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Syzygy Order of Big Polygon Spaces

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Abstract

For a compact smooth manifold with a torus action, its *equivariant cohomology* is a finitely generated module over a polynomial ring encoding information about the space and the action. To such a module, we can associate a purely algebraic notion called *syzygy order*. The syzygy order of equivariant cohomology is closely related to the exactness of *Atiyah-Bredon sequence* in equivariant cohomology. In this thesis we study a family of compact orientable manifolds with torus actions called *big polygon spaces*. We compute the syzygy orders of their equivariant cohomologies. The main tool used is a *quotient criterion for syzygies in equivariant cohomology*. We also generalize a *lacunary principle* for *Morse-Bott functions* to *manifolds with corners* in the process of computation. Some applications of the main result are discussed in the end.

Keywords: Big polygon spaces, equivariant cohomology, syzygy, quotient criterion, manifolds with corners, Morse theory

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Contents

Chapter 1

Introduction

The subject of transformation groups is concerned with studying symmetry of spaces. One particularly interesting aspect of it is the study of manifolds with compact Lie group action. These spaces appear everywhere in geometry, topology and representation theory. For example, the action of classical compact Lie groups like orthogonal groups and unitary groups on vector spaces is an important subject of representation theory. In general, a smooth manifold X with a smooth action of a compact Lie group G is called a *G-manifold*.

Equivariant cohomology was introduced into the study of compact Lie group actions on manifolds by Borel in [5, Chapter IV, §3]. For any compact Lie group G there is a contractible space EG on which G acts freely. The orbit space of EG is denoted BG and is called a classifying space of G. For a G-manifold X, the Borel construction of the ordinary cohomology of a constructed space X_G:

$$H^*(X_G) \tag{1.1}$$

where

$$X_G := (EG \times X)/G \tag{1.2}$$

and the group action on $EG \times X$ is defined as $g(e, x) = (eg^{-1}, gx)$. Since we will mainly be concerned with equivariant cohomology with real coefficients in this thesis, we will be taking cohomology with real coefficients everywhere.

The case when G is a finite group and X is a point is already very interesting. The equivariant cohomology recovers the group cohomology of the finite group G in this case. In general, the equivariant cohomology $H_G^*(X)$ is a graded $H^*(BG)$-module with module structure induced on equivariant cohomology by the following map

$$X_G = (EG \times X)/G \to EG/G = BG. \tag{1.3}$$

In general even the cohomology rings $H^*(BG)$ for a disconnected compact Lie group are unknown. However, the following reduction formula from [21, Chapter III, example 3] shows that we can first try to study equivariant cohomology for manifolds with actions of compact connected Lie groups:

$$H_G^*(X) \simeq H_{G^0}^*(X)^\Gamma \tag{1.4}$$

where G^0 is the identity component of G and $\Gamma = G/G^0$ is a finite group. The right hand side of the above equation is the fixed elements of the G^0-equivariant cohomology of X under a Γ-action where Γ acts as deck transformation of the covering map $X_{G^0} \to X_G$.

There is one more reduction one can do to equivariant cohomology after the above reduction. Let G be a compact connected Lie group. We have the following reduction formula [21, Chapter III, Proposition 1]:

$$H_G^*(X) \simeq H_T^*(X)^W \tag{1.5}$$

where T is a maximal torus of G which is a maximal connected abelian subgroup of G. The finite group W is the Weyl group $N(T)/T$ where $N(T)$ is the normalizer of T in G. Maximal tori and Weyl groups play a central role in the representation theory of compact Lie group and this reduction is analogous to the one in representation theory.

Formulas (1.4) and (1.5) suggest that to understand G-equivariant cohomology for any compact Lie group G we could first try to study the T-equivariant cohomology where T is a maximal torus of the identity component of G. Since the rest of this thesis is all about T-equivariant cohomology, we will restrict our interest to torus action on manifolds. We will also assume that the space X is compact from now on.

Let $T = (S^1)^r$ be a torus. The classifying space of T can be taken to be $(\mathbb{C}P^\infty)^r$ and thus we have

$$H^*(BT) = H^*((\mathbb{C}P^\infty)^r) \simeq \mathbb{R}[t_1, \ldots, t_r] \tag{1.6}$$

where every $t_i \in H^2(BT)$ is the pullback of the first Chern class of the universal complex line bundle over $\mathbb{C}P^\infty$ via the projection map $(\mathbb{C}P^\infty)^r \to \mathbb{C}P^\infty$ onto the ith coordinate. This ring $H^*(BT)$ will be denoted R.

For a T-manifold X, we have seen that $H_T^*(X)$ is a graded module over the polynomial ring R. One natural task is to study the case when $H_T^*(X)$ is free over R. This freeness condition is equivalent to the condition of equivariant formality proposed in [19]. A neat proof of this equivalence can be found at [18, Proposition 2.2].

There are various results under the assumption that $H_T^*(X)$ is a free R-module. Among them, the most interesting results for us can be dated back to the work of Chang-Skjelbred [9], Atiyah [3] and Bredon [6]. Let X_i denote the set of points of X that are fixed by a codimension-i subtorus of T. Note that X_0 is the set of fixed points. Chang and Skjelbred showed that if $H_T^*(X)$ is free over R, then the following sequence is exact:

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0). \tag{1.7}$$

The nontrivial maps in the above sequence are respectively map induced by the inclusion of fixed point set and connecting homomorphism in equivariant cohomology for the pair (X_1, X_0).

Roughly at the same time, under the same freeness condition, Atiyah and Bredon proved the exactness of the following more general sequence called Atiyah-Bredon sequence which will be described in detail in (3.2.4)

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \to \cdots \to H_T^{*+r}(X_r, X_{r-1}) \to 0. \tag{1.8}$$

The first four terms and maps between them are exactly the same as in the Chang-Skjelbred sequence and the rest of the maps are connecting homomorphisms in equivariant cohomology associated to various triples (X_{i+1}, X_i, X_{i-1}).

It turns out that the exactness of the whole Atiyah-Bredon sequence is rather too strong a condition in most applications. For example, GKM theory proposed in [19, Theorem 1.2.2] only requires the exactness of the Chang-Skjelbred sequence and that is only the first several terms of the Atiyah-Bredon sequence. More precisely, under additional assumptions on the torus action, the GKM theory makes use of the exact sequence in (1.7) and compute the equivariant cohomology as the kernel of the map $H_T^*(X_0) \to H_T^{*+1}(X_1, X_0)$ which requires only information on fixed point set and 1-dimensional orbit. That already allows efficient computation of the equivariant cohomology.

Based on these observations, a question was raised by Allday, Franz and Puppe in [1, Introduction]: Under what condition is the Atiyah-Bredon sequence exact from the left up to the i-th position?

They solved this question in [1, Theorem 1.1] by introducing a purely algebraic notion called syzygy into the study of equivariant cohomology. The notion of syzygy was originally introduced by Hilbert to study ideals of a polynomial ring. A finitely generated module M over the polynomial ring $R = k[x_1, \ldots, x_r]$ is defined to be an *n-th syzygy* if there is an exact sequence

$$0 \to M \to F_n \to \cdots \to F_1 \tag{1.9}$$

where F_i's are finitely generated free R-modules. The largest integer n such that M is an n-th syzygy is called the *syzygy order* of M. We will see later that being a certain kind of syzygy implies certain kind of torsionfreeness. For example, the famous Hilbert syzygy theorem implies that being an r-th syzygy is equivalent to being free.

The theorem of Allday, Franz and Puppe states that the exactness of the Atiyah-Bredon sequence below a certain position is equivalent to the condition that the equivariant cohomology $H_T^*(X)$ is a certain kind of syzygy. This theorem will be described in Theorem 3.2.2.

While their theorem makes it a purely algebraic criterion for exactness of the Atiyah-Bredon sequence in equivariant cohomology, it raises new questions on computation of syzygy order of the equivariant cohomology. The main goal of this thesis is to compute the syzygy order of equivariant cohomology for the following family of T-spaces.

Definition 1.1 ([16, (1.4)(1.5)])**.** A vector $l = (l_1, l_2, \ldots, l_r) \in \mathbb{R}^r$ is *generic* if it cannot be split into two groups of equal sum. For a generic l and integers $a, b \geq 1$, a *big polygon space* denoted $X_{a,b}(l)$ is the subspace of $\mathbb{C}^{(a+b)r}$ defined by the following equations:

$$\begin{aligned}(u_1, \ldots, u_r, z_1, \ldots, z_r) &\in \mathbb{C}^{(a+b)r} \text{ where} \\ u_j \in \mathbb{C}^a, z_j &\in \mathbb{C}^b \quad (1 \leq j \leq r), \\ \|u_j\|^2 + \|z_j\|^2 &= 1, \\ l_1 u_1 + \cdots + l_r u_r &= 0.\end{aligned} \tag{1.10}$$

The torus $T = (S^1)^r$ acts on it by scalar multiplication on the variables z_j's as in [16, (1.6)],

$$(g_1, \ldots, g_r) \cdot (u_1, \ldots, u_r, z_1, \ldots, z_r) = (u_1, \ldots, u_r, g_1 z_1, \ldots, g_r z_r). \tag{1.11}$$

It was proved in [16, Lemma 2.1(i)] that if l is generic, then the space $X_{a,b}(l)$ is an orientable compact connected T-manifold. The equivariant cohomology and its syzygy order are also studied in [16].

The space defined in Definition 1.1 is called a big polygon space because the fixed point set of it consists exactly of a so-called space of polygons $E_{2a}(l)$ defined in [13] as follows. Given a generic length vector $l = (l_1, \ldots, l_r)$ as in Definition 1.1 and an integer $d \geq 1$, a space of polygons $E_d(l)$ is the space of all closed n-gons (allowing self-intersection) in $\mathbb{R}^d$ with sides of lengths $|l_i|$'s up to translation in $\mathbb{R}^d$. One can think of a space of polygons as the space of all configurations of loops of r linked robot arms with lengths given by the length vector $l = (l_1, \ldots, l_r)$ starting at the origin point $0 \in \mathbb{R}^d$. These spaces are themselves interesting because they appear in the study of configuration spaces of mechanical linkages. More intuition and details can be found in [14].

The significance of big polygon spaces in equivariant cohomology lies in the following corollary of the theorem of Allday-Franz-Puppe.

Corollary 1.2 ([1, Corollary 1.4])**.** *Let X be a compact orientable T-manifold. If $H_T^*(X)$ is a syzygy of order $\geq r/2$, then it is a free R-module.*

Franz showed in [16, Theorem 1.2] that if $r = 2m + 1$ and l is the length vector $(1, \ldots, 1)$, then the syzygy order of $H_T^*(X_{a,b}(l))$ is m and $H_T^*(X_{a,b}(l))$ is not a free R-module. This example shows that the lower bound in Corollary 1.2 is sharp.

The syzygy order of equivariant cohomology for a big polygon space $X_{a,b}(l)$ in general case was conjectured in [16, Conjecture 6.6] to be an integer depending only on the combinatorial property of the length vector l. To state the conjecture, we need to introduce this number $\mu(l)$.

Definition 1.3. Given a generic length vector $l = (l_1, \ldots, l_r)$, a subset I of $\{1, \ldots, r\}$ is called *short* if

$$\sum_{i \in I} l_i < \sum_{j \in I^{\complement}} l_j. \tag{1.12}$$

If the inequality above is reversed, then the subset I is called *long*. Two generic length vectors are called *equivalent* if they induce the same notion of long and short on subsets of $\{1, \ldots, r\}$. We define the following number for any subset $I \subseteq \{1, \ldots, r\}$:

$$\sigma_l(I) := \#\{j \in I : I - i \text{ short}\} \tag{1.13}$$

where # denotes the number of elements in the set. Then we can define

$$\mu(l) := \min\{\sigma_l(I) : I \text{ is long and } \sigma_l(I) > 0\}. \tag{1.14}$$

Our main goal in this thesis is to give a proof of the following conjecture in [16, Conjecture 6.6].

Main result. *Assume $a, b, r \geq 1$, then we have*

$$\operatorname{syzord} H_T^*(X_{a,b}(l)) = \mu(l) - 1. \tag{1.15}$$

This conjecture has been verified for $r \leq 9$ using computer with a complete list of nonequivalent length vectors $l \in \mathbb{R}^n$ for $n \leq 9$ in [20] that gives a complete list of equivariant diffeomorphism types of big polygon spaces when $r \leq 9$. Recently we also verified this conjecture for $r = 10$ with a complete list of nonequivalent length vectors $l \in \mathbb{R}^{10}$ provided by Dirk Schuetz. We will prove the conjecture for general r and l.

In general it is not easy to prove that the equivariant cohomology is of a certain kind of syzygy without computing the equivariant cohomology. Moreover, even if equivariant cohomology can be computed explicitly, it is still not clear how to compute the syzygy order. For example, the equivariant cohomology of big polygon spaces were computed explicitly in [16, Lemma 4.4, lemma 4.5, proposition 4.6] but the syzygy order of a big polygon space is still not known in general. Our result is the first attempt to compute the syzygy order of equivariant cohomology for a large family of compact orientable manifolds. To solve this problem, a new criterion for syzygy order of equivariant cohomology was proposed in [17] by Franz and will be used in this thesis. The advantage of this new criterion is that it allows computation of syzygy order of equivariant cohomology without actually computing the equivariant cohomology itself. This criterion is called a *quotient criterion for syzygies in equivariant cohomology*. Let us briefly introduce the main idea proposed in [17]. Details will be given in Section 3.3.

With some assumptions on the local behavior of the torus action T on manifold X called *locally standard* in Definition 3.3.1, the orbit space X/T has a structure of a smooth manifold with corners. A smooth manifold with corners of dimension n is locally modelled by open subsets of $(\mathbb{R}_+)^n$ where $\mathbb{R}_+$ is the positive real line including 0. Just like $(\mathbb{R}_+)^n$, a manifold with corners has a natural face structure. Based on this face structure, a cochain complex of real vector spaces $B^*(P)$ for every face P of the orbit space was defined in [17]. This cochain complex is constructed using homology of faces of the orbit space. The equivariant cohomology $H_T^*(X)$ is a certain kind of syzygy if and only if this new cochain complex is exact in some degrees for every face P. This is the main result of [17]. Details of the quotient criterion for syzygies in equivariant cohomology will be stated in Theorem 3.3.2.

We are going to apply this criterion to the T-manifold $X_{a,b}(l)$ and prove our main result.

As a first example, let us look at the face structure of the orbit space of one big polygon space $X = X_{1,1}((1,1,1))$, that is, $a = b = 1$ and length vector $l = (1,1,1)$. We will show later that the orbit space of X has the following face structure

$$\begin{array}{ccccc} & & X/T & & \\ & \diagup & | & \diagdown & \\ F_1 & & F_2 & & F_3 \\ | & \times & & \times & | \\ F_{12} & & F_{13} & & F_{23} \\ & \diagdown & | & \diagup & \\ & & F_{123} & & \end{array} \tag{1.16}$$

where F_I denotes the following subset of orbits in X/T for $I \subseteq \{1,2,3\}$:

$$\{(u_1, u_2, u_3, z_1, z_2, z_3) \in X : \|u_i\| = 1 \text{ for } i \in I\}/T. \tag{1.17}$$

In order to apply the quotient criterion, we have to compute homology of every face in (1.16) and maps between homology induced by this face structure.

In the process of computing the homology of faces of the orbit space, we need a *lacunary principle for Morse-Bott functions on manifolds with corners* which can be used to compute homology of a manifold with corners and thus is of independent interest.

This thesis is organized in the following way: In Chapter 2 we introduce syzygies and several equivalent notions. In Chapter 3 we review equivariant cohomology and the theorem of Allday-Franz-Puppe relating the exactness of the Atiyah-Bredon sequence in equivariant cohomology to the notion of syzygy in equivariant cohomology. In Chapter 3 we will also describe the quotient criterion mentioned above in detail. In Chapter 4 we describe some properties of the big polygon spaces. In Chapter 5 we introduce the machinery we will need to compute homology of manifolds with corners including a lacunary principle for Morse-Bott functions on manifolds with corners. The Chapter 6 is where we apply these tools to compute homology of all the faces of the orbit space $X_{a,b}(l)/T$. In Chapter 7 we state the main theorem and prove the main theorem. In Chapter 8 we include two applications of the result we proved.

Chapter 2

Syzygies

According to [12, Preface], the notion of syzygies originates in astronomy and was first introduced into mathematics by Sylvester. The word "syzygy" means the alignment of sun, earth and moon in astronomy. It was then introduced by Hilbert into the studies of graded free resolutions of graded modules over polynomial rings.

The main goal of this section is to review the notion of an n-th syzygy and several equivalent notions in commutative algebra. Throughout this section, we assume that R is the polynomial ring $k[t_1, \ldots, t_r]$ over a field k.

We first define the notion of a syzygy.

Definition 2.1. A finitely generated R-module M is called an *n-th syzygy* for some $n \geq 0$ if there is an exact sequence of R-modules

$$0 \to M \to F_n \to F_{n-1} \to \cdots \to F_1 \tag{2.1}$$

with F_i's being finitely generated free modules. The largest such n is called the *syzygy order* of M and is denoted $\operatorname{syzord} M$. If the above exact sequence can be infinite, then we set $\operatorname{syzord} M = \infty$.

Remark 2.2. The syzygy order of a finitely generated free module is ∞. For example, for

$M = R$, the following sequence is exact:

$$0 \to R \to R \to 0 \to 0 \to \cdots . \tag{2.2}$$

If we let N be the cokernel of the rightmost map in (2.1) and add it to the right end of (2.1), then (2.1) becomes

$$0 \to M \to F_n \to F_{n-1} \cdots \to F_1 \to N \to 0. \tag{2.3}$$

Then M is exactly an n-th syzygy module of the R-module N according to the usual definition of a syzygy. In particular, a first syzygy M fits into the following exact sequence:

$$0 \to M \to F_1 \to N \to 0 \tag{2.4}$$

where F_1 is a finitely generated free module and M represents the relations between a set of generators of N. Syzygy defined in this way depends on the module N and the free resolution of N while our definition of syzygy in (2.1) is a property of the module itself. We will see later that being a certain kind of syzygy implies some useful properties of the module itself.

Perhaps the most important examples of syzygies come from Koszul complex.

Definition 2.3. [7, 1.6] Given a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in R, the *Koszul complex* $K_*(\boldsymbol{x})$ of $\boldsymbol{x}$ is defined as the following chain complex:

$$0 \to \bigwedge^n R^n \xrightarrow{d^{(n)}} \bigwedge^{n-1} R^n \to \cdots \to \bigwedge^2 R^n \xrightarrow{d^{(2)}} \bigwedge^1 R^n \xrightarrow{d^{(1)}} R. \tag{2.5}$$

With a basis $e_1, \ldots, e_n$ of R^n, the differential map $d^{(i)} : \bigwedge^i R^n \to \bigwedge^{i-1} R^n$ is defined on generators $\{e_{k_1} \wedge \cdots \wedge e_{k_i}\}_{1 \le k_1 < \cdots < k_i \le n}$ as

$$d^{(i)}(e_{k_1} \wedge \cdots \wedge e_{k_i}) := \sum_{j=1}^{i} (-1)^{j-1} x_{k_j} \cdot e_{k_1} \wedge \cdots \hat{e}_{k_j} \wedge \cdots \wedge e_{k_i} \tag{2.6}$$

where $\hat{e}_j$ means that the term is missing in the wedge product.

Every component in the Koszul complex of a sequence $\boldsymbol{x}$ is a finitely generated free module. To introduce a sufficient condition under which $K_*(\boldsymbol{x})$ is exact, we need the following definition.

Definition 2.4. For an R-module M, a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in R is an *M-sequence* if the following conditions are satisfied:

(1) $(x_1, \ldots, x_n)M \neq M$.

(2) For any $1 \leq i \leq n$, x_i is not a zerodivisor of $M/(x_1, \ldots, x_{i-1})M$.

Proposition 2.5 ([7, Corollary 1.6.14])**.** *If $\boldsymbol{x}$ is an R-sequence, then $K_*(\boldsymbol{x})$ is a free resolution of $R/(x_1, \ldots, x_n)$, that is, the sequence (2.5) is exact and the cokernel of the rightmost map in (2.5) is $R/(x_1, \ldots, x_n)$.*

For example, if we let $\boldsymbol{t} = t_1, \ldots t_r$ be the sequence of variables in $R = k[t_1, \ldots, t_r]$, then $\boldsymbol{t}$ is an R-sequence and the Koszul complex $K_*(\boldsymbol{t})$ is a free resolution of $R/(t_1, \ldots, t_r) = k$. Then by the definition of a syzygy, the image of $d^{(j)}$ is a j-th syzygy. In particular, the maximal ideal $(t_1, \ldots, t_r)$ is a first syzygy.

We have seen that the first syzygy represents relations between a set of generators of some finitely generated R-module. In general an n-th syzygy represents relations after n steps in a free resolution of some R-module. The Hilbert syzygy theorem implies that being a certain kind of syzygy means something useful about the module itself.

Theorem 2.6 ([7, Corollary 2.2.14(a),(c)])**.** *Every finitely generated graded module over the polynomial ring $R = k[t_1, \ldots, t_r]$ has a graded free resolution of length $\leq r$. In fact, every finitely generated R-module has a free resolution of length $\leq r$, that is, for any finitely generated R-module N, there is an exact sequence*

$$0 \to F_r \to F_{r-1} \to \cdots \to F_1 \to N \to 0 \tag{2.7}$$

where F_i's are finitely generated free R-modules(possibly zero).

Using Hilbert's syzygy theorem, one can show that if M is a finitely generated r-th syzygy, then M is free. Hilbert's syzygy theorem relates the notion of syzygies to that of freeness. It turns out that a more general notion of torsionfreeness corresponds exactly to the notion of syzygies.

Definition 2.7. [8, 16.E] A module M over R is called *n-torsionfree* if every R-sequence of length at most n is an M-sequence.

The following proposition of Auslander-Bridger relates the notion of torsionfreeness to that of syzygies. We use M^* to denote the dual $\mathrm{Hom}_R(M, R)$ of an R-module M. There is a natural map $h_M : M \to M^{**}$ that sends every $m \in M$ to the evaluation at m.

Proposition 2.8 ([4, Chapter 2, Theorem 2.17],[1, Proposition 2.3])**.** *The following are equivalent for any finitely generated R-module M and any $n \geq 1$:*

(1) M is an n-th syzygy.

(2) M is n-torsionfree.

(3) One of the following conditions holds, depending on n:

a. $n = 1$: M is torsionless, that is, h_M is injective.

b. $n = 2$: M is reflexive, that is, h_M is an isomorphism.

c. $n \geq 3$: M is reflexive and $\mathrm{Ext}^i_R(M^*, R) = 0$ for $i = 1, \ldots, n-2$.

The above proposition holds for more general rings and modules. For example, one can consult [8, 16.E].

Chapter 3

Equivariant cohomology

Cohomologies throughout this chapter are assumed to be singular cohomologies with coefficients in the real field $\mathbb{R}$.

3.1 Equivariant cohomology

All groups G in this section are assumed to be compact Lie groups.

Definition 3.1.1. A *G-space* X is a topological space with a G-action such that the following action map is continuous:

$$G \times X \to X \quad (g, x) \mapsto g \cdot x. \tag{3.1.1}$$

If X is a smooth manifold and the map in (3.1.1) is smooth, then X is called a *G-manifold*.

