6-20-2019 2:00 PM

A computationally efficient methodology in pricing a guaranteed minimum accumulation benefit

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A thesis submitted in partial fulfillment of the requirements for the Master of Science degree in Statistics and Actuarial Sciences

Abstract

In this thesis, we consider a framework under which three correlated factors, namely, financial, mortality and lapse risks, are modelled in an integrated way. This modelling framework supports the valuation of a guaranteed minimum accumulation benefit (GMAB). The change-of-measure approach is employed to come up with a compact and implementable valuation expressions. We provide a numerical demonstration to confirm the efficiency and accuracy of our proposed pricing methodology. In particular, our approach on average takes only 0.07% of the computing time entailed by the Monte-Carlo (MC) simulation technique. Furthermore, the standard errors of our approach's results are lower than those obtained from MC-based computations. When there are no renewal options in a GMAB contract, we get the special case of a guaranteed minimum maturity benefit for which a closed-form pricing solution is derived.

Keywords: Variable annuities, investment guarantee, stochastic model, change of probability measures

Lay Summary

When a customer comes to an insurance company to learn something about one specific insurance product, the insurer will be asked to provide the corresponding purchase price. After obtaining the customer's essential information, they start to calculate the price. However, if they can't give a response within a short time, they would provide a negative customer service experience, which consequently might force the customer to switch to another company. Therefore, it is important for the insurer to have a quick-response evaluation system in order to get an edge over the competition. This thesis will provide such an evaluation framework in the valuation of a specific insurance product, called the guaranteed minimum accumulation benefit (GMAB).

Acknowledgements

I would like to convey my profound gratitude to my supervisor, Dr. Rogemar Mamon, for his enthusiasm, scientific guidance, and for sharing his expertise and experiences throughout my research training at Western. His careful editing and research insights contributed enormously to the production of this thesis.

I would like to express my sincere thanks to my thesis examiners, Dr. Hao Yu, Dr. Marcos Escobar-Anel and Dr. Simon Bonner, for taking the time out of their their busy schedule to read my thesis. Their constructive and useful comments are very much appreciated.

I take pleasure in thanking my colleagues and friends, Dr. Yixing Zhao and doctoral candidates Xing Gu, Yifan Li and Junhe Chen, for their valuable assistance and ideas.

Last but not least, my heartfelt appreciation goes to my family, for their love and constant support. This thesis would not have been possible without their encouragement.

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Chapter 1

Introduction

With a population expected to live much longer into the future, the popularity of a variable annuity has grown rapidly over the years. According to the First-Quarter 2019 U.S. Retail Annuity Sales Survey conducted by the LIMRA Secure Retirement Institute (LIMRA SRI) [12], variable annuity (VA) sales from January-March 2019 totaled $22.8 billion. These represent 37.5% of overall annuity sales; it is the highest figure for a first-quarter total annuity sales going back for a decade.

A variable annuity is a tax-deferred contract between a policyholder and an insurance company. The benefits to the policyholder will depend on the performance of the investment funds provided by the insurance company; typically, the benefit is the greater of the account value and the guaranteed amount. Contracts typically contain certain guarantee riders offered by the insurance company in order to afford different types of financial protection. There are two major types of guarantee riders: guaranteed minimum death benefits (GMDB) and guaranteed minimum living benefits (GMLB). The GMLB consists of three main subcategories: guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB), and guaranteed mnimum withdrawl benefits (GMWB). A detailed overview of a variable annuity is given in Gan [6].

Even though GMAB is a simple living benefit, it differs from the other living benefit riders in terms of the risk posed to the insurance company. It is crucial for an insurance company to scrutinise the contracts with a GMAB rider. This is because there is a need to follow up the detailed fund performance information, reset the guarantee amounts, and pay the difference amounts to the segregated fund at renewal dates.

Bauer et al. [2] provided a comprehensive mathematical model for modelling and valuation of many types of variable annuity riders. A unifying framework is proposed in Bacinello et al. [1] for valuing variable annuity guarantees using a Monte-Carlo (MC) method. In Doyle and Groendyke [5], the use of neural networks is explored to price variable annuity guarantees. Nonetheless, many papers dealing with this problem do not take into consideration the correlation between interest and mortality rates, and they do not consider lapsation as a risk factor as well. Although this paper employs the modelling framework in Zhao and Mamon [21], which synthesises interest, mortality and lapse rates altogether for a guaranteed annuity option pricing, the efficient valuation of GMAB has its own peculiarity and challenges, which requires a separate and focused analysis being addressed by this methodological and empirical study.

The remainder of this paper is organised as follows. Chapter 2 presents the modelling framework for the valuation of GMAB. The detailed description of the GMAB contract is laid out in Chapter 3. In Chapter 4, we introduce a sequence of probability-measure changes to facilitate the proposed pricing methodology. More specifically, certain mathematical techniques are applied to obtain analytical pricing solutions. We demonstrate a numerical implementation in Chapter 5 illustrating the advantages of our proposed approach. Finally, Chapter 6 concludes.

Chapter 2

Modelling framework

We assume that our valuation framework is supported by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$. Here, $\{\mathcal{F}_t\}$ is the joint filtration generated by the interest rate r_t, force of mortality μ_t and lapse rate l_t, and Q is a risk-neutral probability measure.

2.1 Interest rate model

As specified above, it is supposed that Q exists and the dynamics of r_t is given by the Vasiček model

$$\mathrm{d}r_t = a(b - r_t)\mathrm{d}t + \sigma_1 \mathrm{d}X_t, \tag{2.1}$$

where a, b and σ_1 are positive constants, and X_t is a standard Brownian motion (BM) under Q. Such a Q is equivalent to an objective measure P, under which the realisations or some proxies for the realisations of our underlying variables are observed.

Apparently, this model can generate negative interest rate values; nonetheless this feature accommodates the occurrence of negative rates in situation when monetary authorities have to combat deflation by encouraging people and businesses to spend money rather than keep it safe in the banks. For instance, the European Central Bank introduced a negative interest-rate policy in 2014 whilst the Bank of Japan did the same in 2016 to stimulate its economy and overcome persistent deflationary pressures in its economy. The price $B(t, T)$ of a T-maturity zero-coupon bond at time $t < T$ (cf Mamon [18]) is given by

$$B(t,T) = \mathbb{E}^{Q}\left[e^{-\int_t^T r_u \mathrm{d}u}\middle|\mathcal{F}_t\right] = e^{-A(t,T)r_t + D(t,T)}, \tag{2.2}$$

where

$$A(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

and

$$D(t,T) = \left(b - \frac{\sigma_1^2}{2a^2}\right)[A(t,T) - (T-t)] - \frac{\sigma_1^2 A(t,T)^2}{4a}.$$

2.2 Mortality model

The force of mortality $\mu_{x,t}$ at time t for an individual aged x at time 0 is governed by a non-mean reverting OU process, as proposed to Luciano and Vigna [17], and it has dynamics

$$\mathrm{d}\mu_t = c\mu_t \mathrm{d}t + \xi \mathrm{d}Y_t, \tag{2.3}$$

where c and ξ are positive constants, and Y_t is a standard BM. Noting that our emphasis is the dynamics with respect to the passage of time, we shall simply use μ_t to represent $\mu_{x,t}$ in the succeeding discussion to avoid clutter of notation. Then, we recall the survival function

$$S(t,T) = \mathbb{E}^{Q}\left[e^{-\int_t^T \mu_u du}\middle|\mathcal{F}_t\right].$$

2.3 Lapse rate model

Lapse risk is the possibility that policyholders terminate their policies that arises from surrendering or stopping to pay premiums, which could cause huge losses and liquidity problem to the insurers. Therefore, it is another essential factor in pricing insurance products. Let l_t be the lapse rate at time t, and assume that it evolves as a mean-reverting process similar to the setting in Zhao and Mamon [21]. That is,

$$\mathrm{d}l_t = h(m + pr_t - l_t)\mathrm{d}t + \zeta \mathrm{d}Z_t, \tag{2.4}$$

where h, m, p and ζ are positive constants, and Z_t is a standard BM.

2.4 Model dependence

The works by Liu et al. [15] argued that the correlation between interest rate and mortality rate has significant effect in pricing annuity products, thus it must be incorporated in our valuation framework. In particular, as noted in the findings of Dhaene et al. [4], dependence modelling in a risk-neutral pricing world is necessary to give allowance to correlated financial and acturial risks despite their being independent in the real world. Secondly, Kuo et al. [14] used the co-integration technique in the investigation of the contending-lapse-rate hypotheses that tackles the tension between the emergency fund hypothesis and the interest rate hypothesis. It was found that the interest rate has a statistically significant power in explaining the long-term behaviour of the lapse rate as over the long run, it causes lapse rate's variations. Hence, the correlation between interest rate and lapse rate must be considered. Thirdly, a contract policy's lapsation could be linked to mortality or morbidity-adverse selection. This means that policy holders who are in adverse health or have other insurability problems tend not to lapse their policies; this is because they will have difficulty finding comparable insurance coverage at the same premium level. Thus, we need to take into consideration the interaction between mortality rate and lapse rate. Simply put, decisions on whether to continue life insurance policies are influenced by the insureds' perceived likelihood of survival.

We assume that X_t, Y_t and Z_t are correlated and their dependence is modelled as

$$\mathrm{d}X_t \mathrm{d}Y_t = \rho_{12}\mathrm{d}t, \quad \mathrm{d}X_t \mathrm{d}Z_t = \rho_{13}\mathrm{d}t \quad \text{and} \quad \mathrm{d}Y_t \mathrm{d}Z_t = \rho_{23}\mathrm{d}t.$$

Their explicit specifications are as follows:

$$\begin{aligned}
\mathrm{d}X_t &= \mathrm{d}W_t^1, \\
\mathrm{d}Y_t &= \rho_{12}\mathrm{d}W_t^1 + \sqrt{1-\rho_{12}^2}\mathrm{d}W_t^2, \\
\mathrm{d}Z_t &= \rho_{13}\mathrm{d}W_t^1 + \rho_{23}'\mathrm{d}W_t^2 + \sqrt{1-\rho_{13}^2-\rho_{23}'^2}\mathrm{d}W_t^3,
\end{aligned}$$

where W_t^1, W_t^2 and W_t^3 are independent standard BMs and

$$\rho_{23}' = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}. \tag{2.5}$$

Note that we need to choose proper correlation values for ρ_{12}, ρ_{13}, and ρ_{23} such that $|\rho_{23}'| \leq 1$.

Chapter 3

Contract description

In this chapter, we present the detailed contract description of a GMAB.

3.1 Guaranteed Minimum Accumulation Benefit

Denote by $M(t,T)$ the fair value at time t of a \$1 pure endowment payable at maturity T under a two-decrement model (both mortality and lapse rates are considered). From the risk-neutral pricing principle,

$$M(t,T) = \mathbb{E}^{Q}\left[e^{-\int_t^T r_u \mathrm{d}u} e^{-\int_t^T \mu_u \mathrm{d}u} e^{-\int_t^T l_u \mathrm{d}u} \middle| \mathcal{F}_t \right]. \tag{3.1}$$

The value of $M(t,T)$ is needed in our succeeding analysis of a GMAB, which is a contract that guarantees the policyholder a specific monetary amount at maturity, provided that the policyholder is still alive at the contract's maturity. Moreover, the policyholder has the option to renew the contract at some renewal dates, at a new guarantee level. Further descriptions on the design of a GMAB can be found in [9].

In this thesis, we assume two renewals at T_1 and T_2, and the maturity at T_3 (clearly this can be adapted to more renewals). The guaranteed value G_t is assumed to have a roll-up feature, i.e.,

$$G_t = P_0 e^{\delta t},$$

where P_0 is the contract's initial single premium, and δ is a predetermined roll-up rate; when $\delta = 0$ we are in the situation called return of premium. The segregated fund F_t is linked to the performance of a stock index S_t and this is expressed as

$$F_t = F_0 \frac{S_t}{S_0} e^{-\alpha t},$$

where α is the constant continuously compounded management charge rate, and $F_0 = S_0 = P_0$. The stock index S_t follows a geometric BM; so

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma_2 S_t \mathrm{d}W_t^4,$$

where σ_2 is a positive constant, and W_t^4 is a standard BM independent of W_t^1, W_t^2 and W_t^3. Applying Itô's lemma, it can be shown that the dynamics of the fund value F_t satisfies

$$\mathrm{d}F_t = (r_t - \alpha)F_t \mathrm{d}t + \sigma_2 F_t \mathrm{d}W_t^4. \tag{3.2}$$

At renewal T_1, if the fund value F_{T_1} is more than the guarantee G_{T_1}, then the guarantee is reset to equal the fund value at T_1. On the other hand, if the guarantee is greater than the fund value, then the insurance company pays the difference into the fund so that the next period starts with the fund value and guarantee being equal. This process is repeated at time T_2. At the contract maturity T_3, the insurance company must pay the difference between G_{T_3} and F_{T_3} if the guarantee exceeds the fund value at time T_3. Since the segregated fund may increase at the renewal dates, we distinguish between the fund before and after the payout by the insurance company; we denote by $F_{T_k^-}$ the fund immediately before renewal and by $F_{T_k^+}$ the fund immediately after renewal. That is, if H_{T_k} is the payout at renewal T_k, then

$$F_{T_k^+} = F_{T_k^-} + H_{T_k}.$$

Therefore the fair value of a GMAB at time 0 is

$$\begin{aligned} P_{\text{GMAB}} =& \mathbb{E}^Q \Big[e^{-\int_0^{T_1} r_u \mathrm{d}u} e^{-\int_0^{T_1} \mu_u \mathrm{d}u} e^{-\int_0^{T_1} l_u \mathrm{d}u} H_{T_1} + e^{-\int_0^{T_2} r_u \mathrm{d}u} e^{-\int_0^{T_2} \mu_u \mathrm{d}u} e^{-\int_0^{T_2} l_u \mathrm{d}u} H_{T_2} \\ & + e^{-\int_0^{T_3} r_u \mathrm{d}u} e^{-\int_0^{T_3} \mu_u \mathrm{d}u} e^{-\int_0^{T_3} l_u \mathrm{d}u} H_{T_3} \Big| \mathcal{F}_0 \Big]. \end{aligned} \tag{3.3}$$

3.2 Guaranteed Minimum Maturity Benefit

In addition, if the GMAB policyholder wishes not to renew the contract before maturity T_3, then this contract is simplified into a guaranteed minimum maturity benefit (GMMB), with only one payoff of $\max(G_{T_3} - F_{T_3}, 0)$ at maturity T_3. The fair value of GMMB at time 0 is

$$P_{\mathrm{GMMB}} = \mathbb{E}^{Q}\left[e^{-\int_0^{T_3} r_u \mathrm{d}u} e^{-\int_0^{T_3} \mu_u \mathrm{d}u} e^{-\int_0^{T_3} l_u \mathrm{d}u} \max(G_{T_3} - F_{T_3}, 0) \middle| \mathcal{F}_0\right]. \tag{3.4}$$

Chapter 4

Derivation of valuation formula

Probability measure changes are employed to carry out the evaluation of the expected discounted benefit. The forward measure, survival measure and endowment risk-adjusted measure are introduced in the context of GMAB.

