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Some Recent Developments on Pareto-optimal Reinsurance

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Abstract

This thesis focuses on developing Pareto-optimal reinsurance policy which considers the interests of both the insurer and the reinsurer. The optimal insurance/reinsurance design has been extensively studied in actuarial science literature, while in early years most studies were concentrated on optimizing the insurer's interests. However, as early as 1960s, Borch argued that "an agreement which is quite attractive to one party may not be acceptable to its counterparty" and he pioneered the study on "fair" risk sharing between the insurer and the reinsurer. Quite recently, the question of how to strike a balance in risk sharing between an insurer and a reinsurer has drawn considerable attention. This thesis contributes to the existing study in terms of the following aspects: first, we derive the set of Pareto-optimal reinsurance policies within risk mimization framework; second, we obtain the set of Pareto-optimal reinsurance policies within expected utility maximization framework. In addition, we uniquely identify the policy according to classical bargaining models; third, we blend risk minimization criterion and expected utility maximization criterion and study the so called Pareto-optimal reinsurance policy with maximal synergy.

The thesis is structured as follows. Chapter 1 introduces the problem and reviews the most recent advances on related topics. Chapter 2 applies a geometric approach to derive the Pareto-optimal reinsurance policy under Value-at-Risk minimization criterion. The geometric approach visualize the process of seeking the solution which greatly simplifies the mathematical proof. As a natural extention, Chapter 3 studies the design of Pareto-optimal reinsurance policy by assuming that distortion risk measures are employed to measure the risks of the insurer and the reinsurer. The optimal reinsurance policy is derived through three methods: Lagrange dual method, generalized Neyman-Pearson lemma and dynamic control approach. Chapter 4 studies the problem through maximizing the weighted expected utility and applies the results from classical bargaining models to identify the "best" policy on the Pareto efficient frontier. Chapter 5 revisits the problem by considering a mixture of risk minimization and expected utility maximization criteria. Chapter 6 gives potential directions for future research.

Keywords: Pareto-optimal reinsurance policy, risk mimization, expected utlity maximization, bargaining model, Lagrange dual method

Summary for Lay Audience

A decision is made to maximize the decision-maker's interests. In a reinsurance setting, the insurer is always treated as the decision-maker in most past actuarial literature. It is understandable a reinsurance treaty may be reached if the reinsurer's interests are totally out of consideration. This thesis proposed several models to address this concern.

Co-Authorship Statement

The materials presented in this thesis are based on four joint-authored research articles. The first three articles are published on *Risks*, *European Actuarial Journal* and *Insurance: Mathematics and Economics*, the fourth article has been submitted to *Scandinavian Actuarial Journal* and is now under review. I am the lead author for all of these articles. I gratefully acknowledge my supervisors Dr. Jiandong Ren and Dr. Hanping Hong for their contributions to all of these articles. The first article is co-authored with Dr. Ricardas Zitikis and I am truly indebted to him for his significant contributions. My colleague Dr. Chen Yang is acknowledged for his significant contributions to the third article.

Dedicated to my parents Guoliang Jiang and Hefang Chen and my lovely wife Qian Huang for their love, endless support and sacrifices.

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Chapter 1

Introduction

1.1 Literature review

Reinsurance policy is an agreement between an insurer and a reinsurer in which the insurer pays the reinsurer a premium in exchange of indemnity for unpredictable large loss. Optimal insurance/reinsurance design is one of the core problems in actuarial science. On one hand, it addresses how to efficiently transfer the risk to improve the insurer's capacity to bear risk; on the other hand, it provides the reinsurer opportunities to make profit. Figure 1.1 illustrates how the reinsurance mechanism works.

Let Ω be the sample space and $X : \Omega \to [0, M]$ be the initial loss faced by the insurer. Let $I(X)$ denote the ceded loss and P denote the premium charged by the reinsurer. After the reinsurance transcation the insurer's total loss is

$$C_I = X - I(X) + P$$

and the reinsurer's total loss is

$$R_I = I(X) - P.$$

Some commonly used ceded functions are

- stop-loss or excess-of-loss function

  $$I(x) = (x - d)_+$$

  where $d \geq 0$ is the deductible point and $y_+ = \max\{0, y\}$;

- quota-share function

  $$I(x) = ax$$

  where $a \in [0, 1]$;

- limited stop-loss function

  $$I(x) = d_1 \wedge (x - d_2)_+$$

  where $d_1 \geq 0$ is the limit, $d_2 \geq 0$ is the deductible point and $x \wedge y = \min\{x, y\}$.

Some commonly used principles for determining the premium P are

- expectation principle
$$P = (1 + \theta)\mathbf{E}[I(X)]$$
where $\theta \geq 0$ is called the safety loading and $\mathbf{E}[\cdot]$ denotes the expectation;

- distortion principle
$$P = H_g(I(X)) = \int_0^M g\left(S_{I(X)}(x)\right) dx$$
where $g(\cdot)$ is called the distortion function and $S_{I(X)}(\cdot)$ denotes the survival function of $I(X)$, i.e., $S_{I(X)}(x) = \mathbf{P}(I(X) > x)$;

- mean-variance principle
$$P = (1 + \theta_1)\mathbf{E}[I(X)] + \theta_2\mathbf{V}[I(X)]$$
where $\theta_1, \theta_2 \geq 0$ and $\mathbf{V}[\cdot]$ denotes the variance.

The aforementioned premium principles can reflect the reinsurer's aversion towards the insolvency, tail risk and uncertainty.

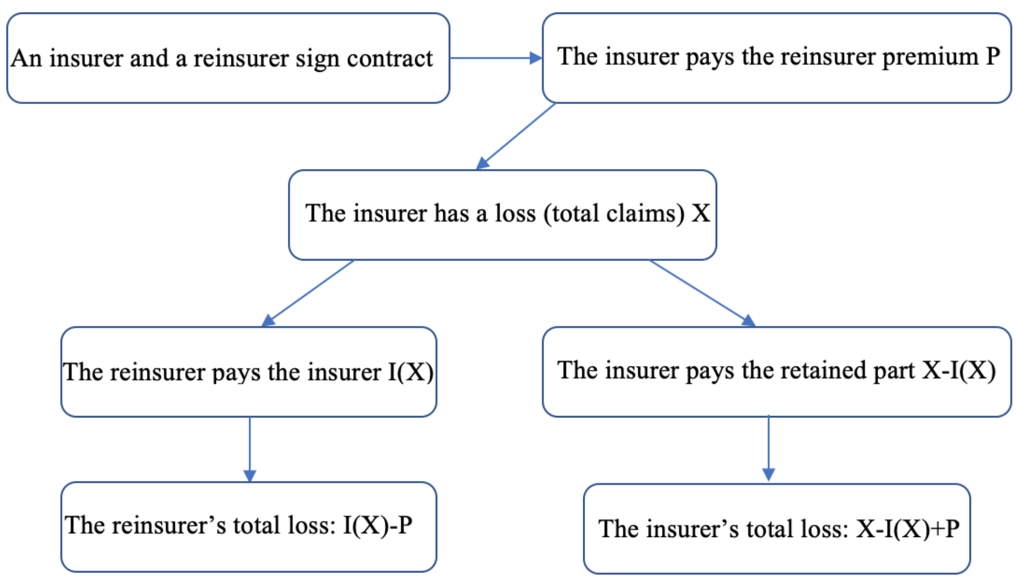


Figure 1.1: The reinsurance agreement.

In the actuarial science literature, the reinsurance policy is designed to optimize the decision maker's interests. Two commonly adopted optimality criteria are risk minimization and expected utility (EU) maximization. For risk minimization, Borch (1960a) showed that the stop-loss ceded function minimized the variance of the insurer's total loss under the expectation premium principle. By restricting the ceded function to be increasing and convex, Cai et al. (2008) showed that the stop-loss function minimized the Value-at-Risk (VaR) of the insurer's total loss. The result in Cai et al. (2008) was later reexamined by Cheung (2010) but using a much simpler geometric method. To prevent ex post moral hazard, Chi and Tan (2011) confined themselves to the set of 1-Lipschitz continuous ceded functions and proved that the limited stop-loss function minimized the VaR of the insurer's total loss. With respect to other popular risk measures, such as expectile, distortion risk measure (DRM) and coherent risk measure, we refer interested readers to Cai and Weng (2014); Zhuang et al. (2016); Cheung et al. (2017) and the references therein.

The optimal reinsurance policy design under EU maximization was pioneered by Arrow (1963). It was shown that stop-loss ceded function maximized the insurer's expected utility under the expectation premium principle. Dana and Scarsini (2007) studied the optimal reinsurance problem in the presence of a background risk. Assuming that the background risk is stochastically increasing with respect to the initial loss and under some regularity conditions, they showed that the optimal ceded function could be of stop-loss form, generalized deductible form and quota-share form. Bernard and Ludkovski (2012) investigated the impact of the counterparty's default risk on the optimal ceded function and showed that more reinsurance coverage is required by the insurer if the reinsurer's average recovery rate is lower. The aforementioned works assume that both the insurer and the reinsurer share the same information regarding the loss distribution. Ghossoub (2017) explored the optimal reinsurance policy when the insurer and the reinsurer had heterogeneous beliefs and proved that the optimal ceded function took the variable deductible form.

1.2 The model of interest

As early as 1960s, Borch already realized that "a contract which is quite attractive to one party may not be acceptable by its counterparty" and thereafter pioneered the study of "fair" risk sharing among the insurers and reinsurers. In Borch (1960b) a set of ceded functions was first identified by minimizing the sum of the variances, then the optimal ceded function was determined by maximizing the product of the EU gains according to the Nash bargaining model. The Pareto-optimal reinsurance policy was first studied by Raviv (1979) in which the insurer's EU was maximized subject to the reinsurer's EU constraint, the optimal ceded function was characterized by a differential equation and a boundary condition. Golubin (2006b) revisited Raviv's problem and solved it using calculus of variation approach, which was verified to be quite efficient to make the original infinite dimensional optimization problem tractable.

One branch of extentions of the above bilateral problem is called optimal risk sharing, in which the interests of all the parties are taken into account simultaneously. For results in this area, see for example, see Borch (1960c); Aase (2002); Ludkovski and Young (2009); Asimit et al. (2013), and the references therein. Another branch of extentions studies optimal reinsurance design in a setting which involves one insurer and multiple reinsurers or multiple insurers and one reinsurer. For results in this area, we refer to Asimit et al. (2017); Asimit and Boonen (2018) and the references therein.

In this thesis, we investigate the Pareto-optimal reinsurance policy within both risk minimization and EU maximization framework. we first explore the Pareto-optimal reinsurance policy when both the insurer and the reinsurer apply VaR to measure their risks. The VaR based model is then naturally extended to DRM based model, in which we further impose some individual risk constraints and investigate the geometric interpretations of our results. Our results generalize those in the literature which only consider the interests of the insurer. Within EU maximization framework, we follow the idea in Golubin (2006b) but apply a different approach which makes the solution much easier to derive. In addition, we impose rationality conditions in our model and identify the “best” solution according to the Nash bargaining model and the Kalai-Smorodinsky model. At last, we present a model which balances the EUs of the insurer and the reinsurer under the constraint that the sum of their risks reaches the minimum. This way, we mix the two criteria – risk minimization and EU maximization into one model and make the derived reinsurance policy more reasonable.

Chapter 2

Pareto-optimal Reinsurance Policies under Value-at-Risk

2.1 Introduction

Reinsurance is a transaction whereby one insurance company (the reinsurer) agrees to indemnify another insurance company (the reinsured, cedent or primary company) against all or part of the loss that the latter sustains under a policy or policies that it has issued. For this service, the ceding company pays the reinsurer a premium, and there are many premium calculation principles (e.g., Denuit et al. (2006); Young (2004); see also Furman and Zitikis (2008, 2009)).

Mathematically, let X be the loss for an insurer from a policy or a group of policies. Assume that under a reinsurance treaty, a reinsurer covers the ceded part of the loss, say $f(X)$, where $0 \le f(X) \le X$, for a premium P_f. The primary insurer's retained loss is denoted by $I_f(X) = X - f(X)$. Commonly-used forms of reinsurance treaties are the excess-of-loss treaty, where $f(X) = (X-d)_+$ with deductible level (attaching point) $d > 0$; and the quota-share treaty, where $f(X) = aX$ with a constant (share) $0 \le a \le 1$.

Optimal forms of reinsurance have been studied extensively in the literature. Most of the results obtained are from the cedent's point of view. That is, the question asked is: for a given premium principle, what is the optimal functional form and/or parameter values of the ceded function f, such that the cedent's expected utility is maximized or its risk is minimized? For example, by maximizing the cedent's expected utility, Arrow (1973) concluded that "given a range of alternative possible reinsurance contracts, the reinsured would prefer a policy offering complete coverage beyond a deductible." Borch (1960b) showed that for a fixed premium and expected reinsurance payments, the variance of the cedent's losses is minimized by the excess-of-loss reinsurance policy. In recent years, various solutions to the optimal reinsurance problem have been obtained where the value-at-risk (VaR) and the tail-value-at-risk (TVaR) have been used to measure the cedent's risk level (e.g., Asimit et al. (2013); Assa (2015); Bernard and Tian (2009); Cai and Tan (2007); Cai et al. (2008); Cheung (2010); Chi and Tan (2011) and the references therein).

Borch (1969) argues that "there are two parties to a reinsurance contract, and that an arrangement which is very attractive to one party may be quite unacceptable to the other." However, as pointed out by Hürlimann (2011), optimal forms of ceded functions considering both the cedent and the reinsurer had scarcely been discussed until quite recently. For example, Ignatov et al. (2004) study the optimal reinsurance contracts under which the finite horizon joint survival probability of the two parties is maximized. Kaishev and Dimitrova (2006) derive explicit expressions for the probability of joint survival up to a finite time of the cedent and the reinsurer, under an excess of loss reinsurance contract with a limiting and a retention level. Golubin (2006b) studies the problem of designing the Pareto-optimal reinsurance policy by maximizing a weighted average of the expected utility of the insurer and the reinsurer. Dimitrova and Kaishev (2010) introduce an efficient frontier type approach to setting the limiting and the retention levels, based on the probability of joint survival. Cai et al. (2013) analyse the optimal reinsurance policies that maximize the joint survival probability and the joint profitable probability of the two parties and derive sufficient conditions for optimal reinsurance contracts within a wide class of reinsurance policies and under a general reinsurance premium principle. Using the results of Cai et al. (2013), Fang and Qu (2014) derive optimal retentions of combined quota-share and excess-of-loss reinsurance that maximize the joint survival probability of the two parties. Cai et al. (2015) study the optimal forms of reinsurance policies that minimize the convex combination of the VaRs of the cedent and the reinsurer under two types of constraints that describe the interests of the two parties. For the determination of the optimal excess of loss contract considering the dependency between the losses of the insurer and the reinsurer, we refer to Castañer and Claramunt Bielsa (2016) and the references therein.

A closely-related problem to optimal reinsurance is the so-called optimal transfer of risks among partners, where everybody's interests are considered simultaneously. The usual approach is to identify Pareto-optimal treaties, whereby no agent can be made better off without making another agent worse off. For results in this area, we refer to, e.g., Aase (2002); Asimit et al. (2013); Borch (1960b); Ludkovski and Young (2009) and the references therein.

In this paper, we determine Pareto-optimal reinsurance policies under which one party's risk, measured by its VaR, cannot be reduced without increasing that of the other party in the reinsurance contract. We consider two classes of ceded functions:

$$C^1 := \{f : f \text{ is convex, non-decreasing and } 0 \leq f(x) \leq x \text{ for all } x\}$$

and:

$$C^2 := \{f : f \text{ and } I_f \text{ are non-decreasing and } 0 \leq f(x) \leq x \text{ for all } x\}.$$

Note the inclusion $C^1 \subset C^2$, which has been verified by Chi and Tan (2011). Furthermore, for every $f \in C^2$, both f and I_f are Lipschitz continuous, and they are comonotonic.

The requirements that the ceded function f is non-decreasing and that the bounds $0 \leq f(x) \leq x$ hold for all x are needed in C^1 and C^2 to avoid the moral hazard problem in reinsurance. The additional requirement of the convexity of f in C^1 essentially requires that $f(x)$ approaches infinity linearly when $x \to \infty$ and thus disallows the popular layered reinsurance policies. Nevertheless, this class includes the important quota-share and the excess-of-loss reinsurance policies. Note also that both classes are of interest in the more general context of economic theory with two agents having conflicting interests. Optimal reinsurance problems with admissible classes C^1 and C^2 have been studied extensively in the literature, and we refer to Chi and Tan (2011) for an informative review.

For simplicity of discussion, we assume that the reinsurance premiums are determined by the expected premium principle:

$$P_f = (1+\theta)\mathbf{E}[f(X)] \tag{2.1}$$

where $\theta > 0$ is the safety loading. Hence, the cedent's total loss becomes:

$$C_f = X - f(X) + (1+\theta)\mathbf{E}[f(X)],$$

and the reinsurer's total loss under the reinsurance contract is:

$$R_f = f(X) - (1+\theta)\mathbf{E}[f(X)].$$

In this paper, we use VaR to measure the insurer's and reinsurer's risk level. A natural starting point for measuring the (joint) risk of the cedent and the reinsurer is a bivariate risk measure, such as the bivariate VaR (Embrechts and Puccetti (2006)) of the pair C_f and R_f. However, since the ceded loss $f(X)$ and the retained loss $I_f(X)$ are comonotonic (see Dhaene et al. (2002b,c) for a very detailed discussion of the concept of comonotonicity with applications), the set of values of the bivariate VaRs of C_f and R_f is determined by values of the univariate VaR of C_f and R_f. Therefore, the Pareto-optimal reinsurance policies could be determined by minimizing a linear combination of the univariate VaRs of C_f and R_f. We note in this regard that the optimization criterion of minimizing linear combinations of the risks of the cedent and the reinsurer was adopted by Asimit et al. (2013); Cai et al. (2015). Our arguments provide an additional economic meaning to such criteria.

Although VaR is not sub-additive in general, it was shown that it is sub-additive in the deep right tail in many cases of interest (Daníelsson et al., 2013). General results related to optimal forms of reinsurance (risk exchanges) using the so-called distortion risk measures exist in the literature, and we refer to Asimit et al. (2013); Assa (2015); Ludkovski and Young (2009). The distortion risk measures are very general and include VaR, TVaR and proportional hazards transforms as special cases. The feature of the current paper is that we extend the geometric approach of Cheung (2010) to our optimization problem that considers the interests of the two parties. The geometric proofs facilitate intuition and enable us to avoid lengthy and complex mathematical arguments. We derive closed-form and user-friendly formulas for the optimal reinsurance policies and thus provide a convenient route for practical implementation of our results.

The rest of the paper is organized as follows. Section 2.2 provides preliminaries and shows (Ludkovski and Young, 2009) that the form of Pareto-optimal reinsurance policies can be determined by minimizing linear combinations of the cedent's and the reinsurer's risks. In Sections 2.3 and 2.4, we determine optimal reinsurance forms and derive the corresponding optimal parameters when the feasible classes of ceded functions are C^1 and C^2, respectively. There, we also provide illustrative numerical examples. Section 2.5 provides further insights regarding the results of our numerical examples. Section 2.6 concludes the paper.

2.2 Preliminaries

Let F_X and S_X denote the cumulative distribution function (c.d.f.) and the survival function of X, respectively. Furthermore, let F_{C_f} and F_{R_f} denote the c.d.f.'s of C_f and R_f, respectively.

Then, the individual VaRs of the cedent and the reinsurer under the reinsurance contract are:

$$\mathrm{VaR}_\alpha(C_f) = \inf\{x : F_{C_f}(x) \geq \alpha\}$$

and:

$$\mathrm{VaR}_\alpha(R_f) = \inf\{x : F_{R_f}(x) \geq \alpha\},$$

respectively. To consider the risk of both the cedent and the reinsurer, we propose to use the bivariate lower orthant VaR introduced by Embrechts and Puccetti (2006), which is:

$$\underline{\mathrm{VaR}}_\alpha(C_f, R_f) = \partial\{(y, z) \in \mathbf{R}^2 : F_{C_f,R_f}(y, z) \geq \alpha\}.$$

For any ceded function $f \in C^2$, the random variables C_f and R_f are comonotonic, and so:

$$\begin{aligned}\underline{\mathrm{VaR}}_\alpha(C_f, R_f) &= \partial\{(y, z) \in \mathbf{R}^2 : \min\{F_{C_f}(y), F_{R_f}(z)\} \geq \alpha\} \\ &= \partial\{(y, z) \in \mathbf{R}^2 : F_{C_f}(y) \geq \alpha, F_{R_f}(z) \geq \alpha\}.\end{aligned}$$

Therefore, when the "joint" risk of the cedent and the reinsurer is measured by their bivariate lower orthant VaR, one could work with the marginal VaRs of C_f and R_f, instead of the much more complicated joint VaR.

In the following, we assume that the probability levels in the VaRs used by the cedent and the reinsurer are possibly different, say α_c and α_r, respectively, and then determine the Pareto-optimal reinsurance policies (ceded functions f) in the sense that one party's risk, measured by its VaR, cannot be reduced without increasing the other party's VaR. Mathematically, let f^* denote a ceded function in an admissible set C, such as C^1 or C^2. Let the corresponding cedent's and reinsurer's total losses under the ceded function f^* be denoted by C_{f^*} and R_{f^*}, respectively. Then, f^* is a Pareto-optimal reinsurance policy if there is no ceded function $f \neq f^*$ belonging to the admissible set C, such that:

$$\mathrm{VaR}_{\alpha_c}(C_f) \leq \mathrm{VaR}_{\alpha_c}(C_{f^*})$$

and:

$$\mathrm{VaR}_{\alpha_r}(R_f) \leq \mathrm{VaR}_{\alpha_r}(R_{f^*}),$$

with at least one of the inequalities being strict. To find the Pareto-optimal reinsurance policies, we utilize the following proposition.

Proposition 2.2.1 *All Pareto-optimal reinsurance policies f in C^i, $i \in \{1, 2\}$, can be determined by solving the problem:*

$$\min_{f \in C^i} \Big\{\beta \mathrm{VaR}_{\alpha_c}(C_f) + (1 - \beta)\mathrm{VaR}_{\alpha_r}(R_f)\Big\}, \tag{2.2}$$

where $0 \leq \beta \leq 1$.

Proof Similar to the discussion on page 90 of Gerber (1979), one method to find Pareto-optimal decisions is to choose two positive constants k_1, k_2 and find:

$$\min_{f\in\mathcal{C}^i}\Big\{k_1 \mathrm{VaR}_{\alpha_c}(C_f) + k_2 \mathrm{VaR}_{\alpha_r}(R_f)\Big\}.$$

Without loss of generality, we set $k_1 = \beta$ and $k_2 = 1-\beta$ with $0 \le \beta \le 1$. In more detail, let g be a function belonging to $\mathcal{C}^i$ and minimizing (2.2), then there cannot exist in $\mathcal{C}^i$ any function $f \neq g$ such that $\mathrm{VaR}_{\alpha_c}(C_f) \le \mathrm{VaR}_{\alpha_c}(C_g)$ and $\mathrm{VaR}_{\alpha_r}(R_f) \le \mathrm{VaR}_{\alpha_r}(R_g)$ with at least one of the inequalities being strict, because otherwise, we would have:

$$\beta \mathrm{VaR}_{\alpha_c}(C_f) + (1-\beta)\mathrm{VaR}_{\alpha_r}(R_f) < \beta \mathrm{VaR}_{\alpha_c}(C_g) + (1-\beta)\mathrm{VaR}_{\alpha_r}(R_g).$$

This is a contradiction to the assumed property of function g.

Furthermore, for any two ceded functions $f_1, f_2 \in \mathcal{C}^i$, the family $\{f_\gamma, 0 \le \gamma \le 1\}$ of ceded functions defined by $f_\gamma(x) = \gamma f_1(x) + (1-\gamma) f_2(x)$, is a subset of $\mathcal{C}^i$ and satisfies:

$$\mathrm{VaR}_{\alpha_c}(C_{f_\gamma}) = \gamma \mathrm{VaR}_{\alpha_c}(C_{f_1}) + (1-\gamma)\mathrm{VaR}_{\alpha_c}(C_{f_2}) \tag{2.3}$$

and:

$$\mathrm{VaR}_{\alpha_r}(R_{f_\gamma}) = \gamma \mathrm{VaR}_{\alpha_r}(R_{f_1}) + (1-\gamma)\mathrm{VaR}_{\alpha_r}(R_{f_2}). \tag{2.4}$$

Equation (2.3) is satisfied because:

$$\begin{aligned}\mathrm{VaR}_{\alpha_c}(C_{f_\gamma}) &= \mathrm{VaR}_{\alpha_c}(I_{f_\gamma}(X) + P_{f_\gamma}) \\ &= \mathrm{VaR}_{\alpha_c}(\gamma C_{f_1} + (1-\gamma) C_{f_2}) \\ &= \gamma \mathrm{VaR}_{\alpha_c}(C_{f_1}) + (1-\gamma)\mathrm{VaR}_{\alpha_c}(C_{f_2}),\end{aligned}$$

where the last equality is due to the fact that C_{f_1} and C_{f_2} are non-decreasing functions of the same random variable X and therefore comonotonic. Similarly, Equation (2.4) is satisfied. Therefore, Condition C on page 90 of Gerber (1979) is satisfied, and we conclude that all Pareto-optimal reinsurance policies in $\mathcal{C}^i$ can be found by solving Problem (2.2).

In view of Proposition 2.2.1, throughout the rest of this paper, we seek optimal reinsurance policies by solving the optimization problem:

$$\min_{f\in\mathcal{C}^i}\Big\{\beta \mathrm{VaR}_{\alpha_c}(C_f) + (1-\beta)\mathrm{VaR}_{\alpha_r}(R_f)\Big\}$$

for $i \in \{1, 2\}$, which is equivalent to minimizing:

$$\mathcal{H}(f) = \beta \mathrm{VaR}_{\alpha_c}(-f(X) + P_f) + (1-\beta)\mathrm{VaR}_{\alpha_r}(f(X) - P_f). \tag{2.5}$$

As shown by Chi and Tan (2011), we have $\mathcal{C}^1 \subset \mathcal{C}^2$, and every function $f \in \mathcal{C}^2$ is Lipschitz-continuous and, hence, continuous. Consequently (e.g., Dhaene et al. (2002b)), for every $f \in \mathcal{C}^2$, we have $\mathrm{VaR}_\alpha(f(X)) = f(\mathrm{VaR}_\alpha(X))$, and thus, with $a_c = \mathrm{VaR}_{\alpha_c}(X)$ and $a_r = \mathrm{VaR}_{\alpha_r}(X)$, the optimization problem becomes:

$$\min_{f\in\mathcal{C}^i}\mathcal{H}(f) = \min_{f\in\mathcal{C}^i}\Big\{-\beta\cdot f(a_c) + (1-\beta)\cdot f(a_r) + (2\beta-1)(1+\theta)\mathbf{E}[f(X)]\Big\},\ i = 1, 2. \tag{2.6}$$

Since we allow $S_X(0) < 1$, the relationships between the probability levels α_c and α_r, as well as $S_X(0)$ need to be discussed. Namely, we have the following observations:

1. If $1 - \alpha_c \geq S_X(0)$ and $1 - \alpha_r \geq S_X(0)$, then $a_c = a_r = 0$. Thus,

   - when $\beta > 1/2$, the solution to Problem (2.6) is $f^*(x) = 0$ for all x;
   - when $\beta < 1/2$, the solution is $f^*(x) = x$;
   - when $\beta = 1/2$, the objective function is always zero.

2. If $1 - \alpha_c < S_X(0)$ and $1 - \alpha_r \geq S_X(0)$, then $a_c > 0$ and $a_r = 0$. Thus,

   - when $\beta < 1/2$, the optimal ceded function is $f^*(x) = x$;
   - when $\beta > 1/2$, the form of the optimal ceded function is similar to the case when $\beta = 1$, with only the risk and the profit of the cedent considered (the solution for the latter case can be found in Case 2 of Sections 2.3.2 and 2.4.2 below);
   - when $\beta = 1/2$, the optimal ceded function is $f^*(x) = x$.

3. If $1 - \alpha_c \geq S_X(0)$ and $1 - \alpha_r < S_X(0)$, then $a_c = 0$ and $a_r > 0$. Thus,

   - when $\beta > 1/2$, the solution to Problem (2.6) is $f^*(x) = 0$ for all x;
   - when $\beta < 1/2$, the form of the optimal ceded function is similar to the case when $\beta = 0$, with only the risk and the profit of the reinsurer being considered (the solution for the latter case can be found in Case 3 of Sections 2.3.2 and 2.4.2 below).
   - when $\beta = 1/2$, the optimal ceded function is $f^*(x) = 0$ for all x.

Throughout the rest of this paper, we only consider the optimal forms of reinsurance policies under the conditions $1 - \alpha_c < S_X(0)$ and $1 - \alpha_r < S_X(0)$.

Now, we are ready to determine the forms of the Pareto-optimal reinsurance policy under VaR, the task that makes up the contents of the following two sections. Namely, in Section 2.3, we consider the case when the admissible set of ceded functions is C^1 and in Section 2.4 when the admissible set is C^2. As noted earlier, both classes are of interest in the broad context of economic theory, with the class C^2 being more relevant to reinsurance policies. Nevertheless, the class C^1 includes the important quota share and excess-of-loss reinsurance policies that provide natural reference points for analysing the optimal reinsurance policies in C^2.

2.3 Optimal Reinsurance Policies When f is convex

In this section, we determine optimal insurance policies under the condition that $f \in C^1$, which means that f is convex and non-decreasing and the retained loss function $I_f(x) = x - f(x)$ is non-decreasing. These conditions are also assumed by Cai et al. (2008); Cheung (2010), where they in fact require that f is Lipschitz-continuous (cf., e.g., Section 2 of Chi and Tan (2011)) and that $f(x)$ linearly tends to infinity when $x \to \infty$.

2.3.1 Functional Form of the Ceded Function

Here, we determine the functional form of the solution to the minimization problem:

$$\min_{f\in C^1} \mathcal{H}(f) = \min_{f\in C^1} \Big\{ -\beta \cdot f(a_c) + (1-\beta)\cdot f(a_r) + (2\beta-1)(1+\theta)\mathbf{E}[f(X)] \Big\}. \tag{2.7}$$

We subdivide our following analysis into three cases.

Case 1: $\beta > 1/2$

In this case, the coefficients in front of $f(a_r)$ and $\mathbf{E}[f(X)]$ on the right-hand side of Equation (2.7) are positive, and the coefficient in front of $f(a_c)$ is negative. Thus, for any ceded function f, we determine the functional form of a ceded function f^*, such that $f^*(a_c) = f(a_c)$ and:

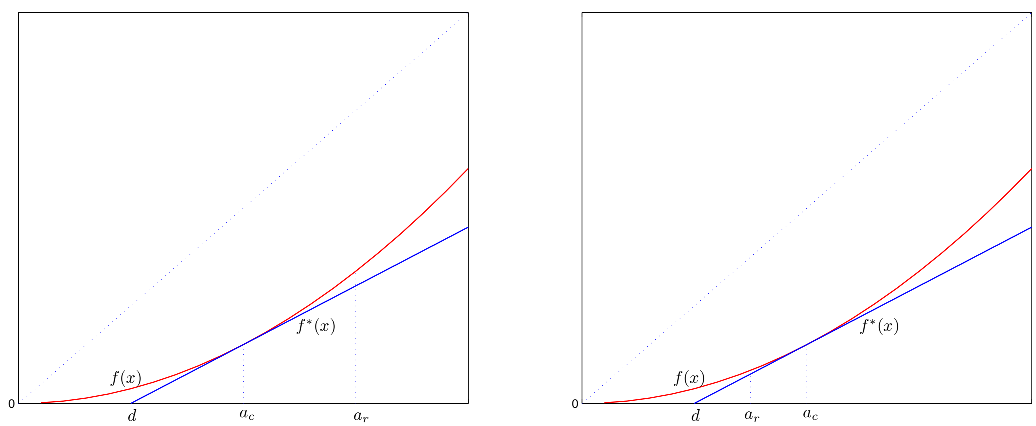
$$(1-\beta)f^*(a_r) + (2\beta-1)P_{f^*} \leq (1-\beta)f(a_r) + (2\beta-1)P_f.$$

This requires $f^*(a_r)$ and also the entire function f^* to be as small as possible.

As we see from Figure 2.1, the convexity of f implies that the above requirements are satisfied by the ceded function:

$$f^*(x) = c(x-d)_+, \tag{2.8}$$

where $c \in (f'(a_c-), f'(a_c+))$ and $d \in [0, a_c]$ are any constants. Since the slope of f should not exceed one, we must have $c \in [0,1]$.



(a) $a_c < a_r$ (b) $a_r < a_c$

Figure 2.1: Optimal ceded functions in C^1: Case 1.

Remark It is clear from the above proof that the result for the optimal form of reinsurance policy is valid as long as $P_{f_1} \leq P_{f_2}$ whenever $f_1(x) \leq f_2(x)$ for all x. Obviously, this condition is satisfied by the distortion premium calculation principle (e.g., Young (2004)), which has been assumed in, for example, Assa (2015); Ludkovski and Young (2009), among others. For a discussion of the validity of this condition in the case of the weighted premium calculation principle, we refer to Furman and Zitikis (2008). In the current paper, we adopt the simplest expectation premium principle (Equation (2.1)) for the simplicity of presentation.

Case 2: $\beta < 1/2$

In this case, the coefficient in front of $f(a_r)$ on the right-hand side of Equation (2.7) is positive, and those in front of $f(a_c)$ and $\mathbf{E}[f(X)]$ are negative. Therefore, to solve Problem (2.7), for any ceded function f, we search for a function f^*, such that $f^*(a_r) = f(a_r)$ and:

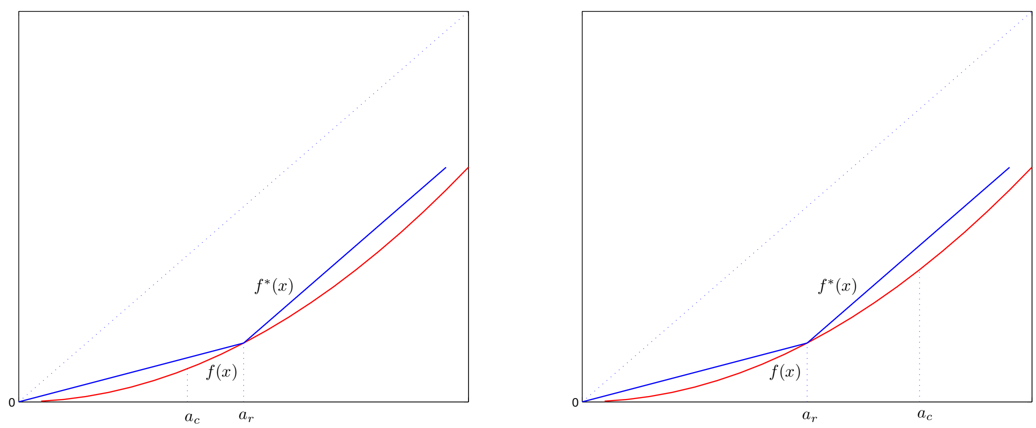
$$\beta f^*(a_c) + (1-2\beta)P_{f^*} \geq \beta f(a_c) + (1-2\beta)P_f,$$

which requires $f^*(a_c)$ and also the entire function f^* to be as large as possible.

As we see from Figure 2.2, the convexity of f implies that the above requirements are satisfied by the ceded function:

$$f^*(x) = \begin{cases} \eta x & \text{when} \quad 0 \leq x < a_r, \\ x - (1-\eta)a_r & \text{when} \quad x \geq a_r, \end{cases} \tag{2.9}$$

where $\eta \in [0, 1]$ can be any constant.



(a) $a_c < a_r$ (b) $a_r < a_c$

Figure 2.2: Optimal ceded functions in C^1: Case 2.

Case 3: $\beta = 1/2$

In this case, Problem (2.7) simplifies to:

$$\min_{f\in C^1} \{f(a_r) - f(a_c)\}. \tag{2.10}$$

Since f is non-decreasing, we have that when $\alpha_c < \alpha_r$, then Problem (2.10) is solved by any ceded function f^*, which is constant on the interval $[a_c, a_r]$. Since f^* has to be convex, this in turn requires f^* to be constant on $[0, a_c]$. Since $f^*(0) = 0$, we conclude that any function f^* in C^1 with $f^*(x) = 0$ on $[0, a_r]$ is Pareto-optimal.

When $\alpha_c > \alpha_r$, then because the slope of the ceded function is no more than one, Problem (2.10) is solved by f^*, which increases at the rate of one in the interval $[a_r, a_c]$, which in turn requires f^* to increase at the rate of one for all $x > a_c$ because of the convexity assumption. In summary, any function f^* in C^1 with $f^{*\prime}(x) = 1$ on $[a_r, \infty)$ is Pareto-optimal.

When $\alpha_c = \alpha_r$, the objective function is constant.

2.3.2 Parameter Values of the Optimal Ceded Function

When $\beta > 1/2$, then the optimal ceded function f^* is given by Equation (2.8) for which the parameters c and d need to be determined. When $\beta < 1/2$, then the optimal ceded function is given by Equation (2.9) for which the parameter η needs to be determined. We accomplish these tasks below by subdividing our considerations into four cases.

Case 1: $\beta > 1/2$ **and** $\alpha_c < \alpha_r$

In this case, the optimal ceded function is given by Equation (2.8) with $d < a_c < a_r$, and optimization Problem (2.7) becomes:

$$\min_{(c,d)\in[0,1]\times[0,a_c]} g_1(c,d),$$

where:

$$g_1(c,d) = c\left(-\beta(a_c - d) + (1-\beta)(a_r - d) + (2\beta - 1)(1+\theta)\int_d^\infty S_X(t)dt\right).$$

Following Cai et al. (2008); Cheung (2010), we use the notations:

$$\begin{aligned} \theta^* &= \frac{1}{1+\theta}, \\ d^* &= S_X^{-1}(\theta^*), \\ Q(\beta, a_c, a_r) &= \frac{\beta a_c - (1-\beta)a_r}{2\beta - 1}, \\ U(x) &= S_X^{-1}(x) + (1+\theta)\int_{S_X^{-1}(x)}^\infty S_X(t)dt. \end{aligned}$$

Theorem 2.3.1 *Under the conditions $\beta > 1/2$ and $\alpha_c < \alpha_r$, the optimal ceded function is $f^*(x) = c(x-d)_+$ with the following parameters:*

1. $c = 1$ *and* $d = d^*$ *when* $\theta^* < S_X(0)$ *and* $U(\theta^*) < Q(\beta, a_c, a_r)$*;*

2. $c \in [0,1]$ *is any constant and* $d = d^*$ *when* $\theta^* < S_X(0)$ *and* $U(\theta^*) = Q(\beta, a_c, a_r)$*;*

3. $c = 1$ *and* $d = 0$ *when* $\theta^* \geq S_X(0)$ *and* $(1+\theta)\mathbf{E}[X] < Q(\beta, a_c, a_r)$*;*

4. $c \in [0,1]$ *is any constant and* $d = 0$ *when* $\theta^* \geq S_X(0)$ *and* $(1+\theta)\mathbf{E}[X] = Q(\beta, a_c, a_r)$*.*

If none of the above conditions are satisfied, then $f^*(x) = 0$ for all x.