Definition 3.1.2. Given two G-spaces X and Y, a map $f : X \to Y$ is called *G-equivariant* if the G-action commutes with the map, that is, we have the following equation:

$$g \cdot f(x) = f(g \cdot x), \quad \forall g \in G, x \in X. \tag{3.1.2}$$

Definition 3.1.3. Let X and Y be two G-spaces. Two G-equivariant continuous maps f, h :

$X \to Y$ are *G-homotopic* to each other if there is a homotopy between f and h

$$H : X \times I \to Y \tag{3.1.3}$$

such that H is G-equivariant in the following sense:

$$H(g \cdot x, t) = g \cdot H(x, t) \quad \forall g \in G, x \in X, t \in I. \tag{3.1.4}$$

We define principal G-bundle now.

Definition 3.1.4. [10, I.8] A *principal G-bundle* over a base space B is a triple (X, B, p) consisting of a G-space X with free G-action called the total space and a surjective continuous map $p : X \to B$ such that

(1) $p(g \cdot x) = p(x) \quad \forall x \in X, g \in G,$

(2) For each $b \in B$, there exists an open neighborhood U of b in B and a G-equivariant homeomorphism $\phi : p^{-1}(U) \to G \times U$ such that the following diagram commutes.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & G \times U \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_U} & \\ & U & \end{array} \tag{3.1.5}$$

where pr_U is the projection onto U.

Remark 3.1.5. Given a principal bundle $\xi = (X, B, p)$ and a continuous map $f : B' \to B$, one can construct a principal bundle $f^*(\xi)$ over B' called the pullback of ξ by f.

Definition 3.1.6. Two principal G-bundles (X_1, B_1, p_1) and (X_2, B_2, p_2) are *isomorphic* if there exists a G-equivariant homeomorphism $\psi : X_1 \to X_2$.

Definition 3.1.7. [22, Chapter 4, Definition 9.1] An open cover $\{U_i\}_{i \in S}$ of a topological space B is called *numerable* if there exists a locally finite partition of unity $\{\mu_i\}_{i \in S}$ such that each μ_i has support in U_i.

Remark 3.1.8. [22, Chapter 4, Section 9] A Hausdorff space is paracompact if and only if each open cover is numerable.

Definition 3.1.9. [22, Chapter 4, Definition 9.2] A principal G-bundle (X, B, p) is called *numerable* if there is a numerable open cover $\{U_i\}_{i\in S}$ of B such that for each $i \in S$ there is a G-equivariant homeomorphism $\phi_i : p^{-1}(U_i) \to G \times U_i$ that makes the diagram (3.1.5) commute with $U = U_i$.

For every compact Lie group G, Milnor constructed in [26] a principal G-bundle $\omega_G = (EG, BG, \pi)$ with paracompact base space satisfying the following conditions:

(1) [22, Chapter 4, Theorem 12.2] For each numerable principal G-bundle $\xi = (X, B, p)$, there exists a map $f : B \to BG$ such that the two principal bundles ξ and $f^*(\omega_G)$ are isomorphic.

(2) [22, Chapter 4, Theorem 12.4] Let $f_0, f_1 : B \to BG$ be two maps. Then $f_0^*(\omega_G)$ and $f_1^*(\omega_G)$ are isomorphic if and only if f_0 and f_1 are homotopic.

Remark 3.1.10. Any numerable principal G-bundle satisfying the two conditions above is called a *universal* principal G-bundle and is unique up to G-homotopy in the sense that the total spaces of two such bundles are G-homotopy equivalent. The total space EG of such universal principal G-bundle is always contractible. The space BG is called a *classifying space* of G and is unique up to homotopy.

Now we can state the Borel's construction of equivariant cohomology.

Given a G-space X, Borel constructed in [5, Chapter IV, §3] the following space:

$$X_G := EG \times_G X = (EG \times X)/G \tag{3.1.6}$$

where the orbit space is taken under the following G-action on $EG \times X$:

$$g \cdot (e, x) = (e \cdot g^{-1}, g \cdot x) \quad \forall g \in G, (e, x) \in EG \times X. \tag{3.1.7}$$

Note that we are assuming that G acts on X from left and G acts on EG from right.

Let G-Top be the category of G-spaces with morphisms being continuous G-equivariant maps. The above construction gives a functor from the category of G-spaces to the category of topological spaces.

$$-_G : G\text{-Top} \to \text{Top}. \tag{3.1.8}$$

Since a G-homotopy equivalence induces an ordinary homotopy equivalence on the orbit space, it follows from Remark 3.1.10 that the homotopy type of X_G is independent of the choice of the universal principal G-bundle. So we can define the equivariant cohomology as the ordinary cohomology of X_G

$$H_G^*(X) := H^*(X_G). \tag{3.1.9}$$

Remark 3.1.11. [10, Chapter 3, Section 1] Let us list some of the formal properties of equivariant cohomology.

(1) A G-equivariant map $f : X \to Y$ induces a map $(f_G)^* : H_G^*(Y) \to H_G^*(X)$. If two G-equivariant map $f, h : X \to Y$ are G-homotopic to each other, then $(f_G)^* = (h_G)^*$. So the equivariant cohomology is G-homotopy invariant.

(2) The following map:

$$X \to \{pt\} \tag{3.1.10}$$

from a G-space to a point with trivial G-action induces a homogeneous map of graded $\mathbb{R}$-algebras

$$H^*(BG) = H^*(\text{pt}_G) \to H^*(X_G) = H_G^*(X). \tag{3.1.11}$$

This makes $H_G^*(X)$ into a graded $H^*(BG)$-algebra.

(3) If $Z \subseteq Y \subseteq X$ is a sequence of inclusions of G-spaces, then we call the triple (X, Y, Z) a *G-triple*. If (X, Y, Z) is a G-triple, then we have the inclusion $Z_G \subseteq Y_G \subseteq X_G$ and the

following long exact sequence associated to this G-triple (X, Y, Z):

$$\cdots \to H_G^n(X, Y) \to H_G^n(X, Z) \to H_G^n(Y, Z) \xrightarrow{\partial} H_G^{n+1}(X, Y) \to \cdots \tag{3.1.12}$$

where the relative equivariant cohomology associated to a G-pair in the above sequence is defined to be the relative ordinary cohomology associated to the corresponding pair of the Borel's construction $-_G$. The map ∂ in the above sequence is called the connecting homomorphism.

In particular, if $Z = \emptyset$, then the sequence in (3.1.12) becomes

$$\cdots \to H_G^n(X, Y) \to H_G^n(X) \to H_G^n(Y) \xrightarrow{\partial} H_G^{n+1}(X, Y) \to \cdots \tag{3.1.13}$$

Let us give some examples of computations of equivariant cohomology.

Example 3.1.12. [10, Chapter 3, (1.11)] If X is a compact G-manifold and G acts freely on X, then we have

$$H_G^*(X) \simeq H^*(X/G) \quad \text{as } H^*(BG)\text{-algebra} \tag{3.1.14}$$

where the $H^*(BG)$-algebra structure on $H^*(X/G)$ is induced by the classifying map $X/G \to BG$ of the principal G-bundle $X \to X/G$.

Example 3.1.13. Let X be a G-space. If G acts trivially on X, then $X_G = BG \times X$ and by Künneth formula we have

$$H_G^*(X) \simeq H^*(BG) \otimes_k H^*(X) \quad \text{as } H^*(BG)\text{-module.} \tag{3.1.15}$$

3.2 Equivariant cohomology and syzygy

From now one, we will focus on torus actions on smooth manifolds. We assume that $T = (\mathbb{S}^1)^r$ and $R = H^*(BT)$ throughout this section. We also assume that all spaces X in this section are compact T-manifolds.

Example 3.2.1. [10, Chapter 3, Proposition 2.2] The classifying space BT of a torus T can be chosen to be $(\mathbb{C}P^\infty)^r$ whose cohomology group is

$$H^*(BT) = H^*((\mathbb{C}P^\infty)^r) \simeq \mathbb{R}[t_1, \ldots, t_r] \tag{3.2.1}$$

where every $t_i \in H^2(BT)$ is the pullback of the first Chern class of the universal complex line bundle over $\mathbb{C}P^\infty$ via the projection map $(\mathbb{C}P^\infty)^r \to \mathbb{C}P^\infty$ onto the i-th coordinate.

For a T-manifold X, there is a filtration by dimension of orbits

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{r-1} \subseteq X_r = X \tag{3.2.2}$$

where X_i is the set of points in X that are fixed by a subtorus of T of codimension i. For example, X_0 is the set of fixed points and X_{r-1} is the set of points that can be fixed by a circle in T.

Chang-Skjelbred showed in [9] that the following Chang-Skjelbred sequence is exact if the equivariant cohomology $H_T^*(X)$ is free over $R = H^*(BT)$

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X_0) \longrightarrow H_T^{*+1}(X_1, X_0). \tag{3.2.3}$$

The map $H_T^*(X) \to H_T^*(X_0)$ is induced by the inclusion of fixed point set. The map $H_T^*(X_0) \to H_T^{*+1}(X_1, X_0)$ is the connecting homomorphism for the T-pair (X_1, X_0).

Atiyah and Bredon proved in [2] and [6] a stronger result under the same freeness hypothesis on equivariant cohomology. They showed that if the equivariant cohomology $H_T^*(X)$ is free over R, then the following Atiyah-Bredon sequence $AB^*(X)$ is exact.

$$\begin{array}{ccccccccccccc} 0 \to & H_T^*(X) & \to & H_T^*(X_0) & \to & H_T^{*+1}(X_1, X_0) & \to & H_T^{*+2}(X_2, X_1) & \to & \cdots & \to & H_T^{*+r}(X_r, X_{r-1}) & \to 0 \\ & \| & & \| & & \| & & \| & & & & \| & \\ 0 \to & AB^{-1}(X) & \to & AB^0(X) & \longrightarrow & AB^1(X) & \longrightarrow & AB^2(X) & \longrightarrow & \cdots & \longrightarrow & AB^r(X) & \longrightarrow 0 \end{array} \tag{3.2.4}$$

The first four terms and the maps between them are the same as in (3.2.3). All the other maps are connecting homomorphisms for T-triples (X_{i+1}, X_i, X_{i-1})'s:

$$H_T^{*+i}(X_i, X_{i-1}) \to H_T^{*+i+1}(X_{i+1}, X_i). \tag{3.2.5}$$

Syzygies were introduced into the study of equivariant cohomology by Allday-Franz-Puppe to find the condition under which the Atiyah-Bredon sequence is exact. They proved the following theorem in [1, Theorem 1.1].

Theorem 3.2.2 ([1, Theorem 1.1]). *Let X be a compact T-manifold and $j \geq 0$. Then the Atiyah-Bredon sequence is exact at all positions $i \leq j-2$ if and only if $H_T^*(X)$ is a jth syzygy, that is, $H^i(AB^*(X)) = 0$ for $i \leq j-2$ if and only if $H_T^*(X)$ is a j-th syzygy.*

An immediate corollary of Theorem 3.2.2 and Proposition 2.8 is that the Chang-Skjelbred sequence is exact if and only if the R-module $H_T^*(X)$ is reflexive.

For spaces with Poincaré duality like compact orientable manifolds, if roughly the first half of the Atiyah-Bredeon sequence is exact, then so is the rest.

Corollary 3.2.3 ([1, Corollary 1.4]). *Let X be a compact orientable T-manifold. If $H_T^*(X)$ is a syzygy of order $\geq r/2$ where r is the rank of T, then it is free over R.*

Franz showed in [16, Theorem 1.2] that this lower bound is sharp for some special cases of a family of orientable compact T-manifolds called big polygon spaces. These spaces are defined in Definition 1.1.

3.3 A quotient criterion for syzygy order of equivariant cohomology

Franz developed in [17] a criterion for the equivariant cohomology to be a certain kind of syzygy with some assumptions on the manifolds and actions.

Definition 3.3.1. Two T-manifolds M, N are *weakly equivariantly diffeomorphic* if there is a smooth map $f : M \to N$ such that under an automorphism of T, f is an equivariant diffeomorphism. We say that the T-action on a T-manifold X is *locally standard* if $n = \dim X \geq 2r$ and every point of X has an T-invariant neighbourhood U *weakly equivariantly diffeomorphic* to an open subset $W \subseteq \mathbb{C}^r \times \mathbb{R}^{n-2r}$ invariant under the standard T-action on $\mathbb{C}^r \times \mathbb{R}^{n-2r}$.

The following assumptions were assumed in [17, Section 4] in order to apply the criterion.

(1) X is non-empty and connected,

(2) $H^*(X)$ is finite-dimensional,

(3) the action is locally standard.

Note that if X is compact, then condition (2) above is satisfied.

The orbit space X/T of a T-manifold X with locally standard T-action is a smooth manifold with corners which will be defined in detail in Definition 5.1.2. The smooth structure of X/T is obtained by inducing the functional structure (sheaf of smooth functions) on X/T from that of X, that is, a function f on X/T is smooth if and only if $f \circ \pi$ is smooth on X where π is the quotient map $X \to X/T$. To state the quotient criterion, we need to introduce briefly some properties of manifolds with corners. Details will be given in Chapter 5.

Every manifold with corners M admits a natural stratification

$$M = \Sigma_0(M) \supseteq \Sigma_1(M) \supseteq \cdots \supseteq \Sigma_n(M) \tag{3.3.1}$$

where $n = \dim M$. The set $\Sigma_k(M) - \Sigma_{k+1}(M)$ is called the *codimension-k stratum* of M. An elementary example is $(\mathbb{R}_+)^n$ where $\mathbb{R}_+$ is the positive real line including 0 and the natural stratification is given by its open faces of various dimensions. The closure of a connected component of the codimension-k stratum is called a *codimension-k face* of M.

For a face P of X/T the following cochain complex $B^*(P)$ and differential are defined in [17].

$$B^i(P) = \bigoplus_{\substack{Q \leq P \\ \mathrm{rank}\, Q = i}} H_*(Q) \tag{3.3.2}$$

where $Q \leq P$ means that Q is a face contained in P and rank Q is the corank of the common isotropy group of points in X lying over the interior of Q. Every direct summand $H_*(Q)$ is the homology group of Q, that is

$$H_*(Q) = \bigoplus_{i=0}^{\infty} H_i(Q). \tag{3.3.3}$$

For $\sigma \in H_*(Q) \subset B^i(P)$, the differential is defined as

$$d\sigma = \sum_{\substack{Q \leq O \leq P \\ \mathrm{rank}\, O = i+1}} \pm(\iota_{QO})_*(\sigma) \tag{3.3.4}$$

where $(\iota_{QO})_*$ is the map on homology induced by inclusion $\iota_{QO} : Q \to O$ and the sign $\pm$ is determined by an ordering of faces and will be made explicit in concrete computation we are going to work on. We state the main theorem in [17].

Theorem 3.3.2. *Let X be a T-manifold with locally standard T-action satisfying assumptions in the beginning of this section and $j \geq 0$. Then $H_T^*(X)$ is a j-th syzygy if and only if $H^i(B^*(P)) = 0$ for all faces P of X/T and all $i > \max(\mathrm{rank}\, P - j, 0)$.*

Example 3.3.3. Let us look at the example $X = X_{1,1}((1, 1, 1))$, that is, the big polygon space mentioned in Definition 1.1 and (1.17) with $a = b = 1$ and $l = (1, 1, 1)$. We have depicted the face structure of the orbit space of X in (1.16). If we take a face $P = F_1$ with F_1 defined in

(1.17), then the cochain complex $B^*(P)$ defined in (3.3.2) looks like

$$0 \to H_*(F_{123}) \to H_*(F_{12}) \oplus H_*(F_{13}) \to H_*(F_1) \to 0 \tag{3.3.5}$$

with arrows all induced by inclusion of faces. If we take $P = F_{12}$, then the cochain complex $B^*(P)$ looks like

$$0 \to H_*(F_{123}) \to H_*(F_{12}) \to 0. \tag{3.3.6}$$

It is actually not hard to compute every term in (3.3.5), (3.3.6) and maps in them. It is also not hard to show that the rightmost non-zero maps in (3.3.5) and (3.3.6) are surjective. According to Theorem 3.3.2 this implies that the equivariant cohomology of X is a first syzygy.

This criterion gives us a way to compute the syzygy order of the equivariant cohomology without computing the equivariant cohomology itself. The main goal of this thesis is to apply this criterion to the big polygon space in Definition 1.1 and compute its syzygy order.

To compute the syzygy order of $H_T^*(X_{a,b}(l))$ using Theorem 3.3.2, there are three steps. First we want to compute $H_*(F)$ for each face F of $X_{a,b}(l)/T$. Next we want to understand the maps $\iota_* : H_*(F) \to H_*(G)$ induced by the inclusions $F \subseteq G$ for all faces G containing F. In the end, we will compute $H^*(B^*(F))$ for every face F and apply Theorem 3.3.2. These tasks will be done in Chapter 6 and Chapter 7.

Chapter 4

Properties of big polygon spaces

4.1 Basic properties of big polygon spaces

We are gong to define big polygon spaces and state some properties of them. Almost all the results stated here come from [16].

Definition 4.1.1. [16] A vector $l = (l_1, \dots, l_r) \in \mathbb{R}^r$ is called a *generic length vector* if it cannot be split into two groups of equal sum. For a given generic length vector l, a subset I of $[r] = \{1, \dots, r\}$ is called *short* if

$$\sum_{i \in I} l_i < \sum_{j \in I^\complement} l_j. \tag{4.1.1}$$

The subset I is called *long* if the inequality above is reversed. Two generic length vectors l and l' are defined to be *equivalent* and denoted $l \sim l'$ if they induce the same notion of 'long' and 'short' on subsets of $[r]$.

Remark 4.1.2. We can think of a length vector l as a point in $\mathbb{R}^r$. If a length vector l is not generic, that is, there is $I \subseteq [r]$ such that

$$\sum_{i \in I} l_i = \sum_{j \in I^\complement} l_j, \tag{4.1.2}$$

then l is a point on the hyperplane in $\mathbb{R}^r$ determined by the above equation. A generic length vector is a point in the complement of all such hyperplanes. A connected component of the complement of these hyperplanes is called a *chamber* in [16] and two length vectors are equivalent if and only if they lie in the same chamber. Note that the number of different such hyperplanes is 2^{r-1}. So the number of chambers grows very fast when r increases.

For a generic length vector, we can define a space called a big polygon space.

Definition 4.1.3. For a generic length vector $l = (l_1, \dots, l_r) \in \mathbb{R}^r$ and $a, b \geq 1$, a *big polygon space* denoted $X_{a,b}(l)$ is the subspace of $\mathbb{C}^{(a+b)r}$ defined by the equations

$$\begin{aligned} (u_1, \dots, u_r, z_1, \dots, z_r) &\in \mathbb{C}^{(a+b)r} \text{ where} \\ u_j \in \mathbb{C}^a, z_j &\in \mathbb{C}^b \quad (1 \leq j \leq r), \\ \|u_j\|^2 + \|z_j\|^2 &= 1, \\ l_1 u_1 + \cdots + l_r u_r &= 0. \end{aligned} \tag{4.1.3}$$

The torus $T = (\mathbb{S}^1)^r$ acts on $X_{a,b}(l)$ by scalar multiplication on the variables z_j's as in [16, (1.6)],

$$(g_1, \dots, g_r) \cdot (u_1, \dots, u_r, z_1, \dots, z_r) = (u_1, \dots, u_r, g_1 z_1, \dots, g_r z_r). \tag{4.1.4}$$

Spaces defined in Definition 4.1.3 are called big polygon spaces because the fixed point set of a big polygon space is a space of polygons $E_{2a}(l)$ defined as follows.

Definition 4.1.4. [15] Let $l = (l_1, \dots, l_r) \in \mathbb{R}^r$ be a generic length vector. A *space of polygons* is defined as

$$E_d(l) = \Big\{(u_1, \dots, u_r) \in \mathbb{S}^{d-1} : \sum_{i=1}^{r} l_i u_i = 0\Big\} \tag{4.1.5}$$

One can think of a space of polygons as closed n-gons (allowing self-intersection) in $\mathbb{R}^d$ with sides of lengths $|l_1|, \dots, |l_r|$ up to translation in the Euclidean space. The topology of these spaces was studied in [15].

We state some properties of a big polygon space.

Lemma 4.1.5 ([16, Lemma 2.1]). *Let $a, b, r \geq 1$. Let l and l' be two generic length vectors in $\mathbb{R}^r$.*

(1) A big polygon space $X_{a,b}(l)$ is an orientable compact connected T-manifold. Its dimension is $(2a + 2b - 1)r - 2a$.

(2) If l' is obtained from l by changing the sign of some components and/or by permuting them, then $X_{a,b}(l)$ and $X_{a,b}(l')$ are equivariantly diffeomorphic with respect to the corresponding permutation of components of T, that is under the corresponding permutation of components of T, $X_{a,b}(l)$ and $X_{a,b}(l')$ are weakly equivariantly diffeomorphic.

(3) If $l \sim l'$, then $X_{a,b}(l)$ and $X_{a,b}(l')$ are equivariantly diffeomorphic.

Following [16, Assumption 2.2 and discussion after that], the assumption below is made throughout the rest of this thesis. We can see from Lemma 4.1.5(2) and (3) that by requiring this assumption we do not lose any equivariantly diffeomorphic types.

Assumption. *We assume that* $0 < l_1 \leq l_2 \leq \cdots \leq l_r$.

In the concrete computation we are going to carry out later, we are going to use a lot the assumption that l_i's are strictly positive.

Since we are going to apply the quotient criterion stated in Theorem 3.3.2, we have to verify that big polygon spaces satisfy assumptions stated in Section 3.3. We have seen in Lemma 4.1.5 that a big polygon space is a compact connected manifold. So the only thing we have to show is that a big polygon space is locally standard. However, we are only going to show that the T-action is locally standard for a subfamily of big polygon spaces, that is, those big polygon spaces with nonempty fixed point set.

We first need some tools from [17, Section 4] to show that a torus action is locally standard.

Definition 4.1.6. [17, Discussion after lemma 4.1] A *characteristic circle* for a T-manifold X is a circle $K \subseteq T$ that occurs as the isotropy group of some $x \in X$. Connected components of

X^K which consists of points fixed by K are called *characteristic submanifolds* of X. If they are all of codimension 2, then we say that the T-action on X is *regular*.

Lemma 4.1.7 ([17, Lemma 4.2]]). *Let X be a regular T-manifold such that $X^T \neq \emptyset$, then the T-action is locally standard.*

We show in the following lemma that the T-action is locally standard for a subfamily of big polygon spaces.

Lemma 4.1.8. *(1) For a generic length vector $l \in \mathbb{R}^r$, if the fixed point set of $X_{a,b}(l)$ under the T-action is not empty, then the T-action is locally standard.*

(2) The fixed point set of $X_{a,b}(l)$ is not empty if and only if $l_r < \sum_{i \neq r} l_i$ where the sum is over $i \in \{1, \ldots, r-1\}$, which means that there is no dominant length.

Proof. Since the torus acts on $X_{a,b}(l)$ by scalar multiplication on the variables z_j's, a characteristic circle of $X_{a,b}(l)$ are just coordinate circles of the torus T. Characteristic manifolds are connected components of those points with one of the z_j's being 0 and they are of codimension 2. Then the first part of the lemma follows from a direct application of Lemma 4.1.7.