4.1 The forward measure

We choose the bond price $B(t,T)$ as a numéraire (where T is an arbitrary number), and then we define the forward measure $\widetilde{Q}$ equivalent to the risk-neutral measure Q via the Radon-Nikodým derivative

$$\left.\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q}\right|_{\mathcal{F}_T} = \Lambda_T^1 := \frac{e^{-\int_0^T r_u \mathrm{d}u} B(T,T)}{B(0,T)}.$$

By the Bayes' rule for conditional expectation,

$$M(t,T) = \mathbb{E}^Q\left[e^{-\int_t^T r_u \mathrm{d}u} e^{-\int_t^T \mu_u \mathrm{d}u} e^{-\int_t^T l_u \mathrm{d}u}\middle|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_u \mathrm{d}u} e^{-\int_t^T l_u \mathrm{d}u}\middle|\mathcal{F}_t\right]. \tag{4.1}$$

Following the generalised results given in Mamon [18], the respective $\widetilde{Q}$ dynamics of r_t, μ_t and l_t are given by

$$\begin{aligned}
\mathrm{d}r_t &= [ab - \sigma_1^2 A(t,T) - ar_t]\mathrm{d}t + \sigma_1 \mathrm{d}\widetilde{W}_t^1, \\
\mathrm{d}\mu_t &= [-\rho_{12}\sigma_1\xi A(t,T) + c\mu_t]\mathrm{d}t + \xi\left(\rho_{12}\mathrm{d}\widetilde{W}_t^1 + \sqrt{1-\rho_{12}^2}\mathrm{d}\widetilde{W}_t^2\right), \\
\mathrm{d}l_t &= [hm + pr_t - \rho_{13}\sigma_1\zeta A(t,T) - hl_t]\mathrm{d}t + \zeta\left(\rho_{13}\mathrm{d}\widetilde{W}_t^1 + \rho'_{23}\mathrm{d}\widetilde{W}_t^2 + \sqrt{1-\rho_{13}^2-\rho_{23}'^2}\mathrm{d}\widetilde{W}_t^3\right),
\end{aligned}$$

where $\widetilde{W}_t^1$, $\widetilde{W}_t^2$ and $\widetilde{W}_t^3$ are standard BMs under $\widetilde{Q}$.

From Liu et al. [16], we have

$$S(t,T) = \mathbb{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_u \mathrm{d}u}\middle|\mathcal{F}_t\right] = e^{-\mu_t \widetilde{G}(t,T)+\widetilde{H}(t,T)}, \tag{4.2}$$

where

$$\widetilde{G}(t,T) = \frac{e^{c(T-t)}-1}{c}$$

and

$$\widetilde{H}(t,T) = \left(\frac{\rho_{12}\sigma_1\xi}{ac} - \frac{\xi^2}{2c^2}\right)\left[\widetilde{G}(t,T)-(T-t)\right] + \frac{\rho_{12}\sigma_1\xi}{ac}\left[A(t,T)-\phi(t,T)\right] + \frac{\xi^2}{4c}\widetilde{G}(t,T)^2$$

with

$$\phi(t,T) = \frac{1-e^{-(a-c)(T-t)}}{a-c}.$$

4.2 The survival measure

In order to obtain an explicit solution to equation (4.1), we define a new measure $\bar{Q}$ equivalent to the forward measure $\widetilde{Q}$, with $S(t,T)$ as the associated numéraire, by considering

$$\left.\frac{\mathrm{d}\bar{Q}}{\mathrm{d}\widetilde{Q}}\right|_{\mathcal{F}_T} = \Lambda_T^2 := \frac{e^{-\int_0^T \mu_u \mathrm{d}u} S(T,T)}{S(0,T)}.$$

By the Bayes' rule for conditional expectation,

$$\mathbb{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_u \mathrm{d}u} e^{-\int_t^T l_u \mathrm{d}u}\middle|\mathcal{F}_t\right] = S(t,T)\mathbb{E}^{\bar{Q}}\left[e^{-\int_t^T l_u \mathrm{d}u}\middle|\mathcal{F}_t\right]. \tag{4.3}$$

Linking equations (4.1) and (4.3), we have

$$M(t,T) = \mathbb{E}^{Q}\left[e^{-\int_t^T r_u \mathrm{d}u} e^{-\int_t^T \mu_u \mathrm{d}u} e^{-\int_t^T l_u \mathrm{d}u}\middle|\mathcal{F}_t\right] = B(t,T)S(t,T)\mathbb{E}^{\bar{Q}}\left[e^{-\int_t^T l_u \mathrm{d}u}\middle|\mathcal{F}_t\right]. \tag{4.4}$$

Following the results given in Zhao and Mamon [21], we have

$$\mathbb{E}^{\overline{Q}}\left[e^{-\int_t^T l_u \mathrm{d}u}\middle|\mathcal{F}_t\right] = e^{-\overline{I}(t,T)l_t - \overline{K}(t,T)r_t + \overline{J}(t,T)}, \tag{4.5}$$

where

$$\overline{I}(t,T) = e^{\overline{b}(t)}\overline{\gamma}(t) = \frac{1-e^{-h(T-t)}}{h}, \quad \overline{K}(t,T) = \frac{hp}{h-a}\left(A(t,T)-\overline{I}(t,T)\right),$$

and $\overline{J}(t,T)$ satisfies the differential equation

$$\frac{\partial \overline{J}}{\partial t} - \overline{Im}_t - \overline{Kb}_t + \frac{1}{2}\left(\zeta^2\overline{I}^2 + \sigma_1^2\overline{K}^2 + 2\rho_{13}\zeta\sigma_1\overline{IK}\right) = 0$$

with

$$\overline{m}_t = hm - \rho_{13}\sigma_1\zeta A(t,T) - \rho_{23}\xi\zeta\widetilde{G}(t,T) \quad \text{and} \quad \overline{b}_t = ab - \sigma^2 A(t,T) - \rho_{12}\sigma_1\xi\widetilde{G}(t,T).$$

Combining equations (2.2), (4.2), (4.4) and (4.5) together, we get

$$M(t,T) = e^{-\left(\left(A(t,T)+\overline{K}(t,T)\right)r_t+\widetilde{G}(t,T)\mu_t+\overline{I}(t,T)l_t\right)+D(t,T)+\widetilde{H}(t,T)+\overline{J}(t,T)}. \tag{4.6}$$

4.3 The endowment-risk-adjusted measure

In order to determine the P_{GMAB} and P_{GMMB} values, another measure called the endowment-risk-adjusted measure $\widehat{Q}_k$ will be defined, with $M(t,T_k)$ as the associated numéraire, through

$$\left.\frac{\mathrm{d}\widehat{Q}_k}{\mathrm{d}Q}\right|_{\mathcal{F}_{T_k}} = \Lambda^{3(k)}_{T_k} := \frac{e^{-\int_0^{T_k} r_u \mathrm{d}u} e^{-\int_0^{T_k} \mu_u \mathrm{d}u} e^{-\int_0^{T_k} l_u \mathrm{d}u} M(T_k,T_k)}{M(0,T_k)}.$$

By the Bayes' rule for conditional expectation, equation (3.3) can be rewritten as

$$\begin{aligned}
P_{\text{GMAB}} =& \mathbb{E}^Q\Big[e^{-\int_0^{T_1} r_u \mathrm{d}u} e^{-\int_0^{T_1} \mu_u \mathrm{d}u} e^{-\int_0^{T_1} l_u \mathrm{d}u} H_{T_1} + e^{-\int_0^{T_2} r_u \mathrm{d}u} e^{-\int_0^{T_2} \mu_u \mathrm{d}u} e^{-\int_0^{T_2} l_u \mathrm{d}u} H_{T_2} \\
&+ e^{-\int_0^{T_3} r_u \mathrm{d}u} e^{-\int_0^{T_3} \mu_u \mathrm{d}u} e^{-\int_0^{T_3} l_u \mathrm{d}u} H_{T_3} \Big| \mathcal{F}_0\Big] \\
=& M(0,T_1)\mathbb{E}^{\widehat{Q}_1}\left[H_{T_1}\middle|\mathcal{F}_0\right] + M(0,T_2)\mathbb{E}^{\widehat{Q}_2}\left[H_{T_2}\middle|\mathcal{F}_0\right] + M(0,T_3)\mathbb{E}^{\widehat{Q}_3}\left[H_{T_3}\middle|\mathcal{F}_0\right].
\end{aligned} \tag{4.7}$$

Equation (3.4) can be rewritten as

$$\begin{aligned}
P_{\text{GMMB}} &= \mathbb{E}^Q\left[e^{-\int_0^{T_3} r_u \mathrm{d}u} e^{-\int_0^{T_3} \mu_u \mathrm{d}u} e^{-\int_0^{T_3} l_u \mathrm{d}u} \max(G_{T_3} - F_{T_3}, 0) \middle| \mathcal{F}_0\right] \\
&= M(0,T_3)\mathbb{E}^{\widehat{Q}_3}\left[(\max(G_{T_3} - F_{T_3}, 0)\middle|\mathcal{F}_0\right].
\end{aligned} \tag{4.8}$$

Calculations leading to the dynamics of $\Lambda^{3(k)}_t$ under Q show

$$\begin{aligned}
\mathrm{d}\Lambda^{3(k)}_t = -\Lambda^{3(k)}_t\Big[&\left(\sigma_1 A(t,T_k) + \rho_{12}\xi\widetilde{G}(t,T_k) + \rho_{13}\zeta\overline{I}(t,T_k) + \sigma_1\overline{K}(t,T_k)\right)\mathrm{d}W^1_t \\
&+ \left(\xi\widetilde{G}(t,T_k)\sqrt{1-\rho_{12}^2} + \rho'_{23}\zeta\overline{I}(t,T_k)\right)\mathrm{d}W^2_t + \zeta\overline{I}(t,T_k)\sqrt{1-\rho_{13}^2-\rho'^2_{23}}\mathrm{d}W^3_t\Big];
\end{aligned} \tag{4.9}$$

see Appendix A for more details.

By the Girsanov's Theorem,

$$
\begin{aligned}
\mathrm{d}\widehat{W}_t^{1(k)} &= \mathrm{d}W_t^1 + \Big(\sigma_1 A(t,T_k) + \rho_{12}\xi\widetilde{G}(t,T_k) + \rho_{13}\zeta\bar{I}(t,T_k) + \sigma_1\overline{K}(t,T_k)\Big)\,\mathrm{d}t,\\
\mathrm{d}\widehat{W}_t^{2(k)} &= \mathrm{d}W_t^2 + \Big(\xi\widetilde{G}(t,T_k)\sqrt{1-\rho_{12}^2} + \rho'_{23}\zeta\bar{I}(t,T_k)\Big)\,\mathrm{d}t,\\
\mathrm{d}\widehat{W}_t^{3(k)} &= \mathrm{d}W_t^3 + \zeta\bar{I}(t,T_k)\sqrt{1-\rho_{13}^2-\rho_{23}'^2}\,\mathrm{d}t,\\
\mathrm{d}\widehat{W}_t^{4(k)} &= \mathrm{d}W_t^4,
\end{aligned}
$$

where $\widehat{W}_t^{1(k)}$, $\widehat{W}_t^{2(k)}$, $\widehat{W}_t^{3(k)}$ and $\widehat{W}_t^{4(k)}$ are $\widehat{Q}_k-$standard BMs.

So, the respective $\widehat{Q}_k$ dynamics of r_t, μ_t, l_t and F_t are

$$
\begin{aligned}
\mathrm{d}r_t =& \Big(ab - \sigma_1^2 A(t,T_k) - \rho_{12}\sigma_1\xi\widetilde{G}(t,T_k) - \rho_{13}\sigma_1\zeta\bar{I}(t,T_k) - \sigma_1^2\overline{K}(t,T_k) - ar_t\Big)\,\mathrm{d}t + \sigma_1\mathrm{d}\widehat{W}_t^{1(k)},\\
\mathrm{d}\mu_t =& \Big(-\rho_{12}\sigma_1\xi A(t,T_k) - \xi^2\widetilde{G}(t,T_k) - \rho_{23}\xi\zeta\bar{I}(t,T_k) - \rho_{12}\sigma_1\xi\overline{K}(t,T_k) + c\mu_t\Big)\,\mathrm{d}t + \xi\rho_{12}\mathrm{d}\widehat{W}_t^{1(k)}\\
&+ \xi\sqrt{1-\rho_{12}^2}\,\mathrm{d}\widehat{W}_t^{2(k)},\\
\mathrm{d}l_t =& \Big(-\rho_{13}\sigma_1\zeta A(t,T_k) - \zeta^2\bar{I}(t,T_k) - \rho_{23}\xi\zeta\widetilde{G}(t,T_k) - \rho_{13}\sigma_1\zeta\overline{K}(t,T_k) + hpr_t - hl_t\Big)\,\mathrm{d}t\\
&+ \zeta\rho_{13}\mathrm{d}\widehat{W}_t^{1(k)} + \zeta\rho'_{23}\mathrm{d}\widehat{W}_t^{2(k)} + \zeta\sqrt{1-\rho_{13}^2-\rho_{23}'^2}\,\mathrm{d}\widehat{W}_t^{3(k)},\\
\mathrm{d}F_t =& (r_t - m)F_t dt + \sigma_2 F_t \mathrm{d}\widehat{W}_t^{4(k)}.
\end{aligned}
$$

4.4 Valuation formula

The $\widehat{Q}_k$ dynamics of r_t in the previous section has the representation

$$
\begin{aligned}
r_t =& e^{-at}r_0 + \frac{\sigma_1^2 e^{-aT_k}}{2a^2}\left(1 + \frac{hp}{h-a}\right)(e^{at} - e^{-at})\\
&+ \left(b - \frac{\sigma_1^2}{a^2} + \frac{\rho_{12}\sigma_1\xi}{ac} - \frac{\rho_{13}\sigma_1\zeta}{ah} - \frac{\sigma_1^2 hp}{(h-a)a^2} + \frac{\sigma_1^2 p}{(h-a)a}\right)(1 - e^{-at})\\
&+ \frac{\sigma_1 e^{-hT_k}}{a+h}\left(\frac{\rho_{13}\zeta}{h} - \frac{\sigma_1 p}{h-a}\right)\left(e^{ht} - e^{-at}\right)\\
&- \frac{\rho_{12}\sigma_1\xi e^{cT_k}}{c(a-c)}\left(e^{-ct} - e^{-at}\right) + \sigma_1 e^{-at}\int_0^t e^{au}\,\mathrm{d}\widehat{w}_u^{1(k)}.
\end{aligned}
\tag{4.10}
$$

Furthermore,

$$\begin{aligned}\int_{t_1}^{t_2} r_u \mathrm{d}u =& r_0\left(\frac{e^{-at_1}-e^{-at_2}}{a}\right)+\frac{\sigma_1^2 e^{-aT_k}}{2a^2}\left(1+\frac{hp}{h-a}\right)\left(\frac{e^{at_2}-e^{at_1}}{a}-\frac{e^{-at_1}-e^{-at_2}}{a}\right)\\&+\left(b-\frac{\sigma_1^2}{a^2}+\frac{\rho_{12}\sigma_1\xi}{ac}-\frac{\rho_{13}\sigma_1\zeta}{ah}-\frac{\sigma_1^2 hp}{(h-a)a^2}+\frac{\sigma_1^2 p}{(h-a)a}\right)\\&\times\left((t_2-t_1)-\frac{e^{-at_1}-e^{-at_2}}{a}\right)\\&+\frac{\sigma_1 e^{-hT_k}}{a+h}\left(\frac{\rho_{13}\zeta}{h}-\frac{\sigma_1 p}{h-a}\right)\left(\frac{e^{ht_2}-e^{ht_1}}{h}-\frac{e^{-at_1}-e^{-at_2}}{a}\right)\\&-\frac{\rho_{12}\sigma_1\xi e^{cT_k}}{c(a-c)}\left(\frac{e^{-ct_1}-e^{-ct_2}}{c}-\frac{e^{-at_1}-e^{-at_2}}{a}\right)+\sigma_1\int_{t_1}^{t_2}\int_0^u e^{-au}e^{as}\mathrm{d}\widehat{W}_s^{1(k)}\mathrm{d}u. \end{aligned} \tag{4.11}$$

By Fubini's Theorem, the last integral in equation (4.11) can be rewritten as

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^u e^{-au}e^{as}\mathrm{d}\widehat{W}_s^{1(k)}\mathrm{d}u &= \int_0^{t_1}\int_{t_1}^{t_2} e^{-au}e^{as}\mathrm{d}u\mathrm{d}\widehat{W}_s^{1(k)}+\int_{t_1}^{t_2}\int_s^{t_2} e^{-au}e^{as}\mathrm{d}u\mathrm{d}\widehat{W}_s^{1(k)}\\&=\int_0^{t_1} e^{as}\left(\int_{t_1}^{t_2} e^{-au}\mathrm{d}u\right)\mathrm{d}\widehat{W}_s^{1(k)}+\int_{t_1}^{t_2} e^{as}\left(\int_s^{t_2} e^{-au}\mathrm{d}u\right)\mathrm{d}\widehat{W}_s^{1(k)}\\&=\int_0^{t_1} e^{as}\left(\frac{e^{-at_1}-e^{-at_2}}{a}\right)\mathrm{d}\widehat{W}_s^{1}+\int_{t_1}^{t_2} e^{as}\left(\frac{e^{-as}-e^{-at_2}}{a}\right)\mathrm{d}\widehat{W}_s^{1}\\&=\left(\frac{e^{-at_1}-e^{-at_2}}{a}\right)\int_0^{t_1} e^{as}\mathrm{d}\widehat{W}_s^{1(k)}+\int_{t_1}^{t_2}\left(\frac{1-e^{-a(t_2-s)}}{a}\right)\mathrm{d}\widehat{W}_s^{1(k)}.\end{aligned}$$

