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Graded Character Rings, Mackey Functors and Tambara Functors

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Abstract

Let G be a finite group. The ring $R_{\mathbb{K}}(G)$ of virtual characters of G over the field $\mathbb{K}$ is a λ-ring; as such, it is equipped with the so-called Γ-filtration, first defined by Grothendieck. In the first half of this thesis, we explore the properties of the associated graded ring $R^*_{\mathbb{K}}(G)$, and present a set of tools to compute it through detailed examples. In particular, we use the functoriality of $R^*_{\mathbb{K}}(-)$, and the topological properties of the Γ-filtration, to explicitly determine the graded character ring over the complex numbers of every group of order at most 8, as well as that of dihedral groups of order $2p$ for p prime.

In the second half, we study the interplay between the graded character ring of a group and those of its subgroups: while restriction of representations gives rise to a well-defined graded ring homomorphism, induction does not preserve the Γ-filtration, thus $R^*_{\mathbb{K}}(-)$ is not a Mackey functor. We introduce a modified filtration that remedies this, and explore ways to compute the associated graded ring. We then turn to tensor induction of representations, and show that in the case of complex characters of abelian groups, both inductions preserve the filtration. Therefore, the restriction of $R^*_{\mathbb{C}}(-)$ to abelian groups is a Tambara functor.

Keywords: Virtual characters, finite groups, λ-rings, Grothendieck filtration, Mackey functors, Tambara functors

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Contents

Chapter 1

Introduction

1.1 Motivation

Let G be a finite group and $\mathbb{K}$ a field of characteristic zero. Let $R_{\mathbb{K}}(G)$ be the *ring of virtual characters* (or *character ring*) of G, generated by the irreducible characters of G over $\mathbb{K}$, which has a ring structure coming from the tensor product of representations. The ring $R_{\mathbb{K}}(G)$, with the operations $\{\lambda^n : R_{\mathbb{K}}(G) \to R_{\mathbb{K}}(G)\}$ induced by exterior powers of representations, together satisfy the axioms of a λ-ring.

Grothendieck used the theory of λ-rings in the late 1950s to provide a categorical framework for the Riemann-Roch theorem (see [Ber71]). With each λ-ring R, he associated a filtration (hereafter the Grothendieck filtration, or Γ-filtration); the associated graded ring gr^*R is equipped with so-called algebraic Chern classes, which satisfy the properties of the eponymous construction in algebraic topology. To underline the importance of this construction, let us mention that, when X is a smooth algebraic variety (say, over the complex numbers), there is an isomorphism

$$gr^*K(X) \otimes \mathbb{Q} \cong CH^*X \otimes \mathbb{Q},$$

where CH^*X is the Chow ring of X and $K(X)$ is the Grothendieck group of algebraic vector bundles over X (see for example [Ful98, Ex. 15.2.16]). If X is a reasonable topological space and $K(X)$ is, this time, its topological K-theory, then

$$gr^*K(X) \otimes \mathbb{Q} \cong H^{2*}(X, \mathbb{Q}),$$

where $H^{2*}(X, \mathbb{Q})$ is the even part of the singular cohomology of X (see [Ati89, Prop. 3.2.7]). Both of these isomorphisms are compatible with Chern classes.

Character rings are natural examples of λ-rings; despite this, examples of graded character rings in the literature are few and far between. In the sequel, we write $R^*_{\mathbb{K}}(G)$ for $gr^*(R_{\mathbb{K}}(G))$. The first mention of an explicit computation appears in a 2001 preprint by Beauville ([Bea01]), and states that for a complex connected reductive group G, the graded ring $R^*_{\mathbb{C}}(G) \otimes \mathbb{Q}$ is simply described in terms of a maximal torus and its Weyl group.
As in all of the above results, the graded ring is tensored with $\mathbb{Q}$; but in [GM14], Guillot and Mináč showed that when G is a finite group, the ring $R^*_{\mathbb{K}}(G) \otimes \mathbb{Q}$ is zero in positive degree. Determining $R^*_{\mathbb{K}}(G)$ in this case is hard, as few tools have been developed to do so; but it is not hopeless, and [GM14] contains computations of $R^*_{\mathbb{C}}(G) \otimes \mathbb{F}_2$ for some 2-groups using elementary tools. Another successful approach is offered in [Yag15]: using the isomorphism between the character ring $R(G)$ of a group and the topological K-theory of its classifying space $K(BG)$, Yagita computes some well-chosen examples of the graded character ring $R^*(G)$ via the Atiyah-Hirzebruch spectral sequence (see

[Ati61]), which converges to the Grothendieck filtration of the character ring in these (but not all) cases. This is a powerful method; however, the elementary approach in [GM14] presents the advantage of being easily checkable, and yielding explicit results in terms of characters of the group.

This thesis presents results obtained in an attempt to better understand graded character rings of finite groups, through a two-sided approach: first, by considering explicit examples (*What do graded character rings look like?*), and second, by focusing on their general structure (*How do they behave?*). The first question is treated in Chapter 2, where we develop several elementary computation techniques to explicitly determine $R^*_{\mathbb{K}}(G)$, using functorial and topological properties of $R^*_{\mathbb{K}}(-)$. The body of examples thus obtained will grant us the necessary intuition to take a deep dive into the general theory in Chapter 3, where we take a closer look at the interplay between $R^*_{\mathbb{K}}(G)$ and $R^*_{\mathbb{K}}(H)$ for subgroups H of G.

The theory of graded character rings is rich and intricate, and full of surprising results. We hope the work presented here will encourage the reader to explore it further.

1.2 The concrete side: computing graded character rings

Determining $R^*_{\mathbb{K}}(G)$ explicitly is an arduous but fascinating endeavour, as even the most "innocent" groups lead to remarkable results; throughout Chapter 2, we build a toolbox of computation tricks and techniques, while gaining insight into the bigger picture. For example, the following computation (presented below as Proposition 2.3.3) shows that there is no "Künneth formula" for $R^*_{\mathbb{C}}(G)$:

Theorem 1.2.1. *Let C_p be a cyclic group of prime order p. Then:*

$$R^*_{\mathbb{C}}(C_p^k) = \frac{\mathbb{Z}[x_1, \cdots, x_k]}{(px_i, x_i x_j^p - x_i^p x_j)}, \qquad \textit{with } |x_i| = 1.$$

In fact, there is no known general formula for the graded character ring of a product of two groups; this makes determining graded character rings of some "easy" groups surprisingly difficult, a consequence that is in turn illustrated by the rather sophisticated computation of $R^*_{\mathbb{C}}(C_4 \times C_4)$, the very last one that we present in Chapter 2. A Künneth formula does hold, however, for products of groups of coprime order (see Corollary 2.3.2); then the ring $R^*_{\mathbb{C}}(G \times H)$ can be expressed as the tensor product $R^*_{\mathbb{C}}(G) \otimes R^*_{\mathbb{C}}(H)$. This means that the computation of (complex) graded character rings of abelian groups, for instance, is reduced to that of $R^*_{\mathbb{C}}(-)$ on abelian p-groups.

Even under this restriction, the structures appearing are strikingly complex. For example, the main theorem of [Qui68] can be adapted to show that for an abelian group G and for each prime p, there is an explicit, surjective morphism:

$$R^*_{\mathbb{C}}(G) \otimes \mathbb{F}_p \to gr_\bullet \mathbb{F}_p G,$$

where $gr_\bullet \mathbb{F}_p G$ is the graded ring associated to the filtration by powers of the augmentation ideal of $\mathbb{F}_p G$. In particular, the following result is a direct corollary of Theorem 2.4.3:

Theorem 1.2.2. *Let G be an abelian p-group of the form $C_{p^{i_1}} \times \cdots \times C_{p^{i_n}}$. Then $R^*_{\mathbb{C}}(G)$ is generated by elements $x_1, \ldots, x_n$ of degree 1 such that any monomial in any relation between these in $R^*_{\mathbb{C}}(G) \otimes \mathbb{F}_p$ features some $x_k^{p^{i_k}}$, for some index k.*

This result is illustrated by Theorem 1.2.1 above, and again by the example of $C_4 \times C_4$:

Proposition 1.2.3.

$$R^*_{\mathbb{C}}(C_4 \times C_4) = \frac{\mathbb{Z}[x,y]}{(4x, 4y, 2x^2y + 2xy^2, x^4y^2 - x^2y^4)}$$

with $|x| = |y| = 1$*, therefore*

$$R^*_{\mathbb{C}}(C_4 \times C_4) \otimes \mathbb{F}_2 = \frac{\mathbb{F}_2[x,y]}{(x^4y^2 + x^2y^4)}.$$

(This is Proposition 2.6.2 in the text.) Notice how relations modulo 2 involve either x^4 or y^4 in each monomial. The computation of $R^*_{\mathbb{C}}(C_4 \times C_4)$ uses every technique and tool presented in Chapter 2: relations are found via algebraic manipulation of virtual characters in $R_{\mathbb{C}}(G)$, and by studying the restriction of characters of $C_4 \times C_4$ to various subgroups. To conclude the computation, we resort to the topological properties of the Grothendieck filtration, presented in detail in Section 2.5.

Theorem 1.2.4. *Let G be a p-group. Then for each $g \in G$, the evaluation morphism $ev_g : R_{\mathbb{C}}(G) \to \mathbb{Z}[\mu_m]$, where μ_m is an appropriate choice of root of unity, is continuous with respect to the topology induced by the Grothendieck filtration and the p-adic topology, respectively.*

This means that if for some large M, a virtual character χ is in $\Gamma^M(G)$, the M-th ideal in the Grothendieck filtration of G, then $\mathrm{ev}_g(\chi)$ must be divisible by a large power of p; we use this fact to solve questions of order and nilpotency in $R^*_{\mathbb{C}}(G)$.

Of course, the abovementioned techniques can be applied to computing graded character rings of non-abelian groups. In fact, Chapter 2 contains computations of $R^*_{\mathbb{C}}(G)$ for every group G of order less than 16, as well as for all dihedral groups of order $2p$, for p prime. Two particularly interesting examples among them are those of the dihedral group D_4 of order 8, and the quaternion group Q_8: because it takes λ-operations into account, the graded character ring is able to distinguish non-isomorphic groups with the same character tables, as shown by comparing the results of Proposition 2.3.5 and Theorem 2.5.4.

Proposition 1.2.5.

$$R^*_{\mathbb{C}}(D_4) = \frac{\mathbb{Z}\,[x, y, b]}{(2x, 2y, 4b, xy, xb - yb)}$$

with $|x| = |y| = 1$ *and* $|b| = 2$, *and*

$$R^*_{\mathbb{C}}(Q_8) = \frac{\mathbb{Z}[x, y, u]}{(2x, 2y, 8u, x^2, y^2, xy - 4u)}$$

where $|x| = |y| = 1$ *and* $|u| = 2$.

Thus graded representation rings not only benefit from a complicated and mysterious theory, they are also a rather fine invariant of groups.

Finally, let us briefly address the matter of the base field: all computations mentioned so far pertain to complex representations. Over other fields, the situation can become much more complicated; we do claim one intriguing result over the rationals:

Proposition 1.2.6.

$$R^*_{\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) = \frac{\mathbb{Z}[x]}{(px)}$$

with $|x| = p - 1$.

Note that $R^*_{\mathbb{Q}}(C_p)$ is concentrated in degrees multiple of $(p-1)$; this is actually true of every p-group over the rationals (see Proposition 2.2.2). Strikingly, even $R^*_{\mathbb{Q}}(C_N)$ for composite N is not known.

The general behaviour of graded character rings of finite groups can be glimpsed through the cracks of Chapter 2. It is the subject of Chapter 3.

1.3 The abstract side: Mackey functors and Tambara functors

Regarding the general structure and behavior of graded character rings, much of the work presented in Chapter 3 boils down to the following question: for each $H \leq G$, the restriction and induction maps going between $R_{\mathbb{K}}(G)$ and $R_{\mathbb{K}}(H)$ turn $R_{\mathbb{K}}(-)$ into a *Mackey functor*, a particularly widespread type of algebraic structure (group cohomology and algebraic K-theory are examples of Mackey functors). Is $R^*_{\mathbb{K}}(-)$ also a Mackey functor?

The answer is, unfortunately, negative: while graded character rings are functorial, and thus restriction induces a well-defined ring homomorphism on $R^*_{\mathbb{K}}(-)$, the induction map does not preserve the Grothendieck filtration. An analogue to Cartan and Eilenberg's result on stable elements in cohomology ([CE99, Th. XII.10.1]) states that, if S is any Mackey functor, the following result applies:

Proposition 1.3.1. *If $H := Syl_p(G)$ is abelian, then*

$$\mathrm{Res}^G_H : S(G)_{(p)} \longrightarrow S(H)^{N_G(H)}$$

is an isomorphism.

Here $Syl_p(G)$ denotes a p-Sylow of G, and $S(H)^{N_G(H)}$ is the set of elements of $S(H)$ that are invariant under the action of the normalizer of H in G. In the example of the alternating group A_4 of order 12, the surjectivity condition fails when restricting to the 2-Sylow $C_4 \times C_4$, and thus the graded character ring functor $R^*_{\mathbb{C}}(-)$ is not a Mackey functor. This is Lemma 3.2.4 and Theorem 3.2.5 in the text.

It is possible to "Mackeyfy" graded character rings by modifying the Grothendieck filtration. We define in Section 3.3 the *saturated filtration* $\{F^n(G)\}_{n\geq 0}$ as the minimal filtration that is preserved by induction of characters and contains the Grothendieck filtration, that is:

$$F^n(G) = \sum_{H\leq G} \mathrm{Ind}_H^G(\Gamma^n(H)),$$

where $\Gamma^n(H)$ is the n-th ideal in the Grothendieck filtration of H. We call its associated graded ring the *saturated ring* of G and denote it by $\mathcal{R}^*_{\mathbb{K}}(G)$ (as opposed to $R^*_{\mathbb{K}}(G)$ for the usual graded ring). Fortunately, restriction of representations also preserves this filtration, and thus:

Theorem 1.3.2. *The* ***saturated graded ring****:*

$$\mathcal{R}^*_{\mathbb{K}}(-) := \bigoplus_{n\geq 0} F^n(-)/F^{n+1}(-)$$

is a Mackey functor.

(See Theorem 3.3.2). At a first glance, there is no guarantee that $\mathcal{R}^*_{\mathbb{K}}(-)$ is not trivial in some way or other: a lot of the information contained in the Grothendieck filtration could be lost through this modification. The following corollary to [Ati61, Th. 6.1] reassuringly states that some of the information remains: the generators of the saturated graded ring $\mathcal{R}^*(-)$ are also topological generators for $R_{\mathbb{K}}(G)$.

Theorem 1.3.3. *The filtrations $\{F^n\}_n$ and $\{\Gamma^n\}_n$ induce the same topology on $R_{\mathbb{K}}(G)$ as the I-adic filtration, where $I = \ker(\varepsilon)$ is the kernel of the degree map.*

This means, in particular, that induction is continuous with respect to the I-adic topology, and extends to a map $\widehat{\mathrm{Ind}_H^G} : \widehat{R}(H) \to \widehat{R}(G)$ on completed rings. This, combined with the stable element result, gives us Theorem 3.3.10, an analogue to Artin's theorem:

Theorem 1.3.4. *Let X be a family of subgroups of a finite group G. Let*

$$\widehat{\mathrm{Ind}} : \bigoplus_{H \in X} \widehat{R}(H) \to \widehat{R}(G)$$

be the morphism defined on each $\widehat{R}(H)$ by $\widehat{\mathrm{Ind}_H^G}$. If X contains a p-Sylow subgroup of G for every prime p, then $\widehat{\mathrm{Ind}}$ is surjective.

What information is gained? Subgroups of G feature prominently in the definition of the saturated filtration, so that the ring $\mathcal{R}^*(G)$ might remember some of the subgroup structure of G, and it might distinguish groups with the same character table and power maps.

There are many examples of groups G such that the two filtrations coincide, and the natural map $R^*_{\mathbb{K}}(G) \to \mathcal{R}^*_{\mathbb{K}}(G)$ is an isomorphism (these two facts are actually equivalent, as we show in Corollary 3.3.4); we call them **saturated groups**. The following result combines Proposition 3.4.1, Proposition 3.4.4, and Proposition 3.4.6:

Theorem 1.3.5. *Groups of order less than 12, as well as abelian groups, and dihedral groups of order* $2p$ *for* p *prime, are saturated. In particular, the restriction of* $R^*_{\mathbb{C}}(-)$ *to abelian groups is a Mackey functor.*

The saturated filtration would be of little use if one could only compute the saturated rings of saturated groups. This is where the stable elements method comes into play; it allows us to deduce the graded ring of a group from those of its Sylow subgroups, as we do in Theorem 3.4.7:

Proposition 1.3.6. *Let* $G = PSL(2,p)$ *be the projective special linear group over* $\mathbb{F}_p$*, where* p *is an odd prime such that* $p \equiv 3, 5 (\mathrm{mod}\ 8)$*. Write:*

$$|G| = 4 \cdot p \cdot l_1^{i_1} \cdots l_n^{i_n} \cdot r_1^{j_1} \cdots r_m^{j_m}, \qquad \textit{with} \quad l_k|(p-1), \quad r_k|(p+1).$$

Then:

$$\mathcal{R}^*(G) \cong \frac{\mathbb{Z}[x_1, \cdots, x_n, y_1, \cdots y_m, z, t, u]}{(l_k^{i_k} x_k, r_k^{j_k} y_k, 2z, 2t, pu, z^3 - t^2)}$$

with $|x_k| = |y_k| = |z| = 2$*,* $|t| = 3$*,* $|u| = (p-1)/2$*.*

Remarkably, the above result is obtained without using any information about the character table of $PSL(2,p)$. Saturated rings are thus particularly interesting from an inverse problem point of view: knowing $\mathcal{R}^*_{\mathbb{K}}(G)$, what can we deduce about $R_{\mathbb{K}}(G)$?

The last problem we treat in Chapter 3 is that of tensor induction, a multiplicative map $R_{\mathbb{K}}(H) \to R_{\mathbb{K}}(G)$. Mackey functors equipped with such a multiplicative map (and satisfying certain axioms) are called *Tambara functors*. In group cohomology, this role is played by the Evens norm, wich, applied to the subgroup inclusion $G \hookrightarrow G \times C_p$ (for p prime), can be used to define Steenrod operations. The (ungraded) character ring $R_{\mathbb{K}}(G)$ with tensor induction is also Tambara functor, as we prove in Section 3.5.

Section 3.6 explores the interaction between tensor induction and the Grothendieck filtration; to this effect, one needs to understand how the tensor induction map (hereafter "norm map") can be extended to virtual characters. This is a remarkably complex problem, as there is no known formula for the norm of the sum of two characters, even when those come from actual representations. We follow Tambara's account and, restricting first to normal subgroups of prime index, then to abelian groups, we obtain such a formula. This is the key to prove Corollary 3.6.10:

Theorem 1.3.7. *The restriction of $R^*_{\mathbb{C}}(-)$ to finite abelian groups is a Tambara functor.*

As an application, we propose in Section 3.7 to compute, for any abelian group G, the norm of any degree-one Chern class from $R^*_{\mathbb{C}}(G)$ to $R^*_{\mathbb{C}}(G \times C_p)$. This brings us one step closer to defining Steenrod operations on graded character rings.

More general cases, as that of $R^*_{\mathbb{K}}(-)$ for abelian groups and general $\mathbb{K}$, or that of $\mathcal{R}^*_{\mathbb{K}}(-)$ for general groups, remain open.

Chapter 2

Computing graded character rings

We start our study of graded character rings with a practical, computational approach. The main definitions are introduced in Section 2.1; each of sections 3 to 6 is focused on a different computational tool. We show in Section 2.2 two elementary computations, concerning cyclic groups over any algebraically closed $\mathbb{K}$ (after [GM14]), and over the rationals, and we will then restrict ourselves to the case $\mathbb{K} = \mathbb{C}$. In Section 2.3, we put the cyclic group example to good use: we show that restriction of characters is a well-defined homomorphism and apply it to elementary abelian groups as well as some dihedral groups. In Section 2.4, using a result of Quillen in [Qui68], we construct the aforementioned morphism $R^*_{\mathbb{C}}(G) \otimes \mathbb{F}_p \to gr_\bullet \mathbb{F}_p G$. In Section 2.5, we look at the continuity of evaluation of characters with respect to the topology induced by the Grothendieck filtration, and the p-adic topology. We apply our results to graded character rings of p-groups: first in Section 2.5 for the quaternion group of order 8, and second in Section 2.6 to some abelian 2-groups.

2.1 Definitions and first properties

We recall some facts about the Grothendieck filtration on λ-rings, in the context of character rings. A concise treatment of the basic facts about λ-rings can be found in [AT69]. Let G be a finite group, and let $\mathbb{K}$ be a field of characteristic zero. The *ring of virtual characters* (or *character ring*) $R_{\mathbb{K}}(G)$ of G is the augmented ring generated by irreducible characters of G over $\mathbb{K}$; the augmentation $\varepsilon : R_{\mathbb{K}}(G) \to \mathbb{Z}$ is the degree map. Note that, since $\mathbb{K}$ has characteristic zero, representations up to isomorphism are determined by their characters. Thus $R_{\mathbb{K}}(G)$ is also the Grothendieck ring on the category of $\mathbb{K}G$-modules, and we use the terms "character" and "representation" interchangeably when there is no risk of confusion. For instance, if χ is a character of G, by "the n-th exterior power $\lambda^n(\chi)$ of χ", we mean "the character associated to the n-th exterior power of the representation affording χ". The maps $\{\lambda^n\}_n$ satisfy for all characters χ, τ:

(i) $\lambda^0(\chi) = 1$

(ii) $\lambda^1(\chi) = \chi$

(iii) $\lambda^k(\chi + \tau) = \sum_{i+j=k} \lambda^i(\chi)\lambda^j(\tau)$

The addition formula above allows us to extend λ^n to $R_{\mathbb{K}}(G)$, by defining each $\lambda^n(-\chi)$ by the equation $\lambda^n(\chi + (-\chi)) = 0$ for $n > 0$. We say that $R_{\mathbb{K}}(G)$, together with the maps $\{\lambda^n\}$, is a pre-λ-ring. Since the λ-operations also satisfy axioms [AT69, §1 (12)-(14)], we see that $R_{\mathbb{K}}(G)$ is a λ-ring. We define:

$$\lambda_T : \begin{cases} R(G) & \to 1 + T \cdot R(G)[[T]] \\ \rho & \mapsto 1 + \sum_{i=1}^{\infty} \lambda^i(\rho) T^i \end{cases}.$$

Call x a *line element* if $\lambda_T(x) = 1 + xT$. Alternatively, x is a line element whenever it is a one-dimensional representation of G.