Let us prove the second part of the lemma. If r is 1 or 2, then the fixed point set is always empty since the length vector is generic. The second part of the lemma is trivially true in these cases. For $r \geq 3$, since the fixed point set consists of the space of polygons described in Definition 4.1.4, it is not empty if and only if $l_r < \sum_{i \neq r} l_i$ by [14, Lemma 1.1]. □

4.2 Equivariant cohomology of big polygon spaces

The equivariant cohomology of $X_{a,b}(l)$ was computed in [16, Lemma 4.5]. For our purpose we only describe some properties of the syzygy order of the equivariant cohomology. We briefly review the result of the computation carried out in [16].

Let $R = H^*(BT) = \mathbb{R}[t_1, \ldots, t_r]$ be the polynomial ring. Equivariant homology was defined in [1, Section 3.3] using the R-dual chain complex of the singular cochain complex of the

Borel's construction X_T and is an R-module. Note that the equivariant homology defined in this way is not the homology of the Borel's construction. For our purpose we can just think of equivariant homology as an R-module associated to a T-manifold.

In our case, let $V = \mathbb{S}^{2a+2b-1} \subseteq \mathbb{C}^a \times \mathbb{C}^b$ with T-action on V^r given by (4.1.4), there is an R-module homomorphism

$$\iota_*^T : H_*^T(V^r - X_{a,b}(l)) \to H_*^T(V^r). \tag{4.2.1}$$

The map ι_*^T as a homomorphism between R-modules will be described below. It was shown in [16, Lemma 4.5] that

(1) $H_*^T(V^r)$ is a free R-module with basis $[V_J]_T$, $J \in [r]$,

(2) $H_*^T(V^r - X_{a,b}(l))$ is a free R-module with basis $[V_J]_T$ and $[W_J]_T$ for every short subset $J \subseteq [r]$.

where $[V_J]_T$ and $[W_J]_T$ are fundamental classes of some subspaces V_J, W_J in V^r. One can find more details on how $[V_J]_T$ and $[W_J]_T$ are defined in [16, Lemma 4.5]. For the purpose of this thesis, it suffices to think of them as free generators of respective equivariant homology. The map ι_*^T is described in [16, Proposition 4.6] on basis of $H_*^T(V^r - X_{a,b}(l))$ as:

$$\iota_*^T[V_J]_T = [V_J]_T, \tag{4.2.2}$$

$$\iota_*^T[W_J]_T = \sum_{j \notin J} (-1)^{[j:J]} t_j^b [V_{J\cup j}]_T. \tag{4.2.3}$$

where t_j is a variable in $R = \mathbb{R}[t_1, \ldots, t_r]$ and $[j : J]$ is the number of elements in J that are strictly less than j.

We have the following exact sequence of graded R-module according to [16, Lemma 4.4]:

$$0 \to (\operatorname{coker} \iota_*^T)[rd] \to H_T^*(X_{a,b}(l)) \to (\ker \iota_*^T)[rd-1] \to 0 \tag{4.2.4}$$

where $[\cdot]$ means a shifting of degree and $d = 2a - 2b - 1$. The above exact sequence follows from a natural Poincaré duality between equivariant cohomology and equivariant homology. The following result was stated in [16]:

Lemma 4.2.1 ([16, Lemma 6.2])**.** $\operatorname{syzord} H_T^*(X_{a,b}(l)) = \operatorname{syzord} \operatorname{coker} \iota_*^T$.

An observation can then be drawn from Lemma 4.2.1.

Lemma 4.2.2. *The syzygy order of $H_T^*(X_{a,b}(l))$ is independent of the integer a.*

Proof. We can see from Lemma 4.2.1 and the explicit description of ι_*^T before Lemma 4.2.1 that for a fixed generic length vector l, the domain and codomain of ι_*^T as R-module are independent of the integer a although as graded R-module they depend on a because of the shifting of degrees in (4.2.4). Since syzygy order is an invariant of R-module, it is independent of a. From Lemma 4.2.1 our lemma follows. □

Chapter 5

Preliminaries on manifolds with corners

In this chapter we first collect some basic properties of manifolds with corners and then generalize a lacunary principle for Morse-Bott functions to manifolds with corners. We will later use them to compute and find basis of homology for manifolds with corners.

5.1 Basics on manifolds with corners

We start with some basics on manifolds with corners. Many of the following definitions and theorems are cited from [13, Appendix A].

Definition 5.1.1. An *n-dimensional topological manifold with boundary M* is a second countable Hausdorff space where every point $p \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

Definition 5.1.2. [13, Section A.1] An n-dimensional *manifold with corners M* is an n-dimensional topological manifold with boundary with a smooth structure given by a C^∞-atlas $\{(U, \phi_U)\}$ covering M where every $U \subseteq M$ is an open subset and $\phi_U : U \to \phi_U(U) \subseteq \mathbb{R}^n_+$ is a homeomorphism, with $\phi_U(U)$ being an open subset of $\mathbb{R}^n_+$ where

$$\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i = 1, \ldots, n\}. \tag{5.1.1}$$

Any two local charts (U, ϕ_U) and (V, ϕ_V) are assumed to be C^∞-compatible in the sense that the transition map

$$\phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V) \tag{5.1.2}$$

is C^∞ which means that it can be extended to a C^∞-map in an open set of $\mathbb{R}^n$ containing $\phi_U(U \cap V)$.

We are going to define the tangent space T_pM of a manifold with corners M at some point $p \in M$ as the set of derivations on smooth functions on M.

Definition 5.1.3. Given two manifolds with corners M and N of dimension m and n with C^∞-atlases $\{(U, \phi_U)\}$ and $\{(V, \psi_V)\}$, a map $f : M \to N$ is *smooth* if $\psi_V|_{V \cap f(U)} \circ f \circ \phi_U^{-1}$ is smooth on $\phi_U(U) \subseteq \mathbb{R}^n_+$ for every U, V in the atlas of M, N. A *diffeomorphism* between two manifolds with corners is a smooth bijective map between them whose inverse is also smooth.

Definition 5.1.4. Given a manifold with corners M, a *smooth function on M* is a smooth map $f : M \to \mathbb{R}$ with the standard chart on $\mathbb{R}$. The set of smooth functions on M is denoted $C^\infty(M)$.

Definition 5.1.5. Given a manifold with corners M, an $\mathbb{R}$-linear map $X : C^\infty(M) \to \mathbb{R}$ is a *derivation at $p \in M$* if it satisfies

$$X(fg) = f(p)X(g) + g(p)X(f) \text{ for all } f, g \in C^\infty(M). \tag{5.1.3}$$

The set of all derivations at p is a vector space of dimension n. This vector space is called the *tangent space to M at p* and denoted T_pM. Every element of T_pM is called a tangent vector to M at p.

Definition 5.1.6. Given two manifolds with corners M, N and a smooth map $f : M \to N$, the *differential* of f at a point $p \in M$ is defined to be the linear map $df_p : T_pM \to T_{f(p)}N$ as follows

$$(df_p(X))(g) = X(g \circ f), \forall X \in T_pM, g \in C^\infty(N). \tag{5.1.4}$$

We are going to define the cone of tangent directions. It is a subset of the tangent space.

Definition 5.1.7. M is a manifold with corners. A *smooth curve* starting at a point $p \in M$ is a smooth map $\gamma : [0, 1) \to M$ such that $\gamma(0) = p$.

Definition 5.1.8. M is a manifold with corners. The *velocity vector at* $p \in M$ of a smooth curve $\gamma : [0, 1) \to M$ starting at p is the derivation $X \in T_pM$ defined by

$$X(f) := (f \circ \gamma)'(0) = \lim_{t \to 0^+} \frac{f \circ \gamma(t) - f \circ \gamma(0)}{t}, \forall f \in C^\infty(M). \tag{5.1.5}$$

Definition 5.1.9. [13, Section A.2] Given a manifold with corners M, the *cone of tangent directions* to M at p is denoted C_pM and is the subset of T_pM defined as follows. A tangent vector $X \in T_pM$ belongs to C_pM if and only if there exists a smooth curve $\gamma : [0, 1) \to M$ starting at $p \in M$ with X being the velocity vector of γ at p.

Based on the following lemma, we have a natural stratification on a manifold with corners.

Lemma 5.1.10. *Assume that* $0 \leq l, s \leq n$. *If there is a diffeomorphism*

$$\phi : (\mathbb{R}^s_+ \times \mathbb{R}^{n-s}, 0) \xrightarrow{\simeq} (\mathbb{R}^l_+ \times \mathbb{R}^{n-l}, 0) \tag{5.1.6}$$

with the standard C^∞*-atlases on both sides, then* $s = l$.

Proof. We only have to prove that $s \geq l$. Let $M = \mathbb{R}^s_+ \times \mathbb{R}^{n-s}$ and $N = \mathbb{R}^l_+ \times \mathbb{R}^{n-l}$. Since ϕ is a diffeomorphism, the differential $d\phi_0 : T_0M \to T_0N$ is an isomorphism of vector spaces.

Furthermore, since ϕ is a diffeomorphism, any curve in M is mapped to a curve in N and thus any tangent vector in the cone C_0M is mapped to a tangent vector in C_0N by $d\phi_0$. Since C_0M contains a linear subspace of T_0M of dimension $n - s$, C_0N also contains a linear subspace of T_0N of dimension $n - s$. Since the maximal subspace of T_0N contained in C_0N is of dimension $n - l$, we have $s \geq l$. □

Thus we have the following well-defined stratification of a manifold with corners.

Definition 5.1.11. [13, (A.1)] Given a manifold with corners M of dimension n, there is a canonical stratification

$$M = \Sigma_0(M) \supseteq \Sigma_1(M) \supseteq \cdots \supseteq \Sigma_n(M). \tag{5.1.7}$$

The *codimension-s stratum* $\Sigma_s(M) - \Sigma_{s+1}(M)$ of M consists of all points $p \in M$ having a neighborhood $U \subseteq M$ such that the pair (U, p) is diffeomorphic to $(\mathbb{R}^s_+ \times \mathbb{R}^{n-s}, 0)$. A *codimension-s face* of M is the closure of a connected component of $\Sigma_s(M) - \Sigma_{s+1}(M)$ in M. A codimension-1 face of M is called a *facet* of M.

Example 5.1.12. Let M be $\mathbb{R}^s_+ \times \mathbb{R}^{n-s}$ with the obvious chart. Then a codimension-k face of M is described by the following set for some $I \subseteq \{1, \ldots, s\}$ such that $|I| = k$:

$$F_I := \{(x_1, \cdots, x_n) : x_i = 0 \text{ for } i \in I\}. \tag{5.1.8}$$

Remark 5.1.13.
1. It follows from the definition of a stratum that every codimension-s stratum of a manifold with corners of dimension n is a smooth manifold of dimension $n - s$.
2. A face of a manifold with corners may not be a manifold with corners itself. One such example can be found in [13, A.3].

Corollary 5.1.14. *A diffeomorphism between manifolds with corners preserves strata and faces, that is, it maps the codimension-s stratum onto the codimension-s stratum and a codimension-s face onto a codimension-s face.*

Proof. The first assertion follows directly from Lemma 5.1.10. Since a diffeomorphism maps a connected component to a connected component and a face is just the closure of a connected component of some stratum, the second assertion follows. □

In addition to the notion of tangent space, we have a notion of tangent space to the stratum of a manifold with corners.

Definition 5.1.15. Let M be a manifold with corners of dimension n and $p \in S_p := \Sigma_s(M) - \Sigma_{s+1}(M)$ be a point in the codimension-s stratum S_p of M. Since S_p is a smooth manifold, the tangent space T_pS_p can be defined as the set of derivations on $C^\infty(S_p)$ at p and is a vector space of dimension $n - s$. We can identify T_pS_p as a subspace of T_pM in the following sense:

$$X(f) := X(f|_{S_p}), \forall X \in T_pS_p, f \in C^\infty(M). \tag{5.1.9}$$

This subspace of T_pM is called the *tangent space to the stratum* of M at p and is denoted $T_p^S M$.

Remark 5.1.16. 1. Since every stratum is a smooth manifold, we have for any $p \in M$ that

$$T_p^S M \subseteq C_pM. \tag{5.1.10}$$

2. $T_p^S M$ is the maximal subspace of T_pM contained in C_pM.

The next definition is slightly different from the definition in [13, Definition A.4]. The difference is that there the author only defined neat submanifolds of a subclasses of manifolds with corners called regular manifolds with corners. But the same definition can be extended to any manifolds with corners.

Definition 5.1.17. [13, Definition A.4] A subset N of a manifold with corners M is called a *neat submanifold* of M of codimension k if each point $p \in N$ has a neighborhood U in M such that $(U, U \cap N, p)$ is diffeomorphic to $(\mathbb{R}_+^s \times \mathbb{R}^{n-s}, \mathbb{R}_+^s \times \mathbb{R}^{n-k-s}, 0)$ for some s depending on the point p where $n = \dim M$ and $\mathbb{R}^{n-k-s}$ is the subspace of $\mathbb{R}^{n-s}$ where the last k coordinates vanish.

Remark 5.1.18. 1. By this definition, empty set is also a neat submanifold of M.

2. A neat submanifold N of a manifold with corners M is again a manifold with corners with the obvious local charts given in Definition 5.1.17 and we have

$$(\Sigma_s(M) - \Sigma_{s+1}(M)) \cap N = \Sigma_s(N) - \Sigma_{s+1}(N). \tag{5.1.11}$$

3. From definition we can see that locally M looks like $N \times \mathbb{R}^k$ so we can define the normal bundle of N in M as in [13, (A.3)].

We will need the following corollary of [27, Theorem 3]. The original theorem is very general and we just need this special case of it.

Proposition 5.1.19. *Let M be a manifold with corners, N be a smooth manifold without boundary and $f : M \to N$ be a smooth map. Fixing a point $q \in N$, if for any point $p \in f^{-1}(q)$ we have $df_p(T_p^S M) = T_q N$, then $f^{-1}(q)$ is a neat submanifold of M and the normal bundle of $f^{-1}(q)$ in M is trivial.*

Proof. This is just a special case of [27, Theorem 3]. Since $q \in N$ is just a point of N, the condition of preserving local facets in [27, Theorem 3] is null. Since N is a smooth manifold, the condition $df_p(T_p^S M) = T_q N$ implies that $df_p(T_p^S M) = T_q^S N$ and $df_p(T_p M) = T_q N$ which in further is equivalent to the condition in [27, Theorem 3] that f intersects N transversally and stratum transversally. So $f^{-1}(q)$ is a neat submanifold of M by [27, Theorem 3].

The last assertion follows because the normal bundle of $f^{-1}(q)$ in M is the pullback of the normal bundle of q in N by f, which is trivial. □

In [13, Appendix B] the author summarized a Morse-Bott theory for a subclass of manifolds with corners called regular manifolds with corners.

Definition 5.1.20. [13, Proposition A.3] A *regular manifold with corners M* is a manifold with corners where for any integer k and $s \geq k$, any codimension-k face $F \subseteq M$ and any $p \in F \cap (\Sigma_s(M) - \Sigma_{s+1}(M))$, there is a neighborhood $U \subseteq M$ of p such that the triple $(U, U \cap F, p)$ is diffeomorphic to $(\mathbb{R}_+^s \times \mathbb{R}^{n-s}, \mathbb{R}_+^{s-k} \times \mathbb{R}^{n-s}, 0)$ where $n = \dim M$ and $U \cap F$ is mapped under this diffeomorphism onto the subspace of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$ where the first k coordinates vanish.

Remark 5.1.21.

1. The following conditions are equivalent for a manifold with corners M, cf. [13, Proposition A.3]:

   (a) M is regular.

(b) Every face is a manifold with corners with atlas obtained from M.

(c) Every codimension-k face F has a neighbourhood $U \subseteq M$ such that (U, F) is diffeomorphic to $(\mathbb{R}_+^k \times F, F)$ by a diffeomorphism identical on F.

2. An example of a manifold with corners that is not regular can be found in [15, Figure A.2].

Next we introduce another subclass of manifolds with corners and show that it is the same as the class of regular manifolds with corners.

Definition 5.1.22. A manifold with corners is called *nice* if every codimension-l face is contained in l different facets.

Proposition 5.1.23. *A manifold with corners is regular if and only if it is nice.*

We first prove a lemma.

Lemma 5.1.24. *Assume that M is a manifold with corners of dimension n and F is a codimension-k face of M. Let $p \in F \cap (\Sigma_s(M) - \Sigma_{s+1}(M))$ be a point in the codimension-s stratum of M and ϕ be a local chart in a neighborhood U of p*

$$\phi : (U, p) \xrightarrow{\cong} (\mathbb{R}_+^s \times \mathbb{R}^{n-s}, 0). \tag{5.1.12}$$

Then $\phi(U \cap F)$ is the union of codimension-k faces of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$. These faces are described in Example 5.1.12. In particular, if M is regular, then $\phi(U \cap F)$ is exactly one of the codimension-k faces of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$.

Proof. Since U is open, by definition of faces, $U \cap F$ is the union of several codimension-k faces of U. Since ϕ is a diffeomorphism onto $\phi(U)$, by Corollary 5.1.14, a codimension-k face of U is mapped onto a codimension-k face of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$ and thus $\phi(U \cap F)$ is the union of several codimension-k faces of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$.

If M is regular, then F is a manifold with corners of dimension $n-k$. If $\phi(U \cap F)$ contains two codimension-k faces of $\mathbb{R}^s_+ \times \mathbb{R}^{n-s}$, then we have

$$\phi(U \cap F) \supseteq F_I \cup F_J \tag{5.1.13}$$

where F_I and F_J are as in Example 5.1.12 for some $I, J \subseteq \{1, \ldots, s\}$, $I \neq J$ and $|I| = |J| = k$. Since F_I and F_J are both manifolds with corners and both T_0F_I and T_0F_J can be identified as subspaces of $T_0(\mathbb{R}^s_+ \times \mathbb{R}^{n-s})$ of dimension $n-k$, the subspace $T_0\phi(U \cap F)$ of $T_0(\mathbb{R}^s_+ \times \mathbb{R}^{n-s})$ contains T_0F_I and T_0F_J and thus has dimension at least $n-k+1$, which leads to a contradiction to the dimension of F. □

Remark 5.1.25. In a neighbourhood $U \ni p$ as in Lemma 5.1.24, a face F of M breaks up into several faces of U. If M is not regular, then $U \cap F$ can consist of more than one faces of U. However, if M is regular, then $U \cap F$ consists of exactly one face of U. So if M is regular, then a face of M containing p corresponds to a unique face of U of the same codimension. Since two different faces of M of the same codimension cannot intersect on the respective interior, this correspondence is one to one.

Now we can prove Proposition 5.1.23.

Proof of Proposition 5.1.23. ($\Rightarrow$): Let F be a codimension-k face of M which is the closure of a connected component C of the codimension-k stratum $\Sigma_k(M) - \Sigma_{k+1}(M)$. Let us show that C is contained in k different facets. Since every facet is closed, that will imply that F is contained in k different facets.

Since M is regular, for any point $x \in C$, it follows from Lemma 5.1.24 that there is a local chart (U_x, ϕ_x) near x such that

$$\phi_x(U_x \cap F) = \{(x_1, \ldots, x_n) \in \mathbb{R}^k_+ \times \mathbb{R}^{n-k} : x_1 = \cdots = x_k = 0\}. \tag{5.1.14}$$

If we define for any $i \in \{1, \dots, k\}$ that

$$G_i := \{(x_1, \dots, x_n) \in \mathbb{R}_+^k \times \mathbb{R}^{n-k} : x_i = 0\}, \tag{5.1.15}$$

then the collection $\{G_i\}_{i=1}^k$ consists of all the facets of $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$ and we have

$$\phi_x(U_x \cap F) = \bigcap_{i=1}^k G_i \tag{5.1.16}$$

which implies that $\phi_x(U_x \cap F)$ is contained in k different facets of $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$.

Now, since M is regular and every G_i is a facet of $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$, by Remark 5.1.25 we can define

$$F_i(x) := \text{the facet of } M \text{ containing } \phi_x^{-1}(G_i) \tag{5.1.17}$$

and we have

$$\begin{aligned} \phi_x^{-1}(G_i) &= U_x \cap F_i(x), \\ F_i(x) &\neq F_j(x) \text{ if } i \neq j. \end{aligned} \tag{5.1.18}$$

It follows from (5.1.16) and (5.1.18) that we have

$$U_x \cap F = \phi_x^{-1}(\bigcap_{i=1}^k G_i) = \bigcap_{i=1}^k (U_x \cap F_i(x)) \tag{5.1.19}$$

which implies that for any $x \in C$, there is a neighborhood of x in F contained in k different facets of M. What we have shown is that locally a codimension-k face is contained in k different facets of M. We are going to show that this collection of different facets $\{F_i(x)\}_{i=1}^k$ remains the same for any point $x \in C$ so that we can extend this local property to the entire C.

Since C is a connected component of $\Sigma_k(M) - \Sigma_{k+1}(M)$, we can find a curve $\gamma : [0, 1] \to C$ such that $\gamma(0) = p$ and $\gamma(1) = q$. We are going to show that for every $t \in [0, 1]$ the collection of different facets $\{F_1(\gamma(t)), \dots, F_k(\gamma(t))\}$ remains the same. To this end, we assume that we have found $0 = t_0 < \cdots < t_l = 1$ such that there are local charts $\{(U_{\gamma(t_j)}, \phi_{\gamma(t_j)})\}_{j=0}^l$ as in (5.1.14) covering $\gamma([0,1])$. Now for any $j \in \{0, \ldots, l-1\}$ we want to show that

$$\{F_1(\gamma(t_j)), \ldots, F_l(\gamma(t_j))\} = \{F_1(\gamma(t_{j+1}), \ldots, F_l(\gamma(t_{j+1}))\}. \tag{5.1.20}$$

Since the following map is a diffeomorphism

$$\phi^{-1}_{\gamma(t_{j+1})} \circ \phi_{\gamma(t_j)} : U_{\gamma(t_j)} \cap U_{\gamma(t_{j+1})} \to U_{\gamma(t_j)} \cap U_{\gamma(t_{j+1})}, \tag{5.1.21}$$

according to Corollary 5.1.14, a point in the codimension-1 stratum of $U_{\gamma(t_j)} \cap U_{\gamma(t_{j+1})}$ is mapped to a point in the codimension-1 stratum of itself under this diffeomorphism. Thus we may assume for any i that there is some i' such that we have

$$(U_{\gamma(t_j)} \cap U_{\gamma(t_{j+1})} \cap \mathring{F}_i(\gamma(t_j))) \cap (U_{\gamma(t_j)} \cap U_{\gamma(t_{j+1})} \cap \mathring{F}_{i'}(\gamma(t_{j+1}))) \neq \varnothing \tag{5.1.22}$$

and thus

$$\mathring{F}_i(\gamma(t_j)) \cap \mathring{F}_{i'}(\gamma(t_{j+1})) \neq \varnothing. \tag{5.1.23}$$

Since both $\mathring{F}_i(\gamma(t_j))$ and $\mathring{F}_{i'}(\gamma(t_{j+1}))$ are connected components of the codimension-1 stratum of M and they intersect on a nonempty set, we have

$$\mathring{F}_i(\gamma(t_j)) = \mathring{F}_{i'}(\gamma(t_{j+1})) \tag{5.1.24}$$

and thus

$$F_i(\gamma(t_j)) = F_{i'}(\gamma(t_{j+1})). \tag{5.1.25}$$

The left hand side of (5.1.20) is thus a subcollection of the right hand side. We can show the inverse by looking at $\phi^{-1}_{\gamma(t_j)} \circ \phi_{\gamma(t_{j+1})}$.

We have shown that for any point $x \in C$, the collection of k different facets in (5.1.17) remains the same and thus the codimension-k face F is contained in k different facets of M. We have shown that M is nice.

