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Valuation of Multiple Exercise Option Using a Modified Longstaff and Schwartz Approach

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Abstract

In this work we study the problem of pricing multiple exercise options, a class of early exercise options that are traded in the energy market, using a modified Longstaff and Schwartz approach. Recent work by Letourneau and Stentoft (2014) shows American option price estimator bias is reduced by imposing additional structure on the regressions used in Monte Carlo pricing algorithms. We extend their methodology to the Monte Carlo valuation of multiple exercise options by requiring additional structure on the regressions used to estimate continuation values. The resulting price estimators have reduced bias, particularly for small sample sizes, and results hold across a variety of option types, maturities and moneyness. A comparison of the original Longstaff and Schwartz approach to the modified Longstaff and Schwartz approach demonstrates the strengths of the developed numerical technique.

Keywords

Modified Longstaff and Schwartz, Monte Carlo, Regression, Multiple Exercise Option

To my wife and my parents

for their endless love, encouragement and support.

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List of Abbreviations Nomenclature and Symbols

Abbreviations

ALP	Approximate Linear Programming
BM	Brownian Motion
BSM	Black Scholes Model
CLT	Central Limit Theorem
GBM	Geometric Brownian motion
ICLS	Inequality Constrained Least Squares
iid	Independent and Identically Distributed
LSMC	Lease Squares Monte Carlo
LSMV	Least Squares Monte Carlo Value
MC	Monte Carlo
OEP	Optimal Exercise Price
OLS	Ordinary Least Squares
RMSE	Root Mean Square Error
PDE	Partial Differential Equation
PRNG	Pseudorandom number generator

Nomenclature

A	Matrix

B	Option value
D	Differentiation matrix
E	High biased estimator
f	Estimation function
h	Exercise payoff
H	Holding value
i	Time indicator
K	Strike price
L	Low biased estimator
M	Number of paths
N	Number of rights remaining
P	Probability
Q	Vector
S	Price of underlying asset
t	Time
Δt	Time step
T	Maturity time

w	Weiner process
y	Dependent variable
X_i	Position at t_i
Z	Standard normal distribution
μ	Rate of return
σ	Variance
ε	White noise
$\emptyset$	Density function
Φ	Cumulative density function
β	Estimator
Γ	Univariate grid

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Chapter 1

1 Introduction

In this dissertation, we apply least squares Monte Carlo (LSMC) method with inequality constraints for regression to price multiple exercise options. If one wants to solve this model numerically, one has to solve it with forest of trees. This is the motivation of this dissertation.

1.1 Background and Significance

An option is a financial derivative that represents a contract sold by an option writer to an option holder. This derivative offers the buyer the right, but not the commitment, to buy or sell a security (or other financial asset) at an agreed-upon price, named the strike price, during a specific interval of time or on a certain date, called the exercise date. A call option gives an investor the right, but not the obligation, to buy a security at a specified price within a specific time period. Call options give the right to buy at a certain price, so the buyer desires the stock to go up. A put option gives an investor the right, but not the obligation, to sell a security or other instrument at a specified price within a specific time period. Unlike a call option owner, put option buyers desire the stock to go down.

In this dissertation, the Monte Carlo valuation of multiple exercise options in discrete time is studied. Multiple exercise options are considered as a combination of put and call rights. A modified Longstaff and Schwartz approach is used which uses constrained regression to obtain the hold value at each time. For put options, it is obvious that the hold value function is convex with respect to the underlying asset and the slope is between -1 and 0. The call option hold value function is convex with respect to the underlying asset as well but the slope is bounded between 0 and +1. It is shown that applying these constraints gives better high and low biased estimators compared to the Longstaff and Schwartz approach.

In the rest of section 1 we review the Monte Carlo method, Monte Carlo for European options, American-style options, detailed description of least squares Monte Carlo, bias of Monte Carlo estimators and detailed description of inequality constrained least squares (ICLS) method. In chapter 2, multiple exercise options, pricing algorithms and a description of ICLS method for multiple exercise options are reviewed. Chapter 3 presents the numerical results.

1.2 Monte Carlo Method

An alternative to the numerical PDE methods is Monte Carlo, which is straightforward and easy to apply, and has application on diverse divisions of mathematics. The recent rise in the complexity of derivative securities pricing has directed a requirement to evaluate high-dimensional integrals. Growth of the problem dimension pushed Monte Carlo methods to be more desirable compared to other numerical integration methods such as quadrature. The Monte Carlo method for pricing derivatives uses the probability distribution of the underlying security and the law of large numbers. In this approach, the first step is simulating sample paths for the underlying state variables for a time period using the risk-neutral measure. The option payoff is calculated for each sample path. The simulated option payoffs are discounted and then averaged across sample paths yielding a price estimate.

Monte Carlo simulation was launched into quantitative finance by Boyle[1]. Monte Carlo simulation is the most important method for pricing complicated financial derivatives particularly for payoffs that are path-dependent or are influenced by multiple factors.

Let’s work with a single underlying asset to describe Monte Carlo in more detail. To generate sample paths assume that the underlying asset (stock) follows a risk-neutral process as

$$dS = rS\,dt + \sigma S dW, \tag{1.1}$$

where S is the asset price, r is the expected return under the risk neutral measure, σ is the volatility, t is time and W is a Weiner process. Dividing the time interval into N equally spaced subintervals of length Δt allows us to simulate a discrete time version of the sample path of S using the equation

$$S(t + \Delta t) - S(t) = rS\,\Delta t + \sigma S \varepsilon_t \sqrt{\Delta t} \tag{1.2}$$

where ε_t are iid N(0,1) random variables and $S(t)$ stands for the value of underlying asset at time t. Equation (1.2) is an Euler discretization of the SDE in (1.1). It is more accurate to use the solution to (1.1) to simulate sample paths of S. Specifically $\ln S$ follows

$$d \ln S = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW. \tag{1.3}$$

So that

$$\ln S(t + \Delta t) - \ln S(t) = \left(r - \frac{\sigma^2}{2}\right) \Delta t + \sigma \varepsilon_t \sqrt{\Delta t} \tag{1.4}$$

or

$$S(t + \Delta t) = S(t) \exp\left[\left(r - \frac{\sigma^2}{2}\right) \Delta t + \sigma \varepsilon_t \sqrt{\Delta t}\right], \tag{1.5}$$

where ε_t’s are iid N(0,1) random variables.

The best advantage of Monte Carlo simulation is that this method can be used when the payoff is path dependent or depends on multiple factors.

1.2.1 Simulating Random Numbers

Monte Carlo methods utilize chains of random numbers to work out problems. Using random numbers has some benefits. It makes possible to simulate trajectories of a stochastic process with a variable that changes randomly in time such as given in equation (1.1). Additionally complex processes can be simulated providing insights for which there are no analytical solutions. Furthermore, Monte Carlo is a tool for evaluating multi-dimensional integrals.

Monte Carlo methods need a trustworthy technique to generate random numbers. Because it is not easy to make absolutely random numbers, generators mainly generate pseudo-random numbers which emulate the behavior of real random numbers and are made in a predictable and deterministic way. As Monte Carlo simulations call for many random numbers, the pseudo-random numbers are better to be generated quickly and satisfy some statistical tests of randomness.

Additionally, psedo-random number generators (PRNG) all eventually repeat themselves, with the period being the length of the unrepeated sequence. The output of a good PRNG is a sequence, $u_1, u_2, \ldots$, that

- has a long period
- is generated efficiently
- satisfies uniformity properties. That is, want $u_1, u_2, \ldots$, to be iid uniform[0,1] random variables. A battery of statistical tests is used to check the uniformity properties. See [2] for example, for further information.

Sometimes it is necessary to use the same sequence of random numbers in the simulation process such as when estimating differences in the function at different parameter settings. As such, it is necessary to control the seed used in the PRNG to be able to reproduce the same sequence of random numbers. Applying parallel processing computing technique is a fantastic benefit of the Monte Carlo method. To use this technique it is important to be able to skip ahead to another part of the sequence. This allows independent sequences of random numbers to be used by the different parallel processes. Finally a random number generating algorithm should be able to run on all computing platforms.

The output of random number generators are uniform [0,1] random numbers but most Monte Carlo simulations require sampling from non-uniform distributions. Methods for generating observations from non-uniform distribution include inverse transform and acceptance-rejection. See [2] for more details.

For many financial applications, the simulation of standard normal random variables is required. If $Z \sim N(0,1)$, then this can be transferred to $X \sim N(\mu, \sigma^2)$ using $X = \mu + \sigma Z$, thus normal random variables with arbitrary mean and variance can be simulated from $N(0,1)$ random variables and then transforming.

The Mersenne-Twister random number generator has been used in this thesis. This is a twisted generalized feedback shift register generator with a very long period of $2^{19937} -$ 11. This PRNG is k-distributed to 32-bit accuracy for every $1 \leq k \leq 623$ and passes numerous tests for statistical randomness.

1.2.2 Monte Carlo Estimate

Suppose we are interested in computing

$$f = \mathrm{E}[\mathrm{g(X)}] = \int \mathrm{g(x)h(x)dx} \tag{1.6}$$

where g is some function and X is a random variable having probability density h. The Monte Carlo estimator of f is

$$\widehat{f_M} = \frac{1}{\mathrm{M}} \sum_{\mathrm{i=1}}^{\mathrm{M}} \mathrm{g}(\mathrm{x}_\mathrm{i}^*) \tag{1.7}$$

where $\mathrm{x}_1^*, \mathrm{x}_2^*, \ldots, \mathrm{x}_\mathrm{M}^*$ are iid simulated values from the probability density h.

It is easily seen that $\mathrm{E}\left[\widehat{f_M}\right] = \mathrm{f}$ and that $\mathrm{var}\left[\widehat{f_M}\right] = \frac{\mathrm{var(g(x))}}{\mathrm{M}} = \frac{\sigma^2}{\mathrm{M}}$.

The estimated standard error of $\widehat{f_M}$ is then

$$\frac{\hat{\sigma}}{\mathrm{M}}, \tag{1.8}$$

where $\hat{\sigma}$ is the standard deviation of the simulated values $\mathrm{g}(\mathrm{x}_1^*), \ldots, \mathrm{g}(\mathrm{x}_\mathrm{M}^*)$. A 95% confidence interval for f is easily constructed as

$$\widehat{f_M} - \frac{1.96\hat{\sigma}}{\sqrt{\mathrm{M}}} < f < \widehat{f_M} + \frac{1.96\hat{\sigma}}{\sqrt{\mathrm{M}}}. \tag{1.9}$$

This shows that the uncertainty of simulation is inversely related with the square root of the number of paths. So to improve the accuracy by a factor of 5, the number of trials should increase by a factor of 25.

In this setting, the Monte Carlo Estimator is unbiased, that is

$$\mathrm{bias}(\widehat{f_M}, f) \equiv E[\widehat{f_M}] - f = 0.$$

Additionally, $\widehat{f_M}$ is a consistent estimator of f, that is, for any $\varepsilon > 0$, $\lim_{M\to\infty} P(|\widehat{f_M} - f| > \varepsilon) = 0$. In what follows, MC estimators for early-exercise options generally yield price estimators that are biased yet consistent for the true price.

1.2.3 Simulation of sample paths

Another important concern in Monte Carlo simulation is generating appropriate sample paths. Usually in quantitative finance applications, geometric Brownian motion (GBM) is used as the stochastic process. Geometric Brownian motion is the product of exponentiating Brownian motion (BM) and as a result the methods for simulating BM are methods for simulating GBM as well. Let's take a random process continuous in time, a function W(t) which for each time $t \geq 0$ is a random variable. The standard Brownian motion process is a stochastic process W(t), for $t \geq 0$, with the following properties:

1) Each increment W(t)−W(s) over any time period of length t-s is normally distributed with mean 0 and variance t-s,

$$W(t) - W(s) \sim N(0, t - s) \tag{1.10}$$

2) The increments $W(t_m) - W(t_{m-1}), \ldots, W(t_1) - W(t_0)$, are independent for all $0 \leq t_0 \leq \cdots \leq t_m \leq T$.
3) W(0)=0
4) W(t) is continuous for all t

Discretizing time, sample paths of W can be generated by taking $Z_1, \ldots, Z_m$ iid N(0,1) random variables and starting from W(0)=0,

$$W(t_{i+1}) = W(t_i) + \sqrt{\Delta t}\, Z_{i+1} \quad {}_{,i=0,\ldots,m-1}. \tag{1.11}$$

where $\Delta t = \frac{\mathrm{T}}{\text{Number of exercise opportunities}}$. Now a process with drift and different variance can be made using constants μ and σ and setting $\mathrm{X(t)} = \mu\mathrm{t} + \sigma\mathrm{W(t)}$. The parameter μ is the drift parameter and σ is the volatility. So the dynamics of this process are

$$\mathrm{dX(t)} = \mu\mathrm{dt} + \sigma\mathrm{dW(t)} \quad (1.12)$$

and path values may be generated using the starting value X(0) and the Euler discretization

$$\mathrm{X}(\mathrm{t}_{\mathrm{i}+1}) = \mathrm{X}(\mathrm{t}_{\mathrm{i}}) + \mu(\Delta t) + \sqrt{\Delta t}\mathrm{Z}_{\mathrm{i}+1} \quad (1.13)$$

for i= 1, ..., m. In general the drift and diffusion can be functions of time and the current value of the process. That is X, follows the SDE starts at X(0) and

$$\mathrm{dX(t)} = \mu\big(\mathrm{t}, \mathrm{X(t)}\big)\mathrm{dt} + \sigma\big(\mathrm{t}, \mathrm{X(t)}\big)\mathrm{dW(t)}. \quad (1.14)$$

With the starting value X(0), sample paths of X can be simulated using

$$\mathrm{X}(\mathrm{t}_{\mathrm{i}+1}) = \mathrm{X}(\mathrm{t}_{\mathrm{i}}) + \mu\big(\mathrm{t}_{\mathrm{i}}, \mathrm{X}(\mathrm{t}_{\mathrm{i}})\big)(\Delta t) + \sigma\big(\mathrm{t}_{\mathrm{i}}, \mathrm{X}(\mathrm{t}_{\mathrm{i}})\big)\sqrt{\Delta t}\mathrm{Z}_{\mathrm{i}+1}. \quad (1.15)$$

With $\mu(\mathrm{t}, \mathrm{x}) = \tilde{\mu}\mathrm{x}$ and $\sigma(\mathrm{t}, \mathrm{x}) = \tilde{\sigma}\mathrm{x}$, where $\tilde{\mu}$ and $\tilde{\sigma}$ are constants, X is GBM, with dynamics

$$\mathrm{dX(t)} = \tilde{\mu}\mathrm{X(t)dt} + \tilde{\sigma}\mathrm{X(t)dW(t)} \quad (1.16)$$

or

$$\frac{\mathrm{dX(t)}}{\mathrm{X(t)}} = \tilde{\mu}\mathrm{dt} + \tilde{\sigma}\mathrm{dW(t)} \quad (1.17)$$

The solution of the above stochastic differential equation with initial value of X(0) could be found by using Ito's lemma as

$$\mathrm{X(t)} = \mathrm{X(0)}\exp\left(\left(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2\right)\mathrm{t} + \tilde{\sigma}\mathrm{W(t)}\right) \quad (1.18)$$

and the path can be simulated by using the recursive relation

$$X(t_{i+1}) = X(t_i)\exp\left(\left(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2\right)(\Delta t) + \tilde{\sigma}\sqrt{\Delta t}Z_{i+1}\right) \tag{1.19}$$

where $Z_1, \dots, Z_m$ are iid N(0,1) random variables.

1.3 Monte Carlo for European Options

A European option is an option that can be exercised only at the maturity time. Pricing of European options is less complicated compared to pricing American options because the option holder has no opportunity to exercise before maturity. A European call option is a contract between two parties that gives the buyer the right to purchase a stock at the future maturity time (T) at a determined strike price (K) agreed in the contract. If the buyer decides to exercise the option at maturity time, the seller has to sell the stock at a price K to the buyer. The holder's payoff function is

$$f(S_T) = \max(S_T - K, 0) \tag{1.20}$$

Equation (1.20) presents the value of the call option at time T because if S(T) >K, the holder makes a profit of S(T)-K. On the other hand if S(T)<K, the holder does not exercise the option hence it expires worthless.