We see that under $\widehat{Q}_k$, $\int_{t_1}^{t_2} r_u du$ follows a normal distribution with the following moments:

$$\begin{aligned}\mathbb{E}^{\widehat{Q}_k}\left[\int_{t_1}^{t_2} r_u \mathrm{d}u\right] =& r_0\left(\frac{e^{-at_1}-e^{-at_2}}{a}\right)+\frac{\sigma_1^2 e^{-aT_k}}{2a^2}\left(1+\frac{hp}{h-a}\right)\left(\frac{e^{at_2}-e^{at_1}}{a}-\frac{e^{-at_1}-e^{-at_2}}{a}\right)\\&+\left(b-\frac{\sigma_1^2}{a^2}+\frac{\rho_{12}\sigma_1\xi}{ac}-\frac{\rho_{13}\sigma_1\zeta}{ah}-\frac{\sigma_1^2 hp}{(h-a)a^2}+\frac{\sigma_1^2 p}{(h-a)a}\right)\\&\times\left((t_2-t_1)-\frac{e^{-at_1}-e^{-at_2}}{a}\right)\\&+\frac{\sigma_1 e^{-hT_k}}{a+h}\left(\frac{\rho_{13}\zeta}{h}-\frac{\sigma_1 p}{h-a}\right)\left(\frac{e^{ht_2}-e^{ht_1}}{h}-\frac{e^{-at_1}-e^{-at_2}}{a}\right)\\&-\frac{\rho_{12}\sigma_1\xi e^{cT_k}}{c(a-c)}\left(\frac{e^{-ct_1}-e^{-ct_2}}{c}-\frac{e^{-at_1}-e^{-at_2}}{a}\right),\end{aligned}$$

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[\int_{t_1}^{t_2} r_u \mathrm{d}u\right] =& \mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[\sigma_1\left(\frac{e^{-at_1}-e^{-at_2}}{a}\right)\int_0^{t_1} e^{as}\mathrm{d}\widehat{W}_s^{1(k)} + \sigma_1\int_{t_1}^{t_2}\left(\frac{1-e^{-a(t_2-s)}}{a}\right)\mathrm{d}\widehat{W}_s^{1(k)}\right] \\
=& \mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[\sigma_1\left(\frac{e^{-at_1}-e^{-at_2}}{a}\right)\int_0^{t_1} e^{as}\mathrm{d}\widehat{W}_s^{1(k)}\right] \\
&+ \mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[\sigma_1\int_{t_1}^{t_2}\left(\frac{1-e^{-a(t_2-s)}}{a}\right)\mathrm{d}\widehat{W}_s^{1(k)}\right] \\
=& \sigma_1^2\left(\frac{e^{-at_1}-e^{-at_2}}{a}\right)^2\left(\frac{e^{2at_1}-1}{2a}\right) \\
&+ \frac{\sigma_1^2}{a^2}\left[(t_2-t_1) - \frac{2\left(1-e^{-a(t_2-t_1)}\right)}{a} + \frac{\left(1-e^{-2a(t_2-t_1)}\right)}{2a}\right].
\end{aligned}
$$

From the $\widehat{Q}_k$ dynamics of F_t, we have

$$
F_{t_2} = F_{t_1}\exp\left[\int_{t_1}^{t_2}\left(r_t - \alpha - \frac{1}{2}\sigma_2^2\right)\mathrm{d}t + \sigma_2\left(\widehat{W}_{t_2}^{4(k)} - \widehat{W}_{t_1}^{4(k)}\right)\right].
$$

Let $Y_{t_1,\,t_2}^{(k)} = \int_{t_1}^{t_2}\left(r_t - \alpha - \frac{1}{2}\sigma_2^2\right)\mathrm{d}t + \sigma_2\left(\widehat{W}_{t_2}^{4(k)} - \widehat{W}_{t_1}^{4(k)}\right)$. It may be verified that $Y_{t_1,\,t_2}^{(k)}$ is normally distributed, whose mean and variance under $\widehat{Q}_k$ can be expressed as follows:

$$
\begin{aligned}
\mu_{t_1,\,t_2}^{(k)} :=& \mathbb{E}^{\widehat{Q}_k}\left[Y_{t_1,\,t_2}^{(k)}\right] = \mathbb{E}^{\widehat{Q}_k}\left[\int_{t_1}^{t_2}\left(r_t - \alpha - \frac{1}{2}\sigma_2^2\right)\mathrm{d}t + \sigma_2\left(\widehat{W}_{t_2}^{4(k)} - \widehat{W}_{t_1}^{4(k)}\right)\right] \\
=& \mathbb{E}^{\widehat{Q}_k}\left[\int_{t_1}^{t_2} r_t\mathrm{d}t\right] - \alpha(t_2-t_1) - \frac{1}{2}\sigma_2^2(t_2-t_1).
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
\left(\sigma_{t_1,\,t_2}^{(k)}\right)^2 :=& \mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[Y_{t_1,\,t_2}^{(k)}\right] = \mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[\int_{t_1}^{t_2}\left(r_t - \alpha - \frac{1}{2}\sigma_2^2\right)\mathrm{d}t + \sigma_2\left(\widehat{W}_{t_2}^{4(k)} - \widehat{W}_{t_1}^{4(k)}\right)\right] \\
=& \mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[\int_{t_1}^{t_2} r_t\mathrm{d}t\right] + \mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[\sigma_2\left(\widehat{W}_{t_2}^{4(k)} - \widehat{W}_{t_1}^{4(k)}\right)\right] \\
=& \mathbb{V}\mathrm{ar}^{\widehat{Q}_k}\left[\int_{t_1}^{t_2} r_t\mathrm{d}t\right] + \sigma_2^2(t_2-t_1).
\end{aligned} \tag{4.13}
$$

In addition, the probability density function (pdf) of $Y_{t_1,\,t_2}^{(k)}$ is given by

$$
f^{(k)}(y) = \frac{1}{\sqrt{2\pi}\sigma_{t_1,\,t_2}^{(k)}}\exp\left[-\frac{\left(y-\mu_{t_1,\,t_2}^{(k)}\right)^2}{2\left(\sigma_{t_1,\,t_2}^{(k)}\right)^2}\right]. \tag{4.14}
$$

Lemma 4.4.1. *Let* $E^{(k)}(t_1,t_2) := \mathbb{E}^{\widehat{Q}_k}\left[\max(e^{\delta(t_2-t_1)} - e^{Y^{(k)}_{t_1,\,t_2}}, 0)\middle|\mathcal{F}_0\right]$. *The analytic representation for the conditional expectation* $E^{(k)}(t_1,t_2)$ *is*

$$E^{(k)}(t_1,t_2) = e^{\delta(t_2-t_1)}\Phi\left(\frac{\delta(t_2-t_1)-\mu^{(k)}_{t_1,\,t_2}}{\sigma^{(k)}_{t_1,\,t_2}}\right) - e^{\mu^{(k)}_{t_1,\,t_2}+\frac{1}{2}\left(\sigma^{(k)}_{t_1,\,t_2}\right)^2}\Phi\left(\frac{\delta(t_2-t_1)-\mu^{(k)}_{t_1,\,t_2}-\left(\sigma^{(k)}_{t_1,\,t_2}\right)^2}{\sigma^{(k)}_{t_1,\,t_2}}\right).$$

Proof We examine and evaluate one by one the two terms in $E^{(k)}(t_1,t_2)$.

$$\begin{aligned}E^{(k)}(t_1,t_2) =&\mathbb{E}^{\widehat{Q}_k}\left[\max(e^{\delta(t_2-t_1)} - e^{Y^{(k)}_{t_1,\,t_2}}, 0)\middle|\mathcal{F}_0\right] = \mathbb{E}^{\widehat{Q}_k}\left[\left(e^{\delta(t_2-t_1)} - e^{Y^{(k)}_{t_1,\,t_2}}\right)\mathbb{1}_{\left\{\delta(t_2-t_1)\geq Y^{(k)}_{t_1,\,t_2}\right\}}\middle|\mathcal{F}_0\right]\\ =&\mathbb{E}^{\widehat{Q}_k}\left[e^{\delta(t_2-t_1)}\mathbb{1}_{\left\{\delta(t_2-t_1)\geq Y^{(k)}_{t_1,\,t_2}\right\}}\middle|\mathcal{F}_0\right] - \mathbb{E}^{\widehat{Q}_k}\left[e^{Y^{(k)}_{t_1,\,t_2}}\mathbb{1}_{\left\{\delta(t_2-t_1)\geq Y^{(k)}_{t_1,\,t_2}\right\}}\middle|\mathcal{F}_0\right].\end{aligned}$$

The first term can then be expressed as

$$\begin{aligned}\mathbb{E}^{\widehat{Q}_k}\left[e^{\delta(t_2-t_1)}\mathbb{1}_{\left\{\delta(t_2-t_1)\geq Y^{(k)}_{t_1,\,t_2}\right\}}\middle|\mathcal{F}_0\right] &= \int_{-\infty}^{\delta(t_2-t_1)} e^{\delta(t_2-t_1)} f(y)\mathrm{d}y\\ &= \int_{-\infty}^{\delta(t_2-t_1)} e^{\delta(t_2-t_1)}\frac{1}{\sqrt{2\pi}\sigma^{(k)}_{t_1,\,t_2}}\exp\left(-\frac{\left(y-\mu^{(k)}_{t_1,\,t_2}\right)^2}{2\left(\sigma^{(k)}_{t_1,\,t_2}\right)^2}\right)\mathrm{d}y\\ &\overset{z=\frac{y-\mu^{(k)}_{t_1,\,t_2}}{\sigma^{(k)}_{t_1,\,t_2}}}{=} \int_{-\infty}^{\frac{\delta(t_2-t_1)-\mu^{(k)}_{t_1,\,t_2}}{\sigma^{(k)}_{t_1,\,t_2}}} e^{\delta(t_2-t_1)}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z\\ &=e^{\delta(t_2-t_1)}\Phi\left(\frac{\delta(t_2-t_1)-\mu^{(k)}_{t_1,\,t_2}}{\sigma^{(k)}_{t_1,\,t_2}}\right),\end{aligned}$$

where $\Phi(x)$ is the standard normal cumulative density function. The second term can be expressed as

$$
\begin{aligned}
\mathbb{E}^{\widehat{Q}_k}\left[e^{Y^{(k)}_{t_1,\,t_2}}\mathbb{1}_{\{\delta(t_2-t_1)\geq Y^{(k)}_{t_1,\,t_2}\}}\middle|\mathcal{F}_0\right] &= \int_{-\infty}^{\delta(t_2-t_1)} e^y f^{(k)}(y)\mathrm{d}y \\
&= \int_{-\infty}^{\delta(t_2-t_1)} e^y \frac{1}{\sqrt{2\pi}\sigma^{(k)}_{t_1,\,t_2}} \exp\left(-\frac{\left(y-\mu^{(k)}_{t_1,\,t_2}\right)^2}{2(\sigma^{(k)}_{t_1,\,t_2})^2}\right)\mathrm{d}y \\
&\overset{z=\frac{y-\mu^{(k)}_{t_1,\,t_2}}{\sigma^{(k)}_{t_1,\,t_2}}}{=} \int_{-\infty}^{\frac{\delta(t_2-t_1)-\mu^{(k)}_{t_1,\,t_2}}{\sigma^{(k)}_{t_1,\,t_2}}} \exp\left(\mu^{(k)}_{t_1,\,t_2}+z\sigma^{(k)}_{t_1,\,t_2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z \\
&= \exp\left(\mu^{(k)}_{t_1,\,t_2}+\frac{1}{2}\left(\sigma^{(k)}_{t_1,\,t_2}\right)^2\right)\int_{-\infty}^{\frac{\delta(t_2-t_1)-\mu^{(k)}_{t_1,\,t_2}}{\sigma^{(k)}_{t_1,\,t_2}}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(z-\sigma^{(k)}_{t_1,\,t_2}\right)^2}{2}\right)\mathrm{d}z \\
&\overset{u=z-\sigma^{(k)}_{t_1,\,t_2}}{=} \exp\left(\mu^{(k)}_{t_1,\,t_2}+\frac{1}{2}\left(\sigma^{(k)}_{t_1,\,t_2}\right)^2\right)\int_{-\infty}^{\frac{\delta(t_2-t_1)-\mu^{(k)}_{t_1,\,t_2}-(\sigma^{(k)}_{t_1,\,t_2})^2}{\sigma^{(k)}_{t_1,\,t_2}}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)\mathrm{d}u \\
&= \exp\left(\mu^{(k)}_{t_1,\,t_2}+\frac{1}{2}\left(\sigma^{(k)}_{t_1,\,t_2}\right)^2\right)\Phi\left(\frac{\delta(t_2-t_1)-\mu^{(k)}_{t_1,\,t_2}-\left(\sigma^{(k)}_{t_1,\,t_2}\right)^2}{\sigma^{(k)}_{t_1,\,t_2}}\right).
\end{aligned}
$$

Hence, $E^{(k)}(t_1, t_2)$ has the explicit form, as desired . ■

4.4.1 Guaranteed Minimum Maturity Benefit

With equation (4.6), it only remains to calculate $\mathbb{E}^{\widehat{Q}_3}\left[\max(G_{T_3}-F_{T_3},0)\middle|\mathcal{F}_0\right]$ in order to evaluate equation (4.8) fully.

$$
\begin{aligned}
\mathbb{E}^{\widehat{Q}_3}\left[\max(G_{T_3}-F_{T_3},0)\middle|\mathcal{F}_0\right] =&\mathbb{E}^{\widehat{Q}_3}\left[\max(P_0e^{\delta T_3}-F_0e^{\int_0^{T_3}\left(r_t-\alpha-\frac{1}{2}\sigma_2^2\right)\mathrm{d}t+\sigma_2\widehat{W}^{4(3)}_{T_3}},0)\middle|\mathcal{F}_0\right] \\
=&P_0\mathbb{E}^{\widehat{Q}_3}\left[\max(e^{\delta T_3}-e^{Y^{(3)}_{0,\,T_3}},0)\middle|\mathcal{F}_0\right] = P_0E^{(3)}(0,T_3). \qquad (4.15)
\end{aligned}
$$

Plugging in (4.15) into (4.8) with the aid of Lemma 4.4.1 gives the following result.

Theorem 4.4.2. *The price of a GMMB at time 0 is*

$$
\begin{aligned}
P_{GMMB} =&P_0M(0,T_3)\left[e^{\delta T_3}\Phi\left(\frac{\delta T_3-\mu^{(3)}_{0,\,T_3}}{\sigma^{(3)}_{0,\,T_3}}\right)\right. \\
&\left.- e^{\mu_{0,\,T_3}+\frac{1}{2}\left(\sigma^{(3)}_{0,\,T_3}\right)^2}\Phi\left(\frac{\delta T_3-\mu^{(3)}_{0,\,T_3}-\left(\sigma^{(3)}_{0,\,T_3}\right)^2}{\sigma^{(3)}_{0,\,T_3}}\right)\right]. \qquad (4.16)
\end{aligned}
$$

4.4.2 Guaranteed Minimum Accumulation Benefit

What remains to be done to implement equation (4.7) is the evaluation of $\mathbb{E}^{\widehat{Q}_1}\left[H_{T_1}\middle|\mathcal{F}_0\right]$, $\mathbb{E}^{\widehat{Q}_2}\left[H_{T_2}\middle|\mathcal{F}_0\right]$ and $\mathbb{E}^{\widehat{Q}_3}\left[H_{T_3}\middle|\mathcal{F}_0\right]$.