($\Leftarrow$): Assume that M is a nice manifold with corners of dimension n, let us verify Definition 5.1.20. For a codimension-k face F of M and a point $p \in F \cap (\Sigma_s(M) - \Sigma_{s+1}(M))$ for some $s \geq k$, there is a local chart around p

$$\phi : (U, p) \xrightarrow{\cong} (\mathbb{R}_+^s \times \mathbb{R}^{n-s}, 0). \tag{5.1.26}$$

It follows from Lemma 5.1.24 that $\phi(U \cap F)$ is the union of a collection of codimension-k faces of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$. Since M is nice, F is contained in k different facets of M whose intersections with U are mapped to k different facets of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$ because two different faces cannot intersect on the interior. So $\phi(U \cap F)$ is exactly the following codimension-k face of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$:

$$\{(x_1, \ldots, x_n) \in \mathbb{R}_+^s \times \mathbb{R}^{n-s} : x_1 = \cdots = x_k = 0\}. \tag{5.1.27}$$

So (U, ϕ) is the required local chart in Definition 5.1.20.

□

Remark 5.1.26. It is also clear from the proof of ”regular ⇒ nice” that a smooth manifold with corners is nice if and only if every point in a codimension-k stratum is contained in k different facets.

Lemma 5.1.27. *An open subset of a regular manifold with corners is a regular manifold with corners.*

Proof. Since a manifold with corners is regular if and only if it is nice, the assertion follows clearly from Remark 5.1.26. □

Corollary 5.1.28. *If M is a regular manifold with corners and N is a neat submanifold of M, then N is a regular manifold with corners.*

Proof. Since being regular is equivalent to being nice, let us assume that M is a nice manifold with corners of dimension n and N is a neat submanifold of M of codimension k. Let us show that N is regular.

Let f be a codimension-l face of N and $p \in f \cap (\Sigma_s(M) - \Sigma_{s+1}(M))$, then there is a local chart

$$\phi : (U, U \cap N, p) \xrightarrow{\cong} (\mathbb{R}_+^s \times \mathbb{R}^{n-s}, \mathbb{R}_+^s \times \mathbb{R}^{n-k-s}, 0). \tag{5.1.28}$$

By Lemma 5.1.24, $\phi(U \cap f)$ is the union of a collection of codimension-l faces of $\mathbb{R}_+^s \times \mathbb{R}^{n-k-s}$. It is also clear from Remark 5.1.18(2) that f is contained in a codimension-l face F of M.

Since M is nice, there are l different facets $\{F_1, \ldots, F_l\}$ of M containing F. The intersections $U \cap F_i$'s are then mapped to distinct facets of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$ under ϕ and the image of the intersection

$$\phi\left(\bigcap_{i=1}^{l}(U \cap F_i)\right) \supseteq \phi(U \cap f) \tag{5.1.29}$$

is exactly one codimension-l face of $\mathbb{R}_+^s \times \mathbb{R}^{n-s}$ which contains only one codimension-l face of $\mathbb{R}_+^s \times \mathbb{R}^{n-k-s}$. So $\phi(U \cap f)$ is exactly one face of codimension-l of $\mathbb{R}_+^s \times \mathbb{R}^{n-k-s}$ and $(U \cap N, \phi_{U \cap N})$ is the required local chart in Definition 5.1.20. □

Corollary 5.1.29. *If M is a regular manifold with corners and F is a face of M, then F is a regular manifold with corners.*

Proof. Similar to Remark 5.1.18(2), for a face F of M of codimension k, we have for any integer s that

$$\Sigma_s(F) - \Sigma_{s+1}(F) = (\Sigma_{s+k}(M) - \Sigma_{s+k+1}(M)) \cap F. \tag{5.1.30}$$

The proof is then similar to that of Corollary 5.1.28. □

Similar to the situation of smooth manifolds, we have the following Thom isomorphism for homology. We have to define Thom class first.

Definition 5.1.30. [11, 17.9] Let B a topological space and $\xi : E \to B$ be a real vector bundle over B of rank k. Let $F_b \in E$ denote the fibre of ξ over a point $b \in B$ and F_b^0 denote $F_b - \{0\}$. Let E^0 denote the complement of the image of zero section in E. A *Thom class* of this vector bundle is a cohomology class $t \in H^k(E, E^0; \mathbb{Z})$ such that the restriction to each fibre $t_b \in H^k(F_b, F_b^0; \mathbb{Z})$ is a basis of $H^k(F_b, F_b^0; \mathbb{Z})$.

Theorem 5.1.31 ([11, Theorem 17.9.4])**.** *Let $\xi : E \to B$ be a real vector bundle over a topological space B. There exists a Thom class of this vector bundle if and only if ξ is orientable.*

Proposition 5.1.32. *Let B be a topological space and E be an oriented vector bundle over B of rank k with a metric. Let $D(E)$ and $S(E)$ be the associated disc bundle and sphere bundle. For any coefficient ring R we have*

$$H_n(D(E), S(E); R) \simeq H_{n-k}(B; R). \tag{5.1.31}$$

The isomorphism is given by transpose of Thom isomorphism in cohomology. Explicitly, let $t \in H_k(E, E^0; \mathbb{Z})$ be a Thom class, then the isomorphism is given by:

$$H_{n+k}(D(E), S(E); R) \xrightarrow{\simeq} H_n(B; R) \tag{5.1.32}$$

$$c \to \pi_*(t \cap i_*(c)). \tag{5.1.33}$$

where π is the projection of the vector bundle and i is the inclusion $(D(E), S(E)) \to (E, E^0)$.

Proof. We have the following chain of maps:

$$H_{n+k}(D(E), S(E); R) \xrightarrow{i_*} H_{n+k}(E, E^0; R) \xrightarrow{t\cap} H_n(E; R) \xrightarrow{\pi_*} H_n(B; R), \tag{5.1.34}$$

Let us show that each map in the above chain is an isomorphism.

Let us look at the first map in (5.1.34). Since the inclusions $D(E) \to E$ and $S(E) \to E^0$ induce isomorphisms

$$H^*(D(E); R) \simeq H^*(E; R) \tag{5.1.35}$$

$$H^*(S(E); R) \simeq H^*(E^0; R), \tag{5.1.36}$$

it follows from the long exact sequence associated to the pairs $(D(E), S(E))$ and (E, E^0) that the inclusion $i : (D(E), S(E)) \to (E, E^0)$ induces an isomorphism

$$i_* : H_n(D(E), S(E); R) \xrightarrow{\simeq} H_n(E, E^0; R). \tag{5.1.37}$$

It follows from [11, Theorem 18.1.2] that the second map in (5.1.34) is an isomorphism.

The last map in (5.1.34) is isomorphism because B is a deformation retract of E. □

Definition 5.1.33. Let M be a manifold with corners of dimension n and $N \subseteq M$ be a neat submanifold of M of codimension k. As pointed out in Remark 5.1.18, we can define the normal bundle of N in M and it is denoted $\nu(N; M)$. Since N is second countable, we can always find a vector bundle metric on $\nu(N; M)$ and denote the associated disc and sphere bundle as $D_M(N)$ and $S_M(N)$.

It was pointed out in [13, Section A.4] that neat submanifold of a regular manifold with corners admits a tubular neighborhood. Combining this fact with Proposition 5.1.32 we get

Corollary 5.1.34. *Let M be a regular manifold with corners and N be a neat submanifold of M. If the normal bundle $\nu(N; M)$ is orientable, then we have the following isomorphims for any coefficient ring R:*

$$H_{*-k}(N; R) \simeq H_*(D_M(N), S_M(N); R) \simeq H_*(M, M - N; R) \tag{5.1.38}$$

where the first isomorphism is the inverse of the map (5.1.32) and the second isomorphism is induced by inclusion.

Proof. The first map in (5.1.38) is an isomorphism because of Proposition 5.1.32.

The second map is the composition of the following maps:

$$H_*(D_M(N), S_M(N); R) \xrightarrow{i_*} H_*(\nu(N; M), \nu^0(N; M); R) \to H_*(M, M - N; R) \tag{5.1.39}$$

where $i : (D_M(N), S_M(N)) \to (\nu(N; M), \nu^0(N; M))$ is the inclusion.

We have shown in the proof of Proposition 5.1.32 that the first map in (5.1.39) is an isomorphism. It follows from the embedding of $\nu(N; M)$ as a tubular neighborhood of N in M and excision that the second map in (5.1.39) is an isomorphism. □

5.2 Morse-Bott theory on manifolds with corners

Now we want to compute the homology of a manifold with corners using Morse-Bott theory. In [13, Appendix B] a Morse-Bott theory for manifolds with corners was developed and we are going to slightly generalize it to prove a lacunary principle for Morse-Bott functions on manifolds with corners.

First we need a notion of critical points and critical submanifolds as in the usual situation of Morse-Bott theory for smooth manifolds.

Definition 5.2.1. [13, Definition B.1] Let M be a manifold with corners and $f \in C^\infty(M)$. A point $p \in M$ is called a *critical point* of f if the cone of tangent direction C_pM is disjoint from $\{X \in T_pM : X(f) < 0\}$.

Remark 5.2.2. [13, (B.1)] If p is a critical point of f, then

$$df_p|_{T_p^S M} = 0 \tag{5.2.1}$$

where $T_p^S M$ is the tangent space to the stratum defined in Definition 5.1.15.

Next we want to define Hessian of a smooth function on M at a critical point.

Definition 5.2.3. [13, B.3] Let M be a manifold with corners and $f \in C^\infty(M)$. Let $p \in M$ be a critical point of f. The *Hessian of f at p* is a symmetric bilinear form

$$H_p(f) : T_p^S M \times T_p^S M \to \mathbb{R} \tag{5.2.2}$$

defined by

$$H_p(f)(X,Y) := X(\tilde{Y}(f)) = Y(\tilde{X}(f)) \tag{5.2.3}$$

where $\tilde{X}$ and $\tilde{Y}$ are smooth vector fields defined in a neighborhood of p in the stratum S_p containing p such that $\tilde{X}(p) = X$ and $\tilde{Y}(p) = Y$.

If M is a manifold with corners and N is a neat submanifold of M, then it can be observed from the definition of a neat submanifold that we have canonical isomorphism

$$T_p^S M/T_p^S N \simeq T_p M/T_p N = \nu_p(N;M) \tag{5.2.4}$$

where $\nu_p(N;M)$ is the fibre of the normal bundle of N in M at point $p \in N$. Furthermore, it was shown in [13, B.4] that if N consists of critical points of a smooth function $f \in C^\infty(M)$, then the Hessian defined in Definition 5.2.3 induces a linear map

$$H_p(f) : T_p^S M/T_p^S N \otimes T_p^S M/T_p^S N \to \mathbb{R}, \forall p \in N \tag{5.2.5}$$

and thus a linear map

$$H_p(f) : \nu_p(N;M) \otimes \nu_p(N;M) \to \mathbb{R} \tag{5.2.6}$$

which depends smoothly on the point p. We will call the following map Hessian of the function f:

$$H(f) : \nu(N;M) \otimes \nu(N;M) \to \mathbb{R}. \tag{5.2.7}$$

Definition 5.2.4. [13, Definition B.7] Let M be a regular manifold with corners and $f \in C^\infty(M)$. A neat submanifold N of a face F of M is a *nondegenerate critical submanifold* of f if the following three conditions are satisfied

1. All points of N are critical points of f according to Definition 5.2.1.

2. The Hessian $H(f) : \nu(N;F) \otimes \nu(N;F) \to \mathbb{R}$ of f is nondegenerate, that is, for any $p \in N$, the Hessian $H_p(f) : \nu_p(N;M) \otimes \nu_p(N;M) \to \mathbb{R}$ is nondegenerate.

3. For each point $p \in N$, one has $C_pM \cap \ker(df_p : T_pM \to \mathbb{R}) = T_p^S M$, that is, when restricted to C_pM, df_p only vanishes on the tangent space to the stratum.

Definition 5.2.5. [13, (B.7)] Let M be a regular manifold with corners and f be a smooth function on M. Suppose that C is a connected nondegenerate critical submanifold of f which is a neat submanifold of a face F of M, then the normal bundle of C in F splits naturally as the Whitney sum of positive and negative subbundles

$$\nu(C; F) = \nu^+(C; F) \oplus \nu^-(C; F). \tag{5.2.8}$$

We call $\nu^-(C; F)$ the *negative normal bundle* of C in F. The rank of $\nu^-(C; F)$ is called the *index* of the critical submanifold C and is denoted $ind_f(C)$. Given a metric on the normal bundle, we can define the disc bundle and sphere bundle associated to the negative normal bundle under that metric and denote them $D(C)$ and $S(C)$ when there is no ambiguity on the face F.

Definition 5.2.6. [15, Definition 2.1] Let $k \geq 2$ be an integer. A manifold with corners C is called *k-lacunary* if the homology groups $H_*(C; \mathbb{Z})$ are free abelian groups and are trivial in all dimensions not divisible by k.

Remark 5.2.7. It follows from a universal coefficient theorem for homology that if C is k-lacunary for some $k \geq 2$, then for any coefficient ring R, the homology groups $H_*(C; R)$ are free R-modules and are trivial in all dimensions not divisible by k.

Definition 5.2.8. Let M be a regular manifold with corners. A smooth function $f : M \to \mathbb{R}$ is called a *Morse-Bott function* on M if the critical point set of f consists of a collection of nondegenerate critical submanifolds. The set of connected nondegenerate critical submanifolds of f is denoted $Crit(f)$.

We have a lacunary principal for Morse-Bott functions on manifolds with corners similar to the one developed in [15, Proposition 2.2]. The proof is also similar.

Proposition 5.2.9 (A lacunary principle for Morse-Bott functions on manifolds with corners)**.** *Let M be a compact regular manifold with corners and $k \geq 2$ be an integer. R is any coefficient ring. Let f be a Morse-Bott function on M. If each $C \in Crit(f)$ is k-lacunary and $ind_f(C)$ is divisible by k, then the Morse-Bott function is perfect, i.e.*

$$H_*(M;R) \simeq \bigoplus_{C \in Crit(f)} H_{*-ind_f(C)}(C;R). \tag{5.2.9}$$

Now we start proving Proposition 5.2.9. We first need a lemma to prove that there are only finitely many connected critical submanifolds in order to prove Proposition 5.2.9 by induction.

Lemma 5.2.10. *If f is a Morse-Bott function on a manifold with corners M, then the set of critical points is closed in M.*

Proof. Let us prove the lemma by showing that if $\{p_i\}_{i=1}^{\infty}$ is a sequence of critical points in M that accumulate to a point $p \in M$, then p is also a critical point.

Assume that M is a manifold with corners of dimension n and $p \in \Sigma_s(M) - \Sigma_{s+1}(M)$ for some s. Then there is a local chart (U, ϕ) containing p such that

$$\phi : (U, p) \xrightarrow{\simeq} (\mathbb{R}_+^s \times \mathbb{R}^{n-s}, 0). \tag{5.2.10}$$

Without loss of generality, we can assume $\{p_i\}_{i=1}^{\infty} \subseteq U$.

Let us verify for p the definiton of a critical point in [13, Definiton B.1]. It is clear from the local chart (5.2.10) that for any $X \in C_p(M)$, there is a vector field $\tilde{X}$ on U such that

$$\tilde{X}(u) \in C_u(M), \forall u \in U \text{ and} \tag{5.2.11}$$

$$\tilde{X}(p) = X. \tag{5.2.12}$$

Since df_u is varying smoothly on u and every p_i is a critical point, we have from definition of a critical point that

$$df_p(X) = \lim_{i\to\infty} df_{p_i}(\tilde{X}(p_i)) \geq 0. \quad (5.2.13)$$

Since this is true for any $X \in C_p(M)$, p is a critical point. □

Corollary 5.2.11. *A Morse-Bott function on a compact regular manifold with corners has only finitely many connected nondegenerate critical submanifolds.*

Proof. Let $\{C_i\}_{i\in I}$ be the collection of all connected critical submanifolds of some Morse-Bott function f on M. By [13, Corollary B.11], every connected nondegenerate critical submanifold C_i can be seperated from other connected nondegenerate critical submanifolds by an open set U_i, that is, there is a collection of open sets $\{U_i\}_{i\in I}$ such that

$$U_i \cap C_j = \emptyset \text{ if } i \neq j. \quad (5.2.14)$$

Let U be the complement of the critical point set. By Lemma 5.2.10, U is open. We now have a cover of M by open sets.

$$M = \left(\bigcup_{i\in I} U_i\right) \cup U. \quad (5.2.15)$$

Since M is compact, there is a finite cover and thus I is finite. □

We will need a corollary of [13, Theorem B.9] to compute the homology using Morse-Bott theory. We will use M^y to denote $f^{-1}((-\infty, y])$ in the next lemma.

Lemma 5.2.12 (A modest generalization of theorem B.9 in [13])**.** *Let M be a regular manifold with corners and $f : M \to \mathbb{R}$ be a smooth function. Suppose that $a, b \in \mathbb{R}$ are regular values of f, the interval $[a, b]$ contains only one critical value c, $f^{-1}([a, b])$ is compact and the set of all critical points in $f^{-1}(c)$ forms finitely many connected nondegenerate critical submanifolds $C_1, \ldots, C_k$ which are neat submanifolds of faces $F_1, \ldots, F_k$. Then M^b is homotopy equivalent to the space obtained from M^a by attaching disk bundles $D(C_i)$ to M^a along $S(C_i)$ for $i = 1, \ldots, k$.*

Proof. Since M is Hausdorff and each C_i is compact, there are open sets U_i's seperating C_i's, that is, there are open sets U_i's satisfying

$$C_i \subseteq U_i \text{ and} \tag{5.2.16}$$

$$U_i \cap U_j = \emptyset \quad \text{for } i \neq j. \tag{5.2.17}$$

The rest of the proof is similar to that of [13, B.7, Proof of theorem B.9]. We can further shrink these U_i's such that they satisfy the same property as the open set U_1 in [13, B.7, Proof of theorem B.9]. Since U_i's are disjoint open sets, as in [13, B.7, Proof of theorem B.9] one can construct a function $\Phi : M \to \mathbb{R}$ such that Φ coincides with f outside $\bigcup_{i=1}^{k} U_i$ and in each U_i the function Φ is given by the same formula as in [13, Before assertion 1 in proof of Theorem B.9].

Then the assertion 1, assertion 2 and assertion 3 in [13, Proof of Theorem B.9] still hold for this function Φ. The assertion 4 is still true once we substitute the set $M^{c-\epsilon} \cup D(N)$ there by $M^{c-\epsilon} \cup \left(\bigcup_{i=1}^{k} D(C_i) \right)$. □

With the preparation above, now we can prove Proposition 5.2.9. The proof is essentially the same as the proof of [15, Proposition 2.2].

proof of Proposition 5.2.9. We suppress the homology coefficient R in the proof.

Since $Crit(f)$ is finite by Corollary 5.2.11, there are only finitely many critical values. Thus we can find regular values $t_0 < t_1 < \cdots < t_k$ of f such that all critical values lie in (t_0, t_k) and each interval (t_i, t_{i+1}) contains exactly one critical value of f. Let us denote $f^{-1}((-\infty, t_i])$ by M^i and prove the following statement by induction on i:

$$H_*(M^i) \simeq \bigoplus_{\substack{C \in Crit(f) \\ f(C) < t_i}} H_{*-ind_f(C)}(C) \tag{5.2.18}$$

When $i = 1$, the critical value $c < t_1$ of f is the minimal value of f on M and thus for a connected nondegenerate critical submanifold C of some face F with $f(C) = c$, we have

$ind_f(C) = 0$ because f can only have value greater than or equal to c in a neighborhood of C and thus $\nu^-(C; F)$ has rank 0. It then follows from Lemma 5.2.12 that M^1 is homotopy equivalent to the disjoint union of $C \in Crit(f)$ with $f(C) = c$. So we have

$$H_*(M^1) \simeq \bigoplus_{\substack{C \in Crit(f) \\ f(C) < t_1}} H_*(C). \tag{5.2.19}$$

Let us assume that we have proved (5.2.18) for $i - 1$ and let us try to prove it for i.

We have the following exact sequence for the pair (M^i, M^{i-1}):

$$H_{*+1}(M^i, M^{i-1}) \to H_*(M^{i-1}) \to H_*(M^i) \to H_*(M^i, M^{i-1}) \to H_{*-1}(M^{i-1}). \tag{5.2.20}$$

Since there is exactly one critical value in the interval (t_{i-1}, t_i), it follows from Lemma 5.2.12 and excision that we have

$$H_*(M^i, M^{i-1}) \simeq \bigoplus_{\substack{C \in Crit(f) \\ t_{i-1} < f(C) < t_i}} H_{*-ind_f(C)}(D(C), S(C)) \tag{5.2.21}$$

where $D(C)$ and $S(C)$ are defind in Definition 5.2.5.

Since every $C \in Crit(f)$ of some face F is k-lacunary with $k \geq 2$, we have $H^1(C) = 0$ and thus the first Stiefel-Whitney class of the $\nu^-(C; F)$ vanishes. So $\nu^-(C; F)$ is orientable and we can apply the isomorphism in Proposition 5.1.32 to get

$$H_*(M^i, M^{i-1}) \simeq \bigoplus_{t_{i-1} < f(C) < t_i} H_{*-ind_f(C)}(C). \tag{5.2.22}$$

Now, since by assumption every critical submanifold is k-lacunary and $ind_f(C)$ is divisible by k, every summand on the right hand side of (5.2.22) is concentrated in degrees divisible by k and thus $H_*(M^i, M^{i-1})$ is concentrated in degrees divisible by k. Then in the exact sequence (5.2.20) with degree $*$ divisible by k, the two terms on both ends vanish and we have the following exact sequence for degree $*$ divisible by k:

$$0 \to H_*(M^{i-1}) \to H_*(M^i) \to H_*(M^i, M^{i-1}) \to 0. \tag{5.2.23}$$

Since by definition of k-lacunary and (5.2.21), $H_*(M^i, M^{i-1})$ is free and thus the exact sequence (5.2.23) splits. So we have

$$H_*(M^i) \simeq H_*(M^{i-1}) \oplus H_*(M^i, M^{i-1}). \tag{5.2.24}$$

The induction hypothesis and (5.2.22) then give us (5.2.18). □

We need another proposition similar to [15, Proposition 2.3] to get a homology basis of a manifold with corners.

We first prove a lemma.

Lemma 5.2.13. *Let M be a regular manifold with corners and F is a codimension-s face of M, then F is a smooth manifold without boundary if and only if F does not intersect $\Sigma_{s+1}(M)$. Any two such faces of M are disjoint.*

Proof. Assume that $\dim M = n$. Let us first show that if F is a smooth manifold without boundary, then F does not intersect $\Sigma_{s+1}(M)$. Assume that there is $p \in F \cap (\Sigma_l(M) - \Sigma_{l+1}(M))$ for some $l > s$. Since M is regular, there is local chart

$$\phi : (U, F, p) \xrightarrow{\simeq} (\mathbb{R}^l_+ \times \mathbb{R}^{n-l}, \mathbb{R}^{l-s}_+ \times \mathbb{R}^{n-l+s}, 0). \tag{5.2.25}$$

Since $l > s$, we can see from the above chart that F is not a smooth manifold without boundary and that is a contradiction.

Now let us show that if F does not intersect $\Sigma_{s+1}(M)$, then F is a smooth manifold without boundary. We assume that F is the closure of a connected component N of $\Sigma_s(M) - \Sigma_{s+1}(M)$.