Proof We only prove Part (1) because the proofs of the other parts are similar. To minimize function $g_1(c,d)$ over $(c,d) \in [0,1] \times [0,a_c]$, we first take the derivative of $g_1(c,d)$ with respect to d and have:

$$\frac{\partial g_1(c,d)}{\partial d} = c(2\beta-1)\Big(1-(1+\theta)S_X(d)\Big),$$

which is an increasing function in d. Consequently, the function $g_1(c,d)$ is convex in d. Since $\theta^* < S_X(0)$, the derivative $\partial g_1(c,d)/\partial d$ is negative at $d = 0$ and is equal to zero at d^*. It is easy to show that $a_c < a_r$ if and only if $a_c > Q(\beta, a_c, a_r)$. Then, the condition $U(\theta^*) < Q(\beta, a_c, a_r)$ indicates that $d^* < U(\theta^*) < a_c$, and so, the deductible level d^* minimizes the function $g_1(c,d)$ when $c > 0$.

Next, setting $d = d^*$, we have:

$$g_1(c,d^*) = c(2\beta-1)\Big(U(\theta^*) - Q(\beta,a_c,a_r)\Big) < 0. \tag{2.11}$$

Because $U(\theta^*) < Q(\beta, a_c, a_r)$ by assumption, $g_1(c,d^*)$ is minimized at $c = 1$. Overall, assuming $c > 0$, function $g_1(c,d)$ is minimized at $(c,d) = (1,d^*)$. Noting that $g_1(0,d) = 0 > g_1(1,d^*)$, we obtain the desired result.

Remark We have the following observations:

- When $\beta = 1$, then only the cedent is considered. In this case, $Q(\beta,a_c,a_r) = a_c$ and $f^*(x) = (x-d^*)_+$ when $U(\theta^*) < a_c$. Therefore, when $U(\theta^*) > a_c$, then $f^*(x) = 0$ for all x, and the primary insurance company will not purchase any reinsurance policy. This result agrees with those derived by Cai et al. (2008); Cheung (2010).

- When $\beta \searrow 1/2$, then $Q(\beta,a_c,a_r) \sim (a_c - a_r)/(4\beta-2) < 0$, and the optimal value of c is zero. Therefore, $f^*(x) = 0$ for all x.

- The value of d^* in the excess-of-loss reinsurance policy does not depend on the choice of β whenever $U(\theta^*) \leq Q(\beta,a_c,a_r)$.

Case 2: $\beta > 1/2$ **and** $\alpha_c > \alpha_r$

In this case, the optimal ceded function is given by Equation (2.8) with $d < \alpha_c$. The order between d and a_r is not, however, determined. Therefore, the optimization problem is:

$$\min_{(c,d)\in[0,1]\times[0,a_c]} g_2(c,d),$$

where:

$$g_2(c,d) = c\left(-\beta(a_c-d) + (1-\beta)(a_r-d)_+ + (2\beta-1)(1+\theta)\int_d^\infty S_X(t)dt\right),$$

which is a continuous function in c and d. Note, however, that the left-hand derivative $\partial g_2(c,d)/\partial d|_{d=a_r-}$ is not equal to the right-hand derivative $\partial g_2(c,d)/\partial d|_{d=a_r+}$. With the additional notations:

$$\theta^*_\beta = \frac{\beta}{(2\beta-1)(1+\theta)},$$

$$d^*_\beta = S_X^{-1}(\theta^*_\beta),$$

$$U_\beta(x) = S_X^{-1}(x) + \frac{1}{\theta^*_\beta}\int_{S_X^{-1}(x)}^\infty S_X(t)dt,$$

we have the following theorem.

Theorem 2.3.2 *Under the conditions $\beta > 1/2$ and $\alpha_c > \alpha_r$, the optimal ceded function is $f^*(x) = c(x-d)_+$ with the following parameters:*

1. $c = 1$ *and* $d = d^*$ *when* $1-\alpha_r < \theta^* < S_X(0)$ *and* $Q(\beta, a_c, a_r) > U(\theta^*)$*;*

2. $c \in [0,1]$ *is any constant and* $d = d^*$ *when* $1-\alpha_r < \theta^* < S_X(0)$ *and* $Q(\beta, a_c, a_r) = U(\theta^*)$*;*

3. $c = 1$ *and* $d = a_r$ *when* $\theta^* < 1-\alpha_r < \theta^*_\beta$*, and* $a_c > U_\beta(1-\alpha_r)$*;*

4. $c \in [0,1]$ *is any constant and* $d = a_r$ *when* $\theta^* < 1-\alpha_r < \theta^*_\beta$*, and* $a_c = U_\beta(1-\alpha_r)$*;*

5. $c = 1$ *and* $d = d^*_\beta$ *when* $1-\alpha_c < \theta^*_\beta < 1-\alpha_r$ *and* $a_c > U_\beta(\theta^*_\beta)$*;*

6. $c \in [0,1]$ *is any constant and* $d = d^*_\beta$ *when* $1-\alpha_c < \theta^*_\beta < 1-\alpha_r$ *and* $a_c = U_\beta(\theta^*_\beta)$*;*

7. $c = 1$ *and* $d = 0$ *when* $\theta^* \geq S_X(0)$ *and* $Q(\beta, a_c, a_r) > (1+\theta)\mathbf{E}[X]$*;*

8. $c \in [0,1]$ *is any constant and* $d = 0$ *when* $\theta^* \geq S_X(0)$ *and* $Q(\beta, a_c, a_r) = (1+\theta)\mathbf{E}[X]$.

If none of the conditions above are satisfied, then $f^*(x) = 0$ *for all* x.

Proof We prove Parts (1), (3) and (5) only, because the proofs of the other parts are similar.

Part (1): The derivative of $g_2(c,d)$ with respect to d is given by:

$$\frac{\partial g_2(c,d)}{\partial d} = \begin{cases} c(2\beta-1)\big(1-(1+\theta)S_X(d)\big) & \text{when} \quad d < a_r, \\ c(2\beta-1)\Big(\frac{\beta}{2\beta-1}-(1+\theta)S_X(d)\Big) & \text{when} \quad d > a_r. \end{cases}$$

Assuming $c > 0$, we have that $\partial g_2(c,d)/\partial d$ is increasing in $d > 0$. The condition $1-\alpha_r < \theta^* < S_X(0)$ ensures that $\partial g_2(c,d)/\partial d$ is negative at $d = 0$, increases to zero at $d^* = S_X^{-1}(\theta^*) < a_r$ and becomes positive for $d > d^*$. Therefore, the objective function is minimized at $d = d^*$. At $d = d^*$, the derivative $\partial g_2(c,d)/\partial c$ is given by Formula (2.11). Therefore, as in the proof of Theorem 2.3.1, the condition $U(\theta^*) < Q(\beta, a_c, a_r)$ ensures that $g_2(c,d)$ is minimized at $(c,d) = (1, d^*)$.

Part (3): When $\theta^* < 1-\alpha_r < \theta^*_\beta$, the derivative $\partial g_2(c,d)/\partial d$ is negative for $d < a_r$ and positive for $d > a_r$. Therefore, the function $g_2(c,d)$ is minimized at $d = a_r$, assuming $c > 0$. Next, since:

$$g_2(c, a_r) = c\beta\Big(U_\beta(1-\alpha_r) - a_c\Big)$$

and $U_\beta(1-\alpha_r) < a_c$ by assumption, the function $g_2(c,d)$ is minimized at $(c,d) = (1, a_r)$ with $g_2(1, a_r) < 0$. Noting that $g_2(0, a_r) = 0 > g_2(1, a_r)$, the desired result follows.

Part (5): Since $\theta^* < \theta^*_\beta$, the assumption $\theta^*_\beta < 1 - \alpha_r$ implies $\theta^* < 1 - \alpha_r$. Therefore, the derivative $\partial g_2(c,d)/\partial d$ is negative for $d < a_r$, equal to zero at $d = d^*_\beta \in (a_r, a_c)$ and positive afterwards. Therefore, the objective function is minimized at $d = d^*_\beta$. Note that the condition $a_c > U_\beta(\theta^*_\beta)$ implies $d^*_\beta < a_c$. Furthermore, since:

$$\begin{aligned} g_2(c, d^*_\beta) &= c\left(-\beta(a_c - d^*_\beta) + (2\beta - 1)(1+\theta)\int_{d^*_\beta}^{\infty} S_X(t)dt\right) \\ &= c\beta\left(U_\beta(\theta^*_\beta) - a_c\right) \end{aligned}$$

and $U_\beta(\theta^*_\beta) < a_c$ by assumption, the objective function $g_2(c,d)$ is minimized at $(c,d) = (1, d^*_\beta)$ when $c > 0$. Noting that $g_2(0,d) = 0 > g_2(1, d^*_\beta)$, the desired result follows.

Remark We have the following observations:

- When $\beta = 1$, then $\theta^*_\beta = \theta^*$ and $U_\beta(x) = U(x)$. Thus, the result is exactly the same as in the first bullet at the end of Case 1 above. The value of α_r makes no difference here because only the cedent's risk is considered when $\beta = 1$.

- When $\beta \searrow 1/2$, then $Q(\beta, a_c, a_r) \sim (a_c - a_r)/(4\beta - 2) \nearrow \infty$, $\theta^*_\beta \nearrow \infty$ and $U_\beta(\theta^*_\beta) = 0$. Therefore, Parts (1) and (3) of Theorem 2.3.2 apply. We have:

$$f^*(x) = \begin{cases} (x - d^*)_+ & \text{when} \quad \theta^* > 1 - \alpha_r, \\ (x - a_r)_+ & \text{when} \quad \theta^* < 1 - \alpha_r. \end{cases}$$

Case 3: $\beta < 1/2$ **and** $\alpha_c < \alpha_r$

With the optimal ceded function f^* given by Equation (2.9), Problem (2.7) reduces to:

$$\min_{\eta \in [0,1]} g_3(\eta),$$

where:

$$g_3(\eta) = -\beta\eta a_c + (1-\beta)\eta a_r + (2\beta - 1)(1+\theta)\left(\eta\int_0^{a_r} x dF_X(x) + \int_{a_r}^{\infty} (x - a_r + \eta a_r) dF_X(x)\right).$$

Taking the derivative of $g_3(\eta)$ with respect to η, we have:

$$\begin{aligned} g_3'(\eta) &= -\beta a_c + (1-\beta)a_r + (2\beta - 1)(1+\theta)\left(\int_0^{a_r} x dF_X(x) + a_r S_X(a_r)\right) \\ &= (1 - 2\beta)\left(Q(\beta, a_c, a_r) - (1+\theta)\mathbf{E}[X \wedge a_r]\right), \end{aligned} \tag{2.12}$$

where $X \wedge a_r = \min\{X, a_r\}$. Therefore, $g_3(\eta)$ achieves its minimum at $\eta = 1$ when the quantity on the right-hand side of Equation (2.12) is negative. Otherwise, the minimum is at $\eta = 0$. Consequently, we have the following theorem.

Theorem 2.3.3 *Under the conditions $\beta < 1/2$ and $\alpha_c < \alpha_r$, the optimal ceded function is:*

$$f^*(x) = \begin{cases} \eta x & when \quad 0 \le x < a_r, \\ x - (1-\eta)a_r & when \quad x \ge a_r, \end{cases}$$

with the parameter:

$$\eta = \begin{cases} 1 & when \quad (1+\theta)\mathbf{E}[X \wedge a_r] > Q(\beta, a_c, a_r), \\ 0 & when \quad (1+\theta)\mathbf{E}[X \wedge a_r] < Q(\beta, a_c, a_r), \\ \text{any constant} \in [0,1] & when \quad (1+\theta)\mathbf{E}[X \wedge a_r] = Q(\beta, a_c, a_r). \end{cases}$$

Remark A few observations follow:

- When $\beta \nearrow 1/2$, then $g_3'(\eta) \to (a_r - a_c)/2 > 0$. In this case, $\eta^* = 0$ and the optimal reinsurance policy is $f^*(x) = (x - a_r)_+$.
- When $\beta = 0$ and only the reinsurer's risk is considered, Theorem 2.3.3 holds with $Q(\beta, a_c, a_r) = a_r$.

Case 4: $\beta < 1/2$ and $\alpha_c > \alpha_r$

With the optimal reinsurance function f^* given by Equation (2.9), Problem (2.7) becomes:

$$\min_{\eta \in [0,1]} g_4(\eta),$$

where

$$\begin{aligned} g_4(\eta) &= -\beta(a_c - a_r + \eta a_r) + (1-\beta)\eta a_r \\ &\quad +(2\beta - 1)(1+\theta)\Big(\eta \int_0^{a_r} x dF_X(x) + \int_{a_r}^{\infty} (x - a_r + \eta a_r) dF_X(x)\Big). \end{aligned}$$

Taking the derivative of $g_4(\eta)$ with respect to η, we get:

$$g_4'(\eta) = (1-2\beta)\Big(a_r - (1+\theta)\mathbf{E}[X \wedge a_r]\Big),$$

which yields the following theorem.

Theorem 2.3.4 *Under the conditions $\beta < 1/2$ and $\alpha_c > \alpha_r$, the optimal ceded function is:*

$$f^*(x) = \begin{cases} \eta x & when \quad 0 \le x < a_r, \\ x - (1-\eta)a_r & when \quad x \ge a_r, \end{cases}$$

with the parameter:

$$\eta = \begin{cases} 1 & when \quad (1+\theta)\mathbf{E}[X \wedge a_r] > a_r, \\ 0 & when \quad (1+\theta)\mathbf{E}[X \wedge a_r] < a_r, \\ \text{any constant} \in [0,1] & when \quad (1+\theta)\mathbf{E}[X \wedge a_r] = a_r. \end{cases}$$

Note that Theorems 2.3.3 and 2.3.4 are quite similar, with the role of $Q(\beta, a_c, a_r)$ in the former theorem played by a_r in the latter one.

2.3.3 An illustrative Example

In this section, we construct a numerical example to illustrate the Pareto optimality of the reinsurance policies that we derived above. Specifically, we assume that the loss variable X follows the exponential distribution with the survival function $S_X(x) = e^{-0.001x}$ for $x \geq 0$. Let the safety loading parameter be $\theta = 0.2$. Then, $\theta^* = 1/(1+\theta) = 0.833$, $d^* = S_X^{-1}(\theta^*) = 182.3$ and $U(\theta^*) = 1182.3$. We discuss two scenarios.

Scenario A: $\alpha_c = 0.95$ **and** $\alpha_r = 0.99$

In this case, $a_c = 2995.7$ and $a_r = 4605.2$. Applying Theorems 2.3.1 and 2.3.3, we have:

$$f_{1A}^*(x) = \begin{cases} (x-4605.2)_+ & \text{when} \quad \beta \in [0, 0.5), \\ 0 & \text{when} \quad \beta \in (0.5, 0.654), \\ (x-182.3)_+ & \text{when} \quad \beta \in (0.654, 1]. \end{cases}$$

When $\beta = 0.5$, then:

$$f_{1A}^*(x) = \begin{cases} 0 & \text{when} \quad x \leq 4605.2, \\ \text{unspecified} & \text{when} \quad x > 4605.2. \end{cases}$$

When $\beta = 0.654$, then:

$$f_{1A}^*(x) = c(x-182.3)_+$$

for any constant $c \in [0,1]$. The values of $\text{VaR}(C_{f_{1A}^*})$ versus $\text{VaR}(R_{f_{1A}^*})$ are reported in Table 2.1.

	$\mathbf{VaR}_{\alpha_c}(C_{f_{1A}^*})$	$\mathbf{VaR}_{\alpha_r}(R_{f_{1A}^*})$
$\beta \in [0, 0.5)$	3005.73	-10
$\beta = 0.5$	between 2995.73 and 3005.73	between -10 and 0
$\beta \in (0.5, 0.654)$	2995.73	0
$\beta = 0.654$	between 1182.32 and 2995.73	between 0 and 3422.85
$\beta \in (0.654, 1]$	1182.32	3422.85

Table 2.1: VaRs of the cedent and the reinsurer when $f \in C^1$.

We have the following observations:

- For $\beta \in (0.654, 1]$, the insurer is "more important". As a result, it retains the "good" risk in the layer of losses $(0, S_X^{-1}(\theta^*))$ and cedes the rest. For $\beta \in [0, 0.5)$, the reinsurer is "more important", and it assumes the risk above a_r. As a result, the chance of a payment is so small that its VaR does not increase; it actually reduces to -10 because of the collected premium. For $\beta \in (0.5, 0.654)$, no agreement is reached between the two parties.

- From Table 2.1, we see that when β gets larger and the cedent becomes increasingly important, then $\text{VaR}_{\alpha_c}(C_{f_{1A}^*})$ decreases, whereas $\text{VaR}_{\alpha_r}(R_{f_{1A}^*})$ increases.

- When $\beta = 0.5$ and $\beta = 0.654$, the optimal ceded functions are only partially specified, and the risk of the two parties varies in some range. For example, when $\beta = 0.5$, then $\text{VaR}_{\alpha_c}(C_{f^*_{1A}})$ is maximized by choosing $f^*_{1A}(x) = (x-4605.2)_+$ because the cedent is choosing a maximal ceded function and paying a maximal reinsurance premium (within the partially-specified optimal ceded functions). However, its VaR does not reduce with such a high deductible value. On the other hand, $\text{VaR}_{\alpha_c}(C_{f^*_{1A}})$ is minimized with $f^*_{1A}(x) = 0$, within the partially-specified optimal ceded functions.

Scenario B: $\alpha_c = 0.99$ **and** $\alpha_r = 0.95$

In this case, we have $a_c = 4605.2$ and $a_r = 2995.7$. Applying Theorems 2.3.2 and 2.3.4, we have:

$$f^*_{1B}(x) = \begin{cases} (x - 2995.7)_+ & \text{when} \quad \beta \in [0, 0.5), \\ (x - 182.3)_+ & \text{when} \quad \beta \in (0.5, 1]. \end{cases} \tag{2.13}$$

When $\beta = 0.5$,

$$f^*_{1B}(x) = \begin{cases} x - d & \text{when} \quad x \geq 2995.7, \\ \text{unspecified} & \text{when} \quad x < 2995.7, \end{cases} \tag{2.14}$$

where $d \in [0, 2995.7]$ can be any constant. The values of $\text{VaR}(C_{f^*_{1B}})$ versus $\text{VaR}(R_{f^*_{1B}})$ are reported in Table 2.2.

	$\text{VaR}_{\alpha_c}(C_{f^*_{1B}})$	$\text{VaR}_{\alpha_r}(R_{f^*_{1B}})$
$\beta \in [0, 0.5)$	3055.73	−60
$\beta = 0.5$	between 1182.32 and 3055.73	between −60 and 1813.41
$\beta \in (0.5, 1]$	1182.32	1813.41

Table 2.2: VaRs of the cedent and the reinsurer when $f \in \mathcal{C}^1$.

2.4 Optimal Reinsurance Policy When both f and R_f are non-decreasing

In this section, we determine optimal reinsurance policies when $f \in \mathcal{C}^2$, that is when both f and the retained loss function I_f are non-decreasing. Comparing this situation with the earlier $f \in \mathcal{C}^1$, we can now deal with non-convex ceded functions, such as $f(x) = \min\{x, l\}$ for any retention level $l > 0$. Mathematically, the problem becomes:

$$\min_{f \in \mathcal{C}^2} \mathcal{H}(f) = \min_{f \in \mathcal{C}^2} \Big\{ -\beta \cdot f(a_c) + (1-\beta) \cdot f(a_r) + (2\beta - 1)(1+\theta)\mathbf{E}[f(X)] \Big\}. \tag{2.15}$$

As pointed out in Section 2.1, solutions to similar problems exist in the literature, and we refer to Asimit et al. (2013); Assa (2015); Ludkovski and Young (2009). for details and further references. Our contribution in this paper is to generalize the geometric arguments of Cheung (2010) to the situation when the interests of both the cedent and the reinsurer are taken into account, and we do so in such a way that allows us to avoid lengthy mathematical arguments and consequently helps us to gain useful intuition. In addition, for all scenarios considered, we are able to provide explicit recipes for determining optimal reinsurance policies.

In Section 2.4.1 below, we derive optimal forms of ceded functions, and in Section 2.4.2, we determine parameter values of the optimal functions. Section 4.3 contains an illustrative numerical example, which is a continuation of that of Section 3.3. Throughout the rest of this section, we assume $1-\alpha_c < S_X(0)$ and $1-\alpha_r < S_X(0)$.

2.4.1 Functional Form of the Ceded Function

We have subdivided our considerations into three cases.

Case 1: $\beta > 1/2$

Similarly to Case 1 of Section 2.3.1, we determine the functional form of the ceded function f^* in the following manner. For any $f \in C^2$, we seek f^*, such that $f^*(a_c) = f(a_c)$ and:

$$(1-\beta)f^*(a_r) + (2\beta-1)P_{f^*} \leq (1-\beta)f(a_r) + (2\beta-1)P_f.$$

This requires $f^*(a_r)$, as well as the entire function f^* to be as small as possible for a fixed value of $f^*(a_c)$.

As we see from Figure 2.3, because f is non-decreasing with a slope not exceeding one, the aforementioned requirements are satisfied by the function:

$$\begin{aligned} f^*(x) &= \min\{(x-d)_+, a_c - d\} \\ &= \begin{cases} 0 & \text{when} \quad x \leq d, \\ x-d & \text{when} \quad d < x < a_c, \\ a_c - d & \text{when} \quad x \geq a_c, \end{cases} \end{aligned} \tag{2.16}$$

where $d \in [0, a_c]$ can be any constant. The optimal value of d will be determined in Section 2.4.2 below. In reinsurance jargon, the above specified optimal form of the reinsurance policy is for the reinsurer to provide coverage over the layer (d, a_c).

Case 2: $\beta < 1/2$

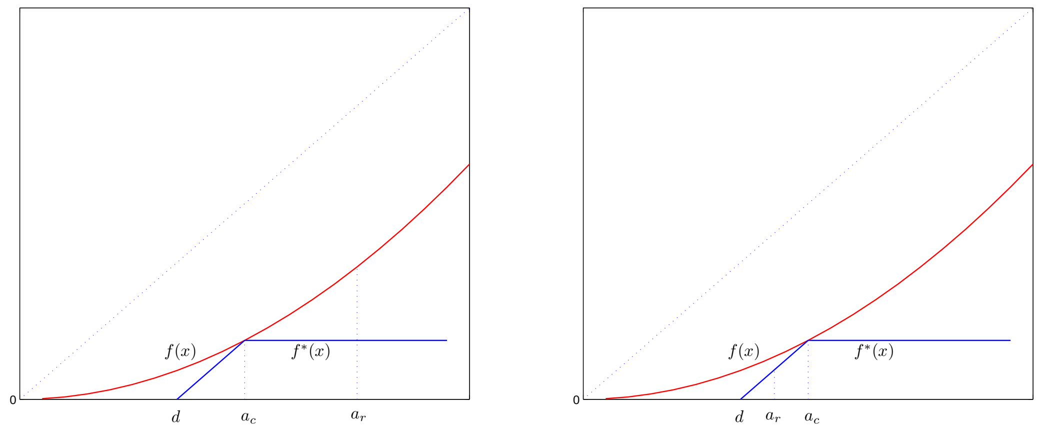
Similarly to Case 2 of Section 2.3.1, since the coefficients in front of $f(a_c)$ and P_f in objective Function (2.15) are negative, the optimal reinsurance policy is found by seeking f^*, such that $f^*(a_r) = f(a_r)$ and:

$$\beta f^*(a_c) + (1-2\beta)P_{f^*} \geq \beta f(a_c) + (1-2\beta)P_f.$$

As we see from Figure 2.4, these requirements are satisfied by the function:

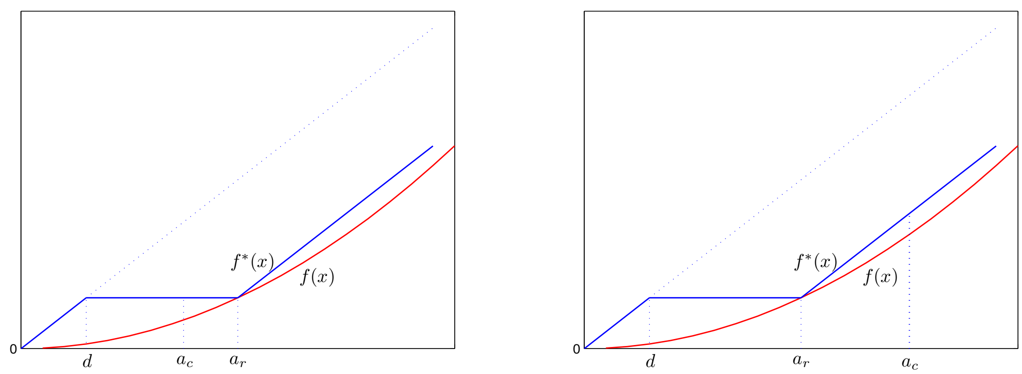
$$\begin{aligned} f^*(x) &= \min\{x, d\}\mathbf{1}_{\{x\leq a_r\}} + (x - a_r + d)\mathbf{1}_{\{x>a_r\}} \\ &= \begin{cases} x & \text{when} \quad 0 \leq x \leq d, \\ d & \text{when} \quad d < x \leq a_r, \\ x - a_r + d & \text{when} \quad x > a_r, \end{cases} \end{aligned} \tag{2.17}$$

where $d \in [0, a_r]$ can be any constant. Hence, the optimal form of the reinsurance policy is for the reinsurer to provide a coverage except for the layer (d, a_r). In other words, the insurer retains losses in the layer (d, a_r).



(a) $a_c < a_r$ (b) $a_r < a_c$

Figure 2.3: Optimal ceded functions in C^1: Case 1.



(a) $a_c < a_r$ (b) $a_r < a_c$

Figure 2.4: Optimal ceded functions in C^1: Case 2.

Case 3: $\beta = 1/2$

In this case, the minimization problem (2.15) simplifies to:

$$\min_{f \in C^2} \{f(a_r) - f(a_c)\}\,.$$

When $\alpha_c < \alpha_r$, because the ceded function is non-decreasing, this requires f^* to be constant on the interval (a_c, a_r). Therefore, any function f^* in C^2 with $f^*(x) = c$ on (a_c, a_r), where $c \in [0, a_c]$ is a constant, is Pareto-optimal.

When $\alpha_c > \alpha_r$, because the slope of the ceded function cannot exceed one, the function f^* increases at the rate of one on the interval (a_r, a_c). Therefore, any function f^* in C^2 with $f^{*\prime}(x) = 1$ on (a_r, a_c) is Pareto-optimal.

Finally, when $\alpha_c = \alpha_r$, then the objective function is always constant.

2.4.2 Parameter Values of the Optimal Ceded Function

In this section, we obtain parameter values of the optimal ceded functions that we derived in Section 2.4.1. Four cases are considered separately.

Case 1: $\beta > 1/2$ **and** $\alpha_c < \alpha_r$

Let:

$$\theta^* = \frac{1}{1+\theta}$$

and:

$$d^* = S_X^{-1}(\theta^*).$$

Theorem 2.4.1 *Under the conditions* $\beta > 1/2$ *and* $\alpha_c < \alpha_r$*, the optimal ceded function is* $f^*(x) = \min\{(x-d)_+, a_c - d\}$ *with the parameter:*

1. $d = d^*$ *when* $1 - \alpha_c < \theta^* < S_X(0)$;

2. $d = 0$ *when* $\theta^* \geq S_X(0)$.

In addition, when $\theta^* \leq 1 - \alpha_c$*, then* $f^*(x) = 0$ *for all x.*

Proof With the function f^* given by Equation (2.16), optimization Problem (2.15) becomes:

$$\min_{d\in[0,a_c]} g_5(d),$$

where:

$$g_5(d) = (2\beta - 1)\left((1+\theta)\int_d^{a_c} S_X(x)dx - a_c + d\right).$$

The derivative:

$$g_5'(d) = (2\beta - 1)(1 - (1+\theta)S_X(d))$$

is increasing in d. Therefore, when $1-\alpha_c < \theta^* < S_X(0)$, then $g_5(d)$ is minimized at $0 < d^* < a_c$. When $\theta^* > S_X(0)$, then $g_5(d)$ is minimized at $d = 0$. Finally, when $\theta^* < 1 - \alpha_c$, then $g_5(d)$ is minimized at $d = a_c$, and so, $f^*(x) = 0$.

Case 2: $\beta > 1/2$ **and** $\alpha_c > \alpha_r$

With the function f^* given by Equation (2.16), optimization problem (2.15) reduces to:

$$\min_{d\in[0,a_c]} g_6(d),$$

where:

$$g_6(d) = -\beta(a_c - d) + (1-\beta)(a_r - d)_+ + (2\beta - 1)(1+\theta)\int_d^{a_c} S_X(x)dx.$$

Let:

$$\theta^*_\beta = \frac{\beta}{(2\beta-1)(1+\theta)}$$

and

$$d^*_\beta = S_X^{-1}(\theta^*_\beta).$$

We calculate the derivative:

$$g_6'(d) = \begin{cases} (2\beta-1)(1+\theta)(\theta^* - S_X(d)) & \text{when} \quad d < a_r, \\ (2\beta-1)(1+\theta)(\theta^*_\beta - S_X(d)) & \text{when} \quad d > a_r, \end{cases}$$

which is an increasing function in d, and so, we have the following theorem.

Theorem 2.4.2 *Under the conditions $\beta > 1/2$ and $\alpha_c > \alpha_r$, the optimal ceded function is $f^*(x) = \min\{(x-d)_+, a_c - d\}$ with the parameter:*

1. $d = d^*$ *when* $1 - \alpha_r < \theta^* < S_X(0)$*;*
2. $d = a_r$ *when* $\theta^* < 1 - \alpha_r < \theta^*_\beta$*;*
3. $d = d^*_\beta$ *when* $\theta^* < 1 - \alpha_r$ *and* $1 - \alpha_c < \theta^*_\beta < 1 - \alpha_r$*;*
4. $d = 0$ *when* $\theta^* \geq S_X(0)$.

If none of the above conditions are satisfied, then $f^*(x) = 0$ for all x.

Proof We use similar arguments to those in Theorem 2.3.2. We illustrate them here by proving Part (1) only. When $1 - \alpha_r < \theta^* < S_X(0)$, the derivative $g_6'(d)$ reaches zero at $d^* = S_X^{-1}(\theta^*) \in (0, a_r)$ and then remains positive for $d > d^*$. Therefore, $g_6(d)$ reaches its minimum at $d^* = S_X^{-1}(\theta^*)$. With this, we conclude the proof of Theorem 2.4.2.

Case 3: $\beta < 1/2$ **and** $\alpha_c < \alpha_r$

With the function f^* given by Equation (2.17), optimization Problem (2.15) reduces to:

$$\min_{d \in [0, a_r]} g_7(d),$$

where the objective function is:

$$g_7(d) = \begin{cases} -\beta d + (1-\beta)d + (2\beta-1)(1+\theta)\left(\int_0^d S_X(x)dx + \int_{a_r}^\infty S_X(x)dx\right) & \text{when} \quad d < a_c, \\ -\beta a_c + (1-\beta)d + (2\beta-1)(1+\theta)\left(\int_0^d S_X(x)dx + \int_{a_r}^\infty S_X(x)dx\right) & \text{when} \quad d > a_c. \end{cases}$$

Thus:

$$g_7'(d) = \begin{cases} (1-2\beta)(1+\theta)(\theta^* - S_X(d)) & \text{when} \quad d < a_c, \\ (1-2\beta)(1+\theta)(\theta^*_\beta - S_X(d)) & \text{when} \quad d > a_c, \end{cases}$$

which leads us to the following theorem, whose proof is similar to that of Theorem 2.3.3 and thus omitted.

Theorem 2.4.3 *Under the conditions $\beta < 1/2$ and $\alpha_c < \alpha_r$, the optimal ceded function is:*

$$f^*(x) = \min\{x, d\}\mathbf{1}_{\{x\le a_r\}} + (x - a_r + d)\mathbf{1}_{\{x>a_r\}}$$

with the parameter:

1. $d = d^*$ *when* $1 - \alpha_c < \theta^* < S_X(0)$*;*
2. $d = a_c$ *when* $\theta^* < 1 - \alpha_c < \theta^*_\beta$*;*
3. $d = d^*_\beta$ *when* $\theta^* < 1 - \alpha_c$ *and* $1 - \alpha_r < \theta^*_\beta < 1 - \alpha_c$*;*
4. $d = a_r$ *when* $\theta^* < 1 - \alpha_c$ *and* $\theta^*_\beta < 1 - \alpha_r$*;*
5. $d = 0$ *when* $\theta^* \ge S_X(0)$.

If none of the above conditions are satisfied, then $f^*(x) = 0$ *for all* x.

Case 4: $\beta < 1/2$ **and** $\alpha_c > \alpha_r$

With the function f^* given by Equation (2.17), optimization Problem (2.15) reduces to:

$$\min_{d\in[0,a_r]} g_8(d),$$

where:

$$g_8(d) = -\beta(a_c - a_r + d) + (1-\beta)d + (2\beta - 1)(1+\theta)\left(\int_0^d S_X(x)dx + \int_{a_r}^{\infty} S_X(x)dx\right).$$

Thus,

$$g_8'(d) = (1-2\beta)(1+\theta)(\theta^* - S_X(d)),$$

which gives us the following theorem.

Theorem 2.4.4 *Under the conditions $\beta < 1/2$ and $\alpha_c > \alpha_r$, the optimal ceded function is:*

$$f^*(x) = \min\{x, d\}\mathbf{1}_{\{x\le a_r\}} + (x - a_r + d)\mathbf{1}_{\{x>a_r\}}$$

with the parameter:

1. $d = d^*$ *when* $1 - \alpha_r < \theta^* < S_X(0)$*;*
2. $d = a_r$ *when* $\theta^* < 1 - \alpha_r < S_X(0)$*;*
3. $d = 0$ *when* $\theta^* \ge S_X(0)$.

If none of the above conditions are satisfied, then $f^*(x) = 0$ *for all* x.

2.4.3 The illustrative Example Continued

In this subsection, we continue the illustrative example of Section 2.3.3, but now assume that the admissible class of ceded functions is C^2

Scenario A: $\alpha_c = 0.95$ **and** $\alpha_r = 0.99$

Applying Theorems 2.4.1 and 2.4.3, we have:

$$f^*_{2A}(x) = \begin{cases} \min\{x, 182.3\}\mathbf{1}_{\{x\le 4605.2\}} + (x - 4422.9)\mathbf{1}_{\{x>4605.2\}} & \text{when} \quad \beta \in [0, 0.5), \\ \min\{(x - 182.3)_+, 2813.4\} & \text{when} \quad \beta \in (0.5, 1]. \end{cases}$$

When $\beta = 0.5$, then:

$$f^*_{2A}(x) = \begin{cases} d & \text{when} \quad 2995.7 \le x \le 4605.2, \\ \text{unspecified} & \text{otherwise}, \end{cases}$$

where $d \in [0, 2995.7]$ can be any constant.

The values of $\text{VaR}(C_{f^*_{2A}})$ versus $\text{VaR}(R_{f^*_{2A}})$ are reported in Table 2.3.

	$\mathbf{VaR}_{\alpha_c}(C_{f^*_{2A}})$	$\mathbf{VaR}_{\alpha_r}(R_{f^*_{2A}})$
$\beta \in [0, 0.5)$	3025.41	−29.68
$\beta = 0.5$	between 1122.32 and 3025.41	between −29.68 and 1873.41
$\beta \in (0.5, 1]$	1122.32	1873.41

Table 2.3: VaRs of the cedent and the reinsurer when $f \in \mathcal{C}^2$.

We have the following observations:

- Since the cedent and the reinsurer have more choices when $f \in \mathcal{C}^2$, their VaRs under the optimal reinsurance policy f^*_{2A} are lower than the corresponding ones under f^*_{1A}. In particular, the reinsurer's risk is reduced significantly even when $\beta = 1$.
- For $\beta \in [0, 0.5)$, the reinsurer assumes the "good" risk in the layer $(0, S_X^{-1}(\theta^*))$, as well as losses greater than 4422.9. The former layer creates profit, and the latter layer does not contribute to its VaR because the chance of penetration is too small compared with the probability level α_r used in its VaR.
- For $\beta \in (0.5, 1)$, the insurer retains the "good" risk in the layer $(0, S_X^{-1}(\theta^*))$, as well as the losses greater than 2813.4. The former layer creates profit, and the latter layer does not contribute to its VaR because the chance of penetration is too small compared with the probability level α_c used in its VaR.

Scenario B: $\alpha_c = 0.99$ **and** $\alpha_r = 0.95$

Applying Theorems 2.4.2 and 2.4.4, we have:

$$f^*_{2B}(x) = \begin{cases} \min\{x, 182.3\}\mathbf{1}_{\{x\le 2995.7\}} + (x - 2813.4)\mathbf{1}_{\{x>2995.7\}} & \text{when} \quad \beta \in [0, 0.5), \\ \min\{(x - 182.3)_+, 4422.85\} & \text{when} \quad \beta \in (0.5, 1]. \end{cases}$$

When $\beta = 0.5$,

$$f^*_{2B}(x) = \begin{cases} x - d & \text{when} \quad x \in [2995.7, 4605.2], \\ \text{unspecified} & \text{when} \quad x \in [0, 2995.7) \cup (4605.7, \infty), \end{cases} \tag{2.18}$$

where $d \in [0, 2995.7]$ can be any constant. The values of $\text{VaR}(C_{f^*_{2B}})$ versus $\text{VaR}(R_{f^*_{2B}})$ are reported in Table 2.4.