The first expectation can be expressed as

$$\begin{aligned}\mathbb{E}^{\widehat{Q}_1}\left[H_{T_1}\middle|\mathcal{F}_0\right] =&\mathbb{E}^{\widehat{Q}_1}\left[\max\left(G_{T_1^-}-F_{T_1^-},0\right)\middle|\mathcal{F}_0\right]\\ =&\mathbb{E}^{\widehat{Q}_1}\left[\max(P_0e^{\delta T_1}-F_0e^{\int_0^{T_1}(r_t-\alpha-\frac{1}{2}\sigma_2^2)\mathrm{d}t+\sigma_2\widehat{W}^{4(1)}_{T_1}},0)\middle|\mathcal{F}_0\right]\\ =&P_0\mathbb{E}^{\widehat{Q}_1}\left[\max(e^{\delta T_1}-e^{Y^{(1)}_{0,\,T_1}},0)\middle|\mathcal{F}_0\right]=P_0E^{(1)}(0,T_1).\end{aligned}\tag{4.17}$$

Furthermore, the second expectation can be expressed as

$$\begin{aligned}\mathbb{E}^{\widehat{Q}_2}\left[H_{T_2}\middle|\mathcal{F}_0\right] =&\mathbb{E}^{\widehat{Q}_2}\left[\max\left(G_{T_2^-}-F_{T_2^-},0\right)\middle|\mathcal{F}_0\right]\\ =&\mathbb{E}^{\widehat{Q}_2}\left[\max(G_{T_1^+}e^{\delta(T_2-T_1)}-F_{T_1^+}e^{\int_{T_1}^{T_2}(r_t-\alpha-\frac{1}{2}\sigma_2^2)\mathrm{d}t+\sigma_2\left(\widehat{W}^{4(2)}_{T_2}-\widehat{W}^{4(2)}_{T_1}\right)},0)\middle|\mathcal{F}_0\right]\\ =&\mathbb{E}^{\widehat{Q}_2}\left[F_{T_1^+}\max\left(e^{\delta(T_2-T_1)}-e^{Y^{(2)}_{T_1,\,T_2}},0\right)\middle|\mathcal{F}_0\right]\\ =&\mathbb{E}^{\widehat{Q}_2}\left[(F_{T_1^-}+H_{T_1})\max\left(e^{\delta(T_2-T_1)}-e^{Y^{(2)}_{T_1,\,T_2}},0\right)\middle|\mathcal{F}_0\right]\\ =&\mathbb{E}^{\widehat{Q}_2}\left[\max\left(G_{T_1^-},F_{T_1^-}\right)\max\left(e^{\delta(T_2-T_1)}-e^{Y^{(2)}_{T_1,\,T_2}},0\right)\middle|\mathcal{F}_0\right]\\ =&P_0\mathbb{E}^{\widehat{Q}_2}\left[\max\left(e^{\delta T_1},e^{Y^{(2)}_{0,\,T_1}}\right)\max\left(e^{\delta(T_2-T_1)}-e^{Y^{(2)}_{T_1,\,T_2}},0\right)\middle|\mathcal{F}_0\right].\end{aligned}\tag{4.18}$$

Note that the last expectation in (4.18) depends only on the the value of $Y^{(2)}_{0,\,T_1}$ and $Y^{(2)}_{T_1,\,T_2}$. This tells us that the simulated pair $(Y^{(2)}_{0,\,T_1},Y^{(2)}_{T_1,\,T_2})$ completes the calculation of $\mathbb{E}^{\widehat{Q}_2}\left[H_{T_2}\middle|\mathcal{F}_0\right]$. It may be verified that under $\widehat{Q}_2$, this pair $(Y^{(2)}_{0,\,T_1},Y^{(2)}_{T_1,\,T_2})$ is a bivariate normal random variable, with the following moments:

$$\mathbb{E}^{\widehat{Q}_2}\left[Y^{(2)}_{0,\,T_1}\right]=\mu^{(2)}_{0,\,T_1};\quad \mathbb{E}^{\widehat{Q}_2}\left[Y^{(2)}_{T_1,\,T_2}\right]=\mu^{(2)}_{T_1,\,T_2};\tag{4.19}$$

$$\mathbb{V}\mathrm{ar}^{\widehat{Q}_2}\left[Y^{(2)}_{0,\,T_1}\right]=\left(\sigma^{(2)}_{0,\,T_1}\right)^2;\quad \mathbb{V}\mathrm{ar}^{\widehat{Q}_2}\left[Y^{(2)}_{T_1,\,T_2}\right]=\left(\sigma^{(2)}_{T_1,\,T_2}\right)^2;\tag{4.20}$$

$$\text{and}\ \ \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[Y^{(2)}_{0,\,T_1},Y^{(2)}_{T_1,\,T_2}\right]=\frac{\sigma_1^2}{2a^3}\left(e^{-aT_1}-e^{-aT_2}\right)\left(e^{aT_1}+e^{-aT_1}-2\right).\tag{4.21}$$

The accompanying details of the calculation for the covariance in (4.21) are given in Appendix B.

Finally, the last expectation can be expressed as

$$\begin{aligned}
\mathbb{E}^{\widehat{Q}_3}\left[H_{T_3}\middle|\mathcal{F}_0\right] =&\mathbb{E}^{\widehat{Q}_3}\left[\max\left(G_{T_3}-F_{T_3},0\right)\middle|\mathcal{F}_0\right] \\
=&\mathbb{E}^{\widehat{Q}_3}\left[\max(G_{T_2^+}e^{\delta(T_3-T_2)}-F_{T_2^+}e^{\int_{T_2}^{T_3}\left(r_t-\alpha-\frac{1}{2}\sigma_2^2\right)\mathrm{d}t+\sigma_2\left(\widehat{W}^{4(3)}_{T_3}-\widehat{W}^{4(3)}_{T_2}\right)},0)\middle|\mathcal{F}_0\right] \\
=&\mathbb{E}^{\widehat{Q}_3}\left[F_{T_2^+}\max\left(e^{\delta(T_3-T_2)}-e^{Y^{(3)}_{T_2,\,T_3}},0\right)\middle|\mathcal{F}_0\right] \\
=&\mathbb{E}^{\widehat{Q}_3}\left[(F_{T_2^-}+H_{T_2})\max\left(e^{\delta(T_3-T_2)}-e^{Y^{(3)}_{T_2,\,T_3}},0\right)\middle|\mathcal{F}_0\right] \\
=&\mathbb{E}^{\widehat{Q}_3}\left[\max\left(G_{T_2^-},F_{T_2^-}\right)\max\left(e^{\delta(T_3-T_2)}-e^{Y^{(3)}_{T_2,\,T_3}},0\right)\middle|\mathcal{F}_0\right] \\
=&\mathbb{E}^{\widehat{Q}_3}\left[F_{T_1^+}\max\left(e^{\delta(T_2-T_1)},e^{Y^{(3)}_{T_1,\,T_2}}\right)\max\left(e^{\delta(T_3-T_2)}-e^{Y^{(3)}_{T_2,\,T_3}},0\right)\middle|\mathcal{F}_0\right] \\
=&\mathbb{E}^{\widehat{Q}_3}\left[\max\left(G_{T_1^-},F_{T_1^-}\right)\max\left(e^{\delta(T_2-T_1)},e^{Y^{(3)}_{T_1,\,T_2}}\right)\max\left(e^{\delta(T_3-T_2)}-e^{Y^{(3)}_{T_2,\,T_3}},0\right)\middle|\mathcal{F}_0\right] \\
=&P_0\mathbb{E}^{\widehat{Q}_3}\Bigg[\max\left(e^{\delta T_1},e^{Y^{(3)}_{0,\,T_1}}\right)\max\left(e^{\delta(T_2-T_1)},e^{Y^{(3)}_{T_1,\,T_2}}\right) \\
&\qquad\times\max\left(e^{\delta(T_3-T_2)}-e^{Y^{(3)}_{T_2,\,T_3}},0\right)\Bigg|\mathcal{F}_0\Bigg]. \qquad (4.22)
\end{aligned}$$

Again we can see that the last expectation in (4.22) depends only on the the value of $Y^{(3)}_{0,\,T_1}$, $Y^{(3)}_{T_1,\,T_2}$ and $Y^{(3)}_{T_2,\,T_3}$, therefore we just need to simulate $(Y^{(3)}_{0,\,T_1}, Y^{(3)}_{T_1,\,T_2}, Y^{(3)}_{T_2,\,T_3})$, which is a multivariate normal random variable, under $\widehat{Q}_3$, with the following moments:

$$\mathbb{E}^{\widehat{Q}_3}\left[Y^{(3)}_{0,\,T_1}\right]=\mu^{(3)}_{0,\,T_1};\quad \mathbb{E}^{\widehat{Q}_3}\left[Y^{(3)}_{T_1,\,T_2}\right]=\mu^{(3)}_{T_1,\,T_2};\quad \mathbb{E}^{\widehat{Q}_3}\left[Y^{(3)}_{T_2,\,T_3}\right]=\mu^{(3)}_{T_2,\,T_3}; \qquad (4.23)$$

$$\mathbb{V}\mathrm{ar}^{\widehat{Q}_3}\left[Y^{(3)}_{0,\,T_1}\right]=\left(\sigma^{(3)}_{0,\,T_1}\right)^2;\quad \mathbb{V}\mathrm{ar}^{\widehat{Q}_3}\left[Y^{(3)}_{T_1,\,T_2}\right]=\left(\sigma^{(3)}_{T_1,\,T_2}\right)^2;\quad \mathbb{V}\mathrm{ar}^{\widehat{Q}_3}\left[Y^{(3)}_{T_2,\,T_3}\right]=\left(\sigma^{(3)}_{T_2,\,T_3}\right)^2; \quad (4.24)$$

$$\mathbb{C}\mathrm{ov}^{\widehat{Q}_3}\left[Y^{(3)}_{0,\,T_1},Y^{(3)}_{T_1,\,T_2}\right]=\frac{\sigma_1^2}{2a^3}\left(e^{-aT_1}-e^{-aT_2}\right)\left(e^{aT_1}+e^{-aT_1}-2\right); \qquad (4.25)$$

$$\mathbb{C}\mathrm{ov}^{\widehat{Q}_3}\left[Y^{(3)}_{0,\,T_1},Y^{(3)}_{T_2,\,T_3}\right]=\frac{\sigma_1^2}{2a^3}\left(e^{-aT_2}-e^{-aT_3}\right)\left(e^{aT_1}+e^{-aT_1}-2\right); \qquad (4.26)$$

$$\text{and}\quad \mathbb{C}\mathrm{ov}^{\widehat{Q}_3}\left[Y^{(3)}_{T_1,\,T_2},Y^{(3)}_{T_2,\,T_3}\right]=\frac{\sigma_1^2}{2a^3}\left(e^{-aT_2}-e^{-aT_3}\right)\left(e^{aT_2}+e^{-aT_2}-e^{aT_1}-e^{-aT_1}\right). \qquad (4.27)$$

See Appendix B for the calculation details of the covariances in (4.25), (4.26) and (4.27).

Plugging in (4.17), (4.18) and (4.22) into (4.7) with the help of Lemma 4.4.1 gives the following result.

Theorem 4.4.3. *The value of a GMAB at time 0 is*

$$\begin{aligned} P_{GMAB} =& P_0 M(0,T_1)\left[e^{\delta T_1}\Phi\left(\frac{\delta T_1-\mu^{(1)}_{0,\,T_1}}{\sigma^{(1)}_{0,\,T_1}}\right)-e^{\mu_{0,\,T_1}+\frac{1}{2}\left(\sigma^{(1)}_{0,\,T_1}\right)^2}\Phi\left(\frac{\delta T_1-\mu^{(1)}_{0,\,T_1}-\left(\sigma^{(1)}_{0,\,T_1}\right)^2}{\sigma^{(1)}_{0,\,T_1}}\right)\right] \\ &+P_0 M(0,T_2)\mathbb{E}^{\widehat{Q}_2}\left[\max\left(e^{\delta T_1},e^{Y^{(2)}_{0,\,T_1}}\right)\max\left(e^{\delta(T_2-T_1)}-e^{Y^{(2)}_{T_1,\,T_2}},0\right)\middle|\mathcal{F}_0\right] \\ &+P_0 M(0,T_3)\mathbb{E}^{\widehat{Q}_3}\Bigg[\max\left(e^{\delta T_1},e^{Y^{(3)}_{0,\,T_1}}\right)\max\left(e^{\delta(T_2-T_1)},e^{Y^{(3)}_{T_1,\,T_2}}\right) \\ &\qquad\times\max\left(e^{\delta(T_3-T_2)}-e^{Y^{(3)}_{T_2,\,T_3}},0\right)\Bigg|\mathcal{F}_0\Bigg]. \end{aligned} \tag{4.28}$$

Chapter 5

Numerical illustration

In this chapter, a numerical experiment is included to showcase the efficiency of our proposed methodology.

5.1 Numerical scheme

Direct computation, which refers to the brute-force implementation of the MC method, of P_{GMAB} and P_{GMMB} by using equations (3.3) and (3.4), respectively, entails the the evolutions of r_t, μ_t, l_t and F_t over the time period $[0, T_k]$. We subdivide each year into $N = 252$ subintervals of same length $\Delta t = \frac{1}{N}$, and let $t_i = i\Delta t$ for $i = 0, \ldots, NT_k$. Based on the Euler–Maruyama discretisation scheme, the respective sample paths of r_t, μ_t, l_t and F_t, under measure Q, are generated by the discretisations:

$$
\begin{aligned}
r_{t_i} =& r_{t_{i-1}} + a(b - r_{t_{i-1}})\Delta t + \sigma_1 \sqrt{\Delta t}\varepsilon^1_{t_i}, \\
\mu_{t_i} =& \mu_{t_{i-1}} + c\mu_{t_{i-1}}\Delta t + \xi \sqrt{\Delta t}\left(\rho_{12}\varepsilon^1_{t_i} + \sqrt{1 - \rho^2_{12}}\varepsilon^2_{t_i}\right), \\
l_{t_i} =& l_{t_{i-1}} + h(m + pr_{t_{i-1}} - l_{t_{i-1}})\Delta t + \zeta \sqrt{\Delta t}\left(\rho_{13}\varepsilon^1_{t_i} + \rho'_{23}\varepsilon^2_{t_i} + \sqrt{1 - \rho^2_{13} - \rho'^2_{23}}\varepsilon^3_{t_i}\right), \\
F_{t_i} =& F_{t_{i-1}} + (r_{t_{i-1}} - \alpha)F_{t_{i-1}}\Delta t + \sigma_2 F_{t_{i-1}} \sqrt{\Delta t}\varepsilon^4_{t_i},
\end{aligned}
$$

where $\{\varepsilon^1_{t_i}\}$, $\{\varepsilon^2_{t_i}\}$, $\{\varepsilon^3_{t_i}\}$ and $\{\varepsilon^4_{t_i}\}$ are four independent sequences of standard normal random variables. Recall that we must reset the fund value F_t at renewal dates, that is, F_{T_1} and F_{T_2}, before generating the next step values.

The integrals in equations (3.3) and (3.4) can be approximated using the trapezoidal rule over the interval $[0, t]$, which is partitioned into h subintervals. Hence,

$$\int_0^t f(u)\mathrm{d}u \approx \frac{\Delta t}{2}\left[f_0 + f_h + 2\sum_{k=1}^{h-1} f_k\right],$$

giving numerical values for the product $e^{-\int_0^{T_k} r_u du} e^{-\int_0^{T_k} \mu_u du} e^{-\int_0^{T_k} l_u du}$ with f_u denoting a generic notation for r_u, μ_u and l_u.

Under our proposed approach, we calculate P_{GMMB} using equation (4.16), which is a pricing solution in closed form. The P_{GMAB} value will be determined by equation (4.28), which only requires the simulation of two multivariate normal random variables, but not the trajectory of r_t, μ_t, l_t and F_t.