We have seen in Remark 5.1.13 that the $\Sigma_s(M) - \Sigma_{s+1}(M)$ is a smooth manifold without boundary. Thus a connected componenet N of $\Sigma_s(M) - \Sigma_{s+1}(M)$ is also a smooth manifold without boundary. Let us show that N is closed in M.

Let us show that if $\{p_i\}_{i=1}^{\infty}$ is a sequence of points in $F \cap (\Sigma_s(M) - \Sigma_{s+1}(M))$ and it converges to a point p, then $p \in F \cap (\Sigma_s(M) - \Sigma_{s+1}(M))$. Indeed, assume that $p \in \Sigma_l(M) - \Sigma_{l+1}(M)$ for some l, then there is a local chart (U, ϕ) containing p and we have

$$\phi : (U, p) \xrightarrow{\cong} (\mathbb{R}_+^l \times \mathbb{R}^{m-l}, 0). \tag{5.2.26}$$

Then every point in U belongs to a stratum of codimension less than or equal to l. Since p_i's accumulate to p and every p_i belongs to the stratum of codimension s, we have $l \geq s$. Since F does not intersect $\Sigma_{s+1}(M)$, the point p can only belong to $\Sigma_s(M) - \Sigma_{s+1}(M)$ and thus N is closed.

Since the face F is the closure of N, we have $F = N$ and thus F is a smooth manifold without boundary. It follows directly from our proof that any two such faces of M are disjoint.

□

Proposition 5.2.14. *In addition to the assumption of Proposition 5.2.9, we assume that every $C \in Crit(f)$ is an orientable submanifold of some face F_C such that F_C is a smooth manifold without boundary. We assume in further that for every $C \in Crit(f)$ we have an orientable closed submanifold W_C of F_C and a finite collection V_C of orientable closed submanifolds of W_C such that the following conditions are satisfied:*

1. *The restriction $f|_{F_C}$ is a perfect Morse-Bott function on F_C for every $C \in Crit(f)$.*

2. *For every $C \in Crit(f)$, we have $C \subseteq W_C$ and $\dim W_C = ind_f(C) + \dim C$.*

3. *The function $f|_{W_C}$ is nondegenerate and achieves maximum on C.*

4. *Each $V \in V_C$ is transversal to C as a submanifold of W_C.*

5. *The set of homology classes $[V \cap C] \in H_*(C; R)$ for all $V \in V_C$ forms a basis of $H_*(C; R)$ where $[\cdot]$ denotes the fundamental class.*

Then the set of homology classes $[V] \in H_*(M; R)$ *for all* $V \in V_C$ *and all* $C \in Crit(f)$ *form a basis of the free* R*-module* $H_*(M; R)$.

Proof. We suppress the homology and homology coefficient R to simplify the notation.

We are going to prove this proposition as in [15, Proposition 2.3] using induction. As in the proof of Proposition 5.2.9 we assume that we have regular values $t_0 < t_1 < \cdots < t_k$ such that all critical values lie in (t_0, t_k) and each interval (t_i, t_{i+1}) contains only one critical value of f. We also denote $f^{-1}((-\infty, t_i])$ by M^i. We are going to prove the following statement for every $i = 1, \cdots, k$:

$$\begin{array}{c}\text{The set of homology classes } [V] \in H_*(M^i) \text{ for all } V \in V_C \text{ and } C \in Crit(f) \\ \text{with } f(C) < t_i \text{ forms a basis of the free } R\text{-module } H_*(M^i).\end{array} \tag{*}$$

Note that $W_C \subseteq M^i$ if $f(C) < t_i$ because f achieves maximum on C in W_C.

For $i = 1$, the interval $(-\infty, t_1]$ contains only one critical value c_1 of f which is also the minimal value of f on M. For any $C \in Crit(f)$ such that $f(C) = c_1$, $f|_{W_C}$ reaches both minimum and maximum on C which implies that $W_C = C$ and $V \subseteq C$ for any $V = V_C$. Since c_1 is the minimal value of f on M, we have $M^0 = \emptyset$. By [13, Theorem B.4] the disjoint union

$$\bigsqcup_{f(C)=c_1} C \tag{5.2.27}$$

is a deformation retract of M^1. Since $[V \cap C] = [V]$ in this case, it follows from condition(5) that (*) is true in this case.

Now we assume that the statement holds for i, let us prove it for $i + 1$. It follows from the exact sequence (5.2.23) that we only have to show that the image of all $[V] \in H_*(M^{i+1})$ in (*) under the map $H_*(M^{i+1}) \to H_*(M^{i+1}, M^i)$ form a basis of $H_*(M^{i+1}, M^i)$.

Let us denote

$$F := \bigcup_{C \in Crit(f)} F_C. \tag{5.2.28}$$

According to Lemma 5.2.13, F is the disjoint union of several smooth manifolds without boundary.

We have seen in Definition 5.2.1 and Remark 5.2.2 that critical points of f consist of critical points of each stratum satisfying some additional condition. Since all critical points of f are in F which is the disjoint union of several smooth manifolds without boundary, this observation implies that critical points of f is a subset of critical points of $f|_F$ as a function on F. Since $f|_{F_C}$ is also a Morse-Bott function for every $C \in Crit(f)$, we have $Crit(f) \subseteq Crit(f|_F)$ and the following inclusion map:

$$\bigoplus_{\substack{C \in Crit(f) \\ t_i < f(C) < t_{i+1}}} H_{*-ind_f(C)}(C) \hookrightarrow \bigoplus_{\substack{C \in Crit(f|_F) \\ t_i < f|_F(C) < t_{i+1}}} H_{*-ind_f(C)}(C). \tag{5.2.29}$$

Furthermore, it was proved in [15, Proof of Proposition 2.2] that via Morse-Bott theory on smooth manifolds, we have the following isomorphism induced by inclusions:

$$\bigoplus_{\substack{C \in Crit(f|_F) \\ t_i < f|_F(C) < t_{i+1}}} H_{*-ind_f(C)}(C) \xrightarrow{\simeq} H_*(F^{i+1}, F^i) \tag{5.2.30}$$

where $F^i = M^i \cap F$. Thus we have the following diagram:

$$\begin{array}{ccc} \bigoplus_{\substack{C \in Crit(f|_F) \\ t_i < f|_F(C) < t_{i+1}}} H_{*-ind_f(C)}(C) & \longleftarrow & \bigoplus_{\substack{C \in Crit(f) \\ t_i < f(C) < t_{i+1}}} H_{*-ind_f(C)}(C) \\ \downarrow \simeq & \swarrow & \downarrow \simeq \\ H_*(F^{i+1}, F^i) & \longrightarrow & H_*(M^{i+1}, M^i). \end{array} \tag{5.2.31}$$

The lower right triangle is trivially commuting. The upper left triangle is commuting because the embeddings of the disc and sphere bundle of C in F given by the generalized Morse-Bott lemma in [13, Lemma B.10] is exactly the same as the embeddings given by the usual Morse-Bott lemma on smooth manifold F. Passing to $H_*(C)$ using Thom isomorphism, we can see that the upper left triangle in the diagram (5.2.31) commutes.

For any $V \in V_C$ as in the assumption of this proposition, it was shown in [15, Proof of Proposition 2.3] that inverse image of $[V, V \cap F^i] \in H_*(F^{i+1}, F^i)$ under the left vertical map in (5.2.31) is $[V \cap C] \in H_*(C)$. Thus the inverse image of $[V, V \cap M^i] \in H_*(M^{i+1}, M^i)$ under the right horizontal map in (5.2.31) is also $[V \cap C] \in H_*(C)$. By assumption (5) the set of $[V \cap C] \in H_*(C)$ for all $V \in V_C$ forms a basis of $H_*(C)$. We have shown the claim (*).

□

The lacunary principle we proved works only for compact regular manifolds with corners. However, the spaces we are going to apply our lacunary principle on are usually non-compact. A common practice is to deformation retract onto a compact space. To apply lacunary principle on the deformation retract, we have to show that this deformation retract is a regular manifold with corners.

For a smooth function $f : M \to \mathbb{R}$ on a manifold with corners M and a value $q \in \mathbb{R}$ satisfying some regularity condition, we first show in the following lemma that the sublevel set $f^{-1}((-\infty, q])$ is a manifold with corners for any $q \in \mathbb{R}$ and the level set $f^{-1}(q)$ has a tubular neighborhood such that f on this tubular neighborhood is exactly a projection map. This local property of level set is similar to [23, Chapter III, Proposition 5.1].

Lemma 5.2.15. *Let M be a manifold with corners and $f : M \to \mathbb{R}$ be a smooth function on M. Assume that $df_p(T^S_p M) = T_q\mathbb{R}$ for any $p \in f^{-1}(q)$. Then $f^{-1}((-\infty, q])$ is a manifold with corners. Furthermore, if $f^{-1}(q)$ is compact, then there is a neighborhood U of q in $\mathbb{R}$ and embedding $f^{-1}(q) \times U \to M$ such that the diagram*

$$\begin{array}{ccc} f^{-1}(q) \times U & \xrightarrow{\ j\ } & M \\ & {\scriptstyle pr_U} \searrow \quad \swarrow {\scriptstyle f} & \\ & U & \end{array} \tag{5.2.32}$$

commutes where pr_U is the projection onto the component U.

Proof. Since we have $df_p(T^S_p M) = T_q\mathbb{R}$ for any $p \in f^{-1}(q)$, it follows from Proposition 5.1.19 that $f^{-1}(q)$ is a neat submanifold of M and its normal bundle in M is trivial. Thus $f^{-1}(q)$ has a tubular neighborhood in M that is diffeomorphic to $f^{-1}(q) \times \mathbb{R}$ with $f^{-1}(q)$ mapped identically to $f^{-1}(q) \times \{0\}$. Then an application of [27, Theorem 3] shows that $f^{-1}((-\infty, q])$ is a manifold with corners.

For the second part of the lemma, the proof is similar to that of [23, Chapter III, Proposition 5.1]. We start with a tubular neighborhood $f^{-1}(q) \times \mathbb{R}$ of $f^{-1}(q)$ in M. It is enough to show the following statement similar to [23, Chapter III, Proposition 5.1, statement (*)]:

> There is a neighborhood U of q such that for every $p \in f^{-1}(q)$ the map f restricted to $U_p = f^{-1}(U) \cap (p \times \mathbb{R})$ is a diffeomorphism of U_p onto U. (*)

To prove (*), we consider the following smooth map between manifolds with corners:

$$h : f^{-1}(q) \times \mathbb{R} \to f^{-1}(q) \times \mathbb{R} \tag{5.2.33}$$

$$h(q, v) = (q, f(q, v)). \tag{5.2.34}$$

Since our f satisfies $df_p(T_p^S M) = T_{f(p)}\mathbb{R}$ for any $p \in M$, the differential of h has maximal rank at $(p, 0)$. Furthermore, since a smooth map between manifolds with corners can be locally extended to a smooth map between local charts that does not involve corners, it follows from the inverse function theorem that the map h is locally a diffeomorphism. The rest of the proof is exactly the same as that of [23, Chapter III, Proposition 5.1]. □

Now we show that the sublevel set is a regular manifold with corners.

Corollary 5.2.16. *Let M be a regular manifold with corners and $f : M \to \mathbb{R}$ be a smooth map of manifolds with corners. Assume that $df_p(T_p^S M) = T_q\mathbb{R}$ for any $p \in f^{-1}(q)$. If $f^{-1}(q)$ is compact, then $f^{-1}((-\infty, q])$ is a regular manifold with corners.*

Proof. We have shown in Lemma 5.2.15 that $f^{-1}((-\infty, q])$ is a manifold with corners, we have to show that it is regular. By Proposition 5.1.23 it suffices to prove that $f^{-1}((-\infty, q])$ is a nice manifold with corners. By Remark 5.1.26, we just have to show the following statement.

$$\begin{aligned}&\text{Every point } p \text{ of } f^{-1}((-\infty, q]) \text{ in the codimension-}k \text{ stratum of } f^{-1}((-\infty, q])\\ &\text{is contained in } k \text{ different facets of } f^{-1}((-\infty, q]).\end{aligned} \tag{*}$$

For $p \in f^{-1}((-\infty, q))$, (*) is trivially true because M is regular and $f^{-1}((-\infty, q))$ is an open subset of M.

Let $N = f^{-1}(q)$. Let us look at the case when $p \in N$. We choose a small enough $\epsilon > 0$ such that $(q - \epsilon, q + \epsilon)$ is contained in the neighborhood U of q in Lemma 5.2.15. Since the normal bundle of N in M is trivial by Proposition 5.1.19, $f^{-1}((q - \epsilon, q + \epsilon))$ is diffeomorphic to $N \times (-\epsilon, \epsilon)$. The set $f^{-1}((q - \epsilon, q])$ is then diffeomorphic to $N \times (-\epsilon, 0]$. We will identify $f^{-1}((q - \epsilon, q + \epsilon))$ with $N \times (-\epsilon, \epsilon)$ and $f^{-1}((q - \epsilon, q])$ with $N \times (-\epsilon, 0]$.

Since $p \in N$ is in the codimension-k stratum of $f^{-1}((-\infty, q])$, p is in the codimension-$(k-1)$ stratum of $f^{-1}((-\infty, q + \epsilon))$. Since $f^{-1}((-\infty, q + \epsilon))$ is regular, p is contained in $k - 1$ different facets of $f^{-1}((-\infty, q + \epsilon))$. The restriction of these facets to $N \times (-\epsilon, \epsilon)$ breaks up to different facets of $N \times (-\epsilon, \epsilon)$. Different facets of $N \times (-\epsilon, \epsilon)$ are just the product of different facets of N and $(-\epsilon, \epsilon)$. So different facets of $N \times (-\epsilon, 0]$ are just the product of different facets of N and $(-\epsilon, 0]$ and they are contained in the restriction of different facets of $f^{-1}((-\infty, q + \epsilon))$. We've got $k - 1$ different facets of $f^{-1}((-\infty, q])$ that contains p.

We need one more different facet that contains p. Let us show that closure of the connected component containing p of the codimension-0 stratum in N is a facet of $f^{-1}((-\infty, q])$. It suffices to show that the codimension-0 stratum in N is disjoint from codimension-1 stratum in M. But this is obvious because a point in the codimension-0 stratum in N is in the codimension-0 stratum of M. □

Chapter 6

Homologies of faces of the orbit space

In this chapter, we first give some examples of manifolds with corners. These examples are related to the computation we are going to carry on. Then we start computing homology of each face of the orbit space of a big polygon space $X_{a,b}(l)/T$ and the maps connecting them induced by inclusions of faces. Homologies in this chapter will always be singular homology with real coefficient.

Before we start the computation, recall that we made the following assumption in Chapter 4:

Assumption. *The length vector* $l = (l_1, \dots, l_r)$ *satisfies the condition* $0 < l_1 \leq l_2 \leq \cdots \leq l_r$.

Throughout this chapter, we are also going to make the following assumption. The reason is that in Lemma 4.1.8 we proved that the T-action on $X_{a,b}(l)$ is locally standard if the fixed point set is not empty and we only want to work with locally standard actions from now on:

Assumption. *We assume that the fixed point set of* $X_{a,b}(l)$ *under* T*-action is not empty. According to Lemma 4.1.8(1), the* T*-action on* $X_{a,b}(l)$ *is locally standard.*

6.1 Examples of manifolds with corners

We want to give some examples of manifolds with corners. We will prove that they are manifolds with corners in Lemma 6.1.3. The reason of giving these examples is that they are closely related to the orbit space $X_{a,b}(l)/T$. We will construct a manifold with corners $F_\varnothing$ that has the same face structure as that of $X_{a,b}(l)/T$. Furthermore, two corresponding faces of $F_\varnothing$ and $X_{a,b}(l)/T$ are homeomorphic to each other. So in order to compute the homology of some face of $X_{a,b}(l)/T$ we only have to compute homology of the corresponding face of $F_\varnothing$. Note that we are not going to answer the question that whether the smooth structure of $X_{a,b}(l)/T$ is equivalent to that of $F_\varnothing$ because homology is a topological invariant and it suffices to look at the face structure and topology of each face.

Example 6.1.1. The following subspace of $\mathbb{C}^{ar}$ has a smooth structure as a manifold with corners and is denoted $F_\varnothing$.

$$F_\varnothing := \left\{(u_1, \ldots, u_r) \in \mathbb{C}^{ar} : \sum_{i=1}^{r} l_i u_i = 0; \|u_i\| \leq 1, \forall i = 1, \ldots, r\right\}. \tag{6.1.1}$$

A codimension-k face of $F_\varnothing$ is F_I defined by the following equation for some subset $I \subseteq [r]$ of cardinality k:

$$F_I := \{(u_1, \ldots, u_r) \in F_\varnothing : \|u_i\| = 1, \forall i \in I\}. \tag{6.1.2}$$

Example 6.1.2. The following subspace $M_\varnothing$ of $\mathbb{C}^{ar}$ is a product of closed unit balls and has a smooth structure as a manifold with corners.

$$M_\varnothing := \{(u_1, \ldots, u_r) \in \mathbb{C}^{ar} : \|u_i\| \leq 1, \forall i = 1, \ldots, r\}. \tag{6.1.3}$$

A codimension-k face of $M_\varnothing$ is M_I defined by the following equation for some subset $I \subseteq [r]$ of cardinality k:

$$M_I := \{(u_1, \ldots, u_r) \in M_\varnothing : \|u_i\| = 1, \forall i \in I\}. \tag{6.1.4}$$

Note that F_I in Example 6.1.1 is a subspace of M_I.

We now prove that the examples above are indeed manifolds with corners.

Lemma 6.1.3. *For any subset I of $[r]$,* *(a) M_I is a regular manifold with corners. A face of M_I is M_J for some $J \supseteq I$.*

(b) F_I is a neat submanifold of M_I. Its normal bundle in M_I is trivial.

(c) $M_I - F_I$ is a regular manifold with corners.

Proof. (a): Every M_I is a product of spheres and closed balls. Since every sphere or closed ball has a standard chart as a manifold with corners, M_I with the product of these charts is a manifold with corners. It is also clear from this chart that M_I is regular and a face of M_I is M_J for some $J \supseteq I$.

(b): Let $f : M_I \to \mathbb{C}^a$ be

$$f(u) := \sum_{i=1}^{r} l_i u_i. \tag{6.1.5}$$

Then F_I is $f^{-1}(0)$. We want to verify the conditions in Proposition 5.1.19 to prove that F_I is a neat submanifold of M_I.

Fixing an integer k, a codimension-k stratum S_k of M_I is of the form

$$S_k = \bigcup_{\substack{|J|=r-k \\ J\supseteq I}} M_J - \bigcup_{\substack{|K|=r-k+1 \\ K\supseteq I}} M_K. \tag{6.1.6}$$

For a point $p \in M_J \subseteq S_k$, the tangent space to the stratum at p is

$$T_p^S M_I = \{(v_1, \ldots, v_r) \in \mathbb{C}^{ar} : \langle u_i, v_i\rangle = 0,\ \forall i \in J\}. \tag{6.1.7}$$

where the inner product is the usual real inner product on $\mathbb{C}^a \simeq \mathbb{R}^{2a}$.

The differential df_p resticted to $T_p^S M_I$ is denoted df_p^S. Explicitly, it is

$$df_p^S(v) = \sum_{i=1}^{r} l_i v_i, v = (v_1, \ldots, v_r) \in T_p^S M_I. \tag{6.1.8}$$

We are going to show that for any $p \in S_k$ we have

$$df_p^S(T_p^S M_I) = T_0\mathbb{C}^a \simeq \mathbb{C}^a \tag{6.1.9}$$

as real vector spaces.

If $J \neq [r]$, then for some integer $i \in [r] - J$ and any vector $v_i \in \mathbb{C}^a$, the following vector

$$v = (0, \ldots, 0, v_i, 0, \ldots, 0) \tag{6.1.10}$$

is in $T_p^S M_I$ and we have

$$df_p^S(v) = l_i v_i. \tag{6.1.11}$$

So df_p^S is surjective onto $T_0\mathbb{C}^a$ since $l_i > 0$ and v_i is arbitrary.

If $J = [r]$, then either u has a zero component or all components of u are non-zero. If $u_i = 0$, then v_i in (6.1.7) can be any vector in $\mathbb{C}^a$, by the same argument as above, we can show that df_p^S is surjective.

If all coordinates of p are non-zero, then every component v_i of v in (6.1.7) is in a hyperplane in $\mathbb{C}^a$. Since the length vector l is generic and $\sum_{i=1}^r l_i u_i = 0$, not all the coordinates of u are on the same line. Thus there are i, j such that v_i and v_j are from different hyperplanes. The linear combination of vectors from these two different hyperplanes generates the whole space $\mathbb{C}^a$. That proves the surjectivity of df_p^S in this case.

The above argument shows that the assumption of Proposition 5.1.19 is satisfied. So F_I is a neat submanifold of M_I and its normal bundle in M_I is trivial.

(c): $M_I - F_I$ is an open subset of M_I. With the atlas inherited from M_I, it is a manifold with corners. Since M_I is regular, an open subset of it is also regular by Lemma 5.1.27. □

Remark 6.1.4. Note that we didn't show that a codimension-k face of $F_\varnothing$ is F_I for some subset $I \subseteq [r]$ of cardinality k as claimed in Example 6.1.1. The reason is that it is a little bit tricky to show that the interior of F_I in $F_\varnothing$ is connected which is required by definition of a face. We will delay the proof of this claim until Corollary 6.2.10.

The ultimate goal of this chapter is to compute the homology of faces of the orbit space $X_{a,b}(l)/T$. We first establish the following homeomorphism.

Lemma 6.1.5. *The map*

$$\psi : (\mathbb{S}^{2a+2b-1})^r \to M_\varnothing \tag{6.1.12}$$

$$(u_1, \ldots, u_r, z_1, \ldots, z_r) \mapsto (u_1, \ldots, u_r) \tag{6.1.13}$$

induces a homeomorphism $\bar{\psi}$ between the quotient space $(\mathbb{S}^{2a+2b-1})^r/T$ and $M_\varnothing \subseteq \mathbb{C}^{ar}$ with subspace topology.

Proof. ψ is obviously continuous and induces a continuous bijection between $(\mathbb{S}^{2a+2b-1})^r/T$ and $M_\varnothing \subseteq \mathbb{C}^{ar}$. Since $(\mathbb{S}^{2a+2b-1})^r/T$ is compact and $M_\varnothing$ is Hausdorff, ψ is a homeomorphism. □

Since $X_{a,b}(l)/T$ is a subspace of $(\mathbb{S}^{2a+2b-1})^r/T$ and the image of $X_{a,b}(l)/T$ under $\bar{\psi}$ is $F_\varnothing$, $X_{a,b}(l)/T$ is homeomorphic to $F_\varnothing$. We want to show that a face of $X_{a,b}(l)/T$ is mapped to a face of $F_\varnothing$ of the same codimension.

Lemma 6.1.6. *The homeomorphism $\bar{\psi}$ induces a bijection between faces of $X_{a,b}(l)/T$ and faces of $F_\varnothing$ of the same codimension. Two corresponding faces are homeomorphic to each other under this map.*

Proof. A face of codimension-k is the closure of a connected component of the codimension-k stratum of $X_{a,b}(l)/T$. Since $X_{a,b}(l)/T$ is homeomorphic to $F_\varnothing$ via $\bar{\psi}$, we only have to show that $\bar{\psi}$ maps a point in the codimension-k stratum of $X_{a,b}(l)/T$ to a point in the codimension-k stratum of $F_\varnothing$.

