Development Of Material Model Subroutines For Linear And Nonlinear Response Of Elastomers

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Abstract

The nature of elastomers has been extensively studied ever since the vulcanization of rubber in the 19th century. Elastomers have been heavily employed in various fields, such as automobile, aerospace, robotics, biomimetics, dynamics and energy harvesting. Due to their molecular nature, these materials display hyperelastic and viscous response when deformed. Their response has been studied in a number of works, which tend to explain their nature through the theory of polymer dynamics or using rheological models. As elastomers are designed as actuators, generators or artificial tissues with complex geometries, the need for finite element analysis to study their response is becoming more essential. The purpose of this study is to develop user defined subroutines (UMATs) to capture the viscoelastic and hyperelastic response of elastomeric materials in ABAQUS CAE. Two UMATs have been formulated based on the existing theory of linear viscoelasticity and hyperelasticity. The developed UMATs are then tested using ABAQUS CAE software on their accuracy, robustness and versatility. The UMAT for predicting the linear response of elastomers can capture the time and rate-dependent response of material. However, it showed some offset from experimental results since the material is nonlinear in nature. The UMAT for predicting the nonlinear behavior of elastomers quite accurately capture the hyperelastic response of the material.

Keywords

Elastomers, viscoelasticity, hyperelasticity, user material subroutine (UMAT), finite element analysis.
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1 Introduction

Ever since the discovery of vulcanization of rubber in 1893 [1] to the introduction of dielectric elastomers at the end of the 20th century [2], elastomers have been an important part of research and scientific discovery. Elastomers consist of polymer networks formed by cross-linking highly mobile and flexible polymer chains. These materials have been extensively employed in various engineering applications such as vibration damping, automotive industry, robotics, biomechanics, and aerospace, just to name a few [3]. Their capability of undergoing large deformation has made their mechanics an active research area for many years [1]. The capability of elastomers to sustain strains of over a few hundred percent and the exhibition of nonlinear time and rate dependent response is due to their hyperelastic and viscous nature. Hence, their mechanics has been a prime focus in the research community [4]. Substantial efforts have been devoted to the development of rheological models to understand the finite-deformation and viscoelastic response of these materials. Their applications in the field of actuation and energy harvesting has made the demand of understanding the experimental responses of elastomeric materials through computational methods a necessity. Finite element analysis of complex geometries and complicated stress fields observed in various applications of these materials has become the recent target of many research studies. The motivation behind the present study stems itself from the growing demand of numerical techniques to model the mechanical behavior of elastomeric materials and to simulate their linear and nonlinear responses. To serve this purpose, customized material models are usually added into the libraries of finite element simulation software like ABAQUS CAE through user-defined subroutines called UMATs. The primary objective of this work is to formulate finite element (FE) models to study the linear and nonlinear responses of elastomeric materials through developing UMAT codes in ABAQUS. The FE models are further validated through available experimental data.

1.1 Elastomers

Polymers are materials which are composed of long molecular chains of covalently bonded atoms, with each chain being a repetition of much smaller units called monomers [5]. Elastomers are types of polymers which are formed by cross-linking these molecular chains.
Most common types of elastomers include rubbers, polyurethanes (materials for making fabric and plastics), polybutadiene (materials for making wheels or tires), neoprene (materials for insulations and belts) and silicone [6].

Elastomers are among the most extensively used materials nowadays. With the discovery of cutting edge electroactive materials like dielectric elastomers, their applications in the field of robotics, biomimetics, dynamics and energy harvesting have made them immensely popular both in academia research fields and industry [7].

1.2 Characteristics of elastomers

Almost without exception, elastomers belong to a class of polymers which are viscoelastic in nature[8]. The strong and flexible cross-linked ground polymer networks in these materials are the main reason for them undergoing large deformation (hyperelasticity). Whereas elastomeric materials demonstrate a time and rate dependent response attributed to the diffusion of these polymer chains [3]. The classical theory of linear elasticity deals with the properties of materials which are elastic in nature, i.e., the stress-strain relation is of direct proportionality, as per Hooke’s law [9]. On the other hand, the classical theory of hydrodynamics, deals with the properties of viscous liquids, for which the stress is rather proportional to the strain rate instead of the direct relation between the stress and the strain. Both these models are idealizations for materials with infinitesimal strains[9]. Deviations from Hookean and Newtonian behavior are observed when dealing with finite deformation, especially for materials which can undergo substantial dimensional changes without fracturing, such as polymeric solutions and undiluted uncross-linked polymers [10]. Viscoelastic materials possess properties of both viscous and elastic materials, for example, when a sinusoidal oscillating load is applied on elastomeric materials, their strain will neither be in phase as in elastic materials, nor will be 90° out of phase as in viscous liquids, it will be somewhere in between [10].

The viscoelastic nature of elastomers has been immensely studied and has been the subject for research using different models, details of which will be discussed in the subsequent chapters. In this section some important characteristics of elastomers will be discussed.

a) The predominant characteristics of elastomers is their ability of elastic recovery after deformation. Even for deformation many times more than their original dimensions,
under ideal circumstances, the material will return to their original dimensions after the removal of stress [1].

b) Elastomers have the capability to undergo large deformation, even more than 300% in some cases [6], this is one of the main reasons they are used for a variety of actuating or dynamic purposes;

c) Elastomers are characterized by high fracture toughness under static or dynamic stresses and better abrasion resistance than that of some metals like steel [1].

d) Dielectric elastomers, one type of elastomers, possess electromechanical coupling properties, which enables them for applications in transduction technology as actuators, sensors and energy harvesters [7]. Due to their capability of sustaining large deformation, they have applications in actuation. The efficiency of electromechanical coupling for elastomeric materials is higher compared to some other traditional transduction materials such as piezoelectric [11]. Materials such as silicone rubbers have shown strains over 30%, actuation pressures of 1.3 MPa and energy densities of 0.22 J/cm³ [12] for example;

e) Some elastomers have compatible properties in terms of surface chemistry to biomaterials, making them suitable for making artificial tissues [6];

f) Some elastomers like Rubbers are also capable of adhering to most other materials, enabling different hybrid constructions. In combination with fibers, such as rayon, polyamide, polyester, glass or steel-cord, the tensile strength of the hybrid is increased considerably with a reduction in extendibility. By joining elastomers with metals, components which combine the elasticity of elastomers with the rigidity of metals can be achieved [1].

1.3 Applications of elastomers

Due to the characteristics described above, there is a vast multitude of applications in which elastomers can be used. Below are just some of the generic applications of these materials to prove how vital it is to develop an understanding of their mechanics. Rubbers are some of the most abundantly used elastomeric materials on the planet [1]. Elastomers have excellent sealing capability, in fact, the very first use of rubber-like materials was to form a coating over shoes to make them water resistant [1]. Due to their effectiveness as sealants, these materials are used as gaskets, oil pipes, steam hoses, silicone gels, and electrical cables etc. Due to their viscoelastic nature, elastomers are used as vibration isolators, dampers, tires, and shock
Elastomers are characterized by great fracture toughness under static or dynamic stresses and by better abrasion resistance than that of steel. Due to these features, elastomers are used in automotive parts, aerospace and industrial facilities for loading and unloading heavy materials. It was discovered in the late 20th century that elastomers show great dielectric capabilities. Their efficient electromechanical coupling and the capability to undergo large deformation have attracted much attention. Even more than 500% of stretching strain[11] can be achieved in theory. Therefore, dielectric elastomers are excellent candidates for actuators. These actuation devices can be used in switches, robotics, electronics and various automation devices, just to name a few. In addition to their large deformation capability, these materials do not produce any noise as compared to mechanical actuators [13], which aids in better acoustic control. Properties like high specific energy density [13], high speed response [14], and good overall performance makes these materials highly desirable for use as actuators in robotics and aero-space applications. Elastomers have also shown promise in harvesting energy from a variety of sources such as ocean waves, wind, water currents and human motion [15]. Due to their high energy density and conversion efficiency, these materials can be used as robust and simple “direct drive” generators [16]. Several types of energy harvesters which use dielectric elastomers have been tested bearing positive results. One such type of energy harvesters is buoy-mounted generator. This generator performed quite successfully during sea trials. Nowadays, the use of larger amounts of dielectric elastomer materials to generate energy in the range of megawatts is being studied and further investigated [15]. Elastomers are being engineered for both small-scale energy scavenging [17] and large scale energy generation [18]. Unlike piezoceramics, elastomers have the potential for large scale energy harvesting as they are highly stretchable and having excellent force coupling efficiency (which increases the electromechanical conversion) [19-21]. Elastomers are also used as artificial human tissue due to their capability of deforming under applied electric field and softness in nature. The changes in configuration for various organs are caused by electrical signals and impulses generated from the brain, like the expanding and contracting of pupils, or beating of the heart or any motion performed by the hands, arms or feet. Now, consider a material (dielectric elastomer) which can react in an analogous manner under applied electric field as human organs, this has led to these materials being used as artificial muscles in robotics and bioengineering.
1.4 Objectives

As discussed in previous sections, due to their large deformation and transduction capabilities, elastomers have shown tremendous promise in a large variety of applications. However, owing to their complex nonlinear nature which is strongly affected by material properties such as viscosity and hyperelasticity, understanding the response of these materials, under applied loading either mechanical or electrical, has challenged and perplexed researchers for decades [20].

Most of the work done in the past, focusing on studying the hyperelastic and viscoelastic behavior of elastomers, is based on models which attempt to predict the behavior of these materials [20]. Simple models have been developed to provide understanding of the responses of elastomers, using mechanical analogies [8-9, 22, 23]. However, for studying the structures with complex geometries, under complicated loading conditions, and with complex material behaviors, one needs to resort to numerical solution techniques. In recent years, several works have tried to form a computational basis for these materials, which can be used to develop FEA models [24-28]. However, these models are complex and mostly confined by specific functionality, or do not cover a wider range of testing capabilities or are limited by the lack of understanding the various failure modes and processes that take place when electric field is applied to these materials.

In the work of Zhou [3], a micro-macro constitutive model was developed to study the viscoelastic behavior and relaxation process of elastomeric materials. The model is capable of showing the variation in the diffusion process of polymer chains when under deformation and suggests the use of hyperelastic models to capture nonlinear viscoelastic behavior of elastomeric materials. The long-term goal of this work is to develop user defined subroutines (UMAT) based on this model and their testing using finite element software like ABAQUS/CAE. The task of the present research work is to develop UMAT for ABAQUS/CAE which can capture the response of elastomeric materials. For this purpose, an attempt has been made to discuss the procedure for constructing a finite element model to develop UMAT subroutines for studying both linear and nonlinear responses of elastomers. Following is a description of what will be covered in this thesis:

1) Examining the linear viscoelastic behavior of elastomers; formulation of finite element models and their implementation in ABAQUS CAE software.
2) Examining non-linear hyperelastic response of elastomeric materials; formulation of finite element models and their implementation in ABAQUS CAE software.

The FE models will be developed based on standard routines and procedures, which provides a framework for future implementation of complex viscoelastic models in the ABAQUS software for complex elastomeric structures, even under coupled field loading conditions.

1.5 Thesis Structure

Following the general introduction and objectives in Chapter 1, a literature review and basic understanding of different material models used to define the hyperelastic and viscoelastic response of elastomeric materials will be discussed in Chapter 2. Chapter 3 will discuss in detail the linear viscoelastic modeling of elastomeric materials; a finite element formulation will be developed which will help in developing a UMAT for ABAQUS/CAE SIMULA. This UMAT will be used to simulate experimental data available for an elastomeric material and results will be discussed. In Chapter 4 the non-linear hyperelastic response of these materials will be discussed utilizing the strain energy density to formulate a computational model based on which a UMAT for ABAQUS/CAE SIMULA will be developed and tested using already existing experimental data. Finally, Chapter 5 concludes the thesis and provides suggestions for the future work.
CHAPTER 2

2 Literature review on fundamental material models for elastomers

To analyze the response behavior of elastomers and provide guidelines for their design, extensive studies have been carried out over the years. From research papers on mechanics of elastomers to books on their viscoelasticity and hyperelasticity, there are abundant investigations which attempt to explain the response of elastomers. The coming section presents a review of relevant literature on the modeling of the linear and non-linear deformation of elastomeric materials and the mechanical analogies developed in the past decades for understanding the response of these materials.

2.1 Hyperelastic models for elastomeric materials

The accurate modeling of constitutive laws governing the response of any material is of key importance, so that experimental results can be predicted. Experimental measurements of the stress-strain relationships for rubbers gave Mooney [22] and Rivlin [23] enough evidence that the linear theories of elasticity, especially the Hooke’s law [29] used since the 17th century [30], was no longer an adequate approach to understand the mechanical properties of elastomeric materials when undergoing large deformation. These observations can be simply witnessed by looking at a stress to strain curve for any elastomeric material [31].

Hence, to form an understanding of the non-linear behavior exhibited by elastomeric materials which undergo large deformation, a classical theory of nonlinear elasticity was developed with details outlined in the works of R.W. Ogden [32] and G. A. Holzapfel [33]. Computational formulations have also been developed using the non-linear continuum approaches by Holzapfel [33] and K Mish et al. [29].

The theory of nonlinear elasticity which contributes to the development of constitutive models for hyperelastic materials such as elastomers [33], is mainly classified into two categories, i.e., physical (mechanistic) and phenomenological models. If a model is derived based on the arguments about the underlying structure of the material, it is regarded as a mechanistic model. Whereas if a mathematical model is formed and tailored to fit experimentally observed behavior, it is known as a phenomenological model. Despite the
general definition of these two types of models, clear distinction cannot be made between mechanistic and phenomenological models, as later some phenomenological models are shown to have interpreted the physical nature of materials. For these models, the nonlinear stress-strain relationship is derived from a strain energy function, which should be selected on the basis of the macromolecular structure of the material [20, 33, 30]. Over the years, there have been developed quite a few approaches in literature, which use the strain energy density function to develop the constitutive models for hyperelastic materials [36-37]. All the models developed for describing the hyperelastic behavior of elastomers are mainly based on three approaches: statistical mechanics treatments, invariant-based continuum mechanics treatments and stretch-based continuum mechanics treatments as mentioned by Boyce and Arruda [36].

