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Localization Theory in an Infinity Topos

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Abstract

Inspired by recent work in homotopy type theory [CORS18], we develop the theory of *reflective subfibrations* on an ∞-topos $\mathcal{E}$. A reflective subfibration $L_\bullet$ on $\mathcal{E}$ is a pullback-compatible assignment of a reflective subcategory $\mathcal{D}_X \subseteq \mathcal{E}_{/X}$ with associated localization functor L_X, for every $X \in \mathcal{E}$. Reflective subfibrations abound in homotopy theory, albeit often disguised, e.g., as stable factorization systems. The added properties of a reflective *subfibration* $L_\bullet$ on $\mathcal{E}$ compared to a mere reflective *subcategory* of $\mathcal{E}$ are crucial for most of our results. For example, we can prove that L-local maps (i.e., those $p \in \mathcal{D}_X$ for some $X \in \mathcal{E}$) admit a classifying map. The existence of such a classifying map is a powerful tool that we exploit to show that there is a reflective subfibration $L'_\bullet$ whose local maps are exactly the *L-separated* maps, that is, those maps with L-local diagonal. We investigate some interactions between $L_\bullet$ and $L'_\bullet$ and explain when the two reflective subfibrations coincide. Finally, we show the existence of reflective subfibrations associated to sets of maps in $\mathcal{E}$ and describe some of their properties.

Keywords: reflective subfibration, local class of maps, classifying map, separated map, higher topos theory, localization theory, homotopy type theory.

Summary for lay audience

In Mathematics, one often faces two important needs.

1. *Simplify* some problems, by discriminating between the properties that are relevant and those that are ancillary for a certain issue.

2. *Present* some mathematical objects of interest in different ways, so as to highlight different aspects of these objects.

A powerful mathematical tool to answer these needs is *localization theory*, which singles out the *local properties* and *objects* that are pertinent to the study of a problem.

Localization theory is particularly useful when studying *spaces*. Classically, this study was carried out *geometrically* in *homotopy theory*, a branch of Mathematics that classifies spaces by looking at whether or not one can be deformed into the other. More recently, localizations of spaces have been studied *logically* in *homotopy type theory*, a syntatic language for reasoning formally about spaces.

This works merges the latter approach with the former, providing a new framework for understanding spaces through their localizations. The most important features of our approach are: a simultaneous treatment of the localization of both spaces and maps between them; and the usage of a modern language that allows the abstraction of the notion of "space" to mean anything that can be thought of as having points, paths between these points, paths between these paths, and so on. In our work, we recover all classically studied examples of localizations of spaces, and give new insights to their properties.

In bridging the logical and the geometrical approaches to localizations of spaces, we establish a dictionary between the two approaches that can help in evaluating the advantages and disadvantages of both, and merging the two communities together.

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Contents

Chapter 1

Introduction

1.1 Overview

This thesis is concerned with a systematic study of the theory of localization in an ∞-topos, inspired by recent work in homotopy type theory ([CORS18] and [RSS17]). We provide here the motivations behind this study, its structure, and how it fits into what is already known on the subject.

In Section 1.2, we provide some background on localizations in homotopy theory and category theory that motivate our work. In Section 1.3, we explain our framework for the study of localization in the context of an ∞-topos, and highlight the main themes of our work. In Section 1.4, we put our work in context by relating it with other work on the subject of localization, both in homotopy type theory and in higher topos theory. In particular, we discuss the relationship between this work and the paper [CORS18]. In Section 1.5, we describe in some detail the content of each chapter of this thesis, highlighting our main results. Finally, in Section 1.6, we explain some conventions and notation concerning ∞-categories that we use throughout this work.

1.2 Motivation for the study of localization

Localization of spaces is a classical topic in homotopy theory. One of its main features is that it provides tools and techniques analogous to those provided in algebra by localizations of rings and modules. These techniques include:

(i) *simplifying* or solving complicated problems by working locally;

(ii) formulating and proving *local-to-global principles*, that is, isolating those properties that hold globally if and only if they hold locally;

(iii) providing *fracture or decomposition theorems* that recover an object of study from its localizations.

Localizations of simple spaces at primes provide striking examples of all the above features. A space X is *simple* if, for every point $x \in X$, $\pi_1(X,x)$ is abelian and acts trivially on $\pi_n(X,x)$, for all $n \geq 2$. Given a set of primes T, localization at T provides a universal T*-localization map* $X \to X_T$ inducing the algebraic localization $\pi_n(X,x) \to \pi_n(X,x) \otimes \mathbb{Z}_T$, for every $n \geq 1$ and every $x \in X$. Here, $\mathbb{Z}_T$ is the localization of $\mathbb{Z}$ at the multiplicative set generated by the primes *not* in T. In particular, X_T is a T*-local space*, that is, a space such that every homotopy group is a $\mathbb{Z}_T$-module. (This is the correct notion of T-local space since we are assuming our spaces to be simple.) When p is a prime number, we write $X_{\{p\}} = X_p$ and talk about p*-local spaces*, whilst, when $T = \emptyset$, we write $X_\emptyset = X_0$ and talk about *rational spaces*. One can then prove the following results, where all spaces are assumed to be simple. (All results are taken from [Sul05, §2].)

(i) Classifying rational H-spaces (that is, spaces equipped with an up-to-homotopy associative and unital operation) is easy: up to homotopy equivalence, they are products of Eilenberg Mac-Lane spaces.

(ii) X is an H-space if and only if all of its p-localizations X_p are H-spaces and, for every pair of primes p, q, the rationalization maps $X_p \to X_0 \leftarrow X_q$ induce a ring isomorphism $H_*(X_p; \mathbb{Q}) \cong H_*(X_q; \mathbb{Q})$.

(iii) A space X is the homotopy limit of its localizations, in the sense that it is the homotopy limit of the diagram $\{X_p \to X_0\}_{p \text{ a prime}}$, where $X_p \to X_0$ is the rationalization map.

One might then wonder whether powerful statements like those above can be adapted to homotopy-theoretic settings other than spaces, such as simplicial sheaves on a site.

There is another interesting perspective on localization that is perhaps more familiar to category theorists. Namely, one can look at localization as a way to *present* certain objects of interest, rather than as a tool for simplifying our objects of interest and making them easier to understand. For example, once we are given a (small) site $\mathcal{C}$, what we are really interested in is the study of sheaves, rather than presheaves, over $\mathcal{C}$, and the process of inverting the covering sieves in the category of presheaves is better understood as a presentation of the category of sheaves, rather than as a simplification of the category of presheaves. Similarly, one is intrinsically interested in studying stacks over $\mathcal{C}$, for example in algebraic geometry. Now, one can present stacks as the (fibrant objects in the) localization of the Jardine model structure on $\mathsf{sPre}(\mathcal{C})$ (the category of simplicial presheaves over $\mathcal{C}$) that inverts the map $S^2 \to \Delta^0$ ([Hol08, Thm. 5.4]). This result provides a point of view that has arguably more to do with the idea of *generalizing* stacks to n-stacks ([Hol08, Def. 1.8]), rather than simplifying simplicial (pre)sheaves.

1.3 Our setting: reflective subfibrations

It is thus interesting to develop a unifying framework for the systematic study of localization theories, with the goal of encompassing and generalizing some of the examples that are already understood, while also providing new insights on them.

In the history of homotopy theory there have certainly been a few successful approaches in this sense. For example, one can consider augmented idempotent homotopical functors, popularized by Adams and Farjoun ([AF10], [Far96]), or one can look at Bousfield localizations of model categories, as developed by Bousfield, Kan and Quillen (see [Hir03] for a complete treatment). Our approach, based on the theory of *reflective subfibrations* on an ∞-topos $\mathcal{E}$, differs from these and other takes on localization as it employs the modern language of ∞-*category theory*, it emphasizes *localization of maps* rather than localization of objects (the latter being a special case of the former), and it naturally shares ties with *homotopy type theory*, the internal language of higher topos theory.

Our setting for localization is best understood and loses its artificial vibe if one looks at the following classical example of localization of spaces. Namely, for a fixed set $S = \{f_i \colon A_i \to B_i\}_{i \in I}$ of maps between spaces, one wishes to study those spaces that see maps in S as weak equivalences, that is, the spaces X for which the maps $X^{f_i} \colon X^{B_i} \to X^{A_i}$ $(i \in I)$ are weak equivalences. These are called S-*local spaces.* There are at least two significant examples of this construction in classical homotopy theory:

(a) given a fixed prime number p, when S consists of all the $q-$th multiplication maps $S^1 \xrightarrow{\cdot q} S^1$ where q is a prime other than p, S-local simple spaces are just the p-local simple spaces introduced earlier ([MP12, § 6.1]);

(b) if A is any space and S consists of the unique map $A \to *$ to a one-point space, S-local spaces are best known as A-*null spaces.* They are those spaces X for which every map $A \to X$ is homotopically equivalent to a (unique up to homotopy) constant map $A \to X$.

In this situation, classical localization theory provides, for each space X, a functorial choice of a space LX and of an S-*localization map* $\eta(X) \colon X \to LX$, such that:

- each LX is an S-local space;

- $\eta(X)$ is the homotopically initial map into an S-local space, i.e., for every S-local space Y, $Y^{\eta(X)}\colon Y^{LX} \to Y^X$ is a weak equivalence.

In other words, if we let $\mathcal{D}$ be the full subcategory on the S-local spaces, $\mathcal{D}$ is a *reflective subcategory* of $\mathcal{D}$ "up to homotopy", in the sense that $\eta(X)$ is only homotopically initial, and not strictly initial, among maps from X into an S-local space. Now, it turns out that this localization also makes sense for maps of spaces and it is done *fiberwise*. Namely, to a map of spaces $p\colon E \to X$, we can functorially associate a map $L_X(p)$ over X with the property that the homotopy fiber of $L_X(p)$ over each $x \in X$ is the S-localization of $\mathrm{hofib}_x(p)$. Hence, maps whose homotopy fibers are S-local spaces are called *S-local maps*. In general, they form a much larger, and much more interesting, class of maps than the maps between S-local spaces. For example, the universal covering map $\mathbb{R} \to S^1$ is S^1-null (i.e., it has 0-truncated fibers), even though S^1 is not S^1-null (it is not 0-truncated). For each space X, we can then consider the category $\mathsf{Spaces}_{/X}$ of maps of spaces over X. The S-local maps with codomain X form a reflective subcategory $\mathcal{D}_X$ of $\mathsf{Spaces}_{/X}$ (again "up to homotopy"). These reflective subcategories are not unrelated one to the other. For example, from the fiberwise description of S-local maps, it is immediate to see that, given a homotopy pullback square of spaces

$$\begin{tikzcd} M \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ Y \arrow[r] & X \end{tikzcd}$$

if p is S-local, then so is q. (We are using that a homotopy commutative square of spaces is a homotopy pullback square if and only if the induced maps on fiber sequences are weak equivalences.) In fact, we could almost argue that the fiberwise definition of S-local maps is given *precisely* to ensure this stability under homotopy pullbacks.

A *reflective subfibration* $L_\bullet$ (Definition 3.1.1) is defined so as to generalize the above setting of a pullback-stable system of reflective subcategories. This generalization proceeds in at least two ways. First, one can define reflective subfibrations on an arbitrary (Grothendieck) ∞-*topos* $\mathcal{E}$ and not just on spaces (the terminal ∞-topos). Roughly, an ∞-*category* $\mathcal{E}$ ([Lur09]) is a category for which, between every two objects X and Y, there is a *space* of maps (as opposed to a *set* of maps). These spaces of maps give notions of morphisms from X to Y (the 1-morphisms) and of morphisms between such morphisms (the 2-morphisms) and of morphisms of morphisms between these morphisms (the 3-morphisms), and so on. The definition of ∞-category implies that all n-morphisms in $\mathcal{E}$ for $n \geq 2$ are invertible. Moreover, ∞-category theory is a conservative extension of ordinary category theory, where all categorical constructions, such as limits, colimits, and adjunctions, still make sense. Because of their intrinsic "space-like" nature, ∞-categories are also contexts in which to develop homotopy theory, which will then inherently be a *categorical homotopy theory* that allows a treatment of homotopy theory via universal properties. An ∞-*topos* is a special kind of ∞-category $\mathcal{E}$ that satisfies properties analogous to the classical Giraud axioms for ordinary Grothendieck topoi (see [MM94, App. A] for an account). These axioms make an ∞-topos $\mathcal{E}$ a more suitable setting for homotopy theory than a bare ∞-category, as they render $\mathcal{E}$ more similar to a "category of spaces". (For example, every object X in an ∞-topos has homotopy groups $\pi_n(X)$.)

Another aspect in which reflective subfibrations generalize the framework of S-local spaces is that they allow for more general systems of reflective subcategories $\mathcal{D}_X \subseteq \mathcal{E}_{/X}$ (for $\mathcal{E}$ an ∞-topos and $X \in \mathcal{E}$) than those associated to a set S of maps in $\mathcal{E}$. For example, in our work we show that we can construct reflective subfibrations out of the datum of a *stable factorization system* on $\mathcal{E}$ or of a *left exact localization* on $\mathcal{E}$ (Theorem 4.2.5 and Proposition 4.2.8). In fact, we can even use the theory of reflective subfibrations to prove new results about stable factorization systems (Proposition 6.1.2).

Given a reflective subfibration $L_\bullet$ on $\mathcal{E}$ with associated reflective subcategories $\mathcal{D}_X \subseteq \mathcal{E}_{/X}$, we write L_X for the induced localization functor, namely, the composition of the reflector of $\mathcal{E}_{/X}$ into $\mathcal{D}_X$ with the inclusion of $\mathcal{D}_X$ into $\mathcal{E}_{/X}$. The collection of all objects in $\mathcal{D}_X$, as X varies in $\mathcal{E}$, form the L-*local maps*, and the objects of $\mathcal{D} := \mathcal{D}_1$ are called L-*local objects*. L-local objects and maps satisfy many enjoyable properties. For example, if $X \in \mathcal{E}$ is an L-local object, then every loop object $\Omega(X, x)$ is also an L-local object. (In an ∞-topos $\mathcal{E}$, $\Omega(X, x)$ is defined as the pullback of $1 \xrightarrow{x} X$ along itself, where 1 is the terminal object of $\mathcal{E}$. This recovers the homotopy type of ordinary loop spaces when $\mathcal{E}$ is the ∞-topos of spaces.) However, the localization *functors* L_X do not commute with loop objects: if (X, x) is a pointed object in $\mathcal{E}$, in general $L(\Omega(X, x))$ is not equivalent to $\Omega(LX, Lx)$. We can fix this misbehavior and compute $L(\Omega(X, x))$ by means of another reflective subfibration $L'_\bullet$, related to $L_\bullet$. The main purpose of our work is the study of such an $L'_\bullet$, which we tackle in Chapter 5. There, we consider the class of L-*separated maps* (Definition 5.1.1). If $\mathcal{E} = \mathsf{Spaces}$ and we look at L-separated *objects*, we get that L-separated spaces are the spaces with L-local loop spaces. We develop a great deal of the theory of L-separated maps, culminating in our main result, the existence of a reflective subfibration $L'_\bullet$ on $\mathcal{E}$ for which the L'-local maps are the L-separated maps. As far as we know, this is a novel result in the context of homotopy theory and higher topos theory (but not in homotopy type theory — see below). In Corollary 6.1.3, we then prove that, for every pointed object (X, x) of $\mathcal{E}$, $L(\Omega(X, x)) \simeq \Omega(L'X, L'x)$.

1.4 Relationship to other work

We mentioned earlier that our take on localization theory is inspired by *homotopy type theory*, a dependent type theory with homotopy-theoretic features ([UF13]). More precisely, the definition of reflective subfibration on an ∞-topos $\mathcal{E}$ appears in [RSS17] as the external notion (in higher topos theory, HTT) that captures the internal description (in homotopy type theory, HoTT) of a *reflective subuniverse*. The work in [RSS17] is mainly concerned with an important subclass of reflective subuniverses, namely the Σ-*closed* ones, also known as *modalities*. These correspond to *composing* reflective subfibrations on $\mathcal{E}$, that is, reflective subfibrations $L_\bullet$ for which the composite of two L-local maps is again an L-local map. In the context of higher topos theory, the authors of [ABFJ17a] use the term "modality" as a synonym for a stable factorization system on an ∞-topos $\mathcal{E}$, and they carry out a systematic study of these factorization systems. On the other hand, the authors of [CORS18] shift their focus back to the general setting of reflective subuniverses, motivated by the study in homotopy type theory of localizations at primes (which are not modalities). We can then depict our work as the (homotopy) pushout square

$$\begin{tikzcd}
\{\text{modalities in HoTT, [RSS17]}\} \arrow[r] \arrow[d] & \{\text{localizations in HoTT, [CORS18]}\} \arrow[d] \\
\{\text{modalities in HTT, [ABFJ17a]}\} \arrow[r] & \{\text{localizations in HTT, this thesis}\} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

A few more words are in order about the relationship between our work and [CORS18]. Many of our results, including Theorem 5.3.3, are formulated and proved there in the framework of homotopy type theory. In fact, [CORS18] was used as a road-map for our work, which is partly about translating results present there into the language of higher topos theory. The basics of this translation are fairly simple and constitute the core of the analogy between HoTT and HTT. For example, if $\mathcal{U}$ is a universe in type theory and $\mathcal{E}$ is our preferred ∞-topos, a type family $P\colon X \to \mathcal{U}$ corresponds to a map $p\colon E \to X$ in $\mathcal{E}$, thought of as having homotopy fiber over $x \in X$ given by $P(x)$. In particular, the type family $\mathrm{Id}_X\colon X \times X \to \mathcal{U}$, associating to each $x, y : X$ the identity type $\mathrm{Id}_X(x, y)$, corresponds to the diagonal map $\Delta X\colon X \to X \times X$. Since the functor $X \times (-)\colon \mathcal{E} \to \mathcal{E}_{/X}$ has both a left adjoint $\sum_X$ and a right adjoint $\prod_X$, the type-theoretic Sigma-type $\Sigma_{x:X} P(x)$ and Pi-type $\Pi_{x:X} P(x)$ are simply the objects $E = \sum_X p$ and $\prod_X p$ in $\mathcal{E}$, respectively. However, mastering this translation process takes some care, and we often encountered difficulties that affected both the statements and the proofs of our results as compared to [CORS18]. There are several causes for these difficulties.

First, in HoTT, terms have types and one can often prove statements about all terms of a type by proving them for one generic term of that type. (This is the analog of set-theoretic statements of the form "for all $x \in X$, it happens that..." whose proofs usually start with "let $x \in X$".) In HTT, the only sensible thing to do, in general, is to work globally, so that statements and proofs concerning all terms of a type are usually made "term free" by working in slice categories. One can get a good sense of the potential consequences of this slicing process by comparing the statement and proof of our Proposition A.3.4 with the analogous result in HoTT, [CORS18, Lemma 2.23].

A further point of difference is given by the fact that the natural starting point for localization in HoTT is a reflective *subuniverse* (an internal notion), whilst the natural starting point in homotopy theory is a reflective *subfibration* (an external notion). In particular, results that are naturally stated in terms of universes in HoTT do not always appear as meaningful in the context of HTT (or, anyway, one might wish to find a better phrasing). For example, Proposition 5.2.6 and Corollary 5.2.8 are the results of our attempt to give a more natural formulation of [CORS18, Lemma 2.19]. Prior to conversations we had with the first and the last author of [CORS18], that lemma was only saying that a type family $P\colon X \to \mathcal{U}_L$ extends along an L'-localization $X \to L'X$ (see also [CORS18, Rmk. 2.35]). This last example shows how the differences between the HoTT and the HTT settings give occasions for one point of view to inform and complement the other. Statements and proofs that are present here but not in [CORS18] should then be taken in this spirit. For example, this applies to the results showing that L-local and L-separated maps form local classes of maps in $\mathcal{E}$ (Proposition 3.2.3, Remark 3.2.7 and Proposition 5.1.7), as well as to the discussion of Section 6.2 about reflective subfibrations for which $L_\bullet = L'_\bullet$.

Finally, the translation process from HoTT to HTT often turns strict equalities in the former setting into canonical equivalences in the latter, and one must keep track of these equivalences. A typical example of this phenomenon is the Beck-Chevalley condition, which provides canonical equivalences involving pullback functors and their adjoints as in

$$\sum_k h^* \xrightarrow{\simeq} g^* \sum_f \quad \text{and} \quad f^* \prod_g \xrightarrow{\simeq} \prod_h k^*,$$

for suitable maps h, k, g, f in $\mathcal{E}$ (see Lemma A.1.3). In type theory, the corresponding statements are judgmental (that is, strict) equalities which say that substitution of terms commute with Σ and Π operations. This bookkeeping process for equivalences becomes particularly evident in our proof of Theorem 5.3.3, which is more complicated than the HoTT proof given in [CORS18, Thm. 2.25].

In April 2019, after most of the results in this thesis were accomplished, M. Shulman gave a proof of the conjecture that every ∞-topos models homotopy type theory ([Shu19]). Hence, all statements proven in HoTT can be translated into true statements in any ∞-topos $\mathcal{E}$[1]. This applies, in particular, to the results given in [CORS18]. However, we believe that this recent development does not invalidate our work entirely, for several reasons. First of all, our work on localization in HTT can not be *immediately* recovered from the analogous work in HoTT since the starting points are different (reflective subfibrations in HTT, reflective subuniverses in HoTT). Hence, the already-mentioned translations that accompany our work still need to be done in practice, at least if one wants to obtain an external understanding of the internal statements provided by HoTT. Secondly, not all proofs we give here are *direct* translations of the HoTT ones, as some type-theoretic arguments do not have an obvious counterpart in the HTT setting, so that we were often forced to employ different proof techniques. Furthermore, even for those arguments that parallel more closely the ones in [CORS18], we believe our proofs can give some working-knowledge on how to use and adapt HoTT reasoning to prove theorems in an ∞-topos $\mathcal{E}$, in a spirit similar to [ABFJ17a] and [Rez15]. Finally, as remarked earlier, several results here are not present in [CORS18].

[1] Modulo the initiality conjecture for homotopy type theory.

1.5 Content and structure

We give below a summary of the structure and of the content of this thesis which consists of seven chapters (including this introduction) and an appendix.

In Chapter 2, we follow [Lur09] and [GK17] to give an overview of the theory of local classes of maps in an ∞-topos $\mathcal{E}$. A class S of maps in $\mathcal{E}$ is said to be *local* (Definition 2.1.2) if it is closed under coproducts, stable under pullbacks, and satisfies a descent condition. Up to size issues, a class S of maps in $\mathcal{E}$ is local exactly if it admits a *classifying map* $p \in S$ (Definition 2.1.4), so that every other map of S is a pullback of p. In particular, the class of *all* maps in $\mathcal{E}$ is local and so it admits a classifying map $u\colon \widetilde{\mathcal{U}} \to \mathcal{U}$. The base space $\mathcal{U}$ plays the role in $\mathcal{E}$ of a given universe in homotopy type theory. Classifying maps $p\colon E \to X$ enjoy an important property called *univalence* (Definition 2.2.8) which, roughly, means that equivalences between (homotopy) fibers of p are completely determined by paths in X. In order to formulate univalence, one needs to introduce an *object of equivalences* between any two objects X and Y of $\mathcal{E}$, defined in [GK17, Thm. 2.10]. We give an alternative characterization of it in Lemma 2.2.4.

In Chapter 3, we apply the results of the previous chapter to a class of maps associated to a *reflective subfibration* $L_\bullet$ on $\mathcal{E}$. We show that L-local maps (Definition 3.1.1) form a local class of maps in $\mathcal{E}$ (Proposition 3.2.3) and use this fact to characterize reflective subfibrations on spaces as “fiberwise localizations” (Corollary 3.2.4). It follows that L-local maps admit a univalent classifying map $u^L\colon \widetilde{\mathcal{U}}^L \to \mathcal{U}^L$. By [GK17, Cor. 3.10], there is a monomorphism $\mathcal{U}^L \rightarrowtail \mathcal{U}$ and u^L is the pullback of u along this mono. This fact links the notion of reflective subfibration on $\mathcal{E}$ with that of reflective subuniverse in HoTT.

In Chapter 4, we introduce and study *L-connected maps*. These are maps in $\mathcal{E}$ whose fibers have trivial L-localization (Definition 4.1.1). We establish a few technical lemmas about these maps, aimed at proving the following result.

Theorem 4.2.5. Let $\mathcal{E}$ be an ∞-topos.

1. Let $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ be a stable factorization system on $\mathcal{E}$. There exists a modality $L^{\mathcal{F}}_{\bullet}$ on $\mathcal{E}$ whose local maps are exactly the maps in $\mathcal{R}$.

2. Let $L_{\bullet}$ be a modality on $\mathcal{E}$. Let $\mathcal{L}$ be the class of L-connected maps and $\mathcal{R}$ the class of L-local maps. Then $\mathcal{F}_L = (\mathcal{L}, \mathcal{R})$ is a stable factorization system on $\mathcal{E}$.

Moreover, the assignments $\mathcal{F} \mapsto L^{\mathcal{F}}_{\bullet}$ and $L_{\bullet} \mapsto \mathcal{F}_L$ are inverse to one another.

Here, a *modality* is a reflective subfibration $L_{\bullet}$ on $\mathcal{E}$ such that the composition of two L-local maps is again L-local. The above theorem reconciles this meaning of the term "modality" with the one used in [ABFJ17a] to refer to a *stable factorization system* (Definition 4.2.1). In a different flavour, this result is internally proven in HoTT in [RSS17]. Chapter 4 is tangential to our main results of Chapter 5 and can be skipped at a first reading, except for the main concept of an L-connected map.

In Chapter 5, we get to the core of our work. For a general reflective subfibration $L_{\bullet}$ on $\mathcal{E}$, we introduce *L-separated maps* as those maps in $\mathcal{E}$ whose diagonal is an L-local map (Definition 5.1.1). L-separated maps inherit a lot of pleasant properties from L-local maps. In particular, they also form a local class of maps of $\mathcal{E}$ (Proposition 5.1.7), hence they satisfy a necessary condition to be themselves the local maps for a reflective subfibration on $\mathcal{E}$. Showing that this is indeed the case is the purpose of this chapter. Our path to victory goes through a careful analysis of the interactions between L-separated and L-local maps which culminates in the following characterization of *L'-localization maps*, that is, those maps into an L-separated object that are initial among maps out of a fixed object $X \in \mathcal{E}$ and into an L-separated object (Definition 5.2.2).

Theorem 5.2.10. The following are equivalent, for a map $\eta' \colon X \to X'$ in $\mathcal{E}$:

1. η' is an L'-localization of X;

2. η' is an effective epimorphism and

$$\begin{tikzcd} X \arrow[r, "\Delta\eta'"] \arrow[dr, "\Delta X"'] & X \times_{X'} X \arrow[d] \\ & X \times X \end{tikzcd}$$

   is an L-localization of ΔX.

On the way, we also prove Proposition 5.2.6 which gives a characterization of L-local maps in terms of extensions along L'-localization maps. This is a good example of how one can use the theory of classifying maps (applied to the classifying map of L-local maps) to translate reasoning about universes in homotopy type theory into proofs of results whose statements have nothing to do with universes. After a few more preliminary lemmas, we conclude Chapter 5 by using Theorem 5.2.10 to prove the following result, which combines Theorem 5.3.3 and Corollary 5.3.4.

Theorem. *Let $L_\bullet$ be a reflective subfibration on an ∞-topos $\mathcal{E}$. Then the following hold.*

1. *For every map $p \in \mathcal{E}_{/Z}$, there exists an L'-localization map $\eta'_Z(p) \colon p \to L_Z(p)$.*

2. *There exists a reflective subfibration $L'_\bullet$ on $\mathcal{E}$ for which the L'-local maps are exactly the L-separated maps.*

We remark that the proof of the above theorem makes crucial use of the fact that L-local maps admit a classifying map and that this map is univalent. This result is also significatively more complicated to prove in our setting than in homotopy type theory (see [CORS18, Thm. 2.25]).

In Chapter 6, we present some results that we can state or prove once we know that L-separated maps are associated to a reflective subfibration on $\mathcal{E}$. For example, we give a way to produce new stable factorization systems from old ones (Proposition 6.1.2), and we prove a result that shows how $L'_\bullet$ accounts for the lack of commutativity between $L_\bullet$ and loop functors (Corollary 6.1.3). We also study further relationships between $L_\bullet$ and $L'_\bullet$, proving, for instance, results that link L'- and L-connected maps (Proposition 6.1.9). We make use of the fact that L'-localization is almost left exact (Proposition 6.1.4) to give a characterization of *self-separated* reflective subfibrations. These are reflective subfibrations $L_\bullet$ for which $L_\bullet = L'_\bullet$, in the sense that every L-separated maps is L-local (the contrary being always true). Self-separated reflective subfibrations are linked to the notions of *hypercomplete* and ∞-*connected maps* in an ∞-topos $\mathcal{E}$ (Definition 6.2.2). Self-separated reflective subfibrations can be characterized in terms of special left exact localizations of $\mathcal{E}$, the *quasi-cotopological* localizations (Definition 6.2.5) as explained by the following result, which does not appear in [CORS18].

Theorem 6.2.8. The following are equivalent, for a reflective subfibration $L_\bullet$ of $\mathcal{E}$.

1. $L_\bullet$ is self-separated.

2. $L_\bullet$ is the modality associated to a quasi-cotopological localization of $\mathcal{E}$.

In this case, hypercomplete maps are L-local.

In Chapter 7, given any *set* S of maps in $\mathcal{E}$, we prove the existence of the S-*localization* on $\mathcal{E}$. Namely, we show that we can construct a reflective subfibration $L^S_\bullet$ on $\mathcal{E}$ with the property that an object $X \in \mathcal{E}$ is L^S-local if and only if, for every map $f\colon A \to B$ in S, the map $X^f\colon X^B \to X^A$ is an equivalence in $\mathcal{E}$ (Proposition 7.1.12). We remark that this existence result uses local presentability of the ∞-topos $\mathcal{E}$. When $S = \{A \to 1\}$ for an object A of $\mathcal{E}$, the corresponding S-localization is called A-*nullification*. We show that it is always a modality (Proposition 7.1.9). We conclude the chapter with two sections where we explore some properties of S-localizations in the case where: a) the maps in S all belong to the left class of a cartesian factorization system on $\mathcal{E}$ (Section 7.3); or (b) all the maps in S are between 0-connected types (Section 7.4).

Finally, in Appendix A, we collect some results about locally cartesian closed ∞-categories that we need in our work, but that we could not naturally fit anywhere in the main body without interrupting the flow of our arguments. Some of the results there are well known, but for others we could not find any reference in the literature. An example of the results in the latter group is given by Proposition A.2.1, where we prove the topos-theoretic version of the function extensionality axiom from HoTT. Section A.3 is a nice little compendium of abstract nonsense that allows us to prove Proposition A.3.4, a result about unique extensions of maps that is crucial for the proof of Theorem 5.2.10.

1.6 Higher categorical conventions

We gather here miscellaneous background concepts and notation about ∞-categories that we use throughout our work.

Infinity categories. In this work, we use (more or less explicitly) many results about the category theory of ∞-categories that are proven in [Lur09], where an *∞-category* is taken to be a quasicategory, that is, a simplicial set which has the right lifting property with respect to inner-horn inclusions. However, all these results are basic facts that every model of ∞-category theory ought to provide, and the specific incarnation of ∞-categories as quasicategories only comes into play in the proofs that these results actually hold in that model. Here are a few examples of basic "model-independent" results we use, where $\mathcal{C}$ is an ∞-category.

- For every $X, Y \in \mathcal{C}$, there is an ∞-groupoid $\mathcal{C}(X, Y)$ of maps in $\mathcal{C}$ from x to y. Here, an *∞-groupoid* is thought of as an abstract homotopy type, but can be incarnated as a Kan complex. Given two maps $f, g\colon X \to Y$, one writes $f = g$ if f and g are *homotopic*, that is, if they are in the same path-component of $\mathcal{C}(X, Y)$.
- Composition of maps is defined, associative and unital up to coherent higher homotopies. Maps in $\mathcal{C}$ having a two-sided inverse with respect to this composition

are called *equivalences*.

- For every object $A \in \mathcal{C}$, every map $f\colon Y \to X$ gives rise to a map of ∞-groupoids $\mathcal{C}(A, f)\colon \mathcal{C}(A, Y) \to \mathcal{C}(A, X)$, well-defined and natural up to homotopy, which represents composition with f. These maps detect equivalences, in that each map $\mathcal{C}(A, f)$ is an equivalence of ∞-groupoids for every $A \in \mathcal{C}$ if and only if f is an equivalence in $\mathcal{C}$.

- For every $X \in \mathcal{C}$, there is a slice ∞-category $\mathcal{C}_{/X}$ whose objects are maps in $\mathcal{C}$ with codomain X and whose morphisms are homotopy commutative triangles between these maps. There is also a forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ which is conservative, i.e., it reflects equivalences. If $f\colon Y \to X$ is a map in $\mathcal{C}$, there is a natural equivalence of ∞-categories $(\mathcal{C}_{/X})_{/f} \simeq \mathcal{C}_{/Y}$.

- There are notions of limits, colimits and adjunctions that have the right homotopical universal property and can be characterized representably through the mapping spaces $\mathcal{C}(\bullet, \bullet)$.

Uniqueness of maps. Given an ∞-category $\mathcal{C}$, when we make statements about the existence and uniqueness of a map in $\mathcal{C}$ satisfying certain conditions, we always mean that the space of maps verifying those conditions is contractible.

Slice categories. Given an ∞-category $\mathcal{C}$, we often depict a map $m\colon p \to q$ in a slice category $\mathcal{C}_{/Z}$ as a commuting triangle in $\mathcal{C}$ of the form

$$\begin{tikzcd} E \arrow[rr, "m"] \arrow[dr, "p"'] & & M \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

leaving the interior 2-simplex implicit. We will often carry over this implicitness to other maps in slice categories that are constructed from m, at least as long as the context is enough to disambiguate. For example, if the implicit 2-simplex of m above is σ, then the implicit 2-simplex of the map in $\mathcal{C}_{/Z^2}$ given by

$$\begin{tikzcd} E \arrow[rr, "m"] \arrow[dr, "{(p,p)}"'] & & M \arrow[dl, "{(q,q)}"] \\ & Z^2 & \end{tikzcd}$$

is (σ, σ).

If p and q are objects in a slice category $\mathcal{C}_{/Z}$, we write $p \times^Z q$ to mean the product object of p and q in $\mathcal{C}_{/Z}$.

Orthogonality relation. We will make extensive use of the orthogonality relation between maps in an ∞-category $\mathcal{C}$.

Definition 1.6.1. Let $f\colon A \to B$, $g\colon X \to Y$ be maps in an ∞-category $\mathcal{C}$. We say that f is *left orthogonal* to g and that g is *right orthogonal* to f if the following square of ∞-groupoids is a homotopy pullback square

$$\begin{tikzcd} \mathcal{C}(B,X) \arrow[r, "{\mathcal{C}(B,g)}"] \arrow[d, "{\mathcal{C}(f,X)}"'] & \mathcal{C}(B,Y) \arrow[d, "{\mathcal{C}(f,Y)}"] \\ \mathcal{C}(A,X) \arrow[r, "{\mathcal{C}(A,g)}"'] & \mathcal{C}(A,Y) \end{tikzcd}$$

When this happens, we write $f \perp g$.

Remark 1.6.2. The pullback condition in Definition 1.6.1 can be restated as asking that the induced map of ∞-groupoids

$$\varphi\colon \mathcal{C}(B,X) \to \mathcal{C}(A,X) \times_{\mathcal{C}(A,Y)} \mathcal{C}(B,Y)$$

is an equivalence. Equivalently, this means that each homotopy fiber of φ is contractible. We can phrase this condition by saying that for every solid commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a unique dotted filler.