The inverse image of a point in the codimension-k stratum of $X_{a,b}(l)/T$ under the quotient map consists of points in $X_{a,b}(l)$ with exactly k vanishing z_i coordinates. The image of these points under ψ are points in $F_\varnothing$ with exactly k of the u_i coordinates such that $\|u_i\| = 1$ which is clearly in the codimension-k stratum. Thus a point in the codimension-k stratum of $X_{a,b}(l)/T$ is mapped to a point in the codimension-k stratum of $F_\varnothing$. □

To sum up, we have shown the following proposition.

Proposition 6.1.7. *The two manifolds with corners $X_{a,b}(l)/T$ and $F_\varnothing$ have the same face structure and the corresponding faces under $\bar{\psi}$ are homeomorphic.*

6.2 Homologies of faces of the orbit space

We want to compute homologies of faces of the orbit space of a big polygon space $X_{a,b}(l)$ in this section. According to Proposition 6.1.7, to compute homology of faces of $X_{a,b}(l)/T$, it suffices to compute the homologies of faces of $F_\varnothing$ which are F_I's as described in Example 6.1.1 (Note that we haven't proved that these F_I's are faces of $F_\varnothing$. The proof will be delayed until Corollary 6.2.10). To do that, we will first compute the homology of the complement of F_I in M_I and then use Corollary 5.1.34 to derive the homology of F_I. The computation in this section is very similar to the computation in [15] and [16]. The difference is that in [15] and [16] people apply Morse-Bott theory on smooth manifolds and here we are going to apply Morse-Bott theory on manifolds with corners.

With Proposition 6.1.7 and Lemma 6.1.3, we are now going to apply Morse-Bott theory we summarized before to $M_I - F_I$ for every $I \subseteq [r]$. The theory we are going to use is summarized in Section 5.2.

Let $f(u) := -\|\sum_{i=1}^r l_i u_i\|^2$ be a function defined on M_I. The face F_I is where f reaches global maximum 0. Although $M_I - F_I$ is not compact, there is some small regular value $\epsilon > 0$ such that $[-\epsilon, 0)$ contains no critical value of f and thus $M_I - F_I$ deformation retracts onto $f^{-1}((-\infty, -\epsilon])$. Since $-\epsilon$ is a regular value of f_I, $f^{-1}((-\infty, -\epsilon])$ is a compact regular manifold with corners and the set of critical points of f restricted to $f^{-1}((-\infty, -\epsilon])$ agrees with that of f. So we can apply Morse-Bott theory to $f^{-1}((-\infty, -\epsilon])$ in order to compute the homology of $M_I - F_I$.

We describe the critical submanifolds of f in the following lemma.

Lemma 6.2.1. *Let $f(u) := -\|\sum_{i=1}^r l_i u_i\|^2$ be a function defined on $M_I - F_I$. The set of critical* *points of f consists of P_K's for $K \subseteq I$ where*

$$P_K := \{u : \|u_i\| = 1, u_i = u_j = -u_k, \forall i, j \in K, k \notin K\}. \tag{6.2.1}$$

Every such P_J is a connected critical submanifold of the face $M_{[r]} - F_{[r]}$. It is diffeomorphic to a sphere $\mathbb{S}^{2a-1}$ and its Morse-Bott index equals $(2a-1)|K|$.

Proof. Assume that $p = (u_1, \ldots, u_r)$ is a point in the codimension-k stratum S_k of $M_I - F_I$. The differential df_p restricted to $T_p^S(M_I - F_I)$ is

$$df_p(v) = -2\Big\langle \sum_{i=1}^r l_i u_i, \sum_{i=1}^r l_i v_i \Big\rangle \quad \text{for } v = (v_1, \ldots, v_r) \in T_p^S(M_I - F_I). \tag{6.2.2}$$

Let us locate all the critical points of f.

The first observation is that critical points of f can only appear in the highest stratum $M_{[r]} - F_{[r]}$. Indeed, we can write out S_k explicitly as described in (6.1.6)

$$S = \bigcup_{|K|=k} (M_K - F_K) - \bigcup_{|J|=k+1} (M_J - F_J). \tag{6.2.3}$$

If p is a point in $M_K - F_K$ for some K in the above union, then we have as in (6.1.7) that

$$T_p^S(M_I - F_I) = \Big\{(v_1, \ldots, v_r) \in \mathbb{C}^{ar} : \langle u_i, v_i \rangle = 0, \forall i \in K\Big\}. \tag{6.2.4}$$

If $k < r$, then $K \neq [r]$. For an integer $j \in [r] - K$ the following vector

$$v = (0, \ldots, 0, v_j, 0, \ldots, 0) \tag{6.2.5}$$

is in $T_p^S(M_I - F_I)$ for any $v_j \in \mathbb{C}^a$. Since $p \in M_K - F_K$, we have $\sum_{i=1}^r l_i u_i \neq 0$ and thus

$$df_p(v) = \Big\langle \sum_{i=1}^r l_i u_i, \sum_{j=1}^r l_j v_j \Big\rangle = \Big\langle \sum_{i=1}^r l_i u_i, l_j v_j \Big\rangle \tag{6.2.6}$$

is not constantly equal to 0 on $T_p^S(M_I - F_I)$. By Remark 5.2.2, p is not critical.

Thus all the possible critical points are in $M_{[r]} - F_{[r]}$. Furthermore, Remark 5.2.2 and Definition 5.2.4(2) implies that a nondegenerate critical submanifold of f has to be a nondegenerate critical submanifold of the stratum in the sense of smooth manifolds. All such critical submanifolds were found in [15, Lemma 4.3] to be P_K's for any $K \subseteq [r]$. These P_K's satisfy Remark 5.2.2 and Definition 5.2.4(2). So we have to check that among all of these P_K's, what are those satisfying Definition 5.2.4(1) and (3)?

Let us check Definition 5.2.4(1) first. For a point $p \in P_K \subseteq M_{[r]} - F_{[r]}$, the tangent cone of $M_I - F_I$ at p consists of all the tangent vectors pointing inward $M_I - F_I$. Explicitly, they are

$$C_p(M_I - F_I) = \{(v_1, \ldots, v_r) : \langle u_i, v_i\rangle = 0 \text{ for } i \in I, \langle u_i, v_i\rangle \leq 0 \text{ for } i \notin I\}. \tag{6.2.7}$$

Since all the coordinates of $p \in P_K$ lie on the same line, let us assume

$$u_j = \begin{cases} e & \text{if } j \in K, \\ -e & \text{if } j \notin K. \end{cases} \tag{6.2.8}$$

Then for any $v \in C_p(M_I - F_I)$ we have

$$\begin{aligned} df_p(v) &= -2\Big\langle \sum_{j=1}^r l_j u_j, \sum_{i=1}^r l_i v_i \Big\rangle \\ &= -2\Big\langle \Big(\sum_{j\in K} l_j\Big)e - \Big(\sum_{j\notin K} l_j\Big)e, \sum_{i\in I} l_i v_i + \sum_{i\notin I} l_i v_i \Big\rangle \quad \text{by (6.2.8)} \\ &= -2\Big(\sum_{j\in K} l_j - \sum_{j\notin K} l_j\Big)\Big\langle e, \sum_{i\notin I} l_i v_i \Big\rangle \quad \substack{\text{because } \langle e, v_i\rangle = 0 \text{ for } i\in I \\ \text{by (6.2.7)}}. \end{aligned} \tag{6.2.9}$$

Since $P_K = P_{K^\complement}$, let us assume that K is short without loss of generality. We recall that short subset is defined in Definition 4.1.1.

If $K \subseteq I$, then for any $v \in C_p(M_I - F_I)$, we have by (6.2.7) and (6.2.8) that

$$\langle e, v_i\rangle \geq 0, \quad \forall i \notin I. \tag{6.2.10}$$

Then (6.2.9) implies that for any $v \in C_p(M_I - F_I)$ we have

$$df_p(v) = -2\Big(\sum_{j\in K} l_j - \sum_{j\notin K} l_j\Big)\Big\langle e, \sum_{i\notin I} l_i v_i\Big\rangle \geq 0 \quad \text{because } K \text{ is short and by (6.2.10)} \tag{6.2.11}$$

and so $C_p(M_I - F_I) \cap \{X \in T_p(M_I - F_I) : df_p(X) < 0\} = \emptyset$. Every point in P_K with $K \subseteq I$ satisfies Definition 5.2.4(1) and thus is a critical point.

If $K \nsubseteq I$, then there is $i \in K$ such that $i \notin I$. We can take some $v_i \in \mathbb{C}^a$ such that

$$\langle e, v_i\rangle < 0. \tag{6.2.12}$$

The vector

$$v = (0, \ldots, v_i, \ldots, 0) \tag{6.2.13}$$

is then in the tangent cone by (6.2.7) and

$$\begin{aligned} df_p(v) &= -2\Big(\sum_{j\in K} l_j - \sum_{j\notin K} l_j\Big)\Big\langle e, \sum_{i\notin I} l_i v_i\Big\rangle \\ &= -2\Big(\sum_{j\in K} l_j - \sum_{j\notin K} l_j\Big)\langle e, v_i\rangle \\ &< 0 \text{ because } K \text{ is short and } \langle e, v_i\rangle < 0. \end{aligned} \tag{6.2.14}$$

Thus $C_p(M_I - F_I) \cap \{X \in T_p(M_I - F_I) : df_p(X) < 0\} \neq \emptyset$. Every point in P_K with $K \nsubseteq I$ does not satisfy Definition 5.2.4(1) and thus is not a critical point.

We have shown that points in P_K satisfy Definition 5.2.4(1) if and only if K is short and $K \subseteq I$. To show that such P_K's satisfy Definition 5.2.4(3), let us compute $C_p \cap \ker[df_p : T_p(M_I - F_I) \to \mathbb{R}]$.

Note that all the P_K's lie in the highest stratum which is

$$S = M_{[r]} - F_{[r]}. \tag{6.2.15}$$

Since $K \subseteq I$, it follows from (6.2.10) and (6.2.11) that for $v \in C_p(M_I - F_I)$, $df_p(v) = 0$ if and only if

$$\langle e, v_i \rangle = 0, \forall i \notin I. \tag{6.2.16}$$

Together with (6.2.7), that implies that $v \in T_p^S(M_I - F_I)$ because

$$T_p^S(M_I - F_I) = \{(v_1, \dots, v_r) : \langle u_i, v_i \rangle = 0, \forall i\}. \tag{6.2.17}$$

Thus $C_p \cap \ker[df_p : T_p(M_I - F_I) \to \mathbb{R}] = T_p^S(M_I - F_I)$.

The Morse-Bott index of each P_J is defined to be the Morse-Bott index of P_K as a submanifold of the face $M_{[r]} - F_{[r]}$ and was computed in [15, Lemma 4.3] to be $(2a-1)|K|$. □

Remark 6.2.2. Note that $M_{[r]} - F_{[r]}$ is the stratum of the highest codimension of $M_I - F_I$ and thus a smooth manifold. Furthermore, $M_{[r]} - F_{[r]}$ is connected by [15, Proposition 4.4] and thus a face of the highest codimension in $M_I - F_I$.

We have the following corollary to describe a homology basis of $M_I - F_I$.

Corollary 6.2.3. *Let $K \subseteq [r]$. Let $pt \in \mathbb{S}^{2a-1}$ be any point. Let*

$$\begin{aligned} W_K &= \{(u_1, \dots, u_r) \in (\mathbb{S}^{2a-1})^r : u_i = u_j, \forall i, j \notin K\}, \\ V_K &= \{(u_1, \dots, u_r) \in (\mathbb{S}^{2a-1})^r : u_i = pt, \forall i \notin K\}. \end{aligned} \tag{6.2.18}$$

Every W_K and V_K is a product of spheres and thus orientable. Assume $a \geq 2$, then $H_*(M_I - F_I)$ is concentrated in degrees divisible by $(2a-1)$. Moreover, $H_*(M_I - F_I)$ is freely generated by $[W_K]$ and $[V_K]$ for all short subsets K of I where $[\cdot]$ denotes the fundamental class fixing an orientation. To make it clear that it is a basis of $H_*(M_I - F_I)$, we denote this basis $[W_K]^I$ and $[V_K]^I$.

Proof. We can choose some $\epsilon > 0$ small enough such that ϵ is a regular value of f and there is no critical value in $[-\epsilon, 0)$. We can further require that $f^{-1}([-\epsilon, 0))$ does not intersect $M_{[r]} - F_{[r]}$ because the length vector is generic.

Since $-\epsilon$ is regular, it is not hard to show that for every $p \in f^{-1}(-\epsilon)$, the differential of f satisifies $df_p(T_p^S(M_I - F_I)) = T_{-\epsilon}\mathbb{R}$. Since $M_I - F_I$ is regular, it then follows from Corollary 5.2.16 that the sublevel set $f^{-1}((-\infty, -\epsilon])$ is a regular manifold with corners. Since $-\epsilon$ is a regular value of f and $f^{-1}(-\epsilon)$ stays away from $M_{[r]} - F_{[r]}$, the critical point set of f on $f^{-1}((-\infty, -\epsilon])$ still forms critical submanifolds described in Lemma 6.2.1 and the Morse-Bott index of each of them remains the same.

By deformation retracting onto the regular manifolds with corners $f^{-1}((-\infty, -\epsilon])$ without affecting the set of critical points, the compactness assumption in Proposition 5.2.9 is satisified. From the description of critical submanifolds of f in Lemma 6.2.1, other assumptions of Proposition 5.2.9 are satisfied.

Furthermore, Let W_{P_K} be W_K and $V_{P_K} = \{V_K, W_K\}$. The manifold W_K is obviously transversal to P_K in W_K. The manifold V_K intersects P_K at one point and is transversal to P_K because of the way V_K and P_K are embedded in W_K. So the homology classes $[W_K \cap P_K] = [P_K]$ and $[V_K \cap P_K] = \{pt\}$ form a basis of the homology of P_K which is a sphere. By Proposition 5.2.14, $H_*(M_I - F_I)$ is freely generated by $[W_K]$'s and $[V_K]$'s for all short subsets K of I once we fix orientations of W_K's and V_K's. □

Since both $H_*(M_I)$ and $H_*(M_I - F_I)$ are concentrated in degrees that are divisible by $2a-1$, the long exact sequence associated to the pair $(M_I, M_I - F_I)$ at degree $(2a-1)k$ is

$$\begin{aligned} 0 \longrightarrow H_{(2a-1)k+1}(M_I, M_I - F_I) \longrightarrow H_{(2a-1)k}(M_I - F_I) \xrightarrow{i_*} \\ H_{(2a-1)k}(M_I) \longrightarrow H_{(2a-1)k}(M_I, M_I - F_I) \longrightarrow 0. \end{aligned} \tag{6.2.19}$$

Remark 6.2.4. Since we will see product of spaces a lot in computations, we want to fix a standard orientation on the product: Whenever there is a product of oriented spaces indexed by a set of integers (for example: $(\mathbb{S}^{2a-1})^K$ where $K \subseteq [r]$) or we write the product space explicitly as $M \times N$, we always orient this product space via the coordinates with ascending indices. We will call it the standard orientation of the product space.

In order to compute the map i_* in (6.2.19), we have to fix a basis of $H_*(M_I)$ and $H_*(M_I - F_I)$.

Since M_I is a product of spheres and discs, by Künneth formula, $H_*(M_I)$ is concentrated in degrees divisible by $2a-1$. We fix an orientation of $\mathbb{S}^{2a-1}$ and orient M_I according to Remark 6.2.4. The cross products of homology basis of each factor give us a basis of $H_{(2a-1)k}(M_I)$. We denote this basis by $\{x^I_K : K \subseteq I, |K| = k\}$. Explicitly, x^I_K represents the embedding $(\mathbb{S}^{2a-1})^K \to M_I$ with the standard orientation on the product of spheres mentioned in Remark 6.2.4.

To fix a basis of $H_*(M_I - F_I)$, we want to fix an orientation on each V_K and W_K according to Remark 6.2.4. Fixing an orientation of $\mathbb{S}^{2a-1}$, we orient V_K in Corollary 6.2.3 via the diffeomorphism $(\mathbb{S}^{2a-1})^K \to V_K$ onto the coordinates with increasing indices in K. We orient W_K in Corollary 6.2.3 via the diffeomorphism $\mathbb{S}^{2a-1} \times (\mathbb{S}^{2a-1})^K \to W_K$ where the first copy of sphere is diagonally mapped to the coordinates with indices not in K and the rest are mapped to coordinates with increasing indices in K. We take the fundamental classes of V_K's and W_K's under these orientations.

The map i_* in (6.2.19) is then described in the following lemma.

Lemma 6.2.5. *With the basis of $H_*(M_I)$ and $H_*(M_I - F_I)$ mentioned above, for any short $K \subseteq I$ we have*

$$\begin{aligned} i_*[V_K]^I &= x^I_K, \\ i_*[W_K]^I &= \sum_{\forall i \in I-K} (-1)^{[i:K]} x^I_{K\cup i} \end{aligned} \tag{6.2.20}$$

where the notation $[i : K]$ for $i \in [r]$ and $K \subseteq [r]$ is the number of elements in K that are strictly less than i.

Proof. The embedding of V_K in M_I is exactly represented by the homology class x_K since both $i_*([V_K]^I)$ and x^I_K are represented by the same embedding $(\mathbb{S}^{2a-1})^K \to M_I$.

To compute $i_*([W_K]^I)$, we decompose the inclusion map

$$\mathbb{S}^{2a-1} \times (\mathbb{S}^{2a-1})^K \xrightarrow{\simeq} W_K \to M_I \tag{6.2.21}$$

into two maps

$$\mathbb{S}^{2a-1} \times (\mathbb{S}^{2a-1})^K \hookrightarrow (\mathbb{S}^{2a-1})^{I-K} \times (\mathbb{S}^{2a-1})^K \times (\mathbb{D}^{2a})^{I^\complement} \xrightarrow{\simeq} M_I \tag{6.2.22}$$

where the first map is the cross product of the diagonal map

$$\Delta : \mathbb{S}^{2a-1} \to (\mathbb{S}^{2a-1})^{I-K}, \tag{6.2.23}$$

the identity map

$$id : (\mathbb{S}^{2a-1})^K \to (\mathbb{S}^{2a-1})^K \tag{6.2.24}$$

and a map to an arbitrarily chosen point in $(\mathbb{D}^{2a})^{I^\complement}$. The second map in (6.2.22) is a permutation of coordinates.

First observe that

$$\Delta_*(x) = \sum_{k\in I-K} x_k \tag{6.2.25}$$

where $x_k \in H_{2a-1}((\mathbb{S}^{2a-1})^{I-K})$ represents the embedding $\mathbb{S}^{2a-1} \to (\mathbb{S}^{2a-1})^{I-K}$ in the k-th coordinate. That is because the composition of Δ and the projection onto the k-th coordinate gives us the identity map on $\mathbb{S}^{2a-1}$.

Since Künneth formula is natural with respect to the components in the cross product, we have

$$(\Delta \times id)_*(x \times x^I_K) = \sum_{k\in I-K} x_k \times x^I_K \tag{6.2.26}$$

where x^I_K is the generator of $H_{(2a-1)k}((\mathbb{S}^{2a-1})^K)$ given the standard orientation of $(\mathbb{S}^{2a-1})^K$ as in Remark 6.2.4.

The image of $x_i \times x_K$ under the second map in (6.2.22) is $(-1)^{[i:K]} x^I_{K\cup i}$ because the orientation we choose for $x_i \times x_K$ is different from that of $x_{K\cup i}$. Indeed, the orientation that gives $x^I_{K\cup i}$ corresponds to the ordered set $K \cup i$ of increasing integers while the orientation of $x_i \times x^I_K$ corresponds to the ordered set with i followed by the set K of increasing integers. One of these two orientations is obtained from another by flipping coordinates $[i : K]$ times, which results in the sign $(-1)^{[i:K]}$.

We have the desired result. □

Lemma 6.2.6. *Assume $a \geq 2$ and $I \subseteq [r]$, then from the exact sequence (6.2.19) we have*

$$\begin{aligned} H_{(2a-1)k+1}(M_I, M_I - F_I) &\simeq \ker i_*, \\ H_{(2a-1)k}(M_I, M_I - F_I) &\simeq \operatorname{coker} i_* \end{aligned} \tag{6.2.27}$$

and $\ker i_* \subseteq H_{(2a-1)k}(M_I - F_I)$ *is isomorphic to the subspace of* $H_{(2a-1)k}(M_I - F_I)$ *consisting of elements of the form*

$$\sum_{\substack{short\ K\subseteq I \\ |K|=k-1}} m_K[W_K]^I \tag{6.2.28}$$

such that

$$\sum_{\substack{i\in H \\ H-i\ short}} (-1)^{[i:H]} m_{H-i} = 0, \quad \forall\ long\ H \subseteq I, |H| = k. \tag{6.2.29}$$

where notation $[i : H]$ is defined as in Lemma 6.2.5.

Proof. From Corollary 6.2.3, $H_{(2a-1)k}(M_I - F_I)$ consists of elements of the form

$$\sum_{K'} n_{K'}[V_{K'}]^I + \sum_K m_K[W_K]^I \tag{6.2.30}$$

where the first sum is over all short subsets K' of I with k elements and the second sum is over all short subsets K of I with $k-1$ elements. We will assume these conditions on number of elements in K and K' whenever they appear in the index. We also assume that H is a long subset of I containing k elements whenever it appears in the index.

Using Lemma 6.2.5 we can compute the image of such an element under i_* as

$$
\begin{aligned}
& i_*\Big(\sum_{K'} n_{K'}[V_{K'}]^I + \sum_K m_K[W_K]^I\Big) \\
&= \sum_{K'} n_{K'} x^I_{K'} + \sum_K m_K\Big(\sum_{i\in I-K}(-1)^{[i:K]} x^I_{K\cup i}\Big) \\
&= \sum_{K'} n_{K'} x^I_{K'} + \sum_K\Big(\sum_{\substack{i\in I-K\\ K\cup i \text{ short}}}(-1)^{[i:K]} m_K x^I_{K\cup i} + \sum_{\substack{i\in I-K\\ K\cup i \text{ long}}}(-1)^{[i:K]} m_K x^I_{K\cup i}\Big) \\
&= \sum_{K'}\Big(n_{K'} + \sum_{i\in K'}(-1)^{[i:K']} m_{K'-i}\Big)x^I_{K'} + \sum_H\Big(\sum_{\substack{i\notin H\\ H-i \text{ short}}}(-1)^{[i:H-i]} m_{H-i}\Big)x^I_H \\
&= \sum_{K'}\Big(n_{K'} + \sum_{i\in K'}(-1)^{[i:K']} m_{K'-i}\Big)x^I_{K'} + \sum_H\Big(\sum_{\substack{i\in H\\ H-i \text{ short}}}(-1)^{[i:H]} m_{H-i}\Big)x^I_H.
\end{aligned}
\tag{6.2.31}
$$

Thus an element $\sum_{K'} n_{K'}[V_{K'}]^I + \sum_K m_K[W_K]^I$ in $H_{(2a-1)k}(M_I - F_I)$ is in $\ker i_*$ if and only if the following equations hold:

$$
\begin{aligned}
n_{K'} &= -\Big(\sum_{i\in K'}(-1)^{[i:K']} m_{K'-i}\Big), \quad \forall \text{ short } K' \subseteq I \\
\sum_{\substack{i\in H\\ H-i \text{ short}}} &(-1)^{[i:H]} m_{H-i} = 0, \quad \forall \text{ long } H \subseteq I, |H| = k.
\end{aligned}
\tag{6.2.32}
$$

We can see that coefficients $n_{K'}$'s are completely determined by m_K's, thus we have the desired result. □

We didn't give an explicit description of coker d because it is quite tedious to write and we will see that this part will not contribute to the syzygy order of the big polygon spaces.