	$\mathbf{VaR}_{\alpha_c}(C_{f^*_{2B}})$	$\mathbf{VaR}_{\alpha_r}(R_{f^*_{2B}})$
$\beta \in [0, 0.5)$	3073.43	−77.67
$\beta = 0.5$	between 1170.33 and 3073.43	between −77.67 and 1825.38
$\beta \in (0.5, 1]$	1170.33	1825.38

Table 2.4: VaRs of the cedent and the reinsurer when $f \in C^2$.

2.5 A Numerical Comparison of the Optimal Reinsurance Policies in different classes

In Sections 2.3.3 and 2.4.3, we derived the Pareto-optimal reinsurance policies in C^1 and C^2, respectively. In this section, we compare the two cases.

In Figure 2.5, we depict f^*_{1A} and f^*_{2A} obtained for Scenario A with the proportional reinsurance $f_1(x) = ax$ when a varies from zero to one and also with the excess-of-loss reinsurance $f_2(x) = (x - d)_+$ when the deductible level d varies from zero to $4605.2 = \max\{a_c, a_r\}$. The following can be concluded from the figure.

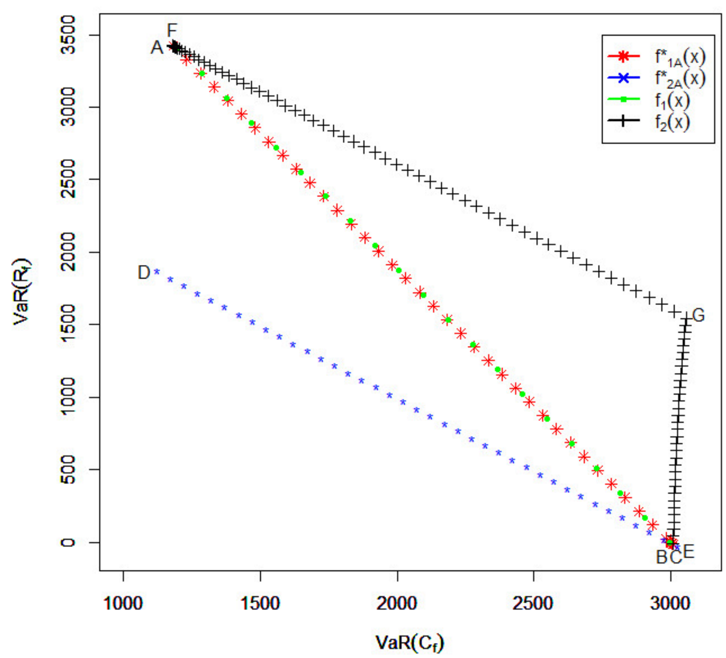


Figure 2.5: VaRs of the cedent and the reinsurer under different policies: Scenario A.

- The efficient frontier for the VaRs of the two parties with $f \in C^1$ is represented by the path from $A = (1182.32, 3422.85)$ to $B = (2995.72, 0)$ and then to $C = (3005.73, -10)$. Note that the points between A and B represent the VaRs of the two parties resulting from the optimal policies obtained with $\beta = 0.5$. The points between B and C represent the VaRs of the two parties resulting from the optimal policies obtained with $\beta = 0.654$.

- The efficient frontier for the VaRs of the two parties when $f \in C^2$ is represented by the path from $D = (1122.32, 1873.41)$ to $E = (3025.41, -29.68)$.
- For the quota-share reinsurance with $f_1(x) = ax$ where a ranges from zero to one, the VaRs of the two parties go from B to $F = (1200, 3405.2)$. When $f \in C^1$, the quota-share reinsurance policy is quite close to the efficient frontier.
- For the excess-of-loss reinsurance $f_2(x) = (x - d)_+$ with d ranging from zero to $a_r = 4605.2$, the VaRs of the two parties go along the path $F \to A \to G \to C$ with $G = (3055.47, 1545.43)$.

From Figure 2.5, we conclude that if the reinsurer worries about the right-hand tail more than the primary insurer ($\alpha_c < \alpha_r$), then the difference between the efficient frontiers obtained for $f \in C^1$ and $f \in C^2$ is significant. This means that the convexity requirement in the definition of C^1 is quite restrictive to the reinsurer, and the coverage with an upper limit (which is not allowed in C^1) is valuable. In the case when the convexity of the ceded function must be required, quota-share policies are quite efficient.

In Figure 2.6, we compare f^*_{1B} and f^*_{2B} obtained for Scenario B with the quota-share reinsurance policies $f_1(x) = ax$ when a ranges from zero to one and the excess-of-loss reinsurance policies $f_2(x) = (x - d)_+$ when the deductible d ranges from zero to $4605.2 = \max\{a_c, a_r\}$.

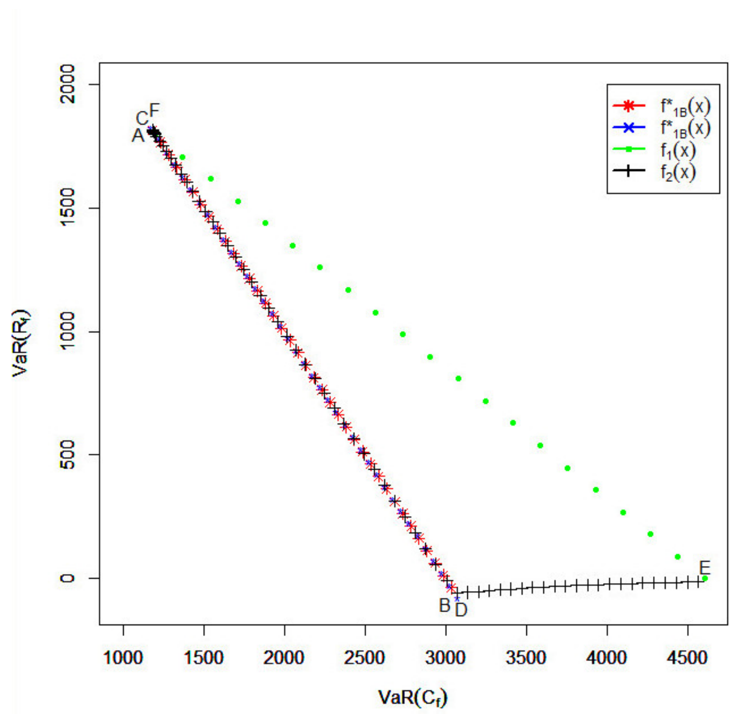


Figure 2.6: VaRs of the cedent and the reinsurer under different policies: Scenario B.

In particular, we observe the following:

- The efficient frontier for the VaRs of the two parties with $f \in C^1$ is represented by the path from $A = (1182.32, 1813.41)$ to $B = (3055.73, -60)$.

- The efficient frontier for the VaRs of the two parties when $f \in C^2$ is represented by the path from $C = (1170.33, 1825.38)$ to $D = (3073.43, -77.67)$. In fact, it can algebraically be shown that the path from B to A is actually a part of the path from D to C. That is, by allowing $f \in C^2$, the efficient frontier is extended from the path $B \to A$ to the path $D \to C$.
- For the quota-share reinsurance with the parameter a ranging from zero to one, the VaRs of the two parties are represented by the path from $E = (4605.7, 0)$ to $F = (1200, 1795.7)$. We see that when $\alpha_c > \alpha_r$, the quota-share reinsurance policies are not efficient.
- For the excess-of-loss reinsurance with the parameter d ranging from zero to $a_c = 4605.2$, the VaRs of the two parties change along the path $F \to A \to B \to E$. We see that setting $d \in (0, a_r)$ is quite efficient, whereas setting $d \in (a_r, a_c)$ is not.

From Figure 2.6, we conclude that if the primary insurer worries about the right-hand tail more than the reinsurer ($\alpha_c > \alpha_r$), then the excess-of-loss policies with the deductible level ranging from $S_X^{-1}(\theta^*)$ to a_r provide a good part of the efficient frontier. The quota-share policies are in general inefficient.

2.6 Conclusions

In this paper, we have extended the geometric approach of Cheung (2010) to obtain the optimal reinsurance policies accommodating both the cedent's and the reinsurer's interests. Specifically, we have derived the forms of optimal reinsurance functions and also specified their parameter values within two classical sets of admissible ceded functions. We have adopted the same value-at-risk measure for assessing risks of the two parties, but at possibly different probability levels. Illustrative numerical examples have been constructed to illuminate our theoretical findings and their practical implications.

Chapter 3

On Pareto-Optimal Reinsurance With Constraints Under Distortion Risk Measures

3.1 Introduction

Reinsurance is an agreement between an insurance company (insurer, cedent) and a reinsurance company (reinsurer), whereby the reinsurer agrees to pay a share of the losses incurred by the insurer for a premium. Particularly, let the losses of the insurer in a time period be denoted by a non-negative variable X, the ceded part of losses be $f(X)$, and the retained part of losses be $I_f(X) = X - f(X)$. Let the reinsurance premium be $P_f = H(f(X))$, where H is a premium principle, which maps a random variable to a real number. Under such setting, the total losses of the insurer and the reinsurer are $C_f = X - f(X) + P_f$ and $R_f = f(X) - P_f$, respectively. Commonly used forms of reinsurance treaties are the excess-of-loss treaty, where $f(X) = (X - d)_+$ with $d > 0$ being the deductible level (attaching point); and the quota-share treaty, where $f(X) = aX$ with $0 \leq a \leq 1$ being the share that the reinsurer assumes.

Optimal forms of reinsurance have been extensively studied in actuarial science and risk management literature. Borch (1960a) showed that with $P_f = (1 + \theta)\mathbf{E}[f(X)]$, where $\theta > 0$ is a risk loading factor, the variance of the insurer's losses is minimized by an excess-of-loss reinsurance treaty. Arrow (1963) proved that a risk-aversive insurer's expected utility is maximized by an excess-of-loss reinsurance treaty. Cai and Tan (2007) proposed to measure the insurer's risk by Value at Risk (VaR) and Tail Value at Risk (TVaR) in designing optimal reinsurance treaties. Many results in this aspect have been obtained thereafter. See for example, Cai et al. (2008), Bernard and Tian (2009), Cheung (2010), Chi and Tan (2011) and references therein. Optimal reinsurance treaties considering other objective functions, such as minimizing the risk-adjusted value of insurer's liabilities, maximizing the survival probability, minimizing the insurer's risk quantified by distortion risks measures or expectile, are discussed recently in Chi (2012), Assa (2015), Cai and Weng (2014), Zheng et al. (2015) and Weng and Zhuang (2016). Strikingly, in most cases, the optimal form of the ceded function $f(x)$ is piecewise linear, as was noted in Assa (2015).

The above mentioned studies focused on designing optimal reinsurance policies from the insurer's point of views, either maximizing its expected utility or minimized its risk. Borch (1960b) proposed to consider the interests of both parties in the reinsurance treaty, and he suggested that one type of such optimal reinsurance treaties could be designed to reduce the variance of both parties. Consequently, the covariance between the ceded losses and the retained losses should be maximized, which indicates that pro-rata reinsurance policies are optimal. Raviv (1979) assumed that both the insurer and the reinsurer are risk aversive and studied the Pareto-optimal reinsurance policies whereby one party's expected utility cannot be increased without decreasing that of the other party. Golubin (2006b) extended the results in Raviv (1979) by assuming that the reinsurance premiums are based on the actuarial value of ceded losses. Ignatov et al. (2004) studied the optimal reinsurance treaties maximizing the finite horizon joint survival probability of the two parties. Hürlimann (2011) proposed minimizing the sum of variances of the two parties. Cai et al. (2015) studied the optimal forms of reinsurance policies that minimize the convex combinations of the VaRs of the cedent and the reinsurer under two types of constraints that describe the interests of the two parties.

A closely related problem to optimal reinsurance is the so–called optimal risk sharing among partners, where each partner's interests are considered simultaneously. For results in this area, see for example, Borch (1960c), Aase (2002), Ludkovski and Young (2009), Asimit et al. (2013), and the references therein.

In this paper, we consider the interests of both the insurer and the reinsurer and study the Pareto-optimal insurance treaties under the constraints that risks of the insurer and the reinsurer cannot exceed some limits. This model generalizes that of Assa (2015) and Zhuang et al. (2016) because of the presence of the constraints regarding the risks of the insurer and the reinsurer. Our model is somewhat similar to that of Ludkovski and Young (2009), however, we show that the constrained optimization problem may be solved equivalently by the Lagrange multiplier method, which was used in Ludkovski and Young (2009); the generalized Neyman-Pearson method, which was used in Golubin (2006b) and Lo (2017); as well as the optimal control theory, which was used by Raviv (1979). To illustrate the practical applications of our main results, we derive explicit Pareto-optimal ceded functions when the risks are quantified by VaR and TVaR. Numerical examples are given. In particular, to illustrate the applicability of our main result, a numerical example where Range Value-at-Risk (RVaR) is used as risk measures of the insurer and the reinsurer is provided.

The rest of the paper is organized as follows. In Section 2, we describe the model and derive the optimal reinsurance policies using the Neyman-Pearson, the Lagrange multiplier, and the dynamic control methods. In Section 3, we apply the results to the scenarios when the risks of the insurer and the reinsurer are measured by the Value at Risk (VaR) and the Tail Value at Risk (TVaR). Section 4 provides numerical examples.

3.2 The Model

We begin by providing a very brief description of distortion risk measures.

3.2.1 Distortion risk measure

The distortion risk measure of a non-negative random variable X with survival function S_X is defined by $H_g(X) = \int_0^\infty g(S_X(t))dt$, where the distortion function $g : [0,1] \to [0,1]$ is non-decreasing and satisfies $g(0) = 0$ and $g(1) = 1$ (see for example, Denuit et al. (2006), Balbás et al. (2009) and references therein). An important property of distortion risk measures is the additivity for comonotonic risks. That is, for any distortion function g and comonotonic random variables X and Y,

$$H_g(X+Y) = H_g(X) + H_g(Y). \tag{3.1}$$

For detailed discussions of comonotonic random variables and the distortion risk measures, see for example, Dhaene et al. (2002b,c), and the references therein.

3.2.2 Model setup

We assume that the admissible set of ceded functions is given by

$$C := \{f : f \text{ and } I_f \text{ are non–decreasing and } 0 \le f(x) \le x \text{ for all } x\}. \tag{3.2}$$

This means that the reinsurer's payments to the insurer cannot exceed the insurer's losses, and that the ceded and retained loss functions are non-decreasing. These conditions are required to reduce moral hazards in reinsurance transactions. It was shown in Chi and Tan (2011) that all functions $f \in C$ are Lipschitz continuous and consequently differentiable almost everywhere.

Assume that the reinsurance premiums are calculated by $P_f = cH_{g_p}(f(X))$, where $g_p(\cdot)$ is a distortion function and $c \ge 1$ is a risk loading factor. Assume that the risks of the insurer and the reinsurer are quantified by distortion risk measures characterized by distortion functions $g_1(\cdot)$ and $g_2(\cdot)$, respectively.

Before we proceed, we state a property of the obtainable set of the risks of the insurer and the reinsurer.

Proposition 3.2.1 *Let $\mathcal{W} = \{(H_{g_1}(C_f), H_{g_2}(R_f))\}$ denote the obtainable set of the risks of the insurer and the reinsurer with all admissible ceded functions $f \in C$. Then the set $\mathcal{W}$ is convex.*

Proof For any two points in the set $\mathcal{W}$, say $O_1 = (H_{g_1}(C_{f_1}), H_{g_2}(R_{f_1}))$ and $O_2 = (H_{g_1}(C_{f_2}), H_{g_2}(R_{f_2}))$, let $O = \alpha O_1 + (1-\alpha)O_2$, where $\alpha \in [0,1]$, is a point on the line connecting O_1 and O_2. Then using the commonotonic linearity of the distortion risk measures,

$$O = (H_{g_1}(C_{\alpha f_1 + (1-\alpha) f_2}), H_{g_2}(R_{\alpha f_1 + (1-\alpha) f_2})). \tag{3.3}$$

Since $\alpha f_1 + (1-\alpha) f_2 \in C$, we must have that $O \in \mathcal{W}$. Therefore the set $\mathcal{W}$ is convex.

The main goal of this paper is to seek Pareto-optimal reinsurance policies, under which one party's risk cannot be further reduced without increasing that of the other party. Mathematically, this means that f^* is a Pareto-optimal ceded function if and only if there is no ceded function $f(x) \ne f^*(x)$ belonging to the admissible set C such that

$$H_{g_1}(C_f) \le H_{g_1}(C_{f^*}) \quad \text{and} \quad H_{g_2}(R_f) \le H_{g_2}(R_{f^*}) \tag{3.4}$$

are satisfied with at least one of the inequality being strict.

A general approach to identify Pareto optimal reinsurance policies is to solve the following problem (see for example, Gerber (1979) and Cai et al. (2017)):

Problem 3.2.2 (Unconstrained Problem)

$$\min_{f\in C} \quad \beta H_{g_1}(C_f) + (1-\beta)H_{g_2}(R_f), \qquad \beta \in [0,1]. \tag{3.5}$$

Since the set $\mathcal{W}$ is convex, applying the result on page 90 of Gerber (1979), we conclude that all Pareto optimal reinsurance policies may be obtain by solving Problem 3.2.2.

In addition, we assume that the companies' management and/or government regulators require that risk levels of the insurer and the reinsurer cannot exceed some monetary levels L_1 and L_2 respectively. More precisely, every admissible ceded function $f(x)$ has to satisfy the risk constraints of the insurer and the reinsurer:

$$H_{g_1}(C_f) \le L_1 \quad \text{and} \quad H_{g_2}(R_f) \le L_2. \tag{3.6}$$

Therefore, in the following we consider the optimization problem with constraints:

Problem 3.2.3 (Main Problem)

$$\begin{aligned} \min_{f\in C} \quad & \beta H_{g_1}(C_f) + (1-\beta)H_{g_2}(R_f), \qquad \beta \in [0,1], \\ s.t \quad & H_{g_1}(C_f) \le L_1 \\ & H_{g_2}(R_f) \le L_2 \end{aligned} \tag{3.7}$$

Remark 3.2.1 *Let $\tilde{f}$ be any of the optimal solution of the unconstrained problem 3.2.2. If $H_{g_1}(C_{\tilde{f}}) > L_1$ and $H_{g_2}(R_{\tilde{f}}) > L_2$, then problem 3.2.3 has no solution because there exists no admissible ceded function $f(x)$ such that $H_{g_1}(C_f) \le H_{g_1}(C_{\tilde{f}})$ and $H_{g_2}(R_f) \le H_{g_2}(R_{\tilde{f}})$.*

Due to the comonotonic additivity of distortion risk measure, Problem 3.2.3 may be written as

$$\begin{aligned} \min_{f\in C} \quad & -\beta H_{g_1}(f(X)) + (1-\beta)H_{g_2}(f(X)) + (2\beta-1)cH_{g_p}(f(X)), \qquad \beta \in [0,1], \\ \text{s.t} \quad & -H_{g_1}(f(X)) + cH_{g_p}(f(X)) \le L_1 - H_{g_1}(X), \\ & H_{g_2}(f(X)) - cH_{g_p}(f(X)) \le L_2. \end{aligned} \tag{3.8}$$

Furthermore, as proved in Lemma 2.1 of Cheung and Lo (2017) as well as in Lemma 2.2 of Lo (2017), for any distortion function g,

$$H_g(f(X)) = \int_0^\infty g[S_X(t)]df(t). \tag{3.9}$$

Consequently, Problem (3.8) can be expressed as

Problem 3.2.4 (Another form of Problem 3.2.3)

$$\begin{aligned} \min_{f\in C} \quad & \int_0^\infty h_0(t)f'(t)dt, \\ s.t. \quad & \int_0^\infty h_1(t)f'(t)dt \le L_1 - H_{g_1}(X) = L_1^*, \\ & \int_0^\infty h_2(t)f'(t)dt \le L_2, \end{aligned} \tag{3.10}$$

where

$$h_0(t) = -\beta g_1(S_X(t)) + (1-\beta)g_2(S_X(t)) + (2\beta - 1)cg_p(S_X(t)), \tag{3.11}$$

$$h_1(t) = cg_p(S_X(t)) - g_1(S_X(t)), \tag{3.12}$$

and

$$h_2(t) = g_2(S_X(t)) - cg_p(S_X(t)), \tag{3.13}$$

In the following two subsections, we solve Problem 3.2.4 by applying the generalized Neyman-Pearson Lemma, the Lagrange multiplier method, and the optimal control theory.

3.2.3 The generalized Neyman-Pearson Lemma and the Lagrange Multiplier Method

Problem 3.2.4 reminds us of the generalized Neyman-Pearson (or the Dantzig-Wald) problem (eg. Lehmann and Romano (2006), Rustagi (2014)):

$$\begin{aligned} \mathbf{V}(\phi) = \min_{\phi\in\Phi} \quad & \int f_{m+1}(t)\phi(t)dt, \\ \text{s.t} \quad & \int f_i(t)\phi(t)dt \le \alpha_i, \quad i = 1, 2, \ldots, m, \end{aligned} \tag{3.14}$$

where $f_1, f_2 \ldots, f_m, f_{m+1}$ are real-valued integrable functions and the set $\Phi = \{\phi : 0 \le \phi(x) \le 1 \text{ for all } x\}$ contains all admissible functions.

According to the generalized Neyman-Pearson Lemma (page 77 of Lehmann and Romano (2006)), let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^{m+}$, the solution to Problem (3.14) is given in the following form:

$$\phi_\lambda(t) = \begin{cases} 1 & \text{when } f_{m+1}(t) + \sum_{i=1}^m \lambda_i f_i(t) < 0, \\ \zeta(t) & \text{when } f_{m+1}(t) + \sum_{i=1}^m \lambda_i f_i(t) = 0, \\ 0 & \text{when } f_{m+1}(t) + \sum_{i=1}^m \lambda_i f_i(t) > 0, \end{cases} \tag{3.15}$$

where $\zeta(t)$ is any function such that $\phi_\lambda(t) \in \Phi$ and that all the constraints in the problem (3.14) are satisfied.

On the other hand, the Lagrange dual problem of (3.14) is

$$\mathbf{W}(\phi, \lambda) = \min_{\phi\in\Phi, \lambda\in\mathbb{R}^m} \quad \int f_{m+1}(t)\phi(t)dt + \sum_{i=1}^m \lambda_i \left(\int f_i(t)\phi(t)dt - \alpha_i \right). \tag{3.16}$$

Since $\int f_i(t)\phi(t)dt - \alpha_i \le 0$ for $i = 1, \ldots, m$, in order to minimize (3.16), we must have that $\lambda \in \mathbb{R}^{m+}$.

Furthermore, (3.16) can be rearranged as

$$\mathbf{W}(\phi, \lambda) = \min_{\phi\in\Phi, \lambda\in\mathbb{R}^{m+}} \quad \int \left(f_{m+1}(t) + \sum_{i=1}^m \lambda_i f_i(t) \right) \phi(t)dt - \sum_{i=1}^m \lambda_i \alpha_i, \tag{3.17}$$

from which it can be seen that the optimal solution for ϕ takes the form (3.15).

The detailed discussion of the generalized Neyman-Pearson problem (3.14) and its dual (3.16) can be found in Luenberger (1969) and Meeks and Francis (1973), but in more abstract settings. Particularly, directly applying Property (2.5) of Meeks and Francis (1973) yields the following Lemma.

Lemma 3.2.5 *Let $\phi_\lambda(t)$ be defined in (3.15). If there exists a $\lambda^* \in \mathbb{R}^{m+}$ such that the constraints*

$$\int f_i(t)\phi_{\lambda^*}(t)dt \le \alpha_i, \quad i = 1, 2, \ldots, m, \tag{3.18}$$

and the complementary slackness conditions

$$\lambda_i^*(\int f_i(t)\phi_{\lambda^*}(t)dt - \alpha_i) = 0, \quad i = 1, 2, \cdots, m \tag{3.19}$$

are satisfied, then $\phi_{\lambda^*}(t)$ is the solution to (3.14) and $(\phi_{\lambda^*}(t), \lambda^*)$ is the solution to (3.16). In addition, $\mathbf{V}(\phi^*) = \mathbf{W}(\phi^*, \lambda^*)$.

We note that the generalized Neyman-Pearson Lemma was also applied in Lo (2017) to solve an optimal reinsurance problem considering the insurer's budget constraint and the presence of the reinsurer's risk constraint.

3.2.4 Perspective of Optimal Control Theory

Following Raviv (1979), the generalized Neyman-Pearson problem may be analyzed using optimal control theory. Particularly, assumes that the limits of the integrals in problem (3.14) is $[0, T]$, where T is fixed but arbitrarily large, we consider the problem:

$$\begin{aligned} \mathbf{V}(\phi) = \min_{\phi \in \Phi} \quad & \int_0^T f_{m+1}(t)\phi(t)dt, \\ \text{s.t} \quad & \int_0^T f_i(t)\phi(t)dt \le \alpha_i, \quad i = 1, 2, \ldots, m, \end{aligned} \tag{3.20}$$

Let $\phi(t)$ be control variable and

$$z(t) = \big(z_1(t), \ldots, z_m(t)\big) = \left(\int_0^t f_1(s)\phi(s)ds, \ldots, \int_0^t f_m(s)\phi(s)ds\right) \tag{3.21}$$

be the set of state variables, the above problem can be rewritten as the following finite horizon dynammic optimization problem

$$\begin{aligned} \mathbf{V}(\phi) = \min_{\phi \in \Phi} \quad & \int_0^T f_{m+1}(t)\phi(t)dt, \\ z_i'(t) = f_i(t)\phi(t), \quad & i = 1, \ldots, m, \\ z_i(0) = 0, \quad & i = 1, \ldots, m, \\ z_i(T) \le \alpha_i, \quad & i = 1, \ldots, m. \end{aligned} \tag{3.22}$$

With the adjoint variables $\lambda(t) = (\lambda_1(t), \ldots, \lambda_m(t))$, the Hamiltonian of the problem (3.22) is given by

$$H(z, \phi, \lambda, t) = f_{m+1}(t)\phi(t) + \sum_{i=1}^m \lambda_i(t) f_i(t)\phi(t). \tag{3.23}$$

Since the Hamiltonian doesn't contain the state variables, the adjoint equations becomes

$$-\lambda_i'(t) = \frac{\partial H(z, \phi, \lambda, t)}{\partial z_i} = 0, \quad i = 1, \dots, m. \tag{3.24}$$

Therefore, adjoint variables $\lambda(t)$ are constant on $[0, T]$. Then we may write $\lambda(t) = \lambda = (\lambda_1, \dots, \lambda_m)$. By the Pontryagin's minimum principle (Kamien and Schwartz, 2012), a necessary condition for $\phi^*(t)$ to be an optimal solution is that

$$H(z^*, \phi^*, \lambda, t) \leq H(z^*, \phi, \lambda, t) \quad \text{for any } \phi \in \Phi, \tag{3.25}$$

where z^* is the optimal state variables of problem (3.22). Noticing that the Hamiltonian (3.23) is exactly the same as the integrand in (3.17), we immediately conclude that $\phi^*(t)$ is given by (3.15). At last, the slackness conditions in Theorem 3.2.5 are given by the transversality conditions in Pontryagin's minimum principle: $\lambda_i \geq 0$ and $\lambda_i(z_i^*(T) - \alpha_i) = 0$, for $i = 1, 2, \dots, m$.

Remark 3.2.2 *Pontryagin's minimum principle only provides a necessary condition to find the optimal control variable $\phi(t)$. However, as shown in Seierstad and Sydsaeter (1977), if the Hamiltonian $H(z, \phi, \lambda, t)$ is concave in z, then it is also the sufficient condition. In our case, as the Hamiltonian doesn't contain the state variable $z(t)$, the concavity condition is satisfied trivially. Therefore, the sufficiency and necessity of the conditions in theorem 3.2.5 is verified from the perspective of optimal control theory.*

Replacing the $\phi(t)$ in problem (3.14) by the derivative of the ceded function, $f'(t)$, we arrive at the following theorem:

Theorem 3.2.6 *Let $h(t) = h_0(t) + \lambda_1^* h_1(t) + \lambda_2^* h_2(t)$, where the functions $h_0(t)$, $h_1(t)$, and $h_2(t)$ were defined in (3.11), (3.12), (3.13) respectively. Then a solution to Problem 3.2.4 is given by a function $f_{\lambda^*}^*(t)$ characterized by*

$${f^*}'_{\lambda^*}(t) = \begin{cases} 1 & \text{when } h(t) < 0, \\ \zeta(t) & \text{when } h(t) = 0, \\ 0 & \text{when } h(t) > 0, \end{cases} \tag{3.26}$$

where and $\zeta(t) \in [0, 1]$ is a function such that $f_{\lambda^*}^* \in C$ and that all the constraints in Problem 3.2.4 are satisfied. The Lagrange constants $\lambda^* = (\lambda_1^*, \lambda_2^*)$ are nonnegative and determined by the slackness conditions

$$\lambda_1^* \left(\int h_1(t) {f^*}'_{\lambda^*}(t) dt - L_1^* \right) = 0 \tag{3.27}$$

and

$$\lambda_2^* \left(\int h_2(t) {f^*}'_{\lambda^*}(t) dt - L_2 \right) = 0. \tag{3.28}$$

If the solution $f_{\lambda^*}^*(t)$ violates constraints (3.7), then no solution exists for Problem 3.2.4.

Remark 3.2.3 *The above methodologies apply to problems with constraints that are more general than those in Problem 3.2.4. For example, when the amount spent on reinsurance is limited, one may require $P_f = cH_{g_p}(f(X)) \leq L_3$, as in Zhuang et al. (2016).*

3.2.5 Geometric interpretations

This subsection provides geometric interpretations to the analytical result obtained in Section 5.1 and 3.2.4.

Let $\mathcal{B}$ denote the Pareto-optimal set (or efficient frontier) of $\mathcal{W}$, that is, for any $b = (H_{g_1}(C_{f_b}), H_{g_2}(R_{f_b})) \in \mathcal{B}$, then there does not exist a $w = (H_{g_1}(C_{f_w}), H_{g_2}(R_{f_w})) \in \mathcal{W}$, such that $H_{g_1}(C_{f_w}) \leq H_{g_1}(C_{f_b})$ and $H_{g_2}(R_{f_w}) \leq H_{g_2}(R_{f_b})$. Since the set $\mathcal{W}$ is convex, the efficient frontier $\mathcal{B}$ may be found by minimizing linear combinations of the risks,

$$\beta H_{g_1}(C_f) + (1-\beta) H_{g_2}(R_f), \tag{3.29}$$

for all $f \in C$ and $0 \leq \beta \leq 1$. Geometrically, as in Figure 3.1, let x denote the risk of the insurer and y the risk of the reinsurer, then, as in Linear Programming problems, the point on the efficient frontier corresponding to a specific value of β may be determined by moving the line $u = \beta x + (1-\beta)y$ (increasing the value of u) until it first reaches $\mathcal{W}$. Changing the value of β in $[0, 1]$ yields all solutions to Problem 3.2.2 (whole efficient frontier).

Now we consider Problem 3.2.3, which has constraints on the insurer and the reinsurer's risk levels. Let $\mathcal{S} = \{(x, y) : 0 \leq x \leq L_1, 0 \leq y \leq L_2, \}$ be the feasible region of the insurer and the reinsurer's risk levels. Consider a point $A = (H_{g_1}(C_{\tilde{f}}), H_{g_2}(R_{\tilde{f}}))$ on the efficient frontier $\mathcal{B}$ that corresponds to a specific value of β, the following scenarios are possible:

1. If $H_{g_1}(C_{\tilde{f}}) > L_1$ and $H_{g_2}(R_{\tilde{f}}) > L_2$, then as shown in Figure 3.1, the obtainable set $\mathcal{W}$ and the feasible set $\mathcal{S}$ do not overlap. Thus Problem 3.2.3 has no solution. This verifies Remark 3.2.1.

2. If $H_{g_1}(C_{\tilde{f}}) \leq L_1$ and $H_{g_2}(R_{\tilde{f}}) \leq L_2$, then both risk constraints are satisfied and $\tilde{f}$ is a solution to the problem (3.2.3). This situation is shown in Figure 3.2.

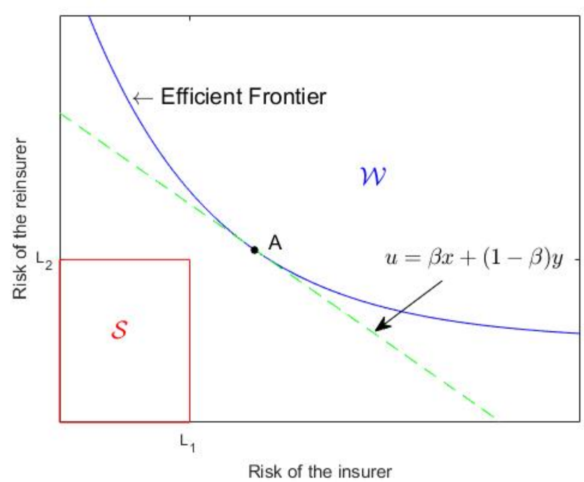


Figure 3.1: Efficient frontier of the risks of the insurer and the reinsurer and the risk constraints: both constraints are violated at point A.

3. If the point $A = (H_{g_1}(C_{\tilde{f}}), H_{g_2}(R_{\tilde{f}}))$ only satisfies one constraint, for example, $H_{g_1}(C_{\tilde{f}}) \leq L_1$ and $H_{g_2}(R_{\tilde{f}}) > L_2$ as shown in Figure 3.3. Then one may move the line $u = \beta x + (1-\beta)y$

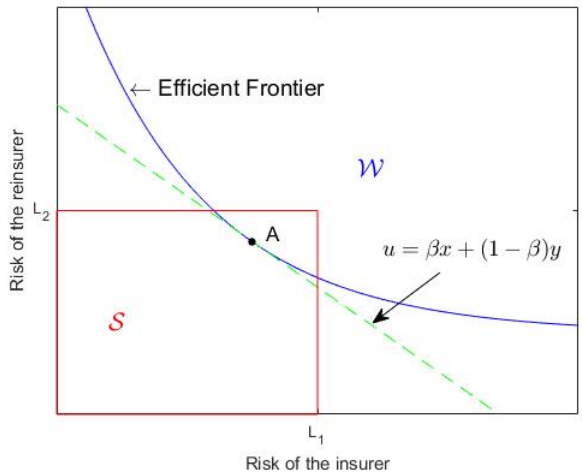
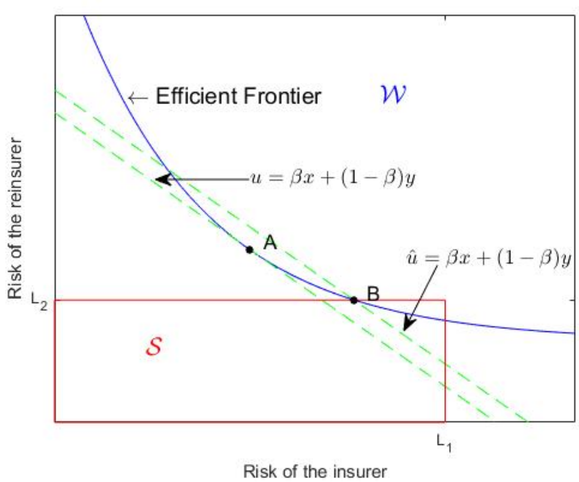
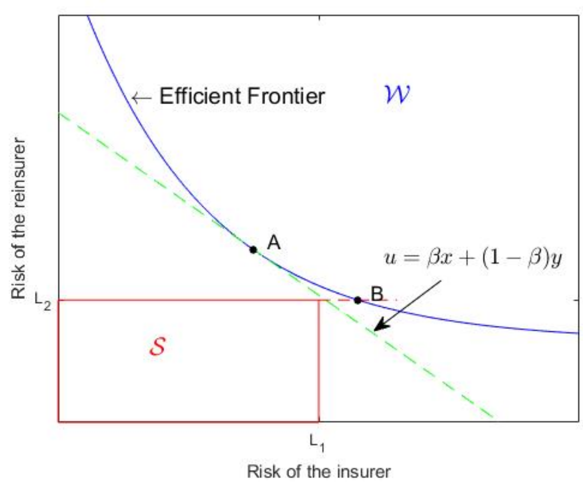


Figure 3.2: Efficient frontier of the risks of the insurer and the reinsurer and the risk constraints: both constraints are satisfied at point A.

(increasing the value of u) until the intersection of the efficient frontier $\mathcal{B}$ and the line $y = L_2$ (point B), where the reinsurer takes less risks and the insurers assumes more. Note that this procedure corresponds to selecting a Langrange multiplier $\lambda_2 > 0$, such that the slackness condition (3.28) is satisfied. In addition, one has to check whether the risk constraints of the insurer is satisfied at point B. If yes, then the ceded function corresponding to point B is the optimal solution to Problem 3.2.3 (see Figure 3.3a). If no, then the problem has no solution (see Figure 3.3b).



(a) Both constraints are satisfied at point B



(b) Constraint on reinsurer's risk violated at point B–no solution exist.

Figure 3.3: Efficient frontier of the risks of the insurer and the reinsurer and the risk constraints: Risk constraint of the insurer is violated at point A.

In the following section, we illustrate the applications of Theorem 3.2.6 by assuming that the risks of the insurer and the reinsurer are measured by VaR and TVaR. Due to the importance of the function $h(t)$, we refer to it as the "key function" in the subsequent discussions.

3.3 Special Cases

In this section, we derive the solution to Problem (3.2.3) when the risks of the insurer and the reinsurer are determined by the Value-at-Risk (VaR) and the Tail Value-at-Risk (TVaR). To simplify algebraic calculations, we employ the expectation premium principle in the following discussions. However, we note that results obtained in the previous sections apply when the premium is determined by any distortion risk measure.

