consider the full subcategory $\mathcal{B}_n(U) \subset \mathrm{Bun}_{\mathfrak{X},t}(U)$ consisting of objects the parabolic bundles $E_\bullet$ over $X \times U$ such that $R^1 p_{U*} E_{\frac{i}{r}}(n) = 0$ for all $0 \leq i \leq r - 1$.

Since p_U is projective there exists an n such that $(R^1 p_U)_* E_{\frac{i}{r}}(n) = 0$ for all $0 \leq i \leq r$ by [Har77], Ch-III, Theorem 8.8. Hence $\mathcal{B}_n$ is a non-empty open substack of $\mathrm{Bun}_{\mathfrak{X},t}$ for some n.

Consider the following Cartesian square:

$$
\begin{array}{ccc}
U \times X & \xrightarrow{\ \lambda \times 1\ } & \mathrm{Bun}_{\mathfrak{X},t} \times X = \mathrm{PBun}_X^t \times X \\
\downarrow & & \downarrow {\scriptstyle p_{\mathrm{Bun}_{\mathfrak{X},t}}} \\
U & \xrightarrow{\qquad \lambda \qquad} & \mathrm{Bun}_{\mathfrak{X},t}
\end{array}
$$

By [Sta18, Tag 072F] we know that $\lambda^*(R^1 p_{\mathrm{Bun}_{\mathfrak{X},t}})_* \Sigma_i(n) = (R^1 p_U)_* (\lambda \times 1)^* \Sigma_i(n)$. But we have $(\lambda \times 1)^* \Sigma_i(n) = E_{\frac{i}{r}}(n)$ by the previous remark. So $\lambda^*(R^1 p_{\mathrm{Bun}_{\mathfrak{X},t}})_* \Sigma_i(n) = (R^1 p_U)_* E_{\frac{i}{r}}(n)$. Let $p_{\mathcal{B}_n} : \mathcal{B}_n \times X \to \mathcal{B}_n$ be the canonical projection. We have,

$$R^1 p_{\mathcal{B}_n *} \Sigma_i(n) = (R^1 p_{\mathrm{Bun}_{\mathfrak{X},t}})_* \Sigma_i(n))|_{\mathcal{B}_n} = 0.$$

Hence $p_{\mathcal{B}_n *} \Sigma_i(n)$ is a vector bundle on $\mathcal{B}_n$ by a version of cohomology and base change for stacks.

Given a point $\lambda : \mathrm{spec}(K) \to \mathcal{B}_n$. We have the following Cartesian square:

$$
\begin{array}{ccc}
\mathrm{spec}(K) \times X & \xrightarrow{\mu := \lambda \times 1} & \mathcal{B}_n \times X \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_{\mathcal{B}_n}} \\
\mathrm{spec}(K) & \xrightarrow{\quad \lambda \quad} & \mathcal{B}_n
\end{array}
$$

Furthermore, rank $((p_{\mathcal{B}_n})_* \Sigma_i(n)) = \dim_K(\lambda^* p_*(\Sigma_i(n))) = \dim_K(\pi_* \mu^* \Sigma(n)) = \dim_K(\pi_*(E_i(n))) = \dim_K H^0(X, E_{\frac{i}{r}}(n))$. So $\mathcal{E}_i := (p_{\mathcal{B}_n})_* \Sigma_i(n)$ are vector bundles of required rank on $\mathcal{B}_n$.

$\square$

Remark 6.3. In the above situation since $H^1(E_{\frac{i}{r}}(n)) = 0$, by Riemann-Roch

$$\dim_K H^0(X, E_{\frac{i}{r}}(n)) = (1-g)r + \deg(E_{\frac{i}{r}}(n)) = (1-g)r + \deg(E_{\frac{i}{r}}) + nr$$

Remark 6.4. The $\mathcal{E}_i$ defined in the previous theorem are twisted sheaves on $\mathcal{B}_n$.

Remark 6.5. The previous theorem is true for any root stack $\mathfrak{X} = X_{(\mathcal{L},s,r)}$. Suppose $\mathcal{L} = \mathcal{O}(D), \mathcal{L}' = \mathcal{O}(D')$ are line bundles with canonical sections s, s' such that D and D' have disjoint supports. Then by Lemma 3.10, section-3.1 in [Bor07], we have $\mathrm{Bun}^t_{X_{(\mathcal{L},s,r)}} \times_{\mathrm{Bun}^{t_0}_X} \mathrm{Bun}^{t'}_{(\mathcal{L}',s',r)} = \mathrm{Bun}^{t''}_{X_{(\mathcal{L} \otimes \mathcal{L}', s \otimes s', r)}}$. This is just saying that, two parabolic bundles (with the same underlying vector bundle) each with parabolic structures at two disjoint divisors, is equivalent to a parabolic bundle with parabolic structure at the sum of the divisors. Here $t = (r, d, \mathcal{K}), t' = (r, d', \mathcal{K}')$ and $t_0 = (r, d_0)$ are the types of the respective

bundles. Here d and d' are parabolic degrees and d_0 usual degree of vector bundles. The parabolic degree and the degree of the underlying vector bundle are related by 4.18.