When $\mathcal{C}$ has a terminal object 1, given an object X in $\mathcal{C}$, we simply write $f \perp X$ to mean $f \perp (X \to 1)$. To minimize the risk of confusion, we use the symbol $\perp_X$ when we want to denote the orthogonality relation in the slice $\infty-$category $\mathcal{C}_{/X}$. So, for example, if $\alpha\colon p \to q$ is a map in $\mathcal{C}_{/X}$ and r is an object in $\mathcal{C}_{/X}$, $\alpha \perp_X r$ means that α is left orthogonal to the map $r \to \mathrm{id}_X$ in $\mathcal{C}_{/X}$.

Given a class $\mathcal{M}$ of maps in $\mathcal{C}$, we write ${}^{\perp}\mathcal{M}$ for the class of maps in $\mathcal{C}$ that are left orthogonal to every map in $\mathcal{M}$, and we write $\mathcal{M}^{\perp}$ for the class of maps in $\mathcal{C}$ that are right orthogonal to every map in $\mathcal{M}$.

Infinity topoi. By an ∞-*topos* we mean an ∞-category $\mathcal{E}$ with the following properties.

(a) $\mathcal{E}$ is a locally presentable ∞-category ([Lur09, Def. 5.5.0.1]).

(b) Colimits in $\mathcal{E}$ are universal ([Lur09, Def. 6.1.1.2]).

(c) The class of all maps in $\mathcal{E}$ is a local class of maps (see Definition 2.1.2).

This characterization of ∞-topoi follows from [Lur09, Thm. 6.1.6.8] and is also given in [ABFJ17a, Def. 2.2.3]. We spell out some consequences of this definition that will be used throughout our work. Let $\mathcal{E}$ be an ∞-topos.

(i) $\mathcal{E}$ *admits small limits and colimits*. In particular, it has a terminal object 1 and an initial object 0.

(ii) $\mathcal{E}$ is a *locally cartesian closed* ∞-category. This means that, for every map $f\colon Y \to X$, the pullback functor

$$f^*\colon \mathcal{E}_{/X} \longrightarrow \mathcal{E}_{/Y}$$

has a left and a right adjoint, which we denote by

$$\sum_f : \mathcal{E}_{/Y} \longrightarrow \mathcal{E}_{/X} \quad \text{and} \quad \prod_f : \mathcal{E}_{/Y} \longrightarrow \mathcal{E}_{/X},$$

respectively. Given an object $p \in \mathcal{E}_{/Y}$, $\sum_f(p)$ is just the composite map $f \circ p$. When f is the unique map $A \to 1$ for $A \in \mathcal{E}$, we simply write $\sum_A$ and $\prod_A$. In this case, $\sum_A$ is the forgetful functor $\mathcal{E}_{/A} \to \mathcal{E}$. We may refer to left and right adjoints to pullback functors as *dependent sums* and *dependent products* respectively.

(iii) $0 \simeq 1$ in $\mathcal{E}$ if and only if $\mathcal{E}$ is equivalent to Δ^0, the terminal ∞-category.

(iv) *Coproducts* in $\mathcal{E}$ are *disjoint*, as in [Lur09, § 6.1.1 and Lemma 6.1.5.1].

(v) For every $n \geq (-2)$, one can define the classes of n*-connected* and n*-truncated* maps ([Lur09, Def. 5.5.6.1 and Def. 6.5.1.10]). These forms the left and the right class of a factorization system in $\mathcal{E}$, respectively (see Definition 4.2.1 and [ABFJ17a, Prop. 3.4.6]). (-1)-connected maps are also called *effective epimorphisms* and (-1)-truncated maps are also known as *monomorphisms*. We write $X \twoheadrightarrow Y$ and $X \rightarrowtail Y$ to denote an effective epimorphism and a monomorphism, respectively.

(vi) For every $X \in \mathcal{E}$, the slice $\infty-$category $\mathcal{E}_{/X}$ is again an ∞-topos.

Chapter 2

Univalence for local classes of maps

In this chapter, we gather some background material and a few new results that will be needed in the rest of our work.

In Section 2.1, we follow [Lur09] and describe the theory of local classes of maps and their classifying maps which will be crucial when working with reflective subfibrations. In Section 2.2, we give an account of the notions of objects of equivalences and univalent maps, as developed in [GK17]. Lemma 2.2.4 provides an internal description of $\mathrm{Eq}(X, Y)$ — the object of equivalences between two objects X, Y of an ∞-topos $\mathcal{E}$ — that does not appear in [GK17].

2.1 Local classes and classifying maps

We introduce here local classes of maps in an ∞-topos $\mathcal{E}$ and explain that, modulo size issues, they are exactly the classes S of maps in $\mathcal{E}$ that admit a classifying map, that is, a map p in S such that every other map in S is a pullback of p. Our exposition follows [Lur09]. We fix an ∞-topos $\mathcal{E}$ throughout.

Proposition 2.1.1 ([Lur09, Prop. 6.2.3.14])**.** *Let S be a class of maps in $\mathcal{E}$ and suppose S is closed under small coproducts in $\mathcal{E}^{\bullet\to\bullet}$ and stable under pullbacks. Then the following are equivalent for S.*

1. *Given any pullback square in* $\mathcal{E}$

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where f *is an effective epimorphism,* p *is in* S *if and only if* q *is in* S*.*

2. *Given any commutative cube in* $\mathcal{E}$

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd, "f"'] & & B \arrow[dr] \arrow[dd, "g"' near start] & \\ & X \arrow[rr] \arrow[dd, "h" near start] & & Y \arrow[dd, "k"] \\ C \arrow[rr] \arrow[dr] & & D \arrow[dr] & \\ & Z \arrow[rr] & & W \end{tikzcd}$$

if the back and the left faces are pullback squares, the top and the bottom faces are pushout squares and $f, g, h \in S$*, then the right and the front faces are pullback squares and* $k \in S$*.*

Definition 2.1.2. A class S of maps in $\mathcal{E}$ which is closed under small coproducts, stable under pullbacks and satisfies one of the equivalent conditions of Proposition 2.1.1 is called *local*.

Example 2.1.3. The class of *all* maps in an ∞-topos is local.

Local classes of maps in an ∞-topos are important because, up to size issues, they are the classes of maps that admit a classifying map.

Definition 2.1.4. Let S be a pullback-stable class of maps in an ∞-topos $\mathcal{E}$. Denote by $\mathrm{Cart}(S)$ the sub-∞-category of $\mathcal{E}^{\bullet \to \bullet}$ having the maps in S as objects and pullback squares as morphisms. A *classifying map* for S, if it exists, is a terminal object $p \colon E \to X$ of $\mathrm{Cart}(S)$.

Thus, a classifying map $p\colon E \to X$ for S is a map in S such that every other map in S is a pullback of p in an essentially unique way.

Example 2.1.5 ([Lur09, Prop. 6.1.6.3])**.** The class of all monomorphisms in an ∞-topos $\mathcal{E}$ has a classifying map.

Definition 2.1.6. Let κ be a regular cardinal. A map $f\colon X \to Y$ in $\mathcal{E}$ is said to be *relatively κ-compact* if for every pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

where Z is κ-compact, W is also κ-compact.

The notion of relatively κ-compact maps is what gives the smallness condition needed to prove the following result.

Proposition 2.1.7 ([Lur09, Prop. 6.1.6.7])**.** *Let S be a local class of maps in an ∞-topos $\mathcal{E}$. Then, there are arbitrarily large regular cardinals κ such that the class S_κ of maps in S that are relatively κ-compact is local and has a classifying map.*

We record in the following observation a few remarks about the “arbitrarily large” cardinal κ appearing in the statement of Proposition 2.1.7.

Remark 2.1.8.

1. The regular cardinal κ in $\mathcal{E}$ has to be large enough for:

   (a) $\mathcal{E}$ to be locally κ-presentable;

   (b) the pullback functor
   $$p\colon \mathcal{E}^{\Lambda^2_2} \longrightarrow \mathcal{E}^{(\Lambda^2_2)^{\triangleleft}}$$
   to preserve κ-filtered colimits;

(c) the restriction of p to κ-compact objects in $\mathcal{E}^{\Lambda^2_2}$ to land in $\mathcal{E}_\kappa^{(\Lambda^2_2)^\triangleleft}$, where $\mathcal{E}_\kappa$ denotes the full subcategory of κ-compact objects in $\mathcal{E}$.

2. When κ is also taken to be strongly inaccessible, then the class of relatively κ-compact maps in $\mathcal{E}$ is closed under dependent products, in the sense that, if $f\colon X \to Y$ and $g\colon E \to X$ are *both* relatively κ-compact, then $\Pi_f(g)$ is also relatively κ-compact (see [GK17, Lemma 4.17]).

Notation 2.1.9. For every regular cardinal κ as in Remark 2.1.8 (1) and (2), we denote by

$$\widetilde{\mathcal{U}_\kappa} \to \mathcal{U}_\kappa$$

the classifying map for κ-compact maps in $\mathcal{E}$. Here, $\mathcal{U}$ stands for "*universe*",where the terminology is borrowed from homotopy type theory (see [UF13, § 1.3]). If S is a local class of maps, we denote by

$$\widetilde{\mathcal{U}^S_\kappa} \to \mathcal{U}^S_\kappa$$

the classifying map for S_κ.

Note that, by definition, there is a pullback square

$$\begin{tikzcd}
\widetilde{\mathcal{U}^S_\kappa} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}_\kappa} \arrow[d] \\
\mathcal{U}^S_\kappa \arrow[r, "s_\kappa"'] & \mathcal{U}_\kappa
\end{tikzcd} \tag{2.1}$$

where the map s_κ is unique up to equivalence.

2.2 Objects of equivalences and univalence

Following [GK17], we explain here how one can associate an object $\mathrm{Eq}(X, Y)$ to every pair of objects $X, Y \in \mathcal{E}$ so that the global elements of $\mathrm{Eq}(X, Y)$ are the equivalences from X to Y. We use such objects of equivalences to define and characterize univalent maps in an ∞-topos.

Every ∞-topos $\mathcal{E}$ is, in particular, a cartesian closed ∞-category. One way of saying what this means is that, for every $X, Y \in \mathcal{E}$, the functor

$$\mathcal{E}(X \times (-), Y) \colon \mathcal{E}^{\mathrm{op}} \longrightarrow \infty\mathsf{Gpd}$$

is represented by an object $Y^X \in \mathcal{E}$, which we call the *internal hom* from X to Y. For every $X \in \mathcal{E}$ there is then a functor $(-)^X \colon \mathcal{E} \to \mathcal{E}$ which is right adjoint to $X \times (-)$. We denote by

$$\mathrm{ev}_{X,Y} \colon Y^X \times X \to Y$$

the counit of the adjunction at $X, Y \in \mathcal{E}$. Note also that, for every $T \in \mathcal{E}$, $\mathcal{E}(X \times T, Y)$ can equivalently be described as

$$\mathcal{E}_{/T}(X \times T, Y \times T),$$

where $X \times T$ is shorthand notation for the projection $X \times T \to T$, seen as an object of $\mathcal{E}_{/T}$.

Now, $\mathcal{E}$ is in fact *locally* cartesian closed, so for every $X \in \mathcal{E}$ and every $p, q \in \mathcal{E}_{/X}$, we get an internal hom q^p in $\mathcal{E}_{/X}$.

Notation 2.2.1. If $E = \mathrm{dom}(p)$ and $M = \mathrm{dom}(q)$, we write $[E, M]_X$ for the domain of q^p. In this way, we can consider q^p as a map

$$[E, M]_X \xrightarrow{q^p} X$$

in $\mathcal{E}$.

We now want to find a subobject of Y^X (or, more generally, of q^p), whose global elements correspond to equivalences $X \to Y$. To this end, let $J(\mathcal{E})$ be the *core* of

$\mathcal{E}$, that is, the maximal ∞-subgroupoid of $\mathcal{E}$. It can be explicitly described as the (strict) pullback of ∞-categories

$$\begin{tikzcd} J(\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d] \\ J(\mathrm{Ho}(\mathcal{E})) \arrow[r] & \mathrm{Ho}(\mathcal{E}) \end{tikzcd}$$

where $J(\mathrm{Ho}(\mathcal{E}))$ denotes the usual, 1-categorical, core functor. Since for every ∞-category $\mathcal{C}$ and every $X, Y \in \mathcal{C}$, $\mathrm{Ho}(\mathcal{C})(X,Y) \simeq \pi_0(\mathcal{C}(X,Y))$, it then follows that there is a pullback square

$$\begin{tikzcd} J(\mathcal{E})(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}(X,Y) \arrow[d] \\ \pi_0(J(\mathcal{E})(X,Y)) \arrow[r] & \pi_0(\mathcal{E}(X,Y)) \end{tikzcd} \tag{2.2}$$

and that the map $J(\mathcal{E})(X,Y) \to \mathcal{E}(X,Y)$ is a monomorphism. The assignment

$$T \mapsto J(\mathcal{E}_{/T})(X \times T, Y \times T)$$

thus defines a subfunctor of $\mathcal{E}(X \times (-), Y)$. It turns out this functor is itself representable.

Proposition 2.2.2 ([GK17, Thm. 2.10])**.** *For every $X, Y \in \mathcal{E}$, there is a subobject $\mathrm{Eq}_{\mathcal{E}}(X,Y)$ of Y^X such that, for every $T \in \mathcal{E}$, there is an equivalence of ∞-groupoids*

$$\mathcal{E}(T, \mathrm{Eq}_{\mathcal{E}}(X,Y)) \simeq J(\mathcal{E}_{/T})(X \times T, Y \times T),$$

natural in $T \in \mathcal{E}$. Furthermore, this is also true "locally", that is, for every two objects p, q in a slice category $\mathcal{E}_{/X}$.

Notation 2.2.3. For $p \colon E \to X$ and $q \colon M \to X$, we write $\mathrm{Eq}_{/X}(E, M)$ for the domain of $\mathrm{Eq}_{\mathcal{E}_{/X}}(p,q)$, so that $\mathrm{Eq}_{\mathcal{E}_{/X}}(p,q)$ is a map

$$\mathrm{Eq}_{/X}(E,M) \xrightarrow{\mathrm{Eq}_{\mathcal{E}_{/X}}(p,q)} X$$

in $\mathcal{E}$. We will often just write $\mathrm{Eq}(p,q)$ for $\mathrm{Eq}_{\mathcal{E}_{/X}}(p,q)$, if no risk of confusion arises.

In the following, it will be useful to have another, more explicit, description of $\mathrm{Eq}(X,Y)$. For $X,Y \in \mathcal{E}$, there is a map $c_{X,Y} \colon X^Y \times Y^X \to X^X$ obtained as the adjunct map to the composite

$$X^Y \times Y^X \times X \xrightarrow{X^Y \times \mathrm{ev}_{X,Y}} X^Y \times Y \xrightarrow{\mathrm{ev}_{Y,X}} X$$

Lemma 2.2.4. *For every $X,Y \in \mathcal{E}$, there is a pullback square*

$$\begin{tikzcd}
\mathrm{Eq}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^Y \times Y^X \times X^Y \arrow[d, "(c_{X,Y}\circ \mathrm{pr}_{12},\ c_{Y,X}\circ \mathrm{pr}_{13})"] \\
1 \arrow[r, "(\mathrm{id}_X, \mathrm{id}_Y)"'] & X^X \times Y^Y
\end{tikzcd}$$

where pr_{12} *(resp.* pr_{13}*) is the projection $Y^X \times X^Y \times X^Y \to Y^X \times X^Y$ onto the first two components (resp. onto the first and last components). This is also true "locally" for every $p,q \in \mathcal{E}_{/X}$.*

Note that, on global elements, the right vertical map sends $f \in \mathcal{E}(X,Y)$ and $g,h \in \mathcal{E}(Y,X)$ to $(gf, fh) \in \mathcal{E}(X,X) \times \mathcal{E}(Y,Y)$.

Proof. The statement for slice categories is proven exactly as the one for $\mathcal{E}$, so we just prove the latter. By [GK17, Lemma 2.8], it follows that there is an essentially unique inversion map $i\colon \mathrm{Eq}(X,Y) \to \mathrm{Eq}(Y,X)$, such that, for every $T \in \mathcal{E}$, the following diagram commutes

$$\begin{tikzcd}
 & J(\mathcal{E}_{/T})(X\times T, Y\times T) \arrow[r] \arrow[d, "(\mathrm{id},i)"'] & 1 \arrow[d, "\mathrm{id}_{X\times T}"] \\
 & J(\mathcal{E}_{/T})(X\times T, Y\times T)\times J(\mathcal{E}_{/T})(Y\times T, X\times T) \arrow[r] \arrow[d] & J(\mathcal{E}_{/T})(X\times T, X\times T) \arrow[d] \\
 & \mathcal{E}_{/T}(X\times T, Y\times T)\times \mathcal{E}_{/T}(Y\times T, X\times T) \arrow[r] & \mathcal{E}_{/T}(X\times T, X\times T)
\end{tikzcd}$$

where the middle and bottom horizontal arrows are composition maps and the unlabelled vertical maps are monomorphisms. In fact, for each T, i picks out an inverse for every equivalence $X \times T \to Y \times T$ over T. Now, let P be the pullback in $\mathcal{E}$ of the cospan displayed in the statement of the lemma. If we still denote by i the composite

$$\mathrm{Eq}(X,Y) \xrightarrow{i} \mathrm{Eq}(Y,X) \rightarrowtail X^Y,$$

there is a map

$$\mathrm{Eq}(X,Y) \to X^Y \times Y^X \times X^Y$$

given by the monomorphism $\mathrm{Eq}(X,Y) \rightarrowtail Y^X$ on the second component and by i on the other two components. The above-mentioned result from [GK17] then implies that this map determines a morphism $\varphi\colon \mathrm{Eq}(X,Y) \to P$.

We show that φ is an equivalence by verifying that $\mathcal{E}(T,\varphi)$ is an equivalence of ∞-groupoids for every $T \in \mathcal{E}$. For ease of exposition, we show this for $T = 1$ only; the same proof below goes through for a generic $T \in \mathcal{E}$ by working with $\mathcal{E}_{/T}(X\times T, Y\times T)$ rather than $\mathcal{E}(X,Y)$. There is a (homotopy) pullback in $\infty\mathsf{Gpd}$

$$\begin{tikzcd}
\mathcal{E}(1,P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}(Y,X)\times\mathcal{E}(X,Y)\times\mathcal{E}(Y,X) \arrow[d] \\
1 \arrow[r] & \mathcal{E}(X,X)\times\mathcal{E}(Y,Y)
\end{tikzcd}$$

whereas $\mathcal{E}(1,\mathrm{Eq}(X,Y)) \simeq J(\mathcal{E})(X,Y)$ can be described as the (strict) pullback (2.2).

Note that, by the description of equivalences in an ∞-category as those maps having a left and a right inverse, the composite map

$$\mathcal{E}(1,P) \to \mathcal{E}(Y,X) \times \mathcal{E}(X,Y) \times \mathcal{E}(Y,X) \to \mathcal{E}(X,Y)$$

automatically lands in $J(\mathcal{E})(X,Y)$, giving a map $\psi\colon \mathcal{E}(1,P) \to J(\mathcal{E})(X,Y)$ which takes (g,f,h) (for $f\colon X \to Y$, $gf \simeq \mathrm{id}$ and $fh \simeq \mathrm{id}$) to just f. On the other hand, the map $\widetilde{\varphi}\colon J(\mathcal{E})(X,Y) \to \mathcal{E}(1,P)$ induced by φ sends an equivalence $f\colon X \to Y$ to $(i(f), f, i(f))$, for a chosen inverse $i(f)$ of f. It follows that $\psi \circ \widetilde{\varphi} \simeq \mathrm{id}$. We conclude by observing that ψ is an equivalence. For, if $f \in J(\mathcal{E})(X,Y)$, by definition of $\mathcal{E}(1,P)$ we get that

$$\mathrm{hofib}_f(\psi) \simeq \mathrm{hofib}_{\mathrm{id}_Y}((-)\circ f) \times \mathrm{hofib}_{\mathrm{id}_X}(f \circ (-))$$

and both factors on the right are contractible, since both the maps

$$f \circ (-)\colon \mathcal{E}(Y,X) \to \mathcal{E}(Y,Y) \text{ and } (-)\circ f\colon \mathcal{E}(Y,X) \to \mathcal{E}(X,X)$$

are equivalences of ∞-groupoids when f is an equivalence. □

Remark 2.2.5. Given $f\colon X \to Y$, we denote by $\mathrm{biinv}(f)$ the corner in the pullback square

$$\begin{tikzcd} \mathrm{biinv}(f) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Eq}(X,Y) \arrow[d, tail] \\ 1 \arrow[r, "f"'] & Y^X \end{tikzcd}$$

Thus, $\mathrm{biinv}(f)$ is a (-1)-truncated object of $\mathcal{E}$ (i.e., it is a proposition) and we have that $\mathrm{biinv}(f) \simeq 1$ exactly when f is an equivalence in $\mathcal{E}$. (In fact, $\mathcal{E}(T, \mathrm{biinv}(f))$ is either empty or contractible precisely depending on whether or not $f \times T$ is an equivalence in $\mathcal{E}_{/T}$, for $T \in \mathcal{E}$.)

Definition 2.2.6 ([GK17, §3.1])**.** The *object of equivalences* for $p\colon E \to X$ is the object of $\mathcal{E}_{/X\times X}$ given by

$$\mathrm{Eq}_{\mathcal{E}_{/X\times X}}(p\times \mathrm{id}_X, \mathrm{id}_X\times p)\colon\ \mathrm{Eq}_{/X\times X}(p\times \mathrm{id}_X, \mathrm{id}_X\times p)\to X\times X$$

where $p\times \mathrm{id}_X\colon E\times X\to X\times X$ and similarly for $\mathrm{id}_X\times p$. We write the object of equivalences for p as

$$\mathrm{Eq}_{/X}(p)\colon\ \mathrm{Eq}_{/X}(E)\to X\times X$$

Note that, given a global element $(x,y)\colon 1\to X\times X$, a global element of the fiber of $\mathrm{Eq}_{/X}(p)$ over (x,y) is given by an equivalence $\mathrm{fib}_x(p)\to \mathrm{fib}_y(p)$.

Remark 2.2.7. There are pullback squares in $\mathcal{E}$

$$\begin{tikzcd} E\times X \arrow[r] \arrow[d, "p\times\mathrm{id}_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ X\times X \arrow[r, "\mathrm{pr}_1"'] & X \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} X\times E \arrow[r] \arrow[d, "\mathrm{id}_X\times p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ X\times X \arrow[r, "\mathrm{pr}_2"'] & X \end{tikzcd}$$

If $\Delta_X\colon X\to X\times X$ is the diagonal of X, we then get that

$$(\Delta_X)^*(p\times \mathrm{id}_X)=p=(\Delta_X)^*(\mathrm{id}_X\times p)$$

By the definition of Eq, it follows that the identity map

$$\mathrm{id}_p\in J(\mathcal{E}_{/X})(p,p)$$

induces a map $\mathsf{idtoequiv}\colon X\to \mathrm{Eq}_{/X}(E)$ over $X\times X$ as in

$$\begin{tikzcd} X \arrow[rr, "\mathsf{idtoequiv}"] \arrow[dr, "\Delta X"'] & & \mathrm{Eq}_{/X}(E) \arrow[dl, "\mathrm{Eq}_{/X}(p)"] \\ & X\times X & \end{tikzcd} \tag{2.3}$$

Definition 2.2.8. [GK17, §3.2] A *univalent map* is a map $p\colon E\to X$ in $\mathcal{E}$ for which the associated map $\mathsf{idtoequiv}\colon X\to \mathrm{Eq}_{/X}(E)$ is an equivalence in $\mathcal{E}_{/X\times X}$.

The following result tells us that univalent maps abound and it will be crucial in the proof of one of our main results (see Theorem 5.3.3).

Proposition 2.2.9 ([GK17, Prop. 3.8])**.** *Every classifying map p is univalent.*

Remark 2.2.10. The Fundamental Theorem of ∞-Topos Theory states that, for every $X \in \mathcal{E}$, $\mathcal{E}_{/X}$ is again an ∞-topos ([Lur09, Prop. 6.3.5.1]). Hence, all the results in this section apply equally well to $\mathcal{E}_{/X}$. In particular, for every local class S of maps in $\mathcal{E}_{/X}$, there are arbitrarily large regular cardinals κ such that there is a univalent classifying map for S_κ over X as in:

$$\begin{tikzcd} \widetilde{\mathcal{U}_\kappa^{S/X}} \arrow[rr, "p"] \arrow[dr] & & \mathcal{U}_\kappa^{S/X} \arrow[dl] \\ & X & \end{tikzcd}$$

We end this section with the following observation about truncated univalent map.

Lemma 2.2.11. *Let $p\colon E \to X$ be a univalent and n-truncated map in an ∞-topos $\mathcal{E}$, for $n \geq (-1)$. Then both E and X are $(n+1)$-truncated.*

Proof. If X is $(n+1)$-truncated, then so is E because p is n-truncated by hypothesis. In order to show that X is $(n+1)$-truncated, we can show that the diagonal ΔX is n-truncated. By univalence, ΔX is equivalent to $\mathrm{Eq}_{/X}(p)$, which is a subobject of $(\mathrm{id}_X \times p)^{(p \times \mathrm{id}_X)}$ in $\mathcal{E}_{/X^2}$. Therefore, it suffices to show that $(\mathrm{id}_X \times p)^{(p \times \mathrm{id}_X)}$ is n-truncated, because a subobject of an n-truncated object is n-truncated. Since $\mathrm{id}_X \times p$ is a pullback of p, it is an n-truncated object of $\mathcal{E}_{/X^2}$. But then

$$(\mathrm{id}_X \times p)^{(p \times \mathrm{id}_X)} = \prod_{p \times \mathrm{id}_X} (p \times \mathrm{id}_X)^*(\mathrm{id}_X \times p)$$

is also n-truncated, because dependent products, like any right adjoint, preserve n-truncated objects (see [Lur09, Prop. 5.5.6.16]). $\square$

Corollary 2.2.12. *Let $p\colon E \to X$ be the classifying map of monomorphisms in an ∞-topos. Then X is 0-truncated and E is contractible. In particular, p is the subobject classifier of $\tau_{\leq 0}(\mathcal{E})$, the ordinary 1-topos of 0-truncated objects of $\mathcal{E}$.*

Proof. We show that E is contractible. By Lemma 2.2.11 above, X is 0-truncated. Since id_1 is a monomorphism, there is a pullback square

$$\begin{tikzcd}
1 \arrow[r, "*"] \arrow[d, "\mathrm{id}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
1 \arrow[r, "\ulcorner 1 \urcorner"'] & X
\end{tikzcd}$$

for some map $\ulcorner 1 \urcorner\colon 1 \to X$, which is a monomorphism because X is 0-truncated. In particular, since E has a global element, it is (-1)-connected, that is, the map $E \to 1$ is an effective epimorphism $E \twoheadrightarrow 1$. The composite square

$$\begin{tikzcd}
E \arrow[r, two heads] \arrow[d, "\mathrm{id}_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[r, "*"] \arrow[d, "\mathrm{id}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
E \arrow[r, two heads] & 1 \arrow[r, "\ulcorner 1 \urcorner"'] & X
\end{tikzcd}$$

shows that id_E is the pullback of p along $E \twoheadrightarrow 1 \xrightarrow{\ulcorner 1 \urcorner} X$. Because p is a monomorphism, id_E is also the pullback of p along itself. Since p is a classifying map, we then get a commutative diagram

$$\begin{tikzcd}
E \arrow[rr, tail, "p"] \arrow[dr] & & X \\
& 1 \arrow[ur, tail, "\ulcorner 1 \urcorner"'] &
\end{tikzcd}$$

and then $E \twoheadrightarrow 1$ has to be an equivalence, as needed. $\square$

Notation 2.2.13. In light of Corollary 2.2.12 above, we follow the traditional convention in topos theory and denote by $t\colon 1 \rightarrowtail \Omega$ the classifying map for monomorphisms in our ∞-topos $\mathcal{E}$.

Remark 2.2.14. Given Corollary 2.2.12, one might wonder whether the result about the truncation level of the total space E in Lemma 2.2.11 can be sharpened. We provide an example of a 0-truncated univalent map $p\colon E \to X$, where E is 1-truncated, but not 0-truncated. Let $J(\mathcal{N})$ be the (1-)groupoid of natural numbers and bijections among them. Let $J(\mathcal{N})_*$ be the groupoid of pointed objects in $J(\mathcal{N})$. Upon applying the nerve functor, the forgetful functor $J(\mathcal{N})_* \to J(\mathcal{N})$ induces a 0-truncated map

$$N(J(\mathcal{N})_*) \to N(J(\mathcal{N}))$$

between 1-truncated objects in $\infty\mathsf{Gpd}$. This map is univalent, because it is a classifying map: it classifies the 0-truncated maps with finite fibers. The fiber of the diagonal map $\Delta(N(J(\mathcal{N})_*))$ over two points $(n,i),(m,j)$ in $N(J(\mathcal{N})_*)$ is $J(\mathcal{N})_*((n,i),(m,j))$, the set of pointed bijections from n to m. Hence, $N(J(\mathcal{N})_*)$ is 1-truncated, but not 0-truncated.

Chapter 3

Reflective subfibrations and classifying maps

In this chapter, we introduce the notion of reflective subfibrations $L_\bullet$ on an ∞-topos $\mathcal{E}$, which constitute our main object of study. Essentially, this is a collection of pullback-stable reflective subcategories $\mathcal{D}_X$ of $\mathcal{E}_{/X}$, with reflector L_X. We call the objects of $\mathcal{D}_X$, as X varies in $\mathcal{E}$, *L-local maps*. The definition is taken directly from [RSS17, §A.2].

In Section 3.1, we discuss some properties of reflective subfibrations that follow from their definition and from the general theory of reflective subcategories.

In Section 3.2, we show that, for a reflective subfibration $L_\bullet$ on $\mathcal{E}$, the class of L-local maps form a local class of maps (Proposition 3.2.3). Therefore, L-local maps admit a univalent classifying map (see Theorem 3.2.6). We can use this observation to link the concept of reflective subfibration on $\mathcal{E}$ to the notion of a reflective subuniverse in homotopy type theory, as given in [CORS18] and in [RSS17].

3.1 Reflective subfibrations

We give here the definition of reflective subfibrations on an ∞-topos $\mathcal{E}$ and establish some notation about them that we will use throughout our work. We also derive some immediate properties of reflective subfibrations.

Definition 3.1.1 ([RSS17, §A.2])**.** Let $\mathcal{E}$ be an ∞-topos.

1. A *system of reflective subcategories (srs)* $L_\bullet$ on $\mathcal{E}$ is the assignment, for each $X \in \mathcal{E}$, of an ∞-category $\mathcal{D}_X$ such that:

   - Each $\mathcal{D}_X$ is a reflective ∞-subcategory of $\mathcal{E}_{/X}$, with associated localization functor $L_X =: \mathcal{E}_{/X} \to \mathcal{E}_{/X}$. This is the composite of the reflector of $\mathcal{E}_{/X}$ into $\mathcal{D}_X$ and the inclusion of $\mathcal{D}_X$ into $\mathcal{E}_{/X}$. When $X = 1$, we write $\mathcal{D}$ for $\mathcal{D}_1$ and L for L_1. $\mathcal{D}$ is called the *underlying reflective subcategory* of the srs $L_\bullet$.

   - For every morphism $f\colon X \to Y$ in $\mathcal{E}$, the pullback functor $f^*\colon \mathcal{E}_{/Y} \to \mathcal{E}_{/X}$ restricts to a functor $\mathcal{D}_Y \to \mathcal{D}_X$ which we still denote by f^*.

   Note that an srs gives in particular a subfunctor of the functor $\mathcal{C}^{\mathrm{op}} \to \infty\mathsf{CAT}$ sending $X \in \mathcal{E}$ to $\mathcal{E}_{/X}$. Here $\infty\mathsf{CAT}$ is the ∞-category of ∞-categories.

2. An srs $L_\bullet$ on $\mathcal{E}$ is a *reflective subfibration* on $\mathcal{E}$, if, for any $f\colon X \to Y$ in $\mathcal{E}$ and any $p \in \mathcal{E}_{/Y}$, the induced map $L_X(f^*p) \to f^*(L_Y p)$ is an equivalence.

3. An srs $L_\bullet$ on $\mathcal{E}$ is *composing* if, whenever $p\colon X \to Y$ is in $\mathcal{D}_Y$ and $q\colon Y \to Z$ is in $\mathcal{D}_Z$, the composite qp is in $\mathcal{D}_Z$. In particular, if $X \in \mathcal{D}$, then $\mathcal{D}_X \subseteq \mathcal{D}_{/X}$.

4. A *modality* on $\mathcal{E}$ is a composing reflective subfibration $L_\bullet$ on $\mathcal{E}$.

Remark 3.1.2. For every object $X \in \mathcal{E}$ and every map $f\colon Y \to X$, we have that $(\mathcal{E}_{/X})_{/f} \simeq \mathcal{E}_{/Y}$ (see [Kap14, Lemma 4.18]). Therefore, for each $X \in \mathcal{E}$, a reflective subfibration $L_\bullet$ induces a reflective subfibration $L_\bullet^{/X}$ of $\mathcal{E}_{/X}$ by taking $\mathcal{D}_f^{/X}$ to be

$\mathcal{D}_Y$. It follows that all the results we give below about reflective subfibrations on an ∞-topos also hold "locally" in the ∞-topos $\mathcal{E}_{/X}$, for $X \in \mathcal{E}$.

From now on, we fix a reflective subfibration $L_\bullet$ on our favorite ∞-topos $\mathcal{E}$.

Notation 3.1.3. We adopt the following notation for the rest of this work.

- A morphism $p \colon E \to X$ is called *L-local* if, seen as an object of $\mathcal{E}_{/X}$, it is in $\mathcal{D}_X$. We call $E \in \mathcal{E}$ an *L-local object* if $E \to 1$ is an L-local map.
- For $X \in \mathcal{E}$, S_X denotes the class of all *L_X-equivalences*, i.e., maps $\alpha \colon p \to q$ in $\mathcal{E}_{/X}$ such that $L_X(\alpha)$ is an equivalence. Equivalently, $S_X = {}^\perp\mathcal{D}_X$, where ${}^\perp\mathcal{D}_X$ denotes the class of maps in $\mathcal{E}_{/X}$ which are left orthogonal to maps in $\mathcal{D}_X$ (see Definition 1.6.1). When it is clear that α is a map in $\mathcal{E}_{/X}$, we will often drop the explicit reference to the object X in our terminology, and just talk about *L-equivalences*.
- Given $p \in \mathcal{E}_{/X}$, we write $\eta_X(p) \colon p \to L_X(p)$ for the reflection (or localization) map of p into $\mathcal{D}_X$. Note that $\eta_X(p) \in S_X$. For $X \in \mathcal{E}$, we set $\eta(X) := \eta_1(X)$.

Recall from Section 1.6 our notation for the adjoints to pullback functors. The following remarks will be used extensively throughout.

Lemma 3.1.4. *Given $f \colon X \to Y$, we have:*

(i) $f^*(S_Y) \subseteq S_X$, that is, if $\alpha \colon p \to q$ is an L_Y-equivalence, then the induced map $f^*(p) \to f^*(q)$ on pullbacks is an L_X-equivalence;

(ii) $\Sigma_f(S_X) \subseteq S_Y$.