Here, focus is given to Green’s hyperelastic materials [37], for which the stress-strain relation is derived from a strain energy density function. The issue lies in the complexities posed by the nonlinearity of the stress to the stretch relations [33]. The strain energy function is defined in terms of invariants $I_1$, $I_2$ and $I_3$, which could be the invariants of the stretch (deformation) or strain tensor [22, 23, 33]. For the statistical mechanic approaches, it is assumed that the elastomeric material is a structure composed of randomly oriented long polymer chains as described by Treloar [38]. When the elongation in the polymer chain is significantly less than its fully extended length, the strain energy density of the material can be described with the Gaussian model as described by Treloar [39], i.e.,

$$W_o(F) = \frac{1}{2} N k \theta (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \quad (2.1)$$

where $N$ is the number of polymer chains, $k$ is the Boltzmann’s constant and $\theta$ is the absolute temperature; $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the principal stretches.

Here we will first briefly elaborate some fundamentals on the deformation of the solids. The deformation of the elastomers is described by the deformation gradient defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad (2.2)$$

where $\mathbf{x}$ describes the vector representing material points in deformed configuration and $\mathbf{X}$ is the vector for the material point in the reference (undeformed and unstressed) configuration. The right or left Cauchy-Green strain tensors, i.e., $C_{ij}$ and $B_{ij}$ are calculated as
\[ C = F^T F, \quad B = F F^T. \] (2.3)

The conventional three invariants in terms of both the right and left C-G strain tensors are given as
\[
I_1 = tr(C), \quad I_2 = \frac{1}{2} (tr(C))^2 - tr(C^2), \quad I_3 = det(C),
\]
\[
I_1 = tr(B), \quad I_2 = \frac{1}{2} (tr(B))^2 - tr(B^2), \quad I_3 = det(B). \] (2.4)

Alternatively, these invariants of the deformation tensor can also be expressed in terms of the principal stretches as
\[
I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\
I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, \\
I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2. \] (2.5)

However, when the elongation of the polymer chains approaches to the extensibility of the polymer chain, the prediction by the Gaussian model significantly differs from the observation in experiments \[20\]. In order to avoid the limitation of the Gaussian model, a more focused approach was discussed by Wang and Guth \[40\], which takes into account more accurate individual chain statistics. This material model proposes an assumption of a representative network structure, which is called 3-Chain model \[40\]. Similar models were introduced later, such as the 4-Chain tetrahedral model introduced by Flory and Rehner \[41\] and the 8-Chain model introduced by Arruda and Boyce recently \[42\]. In the chain models, a chain stretch is defined which is dependent on the \[\lambda_1\], as
\[
\lambda_{\text{chain}} = \left( \frac{1}{3} * (I_1) \right)^{1/2}. \] (2.6)

For example, the strain energy density function for the 8-chain model is given as,
\[
W_{8\text{-chain}} = N \times k \times \theta \times \sqrt{n} \times \left[ \beta_{\text{chain}} \lambda_{\text{chain}} - \sqrt{n} \times \ln \left( \frac{\sinh(\beta_{\text{chain}})}{\beta_{\text{chain}}} \right) \right], \] (2.7)

here
\[
\beta_{\text{chain}} = L^{-1} \left( \frac{\lambda_{\text{chain}}}{\sqrt{n}} \right), \] (2.8)

where \( n \) is the number of links in the chain and \( L^{-1} \) is the inverse Langevin function. Although these chain models adopting the non-Gaussian treatments can cater large deformation of the
material close to its extensibility, they deviate from experimental results [20] under small or moderate deformation. It should be noted that the Gaussian model and chain models are examples of mechanistic models as they are based on molecular chain theories.

Rivlin [23] developed a generalized Rivlin model with the strain energy density function defined in terms of these three invariants. Elastomeric materials are generally incompressible, i.e., they show little to no change in volume under deformation [30, 31, 34, 35], with \( \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1 \). Hence \( I_3 = 1 \). Therefore, the generalized Rivlin model is simplified as

\[
W_R = \sum_{i,j=1}^{\infty} C_{ij} \cdot (I_1 - 3)^i \cdot (I_2 - 3)^j,
\]

where \( C_{ij} \) are material constants. The Neo-Hookean model retained only the first invariant of the deformation tensor [43], i.e.,

\[
W_{NH} = C_{10} \cdot (I_1 - 3).
\]

The Mooney- Rivlin Model [43] is also a special case of the generalized Rivlin model and the strain energy density function is defined as

\[
W_{MR} = C_{10} \cdot (I_1 - 3) + C_{01} \cdot (I_2 - 3).
\]

Yeoh [43] discovered that a model based on the higher powers of the first invariant of the deformation tensor \( I_1 \) gave better results for moderate to large deformation. For this model, the strain energy density function is given as

\[
W_Y = C_{10} \cdot (I_1 - 3) + C_{20} \cdot (I_2 - 3)^2 + C_{30} \cdot (I_3 - 3)^3.
\]

It should be mentioned that none of the models described in equations (2.9) to (2.12) account for the extensibility limit of the material. However, for a real polymer network, there is a limit to which a polymer chain can be extended [20]. To account for this, Gent [24] proposed a model which is an alternative to the model with higher orders of \( I_1 \). In this model, a material parameter \( J_{lim} \) accounting for the stretching limit of the polymer network is introduced, which is in the form of

\[
W_{Gent} = - \frac{G \cdot J_{lim}}{2} \cdot \ln \left[ 1 - \frac{I_1 - 3}{J_{lim}} \right].
\]

Here the macro-scale shear modulus \( G \) of the material and the material extensibility parameter \( J_{lim} \) are related to the polymerization degree of chains. It is vital to note that due to the
logarithmic function in the equation the stretch ratio must satisfy \(1 - \frac{I_1 - 3}{J_{lim}}\) > 0 and the maximum stretch ratios are limited by the value of \(J_{lim}\). For the limiting case when \(J_{lim}\) approaches infinity, this model reduces to the Neo-Hookean model. The interesting thing about the Gent model is that it does not involve a number of fitting constants as in the models from equation (2.9) to (2.12). Since the Gent model has a much simpler mathematical form, it has commonly been adopted to describe the constitutive behavior of hyperelastic materials.

In addition to the models based on statistical mechanics and invariant-based continuum mechanics treatment have been discussed, A stretch-based continuum mechanics treatment has also been employed to model the hyperelastic behavior of elastomers with the strain energy density function defined in terms of the principal stretches. One example is the Ogden model [44], which gives the strain energy density as

\[
W_o = \sum_n \frac{\mu_n}{\alpha_n} (\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3). \quad (2.14)
\]

here \(\mu_n\) and \(\alpha_n\) are material constants that can be determined from experimental data fitting. For particular values of those material constants and some constraint conditions, the Ogden model can be reduced to either the Neo-Hookean solid or the Mooney-Rivlin material [25].

The defined strain energy density functions are used to determine the stresses acting on the material, which are often expressed in terms of the strain invariants [22, 24, 43]. The advantage of using strain invariants is that they are independent of the frame of reference used to define the sample geometry of the material under consideration. From equation (2.9) to (2.13), the strain energy is a function of the strain invariants, given as

\[
W = f(I_1, I_2, I_3). \quad (2.15)
\]

Since the bulk modulus of elastomers is often three orders of magnitude larger than the shear modulus [26] in general, which makes elastomers nearly incompressible. Hence, \(I_3\) is taken as unity, reducing equation (2.15) to

\[
W = f(I_1, I_2). \quad (2.16)
\]

From the defined strain energy density function, the first Piola-kirchhoff stress is defined as

\[
P_{ij} = \frac{\partial W}{\partial F_{ij}} \quad (2.17)
\]

Here the subscript \(ij\) defines vector in tensor notation where \(i\) denotes the plane perpendicular to which the force is acting and \(j\) represents the direction in which the force is acting. For a
three-dimensional cartesian coordinate system the values of $i$ and $j$ are from 1 to 3. Correspondingly, the Cauchy stress is determined as

$$\sigma_{ij} = \frac{1}{J} F_{ik} \frac{\partial W}{\partial F_{jk}}.$$  \hspace{1cm} (2.18)

The Cauchy stress according to the strain energy density function in equation (2.16) is given as

$$\sigma_{ij} = \frac{2}{J} \left[ \frac{1}{J^2} \left( \frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) B_{ij} - \left( I_1 \frac{\partial W}{\partial I_1} + 2I_2 \frac{\partial W}{\partial I_2} \right) \delta_{ij} \right] - \frac{1}{J^2} \frac{\partial W}{\partial I_2} B_{ik} B_{kj} \right]$$

$$+ \frac{\partial W}{\partial J} \delta_{ij}. \hspace{1cm} (2.19)$$

Here the bar on top of strain invariants indicates isochoric invariants which are defined as $\bar{I}_1 = J^{-2/3} I_1$ and $\bar{I}_2 = J^{-4/3} I_2$. When the strain energy density function is defined, the stresses can be formulated accordingly. For example, the Cauchy stress for the Ogden model defined by equation (2.14) is given as

$$\sigma_{ij} = \frac{\lambda_1}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_1} b^{(1)}_i b^{(2)}_j + \frac{\lambda_2}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_2} b^{(1)}_i b^{(2)}_j + \frac{\lambda_3}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_3} b^{(1)}_i b^{(2)}_j,$$  \hspace{1cm} (2.20)

where $b^{(1)}_i$ and $b^{(2)}_j$ are eigen vectors from spectral decomposition of $B$.

### 2.2 Viscoelastic models for elastomeric materials

As discussed in chapter 1, the nature of elastomers is viscoelastic. Most of the basic constitutive models for elastomers are based on theories of linear and nonlinear viscoelasticity [8]. Even though most developments in the theory of viscoelasticity are recent, the basic linear and isothermal field theory has been available for a much longer time [27]. While there were several early contributions by Maxwell, Kelvin and Voigt, the classical theory for linear viscoelasticity was first presented in 1874 by Boltzmann [28]. He presented the first formulation for three-dimension isotropic viscoelasticity. Maxwell suggested the superposed elastic-viscous formulation for the stress relaxation under constant strain and Voigt introduced a similar formulation for creep under constant stress [45].

Viscoelastic relations may be expressed in both integral and differential forms. Integral form is a very general representation of the nature of viscoelasticity and is more suitable for theoretical studies [46]. Differential forms are derived from rheological models that provide a more direct physical interpretation of viscoelastic behavior [9]. In this section the commonly
used rheological models for describing linear viscoelasticity are discussed and their differential formulation is derived.

The behavior of a viscoelastic material is majorly defined by two tests: the stress relaxation test and the creep strain test. From these tests, moduli are deduced from fitting the experimental data which can then be used to describe several viscoelastic responses of these materials [10]. Dynamic tests are also conducted to define frequency related moduli, which is an important feature of viscoelastic materials [47]. The responses of viscoelastic materials are discussed in detail in the literature [8-10, 26, 47]. The fundamentals for understanding the rheological models in these references are discussed below.

2.2.1 Element under a simple strain

We start with a simple shear test, where the two opposite faces of a material element are displaced by sliding one face relative to the other, resulting in pure shear stress. Boltzmann [28] provided a simple superposition principle, according to which the effects of mechanical history for the material element are linearly additive. It means that all the strain histories for the material over time can be integrated, giving the constitutive equation as [8],

\[ \sigma_{21}(t) = \int_{-\infty}^{t} G(t - t') \dot{\gamma}_{21}(t') dt', \]  

(2.21)
here \( \dot{\gamma}_{21} = \frac{\partial \gamma_{21}}{\partial t} \) is the shear strain rate, \( G(t) \) is the relaxation modulus, and the integration is carried out over all past times \( t' \) to the present time \( t \). Two important things must be noted, if \( G(t) \) approaches zero as \( t \) approaches infinity, equation (2.21) describes the nature of a viscoelastic liquid; while if \( G(t) \) remains finite for large \( t \), this will describe the nature of a viscoelastic solid [10]. Another way to describe the viscoelastic behavior of the material is in terms of the stress rate rather than the strain rate [10], i.e.,

\[
\gamma_{21}(t) = \int_{-\infty}^{t} J(t - t') \dot{\sigma}_{21}(t') dt',
\]

(2.22)

with \( \dot{\sigma}_{21} \) being the rate of stress. \( J(t) \) is the creep compliance.

The above relations are of key significance in understanding the responsive nature of viscoelastic materials. If the shear relaxation modulus or the creep compliance of any material is known, the responses for any experiment in shear can be predicted, as long as the deformation is sufficiently small [10].

2.2.1.1 Stress relaxation

In a relaxation test, a strain \( \gamma \) is introduced at a constant rate of \( \dot{\gamma} = \gamma / \xi \) to a viscoelastic material for a very brief duration of time \( \xi \), until it reaches a maximum value at time \( t_0 \) and then is maintained. It is observed that although the strain on the material is maintained, the stress reaches a maximum value at time \( t_0 \) but then reduces gradually until it reaches a minimum value and then becomes constant with time. This phenomenon is called stress relaxation as graphically described in figure 2-2 below. For such a relaxation test, equation (2.21) can be written as

\[
\sigma_{21}(t) = \int_{t_0 - \xi}^{t_0} G(t - t') \left( \gamma_{21} / \xi \right) dt'.
\]

(2.23)

Applying the mean theorem for integration [48], equation (2.23) can be deduced as

\[
\sigma_{21}(t) = \gamma_{21} G(t - t_0 + \epsilon \xi), \quad 0 \leq \epsilon \leq 1.
\]

(2.24)

Setting \( t_0 \) to zero results in

\[
\sigma_{21}(t) = \gamma_{21} G(t + \epsilon \xi).
\]

(2.25)
This equation shows that the modulus $G(t)$ is a function of the loading interval. The important thing to be noted is that if the elemental body is perfectly elastic, the relaxation shear modulus $G$ is simply defined as $\sigma/\gamma$.