Value at Risk

Let X be a random variable with distribution function $F_X(\cdot)$ and survival function $S_X(\cdot)$. The VaR of X at confidence level α is defined as

$$VaR_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x : F_X(x) \geq \alpha\}. \tag{3.30}$$

Note that $VaR_\alpha(\cdot)$ belongs to the family of distortion risk measure with distortion function

$$g_{V,\alpha}(x) = \begin{cases} 0, & 0 \leq x < 1-\alpha, \\ 1, & 1-\alpha \leq x \leq 1. \end{cases} \tag{3.31}$$

Suppose that the insurer and the reinsurer apply confidence levels α_c and α_r respectively in their VaR evaluations. Then the corresponding distortion function are given by g_{V,α_c} and g_{V,α_r}. To simplify discussions, we assume that $\alpha_c > \alpha_r$, which means the insurer is more concerned about the tail of the risk than the reinsurer. The derivation for the case $\alpha_c < \alpha_r$ is similar and therefore omitted in this paper.

Let $P_f = c\mathbf{E}[f(X)] = cH_{g_p}(f(X))$ with the distortion function given by $g_p(x) = x$. Assume that $1 \leq c < \frac{1}{1-\alpha_r}$, which means that the risk loading factor is not extremely high, since the confidence level α_r is usually close to one.

With the above setup, denoting $x = S_X(t)$, the key function $h(t)$ becomes

$$\begin{aligned} h_V(t) \equiv w(x) &= -(\beta+\lambda_1)g_{\alpha_c}(x) + (1-\beta+\lambda_2)g_{\alpha_r}(x) + (2\beta-1+\lambda_1-\lambda_2)cg_p(x) \\ &= \begin{cases} (2\beta-1+\lambda_1-\lambda_2)cx, & x \in [0, 1-\alpha_c), \\ (2\beta-1+\lambda_1-\lambda_2)cx - (\beta+\lambda_1), & x \in [1-\alpha_c, 1-\alpha_r), \\ (2\beta-1+\lambda_1-\lambda_2)(cx-1), & x \in [1-\alpha_r, 1], \end{cases} \end{aligned} \tag{3.32}$$

which has two non-zero roots $x_1 = \frac{1}{c} \cdot \frac{\beta+\lambda_1}{2\beta-1+\lambda_1-\lambda_2}$ and $x_2 = \frac{1}{c}$.

In the following discussions, we assume that $S_X(\cdot)$ is strictly decreasing and introduce the following notation:

$$a_c = S_X^{-1}(1-\alpha_c) = VaR_{\alpha_c}(X),$$
$$a_r = S_X^{-1}(1-\alpha_r) = VaR_{\alpha_r}(X),$$

$$\xi = S_X^{-1}(x_1),$$

and

$$\eta = S_X^{-1}(x_2).$$

The relative locations of the roots x_1 and x_2 and the jump points $1-\alpha_c$ and $1-\alpha_r$ determine the sign of $w(x)$ (thus $h_V(t)$). Accordingly, our analysis is divided into the following three cases.

Case A: $(2\beta - 1) + \lambda_1 - \lambda_2 > 0$

In this case, as shown in in Figure 3.4, $w(x)$ is piecewise linearly increasing, with a downward jump point at $x = 1 - \alpha_c$ and an upward jump at $x = 1 - \alpha_r$. Because of the assumption $1 \leq c < \frac{1}{1-\alpha_r}$, we have that $1 - \alpha_r < x_2 < x_1$.

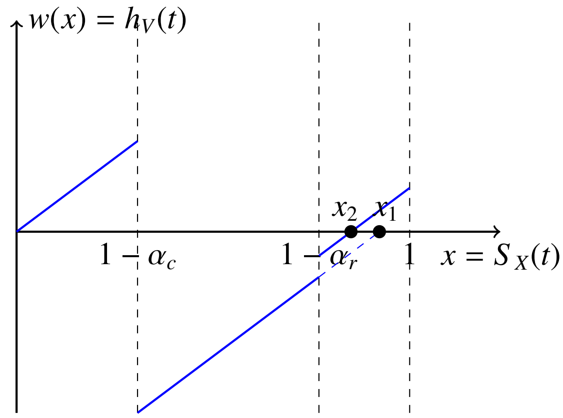


Figure 3.4: The key function $h_V(t)$ when $2\beta - 1 + \lambda_1 - \lambda_2 > 0$

Let $f^*_{V_1}(t)$ denote the optimal ceded function for this case. Then according to Figure 3.4, it must satisfy that $\frac{d}{dt}f^*_{V_1}(t) = 1$ for $S_X(t) \in (1-\alpha_c, x_2)$, or equivalently when $t \in [\eta, a_c)$, and $\frac{d}{dt}f^*_{V_1}(t) = 0$ otherwise. Thus, the optimal ceded function is

$$f^*_{V_1}(t) = \min\{(t-\eta)_+, a_c - \eta\}. \tag{3.33}$$

Applying $f^*_{V_1}(t)$ to the risk constraints of the insurer and the reinsurer yields

$$c\int_{\eta}^{a_c} S_X(t)dt + \eta \leq L_1, \tag{3.34}$$

and

$$a_r - \eta - c\int_{\eta}^{a_c} S_X(t)dt \leq L_2. \tag{3.35}$$

When both constraints are satisfied, $\lambda_1 = \lambda_2 = 0$. Then the condition $(2\beta-1)+\lambda_1-\lambda_2 > 0$ means that $\beta > 1/2$, implying that the insurer has more power in the negotiation. Therefore, we conclude that if $\beta > 1/2$ and the two constraints (3.34) and (3.35) are satisfied, then $f^*_{V_1}(t)$ is the optimal ceded function.

If the constraint (3.34) is satisfied but (3.35) is not, then we may set $\lambda_1 = 0$ and $\lambda_2 > 0$ such that $(2\beta-1)+\lambda_1-\lambda_2 \leq 0$. Then the optimal ceded function could be sought in Case B or Case C.

If the constraints (3.35) is satisfied but (3.34) is not, (this is the unrealistic case when the limit L_1 is extremely low), then the problem has no solution, because setting $\lambda_1 > 0$ still results in $(2\beta - 1) + \lambda_1 - \lambda_2 > 0$ and the $f^*_{V_1}(t)$ is the only option, with which (3.34) is violated.

If neither of the constraints (3.34) and (3.35) is satified, then the problem has no solution, due to Remark 3.2.1.

Case B: $(2\beta - 1) + \lambda_1 - \lambda_2 < 0$

For this case, as shown in Figure 3.5, $h_V(t) = w(x)$ decreases piecewisely with a downward jump at $x = 1 - \alpha_c$ and an upward jump at $1 - \alpha_r$. In addition, $h_V(t) < 0$ for $S_X(t) \in [0, 1 - \alpha_r) \cup (x_2, 1]$ and $h_V(t) > 0$ otherwise.

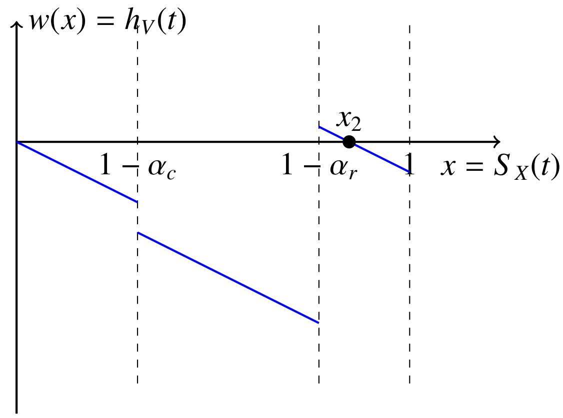


Figure 3.5: The key function $h_V(t)$ when $2\beta - 1 + \lambda_1 - \lambda_2 < 0$.

Therefore, the optimal ceded function is given by

$$f^*_{V_2}(t) = \min\{t, \eta\}\mathbb{1}_{t<a_r} + (t - a_r + \eta)_+\mathbb{1}_{t\geq a_r}. \tag{3.36}$$

Applying $f^*_{V_2}(t)$ to the risk constraints of the insurer and the reinsurer yields:

$$c\left\{\int_0^{\eta} S_X(t)dt + \int_{a_r}^{a_c} S_X(t)dt\right\} - \eta + a_r \leq L_1, \tag{3.37}$$

and

$$\eta - c\left\{\int_0^{\eta} S_X(t)dt + \int_{a_r}^{a_c} S_X(t)dt\right\} \leq L_2. \tag{3.38}$$

If both constraints are satisfied, then $\lambda_1 = \lambda_2 = 0$, and we conclude that if $\beta < 1/2$ and the constraints (3.37) and (3.38) are satisfied, then $f^*_{V_2}(t)$ is the optimal ceded function.

When either of the constraints is not satisfied, the discussions are similar to those in Case A. Therefore they are omitted.

Case C: $2\beta - 1 + \lambda_1 - \lambda_2 = 0$

Figure 3.6 illustrate the shape of $h_V(t)$ for this case, from which it is seen that the optimal ceded function $f^*_{V_3}(t)$ has the form

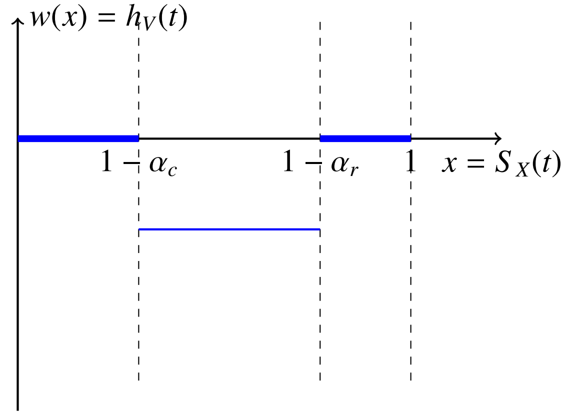


Figure 3.6: The key function $h_V(t)$ when $2\beta - 1 + \lambda_1 - \lambda_2 = 0$.

$$f^*_{V_3}(t) = \begin{cases} f^*_{V_3,1}(t), & t \in [0, a_r], \\ t - a, & t \in [a_r, a_c], \\ f^*_{V_3,2}(t), & t \in [a_c, \infty), \end{cases} \tag{3.39}$$

where $a \in [0, a_r]$, $f^*_{V_3,1}(t)$ and $f^*_{V_3,2}(t)$ are any functions so that $f^*_{V_3}(t)$ is in C. Note that smallest possible $f^*_{V_3}(t)$ is $f^*_{V_3,\min}(t) = \min\{(t-a_r)_+, a_c - a_r\}$ and the largest possible $f^*_{V_3}(t)$ is $f^*_{V_3,\max}(t) = t$.

This result shows that when $\beta = 1/2$, the insurer and the reinsurer only care about the risk transfer but not the premium transfer (see equation (3.8)), then the two parties could negotiate any ceded function of the form $f^*_{V_3}(t)$ as long as the risk constraints on the insurer and the reinsurer are satisfied. Intuitively, because of the properties of the VaR risk measure, the reinsurer disregards losses above a_r and the insurer disregards losses above a_c. Then it is optimal for the reinsurer to cover the losses in the layer (a_r, a_c).

More importantly, this case also includes the scenario when $\beta \neq 1/2$ but one of the risk constraints is not satisfied. For example, in Case A, if constraint (3.35) is not satisfied, then one can choose $\lambda_2^* = 2\beta - 1$. Consequently, the optimal ceded function could be any function with the form $f^*_{V_3}(t)$ such that $H_{g_2}(R_{f^*_{V_3}}) = L_2$. In fact, one can set the ceded function to

$$f^*_{V_3,a^*}(t) = \min\{(t - a^*)_+, a_c - a^*\}, \tag{3.40}$$

where $a^* \in (\eta, a_r]$ satisfies

$$a_r - a^* - c \int_{a^*}^{a_c} S_X(t)dt = L_2. \tag{3.41}$$

Remark 3.3.1 *The ceded function (3.40) is a continuation of (3.33). Since $\beta > 1/2$, it would be optimal for the insurer if the reinsurer to cover the layer (η, a_c). However, because of the risk constraint of the reinsurer, it cannot provide such a coverage. However, the reinsurer could still provide as much coverage as possible. That is, it covers the layer (a^*, a_c) with $a^* > \eta$ and satisfies (3.41).*

Remark 3.3.2 *When no risk constraints are imposed, the optimal ceded functions may be obtained by using the results in this section by setting $\lambda_1 = \lambda_2 = 0$. Particularly, we conclude* *that when $\beta > 1/2$, the optimal ceded function is given by $f^*_{V_1}$ defined in (3.33); when $\beta < 1/2$, the optimal ceded function is given by $f^*_{V_2}$ defined in (3.36); when $\beta = 1/2$, the optimal ceded function is $f^*_{V_3}$ defined in (3.39).*

The case of $\beta = 1/2$ deserves more discussion. For this case, the objective function in (3.8) reduces to

$$\frac{1}{2}(\mathrm{VaR}_{\alpha_r}(f(X)) - \mathrm{VaR}_{\alpha_c}(f(X))) = \frac{1}{2}(f(a_r) - f(a_c)), \tag{3.42}$$

which is trivially minimized by $f^*_{V_3}$. In fact, the minimized objective function is given by $\frac{1}{2}(a_r - a_c)$. Consequently,

$$\frac{1}{2}(\mathrm{VaR}_{\alpha_c}(C_{f^*_{V_3}}) + \mathrm{VaR}_{\alpha_r}(R_{f^*_{V_3}})) = \frac{1}{2}a_r. \tag{3.43}$$

This is, with the optimal reinsurance policy, the total risk of the insurer and the reinsurer is a_r! This makes sense because without reinsurance, the total risk of the loss X is $a_c > a_r$ (by the assumption of Section 3), and it is impossible to reduce the total risk level below a_r.

Remark 3.3.3 *In practical situations, since α_r and α_c are usually taken to be close to 1, the condition $1 < c < \frac{1}{1-\alpha_r}$ is satisfied for most commonly used safety loading factor c. If c is beyond this range, our methodology still applies. However, understandably, different optimal reinsurance policies will be obtained. For the completeness of our discussion, an example is provided in which assuming $c > \frac{1}{1-\alpha_c}$. In this case, referring to equation (3.32), we see that $x_2 < 1 - \alpha_c$. If*

$$x_1 = \frac{\beta + \lambda_1}{2\beta - 1 + \lambda_1 - \lambda_2} \cdot \frac{1}{c} < 1 - \alpha_c, \tag{3.44}$$

then as shown in Figure (3.7), the key function $h_V(t)$ (or $w(x)$) is always greater than zero. As a result, the optimal ceded function is

$$f^*_{V4}(t) = 0. \tag{3.45}$$

In particular, if the risk constraints of the insurer and the reinsurer are satisfied, then $\lambda_1 = \lambda_2 = 0$, and we conclude that if

$$x_1 = \frac{\beta}{2\beta - 1} \cdot \frac{1}{c} < 1 - \alpha_c, \tag{3.46}$$

then the optimal ceded function is given by f^*_{V4}. Notice that Equation (3.46) implies that

$$\beta > \frac{c(1-\alpha_c)}{2c(1-\alpha_c) - 1} > \frac{1}{2}$$

because $c(1 - \alpha_c) > 1$. Intuitively, if the insurer is more important than the reinsurer in the negotiation but the premium rate is very large, then the insurer will not purchase any reinsurance coverage, unless its original risk is greater than the risk constraint (imposed by the regulators).

Remark 3.3.4 *In Section 4.3 of Lo (2017), a Pareto-optimal reinsurance problem is studied under the VaR setting. It was shown that the "exact offsetting" property of the VaR enables one to reduce the two-constraint problem to a one-constraint problem. In our case, if writing $c = 1 + \theta$, then the assumptions $1 < c < \frac{1}{1-\alpha_r}$ and $\alpha_r < \alpha_c$ lead to $\frac{\theta}{1+\theta} < \alpha_r < \alpha_c$, which agrees*

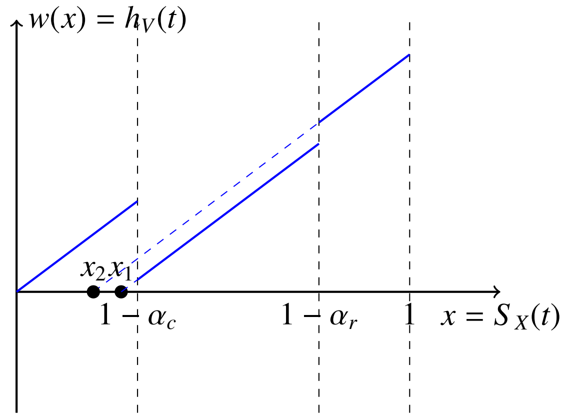


Figure 3.7: The key function $h_V(t)$ when $2\beta - 1 + \lambda_1 - \lambda_2 > 0$ and $x_2 < x_1 < 1 - \alpha_c$

with the conditions of the Proposition 4.3 in Lo (2017). Under the assumption that S_X is strictly decreasing (or F_X is strictly increasing), our results are consistent with those in Lo (2017). In particular, when $\beta > 1/2$, our equations (3.33) and (3.39), for the cases when both constraints are satisfied and when the risk constraint of the reinsurer is violated respectively, agree with Part (a) and (b) of Proposition 4.3 in Lo (2017); when $\beta < 1/2$, our equations (3.36) and (3.39) agree with part (a) and (b) of Corollary 4.5 in Lo (2017); when $\beta = 1/2$, our equation (3.39) agrees with the Proposition 4.6 (b) in Lo (2017) when both constraints are satisfied.

Tail Value at Risk

The TVaR of a random variable X with given a confidence level α is defined as

$$\text{TVaR}_\alpha(X) = \mathbf{E}[X \mid X \geq VaR_\alpha(X)]. \tag{3.47}$$

The corresponding distortion function is

$$g_{T,\alpha}(x) = \begin{cases} \dfrac{x}{1-\alpha}, & 0 \leq x \leq 1-\alpha, \\ 1, & 1-\alpha \leq x \leq 1. \end{cases} \tag{3.48}$$

Suppose that the insurer and the reinsurer apply confidence levels α_c and α_r respectively in their TVaR evaluations. Then the corresponding distortion functions are given by g_{T,α_c} and g_{T,α_r}. As in the VaR case, we assume the expectation premium principle so that $P_f = c\mathbf{E}[f(X)]$ with $1 \leq c < \frac{1}{1-\alpha_r}$, in addition, $\alpha_r < \alpha_c$.

With the above setup, denoting $x = S_X(t)$, the key function $h(t)$ becomes

$$\begin{aligned} h_T(t) \equiv w_T(x) &= -(\beta+\lambda_1)g_{T,\alpha_c}(x) + (1-\beta+\lambda_2)g_{T,\alpha_r}(x) + (2\beta-1+\lambda_1-\lambda_2)cg_p(x) \\ &= \begin{cases} w_{T,1}(x), & x \in [0, 1-\alpha_c], \\ w_{T,2}(x), & x \in, [1-\alpha_c, 1-\alpha_r], \\ w_{T,3}(x), & x \in [1-\alpha_r, 1], \end{cases} \end{aligned} \tag{3.49}$$

where

$$w_{T,1}(x) = (2\beta-1+\lambda_1-\lambda_2)cx - \frac{\beta+\lambda_1}{1-\alpha_c}x + \frac{1-\beta+\lambda_2}{1-\alpha_r}x,$$

$$w_{T,2}(x) = (2\beta - 1 + \lambda_1 - \lambda_2)cx - (\beta + \lambda_1) + \frac{1 - \beta + \lambda_2}{1 - \alpha_r}x$$

and

$$w_{T,3}(x) = (2\beta - 1 + \lambda_1 - \lambda_2)cx - (\beta + \lambda_1) + (1 - \beta + \lambda_2).$$

The function $w_T(x)$ is piecewise linear, continuous and has two roots:

$$x_1 = \frac{\beta + \lambda_1}{(1 - \beta + \lambda_2)(1/(1 - \alpha_r) - c) + (\beta + \lambda_1)c}$$

and $x_2 = \frac{1}{c}$. Thus its sign is determined by three quantities: $w_T(1-\alpha_c)$, $w_T(1-\alpha_r)$ and $w_T(1)$. It is easy to verify that the slope of $w_{T,2}$ is greater than those of $w_{T,1}$ and $w_{T,3}$, and that $w_{T,3}(1-\alpha_r)$ and $w_{T,3}(1)$ have different signs if $2\beta - 1 + \lambda_1 - \lambda_2 \neq 0$ ($w_{T,3}$ crosses the horizontal line). In the following, as in the VaR case, we divide our analysis into the following cases.

Case A: $2\beta - 1 + \lambda_1 - \lambda_2 > 0$

In this case, as shown in Figure 3.8(a), $w_T(1 - \alpha_c) < 0$, $w_T(1 - \alpha_r) < 0$ and $w_T(1) > 0$. In addition, $w_T(x) < 0$ for $0 < x < x_2$ and $w_T(x) > 0$ otherwise. Thus, the optimal ceded function $f^*_{T,1}(t)$ is given by

$$f^*_{T_1}(t) = (t - \eta)_+. \tag{3.50}$$

Applying $f^*_{T_1}$ to the risk constraints of the insurer and the reinsurer yields

$$c\mathbf{E}[(X - \eta)_+] + \eta \leq L_1, \tag{3.51}$$

and

$$TVaR_{\alpha_r}(X) - \eta - c\mathbf{E}[(X - \eta)_+] \leq L_2. \tag{3.52}$$

If both constraints are satisfied, then $\lambda_1 = \lambda_2 = 0$ and the condition $2\beta - 1 + \lambda_1 - \lambda_2 > 0$ becomes $\beta > 1/2$. So we conclude that when $\beta > 1/2$ and the two constraints (3.51) and (3.52) are satisfied, the optimal ceded function is $f^*_{T_1}$.

If constraint (3.51) is satisfied but (3.52) is not, then we may set $\lambda_1 = 0$ and $\lambda_2 > 0$ such that $(2\beta - 1) + \lambda_1 - \lambda_2 \leq 0$. Then the optimal ceded function could be sought in Case B or Case C.

If the constraint (3.52) is satisfied but (3.51) is not, (this is the unrealistic case when the limit L_1 is extremely low), then the problem has no solution, because setting $\lambda_1 > 0$ still results in $(2\beta - 1) + \lambda_1 - \lambda_2 > 0$ and the resulting optimal ceded function $f^*_{V_1}(t)$ violate the constraint (3.51).

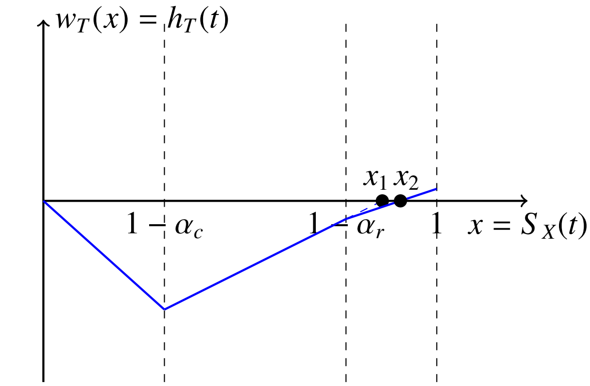
If neither of the constraints (3.51) and (3.52) is satified, then the problem has no solution, due to Remark 3.2.1.

Case B: $2\beta - 1 + \lambda_1 - \lambda_2 = 0$

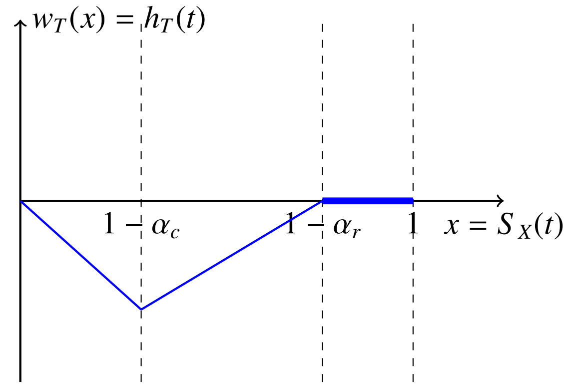
In this case, $w_T(1 - \alpha_c) < 0$, $w_T(1 - \alpha_r) = 0$, and $w_T(1) = 0$. The shape of the key function $h_T(t)$ is shown in Figure 3.8(b), based on which it is seen that the optimal ceded function $f^*_{T_2}(t)$ is given by

$$f^*_{T_2}(t) = \begin{cases} f^*_{T_2,1}(t), & t \in [0, a_r], \\ t - a_r + a, & t \in [a_r, \infty), \end{cases} \tag{3.53}$$

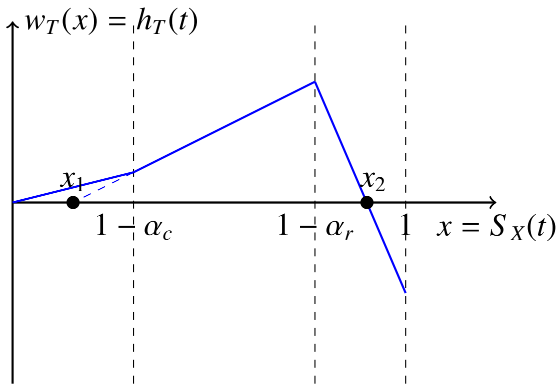
where $f^*_{T_2,1}(t)$ is such that $f^*_{T_2}(t) \in C$ and a is an arbitrary constant in $(0, a_r)$.



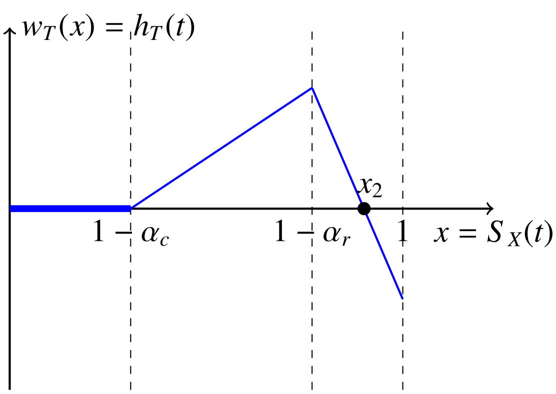
(a) $2\beta - 1 + \lambda_1 - \lambda_2 > 0$



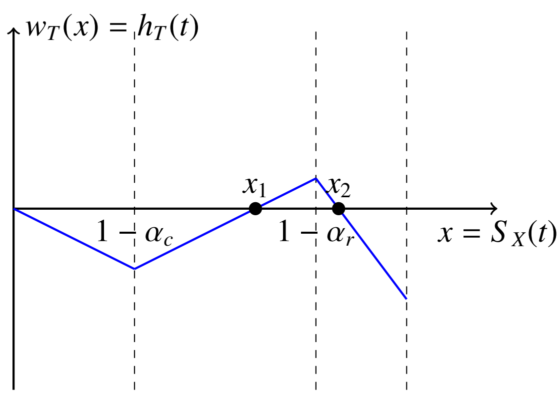
(b) $2\beta - 1 + \lambda_1 - \lambda_2 = 0$



(c) $2\beta-1+\lambda_1-\lambda_2 < 0$ and $w_T(1-\alpha_c) > 0$



(d) $2\beta-1+\lambda_1-\lambda_2 < 0$ and $w_T(1-\alpha_c) = 0$



(e) $2\beta-1+\lambda_1-\lambda_2 < 0$ and $w_T(1-\alpha_c) < 0$

Figure 3.8: The key function $h_T(t)$ in the TVaR case.

The result shows that when $\beta = 1/2$, the insurer and the reinsurer could negotiate any ceded function of the form $f^*_{T_2}(t)$ as long as the constraints on risks of the insurer and the reinsurer are satisfied.

More importantly, this case also include the scenarios when $\beta \neq 1/2$ but one of the constraints are active. For example, in Case A, if constraint (3.52) is not satisfied, the one can choose $\lambda_2^* = 2\beta - 1$, then the optimal ceded function could be any function of form $f_{T_2}^*(t)$ such that $H_{g_2}(R_{f_{T_2}^*}) = L_2$. In fact, one can set the ceded function to

$$f_{T_2,a^*}^*(t) = (t - a^*)_+, \tag{3.54}$$

where $a^* \in (\eta, a_r]$ satisfies

$$TVaR_{\alpha_r}(X) - a^* - c\mathbf{E}[(X - a^*)_+] = L_2. \tag{3.55}$$

Remark 3.3.5 *Analogue to Remark 3.3.1, the ceded function (3.54) is a continuation of (3.50). When $\beta > 1/2$, it would be optimal for the insurer if the reinsurer was to cover the layer (η, ∞). However, because of the risk constraint of the reinsurer, it cannot provide such coverage. As a result, the reinsurer could provide as much coverage as possible. As a result, it covers the layer (a^*, ∞) with a^* satisfies (3.55).*

Case C: $2\beta - 1 + \lambda_1 - \lambda_2 < 0$

In this case, $w_T(1 - \alpha_r) > 0$ and $w_T(1) < 0$. However, the sign of $w_T(1 - \alpha_c)$ is not fixed. Therefore, we further divide the analysis into the following three sub-cases.

Case C.1: $2\beta - 1 + \lambda_1 - \lambda_2 < 0$ and $w_T(1 - \alpha_c) > 0$

Note that the condition $w_T(1 - \alpha_c) > 0$ can be rewritten more explicitly as

$$\frac{\beta + \lambda_1}{1 - \beta + \lambda_2} < \frac{\frac{1}{1-\alpha_r} - c}{\frac{1}{1-\alpha_c} - c}, \tag{3.56}$$

which in fact implies $2\beta - 1 + \lambda_1 - \lambda_2 < 0$, because we assumed that $\alpha_r < \alpha_c$.

The shape of the function $h_T(t)$ in this case is shown in Figure 3.8(c), according to which the optimal ceded function $f_{T_3}^*$ is given by

$$f_{T_3}^*(t) = \min\{t, \eta\}. \tag{3.57}$$

Applying $f_{T_3}^*$ to the risk constraints of the insurer and the reinsurer yields

$$c \int_0^\eta S_X(t)dt - \eta \le L_1 - TVaR_{\alpha_c}(X), \tag{3.58}$$

and

$$\eta - c \int_0^\eta S_X(t)dt \le L_2. \tag{3.59}$$

If the two constraints are satisfied then $\lambda_1 = \lambda_2 = 0$. We conclude that when

$$\frac{\beta}{1 - \beta} < \frac{\frac{1}{1-\alpha_r} - c}{\frac{1}{1-\alpha_c} - c}, \tag{3.60}$$

and the constraints (3.58) and (3.59) are satisfied, $f_{T_3}^*$ is the optimal ceded function. When at least one of the constraints is not satisfied, the analysis is similar to that in Case A and therefore omitted.

Note that in reasonable situations, the constraint (3.59) should be satisfied. In addition, when $\beta = 0$, only the reinsurer is considered, then the condition (3.60) is automatically satisfied. So $f^*_{T_3}$ is the optimal ceded function as long as (3.58) and (3.59) are satisfied.

Case C.2: $2\beta - 1 + \lambda_1 - \lambda_2 < 0$ and $w_T(1-\alpha_c) = 0$

Notice that the condition $w_T(1-\alpha_c) = 0$ can be rewritten more explicitly as

$$\frac{\beta+\lambda_1}{1-\beta+\lambda_2} = \frac{\frac{1}{1-\alpha_r} - c}{\frac{1}{1-\alpha_c} - c}. \tag{3.61}$$

The shape of $h_T(t)$ is shown in Figure 3.8(d), according to which the optimal ceded function is given by

$$f^*_{T_4}(t) = \begin{cases} t, & t \in [0, \eta], \\ \eta, & t \in [\eta, a_c], \\ f^*_{T_4,2}(t), & t \in [a_c, \infty), \end{cases} \tag{3.62}$$

where $f^*_{T_4,2}(t)$ is such that $f^*_{T_4}(t) \in C$. Notice that $f^*_{T_4}(t) \geq f^*_{T_3}(t)$ for all $t \geq 0$.

This case includes the scenario when the constraint (3.58) is not satisfied in Case C.1. Then the reinsurer is required to assume more risk, resulting in $f^*_{T_4}(t)$.

Case C.3: $2\beta - 1 + \lambda_1 - \lambda_2 < 0$ and $w_T(1-\alpha_c) < 0$

The two conditions for this case can be written as

$$1 > \frac{\beta+\lambda_1}{1-\beta+\lambda_2} > \frac{\frac{1}{1-\alpha_r} - c}{\frac{1}{1-\alpha_c} - c}, \tag{3.63}$$

which implies that the insurer has more negotiation power than in Case C.1. The shape of $h_T(t)$ for this case is shown in Figure 3.8(e), based on which the optimal ceded function is given by

$$f^*_{T,5}(t) = \begin{cases} t, & t \in [0, \eta], \\ \eta, & t \in [\eta, \xi], \\ t - \xi + \eta, & t \in [\xi, \infty), \end{cases} \tag{3.64}$$

where $\xi \in (a_r, a_c)$. Notice that $f^*_{T_5}(t) \geq f^*_{T_4}(t)$ for all $t \geq 0$.

Applying $f^*_{T_5}(t)$ to the two constraints yields

$$\kappa_I(\xi) = \xi + c\mathbf{E}[(X-\xi)_+] - \eta + c\int_0^{\eta} S_X(t)dt \leq L_1, \tag{3.65}$$

and

$$\kappa_R(\xi) = (\frac{1}{1-\alpha_r} - c)\mathbf{E}[(X-\xi)_+] + \eta - c\int_0^{\eta} S_X(t)dt \leq L_2, \tag{3.66}$$

The value of the parameter ξ is determined in the following manner:

- If $\lambda_1 = \lambda_2 = 0$, the first root of the function W_T becomes

$$x_1 = x_{1,0} = \frac{1}{\frac{1-\beta}{\beta}(\frac{1}{1-\alpha_r} - c) + c}.$$

Then we conclude that $\xi = \xi_0 = S_X^{-1}(x_{1,0})$ if the constraints (3.65) and (3.66) are both satisfied.

Notice that when β increases and the insurer becomes more important, ξ_0 decreases and the reinsurer has to cover more losses in the right tail.

- If the constraint (3.66) is satisfied but (3.65) is not with $\xi = \xi_0$, then one seeks $\xi_1 \in (a_r, a_c)$ such that $\kappa_I(\xi_1) = L_1$ and $\kappa_R(\xi_1) \leq L_2$. If no such ξ_1 exists, then one needs to select λ_1 so large that $\frac{\beta+\lambda_1}{1-\beta+\lambda_2} \geq 1$. Consequently, the optimal ceded function may be found in cases A or B.
- The constraint (3.66) should be satisfied if L_2 is reasonably big. In case it is violated but (3.65) satisfied, then one seeks $\xi_2 \in (a_r, a_c)$ such that $\kappa_I(\xi_2) \leq L_1$ and $\kappa_R(\xi_1) = L_2$. If no such ξ_2 exists, one should increase λ_2 such that $\frac{\beta+\lambda_1}{1-\beta+\lambda_2} \leq \frac{\frac{1}{1-\alpha_r}-c}{\frac{1}{1-\alpha_c}-c}$ and the optimal ceded function may be found in the Case C.1 and Case C.2.
- If both constraints (3.65) and (3.66) are violated with $\xi = \xi_0$, then the problem has no solution.

Remark 3.3.6 *Under a similar setting when risks are measured by TVaR, Cai et al. (2017) derived the optimal ceded functions for all possible values of c, α_c and α_r, however without considering risk constraints, in their Theorem 3.2. Our results in this section generalizes the results by considering the risk constraints. For presentational simplicity, we imposed the additional reasonable assumption of $c < \frac{1}{1-\alpha_r} < \frac{1}{1-\alpha_c}$. If this assumption is not satisfied, the optimal ceded functions can still be derived by using the general result in Theorem 3.2.6. Detailed analysis is omitted here.*

Remark 3.3.7 *If no constraint is imposed, for the case of $\beta > 1/2$, our equation (3.50) agrees with Theorem 3.2 (xii) of Cai et al. (2017); for the case of $\beta = 1/2$, our equation (3.53) agrees with their Theorem 3.2 (ix); for the case of $\beta < 1/2$; our equations (3.57), (3.62) and (3.64) agree with their Theorem 3.2 (iv), (v) and (vi).*

Remark 3.3.8 *A concept of the best Pareto-optimal reinsurance contract is introduced in Cai et al. (2017), in which the insurer and reinsurer need to reduce their risks to certain levels and the reinsurer need to keep certain expected net profit margin. Specifically, in a best Pareto-optimal reinsurance contract, the following three criterion (with the notations used in the current paper) should be satisfied.*

$$TVaR_{\alpha_c}(C_f) \leq \gamma TVaR_{\alpha_c}(X), \tag{3.67}$$

$$TVaR_{\alpha_r}(R_f) \leq \kappa TVaR_{\alpha_r}(X) \tag{3.68}$$

and

$$P_f \geq (1+\sigma)\mathbf{E}[f(X)]. \tag{3.69}$$

It can be seen that the constraints (3.67) and (3.68) are similar to the risk constraints in Problem 3.2.3 of the current paper and the constraint (3.69) become $c \geq 1+\sigma$ with our *notation. Therefore, the method developed in the current paper can be applied to calculate the best Pareto-optimal reinsurance contract problem introduced in Cai et al. (2017).*

Remark 3.3.9 *The methodology introduced in this paper can be used to study the Pareto-optimal reinsurance when the risks of the insurer and the reinsurer are measured by other distortion risk measures. For example, one may use the RVaR, which was proposed by Cont et al. (2010) and generalizes VaR and TVaR as its two extreme cases. Particularly, the RVaR with two parameters* $0 < \alpha < \omega < 1$ *for a random variable X is defined by*

$$RVaR_{\alpha,\omega}(X) = \frac{1}{\omega-\alpha}\int_{\alpha}^{\omega} VaR_u(X)du = \frac{1-\alpha}{\omega-\alpha}TVaR_{\alpha}(X) - \frac{1-\omega}{\omega-\alpha}TVaR_{\omega}(X). \tag{3.70}$$

RVaR belong the family of distortion risk measures with the corresponding distortion function

$$g_{R,\alpha,\omega}(x) = \frac{x-1+\omega}{\omega-\alpha}\mathbb{1}_{\{x\in(1-\omega,1-\alpha]\}} + \mathbb{1}_{\{x\in(1-\alpha,1]\}}. \tag{3.71}$$

Therefore, when the risks of the insurer and the reinsurer are measured by RVaR, one may apply the above distortion function to obtain the key function $h(t)$ *defined in Theorem 3.2.6, determine its shape and then the optimal reinsurance policies. We will not pursue the detailed solutions here to avoid lengthy discussions. Nevertheless we will present a numerical example in Section 3.4.*

3.4 Numerical Examples

This section provides two numerical examples illustrating the applications of the results obtained in previous sections. The first example deals with the case when both the insurer and the reinsurer use VaR as their risk measures. The second example discusses the case when TVaR is used.