These two multivariate normal random variables $(Y^{(2)}_{0,\,T_1}, Y^{(2)}_{T_1,\,T_2})$ and $(Y^{(3)}_{0,\,T_1}, Y^{(3)}_{T_1,\,T_2}, Y^{(3)}_{T_2,\,T_3})$ can be generated through equations (4.19)-(4.21) and equations (4.23)-(4.27). Our numerical results are based on 100,000 sample paths generated through the MC method in RStudio. A parallel-simulation technique is employed with the machine (i7-6820HK CPU @ 2.70 GHz, 8 Cores). The parameters used for equations (2.1), (2.3), (2.4) and (3.2) are depicted in Table 5.1.

In Table 5.2, we display the price of a GMAB based on a cohort aged 50 at $t = 0$ and assuming a GMAB's maturity at age 65, with the first and second renewals at ages 55 and 60, respectively. The codes for the results in Table 5.2 are given in Appendix C.

The prices of a GMMB based on the same cohort with same 15-year maturity are given in Table 5.3. The codes for generating the values in Table 5.3 can be found in Appendix D. Both GMAB and GMMB contracts are evaluated at $t = 0$ (age 50), and a wide range of correlation values ρ_{12}, ρ_{13} and ρ_{23} are tested to see their influence on GMAB and GMMB prices.

In Table 5.2 and Table 5.3, the prices calculated under the direct approach and our proposed method are shown in the second and third columns, respectively. Standard errors for the simulated values are given in parentheses. We see that the prices from our proposed methodology are very close to those obtained from the direct approach; i.e., the absolute differences are very small. Moreover, it is worth noting that our proposed approach has lower standard errors than those from the direct approach. This confirms the greater accuracy of our results than those given by the MC method. A significant highlight is the fact that the average computing time using our proposed methodology is only 0.07% and 0.002% of the computing times using the direct approach for the GMAB and GMMB, respectively; this establishes the efficiency of our measure-change method. It can also be observed that under the same maturity $T_3 = 15$ years and correlation values (ρ_{12}, ρ_{13}, ρ_{23}), the GMAB is more expensive than the GMMB; the price difference is solely attributed to the cost of the additional renewal options embedded in the GMAB contract.

Table 5.1: Parameter values

GMAB contract specification				
$T_1 = 5$	$T_2 = 10$	$T_3 = 15$	$\delta = 0.05$	$P_0 = 1$
GMMB contract specification				
$T_3 = 15$	$\delta = 0.05$	$P_0 = 1$		
Interest rate model				
$a = 0.15$	$b = 0.045$	$\sigma_1 = 0.03$	$r_0 = 0.045$	
Mortality model				
$c = 0.1$	$\xi = 0.0003$	$\mu_0 = -0.006$		
Lapse rate model				
$h = 0.12$	$m = 0.02$	$\zeta = 0.01$	$l_0 = 0.02$	$p = 0.5$
Segregated fund model				
$\alpha = 0.01$	$\sigma_2 = 0.05$	$F_0 = 1$		

Table 5.2: GMAB prices calculated using equations (3.3) and (4.28)

$(\rho_{12},\ \rho_{13},\ \rho_{23})$	Direct approach using equation (3.3)	Our proposed approach using equation (4.28)
$(-0.9, -0.9, 0.81)$	0.32564 (0.00106)	0.32466 (0.00046)
$(-0.6, -0.6, 0.36)$	0.33812 (0.00116)	0.33874 (0.00048)
$(-0.3, -0.3, 0.09)$	0.35347 (0.00128)	0.35401 (0.00049)
$(0.0, 0.0, 0.0)$	0.36988 (0.00140)	0.37044 (0.00051)
$(0.3, 0.3, 0.3)$	0.38595 (0.00154)	0.38755 (0.00053)
$(0.6, 0.6, 0.6)$	0.40835 (0.00172)	0.40712 (0.00055)
$(0.9, 0.9, 0.9)$	0.42611 (0.00188)	0.42591 (0.00056)
$(-0.9, 0.81, -0.9)$	0.40849 (0.00171)	0.41059 (0.00055)
$(-0.6, 0.36, -0.6)$	0.38673 (0.00156)	0.38739 (0.00053)
$(-0.3, 0.09, -0.3)$	0.37224 (0.00143)	0.37419 (0.00051)
$(0.81, -0.9, -0.9)$	0.32615 (0.00108)	0.32324 (0.00046)
$(0.36, -0.6, -0.6)$	0.34417 (0.00120)	0.34063 (0.00048)
$(0.09, -0.3, -0.3)$	0.35413 (0.00129)	0.35507 (0.00050)
average computing time	1102.18 secs	0.84 secs

Table 5.3: GMMB prices calculated utilising equations (3.4) and (4.16)

$(\rho_{12},\ \rho_{13},\ \rho_{23})$	Direct approach using equation (3.4)	Our proposed approach using equation (4.16)
$(-0.9, -0.9, 0.81)$	0.21148 (0.00086)	0.21028 (0)
$(-0.6, -0.6, 0.36)$	0.22722 (0.00098)	0.22720 (0)
$(-0.3, -0.3, 0.09)$	0.24488 (0.00113)	0.24529 (0)
$(0.0, 0.0, 0.0)$	0.26543 (0.00130)	0.26460 (0)
$(0.3, 0.3, 0.3)$	0.28561 (0.00147)	0.28543 (0)
$(0.6, 0.6, 0.6)$	0.31016 (0.00168)	0.30748 (0)
$(0.9, 0.9, 0.9)$	0.32697 (0.00185)	0.33081 (0)
$(-0.9, 0.81, -0.9)$	0.30924 (0.00166)	0.31031 (0)
$(-0.6, 0.36, -0.6)$	0.28316 (0.00144)	0.28281 (0)
$(-0.3, 0.09, -0.3)$	0.26827 (0.00132)	0.26804 (0)
$(0.81, -0.9, -0.9)$	0.21694 (0.00090)	0.21753 (0)
$(0.36, -0.6, -0.6)$	0.23331 (0.00102)	0.23149 (0)
$(0.09, -0.3, -0.3)$	0.24579 (0.00113)	0.24712 (0)
average computing time	1002.39 secs	0.03 secs

5.2 Price-sensitivity analyses

We perform a price-sensitivity analysis for the GMAB under some parameter-scenario settings. The results are exhibited in Figure 5.1 and Figure 5.2 and they reveal the impact of individual model parameters on the GMAB price. All plots are based on the correlations $(\rho_{12},\ \rho_{13},\ \rho_{23}) = (0,\ 0,\ 0)$. Appendix E.1 and Appendix E.2 contain the algorithms in coming up with Figure 5.1 and Figure 5.2.

In the upper panel of Figure 5.1, the parameter b is negatively related to the GMAB price. Note that b is the mean-reverting level of the interest rate model, and a higher mean-reverting level implies a higher average of interest rate. Therefore, the higher the mean-reverting level, the greater the effect of the discounting factor $\exp\left(-\int_0^t r_u \mathrm{d}u\right)$ and consequently, the lower the price. The right plot in the upper panel shows that the volatility σ_1 of the interest rate is positively related to the GMAB price. This outcome is consistent with the view that the higher the risk, the higher the associated potential yield. A similar pattern follows in the lower panel of Figure 5.1, where m is the mean-reverting level of the lapse rate model and ζ is the corresponding volatility.

Figure 5.1: GMAB prices under different parameter values

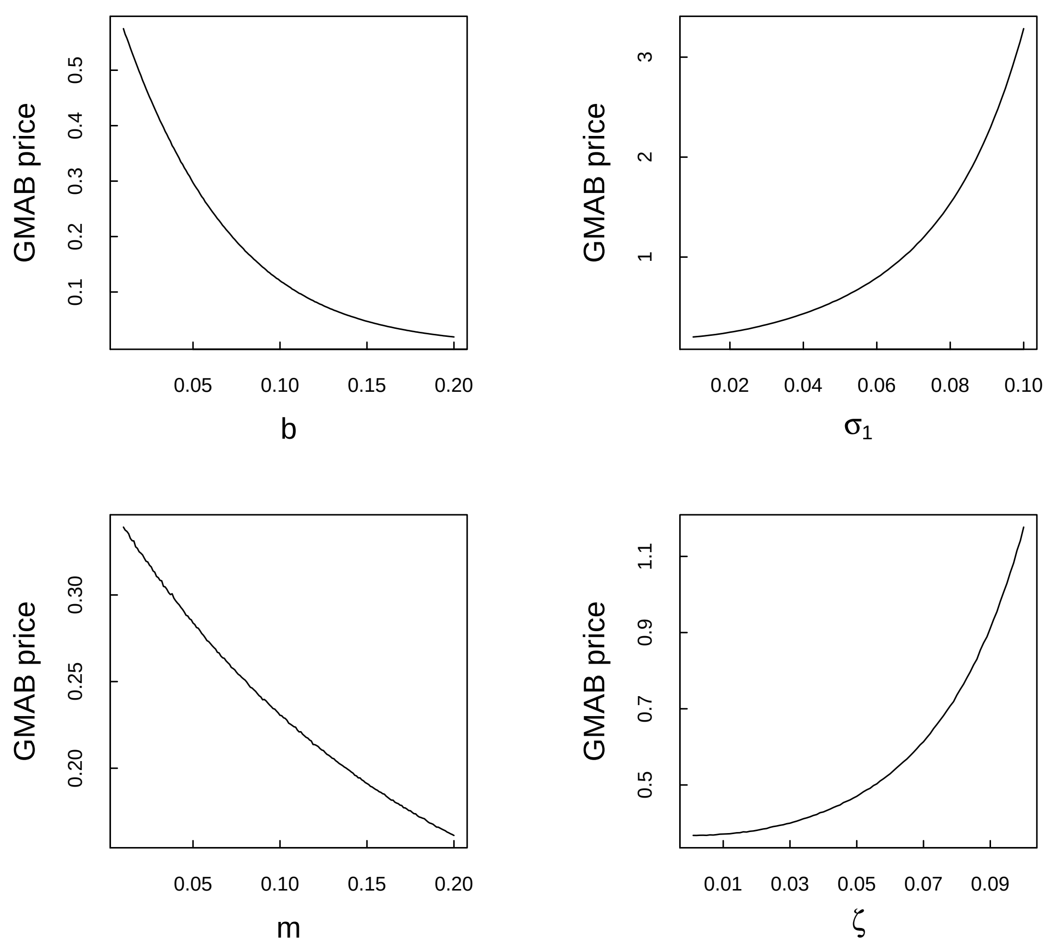
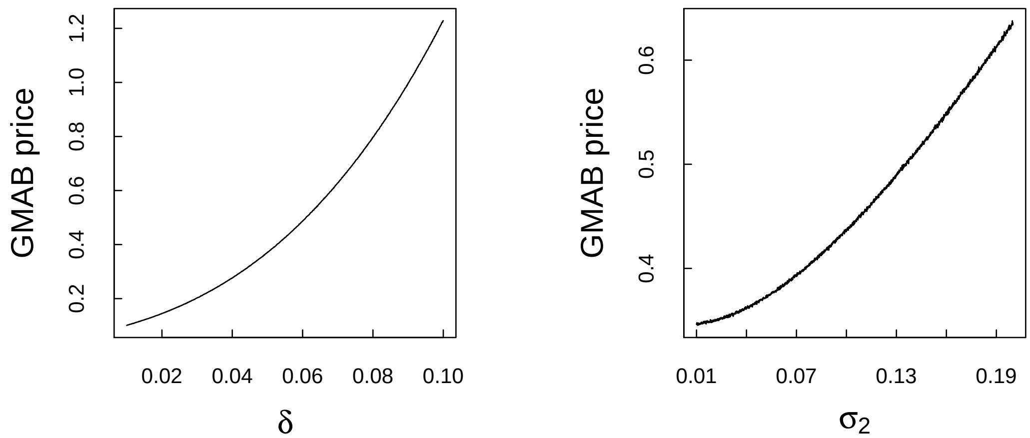
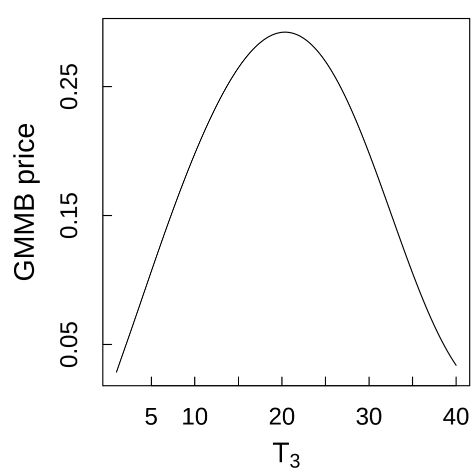


Figure 5.2: GMAB prices under different parameter values



In Figure 5.2, when the roll-up rate δ increases, the GMAB price increases; this is because a higher roll-up rate implies a higher guaranteed value, hence a higher payoff leading to a higher price. Another observation is that the GMAB price increases as the segregated fund's volatility σ_2 increases. Again, this is consistent with the notion that the higher the uncertainty in the performance of the segregated fund, the higher the potential return. Therefore, the GMAB price would have to increase enough to match the corresponding return level.

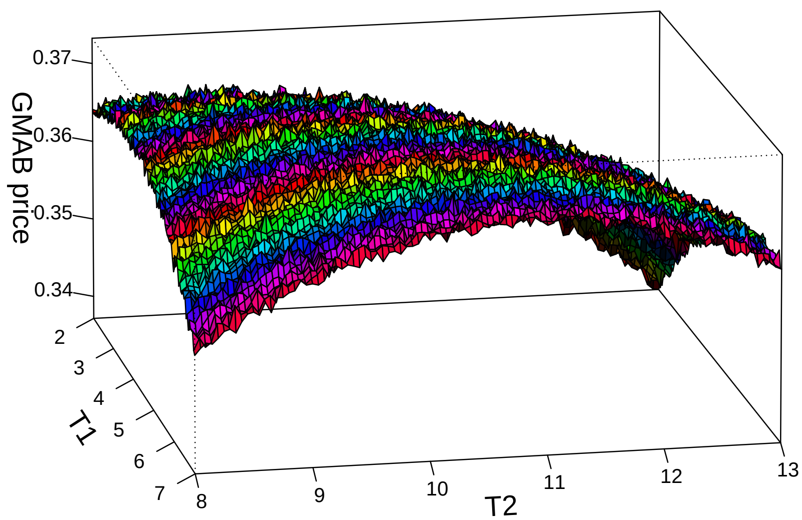
Figure 5.3: GMMB prices with various values of maturity T_3



The price-sensitivity analysis of a GMMB is similar to that of the GMAB. Our investigation of the relationship between the GMMB price and the maturity T_3 discloses an inverted-U pattern; see Figure 5.3. This relationship pattern conveys that the price increases as the uncertainty increases, but after some time the discounting factor has the commanding effect, making the price to decline. Appendix E.3 depicts the codes in generating Figure 5.3.

In Figure 5.4, we display the GMAB prices, with $T_3 = 15$ years, as a function of both T_1 and T_2, where the first renewal is assumed to be between year 2 and year 7 whilst the second renewal is assumed to be between year 8 and year 13. The codes used to produce the results in Figure 5.4 are shown in Appendix E.4.

Figure 5.4: GMAB prices versus varying T_1 and T_2



Chapter 6

Conclusion

Actuarial practice needs a valuation approach that is sophisticated to capture the salient features of the underlying variables yet it must be easily implementable and adaptable to industry's pricing platform. This research responds to this need and constructs a framework whose flexibility could extend to the pricing of other contracts with investment guarantees.

More specifically, we developed an integrated framework for the valuation of a GMAB, where three interrelated risk factors (i.e., interest, mortality, and lapse rates were considered). The change of measure technique was employed to obtain an explicit solution for the pure endowment, and therefore aiding the evaluation of risk-neutral conditional expectation for pricing. In particular, we utilised the forward measure and the survival measure to decompose the pure endowment into the product of the bond price, likelihood of survival, and lapsation probability. The streamlined valuation of a GMAB is finally achieved through the utility of the endowment-risk-adjusted measure. When the option to renew is not present, we successfully derived an analytic solution for the so-called the GMMB contract. Numerical illustrations show that we created a computationally time-saving method with highly significant calculating speed and accuracy when compared to the benchmark chosen, which is the MC simulation method.