Using Risk-neutral valuation, the price of the option is given by

$$C = e^{-rT}E[f(S_T)] \tag{1.21}$$

where r is the continuously compounded risk-free rate. The price can be estimated by Monte Carlo using

$$\hat{C} = e^{-rT}\left(\frac{1}{M}\sum_{j=1}^{M} f\left(S_T^j\right)\right) \tag{1.22}$$

where $S_T^1, S_T^2, \dots, S_T^M$ are iid simulated observations from the risk-neutral distribution of the underlying asset. Like other methods, simulation-based methods such as Monte Carlo were employed to price European options first. Tilley [3] and Barraquand and Martineau [4] incorporated the early exercise feature of American options and used simulation to assign the holding value of the American option.

1.4 American-style Options

An American option is an option that allows the option holder to exercise any time prior and at the maturity date. Because an American option holder has the choice to exercise at any time during the life of the contract, the value of the American option compared to the corresponding European option is higher. Most of the options that are traded on exchanges are American-style.

Nevertheless, European options are normally easier to analyze than American options, and several properties of American options are regularly concluded from those of its European counterparts. Due to this early-exercise feature, the pricing of American options becomes a complicated problem that falls in the general class of optimal stopping, a subclass of optimal control problems. This flexibility makes this problem path dependent and an optimal stopping time problem. So, pricing an American option includes finding the optimal exercise times.

In the last few decades, American options pricing has been examined broadly. The major difficulty is because the American option pricing needs the selection of the optimal exercise boundary with the valuation of the contingent claim. Numerous methods have been proposed in the literature to work out this challenge.

To get the price of an American put option, one should find the optimal discounted expected payoff over all stopping times, τ, in [0,T] which is

$$\sup E[e^{-r\tau}\max(K - S(\tau), 0)] \tag{1.23}$$

where K is the strike price and $S(\tau)$ is the underlying asset value at time τ. To get the American option price numerically, τ should be limited to m exercise opportunities $t_0 < t_1 < \cdots < t_m$.

To obtain the dynamic programming equations for American options, we use the underlying state variable (S) as the related state variable. At each time there is a maximum of one right to exercise and the option holder has the right to exercise or hold the option. When the option holder reaches any exercise opportunity he/she selects between two choices; exercise the right, or keep the right and continue with the option having one right left. A dynamic program has been used to price this option and the recursive equations below have been used,

$$\mathrm{H_i(S_i)} = \mathrm{E}\left[e^{-r(t_{i+1}-t_i)}\mathrm{B_{i+1}(S_i)|Z_i}\right] \tag{1.24}$$

$$\mathrm{B_i(S_i)} = \max\left(\mathrm{h_i(S_i)}, \mathrm{H_i(S_i)}\right) \tag{1.25}$$

where $\mathrm{H_i}(S), \mathrm{B_i}(S)$ and $\mathrm{h_i}(S)$ are respectively the continuation value, option value and exercise payoff at time $\mathrm{t_i}$ and state $\mathrm{Z_i}$ which is the time $\mathrm{t_i}$ information set.

1.4.1 Approximations

Generally there are three major numerical approaches including lattices, trees and simulation methods for pricing the American options. Popularity of lattices is growing mostly in the academic studies. Simplicity and ability of implementing early exercise of American options, is the reason for this popularity. Both lattice and trees are connected with the curse of dimensionality, in the other words they suffer from the exponential growth in computational cost as dimensionality increases. On the other hand simulation based methods don't have the curse of dimensionality, as a result simulation based methods are the best approaches when the problem dimension increases.

Geske and Johnson [5] proposed the first method for this issue using a portfolio of compound European options to replicate the early exercise feature of American options. Bunch and Johnson [6] studied a method to locate the exercise times optimally and enhanced the efficiency of the Geske–Johnson method. They showed that most of the time only two early-exercise dates (including maturity) are required. Barone-Adesi and Whaley [7] presented a quadratic approximation that gives an estimated answer of the Black–Scholes partial differential equation in closed form which is very fast and precise for short and long maturities. Ju and Zhong[8] modified the Barone-Adesi and Whaley method by including a second-order extension and could give accurate answers for middle-term maturities. Later, Li [9] advanced the Ju and Zhong method[8] by the smooth pasting condition and leads to a more accurate solution of the optimal exercise price (OEP). However, all these methods have the limitation that the approximation error cannot be checked, subsequently they cannot be shown to converge.

Kim [10] and Carr, Jarrow, and Myneni [11] derived an integral equation in implicit form to solve the optimal exercise price and made a significant achievement in this problem converting the American option pricing problem into one of finding the optimal exercise price. Later, Ibáñez [12] adapted Kim's approach to show that the prices converge to the true prices by decreasing the size of the time steps. Lately, Broadie and Detemple [13], Laprise, Fu, Marcus, Lim, and Zhang[14] and Chung, Hung, and Wang [15] recommended tight quasi-analytic bounds for American options.

Using the approximate moving boundaries method, Chockalingam and Muthuraman [16] studied the cost of a suboptimal exercise price. This method iteratively finds an approximation of the optimal exercise price. When the maturity time is long enough, more or less all the methods possibly will create considerable pricing errors, consequently convergence to the "true" price relies on increasing the number of iterations (or reducing the time-step size) which leads to significant efficiency losses for these methods.

Boyle [1] initiated the simulation based method to price European options which can be used for a considerable range of assets. These methods have convergence rates that are independent of the number of state variables unlike lattice methods. The main issue for these methods is the computation cost. A major concern for using Monte Carlo to solve the dynamic programming problem is that dynamic programming methods generally work backwards in time, because of the optimal exercise price being simply determined at maturity time. However, simulation methods generally work forward in time**.** Tilley [3] dispelled the dominant belief that simulation based methods are not suitable for pricing American-style options, but simulation based methods are suitable to find optimal exercise price.

1.4.2 Tree base methods / PDE Lattice

Tree-based methods like binomial and trinomial trees can be employed for pricing options, which are mainly well-liked for pricing American options since no closed-form formulas are currently available for these options.

In the tree methods, the price of a European option converges to the Black-Scholes price when the size of the time steps tends to zero. When pricing American options one needs to evaluate whether early exercise is beneficial at each node in the tree compared with holding the option. The advantage of binomial and trinomial trees is that they can be used to value any kind of option and they are extremely straightforward and painless to implement.

Cox and Ross [17] suggested a binomial tree structure to describe the underlying asset paths. This model assumes that the underlying asset takes on one of only two possible prices at each time period. It may seem unrealistic at the first sight but the assumption directs to a formulation that can precisely value options. With many times periods, this model approximates GBM.

The binomial model assumes that the current price, S_0, either increases by a proportion u with the probability q, or decreases by a proportion d with probability 1-q, at each time period. Figure 1.1 shows the movement in one time step schematically.

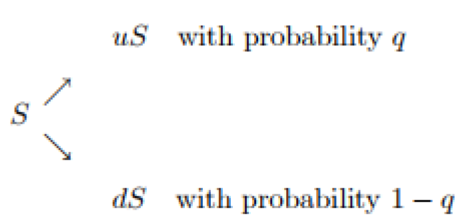


Figure 1.1: Up and down move in one time period

Denote r as the risk-free interest rate for the period which is assumed to be constant. The inequality below should be satisfied to avoid arbitrage opportunities between the stock and the risk-free investment,

$$\mathrm{d} < 1 + r < u. \tag{1.26}$$

Figure 1.2 illustrates the possibilities of a binomial tree for 3 time periods, with the additional assumption that $\mathrm{u} = \frac{1}{\mathrm{d}}$.

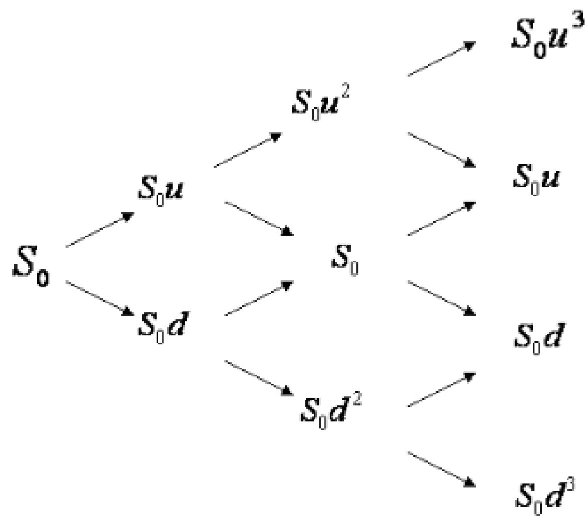


Figure 1.2: Binomial Tree with 3 Periods

We assume a portfolio containing of a long position in Δ shares and a short position in an option with initial value f_0 , f^{u} and f^{d} for up and down move respectively. Then we determine the value of Δ that makes the portfolio risk-free. If there is a downward move in the stock price, the value of the portfolio at the end of time step is

$$\Delta d S_0 - f^d \tag{1.27}$$

If there is an upward move in the stock price, the value of the portfolio at the end of time step is

$$\Delta u S_0 - f^u \tag{1.28}$$

The two are equal when

$$\Delta = \frac{f^u - f^d}{S_0(u-d)} \tag{1.29}$$

For this value of Δ, the portfolio is riskless and, for there to be no arbitrage opportunities, it must earn the risk-free interest rate. Then

$$S_0\Delta - f_0 = (uS_0\Delta - f^u)e^{-r\Delta t} \tag{1.30}$$

where Δt is the length of a time step. Rearranging gives

$$f_0 = e^{-r\Delta t}(\rho f^u + (1-\rho)f^d) \tag{1.31}$$

where,

$$\rho = \frac{e^{r\Delta t} - d}{u-d}. \tag{1.32}$$

Since $0 < \rho < 1$, ρ has the properties of a probability. In fact, this pseudo-probability ρ would equal the true probability q if investors were risk-neutral. This method is simply generalized to any number of time-steps.

There is another significant category on numerical methods in financial modeling, deterministic methods based on PDE which comes from the BSM differential equation [18]:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{1.33}$$

where f is the price of option, S is underlying asset and r is the risk-free interest rate. Solving the PDE becomes more complicated for American options which include variation inequalities.

A variety of techniques has been used in the literature to solve this PDE. One may choose between finite difference methods [19], finite element methods [20], [21], finite volume methods [22], [23] or spectral methods [24], [25].

1.4.3 Monte Carlo Methods for American Options

Broadie and Glasserman [26] introduced a stochastic mesh method for pricing high-dimensional American options when there are a finite, but probably large, number of exercise dates. Their algorithm offered point estimates and confidence intervals and which converges to the correct values as the computational effort increases. Tsitsiklis and Van Roy [27] developed a model for optimal stopping times for discrete-time ergodic Markov processes with discounted rewards. They suggested a stochastic approximation algorithm that adjusts weights of a linear combination of basis functions in order to approximate a value function. They proved that this algorithm converges and that the limit of convergence has some appropriate properties. In another research, Tsitsiklis and Van Roy [28] introduced and analyzed a simulation-based approximate dynamic programming method for pricing complex American-style options, with a possibly high-dimensional underlying state space. They showed that with an arbitrary choice of elements of the state space, the approximation error can grow exponentially with the time horizon (time to expiration).

Of particular interest to this thesis is the regression method of Longstaff and Schwartz [29]. They expressed the possibility of utilizing simulation and regression methods together for pricing American options. Simulation for pricing American options methods, combined with regression on a group of basis functions to extend lower dimensional approximations to higher dimensional dynamic problems. A big advantage of these methods is that their performance does not diminish with dimensionality. In the Least-squares Monte Carlo method a set of basis functions is chosen for regression to estimate continuation values which means only this set of basis functions determines the continuation value estimators. Using a finite set of basis functions initiates an approximation error. Stentoft [30] demonstrated that the LSMC method is the best method among the different suggested numerical methods based on simulation and regression.

In the case of dealing with path dependent options, Monte Carlo is a suitable choice which is one advantage of this method. The ability simulate the underlying asset price path by path then calculate the payoff of each simulated path and employ the average discounted payoff to estimate the price. On the other hand, this advantage makes this method difficult to employ to price the American options because it is complicated to find the continuation value at all times. Figure 1.3 shows the early exercise boundary for a typical American put option.

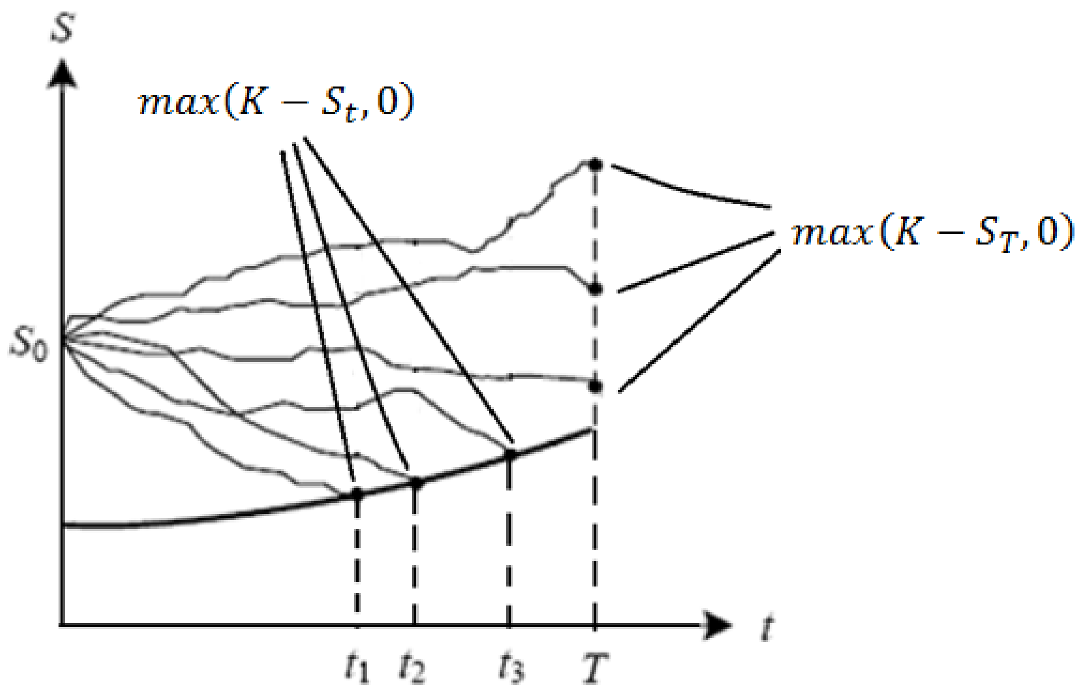


Figure 1.3 Schematic of early exercise boundary for an American put option

The main idea is to find an early exercise boundary at which the American option can behave like a knocked-and-exercised option. If any simulated path never touches this boundary before the maturity time, payoff of this path will be calculated based on the discounted value of the payoff at the maturity time.

1.5 Detailed description of LSMC

Probably after Longstaff and Schwartz [29] used simulation and regression method for pricing American option, this method becomes established. Longstaff and Schwartz method has less accumulated errors and so is less biased. They introduced a strong and easily implemented method for estimating the price of American options by simulation.

This method uses least squares to find the continuation value of the option. This method is appropriate when traditional finite difference and binomial techniques could not be used such as for multifactor models and path dependent situations. The holder of the American option compares the payoff of immediate exercise with the continuation value at each exercise opportunity. The larger of these two values determines whether the owner decides to exercise or hold the option. In the simulation the continuation value is a conditional expectation which is estimated from the cross-sectional data by using least squares. This is the main intuition of LSMC. Here payoffs from continuation are regressed on a set of basis functions where values are determined by the state variables and the fitted value of this regression gives an obvious approximation of the conditional expectation. By moving along exercise times, regression estimation of the conditional expectation function for each exercise date is obtained. From this an estimate of the finest exercise strategy along each path is determined. Using this estimated strategy, American options could then be priced precisely by simulation. This technique is referred as least squares Monte Carlo (LSMC).