3.4.1 Value at Risk

Suppose that the loss random variable X under consideration follows an exponential distribution with mean 1000. Assume that the risk of the insurer and the reinsurer are measured by VaR with probability levels $\alpha_c = 0.99$, $\alpha_r = 0.95$ respectively. Let the reinsurance premium be determined by $P_f = c\mathbf{E}[f(X)]$ where $c = 1.2$. With the setup, we have that $a_c = VaR_{\alpha_c}(X) = 4605.2$, $a_r = VaR_{\alpha_r}(X) = 2995.7$ and $\eta = S_X^{-1}(\frac{1}{c}) = 182.4$.

Let's start by constructing the efficient frontier of the risks of the insurer and the reinsurer when the risk constraints are not imposed. As pointed out in Remark 3.3.2, for $\beta > 1/2$, the optimal ceded function is

$$f^*_{V_1}(t) = \min\{(t-\eta)_+, a_c-\eta\} = \min\{(t-182.4)_+, 4422.8\} \tag{3.72}$$

and the resulting risks are $VaR_{\alpha_c}(C_{f^*_{V_1}}) = 1170.3$ and $VaR_{\alpha_r}(R_{f^*_{V_1}}) = 1825.4$ (point O_1 in Figure 3.9). For $\beta < 1/2$, the optimal ceded function is

$$f^*_{V_2}(t) = \min\{t,\eta\}\mathbb{1}_{t<a_r} + (t-a_r+\eta)_+\mathbb{1}_{t\geq a_r} \tag{3.73}$$

and the resulting risks are $VaR_{\alpha_c}(C_{f^*_{V_2}}) = 3073.4$ and $VaR_{\alpha_r}(R_{f^*_{V_2}}) = -77.7$ (point O_2 in Figure 3.9). For $\beta = 1/2$, the optimal ceded functions can be any function having the form $f^*_{V_3}(t)$ defined by (3.39), and the resulting risks are such that

$$\mathrm{VaR}_{\alpha_c}(C_{f^*_{V_3}}) + \mathrm{VaR}_{\alpha_r}(R_{f^*_{V_3}}) = a_r. \tag{3.74}$$

This actually results in a -45 degree line connecting point O_1 and Point O_2.

Overall, the efficient frontier is given by the straight line O_1O_2 shown in Figure 3.9 when no risk constraints are imposed.

When the risk constraints are present, the optimal ceded function for a particular value of β can be determined by applying the following steps. Let's arbitrarily set $\beta = 0.6$.

- Since $\beta > 1/2$, the optimal ceded function is $f^*_{V_1}(t)$ and the resulting risks is represented by point O_1 if there is no constraints. Therefore, if point O_1 is located inside the feasible region $\mathcal{S}$, that is, if $L_1 > 1170.3$ and $L_2 > 1825.4$, then $f^*_{V_1}$ is the optimal ceded function. This situation is illustrated in Figure 3.10.

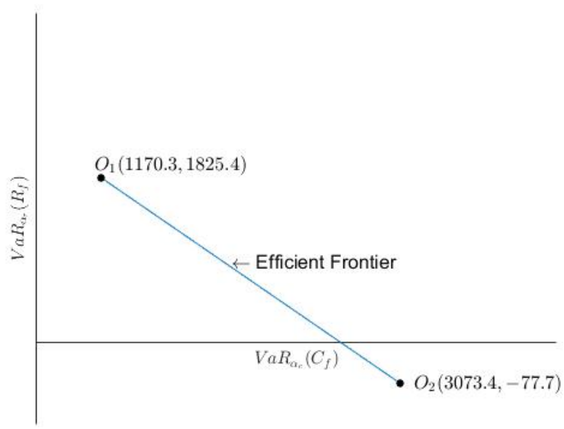


Figure 3.9: The efficient frontier of the risks of the insurer and the reinsurer

- If $L_1 < 1170.3$, then the problem has no solution, as discussed in Case A of Section 3.3. This situation is illustrated in Figure 3.11.

- If $L_1 > 1170.3$ but $L_2 < 1825.4$, more discussions are needed. Essentially, one needs to change the value of λ_2, such that the corresponding optimal ceded function f^*_{V,λ_2} satisfies $VaR_{\alpha_c}(C_{f^*_{V,\lambda_2}}) \le L_1$ and $VaR_{\alpha_r}(R_{f^*_{V,\lambda_2}}) = L_2$. For example, suppose that $L_2 = 1800$, then according to the discussions in Case C of Section 3.3, one may choose $\lambda_2 = 2\beta - 1 = 0.2$, then the optimal ceded function is given by $f^*_{V^*_a}(t) = \min\{(t - a^*)_+, a_c - a^*\}$, where a^* is such that $VaR_{\alpha_r}(R_{f^*_{V^*_a}}) = 1800$. This results in $a^* = 416.5$. Since $VaR_{\alpha_c}(C_{f^*_{V^*_a}}) = 1195.7$, we conclude that if $L_1 > 1195.7$ and $L_2 = 1800$, then $f^*_{V^*_a}(t) = \min\{(t - 416.5)_+, 4188.7\}$ is the optimal ceded function (see the left panel of Figure 3.12). if $L_1 < 1195.7$ and $L_2 = 1800$, then no optimal solution exists (see the right panel of Figure 3.12).

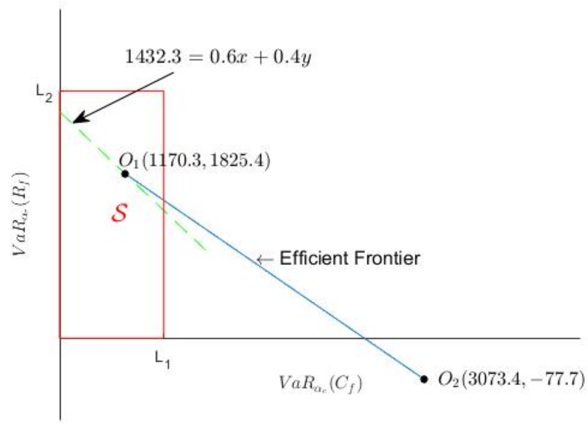


Figure 3.10: Both the insurer and reinsurer's risk constraints are satisfied

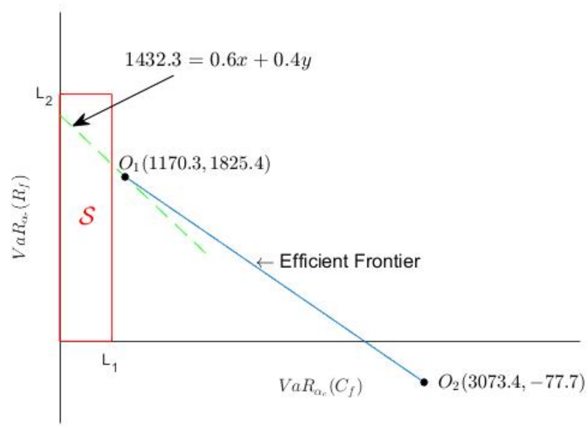


Figure 3.11: The insurer's risk constraint is violated

3.4.2 Tail Value at Risk

In this example, we use the same setup as in Section 3.4.1, however we now assume that the risk of the insurer and the reinsurer are measured by TVaR with probability levels $\alpha_c = 0.99$, $\alpha_r = 0.95$ respectively. The optimal ceded function can be determined by applying the following steps. Further, we assume that $\beta = 0.6$.

- Since $\beta > 1/2$, applying the results in Case A of Section 3.3, the optimal ceded function is $f^*_{T_1}(t) = (t - \eta)_+$ if the risk constraints (3.51) and (3.52) are satisfied simultaneously. Since $TVaR_{\alpha_c}(C_{f^*_{T_1}}) = 1182.3$ and $TVaR_{\alpha_r}(R_{f^*_{T_1}}) = 2813.4$, we conclude that if $L_1 > 1182.3$ and $L_2 > 2813.4$, then $f^*_{T_1}$ is the optimal ceded function.

- If $L_1 < 1182.32$, then the problem has no solution, as discussed in Case A of Section 3.3.

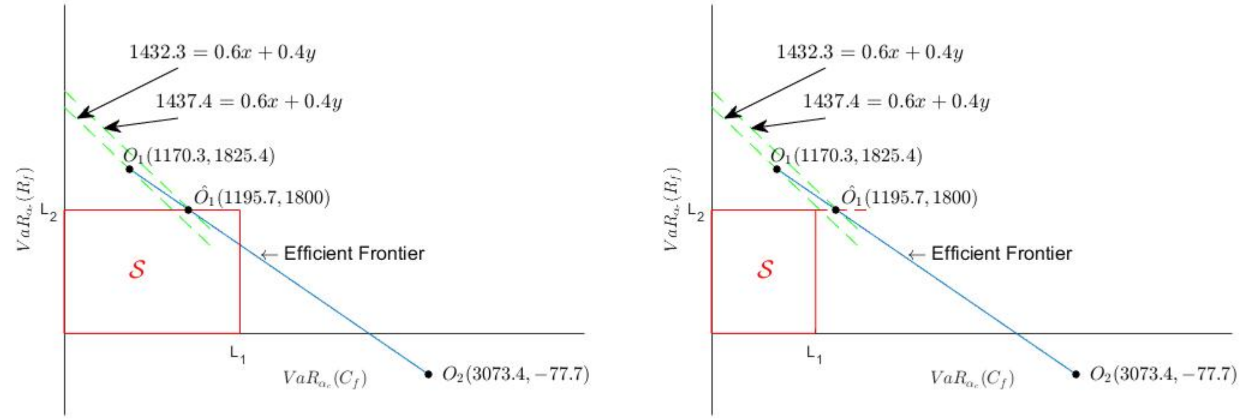


Figure 3.12: The reinsurer's risk constraint is violated

- If $L_1 > 1182.3$ but $L_2 < 2813.4$, more discussions are needed. Again, one needs to change the value of λ_2, such that the corresponding optimal ceded function f^*_{T,λ_2} satisfies $TVaR_{\alpha_c}(C_{f^*_{T,\lambda_2}}) \leq L_1$ and $TVaR_{\alpha_r}(R_{f^*_{T,\lambda_2}}) = L_2$. For example, suppose that $L_2 = 2800$, then according to the discussions in Case B of Section 3.3, one can choose $f^*_{T^*_a}(t) = (t - a^*)_+$, where a^* is such that $TVaR_{\alpha_r}(R_{f_{T^*_a}}) = 2800$. By applying (3.55), this results in $a^* = 350.7$. Since $TVaR_{\alpha_c}(C_{f_{T^*_a}}) = 1195.7$, we conclude that when $L_1 > 1195.7$ and $L_2 = 2800$, then the optimal ceded function is $f^*_{T^*_a}(t) = (t - 350.7)_+$; when $L_1 < 1195.7$ and $L_2 = 2800$, the problem has no solution. Essentially, more capital (either from the insurer or the reinsurer) is needed support the underlying losses.

3.4.3 Range Value at Risk

In this section, we use the same setup as in Section 3.4.2, however we now assume that the risks of the insurer and the reinsurer are measured by RVaR with probability levels $\alpha_c = 0.99$, $\alpha_r = 0.95$ respectively and $\omega_c = \omega_r = 0.995$.

Applying the distortion function in (3.71) and noting that $\alpha_r < \alpha_c < \omega_c = \omega_r$, the key function $w(x) = h(t)$, where $x = S_X(t)$ becomes

$$w(x) = \begin{cases} (2\beta - 1 + \lambda_1 - \lambda_2)cx, & x \in [0, 1 - \omega_c], \\ (1 - \beta + \lambda_2)(\dfrac{x - 1 + \omega_c}{\omega_c - \alpha_r} - cx) - (\beta + \lambda_1)(\dfrac{x - 1 + \omega_c}{\omega_c - \alpha_c} - cx), & x \in (1 - \omega_c, 1 - \alpha_c], \\ (1 - \beta + \lambda_2)(\dfrac{x - 1 + \omega_c}{\omega_c - \alpha_r} - cx) - (\beta + \lambda_1)(1 - cx), & x \in (1 - \alpha_c, 1 - \alpha_r], \\ (2\beta - 1 + \lambda_1 - \lambda_2)(cx - 1), & x \in (1 - \alpha_r, 1]. \end{cases} \tag{3.75}$$

If the two risk constraints are satisfied at the same time, then $\lambda_1 = \lambda_2 = 0$ and $w(x)$ above has two roots: $x_1 \approx 0.005$ and $x_2 = \frac{1}{c} = 0.8333$. In addition, $w(x) > 0$ for $x \in (0, x_1) \cup (x_2, 1)$ and $w(x) \leq 0$ otherwise. Therefore, let $t_1 = S_X^{-1}(0.005) = 5298.3$ and $t_2 = S_X^{-1}(0.8333) = 182.4$, then the optimal ceded function is given by

$$f_R^*(t) = \min\{(t - 182.4)_+, 5298.3\}. \tag{3.76}$$

With this ceded function, the RVaR of the risks of the insurer and the reinsurer are given by $RVaR_{\alpha_c,\omega_c}(C_{f_R^*}) = 1176.4$ and $RVaR_{\alpha_r,\omega_r}(R_{f_R^*}) = 2563.6$ respectively.

With the above calculations, we conclude that:

- If $L_1 \geq 1176.4$ and $L_2 \geq 2563.6$, then f_R^* is the optimal ceded function.
- If $L_1 < 1176.4$ and $L_2 < 2563.6$, the problem has no solution, due to Remark 3.2.1.
- If $L_1 < 1176.4$ and $L_2 \geq 2563.6$, the problem has no solution. This is because if there exists a solution to this problem when L_1-constraint is violated, then according to Theorem 3.2.6, $\lambda_2^* = 0$ and there exists a $\lambda_1^* > 0$, such that resultant optimal ceded function f_{R,λ^*} satisfies $RVaR_{\alpha_c,\omega_c}(C_{f_{R,\lambda^*}}) = L_1$. However, with the parameter values for this example, for all $\lambda_1 > 0$ and $\lambda_2 = 0$, the roots of the key function $w(x)$ are at $x_2 = 0.8333$ and

$$x_1(\lambda_1) = \frac{0.556 + \lambda_1}{110.87 + 198.8\lambda_1}, \tag{3.77}$$

  from which it can be seen that x_1 is greater than but very close to 0.005. Therefore $RVaR_{\alpha_c,\omega_c}(C_{f_{R,\lambda_1}}) \geq 1176.4 > L_1$ for all $\lambda_1 > 0$. Consequently, the problem has no solution.

- If $L_1 \geq 1176.4$ and $L_2 < 2563.6$, more discussions are needed. To be specific, let $L_2 = 2500$. In this case, one needs to set $\lambda_1^* = 0$ and find $\lambda_2^* > 0$ such that resultant optimal ceded function f_{R,λ^*} satisfies $RVaR_{\alpha_r,\omega_r}(R_{f_{R,\lambda^*}}) = L_2$. With the parameter values for this example, one may set $\lambda_2^* = 0.2$. Then $2\beta - 1 + \lambda_1^* - \lambda_2^* = 0$. With this, one can verify that $w(x) = 0$ on $[0, 0.005] \cup (0.05, 1]$ and $w(x) < 0$ otherwise. Then, since $S_X^{-1}(0.005) = 5298.3$ and $S_X^{-1}(0.05) = 2995.7$, the optimal ceded function is given by

$$f_{R,\lambda^*}^*(t) = \begin{cases} f_{R,\lambda^*,1}^*(t), & t \in [0, 2995.7], \\ t - a, & t \in [2995.7, 5298.3], \\ f_{R,\lambda^*,2}^*(t), & t \in [5298.3, \infty), \end{cases} \tag{3.78}$$

  where $a \in [0, 2995.7]$, $f_{R,\lambda^*,1}^*(t)$ and $f_{R,\lambda^*,2}^*(t)$ are any functions such that $f_{R,\lambda^*}^*(t) \in \mathcal{C}$. For example, one can choose $f_{R,\lambda^*}^*(t) = (t - a^*)_+$ where $a^* \in [0, 2995.7]$. Solving the equation $RVaR_{\alpha_r,\omega_r}(R_{f_{R,\lambda^*}^*}) = 2500$ results in $a^* = 542$. In addition, it can be verified that $RVaR_{\alpha_c,\omega_c}(C_{f_{R,\lambda^*}^*}) = 1240$. Therefore, we conclude that: when $L_1 \geq 1240$ and $L_2 = 2500$ then the optimal ceded function is given by $f_{R,\lambda^*}^* = (x - 542)_+$; when $L_1 < 1240$ and $L_2 = 2500$, the problem has no solution.

3.4.4 Conclusions drawn from the numerical examples

In Sections 3.4.1, 3.4.2 and 3.4.3 we analyze the solution to the Problem 3.2.3 using different risk measures. We next compare the resultant optimal ceded functions. For simplicity, we consider the case when the risk constraints of the insurer and reinsurer are satisfied. Under the same model assumptions: $X \sim \exp(1000)$, $c = 1.2$, $\beta = 0.6$, $\alpha_c = 0.99$, $\alpha_r = 0.95$, and $\omega_c = \omega_r = 0.995$, we have that $S_X^{-1}(1/c) = 182.4$, $S_X^{-1}(1-\alpha_c) = 4422.8$, $S_X^{-1}(1-\omega_c) = 5298.3$. The optimal ceded functions are summarized as follows.

$$\begin{aligned} &\text{VaR}: && f_V^* = \min\{(t-182.4)_+, 4422.8\}, \\ &\text{RVaR}: && f_R^* = \min\{(t-182.4)_+, 5298.3\}, \\ &\text{TVaR}: && f_T^* = (t-182.4)_+. \end{aligned} \tag{3.79}$$

The results indicate that for all three risk measures, $S_X^{-1}(1/c) = 182.4$ provides an optimal reinsurance attaching point for the insurer ($\beta > 0.5$ so that insurer is more important in the setting).

With VaR, the insurer does not consider the risk above $S_X^{-1}(1-\alpha_c) = 4422.8$, therefore it does not purchase coverage above it. With RVaR, the insurer does not consider the risk above $S_X^{-1}(1-\omega_c) = 5298.3$, therefore it does not purchase coverage above it. With TVaR, the insurer considers all layers of losses, therefore it purchases coverage for all losses above the optimal attaching point $S_X^{-1}(1/c) = 182.4$.

Chapter 4

On optimal reinsurance treaties in cooperative game under heterogeneous beliefs

4.1 Introduction

Many insurance companies use reinsurance to control their risk levels and the costs of risk capital. As stated in Patrik (2001) and Clark (2014), a reinsurance contract is often a manuscript contract setting forth the unique agreement between the two parties. Each contract must be individually priced to meet the particular needs and risk level of the reinsured. If the two companies conclude a treaty, the treaty must be such that both companies consider themselves better off than without the treaty.

With the above considerations, it is natural to model a reinsurance contract as the result of a two-person bargaining game, in which the two individuals have the opportunity to collaborate for mutual benefit in more than one way (Nash (1950)).

Borch (1960c) was the first to study optimal reinsurance contracts within the context of bargaining games, in which he first derived the set of Pareto-optimal contracts and then identified the one corresponding to the Nash bargaining solution. Borch (1962) generalized Borch (1960c) by considering the reinsurance market as the result of a n-player bargaining game and studied its equilibrium. There are many subsequent important developments in applications of game theory to insurance problems in economics, insurance, and risk theory literature. For example, assuming that the insurer is risk neutral, Kihlstrom and Roth (1982a) studied the effect of the insureds' risk aversion levels on the bargaining results using the Nash bargaining model; Lemaire (1991) gave a comprehensive review of theories of cooperative games and illustrated their applications in reinsurance problems. Aase (2009) characterized a competitive equilibrium as well as a game-theoretical equilibrium in a risk sharing setting. More recently, Boonen (2015) characterized the optimal risk sharing in a competitive equilibrium under the distortion risk measures; Boonen et al. (2016) assumed that the negotiating insurance companies have comonotonic additive utility functions and derived optimal reinsurance contract corresponding to the Nash bargaining solution.

In negotiating a reinsurance contract, the insurer and the reinsurer may have different beliefs regarding the probability distribution of the underlying losses. This difference can result from different subjective beliefs or different analytical models used to evaluate the underlying losses. In this aspect, Wilson (1968) presented detailed analysis and some general results on risk sharing when involved parties have heterogeneous beliefs. Marshall (1992) derived the optimal form of reinsurance policies when the reinsurer is risk neutral and the insurer assigns higher probability to the zero-loss event than the reinsurer. Under a much more general setting of heterogeneous beliefs, Ghossoub (2017) showed that if the decision maker is strictly risk-averse, then optimal indemnity schedules are nondecreasing with respect to the loss. Boonen (2016b) assumed that the premium is determined by the reinsurer using Wang's premium principle and studied the optimal reinsurance contract that maximizes the insurer's dual utility. Boonen et al. (2017) focused on how heterogeneous beliefs between pension funds and insurers regarding mortality rates affect optimal redistribution of longevity risk. They showed that the participants can all benefit by shifting the risk associated with a scenario to the participant who assigns the lowest probability to that scenario. Most recently, Chi (2018) found that full reinsurance above a constant deductible is always optimal for a risk-averse insurer if and only if the reinsurer is more optimistic about the loss distribution than the insurer in the sense of monotone hazard rate order.

In this paper, we study the problem of Pareto-optimal reinsurance as the result of a cooperative game. Similar to the setting in Boonen et al. (2017), we assume that the insurer and the reinsurer "agree to disagree" on the probability distribution of the underlying losses when negotiating the reinsurance policy. We first derive the Pareto-optimal reinsurance contracts, then we specify the optimal reinsurance contract corresponding to the Nash bargaining solution as well as that corresponding to the Kalai-Smorodinsky bargaining solution.

We analyze two scenarios. In the first one, the reinsurance premium is fully negotiable (as in Raviv (1979)); in the second one, the premium is determined by the reinsurer using the actuarial premium principle (as in Golubin (2006b)). Our results for the first scenario generalize those in Borch (1962) for a risk sharing setting. Our results for the second scenario generalize those in Golubin (2006a,b). A very important application of the results for the second situation is that it allows us to identify the Pareto-optimal reinsurance policies when the premium is determined by the Esscher transform principle (Young (2004)).

The rest parts of the paper are organized as follows. Section 2 presents the mathematical setting of the model. Section 3 studies the optimal policies when the premium is fully negotiable. Section 4 analyzes the optimal policies when the premium is determined by the reinsurer using the actuarial premium principle. Section 5 concludes.

4.2 The model

Let random variable X be the insurer's loss under consideration for reinsurance coverage. Assume that X can take values in the interval $[0, M]$ where $M \in [0, +\infty]$. Let the insurer's subjective probability measure of the loss X be $\mathbb{P}_1$ with the corresponding distribution function (CDF), probability density function (PDF) and expectation operator being denoted by F_1, f_1 and $\mathbf{E}_1$ respectively. Let the reinsurer's subjective probability measure of the loss X be $\mathbb{P}_2$ with the corresponding CDF, PDF and the expectation operator being denoted by F_2, f_2 and $\mathbf{E}_2$ respectively. Assume that $f_1(x) > 0$ for $x \in [0, M]$ so that the likelihood ratio $f_2(x)/f_1(x)$ is well defined on $[0, M]$.

Let the utility functions of the insurer and the reinsurer be given by $u(\cdot)$ and $v(\cdot)$ with domain D_u and D_v respectively. Assume that

$$\begin{cases} u'(x) > 0 \\ u''(x) < 0 \end{cases} \quad \text{and} \quad \begin{cases} v'(x) > 0 \\ v''(x) < 0 \end{cases}, \tag{4.1}$$

for all x in their domains. That is, both the insurer and the reinsurer are risk averse. In addition, it is assumed that

$$\lim_{x \to x_u} u'(x) = \lim_{x \to x_v} v'(x) = 0, \tag{4.2}$$

where x_u and x_v are the right end points of D_u and D_v respectively.

Let the insurer and the reinsurer's initial wealth be w_1 and w_2 respectively. Then without the reinsurance treaty, the expected utilities of the insurer and the reinsurer are $\mathbf{E}_1 [u(w_1 - X)]$ and $v(w_2)$, respectively. The point $(\mathbf{E}_1 [u(w_1 - X)], v(w_2))$ is referred to as the disagreement point, following the game theory terminology of Nash (1953) and Lemaire (1991).

A reinsurance treaty is characterized by a ceded loss function $I(x)$, which specifies the amount the reinsurer agrees to pay when the size of the loss is x, and the premium P that the insurer agrees to pay for the loss coverage.

In this paper, we assume that the set of admissible ceded functions is given by

$$C := \left\{ I : [0, M] \to [0, M] \,\middle|\, I \text{ is continuous}, I(0) = 0, 0 \le I(x) \le x \text{ for all } x \ge 0 \right\}.$$

Note that if a ceded function I belongs to set C, the corresponding retained loss function $R_I(x) = x - I(x)$ also belongs to C. Furthermore, for $I_1, I_2 \in C$, it is easy to verify that $\lambda I_1 + (1-\lambda) I_2 \in C$ for $\lambda \in [0, 1]$, therefore C is a convex set.

We assume that there is a maximal amount $\overline{P}$ the insurer is willing to spend on reinsurance contract. Since the insurer is risk averse, we have that $\overline{P} > \mathbf{E}_1[X]$.

Since a reinsurance treaty can be reached only if both parties in the transaction are better off from it, we require that all reinsurance contract to satisfy the rationality constraints

$$\begin{cases} \mathbf{E}_1[u(w_1 - X + I(X) - P)] \ge \mathbf{E}_1 [u(w_1 - X)] \\ \mathbf{E}_2[v(w_2 - I(X) + P)] \ge v(w_2) \end{cases}. \tag{4.3}$$

A reinsurance policy (I^*, P^*) is Pareto-optimal if there is no admissible policy $(I, P) \neq (I^*, P^*)$ that satisfies

$$\mathbf{E}_1 [u(w_1 - X + I(X) - P)] \ge \mathbf{E}_1 [u(w_1 - X + I^*(X) - P^*)]$$

and

$$\mathbf{E}_2 [v(w_2 - I(X) + P)] \ge \mathbf{E}_2 [v(w_2 - I^*(X) + P^*)],$$

with at least one inequality holds strict.

The main objective of this paper is to derive the set of Pareto-optimal reinsurance treaties that satisfy the rationality constraints. Section 4.3 considers the scenario where the premiums are fully negotiated; Section 4.4 studies the case when the premiums are determined by the reinsurer using the expected value premium principle.

4.3 Optimal reinsurance with negotiable premiums

When the premium is negotiated, the Pareto-optimal policies can be determined by considering the following problem (Borch, 1960b; Gerber, 1979; Gerber and Pafumi, 1998).

Problem 4.3.1 *For $k \geq 0$,*

$$\max_{I \in C, P \in [0, \overline{P}]} J(I, P) = \mathbf{E}_1[u(w_1 - X + I(X) - P)] + k\mathbf{E}_2[v(w_2 - I(X) + P)], \tag{4.4}$$

subject to the rationality constraints in (4.3).

In the above equation, the parameter $k \in [0, \infty)$ represents the reinsurer's relative negotiating power. When $k \to 0$, the reinsurer has no negotiation power and its interests are totally ignored; whereas when $k \to \infty$, the insurer's interests are ignored.

To solve Problem 4.3.1, we first disregard the constraints (4.3), and obtain the whole set of Pareto-optimal (Pareto efficient) reinsurance policies, which we denote by $\mathcal{O}$, by solving (4.4) for $k \in [0, \infty)$.

Further, let the set of reinsurance policies that satisfy the rationality constraints (4.3) by

$$\mathcal{M} := \left\{ (I, P) \in C \otimes [0, \overline{P}] \;\middle|\; \begin{array}{c} \mathbf{E}_1[u(w_1 - X + I(X) - P)] \geq \mathbf{E}_1\,[u(w_1 - X)] \\ \text{and } \mathbf{E}_2[v(w_2 - I(X) + P)] \geq v(w_2) \end{array} \right\}.$$

Then the solution to Problem 4.3.1 is given by $\mathcal{O} \cap \mathcal{M}$. If the sets $\mathcal{O}$ and $\mathcal{M}$ are disajoint, then Problem 4.3.1 has no solution.

Before solving Problem 4.3.1, we present a preliminary result.

Proposition 4.3.2 *For arbitrary $I_1, I_2 \in C$, the function $J(\lambda I_1 + (1 - \lambda)I_2, P)$ is concave with respect to $\lambda \in [0, 1]$.*

Proof Replacing the $I(x)$ in equation (4.4) by $\lambda I_1(x) + (1 - \lambda)I_2(x)$, and then taking partial derivative, one can verify that $\frac{\partial^2 J(\lambda I_1(x) + (1-\lambda)I_2(x), P)}{\partial \lambda^2} < 0$ for all $x > 0$ and $\lambda \in [0, 1]$.

4.3.1 Main results

We are now ready to present the main results for this section, The following notations are introduced. Let $LR(x) = f_2(x)/f_1(x)$ denote the likelihood ratio of the reinsurer and insurer's belief about the loss X. Let $R_u(x) = -\dfrac{u''(x)}{u'(x)}$ and $R_v(x) = -\dfrac{v''(x)}{v'(x)}$ denote the indices of absolute risk aversion of the insurer and reinsurer respectively.

A Pareto-optimal policy corresponding to a particular value of k is identified in the following two steps. First we solve

Problem 4.3.2a *For a fixed premium level $P \in [0, \overline{P}]$ and a weight parameter $k \geq 0$,*

$$\max_{I \in C} J(I, P) = \mathbf{E}_1[u(w_1 - X + I(X) - P)] + k\mathbf{E}_2[v(w_2 - I(X) + P)].$$

Secondly, let I_P^* be the solution to Problem 4.3.2a, we identify P^* that solves:

Problem 4.3.2b

$$\max_{P\in[0,\bar{P}]} J(I_P^*, P).$$

The solution I_P^* to Probelm 4.3.2a is given by the following theorem.

Theorem 4.3.3 *For a fixed premium level P and a weight parameter $k \geq 0$, let*

$$K(y,x) = u'(w_1 - x + y - P) - kv'(w_2 - y + P)LR(x),$$

and

$$d(P) := \inf\{x \geq 0 : K(0,x) \geq 0\},$$

with the understanding that $d(P) = \infty$ if the set is empty. Then Problem 4.3.2a is solved by:

$$I_P^*(x) = 0 \quad \text{for} \quad x < d(P)$$

and

$$I_P^*(x) = \min\{x, \max\{0, y_P(x)\}\} \quad \text{for} \quad x \geq d(P) \tag{4.5}$$

where $y_P(x)$ satisfies $K(y_p(x), x) = 0$. Equivalently, $y_P(x)$ satisfies the ordinary differential equation (ODE)

$$y_P'(x) = \frac{R_u(w_1 - P - x + y_P(x)) - [\log(LR(x))]'}{R_u(w_1 - P - x + y_P(x)) + R_v(w_2 + P - y_P(x))} \tag{4.6}$$

with the initial condition $y_P(d(P)) = 0$.

The proof of this theorem is given in the appendix A.

Remark 4.3.1 *Theorem 4.3.3 indicates that the optimal reinsurance policy for a fixed premium level P has a stop-loss form, where the point $d(P)$ can be regarded as the attaching point of the ceded function, above which an amount of* $\min\{x, y_P(x)\}$ *is paid by the reinsurer.*

Remark 4.3.2 *If $LR(x) = 1$, which means the insurer and the reinsurer agree on the distribution of the underlying loss, the optimal ceded function obtained in Theorem 4.3.3 reduces to the results in Raviv (1979) and Gerber and Pafumi (1998). In particular, for this case, equation* (4.6) *indicates that $0 \leq I_P^{*\prime}(x) \leq 1$. This means that for an additional dollar of underlying losses, the reinsurance recovery $I_P^*(x)$ does not exceed a dollar, which is a desirable property because it prevents moral hazards of the insurer.*

Remark 4.3.3 *When the moment generating function of f_1 exists, Esscher transform provides a convenient way to specify f_2, the PDF of the underlying losses according to the reinsurer. Specifically, we let*

$$f_2(x) = \frac{e^{hx} f_1(x)}{M_1(h)},$$

where $M_1(h) = \int_0^\infty e^{hx} f_1(x)dx$.

With $h > 0$, the reinsurer's belief of the distribution of the underlying loss is more adverse than that of the insurer; with $h < 0$, the opposite is true; with $h = 0$, the beliefs are the same of course.

For this case, the term $[\log(LR(x))]'$ in equation (4.6) equals to a constant h. From equation (4.6), it is seen that, comparing with the case of $h = 0$, a positive h leads to smaller ceded functions. This is reasonable because $h > 0$ indicates that the reinsurer's belief about the distribution of the unerlying losses is more adverse, and the room for cooperation is smaller. On the contrary, when $h < 0$, the reinsurer's belief is more favorable and the benefit of cooperation is larger. This leads to a larger ceded function.

Remark 4.3.4 *As in Ghossoub (2017), Problem 4.3.2a can be written as*

$$\max_{I\in C} \int_0^M (u(w_1 - x + I(x) - P) + kv(w_2 - I(x) + P)LR(x))\, dF_1(x),$$

Then a sufficient condition for a ceded function I_P^* to be optimal is that it pointwisely maximizes the integrand for each $x \in [0, M]$. That is, for each $x \in [0, M]$, $I_P^*(x)$ solves

$$\max_{y\in[0,x]} u(w_1 - x + y - P) + kv(w_2 - y + P)LR(x).$$

The pointwise maximization method is applied in Section 4.4.

Having obtained the optimal ceded function I_P^* for a fixed premium P, we next seek the optimal premium level P^* by considering problem 4.3.2b. Since the functional form of the ceded function I_P^* depends on P, analytical expression of P^* is difficult to obtain in general. However, explicit expressions exist for special cases and numerical solutions are always obtainable. This point is shown in examples in the following section.

Having derived the Pareto-optimal reinsurance policy $(I_{P^*}^*, P^*)$ corresponding to a weight parameter k, we can obtain the whole Pareto efficient frontier of the reinsurance policies by varying the weight parameter k. Those policies on the frontier that satisfy the constraints (4.3) are solutions to Problem 4.3.1. Note that whether a Pareto-optimal policy satisfies the constraints (4.3) can be easily checked by direct substitution.

After all Pareto-optimal reinsurance policies have been identified, we next identify the policy in the set of Pareto-optimal reinsurance policies such that the benefits of cooperation are fairly shared by the two parties. To this end, we apply the theory of Nash bargaining model (Nash, 1950) as well as the Kalai-Smorodinsky bargaining model (Kalai and Smorodinsky, 1975).

In particular, based on a set of simple and reasonable axioms, i.e., scale invariance, symmetry, Pareto efficiency, and independence of irrelevant alternatives (IIA), Nash (1950) derived that the unique solution to a two-person bargaining problem is obtained by maximizing the product of utility gains of the two parties. In our context, this means that the unique optimal reinsurance policy in the Nash bargaining model can be obtained by solving

$$\max_{(I,P)\in\mathcal{M}} \{\mathbf{E}_1[u(w_1 - X + I(X) - P)] - \mathbf{E}_1[u(w_1 - X)]\}\{\mathbf{E}_2[v(w_2 - I(X) + P)] - v(w_2)\}. \quad (4.7)$$

Note that in order to guarantee that both the insurer and the reinsurer act rationally, the reinsurance policy should belong to the set $\mathcal{M}$ and thus satisfy the rationality constrains. This point also applies when determining the Kalai-Smorodinsky bargaining solution.

The IIA assumption in the Nash bargaining model stipulates that adding another option or changing the characteristics of a third option does not affect the relative odds between the two options considered. However, this assumption has been criticized to be unrealistic in some situations. As such, in the Kalai-Smorodinsky bargaining model (Kalai and Smorodinsky, 1975), the IIA assumption is replaced by the monotonicity assumption, which states that a player with better options should get a weakly better agreement. Consequently, they proposed that the bargaining solution is the point on the efficient frontier that maintains the ratio of maximal gains. In our context, this says that the optimal reinsurance policy in the Kalai-Smorodinsky bargaining model, denoted by (I_{KS}, P_{KS}), solves

$$D_{KS} := \frac{\mathbf{E}_2\left[v(w_2 - I_{KS}(X) + P_{KS})\right] - v(w_2)}{\mathbf{E}_1\left[u(w_1 - X + I_{KS}(X) - P_{KS})\right] - \mathbf{E}_1\left[u(w_1 - X)\right]} - \frac{v_{max} - v(w_2)}{u_{max} - \mathbf{E}_1\left[u(w_1 - X)\right]} = 0, \quad (4.8)$$

where u_{max} is the maximal utility the insurer may possibly get from the treaty and v_{max} is that for the reinsurer. In our case, $u_{max} = u(w_1)$, which is the case when the insurer cedes all the losses but pays no premium; $v_{max} = v(w_2 + \overline{P})$, which is case when the reinsurer covers no losses but gets the maximal premium.

We note that the theory of Nash bargaining model and Kalai-Smorodinsky bargaining model are widely used in the economics and finance literature for determining "fair" contracts, as well as the "fair" prices, when standardized contracts do not exist and market prices are not available. For a few examples, one is referred to McElroy and Horney (1981), Neslin and Greenhalgh (1983), and more recently Zhou et al. (2015) and Boonen et al. (2017) and references therein.

As stated in the beginning of Section 4.1, reinsurance policies are usually unique and are results of negotiation of an insurer and a reinsurer. Therefore, the application of the bargaining solutions is particularly suitable. In addition, both the IIA assumption of Nash bargaining model and the monotonicity assumption of Kalai-Smorodinsky bargaining model seem to be reasonable in the reinsurance negotiating context. In fact, our numerical examples in Sections 4.3.3 and 4.4.3 show that the optimal reinsurance contracts corresponding to the two kinds of bargaining solutions are very close.

It is known that both the Nash bargaining solution and the Kalai-Smorodinsky bargaining solution locate on the Pareto efficient frontier. Therefore, they may be identified by checking which Pareto-optimal policy (corresponding to the specified value of k) solves (4.7) or (4.8). In the numerical illustrations in the next subsection, we compute and compare the optimal reinsurance policies in the two bargaining models.

4.3.2 Some examples with specific utility functions

We start by assuming that the utility functions of the insurer and the reinsurer are both quadratic.