Proof. Given $q \in \mathcal{E}_{/Y}$ we know that $\eta_X(f^*q) = f^*(\eta_Y(q))$, so that, in particular, $f^*(\eta_Y(q)) \in S_X$. Since, given a map $\alpha \colon q \to q'$ in $\mathcal{E}_{/Y}$, $L_Y(\alpha)$ is the unique map with $L_Y(\alpha) \circ \eta_Y(q) = \eta_Y(q') \circ \alpha$, the first claim follows immediately. The second claim follows by an adjunction argument: if $\alpha \in S_X$, given a map $\beta \colon r \to s$ in $\mathcal{D}_Y$,

$\Sigma_f(\alpha) \perp_Y \beta \iff \alpha \perp_X f^*(\beta)$ and the latter holds since f^* restricts to a functor $\mathcal{D}_Y \to \mathcal{D}_X$. $\square$

Since L-local maps are closed under pullbacks, we can characterize S_X and $\mathcal{D}_X$ as follows.

Proposition 3.1.5. *The following hold for any $X \in \mathcal{E}$.*

(i) $p \in \mathcal{E}_{/X}$ is in $\mathcal{D}_X$ if and only if $\alpha \perp_X (p \to \mathrm{id}_X)$ for each $\alpha \in S_X$.

(ii) If $r \in \mathcal{D}_X$ and $f\colon X \to Y$ is any map in $\mathcal{E}$, then $\prod_f r$ is in $\mathcal{D}_Y$.

(iii) $\mathcal{D}_X$ is an exponential ideal in $\mathcal{E}_{/X}$, i.e., if $r \in \mathcal{D}_X$ and $p \in \mathcal{E}_{/X}$, then the internal hom r^p is also in $\mathcal{D}_X$.

(iv) L_X preserves products and S_X is closed under products in $\mathcal{E}_{/X}$.

(v) A map $\alpha\colon p \to q$ is in S_X if and only if, for each $r \in \mathcal{D}_X$, the map of internal homs $r^\alpha\colon r^q \to r^p$ is an equivalence.

(vi) $p \in \mathcal{E}_{/X}$ is in $\mathcal{D}_X$ if and only if p^α is an equivalence for each $\alpha \in S_X$.

Proof. For the first claim, since $S_X = {}^\perp\mathcal{D}_X$, $\mathcal{D}_X \subseteq (S_X)^\perp$. On the other hand, if p (that is, $p \to \mathrm{id}_X$) is right orthogonal to S_X, then there is a map $\gamma\colon L_X(p) \to p$ with $\gamma \circ \eta_X(p) = \mathrm{id}_p$ and it is easy to see that $\eta_X(p)$ is then an equivalence.

As for (ii), given $r \in \mathcal{D}_X$, $f\colon X \to Y$ and any map $\alpha \in S_Y$, adjointness gives that $\alpha \perp_Y \prod_f r \iff f^*(\alpha) \perp_X r$ and the latter orthogonality condition holds by Lemma 3.1.4 (i) and by the hypothesis that r is in $\mathcal{D}_X$. Since internal homs can be constructed via pullbacks and dependent products, it follows that $\mathcal{D}_X$ is an exponential ideal, establishing (iii). It is straightforward to check that this latter condition is equivalent to L_X preserving and S_X being closed under products in $\mathcal{E}_{/X}$, proving (iv).

As for (v), $\mathcal{D}_X$ being an exponential ideal implies that, for every $p \in \mathcal{E}_{/X}$ and $r \in \mathcal{D}_X$, $r^{\eta_X(p)}\colon r^{L_X(p)} \to r^p$ is an equivalence. Thus, if $\alpha\colon p \to q$ is in S_X, then r^α is equivalent in $\mathcal{E}^{\bullet\to\bullet}$ to the equivalence $r^{L_X(\alpha)}$. Conversely, if r^α is an equivalence for every $r \in \mathcal{D}_X$, then, given a map $\beta\colon r \to s$ in $\mathcal{D}_X$, consider the diagram

$$\begin{tikzcd}
r^q \arrow[dr, dashed] \arrow[drr, bend left, "r^\alpha"] \arrow[ddr, bend right] & & \\
& s^q \times_{s^p} r^p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & r^p \arrow[d] \\
& s^q \arrow[r, "s^\alpha"'] & s^p
\end{tikzcd}$$

Since both s^α and r^α are equivalences, the dotted map is also such, which implies that $\alpha \perp_X \beta$. Finally, (vi) follows immediately from (i), using closure under products of S_X. $\square$

Remark 3.1.6. Proposition 3.1.5 (i) and (v) can be restated as saying that a map $\alpha\colon p \to q$ in $\mathcal{E}_{/X}$ is such that, for every $r \in \mathcal{D}_X$, $\mathcal{E}_{/X}(\alpha, r)$ is an equivalence of ∞-groupoids if and only if r^α is an equivalence in $\mathcal{E}$. In this case, $\alpha \in S_X$. Similarly, $t \in \mathcal{E}_{/X}$ is in $\mathcal{D}_X$ if and only if, for every $\alpha \in S_X$, $\mathcal{E}_X(\alpha, t)$ is an equivalence, if and only if t^α is an equivalence. The external-hom description of L-equivalences is the common one in higher category theory, whereas the internal-hom description is the one available in homotopy type theory. (In fact, homotopy type theory can not even *state* the external description, which provides some added value to the homotopy theoretic approach to localization.) Reflective subfibrations are defined so that these two perspectives on localization coincide.

We will also need the following cancellation property of L-local maps.

Proposition 3.1.7. *Suppose given composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ such that $g, gf \in \mathcal{D}_Z$. Then $f \in \mathcal{D}_Y$.*

Proof. Let $\alpha\colon p \to q$ be a map in $\mathcal{E}_{/Y}$ which is an L_Y-equivalence. To show that f is in $\mathcal{D}_Y$, we need to show that the induced map of ∞-groupoids

$$\mathcal{E}_{/Y}(\alpha, f)\colon \mathcal{E}_{/Y}(q, f) \to \mathcal{E}_{/Y}(p, f)$$

is an equivalence. We will prove this fact by realizing such a map as the comparison map of (homotopy) fiber sequences in a pullback square. Consider the map $\Sigma_g(\alpha)\colon gp \to gq$ and let $\overline{f}\colon gf \to g$ be the map induced by f. We have similar maps $\overline{q}\colon gq \to g$ and $\overline{p}\colon gp \to g$. By Lemma 3.1.4 (ii), $\Sigma_g(\alpha)$ is an L_Z-equivalence. Since both g and gf are in $\mathcal{D}_Z$ by hypothesis, the vertical maps in the commutative square

$$\begin{tikzcd}
\mathcal{E}_{/Z}(gq, gf) \arrow[r, "\mathcal{E}_{/Z}(gq,\overline{f})"] \arrow[d, "\mathcal{E}_{/Z}(\Sigma_g(\alpha),gf)"'] & \mathcal{E}_{/Z}(gq, g) \arrow[d, "\mathcal{E}_{/Z}(\Sigma_g(\alpha),g)"] \\
\mathcal{E}_{/Z}(gp, gf) \arrow[r, "\mathcal{E}_{/Z}(gp,\overline{f})"'] & \mathcal{E}_{/Z}(gp, g)
\end{tikzcd}$$

are equivalences. Hence, the square is a pullback. We can now take the induced map on fiber sequences. By the dual of [Lur09, Lemma 5.5.5.12], we have:

$$\mathrm{hofib}_{\overline{q}}\left(\mathcal{E}_{/Z}(gq,\overline{f})\right) = \left(\mathcal{E}_{/Z}\right)_{/g}(\overline{q},\overline{f}) \simeq \mathcal{E}_{/Y}(q,f)$$

Since $\overline{q}\left(\Sigma_g(\alpha)\right) = \overline{p}$, we similarly get that

$$\mathrm{hofib}_{\overline{q}(\Sigma_g(\alpha))}(\mathcal{E}_{/Z}(gp,\overline{f})) \simeq \mathcal{E}_{/Y}(p,f)$$

and the induced map on fiber sequences is $\mathcal{E}_{/Y}(\alpha, f)$, as required. $\square$

Corollary 3.1.8. *If X, Y are L-local objects, then any map $f\colon X \to Y$ is L-local. In particular, if L is a modality and $X \in \mathcal{D}$, then $\mathcal{D}_X = \mathcal{D}_{/X}$.* $\square$

Corollary 3.1.9. *If $g\colon A \to B$ is L-local, then so is $\Delta g\colon A \to A \times_B A$.*

Proof. Consider the pullback square in $\mathcal{E}$

$$\begin{tikzcd}
A \times_B A \arrow[r] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "g"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

Since $g \in \mathcal{D}_B$, the displayed map pr is in $\mathcal{D}_A$. But $\mathrm{pr} \circ \Delta g = \mathrm{id}_A$, so the claim follows by Proposition 3.1.7, since $\mathrm{id}_A \in \mathcal{D}_A$. $\square$

3.2 Classification of L-local maps

We show here that, for a reflective subfibration $L_\bullet$ on an ∞-topos $\mathcal{E}$, the L-local maps form a local class of maps in $\mathcal{E}$ and, therefore, they admit a univalent classifying map.

Let

$$\mathcal{M}^L := \bigcup_{X \in \mathcal{E}} \mathrm{Ob}(\mathcal{D}_X).$$

Thus, $\mathcal{M}^L$ is the class of all L-local maps. Observe that $\mathcal{M}^L$ is stable under pullbacks: if $p \colon E \to X$ is in $\mathcal{M}^L$ and $f \colon Y \to X$ is an arbitrary map in $\mathcal{E}$, then $f^*(p)$ is in $\mathcal{M}^L$, by Definition 3.1.1.

Lemma 3.2.1. *$\mathcal{M}^L$ is closed under arbitrary small coproducts: if I is a set and $f_j \in \mathcal{D}_{X_j}$ for $j \in I$, then $\coprod_j f_j$ is in $\mathcal{D}_{(\coprod_j X_j)}$.*

Proof. For each $A \in \mathcal{E}$, id_A is L-local since it is the terminal object in $\mathcal{E}_{/A}$. In particular, id_0 is an L-local map, where 0 is the initial object of $\mathcal{E}$. This takes care of closure under empty coproducts. Since colimits in an ∞-topos are universal, there is an equivalence

$$\mathcal{E}_{/\coprod_j X_j} \xrightarrow{\simeq} \prod_j \mathcal{E}_{/X_j}$$

given by taking pullbacks along the inclusions $\iota_j \colon X_j \to \coprod_j X_j$. It follows that, given a map α in $\mathcal{E}_{/\coprod_j X_j}$,

$$\alpha \perp_{\coprod_j X_j} \coprod_j f_j \iff (\iota_k)^*(\alpha) \perp_{X_k} f_k \quad \text{for all } k \in I.$$

(Note that $(\iota_k)^*(\coprod_j f_j) = f_k$ because coproducts in $\mathcal{E}$ are disjoint.) The latter condition is true whenever $\alpha \in S_{(\coprod_j X_j)}$, thanks to Lemma 3.1.4 (i). $\square$

Lemma 3.2.2. *Given any pullback square in $\mathcal{E}$*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where f is an effective epimorphism, p is in $\mathcal{M}^L$ if and only if q is in $\mathcal{M}^L$.

Proof. By [Lur09, Lemma 6.2.3.16], the statement is true if we replace "being in $\mathcal{M}^L$" with "being an equivalence". Suppose $p \in \mathcal{D}_X$ and consider $\eta_Y(q)\colon q \to L_Y(q)$. We know that $f^*(\eta_Y(q)) = \eta_X(f^*(q)) = \eta_X(p)$ and then $f^*(\eta_Y(q))$ must be an equivalence, since $p \in \mathcal{D}_X$. By the opening observation, $\eta_Y(q)$ is also an equivalence, so that $q \in \mathcal{D}_Y$. ◻

We have thus proved the following result.

Proposition 3.2.3. *The class $\mathcal{M}^L = \bigcup_{X\in\mathcal{E}} \mathrm{Ob}(\mathcal{D}_X)$ of all L-local maps for a reflective subfibration on $\mathcal{E}$ is a local class of maps of $\mathcal{E}$.*

We can use the above proposition to characterize reflective subfibrations on ∞-groupoids as fiberwise localizations. The proof of the result below is a typical example of how to use the fact that $\mathcal{M}^L$ is a local class of maps in practice.

Corollary 3.2.4. *If $\mathcal{E} = \infty\mathsf{Gpd}$, a map $p\colon E \to X$ is L-local if and only if, for every $x \in X$, the homotopy fiber $\mathrm{hofib}_x(p)$ is an L-local ∞-groupoid.*

Proof. If $\mathcal{E} = \infty\mathsf{Gpd}$, the canonical map $s\colon \coprod_{x\in X} 1 \longrightarrow X$ is an effective epimorphism since it induces a surjection on path components. Since colimits in an ∞-topos are universal, we have a pullback square

$$\begin{tikzcd} \coprod_{x\in X} \mathrm{hofib}_x(p) \arrow[r] \arrow[d, "s'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ \coprod_{x\in X} 1 \arrow[r, "s"'] & X \end{tikzcd}$$

where s' is the coproduct of the maps $\mathrm{hofib}_x(p) \to 1$. Thus, p is L-local if and only if s' is L-local, by Lemma 3.2.2. Since, for every $x_0 \in X$, the pullback of s along the inclusion $x_0 \colon 1 \to \coprod_{x \in X} 1$ is $\mathrm{hofib}_{x_0}(p) \to 1$, Lemma 3.2.1 and stability under pullbacks of L-local maps give us that s' is L-local if and only if every $\mathrm{hofib}_{x_0}(p)$ is an L-local ∞-groupoid, as required. ☐

Remark 3.2.5. The above corollary can be generalized to any ∞-topos $\mathcal{E}$ upon suitably replacing $\{1\}$ with a set C of κ-compact objects of $\mathcal{E}$ such that, if $\mathcal{C}$ is the full subcategory of $\mathcal{E}$ spanned by the objects in C, every $X \in \mathcal{E}$ is a colimit of the canonical diagram $\mathcal{C}_{/X} \to \mathcal{E}$. (Such a set C always exists for any locally presentable ∞-category, thanks to the proof of (5) $\implies$ (6) in [Lur09, Thm. 5.5.1.1], combined with [Lur09, Lemma 5.1.5.3].) Indeed, in this case, for every $X \in \mathcal{E}$, the canonical map

$$\left(\coprod_{A \in C,\ A \to X} A \right) \to X$$

is an effective epimorphism, by [Lur09, Lemma 6.2.3.13]. By the same argument used in the proof of Corollary 3.2.4, $p \colon E \to X$ is L-local if and only if, for every map $j \colon A \to X$ with $A \in C$, the pullback map $A \times_X E \to A$ is L-local. If every object in C is L-local and L is a modality, this is the same as each object $A \times_X E$ being L-local. For example, if $\mathcal{E} = \mathsf{Pre}(\mathcal{D})$ (the ∞-category of presheaves over a small ∞-category $\mathcal{D}$) and L is a modality such that every representable functor is L-local, then a map $p \colon E \to X$ is L-local if and only if, for every map $\mathcal{D}(-, D) \to X$, the pullback object $\mathcal{D}(-, D) \times_X E$ is L-local.

Thanks to Proposition 2.1.7 and Proposition 2.2.9, Proposition 3.2.3 imply the following result.

Theorem 3.2.6. *Let κ be a regular cardinal as in Remark 2.1.8 and let $\mathcal{M}_\kappa^L$ be the class of maps in $\mathcal{E}$ which are L-local and relatively κ-compact. Then $\mathcal{M}_\kappa^L$ has a classifying map*

$$\widetilde{\mathcal{U}_\kappa^L} \to \mathcal{U}_\kappa^L$$

which is univalent.

Remark 3.2.7. As in (2.1), there is a pullback square

$$\begin{tikzcd} \widetilde{\mathcal{U}_\kappa^L} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}_\kappa} \arrow[d] \\ \mathcal{U}_\kappa^L \arrow[r, "l_\kappa"'] & \mathcal{U}_\kappa \end{tikzcd}$$

Since both $\widetilde{\mathcal{U}_\kappa^L} \to \mathcal{U}_\kappa^L$ and $\widetilde{\mathcal{U}_\kappa} \to \mathcal{U}_\kappa$ are univalent, [GK17, Cor. 3.10] says that l_κ is a monomorphism. Therefore there is a pullback square

$$\begin{tikzcd} \mathcal{U}_\kappa^L \arrow[r] \arrow[d, tail, "l_\kappa"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "t"] \\ \mathcal{U}_\kappa \arrow[r, "\mathsf{IsLocal}_\kappa"'] & \Omega \end{tikzcd}$$

where t is the classifying map for monomorphisms, as in Notation 2.2.13. In other words, $\widetilde{\mathcal{U}_\kappa^L} \to \mathcal{U}_\kappa^L$ (and hence all the relatively κ-compact L-local maps) determines and is determined by the map $\mathsf{IsLocal}_\kappa \colon \mathcal{U}_\kappa \to \Omega$, through which L-local types are introduced in homotopy type theory, where $\mathcal{U}^L$ is called a *subuniverse* of the *universe* $\mathcal{U}$ (see [CORS18, Def. 2.1]). Note that, given a κ-compact object $X \in \mathcal{E}$, the associated characteristic map $1 \to \mathcal{U}_\kappa$ factors through the monomorphism $l_\kappa \colon \mathcal{U}_\kappa^L \rightarrowtail \mathcal{U}_\kappa$ (that is, X is L-local) if and only if the pullback of the composite

$$1 \longrightarrow \mathcal{U}_\kappa \xrightarrow{\mathsf{IsLocal}_\kappa} \Omega$$

along $1 \to \Omega$ gives a (-1)-truncated object of $\mathcal{E}$ which is equivalent to 1.

Chapter 4

L-connected maps

We study here properties of another class of maps associated with a reflective subfibration $L_\bullet$ on an ∞-topos $\mathcal{E}$, the *L-connected maps*.

Section 4.1 contains the important definitions that we will need later on, and a few technical properties of L-conected maps. The hasty reader can ignore the material coming after Remark 4.1.2 and move forward to Chapter 5.

Those willing to read the rest of the chapter will see their patience rewarded in Section 4.2. We prove there that every stable factorization system on $\mathcal{E}$ determines a modality and, conversely, every modality $L_\bullet$ on $\mathcal{E}$ gives rise to a stable factorization system whose left class is given by the L-connected maps and whose right class is given by the L-local maps (Theorem 4.2.5). In the context of homotopy type theory, this correspondence is proven in [RSS17, §1], through some intermediate steps. Although some overlap between our proof and the one in [RSS17] certainly occurs, we did not follow the work there to formulate our arguments. We conclude Section 4.2 by discussing a special kind of modality on $\mathcal{E}$ which is associated to any left exact reflective subcategory of $\mathcal{E}$.

4.1 Definition and basic properties

Given a reflective subfibration $L_\bullet$ on $\mathcal{E}$, we give here the definition of L-connected maps and prove a few technical properties about them that will be used in Section 4.2.

Definition 4.1.1. $f \in \mathcal{E}_{/X}$ is said to be an *L-connected map (in $\mathcal{E}$)* if $L_X(f) \simeq \mathrm{id}_X$. Equivalently, f is L-connected if

$$(f \xrightarrow{\eta_X(f)} L_X(f)) \simeq (f \xrightarrow{f} \mathrm{id}_X)$$

as objects in the arrow category of $\mathcal{E}_{/X}$, where the equivalence is given by id_f and $L_X(f) \to \mathrm{id}_X$. We sometimes refer to this fact by saying that an L-connected map f is *its own reflection map.*

In particular, an L-connected map $f\colon E \to X$ is an L_X-equivalence when seen as a map $f\colon f \to \mathrm{id}_X$ in $\mathcal{E}_{/X}$.

Remark 4.1.2. By taking the reflection of $f \in \mathcal{E}_{/X}$ into $\mathcal{D}_X$ and using stability under pullbacks of reflection maps (see Definition 3.1.1 2.), it follows immediately that L-connected maps are stable under pullbacks along arbitrary maps.

Remark 4.1.3. Suppose $p\colon E \to X$ and $q\colon M \to X$ are maps in $\mathcal{E}$ and let $\alpha\colon p \to q$ be a map from p to q in $\mathcal{E}_{/X}$. Consider the sliced reflective subfibration $L_\bullet^{/X}$ given in Remark 3.1.2. Then, α being $L^{/X}$-connected means that $\Sigma_X(\alpha)\colon E \to M$ is L-connected.

Lemma 4.1.4. *Suppose given composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and suppose that f is L-connected. Then g is L-connected if and only if gf is L-connected.*

Proof. Let $\eta_Z(g)\colon g \to L_Z(g)$ be the reflection map of $g \in \mathcal{E}_{/Z}$ into $\mathcal{D}_Z$. The hypothesis on f means that the map $f\colon f \to \mathrm{id}_Y$ is the reflection map of f into $\mathcal{D}_Y$. By Lemma 3.1.4 (ii), it then follows that $\Sigma_g(f)\colon gf \to g$ is an L_Z-equivalence and so the composite map in $\mathcal{E}_{/Z}$ given by

$$gf \xrightarrow{\Sigma_g f} g \xrightarrow{\eta_Z(g)} L_Z(g)$$

is the reflection map of gf into $\mathcal{D}_Z$. The claim now follows. $\square$

Lemma 4.1.5. *If $L_\bullet$ is a modality on $\mathcal{E}$, then, for every map $f\colon E \to X$, the reflection map $\eta_X(f)\colon f \to L_X(f)$ is L-connected.*

Proof. We prove this result for $X = 1$, the general case having the same proof. Let $\eta(E)\colon E \to LE$ be the reflection map of E and let $n\colon \eta(E) \to L_{LE}(\eta(E))$ be the reflection of $\eta(E)$ into $\mathcal{D}_{LE}$ (that is, $n = \eta_{LE}(\eta(E))$). The situation can be depicted as follows

$$\begin{tikzcd} & & L_{LE}(E) \arrow[d, "L_{LE}(\eta(E))"] \\ E \arrow[rru, bend left, "n"] \arrow[rr, "\eta(E)"] & & LE \end{tikzcd}$$

By Lemma 3.1.4 and since L is a modality, n is an L_1-equivalence into an L-local object and it is therefore equivalent to $\eta(E)$ via the map $L_{LE}(\eta(E))$. Hence, $\eta(E)$ is L-connected. $\square$

Lemma 4.1.6. *Let $L_\bullet$ be a reflective subfibration on $\mathcal{E}$. Then the following hold.*

(i) Suppose $p\colon E \to X$ is a map in $\mathcal{E}$ with the property that $f \perp p$ for every L-connected map f. Then p is an L-local map.

(ii) Suppose that $L_\bullet$ is a modality and let $f\colon A \to B$ be a map in $\mathcal{E}$ with the property that $f \perp p$ for every L-local map p. Then f is an L-connected map.

Proof. We start by proving (i). Consider the reflection map of p into $\mathcal{D}_X$ given by

$$\begin{tikzcd} E \arrow[rr, "\eta_X(p)"] \arrow[dr, "p"'] & & L_X(E) \arrow[dl, "L_X(p)"] \\ & X & \end{tikzcd}$$

Then, by Lemma 4.1.5, $\eta_X(p)$ is L-connected. Therefore, by the hypothesis on p, there is a unique $n\colon L_X(E) \to E$ with $n\eta_X(p) = \mathrm{id}_E$ and $pn = L_X(p)$. In particular, p is a retract of the L-local map $L_X(p)$ and it is therefore an L-local map itself.

As for (ii), consider the reflection map of f into $\mathcal{D}_B$ given by

$$\begin{tikzcd} A \arrow[rr, "\eta_B(f)"] \arrow[dr, "f"'] & & L_B(A) \arrow[dl, "L_B(f)"] \\ & B & \end{tikzcd}$$

The hypothesis on f implies that there is a unique $s\colon B \to L_B(A)$ with $sf = \eta_B(f)$ and $L_B(f)s = \mathrm{id}_B$. In particular, $sL_B(f)$ can be seen as a map $L_B(f) \to L_B(f)$ in $\mathcal{E}_{/B}$. Precomposing this map with $\eta_B(f)$, we deduce that $sL_B(f) = \mathrm{id}$. Hence, s is an equivalence and f is L-connected, by Lemma 4.1.5. □

4.2 Stable factorization systems are modalities

In [ABFJ17a] the term "modality" is used as a synonym for a stable factorization system on an ∞-topos $\mathcal{E}$. We would like to reconcile that terminology with the definition of modality given in Definition 3.1.1. Namely, we want to show that to every stable factorization system on $\mathcal{E}$ one can associate a reflective subfibration which is a modality and, conversely, that every modality on $\mathcal{E}$ gives rise to a stable factorization system.

We start by recalling what a stable factorization system is. The reader might also want to refer back to Definition 1.6.1 and the discussion after it. All the background definitions and results that we report below are taken from [ABFJ17a, §3.1].

Definition 4.2.1. Let $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ be a pair of classes of maps in $\mathcal{E}$.

1. We say that $\mathcal{F}$ is a *factorization system on* $\mathcal{E}$ if $\mathcal{L} = {}^{\perp}\mathcal{R}$, $\mathcal{R} = \mathcal{L}^{\perp}$ and every map in $\mathcal{E}$ admits a factorization into a map in $\mathcal{L}$ followed by a map in $\mathcal{R}$. The classes $\mathcal{L}$ and $\mathcal{R}$ are called the *left class* and the *right class* of the factorization

system $\mathcal{F}$, respectively.

2. We say that a factorization system $\mathcal{F}$ on $\mathcal{E}$ is *stable* if the left class $\mathcal{L}$ is stable under pullbacks. (The right class $\mathcal{R}$ is always stable under pullbacks.)

It follows from the definition of a factorization system $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ that the factorization of a map into a left map followed by a right map is unique up to unique equivalence.

Example 4.2.2. For every $n \geq -2$, the n-truncated maps in an ∞-topos $\mathcal{E}$ form the right class of a stable factorization system, whose left class is given by the n-connected maps (see [ABFJ17a, Prop. 3.5.6] and [Lur09, §6.5.1]).

For reference, we record here the following partial 2-out-of-3 properties satisfied by the left and right classes of a factorization system.

Lemma 4.2.3 ([ABFJ17a, Lemma 3.1.6 (3)])**.** *Let $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ be a factorization system on $\mathcal{E}$ and let f and g be composable morphisms in $\mathcal{E}$.*

(i) If $f \in \mathcal{L}$, then $gf \in \mathcal{L}$ if and only if $g \in \mathcal{L}$.

(ii) If $g \in \mathcal{R}$, then $gf \in \mathcal{R}$ if and only if $f \in \mathcal{R}$.

Given a class $\mathcal{M}$ of maps in $\mathcal{E}$ and an object $X \in \mathcal{E}$, we let $\mathcal{M}_X$ be the class of maps in $\mathcal{E}_{/X}$ that are mapped into $\mathcal{M}$ by the forgetful functor $\mathcal{E}_{/X} \to \mathcal{E}$. We can use this construction to lift factorization systems to slice categories.

Lemma 4.2.4 ([ABFJ17a, Lemma 3.1.7])**.** *Let $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ be a factorization system on $\mathcal{E}$. Then, for every $X \in \mathcal{E}$, $\mathcal{F}_X := (\mathcal{L}_X, \mathcal{R}_X)$ is a factorization system on $\mathcal{E}_{/X}$.*

We are now ready to prove the following result.

Theorem 4.2.5. *Let $\mathcal{E}$ be an ∞-topos.*

1. Let $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ be a stable factorization system on $\mathcal{E}$. There exists a modality $L^{\mathcal{F}}_{\bullet}$ on $\mathcal{E}$ whose local maps are exactly the maps in $\mathcal{R}$.

2. *Let $L_\bullet$ be a modality on $\mathcal{E}$. Let $\mathcal{L}$ be the class of L-connected maps and $\mathcal{R}$ the class of L-local maps. Then $\mathcal{F}_L = (\mathcal{L}, \mathcal{R})$ is a stable factorization system on $\mathcal{E}$.*

Proof. We prove the two statements separately and we begin by proving the first claim.

Assume that $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ is any factorization system on $\mathcal{E}$. Set $\mathcal{D} := \mathcal{R}_{/1}$, the full subcategory of $\mathcal{E}$ spanned by those $X \in \mathcal{E}$ such that the map $X \to 1$ is in $\mathcal{R}$. It follows from uniqueness and functoriality of the $(\mathcal{L}, \mathcal{R})$-factorizations (see [ABFJ17a, § 3.1] and [Lur09, Prop. 5.2.8.17]) that $\mathcal{D}$ is a reflective subcategory of $\mathcal{E}$. For $X \in \mathcal{E}$, the value $L(X) \in \mathcal{D}$ of the reflector and the unit map $\eta(X) \colon X \to L(X)$ are determined by the fact that

$$X \xrightarrow{\eta(X)} L(X) \to 1$$

is the factorization of the map $X \to 1$. In particular, $\eta(X)$ is a map in $\mathcal{L}$, which gives immediately the needed universal property for the unit map. We can then apply the same considerations to the factorization system $(\mathcal{L}_X, \mathcal{R}_X)$ on $\mathcal{E}_{/X}$, hence obtaining a reflective subcategory $\mathcal{D}_X := (\mathcal{R}_X)_{/\mathrm{id}_X}$ of $\mathcal{E}_{/X}$, for every $X \in \mathcal{E}$. Note that, by definition, $p \in \mathcal{E}_{/X}$ is in $\mathcal{D}_X$ if and only if it is in $\mathcal{R}$ when considered as a map in $\mathcal{E}$. Since the class $\mathcal{R}$ is closed under pullbacks along arbitrary maps and under compositions with maps in $\mathcal{R}$ (see Lemma 4.2.3), it follows that the assignment $X \mapsto \mathcal{D}_X$ so defined gives rise to a composing srs $L_\bullet^{\mathcal{F}}$ on $\mathcal{E}$ (see Definition 3.1.1). It is straightforward to see that $L_\bullet^{\mathcal{F}}$ is a reflective subfibration precisely when $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ is a stable factorization system.

We now prove the second claim. Let $L_\bullet$ be a modality on $\mathcal{E}$ and let $\mathcal{F}_L = (\mathcal{L}, \mathcal{R})$ be as in the statement of the theorem. For any $f \colon E \to X$ in $\mathcal{E}$, the reflection of f into $\mathcal{D}_X$ given by

$$\begin{tikzcd} E \arrow[rr, "\eta_X(f)"] \arrow[dr, "f"'] & & L_X(E) \arrow[dl, "L_X(f)"] \\ & X & \end{tikzcd}$$

is an $\mathcal{F}_L$-factorization of f, by Lemma 4.1.5. Both $\mathcal{L}$ and $\mathcal{R}$ contain all equivalences and are closed under composition, by Lemma 4.1.4 and because $L_\bullet$ is a modality. Furthermore, by Remark 4.1.2, the left class is closed under pullbacks, while Lemma 4.1.6 says that $\mathcal{L}^\perp \subseteq \mathcal{R}$ and ${}^\perp\mathcal{R} \subseteq \mathcal{L}$.

Thus, to conclude that $\mathcal{F}_L$ is a factorization system, we just need to show that the reverse inclusions also hold, that is, we need to prove that, for every L-connected map $f\colon X \to Y$ and for every L-local map $p\colon E \to Z$, we have that $f \perp p$. This amounts to showing that the following commutative diagram in $\infty\mathsf{Gpd}$

$$\begin{tikzcd}
\mathcal{E}(Y,E) \arrow[r, "\mathcal{E}(Y,p)"] \arrow[d, "\mathcal{E}(f,E)"'] & \mathcal{E}(Y,Z) \arrow[d, "\mathcal{E}(f,Z)"] \\
\mathcal{E}(X,E) \arrow[r, "\mathcal{E}(X,p)"'] & \mathcal{E}(X,Z)
\end{tikzcd}$$

is a pullback square. Equivalently, we can check that the induced map on fibers is an equivalence. By looking at the fiber over $k \in \mathcal{E}(Y,Z)$, such an induced map is given by

$$\mathcal{E}_{/Z}(\overline{f},p)\colon \mathcal{E}_{/Z}(k,p) \to \mathcal{E}_{/Z}(kf,p),$$

where $\overline{f}$ is given by considering f as a map $kf \to k$ in $\mathcal{E}_{/Z}$. This map fits into the following commutative square in $\infty\mathsf{Gpd}$

$$\begin{tikzcd}
\mathcal{E}_{/Z}(L_Z(k),p) \arrow[r, "\mathcal{E}_{/Z}(\eta_Z(k){,}p)"] \arrow[d, "\mathcal{E}_{/Z}(L_Z(\overline{f}){,}p)"'] & \mathcal{E}_{/Z}(k,p) \arrow[d, "\mathcal{E}_{/Z}(\overline{f}{,}p)"] \\
\mathcal{E}_{/Z}(L_Z(kf),p) \arrow[r, "\mathcal{E}_{/Z}(\eta_Z(kf){,}p)"'] & \mathcal{E}_{/Z}(kf,p)
\end{tikzcd}$$

(Here, $\eta_Z(k)\colon k \to L_Z(k)$ is the reflection of k into $\mathcal{D}_Z$ and similarly for $\eta_Z(kf)$.) Note that the horizontal maps are equivalences because p is L-local by hypothesis. On the other hand, since f is L-connected, the map $f\colon f \to \mathrm{id}_Y$ is an L_Y-equivalence and so $\overline{f} = \Sigma_k(f)$ is an L_Z-equivalence, by Lemma 3.1.4 (ii). Therefore, $L_Z(\overline{f})$ is an equivalence. It follows that $\mathcal{E}_{/Z}(\overline{f},p)$ is an equivalence, since the other three maps in the diagram above are equivalences. This concludes the proof that $f \perp g$, and that

$\mathcal{F}_L$ is a stable factorization system. $\square$

Corollary 4.2.6. *The assignments $\mathcal{F} \mapsto L_\bullet^{\mathcal{F}}$ and $L_\bullet \mapsto \mathcal{F}_L$ determine a bijective correspondence between the class of stable factorization systems on an ∞-topos $\mathcal{E}$ and the class of collections $\{\mathcal{D}_X\}_{X\in\mathcal{E}}$ of reflective subcategories $\mathcal{D}_X \subseteq \mathcal{E}_{/X}$ which form the L-local maps for some modality $L_\bullet$ on $\mathcal{E}$.*

Proof. If $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ is a stable factorization system, the right class of $\mathcal{F}_{L_\bullet^{\mathcal{F}}}$ is again $\mathcal{R}$ and then the left class has to be $\mathcal{L}$ since $\mathcal{L} = {}^{\perp}\mathcal{R}$. Thus, $\mathcal{F} = \mathcal{F}_{L_\bullet^{\mathcal{F}}}$. If $L_\bullet$ is the modality associated to $\{\mathcal{D}_X\}_{X\in\mathcal{E}}$, then, by definition, the reflective subcategories $\widetilde{\mathcal{D}}_X$ associated to the modality $L_\bullet^{\mathcal{F}_L}$ are given by the L-local maps with codomain $X \in \mathcal{E}$. Therefore, $\widetilde{\mathcal{D}}_X = \mathcal{D}_X$, for every $X \in \mathcal{E}$. $\square$

Example 4.2.7. By Example 4.2.2 and applying the above theorem, we get that, for every $n \geq -2$, there is an associated modality $L_\bullet^n$ on $\mathcal{E}$, for which the L-local maps are exactly the n-truncated maps. We call this modality the *n-truncated* modality on $\mathcal{E}$.

We conclude this section by applying Theorem 4.2.5 to construct reflective subfibrations out of a left exact localization of an ∞-topos $\mathcal{E}$.