![Diagram](image)

**Figure 2-2** Relaxation test for a viscoelastic material [10].

### 2.2.1.2 Creep

During a creep experiment an increasing stress with a constant rate of $\dot{\sigma} = \sigma/\xi$ is applied to the element shown in figure 2-1 for a very brief duration of time $\xi$. Then it reaches a maximum value at time $t_0$, and then the stress is kept constant for a sufficiently long time as described in figure 2-3. For this scenario, the material response in strain can be described from equation (2.22) as,

$$\gamma(t) = \int_{t_0-\xi}^{t_0} f(t-t')\sigma/\xi dt'.$$

Here $\gamma$ is the strain and $\sigma$ is the maximum stress. The stress is applied within time $\xi$ and is kept constant after time $t_0$. Applying the mean theorem for integration we get a similar result as obtained in the case of equation (2.24), i.e.,
\[ \gamma(t) = \sigma(t) \ast J(t). \]  \hspace{1cm} (2.27)

For perfectly elastic materials the relation \( J = 1/G \) holds \[10\].

\[\begin{align*}
\text{Figure 2-3 Creep test for a viscoelastic material [10].}
\end{align*}\]

### 2.2.2 Rheological models

In the previous section two major experiments were described to explain the response of a viscoelastic material, i.e., the relaxation test and the creep test. From these tests, the material moduli describing the stress-strain relation can be determined. Knowing these material parameters allows the modeling of a viscoelastic material, which is done by the help of mechanically analogous systems imitating the response of the material under consideration.

These mechanical analogies help simplify the material modeling, dealing with polymers and elastomers with complex polymer chains. These molecular chains and links are responsible for the behavior a material shows when deformed. Instead of dealing with the mechanics of these molecular chains, representative mechanical analogies or rheological models are used.
2.2.2.1 Basic elements: spring and dashpot

The mechanical analogies consist of components such as the Hookean spring element and the Newtonian dashpot or damper [46]. The Spring element describes Hooke’s relation [49] between loading and displacement, i.e.,

\[ \sigma(t) = \mu \varepsilon(t), \] (2.28)

where \( \mu \) is the modulus of elasticity. The dashpot on the other hand is an ideal viscous element and follows Newtonian law for viscosity [50], which states that the force causing the viscous deformation is directly proportional to the rate of deformation, i.e.,

\[ \sigma(t) = \eta \dot{\varepsilon}(t) \] (2.29)

where \( \dot{\varepsilon} = \partial \varepsilon / \partial t \) is the rate of strain and \( \eta \) is the viscosity coefficient.

![Spring and dashpot elements](image)

**Figure 2-4** The spring and damper elements.

Combining the Hookean spring and the Newtonian dashpot, different models are obtained which are used to describe the response of a viscoelastic material. The simplest viscoelastic models are named after scientists J.C. Maxwell and Lord Kelvin [46].

2.2.2.2 Maxwell model

The Maxwell model [9, 46] is a combination of a Hookean spring and a Newtonian dashpot arranged in series as shown in the figure below.
The spring element is assigned with a stiffness $\mu$ (force/displacement) analogous to a modulus contribution $G$ (shear modulus) or $E$ (Young’s modulus) depending on which type of loading is being studied. Also, the dashpot or damper is assigned with a frictional resistance $\eta$ (force/velocity), which is analogous to a viscosity contribution. The Hookean spring element follows the Hooke’s law for stress-strain relation. For the damper the relation between the stress and strain is governed by Newton’s equation for viscous fluids. For the Maxwell element, the stress on spring and the damper is the same while the total strain is the summation of the two components, resulting in

$$\dot{\varepsilon} = \left( \frac{\partial \varepsilon_c}{\partial \sigma} + \frac{1}{\eta} \right) \sigma.$$  \hspace{1cm} (2.30)

If the material is under a constant strain, the stresses decay gradually. However, one limitation of this model is that it cannot capture viscoelastic creep.

### 2.2.2.3 Kelvin-Voigt model

The Kelvin-Voigt model [9, 46] combines a spring and a dashpot in parallel, as shown in figure 2-6. For this model, the strain for the dashpot and the spring are equal, and the total stress for the element is the sum of the stress on the spring and the dashpot, which gives the following stress-strain relation:
Under constant stress, the model can predict the creep scenario. However, the limitation of the model is that its capacity in the prediction of relaxation is less accurate. Once the material parameters are determined through data fitting, these models could be used to describe the linear viscoelastic behavior of materials.

2.2.2.4 Generalized models

Maxwell and Kelvin-Voigt models are good for qualitative and conceptual analysis, but as indicated by Tobolsky and Andrew [45] that single Maxwell or Kelvin-Voigt element is not enough to describe the behavior of polymers or elastomers. Due to the considerable number and variety of molecular chains, the material behavior can be better represented if more than one element in parallel is used. These models may have Maxwell elements in parallel with a spring or Kelvin-Voigt elements in series with either a spring or a damper, and are named as Wiechert’s model or Generalized models [51]. Figure 2-7 shows the generalized Maxwell and Kelvin models for example.
As was discussed in the previous section, for the mechanical analogies being used to better reflect the behavior of any viscoelastic material, the models are put under the same forcing as in one of the tests explained above. Their response is derived and then the results are plotted against the experimental data. Number of chains varies until the experiment data can be captured. Several types of generalized models are used to describe different materials in the literature. One type of the generalized Maxwell models is called Zener model [52], which is simply a single Maxwell element in parallel with a Hookean spring element as shown in figure 2-8 below.

\[ \tau_i = \frac{\eta_i}{\mu_i} \]  

(2.32)
The expansion series used for the solution of generalized models is called Prony series developed by Gaspard Riche de Prony [53].

2.2.3 Modeling of viscoelastic elastomers

One of the earliest works for modeling viscoelastic behavior of materials can be traced back to Tobolsky and Andrews [45], where they worked on a molecular approach to describe the mechanical behavior of such materials when classical theories of solid mechanics and fluid mechanics were insufficient. Tobolsky and Andrew [45] focused majorly on rubbers, carrying out several creep and relaxation tests to validate their models.

Later, Read [54] provided a method for stress analysis of compressible viscoelastic materials. In this paper he proposed that the classical theory of elasticity could be used to describe the time-dependent behavior of viscoelastic materials by utilizing mathematical tools such as Fourier integrals. Such treatments could also be extended to anisotropic materials.

Lee [55] focused on the stress analysis of linear viscoelastic materials such as polymers and plasctics. Motivated by the mathematical problems and the increasing use of inelastic materials. Lee published another work on the stress analysis of linear viscoelastic materials later [56]. It was asserted in this paper that either differential or integral operators can be used to define stress-strain relations for these materials, but it was more convenient to use integral operators for creep and relaxation functions.

With the advent of nonlinear continuum mechanics in the early 1960’s, Coleman and Noll [57] proposed fundamental assumptions for linear viscoelasticity. In their work, they presented the theory of infinitesimal viscoelasticity based on the assumption that at microscopic level the material could be represented by a combination of springs and dashpots connected in complex networks. Further, they discussed the order of error in computing stress when different theories were used for stress-strain constitutive relations. Pipkin [58] suggested nonlinear integrals can be used to approximate the basic constitutive law for viscoelastic solids subjected to small deformation under appropriate assumptions. Based on the work of other researchers such as Green and Rivlin [59] and Noll [60], Pipkin reviewed the derivation of these nonlinear integrals relating stress and strain, for isotropic and incompressible materials. Later Pipkin and Rogers [61], based on previous work, proposed an integral series
representation of nonlinear viscoelastic response to an arbitrary strain or stress history. They tested their working on experimental data for tensile creep tests.

In 1968, Zienkiewicz and coworkers developed general numerical procedures for solving broad range of viscoelastic problems [62]. Their research was mainly concerned with creep analysis of concrete and rock. The finite element method for solving elastic problems has been extended to account for the viscoelasticity. Later Taylor et al. also proposed numerical procedures for solving linear viscoelasticity problems [63], in which they also took into account the thermal effects.

Schapery[64] proposed a three-dimensional nonlinear constitutive model which was aimed at the simulation of nonlinear responses of some metals and plastics. In this work various methods of characterizing nonlinear viscoelastic solids were attempted.

Partom and Schanin [65] presented a nonlinear viscoelastic formulation based on the generalized Maxwell model with linear springs and nonlinear dashpots. Their approach was based on the evolution of the stresses and the internal state variables, which was used to predict creep response for clamped beams under uniaxial loading. Such a procedure was claimed simpler and more straightforward than the multiple and single integral representation adopted in earlier works. Based on their work, Keren et al. [66] developed a finite difference code for a two dimensional axisymmetric problem. Another approach to account for nonlinear viscoelasticity was adopted by Rendell et al. [67]. This approach was based on the coupling model of relaxation, which was found to better relate the viscoelastic behavior and features observed during experiments with molecular theory. The simulations resulting from their model revealed many key features observed through experimental strain histories.

In 1989 Gramoll et al. [68] developed a numerical procedure to solve nonlinear viscoelastic problems of orthotropic materials such as fiber reinforced plastics (FRP) laminate composites. However, their model was not sufficient to accurately predict the evolution of strain and stresses over time. They further improved this numerical procedure by using Newton Raphson method to solve the nonlinear problems, and further calibrated their numerical simulation using experimental results. In the beginning of 1990s, Krishnaswamy et al.[69], presented a finite algorithm to solve both linear and nonlinear problems for viscoelastic response of materials. Their work was mainly focused on failure of material under cracks, and the model was suitable for analyzing the time dependent behavior of cracks in viscoelastic
materials. Kaliske and Rothert [70] also developed a formulation for three dimensional viscoelasticity at small and finite strains based on the generalized Maxwell model for finite element implementation. They presented simulations of time dependent deformations of rubber structures.

Considering the lack of proper understanding and shortcomings in addressing thermo-mechanical coupling, and large deformation in earlier work, Reese and Govindjee [71] proposed a model for finite thermo-viscoelasticity. This model was claimed to be physically reasonable and numerically tractable. Masuero and Creus [72] worked on nonlinearities caused in materials due to cracks and notches by developing nonlinear viscoelastic finite element model to simulate the behavior of the material. This work was based on Schapery’s [62] nonlinear viscoelastic formulation.

A comprehensive review on the nonlinear constitutive laws in viscoelasticity was conducted by Drapaca et al. [73] and Wineman [74]. Drapaca et al. gave a review of classical representation of continuum laws for viscoelastic materials, whereas Wineman reviewed all aspects of modeling in viscoelastic materials from a phenomenological stand point. It was concluded by Wineman that there were no generally accepted well-defined forms of constitutive equations to model nonlinear viscoelastic behavior of solids.
CHAPTER 3

3 Finite element modeling for linear viscoelastic response

In this chapter, formulation for the linear response of viscoelastomers is developed based on a generalized Maxwell rheological model. Moreover, the developed formulation is implemented in ABAQUS CAE as a user material subroutine (UMAT). A comparison between the simulation result from the UMAT and the experimental data of a uniaxial tensile test is also given in this chapter. Furthermore, the robustness of the developed UMAT is tested by simulating viscoelastomers subjected to complex loads.

3.1 Basic theory

The basic theory of linear viscoelasticity is introduced below, which will lead to a better understanding of the formulation for finite element implementation.

3.1.1 Stress and strain

The response of viscoelastomers is strongly affected by its intrinsic viscous and elastic properties. The response of a material is the way the material deforms under the application of a load. Consider a reference configuration (undeformed and stress-free) of an elastomer, the material point in this configuration is identified by the position vector \( \mathbf{X} \). After deformation the current position of this material point is given by \( \mathbf{x} = \chi(\mathbf{X}, t) \), where \( \chi \) describes the motion of material point from the reference to the current configuration. The deformation gradient \( \mathbf{F} \) [75] is defined as follows:

\[
\mathbf{F} = x_{i,j} = \begin{bmatrix}
\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\
\frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\
\frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}
\end{bmatrix} = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{bmatrix}.
\]  

(3.1)

In terms of the displacement field \( \mathbf{u} \), the deformation tensor can also be expressed as
\[ F = \frac{\partial (X + u)}{\partial X} = I + \frac{\partial u}{\partial X}. \] (3.2)

Under deformation, the stress of the material can be expressed as Cauchy stress \( \sigma \) (true stress) or first Piola-Kirchoff stress \( P \) (nominal stress). With the deformation gradient tensor \( F \), these two stresses are interconvertible, i.e.,

\[ \sigma_{ij} = J^{-1} P_{im} F_{mj}. \] (3.3)

Here \( J \) is the determinant of the deformation tensor \( F \) and \( F^{-1}_{ij} \) is the inverse of the deformation tensor \( F \).

### 3.1.2 Boltzmann superposition principle

According to the superposition principle, for all linear systems, the net response caused by multiple stimuli can be regarded as the sum of the responses caused by each individual stimulus. In other words, if an input \( x_1 \) to function \( f \) produces a response \( X \) and an input \( x_2 \) produces a response \( Y \), then input \( (x_1 + x_2) \) produces response \( (X + Y) \) [47], i.e.,

\[ X + Y = f(x_1 + x_2) = f(x_1) + f(x_2). \] (3.4)

### 3.2 Numerical formulation for linear response of elastomers

This section is focused on developing a generic three-dimension formulation for linear response of a viscoelastomer, which is structured based on the work of Kaliske and Rothert [70]. The formulation is derived based on a generalized Maxwell rheological model. First, a one-dimension modeling framework for small strain of viscoelastic solids is revisited. Then a three-dimensional formulation for finite element simulation of viscoelastomers is proposed.

For a single Maxwell element (see figure 2-5), the stress-strain relation (as first given in equation (2.30)) can also be expressed as

\[ \dot{\sigma} + \frac{1}{\tau} \sigma = \mu \dot{\varepsilon}. \] (3.5)

Here \( \tau = \eta/\mu \) is the relaxation time. The homogenous solution for equation (3.5) is
\[ \sigma_h = C \exp \left( -\frac{t}{\tau} \right). \] (3.6)

The value of the constant \( C \) is determined by the initial conditions. When a relaxation test is considered, i.e., \( \varepsilon(0) = \varepsilon(t) = \text{constant} \), figure 3-1 shows the stress response of the material under such a relaxation test [70].