Optimal reinsurance with quadratic utility functions

Let

$$u(x) = -\frac{1}{2}\beta_1 x^2 + x, \quad \beta_1 > 0, \quad x \leq \frac{1}{\beta_1} \quad (4.9)$$

and

$$v(x) = -\frac{1}{2}\beta_2 x^2 + x, \quad \beta_2 > 0, \quad x \leq \frac{1}{\beta_2}. \tag{4.10}$$

The up-bounds $1/\beta_1$ and $1/\beta_2$ can be regarded as saturation points beyond which utility will not increase with additional amount of monetary gains (Gerber and Pafumi, 1998). In practice, they are set to be sufficient large so that the problems are well defined. In the setting of this paper, it is required that $\frac{1}{\beta_1} > w_1$ and $\frac{1}{\beta_2} > w_2 + \bar{P}$.

With the utility functions (4.9) and (4.10), we have

$$K(0, x) = \beta_1 x + \beta_1(P - w_1) + 1 - k(1 - \beta_2(w_2 + P))LR(x).$$

Thus the reinsurance attachment point is given by

$$d(P) = \inf\left\{x \geq 0 : \beta_1 x + \beta_1(P - w_1) + 1 = k\{1 - \beta_2(w_2 + P)\}LR(x)\right\}. \tag{4.11}$$

The equation $K(y_P(x), x) = 0$ in Theorem 4.3.3 becomes

$$-\beta_1\left(w_1 - x + y_P(x) - P\right) + 1 = k\left(-\beta_2\left(w_1 - y_P(x) + P\right) + 1\right)LR(x),$$

from which we obtain

$$y_P(x) = \frac{\beta_1 x + k(\beta_2 w_2 + \beta_2 P - 1)LR(x) + \beta_1(P - w_1) + 1}{\beta_1 + k\beta_2 LR(x)}. \tag{4.12}$$

The optimal reinsurance policy is then given by $I_P^*(x) = \min\{x, \max\{0, y_P(x)\}\}$.

We note the following observations:

- with $LR(x) = 1$,

  $$y_P(x) = \frac{\beta_1}{\beta_1 + k\beta_2} \cdot x + \frac{k(\beta_2 w_2 + \beta_2 P - 1) + \beta_1(P - w_1) + 1}{\beta_1 + k\beta_2}, \tag{4.13}$$

  which shows that the optimal policy is proportional with a deductible. This result agrees with equation (84) in Gerber and Pafumi (1998). The optimal premium P^* is computed by solving Problem 4.3.2b, which yields

  $$P^* = \min\left\{\bar{P}, \max\left\{0, \frac{k(1 - \beta_2 w_2) + \beta_1 w_1 - 1}{\beta_1 + k\beta_2}\right\}\right\}. \tag{4.14}$$

- Since $1/\beta_2 > w_2 + P$, the largest amount of wealth the reinsurer may obtain, we see from (4.13) that $y_P(x)$ is decreasing in k. This suggests that more negotiation power the reinsurer has, the less coverage it will provide for the same amount of premium P. When $k \nearrow \infty$, $y_P(x) < 0$ and therefore $I_P^*(x) = 0$, indicating that the reinsurer receives the premium P and does not pay claims. In addition, we have $P^* = \bar{P}$. This is of course optimal for the reinsurer! On the contrary, when $k \searrow 0$, $y_P(x) > x$ and therefore $I_P^*(x) = x$ for all P and x. In addition the optimal premium is $P^* = 0$. This indicates a situation when the insurer cedes all losses without paying premium P, which is ideal for the insurer. These extreme situations will result in expected utilities of the two parties violating the rationality constraints (4.3). Such contracts will not be reached although they are Pareto-optimal. The situations are different when premium is determined by the actuarial premium principle, as will be discussed in Section 4.4.2.

The optimal reinsurance policies $(I_{P^*}^*, P^*)$ corresponding to different values of k give us all Pareto-optimal policies, among which those satisfying the constraints (4.3) are solutions to Problem 4.3.1. The procedure is shown in the numerical examples in Section 4.3.3.

Optimal reinsurance with exponential utility functions

Let

$$u(x) = \frac{1}{\lambda_1}\left(1 - e^{-\lambda_1 x}\right), \quad \lambda_1 \geq 0, \quad -\infty < x < \infty \tag{4.15}$$

and

$$v(x) = \frac{1}{\lambda_2}\left(1 - e^{-\lambda_2 x}\right), \quad \lambda_1 \geq 0, \quad -\infty < x < \infty. \tag{4.16}$$

For this case,

$$K(0, x) = e^{-\lambda_1(w_1 - x - P)} - k \cdot e^{-\lambda_1(w_1 + P)} LR(x)$$

and the reinsurance attaching point is determined by

$$d(P) = \inf\left\{x \geq 0 : e^{-\lambda_1(w_1 - x - P)} = k \cdot e^{-\lambda_1(w_1 + P)} LR(x)\right\}$$

The equation $K(y_P(x), x) = 0$ in Theorem 4.3.3 becomes

$$e^{-\lambda_1(w_1 - x + y_P(x) - P)} = k \cdot e^{-\lambda_2(w_2 - y_P(x) + P)} LR(x), \tag{4.17}$$

from which we obtain

$$y_P(x) = \frac{\lambda_1 x - \ln LR(x) - \lambda_1(w_1 - P) + \lambda_2(w_2 + P) - \ln k}{\lambda_1 + \lambda_2},$$

and the optimal reinsurance policy is given by $I_P^*(x) = \min\{x, \max\{0, y_P(x)\}\}$.

We note the followings:

- if $LR(x) = 1$,

  $$y_P(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot x + \frac{\lambda_2(w_2 + P) - \lambda_1(w_1 - P) - \ln k}{\lambda_1 + \lambda_2},$$

  which shows that the optimal policy is proportional with a deductible. This result agrees with equation (74) of Gerber and Pafumi (1998). The optimal premium P^* is computed by solving Problem 4.3.2b, which yields $P^* = \max\left\{0, \frac{\lambda_1 w_1 - \lambda_2 w_2 + \ln k}{\lambda_1 + \lambda_2}\right\}$.

- Similar to the case of the quadratic utility function, we observe that $y_P(x)$ is decreasing in k. In addition, when $k \nearrow \infty$, we have for finite value of x, $y_P(x) < 0$ and thus $I_P^*(x) = 0$. On the contrary, when $k \searrow 0$, $I_P^*(x) = x$.

4.3.3 Numerical illustration

This section illustrates our results by carrying out the calculations in details when the utility functions of the both parties are quadratic.

Following the setup in Section 4.3.2, we let $w_1 = 10,000$ and $w_2 = 30,000$; $\beta_1 = 0.00002$ and $\beta_2 = 0.000015$, so that the insurer is more risk aversive than the reinsurer. According to the insurer's belief, the underlying loss X follows an exponential distribution with p.d.f. $f_1(x) = \theta_1 e^{-\theta_1 x}$, $x > 0$, where $\theta_1 = 0.0005$. According to the reinsurer's belief, the underlying loss X follows an exponential distribution with paramter θ_2. To illustrate how the reinsurer's belief affects the optimal reinsurance policies, we consider three scenarios: (1) the reinsurer is more optimistic ($\theta_2 = 0.00051$); (2) the reinsurer shares the same belief as the insurer ($\theta_2 = 0.0005$); (3) the reinsurer is less optimistic ($\theta_2 = 0.000498$).

applying the results in Section 4.3.2, given a premium level P and a weight parameter k, we have

$$y_p(x) = \frac{\beta_1 x + k(\beta_2 w_2 + \beta_2 P - 1)e^{(\theta_1-\theta_2)x}\theta_2/\theta_1 + \beta_1(P - w_1) + 1}{\beta_1 + k\beta_2 e^{(\theta_1-\theta_2)x}\theta_2/\theta_1}, \tag{4.18}$$

and the optimal ceded function is $I_P^*(x) = \min\{x, \max\{0, y_P(x)\}\}$. The optimal premium P^* can be found by numerically solving Problem (4.3.2b).

Pareto efficient frontier

Figure 4.1 shows the Pareto efficient frontier with the relative negotiation power parameter k ranging from 1.51 to 1.54. Please note that utilities have no unit of practical meaning and their numerical values are only used to indicate the order of preferences. The ranges of the values of k with the corresponding reinsurance policies satisfying the rationality constraints (4.3) in the three different scenarios are:

- $\theta_1 = 0.0005$ and $\theta_2 = 0.00051$, $k \in [1.528, 1.531]$,
- $\theta_1 = \theta_2 = 0.0005$, $k \in [1.528, 1.53]$,
- $\theta_1 = 0.0005$ and $\theta_2 = 0.000498$, $k \in [1.528, 1.53]$.

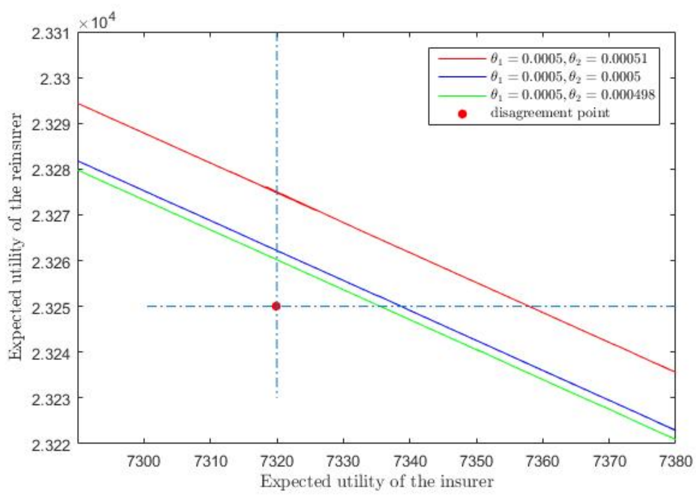


Figure 4.1: The Pareto efficient frontiers.

We observe the followings:

- The ranges of the values of k such that the rationality constraints (4.3) is satisfied are quite narrow in all three cases. Intuitively, the negotiation powers of the two parties have to be "balanced" to reach a reinsurance contract that benefit both parties.

- When the reinsurer is more optimistic than the insurer ($\theta_1 < \theta_2$), there is more room for cooperation, which results in a higher Pareto efficient frontier. On the other hand, when the reinsurer is less optimistic ($\theta_1 > \theta_2$), a lower Pareto efficient frontier is obtained. It could be foreseen that when θ_2 is much smaller than θ_1 and the reinsurer is much more pessimistic than the insurer, no policies on the Pareto efficient frontier will satisfy the constraints (4.3) and thus no reinsurance treaty will be reached.

In the next two subsections, we identify the reinsurance policy corresponding to the Nash bargaining solution and that to the Kalai-Smorodinsky bargaining solution.

The Nash bargaining solution

Since the Nash bargaining solution locates on the efficient frontier, it may be identified by checking which Pareo-optimal policy (corresponding to specified value of k) maximizes the product in (4.7). The utility products are plotted again a range of values of k in the left panel of Figure 4.2. The right panel exhibits the reinsurance policies corresponding to the Nash bargaining solutions for three cases.

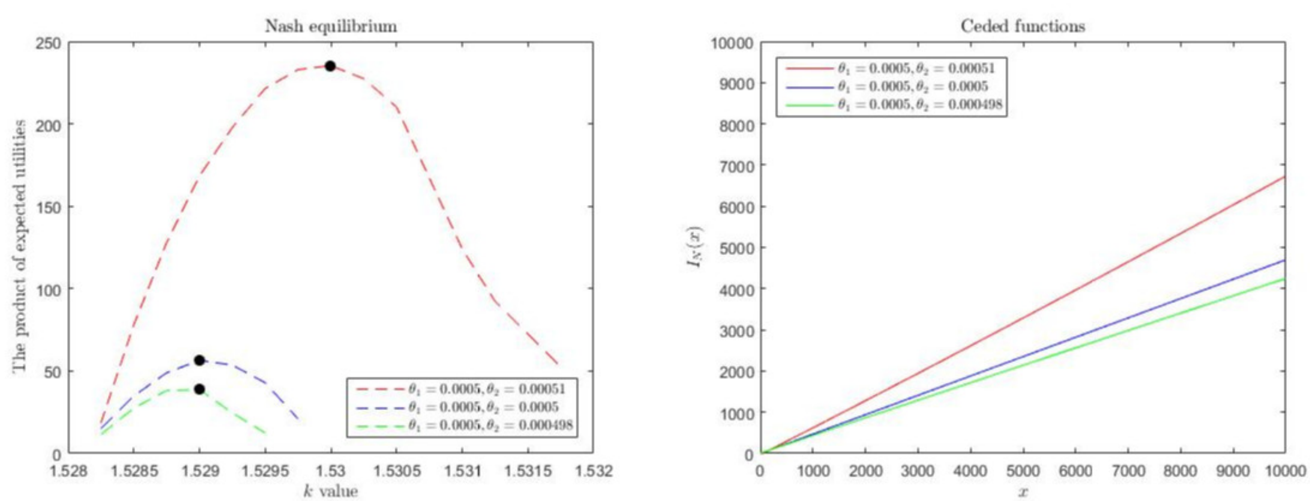


Figure 4.2: The optimal reinsurance policies corresponding to the Nash bargaining solutions

More specifically, the Nash bargaining solutions for the three cases are given by the following:

- for $\theta_1 = 0.0005$ and $\theta_2 = 0.00051$, $k = 1.53$,

$$I_N(x) = \min\left\{x, \max\left\{0, \frac{2 \times 10^{-5}x - 0.83e^{-10^{-5}x} + 0.83}{2 \times 10^{-5} + 2.3 \times 10^{-5}e^{-10^{-5}x}}\right\}\right\} \tag{4.19}$$

  and $P_N = 1311$. Notice that for non-extreme values of the loss x relative to the mean loss of 2000, Equation (4.19) can be approximated quite accurately by its first-order approximation:

$$I_N(x) \approx 0.66x.$$

- for $\theta_1 = \theta_2 = 0.0005$, $k = 1.529$,
$$I_N(x) = 0.47x$$
and $P_N = 936$;

- for $\theta_1 = 0.0005$ and $\theta_2 = 0.000498$, $k = 1.529$,
$$I_N(x) = \min\left\{x, \max\{0, \frac{2\times10^{-5}x - 0.82e^{2\times10^{-6}x} + 0.82}{2\times10^{-5} + 2.29\times10^{-5}e^{2\times10^{-6}x}}\}\right\}, \tag{4.20}$$
which can be approximated by
$$I_N(x) \approx 0.43x$$
and $P_N = 886$.

From the above expressions for the optimal ceded functions, it is clear that when the reinsurer is more optimistic than the insurer, more reinsurance coverages are agreed upon, and vise versa.

Remark 4.3.5 *Examining equation* (4.19) *carefully, one observes that for non-extreme values of x, the optimal ceded function is linear with a slope of less than one. However,* $\lim_{x\to\infty} I_N(x) = x$*, which means that all large losses are ceded. This seems reasonable because the reinsurer is more optimistic about the distribution of the losses. However, there are some range of losses where the slope of the ceded function (the marginal rate of indemnity) is greater than one, which would encourage moral hazards. One possible remedy for such problem is to simply choose the proportional policy. That is, replacing* (4.19) *by its linear approximate* $I_N(x) = 0.66x$*. The effect of such modification is small because the ceded function* (4.19) *deviates from its linear approximation* $I_N(x) = 0.66x$ *only when x is very large. In fact, we argue that this simple linear ceded function is preferable to the more complicated* (4.19) *because it not only prevents moral hazard but also reduces costs related to the handled claims.*

Remark 4.3.6 *The ceded function in* (4.20) *has an opposite problem. For non-extreme values of x, the optimal ceded function is linear with a slope of less than one. However,* $\lim_{x\to\infty} I_N(x) = 0$*, which again seems reasonable because the reinsurer is more pessimistic about the distribution of the losses and less willing to provide coverages (especially for large losses). However, when* $x > 5.64\times10^5$*, the slope of the ceded function (the marginal rate of indemnity) is negative, which again would encourage moral hazards. One possible remedy for such problem is simply to level out the ceded function for* $x > 5.64\times10^5$*. This results in a ceded function of* $I_N(x) = 0.43\times\min\{x, 5.64\times10^5\}$*. The effect of such modification is small because the probability of a loss of greater than* 5.64×10^5 *is extremely small under the insurer's or the reinsurer's belief. Again, we argue that the simpler ceded function is preferable to the more complicated one.*

Kalai-Smorodinsky bargaining model

Since the Kalai-Smorodinsky bargaining solution also locates on the Pareto efficient frontier, it may be identified by checking which Pareo-optimal policy satisfies (4.8).

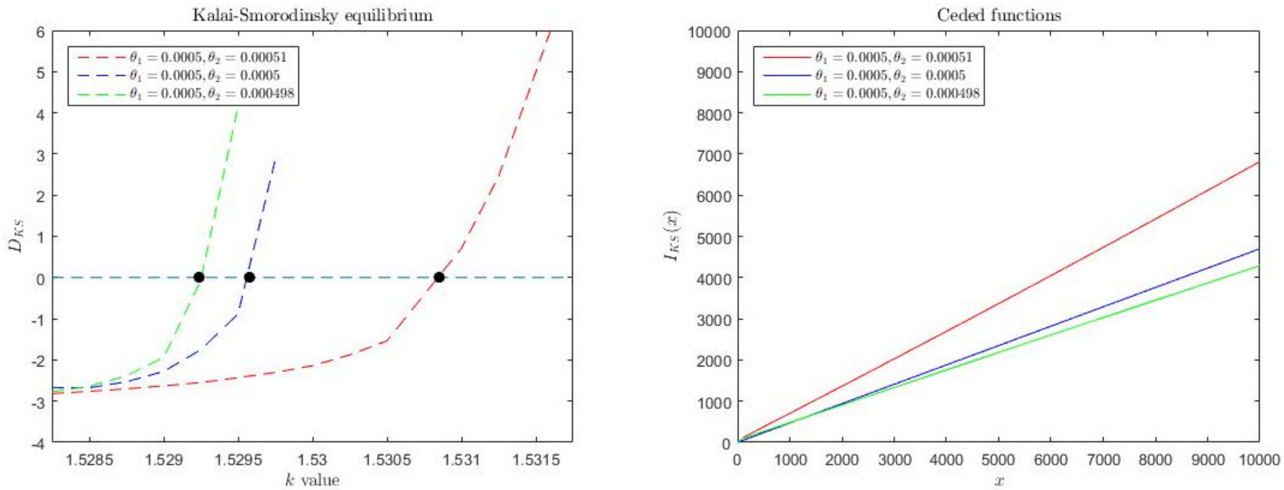


Figure 4.3: The optimal reinsurance policies corresponding to the Kalai-Smorodinsky bargaining solutions.

The values of D_{KS} for different k are plotted in the left panel of Figure 4.3. The reinsurance policies corresponding to the Kalai-Smorodinsky solution in all three cases are shown in the right panel.

The explicit reinsurance policies are:

- for $\theta_1 = 0.0005$ and $\theta_2 = 0.00051$, $k = 1.531$

$$I_{KS}(x) = \min\left\{x, \max\{0, \frac{2 \times 10^{-5}x - 0.83e^{-10^{-5}x} + 0.83}{2 \times 10^{-5} + 2.3 \times 10^{-5}e^{-10^{-5}x}}\}\right\},$$

  which can be approximated by

$$I_{KS}(x) \approx 0.66x$$

  and $P_{KS} = 1311$;

- for $\theta_1 = \theta_2 = 0.0005$, $k = 1.5295$

$$I_{KS}(x) = 0.47x$$

  and $P_{KS} = 936$;

- for $\theta_1 = 0.0005$ and $\theta_2 = 0.000498$, $k = 1.529$

$$I_{KS}(x) = \min\left\{x, \max\{0, \frac{2 \times 10^{-5}x - 0.82e^{2\times10^{-6}x} + 0.82}{2 \times 10^{-5} + 2.29 \times 10^{-5}e^{2\times10^{-6}x}}\}\right\},$$

  which can be approximated by

$$I_{KS}(x) \approx 0.43x,$$

  and $P_{KS} = 886$.

One can see that values of k corresponding to the Nash bargaining solution and the Kalai-Smorodinsky bargaining solution are very close, which leads to the same optimal reinsurance policies. It is comforting to see this agreement because both the IIA assumption of Nash (1950) and the monotonicity assumption of Kalai and Smorodinsky (1975) are reasonable in our application.

4.4 Optimal reinsurance with actuarial premiums

In this section, we obtain the Pareto-optimal reinsurance policies assuming the premium P is set by the reinsurer according to the expected value premium principle. To this end, we solve

Problem 4.4.1 *For $k \geq 0$,*

$$\max_{I\in C, P\in[0,P_m]} J(I,P) = \mathbf{E}_1[u(w_1 - X + I(X) - P)] + k\mathbf{E}_2[v(w_2 - I(X) + P)], \tag{4.21}$$

$$\text{s.t.}\ (1+\theta)\,\mathbf{E}_2\,[I(X)] = P, \quad \theta \geq 0 \tag{4.22}$$

where $P_m = (1+\theta)\mathbf{E}_2\,[X]$ is the the maximum possible premium and θ is a non-negative safety loading for the reinsurer.

Note that when f_2 is the Esscher transform of f_1, our problem becomes finding Pareto-optimal reinsurance policy with premium determined by the Esscher transform principle, which the authors believe that has not been studied in the literature.

To solve Problem 4.4.1, we again utilize a two-step optimization procedure. First, we identify the analytical form of the optimal ceded function $\tilde{I}^*_P$ when the premium P is fixed. This is, we solves

Problem 4.4.1a *For a fixed premium level P and a weight parameter $k \geq 0$,*

$$\max_{I\in C} J(I,P) = \mathbf{E}_1[u(w_1 - X + I(X) - P)] + k\mathbf{E}_2[v(w_2 - I(X) + P)],$$
$$\text{s.t.}\ (1+\theta)\,\mathbf{E}_2\,[I(X)] = P, \quad \theta \geq 0.$$

Second, we search for a P^*, in its reasonable range $(0, P_m)$ that solves

Problem 4.4.1b

$$\max_{P\in[0,P_m]} J(\tilde{I}^*_P, P). \tag{4.23}$$

Note that once all Pareto-optimal reinsurance policies are identified, those satisfy the rationality constraints (4.3) can easily be determined.

4.4.1 Main results

We now solve Problem 4.4.1a by applying a point-wise maximization method proposed in Ghossoub (2017).

Denote the term under the expectation operators in Problem 4.4.1a by

$$M(x, I(x), P) := u(w_1 - x + I(x) - P) + kv(w_2 - I(x) + P)LR(x), \tag{4.24}$$

and the corresponding Lagrange augmented object by

$$N(x, I(x), P, \lambda) := M(x, I(x), P) + \lambda\left[(1+\theta)I(x)LR(x) - P\right]. \tag{4.25}$$

Then we have

Lemma 4.4.2 *A ceded function $\tilde{I}_P^* \in C$ solves Problem 4.4.1a if there exists a $\lambda_P \in \mathbb{R}$ such that the following two conditions are satisfied*

1. *For all $I \in C$ that satisfies* (4.22), $N\left(x, \tilde{I}_P^*(x), P, \lambda_P\right) \geq N\left(x, I(x), P, \lambda_P\right)$ *for all* $x \in [0, M]$.
2. $$(1+\theta)\,\mathbf{E}_2\left[\tilde{I}_P^*(X)\right] = P. \tag{4.26}$$

The proof is given in the appendix A.

Lemma 4.4.2 in fact states that Problem 4.4.1a can be solved pointwisely by maximizing the integrands of the objective function (terms under the expectation operators) augmented by a Langrange multiplier.

To identify $\tilde{I}_P^*(x)$, we next consider the problem

$$\max_{y \in [0,x]} N(x, y, P, \lambda) \tag{4.27}$$

for a fixed x and an arbitrary λ. We use the notation $N_2(\cdot,\cdot,\cdot,\cdot)$ and $N_{22}(\cdot,\cdot,\cdot,\cdot)$ for the first and second partial derivative of N with respective to its second argument respectively, and so forth.

Due to the concavities of utility functions $u(\cdot)$ and $v(\cdot)$, we have

$$N_{22}(x, y, P, \lambda) = u''(w_1 - x + y - P) + kv''(w_2 - y + P)LR(x) < 0.$$

Thus $N(x, y, P, \lambda)$ is strictly concave in y and (4.27) must have a unique solution, which is denoted as $I^*(x; \lambda)$.

The follow two lemmas identify $I^*(x; \lambda)$. Their proofs are given in the appendix A.

Lemma 4.4.3 *The solution to problem (4.27) is given by*

$$I^*(x; \lambda) := \min\{x, \max\{0, y(x, \lambda)\}\}, \tag{4.28}$$

where $y(x, \lambda)$ is the solution to the first-order condition

$$N_2(x, y, P, \lambda) = 0. \tag{4.29}$$

Lemma 4.4.4 *For any $P \in (0, (1+\theta)\mathbf{E}_2(X))$, there exists a $\lambda_P \in \mathbb{R}$ such that*

$$(1+\theta)\,\mathbf{E}_2\left[I^*(X, \lambda_P)\right] = P.$$

Combining Lemmas 4.4.2, 4.4.3 and 4.4.4, we have proved our main result of this Section:

Theorem 4.4.5 *The solution to Problem 4.4.1a is given by*

$$\tilde{I}_P^*(x) = I^*(x;\lambda_P) = \min\left\{x, \max\{0, y(x,\lambda_P)\}\right\},$$

where λ_P is determined by $(1+\theta)\mathbf{E}_2\left[I^*(x;\lambda_P)\right] = P$.

Having obtained the optimal ceded function $\tilde{I}_P^*$ for a fixed premium P, we next seek the optimal premium level P^* by considering Problem 4.4.1b. As in Section 4.4.1, since the functional form of the ceded function $\tilde{I}_P^*$ depends on P, analytical expression of P^* is difficult to obtain in general. However, numerical solutions are always obtainable. This is shown in the examples presented in Section 4.4.3.

Now that we have obtained the Pareto-optimal policy $(\tilde{I}_{P^*}^*, P^*)$ corresponding to any negotiation weight parameter $k \geq 0$, we can obtain the whole Pareto efficient frontier of reinsurance policies. Note that whether a Pareto-optimal policy satisfies the rationality constraints (4.3) can be easily checked by direct substitution.

4.4.2 Some examples with specific utility functions

Optimal reinsurance with quadratic utility functions

Let the utility functions of the insurer and the reinsurer be quadratic and given by (4.9) and (4.10) respectively.Then equation (4.29) becomes

$$-\beta_1(w_1 - x + y(x,\lambda) - P) + 1 = \left[-k\beta_2(w_2 - y(x,\lambda) + P) + k - \lambda(1+\theta)\right] LR(x),$$

which leads to

$$y(x,\lambda) = \frac{\beta_1 x + \left(k\beta_2(w_2 + P) - k + \lambda(1+\theta)\right)LR(x) + 1 - \beta_1(w_1 - P)}{\beta_1 + k\beta_2 LR(x)}. \tag{4.30}$$

Then the optimal ceded function can be obtained by applying Theorem 4.4.5.

Several observations from (4.30) are noted:

- With $LR(x) = 1$, the optimal ceded function is a proportional reinsurance policy with a deductible, where the ceded proportion is $\frac{\beta_1}{\beta_1 + k\beta_2}$. In fact, comparing this with Equation (4.13), we see that the ceded proportion is the same as that when premium level is negotiated rather than determined by a premium principle.

- When $k = 0$ and the reinsurer's interests are not considered,

  $$y(x,\lambda) = x - w_1 + P + \frac{1}{\beta_1}\left(\lambda(1+\theta)LR(x)\right).$$

  and the optimal ceded function takes the variable deductible form

  $$I^*(x;\lambda_P) = \min\left\{x, \max\left\{0, x - d(x,\lambda_P)\right\}\right\},$$

  where $d(x,\lambda_P) = w_1 - P - \frac{1}{\beta_1}\left(\lambda_P(1+\theta)LR(x)\right)$ and λ_P is determined by $(1+\theta)\mathbf{E}_2[I^*(x;\lambda_P)] = P$. The result agrees with Theorem 4.6 in (Ghossoub, 2017).

- when $k \nearrow \infty$, it is difficult to determine the form of $y(x, \lambda)$ because λ is also a function of k. However, the numerical example in Section 4.4.3 indicates that in this case, the reinsurer in fact prefer to cover small losses. This result is in contrast with the one in Section 4.3.2, where the premiums are negotiated.

- If the reinsurer is risk neutral, i.e., $\beta_2 \to 0$, then

$$y(x, \lambda) = x - w_1 + P + \frac{1}{\beta_1}\left((\lambda(1+\theta) - k)LR(x) + 1\right).$$

  The optimal ceded function is again a variable deductible.

Optimal reinsurance with exponential utility functions

Let the utility functions of the insurer and the reinsurer be given by (4.15) and (4.16) respectively. Then, equation (4.29) becomes

$$e^{-\lambda_1(w_1 - x + y(x,\lambda) - P)} = \left[k e^{-\lambda_2(w_2 - y(x,\lambda) + P)} + \lambda(1+\theta)\right] LR(x).$$

In general, there is no analytical solution exists for the above equation, even with $LR(x) = 1$. However, we notice that

- For $k = 0$,

$$y(x, \lambda) = x - w_1 + P - \frac{1}{\lambda_1}\ln\left(\lambda(1+\theta)LR(x)\right)$$

  and the optimal policy is of variable deductible type.

- If the reinsurer is risk neutral (i.e., $\lambda_2 \to 0$ in reinsurer's exponential utility function), then

$$y(x, \lambda) = x - w_1 + P - \frac{1}{\lambda_1}\ln\left((k - \lambda(1+\theta))LR(x)\right),$$

  and the optimal policy is of variable deductible type.

4.4.3 Numerical illustration

In this Section, we apply the exact same setting as in Section 4.3.3, except that the premium is determined by

$$P = (1+\theta)\mathbf{E}_2\left[I(X)\right],$$

where the safety loading $\theta = 0.05$. For this case, applying Theorem 4.4.5 yields

$$I^*(x; \lambda_P) = \min\left\{x, \max\left\{0, y(x; \lambda_P)\right\}\right\},$$

where

$$y(x, \lambda_P) = \frac{\beta_1 x + \left(k\beta_2(w_2 + P) - k + \lambda_P(1+\theta)\right) e^{(\theta_1 - \theta_2)x}\theta_2/\theta_1 + 1 - \beta_1(w_1 - P)}{\beta_1 + k\beta_2 e^{(\theta_1-\theta_2)x}\theta_2/\theta_1}$$

and λ_P is determined by $\mathbf{E}_2\left[I^*(X; \lambda_P)\right] = \frac{P}{1+\theta}$.

Pareto efficient frontier

Figure 4.4 exhibits the Pareto efficient frontier with the negotiation weight parameter k ranging from 0 to 99. For comparison purpose, the Pareto efficient frontiers for the case when the premiums are completely negotiable are also shown with dashed lines. The ranges of k such that the rationality constraints (4.3) are satisfied are given by

- $\theta_1 = 0.005$, $\theta_2 = 0.0051$, $k \in [0.4, 1.6]$,
- $\theta_1 = \theta_2 = 0.005$, $k \in [0, 1.4]$,
- $\theta_1 = 0.005$, $\theta_2 = 0.00498$, $k \in [0, 1.3]$.

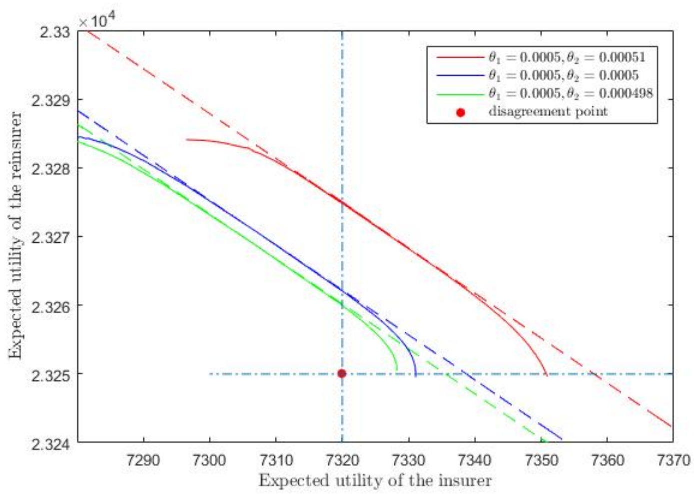


Figure 4.4: The Pareto efficient frontiers.

The following conclusions can be drawn from Figure 4.4:

- The Pareto efficient frontiers obtained when the premiums are fully negotiable dominate those when the premium are determined by the actuarial premium principle.
- Similar to Section 4.3.3, when the reinsurer is more optimistic, there are more rooms for negotiation and a higher Pareto efficient frontier is obtained, and vice versa.

Two special cases

It is interesting to observe the optimal policy when only one party's interest is considered.

Particularly, when $k = 0$, only the interests of the insurer are considered, we have

- For $\theta_1 = 0.0005$ and $\theta_2 = 0.00051$:

$$I^*(x) = \min\left\{x, \max\{0, x - 44960e^{-10^{-5}x} + 40635\}\right\},$$

which can be approximated by

$$I^*(x) \approx \min\{x, \max\{0, 1.45x - 4325\}\}.$$

- For $\theta_1 = \theta_2 = 0.0005$:

$$I^*(x) = \max\{0, x - 3813.9\}.$$

- For $\theta_1 = 0.0005$ and $\theta_2 = 0.000498$:

$$I^*(x) = \min\left\{x, \max\{0, x - 43932e^{2\times 10^{-6}x} + 40258\}\right\},$$

which can be approximated by

$$I^*(x) \approx \max\{0, 0.91x - 3674\}.$$

Remark 4.4.1 *In all cases, the optimal policies include a sizable deductible. When the parties have homogeneous belief ($\theta_1 = \theta_2 = 0.0005$), the reinsurer provide full coverage after the deductible. This result agrees with the classical result in Arrow (1973). When the reinsurer is more optimistic about the loss distribution, it provides more than full coverage after the deductible (marginal indemnity rate greater than one), which might make sense with the "agree to disagree" assumption. However, this policy will encourage moral hazards of the insurer. On the contrary, When the reinsurer is less optimistic about the loss distribution, it will cover less than full coverage after the deductible.*

When $k \to \infty$, only the interests of the reinsurer are considered. For this case, the resultant ceded functions are identical for the three values of θ_2. It is given by

$$I^*(x) = \min\{x, 3472.3\},$$

indicating that the reinsurer would cover small losses if it overpowers the insurer in the negotiation.

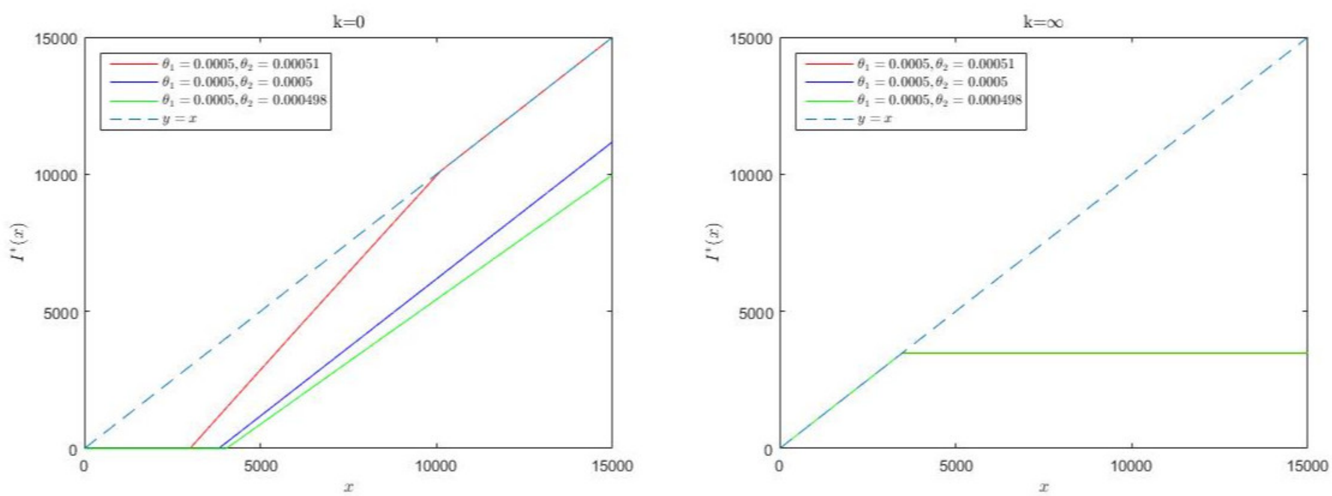


Figure 4.5: The optimal reinsurance policies corresponding to $k = 0$ and $k = \infty$.

Remark 4.4.2 *The optimal reinsurance policies corresponding to $k \searrow 0$ and $k \nearrow \infty$ obtained in this section are rather different from those in Section 4.3.2, where the premiums are negotiated. They seem to be more reasonable, however, such policies still violate the rationality constraints* (4.3) *and would not be agreed upon in practice.*

Nash bargaining model

The reinsurance policy corresponding to the Nash bargaining solution may be identified by evaluating which Pareo-optimal policy maximizes the product in (4.7).

For different values of k, the product of expected utility gains (4.7) is shown in the left panel of Figure 4.6. The corresponding ceded functions are shown in the right panel.