Here the framework for the valuation process is presented. Assume a finite time horizon [0, T] and complete underlying probability space (Ω,F,Q) where Ω is set of all possible outcomes of the stochastic process from time 0 to T, F is a σ-Algebra of subsets of Ω and Q is a risk-neutral probability measure. We are attracted into pricing an American-style option with random cash flows which may occur from time 0 to T.

The aim of the LSMC approach is to obtain a pathwise approximation to the optimal stopping rule yielding a pathwise approximation for the American option value. Although usually American options could be exercised continuously, to express the insight behind the LSMC algorithm, the discussion will be focused on the situation in which the American option can only be exercised at some specific times such as $0 < t_1 \leq t_2 \leq t_{31} \leq \cdots \leq t_K = T$ which includes K exercise times, and be concerned about the finest stopping strategy at each exercise time. By taking K to be sufficiently large, the LSMC technique could be applied to estimate the value of the continuously exercisable options. Earlier than the last expiration date, at any exercise time t, the investor must decide whether to exercise instantly or to keep holding the option and revisit the exercise choice at the next exercise time. The option holder exercises the right once the instant exercise value is greater than or equal to the value of continuation. If the investor decides not to exercise the option until the last exercise time, which is the expiration date of the option, the option will be exercised if it is in the money, or will be expired if it is out of the money.

The investor knows the cash flow from immediate exercise at time t_k which is simply equal to the value of instant exercise. On the other hand, the cash flows from holding the option are not known at that time. The value of holding the option is equal to the value of the option presuming that it is not allowed to be exercised until after the current time. No-arbitrage pricing theory implies that the value of this option is known by calculating the expectation of the leftover discounted cash flows using the risk-neutral pricing measure. The LSMC method employs least squares to estimate the conditional expectation function and moves backwards in time since the path of cash flows made by the option is defined recursively. Cash flows at time t_k could be different from cash flows at time t_{k+1} since it may be optimal to execute at time t_{k+1} so changing all subsequent cash flows along a specific path.

Let's assume $F(S_k, t_k)$ expresses the continuation value of a path having price S_k at time t_k with

$$F(S_k, t_K) = \sum_{i=0}^{\infty} \beta_{k,i} B_{k,i}(S_k) \tag{1.34}$$

where S_k is the price, B_k is the set of basis functions, β_k are coefficients and constant at time t_k and i is related to the basis function. Then

$$\hat{\beta}_k = \left(B_k^T B_k\right)^{-1} B_k^T F_k \tag{1.35}$$

where F is the vector of basis functions. Also define $C(s; t, T)$ to denote the path of cash flows made by the option conditional that it has not been exercised at or prior to time t and the option holder follows the optimal stopping strategy for all s, $t < s \leq T$.

To apply LSMC method, $F(t_{k-1})$ is estimated using the first M basis functions (M<∞) and present this estimation with $F_M(t_{k-1})$ which is approximated by regressing the discounted values of $C(t_k; t_{k-1}, T)$ onto the basis functions only including the paths where the option is in the money at time t_{K-1}. Because taking decision on the exercise of option is concerned, only in the money options are included in the regression as the obvious decision for the out-of-the money options is to hold. The result is that fewer basis functions are needed to attain a precise estimation of the conditional expectation function than when including all of the paths in the regression.

When the conditional expectation function at time t_{K-1} is approximated, by comparing the instant exercise payoff with $\hat{F}_M(S_{k-1}, t_{K-1})$ the early exercise decision at time t_{K-1} could be made for all in-the-money paths w. After the exercise decision is known, the option cash flow paths $C(s; t_{K-2}, T)$ could be estimated. The exercise decision at each exercise time for each path could be identified by repeating the procedure. To value the American option at time zero, we take the first stopping time along each path and discount the payoff from exercise back to time zero, and then take the average over all paths.

The Least-squares Monte Carlo technique has been employed for the valuation of multiple exercise options [31]. A big advantage of this method is that increasing dimensionality does not diminish the performance. In this method a set of basis functions must be chosen to run regressions to approximate continuation values. In theory only an infinite set of basis functions results in the true option value, but usually a finite set of basis functions is employed which brings in an approximation error. The problem is that this approximation error could spread backwards through the exercise opportunities and generate high- and low-biased estimators which may not converge to the identical value [32].

1.6 Bias of Monte Carlo estimator

It is well known that Monte Carlo methods to value American-style options generate price estimators that are biased. Depending on how the price estimator is constructed, it can be possible to determine the sign of the bias. One common strategy for valuation is to construct two estimators, one with positive bias and the other with negative bias, that (on average) bound the true option price. If these estimators are also consistent, then as the sample size increases the difference in the estimators can be made arbitrarily small, hence yielding the option value. In this section we discuss the sources of bias and the construction of price estimators. This discussion is general in that it applies to many Monte Carlo methods (e.g., stochastic tree, stochastic mesh, regression-based technique) used for American-style options.

In general, bias results from i) making incorrect exercise decisions and ii) using the same set of information for exercise decisions and value propagation (back to the next time step in the recursion). Incorrect exercise decisions means the option was held when it should have been exercised or it was exercised when it should have been held.

Suppose that we have simulated a set of M stock price sample paths and consider the recursive dynamic program given by Equations (1.24) and (1.25) replacing the hold value with its simulation-based estimator, $\widetilde{H_i}(s_i)$, in Equation (1.25) yields

$$\widetilde{B}_i(S_i) = \max\left(h_i(S_i), \widetilde{H}_i(S_i)\right) \tag{1.36}$$

This estimator uses the same information to i) make exercise decisions; and ii) propagate the value backwards along the path for the next recursion. Mathematically, it is easy to show that the resulting estimator has positive bias and hence on average overestimates the true price. This bias is termed foresight bias and arises from peering into the future along the path in order to make both the exercise decision and assign value. For example, if by chance the future value of the option along the path produces a higher than average payoff, the hold value will be higher, making it more likely the option will be held and the higher than average value propagated backwards. On the other hand, if by chance the path-wise future option value produces a lower than average payoff, the hold value will be lower, making it more likely the option will be exercised, resulting in the (higher than average) exercise value as the one propagated backwards. Both situations have the effect of pushing up the option value estimator, resulting in an estimator with positive bias.

One way to get rid of the foresight bias is to use independent sets of information for exercise decisions and value propagation. Suppose that in addition to the set of M simulated stock price sample paths, we have another (set of) independent simulated stock price sample path(s). Let * denote the independent sample path. Using the original set of M simulated stock price paths, the hold value estimator is constructed and compared with the exercise value along the independent sample path. If the hold value is less than the exercise value, the option is exercised, otherwise the option is held and the value assigned is the discounted value from the next time step along the independent path. The hold value along the independent path is denoted $\widetilde{H_i}(s_i^*)$, and Equation (1.25) becomes

$$\widetilde{B}_i^* = \begin{cases} h_i(S_i^*) & \text{if } h_i(S_i^*) > \widetilde{H}_i(S_i^*) \\ e^{-r\Delta t_i}\widetilde{B}_{i+1}^* & \text{if } h_i(S_i^*) \leq \widetilde{H}_i(S_i^*) \end{cases} \tag{1.37}$$

This can be repeated for the whole set of independent sample paths, and then the option value estimates across paths are averaged. This estimator is called the out-of-sample or path estimator. Mathematically it is easy to show that the out-of-sample estimator has negative bias, hence underestimates the true option price. This bias is termed sub-optimal exercise bias and it results from making incorrect exercise decisions. Intuitively, along a given path, there is an optimal exercise rule which is the best one can do. Replacing the optimal exercise rule with an estimate results in a sub-optimal exercise rule which gives rise to incorrect exercise decisions. The value propagated is independent of the exercise decision (not peering into the future along the path to make exercise decisions) and hence is not on average higher than it should be as in the case above (in fact due to possibly incorrect exercise decisions at future times, the value propagated along the path is, on average, lower than it should be). All of this implies that the resulting estimator has negative bias.

Another estimator that is commonly used, particularly in the regression-based methods, is called an interleaving estimator by Glasserman [33]. The interleaving estimator avoids the need for simulating a second set of independent sample paths and removes some of the foresight bias that results from using the dynamic program in Equation (1.36). However, these improvements come at the expense of not knowing the sign of the bias, implying there is no knowledge of whether the price estimator is an upper or lower bound for the true price. Using the set of M simulated sample paths, Equation (1.36) in the dynamic program becomes

$$\tilde{B}_i = \begin{cases} h_i(S_i) & \text{if } h_i(S_i) > \tilde{H}_i(S_i) \\ e^{-r\Delta t_i}\,\tilde{B}_{i+1} & \text{if } h_i(S_i) \leq \tilde{H}_i(S_i) \end{cases} \tag{1.38}$$

For this estimator, there is dependence between the set of information used to make exercise decisions and the value propagated. Thus there remains some foresight bias in this estimator. Using the discounted cash flows along each path as the value propagated (in the event that the exercise decision is to hold) results in significantly less foresight bias as compared to the positively-biased estimator. Additionally, sub-optimal exercise bias is present as the estimated exercise rule can induce incorrect exercise decisions. Effectively, there is a tradeoff between the foresight and the sub-optimal exercise biases. For small to moderate sample sizes, the foresight bias can dominate while for large sample sizes, the effect of including the single path in the exercise decision gets washed away in the large number of other paths, resulting in sub-optimal exercise bias dominating the foresight bias. Hence for large sample sizes the interleaving estimator generally has negative bias and for small sample sizes, the sign of the bias cannot, in general, be determined.

Duality methods provide another way to get a positively-biased estimator that is generally less than the estimator given by Equation (1.36), hence providing a tighter upper bound on the true price. Duality methods require as an input a method that generates a negatively-biased estimator, such as the path estimator. Rogers [34], Jamshidian [35], and Haugh [36] are some examples in the literature that use duality to provide an upper bound on the option value.

There have been a number of methods proposed in the literature to reduce estimator bias. By far the vast majority of these have been in the regression-based setting in which various approaches have been used to get a better estimate of the hold-value function. Notable exceptions are the bootstrapping approach proposed by Broadie, Glasserman and Ha [37] for the stochastic tree, the bias estimate suggested by Carriere [38] in the regression setting, and the works by Kan and Reesor [32], Whitehead, Davison, Reesor and Kan [39] and Whitehead, Reesor and Davison [40]. In the latter works, the authors use a central-limit theorem approximation to derive an estimate of the bias, which is subtracted from the price estimator at each step in the dynamic program recursive equations. This method has been shown to be effective for the stochastic tree and mesh and for the regression-based approaches. One recent interesting attempt at reducing bias in the regression setting is given by Cheng and Joshi [41], though the efficacy of this method is unclear.

In this thesis, we work in the regression-based setting and focus on Inequality Constrained Least Squares (ICLS), which has been proposed by Letourneau and Stentoft [42] as a way of reducing bias by imposing monotonicity and convexity constraints on the fitted value function. This work is promising and in this thesis we extend their approach to the valuation of multiple exercise options. In the following subsections, we provide a detailed description of the ICLS approach.

1.7 Detailed description of ICLS

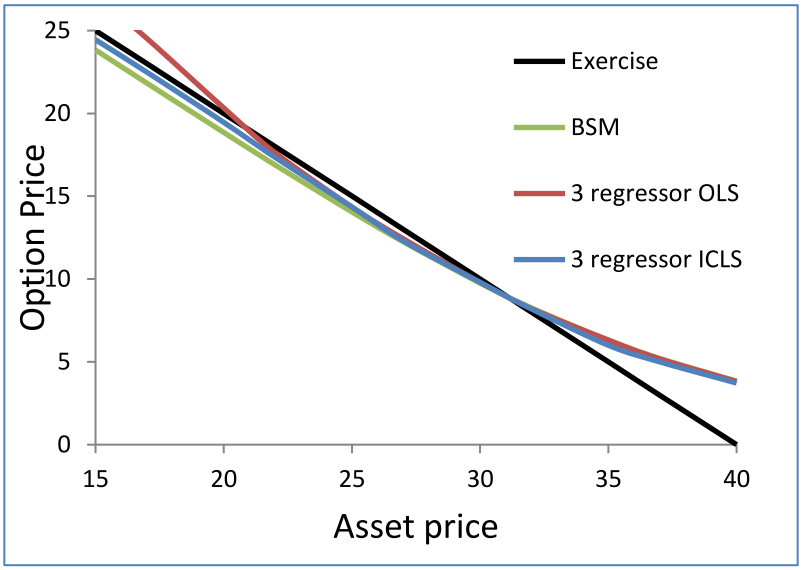
Letourneau and Stentoft [42] showed that refining Longstaff and Schwartz method by imposing proper structure in the regression problem could reduce the bias of the method and improve the results consequently. Interestingly this is true for different maturities, categories of moneyness and type of option payoffs. In the ICLS method monotonicity and convexity of the estimated function is imposed which leads to nice properties of estimation function. The most important result of this structure is not having overfitting. Dynamic programming of ICLS and LSMC are similar but the regression part of these methods is different which would be explained in this section. Least squares regression in matrix form is presented as

$$y = \beta X + \varepsilon \tag{1.39}$$

where β is a vector of unknown parameters and ε is a vector of unobserved disturbances. The Longstaff and Schwartz algorithm introduced an exercise strategy by approximating the holding value using regression then deciding to exercise or hold the option. Their method has three main steps as i) making paths for the underlying asset by taking the proper stochastic model, ii) determine of payoff at maturity, then iii) move one time step back and calculate the expected cash flow for each path. Here the time step is $\Delta t = \frac{\mathrm{T}}{\text{Number of exercise opportunities}}$. The best advantage of the least squares Monte Carlo (LSMC) method is simplicity and the easiness which it could be modified to price a variety of financial products. The LSMC method has decent convergence properties. Stentoft [30] illustrated that the LSMC method converges to the true price when the number of paths and regressors tend to infinity. But in real applications, one uses a finite number of regressors and simulated paths which leads to biased estimates. One way of having a better convergence rate is increasing the number of regressors. Longstaff and Schwartz [29] claimed that the number of regressors should increase until the price estimate starts to decline. Even though more regressors will enhance the flexibility of the estimator, but it will raise the in-sample overfitting and consequently increase the bias, so the final result is not clear.

Letourneau and Stentoft [42] suggested a novel method which uses constraints in the estimation of the holding value function and consequently reduces the chance of making incorrect exercise decisions along a given path. Hence, the resulting estimator will have less bias. Imposing constraints reduces the likelihood of poor exercise decisions that can result from overfitting.

(a)



(b)

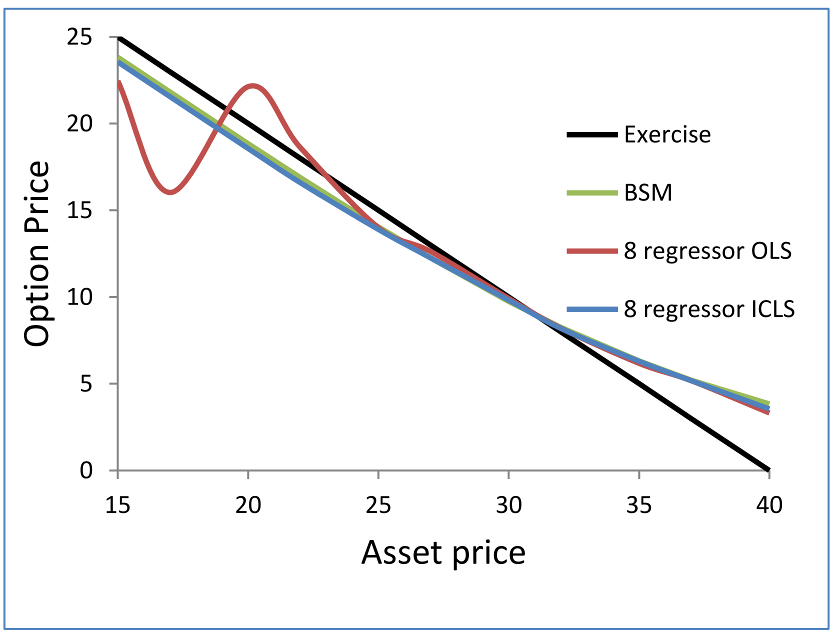


Figure 1.4 Approximation the hold value for a) 3 regressors, b) 8 regressors. The option characteristics are S_0=K=40, r=6% ,σ=40%, t=1 year and S follows geometric Brownian motion. 1000 simulated paths are used.