Suppose $\mathcal{D} \subseteq \mathcal{E}$ is a reflective subcategory, with reflector $a\colon \mathcal{E} \to \mathcal{D}$. Recall that $\mathcal{D}$ is called *left exact* if a is left exact, that is, if it preserves finite limits.

Proposition 4.2.8. *Let $i\colon \mathcal{D} \hookrightarrow \mathcal{E}$ be a left exact reflective subcategory of $\mathcal{E}$ with reflector $a\colon \mathcal{E} \to \mathcal{D}$. Set $L := ia\colon \mathcal{E} \to \mathcal{E}$. Then there exists a modality $L_\bullet$ on $\mathcal{E}$ for which $L_1 = L$ and a map $f\colon X \to Y$ in $\mathcal{E}$ is L-local if and only if the square*

$$\begin{tikzcd} X \arrow[r, "\eta(X)"] \arrow[d, "f"'] & LX \arrow[d, "Lf"] \\ Y \arrow[r, "\eta(Y)"'] & LY \end{tikzcd} \tag{4.1}$$

is a pullback.

Proof. Note that $L = ia$ is left exact, because it is the composite of two left exact functors. By [ABFJ17b, Lemma 2.6.4], L gives rise to a stable factorization system on $\mathcal{E}$. The left class $\mathcal{L}$ of this factorization system consists of the L-equivalences and $\mathcal{R} = \mathcal{L}^{\perp}$ is exactly the class of all maps $f\colon X \to Y$ in $\mathcal{E}$ satisfying the stated pullback condition. We can then conclude by using Theorem 4.2.5 (1). ☐

We will see in Section 5.2 that a pullback-like characterization of L-local maps similar to the above one can be given for any reflective subfibration $L_\bullet$ on $\mathcal{E}$, upon suitably replacing the reflection map $\eta(Y)$ (see Proposition 5.2.6).

Remark 4.2.9. In the context of Proposition 4.2.8, Corollary 4.2.6 implies that the L-connected maps are exactly the L_1-equivalences. This is because, if $\mathcal{R}$ is the class of L-local maps, Corollary 4.2.6 says that ${}^{\perp}\mathcal{R}$ is the class of L-connected maps, whereas the proof of Proposition 4.2.8 says that ${}^{\perp}\mathcal{R}$ is the class of L_1-equivalences. Therefore, Proposition 4.2.8 is really just a special case of the constructions given in Theorem 4.2.5 with a different description of the class of L-connected and L-local maps. In fact, we can note the following. Recall from Definition 4.1.1 that every L-connected map $f\colon Y \to X$ is an L_X-equivalence when seen as a map $f\colon f \to \mathrm{id}_X$. Furthermore, by Lemma 3.1.4 (ii), if $\alpha\colon p \to q$ is a map in $\mathcal{E}_{/X}$ which is an L_X-equivalence, for some $X \in \mathcal{E}$, then it is an L_1-equivalence. It follows that, for the modality $L_\bullet$ of Proposition 4.2.8, the following hold.

(a) The class of L_1-equivalences and the class of L-connected maps coincide.

(b) For every $X \in \mathcal{E}$, a map $\alpha\colon p \to q$ is an L_X-equivalence if and only if $\Sigma_X(\alpha)$ is an L_1-equivalence.

The modalities on $\mathcal{E}$ with these properties correspond to the so-called *lex modalities* in homotopy type theory — see [RSS17, Thm. 3.1].

Chapter 5

L-separated maps

This chapter represents the core of our work. For a reflective subfibration $L_\bullet$ on $\mathcal{E}$, we introduce here the class of *L-separated maps*, that is, those maps in $\mathcal{E}$ whose diagonal is an L-local map, and show that they form the local maps for a reflective subfibration $L'_\bullet$ on $\mathcal{E}$.

In Section 5.1, we provide the formal definition and derive some closure properties of L-separated maps. We also prove that they form a local class of maps in $\mathcal{E}$, thus satisfying a necessary condition for being the local maps of a reflective subfibration on $\mathcal{E}$.

In Section 5.2, we explore some connections between L-local and L-separated maps, culminating in a characterization theorem of *L'-localization maps*. These are the universal maps out of a fixed object and into an L-separated object, and we show they are equivalently those effective epimorphisms whose diagonal is a suitable L-localization map (see Theorem 5.2.10). In Proposition 5.2.6, we also prove a characterization of L-local maps in terms of a pullback condition involving L'-localizations and we deduce from it a (folklore?) description of n-truncated maps in an ∞-topos $\mathcal{E}$ as suitable pullbacks of their $(n+1)$-truncations (Corollary 5.2.8).

Finally, in Section 5.3, we prove the existence of a reflective subfibration $L'_\bullet$ on $\mathcal{E}$ with the property that the L'-local maps are exactly the L-separated maps (Theorem 5.3.3 and Corollary 5.3.4).

The notion of L-separated map, as well as Proposition 5.2.6, Theorem 5.3.3 and a few auxiliary results we prove in this chapter, are expressed in the language of homotopy type theory in [CORS18, §2.2-2.3]. In particular, we borrow from there the main ideas for the proofs of Theorem 5.2.10 and Theorem 5.3.3. However, the details and the techniques used in proofs have been modified, sometimes significatively, to apply to the "term-free" exposition we work with, which often comes with added care needed. This is particularly evident in the proof of Theorem 5.2.10 and in the results of Section 5.3.

5.1 Definition and basic properties

In this section, we wish to study those maps $p\colon E \to X$ whose diagonal map $\Delta p\colon E \to E \times_X E$ is L-local. Before turning this property into a definition, let us stress that there is no ambiguity in what this means. Indeed, recalling Remark 3.1.2, Δp is $L^{/X}$-local as a map $p \to p \times^X p$ in $\mathcal{E}_{/X}$ if and only if it is L-local as a map $E \to E \times_X E$ in $\mathcal{E}$.

Definition 5.1.1. A map $p\colon E \to X$ in $\mathcal{E}$ is called *L-separated* or *L'-local* if the object $\Delta p \in \mathcal{E}_{/E\times_X E}$ is in $\mathcal{D}_{E\times_X E}$, i.e., if Δp is an L-local map.

Remark 5.1.2. Given a space (Kan complex) X, the diagonal map ΔX is, up to equivalence, the path-fibration map $X^{\Delta[1]} \twoheadrightarrow X \times X$. (More generally, this is true whenever we choose a presentation of an ∞-topos by a simplicial model category $\mathcal{M}$ since $X^{\Delta[1]}$ is a path object for $X \in \mathcal{M}$.) Hence, Definition 5.1.1 describes all those spaces X for which the fibers of the path-fibration map (i.e., the spaces $\mathrm{Path}(x, y)$ of paths in X between any two points $x, y \in X$) are L-local. Keeping this intuitive analogy in mind can help in getting a better feeling for many of the results in this section.

Remark 5.1.3. We can make the following elementary observations.

(i) Corollary 3.1.9 is exactly the statement that every L-local map is L-separated.

(ii) The diagonal of every monomorphism is an equivalence, so every monomorphism is L-separated. In particular, any (-1)-truncated object is L-separated.

Example 5.1.4. For every $n \geq -2$, there is a modality $L = L^n_\bullet$ on $\mathcal{E}$ for which the L-local maps are the n-truncated maps in $\mathcal{E}$, by Example 4.2.7. Since a map is $(n+1)$-truncated precisely when its diagonal is n-truncated ([Lur09, Lemma 5.5.6.15]), the L^n-separated maps are exactly the maps in $\mathcal{E}$ which are $(n+1)$-truncated. In particular, note that, for this reflective subfibration, the L-separated maps are themselves the local maps for another reflective subfibration. We will see in Section 5.3 that this is always the case.

L-separated maps share the same closure properties as L-local maps.

Proposition 5.1.5. *The class of L-separated maps is closed under pullback and dependent product along any map; that is, if $f\colon Y \to X$ is any map in $\mathcal{E}$ and $p\colon E \to X$ and $q\colon M \to Y$ are L-separated, then $f^*(p) \in \mathcal{E}_{/Y}$ and $\prod_f q \in \mathcal{E}_{/X}$ are both L-separated. Furthermore, the internal hom p^f is L-separated.*

Proof. We begin by showing that $f^*(p)$ is L-separated. Write $f^*(E)$ for $Y \times_X E$, so that $f^*(p)\colon f^*(E) \to Y$. The composite pullback square in $\mathcal{E}$

$$\begin{tikzcd}
f^*(E) \times_Y f^*(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f^*(E) \arrow[r] \arrow[d, "f^*(p)"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
f^*(E) \arrow[r, "f^*(p)"'] & Y \arrow[r, "f"'] & X
\end{tikzcd}$$

is the same as the composite square

$$\begin{tikzcd}
f^*(E) \times_Y f^*(E) \arrow[r] \arrow[d] & E \times_X E \arrow[r] \arrow[d, "p^*(p)"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
f^*(E) \arrow[r] & E \arrow[r, "p"'] & X
\end{tikzcd}$$

in which the right square is a pullback by definition. It follows that the left square is also a pullback. By an easy application of the pasting lemma for pullbacks, we then get that the square

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d, "\Delta(f^*(p))"'] & E \arrow[d, "\Delta p"] \\ f^*(E) \times_Y f^*(E) \arrow[r] & E \times_X E \end{tikzcd}$$

is a pullback. Hence, $\Delta(f^*(p))$ is the pullback of the L-local map Δp, so it is itself L-local by Definition 3.1.1 (1).

As for stability under dependent products, applying Proposition A.2.2 (and Remark A.2.3) to $q \in \mathcal{E}_{/Y}$ and to $\prod_f$, the diagonal of $\prod_f q$ can be recovered as

$$\prod_{\mathrm{pr}} ((\epsilon_1, \epsilon_2)^*(\Delta q))$$

for suitable maps pr and (ϵ_1, ϵ_2). Since L-local maps are closed under pullbacks (by definition) and dependent products along arbitrary maps (by Proposition 3.1.5(ii)) and since Δq is L-local by hypothesis, we can conclude that $\Delta(\prod_f q)$ is L-local, that is, $\prod_f q$ is L-separated.

The last claim now follows, since $(-)^f \simeq \prod_f f^*(-)$. □

Proposition 3.1.7 also has an exact counterpart for L-separated maps.

Lemma 5.1.6. *Suppose given composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ such that g and gf are L-separated. Then f is L-separated.*

Proof. There are pullback squares

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta g"] \\ X \times_Z X \arrow[r, "\mathrm{id}_X \times_Z f"'] & X \times_Z Y \arrow[r, "f \times_Z \mathrm{id}_Y"'] & Y \times_Z Y \end{tikzcd}$$

in which all the vertical maps are L-local, since Δg is L-local by hypothesis. If we let p be the leftmost vertical map, we have that $p \circ \Delta f = \Delta(gf)$. We can then conclude using Proposition 3.1.7. ☐

Proposition 5.1.7. *The class $\mathcal{M}'$ of all L-separated maps is a local class of maps in $\mathcal{E}$.*

Proof. We established already that $\mathcal{M}'$ is pullback-stable. Suppose given a set-indexed family of L-separated maps $f_i\colon X_i \to Y_i$, for $i \in I$. Let $p\colon P \to \coprod_i X_i$ be the pullback of $\coprod_i f_i$ with itself. Because colimits in $\mathcal{E}$ are universal, $P \simeq \coprod_i \iota_{X_i}^*(p)$, where ι_{X_i} is the coproduct inclusion of X_i. For a fixed $j \in I$, by definition of p and of $\coprod_i f_i$, $\iota_{X_j}^*(p)$ is the same as the pullback of $\coprod_i f_i$ along $\iota_{Y_j} \circ f_j$ and this pullback is just $X_j \times_{Y_j} X_j$:

$$\begin{tikzcd}
X_j \times_{Y_j} X_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_j \arrow[r, "\iota_{X_j}"] \arrow[d, "f_j"'] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_i X_i \arrow[d, "\coprod_j f_j"'] \\
X_j \arrow[r, "f_j"'] & Y_j \arrow[r, "\iota_{Y_j}"'] & \coprod_i Y_i
\end{tikzcd}$$

Here the right square is a pullback because coproducts in $\mathcal{E}$ are disjoint. Thus, $P = \coprod_i X_i \times_{Y_i} X_i$ and it follows that $\Delta(\coprod_i f_i)$ is the map $\coprod_i \Delta(f_i)$, which is L-local because the class of L-local maps is closed under coproducts (see Lemma 3.2.1).

Finally, suppose given a pullback square

$$\begin{tikzcd}
E \arrow[r, two heads, "g"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "q"] \\
X \arrow[r, two heads, "f"'] & Y
\end{tikzcd}$$

where f is an effective epi and p is L-separated. We need to show that q is L-separated too. We have a commutative cube in $\mathcal{E}$

$$\begin{tikzcd}
E \times_Y M \arrow[rr, two heads] \arrow[dr] \arrow[dd] & & M \times_Y M \arrow[dr] \arrow[dd] & \\
& E \arrow[rr, two heads, "g"] \arrow[dd, "p", near start] & & M \arrow[dd, "q"] \\
E \arrow[rr, two heads] \arrow[dr] & & M \arrow[dr, "q"] & \\
& X \arrow[rr, two heads, "f"'] & & Y
\end{tikzcd}$$

Here the bottom, front and right faces are all pullback squares. Since the composite of the back and right faces is also a pullback square, it follows that the back face is a pullback square, which implies all faces are pullback squares. This tells us that $E \times_Y M \twoheadrightarrow M \times_Y M$ is an effective epimorphism and that $E \times_X E = E \times_Y M$. Therefore, the diagonal Δp can be identified with $\mathrm{id}_E \times_Y g$, which is then L-local (since Δp is L-local by the hypothesis that p is L-separated). Therefore, in the pullback square

$$\begin{tikzcd}
E \arrow[r, "g"] \arrow[d, "\mathrm{id}_E \times_Y g"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "\Delta q"] \\
E \times_Y M \arrow[r, two heads] & M \times_Y M
\end{tikzcd}$$

the left vertical map is L-local and the bottom horizontal map is an effective epimorphism. By Lemma 3.2.2, it follows that Δq is also L-local, as required. □

Since every L-local map is L-separated, using Proposition 2.1.7, Proposition 2.2.9 and [GK17, Cor. 3.10] we get the following result.

Corollary 5.1.8. *There are arbitrarily large regular cardinals κ such that the class of relatively κ-compact L-separated maps is classified by a univalent map*

$$u_\kappa^{L'} : \widetilde{\mathcal{U}_\kappa^{L'}} \to \mathcal{U}_\kappa^{L'}.$$

If $u_\kappa^L : \widetilde{\mathcal{U}_\kappa^L} \to \mathcal{U}_\kappa^L$ is the classifying map for relatively κ-compact L-local maps, we have a pullback square

$$\begin{tikzcd}
\widetilde{\mathcal{U}_\kappa^L} \arrow[r] \arrow[d, "u_\kappa^L"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}_\kappa^{L'}} \arrow[d, "u_\kappa^{L'}"] \\
\mathcal{U}_\kappa^L \arrow[r, tail] & \mathcal{U}_\kappa^{L'}
\end{tikzcd}$$

in which the bottom horizontal map is a monomorphism.

5.2 Interactions between L-local and L-separated maps

We study here some relationships between L-local and L-separated maps and prove some important results that characterize L'-localization maps. These results will be used in Section 5.3 to show the existence of $L'_\bullet$, the reflective subfibration on $\mathcal{E}$ whose local maps are the L-separated maps.

Lemma 5.2.1 ([CORS18, Lemma 2.21])**.** *Suppose given a commutative triangle*

$$\begin{tikzcd} E \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & M \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

in which $\Delta q \in \mathcal{D}_{M\times_X M}$ and $\alpha \in \mathcal{D}_M$, that is, q is L-separated and α is L-local. Then Δp is in $\mathcal{D}_{E\times_X E}$, i.e., p is L-separated.

Proof. Since $\alpha\colon E \to M$ is in $\mathcal{D}_M$,

$$(\mathrm{id}_E \times_X \alpha\colon E\times_X E \to E\times_X M) = (E\times_X M \to M)^*(\alpha)$$

is in $\mathcal{D}_{E\times_X M}$. Similarly, the map

$$((\mathrm{id}_E, \alpha)\colon E \to E\times_X M) = (\alpha\times_X \mathrm{id}_M)^*(\Delta q)$$

is in $\mathcal{D}_{E\times_X M}$. But

$$(\mathrm{id}_E \times_X \alpha)\circ \Delta p = (\mathrm{id}_E, \alpha),$$

so we can conclude that Δp is L-local using Proposition 3.1.7. □

Definition 5.2.2. A map $\alpha\colon p \to p'$ in $\mathcal{E}_{/X}$ is called an *L'-localization map* of p if p' is L-separated and, for every other map $\beta\colon p \to q$ where q is L-separated, there is a unique $\psi\colon p' \to q$ with $\psi\circ\alpha = \beta$.

Remark 5.2.3. The above definition is saying that $\alpha\colon p \to p'$ is an L'-localization if p' is L-separated and

$$\mathcal{E}_{/X}(\alpha, q)\colon \mathcal{E}_{/X}(p', q) \to \mathcal{E}_{/X}(p, q)$$

is an equivalence of ∞-groupoids for every L-separated $q \in \mathcal{E}_{/X}$. Since, given an $L-$separated $r \in \mathcal{E}_{/X}$ and any $t \in \mathcal{E}_{/X}$, $r^t \in \mathcal{E}_{/X}$ is again L-separated, this external description of an L'-localization map in terms of equivalences of ∞-groupoids can actually be rephrased internally, by asking that q^α is an equivalence in $\mathcal{E}_{/X}$ for every L-separated map $q\colon Y \to X$. Indeed, $q^\alpha\colon q^{p'} \to q^p$ is an equivalence if and only if every map $\beta\colon f \to q^p$ has a unique lift to $q^{p'}$. This, in turn, happens if and only if, for every object $f \in \mathcal{E}_{/X}$, $(f \times \alpha) \perp_X q$ or, equivalently, $\alpha \perp_X q^f$, which holds true as soon as α is an L'-localization, because q^f is L-separated. Closure under exponentiation of L-separated maps also ensures that, if $\alpha\colon p \to p'$ is an L'-localization of $p \in \mathcal{E}_{/X}$, then $\alpha \times^X \alpha$ is an L'-localization of $p \times^X p$ (where $(-) \times^X (-)$ denotes the product in $\mathcal{E}_{/X}$).

Recall from Definition 4.1.1 the notion of an L-connected map.

Lemma 5.2.4 ([CORS18, Prop. 2.30])**.** *Let $\eta'\colon p \to p'$ in $\mathcal{E}_{/Y}$ be an L'-localization of $p \in \mathcal{E}_{/Y}$, with $\eta'\colon X \to X'$ as a map in $\mathcal{E}$. Then η' is an L-connected map.*

Proof. Let

$$\begin{tikzcd} X \arrow[rr, "\eta_{X'}(\eta')"] \arrow[dr, "\eta'"'] & & L_{X'}X \arrow[dl, "L_{X'}(\eta')"] \\ & X' & \end{tikzcd}$$

be the reflection map of $\eta' \in \mathcal{E}_{/X'}$ into $\mathcal{D}_{X'}$. If we let $r := p' \circ L_{X'}(\eta')$, then we can consider $\eta_{X'}(\eta')\colon p \to r$ and $L_{X'}(\eta')\colon r \to p'$ as maps in $\mathcal{E}_{/Y}$. By Lemma 5.2.1 applied to $L_{X'}(\eta')$, r is L-separated. Hence, by the hypothesis on η', there is a unique map $q\colon p' \to r$ with $q\eta' = \eta_{X'}(\eta')$ as maps $p \to r$ in $\mathcal{E}_{/Y}$. Since

$$L_{X'}(\eta')q\eta' = L_{X'}(\eta')\eta_{X'}(\eta') = \eta',$$

by the universal property of η' we conclude that $L_{X'}(\eta')q = \mathrm{id}_{p'}$. As a consequence, we can consider $qL_{X'}(\eta')$ as a map $L_{X'}(\eta') \to L_{X'}(\eta')$ in $\mathcal{E}_{/X'}$ and we have

$$qL_{X'}(\eta')\eta_{X'}(\eta') = q\eta' = \eta_{X'}(\eta'),$$

from which we conclude that $qL_{X'}(\eta') = \mathrm{id}_r$ and so η' is L-connected. $\square$

Lemma 5.2.5. *Let κ be an arbitrarily large regular cardinal and $u^L_\kappa \colon \widetilde{\mathcal{U}^L_\kappa} \to \mathcal{U}^L_\kappa$ be the classifying map for the class of relatively κ-compact L-local maps. Then $\mathcal{U}^L_\kappa$ is L-separated.*

Proof. We write $\mathcal{U}^L$ for $\mathcal{U}^L_\kappa$ and similarly for $\widetilde{\mathcal{U}^L_\kappa}$ and u^L_κ. Since u^L is univalent (Definition 2.2.8), we have an equivalence

$$\Delta(\mathcal{U}^L) \simeq \mathrm{Eq}_{/\mathcal{U}^L}(u^L)$$

over $\mathcal{U}^L \times \mathcal{U}^L$. By definition, $\mathrm{Eq}_{/\mathcal{U}^L}(u^L)$ is the object of equivalences in $\mathcal{E}_{/\mathcal{U}^L\times\mathcal{U}^L}$ between $\mathrm{id}_{\mathcal{U}^L} \times u^L$ and $u^L \times \mathrm{id}_{\mathcal{U}^L}$, both of which are L-local since u^L is. By Lemma 2.2.4, such an object of equivalences is then the pullback of a cospan of objects in $\mathcal{D}_{\mathcal{U}^L\times\mathcal{U}^L}$ and it is therefore in $\mathcal{D}_{\mathcal{U}^L\times\mathcal{U}^L}$. $\square$

Proposition 5.2.6. *Let $X \in \mathcal{E}$ and let $\eta' \colon X \to X'$ be an L'-localization of X. Then a map $p\colon E \to X$ is L-local if and only if the square*

$$\begin{tikzcd} E \arrow[r, "\eta_{X'}(\eta' p)"] \arrow[d, "p"'] & L_{X'}E \arrow[d, "L_{X'}(\eta' p)"] \\ X \arrow[r, "\eta'"'] & X' \end{tikzcd}$$

is a pullback square in $\mathcal{E}$.

Proof. For the non-trivial implication, assume p is L-local. Let κ be a regular cardinal such that p is relatively κ-compact and the class of relatively κ-compact L-local maps has a classifying map $u^L\colon \widetilde{\mathcal{U}^L_\kappa} \to \mathcal{U}^L_\kappa$. Let $P\colon X \to \mathcal{U}^L_\kappa$ be such that we have a pullback square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}^L_\kappa} \arrow[d, "u^L"] \\ X \arrow[r, "P"'] & \mathcal{U}^L_\kappa \end{tikzcd} \tag{$\dagger$}$$

Since $\mathcal{U}^L_\kappa$ is L-separated, there is a unique map $P'\colon X' \to \mathcal{U}^L_\kappa$ with $P = P'\eta'$. Let $p'\colon E' \to X'$ be the pullback map in

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}^L_\kappa} \arrow[d, "u^L"] \\ X' \arrow[r, "P'"'] & \mathcal{U}^L_\kappa \end{tikzcd}$$

By definition of P', $\eta'\colon X \to X'$ induces a map $n\colon E \to E'$ such that the composite square in

$$\begin{tikzcd} E \arrow[r, "n"] \arrow[d, "p"'] & E' \arrow[r] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}^L_\kappa} \arrow[d, "u^L"] \\ X \arrow[r, "\eta'"'] & X' \arrow[r, "P'"'] & \mathcal{U}^L_\kappa \end{tikzcd} \tag{$\ddagger$}$$

is the square ($\dagger$). It follows that the left square in ($\ddagger$) is also a pullback. Thanks to Lemma 5.2.4, η' is L-connected. Thus, so is n, by Remark 4.1.2. In particular, n is an L-equivalence (i.e., $n\colon n \to \mathrm{id}_{E'}$ is in $S_{E'}$). By composing domain and codomain of $n\colon n \to \mathrm{id}_E$ with p', Lemma 3.1.4 (ii) gives that $n\colon \eta' p \to p'$ is an L-equivalence. Since p' is L-local, it follows that n is the L-localization map of $\eta' p$, as required. $\square$

Remark 5.2.7. As explained in Remark 3.1.2, Proposition 5.2.6 is also true "locally", i.e., when we take our ground ∞-topos to be $\mathcal{E}_{/X}$ instead of $\mathcal{E}$. For the result above, this means specifically that, if

$$\begin{tikzcd} E \arrow[rr, "\eta'_X(p)"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

is an L'-localization of p in $\mathcal{E}_{/X}$, a map

$$\begin{tikzcd} Y \arrow[rr, "m"] \arrow[dr, "q"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

is $L^{/X}$-local (as an object in $(\mathcal{E}_{/X})_{/p}$, so m is in $\mathcal{D}_E$) if and only if

$$\begin{tikzcd} Y \arrow[r, "\eta_{E'}(\eta'_X(p)\circ m)"] \arrow[d, "m"'] & L_{E'}Y \arrow[d, "L_{E'}(\eta'_X(p)m)"] \\ E \arrow[r, "\eta'_X(p)"'] & E' \end{tikzcd}$$

is a pullback square in $\mathcal{E}_{/X}$. (Note that, in the above, $L_{E'}$ should in fact be $L^{/X}_{p'}$, where $L^{/X}_{p'}$ is the reflector of $(\mathcal{E}_{/X})_{/p'}$ onto $\mathcal{D}^{/X}_{p'}$ and $L^{/X}_{\bullet}$ is the reflective subfibration on $\mathcal{E}_{/X}$ induced by $L_{\bullet}$, as in Remark 3.1.2. But, by its own definition, $L^{/X}_{p'} = L_{E'}$.)

The following corollary is probably well-known, though the only explicit reference we could find in the literature is [Rez10, Lemma 8.6], where the statement is proved in the context of model topoi. It might be worth noticing that our proof is completely internal and does not directly use the description of ∞-topoi as left exact localizations of presheaf categories.

Corollary 5.2.8. *For $n \geq -2$, a map $p\colon E \to X$ is n-truncated if and only if $\|p\|_{n+1}$ is n-truncated and the commutative square*

$$\begin{tikzcd} E \arrow[r, "|\cdot|_{n+1}"] \arrow[d, "p"'] & \|E\|_{n+1} \arrow[d, "\|p\|_{n+1}"] \\ X \arrow[r, "|\cdot|_{n+1}"'] & \|X\|_{n+1} \end{tikzcd}$$

is a pullback square.

Proof. By virtue of Example 4.2.7 and Example 5.1.4, we can apply Proposition 5.2.6

where $L_\bullet$ is the n-truncation modality and get a pullback square

$$\begin{tikzcd}
E \arrow[r, "n"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & L_{\|X\|_{n+1}}(E) \arrow[d, "L_{\|X\|_{n+1}}(|\cdot|_{n+1}p)"] \\
X \arrow[r, "|\cdot|_{n+1}"'] & \|X\|_{n+1}
\end{tikzcd}$$

Since $\|X\|_{n+1}$ is $(n+1)$-truncated and $L_{\|X\|_{n+1}}(|\cdot|_{n+1}p)$ is n-truncated, $L_{\|X\|_{n+1}}(E)$ is $(n+1)$-truncated. (This is a specific instance of Lemma 5.2.1.) But n is a pullback of the $(n+1)$-connected map $|\cdot|_{n+1}\colon X \to \|X\|_{n+1}$, so it is itself $(n+1)$-connected. Finally, any $(n+1)$-connected map $m\colon A \to B$ where B is $(n+1)$-truncated is an $(n+1)$-truncation map of A. $\square$

Proposition 5.2.9 ([CORS18, Prop. 2.26])**.** *Let*

$$\begin{tikzcd}
E \arrow[rr, "\eta'_X(p)"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\
& X &
\end{tikzcd}$$

be an L'-localization of $p \in \mathcal{E}_{/X}$. Let $\eta_{E\times_X E}(\Delta p)\colon \Delta p \to r$ be the L-localization of $\Delta p \in \mathcal{E}_{/E\times_X E}$ and consider r' defined by the pullback square

$$\begin{tikzcd}
E \times_{E'} E \arrow[r] \arrow[d, "r'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, "\Delta p'"] \\
E \times_X E \arrow[r, "\eta'_X(p)\times_X \eta'_X(p)"'] & E' \times_X E'
\end{tikzcd} \tag{†}$$

Then there is a natural equivalence $\varphi\colon R \xrightarrow{\simeq} E \times_{E'} E$ over $E \times_X E$ as in

$$\begin{tikzcd}
& E \arrow[dl, "\eta_{E\times_X E}(\Delta p)"'] \arrow[dr, "\Delta(\eta'_X(p))"] & \\
R \arrow[rr, "\varphi"] \arrow[dr, "r"'] & & E \times_{E'} E \arrow[dl, "r'"] \\
& E \times_X E &
\end{tikzcd}$$

Proof. For sake of readability, we write η' and η for $\eta'_X(p)$ and $\eta_{E\times_X E}(\Delta p)$, respectively. The natural map φ is given by the universal property of η, since r' is L-local. (By definition, r' is the pullback of the L-local map $\Delta p'$.) Now, since $\eta' \times_X \eta'$ is the L'-localization map of the product object $p \times^X p$ of $\mathcal{E}_{/X}$, Proposition 5.2.6 applied in $\mathcal{E}_{/X}$ gives that there is a pullback square

$$\begin{tikzcd}
R \arrow[r, "n"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "q"] \\
E \times_X E \arrow[r, "\eta' \times_X \eta'"] & E' \times_X E'
\end{tikzcd}$$

where $n\colon (\eta' \times_X \eta')r \to q$ is the L-localization map of $(\eta' \times_X \eta')r$. Set $m := n\eta\colon E \to T$ and $l := \pi q$, where $\pi\colon E' \times_X E' \to X$ is given by the composite map $E' \times_X E' \to E' \xrightarrow{p'} X$. Note that π is L-separated, because it is the product in $\mathcal{E}_{/X}$ of the L-separated map p' with itself. Hence, since q is L-local, l is L-separated by Lemma 5.2.1. Since $m = n\eta$ is naturally a map $m\colon p \to l$ in $\mathcal{E}_{/X}$, there is a unique $s\colon E' \to T$ over X with $ls = p'$ and $s\eta' = m$.

Now,

$$qs\eta' = qm = qn\eta = (\eta' \times_X \eta')\Delta p = \Delta p'\eta'$$

so that $qs = \Delta p'$ and we can write $s\colon \Delta p' \to q$ as a map over $E' \times_X E'$. Hence, s induces the dotted comparison map ψ of pullback squares in

$$\begin{tikzcd}
E \times_{E'} E \arrow[rr] \arrow[dr, dashed, "\psi"] \arrow[dd, "r'"'] & & E' \arrow[dr, "s"] \arrow[dd, "\Delta p'" near start] & \\
& R \arrow[rr, "n" near start] \arrow[dd, "r" near end] & & T \arrow[dd, "q"] \\
E \times_X E \arrow[rr, "\eta' \times_X \eta'" near start] \arrow[dr, "\mathrm{id}"'] & & E' \times_X E' \arrow[dr, "\mathrm{id}"] & \\
& E \times_X E \arrow[rr, "\eta' \times_X \eta'"'] & & E' \times_X E'
\end{tikzcd}$$

Using the fact that the front face is a pullback, it follows that $\psi \circ \Delta\eta' = \eta$, from which we get $\psi\varphi\eta = \eta$, so that $\psi \circ \varphi = \mathrm{id}$. We now claim that s is an equivalence. This would imply that ψ (and therefore also φ) is an equivalence, since a map of cospans made of equivalences induces an equivalence on pullbacks. We now verify that s is an equivalence. Since $s\colon \Delta p' \to q$ is a map between L-local maps over $E' \times_X E'$, it is enough to show that $s \in S_{E' \times_X E'}$. Now, $\eta'\colon p \to p'$ is L-connected so it is an $L_{E'}$-equivalence (more precisely, $\eta'\colon \eta' \to \mathrm{id}_{E'}$ is in $S_{E'}$). By Lemma 3.1.4 (ii), composing $\eta'\colon \eta' \to \mathrm{id}_{E'}$ with $\Delta p'$ gives that $\eta'\colon (\Delta p')\eta' \to \Delta p'$ is in $S_{E' \times_X E'}$. Similarly, composing domain and codomain of η with $\eta' \times_X \eta'$ turns η into a map in $S_{E' \times_X E'}$ and then $m = n\eta$ is also in $S_{E' \times_X E'}$, since n is an L-equivalence by hypothesis. Since $s\eta' = m$, $s \in S_{E' \times_X E'}$, as needed. $\square$

Our next main result characterizes L'-localization maps in terms of their diagonal maps, providing a useful criterion that we will employ to show that every $p \in \mathcal{E}_{/X}$ has an L'-localization. The proof of the following result uses some general facts about locally cartesian closed ∞-categories that can be found in the Appendix (see Section A.3).

Theorem 5.2.10 ([CORS18, Thm. 2.34])**.** *The following are equivalent for a map*

$$\begin{tikzcd} X \arrow[rr, "\eta'"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Z & \end{tikzcd}$$

in $\mathcal{E}_{/Z}$:

1. *η' is an L'-localization of p;*

2. *η' is an effective epimorphism and the L-localization of Δp is given by*

$$\begin{tikzcd} X \arrow[rr, "\Delta\eta'"] \arrow[dr, "\Delta p"'] & & X \times_{X'} X \arrow[dl] \\ & X \times_Z X & \end{tikzcd}$$

Proof. We prove the theorem when $Z = 1$; the general statement follows from this one by Remark 3.1.2. We start with a map $\eta'\colon X \to X'$ and show first that (1) implies (2). If η' is an L'-localization of X, then thanks to Proposition 5.2.9 we only need to show that η' is an effective epimorphism. Let (π, i) be the (effective epi,mono)-factorization of η', with $i\colon W \to X'$. Since i is a monomorphism, the square

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "\Delta W"'] & X' \arrow[d, "\Delta X'"] \\ W \times W \arrow[r, "i\times i"'] & X' \times X' \end{tikzcd}$$

is a pullback. Hence, since X' is L-separated, so is W. Therefore there is a unique $s\colon X' \to W$ with $s\eta' = \pi$. From $is\eta' = i\pi = \eta'$, we get that $is = \mathrm{id}_{X'}$, i.e., i has a section. Thus, i is both a mono and an effective epi, so it is an equivalence.