There are several possible natural avenues for future research. We may adopt the two-factor Hull-White model [10] instead of the Vasiček model, which is noted for its ability to fit today's term structure of interest rates. Note that the mortality model we adopted ignores the age pattern; so, it may be worthwile to consider the Cairns-Blake-Dowd model [3] in which both age and year factors are taken into account. Moreover, we may include the analysis of a ratchet feature in the guarantee as well as a withdrawal feature in the segregated fund. Lastly, the use of regime-switching set ups (e.g. Gao et al. [7, 8], Zhao and Mamon [20], Xi and Mamon [19], Zhou and Mamon [22], Jalen and Mamon [13], and Ignatieva et al. [11], amongst others) will definitely enrich the methodology in GMAB valuation.

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Appendices

Appendix A

Calculation details for the dynamics of $\Lambda_t^{3(k)}$ under Q

This appendix provides calculation details to support the validity of equation (4.9).

Using equation (4.6), we can rewrite $\Lambda_t^{3(k)}$ as

$$\Lambda_t^{3(k)} := \frac{H_t^{(k)} Y_t^{(k)} M_t^{(k)}}{M(0,T_k)},$$

where

$$\begin{aligned}
H_t^{(k)} =& e^{-\int_0^t r_u \mathrm{d}u} e^{-A(t,T_k)r_t + D(t,T_k)},\\
Y_t^{(k)} =& e^{-\int_0^t \mu_u \mathrm{d}u} e^{-\widetilde{G}(t,T_k)\mu_t + \widetilde{H}(t,T_k)},\\
M_t^{(k)} =& e^{-\int_0^t l_u \mathrm{d}u} e^{-\bar{I}(t,T_k)l_t - \overline{K}(t,T_k)r_t + \bar{J}(t,T_k)}.
\end{aligned}$$

For any $0 \le s < t \le T_k$, we have

$$\begin{aligned}
\mathbb{E}^Q\left[H_t^{(k)}\middle|\mathcal{F}_s\right] =& \mathbb{E}^Q\left[e^{-\int_0^t r_u \mathrm{d}u} e^{-A(t,T_k)r_t + D(t,T_k)}\middle|\mathcal{F}_s\right] = \mathbb{E}^Q\left[e^{-\int_0^t r_u \mathrm{d}u}\mathbb{E}^Q\left[e^{-\int_t^{T_k} r_u \mathrm{d}u}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right]\\
=& \mathbb{E}^Q\left[\mathbb{E}^Q\left[e^{-\int_0^t r_u \mathrm{d}u} e^{-\int_t^{T_k} r_u \mathrm{d}u}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right] = \mathbb{E}^Q\left[\mathbb{E}^Q\left[e^{-\int_0^{T_k} r_u \mathrm{d}u}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right]\\
=& \mathbb{E}^Q\left[e^{-\int_0^{T_k} r_u \mathrm{d}u}\middle|\mathcal{F}_s\right] = e^{-\int_0^s r_u \mathrm{d}u}\mathbb{E}^Q\left[e^{-\int_s^{T_k} r_u \mathrm{d}u}\middle|\mathcal{F}_s\right]\\
=& e^{-\int_0^s r_u \mathrm{d}u} e^{-A(s,T_k)r_s + D(s,T_k)} = H_s^{(k)}.
\end{aligned}$$

So, $H_t^{(k)}$ is a Q-martingale, and the drift coefficient in the Q dynamics of $H_t^{(k)}$ must be 0.

Using Itô's Lemma, we have

$$\begin{aligned}
\mathrm{d}H_t^{(k)} =& e^{-\int_0^t r_u \mathrm{d}u} \mathrm{d}e^{-A(t,T_k)r_t + D(t,T_k)} + e^{-A(t,T_k)r_t + D(t,T_k)} \mathrm{d}e^{-\int_0^t r_u \mathrm{d}u} + \mathrm{d}e^{-A(t,T_k)r_t + D(t,T_k)} \mathrm{d}e^{-\int_0^t r_u \mathrm{d}u} \\
=& -\sigma_1 A(t,T_k) H_t^{(k)} \mathrm{d}W_t^1,
\end{aligned}$$

Similar arguments show that

$$\mathrm{d}Y_t^{(k)} = -\xi \widetilde{G}(t,T_k) Y_t^{(k)} \left(\rho_{12}\sigma_1 A(t,T_k)\mathrm{d}t + \rho_{12}\mathrm{d}W_t^1 + \sqrt{1-\rho_{12}^2}\mathrm{d}W_t^2 \right)$$

and

$$\begin{aligned}
\mathrm{d}M_t^{(k)} =& -M_t^{(k)} \Big[\left(\rho_{13}\zeta \bar{I}(t,T_k) + \sigma_1 \overline{K}(t,T_k) \right) \left(\sigma_1 A(t,T_k) + \rho_{12}\xi \widetilde{G}(t,T_k) \right) \\
& + \zeta \bar{I}(t,T_k) \rho_{23}' \xi \widetilde{G}(t,T_k) \sqrt{1-\rho_{12}^2} \Big] \mathrm{d}t \\
& - M_t^{(k)} \Big[\left(\rho_{13}\zeta \bar{I}(t,T_k) + \sigma \overline{K}(t,T_k) \right) \mathrm{d}W_t^1 + \zeta \bar{I}(t,T_k) \rho_{23}' \mathrm{d}W_t^2 \\
& + \zeta \bar{I}(t,T_k) \sqrt{1-\rho_{13}^2 - \rho_{23}'^2} \mathrm{d}W_t^3 \Big].
\end{aligned}$$

By Itô's Lemma, we have

$$\begin{aligned}
\mathrm{d}H_t^{(k)} Y_t^{(k)} =& Y_t^{(k)} \mathrm{d}H_t^{(k)} + H_t^{(k)} \mathrm{d}Y_t^{(k)} + \mathrm{d}H_t^{(k)} \mathrm{d}Y_t^{(k)} \\
=& -\sigma_1 A(t,T_k) H_t^{(k)} Y_t^{(k)} \mathrm{d}W_t^1 \\
& - \xi \widetilde{G}(t,T_k) H_t^{(k)} Y_t^{(k)} \left(\rho_{12}\sigma_1 A(t,T_k)\mathrm{d}t + \rho_{12}\mathrm{d}W_t^1 + \sqrt{1-\rho_{12}^2}\mathrm{d}W_t^2 \right) \\
& + \rho_{12}\sigma_1 A(t,T_k) \xi \widetilde{G}(t,T_k) H_t^{(k)} Y_t^{(k)} \mathrm{d}t \\
=& -H_t^{(k)} Y_t^{(k)} \left[\left(\sigma_1 A(t,T_k) + \rho_{12}\xi \widetilde{G}(t,T_k) \right) \mathrm{d}W_t^1 + \xi \widetilde{G}(t,T_k) \sqrt{1-\rho_{12}^2} \mathrm{d}W_t^2 \right].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{d}H_t^{(k)}Y_t^{(k)}M_t^{(k)} =& M_t^{(k)}\mathrm{d}H_t^{(k)}Y_t^{(k)} + H_t^{(k)}Y_t^{(k)}\mathrm{d}M_t^{(k)} + \mathrm{d}H_t^{(k)}Y_t^{(k)}\mathrm{d}M_t^{(k)} \\
=& -H_t^{(k)}Y_t^{(k)}M_t^{(k)}\left[\left(\sigma_1 A(t,T_k) + \rho_{12}\xi\widetilde{G}(t,T_k)\right)\mathrm{d}W_t^1 + \xi\widetilde{G}(t,T_k)\sqrt{1-\rho_{12}^2}\mathrm{d}W_t^2\right] \\
& - H_t^{(k)}Y_t^{(k)}M_t^{(k)}\Big[\left(\rho_{13}\zeta\overline{I}(t,T_k) + \sigma_1\overline{K}(t,T_k)\right)\left(\sigma_1 A(t,T_k) + \rho_{12}\xi\widetilde{G}(t,T_k)\right) \\
& \quad + \rho'_{23}\zeta\overline{I}(t,T_k)\xi\widetilde{G}(t,T_k)\sqrt{1-\rho_{12}^2}\Big]\mathrm{d}t \\
& - H_t^{(k)}Y_t^{(k)}M_t^{(k)}\Big[\left(\rho_{13}\zeta\overline{I}(t,T_k) + \sigma\overline{K}(t,T_k)\right)\mathrm{d}W_t^1 + \rho'_{23}\zeta\overline{I}(t,T_k)\mathrm{d}W_t^2 \\
& \quad + \zeta\overline{I}(t,T_k)\sqrt{1-\rho_{13}^2-\rho_{23}'^2}\mathrm{d}W_t^3\Big] \\
& + H_t^{(k)}Y_t^{(k)}M_t^{(k)}\Big[\left(\rho_{13}\zeta\overline{I}(t,T_k) + \sigma\overline{K}(t,T_k)\right)\left(\sigma_1 A(t,T_k) + \rho_{12}\xi\widetilde{G}(t,T_k)\right) \\
& \quad + \rho'_{23}\zeta\overline{I}(t,T_k)\xi\widetilde{G}(t,T_k)\sqrt{1-\rho_{12}^2}\Big]\mathrm{d}t \\
=& -H_t^{(k)}Y_t^{(k)}M_t^{(k)}\Big[\left(\sigma_1 A(t,T_k) + \rho_{12}\xi\widetilde{G}(t,T_k) + \rho_{13}\zeta\overline{I}(t,T_k) + \sigma_1\overline{K}(t,T_k)\right)\mathrm{d}W_t^1 \\
& \quad + \left(\xi\widetilde{G}(t,T_k)\sqrt{1-\rho_{12}^2} + \rho'_{23}\zeta\overline{I}(t,T_k)\right)\mathrm{d}W_t^2 + \zeta\overline{I}(t,T_k)\sqrt{1-\rho_{13}^2-\rho_{23}'^2}\mathrm{d}W_t^3\Big].
\end{aligned}$$

Thus, the dynamics of $\Lambda_t^{3(k)}$ under Q is given by

$$\begin{aligned}
\mathrm{d}\Lambda_t^{3(k)} =& -\Lambda_t^{3(k)}\Big[\left(\sigma_1 A(t,T_k) + \rho_{12}\xi\widetilde{G}(t,T_k) + \rho_{13}\zeta\overline{I}(t,T_k) + \sigma_1\overline{K}(t,T_k)\right)\mathrm{d}W_t^1 \\
& + \left(\xi\widetilde{G}(t,T_k)\sqrt{1-\rho_{12}^2} + \rho'_{23}\zeta\overline{I}(t,T_k)\right)\mathrm{d}W_t^2 + \zeta\overline{I}(t,T_k)\sqrt{1-\rho_{13}^2-\rho_{23}'^2}\mathrm{d}W_t^3\Big].
\end{aligned}$$

■

Appendix B

Calculation details for the covariances in Chapter 4

This appendix provides the computational details to support the validity of equations (4.21), (4.25), (4.26) and (4.27).

We examine and evaluate one by one the four terms in equation (4.21).

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[Y^{(2)}_{0,\,T_1}, Y^{(2)}_{T_1,\,T_2}\right] =& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\int_0^{T_1}\left(r_t-\alpha-\frac{1}{2}\sigma_2^2\right)\mathrm{d}t+\sigma_2\widehat{W}^{4(2)}_{T_1}, \int_{T_1}^{T_2}\left(r_t-\alpha-\frac{1}{2}\sigma_2^2\right)\mathrm{d}t+\sigma_2\left(\widehat{W}^{4(2)}_{T_2}-\widehat{W}^{4(2)}_{T_1}\right)\right]\\
=& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\int_0^{T_1} r_t\mathrm{d}t+\sigma_2\widehat{W}^{4(2)}_{T_1}, \int_{T_1}^{T_2} r_t\mathrm{d}t+\sigma_2\left(\widehat{W}^{4(2)}_{T_2}-\widehat{W}^{4(2)}_{T_1}\right)\right]\\
=& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\int_0^{T_1} r_t\mathrm{d}t, \int_{T_1}^{T_2} r_t\mathrm{d}t\right]+\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\int_0^{T_1} r_t\mathrm{d}t, \sigma_2\left(\widehat{W}^{4(2)}_{T_2}-\widehat{W}^{4(2)}_{T_1}\right)\right]\\
&+\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\sigma_2\widehat{W}^{4(2)}_{T_1}, \int_{T_1}^{T_2} r_t\mathrm{d}t\right]+\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\sigma_2\widehat{W}^{4(2)}_{T_1}, \sigma_2\left(\widehat{W}^{4(2)}_{T_2}-\widehat{W}^{4(2)}_{T_1}\right)\right].
\end{aligned}
$$

Using equation (4.11), the first term can be expressed as

$$\begin{aligned}
\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\int_0^{T_1} r_t \mathrm{d}t, \int_{T_1}^{T_2} r_t \mathrm{d}t\right] =& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\sigma_1 \int_0^{T_1}\int_0^u e^{-au}e^{as}\mathrm{d}\widehat{W}_s^{1(2)}\mathrm{d}u, \sigma_1 \int_{T_1}^{T_2}\int_0^u e^{-au}e^{as}\mathrm{d}\widehat{W}_s^{1(2)}\mathrm{d}u\right] \\
=& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\Bigg[\sigma_1 \int_0^{T_1}\left(\frac{1-e^{-a(T_1-s)}}{a}\right)\mathrm{d}\widehat{W}_s^{1(2)}, \sigma_1\left(\frac{e^{-aT_1}-e^{-aT_2}}{a}\right)\int_0^{T_1} e^{as}\mathrm{d}\widehat{W}_s^{1(2)} \\
&+ \sigma_1\int_{T_1}^{T_2}\left(\frac{1-e^{-a(T_2-s)}}{a}\right)\mathrm{d}\widehat{W}_s^{1(2)}\Bigg] \\
=& \sigma_1^2\left(\frac{e^{-aT_1}-e^{-aT_2}}{a}\right)\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\int_0^{T_1}\left(\frac{1-e^{-a(T_1-s)}}{a}\right)\mathrm{d}\widehat{W}_s^{1(2)}, \int_0^{T_1} e^{as}\mathrm{d}\widehat{W}_s^{1(2)}\right] \\
=& \sigma_1^2\left(\frac{e^{-aT_1}-e^{-aT_2}}{a}\right)\int_0^{T_1}\left(\frac{1-e^{-a(T_1-s)}}{a}\right)e^{as}\mathrm{d}s \\
=& \frac{\sigma_1^2}{a^2}\left(e^{-aT_1}-e^{-aT_2}\right)\int_0^{T_1}\left(e^{as}-e^{-aT_1}e^{2as}\right)\mathrm{d}s \\
=& \frac{\sigma_1^2}{a^2}\left(e^{-aT_1}-e^{-aT_2}\right)\left(\frac{1}{a}e^{as}-\frac{1}{2a}e^{-aT_1}e^{2as}\right)\Bigg|_{s=0}^{s=T_1} \\
=& \frac{\sigma_1^2}{2a^3}\left(e^{-aT_1}-e^{-aT_2}\right)\left(e^{aT_1}+e^{-aT_1}-2\right).
\end{aligned}$$

Moreover, the second term can be expressed as

$$\begin{aligned}
\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\int_0^{T_1} r_t\mathrm{d}t, \sigma_2\left(\widehat{W}_{T_2}^{4(2)}-\widehat{W}_{T_1}^{4(2)}\right)\right] =& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\sigma_1\int_0^{T_1}\int_0^u e^{-au}e^{as}d\widehat{W}_s^{1(2)}\mathrm{d}u, \sigma_2\left(\widehat{W}_{T_2}^{4(2)}-\widehat{W}_{T_1}^{4(2)}\right)\right] \\
=& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\sigma_1\int_0^{T_1}\left(\frac{1-e^{-a(T_1-s)}}{a}\right)\mathrm{d}\widehat{W}_s^{1(2)}, \sigma_2\left(\widehat{W}_{T_2}^{4(2)}-\widehat{W}_{T_1}^{4(2)}\right)\right] \\
=& 0.
\end{aligned}$$

Futhermore, the third term can be expressed as

$$\begin{aligned}
\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\sigma_2\widehat{W}_{T_1}^{4(2)}, \int_{T_1}^{T_2} r_t\mathrm{d}t\right] =& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\sigma_2\widehat{W}_{T_1}^{4(2)}, \sigma_1\int_{T_1}^{T_2}\int_0^u e^{-au}e^{as}\mathrm{d}\widehat{W}_s^{1(2)}\mathrm{d}u\right] \\
=& \mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\Bigg[\sigma_2\widehat{W}_{T_1}^{4(2)}, \sigma_1\left(\frac{e^{-aT_1}-e^{-aT_2}}{a}\right)\int_0^{T_1} e^{as}\mathrm{d}\widehat{W}_s^{1(2)} \\
&+ \sigma_1\int_{T_1}^{T_2}\left(\frac{1-e^{-a(T_2-s)}}{a}\right)\mathrm{d}\widehat{W}_s^{1(2)}\Bigg] = 0.
\end{aligned}$$

Finally, the last term can be expressed as

$$\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[\sigma_2\widehat{W}_{T_1}^{4(2)}, \sigma_2\left(\widehat{W}_{T_2}^{4(2)}-\widehat{W}_{T_1}^{4(2)}\right)\right] = 0.$$

Therefore we have

$$\mathbb{C}\mathrm{ov}^{\widehat{Q}_2}\left[Y_{0,\,T_1}^{(2)}, Y_{T_1,\,T_2}^{(2)}\right] = \frac{\sigma_1^2}{2a^3}\left(e^{-aT_1}-e^{-aT_2}\right)\left(e^{aT_1}+e^{-aT_1}-2\right)$$

as desired.