Figure 1.4 demonstrates the value of an American put option with one year to maturity. Figure 1.4 includes the theoretical holding value of European option using BSM, intrinsic value, ordinary least squares and inequality constrained least squares methods keeping a) 3 and b) 8 regressors. This figure shows the possibility of overfitting and poor fit in the ordinary least squares method for both 3 and 8 regressors. When the intrinsic value is low (asset price is close to strike price), number of simulated paths is adequate, therefore ordinary least squares method and ICLS have the same holding value. Though, in the large moneyness region, not enough paths exist which leads to overfitting for OLS method which consequently raises the incorrect exercise decisions. On the other hand, ICLS method prevents the overfitting by imposing constraints even for a very low number of paths.

In looking at the exercise rule for the maturity, the unconstrained regression approach implies an incorrect exercise decision, as it implies the holder should hold when the underlying is between 15 and 20 (Colour red in Figure 1.4 a). Constraining the fitted regression fixes this problem, as can be seen by looking at the Colour blue curve in the right panel of Figure 1.4 b.

First we introduce notation. Let's define $\Gamma_h = (x_1, x_2, \dots, x_h)'$ to be a univariate grid on $\{x_1 : x_h\}$ with h elements. The first and second difference of f, the estimation function, over grid are defined respectively as

$$\left[\hat{f}(x_{i+1}) - \hat{f}(x_i)\right] \tag{1.40}$$

$$\left[\hat{f}(x_{i+2}) + \hat{f}(x_i) - 2\hat{f}(x_{i+1})\right] \tag{1.41}$$

where $\hat{f}$ is the estimated function over the grid. By checking the second difference to be positive, we could verify if the function is convex over three points of the grid discretely. To check the monotonicity over the grid, the differentiation matrix should be defined as

$$D_k = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & & & \vdots \\ 0 & 0 & -1 & 1 & & \\ \vdots & & & \ddots & \ddots & \\ & & & & & \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} \tag{1.42}$$

where D_k is a $k \times (k+1)$ matrix and k is number of points which we check the monotonicity and convexity of estimated function. The second differentiation matrix could be obtained by $D_{k-1} \times D_k$ as

$$D_{k-1} \times D_k = \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & & \vdots \\ 0 & 0 & 1 & -2 & 1 & \\ \vdots & & & \ddots & \ddots & \\ & & & & & \\ 0 & 0 & \dots & 1 & -2 & 1 \end{pmatrix} \tag{1.43}$$

So the matrix multiplication $D_{k-1} \times \hat{f}(\Gamma_k)$ denotes the first difference over the grid and $D_{k-2} \times D_{k-1} \times \hat{f}(\Gamma_k)$ denotes the second difference. To affect the constraints to the estimated function in the univariate grid, the problem should be established as

$$\min_{B} \left\| \hat{f}(\Gamma_h) - Q \right\| \tag{1.44}$$

such that

$$A \times Q \geq 0 \tag{1.45}$$

where Q is vector of size h and A is a matrix with the finite difference constraints, therefore $A \times Q$ is a vector. Note to impose strict monotonicity over the grid, A should be set as D_{h-1}. On the other hand, to impose the convexity, A should be set as $D_{k-2} \times D_{k-1}$. Nevertheless, when the convexity constraints are satisfied, the slope is monotonically increasing, thus one does not need to check the slope constraints, except at both ends. Consequently, it suffices to use $k - 2$ convexity constraints and 2 slope constraints in the problem. Therefore A is as

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & & \vdots \\ 0 & 0 & 1 & -2 & 1 & \\ \vdots & & & \ddots & \ddots & \\ & & & & & \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix}. \tag{1.46}$$

Here we present some numerical results and compare them with benchmarks. As the first step, some results from Longstaff and Schwartz [29] are reproduced. In this section we price an American put option on a stock when the option has 50 exercise opportunities per year. The interest rate is 6% and strike price is 40. Underlying asset price, volatility of returns and years to expiration are shown by S, σ and T respectively. Our simulation is based on 50,000 sample paths for the stock price process.

Table 1: Comparison of results with Longstaff and Schwartz paper results

S	σ	T	Longstaff		Our results	
			American Option	S.E.	American Option	S.E.
36	0.2	1	4.472	0.01	4.4713	0.0162
36	0.2	2	4.821	0.012	4.8165	0.0167
36	0.4	1	7.091	0.02	7.0905	0.0267
36	0.4	2	8.488	0.024	8.4839	0.0359
38	0.2	1	3.244	0.009	3.2418	0.0137
38	0.2	2	3.735	0.011	3.7280	0.0114
38	0.4	1	6.139	0.019	6.1371	0.0262
38	0.4	2	7.669	0.022	7.6406	0.0317
40	0.2	1	2.313	0.009	2.3038	0.0145
40	0.2	2	2.879	0.01	2.8676	0.0168
40	0.4	1	5.308	0.018	5.3077	0.0201
40	0.4	2	6.921	0.022	6.9007	0.0305
42	0.2	1	1.617	0.007	1.6116	0.0124
42	0.2	2	2.206	0.01	2.2058	0.0186
42	0.4	1	4.588	0.017	4.5803	0.0277
42	0.4	2	6.243	0.021	6.2425	0.0312
44	0.2	1	1.118	0.007	1.1065	0.0069
44	0.2	2	1.675	0.009	1.6745	0.0154
44	0.4	1	3.957	0.017	3.9566	0.0220
44	0.4	2	5.622	0.021	5.6400	0.0292

Comparison between our results and Longstaff and Schwartz [29] showed in table 1 shows very close prices for different American put options. They have used 100,000 samples ((50,000 plus 50,000 antithetic) and this could explain the slight differences in the prices and standard errors. Generally both results are very close and confidence intervals have overlap.

1.7.1 High-biased Estimator

In-sample analysis denotes to approximate the model using available data and then compare the model's fitted values to the same data. This process draws an excessively optimistic representation of the model's forecasting ability. This in-sample over fitting establishes a high bias in the price approximation. To reduce the high bias, must raise the number of simulated paths.

Figure 1.5 shows the estimated values for the ordinary least squares and modified least squares methods which the modified one includes some constraints in the regression. Figure 1.5 includes 9 different options which vary on the initial price and expiry date. Options are American put options and each plot shows the prices attained with polynomials of order 2 to 7. The red line with circles represent the LSMC method using the OLS regressions, while the constrained least squares method is represented by the blue line with squares and the solid black line is the price obtained from binomial method.

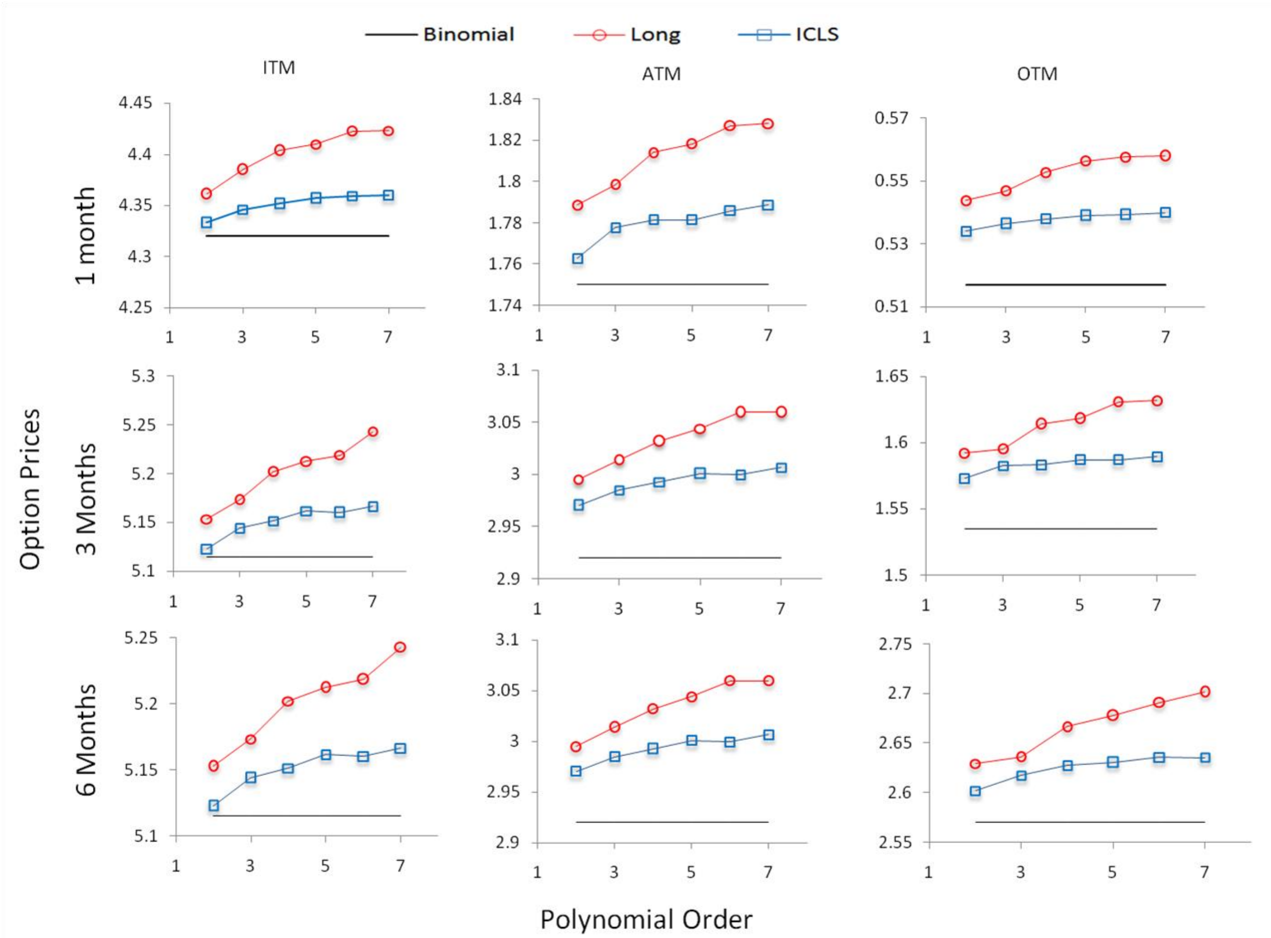


Figure 1.5 American put option pricing using the in-sample LSMS and ICLS methods, ITM, ATM and OTM options are priced for maturities of 1,3 and 6 months with daily exercise. The underlying asset follows a geometric Brownian motion with r= 6%, $\sigma = 40\%$, K=$40, number of paths = 1000 and S_0=$36, $40 and $44 respectively for ITM, ATM and OTM options. All options are priced using polynomials of order 2 to 7, and the regressions are done using the paths are ITM at the current time step. The mean prices of 100 repetitions are shown and the benchmark prices are obtained with the binomial model.

Figure 1.5 is almost the same as results that Letourneau and Stentoft[42] presented, and the only discrepancy is for polynomials of orders 2 and 3. This different could be raised because of using different packages. Letourneau and Stentoft [42] used lsqlin() in Matlab and we used quadprog package in r. Nevertheless, our result is more consistent and accurate.

Results in Figure 1.5 show that the ICLS method has less bias in the in-sample analysis. Both LSMC and ICLS have more bias for polynomial with higher order which could be because of over fitting.

1.7.2 Low-biased Estimator

Here out-of-sample pricing is used to eliminate the over fitting effect of in-sample pricing. Figure 1.6 includes the estimated value of American put option obtained with out-of-sample pricing and shows that ordinary least squares method is always biased low compared to the true value. When polynomial order increases, the bias increases consequently which is because of the over fitting.

Figure 1.6 illustrate that in the out-of-sample approach, ICLS has less bias compared to the OLS and has a higher price. Approximating the conditional expectation and imposed structure in the regression causes less bias in the ICLS. On the other hand, the imposed structure prevents over fitting which is obvious in the prices of the ICLS for different polynomial orders.

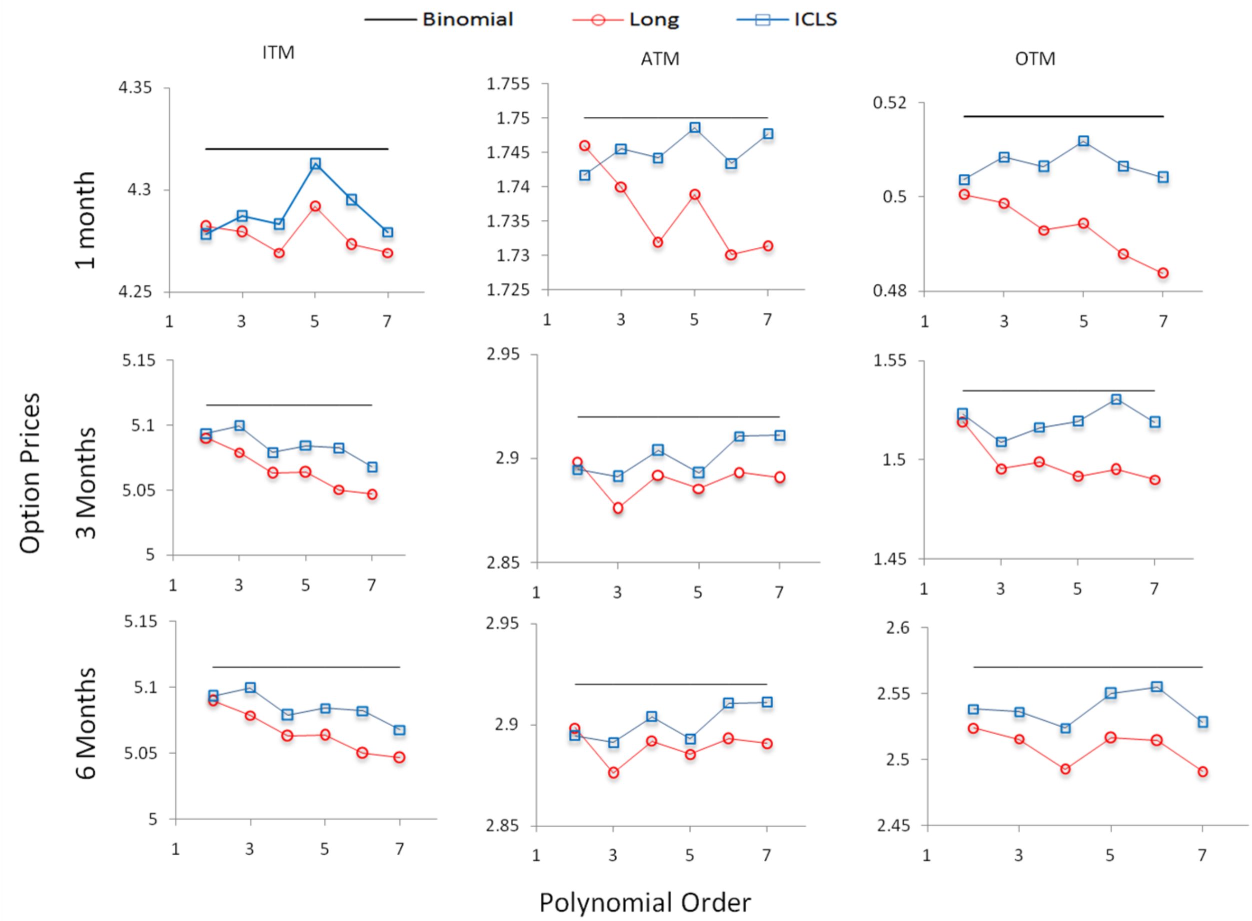


Figure 1.6 American put option pricing using the out-of-sample LSMC and ICLS methods, ITM, ATM and OTM options are priced for maturities of 1,3 and 6 months with daily exercise. The underlying asset follows a geometric Brownian motion with r= 6%, $\sigma = 40\%$, K=$40, number of paths = 1000 and S_0=$36, $40 and $44 respectively for ITM, ATM and OTM options. All options are priced using polynomials of order 2 to 7, and the regressions are done using the paths are ITM at the current time step. The mean prices of 100 repetitions are shown and the benchmark prices are obtained with the binomial model.