Conversely, assume η' is an effective epimorphism and $\Delta\eta'$ is the L-localization of ΔX. In the pullback square

$$\begin{tikzcd} X \times_{X'} X \arrow[r] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "\Delta X'"] \\ X \times X \arrow[r, "\eta'\times\eta'"'] & X' \times X' \end{tikzcd} \tag{$*$}$$

$\eta' \times \eta'$ is also an effective epimorphism and t is L-local by hypothesis. Thus, $\Delta X'$ is also L-local since L-local maps are a local class of maps in $\mathcal{E}$. This shows that X' is L-separated. We now verify that η' has the universal property of an L'-localization map. Let $f\colon X \to Y$ be a map into an L-separated object Y. We need to show that f extends uniquely along η'. We do so by applying Proposition A.3.4 to f and η'. We want to show that

$$E := \sum_{X'\times Y \to X'} \left(\prod_{X\times X'\times Y \to X'\times Y} (\mathrm{pr}_X, X' \times f)^{(\mathrm{pr}_X, \eta'\times Y)} \right)$$

is contractible in $\mathcal{E}_{/X'}$. Applying Lemma A.3.1 and the Beck-Chevalley condition

(Lemma A.1.3) to the pullback squares

$$\begin{tikzcd}
X \times X \times Y \arrow[r, "\mathrm{pr}_{X\times Y}"] \arrow[d, "X\times\eta'\times Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\eta'\times Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta'"'] \\
X \times X' \times Y \arrow[r] & X' \times Y \arrow[r] & X'
\end{tikzcd}$$

we can instead show that

$$E' := \sum_{X\times Y\to X}\left(\prod_{X\times X\times Y\to X\times Y}(X\times\eta'\times Y)^*\left((\mathrm{pr}_X, X'\times f)^{(\mathrm{pr}_X,\eta'\times Y)}\right)\right)$$

is contractible in $\mathcal{E}_{/X}$. We will show that this object of $\mathcal{E}_{/X}$ is equivalent to the object id_X, which is contractible in $\mathcal{E}_{/X}$. Lemma A.1.1 gives that

$$(X\times\eta'\times Y)^*\left((\mathrm{pr}_X, X'\times f)^{(\mathrm{pr}_X,\eta'\times Y)}\right)\simeq$$

$$\simeq((X\times\eta'\times Y)^*(\mathrm{pr}_X, X'\times f))^{(X\times\eta'\times Y)^*((\mathrm{pr}_X,\eta'\times Y))}$$

Notice that

$$(\mathrm{pr}_X, X'\times f) = (f\times \mathrm{pr}_Y)^*(\Delta Y), \quad (\mathrm{pr}_X, \eta'\times Y) = (\eta'\times\mathrm{pr}_{X'})^*(\Delta X')$$

and

$$(f\times\mathrm{pr}_Y)(X\times\eta'\times Y) = (f\times Y)(\mathrm{pr}_1,\mathrm{pr}_3),$$

where $\mathrm{pr}_1\colon X\times X\times Y\to X$ and $\mathrm{pr}_3\colon X\times X\times Y\to Y$ are the projections onto the appropriate factors. Using these, one can see that

$$(X\times\eta'\times Y)^*\left((\mathrm{pr}_X, X'\times f)\right) = (\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)\colon X\times X\to X\times X\times Y,$$

$$(X\times\eta'\times Y)^*\left((\mathrm{pr}_X, \eta'\times Y)\right) = t\times Y\colon (X\times_{X'}X)\times Y\to X\times X\times Y$$

where t is defined in the pullback square $(*)$ above. Therefore,

$$(X \times \eta' \times Y)^* \left((\mathrm{pr}_X, X' \times f)^{(\mathrm{pr}_X, \eta' \times Y)}\right) \simeq (\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)^{t\times Y}.$$

Now, since t is the localization of ΔX in $\mathcal{E}_{/X\times X}$, taking pullbacks along the projection $X \times X \times Y \to X \times X$ gives that $t \times Y$ is the localization of $\Delta X \times Y$ in $\mathcal{E}_{/X\times X\times Y}$. Since $(\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)$ is L-local (as the pullback of the L-local map ΔY), we further have

$$(\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)^{t\times Y} \simeq (\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1)^{\Delta X\times Y} \simeq$$

$$\simeq \prod_{\Delta X\times Y} (\Delta X \times Y)^*(\mathrm{id}_{X\times X}, f\,\mathrm{pr}_1) \simeq \prod_{\Delta X\times Y} (\mathrm{id}_X, f),$$

where $(\mathrm{id}_X, f)\colon X \to X \times Y$. We can now finally conclude because

$$E' \simeq \sum_{X\times Y\to X} \left(\prod_{\mathrm{pr}_{X\times Y}\colon X\times X\times Y\to X\times Y} \left(\prod_{\Delta X\times Y} (\mathrm{id}_X, f) \right)\right) \simeq$$

$$\simeq \sum_{X\times Y\to X} \left(\prod_{\mathrm{pr}_{X\times Y}\circ(\Delta X\times Y)} (\mathrm{id}_X, f) \right) = \sum_{X\times Y\to X} (\mathrm{id}_X, f) = \mathrm{id}_X.$$

□

5.3 Existence of L'-localization

In this section we prove that the class of L-separated maps is always the class of local maps for a reflective subfibration on $\mathcal{E}$. In doing this, the hardest part is constructing an L'-localization map for every $p \in \mathcal{E}_{/Z}$. To this end, we begin by proving a few preliminary results that can be stated independently of the task at hand.

Recall that, if p, q are objects in a slice category $\mathcal{E}_{/Z}$, we write $p \times^Z q$ to mean the product object of p and q in $\mathcal{E}_{/Z}$.

The first result we need is a term-free interpretation of what can be considered an internal Yoneda lemma involving diagonal maps.

Lemma 5.3.1. *Let $t\colon E \to X$ be a map in $\mathcal{E}$ and form the pullback square*

$$\begin{tikzcd} X \times E \arrow[r] \arrow[d, "X\times t"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "t"] \\ X \times X \arrow[r, "\mathrm{pr}_2"'] & X \end{tikzcd}$$

Then there is a map in $\mathcal{E}_{/X^2}$

$$\begin{tikzcd} E \arrow[rr, "(t,\mathrm{id})"] \arrow[dr, "(\Delta X)t"'] & & X \times E \arrow[dl, "X\times t"] \\ & X \times X & \end{tikzcd},$$

inducing an equivalence

$$\beta\colon t \xrightarrow{\simeq} \prod_{\mathrm{pr}_1} (X \times t)^{\Delta X}$$

in $\mathcal{E}_{/X}$, where $\mathrm{pr}_1\colon X \times X \to X$ is the projection onto the first component.

Proof. For any $k\colon M \to X$, there are pullback squares

$$\begin{tikzcd} M \arrow[r, "(\mathrm{id},k)"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \times X \arrow[r] \arrow[d, dashed, "k\times X"] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "k"] \\ X \arrow[r, "\Delta X"'] & X \times X \arrow[r, "\mathrm{pr}_1"'] & X \end{tikzcd}$$

in $\mathcal{E}$, witnessing that the product object $(k\times X)\times^{X^2}(\Delta X)$ in $\mathcal{E}_{/X^2}$ is given by $(\Delta X)k$. Similarly, $(\Delta X)k$ is also the product object $(X \times k) \times^{X^2} (\Delta X)$ in $\mathcal{E}_{/X^2}$. Applying these considerations to $k = t$, we get that

$$(t, \mathrm{id})\colon (\Delta X)t \to X \times t$$

gives a map

$$\beta \colon t \longrightarrow \prod_{\mathrm{pr}_1} (X \times t)^{\Delta X}$$

by adjointness. Using the fact that ΔX is a section of pr_2, and considering the adjoint pairs $\Sigma_{\mathrm{pr}_2} \dashv \mathrm{pr}_2^*$, $\mathrm{pr}_1^* \dashv \prod_{\mathrm{pr}_1}$, we get a chain of natural equivalences

$$\mathcal{E}_{/X}(k, t) \simeq \mathcal{E}_{/X}(\mathrm{pr}_2(\Delta X)k, t) \simeq \mathcal{E}_{/X^2}\left((\Delta X)k, X \times t\right) \simeq$$

$$\simeq \mathcal{E}_{/X^2}\left(k \times X, (X \times t)^{\Delta X}\right) \simeq \mathcal{E}_{/X}\left(k, \prod_{\mathrm{pr}_1}(X \times t)^{\Delta X}\right)$$

where the composite map is given by composition with β. □

Lemma 5.3.2. *Let $X \in \mathcal{E}$ and let $r \colon R \to X^2$ be an object in $\mathcal{E}_{/X^2}$. Let also $\widetilde{X \times r}$ be the composite map $(\tau \times X) \circ (X \times r)$, where $\tau \colon X^2 \simeq X^2$ is the canonical involution. Then the following hold.*

(i) There is a natural equivalence

$$\beta \colon r \xrightarrow{\simeq} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}$$

in $\mathcal{E}_{/X^2}$.

(ii) There is a map $\rho \colon \Delta X \to \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)}$ such that, for any map $\eta \colon \Delta X \to r$ in $\mathcal{E}_{/X^2}$, there is a commutative square

$$\begin{tikzcd}
\Delta X \arrow[r, dashed, "\rho"] \arrow[d, "\eta"'] & \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)} \arrow[d, "\prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\eta \times X)}"] \\
r \arrow[r, "\beta"'] & \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}
\end{tikzcd} \tag{5.1}$$

Proof. The first claim is a special case of Lemma 5.3.1 applied to the map $r =$

$(r_1, r_2)\colon R \to X^2$, seen as a map $r\colon r_2 \to \mathrm{pr}_2$ in $\mathcal{E}_{/X^2}$. Indeed, the following pullback square in $\mathcal{E}$

$$\begin{tikzcd} X^3 \arrow[r, "\mathrm{pr}_{13}"] \arrow[d, "\mathrm{pr}_{23}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^2 \arrow[d, "\mathrm{pr}_2"] \\ X^2 \arrow[r, "\mathrm{pr}_2"'] & X \end{tikzcd}$$

witnesses that $\mathrm{pr}_3\colon X^3 \to X$ is the product object of $\mathrm{pr}_2\colon X^2 \to X$ with itself in $\mathcal{E}_{/X}$ and the displayed maps pr_{13} and pr_{23} give the projection maps out of this product. The map $\Delta X \times X\colon X^2 \to X^3$, seen as a map $\mathrm{pr}_3 \to \mathrm{pr}_3$, is the diagonal of the object $\mathrm{pr}_3 \in \mathcal{E}_{/X}$. Since $\widetilde{X \times r} = \mathrm{pr}_{13}^*(r)$, Lemma 5.3.1 gives the desired natural equivalence $\beta\colon r \simeq \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}$.

For the second part, we describe the map

$$\rho\colon \Delta X \to \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)}$$

and how it makes the square (5.1) commute by looking at its adjunct. Under the adjunction $\mathrm{pr}_{23}^* \dashv \prod_{\mathrm{pr}_{23}}$, giving a square as (5.1) is the same as giving a square

$$\begin{tikzcd} X \times \Delta X \arrow[r, dashed, "\rho'"] \arrow[d, "X \times \eta"'] & (\widetilde{X \times r})^{(r \times X)} \arrow[d, "(\widetilde{X \times r})^{(\eta \times X)}"] \\ X \times r \arrow[r, "\beta'"'] & (\widetilde{X \times r})^{(\Delta X \times X)} \end{tikzcd}$$

where we used that $X \times \Delta X = \mathrm{pr}_{23}^*(\Delta X)$ and similarly for $X \times r$. Taking further adjoints along $(-) \times^{X^2} (\Delta X \times X) \dashv (-)^{\Delta X \times X}$, we need to exhibit a square

$$\begin{tikzcd} (X \times \Delta X) \times^{X^3} (\Delta X \times X) \arrow[r, "(X \times \eta) \times^{X^3} (\Delta X \times X)"] \arrow[d, dashed, "(X \times \Delta X) \times^{X^3} (\eta \times X)"'] & (X \times r) \times^{X^3} (\Delta X \times X) \arrow[d, "\beta^\sharp"] \\ (X \times \Delta X) \times^{X^3} (r \times X) \arrow[r, dashed, "\rho^\sharp"'] & \widetilde{X \times r} \end{tikzcd}$$

The products $(X \times \Delta X) \times^{X^3} (\Delta X \times X)$, $(X \times r) \times^{X^3} (\Delta X \times X)$ and $(X \times \Delta X) \times^{X^3}$

$(r \times X)$ in $\mathcal{E}_{/X^3}$, together with their projections onto the factors, are represented, in order, by the following pullback squares in $\mathcal{E}$

$$\begin{tikzcd}
X \arrow[r, "\Delta X"] \arrow[d, "\Delta X"'] \arrow[dr, "{(\mathrm{id},\mathrm{id},\mathrm{id})}" description] \arrow[dr, phantom, very near start, "\lrcorner"] & X^2 \arrow[d, "\Delta X \times X"] \\
X^2 \arrow[r, "X \times \Delta X"'] & X^3
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "r"] \arrow[d, "{(r_1,\mathrm{id})}"'] \arrow[dr, "{(r_1,r_1,r_2)}" description] \arrow[dr, phantom, very near start, "\lrcorner"] & X^2 \arrow[d, "\Delta X \times X"] \\
X \times R \arrow[r, "X \times r"'] & X^3
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "{(\mathrm{id},r_2)}"] \arrow[d, "r"'] \arrow[dr, "{(r_1,r_2,r_2)}" description] \arrow[dr, phantom, very near start, "\lrcorner"] & R \times X \arrow[d, "r \times X"] \\
X^2 \arrow[r, "X \times \Delta X"'] & X^3
\end{tikzcd}$$

Using Lemma 5.3.1 as in the first part, we know the map $\beta^\sharp$ is given by

$$\begin{tikzcd}
R \arrow[r, "{\beta^\sharp=(r_1,\mathrm{id})}"] \arrow[d, "r"'] \arrow[dr, "{(r_1,r_1,r_2)}" description] & X \times R \arrow[d, "\widetilde{X\times r}"] \\
X^2 \arrow[r, "\Delta X \times X"'] & X^3
\end{tikzcd}$$

We take $\rho^\sharp$ to be given by

$$\begin{tikzcd}
R \arrow[r, "{\rho^\sharp=(r_2,\mathrm{id})}"] \arrow[d, "r"'] \arrow[dr, "{(r_1,r_2,r_2)}" description] & X \times R \arrow[d, "\widetilde{X\times r}"] \\
X^2 \arrow[r, "X \times \Delta X"'] & X^3
\end{tikzcd}$$

Then the composite maps

$$\beta^\sharp \Big((X \times \eta) \times^{X^3} (\Delta X \times X) \Big) \quad \text{and} \quad \rho^\sharp \Big((X \times \Delta X) \times^{X^3} (\eta \times X) \Big)$$

are given by the following composite maps in $\mathcal{E}_{/X^3}$ respectively

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "{(\mathrm{id},\mathrm{id},\mathrm{id})}"'] & R \arrow[r, "{(r_1,\mathrm{id})}"] \arrow[d, "{(r_1,r_1,r_2)}"] & X \times R \arrow[dl, "\widetilde{X\times r}"] \\
& X^3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "{(\mathrm{id},\mathrm{id},\mathrm{id})}"'] & R \arrow[r, "{(r_2,\mathrm{id})}"] \arrow[d, "{(r_1,r_2,r_2)}"] & X \times R \arrow[dl, "\widetilde{X\times r}"] \\
& X^3 &
\end{tikzcd}$$

By using properties of the product $X \times R$ and since η is a section of both r_1 and r_2, one can see that these composite maps are equal since they are both equal to the map

$(\mathrm{id}, \eta)\colon (\mathrm{id}, \mathrm{id}, \mathrm{id}) \to \widetilde{X \times r}$ in $\mathcal{E}_{/X^3}$. (Here all the needed homotopies are obtained by using either degenerate 2-simplices or the 2-simplices defining η as a map $\Delta X \to r$ in $\mathcal{E}_{/X^2}$.) $\square$

Theorem 5.3.3 ([CORS18, Thm. 2.25]). *For every $Y \in \mathcal{E}$, each $f \in \mathcal{E}_{/Y}$ has an L'-localization $\eta'_Y(f)\colon f \to f'$.*

Proof. We prove the result for $Y = 1$. Fix $X \in \mathcal{E}$ and let $\eta\colon \Delta X \to r$ be the L-reflection map of $\Delta X \in \mathcal{E}_{/X^2}$. Let κ be a regular cardinal such that r is relatively κ-compact and the class of relatively κ-compact L-local maps admits a classifying map $u_L^\kappa\colon \widetilde{\mathcal{U}_L^\kappa} \to \mathcal{U}_L^\kappa$. Omitting κ from our notation, we then have pullback squares

$$\begin{tikzcd} R \arrow[r, "\widetilde{r}"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}_L} \arrow[d, "u_L"] \\ X \times X \arrow[r, "\overline{r}"'] & \mathcal{U}_L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathcal{U}_L} \arrow[r] \arrow[d, "u_L"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "u"] \\ \mathcal{U}_L \arrow[r, tail, "\iota"'] & \mathcal{U} \end{tikzcd}$$

We denote the composite pullback square as

$$\begin{tikzcd} R \arrow[r, "\widetilde{r}"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "u"] \\ X \times X \arrow[r, "\ulcorner r \urcorner"'] & \mathcal{U} \end{tikzcd}$$

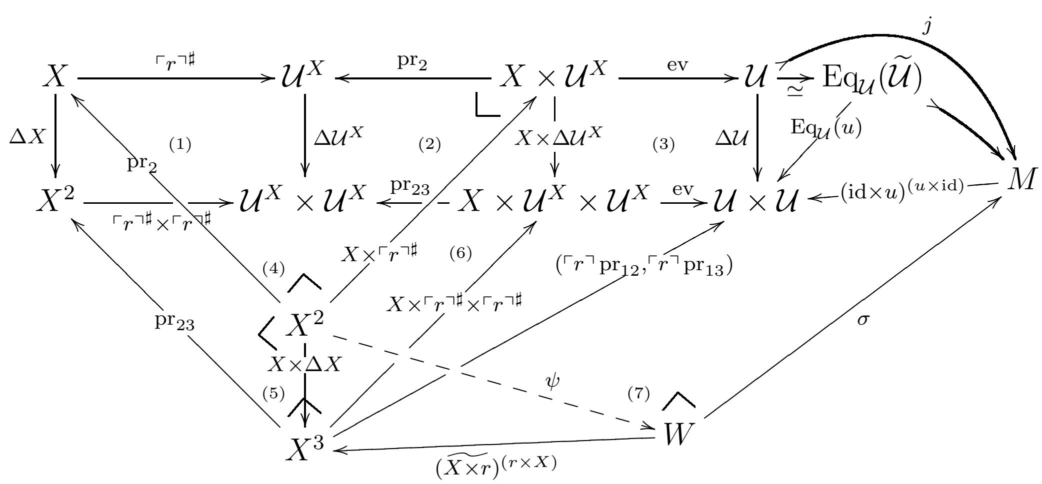
Let (η', i) be the (effective epi,mono)-factorization of $\ulcorner r \urcorner^\sharp\colon X \to \mathcal{U}^X$, the adjunct map to $\ulcorner r \urcorner$. We let $X' := \mathrm{cod}(\eta')$. Note that, if (η'_L, i_L) is the (effective epi,mono)-factorization of $\overline{r}^\sharp$, then $\eta' = \eta'_L$ and $i = \iota^X \circ i_L$ since ι^X is a monomorphism.

Our goal is to verify that the conditions in Theorem 5.2.10 apply to η', thus showing that η' is the L'-localization map of X. The map η' is an effective epimorphism by definition. To show that X' is L-separated, note first that $\mathcal{U}_L$ is L-separated by Lemma 5.2.5, hence so is $\mathcal{U}_L^X$, by Proposition 5.1.5. Since i is a monomorphism, we have that $\Delta X' = (i_L \times i_L)^*(\Delta(\mathcal{U}_L^X))$, which implies that X' is L-separated, because L-local maps are closed under pullbacks.

It remains to show that $\Delta\eta'$ is the L-localization map of ΔX. As usual, we can see $\Delta\eta'$ as a map $\Delta\eta' \colon \Delta X \to t$ in $\mathcal{E}_{/X^2}$, where t is the pullback map $(\eta' \times \eta')^*(\Delta X')$ and it is therefore L-local. Hence, there is a unique map $\varphi \colon r \to t$ with $\varphi\eta = \Delta\eta'$ as maps in $\mathcal{E}_{/X^2}$. We will show that φ is an equivalence.

The strategy we adopt is to, first, construct a monomorphism $\varphi' \colon t \rightarrowtail r$ and, then, show that $\varphi'\varphi \colon r \to r$ is an equivalence by showing that we have $\varphi'\varphi\eta = \eta$. This will imply that φ itself is an equivalence. Note that, by definition of φ, showing that $\varphi'\varphi\eta = \eta$ is the same as showing that $\varphi'\Delta\eta' = \eta$.

Step 1. Construction of φ' and description of $\varphi'\Delta\eta'$ We construct φ' as a composite of various equivalences and a monomorphism. Consider the following diagram.



(D)

The maps labelled as ev are appropriate counits of product $\dashv$ internal-hom adjunctions. The diagram above contains most of the information we need for Step 1. We proceed to explain this diagram, show how it defines φ', and give a description of $\varphi'\Delta\eta'$.

(i) Recall that $\Delta\eta'$ is a map $\Delta X \to t$ in $\mathcal{E}_{/X^2}$, and one can show that t is the pullback map of the cospan in (1) of (D). Because of this, the square (1) determines $\Delta\eta'$.

(ii) Thanks to Function Extensionality (Proposition A.2.1), $\Delta\mathcal{U}^X \simeq \prod_{\mathrm{pr}_{23}} \mathrm{ev}^*(\Delta\mathcal{U})$. Hence,

$$t \simeq (\ulcorner r \urcorner^\sharp \times \ulcorner r \urcorner^\sharp)^* \left(\prod_{\mathrm{pr}_{23}} \mathrm{ev}^*(\Delta\mathcal{U}) \right)$$

(iii) Since the bottom square (5) in (D) is a pullback, we can use the Beck-Chevalley condition (Lemma A.1.3) and obtain an equivalence

$$t \simeq \prod_{\mathrm{pr}_{23}} (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(\Delta\mathcal{U})$$

Since the pullback of $X \times \Delta\mathcal{U}^X$ along $X \times \ulcorner r \urcorner^\sharp \times \ulcorner r \urcorner^\sharp$ is $X \times t$, the square (6) in (D) determines the map $X \times \Delta\eta' \colon X \times \Delta X \to X \times t$ in $\mathcal{E}_{/X^3}$. It follows that the map $X \times \Delta X \to (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(\Delta\mathcal{U})$ determined by the square given as the composite of (3) and (6) is the adjunct of the composite map

$$\Delta X \xrightarrow{\Delta\eta'} t \simeq \prod_{\mathrm{pr}_{23}} (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(\Delta\mathcal{U})$$

(iv) We now consider the map j in $\mathcal{E}_{/\mathcal{U}^2}$ displayed in the top-right corner of (D). Here, M is simply a name for the domain of the map $(\mathrm{id} \times u)^{(u \times \mathrm{id})}$. The map j is defined as the composite of the equivalence $\Delta\mathcal{U} \simeq \mathrm{Eq}_{\mathcal{U}}(u)$, given by univalence (Definition 2.2.8), and the monomorphism $\mathrm{Eq}_{\mathcal{U}}(u) \rightarrowtail (\mathrm{id} \times u)^{(u \times \mathrm{id})}$ which exists since the domain is a subobject of the codomain. Thus, j is a monomorphism as well. Using the fact that (7) in (D) is a pullback square, we then obtain a monomorphism

$$\prod_{\mathrm{pr}_{23}} (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(\Delta\mathcal{U}) \xrightarrow{\prod_{\mathrm{pr}_{23}} (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})^*(j)} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)}.$$

Here, $\widetilde{X \times r}$ is the pullback map $\mathrm{pr}_{13}^*(r) = (\tau \times X)(X \times r)$, where $\tau \colon X^2 \simeq X^2$ is the swapping equivalence, and W is simply a name for the domain of the map $(\widetilde{X \times r})^{(r \times X)}$. Note that the map displayed above is indeed a monomorphism because — being right adjoints — pullback and dependent-product functors preserve monomorphisms. Therefore, we get a composite monomorphism

$$t \rightarrowtail \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)}.$$

The map ψ in $\mathcal{E}_{/X^3}$ given in (D) is determined, as a map $X \times \Delta X \to (\widetilde{X \times r})^{(r\times X)}$, by the composite of the squares (3) and (6) with the 2-simplex representing the map $j\colon \Delta(\mathcal{U}) \rightarrowtail (\mathrm{id}\times u)^{(u\times\mathrm{id})}$. It follows that ψ is the adjunct to the composite

$$\Delta X \xrightarrow{\Delta\eta'} t \rightarrowtail \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)}$$

This means that this latter map is the composite

$$\Delta X \xrightarrow{\gamma} \prod_{\mathrm{pr}_{23}} X \times \Delta X \xrightarrow{\prod_{\mathrm{pr}_{23}} \psi} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)},$$

where γ is the unit of the adjunction $\mathrm{pr}_{23}^* \dashv \prod_{\mathrm{pr}_{23}}$ at ΔX.

(v) Since $\widetilde{X \times r} = \mathrm{pr}_{13}^*(r)$, $\widetilde{X \times r}$ is L-local. Hence, because

$$\eta \times X\colon \Delta X \times X \to r \times X$$

is an L-localization map (it is the pullback along pr_{12} of η), we have an equivalence

$$(\widetilde{X \times r})^{(\eta\times X)}\colon (\widetilde{X \times r})^{(r\times X)} \xrightarrow{\simeq} (\widetilde{X \times r})^{(\Delta X\times X)}.$$

Whence, we have a composite monomorphism

$$t \rightarrowtail \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r\times X)} \simeq \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X\times X)}.$$

(vi) Finally, we have an equivalence

$$\beta\colon r \xrightarrow{\simeq} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}$$

as in Lemma 5.3.2. Composing the monomorphism obtained in (v) with the inverse of β we obtain the needed monomorphism $\varphi'\colon t \rightarrowtail r$. Using what we found in (iv) above, the composite $\varphi'\Delta\eta'$ is then given as the composite

$$\Delta X \xrightarrow{\gamma} \prod_{\mathrm{pr}_{23}} X \times \Delta X \xrightarrow{\prod_{\mathrm{pr}_{23}} \psi} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)} \xrightarrow{\simeq} r,$$

where the displayed equivalence is $\beta^{-1} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\eta \times X)}$.

Step 2. Proof that $\varphi'\Delta\eta' = \eta$. By the work above, it suffices to show that the maps

$$\Delta X \xrightarrow{\eta} r \xrightarrow[\simeq]{\beta} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}$$

and

$$\Delta X \xrightarrow{\gamma} \prod_{\mathrm{pr}_{23}} X \times \Delta X \xrightarrow{\prod_{\mathrm{pr}_{23}} \psi} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)} \xrightarrow[\simeq]{\prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\eta \times X)}} \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}$$

are equal in $\mathcal{E}_{/X^2}$. By Lemma 5.3.2 (ii), there is a map

$$\rho\colon \Delta X \to \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)}$$

making the following diagram commute in $\mathcal{E}_{/X^2}$

$$\begin{tikzcd}
\Delta X \arrow[r, dashed, "\rho"] \arrow[d, "\eta"'] & \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(r \times X)} \arrow[d, "\prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\eta \times X)}"] \\
r \arrow[r, "\beta"'] & \prod_{\mathrm{pr}_{23}} (\widetilde{X \times r})^{(\Delta X \times X)}
\end{tikzcd}$$

Thus, we only need to show that $\rho = \left(\prod_{\mathrm{pr}_{23}} \psi\right)\gamma$. Equivalently, we can show that the adjunct maps

$$\rho', \psi \colon (X \times \Delta X) \to (\widetilde{X \times r})^{(r \times X)}$$

are equal in $\mathcal{E}_{/X^3}$. Since the square (7) in the diagram (D) is a pullback, we only need to show that ρ' and ψ are equal after composing with $g := (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})$ and $\sigma \colon g(\widetilde{X \times r})^{(r \times X)} \to (\mathrm{id} \times u)^{(u \times \mathrm{id})}$, that is, as maps $g(X \times \Delta X) \to (\mathrm{id} \times u)^{(u \times \mathrm{id})}$. Finally, we can further show that $\sigma\rho'$, $\sigma\psi$ are equal in $\mathcal{E}_{/\mathcal{U}^2}$ by showing their adjuncts along the adjunction $(-) \times^{\mathcal{U}^2} (u \times \mathrm{id}) \dashv (-)^{(u \times \mathrm{id})}$ are equal.

In order to describe the adjunct of $\sigma\rho'$, we use Lemma A.1.2 with $f = X \times \Delta X$, $g := (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})$, $p = \mathrm{id} \times u$ and $q = u \times \mathrm{id}$. Consequently, $g^* q = r \times X$, $g^* p = \widetilde{X \times r}$ and the adjunct of $\sigma\rho'$ is given as the composite map

$$g((X \times \Delta X) \times^{X^3} (r \times X)) \xrightarrow{\rho^\sharp} g(\widetilde{X \times r}) = gg^*(\mathrm{id} \times u) \xrightarrow{\epsilon_{(\mathrm{id} \times u)}} \mathrm{id} \times u$$

Recall that, by definition, there is a pullback square

$$\begin{tikzcd} R \arrow[r, "\widetilde{r}"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \arrow[d, "u"] \\ X \times X \arrow[r, "\ulcorner r \urcorner"'] & \mathcal{U} \end{tikzcd} \tag{5.2}$$

Since $(X \times \Delta X) \times^{X^3} (r \times X) = (X \times \Delta X)r$, using the proof of Lemma 5.3.2 and the fact that $g^*(\mathrm{id} \times u) = \widetilde{X \times r}$, we have that $\rho^\sharp \colon (X \times \Delta X)r \to \widetilde{X \times r}$ and $\epsilon_{(\mathrm{id} \times u)} \colon g(\widetilde{X \times r}) \to \mathrm{id} \times u$ are described by the two squares below

$$\begin{tikzcd} R \arrow[r, "\rho^\sharp = (r_2, \mathrm{id})"] \arrow[d, "r"'] \arrow[dr, "(r_1, r_2, r_2)" description] & X \times R \arrow[d, "\widetilde{X \times r}"] \\ X^2 \arrow[r, "X \times \Delta X"'] & X^3 \end{tikzcd} \qquad \begin{tikzcd} X \times R \arrow[r, "\epsilon_{(\mathrm{id} \times u)} = (\ulcorner r \urcorner (r_1 \mathrm{pr}_2, \mathrm{pr}_1), \widetilde{r}\, \mathrm{pr}_2)"] \arrow[d, "\widetilde{X \times r}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \times \widetilde{\mathcal{U}} \arrow[d, "\mathrm{id} \times u"] \\ X^3 \arrow[r, "g = (\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})"'] & \mathcal{U}^2 \end{tikzcd}$$

Hence, the composite $\epsilon_{(\mathrm{id}\times u)}\rho^\sharp$ (the adjunct of $\sigma\rho'$ in $\mathcal{E}_{/\mathcal{U}^2}$) is given by the map

$$\begin{tikzcd}
R \arrow[rr, "(\ulcorner r \urcorner r, \widetilde{r})"] \arrow[dr, "(\ulcorner r \urcorner, \ulcorner r \urcorner) r"'] & & \mathcal{U} \times \widetilde{\mathcal{U}} \arrow[dl, "\mathrm{id}\times u"] \\
& \mathcal{U}^2 &
\end{tikzcd} \tag{5.3}$$

To describe the adjunct of $\sigma\psi$, note that, from the squares (6), (7) and (3) and the definition of j in the diagram (D), $\sigma\psi$ is given as the map in $\mathcal{E}_{/\mathcal{U}^2}$ described by the diagram

$$\begin{tikzcd}
X^2 \arrow[r, "\ulcorner r \urcorner"] \arrow[d, "X\times\Delta X"'] \arrow[dr, "(\ulcorner r \urcorner, \ulcorner r \urcorner)" description] & \mathcal{U} \arrow[r, "j"] \arrow[d, "\Delta\mathcal{U}"] & M \arrow[dl, "(\mathrm{id}\times u)^{(u\times\mathrm{id})}"] \\
X^3 \arrow[r, "g=(\ulcorner r \urcorner \mathrm{pr}_{12}, \ulcorner r \urcorner \mathrm{pr}_{13})"'] & \mathcal{U}^2 &
\end{tikzcd}$$

Then, the adjunct of $j\ulcorner r \urcorner$ in $\mathcal{E}_{/\mathcal{U}^2}$ is the composite $j^\sharp(\ulcorner r \urcorner \times^{\mathcal{U}^2} (u \times \mathrm{id}))$, where $j^\sharp \colon \Delta\mathcal{U} \times^{\mathcal{U}^2} (u \times \mathrm{id}) \to \mathrm{id} \times u$ is the adjunct of j. Using that there are pullback squares

$$\begin{tikzcd}
R \arrow[r, "(\widetilde{r}, \ulcorner r \urcorner r)"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \times \mathcal{U} \arrow[d, "u\times\mathrm{id}"] \\
X^2 \arrow[r, "(\ulcorner r \urcorner, \ulcorner r \urcorner)"'] & \mathcal{U}^2
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{\mathcal{U}} \arrow[r, "(\mathrm{id}, u)"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}} \times \mathcal{U} \arrow[d, "u\times\mathrm{id}"] \\
\mathcal{U} \arrow[r, "\Delta\mathcal{U}"'] & \mathcal{U}^2
\end{tikzcd}$$

we get that $j^\sharp(\ulcorner r \urcorner \times^{\mathcal{U}^2} (u \times \mathrm{id}))$ is the composite map

$$\begin{tikzcd}
R \arrow[r, "\widetilde{r}"] \arrow[dr, "(\ulcorner r \urcorner, \ulcorner r \urcorner) r"'] & \widetilde{\mathcal{U}} \arrow[r, "j^\sharp = (u, \mathrm{id})"] \arrow[d, "(u,u)"] & \mathcal{U} \times \widetilde{\mathcal{U}} \arrow[dl, "\mathrm{id}\times u"] \\
& \mathcal{U}^2 &
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[rr, "(u\widetilde{r}, \widetilde{r})"] \arrow[dr, "(\ulcorner r \urcorner, \ulcorner r \urcorner) r"'] & & \mathcal{U} \times \widetilde{\mathcal{U}} \arrow[dl, "\mathrm{id}\times u"] \\
& \mathcal{U}^2 &
\end{tikzcd} \tag{5.4}$$

One can now see that the maps (5.3) and (5.4) are equal by using the square (5.2) defining $\ulcorner r \urcorner$ (including the implicit given homotopies). Our proof is then complete. □

Once we know that every map in $\mathcal{E}$ has an L'-localization, we can also show that L'-localization form a reflective subfibration on $\mathcal{E}$. The crucial point here is to show pullback-compatibility of L'-reflections. This is necessary when working in higher topos theory, but it is superfluous in homotopy type theory as reflections are automatically stable under pullbacks in that setting.

Corollary 5.3.4. *Given any reflective subfibration $L_\bullet$ of an ∞-topos $\mathcal{E}$, there exists a reflective subfibration $L'_\bullet$ of $\mathcal{E}$ such that the L'-local maps are exactly the L-separated maps. Furthermore, if $L_\bullet$ is a modality, then so is $L'_\bullet$.*

Proof. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{E}$ spanned by the L-separated objects and let $\iota\colon \mathcal{D}' \to \mathcal{E}$ be the inclusion functor. Theorem 5.3.3 constructs, for every $X \in \mathcal{E}$, an L'-localization map $\eta(X)\colon X \to L'(X)$. By definition of L'-localization map, this means that, for every $X \in \mathcal{E}$, the ∞-category defined as the pullback

$$\begin{tikzcd} X_{/\mathcal{D}'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d, "\iota"] \\ X_{/\mathcal{E}} \arrow[r] & \mathcal{E} \end{tikzcd}$$

has an initial object. By [Joy08, §17.4], ι has a left adjoint $L'\colon \mathcal{E} \to \mathcal{D}'$, i.e., $\mathcal{D}'$ is a reflective subcategory of $\mathcal{E}$. The same construction performed on each slice category now gives that, for every $X \in \mathcal{E}$, the full subcategory $\mathcal{D}'_X$ of $\mathcal{E}_{/X}$ on the L-separated $p \in \mathcal{E}_{/X}$ is reflective. On the other hand, Proposition 5.1.5 says that, for every $f\colon X \to Y$, the pullback functor $f^*\colon \mathcal{E}_{/Y} \to \mathcal{E}_{/X}$ restricts to a functor $\mathcal{D}'_Y \to \mathcal{D}'_X$. Therefore, we obtain a system of reflective subcategories $L'_\bullet$ on $\mathcal{E}$. To conclude that we actually get a reflective subfibration, we only need to verify that the L'-reflection maps are compatible with pullbacks.