Remark. In the terminology of [AT69], a ring with λ-operations satisfying the first three axioms above is called a λ-ring, and the additional axioms make it a special λ-ring. These extra axioms describe in particular how λ-operations interact with the ring multiplication. As it turns out, they are equivalent to the so-called "splitting principle", stated below as Proposition 2.1.3, and to which we refer in practice for calculations.

For $x \in R_{\mathbb{K}}(G)$ and $n \in \mathbb{N}$, put

$$\gamma^n(x) = \lambda^n(x + n - 1) = (-1)^n \sum_{i=0}^{n} (-1)^i \lambda^i(x + n),$$

the n-th gamma operation. Let $I = \ker \varepsilon$ be the augmentation ideal, and note that if $x \in I$ then $\gamma^n(x) \in I$. Let Γ^n be the additive subgroup of $R_{\mathbb{K}}(G)$ generated by the monomials

$$\gamma^{i_1}(x_1)\gamma^{i_2}(x_2)\cdots\gamma^{i_k}(x_k),\ x_i \in I,\ \sum_{j=1}^{k} i_j \geq n.$$

One can show that $\Gamma^0 = R(G)$, $\Gamma^1 = I$, and that each Γ^n is a λ-ideal (see [AT69, Prop. 4.1]). Moreover, the Γ-filtration contains the I-adic filtration on $R_{\mathbb{K}}(G)$, that is, $\Gamma^n \supseteq I^n$ for each n. These two filtrations contain the same topological information:

Proposition 2.1.1 ([Ati61, Cor. 12.3])**.** *The topology induced by the Grothendieck filtration coincides with the I-adic topology.*

Define the *graded character ring of* G (with coefficients in $\mathbb{K}$) as:

$$R^*_{\mathbb{K}}(G) = \bigoplus_{i \geq 0} \Gamma^i / \Gamma^{i+1}.$$

The definitions readily imply that $\Gamma^m \cdot \Gamma^n \subset \Gamma^{m+n}$, so this is indeed a graded ring. In the sequel, we simply write $R^*(G)$ whenever $\mathbb{K}$ is clear from the context. Our aim is to compute examples of the graded ring $R^*(G)$ for some finite groups.

Determining generators for $R^*(G)$ is a completely straightforward process. For any $\rho \in R(G)$, let $C_n(\rho) = \gamma^n(\rho - \varepsilon(\rho))$; we define the *$n$-th algebraic Chern class* $c_n(\rho)$ of ρ as the image of $C_n(\rho)$ by the quotient map $\Gamma^n(G) \to R^n(G)$. Define

$$c_T : \begin{cases} R(G) & \to 1 + T \cdot R^*(G)[[T]] \\ \rho & \mapsto 1 + \sum_{i=1}^{\infty} c_i(\rho) T^i \end{cases}.$$

We call $c_T(\rho)$ the *total Chern class* of ρ. Note that if x is a line element, then $c_T(x) = 1 + c_1(x)T$.

Proposition 2.1.2 ([FL85, III.§2])**.** *The total Chern class satisfies the axioms of a Chern class homomorphism as detailed in [FL85, I.§3]. In particular,*

(i) If ρ is the character of a representation of degree n, then $c_k(\rho) = 0$ for $k > n$.

(ii) Whenever ρ and σ are line elements, we have $c_1(\rho\sigma) = c_1(\rho) + c_1(\sigma)$.

(iii) The map c_T is a homomorphism, that is $c_T(x + y) = c_T(x)c_T(y)$. In particular, for all $n \geq 0$:

$$c_n(\rho + \sigma) = \sum_{i=0}^{n} c_i(\rho) c_{n-i}(\sigma).$$

Note that (ii) is seen by remarking that

$$\gamma^1(\rho\sigma - 1) - \left(\gamma^1(\rho - 1) + \gamma^1(\sigma - 1)\right) = \gamma^1(\rho - 1)\gamma^1(\sigma - 1) \in \Gamma^2.$$

Much as it is the case for λ-operations, whenever we need to compute the Chern class of a product, we rely on the splitting principle below.

Proposition 2.1.3 ([FL85, III.§1])**.** *Given representations $\rho_1, \cdots, \rho_k$ of G dimensions $d_1, \cdots, d_k$ respectively, there exists a λ-ring extension R' of $R(G)$ such that $\Gamma^n R' \cap R(G) = \Gamma^n R(G)$ and each $\rho_i = x_{i,1} + \cdots + x_{i,d_i}$ is the sum of d_i line elements in R'.*

Thus for a character ρ of degree n, by (iii) above, ve have in the graded ring $gr^* R'$:

$$c_T(\rho) = c_T(x_1 + \cdots + x_n) = \prod_{i=1}^{n} c_T(x_i) = \prod_{i=1}^{n}(1 + c_1(x_i)T).$$

The graded ring $R^*(G)$ appears as a subring of $gr^* R'$, and we can recover $c_k(\rho)$ as the coefficient of T^k in the above polynomial, that is, the symmetric polynomial of degree k in the n variables $c_1(x_1), \cdots, c_1(x_n)$.

As a first practical example of the splitting principle, consider the following computation. Recall that the determinant of a representation ρ of G of degree n is defined as $\det(\rho) = \lambda^n(\rho)$. In particular, by the splitting principle, if we write $\rho = x_1 + \cdots + x_n$ then we have $\det \rho = \prod x_i$.

Lemma 2.1.4. *For a representation ρ of G, we have $c_1(\rho) = c_1(\det \rho)$.*

Proof. Let R' be an extension of $R(G)$ as in Proposition 2.1.3. The ring R' is a λ-ring, and (ii) and (iii) of Proposition 2.1.2 do apply in full generality. In R', write $\rho = x_1 + \cdots + x_n$ as a sum of line elements. Then, by Proposition 2.1.2(iii):

$$c_T(\rho) = c_T(x_1 + \cdots + x_n) = \prod_{i=1}^{n}(1 + c_1(x_i)T).$$

The coefficient of T is $c_1(\rho) = \sum_{i=1}^{n} c_1(x_i)$. On the other hand, by Proposition 2.1.2(ii):

$$c_1(\det \rho) = c_1\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} c_1(x_i) = c_1(\rho).$$

□

Moreover, the splitting principle, together with property (ii) in Proposition 2.1.2, imply that $c_k(\sigma\tau)$ is a polynomial in the Chern classes of σ and τ. As a direct consequence, we have:

Lemma 2.1.5. *Let $\chi_1, \cdots, \chi_n$ be characters of representations of G of degrees $d_1, \cdots, d_n$ respectively. If the χ_i generate $R(G)$ as a ring, then the classes $c_k(\chi_i)$ for $1 \leq k \leq d_i$ generate $R^*(G)$ as a ring.*

Proof. By definition, each Γ^n is generated by products of Chern classes of virtual characters of G. The result follows from the above discussion. □

We conclude this section with the following improvement on [GM14, Lem. 3.2]:

Proposition 2.1.6. *The graded piece $R^n(G)$ is $|G|$-torsion for $n > 0$.*

Proof. Consider the regular representation $\mathbb{K}G$ of G, with character χ, and let ρ be any character of G. For any $g \in G$,

$$\chi \cdot \rho(g) = (\varepsilon(\rho) \cdot |G|)\delta_{1_G,g}.$$

Pick a virtual character $\rho \in \Gamma^n$ for $n > 0$, and write $\rho = \rho^+ - \rho^-$ with ρ^+, $\rho^- \in R^+(G)$ and $\varepsilon(\rho^+) = \varepsilon(\rho^-)$. Then $\chi \cdot \rho^+ = \chi \cdot \rho^-$ and thus $\chi \cdot \rho = 0$. Looking modulo Γ^{n+1}, we obtain:

$$0 = \chi \cdot \rho = (\chi - |G|)\rho + |G| \cdot \rho = |G| \cdot \rho \pmod{\Gamma^{n+1}},$$

since $\chi - |G| \in I = \Gamma^1$. □

2.2 Computing from the definition: cyclic groups

As introductory examples, we determine the graded character rings of some cyclic groups. In Proposition 2.2.1, we consider cyclic groups over an algebraically closed field: their graded character ring was computed in [GM14] and many of our subsequent examples will rely on it. For the sake of completeness, we reproduce here the calculation of Guillot and Mináč, which is an exercise in the definitions.

In Proposition 2.2.2, we prove a surprising general result about graded character rings of p-groups over the rationals: the classical interplay between Adams operations and rationality (see [Ser77, Th. 13.29]), translates to a condition on the generators of $R^*_{\mathbb{Q}}(G)$. We illustrate this statement in Corollary 2.2.3 with the computation of $R^*_{\mathbb{Q}}(C_p)$, where C_p is a cyclic group of prime order. This constitutes our only incursion outside the field of complex numbers; even the computation for cyclic groups of arbitrary order remains wide open over a general field.

Proposition 2.2.1 ([GM14, Prop. 3.4])**.** *Let C_N be the cyclic group of order*

N. Whenever $\mathbb{K}$ is an algebraically closed field of characteristic prime to N,

$$R^*_{\mathbb{K}}(C_N) = \frac{\mathbb{Z}\left[x\right]}{(Nx)}$$

with $x = c_1(\rho)$ for a one-dimensional representation ρ of C_N that generates $R_{\mathbb{K}}(C_N)$.

Proof. Let ρ be a generating character for $R(G)$. By Lemma 2.1.5, its first chern class $x = c_1(\rho)$ generates $R^*(G)$, and by Proposition 2.1.6 we have $Nx = 0$. It remains to show that there is no additional relation in $R^*(G)$, so suppose that $dx^n = 0$ for some d; that is, $d(\rho - 1)^n \in \Gamma^{n+1}(G)$. We show that necessarily N divides d. Note that since G is cyclic, the augmentation ideal is $I = (\rho - 1)$ and the Grothendieck filtration coincides with the I-adic filtration. Let $X = C_1(\rho)$, then the relation $dx^n = 0$ lifts to $dX^n = X^{n+1}P(X)$ in $R(G)$ for some polynomial $P \in \mathbb{Z}[X]$; we can then lift this relation to $\mathbb{Z}[X]$ as:

$$dX^n = P(X)X^{n+1} + Q(X)\left((X+1)^N - 1\right),$$

for some polynomial $Q(X)$. If $n > 1$, by considering the terms of degree 1 on each side, we conclude that $Q(0) = 0$. We can then divide by X and get a similar equation, with dX^{n-1} on the left. We repeat this process until we reach an equation of the form

$$dX = P(X)X^2 + Q(X)\left((X+1)^N - 1\right).$$

By looking again at terms of degree 1, we see that $d = NQ(0)$, which is what we wanted. □

Before we move on to more involved computations, here is a rather nice application of Proposition 2.1.6 to graded character rings over the rationals. The proof of the following requires the use of Adams operations; they are λ-homomorphisms that exist on any λ-ring, whose precise definition and main properties are outlined in [AT69, §5]. For our purposes, it suffices to know that for a character of G, the k-th Adams operation is defined as $\psi^k(\chi(g)) = \chi(g^k)$.

Proposition 2.2.2. *Let G be a p-group. Then $R^*_{\mathbb{Q}}(G)$ is concentrated in degrees multiple of $(p-1)$.*

Proof. By [AT69, Prop. 5.3], for $x \in \Gamma^n$, we have $\psi^k(x) = k^n x \pmod{\Gamma^{n+1}}$. Moreover, by [Ser77, Th. 13.29], over the rationals, $\psi^k(x) = x$ whenever $(|G|, k) = 1$. In particular, picking any $k \in (\mathbb{Z}/p\mathbb{Z})^\times$ we have $(k^n - 1)x \in \Gamma^{n+1}$. Since x is $|G|$-torsion, we conclude that $x = 0$ whenever $(k^n - 1) \neq 0 \pmod{p}$, that is, whenever n is not a multiple of $(p-1)$. □

A straightforward application of this result is the computation of $R^*_{\mathbb{Q}}(G)$ for G cyclic of prime order p.

Corollary 2.2.3.

$$R^*_{\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) = \frac{\mathbb{Z}[x]}{(px)}$$

with $x = c_{p-1}(\chi)$ where χ is the character of $\mathbb{Q}[\mathbb{Z}/p\mathbb{Z}]$.

Proof. Let $G = \mathbb{Z}/p\mathbb{Z}$. By [Ser77, Prop. 13.30 and Ex. 13.1], the ring $R_{\mathbb{Q}}(G)$ is generated by the characters of permutation representations of the subgroups of G. The only two subgroups of G are the trivial group $\{0\}$ and G itself, so $R_{\mathbb{Q}}(G)$ is generated by χ, the regular representation. So $R^*_{\mathbb{Q}}(G)$ is in turn generated by $c_i(\chi)$ for $1 \leq i \leq p$; by Proposition 2.2.2, it is generated by $c_{p-1}(\chi)$. It remains to show that this generator has additive order p and is non-nilpotent. Consider the natural map $R_{\mathbb{Q}}(G) \to R_{\mathbb{C}}(G)$; it sends χ to

$1 + \rho + \cdots + \rho^{p-1}$, with ρ a generating character of $R_{\mathbb{C}}(G)$. The total Chern class of $1 + \rho + \cdots + \rho^{p-1}$ is

$$c_t\left(\sum_{i=0}^{p-1} \rho^i\right) = \prod_{i=0}^{p-1} c_t(\rho^i) = \prod_{i=0}^{p-1} (1 + ic_1(\rho)),$$

so $c_{p-1}(\rho) = (p-1)! \cdot c_1(\rho)$, which proves our claim using Proposition 2.2.1. □

The graded ring of a general cyclic group over the rationals is not known. In the sequel, unless mentioned otherwise, all graded character rings will be computed over the complex numbers.

2.3 The restriction homomorphism

Graded character rings are computed in two steps: first, we identify a minimal set of generators for $R^*(G)$ using general information on the representation theory of G, and we determine relations in higher degree via Chern class algebra. The second step consists in showing that there are no extra relations in $R^*(G)$, and is usually much less straightforward. In the case of cyclic groups (see Proposition 2.2.1), we used an ad hoc method for this step; in this section we rely on the functoriality of $R^*(-)$ to look at restrictions of representations to subgroups of G. We rely on this technique, and on Proposition 2.2.1, to compute the graded character rings of elementary abelian groups in Proposition 2.3.3. We then turn to the dihedral groups D_p for odd primes p in Proposition 2.3.4, and to D_4 in Proposition 2.3.5. In passing, we prove a Künneth formula for groups of coprime order.

Lemma 2.3.1. *The graded character ring $R^*(-)$ is functorial in G: a group homomorphism $\phi : G \to H$ induces a well-defined ring map $\phi^* : R^*(H) \to$*

$R^*(G)$, which sends each generator $c_n(\rho)$ to $c_n(\rho \circ \phi)$.

In particular, if H is a subgroup of G, the restriction of representation $\mathrm{Res}_H^G : R(G) \to R(H)$ induces a well-defined homomorphism of graded character rings, also denoted $\mathrm{Res}_H^G : R^*(G) \to R^*(H)$, with

$$\mathrm{Res}_H^G c_n(x) = c_n(\mathrm{Res}_H^G(x))$$

for all $x \in R(G), n \in \mathbb{N}$.

Proof. This is clear. $\square$

A powerful consequence of Lemma 2.3.1 is to reduce the computation of graded character rings of abelian groups to that of $R^*(G)$ for p-groups:

Corollary 2.3.2. *Let G and H be groups with coprime order. Then*

$$R^*_{\mathbb{C}}(G \times H) = R^*_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} R^*_{\mathbb{C}}(H)$$

Proof. Let $\pi_G, \pi_H : G \times H \to G, H$ be the projection maps. By [Ser77, Th. 3.10], for any complex irreducible character ρ of $G \times H$, there are irreducible characters σ_G, σ_H of G, H respectively such that $\rho = (\sigma_G \circ \pi_G) \cdot (\sigma_H \circ \pi_H)$. Let $\rho_G = \sigma_G \circ \pi_G$ and $\rho_H = \sigma_H \circ \pi_H$, then $R^*_{\mathbb{C}}(G \times H)$ is generated by classes of the form $c_n(\rho_G \cdot \rho_H)$, which can be written as polynomials in the Chern classes of ρ_G, ρ_H. In other words, the projection maps π_G, π_H induce a surjective homomorphism

$$\pi_G^* \otimes \pi_H^* : R^*_{\mathbb{C}}(G) \otimes R^*_{\mathbb{C}}(H) \to R^*_{\mathbb{C}}(G \times H).$$

Moreover, applying Proposition 2.1.6 to the case where $|G|, |H|$ are coprime, we have that

$$\bigoplus_{i+j=n}\Big(R^i(G)\otimes R^j(H)\Big)\cong R^n_{\mathbb{C}}(G)\oplus R^n_{\mathbb{C}}(H)$$

for any $n\geq 1$. In particular, the surjection above decomposes as $\bigoplus(\pi_G^*)^n\otimes(\pi_H^*)^n : R^n_{\mathbb{C}}(G)\oplus R^n_{\mathbb{C}}(H)\to R^*_{\mathbb{C}}(G\times H)$. The inclusions $\iota_G,\iota_H : G,H\to G\times H$ induce a two-sided inverse $(\iota_G^*,\iota_H^*)=\bigoplus_n(\iota_G^n,\iota_H^n) : R^n_{\mathbb{C}}(G\times H)\to R^n_{\mathbb{C}}(G)\oplus R^n_{\mathbb{C}}(H)$ to this surjection. $\square$

We now use Lemma 2.3.1 to compute of the graded character rings of elementary abelian groups. Let p be a prime number, and let C_p be the cyclic group of order p, with a choice of generator g. Recall that we fixed $\mathbb{K}=\mathbb{C}$.

Proposition 2.3.3. *Let $G=C_p^k$. Then*

$$R^*(G)=\frac{\mathbb{Z}[x_1,\cdots,x_k]}{(px_i,x_i^px_j-x_ix_j^p)}.$$

with $x_i=c_1(\rho_i)$ where ρ_i restricts to a nontrivial one-dimensional representation of the i-th factor C_p.

Proof. Denote by g_i the element $(1,\cdots,1,g,1,\cdots,1)$ with g in i-th position, so that G is generated by $g_1,\cdots,g_k$. Let $\omega=\exp^{2i\pi/p}$, and let ρ_i be the representation of G defined by $\rho_i : g_j\mapsto\omega^{\delta_{ij}}$. The ρ_i's generate $R(G)$, so the elements $x_i:=c_1(\rho_i)$ generate $R^*(G)$. Note that $\rho_i^p=1$ for all i, so $px_i=0$ by Proposition 2.1.2(ii).

The relation $x_i^px_j=x_ix_j^p$ is obtained as follows: let X_i be the standard lift ρ_i-1 of x_i to $R(G)$. Then $(X_i+1)^p=1$, so that

$$\begin{aligned}X_i^p=-\sum_{l=1}^{p-1}\binom{p}{l}X_i^l&=X_i\left(-\sum_{l=0}^{p-2}\binom{p}{l+1}X_i^l\right)\\&=pX_i(-1+\phi(X_i)),\end{aligned}$$

where $\phi(T) \in \mathbb{Z}[T]$ has no constant term. For any i, j, write

$$\begin{aligned} X_i^p X_j \left(-1 + \phi(X_j)\right) &= pX_i X_j \left(-1 + \phi(X_i)\right)\left(-1 + \phi(X_j)\right) \\ &= X_i X_j^p \left(-1 + \phi(X_i)\right). \end{aligned}$$

In $R^{p+1}(G)$, this is:

$$x_i^p x_j = x_i x_j^p,$$

and the generators of $R^*(G)$ satisfy all the required relations.

Let us show that there are no extra relations: the graded piece of rank l is generated by monomials of the form:

$$x_1^{s_1} \cdots x_k^{s_k}, \qquad \sum_{i=1}^{k} s_i = l.$$

Let $S_l \subset \mathbb{Z}_{\geq 0}^k$ be the set of multi-indices $(s_1, \cdots, s_k)$ such that $\sum s_i = l$ and only the first nonzero coordinate of each $s \in S_l$ is (possibly) greater than $p-1$. We must show that the monomials $x^s = x_1^{s_1} \cdots x_k^{s_k}$ are linearly independent. Consider a zero linear combination:

$$\sum_{s \in S_l} a_s x^s = 0 \tag{2.3.1}$$

and let $\psi : R^*(C_p^k) \to R^*(C_p)$ be the restriction to the cyclic group generated by the product $g_1^{t_1} \cdots g_k^{t_k}$ for some $0 \leq t_j \leq p-1$. Then $\psi(x_j) = t_j \cdot z$, where z is the standard one-degree generator of $R^*(C_p)$, and Equation (2.3.1) becomes:

$$\sum_{s \in S_l} a_s t_1^{s_1} \cdots t_k^{s_k} z^l = 0,$$

that is,

$$\sum_{s\in S_l} a_s t_1^{s_1}\cdots t_k^{s_k} = 0 \in \mathbb{F}_p, \tag{2.3.2}$$

for all possible strings $(t_1,\cdots,t_k)$ with $0\le t_j\le p-1$. In particular, grouping terms by powers of t_k in Equation (2.3.2), we get:

$$\begin{cases} a_{(0,\cdots,0,l)}t_k^l = 0 & \text{when } t_1=\cdots=t_{k-1}=0 \\ \sum_{t=0}^{p-1}\left(\sum_{s\in S_{l-i}} b_s t_1^{s_1}\cdots t_{k-1}^{s_{k-1}}\right)t_k^i = 0 & \text{otherwise.}\end{cases} \tag{2.3.3}$$

This implies that the coefficient of x_k^l in Equation (2.3.1) is zero; more generally, the second equation must be true for all values of t_k, from 0 to $p-1$. In other words, the $\left(\sum b_s t_1^{s_1}\cdots t_{k-1}^{s_{k-1}}\right)$ are the entries of a vector in the kernel of the Vandermonde matrix $(t_k^i)_{t_k=1,\cdots,p-1}^{i=1,\cdots,p-1}$, which is invertible in $\mathbb{F}_p$. Therefore

$$\sum_{s\in S_{l-i}} b_s t_1^{s_1}\cdots t_{k-1}^{s_{k-1}} = 0$$

for all combinations $(t_1,\cdots,t_{k-1})$. An immediate induction shows that we must have each $a_s=0$, so the monomials $\{x^s\}_{s\in S_l}$ are linearly independent. □

Note that the relations between the generators of $R^*((C_p)^k)$ appear in degree $p+1$, so the degree of relations goes to ∞ as $p\to\infty$. In Section 2.4 we shed light on this phenomenon, via a general result about the minimal degree of relations in a p-group.