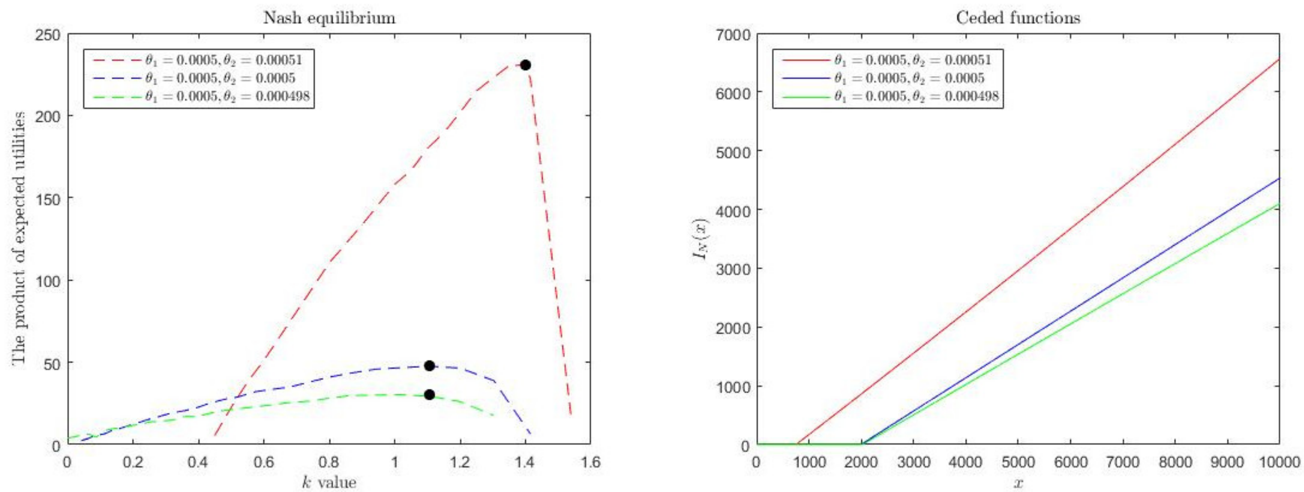


Figure 4.6: The optimal reinsurance policies corresponding to the Nash bargaining solutions

In particular, the explicit reinsurance policies are given by:

- for $\theta_1 = 0.0005$ and $\theta_2 = 0.00051$, $k = 1.39$,

$$I_N(x) = \min\left\{x, \max\{0, \frac{2\times 10^{-5}x - 0.84e^{-10^{-5}x} + 0.82}{2\times 10^{-5} + 2.14\times 10^{-5}e^{-10^{-5}x}}\}\right\}, \tag{4.31}$$

  which can be approximated by

$$I_N(x) \approx \max\{0, 0.69x - 483\}, \tag{4.32}$$

  and $P_N = 979$;

- for $\theta_1 = \theta_2 = 0.0005$, $k = 1.11$,

$$I_N(x) = \max\{0, 0.57x - 1127\}$$

  and $P_N = 418$;

- for $\theta_1 = 0.0005$ and $\theta_2 = 0.000498$, $k = 1.11$,

$$I_N(x) = \min\left\{x, \max\{0, \frac{2\times 10^{-5}x - 0.85e^{2\times 10^{-6}x} + 0.81}{2\times 10^{-5} + 1.53\times 10^{-5}e^{2\times 10^{-6}x}}\}\right\}, \tag{4.33}$$

  which can be approximated by

$$I_N(x) \approx \max\{0, 0.52x - 1050\} \tag{4.34}$$

  and $P_N = 398$.

Remark 4.4.3 *Similar to Section 4.3.3, we observe that more coverage is provided when the reinsurer is more optimistic, and vice versa.*

Remark 4.4.4 *It is seen from* (4.31) *and* (4.33)*, for certain range of extreme values of loss x, the marginal indemnity rate is greater than one when the reinsurer is more optimistic; and it is negative when the reinsurer is more pessimistic. Both situations may encourage moral hazards. For such situations, similar to remarks 4.3.5 and 4.3.6, we propose that the ceded functions in* (4.31) *and* (4.33) *should be replaced by the much simpler versions in* (4.32) *and* (4.34).

Kalai-Smorodinsky bargaining model

Similar to Section 4.3.3, the reinsurance policy corresponding to the Kalai-Smorodinsky bargaining solution may be identified by evaluating which Pareo-optimal policy satisfies (4.8). Different from Section 4.3.3, the maximal utility gains u_{max} and v_{max} are obtained by solving Problem 4.4.1 with $k = 0$ and $k = \infty$ respectively. The values of D_{KS} for different k are plotted in left panel of Figure 4.7. The reinsurance policies corresponding to the KS bargaining solutions are plotted in the right panel.

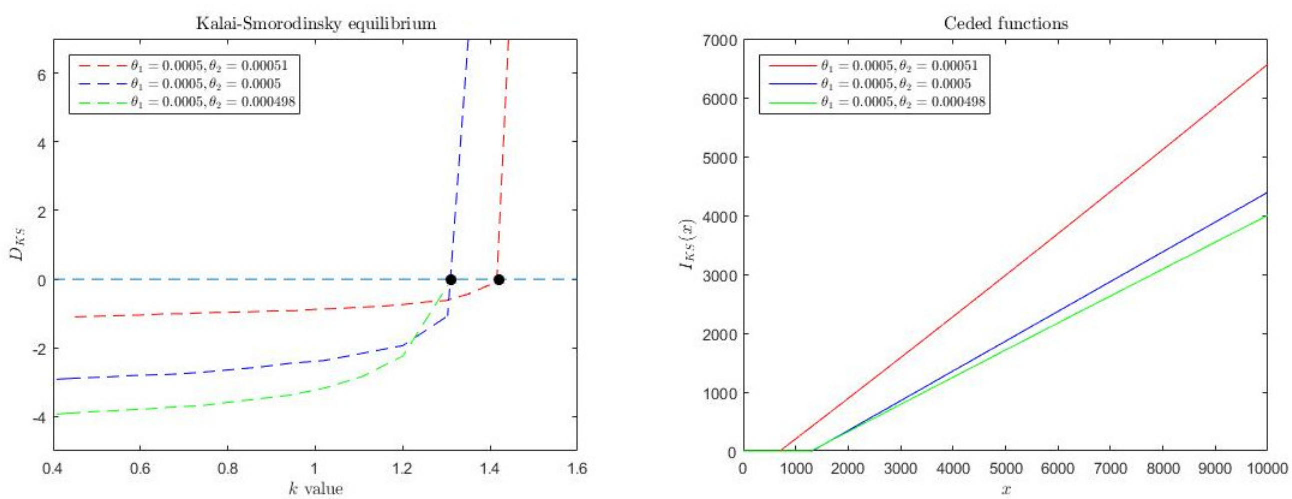


Figure 4.7: The optimal reinsurance policies corresponding to the Kalai-Smorodinsky bargaining solutions

The explicit expressions for the reinsurance policies are:

- for $\theta_1 = 0.0005$ and $\theta_2 = 0.00051$, $k = 1.41$,

$$I_{KS}(x) = \min\left\{x, \max\{0, \frac{2\times 10^{-5}x - 0.84e^{-10^{-5}x} + 0.82}{2\times 10^{-5} + 2.17\times 10^{-5}e^{-10^{-5}x}}\}\right\},$$

  which can be approximated by

$$I_{KS} \approx \max\{0, 0.68x - 456\},$$

  and $P_{KS} = 973$;

- for $\theta_1 = \theta_2 = 0.0005$, $k = 1.31$,

$$I_{KS}(x) = \max\{0, 0.51x - 665\}$$

  and $P_{KS} = 534$;

- for $\theta_1 = 0.0005$ and $\theta_2 = 0.000498$, $k = 1.31$,

$$I_N(x) = \min\left\{x, \max\{0, \frac{2 \times 10^{-5}x - 0.83e^{2\times10^{-6}x} + 0.81}{2 \times 10^{-5} + 1.95 \times 10^{-5}e^{2\times10^{-6}x}}\}\right\},$$

  which can be approximated by

$$I_{KS} \approx \max\{0, 0.46x - 608\},$$

  and $P_N = 512$.

Remark 4.4.5 *It is observed from the numerical examples that the reinsurance policies corresponding to the the Nash and Kalai-Smorodinsky bargaining solutions are rather similar.*

Remark 4.4.6 *From figures 4.2, 4.3, 4.6 and 4.7, it can be seen that the Nash bargaining solution and the Kalai-Smorodinsky solution are unique. In fact, the set of of attainable utility levels resulting from all possible reinsurance policies are likely to be convex, which would guarantee the uniqueness of both solutions. However, the proof of the convexity of the attainable utility levels of the two parties is out of the main focus of the paper and therefore not pursued.*

Comparison of the ceded functions: negotiated premiums vs. actuarial premiums

Analytical comparison of the optimal ceded functions when the premiums are negotiated vs. when the premiums are determined by a preset actuarial principle are difficult. However, our numerical results in Sections 4.3.3 and 4.4.3 seem to indicate that when the premiums are negotiated, the optimal policy form is proportional with no deductible, whereas when the premiums are determined by the expectation premium principle with preset risk loading, the optimal policy form is proportional after sizable deductibles.

4.5 Conclusions

We considered the optimal reinsurance policies as the result of a two-person cooperative game when the two negotiating parties have different beliefs about the distribution of the underlying losses. We first derive all Pareto-optimal reinsurance policies and then identify the reinsurance contract corresponding to the Nash bargaining solution as well as that corresponding to the Kalai-Smorodinsky bargaining solution. In addition, we provide explicit solutions for the optimal policies in our numerical examples, which complements many deep theoretical treatments of this topic in the economics and insurance literature.

We have assumed that the negotiation parties "agree to disagree" on the distribution of the underlying losses. In practice, neither party can be certain about the loss distribution. Therefore, it is important to consider model and parameter uncertainties in contract negotiations.

Decision making under uncertainty is a huge research area and many results exist in the economics literature (Klibanoff et al., 2005; Alary et al., 2013; Gollier, 2014). In the context of optimal reinsurance, Asimit et al. (2017) suggested that one could seek robust Pareto-optimal reinsurance policies when there are the model and parameter uncertainties; Asimit and Boonen (2018) studied the set of Pareto-optimal insurance contracts and the core of an insurance game with multiple insurers when there exist model uncertainties and the insurers have divergent beliefs about the model uncertainties. In future research, it would be very interesting to study the reinsurance negotiation equilibriums under model and parameter uncertainties.

Chapter 5

Reinsurance Policies with Maximal Synergy

5.1 Introduction

A reinsurerace contract is a mechanism for redistributing risks between an insurer and a reinsurer. It is characterized by a pair $(P, I(X))$, where P is the reinsurance premium and $I(X)$ is the ceded function specifying the amount the insurer will get indemnified when it suffers a loss of size X covered by the reinsurance contract.

Extensive results exist for the "optimal" reinsurance policy in the ecomomics and insurance literature. Two types of optimality criteria are commonly used: maximizing the expected utility (EU) or minimizing risks. Classical results on policies that maximize the insurer's expected utility can be found in Borch (1962), Arrow (1963), Arrow (1974) and Raviv (1979). Results on policies that minimize the insurer's risks, measured by variance, the Value ar Risk (VaR), the Tail Value at Risk (TVaR) and general distortion risk measure, are available in Borch (1960a); Aase (2002); Cai and Tan (2007); Assa (2015); Zhuang et al. (2016), and the references therein. Policies that maximize EU under the VaR contraint was studied by Bernard and Tian (2010).

When negotiating a reinsurance contract, the interests of both the insurer and the reinsurer are considered. Therefore, the optimal policies should live in the set of Pareto-optimal reinsurance policies, where one party's expected utility (risk) cannot be increased (reduced) further without reducing (increasing) that of the other party. Results on Pareto-optimal policies that maximize EU can be found in, for example, Borch (1962); Raviv (1979); Gerber and Pafumi (1998); Golubin (2006b); Aase (2009); Results on Pareto-optimal policies that minimize risks can be found in, for example, Cai and Tan (2007); Cai et al. (2017); Jiang et al. (2017, 2018); Asimit and Boonen (2018).

To identify an unique policy from the set of Pareto-optimal policies, one could consider the competitive equilibrium or some bargaining solution in the context of game theory. The optimal policy corresponding to the competitive equilibrium is a policy in the Pareto-optimal set, where the price is determined by the market such that the demand and supply of reinsurance are equal (market is clear). For results about optimal reinsurance in the competitive equilibrium within the framework of EU maximization, see for example, Borch (1962) and Gerber and Pafumi (1998). In the risk minimizing framework, see Embrechts et al. (2018), which identified an

Arrow-Debreu competitive equilibrium when the risks are measured by the so-called Range-Value-at-Risk (RVaR).

The optimal policy corresponding to a bargaining solution is a policy in the Pareto-optimal set, where the benefits of cooperation is distributed among the negotiating parties in accordance to some rationality axioms. For example, Borch (1960c) first identified the set of Pareto-optimal reinsurance policies that maximize the EU of both parties, then an unique policy is determined by making use of Nash's solution for bargaining games. That is, the axioms of "invariant to affine transformations", "independence of irrelevant alternatives", and "symmetry" are assumed. Kihlstrom and Roth (1982b) studied the effects of the insurance buyer's risk aversion on the bargaining outcomes. Much more recently Boonen (2016a) studied the Nash bargaining solution for insurance risk redistribution by assuming the set of admissible policies are regulated so that both parties can benefit from the reinsurance transaction. For results about optimal reinsurance in the bargaining solution within the framework of risk minimization, see for example Asimit and Boonen (2018). An insightful and comprehensive review of reinsurance policy in competitive equilibrium versus bargaining solution was given by Aase (2009).

The optimal reinsurance policy obtained through maximizing the EU and that obtained through minimizing risks are in general different. In practice, insurance companies are likely to consider both EU and risk constraints when negotiating reinsurance policies. One approach to consider both is to maximize EU under some risk constraints, as was done in Bernard and Tian (2010). An alternative approach was in fact used in Borch (1960b), which assumed that the admissible reinsurance policies (in maximizing EU) should be such that the total risk is minimized (the reduction in risk through the reinsurance transaction is maximized). It is further assumed that both the insurer and the reinsurer use variance as their risk measure, therefore, to minimize the total risk, the covariance of the payments of the two parties should be maximized, which necessitate quota share treaties. Among the admissible quote share policies, the set of Pareto-optimal policies that maximize the EU of the two parties are determined. Lastly the Pareto-optimal policy corresponding to the Nash bargaining solution is identified. We note that the criterion of minimizing was further discussed in Hürlimann (2011).

In this paper, we follow the approach in Borch (1960b). However, we assume that the two parties apply distortion risk measures instead of the variance. Both the criteria based on total variance and on total distortion risk measures intend to minimize the total risk in the system. However, the former focuses on the unexpected fluctuations of losses, the later focuses on the solvency issue. Minimizing the total risk in the system is important from the societal point of view, it is also essential in designing "internal reinsurance", where the total distortion risk measure (e.g. VaR or TVaR) determines the solvency capital that should be reserved by the firm before splitted between the fronting company and the reinsurance captive.

In our analysis, we first identify a set of reinsurance policies that minimize the total risk shared by the two parties, then we take this set of policies as admissible and determine the Pareto-optimal policies that maximize the EU of the two parties. Our approach is also somewhat similar to that in Boonen (2016a), where Nash bargaining solutions are determined in some prior-determined set of feasible policies. This approach is also related to the maximal synergy risk sharing (Section 9 in Gerber and Pafumi (1998)), in the sense that we only consider policies that minimize the total risk (synergy maximizing) when maximizing EU. From now on, we refer to the reinsurance policies that minimize the total risk as the synergy-maximizing reinsurance policies.

The remainder of this paper is organized as follows. Section 5.2 reviews the basic concepts of distortion risk measure and describes the objective function of this study. Section 5.3 determines the set of synergy-maximizing reinsurance policies. Section 5.4 identifies the set of Pareto-optimal policies that maximize the EU of the two parties assuming that only the synergy-maximizing policies are admissible. In addition, the policy corresponding to the Nash bargaining solution is determined. Section 5.5 considers the optimal policies when additional risk constraints are imposed. Section 5.6 provides numerical examples. Section 5.7 concludes.

5.2 Background and model formulation

Since this paper assumes that the insurer and and the reinsurer measure their risks by distortion risk measures, we begin by introducing some definitions and notions of distortion risk measures.

5.2.1 Distortion risk measures

The distortion risk measure of a non-negative loss random variable X with distribution function F_X and survival function S_X is defined as $H_g(X) = \int_0^\infty g\,(S_X(x))\,dx$, where the distortion function $g : [0,1] \to [0,1]$ is non-decreasing and satisfies $g(0) = 0$ and $g(1) = 1$ (Denuit et al., 2006). A distortion risk measure has the properties of translation invariance, positive homogeneity, monotonicity and comonotonic additivity (Wang et al., 1997). In addition, it is coherent if and only if the distortion function is concave (Wirch and Hardy, 2001). For detailed discussions of comonotonic random variables and distortion risk measures, see, for example, Dhaene et al. (2002a,b); Balbás et al. (2009) and the references therein.

Two widely used distortion risk measures are Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR). The VaR of X at confidence level α is defined as

$$VaR_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x : F_X(x) \geq \alpha\}, \tag{5.1}$$

where the corresponding distortion function is given by

$$g_{V,\alpha}(x) = \begin{cases} 0, & 0 \leq x < 1-\alpha, \\ 1, & 1-\alpha \leq x \leq 1. \end{cases} \tag{5.2}$$

The TVaR of X at confidence level α is defined by

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_t(X)dt \tag{5.3}$$

with the corresponding distortion function

$$g_{T,\alpha}(x) = \begin{cases} \dfrac{x}{1-\alpha}, & 0 \leq x < 1-\alpha, \\ 1, & 1-\alpha \leq x \leq 1. \end{cases} \tag{5.4}$$

We next describe the objective of this paper.

5.2.2 The Objective function

Let the insurer and the reinsurer have initial wealth w_1 and w_2 and adopt distortion risk measures with distortion functions g_1 and g_2 respectively. Let the underlying loss random variable that is being considered for a reinsurance contract be denoted by X with support $[0, M]$, where $M \leq \infty$. Let the ceded function be denoted by $I(x)$ and the premium be determined by

$$P = (1 + \theta)\mathbf{E}[I(X)]. \tag{5.5}$$

Then with a reinsurance contract characterized by the pair $(P, I(x))$ the insurer's total loss is $L_I = X - I(X) + P$ and the reinsurer's total loss is $L_R = I(X) - P$.

We assume that the set of admissible reinsurance policies is given by

$$C := \left\{ I : [0, M] \to [0, M] \;\middle|\; \begin{array}{l} 0 \leq I(x) \leq x \text{ for all } x \geq 0, \\ 0 \leq I(x_1) - I(x_2) \leq x_1 - x_2 \text{ if } 0 \leq x_2 \leq x_1 \end{array} \right\}.$$

With $I \in C$, both the ceded loss $I(X)$ and the retained loss $X - I(X)$ are non-decreasing with respect to X, so they are comonotonic. The functions belonging to C are 1-Lipschitz continuous and therefore differentiable almost everywhere (Chi and Tan, 2011). These properties make the set of functions a reasonable choice for the admissible set because they satisfy the principle of insurance indemnity and prevent moral hazard.

By the comonotonic additivity of distortion risk measures (Dhaene et al., 2002a), we have for a distortion function g (Ludkovski and Young, 2009; Cheung and Lo, 2017),

$$H_g(X) = H_g(I(X)) + H_g(X - I(X))$$

and

$$H_g(I(X)) = \int_0^\infty g(S_X(x))dI(x). \tag{5.6}$$

Inspired by Borch (1960b), the goal of this paper is to seek Pareto-optimal reinsurance contracts that maximize the EU of the two parties within the admitted set of ceded functions:

$$C_g =: \arg\min_{I \in C} H_{g_1}(L_I) + H_{g_2}(L_R). \tag{5.7}$$

Denote the utility functions of the insurer and reinsurer by u and v respectively. Assume that they are non-decreasing and concave. In addition, assume that

$$\lim_{x\to\infty} u'(x) = \lim_{x\to\infty} v'(x) = 0 \tag{5.8}$$

and

$$\lim_{x\to-\infty} u'(x) = \lim_{x\to-\infty} v'(x) = \infty \tag{5.9}$$

Since a reinsurance treaty can be reached only if both parties in the transaction are better off from it, we require that the EU of the two parties do not decrease because of the reinsurance contract. This is, the optimal policies should satisfy the the rationality constraints

$$\begin{cases} \mathbf{E}_1[u(w_1 - X + I(X) - P)] \geq \mathbf{E}_1\,[u(w_1 - X)] \\ \mathbf{E}_2[v(w_2 - I(X) + P)] \geq v(w_2) \end{cases}. \tag{5.10}$$

Note that the point $(\mathbf{E}[u(w_1 - X)], v(w_2))$ corresponding to the two parties' utilities without reinsurance contract is referred to as the disagreement point in game theory literature (Nash, 1953; Lemaire, 1991).

It is known that in order to determine the Pareto-optimal policies that maximize the EU of the two parties, one could maximize the weighted average of the EU. Therefore, our main problem becomes

Problem 5.2.1 (Main problem)

$$\max_{I \in C_g} \quad \mathbf{E}\left[u(w_1 - X + I(X) - P)\right] + k\mathbf{E}\left[v(w_2 - I(X) + P)\right], \tag{5.11}$$
$$s.t. \quad \textit{rationality constraints } (5.10).$$

Note that the parameter k in the objective function can be interpreted as the relative negotiation power of the reinsurer.

In next section, we characterize the reinsurance contracts in set C_g when both parties apply VaR or TVaR as their risk measures.

5.3 The set of synergy-maximizing policies

Value-at-Risk

Suppose that the insurer and the reinsurer adopt VaR with probability level α_c and α_r respectively to measure their risks. Then the set of synergy-maximizing policies is given by

$$\begin{aligned} &\arg\min_{I \in C} \quad VaR_{\alpha_c}(L_I) + VaR_{\alpha_r}(L_R) \\ =&\arg\min_{I \in C} \quad VaR_{\alpha_c}(X - I(X) + P) + VaR_{\alpha_r}(I(X) - P). \end{aligned}$$

Because of the translation invariance property of the distortion risk measures, the above set is equivalent to

$$\arg\min_{I \in C} \quad VaR_{\alpha_c}(X - I(X)) + VaR_{\alpha_r}(I(X)).$$

In addition,

$$VaR_{\alpha_c}(X - I(X)) = VaR_{\alpha_c}(X) - VaR_{\alpha_c}(I(X)).$$

Further, because $I(x)$ is nondecreasing, $VaR_\alpha(I(X)) = I(VaR_\alpha(X))$. Therefore, the above set simplifies to

$$\arg\min_{I \in C} \quad I(a_r) - I(a_c), \tag{5.12}$$

where $a_c = VaR_{\alpha_c}(X)$ and $a_r = VaR_{\alpha_r}(X)$.

The solution to Problem (5.12) is discussed in the following.

Case 1: $\alpha_c > \alpha_r$

Because $0 \leq I(x) \leq x$, a ceded function, denoted by $I_{V_1}(x)$, solves (5.12) if and only if it has a slope of one in $[a_r, a_c]$. That is,

$$I'_{V_1}(x) = \begin{cases} 1, & x \in [a_r, a_c], \\ \eta(x), & x \notin [a_r, a_c], \end{cases} \tag{5.13}$$

where $\eta(x) \in [0, 1]$ is any function such that $I_{V_1}(x) \in C$. Intuitively, when $\alpha_c \geq \alpha_r$, the losses in layer $[a_r, a_c]$ contribute to the insurer's VaR, but not to that of the reinsurer. So they should be ceded.

Let the set of functions that satisfy (5.13) be denoted by C_{V_1}. Then C_{V_1} is the solution to Problem (5.12). Further, let the set of functions in C_{V_1} that satisfies $I_{V_1}(a_r) = b_1$ for some constant $b_1 \in [0, a_r]$ be denoted by C_{V_1,b_1}. Then upper and lower bounds of C_{V_1,b_1}, denoted by $\overline{I}_{V_1,b_1}(x)$ and $\underline{I}_{V_1,b_1}(x)$ respectively, are given by the following.

- $\overline{I}_{V_1,b_1}(x) = \{x \wedge b_1\} + (x - a_r)_+$.
- $\underline{I}_{V_1,b_1}(x) = (x - (a_r - b_1))_+ \wedge (b_1 + a_c - a_r)$.

A graphical illustration is given in figure 5.1.

Case 2: $\alpha_c < \alpha_r$

Because $0 \leq I(x) \leq x$, The solutions for (5.12), $I_{V_2}(x)$, has slope zero in $[a_c, a_r]$. Therefore, we have

$$I'_{V_2}(x) = \begin{cases} 0, & x \in [a_c, a_r], \\ \eta(x), & x \notin [a_c, a_r], \end{cases} \tag{5.14}$$

where $\eta(x) \in [0, 1]$ is any function such that $I_{V_2}(x) \in C$.

Intuitively, when $\alpha_c \leq \alpha_r$, the losses in the layer $[a_c, a_r]$ contribute to the reinsurer's VaR, but not to the insurer's. Therefore, losses in the layer should be retained.

In this case, the set of functions that satisfy (5.14), denoted by C_{V_2} constitutes the solution to Problem (5.12). Further, let the set of functions in C_{V_2} that satisfies $I_{V_2}(a_c) = b_2$ for some constant $b_2 \in [0, a_c]$ be denoted by C_{V_2,b_2}. Then upper and lower bounds of C_{V_2,b_2}, denoted by $\overline{I}_{V_2,b_2}(x)$ and $\underline{I}_{V_2,b_2}(x)$ respectively, are given by the following.

- $\overline{I}_{V_2,b_2}(x) = \{x \wedge b_2\} + (x - a_r)_+$.
- $\underline{I}_{V_2,b_2}(x) = (x - (a_c - b_2))_+ \wedge b_2$.

Case 3: $\alpha_c = \alpha_r$

In this case, the reduction in total risk due to the reinsurance policy is always zero for every ceded function in C.

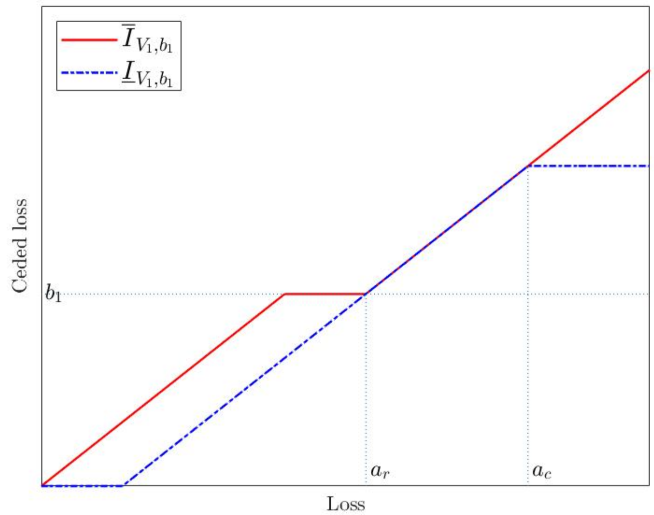


Figure 5.1: An illustration of the upper and lower bounds of C_{V_1,b_1}.

Tail Value-at-Risk

Suppose that the insurer and the reinsurer adopt TVaR, with probability levels with α_c and α_r respectively, to measure their risks. Then the set of synergy-maximizing policies are determined by

$$\begin{aligned} &\arg\min_{I \in C} \quad TVaR_{\alpha_c}(L_I) + TVaR_{\alpha_r}(L_R), \\ = &\arg\min_{I \in C} \quad TVaR_{\alpha_r}(I(X)) - TVaR_{\alpha_c}(I(X)), \end{aligned} \tag{5.15}$$

due to the comonotonic additivity property of TVaR.

By (5.6),

$$\begin{aligned} &\min_{I \in C} \quad TVaR_{\alpha_r}(I(X)) - TVaR_{\alpha_c}(I(X)) \\ = &\min_{I \in C} \quad \int_0^\infty (g_{T,\alpha_r}(S_X(x)) - g_{T,\alpha_c}(S_X(x)))dI(x), \end{aligned} \tag{5.16}$$

whose solutions are obtained in the following.

Case 1: $\alpha_c > \alpha_r$

From (5.4), it is easy to see that

$$g_{T,\alpha_r}(S_X(x)) < g_{T,\alpha_c}(S_X(x)) \quad \text{for } x > a_r$$

and

$$g_{T,\alpha_r}(S_X(x)) = g_{T,\alpha_c}(S_X(x)) \quad \text{for } x \le a_r$$

Thus the solution to Problem (5.16), denoted by $I_{T_1}(x)$, satisfy

$$I'_{T_1}(x) = \begin{cases} 1, & x \in [a_r, \infty), \\ \eta(x), & x \in [0, a_r], \end{cases} \tag{5.17}$$

where $\eta(x) \in [0, 1]$ is any function such that $I_{T_1}(x) \in \mathcal{C}$.

Let the set of functions that satisfy (5.17) be denoted by $\mathcal{C}_{T_1}$. Then it is the solution to Problem (5.15). Further, let the set of functions in $\mathcal{C}_{T_1}$ that satisfies $I_{T_1}(a_r) = b_1$ for some constant $b_1 \in [0, a_r]$ be denoted by $\mathcal{C}_{T_1,b_1}$. Then upper and lower bounds of $\mathcal{C}_{T_1,b_1}$, denoted by $\overline{I}_{T_1,b_1}(x)$ and $\underline{I}_{T_1,b_1}(x)$ respectively, are given by the following.

- Upper bound: $\overline{I}_{T_1,b_1}(x) = (x \wedge b_1) + (x - a_r)_+$.
- Lower bound: $\underline{I}_{T_1,b_1}(x) = (x - (a_r - b_1))_+$.

A graphical illustration is given in Figure 5.2.

Case 2: $\alpha_c < \alpha_r$

In this case, $g_{T,\alpha_r}(S_X(x)) > g_{T,\alpha_c}(S_X(x))$ for $x > a_c$ and $g_{T,\alpha_r}(S_X(x)) = g_{T,\alpha_c}(S_X(x))$ otherwise. Thus, the solutions to Problem (5.15), denoted by $I_{T_2}(x)$ is given by

$$I'_{T_2}(x) = \begin{cases} 0, & x \in [a_c, \infty), \\ \eta(x), & x \in [0, a_c], \end{cases} \tag{5.18}$$

where $\eta(x) \in [0, 1]$ is any function such that $I_{T_2}(x) \in \mathcal{C}$.

The set of functions that satisfy (5.18), denoted by $\mathcal{C}_{T_2}$, constitute the solution to problem (5.15). Further, let the set of functions in $\mathcal{C}_{T_2}$ that satisfies $I_{T_2}(a_r) = b_2$ for some constant $b_2 \in [0, a_c]$ be denoted by $\mathcal{C}_{T_2,b_2}$. Then upper and lower bounds of $\mathcal{C}_{T_2,b_2}$, denoted by $\overline{I}_{T_2,b_2}(x)$ and $\underline{I}_{T_2,b_2}(x)$ respectively, are given by the following.

- Upper bound: $\overline{I}_{T_2,b_2}(x) = x \wedge b_2$.
- Lower bound: $\underline{I}_{T_2,b_2}(x) = (x - (a_c - b_2))_+ \wedge b_2$.

Case 3: $\alpha_c = \alpha_r$

In this case, the reduction in total risk due to the reinsurance policy is always zero for every ceded function in $\mathcal{C}$.

Remark 5.3.1 *It can be easily verified that $\mathcal{C}_{T_1} \subset \mathcal{C}_{V_1}$ and $\mathcal{C}_{T_2} \subset \mathcal{C}_{V_2}$. This means that the TVaR synergy-maximizing policies are also VaR synergy-maximizing. In other words, the TVaR requirement is more stringent than the VaR requirement.*

Remark 5.3.2 *Note that $\alpha_c > \alpha_r$ means that the insurer is more risk averse than the reinsurer, which is more common in the reinsurance negotiation setting. Therefore, to simplify presentation, this will be assumed in the following discussions.*

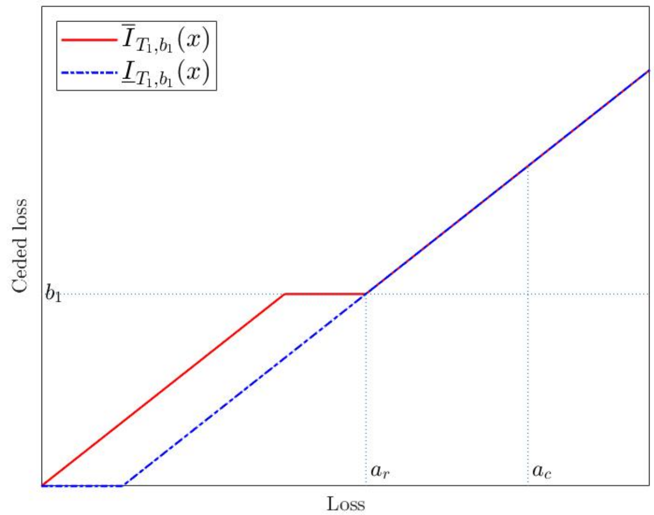


Figure 5.2: Illustrative upper bound and lower bound for $\alpha_c \geq \alpha_r$ under TVaR.

5.4 Pareto-optimal policies

In this section we derive the optimal reinsurance policy that solves **Problem** 5.2.1 when the risk measures are VaR or TVaR.

As stated in Remark 5.3.2, we will assume that $\alpha_c > \alpha_r$, so that insurer is more risk averse than the reinsurer. The analysis for the case of $\alpha_c < \alpha_r$ is similar and so omitted in the paper. Within this scenario and considering the results in Section 3, we next solve Problem 5.2.1 with C_g replaced by C_{V_1} for the VaR case and by C_{T_1} for the TVaR case.

The methods to solve the above two problems are very similar, so our presentation will focus on the VaR case.

Our strategy to solve the problem is to first fix the premium P and $I(a_r) = b$ and determine the functional form of ceded function in $[0, a_r]$ and $[a_c, \infty]$. Then we search for the best P and b.

Specifically, we first derive the solution $I^*_{b,P}$ to

Problem 5.4.0a (VaR synergy-maximizing policy with fixed b and P)

$$\max_{I \in C_{V_1,b}} \quad \mathbf{E}\left[u(w_1 - X + I(X) - P)\right] + k\mathbf{E}\left[v(w_2 - I(X) + P)\right],$$

where the ceded function also satisfies $P = (1+\theta)\mathbf{E}[I(X)]$.

Then we seek (b^*, P^*) that solves

Problem 5.4.0b (Optimal parameters)

$$\max_{P,b} \quad \mathbf{E}\left[u(w_1 - X + I^*_{b,P}(X) - P)\right] + k\mathbf{E}\left[v(w_2 - I^*_{b,P}(X) + P)\right],$$
$$s.t. \quad \textit{rationality constraints}\ (5.10),$$

where $b \in [0, a_r]$ *and* $P \in [\underline{P}_{V_1,b}, \overline{P}_{V_1,b}]$ *with*

$$\underline{P}_{V_1,b} = (1+\theta)\mathbf{E}[\underline{I}_{V_1,b}(X)]$$

and

$$\overline{P}_{V_1,b} = (1+\theta)\mathbf{E}[\overline{I}_{V_1,b}(X)]$$

being the minimum and maximum possible premiums in this scenario.

Theorem 5.4.1 *The solution to Problem 5.4.0a is given by*

$$I^*_{b,P}(x) = \overline{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee y(x,\lambda)\right\}, \tag{5.19}$$

where $y(x,\lambda)$ *is the solution to*

$$u'(w_1 - x + y(x,\lambda) - P) = k \cdot v'(w_2 - y(x,\lambda) + P) - \lambda(1+\theta), \tag{5.20}$$

and the Lagrange multiplier coefficient λ *is such that* $(1+\theta)\mathbf{E}\left[I^*_{b,P}(x)\right] = P$.

The proof of Theorem 5.4.1 is given in the appendix B.

Remark 5.4.1 *If TVaR is the risk measure, the optimal solution can be obtain by modifying Theorem 5.4.1 slightly by replacing* $\overline{I}_{V_1,b}$ *and* $\underline{I}_{V_1,b}$ *with* $\overline{I}_{T_1,b}$ *and* $\underline{I}_{T_1,b}$ *respectively. The search range for Problem 5.4.0b changes accordingly to* $[\underline{P}_{T_1,b}, \overline{P}_{T_1,b}]$, *where*

$$\underline{P}_{T_1,b} = (1+\theta)\mathbf{E}[\underline{I}_{T_1,b}(X)]$$

and

$$\overline{P}_{T_1,b} = (1+\theta)\mathbf{E}[\overline{I}_{T_1,b}(X)].$$

Remark 5.4.2 *Taking derivative with respective to x on both sides of the equation (5.20), one gets*

$$y'(x,\lambda) = \frac{u''(w_1 - x + y(x,\lambda) - P)}{u''(w_1 - x + y(x,\lambda) - P) + kv''(w_2 - y(x,\lambda) + P)}, \tag{5.21}$$

which is the same as the equation (6) in Golubin (2006a). In addition, from (5.21)*, it is seen that the slope of the ceded function is between 0 and 1.*

Without the synergy-maximizing requirement, or equivalently when the admissible set of ceded function is $\mathcal{C}$ *with the upper bound* $\overline{I}^*_{\mathcal{C}}(x) = x$ *and the lower bound* $\underline{I}^*_{\mathcal{C}}(x) = 0$. *Then the optimal reinsurance takes the form*

$$I^*(x,\lambda) = x \wedge \{0 \vee y(x,\lambda)\},$$

where $y(x,\lambda)$ *is the solution of equation (5.20).*

Remark 5.4.3 *For* $k = 0$,

$$y(x,\lambda) = x - w_1 + P + [u']^{-1}(\lambda(1+\theta)),$$

where $[u']^{-1}(\cdot)$ *is the inverse function of* $u'(\cdot)$. *Therefore,*

$$I^*_{b,P}(x) = \overline{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x), x - w_1 + P + [u']^{-1}(\lambda(1+\theta))\right\}.$$

One can see that the optimal ceded function in this case is piecewise linear.

Having determined the optimal form of ceded function by making use of Theorem 5.4.1, one can next seek the optimal parameter values b^* and P^* by solving Problem 5.4.0b. Since the functional form of the ceded function implicitly depends on P and b, analytical expressions are difficult to obtain in general. However, numerical solutions may be obtained because it is a maximization problem over two real parameters. We will illustrate this in the numerical example provided in Section 5.6.

Having derived the Pareto-optimal reinsurance policy $I^*_{b^*,P^*}(x)$ corresponding to a weight parameter k, we can obtain the whole Pareto efficient frontier of the reinsurance policies by varying the weight parameter k. Those policies on the frontier that satisfy the constraints (5.10) are solutions to Problem 5.2.1. Note that whether a Pareto-optimal policy satisfies the constraints (5.10) can be easily checked by direct substitution.

5.4.1 Optimal policies as the Nash bargaining solutions

In many situations, it is desirable to identify an "optimal" reinsurance policy from a set of Pareto-optimal policies. Therefore, we next identify the policy such that the benefits of cooperation are "fairly" shared by the two parties. To this end, we apply the Nash bargaining model (Nash, 1950).