Since F_I is a neat submanifold of M_I and the normal bundle of F_I in M_I is trivial by Lemma 6.1.3, the following lemma follows directly from Corollary 5.1.34.

Lemma 6.2.7. *Assume $I \subseteq [r]$, $a \geq 1$, then we have*

$$H_{*-2a}(F_I) \simeq H_*(M_I, M_I - F_I)$$

and this isomorphism satisfies the following commuting diagram if $J \subseteq I$:

$$\begin{array}{ccc} H_{*-2a}(F_I) & \longrightarrow & H_*(M_I, M_I - F_I) \\ \downarrow & & \downarrow \\ H_{*-2a}(F_J) & \longrightarrow & H_*(M_J, M_J - F_J). \end{array} \tag{6.2.33}$$

So we have the following commuting diagram for any $J \subseteq I \subseteq [r]$ where the degree $*$ is $(2a-1)k$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_{*-2a+1}(F_I) & \longrightarrow & H_*(M_I - F_I) & \xrightarrow{i_*} & H_*(M_I) & \longrightarrow & H_{*-2a}(F_I) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota^{IJ}_{*-2a+1}} & & \downarrow{\scriptstyle \iota^{IJ}_*} & & \downarrow{\scriptstyle \iota^{IJ}_*} & & \downarrow{\scriptstyle \iota^{IJ}_{*-2a}} & & \\ 0 & \longrightarrow & H_{*-2a+1}(F_J) & \longrightarrow & H_*(M_J - F_J) & \xrightarrow{i_*} & H_*(M_J) & \longrightarrow & H_{*-2a}(F_J) & \longrightarrow & 0. \end{array} \tag{6.2.34}$$

The map ι^{IJ}_* will be described in the following lemma. Note that in (6.2.34) we use the same notation ι^{IJ}_* to denote the three maps $H_*(F_I) \to H_*(F_J)$, $H_*(M_I - F_I) \to H_*(M_J - F_J)$ and $H_*(M_I) \to H_*(M_J)$.

Lemma 6.2.8. *Assume that $I \subseteq [r]$ and that K is a short subset of I. Let $[W_K]^I$ and $[V_K]^I$ be as in Corollary 6.2.3. If $J \subseteq I$, then the map $\iota^{IJ}_* : H_*(M_I - F_I) \to H_*(M_J - F_J)$ induced by inclusion map is*

$$\iota^{IJ}_*([W_K]^I) = \begin{cases} [W_K]^J & \text{if } K \subseteq J, \\ 0 & \text{if } K \not\subseteq J, \end{cases} \tag{6.2.35}$$

$$\iota^{IJ}_*([V_K]^I) = \begin{cases} [V_K]^J & \text{if } K \subseteq J, \\ 0 & \text{if } K \not\subseteq J. \end{cases} \tag{6.2.36}$$

Similarly, the map $\iota^{IJ}_* : H_*(M_I) \to H_*(M_J)$ induced by inclusion map is

$$\iota^{IJ}_*(x^I_K) = \begin{cases} x^J_K & \text{if } K \subseteq J, \\ 0 & \text{if } K \not\subseteq J. \end{cases} \tag{6.2.37}$$

Proof. If $K \subseteq J$, $\iota^{IJ}_*([W_K]^I)$ is obviously $[W_K]^J \in H_*(M_J - F_J)$.

If $K \not\subseteq J$, then there is an index $k \in K - J$. Then the inclusion of W_K as a product of spheres into M_J is homotopic to an inclusion of a lower dimensional product of spheres via the following homotopy

$$H : W_K \times I \to M_J - F_J \tag{6.2.38}$$

given by the following formula:

$$\begin{aligned} &H(u,t) := (H_1(u,t), \ldots, H_r(u,t)) \quad \text{where} \\ &H_i(u,t) = \begin{cases} u_i & \text{if } i \neq k, \\ tu_i & \text{if } i = k. \end{cases} \end{aligned} \tag{6.2.39}$$

It is clear that H defined above has image in M_J. We have to show that H has image in $M_J - F_J$, that is, the image of H does not intersect F_J. Since K is short, we have $\|\sum_{i\in K} H_i(u,t)\| < \|\sum_{i\notin K} H_i(u,t)\|$ because

$$\Big\|\sum_{i\in K} H_i(u,t)\Big\| = \Big\|tu_k + \sum_{i\in K, i\neq k} u_i\Big\| \leq \sum_{i\in K} l_i < \sum_{i\notin K} l_i = \Big\|\sum_{i\notin K} H_i(u,t)\Big\| \tag{6.2.40}$$

and thus

$$\sum_{i\in[r]} H_i(u,t) \neq 0 \quad \forall (u,t) \in W_K \times I. \tag{6.2.41}$$

So H indeed has image in $M_J - F_J$. We have shown that $\iota_*^{IJ}([W_K]^I) = 0$ if $K \nsubseteq J$.

We can construct the homotopy also for $[V_K]$ and x_K^I using the same formula above and show that $\iota_*^{IJ}([V_K]^I) = 0$ and $\iota_*^{IJ}(x_K^I) = 0$ if $K \nsubseteq J$. □

Now we can give a description of $H_*(F_I)$ and the map $\iota_*^{IJ} : H_*(F_I) \to H_*(F_J)$ if $J \subseteq I$.

Proposition 6.2.9. *Assume $a \geq 2$, $J \subseteq I \subseteq [r]$, then we have*

(a) $H_*(F_I)$ is concentrated in degrees $(2a-1)k-1$ and $(2a-1)k$ for integers $0 \leq k \leq r-1$.

(b) $H_{(2a-1)k}(F_I)$ is isomorphic to the subspace of $H_{(2a-1)(k+1)}(M_I - F_I)$ consisting of elements of the form

$$\sum_{\substack{K short, |K|=k \\ K\subseteq I}} m_K [W_K]^I \tag{6.2.42}$$

such that the coefficients m_K's satisfy

$$\sum_{\substack{i\in H\\ H-i\ short}} (-1)^{[i:H]} m_{H-i} = 0 \quad \forall\ long\ H \subseteq I, |H| = k+1 \tag{6.2.43}$$

where $[i:H]$ is as in Lemma 6.2.5.

For such an element in $H_{(2a-1)k}(F_I)$, the map induced by the inclusion $F_I \subseteq F_J$ is

$$\iota^{IJ}_{(2a-1)k}\Big(\sum_{\substack{short\ K\subseteq I\\ |K|=k}} m_K[W_K]^I\Big) = \sum_{\substack{short\ K\subseteq J\\ |K|=k}} m_K[W_K]^J. \tag{6.2.44}$$

(c) $\iota^{IJ}_{(2a-1)k-1} : H_{(2a-1)k-1}(F_I) \to H_{(2a-1)k-1}(F_J)$ is surjective for any k.

Proof. Since both $H_*(M_I)$ and $H_*(M_I - F_I)$ are concentrated in degrees $(2a-1)k$ and we have the exact sequence (6.2.19), $H_*(M_I, M_I - F_I)$ is concentrated in degrees $(2a-1)k$ and $(2a-1)k-1$. From Lemma 6.2.7 we get (a).

(b) follows directly from Lemma 6.2.6 and Lemma 6.2.7.

When $J \subseteq I$, the map $H_*(M_I) \to H_*(M_J)$ induced by inclusion $M_I \to M_J$ is surjective. Since the map $H_*(M_J) \to H_{*-2a}(F_J)$ in (6.2.34) is surjective, the vertical map at the right end of (6.2.34) is also surjective. (c) follows. □

A corollary of this proposition is a proof of the claim mentioned in Remark 6.1.4.

Corollary 6.2.10. *A codimension-k face of $F_\varnothing$ is F_I for some subset $I \subseteq [r]$ of cardinality k such that F_I is not empty.*

Proof. Since we have shown in Lemma 6.1.3 that $F_\varnothing$ is a neat submanifold of $M_\varnothing$, it follows from Remark 5.1.18(2) that the codimension-k stratum of $F_\varnothing$ are exactly the intersection of codimension-k stratum of $M_\varnothing$ and $F_\varnothing$. We have written explicitly the codimension-k stratum of $M_\varnothing$ in (6.1.6). The codimension-k stratum S_k of $F_\varnothing$ is thus of the following form

$$S_k = \bigcup_{|I|=r-k} (M_I \cap F_\varnothing) - \bigcup_{|K|=r-k+1} (M_K \cap F_\varnothing) \tag{6.2.45}$$

$$= \bigcup_{|I|=r-k} F_I - \bigcup_{|K|=r-k+1} F_K \tag{6.2.46}$$

$$= \bigsqcup_{|I|=r-k} \Bigg(F_I - \bigcup_{\substack{K \supseteq I \\ |K|=r-k+1}} F_K \Bigg). \tag{6.2.47}$$

where the last union is a disjoint union.

Then it suffices to show that every term in the last disjoint union in (6.2.45) is connected. Note that since F_I is a neat submanifold of M_I and thus a manifold with corners itself, the following term in the last disjoint union in (6.2.45)

$$F_I - \bigcup_{\substack{K \supseteq I \\ |K|=r-k+1}} F_K \tag{6.2.48}$$

is exactly the interior of F_I as a manifold with corners. We will show that it is connected by looking at its 0-th homology.

Since F_I is topological manifold with boundary, we have

$$H_*(\mathring{F}_I) = H_*(F_I) \tag{6.2.49}$$

from [25, Lemma 11.6] or [17, (5.1)] where $\mathring{F}_I$ is the interior of F_I. Since we have shown in Proposition 6.2.9(b) that $H_0(F_I)$ is of dimension 1 if F_I is not empty, we conclude that $\mathring{F}_I$ is connected. □

Chapter 7

Syzygy order of big polygon spaces

7.1 The main result

It was proved in [16, Proposition 6.3] that the syzygy order of a big polygon space $X_{a,b}(l)$ has an upper bound related only to the combinatorial property of the length vector l. To state this upper bound, we need to define such a number for every generic length vector.

Definition 7.1.1. [16, (6.6)(6.7)] For any subset $J \subseteq [r]$, define

$$\sigma_l(J) := \#\{j \in J : J - j \text{ short}\} \tag{7.1.1}$$

where # denotes the number of elements in the set. Then we can define

$$\mu(l) := \min\{\sigma_l(J) : J \text{ is long and } \sigma_l(J) > 0\}. \tag{7.1.2}$$

Remark 7.1.2. Since l is a generic length vector, the complement of a short subset is long. There is a dual version of $\sigma_l(J)$:

$$\tilde{\sigma}_l(J) := \#\{j \in [r] - J : J \cup j \text{ long}\}. \tag{7.1.3}$$

Note that $\tilde{\sigma}_l(J)$ is equal to $\sigma_l(J^{\complement})$.

We also have an equivalent definition of $\mu(l)$ in terms of $\tilde{\sigma}_l$.

$$\mu(l) := \min\{\tilde{\sigma}_l(J) : J \text{ is short and } \tilde{\sigma}_l(J) > 0\}. \tag{7.1.4}$$

The following upper bound on the syzygy order of a big polygon space was given in [16, Proposition 6.3]

Proposition 7.1.3 ([16, Proposition 6.3])**.** *For any $a, b, r \geq 1$ and generic length vector $l \in \mathbb{R}^r$, we have*

$$\operatorname{syzord} H_T^*(X_{a,b}(l)) \leq \mu(l) - 1. \tag{7.1.5}$$

We are going to prove that the above inequality is an equality. This was conjectured in [16, Conjecture 6.6].

Theorem 7.1.4. *For any $a, b, r \geq 1$ and generic length vector $l \in \mathbb{R}^r$, we have*

$$\operatorname{syzord} H_T^*(X_{a,b}(l)) = \mu(l) - 1. \tag{7.1.6}$$

We will prove Theorem 7.1.4 in the rest of this chapter. In Section 7.2 we will prove the easy case of Theorem 7.1.4 when the fixed point set of $X_{a,b}(l)$ is empty. In Section 7.3-Section 7.7 we will prove the case of Theorem 7.1.4 when the fixed point set of $X_{a,b}(l)$ is nonempty.

7.2 Proof of Theorem 7.1.4 when the fixed point set is empty

Let us first prove the case of Theorem 7.1.4 when the fixed point set is empty. The reason to single this case out is that we are not able to show that the T-action on $X_{a,b}(l)$ is locally standard in this case. Fortunately, Theorem 7.1.4 still holds in this case.

Lemma 7.2.1 (Theorem 7.1.4 under the assumption that the fixed point set is empty)**.** *For any $a, b, r \geq 1$ and generic length vector $l \in \mathbb{R}^r$, if the fixed point set of $X_{a,b}(l)$ under T-action is* *empty, then we have*

$$\operatorname{syzord} H_T^*(X_{a,b}(l)) = \mu(l) - 1 = 0. \tag{7.2.1}$$

Proof. Since the fixed point set is empty, the leftmost nontrivial map $H_T^*(X) \to H_T^*(X_0)$ in the Atiyah-Bredon sequence (3.2.4) is a zero map and thus it is not injective. According to Theorem 3.2.2, that implies:

$$\operatorname{syzord} H_T^*(X_{a,b}(l)) = 0. \tag{7.2.2}$$

On the other side, since the fixed point set of $X_{a,b}(l)$ is empty, according to Lemma 4.1.8(2), we have one dominant length l_r. By definition of σ_l in (7.1.1), we have

$$\sigma_l([r]) = 1. \tag{7.2.3}$$

Then by definition of $\mu(l)$ in (7.1.2) we have

$$\mu(l) = 1. \tag{7.2.4}$$

We have proved the lemma. □

7.3 A lemma to Theorem 7.1.4 when the fixed point set is nonempty

When the fixed point set of $X_{a,b}(l)$ is not empty, we have shown in Lemma 4.1.8(1) that the T-action on $X_{a,b}(l)$ is locally standard. Since $X_{a,b}(l)$ is a compact connected smooth manifold, all the assumptions in the beginning of Section 3.3 are satisfied. We can then apply the quotient criterion in Theorem 3.3.2 to $X_{a,b}(l)$. Furthermore, by Lemma 4.2.2 it suffices to prove Theorem 7.1.4 when $a \geq 2$. Also recall that without loss of generality, we can assume that the length vector is positive and weakly increasing, that is, $0 < l_1 \leq l_2 \leq \cdots \leq l_r$.

We have seen in Proposition 6.1.7 that there is a homeomorphism $F_\varnothing \simeq X_{a,b}(l)/T$ preserving face structure. It then suffices to apply Theorem 3.3.2 to $F_\varnothing$. Since all the faces of $F_\varnothing$ are F_J's where $J \subset [r]$, the cochain complex $B^*(P)$ in (3.3.2) is

$$B^i(F_J) = \bigoplus_{\substack{J\subseteq I\\ |I|=r-i}} H_*(F_I) \tag{7.3.1}$$

with differential map described in (3.3.4). For $\sigma \in H_*(F_I) \subseteq B^i(F_J)$, it is

$$d\sigma = \sum_{\substack{J\subseteq L\subseteq I\\ |L|=r-i-1}} \pm \iota_*^{IL}(\sigma) \tag{7.3.2}$$

where ι_*^{IL} is the map induced on homology by the inclusion of faces $F_I \to F_L$ and the signs will be explained shortly in (7.4.1). One might find it helpful to look at Example 3.3.3 in order to understand how we are going to apply the quotient criterion. In Chapter 6 we have computed $H_*(F_i)$'s, identified basis of them and computed ι_*^{IL}'s explicitly.

Based on the quotient criterion Theorem 3.3.2 and the above discussion, to prove Theorem 7.1.4 in the case when the fixed point set of $X_{a,b}(l)$ is nonempty, it suffices to prove the following lemma to Theorem 7.1.4.

Lemma 7.3.1. *For $a \geq 2$, $b, r \geq 1$ and generic length vector $l = (l_1, \cdots, l_r) \in \mathbb{R}^r$ such that $0 < l_1 \leq l_2 \leq \cdots \leq l_r$, if the fixed point set of $X_{a,b}(l)$ is nonempty, then we have*

$$H^i(B^*(F_J)) = 0 \tag{7.3.3}$$

for all $J \subseteq [r]$ and all $i > \max(r - |J| - \mu(l) + 1, 0)$.

The rest of this chapter will be devoted to the proof of Lemma 7.3.1.

7.4 A reduction step

The first step toward Lemma 7.3.1 is a reduction lemma analogous to [16, Lemma 6.2].

Since we are going to explicitly compute the map d in (7.3.2) and construct more cochain complexes based on that, it is better to make it clear what are these $\pm$ signs in (7.3.2).

Assume that $I = J \cup \{i_1, \ldots, i_n\}$ with $i_1 < \cdots < i_n$, the natural ordering of pairwise distinct integers $i_1, \ldots, i_n$ induces a natural ordering on the set of faces of F_J that contain F_I as a facet. For any $\sigma \in H_*(F_I)$, the differential d is

$$d\sigma := \sum_{i \in \{i_1,\ldots,i_n\}} (-1)^{[i:\{i_1,\ldots,i_n\}]} \iota_*^{I,I-i}(\sigma) \tag{7.4.1}$$

where the notation $[i : \{i_1, \ldots, i_n\}]$ was defined in Lemma 6.2.5.

Since we assume $a \geq 2$, for every subset $I \subseteq [r]$, $H_*(F_I)$ is concentrated in degrees $(2a-1)k-1$ and $(2a-1)k$ for integers $0 \leq k \leq r-1$ by Proposition 6.2.9(a). So $H_*(F_I)$ is the direct sum of two subspaces as a graded vector space.

$$H_*(F_I) = \Big(\bigoplus_{0 \leq k \leq r-1} H_{(2a-1)k-1}(F_I) \Big) \oplus \Big(\bigoplus_{0 \leq k \leq r-1} H_{(2a-1)k}(F_I) \Big). \tag{7.4.2}$$

Since differential d in (7.3.2) preserves homological degree, the cochain complex $B^i(F_J)$ in (7.3.1) is the direct sum of two cochain subcomplexes with differentials being just those induced from differential d on each direct summand in (7.3.2). Explicitly, we have

$$B^i(F_J) = \bigoplus_{0 \leq k \leq r-1} P_k^i(F_J) \oplus \bigoplus_{0 \leq k \leq r-1} Q_k^i(F_J) \tag{7.4.3}$$

where

$$P_k^i(F_J) := \bigoplus_{\substack{J \subseteq I \\ |I| = r-i}} H_{(2a-1)k}(F_I), \tag{7.4.4}$$

$$Q_k^i(F_J) := \bigoplus_{\substack{J \subseteq I \\ |I| = r-i}} H_{(2a-1)k-1}(F_I) \tag{7.4.5}$$

and differentials on $P_k^*(F_J)$ and $Q_k^*(F_J)$ have exactly the same formula as in (7.4.1).

As a direct sum of the two cochain subcomplexes, the cochain complex $B^*(F_J)$ is exact at degree i if and only if for any $0 \le k \le r-1$, both $P_k^*(F_J)$ and $Q_k^*(F_J)$ are exact at degree i. We will see that in order to show that $B^*(F_J)$ is exact at some degree, we only have to show that $P_k^*(F_J)$ is exact at that degree. We first show the following lemma.

Lemma 7.4.1. *For $i > 0$, $0 \le k \le r-1$ and any $J \subseteq [r]$, $P_k^*(F_J)$ is exact at degree $i+2$ if and only if $Q_k^*(F_J)$ is exact at degree i.*

To prove Lemma 7.4.1, we are going to construct two more cochain complexes $B_k^*(M_J - F_J)$ and $B_k^*(M_J)$ for any $0 \le k \le r$. The construction is similar to that in (7.3.1) and (7.3.2):

$$\begin{aligned} B_k^i(M_J - F_J) &:= \bigoplus_{\substack{J\subseteq I \\ |I|=r-i}} H_{(2a-1)(k+1)}(M_I - F_I), \\ B_k^i(M_J) &:= \bigoplus_{\substack{J\subseteq I \\ |I|=r-i}} H_{(2a-1)(k+1)}(M_I). \end{aligned} \tag{7.4.6}$$

with differentials given by the same formula as in (7.3.2) and (7.4.1) while the notation ι_*^{IL} means, of course, the maps on homology induced by the inclusion of faces $M_I - F_I \to M_L - F_L$ and $M_I \to M_L$.

We can get the following commuting diagram for any integer i and any $J \subseteq [r]$ by summing over all the necessary summands in (6.2.34):

$$\begin{array}{ccccccccc} 0 \to & P_k^{i+1}(F_J) & \to & B_k^{i+1}(M_J - F_J) & \to & B_k^{i+1}(M_J) & \to & Q_k^{i+1}(F_J) & \to 0 \\ & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & \\ 0 \to & P_k^{i}(F_J) & \to & B_k^{i}(M_J - F_J) & \to & B_k^{i}(M_J) & \to & Q_k^{i}(F_J) & \to 0. \end{array} \tag{7.4.7}$$

The rows of (7.4.7) are exact because rows of (6.2.34) are exact and rows of (7.4.7) are just direct sum of the rows of (6.2.34) over $0 \le k \le r-1$. This diagram (7.4.7) is commuting because differentials on $B_k^*(M_J - F_J)$ and $B_k^*(M_J)$ are compatible with differentials on $P_k^*(F_J)$ and $Q_k^*(F_J)$.

After these constructions and clarifications, we are going to prove the following lemma:

Lemma 7.4.2. *For any subset $J \subseteq [r]$ and any $0 \leq k \leq r-1$, both $B_k^*(M_J - F_J)$ and $B_k^*(M_J)$ are exact at all degrees except at degree 0.*

Proof. Let us show it for $B_k^*(M_J - F_J)$ first. Recall from Corollary 6.2.3 that for any $I \subseteq [r]$, $H_*(M_I - F_I)$ is freely generated by $[W_K]^I$'s and $[V_K]^I$'s for all short subsets $K \subseteq I$ as a vector space.

Since the boundary map of $B_k^*(M_J - F_J)$ is the alternating sum of $\iota_*^{I,I-i}$ as described in (6.2.44) and (7.4.1), we can further decompose $B_k^*(M_J - F_J)$ into subcomplexes in the following way:

$$B_k^*(M_J - F_J) = \bigoplus_{\substack{K\subseteq[r] \text{ short} \\ |K|=k-1}} B_J^*([W_K]) \oplus \bigoplus_{\substack{K\subseteq[r] \text{ short} \\ |K|=k}} B_J^*([V_K]) \tag{7.4.8}$$

where

$$\begin{aligned} B_J^i(W_K) &= \bigoplus_{\substack{I\supset J\cup K \\ |I|=r-i}} \langle [W_K]^I \rangle, \\ B_J^i(V_K) &= \bigoplus_{\substack{I\supset J\cup K \\ |I|=r-i}} \langle [V_K]^I \rangle \end{aligned} \tag{7.4.9}$$

and $\langle [W_K]^I \rangle$ (resp. $\langle [V_K]^I \rangle$) denotes the one dimensional real vector subspace spanned by $[W_K]^I$ (resp. $[V_K]^I$). Differentials on these two complexes are still given by the same formula as in (7.4.1).