Chapter 2

2 Multiple Exercise Options

In chapter 2 we focus mostly on pricing multiple exercise options. First we discuss a detailed description of multiple exercise options and previous models of their valuation. Then we present a detailed description of LSMC and ICLS methods and their algorithms.

2.1 Overview of Some Multiple Exercise Products

Options with multiple exercise opportunities provide more than one exercise right for the option holder which is a generalized version of American options. In some cases, the option holder has more flexibility such as control of the amount that exercised. The valuation of multiple exercise options is a key area of financial modelling, with variety of applications including interest rate derivatives, energy and commodity contracts. These types of options have become more common in the last decade and have a wide range of application from insurance to energy industries. A survey of the literature gives plenty of examples including swing options (Jaillet et al. [43], Chandramouli and Shyam [44]), switching options (Cortazar et al. [45]), portfolio liquidation (Gyurko et al. [46]), chooser flexible caps (Hambly and Meinshausen [31]) and commodity processing and storage (Lari et al. [47]). In particular, our focus is on valuation of multiple exercise options using ICLS method.

The majority of the studies in the literature have concentrated on swing options which have multiple exercise rights, and constraints on the total volume delivered. Swing options have widely been employed in energy markets to help producers deal with the raw materials consumed in energy production facing uncertain demand. Swing options allow the option holder to buy a predetermined number of the underlying asset at a predetermined price while having some control over the time and quantity of the underlying asset. Within the duration of the contract the holder may exercise a specified number swing rights which typically could only be exercised at a predetermined discrete set of times. A swing option is usually distinguished as the swing part of a base-loaded futures contract that provides a prearranged price for an amount of a commodity over a predetermined period of time and this part permits for a flexible delivery amount of the underlying asset above or under the base-loaded contract. Nonetheless, the two parts of the contract can be separated and treated separately for valuation.

Another multiple exercise option is a chooser flexible cap which is an interest rate derivative with multiple exercise opportunities. The number of rights is limited and the buyer does not wait for automatic exercise but chooses when to use a cap. On the other hand, the buyer might decide to use the caplets on a later date which potentially could be more valuable. This conclusion depends on multiple factors such as number of rights left, time to maturity and expected volatility of underlying asset. The chooser flexible cap is an appropriate substitute to the interest rate cap and flexible cap particularly where the buyer believes there is a high chance that rates could rise above the strike. They are most appropriate for buyers enthusiastic in risk management. The chooser flexible cap has some advantages over a traditional cap such as lower premium, flexibility and ability to be customized.

2.2 Multiple Exercise Options Pricing

Similar to American option pricing, multiple exercise option valuation is a stochastic optimal control problem. For both of them the solution provides a value and an optimal exercise policy. For example, take a swing option as the multiple exercise option, the exercise policy is a paired arrangement of stopping times and exercise amounts. In the case of a chooser flexible cap as the multiple exercise option, the exercise policy is an arrangement of stopping times.

A dynamic program has been used to price multiple exercise options and the recursive equations below have been used,

$$\mathrm{H_i}(\mathrm{S_i}, \mathrm{N}) = \mathrm{E}[\mathrm{B_{i+1}}(\mathrm{S_i}, \mathrm{N}) | \mathrm{Z_i}] \tag{2.1}$$

$$\mathrm{B_i}(\mathrm{S_i}, \mathrm{N}) = \max\left(\mathrm{h_i}(\mathrm{S_i}, \mathrm{N}) + \mathrm{H_i}(\mathrm{S_i}, \mathrm{N}-1), \mathrm{H_i}(\mathrm{S_i}, \mathrm{N})\right) \tag{2.2}$$

where $H_i(S_i, N)$, $B_i(S_i, N)$and $h_i(S_i, N)$ are respectively the continuation, option and exercise values at time t_i and state Z_i which is time t_i information set on the tree with N exercise rights left.

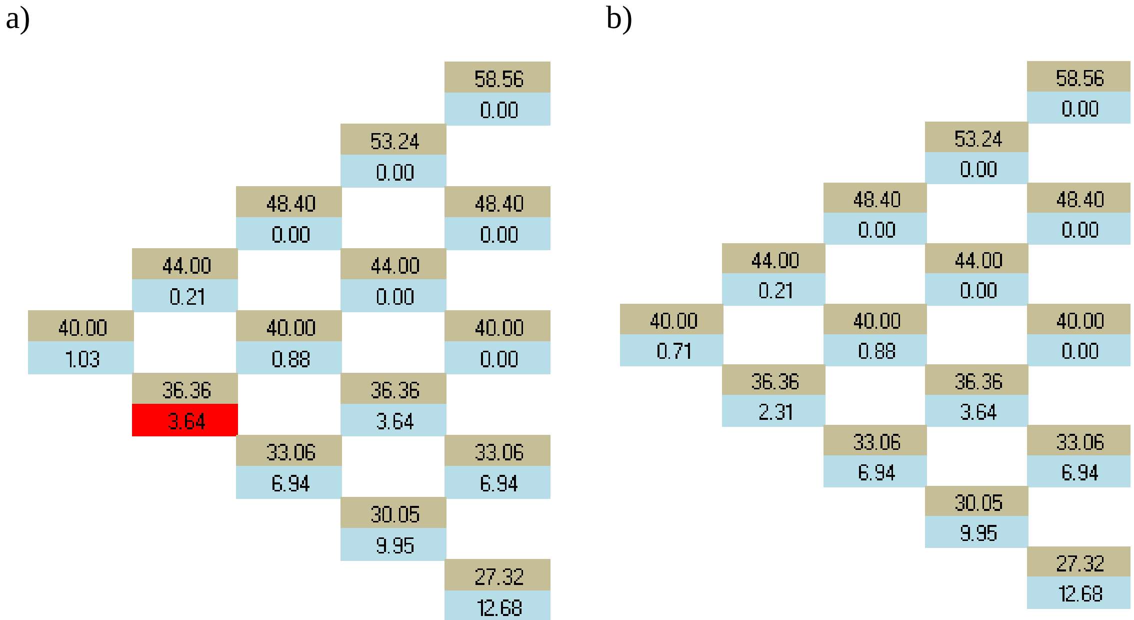


Figure 2.1 Multiple exercise option pricing 2 put rights using binomial trees a) 1 right left b) 2 rights left. Options are priced for maturities of 1 year with no dividend, r= 5%, $u = 1.1$, d=1/u, K=\$40, S_0=\$40 and time steps=5.

Figure 2.1shows the binomial trees of a multiple exercise option with two put rights. Grey and blue cells show the price and option value respectively. Red cell in Figure 2.1a indicates the best time of exercise when 1 right is left then in the tree with 2 rights left, no right could exercise in that time which leads to a lower value for the second right. Note that the total value of this multiple exercise option is summation of the value of both rights value which is 1.74 (= 1.03+0.71).

Lets assume price of multiple exercise option with Np put rights and Nc call rights is V(Np,Nc). The price of call and put rights are distinct which means adding one put right to this option, adds some value which is independent of number of call rights. So

$$V(N_p, N_c) - V(N_{p-1}, N_c) = V(N_p, 0) - V(N_{p-1}, 0) \tag{2.3}$$

then it is easy to conclude

$$V(N_p, N_c) = V(N_p, 0) + V(0, N_c) \tag{2.4}$$

Note that above conditions hold when put right strike price is not more than call right strike price.

2.2.1 Tree base Methods / PDE Lattice

It is obvious that American option has no closed form pricing formula and consequently multiple exercise options and swing option don't have closed form pricing formula also. Therefore a numerical method should be employed to price these options approximately. The numerical methods fall into lattice/tree, numerical PDEs and Monte Carlo solutions.

Multiple exercise options are complicated and there is a lack of literature about their valuation. Most of the focus in the literature is on swing options. Multiple exercise options have specific characteristics that make their valuation different from American options. Similar to American options, multiple exercise options have the right of early exercise therefore methods that have been developed for American options could be modified to value the multiple exercise options.

Lari-Lavassani et al. [48] provided an overview of the valuation of American options via trees related to both widening the number of trees and the stop pricing time strategy. For valuating an American option with dynamic programming techniques, a specific tree could be used. On the other hand, for pricing multiple exercise option a forest of trees should be made which each tree representing a possible combination of rights. The dynamic programming algorithm moves backward in time and is used for pricing American options. This is modified to be able to move both backward in the time and through the trees corresponding to different numbers of exercise rights. In this case the start of the progress would be from the tree with no exercise rights remaining. Pricing of options with multiple exercise opportunities is a stochastic optimal control problem.

The insight behind the valuation of multiple exercise options using forest of trees as follows. The process begins from the expiration date of the option and moves towards the back in time to value the instrument by backward induction in two dimensions: price; and number of exercise rights left. At each exercise opportunity the holder chooses the maximum value of continuing in the current tree (to not exercise a right), or the payoff from exercising a right and continuing to hold an option with one less right. if the choice is to exercise then this "jumps" the valuation algorithm to the tree with one less exercise right. Suppose that up until the current time k out of N rights have been exercised. Exercising of an additional right would leave the option holder with the payoff from immediate exercise plus continuing with an option with N-k-1 rights left.

Besides the forest of trees method, there are other methods for valuing multiple exercise options. PDE valuation methods are some of them and they need clearly defined boundary conditions. The holder of a multiple exercise option cannot exercise two rights at the same time and this feature makes valuing of multiple exercise option with PDE approach more challenging.

The PDE valuation models are more complicated than the BSM therefore to solve the PDE methods numerically, well-posed boundary conditions should be defined. Yan [49] developed a PDE method with well-posed boundary conditions for valuing swing options. To develop the boundary condition he used the energy method and made a two space variable model of Asian options well-posed on a finite domain and used the PDE approaches to approximate the solution. He priced the multiple exercise option with a waiting period as well. In the extreme case, when the waiting time tends to zero, the value of the M rights option price increases to the value of a portfolio of M American options.

Wilhelm and Winter [20] solved a PDE of the excess to the payoff function to bypass the difficulty of early exercise option. In addition they extended a PDE approach to solve the usage based on the BSM. In the lattice and PDE methods one exogenous price process should be specified in the form of stochastic differential equation.

2.2.2 MC Methods

Pricing and hedging early exercise options such as multiple exercise options is an important problem in the finance area and its analysis usually involves solving problem of optimal stopping or optimal control. For uncomplicated contracts, for instance American puts and calls, the related optimal stopping time problems can be solved by usual numerical methods such as binomial trees. But, the computational expenses for these methods grow dramatically as the number of parameters affecting the price of a contract grows. The approach for pricing multiple exercise options numerically, such as trees and finite difference methods for PDEs, have similar issues as the computational increases with time dimension of the problem.

Extensive studies have been done on the high dimensional American option pricing challenge. Tilley [3] started with American style option and then Barraquand [4], Broadie and Glasserman [50] and Tsitsiklis and Van Roy [28] worked on the approximation of the exercise boundary by applying different methods on more complex options with high dimensionality. All of their techniques conclude a non-optimal exercise policy and give a lower bound on the price because the price is the supremum over the return from all possible exercise strategies in these methods.

Generally there is no unbiased estimator for pricing multiple exercise options by simulation. Therefore attempt to limit the option value from above and below as good as possible. Denote random variable $V_0^{\uparrow,M}$ as a positively biased estimator and$V_0^{\downarrow,M}$ as a negatively biased estimator, in that

$$\mu^{\downarrow,M} = \mathrm{E}\left[V_0^{\downarrow,M}\right] \leq V_0^{*,M} \tag{2.5}$$

$$\mu^{\uparrow,M} = \mathrm{E}\left[V_0^{\uparrow,M}\right] \geq V_0^{*,M} \tag{2.6}$$

where $\mu^{\uparrow,M}$ and $\mu^{\downarrow,M}$ are the means of the positively and negatively biased estimator using a sample size of M. Similarly let $\sigma^{\uparrow,M}$ and $\sigma^{\downarrow,M}$ be the standard deviations of the positively and negatively biased estimators from a sample size M. A $(1-\alpha)\%$ confidence interval for the true value of $V_0^{*,M}$ is given by

$$\left[\mu^{\downarrow,M} - Z_\alpha \frac{\sigma^{\downarrow,M}}{\sqrt{M}}\ , \qquad \mu^{\uparrow,M} - Z_\alpha \frac{\sigma^{\uparrow,M}}{\sqrt{M}}\right] \leq V_0^* \tag{2.7}$$

As sample size M increases the width of the confidence interval decreases.

Haugh and Kogan [36] and Rogers [34] improved the Monte Carlo approach by considering the dual problem and constructed a positive biased approximation for the valuation of an American option. Hambly and Meinshausen [31] extended this theory to the multiple exercise options and obtained an expression for the price of the option as the infimum over a choice of stopping times and martingales then developed an algorithm to obtain a positive biased approximation for the price.

Currently the Monte Carlo (MC) method is the most prosperous for pricing early exercise options such as American, multiple exercise and swing options. Hambly and Meinshausen [31] developed the technique to price a straightforward swing call option which includes a right to exercise at each exercise time. Aleksandrov and Hambly [51] recently used Monte Carlo to price a general form of swing call option. Leow [52] studied valuation of swing option with MC using the pricing problem formulated as a stochastic optimal control problem in discrete time and state space. Nadrajah et al. [53] developed least squares Monte Carlo value (LSMV) and approximate linear programming (ALP) methods to value the multiple exercise options with term structure model. They compared the performance of these methods with least squares Monte Carlo continuation method. A drawback of the LSMV and ALP methods is the large computation of high dimensional expectation in the valuation of Markov decision problem.

2.3 Detailed Description of LSMC for Multiple Exercise Options

Because of its simplicity and treating the high dimensionality problem, MC methods are accepted in practical finance including for the valuation of multiple exercise options. As discussed before, handling the early exercise feature is the most challenging issue in the MC methods. Most authors concentrate on the expectation function engaged in the repetition of the dynamic programming using least squares regression for estimating the continuation values. In this section least squares Monte Carlo (LSMC) method for multiple exercise options is discussed.

The method LSMC introduced by Longstaff and Schwartz [29] intends to evaluate continuation values by regression. To price the multiple exercise options with LSMC, an analogous forest of trees method should be employed. The forest of trees method for multiple exercise option pricing is a generalization of the pricing of American options by trees which extends the number of trees linked with the exercise rights and a forest is constructed which includes a tree for every possible combination of rights. The conventional tree-based approach for pricing of options is based on constructing a tree for the option price that identifies the progress of the option settlement price until the expiration date. Although this method is hard to implement for path dependent options such as Asian options, but works properly for multiple exercise options.

In this thesis, we face with the underlying state variable (S) and number of exercise rights left (N) as related state variables. It is assumed that the exercise volume is fixed. When the option holder reaches any exercise opportunity he/she should select between two choices; exercising one right and continuing with another option with N-1 rights, or keep all rights and continuing with an option having N rights left. Dynamic programming has been used to price this multiple exercise option and the algorithm below has been used for pricing one specific tree with N_j number of rights left and LSMC method:

1. Generate M number of independent paths of the underlying asset price from time 0 to maturity time T.

2. Calculate the option value for the last time step $\mathrm{B_T(S_i, N)}$, which is the payoff of each path at time T.

3. By backward induction from i=T-1 to i=0

    a) Estimate the $\mathrm{E[B_{i+1}(S_i, N)|Z_i]}$ using linear regression with the set of all ITM paths and N represents the number of rights remaining.

b) Execute equations (2.1) and (2.2) to decide whether hold the right or exercise one right and jump to another tree with one right less left,

c) This results in an estimator of the optimal exercise policy corresponding to each value for the number of exercise rights remaining and, similar to the case for American style options, results in a high-biased price estimator.

Note that here time steps are equal to $\Delta t = \frac{\mathrm{T}}{\text{Number of exercise opportunities}}$.

4. The low biased estimator of Monte Carlo simulation for tree N rights left is calculated using the estimated policy from above with another set of M independently simulated sample paths.

Obviously constructing the jungle of trees starts from making the tree with 1 possible right (call or put right) because in (2.2) price of the tree with one less right is needed always. Note that

$$\mathrm{H_i}(\mathrm{S_i}, 0) = 0. \tag{2.8}$$

All of the above calculations can be repeated independently and each repetition results in one price. These prices can be averaged and the standard deviation computed to yield a confidence interval for the price.