**Figure 3-1** Relaxation test of a Maxwell element [70]

From figure 3-1, the initial stress on the material is governed by the elastic response, i.e., at time \( t = 0 \) the stress on the material is \( \sigma(0) = \mu \varepsilon(0) \). With this initial condition, constant \( C \) in the homogenous solution can be solved. Since the strain in the experiment is constant, i.e., the strain rate is zero, it leads to the particular solution of equation (3.5) as \( \sigma_p = 0 \). Thus, the solution of equation (3.5) is determined as

\[ \sigma(t) = \mu \exp \left( -\frac{t}{\tau} \right) \varepsilon(0). \] (3.7)

Here the relaxation function is defined as

\[ \Gamma(t) = \mu \exp \left( -\frac{t}{\tau} \right), \] (3.8)

which indicates the viscoelastic characteristics of the material. The above derivation can be extended to a generalized Maxwell model, see figure 2-7 (a). Let the modulus of the spring element parallel to the Maxwell elements be \( \mu_0 \) and the moduli for the spring components of the Maxwell elements be \( \mu_1, \mu_2, \mu_3, \ldots, \mu_N \), where \( N \) is the total number of Maxwell elements as shown in figure 3-2. The viscosities of the damper components in the Maxwell elements are
given as $\eta_1, \eta_2, \eta_3 \ldots \eta_N$. The overall stress $\sigma$ acting on the generalized model is equal to the sum of the stresses acting on the Hookean spring and the parallel Maxwell elements. Adopting the generalized Maxwell model, the stress of the material is expressed as

$$\sigma(t) = \mu_0 \varepsilon(0) + \sum_{i=1}^{N} \mu_i \exp\left(-\frac{t}{\tau_i}\right) \varepsilon(0) = \Gamma(t) \varepsilon(0).$$  \hspace{1cm} (3.9)

Here $\tau_i$ is the relaxation time for each Maxwell element, which equals to the ratio of viscosity constant for each damper to the modulus of the spring in the Maxwell element. The relaxation function becomes

$$\Gamma(t) = \mu_0 + \sum_{i=1}^{N} \mu_i \exp\left(-\frac{t}{\tau_i}\right).$$  \hspace{1cm} (3.10)

A normalized form of the relaxation function is given below as

$$\gamma(t) = \frac{\Gamma(t)}{\mu_0} = 1 + \sum_{i=1}^{N} \gamma_i \exp\left(-\frac{t}{\tau_i}\right).$$  \hspace{1cm} (3.11)

Here $\gamma_i$ is the ratio of $\mu_i$ to $\mu_0$. For incremental strain $\Delta \varepsilon_i$ of the material, the Cauchy stress is expressed as

$$\sigma(t) = \int_{0}^{t} \Gamma(t - s) \frac{\partial \varepsilon}{\partial s} ds.$$  \hspace{1cm} (3.12)

Here the initial response commences at any time $s$. Although only the relaxation of the material is considered, a very similar formulation can also be obtained for the creep test. For equation (3.12), the relaxation function $\Gamma(t - s)$ is given as

$$\Gamma(t - s) = \mu_0 + \sum_{i=1}^{N} \mu_i \exp\left(-\frac{t - s}{\tau_i}\right).$$  \hspace{1cm} (3.13)

Expanding equation (3.12), the stress is written as

$$\sigma(t) = \int_{0}^{t} \mu_0 \frac{\partial \varepsilon(s)}{\partial s} ds + \int_{0}^{t} \sum_{i=1}^{N} \mu_i \exp\left(-\frac{t - s}{\tau_i}\right) \frac{\partial \varepsilon(s)}{\partial s} ds.$$  \hspace{1cm} (3.14)
Since $\mu_0$ is a constant and the sum of all the responses for strain $\partial \varepsilon(s)$ over the time $t$ is $\varepsilon(t)$, equation (3.14) is thus rewritten as

$$\sigma(t) = \mu_0 \varepsilon(t) + \int_0^t \sum_{i=1}^{N} \mu_i \exp\left(-\frac{t - s}{\tau_i}\right) \frac{\partial \varepsilon(s)}{\partial s} \, ds,$$

resulting in

$$\sigma(t) = \sigma_0(t) + \sum_{i=1}^{N} h_i(t) \quad (3.15)$$

Here $\sigma_0(t)$ is the stress for the elastic element (the Hookean spring), and $h_i(t)$ defines the internal stresses in the Maxwell elements, i.e.,

$$h_i(t) = \int_0^t \mu_i \exp\left(-\frac{t - s}{\tau_i}\right) \frac{\partial \varepsilon(s)}{\partial s} \, ds \quad (3.16)$$

From equations (3.15) and (3.16), and $\varepsilon(s) = \sigma_0(s)/\mu_0$, equation (3.17) can be expressed in the other format as,

$$h_i(t) = \int_0^t \gamma_i \exp\left(-\frac{t - s}{\tau_i}\right) \frac{\partial \sigma_0(s)}{\partial s} \, ds. \quad (3.18)$$

In a relaxation test, the internal stress variable $h_i(t)$ approaches to zero if the time approaches infinity $[70]$. This means that under a constant deformation, the stress on the material will eventually relax to the value of the elastic portion ($\lim_{t \to \infty} h_i(t) = 0$).
In the formulation below, a finite difference method is applied to equation (3.18) to determine $h_i(t)$ [70]. Consider a time interval $[t_n, t_{n+1}]$ and a time step $\Delta t = t_{n+1} - t_n$, the multiplicative split of the exponential term leads to

$$\exp\left(\frac{-t_{n+1}}{\tau_i}\right) = \exp\left(\frac{-t_n + \Delta t}{\tau_i}\right) = \exp\left(-\frac{t_n}{\tau_i}\right) \exp\left(-\frac{\Delta t}{\tau_i}\right).$$  \hspace{1cm} (3.19)

For the defined time interval, the deformation is kept constant for each time increment from $t_n$ to $t_{n+1}$. Thus, stress $\sigma_0$ is only a function of time. Integrating equation (3.18) up to time $t_{n+1}$ results in,

$$h_i(t_{n+1}) = \gamma_i \int_0^{t_{n+1}} \exp\left(-\frac{(t_{n+1} - s)}{\tau_i}\right) \frac{d\sigma_0(s)}{ds} ds.$$  \hspace{1cm} (3.20)

The deformation history can be separated into two steps, i.e., from $0 \leq s \leq t_n$ and from $t_n \leq s \leq t_{n+1}$, which further leads to the following formulation,

$$h_i(t_{n+1}) = \gamma_i \int_0^{t_n} \exp\left(-\frac{(t_{n+1} - s)}{\tau_i}\right) \frac{d\sigma_0(s)}{ds} ds + \gamma_i \int_{t_n}^{t_{n+1}} \exp\left(-\frac{(t_{n+1} - s)}{\tau_i}\right) \frac{d\sigma_0(s)}{ds} ds.$$  \hspace{1cm} (3.21)

Since $\Delta t = t_{n+1} - t_n$, equation (3.21) can be rewritten as

$$h_i(t_{n+1}) = \exp\left(-\frac{\Delta t}{\tau_i}\right) \gamma_i \int_0^{t_n} \exp\left(-\frac{t_n - s}{\tau_i}\right) \frac{d\sigma_0(s)}{ds} ds + \gamma_i \int_{t_n}^{t_{n+1}} \exp\left(-\frac{t_{n+1} - s}{\tau_i}\right) \frac{d\sigma_0(s)}{ds} ds.$$  \hspace{1cm} (3.22)

Comparing equation (3.22) and equation (3.18), we can find that

$$h_i(t_n) = \gamma_i \int_0^{t_n} \exp\left(-\frac{(t_n - s)}{\tau_i}\right) \frac{d\sigma_0(s)}{ds} ds.$$  \hspace{1cm} (3.23)

Therefore equation (3.22) is written as

$$h_i(t_{n+1}) = \exp\left(-\frac{\Delta t}{\tau_i}\right) h_i(t_n) + \gamma_i \int_{t_n}^{t_{n+1}} \exp\left(-\frac{t_{n+1} - s}{\tau_i}\right) \frac{d\sigma_0(s)}{ds} ds.$$  \hspace{1cm} (3.24)

The integral in the above equation can be simplified using a similar finite difference approach on $\frac{d\sigma_0(s)}{ds}$, i.e.,

$$\frac{d\sigma_0(s)}{ds} = \lim_{\Delta s \to 0} \left(\frac{\Delta \sigma_0(s)}{\Delta s}\right) = \lim_{\Delta t \to 0} \left(\frac{\sigma_0^{n+1} - \sigma_0^n}{\Delta t}\right).$$  \hspace{1cm} (3.25)

Introducing the time discretized approximation of the second order into the formula, a numerical solution of equation (3.18) is obtained as
\[ h_i^{n+1} = \exp \left( -\frac{\Delta t}{\tau_i} \right) h_i^n + \frac{\gamma_i \left( 1 - \exp \left( -\frac{\Delta t}{\tau_i} \right) \right)}{\Delta t} \left[ \sigma_0^{n+1} - \sigma_i^n \right]. \]  

(3.26)

This indicates that all the values for \( h_i(t) \) are dependent on the preceding values of \( h_i \). If those values are known, subsequent values for any given time step can be determined using the iterative formulation shown in equation (3.26). Utilizing the iterative formulation equation (3.16) becomes

\[ \sigma^{n+1} = \sigma_0^{n+1} + \sum_{i=1}^{N} h_i^{n+1}, \]  

(3.27)

which defines a one-dimensional numerical formulation for a linear response of viscoelastomers. One can observe that the relaxation function has been replaced by an exponential series which is dependent on individual moduli of spring and damper in the model.

To introduce the three-dimension formulation, tensor notation is adopted in equation (3.27), i.e.,

\[ \sigma_{ij}^{n+1} = \sigma_{0ij}^{n+1} + \sum_{i=1}^{N} h_i^{n+1}. \]  

(3.28)

Here the elastic stress is defined as

\[ \sigma_{0ij}^{n+1} = C_{ijkl}^{e} \varepsilon_{kl}^{n+1}. \]  

(3.29)

The superscript \( e \) in \( C_{ijkl}^{e} \) emphasizes that this tensor describes an elastic relation between the stress and the strain. The fourth order tensor \( C_{ijkl} \) is the stiffness matrix which is defined by generalized Hooke’s law for elastic materials [76]. The formulation derived above is based on the following assumptions: 1) the material under consideration is linear in nature, i.e., its stiffness is constant; 2) the deformation the material undergoes is also linear, i.e., Boltzmann superposition, Hookean and Newtonian principles for linearity apply; 3) the material is homogenous and isotropic. Consider that the Voigt notation [77] is convenient to use, it is introduced here and leads to

\[ \sigma_{0M}^{n+1} = C_{MK}^{e} \varepsilon_{K}^{n+1}, \]  

(3.30)

in which the Cauchy stress tensor is written as
\[ \sigma_M = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}, \]  
(3.31)

with \( \sigma_{11} = \sigma_1, \sigma_{22} = \sigma_2, \sigma_{33} = \sigma_3, \sigma_{23} = \sigma_{32} = \sigma_4, \sigma_{13} = \sigma_{31} = \sigma_5, \) and \( \sigma_{12} = \sigma_{21} = \sigma_6. \) Similarly, the strain tensor with Voigt notation is given as

\[ \varepsilon_K = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}, \]  
(3.32)

The stiffness matrix \( C_{MK}^e \) defines the relation between elastic stress and strain, which takes the form for the isotropic material as

\[
C_{KM} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} \\
\end{bmatrix} = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu \\
\end{bmatrix},
\]  
(3.33)

where \( \lambda \) and \( \mu \) are Lame’s constants [78]. Also, \( 3K = 3\lambda + 2\mu \) with \( K \) being the bulk modulus of the material. Thus, the internal stress variables in equation (3.26) can be rewritten as

\[
h_i^{n+1} = \exp\left(-\frac{\Delta t}{\tau_i}\right)h_i^n + \frac{\gamma_i \left(1 - \exp\left(-\frac{\Delta t}{\tau_i}\right)\right)}{\Delta t} \left[ C_{MK}^e \varepsilon_K^{n+1} - C_{MK}^e \varepsilon_K^n \right].
\]  
(3.34)

Expanding equation (3.28), the incremental formulation for the Cauchy stress is given as

\[
\sigma_M^{n+1} = C_{MK}^e \varepsilon_K^{n+1} + \sum_{j=1}^N \exp\left(-\frac{\Delta t}{\tau_j}\right)h_j^n + \frac{\gamma_j}{\tau_j} \left[ C_{MK}^e \varepsilon_K^{n+1} - C_{MK}^e \varepsilon_K^n \right].
\]  
(3.35)
Equation (3.35) provides an iterative formulation for the stress of a viscoelastic material. Using this formulation, a user material subroutine (UMAT) will be developed to solve complex problems using FEM software like ABAQUS CAE.

3.3 Development of UMAT for linear viscoelastic response

A UMAT is a user material subroutine that defines a constitutive relation of the material. Details and format of UMATs can be found in the ABAQUS/STANDARD manuals and subroutine guides [78, 79].

3.3.1 Formulation of UMAT for linear viscoelastic analysis

The development of the UMAT is based on constitutive equation (3.35). Three essentials are required to complete a UMAT, including defining the material parameters, the constitutive equation and the material Jacobian matrix.

Stress-strain constitutive equation

In this case study, we only consider three Maxwell elements in parallel with the Hookean spring. Therefore, the constitutive equation (3.35) is expressed as

$$
\sigma_M^{n+1} = C_{MK}\varepsilon_K^n + \exp\left(-\frac{\Delta t}{\tau_1}\right) h_1^n + \exp\left(-\frac{\Delta t}{\tau_2}\right) h_2^n + \exp\left(-\frac{\Delta t}{\tau_3}\right) h_3^n \\
+ \left\{ \begin{array}{c}
1 + \gamma_1\tau_1 \frac{1 - \exp\left(-\frac{\Delta t}{\tau_1}\right)}{\Delta t} + \gamma_2\tau_2 \frac{1 - \exp\left(-\frac{\Delta t}{\tau_2}\right)}{\Delta t} \\
+ \gamma_3\tau_3 \frac{1 - \exp\left(-\frac{\Delta t}{\tau_3}\right)}{\Delta t}
\end{array} \right\} [C_{MK}^e \Delta \varepsilon_K],
$$

(3.36)

where $\Delta \varepsilon_K = \varepsilon_K^{n+1} - \varepsilon_K^n$.