We now turn to our first non-abelian group, for which the computation combines the restriction map with some basic Chern class algebra. Let p be an odd prime and consider the dihedral group:

$$D_p = \left\langle \tau,\sigma | \tau^2=\sigma^p=1, \tau\sigma\tau=\sigma^{-1}\right\rangle.$$

There are $(p+1)/2$ irreducible representations of D_p:

- Two representations of degree 1, the trivial representation 1 and the signature ε which sends elements of the form σ^j to 1, and elements of the form $\tau\sigma^j$ to -1.
- And $(p-1)/2$ representations $\chi_1, \cdots \chi_{(p-1)/2}$ of degree 2:

$$\chi_k(\sigma^j) = \begin{pmatrix} e^{\frac{2ikj\pi}{p}} & 0 \\ 0 & e^{-\frac{2ikj\pi}{p}} \end{pmatrix}$$
$$\chi_k(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characters of these generate the ring $R(D_p)$. For convenience, define $\chi_0 = 1 + \varepsilon$; we have the following relations:

$$\varepsilon^2 = 1 \tag{2.3.4}$$
$$\varepsilon \cdot \chi_k = \chi_k \tag{2.3.5}$$
$$\chi_k \cdot \chi_l = \chi_{k+l} + \chi_{k-l}. \tag{2.3.6}$$

Proposition 2.3.4. *Let $x = c_1(\chi_1)$ and $y = c_2(\chi_1)$, then*

$$R^*(D_p) = \frac{\mathbb{Z}[x,y]}{(2x, py, xy)}.$$

Proof. Let x and y be as above; note that Lemma 2.1.4 implies that $x = c_1(\chi_1) = c_1(\chi_k) = c_1(\det \chi_k) = c_1(\varepsilon)$ for any k, and $0 = c_1(\varepsilon^2) = 2c_1(\varepsilon) = 2x$. For the other relations, we use Equation (2.3.6) above and apply the total

Chern class c_t to both sides:

$$c_t(\chi_k\chi_l) = c_t(\chi_{k+l})c_t(\chi_{k-l}). \tag{2.3.7}$$

Let $y_i = c_2(\chi_i)$. Expand the right-hand side:

$$\begin{aligned} c_t(\chi_{k+l})c_t(\chi_{k-l}) =&(1 + xT + y_{k+l}T^2)(1 + xT + y_{k-l}T^2) \\ =&1 + 2xT + (x^2 + y_{k+l} + y_{k-l})T^2 \\ &+ (xy_{k+l} + xy_{k-l})T^3 + y_{k+l}y_{k-l}T^4. \end{aligned} \tag{2.3.8}$$

For the left-hand side, we use the splitting principle (Proposition 2.1.3): in some extension of $R(D_p)$, we can write $\chi_k = \rho_1 + \rho_2$ and $\chi_l = \eta_1 + \eta_2$ with ρ_i, η_i of dimension 1, in a way that is compatible with the Γ-filtration. Then:

$$\begin{aligned} c_t(\chi_k\chi_l) =&c_t\left((\rho_1 + \rho_2)(\eta_1 + \eta_2)\right) \\ =&c_t(\rho_1\eta_1)c_t(\rho_1\eta_2)c_t(\rho_2\eta_1)c_t(\rho_2\eta_2) \\ =&(1 + (c_1(\rho_1) + c_1(\eta_1))T) \cdot (1 + (c_1(\rho_1) + c_1(\eta_2))T) \\ &\cdot (1 + (c_1(\rho_2) + c_1(\eta_1))T) \cdot (1 + (c_1(\rho_2) + c_1(\eta_2))T). \end{aligned}$$

Now, let s_1, s_2 (resp. t_1, t_2) be the first and second symmetric polynomials in (ρ_1, ρ_2) (resp. (η_1, η_2)). Then $c_i(\chi_k) = s_i$ and $c_i(\chi_l) = t_i$. The last equality can be rewritten:

$$\begin{aligned} c_t(\chi_k\chi_l) =&1 + 2(s_1 + t_1)T + (t_1^2 + s_1^2 + 3s_1t_1 + 2t_2 + 2s_2)T^2 \\ &+ (s_1^2t_1 + s_1t_1^2 + 2s_1s_2 + 2t_1t_2 + 2s_1t_2 + 2s_2t_1)T^3 \\ &+ (t_2^2 + s_2^2 + s_1s_2t_1 + s_1t_1t_2 + s_1^2t_2 + s_2t_1^2 - 2s_2t_2)T^4. \end{aligned}$$

We replace $s_1 = t_1 = x$ and eliminate all occurrences of $2x$ to obtain

$$c_t(\chi_k\chi_l) = 1 + (x^2 + 2y_k + 2y_l)T^2 + (y_k - y_l)^2T^4. \tag{2.3.9}$$

Comparing coefficients in Equation (2.3.8) and Equation (2.3.9), we obtain:

$$y_{k+l} + y_{k-l} = 2(y_k + y_l) \tag{2.3.10}$$

$$xy_{k+l} + xy_{k-l} = 0 \tag{2.3.11}$$

$$y_{k+l}y_{k-l} = (y_k - y_l)^2. \tag{2.3.12}$$

First look at Equation (2.3.11) with $k = l$. Note that $y_0 = c_2(1 + \varepsilon) = 0$, and thus Equation (2.3.11) yields $x \cdot y_{2k} = 0$ for all k, which is equivalent to $x \cdot y_k = 0$ for all k since indices are understood modulo the odd prime p.
We then show that $py_k = 0$ for all k. Recall that $R^*(D_p)$ is $2p$-torsion, and consider Equation (2.3.10) with $k = l$. Multiplying by p, we obtain $py_{2k} = 0$ for all k. Again, this implies that $py_k = 0$ for all k.
Finally, consider Equation (2.3.10) with $l = k$ and $l = k + 1$. This gives:

$$y_{2k} = 4y_k$$

$$y_{2k+1} = 2(y_k + y_{k+1}) - y_1.$$

Together, these two relations imply that all y_k's are multiples of $y_1 =: y$.
It remains to show that these are the only relations in $R^*(D_p)$, that is, x and y are not nilpotent, and there is no extra dependency relation between them. Restricting x to C_2 and y to C_p shows none of the generators are nilpotent, while restricting both x and y to C_2 eliminates any extra possible relation. $\square$

A similar argument gives $R^*(D_4)$, where D_4 is the dihedral group D_4 of order 8. Note that $R^*(D_4)\otimes\mathbb{F}_2$ is already known and was computed in [GM14, Prop. 3.12]. It has four nontrivial irreducible representations:

- In degree 1, the representations $\rho : r \mapsto -1,\ s \mapsto 1$ and $\eta : r \mapsto 1, s \mapsto -1$ and their product $\rho\eta$,
- And in degree 2, the representation Δ, which sends s to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and r to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

with relations:

$$\rho^2 = \eta^2 = 1 \tag{2.3.13}$$

$$\rho\Delta = \eta\Delta = \Delta \tag{2.3.14}$$

$$\Delta^2 = 1 + \rho + \eta + \rho\eta. \tag{2.3.15}$$

Proposition 2.3.5. *Let $c_1(\rho) = x$; $c_1(\eta) = y$ and $c_2(\Delta) = b$. Then*

$$R^*(D_4) = \frac{\mathbb{Z}\,[x, y, b]}{(2x, 2y, 4b, xy, xb - yb)}.$$

Proof. Note that $c_1(\rho\eta) = x+y$ and $c_1(\Delta) = c_1(\det\Delta) = c_1(\rho\eta)$. So the graded ring is indeed generated by x, y, b. We have $2x = 2y = 0$ from the relations above; and, letting X, Y, B being the standard lifts $C_1(\rho), C_1(\eta), C_2(\Delta)$ of x, y and b to $R(D_4)$, we compute that $XB = YB = XY$. So $XY \in \Gamma^3$, thus $xy = 0$ and $xb = yb$. Finally, applying the total Chern class to Equation (2.3.15) yields the equation:

$$5c_1(\Delta)^2 + 4b = x^2 + 3xy + y^2 = x^2 + y^2 = (x + y)^2$$

and since $c_1(\Delta) = x + y$, we obtain $4b = 0$.

To see these are the only relations, we use the computation of $R^*(D_4) \otimes \mathbb{F}_2 = \frac{\mathbb{Z}[x,y,b]}{(xy, xb-yb)}$ from [GM14]: tensoring with $\mathbb{F}_2$ shows that none of x, y, b is nilpotent and that there are no extra relations between the genrerators. Finally, restriction to $C_4 = \langle r \rangle$ shows that any power b^i of b has additive order 4. $\square$

2.4 Universal enveloping algebras

The aim of this section is to construct, for any abelian p-group G, a map:

$$R^*(G) \otimes \mathbb{F}_p \to gr_\bullet(\mathbb{F}_p G),$$

where $gr_\bullet \mathbb{F}_p G$ is the graded ring associated to the filtration of the group ring $\mathbb{F}_p G$ by powers of its augmentation ideal. To this effect, we apply the main result of [Qui68]: fix a prime p, and let $\{G_n\}$ denote the lower central series of G, defined by $G_1 = G$ and $G_{n+1} = (G_n, G)$. Consider the sequence $\{D_n\}$, where D_n is the n-th mod p dimension subgroup of G:

$$D_n := \prod_{ip^s \geq n} G_i^{p^s}.$$

Then $\{D_n\}$ is a p-filtration of G, that is, it satisfies:

- $(D_r, D_s) \subseteq D_{r+s}$
- $x \in D_r \implies x^p \in D_{pr}$ for all r.

Moreover, $\{D_n\}$ is the fastest descending p-filtration of G (see [DdSMS99, §11.1]). Set $L_\bullet(G) = \bigoplus D_n/D_{n+1}$, then $L_\bullet(G)$ is a p-restricted graded Lie algebra over $\mathbb{F}_p$. On the other hand, if I denotes the augmentation ideal of the group ring $\mathbb{F}_pG$, then

$$F_n := \{x \in G \mid x - 1 \in I^n\}$$

is also a p-filtration of G, thus $F_n \supset D_n$ and there is a map of Lie algebras:

$$\psi : \begin{cases} L_\bullet(G) & \to gr_\bullet(\mathbb{F}_pG) \\ g \pmod{F^n} & \mapsto (g-1) \pmod{I^n}. \end{cases}$$

Theorem 2.4.1 ([Qui68, §1])**.** *The homomorphism $\widehat{\psi}$ from $gr_\bullet(\mathbb{F}_pG)$ to the universal enveloping algebra $U(L_\bullet(G))$ induced by ψ is an isomorphism.*

Now suppose G is an abelian p-group of the form $C_{p^{i_1}} \times \cdots \times C_{p^{i_m}}$. Then $D_n = G^{p^i}$ for p^i the smallest power of p such that $p^i \geq n$. Thus

$$L_n(G) \cong \begin{cases} \{1\}, & n \neq p^i \\ C_p \times \cdots \times C_p, & n = p^i. \end{cases}$$

Lemma 2.4.2. *If G is abelian then $R(G) \otimes \mathbb{F}_p \cong \mathbb{F}_pG$ through an isomorphism that sends the Grothendieck filtration $\{\Gamma_p^n\}_n$ induced on $R(G) \otimes \mathbb{F}_p$ to the I-adic filtration on $\mathbb{F}_pG$.*

Proof. Write G as a product of cyclic groups. The isomorphism that sends each cyclic group generator g to the character $\rho_g : g \mapsto e^{2\pi i/|g|}$, sends $I \subset \mathbb{F}_p$ to Γ_p^1. Since every irreducible character of G has dimension 1, the filtration $\{\Gamma_p^n\}_n$ coincides with the Γ_p^1-adic filtration. $\square$

So $gr_\bullet(R(G)\otimes\mathbb{F}_p)\cong gr_\bullet(\mathbb{F}_pG)\cong U(L_\bullet(G))$ with universal map

$$h:\begin{cases} L_\bullet(G) & \to gr_\bullet(R(G)\otimes\mathbb{F}_p)\\ g & \mapsto C_1(\rho_g) \pmod{\Gamma_p^2}\end{cases},$$

and there is a map ϕ of algebras induced by $L_\bullet(G)\to gr_\bullet(\mathbb{F}_pG)$:

$$\phi: gr_\bullet(R(G)\otimes\mathbb{F}_p)\to gr_\bullet(\mathbb{F}_pG).$$

On the other hand, the map $R(G)\to R(G)\otimes\mathbb{F}_p$ preserves the Γ-filtration and induces a maps $R^*(G)\to gr_\bullet(R(G)\otimes\mathbb{F}_p)$, and thus a map $R^*(G)\otimes\mathbb{F}_p\to gr_\bullet(R(G)\otimes\mathbb{F}_p)$. Composing this latter map with ϕ, we obtain a map

$$R^*(G)\otimes\mathbb{F}_p\to gr_\bullet(\mathbb{F}_pG)$$

satisfying $\phi(c_1(\rho_g))=g-1$.

A straightforward corollary of this is the following:

Theorem 2.4.3. *Let $G=C_{p^{i_1}}\times\cdots\times C_{p^{i_n}}$, and let ρ_k be the generating character of $R(C_{p^{i_k}})$ sending a generator g_k of $C_{p^{i_k}}$ to $e^{2i\pi/p^{i_k}}$. Then there is a well-defined homomorphism:*

$$R^*(G)\otimes\mathbb{F}_p\to\frac{\mathbb{F}_p[u_1,\cdots,u_n]}{(u_1^{p^{i_1}},\cdots,u_n^{p^{i_n}})}$$

sending $c_1(\rho_k)$ to u_k.

Although we do not directly refer to it in the sequel, Theorem 2.4.3 proves useful when "guessing" relations in $R^*(G)$, as illustrated by the graded character rings of abelian 2-groups: in Proposition 2.6.2, we show that

$$R^*(C_4 \times C_4) = \frac{\mathbb{Z}[x,y]}{(4x, 4y, 2x^2y + 2xy^2, x^4y^2 - x^2y^4)}$$

with $x = c_1(\rho_{(1,0)})$, $y = c_1(\rho_{(0,1)})$. By Theorem 2.4.3, modulo 2, nontrivial relations must involve x^4 or y^4. Since one can easily rule out relations of the form $x^4, y^4 = 0$ by restriction to C_4, we know that any extra relation will occur in degree 5 or more. Here, it occurs in degree 6. Again, in Proposition 2.6.1, we show

$$R^*(C_4 \times C_2) = \frac{\mathbb{Z}[x,y]}{(4x, 2y, xy^3 + x^2y^2)}$$

with $x = c_1(\rho_{(1,0)})$, $y = c_1(\rho_{(0,1)})$. We know by Theorem 2.4.3 that any nontrivial relation modulo 2 must involve x^2 or y^4.

2.5 Continuity of characters

In the sequel, we view $R(G)$ as a topological ring, with the topology induced by the filtration $\{\Gamma^n\}$; that is, a subset $U \subseteq R(G)$ is open if for any $x \in U$ there is a t such that $x + \Gamma^t \subseteq U$. If G has exponent m, then each conjugacy class representative $g \in G$ gives rise to a ring morphism:

$$\phi_g : \begin{cases} R(G) \to \mathbb{Z}[\mu_m] \\ \rho \mapsto \chi_\rho(g) \end{cases},$$

where μ_m is a choice of primitive m-th root of unity. We are interested in continuity and density questions with respect to p-adic topologies on $\mathbb{Z}$. Note that, to make any kind of rigorous statement, we need to fix an extension of the p-adic valuation to $\mathbb{Z}[\mu_m]$. However, we are primarily interested in the case where m is a power of p; in that case, as is well-known, there is only one such extension. In particular, Proposition 2.5.3 states that whever G is a p-group, all evaluation morphisms are continuous. We apply this result in Theorem 2.5.4 to the computation of $R^*(Q_8)$.

Suppose we are given additive groups $\widetilde{\Gamma}^n \subseteq \Gamma^n$ $(n \geq 1)$ such that:

A. $\widetilde{\Gamma}^{n+1} \subseteq \widetilde{\Gamma}^n$

B. $\Gamma^n = \widetilde{\Gamma}^n + \Gamma^{n+1}$

(think of $\widetilde{\Gamma}^n$ as an approximation of Γ^n). Then by an immediate induction:

Lemma 2.5.1. *For all $k \in \mathbb{N}$,*

$$\Gamma^n = \widetilde{\Gamma}^n + \Gamma^{n+k}$$

$\square$

Call $\{\widetilde{\Gamma}^n\}_n$ an *admissible approximation* for $\{\Gamma^n\}_n$ if it satisfies conditions (A) and (B).

Remark. Whenever $\{\widetilde{\Gamma}^n\}$ is an admissible approximation, each $\widetilde{\Gamma}^n$ is dense in Γ^n for the Γ-topology.

Proposition 2.5.2. *Let p be a prime number, and suppose the evaluation morphisms*

$$\phi_1, \cdots \phi_k : R(G) \mapsto \mathbb{Z}[\mu_m]$$

are continuous with respect to the topology induced by the filtration $\{\Gamma^n\}$ on $R(G)$, and the p-adic topology on $\mathbb{Z}[\mu_m]$. Then for all $x \in \Gamma^n$, and for all

$M > 0$, there is an element $\widetilde{x} \in \widetilde{\Gamma}^n$ such that for all $i = 1, \cdots, k$:

$$\begin{cases} v_p(\phi_i(\widetilde{x})) = v_p(\phi_i(x)) \text{ whenever } v_p(\phi_i(x)) < +\infty \\ v_p(\phi_i(\widetilde{x})) > M \text{ whenever } \phi_i(x) = +\infty \end{cases}$$

Proof. Let $x \in \Gamma^n$, $M > 0$. Since all the ϕ_i are continuous with respect to the p-adic topology, there exists N such that for all j and for all $y \in \Gamma^N$ we have

$$v_p(\phi_j(y)) > \max\left(\max_{v_p(\phi_i(x))<\infty} v_p(\phi_i(x)),\ M\right).$$

We can then write $x = \widetilde{x} + r$ with $\widetilde{x} \in \widetilde{\Gamma}^n$ and $r \in \Gamma^N$. □

Proposition 2.5.3. *Let G be a p-group. Then the morphisms ϕ_g, for $g \in G$, are all continuous with respect to the p-adic topology on $\mathbb{Z}[\mu_m]$.*

Proof. Fix an element $g \in G$ and let $|G| =: p^n$. By Proposition 2.1.1, it suffices to show that ϕ_g is continuous with respect to the I-adic topology on the left. We show that for any irreducible character ρ of G,

$$v_p\left(\phi_g\left(\rho - \varepsilon\left(\rho\right)\right)\right) > 0,$$

which implies continuity. Since G is a p-group, every character is a sum of p^n-th roots of unity, so

$$\begin{aligned} \phi_g(\rho - \varepsilon(\rho)) &= \rho(g) - \varepsilon(\rho) \\ &= \sum_{l=1}^{\varepsilon(\rho)} (\mu_{p^n}^{i_l} - 1), \end{aligned}$$

and each $(\mu_{p^n}^{i_l} - 1)$ has positive p-valuation. □

The continuity method allows us to solve questions of torsion and nilpotency in $R^*(G)$: if some element $x \in R(G)$ is contained in Γ^M with M large, then $\phi_g(x)$ must be divisible by a large power of p. Here is a concrete example: let $G = Q_8 = \langle\, i, j, k \mid i^2 = j^2 = k^2 = ijk \,\rangle$ be the quaternion group of order 8.

Theorem 2.5.4.

$$R^*(Q_8) = \frac{\mathbb{Z}[x, y, u]}{(2x, 2y, 8u, x^2, y^2, xy - 4u)}$$

where $|x| = |y| = 1$ *and* $|u| = 2$*, with explicit generators as described in Lemma 2.5.5.*

We first show that the generators of $R^*(Q_8)$ do satisfy the relations above; we then prove that these are the only relations in the graded rings. For the second step, we use continuity of characters to show that the additive order of u is 8 .

The group Q_8 has 5 conjugacy classes: $\{1\}$, $\{-1\}$, $\{\pm i\}$, $\{\pm j\}$, $\{\pm k\}$ so 5 irreducible representations on $\mathbb{C}$. They are as follows:

- In dimension 1, the trivial representation, and the characters $\rho_1 : \begin{cases} i \mapsto 1 \\ j \mapsto -1 \end{cases}$, $\rho_2 = -\rho_1$ and $\rho_3 = \rho_1\rho_2$,
- and in dimension 2, the representation Δ:

$$\Delta(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \Delta(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Delta(k) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

These representations satisfy the relations

$$\rho_i^2 = 1 \tag{2.5.1}$$

$$\rho_3 = \rho_1\rho_2 \tag{2.5.2}$$

$$\Delta\rho_i = \Delta \tag{2.5.3}$$

$$\Delta^2 = 1 + \rho_1 + \rho_2 + \rho_3. \tag{2.5.4}$$

Let us first take a look at the generators and relations of Q_8:

Lemma 2.5.5. *The graded ring $R^*(Q_8)$ is generated by the elements*

$$x := c_1(\rho_1), \quad y := c_1(\rho_2), \quad u := c_2(\Delta),$$

which satisfy the relations in Theorem 2.5.4, that is:

$$2x = 2y = 8u = 0, \quad x^2 = y^2 = 0, \quad xy = 4u.$$

Proof. First, we eliminate the redundant generators: $c_1(\rho_3) = c_1(\rho_1\rho_2) = c_1(\rho_1) + c_1(\rho_2)$, and $c_1(\Delta) = c_1(\det(\Delta)) = c_1(1) = 0$. So $R^*(Q_8)$ is indeed generated by x, y and u.

Now since $\rho_1^2 = \rho_2^2 = 1$, we have $2x = 2y = 0$, and the order of Q_8 kills $R^*(Q_8)$ so $8u = 0$. For the relations in degree 2, we apply the total Chern class to Equation (2.5.3), splitting the 2-dimensional representation Δ into $\sigma_1 + \sigma_2$. On the left-hand side we have:

$$\begin{aligned} c_t(\Delta\rho_i) =& c_t(\sigma_1\rho_i)c_t(\sigma_2\rho_i) \\ =& 1 + [c_1(\sigma_1) + c_1(\sigma_2) + 2c_1(\rho_i)]\, T \\ &+ \left[c_1(\sigma_1)c_1(\sigma_2) + c_1(\sigma_1)c_1(\rho_i) + c_1(\sigma_2)c_1(\rho_i) + c_1(\rho_i)^2\right] T^2 \end{aligned}$$

While on the right-hand side:

$$c_t(\Delta) = 1 + [c_1(\sigma_1) + c_1(\sigma_2)]\,T + [c_1(\sigma_1)c_1(\sigma_2)]\,T^2.$$

In degree 2, this yields the relation:

$$c_1(\rho_i)(c_1(\sigma_1) + c_1(\sigma_2)) + c_1(\rho_i)^2 = 0.$$

Bearing in mind that $c_1(\sigma_1)+c_1(\sigma_2) = c_1(\Delta) = 0$, we obtain $x^2 = y^2 = 0$. The relation $xy = 4u$ is obtained by applying c_t to the relation $\Delta^2 = 1+\rho_1+\rho_2+\rho_1\rho_2$ and identifying the terms in degree 2, which yields

$$5c_1(\Delta)^2 + 4u = x^2 + y^2 + 3xy,$$

that is, $4u = xy$. □

In order to prove Theorem 2.5.4, we now only need to show that these are the only relations satisfied by the generators; thus we want to check that we have no extra nilpotency or torsion conditions on u, and that the products xu^i and yu^i are nonzero for any i. For this, we look at the 2-valuation of the characters of Q_8: we define an admissible approximation $(\widetilde{\Gamma}^n)$ that only takes into accounts some generators. This allows us to restrict ourselves when we compute the 2-valuations of our evaluation morphisms, which we use to extract information about torsion in $R^*(G)$.