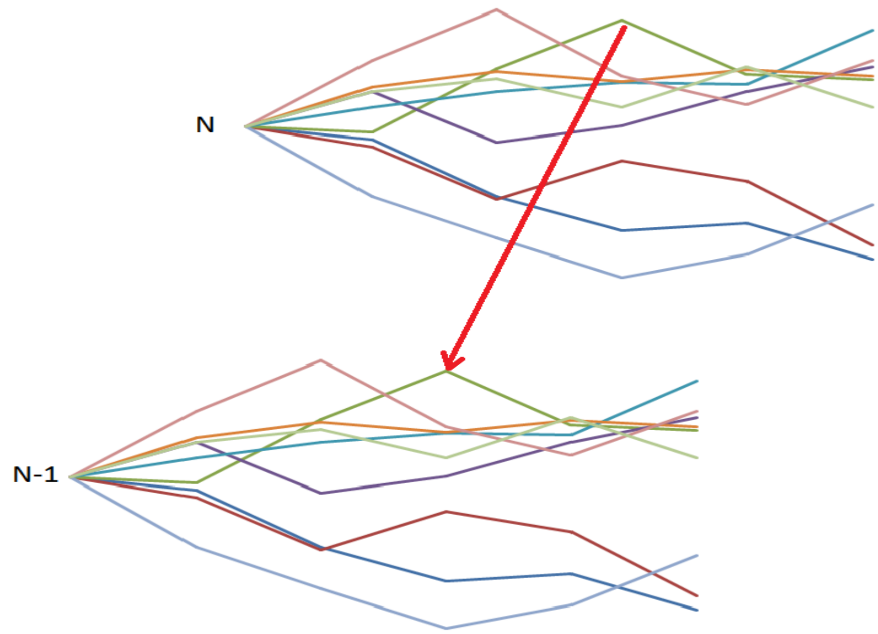


Figure 2.2 Section of a forest of trees with N and N-1 number of rights remaining

Figure 2.2 is a schematic diagram in a forest that illustrates the node in the tree with N remaining rights which decides to exercise one right and move to the other tree with N-1 rights left.

2.4 Detailed Description of ICLS for Multiple Exercise Options

In this section the inequality constrained least squares Monte Carlo (ICLS) method for multiple exercise options is discussed. As reviewed before, the Longstaff and Schwartz method uses regression to obtain an estimate for the hold value of option in the next time step. Letourneau and Stentoft [42] suggested imposing structure in the regression part of the method leading to more accurate prices. Dynamic programming valuation of multiple exercise option using ICLS is similar to LSMC but the regression part is different.

In the valuation process of a multiple exercise option having both call and put rights, there are 3 possibilities at each time step for any Monte Carlo path;

I. Exercise one call right (assuming at least one call right remains and spot price is above strike price) then jump to another tree with one less call rights remaining.

II. Exercise one put right (assuming at least one put right remains and spot price is below strike price) then jump to another tree with one less put right remaining.

III. Do not exercise any rights and stay on the same tree.

Figure 2.3 shows a schematic plot of a call option. The curve of call option has some characteristics including positive convexity and slope between 0 and 1at all prices. Imposing this structure to the regression leads to a better estimation for option price at each time.

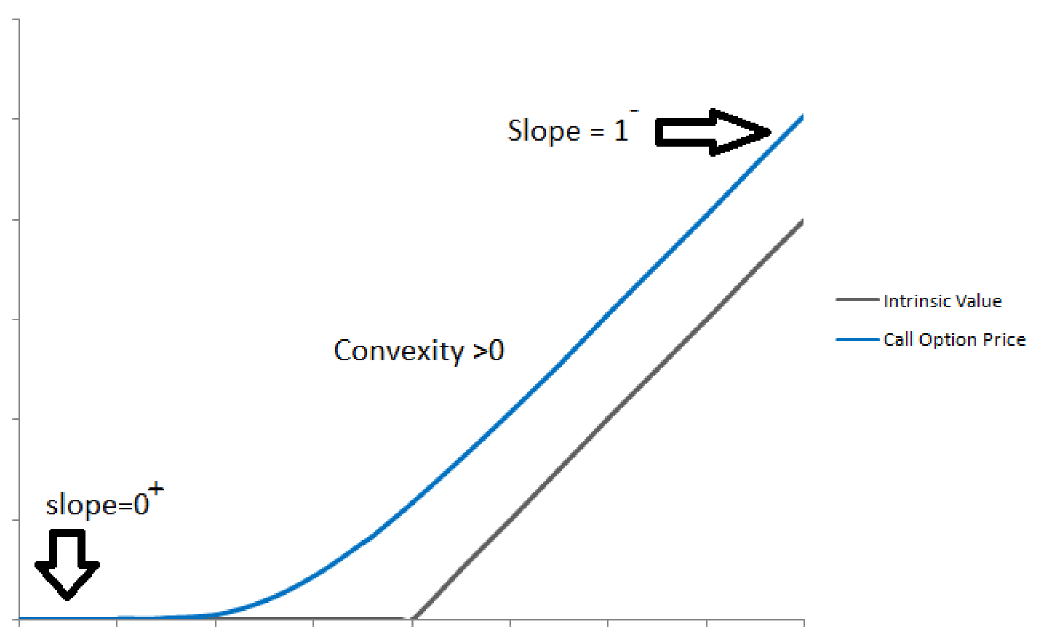


Figure 2.3 Schematic of constraints which ICLS applies to American call options

On the other hand, the curve of price for put option has positive convexity and slope between -1 and 0 at all prices. Figure 2.4 presents these constraints schematically.

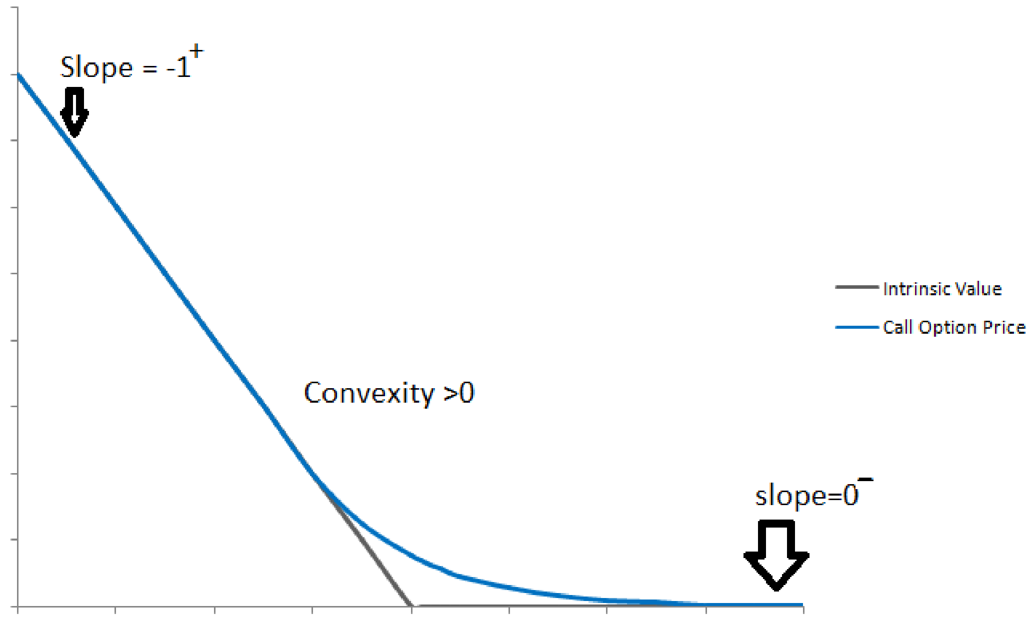


Figure 2.4 Schematic of constraints which ICLS applies to American put options

When there are both call and put rights remaining, for each path there is possibility of exercising one put right or one call right, depends on the price related to that path. Consequently at each time step constraints displayed in Figure 2.3 would be applied to paths that are in the money for call rights and constraints displayed in Figure 2.4 would be applied to paths that are in the money for put rights. Assuming the strike prices is the same for both call and put rights implies that the sets of ITM paths for call rights and ITM paths for put rights are disjoint.

As with the ICLS method for American-style options, high- and low-biased estimators for multiple exercise options can be similarly constructed. Additionally, with independent repeated valuations, confidence intervals for the high- and low-biased estimators can easily be computed. Using the upper and lower confidence limits for the high- and low-biased estimators, respectively, a conservative confidence interval for the true price is obtained.

Chapter 3

3 Pricing Multiple Exercise Options

In the previous chapters, we have provided an introduction to the ICLS and LSMC methods of pricing multiple exercise options. In this chapter, we use ICLS and LSMC methods to price some multiple exercise options using the forest of trees technique.

In the first section, we use our valuation method and compare results to those obtained in other studies to verify our methodology. In the second section, some numerical results and the effects of different parameters on the option value are discussed. The last section presents the processing time required for pricing along with root mean squared errors (RMSE).

The multiple exercise options can be exercised at discrete times up to expiry. In this thesis the volume choices given are constant and don't change. This means that the holder could exercise one of the rights at any time which is chosen from a limited list.

3.1 Verification with Binomial and Other Studies

This section presents verification of our pricing methodology by comparing with prices obtained from different methods.

Figure 3.1and Figure 3.2 show the price of a multiple exercise option using regression polynomials order of 2 and 6 respectively. Figure 3.1 illustrates the effect of sample size on the high and low biased estimators of both ICLS and LSMC when polynomial of second order is chosen as the basis functions for regression. As predicted, increasing the sample size from 20 to 10000 paths leads the high and low biased estimators to converge to the price of binomial method. Prices shown are the average of 100 repetitions of the given sample size. The option has 1 put right and 1 call right. Option prices are computed using the out-of-sample and in-sample LSMC and ICLS methods, ATM options are priced for maturities of 3 years with yearly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, σ=20%, K=\$40 and S_0=\$40, number of paths = 1000. All options are priced using polynomial of order 2 and the regressions are done using the paths that are ITM at the current time step. Both ICLS and LSMC converge to 10.08 which is in agreement with Marshall[54].

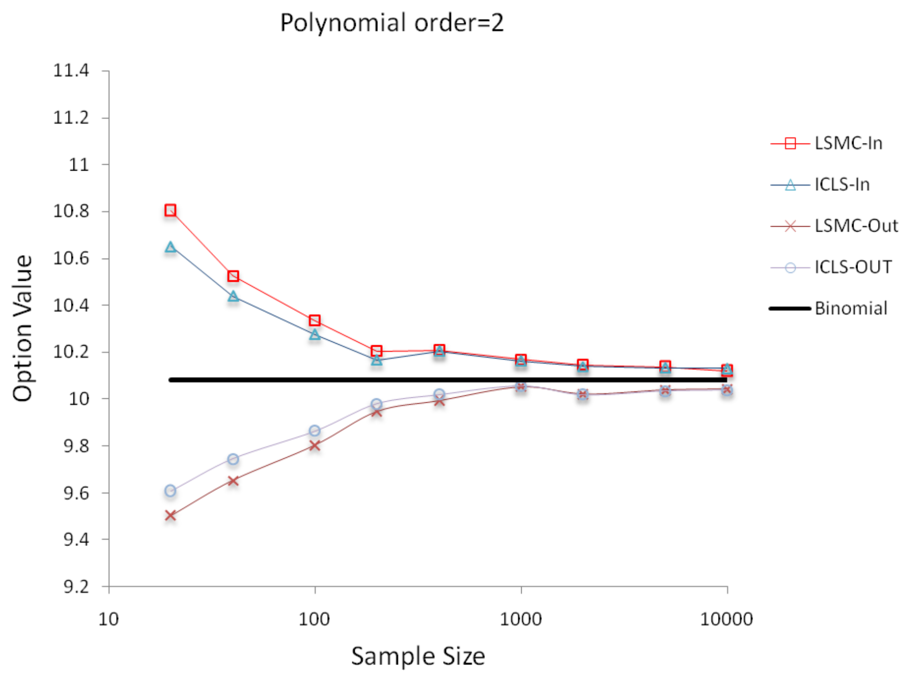


Figure 3.1 Multiple exercise option price versus sample size. The option has 1 put right and 1 call right. Prices are computed using the out-of-sample and in-sample LSMC and ICLS methods, ATM options are priced for maturities of 3 years with yearly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, $\sigma = 20\%$, K=$40 and S_0=$40. All options are priced using polynomial of order 2 and the regressions are done using the paths that are ITM at the current time step. The mean prices of 100 repetitions are shown and the benchmark prices are obtained with the binomial model.

Figure 3.1 shows that when sample size is small using ICLS improves the estimation giving in- and out-of-sample estimators that are closer to the true price. By increasing the sample size, both LSMC and ICLS methods converge to the binomial method and the differences between the estimators vanishes.

Figure 3.2 presents the pricing of the same instrument as in Figure 3.1 but using a regression polynomial of order 6. Figure 3.2 shows that in LSMC method, increasing the polynomial order hurts the approximation a result over fitting. On the other hand, in the

ICLS method increasing the polynomial order leads to tightening the spread between high-biased and low-biased estimators then ICLS does not suffer from over fitting.

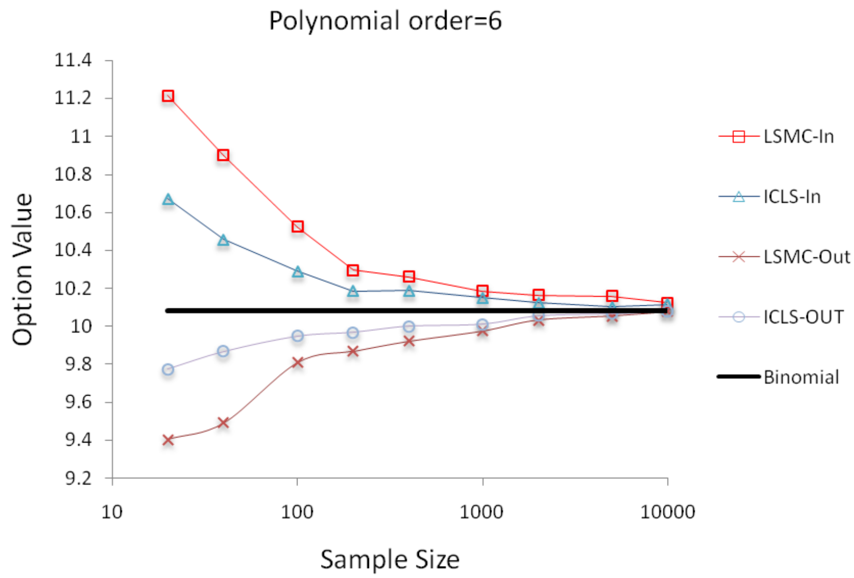


Figure 3.2 Multiple exercise option price versus sample size. The option has 1 put right and 1 call right. Prices are computed using the out-of-sample and in-sample LSMC and ICLS methods, ATM options are priced for maturities of 3 years with yearly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, $\sigma = 20\%$, K=$40 and S_0=$40. All options are priced using polynomial of order 6 and the regressions are done using the paths that are ITM at the current time step. The mean prices of 100 repetitions are shown and the benchmark prices are obtained with the binomial model.

3.2 Numerical Results

In this section additional numerical results are presented, mostly focused on the comparison of LSMC and ICLS methods under parameter settings such as number of rights, number of exercise opportunities and volatility of underlying asset.