State variables

In the development of the generalized formulation we used a state variable approach [46], where we replaced the relaxation function with an approximated exponential series. The internal stresses in the Maxwell elements given by the function $h_i^n$ are defined as state variables in the UMAT. These stresses will be updated in each iteration using the relation defined by equation (3.34).
Stiffness matrix

The Lame’s constants described in previous section satisfy $3K = 3\lambda + 2\mu$. Here $\mu = \mu_0$. So, $\lambda = K - \frac{2}{3}\mu_0$. Hence, as per equation (3.33) the stiffness matrix for the elastic component of the model is given as

\[
C_{KM} = D(i,j) = \begin{bmatrix}
K + \frac{4}{3}\mu_0 & K - \frac{2}{3}\mu_0 & K - \frac{2}{3}\mu_0 & 0 & 0 \\
K - \frac{2}{3}\mu_0 & K + \frac{4}{3}\mu_0 & K - \frac{2}{3}\mu_0 & 0 & 0 \\
K - \frac{2}{3}\mu_0 & K - \frac{2}{3}\mu_0 & K + \frac{4}{3}\mu_0 & 0 & 0 \\
0 & 0 & 0 & \mu_0 & 0 \\
0 & 0 & 0 & 0 & \mu_0 \\
0 & 0 & 0 & 0 & 0 & \mu_0
\end{bmatrix}
\] (3.37)

Jacobian matrix

The material Jacobian which is also known as the tangent modulus is the slope for the stress-strain curve obtained from the constitutive equation. It is the requirement for ABAQUS/STANDARD UMAT only [78]. For a small deformation problem, or a large deformation problem with no or very little volume change (incompressible material), the Jacobian is calculated by $\frac{\partial \Delta\sigma}{\partial \Delta\varepsilon}$ [78, 79]. Here, $\Delta\sigma$ is a small increment in Cauchy stress and $\Delta\varepsilon$ is the increment in strain. The matrix formed may be non-symmetric and depends on the constitutive equation for stress and strain. In this case, equation (3.36) is used to calculate the Jacobian matrix, i.e.,

\[
C^{n+1} = \frac{\partial \Delta\sigma}{\partial \Delta\varepsilon} = \left\{1 + \sum_{j=1}^{3} y_j \frac{1 - \exp\left(-\frac{\Delta\varepsilon}{\tau_j}\right)}{\Delta\varepsilon/\tau_j}\right\}C^{n+1},
\] (3.38)

where $C$ is the stiffness matrix defined in equation (3.37).

UMAT code

The UMAT code is given below, with detailed interpretation given in Appendix.
This UMAT is based on generalized Maxwell model with three Maxwell element chains.

**SUBROUTINE** UMAT(STRESS, STATEV, DDSDDE, SSE, SPD, SCD,
RPL, DDSDDT, DRPLDE, DRPLDT,
STRAN, DSTRAN, TIME, DTIME, TEMP, DTEMP, PREDEF, DPRED, CMNAME,
NDI, NSHR, NTENS, NSTATEV, PROPS, NPROPS, COORDS, DROT, PNEWDT,
CELENT, DFGRD0, DFGRD1, NOEL, NPT, LAYER, KSPT, KSTEP, KINC)

INCLUDE 'ABA_PARAM.INC'

CHARACTER*8 CMNAME
DIMENSION STRESS(NTENS), STATEV(NSTATEV),
DDSDDE(NTENS, NTENS), DDSDDT(NTENS), DRPLDE(NTENS),
STRAN(NTENS), DSTRAN(NTENS), TIME(2), PREDEF(1), DPRED(1),
PROPS(NPROPS), COORDS(3), DROT(3, 3), DFGRD0(3, 3), DFGRD1(3, 3)
DIMENSION D(3, 3), SM1OLD(6), SM2OLD(6), SM3OLD(6),
SM1(6), SM2(6), SM3(6), SM1DOT(6), SM2DOT(6), SM3DOT(6), G(6, 6)
REAL*8 M1, M2, M3

PROPS(1) THROUGH PROPS(3) ARE THE SHEAR MODULI IN PASCALS
M0 = PROPS(1) = 14.59E3
MU1 = PROPS(2) = 11.87E3
MU2 = PROPS(3) = 41.35E3
MU3 = PROPS(4) = 19.75E3
PROPS(5) THROUGH PROPS(7) ARE THE RELAXATION TIMES IN SECONDS
TAU1 = PROPS(5) = 960.4
TAU2 = PROPS(6) = 1.044
TAU3 = PROPS(7) = 19.3
PROPS(8) IS THE BULK MODULUS OF THE MATERIAL = 10^3 * PROPS(1)
PROPS(8) = 14590E3

DEFINING STATE VARIABLES FOR INTERNAL STRESSES FOR MAXWELL ELEMENTS

DO I = 1, 6
SM1OLD(I) = STATEV(I)
ENDDO
DO I = 1, 6
SM2OLD(I) = STATEV(I + 6)
ENDDO
DO I = 1, 6
SM3OLD(I) = STATEV(I + 12)
ENDDO

DEFINING THE VALUES FOR M1, M2 AND M3

M1 = (PROPS(5) * PROPS(2) -
PROPS(5) * PROPS(2) * EXP(-DTIME / PROPS(5))) / (PROPS(1) * DTIME)
M2 = (PROPS(6) * PROPS(3) -
PROPS(6) * PROPS(3) * EXP(-DTIME / PROPS(6))) / (PROPS(1) * DTIME)
M3 = (PROPS(7) * PROPS(4) -
PROPS(7) * PROPS(4) * EXP(-DTIME / PROPS(7))) / (PROPS(1) * DTIME)

DEFINING THE TERMS TO BE USED IN STIFFNESS MATRIX
TERM1 = PROPS(8) + (4.D0 * PROPS(1)) / 3.D0
TERM2 = PROPS(8) - (2.D0 * PROPS(1)) / 3.D0

C
A1 = EXP(-DTIME / PROPS(5))
A2 = EXP(-DTIME / PROPS(6))
A3 = EXP(-DTIME / PROPS(7))

C PRINTING THE VALUES OF M1, M2, M3, TERM1 AND TERM2

WRITE (6,*) 'THIS IS A TEST'
WRITE (6,*) NDI, NSHR, NTENS, NSTATEV, NPROPS
WRITE (6,*) 'values for M1,M2,M3, Term1, Term2'
WRITE (6,*) M1, M2, M3, TERM1, TERM2

C DEFINING THE ELASTIC STIFFNESS MATRIX FOR THE NEO-HOOKEAN ELEMENT

DO K1 = 1, NTENS
    DO K2 = 1, NTENS
        G(K1, K2) = 0.D0
    ENDDO
ENDDO

DO K1 = 1, NDI
    G(K1, K1) = TERM1
ENDDO

DO K1 = 2, NDI
    N = K1 - 1
    DO K2 = 1, N
        G(K2, K1) = TERM2
        G(K1, K2) = TERM2
    ENDDO
ENDDO

N2 = NDI
M = N2 + 1
DO K1 = M, NTENS
    G(K1, K1) = PROPS(1)
ENDDO

C DEFINING STRESS AND STRAIN RELATION

STRESS(1) = G(1, 1) * STRAN(1) + G(1, 2) * STRAN(2) + G(1, 3) * STRAN(3) + SM1OLD(1) + SM2OLD(1) + SM3OLD(1) + (1 + M1 + M2 + M3) * G(1, 1) * DSTRAN(1) + G(1, 2) * DSTRAN(2) + G(1, 3) * DSTRAN(3)

STRESS(2) = G(2, 1) * STRAN(1) + G(2, 2) * STRAN(2) + G(2, 3) * STRAN(3) + SM1OLD(2) + SM2OLD(2) + SM3OLD(2) + (1 + M1 + M2 + M3) * G(2, 1) * DSTRAN(1) + G(2, 2) * DSTRAN(2) + G(2, 3) * DSTRAN(3)

STRESS(3) = G(3, 1) * STRAN(1) + G(3, 2) * STRAN(2) + G(3, 3) * STRAN(3) + SM1OLD(3) + SM2OLD(3) + SM3OLD(3) + (1 + M1 + M2 + M3) * G(3, 1) * DSTRAN(1) + G(3, 2) * DSTRAN(2) + G(3, 3) * DSTRAN(3)

STRESS(4) = G(4, 4) * STRAN(4) + SM1OLD(4) + SM2OLD(4) + SM3OLD(4) + (1 + M1 + M2 + M3) * G(4, 4) * DSTRAN(4)

STRESS(5) = G(5, 5) * STRAN(5) + SM1OLD(5) + SM2OLD(5) + SM3OLD(5) + (1 + M1 + M2 + M3) * G(5, 5) * DSTRAN(5)

STRESS(6) = G(6, 6) * STRAN(6) + SM1OLD(6) + SM2OLD(6) + SM3OLD(6) + (1 + M1 + M2 + M3) * G(6, 6) * DSTRAN(6)

WRITE (6, *) 'OUTPUT OF RESULTS FOR STRAN AND STATEV'
WRITE (6, *) STRAN(1), STRAN(2), STRAN(3), SM1OLD(1)
WRITE (6, *) SM2OLD(1), SM3OLD(1), SM1OLD(2), SM2OLD(2)
WRITE (6, *) SM3OLD(2), SM1OLD(3), SM2OLD(3), SM3OLD(3)
NOW THE STATE VARIABLES WILL BE UPDATED

SM1(1)=SM1OLD(1)+M1*(G(1,1)*DSTRAN(1)+G(1,2)*DSTRAN(2)+
       G(1,3)*DSTRAN(3))
SM2(1)=SM2OLD(1)+M2*(G(1,1)*DSTRAN(1)+G(1,2)*DSTRAN(2)+
       G(1,3)*DSTRAN(3))
SM3(1)=SM3OLD(1)+M3*(G(1,1)*DSTRAN(1)+G(1,2)*DSTRAN(2)+
       G(1,3)*DSTRAN(3))
SM1(2)=SM1OLD(2)+M1*(G(2,1)*DSTRAN(1)+G(2,2)*DSTRAN(2)+
       G(2,3)*DSTRAN(3))
SM2(2)=SM2OLD(2)+M2*(G(2,1)*DSTRAN(1)+G(2,2)*DSTRAN(2)+
       G(2,3)*DSTRAN(3))
SM3(2)=SM3OLD(2)+M3*(G(2,1)*DSTRAN(1)+G(2,2)*DSTRAN(2)+
       G(2,3)*DSTRAN(3))
SM1(3)=SM1OLD(3)+M1*(G(3,1)*DSTRAN(1)+G(3,2)*DSTRAN(2)+
       G(3,3)*DSTRAN(3))
SM2(3)=SM2OLD(3)+M2*(G(3,1)*DSTRAN(1)+G(3,2)*DSTRAN(2)+
       G(3,3)*DSTRAN(3))
SM3(3)=SM3OLD(3)+M3*(G(3,1)*DSTRAN(1)+G(3,2)*DSTRAN(2)+
       G(3,3)*DSTRAN(3))
SM1(4)=SM1OLD(4)+M1*(G(4,1)*DSTRAN(4))
SM2(4)=SM2OLD(4)+M2*(G(4,2)*DSTRAN(4))
SM3(4)=SM3OLD(4)+M3*(G(4,3)*DSTRAN(4))
SM1(5)=SM1OLD(5)+M1*(G(5,1)*DSTRAN(5))
SM2(5)=SM2OLD(5)+M2*(G(5,2)*DSTRAN(5))
SM3(5)=SM3OLD(5)+M3*(G(5,3)*DSTRAN(5))
SM1(6)=SM1OLD(6)+M1*(G(6,1)*DSTRAN(6))
SM2(6)=SM2OLD(6)+M2*(G(6,2)*DSTRAN(6))
SM3(6)=SM3OLD(6)+M3*(G(6,3)*DSTRAN(6))
WRITE(6,*),'THE VALUES FOR SM1OLD, SM2OLD, SM3OLD UPDATED'
WRITE(6,*), SM1DOT(1), SM2DOT(1), SM3DOT(1)

DO K1=1,NTENS
STATEV(K1)=SM1DOT(K1)
STATEV(K1+6)=SM2DOT(K1)
STATEV(K1+12)=SM3DOT(K1)
ENDDO

C CREATING JACCOBIAN MATRIX
C
DO K1 = 1,NTENS
   DO K2= 1, NTENS
      DDSDDE(K2,K1) = 0.D0
   ENDDO
ENDDO
C
DO K1=1,NTENS
   DO K2=1,NTENS
      DDSDDE(K1,K2)=(1+M1+M2+M3)*G(K1,K2)
   ENDDO
ENDDO
C
RETURN
END

3.4 Testing with ABAQUS

The developed UMAT will now be tested using ABAQUS CAE simulation software. A series of test will be performed to analyze the accuracy, robustness and versatility of the developed UMAT.

3.4.1.1 Tensile and shear tests

For the tensile test, the deformation is introduced in the $x$ direction, giving the equations for analytical calculations as

$$
\varepsilon_2 = \varepsilon_3 = -\nu \varepsilon_1.
$$

(3.39)

Elastomers are almost incompressible, so it is safe to assume that their Poisson’s ratio $\nu$ is 0.499. The uniaxial tensile test and simple shear test are performed on a unit element. The element is a cube of 1x1x1 dimensions in mm. The material selected for the case study is VHB4910 with material parameters given in Table 3-1. The element type is C3D8. For the tensile test, the element is restrained as shown in figure 3-3 and the deformation of the material under uniaxial tension is shown in figure 3-4.
Similarly, the same element shown in figure 3-3 is deformed under a simple shear. The element is restricted in $y$ and $z$ directions on the side, top and bottom surfaces respectively. A deformation is applied in $x$ direction on the top surface as shown in figure 3-5 below. The deformation of the material under such a simple shear is shown in figure 3-6.
Figure 3-5 A unit element under a simple shear.

Figure 3-6 Deformed configuration of a unit element under a simple shear.
After the simulations are complete, the stress-strain plots are obtained from ABAQUS CAE and compared with results from analytical calculations using equation 3.36. The comparisons are given in figures 3-7 and 3-8.
Figure 3-8 Comparison of stress-strain for ABAQUS simulation and analytical calculation for shear test.