Let X, Y, U be standard lifts of x, y, u. We consider the approximation $\{\widetilde{\Gamma}^n\}$, where $\widetilde{\Gamma}^n$ is the additive subgroup of $(R(G), +)$ generated by

$$X^{\varepsilon_1}Y^{\varepsilon_2}U^k,\ \text{with } 2k + \varepsilon_1 + \varepsilon_2 \geq n \text{ and } 0 \leq \varepsilon_1 + \varepsilon_2 \leq 1. \qquad (2.5.5)$$

Lemma 2.5.6. *The approximation* $(\widetilde{\Gamma}^n)$ *is admissible.*

Proof. Obviously we have $\widetilde{\Gamma}^{n+1} \subset \widetilde{\Gamma}^n$, so $(\widetilde{\Gamma}^n)$ satisfies condition (A). To check (B), let $Z = \rho_3 - 1$ and $T = \Delta - 2$ be lifts of $c_1(\rho_3)$, $c_1(\Delta)$, respectively. Let α be an additive generator for Γ^n. We know that Γ^n is generated by products of Chern classes of the generating characters of $R(G)$, hence α is of the form:

$$\alpha = X^{\varepsilon_1} Y^{\varepsilon_2} Z^{\varepsilon_3} U^k T^l \in \Gamma^n, \qquad 2k + l + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \geq n.$$

If $\varepsilon_1 \geq 2$, then α contains a factor X^2, but the relation $x^2 = 0$ implies that $X^2 \in \Gamma^3$, so in that case $\alpha \in \Gamma^{n+1}$. The same goes for ε_2, so we can restrict ourselves to monomials such that $\varepsilon_1 + \varepsilon_2 \leq 1$. Similarly, since $c_1(\Delta) = 0$ we have $T \in \Gamma^2$, which means that $l \neq 0$ forces $\alpha \in \Gamma^{n+1}$. We proceed similarly for all factors and obtain

$$\Gamma^n = \widetilde{\Gamma}^n + \Gamma^{n+1},$$

so $(\widetilde{\Gamma}^n)$ satisfies condition (B). □

There are 4 nontrivial evaluation morphisms on $R_{\mathbb{C}}(Q_8)$:

- $\phi_{-1} : \rho_i \mapsto 1, \Delta \mapsto -2,$
- $\phi_i : \rho_1 \mapsto 1, \quad \rho_2, \rho_3 \mapsto -1, \quad \Delta \mapsto 0,$
- $\phi_j : \rho_2 \mapsto 1, \quad \rho_1, \rho_3 \mapsto -1, \quad \Delta \mapsto 0,$
- $\phi_k : \rho_3 \mapsto 1, \quad \rho_1, \rho_2 \mapsto -1, \quad \Delta \mapsto 0.$

We apply those to our X, Y, T, U and obtain:

- $\phi_{-1} : X, Y \mapsto 0, \quad T \mapsto -4, \quad U \mapsto 4,$
- $\phi_i : X \mapsto 0, \quad Y, T \mapsto -2, \quad U \mapsto 2,$

- $\phi_j : X, T \mapsto -2, \quad Y \mapsto 0, \quad T \mapsto 2,$
- $\phi_k : X, Y, T \mapsto -2, \quad T \mapsto 2.$

It is easy to check that, as stated in Proposition 2.5.3, these morphisms are all continuous with respect to the 2-adic topology. We can now wrap up the computation:

Proof of Theorem 2.5.4. We want to show that $R^{2n}(G) = \langle u^n \rangle = \mathbb{Z}/8\mathbb{Z}$, and $R^{2n+1}(G) = \langle xu^n, yu^n \rangle = (\mathbb{Z}/2\mathbb{Z})^2$.
We first look at $R^{2n}(G)$, where we need to show that $4u^n \neq 0$, that is, $4U^n \notin \Gamma^{2n+1}$. We have $\phi_g(4U^n) = 2^{n+2}$, for any $g = i, j, k$. Suppose that $4U^n \in \Gamma^{2n+1}$; then by Proposition 2.5.2 there is an element $\widetilde{X} \in \widetilde{\Gamma}^{2n+1}$ satisfying

$$n + 2 = v_2(\phi_g(4U^n)) = v_2(\phi_g(\widetilde{X})).$$

Write

$$\begin{aligned} \widetilde{X} =& a_1 U^{n+1} + a_2 X U^n + a_3 Y U^n \\ & + U^{n+2} P(X, Y, U) + X U^{n+1} Q(X, Y, U) + Y U^{n+1} S(X, Y, U) \end{aligned}$$

where the a_i's are integers and P, Q, S are polynomials with integer coefficients. Apply ϕ_g for $g = i, j, k$ to this equation, divide each equation by 2^{n+1} and consider the result mod 2. We obtain a system of three equations:

$$0 = a_1 + a_3$$

$$0 = a_1 + a_2$$

$$0 = a_1 + a_2 + a_3$$

which has no nontrivial solution. But if $a_m = 0(\text{mod } 2)$ for $m = 1, 2, 3$ then $v_2(\phi_g(\tilde{x})) > n+2$, which is impossible. Thus $4U^n$ cannot be in Γ^{2n+1}, and the additive order of u^m is indeed 8 for all m.

The same process shows that xu^n, yu^n and $xu^n + yu^n$ are nonzero elements of $R^{2n+1}(Q_8)$ for all n. $\square$

2.6 Abelian 2-groups

Abelian 2-groups are a rich source of examples; we present below the computations of $R^*_{\mathbb{C}}(C_4 \times C_2)$ and $R^*_{\mathbb{C}}(C_4 \times C_4)$. To determine the graded rings in this section, we used all of the tools presented in this chapter. Theorem 2.4.3 gives us a starting point to guess relations in the graded character rings, which we then determine precisely using basic virtual character algebra. Once we have a good candidate for $R^*(G)$, we discard as many extra relations as possible by restricting to various subgroups, and get rid of the last ones by looking at continuity of characters.

Remark. In the sequel, we denote the evaluation of a virtual character X at $g \in G$ by $X|_g$ rather than $\phi_g(X)$. Besides, it will be convenient to use an additive notation throughout, so we denote the cyclic groups by $\mathbb{Z}/4$ and $\mathbb{Z}/2$.

Proposition 2.6.1.

$$R^*(\mathbb{Z}/4 \times \mathbb{Z}/2) = \frac{\mathbb{Z}[x, y]}{(4x, 2y, xy^3 + x^2y^2)}$$

with $|x| = |y| = 1$.

Proof. Let ρ be the generating character of $R(\mathbb{Z}/4)$ sending 1 to i, and σ the nontrivial representation of $\mathbb{Z}/2$. Then $R^*(G)$ is generated by $c_1(\rho) =: x$ and

$c_1(\sigma) =: y$. By restriction to cyclic subgroups, we see that x has additive order 4 and y, additive order 2. Now consider $X = \rho - 1$ and $Y = \sigma - 1$. Then by expanding $(X+1)^4 = 1$ we get

$$4X = -(6X^2 + 4X^3 + X^4)$$

On the other hand, $Y^n = (-2)^{n-1}Y$, so

$$XY^3 = 4XY = -X^4Y - 4X^3Y - 6X^2Y = -X^4Y - X^3Y^3 + 3X^2Y^2.$$

Modulo Γ^5, this is $xy^3 = x^2y^2$. The only other possible extra relations (that cannot be ruled out by restrictions to various subgroups) are: $x^{n-1}y = 0$, $x^{n-2}y^2 = 0$ or $x^{n-1}y = x^{n-2}y^2$ for some n. We use the continuity method to disprove all of these. Let $\widetilde{\Gamma}^n = \langle X^n, Y^n, X^{n-1}Y, X^{n-2}Y^2 \rangle$, then $\{\widetilde{\Gamma}^n\}_n$ is an admissible approximation for $\{\Gamma^n\}_n$.

First suppose that $x^{n-1}y = 0$ for some n, that is $X^{n-1}Y \in \Gamma^{n+1}$. Then for N arbitrarily large, there exists $\widetilde{Z} \in \widetilde{\Gamma}^{n+1}$ such that $X^{n-1}Y = \widetilde{Z} + R$ with $R \in \Gamma^N$; in particular, by Proposition 2.5.2, for any $M > 0$, there is an N such that:

$$\begin{cases} v_2\left(\widetilde{Z}|_{(k,\ell)}\right) = v_2\left(X^{n-1}Y|_{(k,\ell)}\right) \text{ whenever } v_2\left(X^{n-1}Y|_{(k,\ell)}\right) < \infty \\ v_2\left(\widetilde{Z}|_{(k,\ell)}\right) > M \text{ whenever } v_2\left(X^{n-2}Y^2|_{(k,\ell)}\right) = \infty \end{cases},$$

for all $(k,\ell) \in \mathbb{Z}/4 \times \mathbb{Z}/2$.

Write:

$$\begin{aligned}\widetilde{Z} =& a \cdot X^{n+1} + b \cdot X^n Y + c \cdot X^{n-1} Y^2 + d \cdot Y^{n+1} \\ &+ P \cdot X^{n+2} + Q \cdot X^{n+1} Y + S \cdot X^n Y^2 + T \cdot Y^{n+2}\end{aligned}$$

where $a, b, c, d \in \mathbb{Z}$ and $P, Q, S, T \in \mathbb{Z}[X, Y]$. Evaluatin at $(2, 1)$ gives $X|_{(2,1)} = Y|_{(2,1)} = -2$. Then the 2-valuation of $X^{n-1}Y$ is n while the 2-valuation of $\widetilde{Z}$ is at least $n+1$. This shows that such a $\widetilde{Z}$ cannot exist, and thus $x^{n-1}y$ cannot be zero. A similar argument shows that $x^{n-2}y^2 \neq 0$.

The only possible remaining relation is $x^{n-1}y = x^{n-2}y^2$. Let $Z = X^{n-1}Y + X^{n-2}Y^2$, and suppose $Z \in \Gamma^{n+1}$. Fix a large number $M > n+2$, and let $\widetilde{Z} \in \widetilde{\Gamma}^{n+1}$ and $R \in \Gamma^N$ satisfy $Z = \widetilde{Z} + R$ with the usual conditions on the valuation of $\widetilde{X}$. Then:

$$\widetilde{Z}|_{(2,0)} = a \cdot (-2)^{n+1} + P \cdot (-2)^{n+2}$$

while $Z|_{(2,0)} = 0$, so $v_2(Z|_{(2,0)}) = +\infty$. In this case, we have $v_2(\widetilde{Z}) > M > n+2$. This means in particular that $a = 0 \pmod 2$, hence $a = 2a'$ for some a'.

We now evaluate at $(1, 1)$:

$$Z|_{(1,1)} = (i-1)^{n-2} \cdot 4 + (i-1)^{n-1} \cdot (-2).$$

thus $v_2(Z|_{(1,1)}) = (n+1)/2$. On the other hand:

$$\widetilde{Z}|_{(1,1)} = a' \cdot 2 \cdot (i-1)^{n+1} + b \cdot (i-1)^n(-2) + c \cdot (i-1)^{n-1} 4 + d \cdot (-2)^{n+1} + \widetilde{R}$$

where $v_2(\widetilde{R}) \geq \frac{n+2}{2}$. We see that $v_2(\widetilde{Z}|_{(1,1)}) \geq (n+2)/2$, so we cannot have $v_2(\widetilde{Z}|_{(1,1)}) = \widetilde{Z}|_{(1,1)}$, in contradiction with our assumption. This means that

$Z \notin \Gamma^{n+1}$, and thus $x^{n-1}y \neq x^{n-2}y^2$. This completes the proof. $\square$

Remark. Let $G = \mathbb{Z}/2^n \times \mathbb{Z}/2$. Then one can show, as above, that there is in $R^*(G)$ a relation of the form $xy^{n+1} + x^2y^n = 0$. For $n = 3$, it is possible to adapt the argument above and show that $R^*(G) = \frac{\mathbb{Z}[x,y]}{(8x,2y,xy^4+x^2y^3)}$. Is it true in general that

$$R^*(\mathbb{Z}/2^n \times \mathbb{Z}/2) = \frac{\mathbb{Z}[x,y]}{(2^nx, 2y, xy^{n+1} + x^2y^n)}? \tag{2.6.1}$$

Proposition 2.6.2.

$$R^*(\mathbb{Z}/4 \times \mathbb{Z}/4) = \frac{\mathbb{Z}[x,y]}{(4x, 4y, 2x^2y + 2xy^2, x^4y^2 - x^2y^4)}$$

with $|x| = |y| = 1$

Proof. Existence of relations: let X, Y be the usual lifts of x, y. We use that $(X+1)^4 = 1$, that is,

$$X^4 = -4X - 6X^2 - 4X^3 \tag{2.6.2}$$

Then:

$$\begin{aligned} X^4Y &= (-4X - 6X^2 - 4X^3)Y = -4XY - 6X^2Y - 4X^3Y \\ &= X(6Y^2 + 4Y^3 + Y^4) - 6X^2Y - 4X^3Y \\ 6XY^2 - 6X^2Y &= X^4Y - XY^4 + 4X^3Y - 4XY^3 \\ 0 &= 2XY^2 + 2X^2Y (\mathrm{mod}\ \Gamma^4) \end{aligned} \tag{2.6.3}$$

so $2x^2y = 2xy^2$. By repeatedly applying Equation (2.6.2) to $X^6Y - XY^6$, one obtains:

$$X^6Y - XY^6 = 14X^2Y^3 - 14X^3Y^2 + 11X^2Y^4 - 11X^4Y^2 + 5X^3Y^4 - 5X^4Y^3.$$

The expression $14X^2Y^3 - 14X^3Y^2$ is the sum of $8X^2Y^3 - 8X^3Y^2$ which belongs to Γ^7, and $6X^2Y^3 - 6X^3Y^2$ which is simply Equation (2.6.3) multiplied by XY, and thus also belongs to Γ^7. Thus $x^4y^2 = x^2y^4$.

To show there are no extra relations, let $a_0, \cdots a_n \in \mathbb{Z}, n \geq 2$ satisfy:

$$z = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + a_3x^{n-3}y^3 + a_{n-1}xy^{n-1} + a_ny^n = 0$$

or in other words,

$$\begin{aligned} Z &:= a_0X^n + a_1X^{n-1}Y + a_2X^{n-2}Y^2 + a_3X^{n-3}Y^3 + a_{n-1}XY^{n-1} + a_nY^n \\ &= \sum_{k=0}^{n+1} b_kX^{n+1-k}Y^k + \sum_{k=0}^{n+2} P_k(X,Y)X^{n+2-k}Y^k \in \Gamma^{n+1}. \end{aligned} \tag{2.6.4}$$

Suppose, without loss of generality, that $a_0, a_1, a_n, b_0, b_1, b_n \in \{0,1,2,3\}$ and $a_2, a_3, a_{n-1}, b_2, \cdots, b_{n-1} \in \{0,1\}$ while $P_k(X,Y) \in \mathbb{Z}[X,Y]$. Let $\widetilde{Z}$ be the right hand side of the equation, and consider the restriction of Equation (2.6.4) to the following subgroups:

- To $\mathbb{Z}/4 \times 1$: then Equation (2.6.4) becomes $a_0X^n = b_0X^{n+1} + P_0(X,0)X^{n+2} \in \Gamma^{n+1}$. This implies that $a_0 = b_0 = 0 (\text{mod } 4)$, and since we had assumed that $a_0, b_0 \in \{0,1,2,3\}$ we have $a_0 = b_0 = 0$.
- To $1 \times \mathbb{Z}/4$: similarly we obtain $a_n = b_n = 0$.
- To $\langle(1,1)\rangle \cong \mathbb{Z}/4$: the generators X and Y both restrict to the generator

T and we obtain $\sum a_k T^n = 0 (\text{mod } \Gamma^{n+1})$, so

$$a_1 + a_2 + a_3 + a_{n-1} = 0 (\text{mod } 4). \tag{2.6.5}$$

Evaluating at $(1,2)$ yields:

$$v_2(\phi_{(1,2)}(Z)) \geq \min\left(\left(v_2(a_\ell) + \frac{n+\ell}{2}\right)_{\ell=1,2,3}, v_2(a_{n-1}) + \frac{2n-1}{2}\right) \tag{2.6.6}$$
$$v_2(\phi_{(1,2)}(\tilde{Z})) \geq \frac{n+2}{2}$$

with equality if there is a strict minimum in Equation (2.6.6), so we must have $v_2(a_1) \geq 1$, that is $a_1 = 0$ or $a_1 = 2$. By evaluating at $(2,1)$ instead, one shows that $v_2(a_{n-1}) \geq 1$ and thus $a_{n-1} = 0$, since we had assumed $a_{n-1} \in \{0,1\}$. Our equation becomes $a_1 + a_2 + a_3 = 0 (\text{mod } 4)$. If $a_1 = 0$ then $a_2 = a_3 = 0$, so we can assume $a_1 = 2$, and then $a_2 = a_3 = 1$. Thus $Z = 2X^{n-1}Y + X^{n-2}Y^2 + X^{n-3}Y^3$. To rule out this last possibility, consider the automorphism τ of G, which leaves $(1,0)$ invariant and sends $(0,1)$ to $(0,3)$. This induces a well-defined map τ^* of graded rings, which maps y to $-y$. Then

$$\tau^* z = -2x^{n-1}y + x^{n-2}y^2 - x^{n-3}y^3$$

and summing $z + \tau^* z$ we obtain $2x^{n-2}y = 0$, which is impossible since all monomials in $R^*(G)$ have additive order 4. Thus the assumption $a_0 = 2$ is wrong, and this concludes the proof. $\square$

Chapter 3

Mackey functors and Tambara functors

This chapter focuses on the general properties of graded character rings: in Section 3.1, we introduce the necessary notation to consider representation rings from the point of view of equivariant K-theory. In Section 3.2, we show that graded character rings are not Mackey functors, through the example of the alternating group A_4; Section 3.3 introduces the saturated filtration and explores its properties, which we apply in Section 3.4 to compute the saturated ring of, among others, the projective special linear group $PSL(2, p)$ for some primes p. We then move on to Tambara functors in Section 3.5, where we present a proof that equivariant K-theory is a Tambara functor. In Section 3.6, we study the norm of the sum of two characters, and determine a formula which shows that graded character rings of abelian groups are Tambara functors; we conlude in Section 3.7 by a straightforward application of these result to norms in abelian groups of the form $G \times C_p$.

3.1 Definitions and notations

As in Chapter 2, we fix a field $\mathbb{K}$ of characteristic zero and G a finite group. The ring $R_{\mathbb{K}}(G)$ is the ring of virtual characters of G over $\mathbb{K}$, and $R^*_{\mathbb{K}}(G)$ denotes the correponding graded rings.

The definitions of Mackey and Tambara functors, to be given in later sections, are greatly simplified by looking at character rings from the point of view of G-equivariant K-theory. We view a G-set X as a category with an object for each point, and an arrow between two objects $(g, x) : x \to y$ for each $g \in G$ such that $g \cdot x = y$. A vector bundle is then defined as a functor V between X and the category of $\mathbb{K}$-vector spaces and linear maps; that is, it associates to each $x \in X$ a vector space V_x, and to each $g \in G$ linear maps $V_{(g,x)} : V_x \to V_{g \cdot x}$. For an element $e \in V_x$, we write $g \cdot e \in V_{g \cdot x}$ for $V_{(g,x)} e$. A functor V then corresponds to the data of each V_x and $g \cdot e$. Let $K^+_G(X)$ be the semigroup of isomorphism classes of vector bundles over X under direct sum. In the sequel, we restrict ourselves to finite G-sets.

Lemma 3.1.1. *Let X be a transitive G-set with a distinguished point $x \in X$, and let $H = \mathrm{Stab}(x)$. Then there is an isomorphism (depending on x) between $K^+_G(X)$ and the semiring of representations $R^+(H)$.*

Proof. Let W be a representation of H and consider the induced representation $V = \mathrm{Ind}_H^G = \mathbb{K}[G] \otimes_{\mathbb{K}[H]} W$. Define a vector bundle on X as follows: for each $y \in X$, write $y = g \cdot x$ and let $(V)_y = g \cdot W = g \otimes W \subset V$. This depends only on y and the action of g takes V_x to V_{gx}, so this is a vector bundle. Conversely, if V is a vector bundle on X, define $W = V_x$. This is a well-defined H-module (since it is stable by H), so W is a representation. These two constructions are mutually inverse. $\square$

Remark. (i) The isomorphism above depends on x; choosing the point $y = g \cdot x$ as a basepoint instead, one obtains the isomorphic representation of gHg^{-1} which is given by precomposing the action of H on V_x by conjugation with g.

(ii) As a direct corollary, the Grothendieck group $K_G(X)$ of vector bundles and the ring $R(G)$ are isomorphic.

(iii) Since every finite G-set can be written as a disjoint union of transitive G-set, this gives us a way to prove general facts about $K_G^+(X)$ by restricting to representation rings.

This vocabulary allows us to generalize the notions of restriction, transfer and tensor induction of representations. Let $f : X \to Y$ be a map of G-sets, given by a functor between the categories X and Y as described above. We define:

(i) The *restriction* $f^* : K_G^+(Y) \to K_G^+(X)$, as the composition of f and V. In other words:

$$(f^*V)_x := V_{f(x)},$$
$$(f^*V)_{(g,x)} := V_{(g,f(x))}.$$

Note that with the shorthand notation mentioned above, for $e \in V_{f(x)}$, we have $(f^*V)_{(g,x)}e = g \cdot e$, which corresponds to the same element in $f^*(V)$ as in V, only understood in a different fibre. This is particularly intuitive in the case where $f : G/H \to G/K$ corresponds to the inclusion of a subgroup $H \hookrightarrow K$.