Similar arguments show that

$$\mathbb{C}\mathrm{ov}^{\widehat{Q}_3}\left[Y^{(3)}_{0,\,T_1}, Y^{(3)}_{T_1,\,T_2}\right] = \frac{\sigma_1^2}{2a^3}\left(e^{-aT_1} - e^{-aT_2}\right)\left(e^{aT_1} + e^{-aT_1} - 2\right),$$

$$\mathbb{C}\mathrm{ov}^{\widehat{Q}_3}\left[Y^{(3)}_{0,\,T_1}, Y^{(3)}_{T_2,\,T_3}\right] = \frac{\sigma_1^2}{2a^3}\left(e^{-aT_2} - e^{-aT_3}\right)\left(e^{aT_1} + e^{-aT_1} - 2\right),$$

and

$$\mathbb{C}\mathrm{ov}^{\widehat{Q}_3}\left[Y^{(3)}_{T_1,\,T_2}, Y^{(3)}_{T_2,\,T_3}\right] = \frac{\sigma_1^2}{2a^3}\left(e^{-aT_2} - e^{-aT_3}\right)\left(e^{aT_2} + e^{-aT_2} - e^{aT_1} - e^{-aT_1}\right).$$

■

Appendix C

Codes for GMAB evaluation

This appendix provides the R codes used to produce the results in Table 5.2.

C.1 Codes for the direct approach in the GMAB evaluation

```
# Set the parameter values
a=0.15
b=0.045
sigma1=0.03
r0=0.045
c=0.1
xi=0.0003
u0=0.006
h=0.12
m=0.02
zeta=0.01
p=0.5
l0=0.02
sigma2=0.05
```

```
# initial premium
premium=1
# mc is the management charge, which is denoted by alpha in
    the thesis.
mc=0.01

library(MASS)
library(parallel)

# Generate sample path
path<-function(v,en,ri,ui,li,fi){
  tem=matrix(rep(0,4*(252*v+1)),(252*v+1),4)
  tem[1,1]=ri
  tem[1,2]=ui
  tem[1,3]=li
  tem[1,4]=fi
  for (k in 2:(1+252*v)){
    tem[k,1]=tem[k-1,1]+a*(b-tem[k-1,1])*(1/252)+sigma1*
       sqrt(1/252)*en[k-1,1]
    tem[k,2]=tem[k-1,2]+c*tem[k-1,2]*(1/252)+xi*pho12*sqrt
       (1/252)*en[k-1,1]+xi*sqrt(1-(pho12)^2)*sqrt(1/252)*
       en[k-1,2]
    tem[k,3]=tem[k-1,3]+h*(m+p*tem[k-1,1]-tem[k-1,3])*(1/
       252)+zeta*pho13*sqrt(1/252)*en[k-1,1]+zeta*pho23t*
       sqrt(1/252)*en[k-1,2]+zeta*sqrt(1-(pho13)^2-(pho23t)
       ^2)*sqrt(1/252)*en[k-1,3]
    tem[k,4]=tem[k-1,4]+(tem[k-1,1]-mc)*tem[k-1,4]*(1/252)+
       sigma2*tem[k-1,4]*sqrt(1/252)*en[k-1,4]
}
  return(tem)
```

```
}

set.seed(201903)

# GMAB price function
f1<-function(j){
    t1=5            #first renewal date
    t2=10           #second renewal date
    t3=15           #maturity
    delta=0.05
    ans=0
    e=mvrnorm((252*t1),c(0,0,0,0),diag(1,4,4))
    te1=path(t1,e,r0,u0,10,premium)
    discount1=exp(-(1/504)*(2*sum(te1[,1])-te1[1,1]-te1
        [1+252*t1,1]))*exp(-(1/504)*(2*sum(te1[,2])-te1
        [1,2]-te1[1+252*t1,2]))*exp(-(1/504)*(2*sum(te1[,3])
        -te1[1,3]-te1[1+252*t1,3]))
    ans=ans+discount1*max(premium*exp(delta*t1)-te1[1+252*
        t1,4],0)
    GT1=max(te1[1+252*t1,4],premium*exp(delta*t1))
    e=mvrnorm((252*(t2-t1)),c(0,0,0,0),diag(1,4,4))
    te2=path((t2-t1),e,te1[1+252*t1,1],te1[1+252*t1,2],te1
        [1+252*t1,3],GT1)
    discount2=discount1*exp(-(1/504)*(2*sum(te2[,1])-te2
        [1,1]-te2[1+252*(t2-t1),1]))*exp(-(1/504)*(2*sum(te2
        [,2])-te2[1,2]-te2[1+252*(t2-t1),2]))*exp(-(1/504)*
        (2*sum(te2[,3])-te2[1,3]-te2[1+252*(t2-t1),3]))
    ans=ans+discount2*max(GT1*exp(delta*(t2-t1))-te2[1+252*
        (t2-t1),4],0)
    GT2=max(te2[1+252*(t2-t1),4],GT1*exp(delta*(t2-t1)))
```

```
    e=mvrnorm((252*(t3-t2)),c(0,0,0,0),diag(1,4,4))
    te3=path((t3-t2),e,te2[1+252*(t2-t1),1],te2[1+252*(t2-
        t1),2],te2[1+252*(t2-t1),3],GT2)
    ans=ans+discount2*exp(-(1/504)*(2*sum(te3[,1])-te3
        [1,1]-te3[1+252*(t3-t2),1]))*exp(-(1/504)*(2*sum(te3
        [,2])-te3[1,2]-te3[1+252*(t3-t2),2]))*exp(-(1/504)*
        (2*sum(te3[,3])-te3[1,3]-te3[1+252*(t3-t2),3]))*max(
        GT2*exp(delta*(t3-t2))-te3[1+252*(t3-t2),4],0)
  return(ans)
}

# Set the correlation values
pho12=-0.9
pho13=-0.9
pho23=0.81
pho23t=(pho23-pho12*pho13)/sqrt(1-pho12^2)

# Parallel simulation
set.seed(201903)
detectCores()
cl <- makeCluster(4)
clusterExport(cl=cl, varlist=c("mvrnorm","path","r0","u0","
    l0","premium","a","b","sigma1","c","xi","h","m","zeta","
    p","sigma2","mc","pho12","pho13","pho23","pho23t"))
ans=rep(0,100000)

ptm<-proc.time()
ans=parSapply(cl,1:100000,FUN = f1)
mean(ans)
sd(ans)/sqrt(100000)
```

```
proc.time()-ptm

stopCluster(cl)
```

C.2 Codes for our proposed method in the evaluation of GMAB

```
# Set the parameter values
b=0.045
r0=0.045
c=0.1
u0=0.006
h=0.12
p=0.5
l0=0.02
premium=1

library(MASS)

A<-function(t,v,a){
  return((1-exp((-a)*(v-t)))/a)
}

D<-function(t,v,a,b,sigma1){
  return((b-(sigma1)^2/(2*a^2))*(A(t,v,a)-(v-t))-(sigma1)^2
    *(A(t,v,a))^2/(4*a))
}

G<-function(t,v){
```

```
    return((exp(c*(v-t))-1)/c)
}

H<-function(t,v,a,sigma1,xi){
    result=(pho12*sigma1*xi/(a*c)-(xi)^2/(2*c^2))*(G(t,v)-(v-
        t))+pho12*sigma1*xi/(a*c)*(A(t,v,a)-phi(t,v,a))+(xi)^2
        *(G(t,v))^2/(4*c)
    return(result)
}

phi<-function(t,v,a){
    return((1-exp(-(a-c)*(v-t)))/(a-c))
}

I<-function(t,v){
    return(((1-exp((-h)*(v-t)))/h))
}

K<-function(t,v,a){
    return((h*p*(A(t,v,a)-I(t,v))/(h-a)))
}

mbar<-function(t,v,a,sigma1,xi,m,zeta){
    return((h*m-pho13*sigma1*zeta*A(t,v,a)-pho23*xi*zeta*G(t,
        v)))
}

bbar<-function(t,v,a,b,sigma1,xi){
    return((a*b-(sigma1)^2*A(t,v,a)-pho12*sigma1*xi*G(t,v)))
}
```

```

# Numerical methods for the ordinary differential equation
J<-function(t,v,a,b,sigma1,xi,m,zeta){
  u=rep(0,(1+100*(v-t)))
  u[1]=0
  for (i in 2:(1+100*(v-t))){
    u[i]=u[i-1]-0.01*(I(v-(i-1)*0.01,v)*mbar(v-(i-1)*0.01,v
        ,a,sigma1,xi,m,zeta)+K(v-(i-1)*0.01,v,a)*bbar(v-(i
        -1)*0.01,v,a,b,sigma1,xi)-0.5*((zeta)^2*I(v-(i-1)*
        0.01,v)^2+(sigma1)^2*K(v-(i-1)*0.01,v,a)^2+2*pho13*
        zeta*sigma1*I(v-(i-1)*0.01,v)*K(v-(i-1)*0.01,v,a)))
 }
  return(u[100*(v-t)+1])
}

# Pure enowment value
M<-function(v,a,b,sigma1,xi,m,zeta){
  ans=exp(-((A(0,v,a)+K(0,v,a))*r0+G(0,v)*(u0)+I(0,v)*l0)+D
      (0,v,a,b,sigma1)+H(0,v,a,sigma1,xi)+J(0,v,a,b,sigma1,
      xi,m,zeta))
  return(ans)
}

# Mean
miuT<-function(u,v,maturity,mc,sigma2,a,b,sigma1,xi,zeta){
  ans=-mc*(v-u)-0.5*(sigma2)^2*(v-u)+r0*(exp(-a*u)-exp(-a*v
      ))/a+(b-((sigma1)^2)/(a^2)+pho12*sigma1*xi/(a*c)-pho13
      *sigma1*zeta/(a*h)-((sigma1)^2)*p/(a^2))*((v-u)-(exp(-
      a*u)-exp(-a*v))/a)+(sigma1)^2/(2*a^2)*exp(-a*maturity)
      *(1+h*p/(h-a))*((exp(a*v)-exp(a*u))/a-(exp(-a*u)-exp(-
```

```
    a*v))/a)+sigma1/(a+h)*exp(-h*maturity)*(pho13*zeta/h-sigma1*p/(h-a))*((exp(h*v)-exp(h*u))/h-(exp(-a*u)-exp(-a*v))/a)-(pho12*sigma1*xi/(c*(a-c)))*exp(c*maturity)*((exp(-c*u)-exp(-c*v))/c-(exp(-a*u)-exp(-a*v))/a)
  return(ans)
}

# Variance
sigmaT<-function(u,v,maturity,sigma2,a,sigma1){
  ans=sigma1^2*((exp(-a*u)-exp(-a*v))/a)^2*(exp(2*a*u)-1)/(2*a)+((sigma1)^2/(a^2))*((v-u)-2*(1-exp(-a*(v-u)))/a+(1-exp(-2*a*(v-u)))/(2*a))+(sigma2)^2*(v-u)
  return(ans)
}

# Covariance
cov2<-function(t1,t2,maturity,a,sigma1){
  return(sigma1^2/(2*a^3)*(exp(-a*t1)-exp(-a*t2))*(exp(a*t1)+exp(-a*t1)-2))
}

cov312<-function(t1,t2,maturity,a,sigma1){
  return(sigma1^2/(2*a^3)*(exp(-a*t1)-exp(-a*t2))*(exp(a*t1)+exp(-a*t1)-2))
}

cov313<-function(t1,t2,maturity,a,sigma1){
  return(sigma1^2/(2*a^3)*(exp(-a*t2)-exp(-a*maturity))*(exp(a*t1)+exp(-a*t1)-2))
}
```

```
cov323<-function(t1,t2,maturity,a,sigma1){
  return(sigma1^2/(2*a^3)*(exp(-a*t2)-exp(-a*maturity))*(
      exp(a*t2)+exp(-a*t2)-exp(a*t1)-exp(-a*t1)))
}

price1<-function(t1,t2,maturity,mc,sigma2,delta,a,b,sigma1,
    xi,zeta){
  ans=premium*(-exp(miuT(0,t1,t1,mc,sigma2,a,b,sigma1,xi,
      zeta)+0.5*sigmaT(0,t1,t1,sigma2,a,sigma1))*pnorm((
      delta*t1-miuT(0,t1,t1,mc,sigma2,a,b,sigma1,xi,zeta)-
      sigmaT(0,t1,t1,sigma2,a,sigma1))/sqrt(sigmaT(0,t1,t1,
      sigma2,a,sigma1)),mean=0,sd=1)+exp(delta*t1)*pnorm((
      delta*t1-miuT(0,t1,t1,mc,sigma2,a,b,sigma1,xi,zeta))/
      sqrt(sigmaT(0,t1,t1,sigma2,a,sigma1)),mean=0,sd=1))
  return(ans)
}

price2<-function(t1,t2,maturity,delta,e1){
  result=premium*max(exp(delta*t1),exp(e1[1]))*max(exp(
      delta*(t2-t1))-exp(e1[2]),0)
  return(result)
}

price3<-function(t1,t2,maturity,delta,e2){
  result=premium*max(exp(delta*t1),exp(e2[1]))*max(exp(
      delta*(t2-t1)),exp(e2[2]))*max(exp(delta*(maturity-t2
      [3]))-exp(e2),0)
  return(result)
}
```

```
set.seed(2019045)

# GMAB price function
# t1: first renewal date, t2: second renewal date, t3:
    maturity.
# mc is the management charge, which is denoted by alpha in
    the thesis.
priceall<-function(t1,t2,t3,mc,delta,sigma2,a,b,sigma1,xi,m
    ,zeta,n){
  ans=rep(0,n)
  e1=mvrnorm(n,c(miuT(0,t1,t2,mc,sigma2,a,b,sigma1,xi,zeta)
     ,miuT(t1,t2,t2,mc,sigma2,a,b,sigma1,xi,zeta)),matrix(c
     (sigmaT(0,t1,t2,sigma2,a,sigma1),cov2(t1,t2,t3,a,
     sigma1),cov2(t1,t2,t3,a,sigma1),sigmaT(t1,t2,t2,sigma2
     ,a,sigma1)),2,2,byrow = TRUE))
  e2=mvrnorm(n,c(miuT(0,t1,t3,mc,sigma2,a,b,sigma1,xi,zeta)
     ,miuT(t1,t2,t3,mc,sigma2,a,b,sigma1,xi,zeta),miuT(t2,
     t3,t3,mc,sigma2,a,b,sigma1,xi,zeta)),matrix(c(sigmaT
     (0,t1,t3,sigma2,a,sigma1),cov312(t1,t2,t3,a,sigma1),
     cov313(t1,t2,t3,a,sigma1),cov312(t1,t2,t3,a,sigma1),
     sigmaT(t1,t2,t3,sigma2,a,sigma1),cov323(t1,t2,t3,a,
     sigma1),cov313(t1,t2,t3,a,sigma1),cov323(t1,t2,t3,a,
     sigma1),sigmaT(t2,t3,t3,sigma2,a,sigma1)),3,3,byrow =
     TRUE))
  m1=M(t1,a,b,sigma1,xi,m,zeta)*price1(t1,t2,t3,mc,sigma2,
     delta,a,b,sigma1,xi,zeta)
  m2=M(t2,a,b,sigma1,xi,m,zeta)
  m3=M(t3,a,b,sigma1,xi,m,zeta)
  for(i in 1:n){
```

```
    ans[i]=m1+m2*premium*max(exp(delta*t1),exp(e1[i,1]))*max
        (exp(delta*(t2-t1))-exp(e1[i,2]),0)+m3*premium*max(
        exp(delta*t1),exp(e2[i,1]))*max(exp(delta*(t2-t1)),
        exp(e2[i,2]))*max(exp(delta*(t3-t2))-exp(e2[i,3]),0)
  }
  return(c(mean(ans),sd(ans)/sqrt(n)))
}

# Set the correlation values
pho12=-0.9
pho13=-0.9
pho23=0.81

# priceall(t1,t2,t3,mc,delta,sigma2,a,b,sigma1,xi,m,zeta,n)
# GMAB price using our proposed method
ptm<-proc.time()
priceall
    (5,10,15,0.01,0.05,0.05,0.15,0.045,0.03,0.0003,0.02,0.01,100000)

proc.time()-ptm
```

Appendix D

Codes for GMMB evaluation

The results shown in Table 5.3 were generated utilising the codes in this Appendix.