The comparison in figures 3-7 and 3-8 demonstrates that the ABAQUS simulations are in excellent agreement with the analytical results, indicating the accuracy of the developed UMAT.

3.4.1.2 Experimental validation

In the following case study, the simulation results are compared with the experimental data of elastomer VHB 4910 [4, 81]. In the experiments, the elastomers are subjected to a uniaxial cyclic load. In the simulation, the material parameters given in table 3-1 are obtained through fitting of the experimental data of VHB 4910 [4, 81].

In table 3-1, $\mu_0$ is obtained by fitting the experimental data from tensile test with stretch rate of $0.9 \times 10^{-4}$ /s (which can be considered as a quasi-static deformation). The other material parameters are obtained by fitting the cyclic tensile tests performed by Hossain et al. [4] at a stretch rate of 0.01 /s and 0.03 /s as shown in figure 3-9. After the material parameters are obtained, the simulation results are further verified with the cyclic tensile test data at the
stretch rate of 0.05/s, which is shown in figure 3-10. The comparison between the finite element simulation and the experimental data of VHB 4910 shows that the UMAT based on linear viscoelasticity can capture the trend and the loading rate-dependency of the experimental data, while discrepancy still exists due to the fact that there is nonlinearity in nature of the materials undergoing large deformation. Therefore, more accurate material models are needed to describe the response of elastomers under large deformation.

| Table 3-1 Parameters for VHB 4910 obtained after parametric study |
|------------------|------------------|------------------|------------------|
| Parameter       | Value            | Parameter       | Value            | Parameter       | Value            |
| $\mu_0$         | 12.7 KPa         | $\mu_3$        | 11.47 KPa        | $\eta_3$       | 23.65 KPa.s     |
| $\mu_1$         | 23.86 KPa        | $\eta_1$       | 47.73 MPa.s      | K               | 12.7 MPa        |
| $\mu_2$         | 41.55 KPa        | $\eta_2$       | 85.75 KPa.s      |                 |                 |

Figure 3-9 Parametric study for VHB 4910 generalized Maxwell model at different stretch rates.
Lastly, simulation of a rubber specimen [31] (figure 3-11) subjected to cyclic shear is carried out. The test specimen is constrained completely from the bottom surface, and a cyclic deformation in the $x$ direction is applied at the top with a magnitude range of $(+6, -6)$ mm and a rate of $\dot{u}(t) = 40 \text{mm/min}$. The top surface is also constrained in $y$ and $z$ direction. The element used in the ABAQUS is C3D8. The displacement and the Von Mises stress of the specimen are shown in figure 3-12 and 3-13 respectively. In addition, the reaction force $RF$ versus the displacement $u(t)$ is shown in figure 3-14. Due to the material viscoelasticity, significant hysteresis effect is observed from figure 3-14.

Figure 3-10 Experimental vs simulation results for a cyclic test at loading rate of 0.05/s.
Figure 3-11 Complex geometry configuration of a specimen.

Figure 3-12 Displacement profile at time $t=9$ seconds.
Figure 3-13 Von Mises stress at time t=9 seconds.

Figure 3-14 Reaction force vs displacement under a cyclic loading for VHB 4910.

It should be noted that the convergence of the FE simulation is ensured. An example for the convergence test on the complex geometry testing of the UMAT is run. The finite element results change with the size of the meshed elements. When the element is refined to a
certain size, further decreasing the size of the element may no longer have an effect on the FE simulation results, leading to the convergence. Table 3-2 shows the global approximate size, number of elements, and results of stresses for each refinement. Figures 3-15 and 3-16 depict the convergence plots for the principal stress in y-direction and the Von Mises stress against the number of elements respectively. It is observed from these figures that the stress reaches a stable value after the global size is decreased to a certain level. This in turn verifies the robustness of the developed UMAT. A quarter part of the specimen in figure 3-11 with symmetric boundary conditions is used to decrease processing time. The element for meshing is C3D8.

**Table 3-2** Convergence test data for linear viscoelastic UMAT.

<table>
<thead>
<tr>
<th>Global approximate size</th>
<th>Number of elements</th>
<th>Maximum principal stress in y direction</th>
<th>Maximum von Mises Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>7</td>
<td>1.66e-02</td>
<td>1.30e-02</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>1.50e-02</td>
<td>1.30e-02</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1.42e-02</td>
<td>1.26e-02</td>
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<tr>
<td></td>
<td>4</td>
<td>1.60e-02</td>
<td>1.40e-02</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.62e-02</td>
<td>1.46e-02</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.70e-02</td>
<td>1.62e-02</td>
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<tr>
<td></td>
<td>1</td>
<td>1.77e-02</td>
<td>1.62e-02</td>
</tr>
<tr>
<td>0.75</td>
<td>23,828</td>
<td>1.80e-02</td>
<td>1.64e-02</td>
</tr>
</tbody>
</table>

**Figure 3-15** Variation of principal stress in y-direction with number of elements.
3.5 Summary

In this chapter a UMAT has been developed for simulating the viscoelastic response of elastomeric materials. The formulation of the UMAT is based on the generalized Maxwell model. Adopting finite difference method, the relaxation function from the stress-strain relation is replaced by an exponential series [46]. Several different tests are performed on the developed UMAT in ABAQUS/CAE. In the first test, the developed UMAT is tested with a single element under uniform tensile and pure shear loads. The results obtained from the ABAQUS simulations are then verified by analytical calculations. The comparison between the finite element simulation and the experimental data of VHB 4910 shows that the UMAT based on linear viscoelasticity can only capture the experimental data to a certain extent. Therefore, more accurate models may be needed to describe the response of elastomers under large deformation. The third test is for determining the convergence efficiency of the UMAT. The results from convergence test further verify the robustness of the developed UMAT. All the above-mentioned tests show the accuracy, the versatility and the robustness of the UMAT.

Figure 3-16 Variation of maximum von Mises stress with number of elements.
CHAPTER 4

4 Finite element modeling of nonlinear response of elastomers

In this chapter, formulation for nonlinear response of elastomers is developed using hyperelastic models. Based on this formulation framework, a user material subroutine (UMAT) for ABAQUS CAE is defined for finite element simulation. Moreover, the results from the finite element simulation are compared with experimental data of elastomer VHB 4910 to demonstrate the capability of the developed UMAT to predict nonlinear response.

4.1 Constitutive equations for materials under large deformations

Hyperelastic constitutive laws can be applied to materials that undergo large deformation[82]. Those models can capture the nonlinear behavior of materials and are generally used to model elastomeric materials. For elastomeric materials, the stress is commonly calculated from the strain energy density functions as described in section 2.2 of chapter 2. Also, it should be noted that most elastomers can be considered as incompressible and isotropic in nature. There are a number of works which investigate the constitutive relations of elastomers while most of these works are based on a similar framework as described below:

4.1.1 Equation of stresses for nonlinear response

Most hyperelastic models for large deformation are constructed as follows:

1. The stress-strain relation is defined based on the strain energy density function. The strain energy can be a function of the deformation tensor \( U = W(F) \) or of the strain invariants \( U = W(I) \) (\( I_1, I_2 \) and \( I_3 \)). Hence, to obtain the constitutive relation of elastomers, the first step is to adopt a strain energy density function which can best define the material under consideration.

2. For any deformation, the left or the right Cauchy Green strain tensor is then calculated with the deformation gradient \( F. C = F^T. F \) is the right C-G strain tensor, while \( B = F.F^T \) is the left C-G strain tensor.
3. Then the Cauchy stress is calculated by differentiating the strain energy density function with respect to \( \mathbf{F} \) or \( \mathbf{C} \) or \( \mathbf{B} \).

Although most elastomers are regarded as incompressible, the formulation discussed in this section is generic and will be applicable on compressible materials as well. Hence the isochoric deformation gradient tensor is used here, i.e.,

\[
\mathbf{F} = J^{-1/3} \mathbf{F},
\]

where \( J \) is the determinant of the deformation gradient, representing the change in volume per original volume. The isochoric Cauchy left green strain tensor is given as

\[
\mathbf{B} = \mathbf{F} \mathbf{F}^T.
\]

The isochoric strain invariants are defined as

\[
\bar{I}_1 = \text{trace}(\mathbf{B}), \\
\bar{I}_2 = \frac{1}{2}(\bar{I}_1^2 - \text{trace}\mathbf{B}^2) , \\
\bar{I}_3 = \text{det}(\mathbf{B}).
\]

From section 2.2 of chapter 2, the strain energy density functions for hyperelastic materials are either a function of the deformation gradient or the conventional invariants (\( I_3 \) relates to \( J \)).

\[
W(\mathbf{F}) = U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, J),
\]

The Cauchy stress can be obtained by differentiating the strain energy density function in terms of \( \sigma_{ij} \) as described in chapter 2, i.e.,

\[
\sigma_{ij} = \frac{1}{J} F_{ik} \frac{\partial W}{\partial F_{jk}}
\]

On the other hand, if the strain energy density is in terms of \( \bar{I}_1, \bar{I}_2 \) and \( J \), the Cauchy stress can be obtained as

\[
\sigma_{ij} = \frac{2}{J} \left[ \left( \frac{\partial \bar{U}}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial \bar{U}}{\partial \bar{I}_2} \right) \delta_{ij} - \left( \bar{I}_1 \frac{\partial \bar{U}}{\partial \bar{I}_1} + 2\bar{I}_2 \frac{\partial \bar{U}}{\partial \bar{I}_2} \right) \frac{\delta_{ij}}{3} - \frac{\partial \bar{U}}{\partial \bar{I}_2} \delta_{uk} \delta_{kj} \right] + \frac{\partial \bar{U}}{\partial J} \delta_{ij}.
\]

For the development of the UMAT, the Gent strain energy density function [24] for hyperelasticity is adopted as an example, which has the following form:

\[
\bar{U} = \frac{-G J_{lim}}{2} \ln \left( 1 - \frac{\bar{I}_1 - 3}{J_{lim}} \right) + \frac{1}{D_1} (J - 1)^2.
\]

Here \( G \) is the shear modulus for the material, \( J_{lim} \) defines the maximum stretch limit for the molecular strand of the elastomer and \( D_1 \) is a factor which defines the compressibility of the
material [83]. The term $D_1$ is equal to $2/K$ with $K$ being the bulk modulus of the material. Substituting equation (4.6) into (4.5), we can get the expression for the Cauchy stresses, i.e.,

$$
\sigma_{ij} = \frac{2}{J} \left[ \left( \frac{G_{l_{im}}}{2(J_{im} - \bar{I}_1 + 3)} \right) \ast (\bar{B}_{ij} - \bar{I}_1 \delta_{ij}) \right] + \left( \frac{2}{D_1} (J - 1) \right) \ast \delta_{ij}.
$$

(4.7)

Now the UMAT can be developed based on these equations.

### 4.2 Development of UMAT for nonlinear hyperelastic response

There are three basic requirements for a UMAT as explained in chapter 3. First is the defining of material parameters or properties which will be used in the UMAT. Secondly the stress-strain constitutive equations are defined. Finally, a Jacobian matrix is defined. The parameters used in this UMAT are the shear modulus of the material $G$, the extensibility limit $f_{l_{im}}$ and the parameter for compressibility $D_1$ for the Gent model.

#### Constitutive equation for stress and strain

The stress-strain relation is defined by equation (4.7).

#### The Jacobian matrix

The final step is to determine the Jacobian matrix, which is calculated using the relation given by ABAQUS theory manual [79] and Suchocki [83]. The equation for calculating the Jacobian matrix for large deformation is given as

$$
C_{ijkl} = \frac{1}{J} \frac{\delta \tau_{ij}}{\delta D_{kl}},
$$

(4.8)

where $C_{ijkl}$ are the components of the Jacobian matrix. The Kirchhoff stress $\tau_{ij}$ is related to the Cauchy stress $\sigma_{ij}$ as

$$
\tau_{ij} = J \sigma_{ij}.
$$

(4.9)

The virtual rate of deformation $\delta D_{kl}$ is the symmetric part of $\delta L_{kl}$ which is the variation of the gradient of displacement with respect to the current (deformed) configuration, and is given as

$$
\delta L_{ij} = \frac{\partial \delta u}{\partial x}.
$$

(4.10)

Thus, the virtual rate of deformation is given as
\[ \delta \mathbf{D} = \frac{1}{2} (\delta \mathbf{L} + \delta \mathbf{L}^T) = \frac{1}{2} (\delta \mathbf{F} \mathbf{F}^{-1} + (\delta \mathbf{F} \mathbf{F}^{-1})^T). \]  

The virtual rate of deformation can also be written in terms of the virtual volumetric \( \delta \epsilon \) and deviatoric \( \delta e \) strain rates as

\[ \delta \epsilon = \text{trace} (\delta \mathbf{D}), \]

\[ \delta e = \delta \mathbf{D} - \frac{1}{3} \delta \epsilon. \]  

These terms are used to calculate \( \delta \mathbf{\bar{B}}, \delta (\mathbf{\bar{B}} \cdot \mathbf{\bar{B}}), \delta I_1, \delta I_2, \) and \( \delta J \). The expressions for these small variations are given in ABAQUS theory manual [79], and are used to calculate the components of the Jacobian matrix which are given by equation:

\[ C_{ijkl} = \frac{2}{J} \left( \frac{\partial U}{\partial I_i} \right) \]

\[ \times \left[ \frac{1}{2} \left( \delta_{ik} \delta_{jl} + \delta_{jl} \delta_{ik} + \delta_{il} \delta_{jk} + \delta_{jk} \delta_{il} \right) \right] 

\[ + \frac{2}{3} \left( \frac{1}{3} I_i \delta_{ij} \delta_{kl} - \delta_{ij} \bar{B}_{kl} \right) \right] + \frac{2}{J} \left( \frac{\partial^2 U}{\partial I_i^2} \right) \]

\[ \times \left[ \bar{B}_{ij} \delta_{kl} - \frac{1}{3} I_i \left( \bar{B}_{ij} \delta_{kl} + \delta_{ij} \bar{B}_{kl} \right) + \frac{1}{9} I_i^2 \delta_{ij} \delta_{kl} \right] 

\[ + \left( \frac{\partial U}{\partial J} + \frac{\partial^2 U}{\partial J^2} \right) \delta_{ij} \delta_{kl} \]  

The above equations are used to formulate the UMAT given below.