(ii) The *induction* (or transfer) $f_* : K_G^+(X) \to K_G^+(Y)$,

$$f_*(V)_y := \bigoplus_{x \in f^{-1}(y)} (V_x),$$

$$f_*(V)_{(g,y)} := \bigoplus_{x \in f^{-1}(y)} V_{(g,x)}.$$

In shorthand notation we have $g \cdot (\bigoplus_x e_x) = \bigoplus_x g \cdot e_{g^{-1}x}$.

(iii) The *norm* (or tensor induction) $f : K_G^+(X) \to K_G^+(Y)$,

$$f_\sharp(V)_y := \bigotimes_{x \in f^{-1}(y)} V_x,$$

$$f_\sharp V_{(g,y)} := \bigotimes_{x \in f^{-1}(y)} V_{(g,x)}.$$

With $g \cdot (\bigotimes_x e_x) = \bigotimes_x g \cdot e_{g^{-1}x}$.

Note that although we use its vocabulary and definitions, the full extent of equivariant K-theory is beyond our scope. Thus we will mostly assume that X, Y are of the form G/K for some subgroup $K \leq G$, and more often than not we will have $Y = G/G = \{*\}$.

Remark. (i) Equivalently,

$$f_*(V)_y \cong \bigoplus_{x \in f^{-1}(y)/\operatorname{Stab}(y)} \operatorname{Ind}_{\operatorname{Stab}(x)}^{\operatorname{Stab}(y)} V_x.$$

(ii) One can check that applying the norm formula to the case of $X = G/H$ and $Y = \{*\}$ yields the usual tensor induction, as defined eg. in [CR90, §13A]

The restriction and induction maps can be extended to K_G in a straightforward way. By [Tam93, Th. 6.1], so can the tensor induction map. We explore in Section 3.6 how to determine a formula for the tensor induction of virtual characters.

3.2 Graded character rings are not Mackey functors

Graded character rings are functorial (see Lemma 2.3.1); in particular, if H is a subgroup of G, restricting representations from G to H induces a well-defined homomorphism $R^*(G) \to R^*(H)$. Naturally, one wonders whether induction of representations from a subgroup H of G also preserves the Grothendieck filtration, and thus gives rise to a well-defined, additive induction map from $R^*(H)$ to $R^*(G)$. If so, then $R^*(-)$ satisfies the axioms of a cohomological Mackey functor (which we define below). In particular, an analogue to Cartan and Eilenberg's result on stable elements in cohomology ([CE99, Th. XII.10.1]) states that each p-primary component $R^*(G)_p$ of $R^*(G)$ is isomorphic to some subring of the graded character ring of its p-Sylow subgroup. This is not the case, and we produce below an example where this property fails. Thus $R^*(-)$ cannot be a Mackey functor.

A thorough treatment of the theory of Mackey functors is given in [Web]; let us start with the definition. Let R be a commutative ring and G a group, and let $G\mathsf{set}$ be the category of finite G-sets. A *Mackey functor* is a pair (S^*, S_*) of functors from $G\mathsf{set}$ to $R-\mathsf{mod}$, where S^* is contravariant and S_* is covariant, and $S^*(-)$ and $S_*(-)$ are equal on objects. Additionally, we require the following axioms be satisfied:

(i) If

$$\begin{tikzcd} \Omega_1 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \Omega_2 \arrow[d, "\gamma"] \\ \Omega_3 \arrow[r, "\delta"'] & \Omega_4 \end{tikzcd}$$

is a pullback diagram of G-sets, then $S^*(\delta)S_*(\gamma) = S_*(\beta)S^*(\alpha)$.

(ii) For every pair Ω, Ψ of finite G-sets, the morphism $S(\Omega) \oplus S(\Psi) \to S(\Omega \sqcup \Psi)$ obtained by applying S_* to $\Omega \to \Omega \sqcup \Psi \leftarrow \Psi$, is an isomorphism.

Remark. Alternatively, a Mackey functor S can be viewed a a function from the subgroups of G to $R-\mathsf{mod}$, with, for any two subgroups $H \leq K$ and $g \in G$, maps $\mathrm{Res}^H_K : S(H) \to S(K)$, $\mathrm{Ind}^H_K : S(K) \to S(H)$ and $c_g : S(H) \to S({}^gH)$. The maps are required to satisfy the usual axioms governing conjugation, induction and restriction of representations, as detailed in [Web, §2]. If, additionally, the induction and restriction satisfy $\mathrm{Res}^H_K(\mathrm{Ind}^H_K(x)) = [K : H] \cdot x$ then $S(-)$ is called a cohomological Mackey functor.

This second definition makes it easy to check that the (ungraded) character ring $R(-)$ is a Mackey functor. Thus, if induction preserves the filtration, then $R^*(-)$ is also a Mackey functor. The following general result should then be valid for $R^*(-)$.

Proposition 3.2.1 (Cartan-Eilenberg)**.** *Let $H \geq G$ be any subgroup, and suppose S is a Mackey functor. Call an element $x \in S(H)$ stable if*

$$\mathrm{Res}^{{}^gH}_{{}^gH \cap H}(c_g(x)) = \mathrm{Res}^{H}_{{}^gH \cap H}(x)$$

for all $g \in G$. If $G \geq H \geq Syl_p(G)$ where $Syl_p(G)$ is a p-Sylow of G, and

$S(G)_{(p)}$ denotes the p-primary component of $S(G)$, then:

$$\mathrm{Res}^G_H : S(G)_{(p)} \longrightarrow S(H)_{(p)}$$

is injective, and its image consists of the stable elements in $S(H)_{(p)}$.

This result is a consequence of [Web, Cor. 3.7 and Prop. 7.2]; a more elementary proof, in the case of cohomology, can be found in [AM69, Th. 6.6]. In the case of the alternating group A_4 of order 12, we use the following corollary:

Corollary 3.2.2. *If $H := Syl_p(G)$ is abelian, then*

$$\mathrm{Res}^G_H : \mathcal{R}^*(G)_{(p)} \longrightarrow \mathcal{R}^*(H)^{N_G(H)}$$

is an isomorphism.

We show that the condition of surjectivity on stable elements fails. Note that the following computation relies heavily on the techniques developed in Chapter 2, to which we refer the reader for any details. We also use the following corollary to Proposition 2.3.3:

Lemma 3.2.3. *Let C_2 be the cyclic group of order 2, and let ρ_1, ρ_2 be the generating representations for $R_{\mathbb{C}}(C_2 \times 1), R_{\mathbb{C}}(1 \times C_2)$ respectively. Then,*

$$R^*_{\mathbb{C}}(C_2 \times C_2) = \frac{\mathbb{Z}[t_1, t_2]}{(2t_1, 2t_2, t_1^2 t_2 - t_1 t_2^2)}$$

where $t_i = c_1(\rho_i)$.

Let A_4 be generated by the permutations (12)(34) and (123). There are 4 irreducible complex representations of A_4:

- Of dimension 1: the trivial representation 1, and the representations ρ (resp. $\bar{\rho}$) that send (123) to $e^{2i\pi/3}$ (resp. $e^{-2i\pi/3}$) and (12)(34) to 1.
- Of dimension 3: the standard representation θ, which is the quotient of the representation $\bar{\theta}$ acting on $\mathbb{C}^4$ by permutation of the basis vectors, by the trivial representation. The character of θ sends 3-cycles to 0 and (12)(34) to -1.

There are the following relations between the representations:

$$\rho^2 = \bar{\rho} \tag{3.2.1}$$

$$\rho\theta = \theta \tag{3.2.2}$$

$$\theta^2 = 1 + \rho + \bar{\rho} + 2\theta \tag{3.2.3}$$

Additionally $\lambda^2(\theta) = \theta$ (by a direct calculation of the exterior power) and $\det(\theta) = 1$.

Lemma 3.2.4. *Let $x = c_1(\rho)$ and $y = c_2(\theta)$, then*

$$R^*_{\mathbb{C}}(A_4) = \frac{\mathbb{Z}[x, y]}{(3x, 12y, 4y + x^2)}.$$

Proof. The graded character ring $R^*_{\mathbb{C}}(A_4)$ is generated by all Chern classes of irreducible characters of A_4; we start by ridding ourselves of redundant generators. Let $x = c_1(\rho)$ and $y = c_2(\theta)$. Then $x = c_1(\rho) = -c_1(\bar{\rho})$ and $3x = 0$. Moreover $c_1(\theta) = c_1(\det\theta) = c_1(1) = 0$, so x generates $R^1(A_4)$, and y, x^2 generate $R^2(A_4)$. As for the degree 3 generator $c_3(\theta)$, we have:

$$C_3(\theta) = \gamma^3(\theta - 3) = \lambda^3(\theta - 1) = -1 + \theta - \lambda^2(\theta) + 1 = 0,$$

so there is no additional generator in degree 3 and $R^*(A_4)$ is generated by x, y.

We have $3x = 0$ by the above, and $12y = 0$ since the order of A_4 kills $R^*(A_4)$ (see Proposition 2.1.6). We now turn to the relation $4y + x^2 = 0$: applying the total Chern class c_T to both sides of (3.2.3) yields:

$$\begin{aligned} c_T(\theta^2) &= c_T(1 + \rho + \bar{\rho} + 2\theta) \\ &= c_T(\rho)c_T(\bar{\rho})c_T(\theta)^2 \\ &= (1 + xT)(1 - xT)(1 + yT^2)^2 \\ &= 1 + (2y - x^2)T^2 + (y^2 - 2yx^2)T^4 + zy^2T^6. \end{aligned} \tag{3.2.4}$$

On the left-hand side, use the splitting principle (Proposition 2.1.3): we write the character θ as a sum $\theta_1 + \theta_2 + \theta_3$ of linear characters. Looking only at even terms of degree ≤ 6 and keeping in mind that $c_1(\theta) = c_3(\theta) = 0$, we get:

$$\begin{aligned} c_T(\theta^2) &= c_T((\sigma_1 + \sigma_2 + \sigma_3)^2) \\ &= 1 + 6yT^2 + 9y^2T^4 + 4y^3T^6 \end{aligned} \tag{3.2.5}$$

Equating 3.2.4 and 3.2.5 yields $4y = -x^2$. In particular this means that the order of y is a multiple of 3. To obtain more information, we can use the restriction $\mathrm{Res}^{A_4}_{C_2 \times C_2} : R^*(A_4) \to R^*(C_2 \times C_2)$ y to $H := C_2 \times C_2$. By Lemma 3.2.3 :

$$R^*(\mathbb{Z}/2 \times \mathbb{Z}/2) = \frac{\mathbb{Z}[t_1, t_2]}{(2t_1, 2t_2, t_1^2 t_2 - t_1 t_2^2)}.$$

We have $\mathrm{Res}_H(y) = t_1^2 + t_1 t_2 + t_2^2$, which has order 2. So the order of y^i is a multiple of 2, that is, it is either 6 or 12. To conclude, we use the continuity method from Section 2.5. Let $X = C_1(\rho) = \rho - 1$ and $Y = C_2(\theta) = 3 - \theta$, and let

$$\widetilde{\Gamma}^n = \begin{cases} \langle Y^{n/2} \rangle & n \text{ even} \\ \langle XY^{(n-1)/2} \rangle & n \text{ odd} \end{cases}.$$

Then $\widetilde{\Gamma}^n$ is an admissible approximation for Γ. The evaluation $\phi_{(12)(34)}$ sends X to 0 and Y to -4, and thus is continuous with respect to the 2-adic topology on $\mathbb{Z}$. Suppose, for a contradiction, that $6Y^k \in \Gamma^{2k+1} = \widetilde{\Gamma}^{2k+1} + \Gamma^M$ for some large M. Since $2k+1$ is odd, $\widetilde{\Gamma}^{2k+1}$ is generated by XY^k, which evaluates to zero. Thus we must have $v_2(6Y^k) \geq 2k+2$, but $v_2(6Y^k) = 2k+1$. So Y^k has additive order 12. Finally, restricting x to the subgroup generated by (123), and yx to that generated by $(12)(34)$, shows that there are no additional relations. □

Theorem 3.2.5. *$R^*(-)$ is not a Mackey functor.*

Proof. Let $G = A_4$, and consider its normal, abelian 2-Sylow $H = C_2 \times C_2$. Since $\mathrm{Res}^G_H(x) = 0$, the image of $R^*_{\mathbb{C}}(G)$ under the restriction map

$$\mathrm{Res}^G_H : R^*_{\mathbb{C}}(G) \longrightarrow R^*_{\mathbb{C}}(H)$$

is generated by powers of $\mathrm{Res}^G_H(y) = t_1^2 + t_1t_2 + t_2^2$. On the other hand, G acts on $R^*_{\mathbb{C}}(H)$ by cyclic permutations of the elements $t_1, t_2, t_1 + t_2$. The element $z = t_1^3 + t_2^3 + t_1^2 t_2$ is invariant under this action. But z is not a combination of powers of $t_1^2 + t_1t_2 + t_2^2$ since it has odd degree, and thus does not belong to the image of the restriction map. Therefore:

$$\mathrm{Im}(\mathrm{Res}^G_H) \subsetneq R^*_{\mathbb{C}}(H)^{N_G(H)}$$

which means that $R^*(-)$ is not a Mackey functor. □

3.3 Saturated rings

Theorem 3.2.5 tells us that induction of representations is not compatible with the Grothendieck filtration. This prompts us to define a modified filtration, taking into account all images of Chern classes of subgroups of G under the induction map. This new filtration retains much of the information of the Grothendieck filtration: in fact, both induce the same topology on $R(G)$. In the sequel, let H, K denote two arbitrary subgroups of G. On the λ-ring $R(G)$, define the *saturated filtration* $\{F^n\}_n$ as follows:

$$F^n(G) = \sum_{H \leq G} \operatorname{Ind}_H^G(\Gamma^n(H)).$$

This means that $F^n(G)$ is generated by elements of the form:

$$x = \operatorname{Ind}_H^G(\gamma^{i_1}(\rho_1) \cdots \gamma^{i_m}(\rho_m)), \quad i_1 + \cdots + i_m \geq n$$

with each ρ_ℓ an irreducible representation of H. By definition, induction of representations preserves the filtration F.

Remark. In [Rit70], J. Ritter defines *admissible filtrations* on the complex character ring as filtrations which, among other properties, are preserved by induction of representations; he then proves some general results about their topology, and the torsion of the associated graded ring. In this terminology, the saturated filtration is the smallest admissible filtration that contains the Grothendieck filtration.

Lemma 3.3.1. *Let $I = \ker \varepsilon$ be the augmentation ideal.*

(i) Induction and restriction of characters preserve the filtration F.

(ii) $F^i(G) \cdot F^j(G) \subseteq F^{i+j}(G)$.

(iii) $F^0(G) = R(G)$, $F^1(G) = I$.

Proof. (i) We only need to check that restriction does preserve the filtration. Let $x \in F^i(G)$; we prove that if $x = \operatorname{Ind}_H^G(y)$ with $y \in \Gamma^i(H)$ then $\operatorname{Res}_H^G(x) \in F^i(K)$ for all $K \leq G$. We use the Mackey double coset formula ([Ser77, Prop. 7.3.22]): let S be a set of (H, K)-double coset representatives of G. For $s \in S$, let

$$ {}_sH = sHs^{-1} \cap K \leq K $$

and let

$$ y^s(g) = y(s^{-1}gs), \quad \text{for all} \quad g \in {}_sH. $$

Then y^s is a representation of ${}_sH$ and

$$ \operatorname{Res}_K^G \operatorname{Ind}_H^G(y) = \sum_{s \in K\backslash G/H} \operatorname{Ind}_{{}_sH}^K(y^s) $$

Note that each y^s is in $\Gamma^i({}^sH)$ by functoriality of $R^*(-)$. Thus $\operatorname{Res}_K^G(x) \in F^i(K)$.

(ii) It is sufficient to prove that if $\tilde{x} = \operatorname{Ind}_H^G(x)$ and $\tilde{y} = \operatorname{Ind}_K^G(y)$ with $x \in \Gamma^i(H)$ and $y \in \Gamma^j(K)$ then $\tilde{x} \cdot \tilde{y} \in F^{i+j}(G)$. We proceed by induction on the order of G. Suppose $H < G$ is a proper subgroup. By the projection formula ([Ser77, §7.2]):

$$ \tilde{x} \cdot \tilde{y} = \operatorname{Ind}_H^G(x) \operatorname{Ind}_K^G(y) = \operatorname{Ind}_H^G(x \cdot \operatorname{Res}_H^G \operatorname{Ind}_K^G(y)). $$

Since restriction preserves the filtration and $\operatorname{Ind}_K^G(y) \in F^j(G)$, we have

$\mathrm{Res}^G_H \mathrm{Ind}^G_K(y) \in F^j(H)$, so:

$$x \cdot \mathrm{Res}^G_H \mathrm{Ind}^G_K(y) \in F^i(H)F^j(H) \subseteq F^{i+j}(H).$$

where the inclusion is true by induction. In conclusion:

$$\tilde{x} \cdot \tilde{y} = \mathrm{Ind}^G_H(x \cdot \mathrm{Res}^G_H \mathrm{Ind}^G_K(y)) \in F^{i+j}(G).$$

(iii) The fact that $F^0(G) = R(G)$ comes from the fact that $\Gamma^0(G) = R(G)$. For $F^1(G)$, simply observe that, since $\varepsilon(\mathrm{Ind}^G_H(\rho)) = [G : H]\varepsilon(\rho)$:

$$F^1(G) = \sum_{H \leq G} \mathrm{Ind}^G_H(\ker(\varepsilon|_H)) = \ker(\varepsilon) = I.$$

$\square$

Lemma 3.3.1 lets us define the *saturated graded ring* associated to G as:

$$\mathcal{R}^*(G) = \bigoplus_{i \geq 0} F^i(G)/F^{i+1}(G).$$

Note that, as representation rings are of the form $K_G(X)$ for some transitive G-set X, we can extend the definition of this filtration to $K_G(X)$ for a general finite G-set X. Then the above discussion means that for every map of finite G-sets $f : X \to Y$, the maps f_* and f^* defined in Section 3.1 are compatible with the saturated filtration.

Theorem 3.3.2. *The saturated graded ring $\mathcal{R}^*(-) : G\mathsf{set} \to \mathbb{Z} - \mathrm{mod}$ is a Mackey functor.*

Proof. This follows from the above. $\square$

Note that $\mathcal{R}^*$ is actually a Green functor, that is, a Mackey functor with an R-algebra structure compatible with restriction and satisfying the projection formula. We still need to ensure we do not lose too much information by modifying the filtration: after all, we could end up with trivial graded rings. It is not the case, and in fact both filtrations induce the same topology on $R(-)$. Recall that by Proposition 2.1.1, the topology induced by the Grothendieck filtration coincides with the I-adic topology.

Theorem 3.3.3. *The filtrations* $(F^n)_n$ *and* $(\Gamma^n)_n$ *induce the same topology on* $R(G)$*.*

Proof. Let $U \subseteq R(G)$ be open for the F-topology, that is, for any $x \in U$ there is an integer N such that $x + F^N \subseteq U$. Since $\Gamma^N \subseteq F^N$, we also have $x + \Gamma^N \subseteq U$, so U is also open for the Γ-topology.

To prove that a set U open in the Γ-topology is also open in the F-topology, we need to show that for each N, there is an M such that $F^M \subseteq \Gamma^N$. Let $H \leq G$, and recall that $R(H)$ can be viewed as an $R(G)$-module via the restriction homomorphism. Then, by [Ati61, Th. 6.1], the $I(H)$-adic topology is equal to the topology on $R(H)$ induced by the $I(G)$-adic topology. By Proposition 2.1.1, these topologies are also equal to the Γ-topology on H. In particular, for our fixed N, there are some k, m satisfying:

$$\Gamma^N(H) \supset I(G)^k \cdot R(H) \supset \Gamma^m(H).$$

Pick k (and thus m) large enough that we also have $I(G)^k \subset \Gamma^N(G)$. Then

$$\mathrm{Ind}_H^G(\Gamma^m(H)) \subset \mathrm{Ind}_H^G(I(G)^k \cdot R(H)) \subset I(G)^k \subset \Gamma^N(G).$$

Now let

$$M = \max_{H \leq G} \left\{ \min \{ m \mid \operatorname{Ind}_H^G(\Gamma^m(H)) \subset \Gamma^N(G) \} \right\},$$

then $F^M(G) = \sum_{H \leq G} \operatorname{Ind}_H^G(\Gamma^M(H)) \subset \Gamma^N(G)$, which completes the proof. $\square$

Since $\Gamma^n \subseteq F^n$ for all $n \geq 0$, there is a natural map of graded rings:

$$\eta : R^*(G) \longrightarrow \mathcal{R}^*(G)$$

induced by the identity. Here is a neat consequence of Theorem 3.3.3:

Corollary 3.3.4. *If the natural map $\eta : R^*(G) \to \mathcal{R}^*(G)$ is surjective, then it is an isomorphism and the filtrations (F^n) and (Γ^n) are equal.*

Proof. If η is surjective, then $\mathcal{R}^*(G)$ is generated by Chern classes of elements of $R(G)$. Let P_w denote a polynomial in the $C_l(\rho_k)$ of weight w, then any $x \in F^n(G)$ can be written as:

$$x = P_n(C_{i_1}(\rho_1), \cdots, C_{i_k}(\rho_k)) + y_{n+1}$$

where the ρ_j's are irreducible representations of G and $y_n \in F^{n+1}$. Then we also have:

$$\begin{aligned} x &= P_n(C_{i_1}(\rho_1), \cdots, C_{i_k}(\rho_k)) + P_{n+1}(C_{i_1}(\rho_1), \cdots, C_{i_k}(\rho_k)) + y_{n+2} \\ &= \sum_{l=1}^{m} P_{n+l}(C_{i_1}(\rho_1), \cdots, C_{i_k}(\rho_k)) + y_{m+1}, \end{aligned}$$

for any positive m. So x is in $\Gamma^n(G) + F^m(G)$ for all m, that is, x is in the topological closure $\overline{\Gamma^n(G)}$ of $\Gamma^n(G)$. But $\Gamma^n(G)$ is closed in $R(G)$, thus $x \in \Gamma^n(G)$. $\square$

We say that $R^*(G)$ is *saturated* if the natural map η is an isomorphism. A group G is *saturated* (over $\mathbb{K}$) if $R^*_{\mathbb{K}}(G)$ is saturated. For $H \leq G$, if the induction $i_* : R(H) \to R(G)$ is compatible with the filtration (Γ^n), then H is said to be Γ-*compatible* with G.