D.1 Codes for the computation of GMMB value using the direct approach

```
# Set the parameter values
a=0.15
b=0.045
sigma1=0.03
r0=0.045
c=0.1
xi=0.0003
u0=0.006
h=0.12
m=0.02
zeta=0.01
p=0.5
l0=0.02
```

```
sigma2=0.05
premium=1
# mc is the management charge, which is denoted by alpha in
    the thesis.
mc=0.01

library(MASS)
library(parallel)

# Generate the sample path
path<-function(v,en,ri,ui,li,fi){
  tem=matrix(rep(0,4*(252*v+1)),(252*v+1),4)
  tem[1,1]=ri
  tem[1,2]=ui
  tem[1,3]=li
  tem[1,4]=fi
  for (k in 2:(1+252*v)){
    tem[k,1]=tem[k-1,1]+a*(b-tem[k-1,1])*(1/252)+sigma1*
        sqrt(1/252)*en[k-1,1]
    tem[k,2]=tem[k-1,2]+c*tem[k-1,2]*(1/252)+xi*pho12*sqrt
        (1/252)*en[k-1,1]+ xi*sqrt(1-(pho12)^2)*sqrt(1/252)*
        en[k-1,2]
    tem[k,3]=tem[k-1,3]+h*(m+p*tem[k-1,1]-tem[k-1,3])*(1/
        252)+zeta*pho13*sqrt(1/252)*en[k-1,1]+ zeta*pho23t*
        sqrt(1/252)*en[k-1,2]+ zeta*sqrt(1-(pho13)^2-(pho23t)
        ^2)*sqrt(1/252)*en[k-1,3]
    tem[k,4]=tem[k-1,4]+(tem[k-1,1]-mc)*tem[k-1,4]*(1/252)+
        sigma2*tem[k-1,4]*sqrt(1/252)*en[k-1,4]
  }
  return(tem)
```

```
}

# GMMB price function
f<-function(i){
    t3=15
    delta=0.05
    e=mvrnorm((252*t3),c(0,0,0,0),diag(1,4,4))
    te1=path(t3,e,r0,u0,l0,premium)
    ans=exp(-(1/504)*(2*sum(te1[,1])-te1[1,1]-te1[1+252*t3
      ,1]))*exp(-(1/504)*(2*sum(te1[,2])-te1[1,2]-te1
      [1+252*t3,2]))*exp(-(1/504)*(2*sum(te1[,3])-te1
      [1,3]-te1[1+252*t3,3]))*max(premium*exp(delta*t3)-
      te1[1+252*t3,4],0)
    return(ans)
}

# Set the correlation values
pho12=-0.9
pho13=-0.9
pho23=0.81
pho23t=(pho23-pho12*pho13)/sqrt(1-pho12^2)

# Parallel simulation
set.seed(201903)
detectCores()
cl <- makeCluster(4)
clusterExport(cl=cl, varlist=c("mvrnorm","path","r0","u0","
  l0","premium","a","b","sigma1","c","xi","h","m","zeta","
  p","sigma2","mc","pho12","pho13","pho23","pho23t"))

```

```
ptm<-proc.time()
ans=parSapply(cl,1:100000,FUN = f)
mean(ans)
sd(ans)/sqrt(100000)
proc.time()-ptm

stopCluster(cl)
```

D.2 Codes for the computation of the GMMB under our proposed method

```
# Set the parameter values
a=0.15
b=0.045
sigma1=0.03
r0=0.045
c=0.1
xi=0.0003
u0=0.006
h=0.12
m=0.02
zeta=0.01
p=0.5
l0=0.02
sigma2=0.05
premium=1
```

```
# mc is the management charge, which is denoted by alpha in the thesis.
mc=0.01

library(MASS)

A<-function(t,v){
  return((1-exp((-a)*(v-t)))/a)
}

D<-function(t,v){
  return((b-(sigma1)^2/(2*a^2))*(A(t,v)-(v-t))-(sigma1)^2*(A(t,v))^2/(4*a))
}

G<-function(t,v){
  return((exp(c*(v-t))-1)/c)
}

H<-function(t,v){
  result=(pho12*sigma1*xi/(a*c)-(xi)^2/(2*c^2))*(G(t,v)-(v-t))+pho12*sigma1*xi/(a*c)*(A(t,v)-phi(t,v))+(xi)^2*(G(t,v))^2/(4*c)
  return(result)
}

phi<-function(t,v){
  return((1-exp(-(a-c)*(v-t)))/(a-c))
}

```

```
I<-function(t,v){
  return(((1-exp((-h)*(v-t)))/h))
}

K<-function(t,v){
  return((h*p*(A(t,v)-I(t,v))/(h-a)))
}

mbar<-function(t,v){
  return((h*m-pho13*sigma1*zeta*A(t,v)-pho23*xi*zeta*G(t,v)
    ))
}

bbar<-function(t,v){
  return((a*b-(sigma1)^2*A(t,v)-pho12*sigma1*xi*G(t,v)))
}

# Numerical methods for the ordinary differential equation
J<-function(t,v){
  u=rep(0,(1+100*(v-t)))
  u[1]=0
  for (i in 2:(1+100*(v-t))){
    u[i]=u[i-1]-0.01*(I(v-(i-1)*0.01,v)*mbar(v-(i-1)*0.01,v
      )+K(v-(i-1)*0.01,v)*bbar(v-(i-1)*0.01,v) -0.5*((zeta)
      ^2*I(v-(i-1)*0.01,v)^2+(sigma1)^2*K(v-(i-1)*0.01,v)
      ^2+2*pho13*zeta*sigma1*I(v-(i-1)*0.01,v)*K(v-(i-1)*
      0.01,v)))
  }
  return(u[100*(v-t)+1])
}
```

```
# Pure endowment
M<-function(v){
  ans=exp(-((A(0,v)+K(0,v))*r0+G(0,v)*(u0)+I(0,v)*l0)+D(0,v)+H(0,v)+J(0,v))
  return(ans)
}

# Mean
miuT<-function(u,v,maturity){
  ans=-mc*(v-u)-0.5*(sigma2)^2*(v-u)+r0*(exp(-a*u)-exp(-a*v))/a+(b-((sigma1)^2)/(a^2)+pho12*sigma1*xi/(a*c)-pho13*sigma1*zeta/(a*h)-((sigma1)^2)*p/(a^2))*((v-u)-(exp(-a*u)-exp(-a*v))/a)+(sigma1)^2/(2*a^2)*exp(-a*maturity)*(1+h*p/(h-a))*((exp(a*v)-exp(a*u))/a-(exp(-a*u)-exp(-a*v))/a)+sigma1/(a+h)*exp(-h*maturity)*(pho13*zeta/h-sigma1*p/(h-a))*((exp(h*v)-exp(h*u))/h-(exp(-a*u)-exp(-a*v))/a)-(pho12*sigma1*xi/(c*(a-c)))*exp(c*maturity)*((exp(-c*u)-exp(-c*v))/c-(exp(-a*u)-exp(-a*v))/a)
  return(ans)
}

# Variance
sigmaT<-function(u,v,maturity){
  ans=sigma1^2*((exp(-a*u)-exp(-a*v))/a)^2*(exp(2*a*u)-1)/(2*a)+((sigma1)^2/(a^2))*((v-u)-2*(1-exp(-a*(v-u)))/a+(1-exp(-2*a*(v-u)))/(2*a))+(sigma2)^2*(v-u)
  return(ans)
}
```

```
price1<-function(t1,delta){
  ans=premium*(-exp(miuT(0,t1,t1)+0.5*sigmaT(0,t1,t1))*
     pnorm((delta*t1-miuT(0,t1,t1)-sigmaT(0,t1,t1))/sqrt(
     sigmaT(0,t1,t1)),mean=0,sd=1)+exp(delta*t1)*pnorm((
     delta*t1-miuT(0,t1,t1))/sqrt(sigmaT(0,t1,t1)),mean=0,
     sd=1))
  return(ans)
}

# GMMB price function
pricegmmb<-function(maturity,delta){
  return(M(maturity)*price1(maturity,0.05))
}

# Set the correlation values
pho12=-0.9
pho13=-0.9
pho23=0.81

# Calculate the GMMB price
ptm<-proc.time()
pricegmmb(15,0.05)
proc.time()-ptm
```

Appendix E

Codes in conducting price-sensitivity analyses

This appendix presents the R codes for the price-sensitivity analyses found in Section 5.2.

E.1 Codes for Figure 5.1

The following are the codes in producing the results displayed in Figure 5.1.

```
# GMAB price with different values of b
xb=seq(0.01,0.2,0.001)
yb=rep(0,191)
for (i in 1:191) {
  yb[i]=priceall(5,10,15,0.01,0.05,0.05,0.15,0.01+(i-1)/1000,0.03,0.0003,0.02,0.01,100000)
}

# GMAB price with different values of sigma1
xsigma1=seq(0.01,0.1,0.001)
ysigma1=rep(0,91)
```

```
for (i in 1:91) {
  ysigma1[i]=priceall
      (5,10,15,0.01,0.05,0.05,0.15,0.045,0.01+(i-1)/
      1000,0.0003,0.02,0.01,100000)
}

# GMAB price with different values of m
xm=seq(0.01,0.2,0.001)
ym=rep(0,191)
for (i in 1:191) {
  ym[i]=priceall
      (5,10,15,0.01,0.05,0.05,0.15,0.045,0.03,0.0003,0.01+(i
      -1)/1000,0.01,100000)
}

# GMAB price with different values of zeta
xzeta=seq(0.001,0.1,0.001)
yzeta=rep(0,100)
for (i in 1:100) {
  yzeta[i]=priceall
      (5,10,15,0.01,0.05,0.05,0.15,0.045,0.03,0.0003,0.02,0.001+(
      i-1)/1000,100000)
}

# Plot
pdf("figure1.pdf",width = 8,height = 7)
par(mfcol=c(2,2))
par(mar=c(5,5,2,4),tcl=0.3)
plot(xb,yb,type = "l",xlab=expression(b),ylab="GMAB price",
    cex.lab=1.5,cex.axis=1.7,xaxt="n",yaxt="n")
```

```
axis(1,at=seq(0.05,0.3,0.05))
axis(2,at=seq(0.1,0.8,0.1))
plot(xm,ym,type = "l",xlab=expression(m),ylab="GMAB price",
   cex.lab=1.5,cex.axis=1.7,xaxt="n",yaxt="n")
axis(1,at=seq(0.05,0.2,0.05))
axis(2,at=seq(0.1,0.4,0.05))
plot(xsigma1,ysigma1,type = "l",xlab=expression(sigma[1]),
   ylab="GMAB price",cex.lab=1.5,cex.axis=1.7,xaxt="n",yaxt
   ="n")
axis(1,at=seq(0.02,0.1,0.02))
axis(2,at=seq(-1,5,1))
plot(xzeta,yzeta,type = "l",xlab=expression(zeta),ylab="
   GMAB price",cex.lab=1.5,cex.axis=1.7,xaxt="n",yaxt="n")
axis(1,at=seq(0.01,0.12,0.02))
axis(2,at=seq(0.1,1.2,0.2))
dev.off()
```

E.2 Codes for Figure 5.2

The results shown in Figure 5.2 were generated utilising the following codes.

```
# GMAB price with different values of delta
xdelta=seq(0.01,0.1,0.0001)
ydelta=rep(0,901)
for (i in 1:901) {
  ydelta[i]=priceall(5,10,15,0.01,0.01+(i-1)/
     10000,0.05,0.15,0.045,0.03,0.0003,0.02,0.01,100000)
}

```

```
# GMAB price with different values of sigma2
xsigma2=seq(0.01,0.3,0.0001)
ysigma2=rep(0,1901)
for (i in 1:1901) {
  ysigma2[i]=priceall(5,10,15,0.01,0.05,0.01+(i-1)/
      10000,0.15,0.045,0.03,0.0003,0.02,0.01,50000)
}

# Plot
pdf("figure2.pdf",width = 9,height = 4)
par(mfcol=c(1,2))
par(mar=c(5,5,2,4),tcl=0.3)
plot(xdelta,ydelta,type = "l",xlab=expression(delta),ylab="
   GMAB price",cex.lab=1.5,cex.axis=1.7,xaxt="n",yaxt="n")
axis(1,at=seq(0.02,0.1,0.02))
axis(2,at=seq(0.2,1.8,0.2))
plot(xsigma2,ysigma2,type = "l",xlab=expression(sigma[2]),
    ylab="GMAB price",cex.lab=1.5,cex.axis=1.7,xaxt="n",yaxt
    ="n")
axis(1,at=seq(0.01,0.2,0.03))
axis(2,at=seq(0.2,0.7,0.1))
dev.off()
```

E.3 Codes for Figure 5.3

The following are the codes in coming up with Figure 5.3.

```
# GMMB price with different values of maturity
xT=seq(1,40,0.1)
```

```
yT=rep(0,391)
for(i in 1:391){
  yT[i]=pricegmmb(1+(i-1)/10,0.05)
}

# Plot
pdf("figure3.pdf",width = 5,height = 5)
par(mar=c(5,5,3,3),tcl=0.4)
plot(xT,yT,xlab = expression(T[3]),ylab = "GMMB price",
   type="l", cex.lab=1.6,cex.axis=2,xaxt="n",yaxt="n")
axis(1,at=seq(5,40,5),cex.axis=1.3)
axis(2,at=seq(0.05,0.35,0.1),cex.axis=1.3)
dev.off()
```

E.4 Codes for Figure 5.4

The following depict the codes in generating Figure 5.4.

```
library(rsm)

# GMAB price with different values of renewal dates T1 and
   T2
xT1=seq(2,7,0.05)
xT2=seq(8,13,0.05)
yT1T2=matrix(rep(0,101*101),101,101)
for (i in 1:101) {
  for (j in 1:101) {
    yT1T2[i,j]=priceall(2+(i-1)/20,8+(j-1)/
       20,15,0.01,0.05,0.05,0.15,0.045,0.03,0.0003,0.02,0.01,50000)
```

```
        [1]
    }
}

pdf("figure4.pdf",width = 8,height = 6)

# Plot
persp(xT1,xT2,yT1T2,col = rainbow(25),xlab = "Renewal T1",
    ylab = "Renewal T2", zlab = "GMAB price",cex.lab=1.4,
        theta=79, phi=15,r=50,d=0.5,expand=0.5,ltheta=90,
            shade=0.75,ticktype="detailed",nticks=5, box =
            TRUE)

dev.off()
```

Curriculum Vitae

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