Let then $p\colon E \to X$ be an object in $\mathcal{E}_{/X}$ and $f\colon Y \to X$ a map in $\mathcal{E}$. Let

$$\begin{tikzcd} E \arrow[rr, "\eta' := \eta'_X(p)"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

be the L'-localizations of p. We need to show that

$$m := f^*(\eta') \colon f^*(p) \to f^*(p')$$

is the L'-localization of $f^*(p)$ in $\mathcal{E}_{/Y}$. We do so by using Theorem 5.2.10. Set $f^*(E) := Y \times_X E$, $q := f^*(p)$ and $f^*(E') := Y \times_X E'$. Since η' is an L'-localization, it is an effective epimorphism. Since effective epimorphisms are closed under pullbacks, an application of the pasting lemma for pullbacks show that m is also an effective epimorphism. By Proposition 5.1.5, we also know that $f^*(p')$ is L-separated. Therefore, we only need to show that $\Delta(m)$ — as a map in $\mathcal{E}_{/f^*(E)\times_Y f^*(E)}$ — is the L'-localization map of Δq. In $\mathcal{E}_{/Y}$ we have the pullback square (products are products in $\mathcal{E}_{/Y}$)

$$\begin{tikzcd}
q \times_{f^*(p')} q \arrow[r] \arrow[d, "(m\times m)^*(\Delta(f^*(p'))"'] \arrow[dr, phantom, "\lrcorner", very near start] & f^*(p') \arrow[d, "\Delta(f^*(p'))"] \\
q \times q \arrow[r, "m\times m"'] & f^*(p') \times f^*(p')
\end{tikzcd}$$

and Δm is a map $\Delta q \to (m \times m)^*(\Delta(f^*(p'))$ in $\left(\mathcal{E}_{/Y}\right)_{/(q\times q)}$. Using the equivalence

$$\left(\mathcal{E}_{/Y}\right)_{/(q\times q)} \simeq \mathcal{E}_{/f^*(E)\times_Y f^*(E)}$$

and the definition of $m = f^*(\eta')$, one can see that Δm is the map

$$\begin{tikzcd}
f^*(E) \arrow[rr, "\Delta m"] \arrow[dr, "\Delta q"'] & & f^*(E) \times_{f^*(E')} f^*(E) \arrow[dl, "t"] \\
& f^*(E) \times_Y f^*(E) &
\end{tikzcd}$$

in $\mathcal{E}_{/f^*(E)\times_Y f^*(E)}$, where t corresponds to the map $(m\times m)^*(\Delta(f^*(p'))$ above. Similarly, $\Delta\eta'$ is a map $\Delta p \to s$ in $\mathcal{E}_{/E\times_X E}$ (where s is a suitable pull-backed map) and it is the L-localization of Δp by Theorem 5.2.10. We want to show that Δm is a pullback of this L-localization and conclude because $L_\bullet$ is a reflective subfibration. Let $g \colon f^*(E) \to E$

and $g'\colon f^*(E') \to E'$ be the projection maps. As in the proof of Proposition 5.1.5, with a few straightforward applications of the pasting lemma for pullbacks we see that the following are all pullback squares in $\mathcal{E}$

$$\begin{tikzcd}
f^*(E) \times_Y f^*(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \times_X E \arrow[d] \\
f^*(E) \arrow[r, "g"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*(E') \times_Y f^*(E') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \times_X E' \arrow[d] \\
f^*(E') \arrow[r, "g'"'] & E'
\end{tikzcd}$$

$$\begin{tikzcd}
f^*(E) \arrow[r, "g"] \arrow[d, "\Delta(f^*(p))"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\Delta p"] \\
f^*(E) \times_Y f^*(E) \arrow[r] & E \times_X E
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*(E') \arrow[r, "g'"] \arrow[d, "\Delta(f^*(p'))"'] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, "\Delta p'"] \\
f^*(E') \times_Y f^*(E') \arrow[r] & E' \times_X E'
\end{tikzcd}$$

Then in the diagram

$$\begin{tikzcd}
f^*(E) \times_{f^*(E')} f^*(E) \arrow[rr] \arrow[dr] \arrow[dd, "t"'] & & E \times_{E'} E \arrow[dr] \arrow[dd, "s"' near start] & \\
& f^*(E') \arrow[rr, crossing over, "g'"' near start] & & E' \arrow[dd, "\Delta p'"] \\
f^*(E) \times_Y f^*(E) \arrow[rr] \arrow[dr, "m \times_Y m"'] & & E \times_X E \arrow[dr, "\eta' \times_X \eta'"] & \\
& f^*(E') \times_Y f^*(E') \arrow[rr] \arrow[from=uu, crossing over, "\Delta(f^*(p'))" near end] & & E' \times_X E'
\end{tikzcd}$$

the left and right sides are pullbacks (by definition of t and s) and the front square is a pullback by the above. Therefore, the back square is also a pullback. A final application of the pasting lemma now shows that there are pullback squares in $\mathcal{E}$

$$\begin{tikzcd}
f^*(E) \arrow[r, "\Delta m"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & f^*(E) \times_{f^*(E')} f^*(E) \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f^*(E) \times_Y f^*(E) \arrow[d] \\
E \arrow[r, "\Delta\eta'"'] & E \times_{E'} E \arrow[r, "s"'] & E \times_X E
\end{tikzcd}$$

completing the proof that $L'_\bullet$ is a reflective subfibration.

The final claim about L' being a modality when L is follows from the observation that, given composable maps in $\mathcal{E}$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

we have $\Delta(gf) = p\Delta f$, where p is a suitable pullback of Δg (see the proof of Lemma 5.1.6). Therefore, if g is L-separated (so that Δg is L-local), p is L-local. If also f is L-separated and L is a modality, we can then conclude from $\Delta(gf) = p\Delta f$ that gf is L-separated. $\square$

Chapter 6

Consequences of the existence of $L'_\bullet$

In this chapter, we explore a few interactions between $L'_\bullet$ and $L_\bullet$. We apply some of these results to provide a detailed discussion of those reflective subfibrations for which $L_\bullet = L'_\bullet$.

We start Section 6.1 with a few consequences of the existence of and the characterization of L'-localizations. In particular, Proposition 6.1.2 gives a recipe for constructing new stable factorization systems from a given one, and provides an interesting example of how the theory of reflective subfibrations can be used to prove theorems that make no reference to it. Corollary 6.1.3 can serve as a motivation for the study of $L'_\bullet$, as it shows how L'-localization can be used to compute the L-localization of loop objects. We then prove that L'_1 is almost left exact (Proposition 6.1.4) and investigate some relationships between L- and L'-equivalences/connected maps. Statements and some proof ideas parallel the equivalent ones in [CORS18, §2.4].

In Section 6.2, we introduce *self-separated* reflective subfibrations as those $L_\bullet$ for which L-separated maps are automatically L-local. We show that these can only be non-trivial if the ground topos $\mathcal{E}$ is *not* hypercomplete. In this case, every self-separated reflective subfibration arises as one associated to a *(quasi-)cotopological localization* of $\mathcal{E}$ (Theorem 6.2.8), of which the hypercompletion localization is the maximal example. The content of this section does not appear in [CORS18].

6.1 Further interactions between $L_\bullet$ and $L'_\bullet$

Let $L_\bullet$ be a reflective subfibration on $\mathcal{E}$. We gather here some results that further highlight the relationship between the reflective subfibrations $L_\bullet$ and $L'_\bullet$.

We start with a couple of general properties of $L'_\bullet$ that follow from the existence of $L'_\bullet$. Since $L'_\bullet$ is itself a reflective subfibration, we can talk about L'-connected maps. These turn out to be linked to L-connected maps in the expected way.

Proposition 6.1.1. *A map $p\colon E \to X$ is L'-connected if and only if it is an effective epimorphism and $\Delta p\colon E \to E \times_X E$ is L-connected.*

Proof. By definition (see Definition 4.1.1), p is L'-connected if and only if $p\colon p \to \mathrm{id}_X$ is the L'-localization map of $p \in \mathcal{E}_{/X}$. The claim now follows by applying Theorem 5.2.10. □

We can use this corollary to produce new stable factorization systems from a given one. (Recall the definitions and results in Section 4.2.)

Proposition 6.1.2. *Let $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ be a stable factorization system on an ∞-topos $\mathcal{E}$. Let $\mathcal{L}'$ be the class of maps f in $\mathcal{E}$ which are effective epimorphisms and such that $\Delta f \in \mathcal{L}$, and let $\mathcal{R}'$ be the class of maps g in $\mathcal{E}$ such that $\Delta g \in \mathcal{R}$. Then $\mathcal{F}' = (\mathcal{L}', \mathcal{R}')$ is a stable factorization system on $\mathcal{E}$.*

Proof. By Theorem 4.2.5 (1), there is a modality $L^{\mathcal{F}}_\bullet$ on $\mathcal{E}$ associated to $\mathcal{F}$. We can apply Corollary 5.3.4 to $L = L^{\mathcal{F}}_\bullet$ and obtain the modality $L'_\bullet$ of L-separated maps, which then gives rise to a stable factorization system $\mathcal{F}' = \mathcal{F}_{L'}$, again by Theorem 4.2.5. Since the $L^{\mathcal{F}}_\bullet$-connected maps are exactly the maps in $\mathcal{L}$, Proposition 6.1.1 allows us to conclude. □

Let $L^0_\bullet$ be the modality associated to the stable factorization system of 0-connected and 0-truncated maps, as in Example 4.2.2. Consider the circle S^1 in the ∞-topos $\mathcal{E} = \infty\mathsf{Gpd}$ and fix a point in it. Then ΩS^1, the loop space of S^1 with respect to the given point, is 0-truncated. (Recall that, for $X \in \mathcal{E}$ and $x\colon 1 \to X$ a global element of X, $\Omega(X, x) := 1 \times_X 1$.) Therefore, $L^0(\Omega S^1) \simeq \Omega S^1$. On the other hand, $L^0(S^1)$ is a point, because S^1 is 0-connected. This simple observation shows that, in general, L-localization does not commute with taking loop objects. It turns out that L'-localization can be used to fix this misbehaviour and compute localizations of loop objects.

Corollary 6.1.3. *Let $L_\bullet$ be a reflective subfibration on $\mathcal{E}$. Let $X \in \mathcal{E}$ be an object of $\mathcal{E}$ with a global element $x\colon 1 \to X$. Set $\Omega X := \Omega(X, x)$. Then $L(\Omega X) \simeq \Omega(L'X)$, where the loop space of $L'X$ is taken with respect to the global element $1 \xrightarrow{x} X \xrightarrow{\eta'(X)} L'X$.*

Proof. Since ΩX is the pullback along $1 \xrightarrow{(x,x)} X^2$ of ΔX, the localization $L(\Omega X)$ is the pullback along (x, x) of the L-localization of ΔX in $\mathcal{E}_{/X^2}$, by Definition 3.1.1 (2). By Proposition 5.2.9, the L-localization of ΔX is the map $X \times_{L'X} X \to X^2$, which can be obtained as the pullback of $\Delta(L'X)$ along $(\eta'(X))^2$. The claim then follows. $\square$

The next result we prove relates the pullback of a cospan of objects in $\mathcal{E}$ to the pullback of the L'-localized span.

Proposition 6.1.4 ([CORS18, Prop. 2.28])**.** *Let $Y \xrightarrow{g} X \xleftarrow{f} Z$ be a cospan of maps in $\mathcal{E}$ and let $L'Y \xrightarrow{L'g} L'X \xleftarrow{L'f} L'Z$ be the associated cospan of L'-local objects. Then, the natural map $\psi\colon P \to Q$ induced on pullbacks is an L_1-equivalence.*

Proof. The situation can be described by the diagram

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr, dashed, "\psi"] \arrow[dd, "p"'] & & Z \arrow[dr, "\eta'(Z)"] \arrow[dd, "\Delta Z" near end] & \\
& Q \arrow[rr, crossing over] & & L'Z \arrow[dd, "\Delta(L'Z)"] \\
Y \times X \arrow[rr, "g\times f" near end] \arrow[dr, "\eta'(Y)\times\eta'(X)"'] & & Z^2 \arrow[dr, "\eta'(Z)^2"] & \\
& L'Y \times L'X \arrow[rr, "L'g\times L'f"'] \arrow[from=uu, crossing over, "q" near start] & & L'Z^2
\end{tikzcd}$$

where the front and back squares are pullbacks. If we let $\eta\colon \Delta Z \to r_Z$ be the L-reflection map of ΔZ into $\mathcal{D}_{Z^2}$, we can expand the back and the right faces above as in the following diagram

$$\begin{tikzcd}
& P \arrow[rr] \arrow[dl, "\bar\eta"'] \arrow[dd, "p"] & & Z \arrow[dl, "\eta"'] \arrow[dd, "\Delta Z"'] \arrow[dr, "\Delta(\eta'(Z))"] \arrow[rrr, bend left, "\eta'(Z)"] & & & \\
(g\times f)^* R_Z \arrow[rr] \arrow[dr, "(g\times f)^* r_Z"'] & & R_Z \arrow[rr, dashed, "\exists ! \varphi", "\simeq"'] \arrow[dr, "r_Z"'] & & Z \times_{L'Z} Z \arrow[rr] \arrow[dl] & & L'Z \arrow[dl, "\Delta(L'Z)"] \\
& Y \times X \arrow[rr, "g\times f"'] & & Z^2 \arrow[rr, "\eta'(Z)^2"'] & & L'Z^2 &
\end{tikzcd}$$

where the equivalence φ is given by Proposition 5.2.9. Note that $\overline{\eta}$, being the reflection map of p into $\mathcal{D}_{Y\times X}$, is an $(L_{Y\times X})$-equivalence. The bottom half of the diagram above gives a composite pullback square. So we see that $(g\times f)^*(r_z)$ is the pullback of $\Delta(L'Z)$ along

$$(\eta'(Z)^2)(g\times f) = (L'g \times L'f)(\eta'(Y)\times\eta'(X)).$$

Therefore, the composite pullback square factors through q as

$$\begin{tikzcd}
(g\times f)^*(R_Z) \arrow[r, "t"] \arrow[d, "(g\times f)^*(r_Z)"'] & Q \arrow[r] \arrow[d, "q"] & L'Z \arrow[d, "\Delta(L'Z)"] \\
Y \times X \arrow[r, "\eta'(Y)\times\eta'(X)"'] & L'Y \times L'X \arrow[r, "L'g\times L'f"'] & L'Z^2
\end{tikzcd}$$

since the right square is a pullback by definition. Therefore, the left square is also a pullback and the map t is L-connected, because it is a pullback of the L-connected map $\eta'(Y)\times\eta'(X)$ (see Lemma 5.2.4). It follows from these considerations that the map $\psi\colon P \to Q$ is given by the composite $t\overline{\eta}$, where both t and $\overline{\eta}$ are L_1-equivalences (see also Lemma 3.1.4 and Definition 4.1.1). Hence, ψ is also an L_1-equivalence, as needed. ☐

Remark 6.1.5. Although we will not need this stronger result, the proof of Proposition 6.1.4 above actually shows that ψ is an $(L_{L'Y\times L'X})$-equivalence.

We now study interactions between L/L'-equivalences and L/L'-connected maps.

Proposition 6.1.6 ([CORS18, Prop. 2.30]). *Suppose $f\colon Y\to X$ is an L'_1-equivalence. Then f is L-connected.*

Proof. To ease readability, we slightly change our usual notation here. Namely, we let $\eta'_X\colon X\to L'X$ be the L'-localization map of X, and we let $\eta_X(f)\colon f\to L_X(f)$ be the reflection map of $f\in\mathcal{E}_{/X}$ into $\mathcal{D}_X$. We need to show that $L_X(f)$ is an equivalence. Observe that if we apply $\Sigma_{\eta'_X}$ to $\eta_X(f)$, we get an $L_{L'X}$-equivalence $\eta_X(f)\colon \eta'_X f\to\eta'_X L_X(f)$, by Lemma 3.1.4 (ii). Consider the composite map $\eta'_X L_X(f)$ and reflect it into $\mathcal{D}_{L'X}$ to obtain the map

$$\eta_{L'X}(\eta'_X L_X(f))\colon \eta'_X L_X(f)\to L_{L'X}(\eta'_X L_X(f)).$$

Now, the composite in $\mathcal{E}_{/L'X}$ of this map with $\eta_X(f)\colon\eta'_X f\to\eta'_X L_X(f)$ is the reflection map of $\eta'_X f$ into $\mathcal{D}_{L'X}$, because it is an $L_{L'X}$-equivalence into an $L_{L'X}$-local object. Thus, we can write this composite as

$$\eta_{L'X}(\eta'_X f)\colon \eta'_X f\to L_{L'X}(\eta'_X L_X(f))$$

We have a commutative square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\eta'_Y"'] & X \arrow[d, "\eta'_X"] \\ L'Y \arrow[r, "L'f"'] & L'X \end{tikzcd}$$

and $L'f$ is an equivalence by hypothesis. Hence, if we apply $\Sigma_{(L'f)^{-1}}$ to $\eta_{L'X}(\eta'_X f)$ above, we get the reflection map of $(L'f)^{-1}\eta'_X f=\eta'_Y$, as in

$$\eta_{L'X}(\eta'_X f)\colon \eta'_Y\to (L'f)^{-1}(L_{L'X}(\eta'_X L_X(f)))$$

Since η'_Y is L-connected (see Lemma 5.2.4), we must have that the map

$$(L'f)^{-1}(L_{L'X}(\eta'_X L_X(f)))$$

is an equivalence and, then, $L_{L'X}(\eta'_X L_X(f))$ is an equivalence as well. But, by Proposition 5.2.6, there is a pullback square

$$\begin{tikzcd} L_X Y \arrow[r, "\eta_{L'X}(\eta'_X L_X(f))"] \arrow[d, "L_X(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & L_{L'X}(L_X Y) \arrow[d, "L_{L'X}(\eta'_X L_X(f))"] \\ X \arrow[r, "\eta'_X"'] & L'X \end{tikzcd}$$

Therefore, $L_X(f)$ is also an equivalence, as required. □

Remark 6.1.7. Let $p\colon E \to X$ and $q\colon M \to X$ be maps in $\mathcal{E}$. If $\alpha\colon p \to q$ is a map in $\mathcal{E}_{/X}$ which is an L'_X-equivalence, then, as a map in $\mathcal{E}$, $\alpha\colon E \to M$ is an L'_1-equivalence, by Lemma 3.1.4 applied to $L'_\bullet$. Hence, the result above applies to show that $\alpha\colon E \to M$ is L-connected.

Lemma 6.1.8. *Let $f\colon Y \to X$ be an L-connected map, with Y L'-connected. Then X is L'-connected as well.*

Proof. The hypothesis on Y implies that there is a commutative square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\eta'_Y"'] & X \arrow[d, "\eta'_X"] \\ 1 \arrow[r, "L'f"'] & L'X \end{tikzcd}$$

To conclude that X is L'-connected, it suffices to show that $L'f\colon 1 \to L'X$ is an equivalence. Consider the pullback square

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta'_X"] \\ 1 \arrow[r, "L'f"'] & L'X \end{tikzcd}$$

Note that i is an L-local map, because it is also the pullback of the L-local map $\Delta(L'X)$ along $X \simeq X \times 1 \xrightarrow{\eta'_X \times L'f} (L'X)^2$. Since η'_X is an effective epimorphism, the above pullback square implies that $L'f$ is an L-local map. Since $!\colon Y \to 1$ is an L_1-equivalence by hypothesis, after composing it with $L'f\colon 1 \to L'X$, it becomes an $L_{L'X}$-equivalence, as a map $L'f \circ ! = \eta'_X f \to L'f$. Altogether, this means that

$$\begin{tikzcd} Y \arrow[rr, "!"] \arrow[dr, "\eta'_X f"'] & & 1 \arrow[dl, "L'f"] \\ & L'X & \end{tikzcd}$$

gives the reflection map of $\eta'_X f$ into $\mathcal{D}_{L'X}$. But $\eta'_X f$ is L-connected, by Lemma 4.1.4 and Lemma 5.2.4, and so $L'f$ is an equivalence, as needed. $\square$

Proposition 6.1.9 ([CORS18, Prop. 2.31])**.** *Suppose given a commutative triangle*

$$\begin{tikzcd} E \arrow[rr, "a"] \arrow[dr, "p"'] & & M \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

where p is L'-connected. Then a is L-connected if and only if q is L'-connected.

Proof. Recall from Remark 4.1.3 that a being L-connected is the same as the map $a\colon p \to q$ in $\mathcal{E}_{/X}$, given by the commutative triangle above, being $L^{/X}$-connected for the sliced reflective subfibration $L_\bullet^{/X}$ of $\mathcal{E}_{/X}$. Suppose that p is L'-connected. If a is L-connected, then q is L'-connected, by Lemma 6.1.8 applied to $\mathcal{E}_{/X}$. Conversely, if q is L'-connected, then both $q\colon q \to \mathrm{id}_X$ and $p\colon p \to \mathrm{id}_X$ are L'_X-equivalences. But then $a\colon p \to q$ is also an L'_X-equivalence, and so $a\colon E \to M$ is an L'_1-equivalence. Therefore, a is L-connected, by Proposition 6.1.6. $\square$

6.2 Self-separated reflective subfibrations

We study here those reflective subfibrations $L_\bullet$ on $\mathcal{E}$ for which $L_\bullet = L'_\bullet$ and show that they correspond to special kinds of left exact reflective subcategories of $\mathcal{E}$, the *quasi-cotopological localizations* of $\mathcal{E}$.

Definition 6.2.1. A reflective subfibration $L_\bullet$ on an ∞-topos $\mathcal{E}$ is *self-separated* if every L-separated map is L-local.

The existence of self-separated reflective subfibrations on $\mathcal{E}$ is related to a property of an ∞-topos called *hypercompleteness*. This property is discussed in detail in [Lur09, §6.5.2]. For the reader's convenience, we gather here the main aspects of hypercompleteness that we need.

Definition 6.2.2. Let $\mathcal{E}$ be an ∞-topos.

1. A map f in $\mathcal{E}$ is called ∞*-connected* if it is n-connected, for every $n \geq (-2)$. In particular, equivalences are ∞-connected.

2. An object X in $\mathcal{E}$ is called *hypercomplete* if it is local with respect to the class of all ∞-connected maps. This means that, for every ∞-connected map $f\colon A \to B$, the map $X^f\colon X^B \to X^A$ is an equivalence in $\mathcal{E}$. Equivalently, since ∞-connected maps are closed under products, $\mathcal{E}(f, X)\colon \mathcal{E}(B, X) \to \mathcal{E}(A, X)$ is an equivalence in $\infty\mathsf{Gpd}$, for every ∞-connected map f. In particular, n-truncated objects are hypercomplete. A map $p\colon E \to X$ is *hypercomplete* if it is a hypercomplete object of $\mathcal{E}_{/X}$.

3. $\mathcal{E}$ is a *hypercomplete* ∞-topos if every object in $\mathcal{E}$ is hypercomplete. Equivalently, $\mathcal{E}$ is hypercomplete if every ∞-connected map in $\mathcal{E}$ is an equivalence.

Typical hypercomplete ∞-topoi are $\mathcal{E} = \infty\mathsf{Gpd}$ and $\mathcal{E} = \mathsf{Pre}(\mathcal{C})$, the infinity topos of presheaves of ∞-groupoids over a small category $\mathcal{C}$. However, not every ∞-topos is hypercomplete. What we have in general is the following result.

Proposition 6.2.3. *Let $\mathcal{E}$ be an ∞-topos and let $\mathcal{E}^\wedge$ be the full subcategory of $\mathcal{E}$ spanned by the hypercomplete objects.*

1. *$\mathcal{E}^\wedge$ is an accessible, left exact, reflective subcategory of $\mathcal{E}$. In particular, it is an ∞-topos. As such, it is hypercomplete.*

2. *A map in $\mathcal{E}$ is hypercomplete if and only if it is right orthogonal to every ∞-connected map. For $X \in \mathcal{E}$, a map α in the ∞-topos $\mathcal{E}_{/X}$ is ∞-connected if and only if $\Sigma_X(\alpha)$ is ∞-connected in $\mathcal{E}$.*

3. *There exists a modality $L^\wedge_\bullet$ on $\mathcal{E}$ for which, given any $X \in \mathcal{E}$, the $L^\wedge$-equivalences are the ∞-connected maps, and the $L^\wedge$-local maps are the hypercomplete maps. We call this modality the* hypercompletion modality *on $\mathcal{E}$.*

Proof. The first part follows from [Lur09, Prop. 6.5.2.8] (see the discussion right after it) and from [Lur09, Lemma 6.5.2.12]. The second part is [Lur09, Rmk. 6.5.2.21]. For the last part, we can apply the results of Proposition 4.2.8 to the left adjoint $\mathcal{E} \to \mathcal{E}^\wedge$ of the inclusion $\mathcal{E}^\wedge \subseteq \mathcal{E}$. In this way, we obtain a modality $L^\wedge_\bullet$ on $\mathcal{E}$ with the desired $L^\wedge$-equivalences and $L^\wedge$-local maps. □

We are now ready to study self-separated reflective subfibrations.

Lemma 6.2.4. *Let $L_\bullet$ be a self-separated reflective subfibration on $\mathcal{E}$. Then every L-equivalence is an ∞-connected map and every hypercomplete map is L-local. In particular, if $\mathcal{E}$ is hypercomplete, $L_\bullet$ is the trivial reflective subfibration for which the L-equivalences are exactly the equivalences in $\mathcal{E}$, and every map is an L-local map.*

Proof. The self-separated property of $L_\bullet$ means that, for every map $p \in \mathcal{E}$, if Δp is L-local, then p is L-local. Since equivalences are L-local, this implies that every monomorphism is L-local. It follows that every n-truncated map is L-local, due to the recursive characterization of n-truncated maps in terms of their diagonal maps (see [Lur09, Lemma 5.5.6.15]). Since, for every $X \in \mathcal{E}$, L_X-equivalences are left orthogonal to every map in $\mathcal{D}_X$ (see Notation 3.1.3), we get that every L-equivalence is ∞-connected. Since a hypercomplete map $p \in \mathcal{E}_{/X}$ is local with respect to *all* ∞-connected map in $\mathcal{E}_{/X}$, it follows that every hypercomplete map is L-local. □

When $\mathcal{E}$ is not hypercomplete, we can find non-trivial examples of self-separated reflective subfibrations.

Definition 6.2.5. Suppose $i\colon \mathcal{D} \hookrightarrow \mathcal{E}$ is a reflective subcategory of $\mathcal{E}$, with reflector $a\colon \mathcal{E} \to \mathcal{D}$. We say that $L := ia$ is a *quasi-cotopological localization* of $\mathcal{E}$ if it is left exact and, for every map f in $\mathcal{E}$, if Lf is an equivalence, then f is ∞-connected.

Note that the hypercompletion $L^\wedge\colon \mathcal{E} \to \mathcal{E}^\wedge$ is a quasi-cotopological localization.

Remark 6.2.6. In [Lur09, Def. 6.5.2.17], Lurie calls a localization *cotopological* if it is quasi-cotopological and accessible. Accessibility is not needed in our setting, so we dropped that condition from our definition.

Proposition 6.2.7. *Let $L_\bullet$ be the modality associated to a quasi-cotopological localization $L\colon \mathcal{E} \to \mathcal{E}$ (see Proposition 4.2.8). Then $L_\bullet$ is self-separated.*

Proof. We start by remarking that, by the construction of $L_\bullet$ given in Proposition 4.2.8, a map in $\mathcal{E}_{/Z}$ is an L-equivalence if and only if it is an $(L_1 = L)$-equivalence in $\mathcal{E}$. Since L is a quasi-cotopological localization, it follows that, for any $Z \in \mathcal{E}$ and any $p \in \mathcal{E}_{/Z}$, all reflection maps $\eta_Z(p)\colon p \to L_Z(p)$ are ∞-connected. In particular, they are effective epimorphisms. We now show that every L-separated object is L-local, the proof for maps being the same, but done in an appropriate slice category. Suppose that $X \in \mathcal{E}$ is such that ΔX is L-local and let $\eta\colon X \to LX$ be the reflection map of X. Using the definition of L-local maps from Proposition 4.2.8, since $L(X^2) \simeq (LX)^2$, the hypothesis that ΔX is an L-local map means that there is a pullback square

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\Delta X"'] \arrow[dr, phantom, "\lrcorner", very near start] & LX \arrow[d, "\Delta(LX)"] \\ X^2 \arrow[r, "\eta^2"'] & LX^2 \end{tikzcd}$$

The statement that this square is a pullback is precisely the statement that the diagonal of η is an equivalence, i.e., η is a monomorphism. Since η is also an effective epimorphism, it is an equivalence and, then, X is L-local, as needed. $\square$

It turns out that the quasi-cotopological localizations are exactly the self-separated reflective subfibrations on $\mathcal{E}$.

Theorem 6.2.8. *The following are equivalent, for a reflective subfibration $L_\bullet$ on $\mathcal{E}$.*

1. *$L_\bullet$ is self-separated.*

2. *$L_\bullet$ is the modality associated to a quasi-cotopological localization of $\mathcal{E}$.*

In this case, hypercomplete maps are L-local.

Proof. Proposition 6.2.7 is the statement that (2) $\implies$ (1). Thus, we need to show that (1) $\implies$ (2). If every L-separated map is L-local, $L_\bullet$ is a modality, by Lemma 5.2.1. By Proposition 6.1.4, given any cospan $X \to Z \leftarrow Y$ of maps in $\mathcal{E}$, the canonical map $X \times_Z Y \to LX \times_{LZ} LY$ is an L-equivalence and $LX \times_{LZ} LY$ is an L-local object, because $L_\bullet$ is a modality. This means that $LX \times_{LZ} LY \simeq L(X \times_Z Y)$, so $L = L_1 \colon \mathcal{E} \to \mathcal{D} \hookrightarrow \mathcal{E}$ is left exact. On the other hand, if $L_\bullet$ is self-separated, Lemma 6.2.4 says that every L-equivalence is ∞-connected. Therefore, $L \colon \mathcal{E} \to \mathcal{E}$ is a quasi-cotopological localization of $\mathcal{E}$. To conclude, we show that a map $f \colon X \to Y$ is L-local if and only if the square

$$\begin{tikzcd} X \arrow[r, "\eta(X)"] \arrow[d, "f"'] & LX \arrow[d, "L(f)"] \\ Y \arrow[r, "\eta(Y)"'] & LY \end{tikzcd} \tag{Δ}$$

is a pullback square. If this is the case, then the local maps of the given reflective subfibration $L_\bullet$ are the same as the local maps of the reflective subfibration associated to L_1, by Proposition 4.2.8, and therefore the two reflective subfibrations are the same. Suppose that $f \colon X \to Y$ is L-local. Then, by Proposition 5.2.6, there is a pullback square

$$\begin{tikzcd} X \arrow[r, "\eta_{LY}(\eta(Y)f)"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & L_{LY}(X) \arrow[d, "L_{LY}(\eta(Y)f)"] \\ Y \arrow[r, "\eta(Y)"'] & LY \end{tikzcd}$$

Since $Lf \in \mathcal{D}_{LY}$, there is a unique map $\varphi\colon L_{LY}(\eta(Y)f) \to Lf$ with $\varphi\eta_{LY}(\eta(Y)f) = \eta(X)$ and $(Lf)\varphi = L_{LY}(\eta(Y)f)$. Since the maps $\eta_{LY}(\eta(Y)f)$ and $\eta(X)$ are both L_1-equivalences, so is φ. Since $L_\bullet$ is a modality, both LX and $L_{LY}(X)$ are L-local objects, and then $\varphi\colon L_{LY}(X) \to LX$ is an equivalence. It follows that the square (Δ) above is a pullback, as needed. $\square$

Chapter 7

Localization with respect to a map

In this chapter, we study S*-localizations*, that is, reflective subfibrations associated to *sets* $S = \{f_i : A_i \to B_i\}_{i \in I}$ of maps in an ∞-topos $\mathcal{E}$. The local objects for these reflective subfibrations are the S*-local objects*, i.e., those $X \in \mathcal{E}$ such that $X^{f_i} : X^{B_i} \to X^{A_i}$ is an equivalence, for every $i \in I$.

In Section 7.1, we prove the existence of such reflective subfibrations, which we specifically carry out when the set S consists of a single map $f : A \to B$, the general case following by the same arguments (Proposition 7.1.7). We call this reflective subfibration $L^f_\bullet$ on $\mathcal{E}$ the f*-localization*. When f is the unique map $A \to 1$, we show that f-localization is a modality, and we call it A*-nullification* (Proposition 7.1.9). Even though the existence results in Section 7.1 will probably not come as a surprise to a homotopy theorist, we stress that what we are concerned with is the existence of a reflective *subfibration* on $\mathcal{E}$, as opposed to the mere existence of a reflective *subcategory*, which is a well-known fact, due to local presentability of $\mathcal{E}$ (see [Lur09, Prop. 5.5.4.15]).

In Section 7.2, we investigate the effect of L'-localization when $L_\bullet$ is given by f-localization and show that it corresponds to Σf-localization, where Σf is the suspension of f. In Section 7.3 and in Section 7.4, we explore properties of f-localization when f is a map in the left class of a factorization system on $\mathcal{E}$, or when f is a map between 0-connected objects.

Section 7.2 to Section 7.4 parallel [CORS18, §3] closely in terms of statements and proofs, modulo the needed adjustments to the topos-theoretic setting. For example, in Section 7.2, we need to work with *cartesian* factorization systems because, in contrast with the situation in homotopy type theory, the left class of a factorization system on $\mathcal{E}$ need not be closed under products.

7.1 Existence of f-localization

Fix a map $f\colon A \to B$ in our favorite ∞-topos $\mathcal{E}$. We will construct a reflective subfibration on $\mathcal{E}$ out of this datum.

Let us start by recalling from [ABFJ17a, Section 3.3] a few definitions and results that we need. The reader might also want to refer back to Section 4.2 for definitions and results concerning factorization systems.

Definition 7.1.1. A map f is *internally orthogonal* to a map $g\colon X \to Y$ in $\mathcal{E}$ if $(f \times Z) \perp g$ for every object $Z \in \mathcal{E}$. If this holds, we write $f \mathrel{\underline{\perp}} g$.

Definition 7.1.2. A factorization system $(\mathcal{L}, \mathcal{R})$ in an ∞-topos $\mathcal{E}$ is called *cartesian* if, for every $l \in \mathcal{L}$ and every $r \in \mathcal{R}$, $l \mathrel{\underline{\perp}} r$ (rather than simply having $l \perp r$).

Let $\mathcal{R} := \{f\}^{\underline{\perp}}$ be the class of all maps g such that $f \mathrel{\underline{\perp}} g$. Then, if we let $\mathcal{L} := {}^{\perp}\mathcal{R} = {}^{\underline{\perp}}\mathcal{R}$, $(\mathcal{L}, \mathcal{R})$ is a factorization system (see [ABFJ17a, Prop. 3.3.8]). By definition, this is a cartesian factorization system. Thus, as in Theorem 4.2.5, we get that $\mathcal{D} := \mathcal{R}_{/1}$ is a reflective subcategory of $\mathcal{E}$. Using the definitions of $\mathcal{R}$ and $\mathcal{L}$ and the fact that if $l, l' \in \mathcal{L}$, then $l \times l' \in \mathcal{L}$ (see [ABFJ17a, Lemma 3.3.7]), it is easy to see that $\mathcal{D}$ consists precisely of all those $X \in \mathcal{E}$ such that $X^f\colon X^B \to X^A$ is an equivalence in $\mathcal{E}$. Furthermore, $\mathcal{D}$ is an exponential ideal (see Proposition 3.1.5 (iii) for terminology).