Figure 3.3 a) and b) presents multiple exercise option and relative values respectively, compared to a basket of American put options for 1 to 5 put rights using the out-of-sample and in-sample LSMC and ICLS methods. ATM options are priced for maturities of 1 year with weekly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 5%, σ=20%, K=$40, S_0=$40 and number of paths = 1000. All options are priced using polynomial of order 2 and the regressions are done using the paths that are ITM at the current time step. The mean prices of 100 repetitions are shown and the benchmark prices are obtained with the binomial model. Obviously increasing the number of rights widens the range between high-biased and low-biased estimators.

a) b)

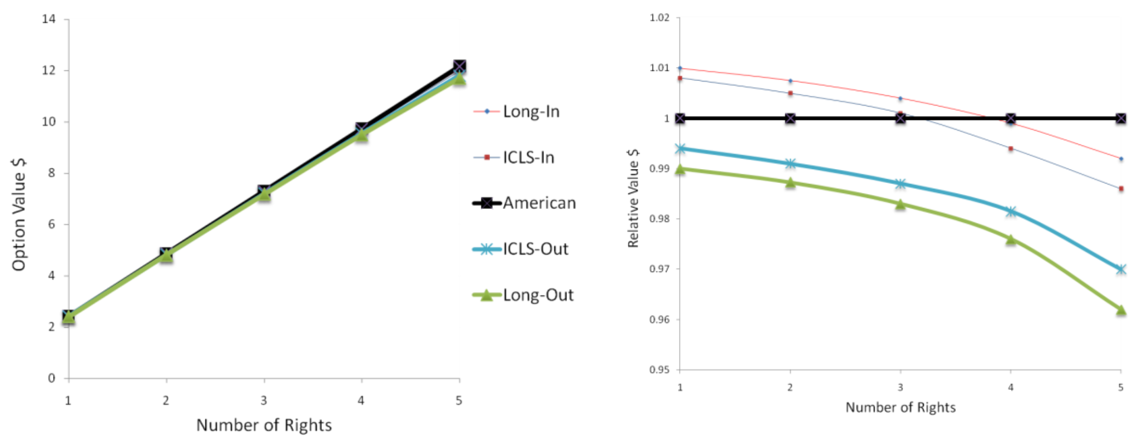


Figure 3.3 Multiple exercise option pricing including 1 to 5 put rights using the out-of-sample and in-sample LSMC and ICLS methods, ATM options are priced for maturities of 1 year with weekly exercise opportunities a) option value b) relative value of option compare to basket of American put options. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 5%, $\sigma = 20\%$, K=$40, S_0=$40 and number of paths = 1000. All options are priced using polynomial of order 2 and the regressions are done using the paths that are ITM at the current time step. The mean prices of 100 repetitions are shown and the benchmark prices are obtained with the binomial model.

Figure 3.4 presents the effect of increasing number of exercise opportunities and compares the LSMC and ICLS methods. This figure presents pricing of multiple exercise option with 5 put rights using the out-of-sample and in-sample LSMC and ICLS methods, ATM options are priced for maturities of 1 year with different number of exercise opportunities from 10 to 100. The underlying asset follows geometric Brownian motion with no dividend, r= 5%, $\sigma = 20\%$, K=$40, S_0=$40, and number of paths 1000 and 100 respectively. All options are priced using polynomial of order 2 and the regressions are done using the paths that are ITM at the current time step. The mean prices of 100 repetitions are shown. The price of both methods increases monotonically although for large number of exercise opportunities the curves tend to get flat and the effect of changing the number of exercise opportunities diminishes. Interestingly for low exercise opportunities ICLS and LSMC methods are closer and the more exercise opportunities brings in more chance of choosing not optimal decision by LSMC, so the difference of these methods increases by exercise opportunities.

a) b)

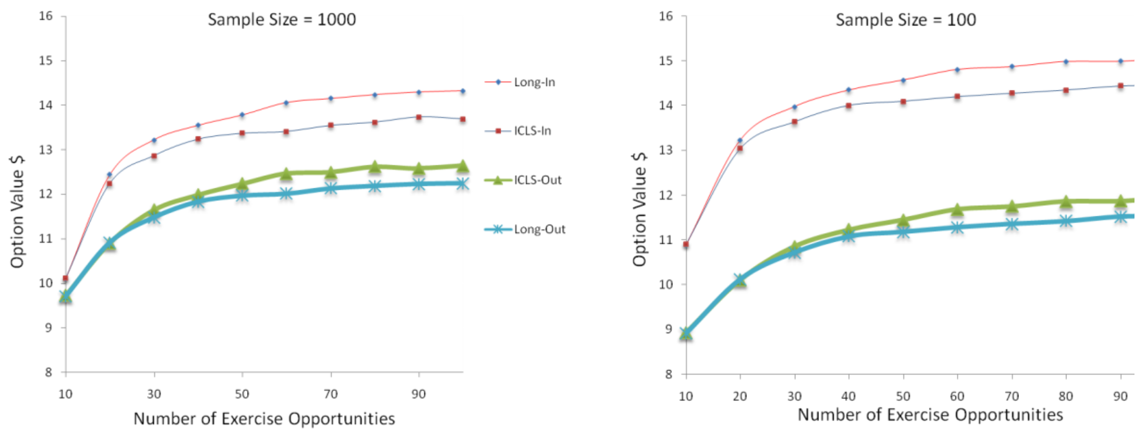


Figure 3.4 Multiple exercise option pricing 5 put rights using the out-of-sample and in-sample LSMC and ICLS methods. ATM options are priced for maturities of 1 year with different number of exercise opportunities from 10 to 100. The underlying asset follows a geometric Brownian motion with no dividend, r= 5%, $\sigma = 20\%$, K=$40, S_0=$40 and number of paths a) 1000 and b) 100. All options are priced using polynomial of order 2 and the regressions are done using the paths that are ITM at the current time step. The mean prices of 100 repetitions are shown.

The main difference of LSMC and ICLS methods is the regression of the estimated value of the option which LSMC uses regular regression but ICLS uses constrained regression, as explained before. The effect of moneyness on the fitted regression value is displayed in Figure 3.5 to Figure 3.8. Additionally, the impact of the polynomial order on constrained versus unconstrained regression and the number of exercise opportunities is also displayed in these figures. The curves illustrate the regions that holding the option is beneficial. Wherever the fitted regression value is above the intrinsic value, one should hold the option but if the fitted regression value is less than the intrinsic value, one should exercise the option because the payoff is more than the discounted expected value of the option at the next time step.

Figure 3.5 illustrates the influence of moneyness and time to maturity parameters on the fitted regression values for LSMC method using polynomial of order 2 as the set of basis functions. In Figure 3.5 the prices of multiple exercise options including a) 1 put right and b) 5 put rights using LSMC method are presented. ATM options are priced for maturities of 1 year. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 6%, $\sigma = 20\%$, K=$40, S_0=$40 and number of paths = 1000. All options are priced using polynomial of order 2 and the regressions are done using the ITM paths at the current time step. Regression values are presented for different remaining time to maturity. Note that moneyness is the relative position of the current price of the underlying asset with respect to the strike price of the option which is $(K - S)^+$ for a put option.

Options with longer time to maturity are farther from the exercise boundary than the near-to-expiry options when the moneyness is low. That means when the moneyness is low and you are close to expiry, there is low possibility of getting higher payoff by holding the option, compared to the case when the option is farther away from expiry. Figure 3.5 illustrates that when moneyness is low, better to hold the option. Similarly, when time-to-maturity is considerable, there is no reason to exercise the option and again should keep the right. When the time is passed enough, there is a middle region which LSMC method recommends to exercise the right. By increasing the moneyness, LSMC would imply an incorrect exercise decision which is the major drawback for this method. For instance, in Figure 3.5 at time step 30 LSMC methods implies no exercise for moneyness more than $13 and only suggests exercising the right between $7 and $13.

a) b)

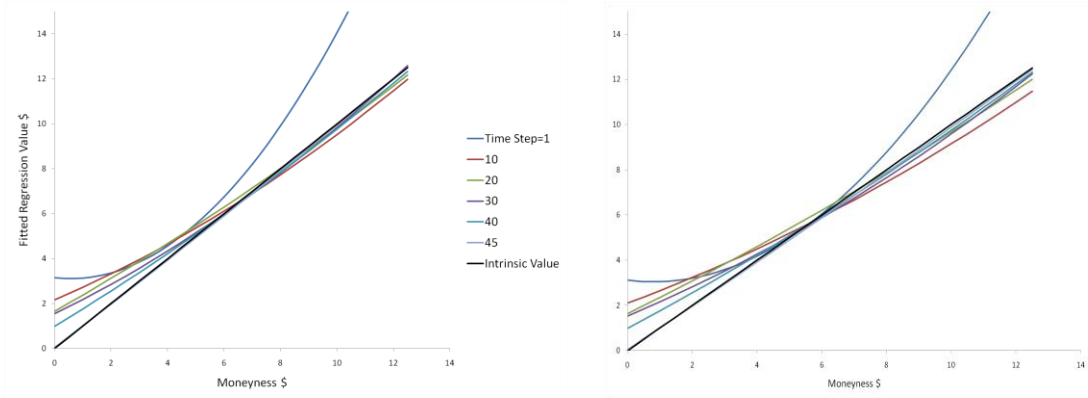


Figure 3.5 Multiple exercise option pricing including a) 1 put right b) 5 put rights using LSMC methods. ATM options are priced for maturities of 1 year. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 6%, $\sigma = 20\%$, K=\$40, S_0=\$40 and number of paths = 1000. All options are priced using polynomial of order 2 and the regressions are done using the paths are ITM at the current time step. Regression values are presented for different remaining time to maturity.

Figure 3.6 presents multiple exercise option pricing including a) 1 put right and b) 5 put rights using ICLS methods. ATM options are priced for maturities of 1 year. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 6%, σ=20%, K=\$40, S_0=\$40 and number of paths = 1000. All options are priced using polynomial of order 2 and the regressions are done using the ITM paths at the current time step. Regression values are presented for different remaining time to maturity. Fitted regression values of ICLS method are displayed in Figure 3.6 for polynomial of order 2. Comparison of Figure 3.6 against Figure 3.5 shows that ICLS fixes the drawback of LSMC and for large moneyness (after enough time steps), exercising the option is recommended because fitted regression values lie under the intrinsic value.

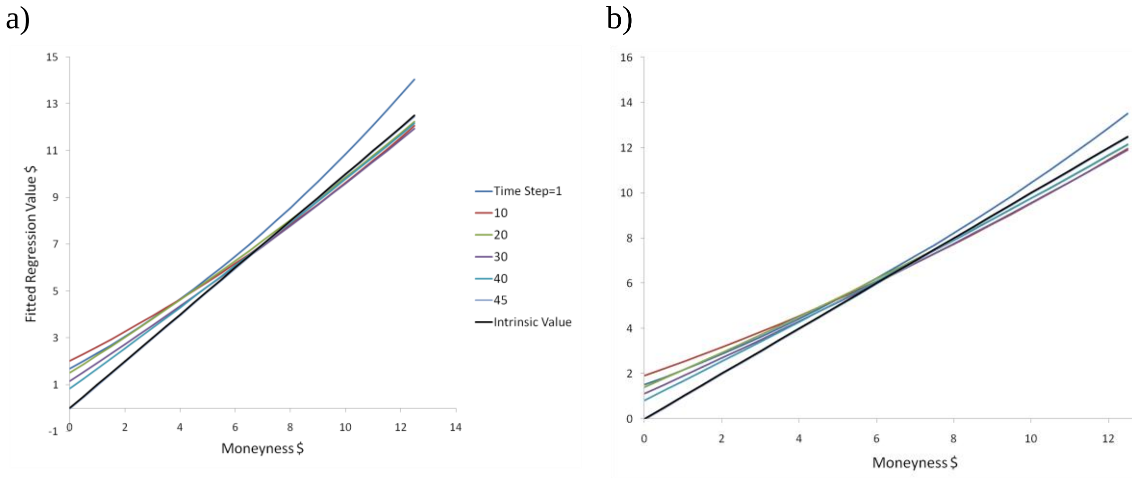


Figure 3.6 Multiple exercise option pricing including a) 1 put right b) 5 put rights using ICLS methods, ATM options are priced for maturities of 1 year. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 6%, $\sigma = 20\%$, K=$40, S_0=$40 and number of paths = 1000. All options are priced using polynomial of order 2 and the regressions are done using the paths are ITM at the current time step. Regression values are presented for different remaining time to maturity.

Figure 3.7 presents multiple exercise option pricing of a) 1 put right and b) 5 put rights using LSMC method. ATM options are priced for maturities of 1 year. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 6%, $\sigma = 20\%$, K=$40, S_0=$40 and number of paths = 1000. All options are priced using polynomial of order 6 and the regressions are done using the ITM paths at the current time step. Regression values are presented for different remaining time to maturity. Figure 3.7 presents the same curves as Figure 3.5 but for polynomial of order 6. The fitted regression values for t=1 is not logical which is because simulated paths are not enough diffused and most of the paths have small moneyness. Still for all moneyness at t=10 and large moneyness at t=20, LSMC implies to hold the option which are incorrect exercise decisions.

Note that fitted values for small times highly depend on the generated random paths and could change dramatically if the seed of the random number generator changes.

a) b)

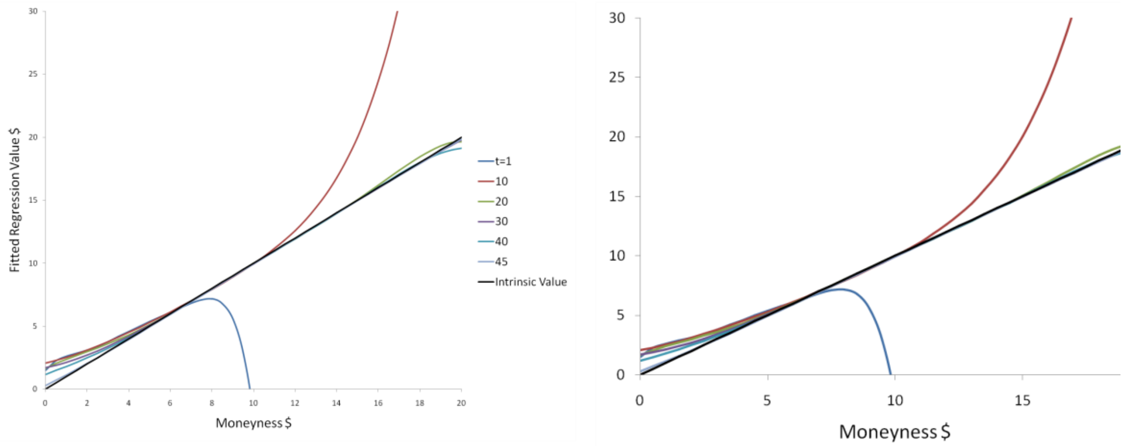


Figure 3.7 Multiple exercise option pricing including a) 1 put right b) 5 put rights using LSMC method. ATM options are priced for maturities of 1 year. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 6%, $\boldsymbol{\sigma = 20\%}$, K=\$40, S_0=\$40 and number of paths = 1000. All options are priced using polynomial of order 6 and the regressions are done using the ITM paths at the current time step. Regression values are presented for different remaining time to maturity.

On the other hand ICLS implies correct and consistent exercise decisions if only paths are enough diffused. Figure 3.8 presents the fitted regression curves for polynomial order 6 ICLS method. In this case the exercise boundary moves toward the lower moneyness when the time passes. For example at times 20 and 45, ICSL implies exercising the option if moneyness is larger than \$7 and \$2.5 respectively.

In Figure 3.7, the fitted regression values are not increasing, convex functions of moneyness, ICLS fixes this and the effect is clearly shown in Figure 3.8. Note that blue curves in Figure 3.8 are not monotonic when moneyness is close to zero because no sample path is in that area in the first time step.