**UMAT code**

The UMAT code is given below, the detailed interpretation for UMAT code is given in the Appendix.

---

**SUBROUTINE** UMAT(STRESS,STATEV,DDSDDE,SSE,SPD,SCD,
RPL,DDSDDT,DRPLDE,DRPLDT,
STRAN,DSTRAN,TIME,DTIME,TEMP,DTEMP,PREDEF,DPRED,CMNAME,
NID,NSHR,NTENS,NSTATEV,PROPS,NPROPS,COORDS,DROT,PNEWDT,
CELENT,DFGRD0,DFGRD1,NOEL,NPT,LAYER,KSPT,KSTEP,KINC)

**INCLUDE 'ABA_PARAM.INC'**

**CHARACTER*80 CMNAME**

**DIMENSION** STRESS(NTENS),STATEV(NSTATEV),
DDSDDE(NTENS,NTENS),DDSDDT(NTENS),DRPLDE(NTENS),
---
PARAMETER (ZERO=0.D0, ONE=1.D0, TWO=2.D0, THREE=3.D0, 
FOUR=4.D0, FIVE=5.D0, SIX=6.D0, SEVEN=7.D0, EIGHT=8.D0)

C DIMENSION BBAR(6), DISTGR(3,3)

C Energy density function: \( W = -G*J_{\text{lim}}/2*\ln((J_{\text{lim}}-I_1+3)/J_{\text{lim}})+(J_1)^2/D_1 \)

C PROPS(1)=G Equilibrium shear modulus
C PROPS(2)=JM Extensibility
C PROPS(3)=D1 Related to the bulk modulus \( D_1=2/K \)

REAL*8 G, JM, D1
REAL TERM1, TERM2, SCALE, DET, TRBBAR, EK, PR

C Material properties
G=PROPS(1)
JM=PROPS(2)
D1=PROPS(3)

C Calculate the Jacobian \( J=\det(F) \)

DET=DFGRD1(1,1)*DFGRD1(2,2)*DFGRD1(3,3)
-DFGRD1(1,2)*DFGRD1(2,1)*DFGRD1(3,3)
IF (NSHR == 3) THEN
DET=DET+DFGRD1(1,2)*DFGRD1(2,3)*DFGRD1(3,1)
+DFGRD1(1,3)*DFGRD1(3,2)*DFGRD1(2,1)
-DFGRD1(1,3)*DFGRD1(3,1)*DFGRD1(2,2)
-DFGRD1(2,3)*DFGRD1(3,2)*DFGRD1(1,1)
END IF

WRITE(6,*), 'THE VALUES OF STRETCH AT EACH INCREMENT'
WRITE(6,*), 'KINC'
WRITE(6,*), 'DFGRD1'
WRITE(6,*), DFGRD1(1,1), DFGRD1(1,2), DFGRD1(1,3)
WRITE(6,*), DFGRD1(2,1), DFGRD1(2,2), DFGRD1(2,3)
WRITE(6,*), DFGRD1(3,1), DFGRD1(3,2), DFGRD1(3,3)

WRITE(6,*), 'THIS IS A TEST'
WRITE(6,*), 'THE VALUE OF KINC, SCALE AND DISTGR'
WRITE(6,*), KINC, SCALE, DISTGR(1,1), DISTGR(1,2)
WRITE(6,*), DISTGR(1,3), DISTGR(2,1), DISTGR(2,2), DISTGR(2,3)
WRITE(6,*), DISTGR(3,1), DISTGR(3,2), DISTGR(3,3)

C

REAL*8 G, JM, D1
REAL TERM1, TERM2, SCALE, DET, TRBBAR, EK, PR

C Material properties
G=PROPS(1)
JM=PROPS(2)
D1=PROPS(3)

C Calculate the Jacobian \( J=\det(F) \)

DET=DFGRD1(1,1)*DFGRD1(2,2)*DFGRD1(3,3)
-DFGRD1(1,2)*DFGRD1(2,1)*DFGRD1(3,3)
IF (NSHR == 3) THEN
DET=DET+DFGRD1(1,2)*DFGRD1(2,3)*DFGRD1(3,1)
+DFGRD1(1,3)*DFGRD1(3,2)*DFGRD1(2,1)
-DFGRD1(1,3)*DFGRD1(3,1)*DFGRD1(2,2)
-DFGRD1(2,3)*DFGRD1(3,2)*DFGRD1(1,1)
END IF

WRITE(6,*), 'THE VALUES OF STRETCH AT EACH INCREMENT'
WRITE(6,*), 'KINC'
WRITE(6,*), 'DFGRD1'
WRITE(6,*), DFGRD1(1,1), DFGRD1(1,2), DFGRD1(1,3)
WRITE(6,*), DFGRD1(2,1), DFGRD1(2,2), DFGRD1(2,3)
WRITE(6,*), DFGRD1(3,1), DFGRD1(3,2), DFGRD1(3,3)

WRITE(6,*), 'THIS IS A TEST'
WRITE(6,*), 'THE VALUE OF KINC, SCALE AND DISTGR'
WRITE(6,*), KINC, SCALE, DISTGR(1,1), DISTGR(1,2)
WRITE(6,*), DISTGR(1,3), DISTGR(2,1), DISTGR(2,2), DISTGR(2,3)
WRITE(6,*), DISTGR(3,1), DISTGR(3,2), DISTGR(3,3)
Calculate the left Cauchy-Green deformation tensor \( \text{BBAR}\ )

\[
\text{BBAR}(1) = \text{DISTGR}(1,1)^2 + \text{DISTGR}(1,2)^2 + \text{DISTGR}(1,3)^2 \\
\text{BBAR}(2) = \text{DISTGR}(2,1)^2 + \text{DISTGR}(2,2)^2 + \text{DISTGR}(2,3)^2 \\
\text{BBAR}(3) = \text{DISTGR}(3,3)^2 + \text{DISTGR}(3,1)^2 + \text{DISTGR}(3,2)^2 \\
\text{BBAR}(4) = \text{DISTGR}(1,3)\text{DISTGR}(2,1) + \text{DISTGR}(1,2)\text{DISTGR}(2,2) \\
\text{BBAR}(5) = \text{DISTGR}(1,1)\text{DISTGR}(3,1) + \text{DISTGR}(1,2)\text{DISTGR}(3,2) \\
\text{BBAR}(6) = \text{DISTGR}(2,1)\text{DISTGR}(3,1) + \text{DISTGR}(2,2)\text{DISTGR}(3,2) \\
\]

IF \( \text{NSHR} == 3 \) THEN

\[
\text{BBAR}(5) = \text{DISTGR}(1,1)\text{DISTGR}(3,1) + \text{DISTGR}(1,2)\text{DISTGR}(3,2) \\
\text{BBAR}(6) = \text{DISTGR}(2,1)\text{DISTGR}(3,1) + \text{DISTGR}(2,2)\text{DISTGR}(3,2) \\
\]

ENDIF

WRITE(6,*) 'THIS IS A TES3'
WRITE(6,*) 'THE VALUE OF KINC, \text{BBAR}(K1)'
WRITE(6,*) KINC, BBAR(1), BBAR(2), BBAR(3), BBAR(4)
WRITE(6,*) BBAR(5), BBAR(6)

Calculate the stress

!!! Calculate \( \text{I1}_{-}\text{Bar} \)

\[
\text{TRBBAR} = (\text{BBAR}(1) + \text{BBAR}(2) + \text{BBAR}(3))/3 \\
\]

!!! Calculate the terms to be used in the constitutive model

\[
\text{TERM1} = \frac{G}{D\text{ET}} \times (\text{ONE} - (3\text{TRBBAR} - \text{THREE})/\text{JM}) ** (-\text{ONE}) \\
\text{TERM2} = \frac{2\text{O}G}{D\text{ET}\text{J}\text{M}} \times (\text{ONE} - (3\text{TRBBAR} - \text{THREE})/\text{JM}) ** (-\text{TWO}) \\
\text{EK} = \frac{\text{TWO}}{\text{D}\text{I}} \times (\text{TWO} - \text{DET} - \text{ONE}) \\
\text{PR} = \frac{\text{TWO}}{\text{D}\text{I}} \times (\text{DET} - \text{ONE}) \\
\]

WRITE(6,*) 'THIS IS A TES4'
WRITE(6,*) 'THE VALUE OF KINC, \text{TRBBAR}, \text{TERM1}, \text{TERM2}, \text{EK}, \text{PR}'
WRITE(6,*) KINC, TRBBAR, TERM1, TERM2, EK, PR

DO \( K1 = 1, \text{NDI} \)

\[
\text{STRESS}(K1) = \text{TERM1} \times (\text{BBAR}(K1) - \text{TRBBAR}) + \text{PR} \\
\]

END DO

DO \( K1 = \text{NDI} + 1, \text{NDI} + \text{NSHR} \)

\[
\text{STRESS}(K1) = \text{TERM1} \times \text{BBAR}(K1) \\
\]

END DO

WRITE(6,*) 'THIS IS A TESS'
WRITE(6,*) 'THE VALUE OF KINC, STRESSES'
WRITE(6,*) KINC
WRITE(6,*) STRESS(1), STRESS(2), STRESS(3)
WRITE(6,*) STRESS(4), STRESS(5), STRESS(6)

Calculate the stiffness matrix (Jacobian matrix)

\[
\text{DDSDDE}(1,1) = \frac{2\text{O}}{\text{THREE}} \times (\text{TERM1} \times (\text{BBAR}(1) + \text{TRBBAR})) + \text{E}\text{K} \\
\text{DDSDDE}(2,2) = \frac{2\text{O}}{\text{THREE}} \times (\text{TERM1} \times (\text{BBAR}(2) + \text{TRBBAR})) + \text{E}\text{K} \\
\text{DDSDDE}(3,3) = \frac{2\text{O}}{\text{THREE}} \times (\text{TERM1} \times (\text{BBAR}(3) + \text{TRBBAR})) + \text{E}\text{K} \\
\text{DDSDDE}(2,1) = \frac{2\text{O}}{\text{THREE}} \times (\text{TERM1} \times (\text{BBAR}(1) + \text{TRBBAR}) \times \text{BBAR}(2) + \text{TRBBAR}) + \text{E}\text{K} \\
\text{DDSDDE}(3,1) = \frac{2\text{O}}{\text{THREE}} \times (\text{TERM1} \times (\text{BBAR}(1) + \text{TRBBAR}) \times \text{BBAR}(3) + \text{TRBBAR}) + \text{E}\text{K} \\
\text{DDSDDE}(3,2) = \frac{2\text{O}}{\text{THREE}} \times (\text{TERM1} \times (\text{BBAR}(2) + \text{TRBBAR}) \times \text{BBAR}(3) + \text{TRBBAR}) + \text{E}\text{K} \\
\]

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4.3 Testing with ABAQUS

The developed UMAT, will now be tested using ABAQUS CAE finite element simulation software as done in chapter 3. The first test will be the comparison of the analytical results with FEA results. The material parameters used here are taken from the work of Zhou et al. [3] (see Table 4-1).
Table 4-1 Material parameters for testing

<table>
<thead>
<tr>
<th>Parameter</th>
<th>VHB 4910</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G^{EQ}$</td>
<td>14.59 KPa</td>
</tr>
<tr>
<td>$J_{lim}$</td>
<td>115.8</td>
</tr>
<tr>
<td>$D1$</td>
<td>$1.371 \times 10^{-7} Pa^{-1}$</td>
</tr>
</tbody>
</table>

In this test a unit cubic element is examined under uniform tensile and pure shear test. To focus on the nonlinear response of elastomers, the test specimen is stretched to an exceptionally large stretch ratio. The experimental data for VHB 4910 is taken from Wang et al. [81]. In the experiment, the material is stretched under a uniform tension to more than 8 times its original dimensions. Finally, a convergence test is carried out similarly to the one performed in chapter 3 to ensure accuracy of the FE simulation.

4.3.1.1 Simple tensile test

For the simple tensile test, the deformation is described as

$$x_1 = \lambda_{11} X_1, \quad \lambda_{22} = \lambda_{33} = \lambda_{11}^{-\frac{1}{2}}.$$  (4.14)

All shear stretches are zero. Giving the relations:

$$F = \tilde{F}, \quad \tilde{F} = \tilde{F}^T,$$

$$\tilde{B}(1) = \lambda_{11}^2, \quad \tilde{B}(2) = \frac{1}{\lambda_{11}}, \quad \tilde{B}(3) = \frac{1}{\lambda_{11}^2},$$

$$\tilde{I}_1 = \lambda_{11}^2 + \frac{2}{\lambda_{11}}.$$

For VHB 4910 (the material is assumed as incompressible), The stress can be calculated using equation $\sigma_{ij} = F_{ik} \frac{\partial w}{\partial F_{jk}} + p \delta_{ij}$, or the equation below,

$$\sigma_{ij} = 2 \left[ \left( \frac{\partial \tilde{U}}{\partial \tilde{l}_1} + \tilde{I}_1 \frac{\partial \tilde{U}}{\partial \tilde{l}_2} \right) \tilde{B}_{ij} - \left( \tilde{I}_1 \frac{\partial \tilde{U}}{\partial \tilde{l}_1} + 2 \tilde{I}_2 \frac{\partial \tilde{U}}{\partial \tilde{l}_2} \right) \frac{\delta_{ij}}{3} - \frac{\partial \tilde{U}}{\partial \tilde{l}_2} \tilde{B}_{ik} \tilde{B}_{kj} \right] + p \delta_{ij}.$$  (4.16)

Here $p$ is the hydrostatic stress and for this case its value is $\sigma_{11}/3$. Using either equation the stress is given as
\[ \sigma_{11} = \frac{G^{EQ}J_{lim}\left(\lambda_{11}^2 - \frac{1}{\lambda_{11}}\right)}{J_{lim} - (\lambda_{11}^2 + \frac{2}{\lambda_{11}}) + 3}. \] (4.17)

**Figure 4-1** Uniaxial deformation of a single element.

For the FE simulation in ABAQUS, the value of the bulk modulus is chosen to be high enough to make the material nearly incompressible. The unit specimen is meshed with a single C3D8 element. The specimen is restrained in \( y, z \) direction and one of the sides in \( x \) direction, and a stretch is applied in the \( x \) direction as shown in figure below.