Definition 7.1.3. An object $X \in \mathcal{E}$ such that $X^f\colon X^B \to X^A$ is an equivalence is called an *f-local* object.

Remark 7.1.4. If we let $L^f \colon \mathcal{E} \to \mathcal{E}$ be the localization functor associated to $\mathcal{D}$, the L^f-equivalences form the class ${}^{\perp}\mathcal{D} = {}^{\perp\!\!\!\perp}\mathcal{D}$. Since $X \in \mathcal{E}$ is f-local if and only if X^s is an equivalence for every L^f-equivalence s, one can use this fact to show that $X \in \mathcal{E}$ is f-local if and only if the map of ∞-groupoids $\mathcal{E}(f, X) \colon \mathcal{E}(B, X) \to \mathcal{E}(A, X)$ is an equivalence.

For any fixed $X \in \mathcal{E}$, we can now consider the map $f \times X \colon A \times X \to B \times X$, seen as a map in $\mathcal{E}_{/X}$:

$$\begin{tikzcd} A \times X \arrow[rr, "f\times X"] \arrow[dr, "\mathrm{pr}_A"'] & & B \times X \arrow[dl, "\mathrm{pr}_B"] \\ & X & \end{tikzcd}$$

Set $\mathcal{R}^X := \{f \times X\}^{\perp\!\!\!\perp}$. As above, we get a cartesian factorization system $(\mathcal{L}^X, \mathcal{R}^X)$ in $\mathcal{E}_{/X}$ with $\mathcal{L}^X := {}^{\perp}\big(\mathcal{R}^X\big) = {}^{\perp\!\!\!\perp}\big(\mathcal{R}^X\big)$. Then $\mathcal{D}_X := \mathcal{R}^X_{/\mathrm{id}_X}$ is a reflective subcategory of $\mathcal{E}_{/X}$.

Definition 7.1.5. We call an object $p \in \mathcal{E}_{/X}$ an f*-local map* if the map

$$p^{(f\times X)} \colon p^{\mathrm{pr}_B} \to p^{\mathrm{pr}_A}$$

is an equivalence.

The reflective subcategory $\mathcal{D}_X$ consists precisely of the f-local maps.

Remark 7.1.6. Using the fact that, if $s \in \mathcal{E}_{/X}$ and $S = \mathrm{dom}(s)$, the product map $(f \times X) \times^X s$ in $\mathcal{E}_{/X}$ is the map

$$\begin{tikzcd} A \times S \arrow[rr, "f\times S"] \arrow[dr, "s\,\mathrm{pr}_A"'] & & B \times S \arrow[dl, "s\,\mathrm{pr}_B"] \\ & X & \end{tikzcd}$$

in $\mathcal{E}_{/X}$, it is easy to see that $\mathcal{R}_X \subseteq \mathcal{R}^X$, where — as in Lemma 4.2.4 — $\mathcal{R}_X$ is the class of maps in $\mathcal{E}_{/X}$ that are in $\mathcal{R}$ when seen as maps in $\mathcal{E}$.

We claim that the assignment $X \mapsto \mathcal{D}_X$ above defines a reflective subfibration on $\mathcal{E}$.

Proposition 7.1.7. *For every map $f\colon A \to B$ in $\mathcal{E}$, there is a reflective subfibration $L^f_\bullet$ on $\mathcal{E}$ for which the local maps are exactly the f-local maps.*

Proof. Fix $f\colon A \to B$ in $\mathcal{E}$. We show the reflective subcategories $\mathcal{D}_X$ of f-local maps constructed as above give rise to a reflective subfibration. To start with, consider a map $g\colon Y \to X$ in $\mathcal{E}$. We need to show that the pullback functor $g^*\colon \mathcal{E}_{/X} \to \mathcal{E}_{/Y}$ restricts to a functor $\mathcal{D}_X \to \mathcal{D}_Y$, i.e., we need to prove that $g^*(p)$ is in $\mathcal{D}_Y$ whenever $p \in \mathcal{D}_X$. By hypothesis, $p^{(f\times X)}$ is an equivalence in $\mathcal{E}_{/X}$, so that $g^*(p^{f\times X})$ is an equivalence in $\mathcal{E}_{/Y}$. But, by Lemma A.1.1, $g^*\left(p^{(f\times X)}\right)$ is equivalent to $(g^*(p))^{g^*(f\times X)}$ (as maps in $\mathcal{E}_{/Y}$). Since $g^*(f\times X) = f\times Y$ as maps in $\mathcal{E}_{/Y}$, we can conclude that $g^*(p)$ is in $\mathcal{D}_Y$. This shows that $X \mapsto \mathcal{D}_X$ is a srs on $\mathcal{E}$.

To prove that it is a reflective subfibration, we verify the condition of Definition 3.1.1 (2). Thus, let $p \in \mathcal{E}_{/X}$ with $E := \mathrm{dom}(p)$. The reflection of p into $\mathcal{D}_X$ is given by the $(\mathcal{L}^X, \mathcal{R}^X)$-factorization of $p \to \mathrm{id}_X$, which we depict as the commutative triangle

$$\begin{tikzcd} E \arrow[rr, "l_p"] \arrow[dr, "p"'] & & L_X(E) \arrow[dl, "r_p"] \\ & X & \end{tikzcd}$$

in $\mathcal{E}$. We need to show that, for $g\colon Y \to X$ in $\mathcal{E}$, $(g^*(l_p), g^*(r_p))$ is the $(\mathcal{L}^Y, \mathcal{R}^Y)$-factorization of $g^*(p) \to \mathrm{id}_Y$ in $\mathcal{E}_{/Y}$, where $g^*(l_p)\colon g^*(p) \to g^*(r_p)$. Note that, by the first part above, we certainly have $g^*(r_p) \in \mathcal{D}_Y$ (that is, $g^*(r_p) \to \mathrm{id}_Y$ is in $\mathcal{R}^Y$). Thus, we only need to show that $g^*(l_p)$ is in $\mathcal{L}^Y$. By definition, this means showing that, for every $m \in \mathcal{R}^Y$, $g^*(l_p) \perp_Y m$. But, by adjointness, this orthogonality condition in $\mathcal{E}_{/Y}$ is equivalent to the orthogonality condition $l_p \perp_X \prod_g m$ in $\mathcal{E}_{/X}$. Since $l_p \in \mathcal{L}^X$ by hypothesis, it suffices to show that, for every $g\colon Y \to X$ and every $m \in \mathcal{R}^Y$, $\prod_g m$ is in $\mathcal{R}^X$.

By Remark 7.1.6, for every object $s \in \mathcal{E}_{/X}$ with $S := \mathrm{dom}(s)$, the product map $(f\times X)\times^X s$ in $\mathcal{E}_{/X}$ is just the map $f\times S\colon s\,\mathrm{pr}_A \to s\,\mathrm{pr}_B$ in $\mathcal{E}_{/X}$. By definition, $\prod_g m$ is in $\mathcal{R}^X$ precisely if $(f\times S) \perp_X \prod_g m$ in $\mathcal{E}_{/X}$ for every $s \in \mathcal{E}_{/X}$ as above. By adjointness, this happens if and only if $g^*(f\times S) \perp_Y m$ in $\mathcal{E}_{/Y}$. An easy application of the pasting lemma for pullbacks shows that, if we denote the domain of $g^*(s)$ by $g^*(S)$, $g^*(f \times S)$ is the map

$$\begin{tikzcd} A \times g^*(S) \arrow[rr, "f \times g^*(S)"] \arrow[dr, "\mathrm{pr}_A"'] & & B \times g^*(S) \arrow[dl, "\mathrm{pr}_B"] \\ & Y & \end{tikzcd}$$

in $\mathcal{E}_{/Y}$. This map is the product map of the object $g^*(s) \in \mathcal{E}_{/Y}$ with the map $f \times Y\colon \mathrm{pr}_A \to \mathrm{pr}_B$ in $\mathcal{E}_{/Y}$. Since $m \in \mathcal{R}^Y = \{f \times Y\}^{\perp}$, we can conclude that $g^*(f \times S) \perp_Y m$, as required. □

Definition 7.1.8. Given a map $f\colon A \to B$ in $\mathcal{E}$, we call the reflective subfibration L^f of Proposition 7.1.7, the *f-local reflective subfibration* on $\mathcal{E}$. When f is the unique map $A \to 1$ for an object $A \in \mathcal{E}$, we call f-local maps A-*null* maps and the f-local reflective subfibration A-*nullification*.

Nullifications are particularly interesting because they always form a modality.

Proposition 7.1.9 (cf. [ABFJ17a, Ex. 3.5.3])**.** *A-nullification is a modality for every $A \in \mathcal{E}$.*

Proof. Using the same notation as in Lemma 4.2.4, we show that, for every $X \in \mathcal{E}$, $\mathcal{R}^X = \mathcal{R}_X$ so that the claim follows from Theorem 4.2.5. By Remark 7.1.6, $\mathcal{R}_X \subseteq \mathcal{R}^X$ holds for every f-local reflective subfibration so we just need to prove the reverse inclusion. Let then $\alpha\colon p \to q$ be a map in $\mathcal{R}^X$ and set $m := \Sigma_X(\alpha)$. Showing that α is in $\mathcal{R}_X$ means proving that $m \in \mathcal{R}$. That is, we have to show that, for every $C \in \mathcal{E}$, every commutative square in $\mathcal{E}$

$$\begin{tikzcd} A \times C \arrow[r, "h"] \arrow[d, "\mathrm{pr}_A"'] & E \arrow[d, "m"] \\ C \arrow[r, "k"'] & M \end{tikzcd} \tag{$*$}$$

has a unique diagonal filler. If we let $\overline{\mathrm{pr}}_A\colon k\,\mathrm{pr}_A \to k$ be the map in $\mathcal{E}_{/M}$ induced by pr_A, this means showing that, for every $k \in \mathcal{E}(C, M)$, in the following comparison diagram of fiber sequences

$$\begin{tikzcd}
\mathcal{E}_{/M}(k,m) \arrow[r] \arrow[d, "\mathcal{E}_{/M}(\overline{\mathrm{pr}}_A, m)"] & \mathcal{E}(C,E) \arrow[r, "\mathcal{E}(C,m)"] \arrow[d, "\mathcal{E}(\mathrm{pr}_A, E)"] & \mathcal{E}(C,M) \arrow[d, "\mathcal{E}(\mathrm{pr}_A, M)"] \\
\mathcal{E}_{/M}(k\,\mathrm{pr}_A, m) \arrow[r] & \mathcal{E}(A\times C, E) \arrow[r, "\mathcal{E}(A\times C, m)"'] & \mathcal{E}(A\times C, M)
\end{tikzcd}$$

the rightmost square is a pullback, that is, the leftmost map is an equivalence. Now, k gives rise to a map $\overline{k}\colon qk \to q$ in $\mathcal{E}_{/X}$ and

$$\mathrm{hofib}_{\overline{k}}\left(\mathcal{E}_{/X}(qk,\alpha)\right) = (\mathcal{E}_{/X})_{/q}(\overline{k},\alpha) \simeq \mathcal{E}_{/M}(k,m)$$

Similarly, $k\,\mathrm{pr}_A$ gives rise to a map $\overline{k\,\mathrm{pr}}_A\colon qk\,\mathrm{pr}_A \to q$ in $\mathcal{E}_{/X}$ and

$$\mathrm{hofib}_{\overline{k\,\mathrm{pr}}_A}\left(\mathcal{E}_{/X}(qk\,\mathrm{pr}_A,\alpha)\right) \simeq \mathcal{E}_{/M}(k\,\mathrm{pr}_A, m).$$

It follows that there is a comparison diagram of fiber sequences

$$\begin{tikzcd}
\mathcal{E}_{/M}(k,m) \arrow[r] \arrow[d, "\mathcal{E}_{/M}(\overline{\mathrm{pr}}_A, m)"] & \mathcal{E}_{/X}(qk,p) \arrow[r, "\mathcal{E}_{/X}(qk,\alpha)"] \arrow[d, "\mathcal{E}_{/X}(\overline{\mathrm{pr}}_A, p)"] & \mathcal{E}_{/X}(qk,q) \arrow[d, "\mathcal{E}_{/X}(\overline{\mathrm{pr}}_A, q)"] \\
\mathcal{E}_{/M}(k\,\mathrm{pr}_A, m) \arrow[r] & \mathcal{E}_{/X}(qk\,\mathrm{pr}_A, p) \arrow[r, "\mathcal{E}_{/X}(qk\,\mathrm{pr}_A,\alpha)"] & \mathcal{E}_{/X}(qk\,\mathrm{pr}_A, q)
\end{tikzcd}$$

We claim that the right square is a pullback so that the induced map on fibers is an equivalence. Indeed, $\overline{\mathrm{pr}}_A$ is the product map in $\mathcal{E}_{/X}$ of id_{qk} with $\mathrm{pr}_2\colon \mathrm{pr}_2 \to \mathrm{id}_X$, where $\mathrm{pr}_2\colon A\times X \to X$. Since $\mathcal{L}^X$ is closed under products and $\mathrm{pr}_2, \mathrm{id}_{qk} \in \mathcal{L}^X$, $\overline{\mathrm{pr}}_A \perp_X \alpha$, which means that the right square above is a pullback. $\square$

Remark 7.1.10. By [ABFJ17a, Lemma 3.3.3], for every $n \geq (-1)$, a map p in $\mathcal{E}$ is n-truncated if and only if $s_{n+1} \perp p$, where $s_{n+1}\colon S^{n+1} \to 1$, and $S^{n+1} := \Sigma^{n+1}(1 \coprod 1)$. (Here, $\Sigma^{n+1}X$ is the $(n+1)$-th *suspension* of $X \in \mathcal{E}$, which is defined recursively by $\Sigma X := 1 \coprod_X 1$ and $\Sigma^{n+1}X := \Sigma(\Sigma^n X)$, for $n \geq 1$.) It then follows that the modality

$L^n_\bullet$ defined in Example 4.2.7 can also be described as the S^{n+1}-nullification.

The results above still hold true if we replace the map $f\colon A \to B$ with an arbitrary *set* $S = \{f_i\colon A_i \to B_i\}_{i\in I}$ of maps in $\mathcal{E}$.

Definition 7.1.11. Let $S = \{f_i\colon A_i \to B_i\}_{i\in I}$ be a set of maps in $\mathcal{E}$. A map p is *S-local* if it is f_i-local for every $i \in I$. We denote by $\mathcal{D}^S_X$ the full subcategory of $\mathcal{E}_{/X}$ spanned by the S-local maps.

Proposition 7.1.12. *Let $S = \{f_i\colon A_i \to B_i\}_{i\in I}$ be a set of maps in $\mathcal{E}$. There exists a reflective subfibration $L^S_\bullet$ on $\mathcal{E}$ whose local maps are exactly the S-local map. In particular, a map is an L^S-equivalence precisely if it is an f_i-equivalence for every $i \in I$.*

Proof. [ABFJ17a, Prop. 3.3.8] applies to give a cartesian factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$ in which $\mathcal{R} = S^{\perp\!\!\!\perp}$. Furthermore, $X \in \mathcal{E}$ belongs to the associated reflective subcategory $\mathcal{D} = \mathcal{R}_{/1}$ if and only if X^{f_i} is an equivalence for every $i \in I$. For every $X \in \mathcal{E}$ we can then take the class of maps in $\mathcal{E}_{/X}$ given by

$$S \times X := \{f_i \times X\colon\ \mathrm{pr}_{A_i} \to \mathrm{pr}_{B_i}\}_{i\in I},$$

consider the associated cartesian factorization system on $\mathcal{E}_{/X}$, and obtain a reflective subcategory $\mathcal{D}_X$ of $\mathcal{E}_{/X}$. An argument essentially the same as the proof of Proposition 7.1.7 shows that this collection of reflective subcategories forms a reflective subfibration $L^S_\bullet$ on $\mathcal{E}$ with the required properties. $\square$

Definition 7.1.13. The reflective subfibration $L^S_\bullet$ on $\mathcal{E}$ whose local maps are the S-local maps is called the *S-local reflective subfibration* on $\mathcal{E}$, or simply the *S-localization* on $\mathcal{E}$. We call a map an *S-equivalence* if it is an L^S-equivalence.

Remark 7.1.14. By the very definition of L^S, properties of $L^S_\bullet$ involving S-local maps and S-equivalences can be recovered from properties of $L^f_\bullet$, for a single map $f\colon A \to B$.

In the following, we will then often state results for the reflective subfibration $L^S_\bullet$ and prove them for $L^f_\bullet$.

7.2 $(L^f_\bullet)'$ and suspensions

When $L_\bullet$ is an f-localization, $L'_\bullet$ is easy to describe and turns out to be another localization with respect to a map. We prove this below and then highlight some interactions between f-local and $\Sigma^n f$-local objects, for $n \geq 1$.

Proposition 7.2.1. *Let S be a set of maps in $\mathcal{E}$. Then $(L^S)' = L^{\Sigma S}$, where ΣS is the set of maps given by the suspensions of the maps in $\mathcal{E}$.*

Proof. We show that the L^S-separated maps are exactly the $L^{\Sigma S}$-local maps. We prove this for objects, the proof for maps being essentially the same, upon replacing $\mathcal{E}$ with a slice ∞-topos $\mathcal{E}_{/Z}$. By Remark 7.1.14, we can reduce to the case where S consists of a single map $f\colon A \to B$. Let $X \in \mathcal{E}$. We want to show that $X^{\Sigma f}\colon X^{\Sigma B} \to X^{\Sigma A}$ is an equivalence if and only if $(\Delta X)^{f\times X^2}\colon \Delta X^{\mathrm{pr}_B} \to \Delta X^{\mathrm{pr}_A}$ is an equivalence. Here, $f \times X^2$ denotes the map $f \times X^2\colon \mathrm{pr}_A \to \mathrm{pr}_B$ in $\mathcal{E}_{/X^2}$, where $\mathrm{pr}_A\colon A \times X^2 \to X^2$ is the projection map, and similarly for pr_B.

Note that, since ΣA comes with two canonical basepoints $S, N\colon 1 \to \Sigma A$, $X^{\Sigma A}$ is naturally an object over X^2. Similarly, $X^{\Sigma f}$ is naturally a map over X^2 and it is an equivalence as such if and only if it is an equivalence as a map in $\mathcal{E}$, since the forgetful functor $\mathcal{E}_{/X} \to \mathcal{E}$ is conservative. We now proceed to show that $X^{\Sigma f}$ is actually $(\Delta X)^{f\times X}$.

Let $[A \times X^2, X^2]$ be the domain of $(\Delta X)^{\mathrm{pr}_A}$, when seen as a map in $\mathcal{E}$. By [ABFJ17b, Lemma 2.5.5], there is a pullback square

$$\begin{tikzcd}
{[A\times X^2, X^2]} \arrow[r] \arrow[d, "(\Delta X)^{\mathrm{pr}_A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^A \arrow[d, "(\Delta X)^A"] \\
X^2 \arrow[r] & (X^2)^A
\end{tikzcd}$$

where the bottom map is induced by $A \to 1$. If we paste this square with the pullback square

$$\begin{tikzcd}
X^A \arrow[r, "\mathrm{id}"] \arrow[d, "(\Delta X)^A"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^A \arrow[d, "\Delta(X^A)"] \\
(X^2)^A \arrow[r, "\simeq"'] & (X^A)^2
\end{tikzcd}$$

we get that $(\Delta X)^{\mathrm{pr}_A}$ is also the pullback of $\Delta(X^A)$ along $c^2\colon X^2 \to (X^A)^2$, where $c\colon X \to X^A$ is induced by $A \to 1$. But

$$(c^2)^*(\Delta(X^A)) = (X \times_{X^A} X \to X^2)$$

and $X \times_{X^A} X \simeq X^{\Sigma A}$, because $\Sigma A = 1 \coprod_A 1$. Therefore, $(\Delta X)^{\mathrm{pr}_A} = (X^{\Sigma A} \to X^2)$. Similarly, $(\Delta X)^{\mathrm{pr}_B} = (X^{\Sigma B} \to X^2)$ and $(\Delta X)^{f \times X}$ is $X^{\Sigma f}$. □

Using Corollary 6.1.3 the above proposition has the following immediate consequence.

Corollary 7.2.2. *For a set S of maps in $\mathcal{E}$ and for $1 \xrightarrow{x} X$ a pointed object, $L^S(\Omega X) \simeq \Omega(L^{\Sigma S} X)$, where the loop space of $L^{\Sigma S} X$ is taken with respect to the global element $1 \xrightarrow{x} X \xrightarrow{\eta(X)} L^{\Sigma S} X$.*

We now study the effect of iterating the construction $(L^f_\bullet) \mapsto (L^f_\bullet)'$, where f is a map between pointed objects. Recall that, if $f\colon A \to B$ is a map in $\mathcal{E}$, the cofiber of f is the object C_f fitting in the pushout square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
1 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C_f
\end{tikzcd}$$

Recall also that an object $Z \in \mathcal{E}$ is A-null if it is $(A \to 1)$-local.

Theorem 7.2.3 ([CORS18, Thm. 3.6])**.** *Let $a\colon 1 \to A$ and $b\colon 1 \to B$ be pointed objects in $\mathcal{E}$ and let $n \geq 1$. Let $f\colon (A, a) \to (B, b)$ be a map of pointed objects. Consider the following three conditions, for an object $Z \in \mathcal{E}$.*

1. Z is f-local.

2. Z is $\Sigma^{n-1}(C_f)$-null.

3. Z is $\Sigma^n f$-local.

Then (1) $\implies$ (2) $\implies$ (3). Furthermore, if the maps $\mathrm{ev}_a\colon Z^A \to Z$ and $\mathrm{ev}_b\colon Z^B \to Z$ are $(n-1)$-connected, the three conditions above are equivalent.

Proof. We start with some general considerations. The maps ev_a and ev_b are defined as the maps induced by a and b respectively. Therefore the map $c_A\colon Z \to Z^A$ induced by $A \to 1$ is a section of ev_a, and, similarly, $c_B\colon Z \to Z^B$ is a section of ev_b. In particular, both ev_a and ev_b are (-1)-connected. On the other hand, since f is a pointed map, Z^f is naturally a map $\mathrm{ev}_b \to \mathrm{ev}_a$ in $\mathcal{E}_{/Z}$. Furthermore, $Z^f \circ c_B = c_A$.

Now, by definition of C_f, there is a pullback square

$$\begin{tikzcd}
Z^{C_f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "c_A"] \\
Z^B \arrow[r, "Z^f"'] & Z^A
\end{tikzcd}$$

We can then expand the above to a long fiber sequence as in the following diagram, where all squares are pullbacks.

$$\begin{tikzcd}
\cdots \arrow[r] & Z^{\Sigma^2 A} \arrow[r] \arrow[d] & Z \arrow[d, "c_{\Sigma B}"] \arrow[dr, "\mathrm{id}_Z"] & & \\
\cdots \arrow[r] & Z^{\Sigma C_f} \arrow[r] \arrow[d] & Z^{\Sigma B} \arrow[r] \arrow[d, "Z^{\Sigma f}"] & Z \arrow[d, "c_{C_f}"] \arrow[dr, "\mathrm{id}_Z"] & \\
 & Z \arrow[r, "c_{\Sigma A}"] \arrow[dr, "\mathrm{id}_Z"'] & Z^{\Sigma A} \arrow[r] \arrow[d] & Z^{C_f} \arrow[r] \arrow[d] & Z \arrow[d, "c_A"] \\
 & & Z \arrow[r, "c_B"'] & Z^B \arrow[r, "Z^f"'] & Z^A
\end{tikzcd} \tag{†}$$

(Note that we used $Z^f \circ c_B = c_A$.)

We first prove that (1) $\implies$ (2). If Z is f-local, then Z^f is an equivalence and then so is $Z^{C_f} \to Z$. It follows that $c_{C_f} \colon Z \to Z^{C_f}$ is also an equivalence, i.e., Z is C_f-null. By Remark 5.1.3 (1) and Proposition 7.2.1, we then get that Z is also $\Sigma^{n-1}C_f$-null.

Assume now that (2) holds. Then, in the portion of (†) given by

$$\begin{tikzcd}
Z^{\Sigma^n B} \arrow[r, "Z^{\Sigma^n f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z^{\Sigma^n A} \arrow[d] \\
Z \arrow[r] & Z^{\Sigma^{n-1} C_f}
\end{tikzcd}$$

the bottom map is an equivalence by hypothesis, and then so is $Z^{\Sigma^n f}$. Therefore, (2) $\implies$ (3).

Suppose now that ev_a and ev_b are $(n-1)$-connected. Then, in (†) the maps c_A and c_B are $(n-2)$-connected, because they are sections of ev_a and ev_b respectively (see [Lur09, Prop. 6.5.1.20]). Since, for any $m \geq (-2)$, m-connected maps are stable under pullbacks, it follows that both of the maps $Z^{\Sigma A} \to Z$ and $Z^{\Sigma A} \to Z^{\Sigma C_f}$ are $(n-2)$-connected. Note that, if $n > 1$, $c_{\Sigma A}$, as a section of $Z^{\Sigma A} \to Z$, is $(n-3)$-connected and then $c_{\Sigma C_f}$, as the composite $Z \xrightarrow{c_{\Sigma A}} Z^{\Sigma A} \to Z^{\Sigma C_f}$, is also $(n-3)$-connected. Proceeding in this way, one obtains that, in the portion of (†) given by

$$\begin{tikzcd}
Z^{\Sigma^n B} \arrow[r, "Z^{\Sigma^n f}"] \arrow[d] & Z^{\Sigma^n A} \arrow[r] \arrow[d] & Z \arrow[d] \\
Z \arrow[r] \arrow[dr, "\mathrm{id}_Z"'] & Z^{\Sigma^{n-1} C_f} \arrow[r] \arrow[d] & Z^{\Sigma^{n-1} B} \arrow[d, "Z^{\Sigma^{n-1} f}"] \\
 & Z \arrow[r, "c_{\Sigma^{n-1} A}"'] & Z^{\Sigma^{n-1} A}
\end{tikzcd}$$

the maps $c_{\Sigma^{n-1}A}$ and $Z^{\Sigma^n A} \to Z^{\Sigma^{n-1} C_f}$ are (-1)-connected (effective epimorphisms). Therefore, if Z is $\Sigma^n f$-local, i.e., $Z^{\Sigma^n f}$ is an equivalence, $Z \to Z^{\Sigma^{n-1} C_f}$ and $Z^{\Sigma^{n-1} f}$ are also equivalences. If $n = 1$, this shows that (3) $\implies$ (1). Otherwise, by induction, Z^f is an equivalence and, then, (3) $\implies$ (1) again. $\square$

7.3 Interactions with factorization systems

We investigate here some properties of $L^S_\bullet$ when the maps in S can all be taken to belong to the left class $\mathcal{L}$ of a *cartesian* factorization system $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$ (Definition 7.1.2). By [ABFJ17a, Lemma 3.5.5], stable factorization systems are in particular cartesian.

Recall that, given any factorization system $\mathcal{F} = (\mathcal{L}, \mathcal{R})$, there is a factorization system $\mathcal{F}_X = (\mathcal{L}_X, \mathcal{R}_X)$ in $\mathcal{E}_{/X}$, for $X \in \mathcal{E}$. Here, $\mathcal{L}_X$ is formed by those maps β in $\mathcal{E}_{/X}$ such that $\Sigma_X(\beta)$ is in $\mathcal{L}$, and analogously for $\mathcal{R}$.

Proposition 7.3.1 ([CORS18, Lemma 3.11])**.** *Let $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ be a cartesian factorization system, and let S be a set of maps in $\mathcal{L}$. If $\alpha\colon p \to q$ is a map in $\mathcal{R}_X$ and q is S-local, then p is S-local.*

Proof. We show this for $S = \{f\colon A \to B\}$. Consider the map $f \times X\colon \mathrm{pr}_A \to \mathrm{pr}_B$ in $\mathcal{E}_{/X}$, where $\mathrm{pr}_A\colon A \times X \to X$ is the projection map, and similarly for pr_B. Assume q is f-local. Thanks to Remark 7.1.4, to show that p is f-local, we need to show that the map of ∞-groupoids

$$\mathcal{E}_{/X}(f \times X, p)\colon \mathcal{E}_{/X}(\mathrm{pr}_B, p) \to \mathcal{E}_{/X}(\mathrm{pr}_A, p)$$

is an equivalence. We have a commutative square in $\infty\mathsf{Gpd}$

$$\begin{tikzcd}
\mathcal{E}_{/X}(\mathrm{pr}_B, p) \arrow[r, "\mathcal{E}_{/X}(f\times X, p)"] \arrow[d, "\mathcal{E}_{/X}(\mathrm{pr}_B, \alpha)"'] & \mathcal{E}_{/X}(\mathrm{pr}_A, p) \arrow[d, "\mathcal{E}_{/X}(\mathrm{pr}_A, \alpha)"] \\
\mathcal{E}_{/X}(\mathrm{pr}_B, q) \arrow[r, "\mathcal{E}_{/X}(f\times X, q)"'] & \mathcal{E}_{/X}(\mathrm{pr}_A, q)
\end{tikzcd}$$

Now, because the class $\mathcal{L}$ is closed under product (see [ABFJ17a, Lemma 3.3.7]), the map in $\mathcal{E}$ given by $f \times X = f \times \mathrm{id}_X$ is in $\mathcal{L}$ and, then, the map $f \times X\colon \mathrm{pr}_A \to \mathrm{pr}_B$ in $\mathcal{E}_{/X}$ is in $\mathcal{L}_X$. Since α is in $\mathcal{R}_X$ by hypothesis and $\mathcal{F}_X = (\mathcal{L}_X, \mathcal{R}_X)$ is a factorization system, we then get that the above square is a pullback. But, since q is f-local by hypothesis, $\mathcal{E}_{/X}(f \times X, q)$ is an equivalence, and then so is $\mathcal{E}_{/X}(f \times X, p)$. This shows that p is f-local, as needed. $\square$

Corollary 7.3.2. *Let S be a set of effective epimorphisms. If $p \in \mathcal{E}_{/X}$ is an S-local map and $t \in \mathcal{E}_{/X}$ is a subobject of p, then t is an S-local map as well.*

The above proposition can be used to show that S-local maps satisfy a weaker version of the composition property required for modalities.

Corollary 7.3.3. *Let S be a class of maps in $\mathcal{E}$ and let $a\colon E \to M$ be a map such that $f \perp a$ for every $f \in S$. Then a is an S-local map and, given any commutative triangle*

$$\begin{tikzcd} E \arrow[rr, "a"] \arrow[dr, "p"'] & & M \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

if q is an S-local map, then so is p.

Proof. We use results and notation discussed in Section 7.1. For a fixed map f in $\mathcal{E}$, let $\mathcal{R} = \{f\}^{\perp}$ and $\mathcal{L} = {}^{\perp}\mathcal{R} = {}^{\perp}\mathcal{R}$. The hypothesis on a (for $S = \{f\}$) says that $a \in \mathcal{R}$ and, since $\mathcal{R}_X \subseteq \mathcal{R}^X$ (see Remark 7.1.6), the map $a \to \mathrm{id}_M$ is in $\mathcal{R}^X$, that is, a is an f-local map. Since $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ is a cartesian factorization system and $f \in \mathcal{L}$, the second claim now follows from Proposition 7.3.1 applied to the given commuting triangle. $\square$

Theorem 7.3.4 ([CORS18, Thm. 3.12])**.** *Let $\mathcal{F} = (\mathcal{L}, \mathcal{R})$ be a cartesian factorization system, and let S be a set of maps in $\mathcal{L}$. For any $X \in \mathcal{E}$ and $p \in \mathcal{E}_{/X}$, the reflection map $\eta_X(p)\colon p \to L^S_X(p)$ of p into $\mathcal{D}^S_X$ is in $\mathcal{L}_X$.*

Proof. Factor $\eta_X(p)$ according to $(\mathcal{L}_X, \mathcal{R}_X)$:

$$\begin{tikzcd} p \arrow[rr, "\eta_X(p)"] \arrow[dr, "l"'] & & L^S_X(p) \\ & i \arrow[ur, "r"'] & \end{tikzcd}$$

By Proposition 7.3.1, i is S-local. Thus, there exists a unique $\bar{l}\colon L^S_X(p) \to i$ with $\bar{l}\eta_X(p) = l$. We then have $r\bar{l}\eta_X(p) = rl = \eta_X(p)$ and so $r\bar{l} = \mathrm{id}_{L^S_X(p)}$, by the universal property of $\eta_X(p)$. On the other hand, the map $\bar{l}r\colon i \to i$ satisfies $\bar{l}rl = \bar{l}\eta_X(p) = l$, i.e., $(\bar{l}r)l = l$ with $l \in \mathcal{L}_X$. By Lemma 4.2.3, $\bar{l}r \in \mathcal{L}_X$. Since $r \in \mathcal{R}_X$ and $r\bar{l}r = r$, we get that $\bar{l}r$ is an equivalence, by uniqueness of factorizations. Thus, r is an equivalence and so $\eta_X(p) = rl$ is in $\mathcal{L}_X$. □

We can use the above theorem to give a sufficient condition for $L^S_\bullet$ to preserve connectedness of maps in $\mathcal{E}$.

Corollary 7.3.5. *For $n \geq -1$, let S be a set of $(n-1)$-connected maps. If $X \in \mathcal{E}$ and $p \in \mathcal{E}_{/X}$ is n-connected, then so is $L^S_X(p)$.*

Proof. By Theorem 7.3.4 with $\mathcal{F}_{n-1}$ =(($n-1$)-connected, ($n-1$)-truncated), the reflection map $\eta_X(p)\colon p \to L^S_X(p)$ is $(n-1)$-connected. Recall now that, if $L^{n-1}_\bullet$ is the modality associated to $\mathcal{F}_{n-1}$, $(L^{n-1}_\bullet)' = L^n_\bullet$, the modality associated to the factorization system of n-connected and n-truncated maps. Since p is n-connected by hypothesis, Proposition 6.1.9 then gives that $L^S_X(p)$ is n-connected as well. □

7.4 Interaction with coproducts

By Lemma 3.2.1, if $L_\bullet$ is a reflective subfibration on $\mathcal{E}$, the class of L-local maps is closed under coproducts, *in the arrow category* $\mathcal{E}^{\bullet\to\bullet}$. In particular, for a set $\{X_j\}_{j\in J}$ of L-local objects in $\mathcal{E}$, this tells us that the induced map $\coprod_{j\in J} X_j \to \coprod_{j\in J} 1$ is L-local, but it does not allow us to conclude that the object $\coprod_{j\in J} X_j$ is L-local. However, if $L = L^S_\bullet$, we can find sufficient conditions on S for this to happen, by studying the relationship between S-local and 0-truncated maps in $\mathcal{E}$.