Remark 6.6. Applying the previous theorem to the situation where $\mathfrak{X}_P := X_{(\mathcal{O}(P),s,r)}$ for each $P \in |D|$, given a point $E_\bullet^{(P)} : \mathrm{spec}(K) \to \mathrm{Bun}_{\mathfrak{X}_P}^{t_P}$ we get vector bundles $\mathcal{E}_i^{(P)}$ on an open dense substack $\mathcal{B}_n^P \subset \mathrm{Bun}_{\mathfrak{X}_P}^{t_P}$ of rank $(1 - g + n)r + \deg(E_{\frac{i}{r}}^P)$ for some natural number n.

Now given $\mathcal{L} = \mathcal{O}(D)$ and $|D| = \{P_1, \ldots, P_k\}$, by Remark 6.5 we may write

$$\mathcal{B}_n := \mathcal{B}_{n_1}^{P_1} \times_{\mathrm{Bun}_X} \mathcal{B}_{n_k}^{P_k} \subset \mathrm{Bun}_{X_{(\mathcal{O}(P_1),s_1,r)}}^{t_{P_1}} \times_{\mathrm{Bun}_X} \times \ldots \times_{\mathrm{Bun}_X} \mathrm{Bun}_{X_{(\mathcal{O}(P_k),s_k,r)}}^{t_{P_k}}$$

$$= \mathrm{Bun}_{X_{(\mathcal{L},s,r)}}^t$$

$$(6.0.1)$$

Pulling back $\mathcal{E}_i^P$ to $\mathcal{B}_n$ we get vector bundles which we also denote by $\mathcal{E}_i^{(P)}$ of rank $a_P r + \deg(E_i^P)$.

Lemma 6.7. *Let $\psi_{\mathcal{G}}$ denote the Brauer class of the residual gerbe $\mathcal{G}$ over $k(E)$. Then the index of this class denoted by $ind(\psi_{\mathcal{G}})$ divides $h := gcd(S)$. Where $S = \{r, d_0, n_i^{(p)}\}$, $0 \le i \le r - 1$ and p varies in $|D|$.*

Proof. We may pull back the above constructed vector bundles $\mathcal{E}_i^P$ to the gerbe $\mathcal{G}$. Now applying Lieblich's Proposition 3.1.2.1 in [Lie08] to previous remark implies that $ind(\psi_{\mathcal{G}})$ divides $(1 - g + n)r + \deg(E_{\frac{i}{r}}^P)$ and hence divides $\deg(E_0^P) - \deg(E_i^P)$ for $0 \le i \le r$. But $\deg(E_0^P) - \deg(E_i^P) = \Sigma_{t=i}^{r-1} n_t^P$. Hence the Lemma follows. $\qquad\square$

Lemma 6.8. *Let E be a vector bundle over the root stack $\mathfrak{X}_K$ of rank $r \ge 1$ for some field $K \supset k$, then*

$$\mathrm{ed}_{k(E)}(E) \le r - 1$$

Proof. Let $\mathcal{G}$ be the residual gerbe associated to E as considered in Chapter-5. The chosen coherent sheaf $\mathcal{F} \in \mathcal{G}(l)$ over $\mathfrak{X}_l$ is locally free of rank r. This is because, by virtue of $\mathcal{G}$ being a gerbe, there exists a field L containing both l and K such that $\mathcal{F} \otimes_l L \cong E \otimes_K L$. Since E is locally free, $\mathcal{F}$ cannot have any torsion and it has the same rank as E. By Wedderburn's theorem there is a decomposition

$$\mathrm{End}(\pi_* \mathcal{F})/\mathrm{j}(\pi_* \mathcal{F}) \cong \Pi_i A_i \qquad (6.0.2)$$

where $A_i = M_{n \times n}(D_i)$ for some division algebras D_i over $k(E)$. By Lemma 2.41 we have that $\pi_* \mathcal{F} \cong \bigoplus_i E_i^{\oplus n_i}$ such that $\mathrm{End}(E_i)/\mathrm{j}(E_i) \cong D_i$ for some vector bundles E_i over $\mathfrak{X}_{k(E)}$. By 2.47 and 2.42 we have

$$\mathrm{ed}_{k(E)}(\mathrm{Mod}_{A_i,1/d}) < \frac{n_i}{d}\dim_{k(E)} D_i \le \frac{n_i}{d}\mathrm{rank}\,(E_i) \qquad (6.0.3)$$

By 2.45, we have

$$\mathrm{ed}_{k(E)}(\mathrm{Mod}_{\mathrm{End}(\pi_* \mathcal{F}),1/d}) < \frac{1}{d}\mathrm{rank}\,(\pi_* \mathcal{F}) = \mathrm{rank}\,(\mathcal{F}) = r \qquad (6.0.4)$$

Using Corollary 5.9 we conclude that $\mathrm{ed}_{k(E)}(E) < r$.