In figure 4-2, the comparison between the results obtained from analytical calculations as per equations given above and the simulation performed in ABAQUS based on the developed UMAT is demonstrated. The excellent agreement of the results indicates the successful implementation of the UMAT into the ABAQUS.
4.3.1.2 Simple Shear

For simple shear, the deformation is described as

\[ x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3. \]  

(4.18)

Giving the relations:

\[
\begin{align*}
F &= \bar{F}, \\
\bar{F} &= \bar{F}^T, \\
\bar{B}(1) &= 1 + \gamma^2, \\
\bar{B}(2) &= 1, \\
\bar{B}(3) &= 1, \\
\bar{B}(4) &= \gamma.
\end{align*}
\]  

(4.19)

The stress can be calculated using any of the two equations discussed in section 4.2.1.1, which leads to

\[ \sigma_{12} = 2 \left( \frac{\partial U}{\partial I_1} \right) \cdot \bar{B}(4) = \frac{G J_{lim} \cdot F_{12}}{(J_{lim} - F_{12}^2)}. \]  

(4.20)

For the FE simulation in ABAQUS, the material’s incompressibility is ensured by the penalty method [83] as done in previous section. The unit specimen is meshed with a single C3D8 element. The element is restrained in \( y \) – direction and loaded in the \( x \) – direction, as shown in figure below:
Figure 4-3 Simple shear for a single element.

In the figure below, we show the comparison between the results obtained from analytical calculations as per equations given above and simulation performed in ABAQUS based on UMAT. Excellent agreement is reached again for this case.

Figure 4-4 Comparison of analytical results with FE results from ABAQUS for a shear test.
4.3.1.3 Experimental verification

In this section a tensile test on material VHB 4910 is considered. For a simple tensile test of a compressible material the following conditions are applied,

\[ F_{22} = F_{33}, \quad \sigma_2 = \sigma_3 = 0. \]  

(4.21)

Since the deformation is produced in the \( x \)-direction, \( F_{11} \) is known. The variation of the stretching stress with the stretch ratio in the \( x \)-direction is plotted in figure 4-5 from both the FE simulation and the experimental data of Wang et al. [81]. Excellent agreement is observed, which demonstrates the accuracy of the developed UMAT and its implementation into ABAQUS.

![Comparison between experimental results for VHB4910 and ABAQUS simulation](image)

**Figure 4-5** Comparison between experimental results for VHB4910 and ABAQUS simulation

4.3.1.4 Convergence Test:

A convergence test has been carried out to ensure the accuracy of the simulation results. In this test the convergence plots for stresses are obtained based on the number of elements used. The
element used in this test is C3D8RH (incompressible hyperelastic materials can only be used with hybrid, plane stress, or 1d elements). The table below gives the relation between number of elements and maximum values of stress in the y-direction and the Von Mises stress.

<table>
<thead>
<tr>
<th>Global approximate size</th>
<th>Number of Elements</th>
<th>Maximum Stress in y direction $\text{KPa}$</th>
<th>Maximum Von Mises Stress $\text{KPa}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>n</td>
<td>16.26</td>
<td>14.61</td>
</tr>
<tr>
<td>7</td>
<td>24</td>
<td>14.77</td>
<td>14.01</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
<td>14.21</td>
<td>13.52</td>
</tr>
<tr>
<td>5</td>
<td>105</td>
<td>15.97</td>
<td>15.05</td>
</tr>
<tr>
<td>4</td>
<td>192</td>
<td>16.19</td>
<td>15.53</td>
</tr>
<tr>
<td>3</td>
<td>420</td>
<td>17.27</td>
<td>16.5</td>
</tr>
<tr>
<td>2</td>
<td>1,287</td>
<td>17.74</td>
<td>16.5</td>
</tr>
<tr>
<td>1</td>
<td>10,353</td>
<td>17.74</td>
<td>16.5</td>
</tr>
</tbody>
</table>

Figures 4-6 and 4-7 show the convergence plots for principal stress in $y$ -direction and von Mises stress against the number of elements.

**Figure 4-6** Variation of principal stress in $y$-direction with number of elements.
4.4 Summary

In this chapter, a UMAT has been developed for capturing the nonlinear response of elastomeric materials. The user subroutine is based on the Gent model for hyperelastic materials. Several different tests are performed on the developed UMAT implemented into ABAQUS/CAE. Firstly, a single cubic element is tested under uniform tensile and pure shear loading conditions. The results obtained from the ABAQUS simulations are then verified by analytical calculations. The second test is a comparison between the simulation results and the experimental data of elastomer VHB 4910. The comparison shows that the UMAT accurately predicts the nonlinear response of VHB 4910. Finally, a finite element convergence test is conducted, which is to determine the optimum mesh size. The convergence plots show that the solutions converge after mesh size is refined to a certain level. All the above-mentioned tests show the accuracy, versatility and robustness of the developed user subroutine.
CHAPTER 5

5 Conclusion and future working

5.1 Conclusion

Elastomeric materials have gained immense popularity in the field of electronics, robotics, energy transduction, and bioengineering. Their capability of undergoing large deformation, efficient electromechanical coupling and abundance has made them an interesting alternative to conventional technologies in transduction. In order to make full potential applications of elastomeric materials with efficient and reliable design, it is crucial to have a better understanding of the behaviors of these materials. Understanding the behaviors of elastomers is challenging due to their nonlinear material nature and time/rate-dependent response. The objective of the current study is to develop UMAT subroutines for capturing the response of elastomeric materials. To that end, two subroutines have been developed, one for studying the linear viscoelastic material response and the other for nonlinear hyperelastic behavior. The contribution of this work includes:

1. Development of a UMAT for predicting the linear viscoelastic response of elastomers. The UMAT is based on a generalized Maxwell rheological model. The stress-strain relation is derived based on a relaxation test. A generalized formulation is provided for finite element implementation of the constitutive relation, which can be applied to any number of Maxwell elements.

2. Development of UMAT for predicting nonlinear hyperelastic response of elastomers. This user subroutine is based on a constitutive equation obtained from the Gent energy density function [85].

3. The developed UMAT subroutines were tested on ABAQUS/CAE SIMULA finite element analysis software. Different tests are carried out to ascertain the compliance, accuracy, versatility and robustness of the developed UMAT subroutines.

Based on the work and simulation results, some concluding remarks of this work are listed below:

1. The user subroutine for predicting the linear response of elastomers can capture the time and rate-dependent response of material. However, it showed some offset from experimental results since the material is nonlinear in nature.
2. The UMAT for predicting the nonlinear behavior of elastomers quite accurately reproduces the experimental response of the material.

5.2 Future work

Based on this work, some suggestions for my future work are given below:

1. UMAT subroutine to capture nonlinear viscoelastic response of elastomers should be developed. This can be modeled in a similar way as done for the hyperelastic response.

2. Refining of user subroutines to accurately predict response of viscoelastomers under different loading conditions.

3. The UMATs should be further developed to account for the electrostatic stress for dielectric elastomers.
APPENDIX

Appendix. Basic Definitions for terms used in UMAT

In this section basic terms are defined which will help in understanding the UMATs:

**PROPS and NPROPS**

PROPS are the user specified array of material constants associated with the user material. These are defined in the UMAT, but the values of these material constants can be updated in ABAQUS CAE interface. NPROPS defines the number of material constants which will be used in the UMAT, this set will indicate how many material inputs are required in ABAQUS CAE interface.

**TIME and DTIME**

TIME defines the time input, and this is the time duration in which the deformation or stress is applied. DTIME is the increment in time, which is also user defined but during analysis ABAQUS CAE will decide which DTIME to choose depending on convergence. But a minimum defined increment is input in ABAQUS CAE below which the analysis is terminated.

**STRAN**

STRAN defines an array containing the total strains at the beginning of the increment. This array is defined by ABAQUS CAE. Inputs for defining STRAN come from the type of boundary conditions specified and time of step defined in ABAQUS CAE interface.

**DSTRAIN**

DSTRAIN is the array of strain increments. This is calculated by ABAQUS CAE based on the time increment and deformation or stress defined in ABAQUS CAE.

**DFGRD0 and DFGRD1**

DFGRD0 defines an array containing the deformation gradients at the beginning of the increment. DFGRD1 defines an array containing the deformation gradient at the end of the increment. The components of this array are set to the identity matrix if non-linear geometric effects are not included in the step definition associated with this increment. Both these arrays
are automatically calculated by ABAQUS CAE and are dependent on input of time, increment and boundary conditions.

**STATEV and NSTATEV**

STATEV defines array containing the solution-dependent state variables. The state variables are passed as the values at the beginning of the increment unless they are updated in user subroutines, in which case the updated values are used. The solution dependent state variables are basically inquiry points in the code and any property which the user wishes to see or trace during the duration of the simulation can be defined in the code as state variables. These help in making comparison between hand calculations and simulation results if used properly. NSTATEV defines the number of solution dependent state variables in the UMAT.

**STRESS**

This array is passed in as the Cauchy stress tensor at the beginning of the increment and must be updated in this routine to be the stress tensor at the end of the increment. The stress is defined in Voigt notation discussed in section 3.2. It is calculated based on the constitutive model defined in the UMAT. This is the output that ABAQUS CAE will give after the analysis is complete.

**DDSDDE**

Defines the Jacobian Matrix of the constitutive model. DDSDDE(i, j) defines the change in the \( i^{th} \) stress component at the time of increment caused by an infinitely small perturbation of the \( j^{th} \) component of the strain increment array. The Jacobian matrix is of key importance for the convergence process and is only used for UMAT subroutines. The ABAQUS CAE calculates the component of Cauchy stress from constitutive model in UMAT and material Jacobian for each Gauss integration point. These are then used by ABAQUS to form an element stiffness matrix; the components of stress are compared with the values calculated with the tangent modulus for the iteration and if the difference is larger than the defined tolerance, the time interval will be reduced. If the solution for both stress calculation and Jacobian is within tolerance the solution is said to be convergent.
CMNAME

This defines the name of the user defined material. It should not start with “ABQ_” as most predefined materials in ABAQUS/ STANDARD library start with “ABQ_” as the leading string for CMNAME.

NDI, NSHR and NTENS

NDI are the number of principal stresses also called direct stresses at a point; NSHR defines the number of engineering shear stress components at the point and NTENS defines the size of the stress or strain component array. NTENS is the sum of NDI and NSHR.

NPT, KSTEP and KINC

NPT defines the integration point number, KSTEP is the step number and KINC is the increment number.
References:


Asim Gillani  
Mechanical Engineer, MESc.  
SUMMARY OF QUALIFICATIONS

▪ MESc. in Mechanical engineering from Western University, London ON. (2016-2018)
▪ Experienced Mechanical Engineer with 7 years of experience specialized in Project design and execution, maintenance and inspection. (2008-2015)
▪ Trained on DOW Life Critical standards such as Vessel entry, Hot and Cold PTW, and fall hazards.
▪ Canadian permanent resident, eligible for applying for P. Eng. status.

Job Related Skills

▪ Development and review of engineering drawing using AutoCAD and SolidWorks.
▪ Analyzing pressure vessel and piping thickness, and remaining life of piping networks, pressure vessels and storage tanks.
▪ Proficient in use of AutoCAD, SolidWorks, ABAQUS CAE and CATEA design and analysis software.
▪ Knowledge of codes and standards ASME, ASTM, API, ISO, OHSA.
▪ Working knowledge of P&IDs, PFD, Data sheets, MSDS, GA Drawings, BOQ, MTO, Isometrics, Panel &JB drawings, Line Lists, Plot and underground piping drawings, Civil foundations.
▪ Experienced in Pumps, compressors, turbines, fans, Tower vessel, steam drums etc.
▪ Proficient in use of MS word, excel, Power point presentations, ERP Oracle and MS project.

PROFESSIONAL EXPERIENCE

Mechanical Engineer, January 2013 - October 2015
Fatima Fertilizer Company Ltd. Pakistan.
▪ Analyzed the remaining life of process equipment, developed and implemented a phasing out plan based on the results.
▪ Developed improved mechanical integrity program aimed at effective monitoring of process equipment for reliability improvement.
▪ Lead FFCL site for MARSH inspection audit, developed pertinent documents, training, internal auditing, gap analysis and successful final audit.
▪ As area engineer, prepared and implemented inspection plans, carried out RCA for failures, developed thickness monitoring regimes, installation of corrosion coupon
Asim Gillani
Mechanical Engineer, MESc.

- assemblies at tanks, introduced advanced inspection techniques, defined inspection regimes for turnarounds. Vibration analysis of rotodynamic equipment.

**Project Maintenance Engineer** August 2008 - December 2012
Fatima Fertilizer Company Ltd. Pakistan.

- Acted as Field Engineer during plant construction to check correct installation for $1 Billion project.
- Design and review of engineering documents, mechanical drawings, erection and construction documents, piping isometrics and P&ID’s using AutoCAD and SolidWorks.
- Facilitated PSSR, commissioning and startup activities while working with contractors.
- Coordinated with OEM and vendors for equipment and machinery maintenance.
- Created maintenance job reports, updated maintenance history and inspection reports.
- Developed BOQ, BOM, project cost estimates and scope for contract work.
- Prepared project SOW (scope) and award recommendations.
- Experienced in computerized maintenance management systems like CMMS and Oracle ERP.

**Achievements and major projects at Fatima Fertilizers:**

- Implemented reengineering of reforming furnace at Fatima Fertilizers, which optimized ammonia production from 1000 MTPD (Metric ton/day) to 1500 MTPD
- Calculated remaining life of equipment after implementing an inspection regime and based on that calculation developed a phasing out plan for piping networks, stationary and rotodynamic equipment saving 10 million $ and is still in place.

**EDUCATION**

**MESc. Mechanical Engineering** September 2016 - October 2018
Western University, London, ON, Canada.

- Research major in developing and implementing customized material model codes called UMATS for FEA software ABAQUS/ CAE, for linear and nonlinear response of viscoelastic elastomeric materials

**BSc. Mechanical Engineering** August 2004 - August 2008
University of Engineering and Technology, Lahore, Pakistan.

- Major in mechanical engineering design.