Lemma 3.3.5. *If the restriction maps $i^* : R(G) \to R(H)$ are surjective for all $H \leq G$, then G is saturated.*

Proof. First note that if i^* is surjective then each $i^*_{\Gamma^M} : \Gamma^M(G) \to \Gamma^M(H)$ is surjective: a monomial in the $\gamma^l(\rho_k)$, with $\rho_k \in R(H)$ is just the image by i^* of $\gamma^l(\sigma_k)$, with $\sigma_i \in R(G)$ satisfying $i^*(\sigma_k) = \rho_k$. So let $\rho \in \Gamma^M(H)$ and pick some $\sigma \in \Gamma^M(G)$ such that $\rho = i^*(\sigma)$. Then:

$$i_*(\rho) = i_* i^*(\sigma) = i_*(1)\sigma \in \Gamma^M(G).$$

So all virtual characters in $F^n(G)$ (which are induced from subgroups of G) are also in $\Gamma^n(G)$, and thus $\mathcal{R}^*(G) = R^*(G)$. □

Remark. (i) In Section 3.3, we use Lemma 3.3.5 to show that Abelian groups are saturated. So $R^*(-)$ is a Mackey functor when restricted to abelian groups.

(ii) We show in Proposition 3.4.6 that the converse of Lemma 3.3.5 is not true: the dihedral group of order D_p for p odd is saturated, but restriction of representations to C_p is not surjective.

The following result implies that the saturated graded ring of G is completely determined by that of its Sylow subgroups. It is a consequence of [Web, Cor. 3.7 and Prop. 7.2]; for a more concrete proof, see for example [AM69, Th. 6.6].

Theorem 3.3.6. *Let $G \geq H \geq Syl_p(G)$ where $Syl_p(G)$ is a p-Sylow of G and let $\mathcal{R}^*(G)_{(p)}$ denote the p-primary component of $\mathcal{R}^*(G)$. Then:*

$$\mathrm{Res}^G_H : \mathcal{R}^*(G)_{(p)} \longrightarrow \mathcal{R}^*(H)_{(p)}$$

is injective, and its image consists of the stable elements in $\mathcal{R}^*(H)_{(p)}$

A similar result to that due to Swan in cohomology (see [Swa60]) can be obtained as a straightforward application of Theorem 3.3.6.

Corollary 3.3.7 (Swan's Lemma)**.** *If $H \trianglelefteq G$ is a normal subgroup such that $H \supseteq Syl_p(G)$, then*

$$\mathcal{R}^*(G)_{(p)} \cong \mathrm{Im}(\mathrm{Res}^G_H) = \mathcal{R}^*(H)^{G/H}_{(p)}$$

Proof. If H is normal, the stability condition becomes $c_g(x) = x$, that is, x is invariant by the action of G/H. □

Corollary 3.3.8. *If $H := Syl_p(G)$ is abelian, then*

$$\mathrm{Res}^G_H : \mathcal{R}^*(G)_{(p)} \longrightarrow \mathcal{R}^*(H)^{N_G(H)}$$

is an isomorphism.

Proof. See [AM69, Th 6.8]. □

Corollary 3.3.9. *Let $H = Syl_p(G)$ be a p-Sylow subgroup. Then the induction map*

$$\mathrm{Ind}^G_H : \mathcal{R}^*(H) \to \mathcal{R}^*(G)_{(p)}$$

is surjective.

Proof. First note that since $\mathcal{R}^*(H)$ is p-torsion, the image of Ind_H^G is indeed contained in $\mathcal{R}^*(G)_{(p)}$. Pick an element $x \in \mathcal{R}^*(G)_{(p)}$, then $\operatorname{Ind}_H^G \operatorname{Res}_H^G(x) = [G : H]x$, and $[G : H]$ is invertible in $\mathcal{R}^*(G)_{(p)}$. $\square$

Note that since the induction map preserves the F-filtration, it is continuous with respect to the topology induced by it (and thus with respect to the Γ and I-adic topologies). In particular, induction extends to a well-defined map of completed rings

$$\widehat{\operatorname{Ind}_H^G} : \widehat{R}(H) \to \widehat{R}(G)$$

and by Corollary 3.3.9 the characters induced from Sylow subgroups of G form a dense subset of the completed ring $\widehat{R}(G)$. In other words, we have the following variant of Artin's theorem (see [Ser77, Th. II.9.17]):

Theorem 3.3.10. *Let X be a family of subgroups of a finite group G. Let*

$$\widehat{\operatorname{Ind}} : \bigoplus_{H \in X} \widehat{R}(H) \to \widehat{R}(G)$$

be the morphism defined on each $\widehat{R}(H)$ by $\widehat{\operatorname{Ind}_H^G}$. If X contains a p-Sylow of G for all p, then the map $\widehat{\operatorname{Ind}}$ is surjective.

Proof. By Corollary 3.3.9, the characters induced from H_p form a dense subset of $\widehat{R}(G)_{(p)}$ for the F-topology, so if X contains a p-Sylow of G for every p then $\widehat{\operatorname{Ind}}$ is surjective.

$\square$

3.4 Computing saturated rings

We now apply Section 3.3 by trying our hand at some computations; a number of the groups mentioned in Chapter 2 (including all abelian groups) are saturated, as we show below. In general, it is much more difficult to compute saturated rings than usual graded character rings, due to the complexity of the saturated filtration. This is where Corollary 3.3.8 comes into play, as we show with the example of the projective special linear group $PSL(2, q)$. For convenience, when the groups $H \leq G$ are clear from the context, we denote the induction $\mathrm{Ind}_H^G : R(H) \to R(G)$ by i_* and the restriction $\mathrm{Res}_H^G : R(G) \to R(H)$ by i^*. For the rest of this section, we fix $\mathbb{K} = \mathbb{C}$.

3.4.1 Saturated groups

Abelian groups, dihedral groups of order $2p$ and the quaternion group of order 8 are all saturated (over $\mathbb{C}$).

Proposition 3.4.1. *Abelian groups are saturated over* $\mathbb{C}$.

Proof. Let G be an abelian group and define $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^*)$. Then any abelian group homomorphism $\phi : G \to H$ induces a map $\widehat{\phi} : \widehat{H} \to \widehat{G}$, which is injective if and only if ϕ is surjective. Additionally, there is a natural isomorphism between G and its double dual $\widehat{\widehat{G}}$ given by associating to g the evaluation at g.
Now if $H \leq G$, then the injection $H \to G$ induces a map $\widehat{\phi} : \widehat{G} \to \widehat{H}$, and also a map $\widehat{\widehat{\phi}} : \widehat{\widehat{H}} \to \widehat{\widehat{G}}$. The latter is injective, which means by the above that $\widehat{\phi}$ is surjective. Thus the characters of H all come from restrictions of characters of G, and G is saturated. $\square$

We now turn to the quaternion group $Q_8 == \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$. The group Q_8 has 5 conjugacy classes: $\{1\}$, $\{-1\}$, $\{\pm i\}$, $\{\pm j\}$, $\{\pm k\}$ so 5 irreducible representations on $\mathbb{C}$. They are as follows:

(i) In dimension 1, the trivial representation, and the characters $\rho_1 : \begin{cases} i \mapsto 1 \\ j \mapsto -1 \end{cases}$, $\rho_2 = -\rho_1$ and $\rho_3 = \rho_1\rho_2$,

(ii) and in dimension 2, the representation Δ:

$$\Delta(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \Delta(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Delta(k) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Let us recall Theorem 2.5.4:

Lemma 3.4.2. *Let ρ_1 be the character of Q_8 defined by $\rho_1(i) = 1$, $\rho_1(j) = -1$, let $\rho_2 = -\rho_1$, and let Δ be irreducible character of degree 2 of Q_8 sending i, j, k to 0. Then*

$$R^*(Q_8) = \frac{\mathbb{Z}[x_1, x_2, y]}{(2x_i, 8y, x_i^2, x_1x_2 - 4y)},$$

where $x = c_1(\rho_1)$, $y = c_1(\rho_2)$ and $y = c_2(\Delta)$.

We also need a result from [GM14]:

Lemma 3.4.3 ([GM14, Prop 3.4])**.** *Let C_N be the cyclic group of order N and ρ a generating representation for $R(C_N)$. Then*

$$R^*(C_N) = \frac{\mathbb{Z}[t]}{(Nt)},$$

where $t = c_1(\rho)$.

Proposition 3.4.4. *The quaternion group Q_8 is saturated.*

Proof. The quaternion group contains one subgroup isomorphic to C_2, which is generated by -1, and three subgroups isomorphic to C_4, which all contain -1 and are generated respectively by i, j and k. Since all these groups are saturated, we only need to check that the maximal saturated subgroup $H = \langle k \rangle \cong C_4$ is Γ-compatible with Q_8, which we do by showing that, if ρ is the generating representation of $R(C_4)$, then each induced character $\mathrm{Ind}_{C_4}^{Q_8}(C_1(\rho)^n)$ is in $\Gamma^n(Q_8)$. Note first that $\mathrm{Ind}_{C_4}^{Q_8}(C_1(\rho)) \in \Gamma^1(Q_8) = I_{Q_8}$. Moreover, the representation Δ restricts on C_4 to $\rho + \rho^{-1}$, and so

$$\mathrm{Res}_{C_4}^{Q_8}(y) = c_2(\rho + \rho^{-1}) = -c_1(\rho)^2 = -t^2.$$

Therefore $C_1(\rho)^2 = \mathrm{Res}(-C_2(\Delta))$, and so

$$i_*(C_1(\rho)^2) = i_*(i^*(-C_2(\Delta))) = -\mathbb{C}[Q_8/C_4] \otimes C_2(\Delta) \in \Gamma^2(Q_8).$$

Thus, for any $n = 2m + l$ with $l = 0, 1$:

$$\begin{aligned} i_*(C_1(\rho)^n) = i_*(C_1(\rho)^{2m+l}) &= i_*\left(C_1(\rho)\right) \cdot i^*(-C_2(\Delta))^m) \\ &= i_*(C_1(\rho)) \cdot (-C_2(\Delta))^m, \end{aligned}$$

which is an element of $\Gamma^l \cdot \Gamma^{2m}$. This means that C_4 is Γ-compatible with Q_8, and therefore Q_8 is saturated. $\square$

With a similar method, we can prove that dihedral groups are saturated. By Proposition 2.3.4:

Lemma 3.4.5. *Let p be an odd prime, and let $D_p = \langle \sigma, \tau \mid \tau^2 = \sigma^p = 1, \tau\sigma\tau = \sigma^{-1} \rangle$ be the dihedral group of order p. Let χ be the irreducible character of D_p of degree 2, sending τ to 0 and σ to $2cos(\frac{2\pi}{p})$, then*

$$R^*(D_p) = \frac{\mathbb{Z}\left[x, y\right]}{(2x, py, xy)},$$

where $x = c_1(\chi)$ *and* $y = c_2(\chi)$.

Proposition 3.4.6. *Let* p *be an odd prime, then the dihedral group* D_p *of order* $2p$ *is saturated.*

Proof. Since $D_p = C_p \rtimes C_2$ and C_p, C_2 are abelian, these are the maximal saturated subgroups of D_p. The signature ε of D_p restricts on C_2 to the representation ρ, which generates $R(C_2)$. Thus C_2 is Γ-compatible with D_p, and we only need to look at C_p.

Since $\mathrm{Res}(Y) = -C_1(\rho)^2$ the same argument as in the proof of Proposition 3.4.4 applies.

□

3.4.2 Projective linear groups

We compute the saturated character ring of $G = PSL(2,p)$, the projective special linear group over $\mathbb{F}_p$, where p is an odd prime such that $p \equiv 3, 5 (\mathrm{mod}\ 8)$. Note that we do not use any information about the character table of G: we only need to know those of its Sylow subgroups, which are all abelian. For each prime l dividing $|G| = \frac{p(p+1)(p-1)}{2}$, let $H_l = Syl_l(G)$ and $N_l = N_G(H_l)$. For each l, we determine the l-Sylow of G and the action of its normalizer, then deduce the stable element subring. There are 4 possible cases:

(i) $l = p$. Then $H_p \cong C_p$ is generated by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The normalizer of H_p is the group:

$$N_p = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in PSL(2,p) \right\},$$

with action

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & a^2 n \\ 0 & 1 \end{pmatrix},$$

inducing $\rho \mapsto \rho^{a^2}$ for a generator ρ of $R(C_p)$. On $\mathcal{R}^*(C_p) \cong \frac{\mathbb{Z}[x]}{(px)}$, this induces $x \mapsto a^2 x$. The subring generated by $x^{\frac{p-1}{2}}$ is stable by this action, and conversely if a is an element of multiplicative order $(p-1)$, then a monomial x^m being stable by the action $x \mapsto a^2 x$ implies that m is a multiple of $\frac{p-1}{2}$. Thus

$$\mathcal{R}^*(H_p)^{N_p} \cong \frac{\mathbb{Z}[u]}{(pu)}, \qquad |u| = \frac{p-1}{2}. \tag{3.4.1}$$

(ii) l is an odd prime dividing $(p-1)$. Then $H_l \cong C_{l^i}$ for some integer i, generated by $\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}$ for some n of order l^i in $\mathbb{F}_p^\times$. A straightforward computation gives that N_l is generated by diagonal matrices (which commute with the elements of H_l) together with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which sends an element $h \in H_l$ to its inverse. The induced action on the representation ring is $\rho \mapsto \rho^{-1}$, which translates as $x \mapsto -x$ in the graded ring. Thus

$$\mathcal{R}^*(H_l)^{N_l} \cong \frac{\mathbb{Z}[x]}{(l^i x)}, \qquad |x| = 2. \tag{3.4.2}$$

(iii) $l = r$ is an odd prime dividing $p+1$. We prove that H_r is cyclic. Note that the r-Sylow of G is isomorphic to that of $G' := PSL(2, p^2)$ since the index of G in G' is coprime to r. Let $\alpha \in \mathbb{F}_{p^2}^\times$ have multiplicative order r^i. The matrix $A' = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ generates a cyclic group H_r' of order r^i

in G', which is thus an r-Sylow subgroup. We have $\alpha \notin \mathbb{F}_p^\times$, however any matrix of G similar to A generates an isomorphic group in G. One can take for example $A = \begin{pmatrix} 0 & -1 \\ 1 & \alpha + \alpha^{-1} \end{pmatrix}$, the companion matrix to the minimal polynomial of α.

The normalizer N'_r of C'_{r^i} in G' is a dihedral group of order $p^2 - 1$, generated by all diagonal matrices together with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which sends A to its inverse. The change of basis sending A to A' allows us to view N_r as a subgroup of N'_r, and thus the elements of N_r act either trivially or by inversion on H_r.

It remains to show that there exists a matrix $S \in G$ such that $S^{-1}AS = A^{-1}$. Let $a = \alpha + \alpha^{-1}$. By a direct calculation, one shows that any matrix of the form $\begin{pmatrix} -x & y \\ ax + y & x \end{pmatrix}$ in $GL(2, p)$ satisfies this property, thus $S \in PSL(2, p)$ exists if and only if there is a pair $(x, y) \in \mathbb{F}_p^2$ such that $-x^2 - axy - y^2 = 1$. This equation is equivalent to $X^2 + 1 = bY^2$, with $X = x + \frac{a^2}{4}y$, $Y = y$ and $b = \frac{a^2}{4} - 1$. There are $(p+1)/2$ squares in $\mathbb{F}_p$ (including 0), so there are $(p+1)/2$ elements of the form $X^2 + 1$, and, if $b \neq 0$ then there are also $(p+1)/2$ elements of the form bY^2. Thus whenever $b \neq 0$, the set of elements of the form $X^2 + 1$ and the set of elements of the form bY^2 have nontrivial intersection, and there is a solution to $x^2 + axy + y^2 = -1$. Now, $b = 0$ if and only if $a^2 = 4$, that is, $a = \pm 2 (\mathrm{mod}\ p)$. But then α is a solution of $t^2 \pm 2t + 1$, that is, $\alpha = \alpha^{-1}$ has multiplicative order 2, in contradiction with our assumption. Thus b is always nonzero, which completes the proof.

We have:

$$\mathcal{R}^*(H_r)^{N_r} \cong \frac{\mathbb{Z}[y]}{(r^i y)}, \qquad |y| = 2. \tag{3.4.3}$$

(iv) $l = 2$. Since $p \equiv 3, 5 (\text{mod } 8)$, the 2-Sylow subgroup of G has order 4. There are two cases:

- if $p \equiv 5 (\text{mod } 8)$, then -1 is a quadratic residue, so let a satisfy $a^2 \equiv -1 (\text{mod } p)$. Then

$$H_2 = \left\langle h_1 := \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, h_2 := \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \right\rangle.$$

We show that $N_2 \cong A_4$. First, we have $C_G(h_1) \cap N_G(H_2) = \{Id\}$, as a direct calculation shows, and similarly for h_2 and $h_1 h_2 =: h_3$. Therefore, if $N \in N_2$ acts nontrivially on H_2, it must permute all 3 nontrivial elements. If $T = \begin{pmatrix} x & -ax \\ x & ax \end{pmatrix}$, with $x^2 = \frac{1}{2a}$, then $T h_1 T^{-1} = h_2$ and $T h_2 T^{-1} = h_3$. Both 2 and a are nonresidues mod p since $p \equiv 5 (\text{mod } 8)$ and if a were a residue, then $PSL(2, p)$ would contain an element of order 4, contradicting $H_2 \cong C_2 \times C_2$. Thus there is an x satisfying $x^2 = 1/2a$. Moreover T is unique up to multiplication by an element of $C_G(H_2) = H_2$, which shows that $N_2 = \langle T, H_2 \rangle \cong A_4$.

- if $p \equiv 3 (\text{mod } 8)$, then -2 is a residue, so let b satisfy $b^2 \equiv -2 (\text{mod } p)$. Then

$$H_2 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} b & 1 \\ 1 & -b \end{pmatrix} \right\rangle$$

Again, we have $N_2 \cong A_4$ acting by cyclic permutations, generated by H_2 together with the matrix $T = \begin{pmatrix} \frac{1}{b} & \frac{1}{b} \\ -\frac{b+2}{2} & \frac{b-2}{2} \end{pmatrix}$.

In both cases the normalizer acts as cyclic permutations on the nontrivial elements of H_2, and thus:

$$\mathcal{R}^*(H_2)^{N_2} \cong \frac{\mathbb{Z}[z,t]}{(2z, 2t, z^3 - t^2)}, \qquad |z| = 2, |t| = 3. \tag{3.4.4}$$

Putting all of this together, we get:

Theorem 3.4.7. *Let $G = PSL(2,p)$ be the projective special linear group over $\mathbb{F}_p$, where p is an odd prime such that $p \equiv 3, 5 (\mathrm{mod}\ 8)$. Write:*

$$|G| = 4 \cdot p \cdot l_1^{i_1} \cdots l_n^{i_n} \cdot r_1^{j_1} \cdots r_m^{j_m}, \qquad \text{with} \quad l_k | (p-1), \quad r_k | (p+1).$$

Then:

$$\mathcal{R}^*(G) \cong \frac{\mathbb{Z}[x_1, \cdots, x_n, y_1, \cdots y_m, z, t, u]}{(l_k^{i_k} x_k, r_k^{j_k} y_k, 2z, 2t, pu, z^3 - t^2)} \tag{3.4.5}$$

with $|x_k| = |y_k| = |z| = 2$, $|t| = 3$, $|u| = (p-1)/2$, and:

(i) $x_k = \mathrm{Ind}_{H_{l_k}}^G (x_k^2)$, $y_k = \mathrm{Ind}_{H_{r_k}}^G (y_k^2)$ *where x_k (resp. y_k) is a generating class of the ring $R^*_{\mathbb{C}}(C_{l_k^{i_k}})$ (resp. $R_{\mathbb{C}}(C_{r_k^{j_k}})$).*

(ii) $u = \mathrm{Ind}_{H_p}^G (u)^{(p-1)/2})$ *where*

(iii) $z = \mathrm{Ind}_{H_2}^G (t_1^2 + t_1 t_2 + t_2^2)$ *and* $t = \mathrm{Ind}_{H_2}^G (t_1^3 + t_1^2 t_2 + t_2^3)$

Remark. For $p = 3$, this is the saturated ring $\mathcal{R}^*(A_4)$.

3.5 Tambara functors, the ungraded case

After discussing whether the graded character ring functor is Mackey, it seems natural to turn to the theory of Tambara functors, which was introduced by Tambara in [Tam93]; they can be understood as Mackey functors $S(-)$ that are equipped, for each subgroup $H \leq G$, with a multiplicative transfer map $S(H) \to S(G)$. In cohomology, this is the Evens norm map (see for example [CTVEZ03, Ch. 6]). In the case of graded character rings, tensor induction of representations is a natural candidate for the role of the multiplicative transfer. We must begin, however, with the ungraded situation: the fact that the multiplicative transfer turns $K_G(X)$ into a Tambara functor is mentioned without proof in both [Str12] and [Tam93], and we propose here a proof for the sake of completeness.

To define Tambara functors, we need the notion of exponential diagrams. Let $G\mathsf{set}/X, G\mathsf{set}/Y$ be the categories of G-sets over X, Y respectively, and let an equivariant map $f : X \to Y$ be given. The pullback functor $G\mathsf{set}/Y \to G\mathsf{set}/X$ has a right adjoint $\Pi_f : G\mathsf{set}/X \to G\mathsf{set}/Y$, which we now describe. Let $p : A \to X$ be a set over X. We construct $q : \Pi_f A \to Y$ as follows:

$$\Pi_f A = \bigsqcup_{y \in Y} \mathrm{sec}_p(f^{-1}(y), A),$$

where we write $\mathrm{sec}_p(U, A)$, given a subset $U \subset X$, for the set of all sections of p over U, that is, maps $s : U \to A$ such that $p \circ s(u) = u$ for all $u \in U$.

Then $\Pi_f A$ is a G-set if we define ${}^g s : f^{-1}(gy) \to A$, $x \mapsto g \cdot s(g^{-1} \cdot x)$, and of course there is an obvious map $\Pi_f A \to Y$. The adjointness property means that, as is easily established,

$$\mathrm{Hom}_{G\mathsf{set}/X}(P(B), A) \cong \mathrm{Hom}_{G\mathsf{set}/Y}(B, \Pi_f A)$$

for all appropriate A, B, where P is the pullback functor. In particular for $B = \Pi_f A$, there is an element $e \in \mathrm{Hom}_{G\mathsf{set}/X}(P(\Pi_f A), A)$ corresponding to the identity of B. It is involved in the following commutative diagram:

$$\begin{tikzcd} X \times_Y \Pi_f A \arrow[r, "e"] \arrow[d, "f'"'] & A \arrow[r, "p"] & X \arrow[d, "f"] \\ \Pi_f A \arrow[rr, "q"'] & & Y \end{tikzcd} .$$

Seeing the pullback as pairs (x, s) with $x \in X$, and $s \in \mathrm{sec}_p(f^{-1}f(x), A)$ such that $f(x) = q(s)$, the map f' is just a projection on the second coordinate. A diagram isomorphic to the one above is called an exponential diagram.