$\square$

Theorem 6.9. *Let X be a non-singular projective curve of genus $g \ge 2$ over an algebraically closed field k and D a reduced divisor with N points in its support. Assume that the char(k) is coprime to N. Let $\mathcal{K} = \{(n_0^{(p)}, \ldots, n_N^{(p)})\}_{p \in |D|}$ be a set of weight vectors. Then*

$$\mathrm{ed}_k(\mathrm{PBun}_{X,t}) \le (g-1)r^2 + 1 + \Sigma_{p|h}(p^{v_p(h)} - 1) + \Sigma_{x \in |D|} \mathrm{Flag}_{\mathcal{K}}(x).$$

where $\mathrm{PBun}_{X,t}$ *denotes the stack of parabolic bundles on* X *of type* $t =$ $(r, d, \mathcal{K})$ *with parabolic structure at* $|D|$ *and weight* $\{i/N\}_{0 \leq i \leq N-1}$ *and* $h =$ $gcd(r, d, n_k^{(p)})_{p \in |D|}$.

Proof. Let E be a vector bundle of type t over $\mathfrak{X}_K$, for some field $K \supset k$. If E is not simple then Lemma 4.39 and 6.8 imply that

$$\mathrm{ed}_k(E) = \mathrm{trdeg}_k(k(E)) + \mathrm{ed}_{k(E)}(E)$$
$$\leq (g-1)(r^2 - r) + 2 + \Sigma_{x \in |D|}\mathrm{Flag}_\mathcal{K}(x) + r - 1 \qquad (6.0.5)$$
$$\leq (g-1)r^2 + 1 + \Sigma_{x \in |D|}\mathrm{Flag}_\mathcal{K}(x)$$

Suppose E is simple, Corollary 4.37 implies

$$\mathrm{trdeg}_k k(E) \leq (g-1)r^2 + 1 + \Sigma_{x \in |D|}\mathrm{Flag}_\mathcal{K}(x)$$

Since by definition $\mathrm{End}_{\mathfrak{X}}(E) = K$ and hence $j(E) = 0$. So $r_i = 0$ for $i > 1$ and $r_1 = \mathrm{rk}(\frac{im(\varphi^0)}{im(\varphi^1)}) = \mathrm{rk}(im(Id)) = \mathrm{rk}(E)$.

Let $\mathcal{G}$ denote the residue gerbe of the point $E : spec(K) \to \mathrm{Bun}_{\mathfrak{X},t}$. The residue field of this point is $k(E)$. We have

$$\mathrm{ed}_{k(E)}(E) = \mathrm{ed}_{k(E)}(\mathcal{G})$$

by definition.

Hence $\mathrm{ed}_k(E) = \mathrm{trdeg}_k k(E) + \mathrm{ed}_{k(E)}(\mathcal{G})$. So we need to estimate $\mathrm{ed}_{k(E)}(\mathcal{G})$. By the correspondence in Theorem 5.8 we also have

$$\mathrm{ed}_{k(E)}(\mathcal{G}) = \mathrm{ed}_{k(E)}(\mathrm{Mod}_{A, \frac{1}{degA}}).$$

The gerbe $\mathcal{G}$ and the central simple algebra $A = \mathrm{End}_{\mathfrak{X}}(\pi_* F)$ defined in Chapter 5 represent the same Brauer class over $k(E)$. Hence by Lemma 6.7 the index of A divides h. By Corollary 5.9 we have $\mathrm{ed}_{k(E)}(\mathrm{Mod}_{A, \frac{1}{degA}}) \leq \Sigma_{p|h}(p^{v_p(h)} - 1)$. Hence we have

$$\mathrm{ed}_{k(E)}(\mathcal{G}) \leq \Sigma_{p|h}(p^{v_p(h)} - 1)$$

The statement of the theorem follows.

$\square$

Chapter 7

Conclusion

In this thesis we computed upper bounds for the stack of parabolic bundles of a given rank, degree and parabolic datum over a non-singular curve of genus $g \geq 2$. To do this we employed the techniques in [BDH18], along with a theorem of Borne [Bor07]. In the process we used a powerful theorem due to Toen [Toe] which allowed us to compute the Euler characterstic of a vector bundle over a root stack. Further directions to this work include the same question of essential dimension of parabolic bundles over elliptic curves, i.e., smooth curves of genus 1. In [BDH18] an explicit vector bundle over an elliptic curve of expected essential dimension is constructed in Section-7. The natural question is, if an analogous parabolic bundle can be constructed.

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