Based on a set of simple and reasonable axioms: scale invariance, symmetry, Pareto efficiency, and independence of irrelevant alternatives, Nash (1950) proposed that the unique solution to a two-person bargaining problem is obtained by maximizing the product of utility gains of the two parties. In our context, this means that the unique optimal reinsurance policy in the Nash bargaining model can be obtained by solving

$$\max_{I \in C_g} \{\mathbf{E}_1\left[u(w_1 - X + I(X) - P)\right] - \mathbf{E}_1\left[u(w_1 - X)\right]\}\{\mathbf{E}_2\left[v(w_2 - I(X) + P)\right] - v(w_2)\}. \quad (5.22)$$

It is known that the Nash bargaining solution locates on the Pareto efficient frontier. Therefore, it may be identified by checking which Pareto-optimal policy (corresponding to different values of k) solves (5.22). A numerical example is presented in Section 5.6.

5.5 Optimal policies with additional risk constraints

In this section, we study the optimal reinsurance policies when the following additional risk constraints are imposed on the two parties:

$$H_{g_1}(L_I) \le L_1, \quad H_{g_2}(L_R) \le L_2. \quad (5.23)$$

We derive the results for the cases when VaR and TVaR are risk measures in the following.

Value-at-Risk

Because the ceded functions $I \in C$ are 1-Lipschitz continuous and VaR satisfies the properties of translation invariance and the commonotonic additivity, we have

$$\begin{aligned} & VaR_{\alpha_c}(X - I(X) + P) \le L_1 \\ \iff \quad & P \le L_1 + I(a_c) - a_c, \end{aligned}$$

and

$$\begin{aligned} & VaR_{\alpha_r}(I(X) - P) \leq L_2 \\ \iff & \quad P \geq I(a_r) - L_2. \end{aligned}$$

Recall that for $I \in C_{V_1,b}$, $I(a_r) = b$ and $I(a_c) - I(a_r) = a_c - a_r$. The above two inequalities become

$$P \in [b - L_2, b + L_1 - a_r]. \tag{5.24}$$

Consequently, the optimal ceded function with the new VaR constraints can be obtained by using the results in Section 4 with very slight modification. That is, we change the search range for P (in Problem 5.4.0b) from $[\underline{P}_{V_1,b}, \overline{P}_{V_1,b}]$ to

$$[\underline{P}_{V_1,b}, \overline{P}_{V_1,b}] \cap [b - L_2, b + L_1 - a_r],$$

with the understanding that the problem has no viable solution if the above set is empty.

Tail Value-at-Risk

In this case, let $t_c = TVaR_{\alpha_c}(X)$, $t_r = TVaR_{\alpha_r}(X)$ and $a_s = VaR_s(X)$ for $0 \leq s \leq 1$. Then

$$\begin{aligned} & TVaR_{\alpha_c}(X - I(X) + P) \leq L_1 \\ \iff & \quad P \leq L_1 + \tfrac{1}{1-\alpha_c} \int_{\alpha_c}^{1} I(a_s) ds - t_c, \end{aligned}$$

and

$$\begin{aligned} & TVaR_{\alpha_r}(I(X) - P) \leq L_2 \\ \iff & \quad P \geq \tfrac{1}{1-\alpha_r} \int_{\alpha_r}^{1} I(a_s) ds - L_2. \end{aligned}$$

Recall that for $I \in C_{T_1,b}$, $I(a_r) = b$ and $I(x) = b + x - a_r$ for $x > a_r$. Therefore, the above two inequalities become

$$P \in [b - L_2 + t_r - a_r, b + L_1 - a_r]. \tag{5.25}$$

Analogue to the VaR case, the optimal ceded function with the additional TVaR risk constraints can be obtained by changing the search range for P (in Problem 5.4.0b) from $[\underline{P}_{T_1,b}, \overline{P}_{T_1,b}]$ to

$$[\underline{P}_{T_1,b}, \overline{P}_{T_1,b}] \cap [b - L_2 + t_r - a_r, b + L_1 - a_r].$$

Remark 5.5.1 *Comparing* (5.24) *and* (5.25)*, we observe that the allowable range of P is narrower in the TVaR case than in the VaR case. This verifies that the TVaR constraints are more stringent than the VaR constraints.*

5.6 Numerical examples

Suppose that the utility functions of the insurer and the reinsurer are given by

$$u(x) = -\frac{1}{2}\beta_1 x^2 + x, \quad x \leq \frac{1}{\beta_1},$$

and

$$v(x) = -\frac{1}{2}\beta_2 x^2 + x, \quad x \leq \frac{1}{\beta_2}.$$

Then solving equation (5.20) yields

$$y(x,\lambda) = \frac{\beta_1 x + k\beta_2(w_2 + P) - k + \lambda(1+\theta) + 1 - \beta_1(w_1 - P)}{\beta_1 + k\beta_2},$$

and the optimal reinsurance policy is given by (5.19). One see that the optimal ceded function is piecewise linear, with slope being either one, zero, or $\frac{\beta_1}{\beta_1 + k\beta_2}$.

More specifically, we next provide numerical solutions to the problem with the following assumptions.

- The insurer and the reinsurer have initial wealth $w_1 = \$10000$ and $w_2 = \$30000$.
- The parameters for quadratic utility functions are $\beta_1 = 0.00002$ and $\beta_2 = 0.000015$, so that the insurer is more risk averse than the reinsurer.
- The insurer and the reinsurer apply VaR (TVaR) as risk measures with probability level $\alpha_c = 0.95$ and $\alpha_r = 0.9$ respectively.
- The underlying loss X follows an exponential distribution with mean \$2000. Then $a_c = VaR_{\alpha_c}(X) = \$5991.5$ and $a_r = VaR_{\alpha_r}(X) = \$4605.2$.

5.6.1 Pareto efficient frontier

We solve Problem 5.2.1 with the admissible ceded functions given by the sets C, C_{V_1} and C_{T_1} and obtain the optimal policies, denoted by I^*, I_V^* , I_T^* respectively, corresponding to the three levels of synergy-maximizing: no constraint, VaR constraint and TVaR constraint. These were done for a range of values of negotiation weight parameter k, thus the Pareto efficient frontiers are obtained. The resultant Pareto efficient frontiers are shown in Figure 5.3. The following observations are noted:

- The Pareto efficient frontier becomes lower and lower in the order of none, VaR, and TVaR synergy-maximizing requirement. This is not surprising because the admissible sets have the relationship $C \supset C_{V_1} \supset C_{T_1}$. Intuitively, TVaR synergy-maximizing requirement restrict the form of ceded function quite significantly and this sacrifices the two parties' EU. Understandably, increasing (quadratic) EU and minimizing risk (VaR, TVaR) can be contradictory objectives and one has to strike a balance.

- the range of values of k so that the rationality constraints (5.10) are satisfied is approximately $[0, 1.4]$ in all three cases. Paticularly, $k = 0$ corresponds to the case where the reinsurer has no negotiation power, which results in the EU of the two parties be at the lower right corner of the plot.

To illustrate the effects of synergy-maximizing constraints on the level of risks, the total risk (in terms of VaR and TVaR respectively) of the two parties corresponding to the optimal policies I^*, I_V^* , I_T^* are shown in Figures 5.4 and 5.5. Two observations are noted:

- TVaR synergy-maximizing policies are also VaR synergy-maximizing. This is reasonable because $C_{V_1} \supset C_{T_1}$.
- Without the synergy-maximizing requirement, the total risk increase with the reinsurer's negotiation power k. This is because we have assumed the reinsurer is more risk tolerant ($\alpha_r < \alpha_c$), maximizing its EU will increase the total risk in the system.

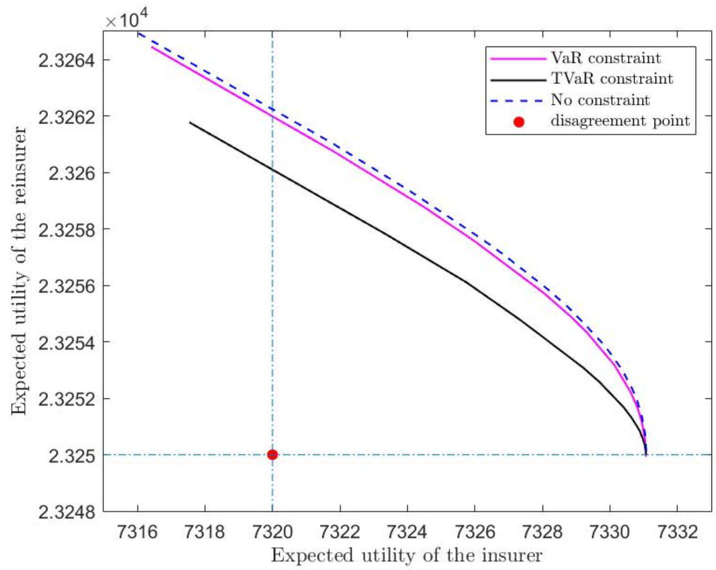


Figure 5.3: EU Pareto efficient frontier.

5.6.2 Nash bargaining solution

To identify an unique "optimal" policy among the Pareto-optimal policies, we adopt the Nash bargaining solution (5.22). Because the Nash solution is on the efficient frontier, it can be numerically determined by seeking the best parameter k so that (5.22) is maximized.

Specifically, the optimal policies are as follows:

- No constraint: $k = 1.1$, $P^* = \$496.08$, $I^*(x) = 0.55(x - 1682.9)_+$.

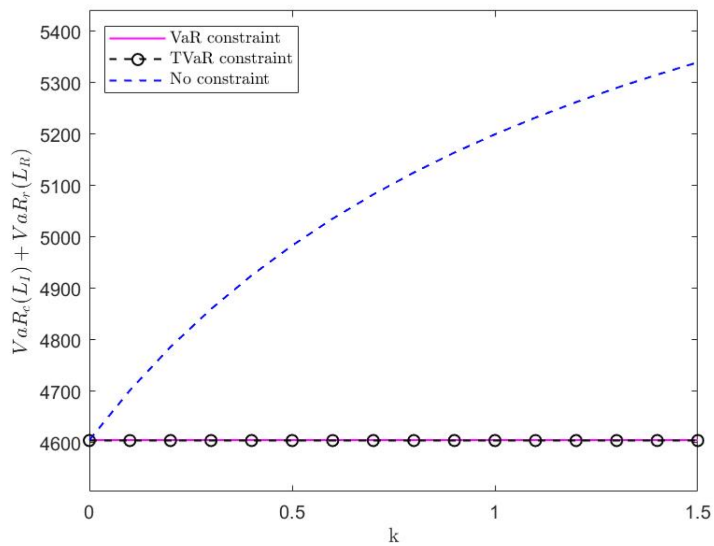


Figure 5.4: The total VaR of the insurer and the reinsurer

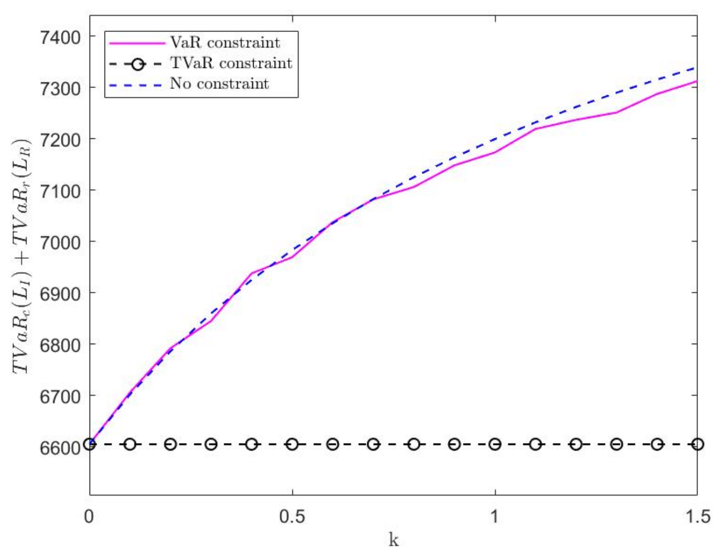


Figure 5.5: The total TVaR of the insurer and the reinsurer

- VaR constraint: $k = 1.1$, $P^* = \$474.98$,

$$I_V^*(x) = \begin{cases} 0.55(x - 1770.9)_+ \wedge 1305.9, & x \in [0, 4605.2], \\ x - 3299.3, & x \in [4605.2, 5991.5], \\ 0.55(x - 1770.9)_+ \vee 2692.2, & x \in [5991.5, \infty). \end{cases} \tag{5.26}$$

- TVaR constraint: $k = 1.3$, $P^* = \$575.83$,

$$I_T^*(x) = \begin{cases} 0.51(x - 1216.1)_+ \wedge 1012, & x \in [0, 4605.2], \\ x - 3585.2, & x \in [4605.2, \infty). \end{cases} \tag{5.27}$$

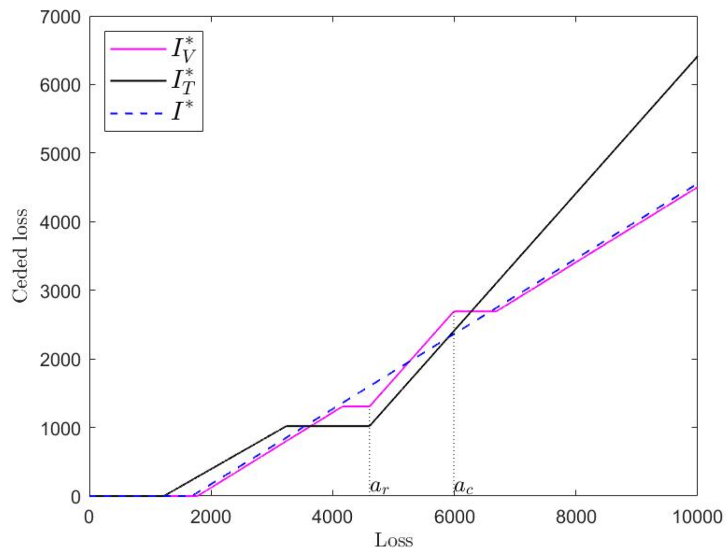


Figure 5.6: Pareto-optimal reinsurance policies corresponding to the Nash bargaining solutions

The ceded functions are plotted in Figure 5.6. We note the following observations:

- Under VaR constraint, the obtained optimal reinsurance policies are quite close to the one obtained without constraint. This happens when the values of α_c and α_r are close. In fact, as commented at the end of Section 3.1, with $\alpha_c = \alpha_r$, no reinsurance policy in the set C can reduce the total risk in the system, in which case $C_{V_1} = C$.
- The TVaR synergy-maximizing constraint (for $\alpha_c > \alpha_r$, even when the values are very close) requires that all the losses greater than a_r are ceded (see Eq. (5.17)). This deviates rather significantly from the policy for EU maximization without constraint (or when $\alpha_c = \alpha_r$), which requires that about 55% of losses are ceded after a deductible of about 1682.9. This results in lower Pareto efficient frontier, as shown in Figure 5.3.

5.6.3 Optimal policies under additional risk constraints

Additional VaR constraint

We assume that an additional risk constraint is imposed on the insurer such that

$$VaR_{\alpha_c}(X - I(X) + P) \leq L_1,$$

where L_1 is $0.6 \times VaR_{\alpha_c}(X) = \$3594.9$ for illustrative purpose. The optimal policy (5.26) derived in Section 5.6.2 results in $VaR_{\alpha_c}(X - I_V^*(X) + P) = \$3774.3$, which violates the imposed risk constraint. Therefore, we adjust the search range for premium as discussed in Section 5.5 and obtain the optimal policy as follows:

$$I_{V_a}^*(x) = \begin{cases} 0.55(x - 1770.9)_+ \wedge 1512.4, & x \in [0, 4605.2], \\ x - 3092.8, & x \in [4605.2, 5991.5], \\ 0.55(x - 1770.9)_+ \vee 2898.7, & x \in [5991.5, \infty). \end{cases} \tag{5.28}$$

The premium for this policy is \$497.21. The ceded function $I^*_{V_a}(x)$ is compared with $I^*_V(x)$ in (5.26) in Figure 5.7. It is seen that $I^*_{V_a}(x)$ provides more coverage for losses around the layer $[a_r, a_c]$, resulting in lower VaR of the insurer. Of course, the premium increases from \$475 for $I^*_V(x)$ to \$497 for $I^*_{V_a}(x)$.

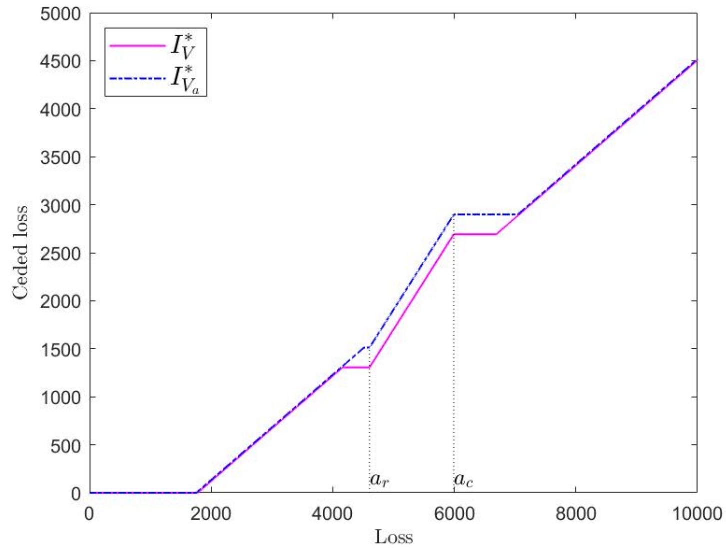


Figure 5.7: Pareto-optimal reinsurance policy corresponding to the Nash bargaining solution with additional VaR constraint of the insurer.

Additional TVaR constraint

Now assume that an additional risk constraint is imposed on the insurer such that $TVaR_{\alpha_c}(X - I(X) + P) \leq L_1$, where L_1 is again set to be $0.6 \times VaR_{\alpha_c}(X) = \$3594.9$. The optimal policy I^*_T in (5.27) results in $TVaR_{\alpha_c}(X - I(X) + P) = \$4161.1$. Thus the additional risk constraint is violated. Therefore, we adjust the search range for premium as discussed in Section 5.5 and obtain the optimal policy as follows:

$$I^*_{T_a}(x) = \begin{cases} 0.53(x - 1484.8)_+ \wedge 1632.7, & x \in [0, 4605.2], \\ x - 2972.5, & x \in [4605.2, \infty). \end{cases} \tag{5.29}$$

The premium for this policy is \$622.39. The two policies are shown in Figure 5.8. As expected, the policy $I^*_{T_a}$ in (5.29) covers more losses in the right tail than I^*_T in (5.27).

5.7 Conclusions

In this paper, we study the Pareto-optimal reinsurance design considering two optimality criteria: EU maximization and risk minimization. We first identify a set of reinsurance policies that minimize the total risk shared by the two parties, then we take this set of policies as admissible and determine the Pareto-optimal policies that maximize the EU of the two parties. The policy

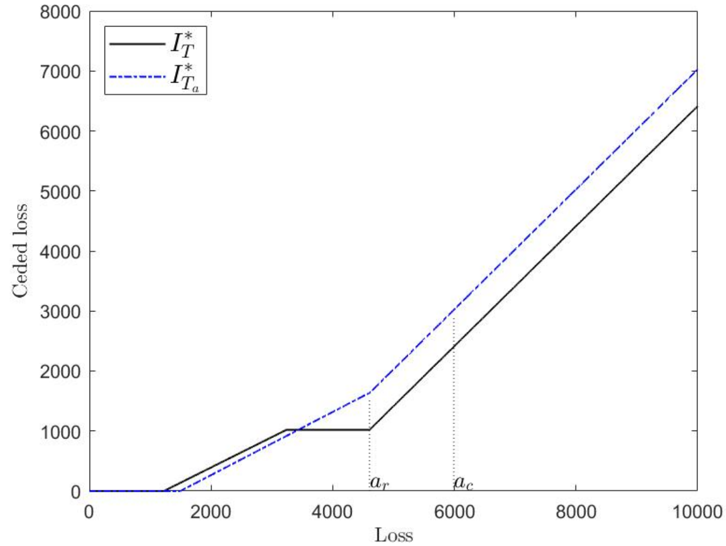


Figure 5.8: Pareto-optimal reinsurance policy corresponding to the Nash bargaining solution with additional TVaR constraint of the insurer.

corresponding to the Nash bargaining solution is identified. In addition, we characterize the optimal policy when additional risk constraints are imposed to the two parties.

Chapter 6

Summary and future research

6.1 Summary

In this thesis, we studied the design of Pareto-optimal reinsurance policy within risk minimization and EU maximization framework. First, a simple model constructed under VaR was studied by applying a geometric approach. Second, considering that VaR belongs to the family of distortion risk measure, we studied a new model under general distortion risk measures together with some additional individual risk constraints. The model solution could be obtained by using Lagrange dual method or generalized Neyman-Pearson lemma or dynamic programming approach. Third, we studied a model by maximizing the weighted average of the EUs under heterogeneous beliefs, which complements the early works done within risk minimization framework. Finally, we blend the risk minimization and EU maximization to study a new reinsurance policy which is named synergy-maximizing policy.

This thesis comprises the following major research contributions: (i) the second chapter extends the existing geometric approach to a game-theoretical setting and shows the effectiveness of this approach in solving such problem if quantile-based risk measures are involved in the model; (ii) the third chapter solves the problem when a much more general risk measure is being applied; (iii) the fourth chapter provides implicit solutions to EU-based problem and identifies the “best” solution located on the Pareto efficient frontier; (iv) the fifth chapter blends different criteria and derives the reinsurance policy which takes care of different aspects in a decision-making process.

6.2 Future research

A natural extention is following the works done by Asimit et al. (2017); Asimit and Boonen (2018) to design the optimal reinsurance policy in a setting which involves multiple insurers or multiple reinsurers.

Another extention is to adjust the admissible set of ceded functions. So far, the most popular admissible set is the set of 1-Lipschitz continuous functions and most scholars believe that if the ceded function belongs to this set the insurer has no incentive to overreport or underreport the loss. However, it is understandable that there exist other ways to eliminate the ex post moral hazard, such as adding a state-verification cost. The state-verifiation cost is a cost shared between the insurer and the reinsurer in some manner such that the actual loss of the insurer is verified by a third party. If adopting such assumption, more ceded functions can be taken into consideration, which may greatly simplify the process to seek the solution. Besides, other assumption such as Vajda condition could be made to reflect the spirit of the reinsurance, which means more proportion should be shared by the reinsurer if the actual loss is larger. All these assumptions can influence the final solution to different extents.

Moreover, quite recently the model ambiguity or model uncertainty has drawn considerable attention. Traditionally, people apply the worst case analysis to derive the so called robust reinsurance policy. In this way, the decision maker is treated as extremely ambiguity averse. However, different decision makers may have different levels of aversion towards the model ambiguity. A novel model (KMM model) of quantifying the ambiguity aversion was proposed by Klibanoff et al. (2005) and got applied in many insurance related areas (Alary et al., 2013; Gollier, 2014; Robert and Therond, 2014). This model can split the ambigutiy aversion and the risk aversion and thus could provide more interpretations for the obtained results. The KMM model can serve as a new frame within which new reinsurance policies can be designed to reflect the decision maker's ambiguity aversion and risk aversion seperately and simultaneously.

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Appendix A

Proofs of Theorems in Chapter 4

Proof of Theorem 4.3.3: Following (Golubin, 2006a,b), we solve Problem 4.3.2a by applying the calculus of variations. Suppose that $I_P^* \in \mathcal{C}$ is the solution to Problem 4.3.2a and $I \in \mathcal{C}$ is arbitrary admissible ceded function, then since the set $\mathcal{C}$ is convex, any convex combination of these two ceded functions $\lambda I_P^* + (1-\lambda)I$, where $\lambda \in [0,1]$, belongs to $\mathcal{C}$. Moreover, the function $J(\lambda I_P^* + (1-\lambda)I, P)$ is maximized at $\lambda = 1$. With Proposition 4.3.2, this indicates that a sufficient and necessary condition for $I_P^* \in \mathcal{C}$ to solve Problem 4.3.2a is

$$\left.\frac{\partial J(\lambda I_P^* + (1-\lambda)I, P)}{\partial \lambda}\right|_{\lambda=1} \geq 0,$$

which leads to

$$\int_0^M K(I_P^*(x), x) I_P^*(x) dF_1(x) \geq \int_0^M K(I_P^*(x), x) I(x) dF_1(x). \tag{A.1}$$

In order for (A.1) to be satisfied for all $I \in \mathcal{C}$, we must have that

$$I_P^*(x) = \begin{cases} 0, & \text{if } K(I_P^*(x), x) < 0 \\ x, & \text{if } K(I_P^*(x), x) > 0. \end{cases} \tag{A.2}$$

Hence if $0 < d(P) < \infty$, then $I_P^*(x) = 0$ for all $x < d(P)$. Otherwise, if $d(P) = \infty$, then $I_P^*(x) \equiv 0$ for all $x \geq 0$, implying that no reinsurance agreement is reached.

In addition, for $x > d(P)$, let $y_P(x)$ be such that $K(y_P(x), x) = 0$ and $0 \leq y_P(x) \leq x$, then $I_P^*(x) = y_P(x)$.

Therefore, we construct the optimal solution to Problem 4.3.2a as

$$I_P^*(x) = \begin{cases} 0, & y_P(x) < 0 \\ y_P(x), & 0 \leq y_P(x) \leq x \\ x, & y_P(x) > x \end{cases}. \tag{A.3}$$

To ensure that (A.3) agrees with (A.2), we next verify that

$$\{x : x > d(P), K(I_P^*(x), x) > 0\} = \{x : x > d(P), y_P(x) > x\}, \tag{A.4}$$

and

$$\{x : x > d(P), K(I_P^*(x), x) < 0\} = \{x : x > d(P), y_P(x) < 0\}. \tag{A.5}$$

Firstly, If there exists $x_0 > d(P)$ such that $y_P(x_0) > x_0$, then by (A.3) we have $I_P^*(x_0) = x_0$ and thus

$$\begin{aligned}K(I_P^*(x_0), x_0) &= u'(w_1 - P) - kv'(w_2 - x_0 + P)LR(x_0)\\ &> u'(w_1 - x_0 + y_P(x_0) - P) - kv'(w_2 - y_P(x_0) + P)LR(x_0)\\ &= 0.\end{aligned}$$

Secondly, if there exists $x_0' > d(P)$ such that $K(I_P^*(x_0'), x_0') > 0$, then we must have that $I_P^*(x_0') < y_P(x_0')$ because $K(y_P(x_0'), x_0') = 0$ and the function K is decreasing in its first argument. Since $I_P^*(x_0') = x_0'$ by (A.2), we have $x_0' < y_p(x_0')$.

Therefore, (A.4) is proved. Equation (A.5) can be proved similarly. Consequently, we conclude that the optimal ceded function is given by (A.3), or equivalently, (4.5).

Lastly, differentiate both sides of $K(y_P(x), x) = 0$ with respective to x yields (4.6). The initial condition is due to the assumed continuity of the ceded function. This completes the proof. ∎

Proof of Lemma 4.4.2: If $N\left(x, \tilde{I}_P^*(x), P, \lambda_P\right) \geq N(x, I(x), P, \lambda_P)$ for arbitrary feasible solution $I \in C$ that satisfies (4.22), then we have for all $x \geq 0$

$$M\left(x, \tilde{I}_P^*(x), P\right) - M(x, I(x), P) \geq \lambda(1+\theta)LR(x)\left(\tilde{I}_P^*(x) - I(x)\right).$$

Thus

$$\begin{aligned}J(\tilde{I}_P^*, P) - J(I, P) &= \int_0^M \left(M\left(x, \tilde{I}_P^*(x), P\right) - M(x, I(x), P)\right) dF_1(x)\\ &\geq \lambda(1+\theta)\Big[\mathbf{E}_2[\tilde{I}_P^*(X)] - \mathbf{E}_2[I(X)]\Big] = 0.\end{aligned}$$

This completes the proof. ∎

Proof of Lemma 4.4.3: Firstly, notice that

$$N_2(x, y, P, \lambda) = u'(w_1 - x + y - P) - kv'(w_2 - y + P)LR(x) + \lambda(1+\theta)LR(x).$$

It is continuous in y and satisfies

$$\lim_{y\to\infty} N_2(x, y, P, \lambda) = u'(\infty) - kv'(-\infty) \leq 0$$

and

$$\lim_{y\to-\infty} N_2(x, y, P, \lambda) = u'(-\infty) - kv'(\infty) \geq 0,$$

by condition (4.2) of the utility functions. Therefore, the solution to (4.29) always exists in $(-\infty, \infty)$, which we denote by $y(x, \lambda)$.

If

$$N_2(x, y, P, \lambda)\,|_{y=0} = u'(w_1 - x - P) - \left[kv'(w_2 + P) - \lambda(1+\theta)\right] LR(x) < 0,$$

then $y(x,\lambda) < 0$ and the solution to problem 4.27 is $I^*(x,\lambda) = \max(0, y(x,\lambda)) = 0$.

On the other hand, if

$$N_2(x,y,P,\lambda)|_{y=x} = u'(w_1 - P) - [kv'(w_2 - x + P) - \lambda(1+\theta)]LR(x) > 0$$

then $y(x,\lambda) > x$ and $I^*(x,\lambda) = \min(x, y(x,\lambda)) = x$.

If $y(x,\lambda) \in [0,x]$ then $I^*(x,\lambda) = y(x,\lambda)$. This ends the proof. ∎

Proof of Lemma 4.4.4: For an arbitrary λ, let

$$\phi(\lambda) := \mathbf{E}_2\left[I^*(X;\lambda)\right].$$

We prove the Lemma by showing that $\phi(\lambda)$ is continuous and nondecreasing with respect to λ. In addition,

$$\lim_{\lambda\to-\infty}\phi(\lambda) = 0, \quad \lim_{\lambda\to\infty}\phi(\lambda) = \mathbf{E}_2[X]. \tag{A.6}$$

Firstly, for any x, $I^*(x;\lambda)$ is continuous in λ. By definition, $|I^*(x;\lambda)| \le x$, so by Lebesgue Dominated Convergence Theorem, $\phi(\lambda) = \mathbf{E}_2[I^*(x;\lambda)]$ is continuous with respect to λ.

Secondly, we show that $\phi(\lambda)$ is nondecreasing by showing that $I^*(x;\lambda)$ is nondecreasing in λ for every x. since $I^*(x;\lambda) = \min\left\{x, \max\{0, y(x,\lambda)\}\right\}$, it suffices to prove $y(x,\lambda)$ is nondecreasing in λ. To this end, taking derivative on both sides of equation (4.29) with respect to λ gives

$$\frac{\partial y(x,\lambda)}{\partial\lambda} = -\frac{(1+\theta)LR(x)}{u''(w_1 - x + y(x,\lambda) - P) + k\cdot v''(w_2 - y(x,\lambda) + P)\cdot LR(x)} \ge 0.$$

Finally, it is easy to see that when $\lambda \to -\infty$, $N_2(x,y,P,\lambda) < 0$ and then $I^*(x,-\infty) = 0$ for any x; when $\lambda \to \infty$, $N_2(x,y,P,\lambda) > 0$ and then $I^*(x,\infty) = x$ for any x. Therefore, (A.6) is true. This ends the proof. ∎

Appendix B

Proofs of Theorems in Chapter 5

Proof of Theorem 5.4.1: Denote terms under the expectation operators in the objective function of Problem 5.4.0a by

$$M(x, I(x), P) := u(w_1 - x + I(x) - P) + k \cdot v(w_2 - I(x) + P), \tag{B.1}$$

and let

$$N(x, I(x), P, \lambda) := M(x, I(x), P) + \lambda\,[(1+\theta)I(x) - P] \tag{B.2}$$

be the Langrange augmented function. We first have the following verification lemma.

Lemma B.0.1 *A ceded function $I^*(x, \lambda^*) \in C_{V_1,b}$ solves Problem 5.4.0a if there exists a constant $\lambda^* \in \mathbf{R}$ such that the following two conditions are satisfied:*

Condition 1: *For all $I \in C_{V_1,b}$ that satisfies $(1+\theta)\mathbf{E}[I(X)] = P$,*

$$N\left(x, I^*(x, \lambda^*), P, \lambda^*\right) \geq N\left(x, I(x), P, \lambda^*\right), \quad x \in [0, M].$$

Condition 2:

$$(1+\theta)\,\mathbf{E}\left[I^*(X, \lambda^*)\right] = P, \tag{B.3}$$

where the ceded function I^* includes a second argument λ to emphasize its dependence on the Langrange coefficient λ.

Proof Since $N\left(x, I^*(x, \lambda^*), P, \lambda^*\right) \geq N\left(x, I(x), P, \lambda^*\right)$ for arbitrary feasible solution $I \in C$, we have for all $x \geq 0$

$$M\left(x, I^*(x, \lambda^*), P\right) - M\left(x, I(x), P\right) \geq \lambda(1+\theta)\left(I^*(x, \lambda^*) - I(x)\right).$$

Thus

$$\begin{aligned} & \mathbf{E}\left[u(w_1 - X + I(X) - P)\right] + k\mathbf{E}\left[v(w_2 - I(X) + P)\right] \\ = \; & \int_0^M \left(M\left(x, I^*(x, \lambda^*), P\right) - M\left(x, I(x), P\right)\right) dF(x) \\ \geq \; & \lambda(1+\theta)\Big[\mathbf{E}[I^*(X, \lambda^*)] - \mathbf{E}[I(X)]\Big] = 0, \end{aligned}$$

which completes the proof. ∎

Lemma B.0.1 in fact states that Problem 5.4.0a can be solved pointwisely by identifying $I^*(x,\lambda^*)$ that maximizes the Lagrange augmented function .

We next consider the problem

$$\max_{y\in[\underline{I}_{V_1,b}(x),\overline{I}_{V_1,b}(x)]} N(x,y,P,\lambda), \tag{B.4}$$

for fixed x, b and λ. We will use the notation $N_1(\cdot,\cdot,\cdot,\cdot)$ for the first partial derivative of N with respective to the first argument and $N_{11}(\cdot,\cdot,\cdot,\cdot)$ for the second derivative and so on.

Due to the concavities of $u(\cdot)$ and $v(\cdot)$, we have

$$N_{22}(x,y,P,\lambda) = u''(w_1 - x + y - P) + kv''(w_2 - y + P) < 0.$$

Thus $N(x,y,P,\lambda)$ is strictly concave in y and there must exist a solution to (4.27), which we denote as $I^*(x;\lambda)$ and determine its form in the next lemma.

Lemma B.0.2 *The solution to problem 4.27 is given by*

$$I^*(x;\lambda) := \overline{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee y(x,\lambda)\right\}, \tag{B.5}$$

where $y(x,\lambda)$ be the solution to the first-order condition

$$N_2(x,y,P,\lambda) = 0. \tag{B.6}$$

Proof Firstly, notice that

$$N_2(x,y,P,\lambda) = u'(w_1 - x + y - P) - kv'(w_2 - y + P) + \lambda(1+\theta)$$

is continuous and nonincreasing in y. Second, because of the assumptions (5.8) and (5.9), $N_2(x,-\infty,P,\lambda) > 0$ and $N_2(x,\infty,P,\lambda) < 0$. Therefore, a solution, $y(x,\lambda)$, to (B.6) always exists in $[-\infty,\infty]$.

If

$$N_2(x,y,P,\lambda)|_{y=\underline{I}_{V_1,b}(x)} = u'(w_1 - x + \underline{I}_{V_1,b}(x) - P) - k\cdot v'(w_2 - \underline{I}_{V_1,b}(x) + P) + \lambda(1+\theta) < 0$$

then $y(x,\lambda) < \underline{I}_{V_1,b}(x)$ and the solution to problem 4.27 is $I^*(x,\lambda) = \underline{I}_{V_1,b}(x)$.

On the other hand, if

$$N_2(x,y,P,\lambda)|_{y=\overline{I}_{V_1,b}(x)} = u'(w_1 - x + \overline{I}_{V_1,b}(x) - P) - k\cdot v'(w_2 - \overline{I}_{V_1,b}(x) + P) + \lambda(1+\theta) > 0$$

then $y(x,\lambda) > \overline{I}_{V_1,b}(x)$ and $I^*(x,\lambda) = \overline{I}_{V_1,b}(x)$.

Finally, if $y(x,\lambda) \in [\underline{I}_{V_1,b}(x), \overline{I}_{V_1,b}(x)]$ then $I^*(x,\lambda) = y(x,\lambda)$. This ends the proof. ∎

Lemma B.0.3 *For any $P \in \left(\underline{P}_{V_1,b}, \overline{P}_{V_1,b}\right)$, there exists a λ^* such that*

$$(1+\theta)\,\mathbf{E}\left[I^*(X,\lambda^*)\right] = P.$$

Proof Let

$$\phi(\lambda) := \mathbf{E}\left[I^*(X;\lambda)\right]$$

where $I^*(x;\lambda)$ is given by Lemma B.0.2. We first show that $\phi(\lambda)$ is continuous non-decreasing with respect to λ and that

$$\lim_{\lambda\to-\infty}\phi(\lambda) = \mathbf{E}[\underline{I}_{V_1,b}(X)] = \underline{P}_{V_1,b}, \quad \lim_{\lambda\to\infty}\phi(\lambda) = \mathbf{E}\left[\overline{I}_{V_1,b}(X)\right] = \overline{P}_{V_1,b}.$$

Since $I^*(x;\lambda) = \overline{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee y(x,\lambda)\right\}$, it suffices to prove $y(x,\lambda)$ is non-decreasing in λ. Note that $y(x,\lambda)$ is the unique solution to equation (B.6), taking derivative on both sides of equation (B.6) with respect to λ gives

$$\frac{\partial y(x,\lambda)}{\partial\lambda} = -\frac{(1+\theta)}{u''(w_1 - x + y(x,\lambda) - P) + k\cdot v''(w_2 - y(x,\lambda) + P)} \geq 0.$$

For each value of x, $I^*(x;\lambda)$ is continuous in λ. By its characterization, $|I^*(x;\lambda)| \leq \overline{I}_{V_1,b}(x)$, so by Lebesgue Dominated Convergence Theorem, $\phi(\lambda) = \mathbf{E}[I^*(x;\lambda)]$ is continuous with respect to λ.

It is easy to see that when $\lambda \to -\infty$, $N_2(x,y,P,\lambda) < 0$ and then $I^*(x,-\infty) = \underline{I}_{V_1,b}(x)$; when $\lambda \to \infty$, $N_2(x,y,P,\lambda) > 0$ and then $I^*(x,\infty) = \overline{I}_{V_1,b}(x)$. This ends the proof. ∎

Combining Lemmas B.0.1, B.0.2 and B.0.3, Theorem 5.4.1 is proved.

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