Definition ([Tam93, §2])**.** Let X, Y be G-sets and $f : X \to Y$ a G-set map. A semi-Tambara functor is a function $S(-)$ associating to X, Y, (semi)-rings $S(X), S(Y)$ and to f three maps $f^* : S(Y) \to S(X)$, $f_*, f_\sharp : S(X) \to S(Y)$ such that the following conditions are satisfied:

(i) f^*, f_*, $f_\sharp$ are homomorphism of rings, additive monoids, multiplicative monoids respectively.

(ii) The triples (S, f^*, f_*) and $(S, f^*, f_\sharp)$ form semi-Mackey functors.

(iii) If

$$\begin{tikzcd} X' \arrow[r, "e"] \arrow[d, "f'"'] & Z \arrow[r, "p"] & X \arrow[d, "f"] \\ Y' \arrow[rr, "q"'] & & Y \end{tikzcd}$$

is an exponential diagram, then the corresponding diagram

$$\begin{tikzcd} S(X') \arrow[d, "f'_\sharp"'] & S(Z) \arrow[l, "e^*"'] \arrow[r, "p_*"] & S(X) \arrow[d, "f_\sharp"] \\ S(Y') \arrow[rr, "q_*"'] & & S(Y) \end{tikzcd}$$

commutes.

If additionally, S associates a ring to a G-set, and f_* is a homomorphism of additive groups, then S is a Tambara functor.

Theorem 3.5.1. *The functor $K_G^+(-)$ with the restriction, induction and tensor inducion maps described in Section 2.1, is a semi-Tambara functor.*

Proof. Since $(K_G^+(-), f^*, f_*)$ is a Mackey functor, so we only concern ourselves with the properties of $f_\sharp$.

(i) The fact that $f_\sharp$ is a homomorphism of multiplicative monoids follows from the properties of the tensor product.

(ii) To show that $(S, f^*, f_\sharp)$ is a Mackey functor, we check both axioms from the definition in Section 3.2. Let

$$\begin{tikzcd} \Omega_1 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \Omega_2 \arrow[d, "\gamma"] \\ \Omega_3 \arrow[r, "\delta"'] & \Omega_4 \end{tikzcd}$$

be a pullback diagram of G-sets, then we want to check that $\delta^*\gamma_\sharp = \beta_\sharp\alpha^*$. Note that because any G-set can be expressed as a disjoint union of orbits, it is sufficient to check this axiom on pullback diagrams of the form:

$$\begin{tikzcd} \Omega \arrow[r, "\alpha"] \arrow[d, "\beta"'] & G/K \arrow[d, "\gamma"] \\ G/H \arrow[r, "\delta"'] & G/J \end{tikzcd}$$

for $H, K \leq J \leq G$. Let W be a vector bundle over G/K and let $V = \delta^*\gamma_\sharp(W)$, then for $x \in G/H$, we have:

$$V_x = (\delta^*\gamma_\sharp(W))_x = \bigotimes_{t \in G/K} W_t,$$

where the tensor product is taken over all of the $tK \in G/K$ such that $\gamma(tK) = \delta(xH)$. The action of g takes V_x to $V_{g \cdot x}$ and can be written $g \cdot \bigotimes_{tK \in G/K} v_{tK} = \bigotimes_{tK \in G/K} g \cdot v_{g^{-1}tK}$. The vector bundle $E := \beta_\sharp \alpha^*(W)$ is defined by

$$E_x = (\beta_\sharp \alpha^*(W))_x = \bigotimes_{s \in G/K} W_s,$$

where the tensor product is taken over all s such that $(sK, xH) \in \Omega$, which is equivalent to requiring $\gamma(tK) = \delta(xH)$. The action of G is given by $g \cdot \bigotimes_{s \in G/K} e_s = \bigotimes_{s \in G/K} g \cdot e_{g^{-1} \cdot s}$, and thus E and V are isomorphic vector bundles.

For the second axiom, consider two finite G-sets Ω, Ψ and the corresponding inclusion maps $i_\Omega, i_\Psi : \Omega, \Psi \to \Omega \sqcup \Psi$. Then the map $f_\sharp : K_G^+(\Omega) \oplus K_G^+(\Psi) \to K_G^+(\Omega \sqcup \Psi)$ whose components are given by $i_{\Omega,\sharp}, i_{\Psi,\sharp}$ should be an isomorphism. This is obviously the case.

(iii) Again, we can assume without loss of generality that $A = G/K, X = G/H, Y = G/J$ where $K \leq H \leq J$. We denote by $\pi_K^H : G/K \to G/H$ the map sending a coset tK to the corresponding coset tH in G/H, and similarly for π^J_K, π^J_H. Then the set $\Pi_{\pi^J_H} G/K$ above G/J is the set of sections $s : {\pi^J_H}^{-1}(yJ) \to G/K$ such that for any $tK \in G/K$ satisfying $tJ = yJ$, we have $s(tH) = tK$. Consider the diagram:

$$\begin{tikzcd} \Pi_{\pi^J_H} G/K \bigtimes_{G/J} G/K \arrow[r, "e"] \arrow[d, "f"'] & G/K \arrow[r, "\pi^H_K"] & G/H \arrow[d, "\pi^J_H"] \\ \Pi_{\pi^J_H} G/K \arrow[rr, "q"'] & & G/J \end{tikzcd}$$

where e, f are projections and q is the map sending a section $s : {\pi^J_H}^{-1}(yJ) \to G/K$ to yJ. The third axiom for Tambara functors says that the corresponding diagram:

$$\begin{tikzcd} K^+_G\left(\Pi_{\pi^J_H} G/K \bigtimes_{G/J} G/K\right) \arrow[d, "f_\sharp"'] & K^+_G(G/K) \arrow[l, "e^*"'] \arrow[r, "\pi^H_{K*}"] & K^+_G(G/H) \arrow[d, "\pi^J_{H\sharp}"] \\ K^+_G\left(\Pi_{\pi^J_H} G/K\right) \arrow[rr, "q_*"'] & & K^+_G(G/J) \end{tikzcd}$$

should commute. For convenience, let $X = \Pi_{\pi^J_H} G/K \bigtimes_{G/J} G/K$. Consider $W \in K^+_G(G/K)$, then on the one hand:

$$V_{yJ} := \left(\pi^J_{H\sharp}\pi^H_{K*}(W)\right)_{yJ} = \bigotimes_{xH \subseteq yJ} \left(\bigoplus_{tK \subseteq xH} W_{tK}\right)$$

and on the other hand:

$$E_{yJ} := \left(q_* f_\sharp e^*(W)\right)_{yJ} = \bigoplus_{s \in q^{-1}(y)} \left(\bigotimes_{(s,tK) \in X} W_{tK}\right).$$

The fact that $V_{yJ} \cong E_{yJ}$ as vector spaces comes from the distributivity property of the tensor product with respect to the direct sum, as well as the definition of the exponential functor $\Pi_{\pi_H^J} : G\mathsf{set}/(G/H) \to G\mathsf{set}/(G/J)$. To construct each term of the sum in E_{yJ}, we pick a section $s : {\pi_H^J}^{-1}(yJ) \to G/K$. Each term is then a product of all spaces of the form W_{tK} with $tK = s(xH)$ for $xH \in {\pi_H^J}^{-1}(yJ)$. Summing over all possible such sections s, we get all possible combinations of factors in V_{yJ}. So E_{yJ} is just a rewriting of V_{yJ}. The action of $g \in G$ is given by

$$g \cdot \left(\bigotimes_{xH \subseteq yJ} \left(\bigoplus_{tK \subseteq xH} w_{tK} \right) \right) \mapsto \bigotimes_{xH \subseteq yJ} \left(\bigoplus_{tK \subseteq xH} g \cdot w_{g^{-1} \cdot tK} \right).$$

On the other hand, the action of G on $\Pi_{\pi_H^J} G/K$ is given by $g \cdot s = {\pi_H^J}^{-1}(gyJ) \to G/K,\ gxH \mapsto g \cdot s(g^{-1}(gxH))$, that is, $g \cdot s$ maps $g \cdot xH$ to $g \cdot s(xH)$. This means that the permutation of the factors induced by the action of g on E is the same as the one on V.

$\square$

The following result by Tambara shows that, in fact, $K_G(X)$ is a Tambara functor. For an abelian monoid M, let γM be the universal abelian group with monoid map $k_M : M \to \gamma M$, and generators $k_M(m)$ for $m \in M$ and relations $k_M(m + m') = k_M(m) + k_M(m')$ for $m, m' \in M$. If M is a semi-ring, then γM has a unique ring structure such that k_M is a semi-ring map.

Theorem 3.5.2 ([Tam93, Th. 6.1])**.** *Let S be a semi-Tambara functor. Then the function which assigns the set $\gamma S(X)$ to each G-set X has a unique structure of a Tambara functor such that the maps $k_{S(X)}$ form a morphism of semi-Tambara functors.*

Corollary 3.5.3. *The functor $K_G(-)$ has the structure of a Tambara functor.*

3.6 The addition formula

A formula for the norm of the sum of two characters would enable us to compute the value of the norm map on negative virtual characters, a necessary step in determining whether the norm map preserves the Grothendieck filtration. Strikingly, there is no known general formula for the tensor induction of a sum of characters, or its cohomological equivalent, the Evens norm of a sum of classes. Below, we first establish a formula for the sum of two positive characters after [Tam93, §4]; we then use this formula to determine $\mathcal{N}_H^G(-\rho)$ for $\rho \in R^+(H)$, in the case of a normal subgroup H of prime index in G, which gives us an explicit expression for the norm of a virtual character in this case. We then prove that in the case of abelian groups, the norm map preserves the Grothendieck filtration, and thus $R^*(G)$ is a Tambara functor on abelian groups.

3.6.1 A general formula for positive representations

The following is an application of [Tam93, §4], where Tambara gives a general addition formula for the norm. Let X, Y be G-sets and let $f : X \to Y$ be a G-map. As usual, we assume $X = G/H$, $Y = G/K$ with $H \leq K \leq G$, and $f = \pi_K^H$. Moreover, we can restrict ourselves to $K = G$. So $Y = G/G = \bullet$, the one point set. Let:

$$V = \{C \mid C \subset G/H\} =: \mathcal{P}(G/H)$$
$$U = \{(xH, C) \mid xH \in C, C \subset G/H\}$$

Then we have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "r"] \arrow[d, "t"'] & G/H \arrow[d, "f"] \\ V \arrow[r, "s"'] & \bullet \end{tikzcd}$$

where r, t are projection maps. Let

$$\chi := t_\sharp r^* : K_G^+(G/H) \to K_G^+(V),$$

then for a vector bundle $W \in K_G^+(G/H)$, the vector bundle $\chi(W) \in K_G^+(V)$ associates to each $C = \{x_1, \cdots, x_n\}$ the vector space $W_{x_1} \otimes \cdots \otimes W_{x_n} \cong W^{\otimes n}$, and to each $g \in G$ the linear map given by:

$$g \cdot \left(\bigotimes_{x_i \in C} w_{x_i} \right) = \bigotimes_{x_i \in C} \left(g \cdot w_{g^{-1} \cdot x_i} \right).$$

So the reader must bear in mind that, in our current notation, for a representation ρ of H corresponding to some vector bundle $x \in K_G^+(G/H)$, we have:

$$f_\sharp(x) = \mathcal{N}_H^G(x) = \chi(x)_{G/H} \tag{3.6.1}$$

Note that, since it involves the map $t_\sharp$, the morphism χ is only defined on $K_G^+(G/H)$ for now. Throughout this section, we determine how to extend χ to vector bundles with (all) negative coefficients, then to all virtual bundles.

We define a ring operation $\vee$ on the group $K_G(V)$ as follows: let $V^{(2)}$ be the G-set of pairs (C_1, C_2) of disjoint subsets of G/H. Let p_1, p_2, m be the G-maps taking (C_1, C_2) to C_1, C_2, $C_1 \sqcup C_2$, respectively. Then for $z, t \in K_G(V)$, we let:

$$(z \vee t) = m_*(p_1^*(z) \cdot p_2^*(t)).$$

This operation does not involve multiplicative norms (that is, it does not involve $f_\sharp$ for some map f), thus it is well-defined on the whole ring $K_G(V)$, and not just the semi-ring $K_G^+(V)$.

Each fiber in a vector bundle is a representation of the stabilizer of the point above which it sits; for purposes of intuition, we point out that, as a representation of $\mathrm{Stab}(C)$, we have

$$(z \vee t)_C = \bigoplus \mathrm{Ind}_{\mathrm{Stab}(C_1, C_2)}^{\mathrm{Stab}(C)}(z_{C_1} \otimes t_{C_2}), \tag{3.6.2}$$

where the direct sum is taken over all orbit representatives under $\mathrm{Stab}(C)$ of pairs (C_1, C_2) such that $C_1 \sqcup C_2 = C$. Since this operation only involves restrictions and inductions, it is defined for virtual characters.

By [Tam93, Prop. 4.4], the map χ is is a morphism from the monoid $(K_G^+(G/H), +)$ to $(K_G(V), \vee)$. In particular, for $\tau, \sigma \in K_G^+(G/H)$, we have:

$$f_\sharp(\sigma + \tau) = \chi(\sigma + \tau)_{G/H} = \left(\chi\left(\sigma\right) \vee \chi\left(\tau\right)\right)_{\{G/H\}}. \tag{3.6.3}$$

We now assume that H is a normal subgroup of G. In terms of representations, we introduce the following notation for purposes of intuition: for a representation $\rho \in R^+(H)$ and $C \subset G/H$, write

$$\rho^{\otimes C} := \chi(\rho)_C. \tag{3.6.4}$$

This is meant to remind us of the following description. Pick a transversal set $T = \{t_1, \cdots t_n\}$ for G/H and let $C \subset G/H$. Then, as a $\mathrm{Stab}\, C$-module, we have:

$$\rho^{\otimes C} = \bigotimes_{t_i} \rho^{t_i},$$

where ρ^{t_i} is the representation ρ conjugated by $t_i \in G$, and the sum is over those t_i, whose image in G/H is in C; the action of $\operatorname{Stab} C$ is obvious. Note that, because H is normal in G, the representation $\rho^{\otimes C}$ does not depend on the choice of transversal set T, since a different coset representative t_i' of t_iH would be t_i^h for some $h \in H$, and ρ is invariant under conjugation by an element of H.

As we recall below, one can extend χ to virtual characters. However, we shall refrain from using the notation $\rho^{\otimes C}$ when ρ is not known to be an actual representation, as it can be misleading. For example, if $\rho \in R^+(H)$, and one writes $(-\rho)^{\otimes C}$ for $\chi(-\rho)_C$, then one is tempted to guess that $\chi(-\rho)_C = \pm\chi(\rho)_C$; while Proposition 3.6.2 establishes just that when H has odd, prime index in G, Proposition 3.6.3 shows that it is erroneous in general.

Putting eqs. (3.6.1) to (3.6.4) together yields:

Proposition 3.6.1. *Let $\sigma, \tau \in R^+(H)$, where $H \trianglelefteq G$. Then:*

$$\mathcal{N}_H^G(\tau + \sigma) = \sum_{C \in \mathcal{O}(V)} \operatorname{Ind}_{\operatorname{Stab} C}^G \left(\tau^{\otimes C} \sigma^{\otimes C'}\right)$$

where C' denotes the complement of C in G/H, and $\mathcal{O}(V)$ is a complete set of orbit representatives of V under the action of G.

Again, this formula does not depend on the choice of orbit representatives, since choosing different representatives boils down to conjugating $\tau^{\otimes C}\sigma^{\otimes C'}$ by some $g \in G$, under which induction of representations is invariant. Note that, so far, the formula in Proposition 3.6.1 is only valid on $R^+(H)$.

Let us point out that, if G is abelian, then $\rho^{\otimes C} = \rho^{\otimes |C|}$ as an H-module;

there results a simplified formula for $\mathcal{N}_H^G(\sigma+\tau)$ in this case, especially when H has prime index in G. What we establish in the sequel is that this simplified formula holds *even when σ and τ are virtual*. This will be Lemma 3.6.6.

A key argument of the proof of [Tam93, Th 6.1], is that the image of χ lies in a subset of $K_G(V)$ that is a group for $\vee$. Thus, defining $\chi(-\tau)$ as the element $b \in K_G(V)$ such that $\chi(\tau) \vee b = 1$, extends χ to virtual characters in a way compatible with the addition formula. With b thus defined, one has $\mathcal{N}_H^G(-\tau) = b_{\{G/H\}}$, and the equation $\mathcal{N}_H^G(\sigma-\tau) = \chi(\sigma) \vee \chi(-\tau)$ is an explicit formula for the norm of any virtual character. Unfortunately, this formula is quite unpractical to apply, as the example below shows.

3.6.2 The prime normal case

We first extend the norm map to negative bundles, which allows us to determine an addition formula in the case where H is a normal subgroup of G of prime index p. We denote (slightly abusively) the extention of χ to negative (and generally, all virtual) characters, by χ as well. Throughout, we use the fact that a vector bundle above a G-set X is entirely determined by its fibre above each point x and the action of $\operatorname{Stab} x$ on it; any equality of vector spaces above a point x is to be understood as a canonical isomorphism of $\operatorname{Stab} x$-modules. We start with the case where p is odd:

Proposition 3.6.2. *Let $H \trianglelefteq G$ with $|G:H| = p$ an odd prime and let $W \in K_G^+(G/H)$. Then for any $C \subseteq G/H$:*

$$\chi(-W)_C = (-1)^{|C|}\chi(W)_C.$$

In particular, for $C = G/H$:

$$f_\sharp(-W) = -f_\sharp(W).$$

Proof. Let $a = \chi(W) \in K_G(V)$. Let $b \in K_G(V)$ satisfy $a \vee b = 1$, that is, $(a \vee b)_C = 0$ for any $C \neq \emptyset$. We proceed by induction on the cardinality of C. Note that, because H is normal of prime index in G, for any $C \subsetneq G/H$ we have $\mathrm{Stab}(C) = H$; thus, the vector space $\chi(W)_C := \otimes_{c \in C} W_c$, is an H-module (but not a G-module). When $C = G/H$, the vector space $\chi(W)_C$ is a G-module. We first treat the case $C \neq G/H$.

Because it is normal in G, the subgroup H stabilizes each $c \in C$ individually, and thus for any $C_1 \sqcup C_2 = C$ we have $a_{C_1} \otimes a_{C_2} \cong a_C$ as H-modules. To declutter notation, we write eg. a_{xy} for $a_{\{x,y\}}$. Let us explicitly state the first few steps of the induction, assuming in each case that $C \subsetneq G/H$. Throughout, we use Equation (3.6.2) repeatedly:

- If $C = \emptyset$ then $(a \vee b)_C = 1$.
- If $|C| = 1$ and (say) $C = \{x\}$ then:

  $$(a \vee b)_C = \mathrm{Ind}^H_{\mathrm{Stab}\, x}(a_x \otimes 1) + \mathrm{Ind}^H_{\mathrm{Stab}\, x}(1 \otimes b_x) = \mathrm{Ind}^H_H(a_x) + \mathrm{Ind}^H_H(b_x)$$

  and so $b_x = -a_x$ as H-modules.
- If $|C| = 2$, say $C = \{x, y\}$, then (omitting tensor product signs for simplicity of notation):

  $$\begin{aligned}(a \vee b)_C &= \mathrm{Ind}^H_H(a_{xy}) + \mathrm{Ind}^H_H(a_x b_y) + \mathrm{Ind}^H_H(a_y b_x) + \mathrm{Ind}^H_H(b_{xy}) \\ &= b_{xy} + a_{xy} - a_x a_y - a_y a_x,\end{aligned}$$

Since $a_{xy} \cong a_x a_y$ as H-modules, we have

$$b_{xy} = a_x a_y = a_{xy}.$$

- If $|C| = 3$, say $C = \{x, y, z\}$, then $a_{xyz} = a_{xy}a_z = a_x a_{yz}$ as H-modules, and:

$$\begin{aligned} b_{xyz} =& \operatorname{Ind}_H^H(-a_{xyz}) + \operatorname{Ind}_H^H(a_x a_{yz}) + \operatorname{Ind}_H^H(a_y a_{xz}) + \operatorname{Ind}_H^H(a_z a_{xy}) \\ &+ \operatorname{Ind}_H^H(a_{xy} a_z) + \operatorname{Ind}_H^H(-a_{xz} a_y) + \operatorname{Ind}_H^H(-a_{yz} a_x) \\ =& - a_{xyz}. \end{aligned}$$

- Suppose $b_C = (-1)^{|C|} a_C$ for $|C| < n$ and take $|C| = n < |G : H|$. Then

$$\begin{aligned} (a \vee b)_C &= \sum_{D \subset C} \operatorname{Ind}_{\operatorname{Stab}(C \setminus D, D)}^{\operatorname{Stab} C} (a_{C \setminus D} b_D) = b_C + \sum_{D \subsetneq C} \operatorname{Ind}_H^H((-1)^{|D|} a_{C \setminus D} a_D) \\ &= b_C + \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} a_C \\ &= b_C + (-1)^{n+1} a_C, \end{aligned}$$

and thus $b_C = (-1)^n a_C$.

The induction is complete.