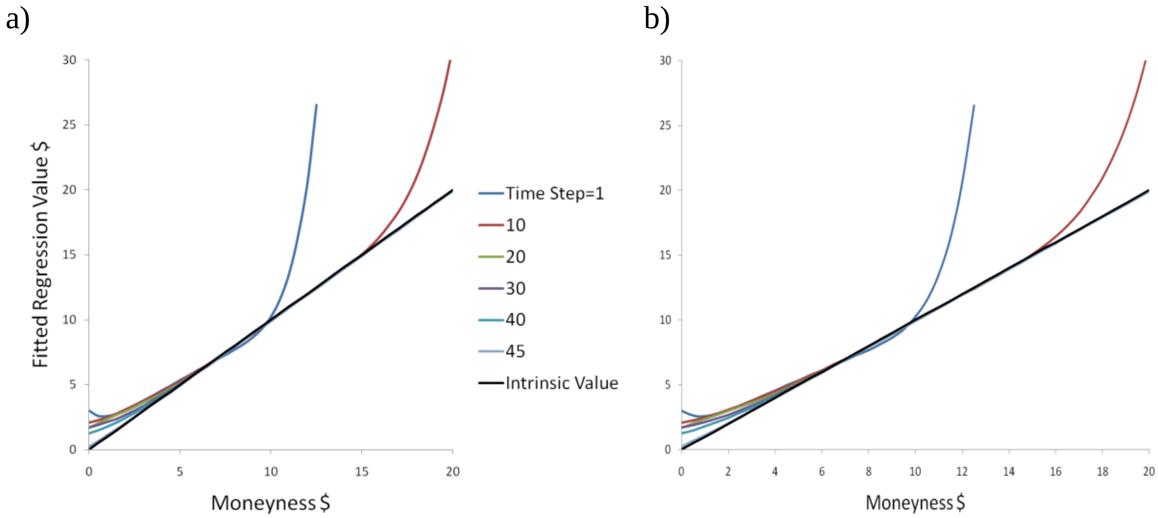


Figure 3.8 Multiple exercise option pricing including a) 1 put right b) 5 put rights using ICLS method. ATM options are priced for maturities of 1 year. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 6%, $\sigma = 20\%$, K=$40, S_0=$40 and number of paths = 1000. All options are priced using polynomial of order 6 and the regressions are done using the ITM paths at the current time step. Regression values are presented for different remaining time to maturity.

3.3 Processing Time and RMSE

This section discusses the processing time and root mean square error of the examples presented in Sections 3.1 and 3.2. All simulations in this section were completed on the same computer with Intel Core i7-6700 and 3.4 GHz processors.

Equation (1.8) simply explains that the standard error monotonically decreases for higher sample paths of Monte Carlo simulation, subsequently equation (1.9) indicates for very large number of sample paths, confidence interval of estimation reaches the exact solution.

Figure 3.9 and Figure 3.10 present the root mean squared error of option values showed in Figure 3.1 and Figure 3.2 respectively. These figures present the root mean squared error (RMSE) of multiple exercise option pricing including 1 put right and 1 call right using the out-of-sample and in-sample LSMC and ICLS methods. ATM options are priced for maturities of 3 years with weekly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, $\sigma = 20\%$, K=$40 and S_0=$40. The number of paths varies from 20 to10^5. All options are priced using polynomial of order 2 and the regressions are done using the ITM paths at the current time step. As expected increasing the sample size decreases the RMSE for both ICLS and LSMC methods, both in-sample and out-of-sample estimators and any polynomial order of fitted regression function. Apparently the polynomial order of the fitted regression does not affect the RMSE but interestingly, increasing the sample size from 1000 to 10^4 (10 times larger), diminishes the RMSE by half.

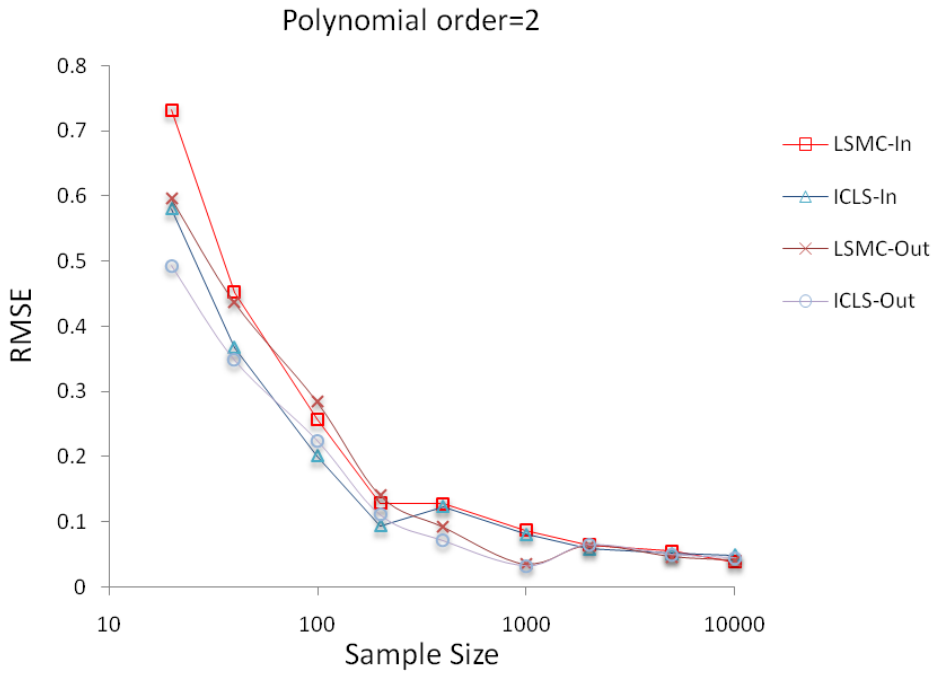


Figure 3.9 Root mean squared error of multiple exercise option pricing including 1 put right and 1 call right using the out-of-sample and in-sample with LSMC and ICLS methods. ATM options are priced for maturities of 3 years with weekly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, $\sigma = 20\%$, K=$40 and S_0=$40. All options are priced using polynomial of order 2 and the regressions are done using the ITM paths at the current time step.

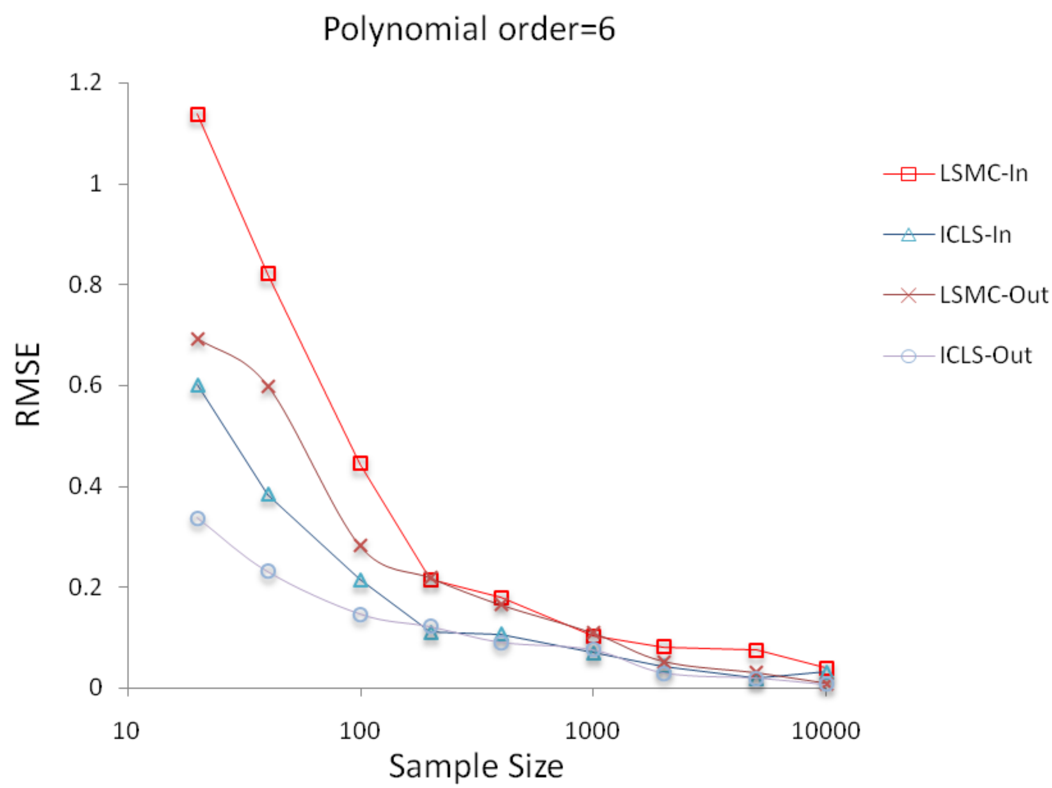


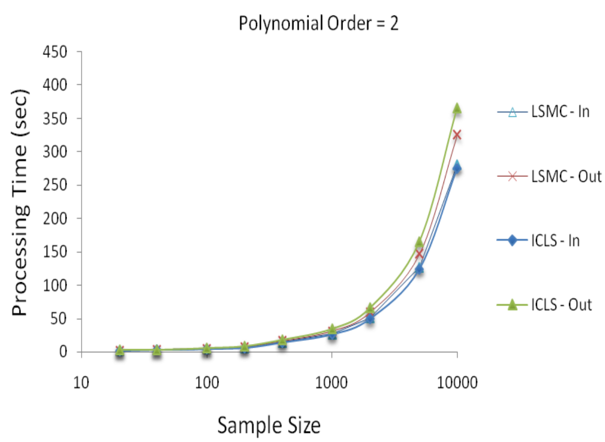
Figure 3.10 Root mean squared error of multiple exercise option pricing including 1 put right and 1 call right using the out-of-sample and in-sample LSMC and ICLS methods. ATM options are priced for maturities of 3 years with weekly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, $\sigma = 20\%$, K=$40 and S_0=$40. All options are priced using polynomial of order 6 and the regressions are done using the ITM paths at the current time step.

Obviously increasing the number of sample paths size is a tradeoff between RMSE and processing time. Although increasing the sample size decreases the RMSE (which is favorable), it also increases the processing time (which is not a favorable event).

Referring to Figure 3.2, estimator precision (bias) depends on the sample size. On the other hand standard error can be controlled through independent repeated valuations. Therefore we can fix bias by choosing a sample size and then control the standard error by doing independent repeated valuations.

Figure 3.11 presents the processing time of multiple exercise option pricing including 1 put right and 1 call right using the out-of-sample and in-sample LSMC and ICLS methods. ATM options are priced for maturities of 3 years with weekly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, σ=20%, K=$40 and S_0=$40. All options are priced as the mean of 100 repetitions using a) polynomial of order 2 and b) polynomial of order 6, while the regressions are done using the ITM paths at the current time step. Figure 3.11 illustrates that the increase of polynomial order slightly increases the processing time but sample size has a significant effect on the processing time. Increasing the sample size from 1000 to 10000 (10 times larger) leads to a processing time with roughly 13 times slower and half RMSE (see Figure 3.9 and Figure 3.10).

a) b)



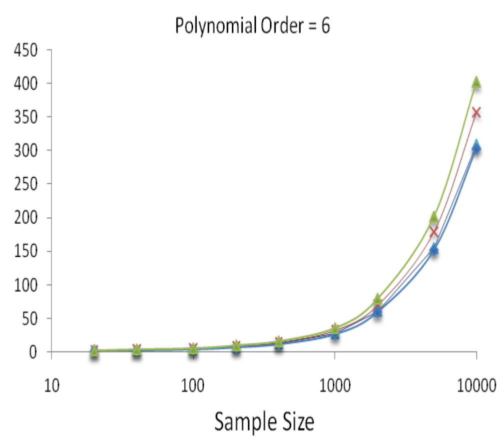


Figure 3.11 Processing time of multiple exercise option pricing including 1 put right and 1 call right using the out-of-sample and in-sample LSMC and ICLS methods. ATM options are priced for maturities of 3 years with weekly exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, $\sigma = 20\%$, K=$40 and S_0=$40. All options are priced a) polynomial of order 2, b) polynomial of order 6; while the regressions are done using the ITM paths at the current time step.

Figure 3.12 presents the processing time of multiple exercise option pricing including 5 call rights using the out-of-sample and in-sample LSMC and ICLS methods. ATM options are priced for maturities of 3 years with 10 to 100 exercise opportunities. The underlying asset follows a geometric Brownian motion with dividend=10%, r= 5%, σ=20%, K=$40 and S_0=$40. All options are priced as the mean of 100 repetitions using a) polynomial of order 2 and b) polynomial of order 6, while the regressions are done using ITM paths at the current time step. Figure 3.12 illustrates that processing time for multiple exercise opportunity including 5 call rights, increases exponentially with the number of exercise opportunities.

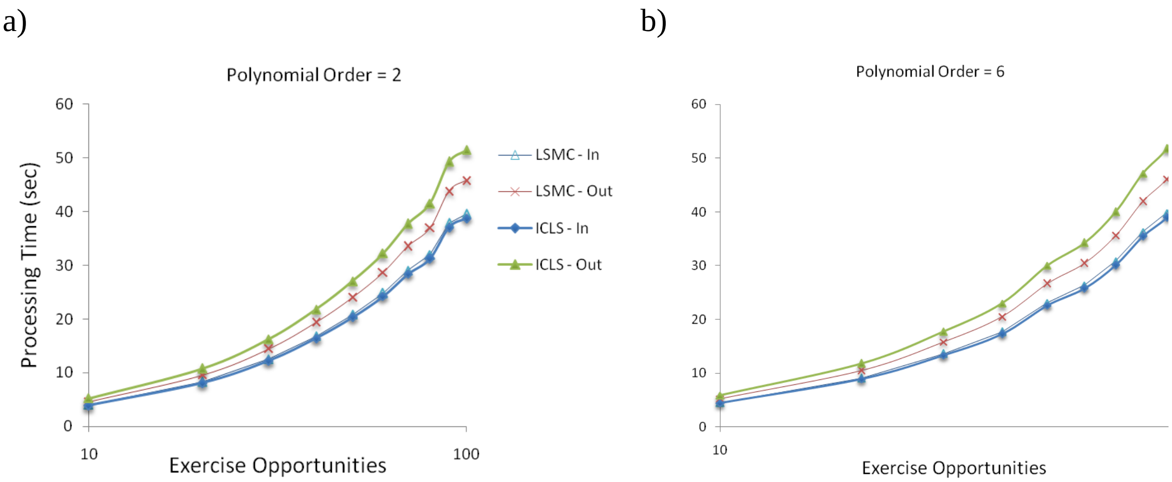


Figure 3.12 Processing time of multiple exercise option pricing including 5 call rights using the out-of-sample and in-sample LSMC and ICLS methods. ATM options are priced for maturities of 1 year with number of exercise opportunities from 10 to 100. The underlying asset follows a geometric Brownian motion with dividend=0%, r= 5%, $\sigma = 20\%$, K=$40, S_0=$40 and number of paths = 1000. All options are priced a) polynomial of order 2, b) polynomial of order 6, while the regressions are done using the ITM paths at the current time step.

Parallel processing uses multiple processors to divide large problems into smaller ones that are worked on in parallel to save time. The current problem is inherently suitable for applying parallel processing because each processor can perform an independent valuation. This can be as straightforward as doing serial farming of the independent repeated valuations. Few communications are needed between the processors as only the parameter setting at the beginning and the valuation results at the end need to be communicated.

As example, if we engage 64 processors, 64 independent valuations of an option can be performed in parallel by sending each processor a single valuation, have them work in parallel. Excluding the very tiny processes that should occur at the end to calculate the mean of all valuations, the computational time for the 64 repeated valuations would be the same as the computational time for a single valuation using a single processor.

3.4 Conclusion

This study employed Inequality Constrained Least Squares Monte Carlo (ICLS) method developed by Letourneau and Stentoft [42]. This is least squares Monte Carlo with inequality constraints for regression to price multiple exercise options. We numerically compared the results from ICLS to the LSMC for multiple exercise options and showed that imposing structure to the regression reduces estimator bias.

The number of regressors is one important choice in both ICLS and LSMC methods. Increasing the number of regressors in LSMC leads to overfitting especially when the sample size is low. Unlike LSMC, constraints in the ICLS method prevent overfitting which leads to smaller estimator bias. We showed that to obtain the same bias for these methods, LSMC should use a sample size 10 times larger compared to ICLS which increases the processing time 13 times compare to ICLS.

Pricing multiple exercise options is a computationally intensive problem and consequently takes considerable processing time compared to single-exercise options. Valuation methods used in this thesis are adaptable to the parallel processing technique because many independent valuations could be performed with different processors requiring minimum communication. As example, using 64 processors in parallel makes the processing time almost 64 times faster.

Future work on this problem is to extend methodology presented here to allow for a multi-dimensional underlying. This extension has been explored by Letourneau and Stentoft [42] for the case of American style option (single exercise). Another potential avenue for future research is using independent sets of samples for each number of exercise rights. Imposing constraints across number of exercise rights could be another extension for this work. As Figure 3.3a shows the value of multiple exercise option increases monotonically by number of exercise rights with negative convexity.

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