Proposition 7.4.1 ([CORS18, Lemma 3.7])**.** *Let S be a set of maps between 0-connected objects. If $p\colon X \to J$ is an S-local map and J is 0-truncated, then X is S-local. In particular, 0-truncated objects are S-local.*

Proof. We show the result for $S = \{f\colon A \to B\}$, where A and B are 0-connected. Note that, since J is 0-truncated, we have an equivalence $J^{\tau_{\leq 0}A} \xrightarrow{\simeq} J^A$, where $\tau_{\leq 0}A$ is the 0-truncation of A. On the other hand, A is 0-connected, so there is an equivalence $\tau_{\leq 0}A \xrightarrow{\simeq} 1$. Altogether, we get that the map $A \to 1$ induces an equivalence $c_A\colon J \xrightarrow{\simeq} J^A$. For the same reasons, we also have an equivalence $c_B\colon J \xrightarrow{\simeq} J^B$. Consider now the map in $\mathcal{E}_{/J}$ given by $f \times J\colon \mathrm{pr}_A \to \mathrm{pr}_B$, where $\mathrm{pr}_A\colon A \times J \to J$ is the projection map, and similarly for pr_B. Write $[A \times J, X]$ (resp., $[B \times J, X]$) for the domain of the internal hom $p^{\mathrm{pr}_A} \in \mathcal{E}_{/J}$ (resp., $p^{\mathrm{pr}_B} \in \mathcal{E}_{/J}$). By [ABFJ17b, Lemma 2.5.5], there are pullback squares in $\mathcal{E}$

$$
\begin{tikzcd}
{[B \times J, X]} \arrow[r] \arrow[d, "p^{\mathrm{pr}_B}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^B \arrow[d, "p^B"] \\
J \arrow[r, "c_B"'] & J^B
\end{tikzcd}
\qquad
\begin{tikzcd}
{[A \times J, X]} \arrow[r] \arrow[d, "p^{\mathrm{pr}_A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^A \arrow[d, "p^A"] \\
J \arrow[r, "c_A"'] & J^A
\end{tikzcd}
$$

Hence, since both c_A and c_B are equivalences, the maps $[B \times J, X] \to X^B$ and $[A \times J, X] \to X^A$ are also equivalences. But now, in the commutative diagram

$$
\begin{tikzcd}
{[B \times J, X]} \arrow[rr, "\simeq"] \arrow[dr, dashed] \arrow[dd, "p^{\mathrm{pr}_B}"'] & & X^B \arrow[dr, "X^f"] \arrow[dd, "p^B"] & \\
& {[A \times J, X]} \arrow[rr, "\simeq"] \arrow[dd, "p^{\mathrm{pr}_A}"] & & X^A \arrow[dd, "p^A"] \\
J \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{id}_J"'] & & J^B \arrow[dr, "J^f"] & \\
& J \arrow[rr, "\simeq"'] & & J^A
\end{tikzcd}
$$

the dotted map is $p^{f \times J}$, which is an equivalence by the hypothesis that p is f-local. By the two-out-of-three property for equivalences, it follows that $X^f\colon X^B \to X^A$ is also an equivalence, that is, X is f-local. The second claim follows by considering $\mathrm{id}_J\colon J \to J$. $\square$

Corollary 7.4.2. *Let S be a set of maps between 0-connected objects. If $\{X_j\}_{j \in J'}$ is* *a set of S-local objects, the coproduct $\coprod_{j\in J'} X_j$ is S-local.*

Proof. Set $J := \coprod_{j\in J'} 1 \in \mathcal{E}$. Then J is 0-truncated since $1 \in \mathcal{E}$ is such and the 0-truncation functor $\tau_{\leq 0}\colon \mathcal{E} \to \mathcal{E}$ preserves coproducts. By Lemma 3.2.1, the map

$$p := \coprod_j (X_j \to 1)\colon \coprod_j X_j \to J$$

is S-local. Therefore, $\coprod_j X_j$ is S-local, by Proposition 7.4.1. □

Corollary 7.4.3 ([CORS18, Cor. 3.8])**.** *Let S be a set of maps between 0-connected objects and let $p\colon X \to J$ be any map into a 0-truncated object J. Let $\psi\colon L^S X \to J$ be the unique map with $\psi\eta(X) = p$. Then $\eta(X)\colon p \to \psi$ is the reflection of p into $\mathcal{D}^S_J$. In particular, the S-localization functor $L^S\colon \mathcal{E} \to \mathcal{E}$ preserves coproducts.*

Proof. Note that the map ψ exists because J is S-local. Let

$$\eta_J(p)\colon p \to L^S_J(p)$$

be the reflection map of p into $\mathcal{D}^S_J$. Then, $\Sigma_J(\eta_J(p))$ is an L^S_1-equivalence, by Lemma 3.1.4, and $\Sigma_J(L^S_J(p))$ is S-local, by Proposition 7.4.1. Hence, $\Sigma_J(\eta_J(p))$ can be taken to be $\eta(X)$ and the first claim follows. For the second claim, suppose $\{X_j\}_{j\in J'}$ is a set of objects in $\mathcal{E}$ and set

$$X := \coprod_{j\in J'} X_j, \quad J := \coprod_{j\in J'} 1 \quad \text{and} \quad p := \coprod_{j\in J'} (X_j \to 1)\colon X \to J.$$

For each $j \in J'$, let $(\eta(X))_j\colon X_j \to (L^S X)_j$ be the map obtained by pulling back $\eta(X)\colon p \to \psi$ along $j\colon 1 \to J$. Since $\eta(X)$ is the reflection map of p into $\mathcal{D}^S_J$, $(\eta(X))_j$ is the reflection map of X_j into $\mathcal{D}^S$, by Definition 3.1.1 (2). By universality of colimits in an ∞-topos, we get that $\coprod_{j\in J'} \eta(X_j)$ is the reflection map of X and then $L^S\left(\coprod_{j\in J'} X_j\right) \simeq \coprod_{j\in J'} L^S(X_j)$, as needed. □

Remark 7.4.4. Proposition 7.4.1 also holds in slice categories $\mathcal{E}_{/Z}$, giving that, if we have a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "a"'] & & J \arrow[dl, "b"] \\ & Z & \end{tikzcd}$$

with p S-local (i.e., $p \in \mathcal{D}_J^S$) and b 0-truncated, then a is also S-local (see also Remark 3.1.2). The proof parallels the one for $\mathcal{E}$, upon noticing that, if A is 0-connected, the projection map $A \times Z \to Z$, seen as an object in $\mathcal{E}_{/Z}$, is 0-connected, because it is the pullback of A along $Z \to 1$. We get similar local versions of the two corollaries above.

Chapter 8

Summary and conclusion

In this work, we have provided a new approach to the study of localization theory in an ∞-topos that bridges the classical homotopy-theoretic study of the subject with recent developments in homotopy type theory ([RSS17] and [CORS18]).

Our approach is based on the notion of a *reflective subfibration* $L_\bullet$ on an ∞-topos $\mathcal{E}$. This is a pullback-compatible assignment of reflective subcategories $\mathcal{D}_X \subseteq \mathcal{E}_{/X}$ of *L-local maps*, for every $X \in \mathcal{E}$. All of the classically studied examples of localizations fit into this framework: localizations (of spaces) at a set of maps (Proposition 7.1.12), stable factorization systems (Theorem 4.2.5), and left exact reflections (Proposition 4.2.8).

We have investigated the properties of many classes of maps associated to a reflective subfibration $L_\bullet$. In particular, we have proved that L-local maps form a local class of maps in $\mathcal{E}$, thus admitting a classifying map (Theorem 3.2.6). We have also shown that *L-connected maps* allow one to completely describe reflective subfibrations with the property that the composite of two L-local maps is local as being associated to stable factorization systems (Theorem 4.2.5).

We have extensively studied the class of *L-separated maps*, that is, those maps in $\mathcal{E}$ whose diagonal is L-local. Our main result is the existence of a reflective subfibration $L'_\bullet$ on $\mathcal{E}$ with the property that the L'-local maps are the L-separated maps

(Corollary 5.3.4). With this existence result at hand, we have managed to prove many results about the interactions between $L_\bullet$ and $L'_\bullet$, including a complete description of those reflective subfibrations $L_\bullet$ for which $L_\bullet = L'_\bullet$ (Theorem 6.2.8).

In adopting homotopy type theoretic ideas to the homotopy theoretic context of an ∞-topos, we have built a hands-on dictionary between homotopy type theory and homotopy theory, highlighting similarities and differences between the two viewpoints, and showcasing both the insights and the difficulties that this translation process determines. We hope that this work will help in merging the two communities together, and in evaluating the advantages and drawbacks of both approaches.

Appendix A

On locally cartesian closed ∞-categories

In this appendix, we prove some miscellaneous facts about locally cartesian closed (lcc) ∞-categories that are needed in our work but do not naturally fit elsewhere. Some of these results are well-known, but others do not seem to appear or be proven in the literature.

In Section A.1, we discuss some results about cartesian-closedness of pullback functors, as well as some interactions between their adjoints. In Section A.2, we formulate a "term-free" version of the type-theoretic axiom known as function extensionality, and we prove that it holds in any lcc ∞-category. Finally, in Section A.3, we prove a criterion for extending a map along another one with the same domain. We formulate this criterion in terms of a certain object of "fiberwise extensions" being contractible.

We fix throughout an lcc ∞-category $\mathcal{C}$.

A.1 Pullback functor and its adjoints

The first set of results we need explore the behaviours of the pullback functors and of their adjoints in $\mathcal{C}$.

Lemma A.1.1. *Let $\mathcal{C}$ be a locally cartesian closed ∞-category. For any morphism $g\colon Y \to X$ in $\mathcal{C}$ the pullback functor*

$$g^*\colon \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$$

is cartesian closed, i.e., for every $p, q \in \mathcal{C}_{/X}$, $g^*(p^q)$ is the exponential object $g^*(p)^{g^*(q)}$ in $\mathcal{C}_{/Y}$.

A proof of the above result for 1-categories can be found in [Joh02, Lemma A.1.5.2] and the same proof carries over to ∞-categories.

Lemma A.1.2. *Let $\epsilon\colon gg^* \to \mathrm{id}_{\mathcal{C}_{/X}}$ be the counit of the adjunction $g \circ (-) \dashv g^*$. Given $X \in \mathcal{C}$, take $p, q \in \mathcal{C}_{/X}$. Suppose given a diagram in $\mathcal{C}$*

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[dr, bend right, "f"'] & W \arrow[r, "\sigma := \epsilon_{p^q}"] \arrow[d, dashed, "(g^*p)^{(g^*q)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "p^q"] \\
& Y \arrow[r, "g"'] & X
\end{tikzcd}$$

Let $\rho^\sharp\colon f \times^Y g^*q \to g^*p$ be the adjunct to ρ in $\mathcal{C}_{/Y}$ and consider the map $\sigma\rho\colon gf \to p^q$ in $\mathcal{C}_{/X}$. Then, $g(f \times^Y g^*q) = gf \times^X q$ and the adjunct of $\sigma\rho$ is given by the composite map

$$g(f \times^Y g^*q) \xrightarrow{\rho^\sharp} gg^*p \xrightarrow{\epsilon_p} p$$

Proof. The fact that $g(f \times^Y g^*q) = gf \times^X q$ is given by the pasting-lemma for pullbacks. By definition, the adjunct of $\sigma\rho$ is the composite

$$gf \times^X q \xrightarrow{\sigma\rho \times^X q} p^q \times^X q \xrightarrow{\mathrm{ev}_{p,q}} p$$

and the adjunct $\rho^\sharp$ is the composite

$$f \times^Y g^*q \xrightarrow{\rho\times^Y g^*q} (g^*p)^{(g^*q)} \times^Y g^*q \xrightarrow{\mathrm{ev}_{g^*p,g^*q}} g^*p.$$

Using that $(g^*p)^{(g^*q)} \times^Y g^*q = g^*(p^q \times^X q)$, the map ev_{g^*p,g^*q} is the map $g^*(\mathrm{ev}_{p,q})$. One then needs to show that the maps $\mathrm{ev}_{p,q}(\sigma\rho \times^X q)$ and $\epsilon_p g^*(\mathrm{ev}_{p,q})(\rho \times^Y g^*q)$ are equal. Consider the diagram below, where all squares are pullbacks

$$\begin{tikzcd}
& g^*(T \times_X Q) \arrow[rr, "\sigma'"] \arrow[dd] \arrow[dr] & & T \times_X Q \arrow[dd] \arrow[dr] & \\
(gf)^*Q \arrow[ur, "m"] \arrow[rr, crossing over] \arrow[dd, "(gf)^*q"'] & & g^*Q \arrow[rr, crossing over] & & Q \arrow[dd, "q"] \\
& W \arrow[rr, "\sigma" near end] \arrow[dr] & & T \arrow[dr, "p^q"] & \\
A \arrow[ur, "\rho"] \arrow[rr, "f"'] & & Y \arrow[rr, "g"'] \arrow[from=uu, crossing over, "g^*q" near end] & & X
\end{tikzcd}$$

Then m (as a map over Y) is $\rho \times^Y g^*q$ and $\sigma' m$ (as a map over X) is $\sigma\rho \times^X q$. The claim now follows by considering the following commutative diagram, where the back, front and bottom faces of the cube (and, hence, also the top face) are pullbacks

$$\begin{tikzcd}
& g^*(T \times_X Q) \arrow[rr, "\sigma'"] \arrow[dd] \arrow[dr, dashed, "g^*(\mathrm{ev}_{p,q})"] & & T \times_X Q \arrow[dd] \arrow[dr, "\mathrm{ev}_{p,q}"] & \\
(gf)^*Q \arrow[ur, "\rho\times^Y g^*q"] \arrow[dd, "(gf)^*q"'] & & g^*P \arrow[rr, crossing over, "\epsilon_p"] & & P \arrow[dd, "p"] \\
& W \arrow[rr, "\sigma" near end] \arrow[dr] & & T \arrow[dr, "p^q"] & \\
A \arrow[ur, "\rho"] \arrow[rr, "f"'] & & Y \arrow[rr, "g"'] \arrow[from=uu, crossing over, "g^*p" near end] & & X
\end{tikzcd}$$

$\square$

Lemma A.1.3 (Beck-Chevalley condition)**.** *Let $\mathcal{C}$ be a locally cartesian closed ∞-*

category and let

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

be a pullback square in $\mathcal{C}$. *Then the canonical natural transformations*

$$\sum_k h^* \longrightarrow g^* \sum_f \quad \text{and} \quad f^* \prod_g \longrightarrow \prod_h k^*$$

are equivalences.

Proof. The first map being an equivalence at every $p \in \mathcal{C}_{/C}$ is a restatement of the pasting lemma for pullbacks. The result for dependent products follows from the one for dependent sums by taking right adjoints, since adjoints compose ($g^*\Sigma_f$ is left adjoint to $f^* \prod_g$ and similarly for $\Sigma_k h^*$). □

A.2 Function extensionality

In homotopy type theory, given two functions $f, g\colon X \to A$ between types X and A, there is a map

$$(f =_{A^X} g) \longrightarrow \prod_{x:X} (f(x) =_A g(x))$$

evaluating a path between $f, g\colon X \to A$ at each $x : X$. The statement that this map is an equivalence (for all types A, X and all functions $f, g\colon A \to X$) is what is commonly known as *function extensionality*.

In our setting, abstracting away from its term-based description, function extensionality can be stated as the following result.

Proposition A.2.1 (Function Extensionality). *Let* $\mathcal{C}$ *be a locally cartesian closed* ∞*-category. Given* $A, X \in \mathcal{C}$, *let* $\mathrm{ev}\colon A^X \times X \to A$ *be the counit of the adjunction*

$(-) \times X \dashv (-)^X$ *and form the pullback*

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\Delta A"] \\
A^X \times A^X \times X \arrow[r, "(\mathrm{ev}_1, \mathrm{ev}_2)"'] & A \times A
\end{tikzcd}$$

Here ev_1 *(resp.* ev_2*) is the composite of the projection*

$$A^X \times A^X \times X \to A^X \times X$$

onto the first (resp. second) and third components with the evaluation map. Let $\mathrm{pr}\colon A^X \times A^X \times X \to A^X \times A^X$ *be the projection map. Then there is a canonical equivalence*

$$\Delta(A^X) \to \prod_{\mathrm{pr}} q$$

in $\mathcal{C}_{/A^X \times A^X}$.

Proof. Let $k\colon E \to A^X \times A^X$ be an object in $\mathcal{C}_{/A^X \times A^X}$. By adjointness, there is a natural equivalence

$$\mathcal{C}_{/A^X \times A^X}\left(k, \prod_{\mathrm{pr}} q\right) \simeq \mathcal{C}_{/A^X \times A^X \times X}(k \times X, q)$$

By the description of hom-spaces in ∞-slice categories (see [Lur09, Lemma 5.5.5.12]) and since Q is a pullback, we get a homotopy pullback square of ∞-groupoids

$$\begin{tikzcd}
\mathcal{C}_{/A^X \times A^X \times X}(k \times X, q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(E \times X, A) \arrow[d, "\mathcal{C}(E \times X, \Delta A)"] \\
* \arrow[r, "(\mathrm{ev}_1, \mathrm{ev}_2) \circ (k \times X)"'] & \mathcal{C}(E \times X, A \times A)
\end{tikzcd}$$

But

$$\mathcal{C}(E, \Delta(A^X)) \simeq \mathcal{C}(E \times X, \Delta A) \simeq \Delta_{\mathcal{C}(E \times X, A)}$$

which means

$$\mathcal{C}_{/A^X \times A^X \times X}(k \times X, q) \simeq \mathrm{hofib}_k(\mathcal{C}(E, \Delta(A^X))) \simeq \mathcal{C}_{/A^X \times A^X}(k, \Delta(A^X)),$$

where the last equivalence is again [Lur09, Lemma 5.5.5.12]. We then get a composite equivalence

$$\mathcal{C}_{/A^X \times A^X}\left(k, \prod_{\mathrm{pr}} q\right) \simeq \mathcal{C}_{/A^X \times A^X}(k, \Delta(A^X))$$

natural in $k \in \mathcal{C}_{/A^X \times A^X}$, as required. $\square$

Proposition A.2.1 can be promoted to a result about diagonals of dependent products, which corresponds to the type-theoretic function extensionality for dependent functions. We now set up what we need to state this generalization of Proposition A.2.1.

Let $p\colon E \to X$ be a map in $\mathcal{C}$ and let

$$\begin{tikzcd} (\prod_X p) \times X \arrow[rr, "\epsilon"] \arrow[dr, "\pi"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

be the component of the counit of the adjunction $(-) \times X \dashv \prod_X$ at $p \in \mathcal{C}_{/X}$. Here π is the projection map onto X. The projection map

$$\left(\prod_X p\right) \times \left(\prod_X p\right) \times X \to X$$

is the product object $\pi \times^X \pi$ in $\mathcal{C}_{/X}$. We can therefore describe the product map $\epsilon \times^X \epsilon\colon \pi \times^X \pi \to p \times^X p$ in $\mathcal{C}_{/X}$ as the map over X given by

$$(\epsilon_1, \epsilon_2)\colon \left(\prod_X p\right) \times \left(\prod_X p\right) \times X \to E \times_X E,$$

where ϵ_1 (resp. ϵ_2) is the composite of the projection

$$\left(\prod_X p\right) \times \left(\prod_X p\right) \times X \to \left(\prod_X p\right) \times X$$

onto the first (resp. the second) and third components with the counit map. The pullback of Δp along $\epsilon \times^X \epsilon$ in $\mathcal{C}_{/X}$ can be described as the pullback square

$$\begin{tikzcd} Q' \arrow[r] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\Delta p"] \\ (\prod_X p) \times (\prod_X p) \times X \arrow[r, "(\epsilon_1, \epsilon_2)"'] & E \times_X E \end{tikzcd} \tag{A.1}$$

in $\mathcal{C}$ and Q' can be naturally regarded as an object over X.

Proposition A.2.2 (Dependent Function Extensionality)**.** *Let $\mathcal{C}$ be a locally cartesian closed ∞-category and let $p\colon E \to X$ be a map in $\mathcal{C}$. Construct q' as in (A.1) and let*

$$\mathrm{pr}\colon \left(\prod_X p\right) \times \left(\prod_X p\right) \times X \to \left(\prod_X p\right) \times \left(\prod_X p\right)$$

be the projection map. Then there is a canonical equivalence

$$\Delta\left(\prod_X p\right) \xrightarrow{\simeq} \prod_{\mathrm{pr}} q'$$

in $\mathcal{C}_{/(\prod_X p) \times (\prod_X p)}$.

The proof of this result is, *mutatis mutandis*, the same as Proposition A.2.1, so we will omit it.

Remark A.2.3. If $\mathcal{C}$ is a locally cartesian closed ∞-category, then so is $\mathcal{C}_{/X}$ for any $X \in \mathcal{C}$. Thus, Proposition A.2.1 and Proposition A.2.2 hold true also in $\mathcal{C}_{/X}$ and give, for maps $p\colon E \to X$, $f\colon Y \to X$ and $q\colon M \to Y$ in $\mathcal{C}$, an alternative description of the diagonal of $p^f \in \mathcal{C}_{/X}$ and of $\Delta\left(\prod_f q\right)$ as a map in $\mathcal{C}_{/Y}$.

A.3 Contractibility

We provide here a criterion for the existence and the uniqueness of extensions of one map along another one with the same domain. This result is linked to the notion of contractibility in $\mathcal{C}$.

Recall that an object $A \in \mathcal{C}$ is *contractible* (or (-2)-*truncated*) if the unique map $A \to 1$ is an equivalence (e.g., see [ABFJ17a, Def. 3.4.1]). When we apply this definition to an object $p \in \mathcal{C}_{/X}$, this means that p is contractible in $\mathcal{C}_{/X}$ exactly when, seen as a map in $\mathcal{C}$, it is an equivalence. Since equivalences in an ∞-topos form a local class of maps, we immediately get the following result, which we record for reference.

Lemma A.3.1. *Let $\mathcal{E}$ be an ∞-topos and let $f\colon Y \to X$ be an effective epimorphism in $\mathcal{E}$. For any $p \in \mathcal{E}_{/X}$, $f^*(p) \in \mathcal{E}_{/Y}$ is contractible if and only if p is.*

Before proving the technical extension result we need, we give a few preliminary lemmas.

The following lemma is a standard exercise in 2-category theory since the notions of slice ∞-categories and of adjunctions between ∞-categories can be completely characterized in the 2-category of ∞-categories — see [RV18, §3 and 4].

Lemma A.3.2. *Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ ($F \dashv G$) be an adjunction and let $D \in \mathcal{D}$. Then there is an induced adjunction on slice categories*

$$\mathcal{C}_{/GD} \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} \mathcal{D}_{/D}$$

where, for $p \in \mathcal{C}_{/GD}$ and $q \in \mathcal{D}_{/D}$, $\bar{F}(p) = \epsilon_D Fp$ and $\bar{G}(q) = Gq$. □

Lemma A.3.3. *Let $p\colon D \to B \times C$ be a map in a locally cartesian closed ∞-category*

$\mathcal{C}$. *Consider the map* $q\colon E \to B \times C^B$ *given by the pullback square*

$$\begin{tikzcd} E \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "p"] \\ B \times C^B \arrow[r, "(\mathrm{pr}_1, \mathrm{ev})"'] & B \times C \end{tikzcd}$$

Then there is an equivalence

$$\left(\prod_B \sum_{B\times C\to B} p\right) \simeq \left(\sum_{C^B} \prod_{B\times C^B \to C^B} q\right)$$

Proof. Let $\mathrm{pr}_B\colon B \times C \to B$ and $\mathrm{pr}_{C^B}\colon B \times C^B \to C^B$ be the projection maps. Note that $\prod_{\mathrm{pr}_{C^B}} q$ is, by definition, a map

$$\prod_{\mathrm{pr}_{C^B}} q\colon \sum_{C^B} \prod_{\mathrm{pr}_{C^B}} q \longrightarrow C^B.$$

On the other hand, we can see p as a map $p\colon \sum_{\mathrm{pr}_B} p \to \mathrm{pr}_B$ in $\mathcal{C}_{/B}$. Setting $\alpha := \prod_B p$, we then get a map

$$\alpha\colon \prod_B \sum_{\mathrm{pr}_B} p \longrightarrow \prod_B \mathrm{pr}_B = C^B.$$

It is therefore sufficient to show that $\alpha \simeq \prod_{\mathrm{pr}_{C^B}} q$ in $\mathcal{C}_{/C^B}$. Let $k\colon Z \to C^B$ be an object in $\mathcal{C}_{/C^B}$. Using Lemma A.3.2 applied to the adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "B\times(-)"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{/B} \arrow[l, shift left, "\prod_B"] \end{tikzcd}$$

we get

$$\mathcal{C}_{/C^B}(k, \alpha) \simeq \left(\mathcal{C}_{/B}\right)_{/\,\mathrm{pr}_B}\left(\kappa^\sharp, p\right)$$

Here $\kappa^\sharp$ is the composite map $B \times Z \xrightarrow{B\times k} B \times C^B \xrightarrow{(\mathrm{pr}_1, \mathrm{ev})} B \times C$, seen as a map from $(B \times Z \xrightarrow{\mathrm{pr}_1} B)$ to pr_B, and thus as an object in $\left(\mathcal{C}_{/B}\right)_{/\,\mathrm{pr}_B}$. Since $\left(\mathcal{C}_{/B}\right)_{/\,\mathrm{pr}_B} \simeq \mathcal{C}_{/B\times C}$

and using the definition of $q = (\mathrm{pr}_1, \mathrm{ev})^* p$, we then obtain

$$\mathcal{C}_{/C^B}(k, \alpha) \simeq \mathcal{C}_{/B\times C}\left(\kappa^\sharp, p\right) = \mathcal{C}_{/B\times C}\left((\mathrm{pr}_1, \mathrm{ev})(B \times k), p\right) \simeq$$

$$\simeq \mathcal{C}_{/B\times C^B}(B \times k, q) = \mathcal{C}_{/B\times C^B}\left((\mathrm{pr}_{C^B})^* k, q\right) \simeq \mathcal{C}_{/C^B}\left(k, \prod_{\mathrm{pr}_{C^B}} q\right),$$

whence $\alpha \simeq \prod_{\mathrm{pr}_{C^B}} q$, as needed. □

Intuitively, the following result is giving a condition for the existence of a unique extension of a map f along another map g in terms of unique extensions along the fibers of g. When we take fibers out of the picture, we get the following odd-looking statement.

Proposition A.3.4 (cf. [CORS18, Lemma 2.23]). *Let $f\colon A \to C$ and $g\colon A \to B$ be two maps in a locally cartesian closed ∞-category $\mathcal{C}$. Form the following pullback squares in $\mathcal{C}$*

$$\begin{tikzcd} A \times C \arrow[r] \arrow[d, "{(\mathrm{pr}_A, g\times C)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\Delta B"] \\ A \times B \times C \arrow[r, "g\times \mathrm{pr}_B"'] & B \times B \end{tikzcd} \qquad \begin{tikzcd} B \times A \arrow[r] \arrow[d, "{(\mathrm{pr}_A, B\times f)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\Delta C"] \\ A \times B \times C \arrow[r, "f\times \mathrm{pr}_C"'] & C \times C \end{tikzcd}$$

Consider the following object in $\mathcal{C}_{/B}$

$$E := \sum_{B\times C\to B}\left(\prod_{A\times B\times C\to B\times C} (\mathrm{pr}_A, B\times f)^{(\mathrm{pr}_A, g\times C)}\right)$$

where the displayed internal hom is taken in $\mathcal{C}_{/A\times B\times C}$. Then the following hold.

(i) If we let $f\colon C^B \to C^A$ denote the composite $C^B \to 1 \xrightarrow{f} C^A$, there is an equivalence

$$\prod_B E \simeq \sum_{C^B} (f, C^g)^*\left(\Delta(C^A)\right) \tag{A.2}$$

where $(f, C^g)\colon C^B \to C^A \times C^A$.

(ii) *The space of global elements of the right-hand side in* (A.2) *is equivalent to the space* $\mathrm{Ext}(f, g)$ *of extensions of* f *along* g*. In particular, if* $\prod_B E$ *is contractible in* $\mathcal{C}_{/B}$*, then there is a unique dotted extension in*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & C \\ B \arrow[ur, dashed] & \end{tikzcd}$$

Proof. We start by proving the first claim. We have

$$(\mathrm{pr}_A, B \times f)^{(\mathrm{pr}_A, g \times C)} = \prod_{(\mathrm{pr}_A, g \times C)} (\mathrm{pr}_A, g \times C)^*(\mathrm{pr}_A, B \times f)$$

Since $(\mathrm{pr}_A, B \times f) = (f \times \mathrm{pr}_C)^*(\Delta C)$ and $(f \times \mathrm{pr}_C)(\mathrm{pr}_A, g \times C) = f \times C$, we get that

$$(\mathrm{pr}_A, g \times C)^*(\mathrm{pr}_A, B \times f) = (\mathrm{id}_A, f) \colon A \to A \times C.$$

Therefore, letting $\mathrm{pr}_{B \times C} \colon A \times B \times C \to B \times C$ be the projection map, we have

$$\prod_{\mathrm{pr}_{B \times C}} (\mathrm{pr}_A, B \times f)^{(\mathrm{pr}_A, g \times C)} = \prod_{\mathrm{pr}_{B \times C}} \left(\prod_{(\mathrm{pr}_A, g \times C)} (\mathrm{id}_A, f) \right) = \prod_{g \times C} (\mathrm{id}_A, f)$$

Using Lemma A.3.3, we then get

$$\prod_B E = \prod_B \sum_{B \times C \to B} \prod_{g \times C} (\mathrm{id}_A, f) \simeq \sum_{C^B} \prod_{\mathrm{pr}_{C^B}} (\mathrm{pr}_1, \mathrm{ev})^* \left(\prod_{g \times C} (\mathrm{id}_A, f) \right) =: E'$$

where $\mathrm{pr}_{C^B} \colon B \times C^B \to C^B$ is the projection map. There are pullback squares

$$\begin{tikzcd} A \times C^B \arrow[r, "{(\mathrm{id}_A,\ \mathrm{ev}(g \times C^B))}"] \arrow[d, "g \times C^B"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \times C \arrow[d, "g \times C"] \\ B \times C^B \arrow[r, "{(\mathrm{pr}_1, \mathrm{ev})}"'] & B \times C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "{(\mathrm{id}_A, f)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\Delta C"] \\ A \times C \arrow[r, "f \times C"'] & C \times C \end{tikzcd}$$

Thus, using the Beck-Chevalley condition, we get

$$E' \simeq \sum_{C^B} \prod_{\mathrm{pr}_{C^B}} \prod_{g \times C^B} \big((f \times C)(\mathrm{id}_A,\ \mathrm{ev}(g \times C^B))\big)^* (\Delta C) \simeq$$

$$\simeq \sum_{C^B} \prod_{A \times C^B \to C^B} \big((f \times C)(\mathrm{id}_A,\ \mathrm{ev}(g \times C^B))\big)^* (\Delta C) \simeq$$

$$\simeq \sum_{C^B} \prod_{A \times C^B \to C^B} (\mathrm{ev}(A \times (f, C^g)))^* (\Delta C) =: E''$$

where the last equivalence is due to the fact that $(f \times C)(\mathrm{id}_A,\ \mathrm{ev}(g \times C^B))$ is equal to the composite map

$$A \times C^B \xrightarrow{A \times (f, C^g)} A \times C^A \times C^A \xrightarrow{\mathrm{ev}} C \times C$$

Using the Beck-Chevalley condition applied to the pullback square

$$\begin{tikzcd}
A \times C^B \arrow[r, "A \times (f, C^g)"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \times C^A \times C^A \arrow[d, "\mathrm{pr}_2"] \\
C^B \arrow[r, "(f, C^g)"'] & C^A \times C^A
\end{tikzcd}$$

we further deduce that

$$E'' = \sum_{C^B} \prod_{A \times C^B \to C^B} (A \times (f, C^g))^* (\mathrm{ev}^*(\Delta C)) \simeq$$

$$\simeq \sum_{C^B} (f, C^g)^* \left(\prod_{\mathrm{pr}_2} \mathrm{ev}^*(\Delta C) \right) \simeq \sum_{C^B} (f, C^g)^* (\Delta(C^A))$$

where the last equivalence is given by Function Extensionality. This concludes the proof of the first part.

For the second part, note that

$$P := \sum_{C^B} (f, C^g)^*(\Delta(C^A))$$

is the pullback object of $C^g \colon C^B \to C^A$ along $f \colon 1 \to C^A$ and thus $\mathcal{C}(1, P)$ is the homotopy fiber of $\mathcal{C}(1, C^g)$ at $f \in \mathcal{C}(1, C^A)$. The latter homotopy fiber gives exactly the needed space of extensions. $\square$

Bibliography

[ABFJ17a] M. Anel, G. Biedermann, E. Finster, and A. Joyal, *A Generalized Blakers-Massey Theorem*, ArXiv e-prints (2017), arXiv:1703.09050v2.

[ABFJ17b] ______, *Goodwillie's calculus of functors and higher topos theory*, ArXiv e-prints (2017), arXiv:1703.09632.

[AF10] J. F. Adams and Z. Fiedorowicz, *Localisation and Completion with an addendum on the use of Brown-Peterson homology in stable homotopy*, ArXiv e-prints (2010), arXiv:1012.5020.

[CORS18] J. D. Christensen, M. Opie, E. Rijke, and L. Scoccola, *Localization in Homotopy Type Theory*, to appear in *Higher Structures*, ArXiv e-prints (2018), arXiv:1807.04155.

[Far96] E. D. Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996, doi:10.1007/BFb0094429.

[GK17] D. Gepner and J. Kock, *Univalence in locally cartesian closed ∞-categories*, Forum Math. **29** (2017), no. 3, 617–652, doi:10.1515/forum-2015-0228.

[Hir03] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.

[Hol08] S. Hollander, *A homotopy theory for stacks*, Israel J. Math. **163** (2008), 93–124, doi:10.1007/s11856-008-0006-5.

[Joh02] P. T. Johnstone, *Sketches of an elephant: a topos theory compendium. Vol. 1*, Oxford Logic Guides, vol. 43, The Clarendon Press, Oxford University Press, New York, 2002.

[Joy08] A. Joyal, *Notes on quasi-categories*, 2008.

[Kap14] K. R. Kapulkin, *Joyal's Conjecture in homotopy type theory*, ProQuest LLC, Ann Arbor, MI, 2014, Thesis (Ph.D.)–University of Pittsburgh.

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[MM94] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994, A first introduction to topos theory, Corrected reprint of the 1992 edition.

[MP12] J. P. May and K. Ponto, *More concise algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2012, Localization, completion, and model categories.

[Rez10] C. Rezk, *Toposes and homotopy toposes*, 2010.

[Rez15] ———, *Proof of the blakers-massey theorem*, 2015.

[RSS17] E. Rijke, M. Shulman, and B. Spitters, *Modalities in homotopy type theory*, ArXiv e-prints (2017), `arXiv:1706.07526`.

[RV18] E. Riehl and D. Verity, *Elements of* ∞*-category theory*, Preprint available at `www.math.jhu.edu/~eriehl/elements.pdf` (2018).

[Shu19] M. Shulman, *All* $(\infty, 1)$*-toposes have strict univalent universes*, ArXiv e-prints (2019), `arXiv:arXiv:1904.07004v1`.

[Sul05] D. P. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry*, K-Monographs in Mathematics, vol. 8, Springer, Dordrecht, 2005, The 1970 MIT notes, Edited and with a preface by Andrew Ranicki.

[UF13] The Univalent Foundations Program, *Homotopy type theory—univalent foundations of mathematics*, The Univalent Foundations Program, Princeton, NJ; Institute for Advanced Study (IAS), Princeton, NJ, 2013.

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Education

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Selected talks

- "Localization theory in an ∞-topos". Geometry & Topology seminar, Western University, 4 February 2019. Invited talk at the AMS Special Session on Structured Homotopy Theory, Ann-Arbor, MI, 20-21 October 2018.
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Selected conferences

- Fall Perspectives on Teaching, Western University, August 2018.
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