We can now treat the case $C = G/H$. Then $\operatorname{Stab}(C) = G$, and $a_D \otimes a_{C \setminus D}$ is not a G-module for any $D \subsetneq C$. Let $\mathcal{O}(V)$ be a set of orbit representatives for the action of G on V, then, as G-modules, we have:

$$\begin{aligned}(a \vee b)_C &= a_C + b_C + \sum_{D \in \mathcal{O}(V)} \operatorname{Ind}_H^G \left((-1)^{|D|} a_D \otimes a_{C \setminus D}\right) \\ &= a_C + b_C + \operatorname{Ind}_H^G \left(\sum_{D \in \mathcal{O}(V)} (-1)^{|D|} a_D \otimes a_{C \setminus D} \right).\end{aligned}$$

One can pair the summands $(-1)^{|D|}(a_D \otimes a_{C\setminus D})$ and $(-1)^{|C\setminus D|}(a_{C\setminus D} \otimes a_D)$, which are isomorphic and of opposite sign (since p is odd). Hence the terms of the sum cancel and we have $b_C = -a_C$, that is

$$f_\sharp(-W) = -f_\sharp(W).$$

□

Note that the first step of the proof shows:

Proposition 3.6.3. *Let $H \leq G$ be a subgroup of index 2 and $W \in K_G^+(G/H)$. Let t be a representative for the non-trivial coset in G/H. Then, in $K_G(\bullet)$,*

$$f_\sharp(-W) = -f_\sharp(W) + f_*(W \otimes W_t)$$

Proof. In this case, the orbit representatives of ordered pairs of disjoint sets of cosets of H in G are $(\{1\},\{t\})$, $(\{1,t\},\emptyset)$ and $(\emptyset,\{1,t\})$. Using notation as in the above proof of Proposition 3.6.2, we have

$$(a \vee b)_{G/H} = \operatorname{Ind}^G_{\mathrm{Stab}(\{1,t\},\emptyset)}(a_{1,t}) + \operatorname{Ind}^G_{\mathrm{Stab}(\{1\},\{t\})}(a_1 \otimes b_t) + \operatorname{Ind}^G_{\mathrm{Stab}(\emptyset,\{1,t\})} b_{1,t}.$$

Since $\mathrm{Stab}(\{1,t\},\emptyset) = \mathrm{Stab}(\emptyset,\{1,t\}) = G$ and $\mathrm{Stab}(\{1\},\{t\}) = H$, this becomes

$$b_{1,t} = -a_{1,t} + \mathrm{Ind}_H^G(a_1 \otimes a_t).$$

□

The above yields a formula for differences of characters, as follows:

Corollary 3.6.4. *Let $\rho, \sigma \in R^+(H)$ for any finite group H, and suppose $H \triangleleft G$ with $|G : H| = p$ prime. Then*

$$\mathcal{N}_H^G(\sigma - \tau) = \begin{cases} \mathcal{N}_H^G(\sigma) - \mathcal{N}_H^G(\tau) + \sum_{C \in \mathcal{O}(V)} (-1)^{p-|C|} \mathrm{Ind}_H^G(\sigma^{\otimes C} \tau^{\otimes C'}) & \text{if } p \text{ is odd} \\ \mathcal{N}_H^G(\sigma) - \mathcal{N}_H^G(\tau) + \mathrm{Ind}_H^G(\tau \otimes \tau^t) - \mathrm{Ind}_H^G(\sigma \otimes \tau^t) & \text{if } p = 2, \end{cases}$$

where t is a representative for the non-trivial coset in G/H when $p = 2$.

Proof. When p is odd, by Proposition 3.6.2, we have $\chi(-\tau)_C = (-1)^{|C|}\chi(\tau)_C$ for $C \subseteq G/H$. Thus

$$\begin{aligned} \mathcal{N}_H^G(\sigma - \tau) &= (\chi(\sigma) \vee \chi(-\tau))_{G/H} \\ &= \mathcal{N}_H^G(\sigma) - \mathcal{N}_H^G(\tau) + \mathrm{Ind}_H^G\left(\sum_{C \in \mathcal{O}(V)} (-1)^{p-|C|} \chi(\sigma)_C \otimes \chi(\tau)_{C'}\right) \\ &= \mathcal{N}_H^G(\sigma) - \mathcal{N}_H^G(\tau) + \sum_{C \in \mathcal{O}(V)} (-1)^{p-|C|} \mathrm{Ind}_H^G(\sigma^{\otimes C} \tau^{\otimes C'}). \end{aligned}$$

If $p = 2$:

$$\begin{aligned} \mathcal{N}_H^G(\sigma - \tau) &= (\chi(\sigma) \vee \chi(-\tau))_{G/H} \\ &= \mathcal{N}_H^G(\sigma) - \mathcal{N}_H^G(\tau) + \mathrm{Ind}_H^G(\tau \otimes \tau^t) - \mathrm{Ind}_H^G(\sigma \otimes \tau^t). \end{aligned}$$

□

3.6.3 The abelian case

The next step in our derivation is to simplify expressions of the type $x^{\otimes C}$, which are *a priori* defined in the case of actual characters but not for virtual ones. In the abelian case however, the group action of G on H is trivial, and the notation $x^{\otimes C}$ can be extended to cover any virtual character x.

Proposition 3.6.5. *Let $H \leq G$ be abelian groups with $|G : H| = p$ a prime number. Let $x \in R(H)$ be any virtual character, then for any $C \subsetneq G/H$:*

$$\chi\left(x\right)_C = \bigotimes_{t \in C} x^t = x^{|C|} \text{ as (virtual) } H\text{-modules.}$$

Recall that the morphism χ was originally only defined for actual characters, then extended to negative characters. Proposition 3.6.5 says that in the abelian prime case, the naive extension of χ to all virtual characters is the right one.

Proof. First note that the statement is trivial when $p = 2$. For p odd, write $x = x^+ - x^-$ with $x^+, x^- \in R^+(H)$. Let $C \subset G/H$, and for any $D \subset C$ let D' be the complement of D in C. Recall that by Proposition 3.6.2, for any $\rho \in R^+(H)$, we have $\chi(-\rho)_C = (-1)^{|C|}\chi(\rho)_C$. Thus:

$$\begin{aligned}
\chi(x)_C &= (\chi(x^+) \vee \chi(-x^-))_C \\
&= \bigoplus_{D \subset C} \chi(x^+)_D \otimes \chi(-x^-)_{D'} \\
&= \bigoplus_{D \subset C} (x^+)^{|C|} \otimes (-1)^{|D'|}(x^-)^{|D'|} \\
&= \bigoplus_{i=0}^{|C|} \binom{p}{i} (x^+)^i(-x^-)^{|C|-i} \\
&= x^{|C|}
\end{aligned}$$

which proves the statement. □

Thus we can extend the addition formula to all virtual characters:

Lemma 3.6.6. *Let $x, y \in R(H)$ be virtual characters and suppose $|G : H| = p$ is prime. Then:*

$$\mathcal{N}_H^G(x+y) = \mathcal{N}_H^G(x) + \mathcal{N}_H^G(y) + \sum_{i=1}^{p-1} \mathrm{Ind}_H^G(x^i y^{p-i}).$$

Proof. Recall that $\mathcal{N}_H^G(x+y) = (\chi(x+y))_{G/H}$. So:

$$\begin{aligned}
\mathcal{N}_H^G(x+y) &= (\chi(x) \vee \chi(y))_{G/H} \\
&= \mathcal{N}_H^G(x) + \mathcal{N}_H^G(y) + \sum_{C \in \mathcal{O}(V)} \mathrm{Ind}_H^G(x^{|C|} y^{|C'|}) \\
&= \mathcal{N}_H^G(x) + \mathcal{N}_H^G(y) + \sum_{i=1}^{p-1} \mathrm{Ind}_H^G(x^i y^{p-i}).
\end{aligned}$$

□

Note that this formula is also valid for $p = 2$.

Lemma 3.6.7. *Let $H \leq G$ be finite abelian groups with $[G : H] = n$, and let $\rho \in R^+(H)$ be a representation of degree 1. If $\overline{\rho} \in R^+(G)$ satisfies $\rho = \mathrm{Res}_H^G(\overline{\rho})$, then:*

$$\mathcal{N}_H^G(\rho) = \overline{\rho}^n.$$

Proof. Recall that $\mathcal{N}_H^G(\rho) = \rho^{\otimes G/H}$, viewed as a representation of G. Since H, G are abelian groups, we have $\rho^g = \rho$ for any $g \in G$, so that $\mathcal{N}_H^G(\rho)$ is given by ρ^n on H. If $g \notin H$, then since $|G : H| = n$ we have $g^n \in H$ and $\mathcal{N}_H^G(\rho)(g) = \rho(g^n)$. Here we use, crucially, the fact that ρ has dimension 1. Thus $\mathcal{N}_H^G \rho$ is determined by its values on H, and if $\mathrm{Res}_H^G(\overline{\rho}) = \rho$ then $\overline{\rho}^n = \mathcal{N}_H^G(\rho)$. □

Let us now restrict to the case $\mathbb{K} = \mathbb{C}$. Then the irreducible characters of G are one-dimensional and we can apply the above result.

Corollary 3.6.8. *Let $H \leq G$ be abelian groups and $|G : H| = p$ be a prime. Let $\sigma, \tau \in R_{\mathbb{C}}^{+}(H)$ be one-dimensional representations, and let $\bar{\sigma}$ (resp. $\bar{\tau}$) satisfy $\operatorname{Res}_H^G \bar{\sigma} = \sigma$ (resp. $\operatorname{Res}_H^G \bar{\tau} = \tau$). Then, if p is odd:*

$$\mathcal{N}_H^G(\sigma + \tau) = \bar{\sigma}^p + \bar{\tau}^p + \mathbb{C}[G/H] \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} \bar{\sigma}^i \bar{\tau}^{p-i},$$
$$\mathcal{N}_H^G(\sigma - \tau) = \bar{\sigma}^p - \bar{\tau}^p + \mathbb{C}[G/H] \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} (-1)^{p-i} \bar{\sigma}^i \bar{\tau}^{p-i}.$$

If $p = 2$:

$$\mathcal{N}_H^G(\sigma + \tau) = \bar{\sigma}^2 + \bar{\tau}^2 - \mathbb{C}[G/H]\bar{\sigma}\bar{\tau},$$
$$\mathcal{N}_H^G(\sigma - \tau) = \bar{\sigma}^2 - \bar{\tau}^2 + \mathbb{C}[G/H]\bar{\tau}^2 - \mathbb{C}[G/H]\bar{\sigma}\bar{\tau}.$$

Proof. If G is abelian then the action of G on H is trivial and $\sigma^{\otimes C} = \sigma^{|C|}$ as $\operatorname{Stab} C$-modules for all $C \subset G/H$, trivially when C is proper, and by Lemma 3.6.7 otherwise. The number of subsets C of G/H of size i is $\binom{p}{i}$ which we divide by p to sum over orbits. Moreover, the induced representation $\operatorname{Ind}_H^G(\rho)$ is $\mathbb{C}[G/H]\bar{\rho}$, for any $\bar{\rho}$ such that $\operatorname{Res}_H^G \bar{\rho} = \rho$. □

3.6.4 Norm and Grothendieck filtration

We can now show our main theorem:

Theorem 3.6.9. *Let $H \leq G$ be abelian with $[G : H] = p$ a prime number. If $x \in \Gamma^n(H)$ then $\mathcal{N}_H^G(x) \in \Gamma^{np}(G)$.*

Proof. Recall that $\Gamma^n(H)$ is generated by elements of the form $(\rho_1 -$

$1)^{i_1} \cdots (\rho_k - 1)^{i_k}$ for ρ_i irreducible characters of H and $\sum i_k \geq n$. Note that, in our case, each ρ_i is one-dimensional.

As a first step, we prove that if ρ is any irreducible character of H, then $\mathcal{N}_H^G(\rho - 1) \in \Gamma^p(G)$. First, assume $p = 2$. In the notation of Corollary 3.6.8, we have:

$$\begin{aligned}
\mathcal{N}_H^G(\rho - 1) &= \bar{\rho}^2 - 1 + \mathbb{C}[G/H] - \mathbb{C}[G/H]\bar{\rho} \\
&= (\bar{\rho} - 1)^2 + 2(\bar{\rho} - 1) - \mathbb{C}[G/H](\bar{\rho} - 1) \\
&= (\bar{\rho} - 1)^2 - (\mathbb{C}[G/H] - 2)(\bar{\rho} - 1) \ \in \Gamma^2(G).
\end{aligned}$$

If p is odd:

$$\mathcal{N}_H^G(\rho - 1) = (\bar{\rho})^p - 1 + \mathbb{C}[G/H] \sum_{i=1}^{p} \frac{1}{p} \binom{p}{i} (-1)^{p-i} (\bar{\rho})^i.$$

Consider the permutation representation $\mathbb{C}[G/H]$ and recall that $\mathbb{C}[G/H] = 1 + \sigma + \cdots + \sigma^{p-1}$ for some linear representation σ of G. Let $Y = \sigma - 1$, then $Y^i \in \Gamma^i(G)$ and:

$$\mathbb{C}[G/H] = \sum_{i=0}^{p-1} (Y + 1)^i = \sum_{i=0}^{p-1} \binom{p}{i+1} Y^i.$$

Note that $(Y + 1)^p - 1 = 0$, and thus

$$pY = -\sum_{i=2}^{p} \binom{p}{i} Y^i \ \in \Gamma^2(G).$$

Thus we can substitute every instance of pY in the right-hand-side of the equation by $-\sum_{i=2}^{p} \binom{p}{i} Y^i$. Iterating, we obtain that $pY \in \Gamma^p(G)$, and thus:

$$\mathbb{C}[G/H] \equiv p \pmod{\Gamma^p(G)}.$$

Therefore

$$\begin{aligned}
\mathcal{N}_H^G(\rho - 1) &= (\bar{\rho})^p - 1 + \mathbb{C}[G/H] \sum_{i=1}^{p} \frac{1}{p} \binom{p}{i} (-1)^{p-i} (\bar{\rho})^i \\
&\equiv (\bar{\rho})^p - 1 + \sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} (\bar{\rho})^i \pmod{\Gamma^p(G)} \\
&\equiv (\bar{\rho} - 1)^p \pmod{\Gamma^p(G)} \\
&\equiv 0 \pmod{\Gamma^p(G)},
\end{aligned}$$

Which completes the first step.

Since the norm map is multiplicative, we have, for any prime p,

$$\mathcal{N}_H^G \left((\rho_1 - 1)^{i_1} \cdots (\rho_k - 1)^{i_k} \right) \in \Gamma^{pn}(G)$$

whenever $\sum i_k \geq n$.

Finally, let $x, y \in R(H)$ be generators of $\Gamma^n(H)$ of the form $(\rho_1 - 1)^{i_1} \cdots (\rho_k - 1)^{i_k}$, as above. Then by Lemma 3.6.6, we have:

$$\mathcal{N}_H^G(x + y) = \mathcal{N}_H^G(x) + \mathcal{N}_H^G(y) + \sum_{i=1}^{p-1} \mathrm{Ind}_H^G(x^i y^{p-i}).$$

The terms $\mathcal{N}_H^G(x), \mathcal{N}_H^G(y)$ are both in $\Gamma^{np}(G)$ as shown above. Each term $x^i y^{p-i}$ is a product of n elements in $\Gamma^n(H)$ and therefore in $\Gamma^{np}(H)$, and since induction preserves the Grothendieck filtration in the case of abelian groups, we have $\mathrm{Ind}_H^G(x^i y^{p-i}) \in \Gamma^{np}(G)$ for all i; this concludes the proof. □

Thus, on the complex field $\mathbb{C}$, tensor induction preserves the Grothendieck filtration. Since it satisfies the compatibility axioms of a Tambara functor on $R_{\mathbb{C}}(-)$, it satisfies them at the graded level, and we have the following corollary:

Corollary 3.6.10. *The restriction of $R^*_{\mathbb{C}}(-)$ to abelian groups is a Tambara functor.* $\square$

3.7 Application: norms in graded character rings of abelian groups

In group cohomology Steenrod operations can be defined via the Evens norm corresponding to the inclusion $G \hookrightarrow G \times C_p$ (see eg. [CTVEZ03, Ch. 7]). We propose here to compute that norm in the case of degree 1 classes in abelian groups. Let G be abelian, and σ a one-dimensional representation of G, with $x := c_1(\sigma)$.

Recall that $R_{\mathbb{C}}(C_p)$ is generated by one character ρ and that

$$R_{\mathbb{C}}(G \times C_p) \cong R_{\mathbb{C}}(G) \otimes R_{\mathbb{C}}(C_p).$$

In $R_{\mathbb{C}}(G \times C_p)$, let $\overline{\rho} = 1 \otimes \rho$ and $\overline{\sigma} = \sigma \otimes 1$. In $R^*_{\mathbb{C}}(G \times C_p)$, let $y = c_1(\overline{\rho})$ and $z = c_1(\overline{\sigma})$.

Proposition 3.7.1. *With notation as above:*

$$\mathcal{N}_G^{G \times C_p}(x) = z^p - zy^{p-1}.$$

Proof. Let $X = \rho - 1 \in R_{\mathbb{C}}(G)$ and $Y = \overline{\rho} - 1, Z = \overline{\sigma} - 1 \in R_{\mathbb{C}}(G \times C_p)$. We compute $\mathcal{N}_G^{G \times C_p}(X)$ in the cases $p = 2$ and p odd.

- Case $p = 2$. By Corollary 3.6.8:

$$\begin{aligned}
\mathcal{N}_G^{G\times C_2}(X) = \mathcal{N}_G^{G\times C_2}(\rho - 1) &= \overline{\rho}^2 - 1^2 + \mathbb{C}[G \times C_2/G] - \mathbb{C}[G \times C_2/G] \cdot \overline{\rho} \\
&= (\overline{\rho} - 1)^2 + 2(\overline{\rho} - 1) - \mathbb{C}[G \times C_2/G](\overline{\rho} - 1) \\
&= (\overline{\rho} - 1)^2 - (\overline{\sigma} + 1 - 2)(\overline{\rho} - 1) \text{ since } \mathbb{C}[G \times C_2/G] = \overline{\sigma} + 1 \\
&= (\overline{\rho} - 1)^2 - (\overline{\sigma} - 1)(\overline{\rho} - 1) \\
&= Z^2 - ZY.
\end{aligned}$$

In the graded ring $R^*_{\mathbb{C}}(G \times C_2)$, this yields

$$\mathcal{N}_G^{G\times C_2}(x) = z^2 - zy.$$

- Case p odd. Let $K = G \times C_p$, then, again by Corollary 3.6.8:

$$\begin{aligned}
\mathcal{N}_G^K(X) &= \mathcal{N}_G^K(\rho - 1) \\
&= \mathcal{N}_G^K(\rho) + \mathcal{N}_G^K(-1) + \mathbb{C}[K/G] \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} \overline{\rho}^i (-1)^{p-i} \\
&= \overline{\rho}^p - 1 + (\mathbb{C}[K/G] - p) \left(\sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} \overline{\rho}^i (-1)^{p-i} \right) + \sum_{i=1}^{p-1} \binom{p}{i} \overline{\rho}^i (-1)^{p-i}.
\end{aligned}$$

Recall that $\overline{\rho} = Z + 1$ and that:

$$\begin{aligned}
\mathbb{C}[K/G] &= 1 + \overline{\sigma} + \cdots + \overline{\sigma}^{p-1} \\
&= \sum_{i=0}^{p-1} (Y + 1)^i.
\end{aligned}$$

thus

$$\begin{aligned}\mathcal{N}_G^K(X) =&(Z+1)^p - 1 + \sum_{i=1}^{p-1} \binom{p}{i} (Z+1)^i (-1)^{p-i} \\ &+ \left(\sum_{i=0}^{p-1} (Y+1)^i - p\right) \left(\frac{1}{p}\sum_{i=1}^{p-1} \binom{p}{i} (Z+1)^i (-1)^{p-i}\right) \\ =&(Z+1-1)^p + \frac{1}{p}\left[\sum_{j=0}^{p-1}\left(\sum_{i=j}^{p-1}\binom{i}{j} Y^j\right) - p\right] \cdot \left[\sum_{i=1}^{p-1}\binom{p}{i}(Z+1)^i(-1)^{p-i}\right].\end{aligned}$$

A straightforward induction shows that

$$\sum_{i=j}^{p-1} \binom{i}{j} = \binom{p}{j+1},$$

so that:

$$\begin{aligned}\mathcal{N}_G^K(X) =&Z^p + \frac{1}{p}\left[\sum_{j=0}^{p}\binom{p}{j+1}Y^j - p\right] \cdot [(Z+1-1)^p - (Z+1)^p + 1] \\ =&Z^p + \frac{1}{p}\left[\sum_{j=1}^{p}\binom{p}{j+1}Y^j\right] \cdot \left[-\sum_{i=1}^{p-1}\binom{p}{i}Z^i\right].\end{aligned}$$

Now recall from the proof of Theorem 3.6.9 that $pY, pZ \in \Gamma^p(G \times C_p)$. Thus:

$$\mathcal{N}_G^{G\times C_p}(X) \equiv Z^p - ZY^{p-1} (\text{mod } \Gamma^{p+1}),$$

which concludes the proof.

$\square$

Chapter 4

Conclusion

In light of the results presented in this thesis, the dearth of previous work on graded character rings seems quite surprising: they are fine invariant of groups that can be computed explicitly; their structure appears complex enough to be challenging, yet mysterious enough to warrant further investigation. Thanks to this apparent lack of interest, however, there are multiple paths left to explore.

On the computational side, the more straightforward path would be to extend the work in Chapter 2 to larger families of groups. Abelian groups are of particular interest, of course, since $R^*_{\mathbb{C}}(-)$ is a Tambara functor on them; as previously mentioned, the Künneth formula reduces the problem to that of computing $R^*_{\mathbb{C}}(-)$ on abelian p-groups. One obstacle to proving the conjecture mentioned in Equation (2.6.1) is that the valuation method is not conclusive when applied to terms whose valuations are too different (eg. a character coming from C_{2^n} for n large will typically have a 2-valuation much smaller than that of a character from C_2). Until more properties are uncovered, a solution seems, as of yet, out of reach. A perhaps more promising problem, for a start, is that of computing $R^*_{\mathbb{C}}(C_4^k)$ for arbitrary k. A combination of the proof of Proposition 2.3.3 with the valuation method might prove fruitful.

Due to time constraints, the subject of changing base fields had to remain barely touched. The example of $R^*_{\mathbb{Q}}(C_p)$ (Corollary 2.2.3) should however be enough to encourage further investigation. The computation could be extended to cyclic groups of composite orders, which might give insight into a possible Künneth formula for groups of coprime order over the rationals. Characters over extensions of the rationals, and real characters, are also intriguing candidates.

On the theoretical side, the properties of Mackey and Tambara functors have been extensively studied and will no doubt provide further insight into the behavior of saturated rings. Despite the strikingly straightforward computations of Section 3.7, norms for abelian groups may yet prove full of surprises. In particular, it would be interesting to study properties of the Evens norm in cohomology and how they translate in terms of tensor-induction.

And while $R^*_{\mathbb{K}}(-)$ is, in general, not even a Mackey functor, the question of whether the norm preserves the saturated filtration (that is, whether $\mathcal{R}^*_{\mathbb{C}}(-)$ is a Tambara functor) is still open. In fact, even the proof that $R^*_{\mathbb{C}}(-)$ is Mackey on abelian groups relies on the properties of the complex field; it is unclear whether this result extends to $R^*_{\mathbb{K}}(-)$ for arbitrary $\mathbb{K}$.

Thus graded character rings are surprising and mysterious; it is our hope that the work presented here was advertisement enough for the study of a theory that has not revealed the last of its secrets.

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Universities of Versailles-St Quentin, Paris Diderot, Strasbourg, UWO

Graded character rings · Conferences
Groups and Algebras in Bicocca for Young mathematicians - Milan, Italy
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