In general, the cochain complex $B_J^*(W_K)$ has the following form:

$$0 \to \langle [W_K]^{[r]} \rangle \to \bigoplus_{\substack{I\supset J\cup K \\ |I|=r-1}} \langle [W_K]^I \rangle \to \cdots \to \bigoplus_{\substack{I\supset J\cup K \\ |I|=|J\cup K|+1}} \langle [W_K]^I \rangle \to \langle [W_K]^{J\cup K} \rangle \to 0. \tag{7.4.10}$$

We can convert this cochain complex into a chain complex by requiring degree i term in $B_J^*(W_K)$ to be in degree $r - 1 - |J \cup K| - i$. The resulting chain complex is isomorphic to the reduced chain complex of a simplicial complex with i-simplices being subsets of $[r] - (J \cup K)$ with $i + 1$ elements. This simplicial complex is just the simplicial complex of one single

$(r - |J \cup K| - 1)$-simplex $[r] - (J \cup K)$. It has trivial reduced homology if $J \cup K \neq [r]$. If $J \cup K = [r]$, then it has nontrivial reduced homology only at degree -1 which corresponds to degree 0 in the cochain complex (7.4.10).

We can repeat the above proof for $B^*(V_K)$ to show that it is exact at all degrees except at degree 0. So $B^*_k(M_I - F_I)$ is exact at all degrees except at degree 0.

For $B^*_k(M_J)$, since $B^*_k(M_J)$ is freely generated by $[x_K]^I$ and boundary map of $B^*_k(M_I)$ is the alternating sum of $\iota_*^{I,I-i}$ as described in (7.4.1), we can decompose $B^*_k(M_J)$ as in (7.4.8) and show that $B^*_k(M_J)$ is exact in all degrees except at degree 0. □

Now we can prove Lemma 7.4.1.

Proof of Lemma 7.4.1. We can splice the bottom horizontal exact sequence in (7.4.7) and get two short exact sequences

$$\begin{aligned} &0 \to P^*_k(F_J) \to B^*_k(M_J - F_J) \to {}^{B^*_k(M_J-F_J)}/_{P^*_k(F_J)} \to 0, \\ &0 \to {}^{B^*_k(M_J-F_J)}/_{P^*_k(F_J)} \to B^*_k(M_J) \to Q^*_k(F_J) \to 0. \end{aligned} \tag{7.4.11}$$

Since both $B^*_k(M_J - F_J)$ and $B^*_k(M_J)$ are exact at all degrees except at degree 0 by Lemma 7.4.2, we can infer from the long exact sequences associated to these two short exact sequences that

$$\begin{aligned} H^i(P^*_k(F_J)) &\simeq H^{i-1}({}^{B^*_k(M_J-F_J)}/_{P^*_k(F_J)}) \quad \text{for } i \geq 2, \\ H^i(Q^*_k(F_J)) &\simeq H^{i+1}({}^{B^*_k(M_J-F_J)}/_{P^*_k(F_J)}) \quad \text{for } i \geq 1. \end{aligned} \tag{7.4.12}$$

So for $i > 0$, $P^*_k(F_J)$ is exact at degree $i + 2$ if and only if $Q^*_k(F_J)$ is exact at degree i. □

Corollary 7.4.3. *If $a \geq 2$ and $i > 0$, then*

$$H^i(B^*(F_J)) = 0 \tag{7.4.13}$$

if and only if $H^i(P^*_k(F_J)) = 0$ for all $0 \leq k \leq r - 1$ where the complex $P^*_k(F_J)$ is defined in (7.4.4).

Proof. By Lemma 7.4.1 and the decomposition (7.4.3), for $i > 0$, $H^i(B^*(F_J)) = 0$ if and only if $H^i(P_k^*(F_J)) = 0$ for all $0 \le k \le r-1$. □

7.5 Outline of the proof of Lemma 7.3.1

We want to give an outline of the proof of Lemma 7.3.1 before we go into the details of the proof.

According to Corollary 7.4.3, in order to prove Lemma 7.3.1 we have to show the following statement

$$H^i(P_k^*(F_J)) = 0, \ \forall J \subseteq [r], i > \max(r - |J| - \mu(l) + 1, 0) \text{ and } \forall k. \qquad (*)$$

Let us fix J, k and have a look at the cochain complex $P_k^*(F_J)$ to get a sense of what this statement means. The cochain complex $P_k^*(F_J)$ was defined in (7.4.4) to be the following cochain complex:

$$\begin{array}{l} 0 \longrightarrow H_{(2a-1)k}(F_{[r]}) \longrightarrow \bigoplus_{i \in [r]-J} H_{(2a-1)k}(F_{[r]-i}) \rightarrow \cdots \\ \hookrightarrow \bigoplus_{\substack{J \subseteq K \\ |K|=|J|+2}} H_{(2a-1)k}(F_K) \rightarrow \bigoplus_{\substack{J \subseteq I \\ |I|=|J|+1}} H_{(2a-1)k}(F_I) \longrightarrow H_{(2a-1)k}(F_J) \longrightarrow 0. \end{array} \qquad (7.5.1)$$

Note that according to our grading on this cochain complex, $P_k^0(F_J) = H_{(2a-1)k}(F_{[r]})$ is the left hand side of this cochain complex and $P_k^{r-|J|}(F_J) = H_{(2a-1)k}(F_J)$ is the right hand side.

The statement $(*)$ is saying that $P_k^*(F_J)$ is exact at either all degrees greater than 0 or positions to the right of the $(\mu(l) - 1)$-th position counting from the rightmost term $H_{(2a-1)k}(F_J)$ if the degree of this position is greater than 0. For example, if $\mu(l) = 2$, then the statement $(*)$ says that $P_k^*(F_J)$ is exact at the rightmost term $H_{(2a-1)k}(F_J)$, that is, the map to $H_{(2a-1)k}(F_J)$ in (7.5.1) is surjective as long as $J \neq [r]$. If $\mu(l) = 3$, then the statement $(*)$ says that $P_k^*(F_J)$ is exact at the first and second term from right as long as $|J| < r - 1$.

The case when $\mu(l) = 2$ is relatively easy. We just have to show that the map

$$\bigoplus_{\substack{J\subseteq I\\ |I|=|J|+1}} H_{(2a-1)k}(F_I) \to H_{(2a-1)k}(F_J) \tag{7.5.2}$$

is surjective. We will show this in Corollary 7.6.3.

However things get much more complicated when we want to move on to the case $\mu(l) = 3$. In this case, in addition to showing the surjectivity of (7.5.2) we have to show the exactness at the second term to the right in (7.5.1) if $|J| < r - 1$, that is, the exactness at the middle term in the following sequence

$$\bigoplus_{\substack{J\subseteq K\\ |K|=|J|+2}} H_{(2a-1)k}(F_K) \xrightarrow{d} \bigoplus_{\substack{J\subseteq I\\ |I|=|J|+1}} H_{(2a-1)k}(F_I) \xrightarrow{d} H_{(2a-1)k}(F_J). \tag{7.5.3}$$

Domain and codomain of the first map in (7.5.3) are both direct sums, making it difficult to verify the exactness condition.

Instead of trying to work directly on (7.5.3), we can first tackle a simplified version. If $x \in H_{(2a-1)k}(F_I)$ is in one of the components of the middle term in (7.5.3) and is a cocycle, then we can try to show that x is a coboundary. That is part of the reason why we want to prove Lemma 7.6.2. Condition (7.6.2) there is equivalent to the cocycle condition on x and the properties (a) and (c) are just saying that x is a coboundary.

However the case $\mu(l) = 3$ is not proved by Lemma 7.6.2 because a cocycle in the middle term of (7.5.3) can be the sum of terms in different components (instead of concentrating in just one of the components). The main idea behind Proposition 7.7.2 is then to show that any cocycle (which may involve several terms in different components) is cohomologous to a cocycle that involves only one term in one of the componenents. With Proposition 7.7.2 and Lemma 7.6.2 we can then prove Lemma 7.3.1 when $\mu(l) = 3$. The generality of Proposition 7.7.2 and Lemma 7.6.2 then allows us to prove Lemma 7.6.2 for any $\mu(l)$.

Next we want to introduce a different way to think about $H_*(F_I)$ and some notation in order to facilitate our proof. For any subset $J \subseteq [r]$, there is a simplicial complex Δ^J (resp. Δ^J_-) consisting of all subsets (resp. short subsets) of J. Let Δ^J_+ be the collection of long sets of J. Note that Δ^J_+ is not a subcomplex of Δ^J.

Recall that the *augmented* simplicial chain complex $C_*(\Delta^J)$ with real coefficients associated to Δ^J is defined as the following chain complex in [24, Theorem 3.5.4]:

$$0 \leftarrow C_{-1}(\Delta^J) \xleftarrow{\epsilon} C_0(\Delta^J) \xleftarrow{\partial_1} C_1(\Delta^J) \xleftarrow{\partial_2} C_2(\Delta^J) \xleftarrow{\partial_3} \cdots \tag{7.5.4}$$

where each $C_k(\Delta)$ is a real vector space with a basis consisting of all k-simplices of Δ^J, that is, subsets of J containing $k+1$ elements. The left end $C_{-1}(\Delta)$ is generated by (-1)-simplex which is the empty set and thus is isomorphic to $\mathbb{R}$ as a real vector space.

Let $C_*(\Delta^J_+)$ be the subspace of $C_*(\Delta^J)$ generated by simplices in Δ^J_+. It follows from Proposition 6.2.9 that if $a \geq 2$, for any integer k we can think of an element in $H_{(2a-1)(k+1)}(F_J)$ as a cochain in the $\mathbb{R}$-dual of $C_*(\Delta^J)$ that vanishes on $C_k(\Delta^J_+)$ and $\partial C_{k+1}(\Delta^J_+)$. Let $C^*(\Delta^J)$ be the $\mathbb{R}$-dual of $C_*(\Delta^J)$. We have the following isomorphism for $a \geq 2$:

$$H_{(2a-1)(k+1)}(F_J) \simeq \{x \in C^k(\Delta^J) | x = 0 \text{ on } C_k(\Delta^J_+) \text{ and } \partial C_{k+1}(\Delta^J_+)\}. \tag{7.5.5}$$

The right hand side of (7.5.5) is denoted $C^k_-(\Delta^J)$. We have the following remark.

Remark 7.5.1. If follows from Proposition 6.2.9(b) that the isomorphism in (7.5.5) gives the following commuting diagram:

$$\begin{array}{ccc} H_{(2a-1)(k+1)}(F_{J\cup i}) & \xrightarrow{\simeq} & C^k_-(\Delta^{J\cup i}) \\ \downarrow{\scriptstyle \iota_*^{J\cup i,J}} & & \downarrow{\scriptstyle \iota^*_{J\cup i,J}} \\ H_{(2a-1)(k+1)}(F_J) & \xrightarrow{\simeq} & C^k_-(\Delta^J) \end{array} \tag{7.5.6}$$

where the right vertical map is induced by $\Delta^J \to \Delta^{J\cup i}$.

From now on we will always identify a subset of $J \cup i$ with the chain represented by it in $C_*(\Delta^{J\cup i})$. Whenever we write ∂H for some subset H of $J \cup i$, we mean the boundary of the chain H in $C_*(\Delta^{J\cup j})$.

7.6 A technical lemma to Lemma 7.3.1

Now we are going to prove a very technical lemma to Lemma 7.3.1. We first need to make some observation on the combinatorial property of the number $\mu(l)$ defined in (7.1.2).

Remark 7.6.1. Assume that $\mu(l) \geq p$. If one facet of a simplex in Δ_+^J which is a subset of $[r]$ is short, then J contains at least p short facets. In particular, assume that $J = \{j_1, \ldots, j_n, m_1, \ldots, m_{p-2}, i\} \subset [r]$ is long and contains a short facet where $j_1 < j_2 < \cdots < j_n$, then $J - j_n$ must be short.

Now we are going to prove the following lemma.

Lemma 7.6.2. *Assume that $a \geq 2$ and that $\mu(l) \geq p \geq 2$. Let J be a subset of $[r]$ with $0 \leq n \leq r - p + 1$ elements. Assume that $J = \{j_1, \ldots, j_n\}$ where $j_1 < \cdots < j_n$ if $n > 0$. Let M be a subset of $[r] - J$ with $p - 2$ elements. Assume that $M = \{m_1, \ldots, m_{p-2}\}$ if $p > 2$. Let $i \in [r] - (J \cup M)$ and $k \geq -1$. Furthermore, we define A as*

$$A := \begin{cases} \{K \subseteq J \cup M \cup i : K \text{ short}, |K| = k+1, M \cup \{j_n, i\} \subseteq K\} & \text{if } n > 0, \\ \{K \subseteq J \cup M \cup i : K \text{ short}, |K| = k+1, M \cup i \subseteq K\} & \text{if } n = 0. \end{cases} \tag{7.6.1}$$

For any $x \in C_-^k(\Delta^{J\cup M})$ such that the following condition is satisfied:

$$\forall m \in M, \quad i^*_{J\cup M,(J\cup M)-m}(x) = 0 \tag{7.6.2}$$

and any tuple of real numbers $(a_K)_{K\in A}$ indexed by elements in A, there is $\tilde{x} \in C_-^k(\Delta^{J\cup M\cup i})$ such that the following three properties are satisfied:

(a) $i^*_{J\cup M\cup i, J\cup M}(\tilde{x}) = x$. *Explicitly, that means* $\tilde{x}(K) = x(K)$ *for any* k*-simplex* K *contained in* $J \cup M$.

(b) $\tilde{x}(K) = a_K$ *for any* $K \in A$.

(c) $i^*_{J\cup M\cup i,(J\cup M\cup i)-m}(\tilde{x}) = 0$ *for any* $m \in M$. *That means* $\tilde{x}(K) = 0$ *for any* K *contained in* $(J \cup M \cup i) - m$.

Before proving Lemma 7.6.2, we state a special case of Lemma 7.6.2.

Corollary 7.6.3. *Assume that* $a \geq 2$ *and that* $\mu(l) \geq 2$. *Let* J *be a subset of* $[r]$ *with* $0 \leq n \leq r-1$ *elements and* $i \in [r] - J$. *Then the map*

$$\iota^*_{J\cup i, J} : C^k_-(\Delta^{J\cup i}) \to C^k_-(\Delta^J) \tag{7.6.3}$$

is surjective.

Proof of Lemma 7.6.2. To avoid unnecessary repetition, whenever we use letter K, we mean a subset $K \subseteq J \cup M \cup i$ containing $k+1$ elements.

We first prove the lemma for $k = -1$. In this case, we have $A = \varnothing$. Since by definiton every $C_{-1}(\Delta^I)$ is generated by the empty set as the (-1)-simplex. We can just set

$$\tilde{x}(\varnothing) = x(\varnothing). \tag{7.6.4}$$

It is trivial to verify that $\tilde{x}$ has desired properties.

Now we fix $k \geq 0$ and $p \geq 2$, let us prove the lemma by induction on n. We start with the base case when $\mathbf{0 \leq n \leq \max(0, k-p+2)}$. To show the lemma in this base case, we have to consider two cases: $k - p + 2 \geq 0$ and $k - p + 2 < 0$.

(1) If $k - p + 2 \geq 0$, then for any $n \leq k - p + 2$, we have $n + p - 2 < k + 1$ and thus $J \cup M$ has less than $k + 1$ elements. The cochain $x \in C^k_-(\Delta^{J\cup M})$ can only be 0.

If $n < k - p + 2$, then we have $n - p - 1 < k + 1$ and thus $J \cup M \cup i$ has less than $k + 1$ elements. We can set $\tilde{x} = 0$ and we are done.

If $n = k - p + 2$, then $n + p - 1 = k + 1$ which means that $J \cup M \cup i$ contains $k + 1$ elements. Since $J \cup M \cup i$ is the only k-simplex of $\Delta^{J \cup M \cup i}$, we can construct the cochain $\tilde{x} \in C^k(\Delta^{J \cup M \cup i})$ by setting

$$\tilde{x}(M \cup i) = \begin{cases} a_{M \cup i} & \text{if } J \cup M \cup i \text{ is short,} \\ 0 & \text{if } J \cup M \cup i \text{ is long.} \end{cases} \tag{7.6.5}$$

The constructed $\tilde{x}$ is obviously in $C^k_-(\Delta^{J \cup M \cup i})$ because $J \cup M \cup i$ is the only subset containing $k + 1$ elements in $J \cup M \cup i$ and there is no subset containing $k + 2$ elements in $J \cup M \cup i$.

We claim that the constructed $\tilde{x}$ satisfies condition (a), (b) and (c). Condition (a) is null because there is no k-simplex in $J \cup M$.

Since $n + p - 2 < k + 1$, $(J \cup M \cup i) - m$ contains less than $k + 1$ elements. So condition (c) is satisfied because $C_k(\Delta^{(J \cup M \cup i) - m}) = 0$ for every $m \in M$.

Condition (b) is satisfied because $J \cup M \cup i$ is the only subset containing $k + 1$ elements in $J \cup M \cup i$.

We have shown that $\tilde{x}$ has desired properties.

If $k - p + 2 < 0$, then we only have to prove the lemma for $n = 0$ in the base case, that is, $J = \varnothing$.

(2) If $k - p + 2 < 0$, then we have $k + 1 \leq p - 2$ and thus $A = \varnothing$.

If $k + 1 < p - 2$, any $x \in C^k_-(\Delta^M)$ satisfying (7.6.2) can only be 0 because any subset of M with less than $k + 1$ elements does not contain the whole M. We can just let $\tilde{x} = 0$ and we are done.

If $k+1=p-2$, then M is the only k-simplex of M. We just set

$$\begin{aligned} \tilde{x}(M) &:= x(M), \\ \tilde{x}((M\cup i)-m) &:= 0, \quad \forall m\in M \end{aligned} \tag{7.6.6}$$

and this $\tilde{x}$ satisfies all the conditions $(a), (b)$ and (c). We have to show that $\tilde{x}\in C^k_-(\Delta^{J\cup i})$, that is, $\tilde{x}$ vanishes on $C^k_+(\Delta^{M\cup i})$ and $\partial C^{k+1}_+(\Delta^{M\cup i})$.

If $M\cup i$ is short, then we have $\tilde{x}\in C^k_-(\Delta^{M\cup i})$ trivially.

If $M\cup i$ is long, then according to Remark 7.6.1, the set M has to be long and thus $\tilde{x}(M)=x(M)=0$. It then follows from (7.6.6) that $\tilde{x}=0$ and thus $\tilde{x}\in C^k_-(\Delta^{M\cup i})$.

Now we assume that the lemma is true for J with $n-1$ elements where $\boldsymbol{n-1\geq \max(0,k-p+2)}$. Let us prove it for J with n elements. Note that we have $n+p-2\geq k+1$ and $n\geq 1$.

For a cochain $x\in C^k_-(\Delta^{J\cup M})$ satisfying the condition (7.6.2) and a tuple $(a_K)_{K\in A}$ as in the assumptions of the lemma, let us construct $\tilde{x}\in C^k_-(\Delta^{J\cup M\cup i})$ such that conditions (1), (2) and (3) are satisifed. There are two steps in this construction.

STEP I: First we construct an $\hat{x}\in C^k(\Delta^{J\cup M\cup i})$ instead of in $C^k_-(\Delta^{J\cup M\cup i})$. To initialize, we set

$$\hat{x}(K) := x(K), \quad \forall K\subseteq J\cup M, \tag{7.6.7}$$

$$\hat{x}(K) := a_K, \quad \forall K\in A, \tag{7.6.8}$$

$$\hat{x}(K) := 0, \quad \forall \text{ long } K\subseteq J\cup M\cup i, \tag{7.6.9}$$

$$\hat{x}(K) := 0, \quad \forall K\subseteq J\cup M\cup i \text{ and } M\nsubseteq K. \tag{7.6.10}$$

There is some overlapping between (7.6.7) and (7.6.10) on $K\subseteq J\cup M$ such that $M\nsubseteq K$. Since $x(K)=0$ for such K by (7.6.2), on this overlapping (7.6.7) and (7.6.10) are compatible with each other.

If $n>1$ (resp. $n=1$), then for any short $K\subseteq J\cup M\cup i$ containing $M\cup\{j_{n-1},i\}$ (resp. $\{i\}$)

but not j_n, the value of $\hat{x}$ on all facets of $K \cup j_n$ except K has been set by (7.6.7) - (7.6.10). So for every such subset K, we can set $\hat{x}(K)$ such that $\hat{x}(\partial(K \cup j_n)) = 0$. Explicitly, it means

$$\hat{x}(K) := (-1)^{[j_n:K\cup j_n]+1} \sum_{k\in K} (-1)^{[k:K\cup j_n]} \hat{x}((K-k) \cup j_n). \tag{7.6.11}$$

If $n > 1$ (resp. $n = 1$), the above steps (7.6.7)-(7.6.11) have set $\hat{x}(K)$ for all subsets $K \subseteq J \cup M \cup i$ except those short subsets containing i but excluding $\{j_{n-1}, j_n\}$ (resp. $\{j_1\}$). We are going to use induction hypothesis to "extend" $\hat{x}$ to this missing part. Let $\tilde{J} := J - j_n$ so that $|\tilde{J}| = n - 1$. We restrict x to $C_k(\Delta^{\tilde{J}\cup M})$ and by (7.6.2), we have

$$\forall m \in M, \quad i^*_{\tilde{J}\cup M,(\tilde{J}\cup M)-m}(x|_{\Delta^{\tilde{J}\cup M}}) = 0. \tag{7.6.12}$$

Since (7.6.11) has set $\hat{x}(K)$ for all short subsets K containing $\{j_{n-1}, i\}$ (resp. $\{i\}$) but excluding j_n, now we have all the ingredient to apply induction hypothesis. By induction hypothesis, there is $\bar{x} \in C^k_-(\Delta^{\tilde{J}\cup M\cup i})$ such that

$$\bar{x}(K) = x(K), \quad \forall \text{ short } K \subseteq \tilde{J} \cup M,$$

$$\bar{x}(K) = \hat{x}(K), \quad \forall K \subseteq \tilde{J} \cup M \cup i \text{ containing } M \cup \{j_{n-1}, i\} \text{ (resp.} M \cup i),$$

$$i^*_{\tilde{J}\cup M\cup i,(\tilde{J}\cup M\cup i)-m}(\bar{x}) = 0, \quad \forall m \in M.$$

Since values of $\hat{x}(K)$ have been set for all K containing j_n, with this $\bar{x}$ we can complete the construction of $\hat{x}$ by setting

$$\forall K \subseteq \tilde{J} \cup M \cup i, \quad \hat{x}(K) := \bar{x}(K). \tag{7.6.13}$$

We can say the above formula "extends" $\hat{x}$ to a cochain in $C^k(\Delta^{J\cup M\cup i})$ because it does not change the value we set for $\hat{x}$ before in (7.6.7) - (7.6.11).

We summarize the above construction by having the following diagram in mind:

$$\begin{array}{ccc} x \in C^k_-(\Delta^{J\cup M}) & \searrow & \\ & & C^k_-(\Delta^{(J-j_n)\cup M}) \ni x|_{\Delta^{(J-j_n)\cup M}} \\ \exists \bar{x} \in C^k_-(\Delta^{(J-j_n)\cup M\cup i}) & \nearrow & \end{array} \tag{7.6.14}$$

where the arrows are all restrictions of cochains to subcomplexes. The construction of $\hat{x}$ goes like this. We start by requiring (7.6.7), (7.6.8) (7.6.9) and (7.6.10). Then we set $\hat{x}(K)$ for K containing $M \cup \{j_{n-1}, i\}$ (resp. $M \cup \{i\}$) but not j_n by (7.6.11). These K's are simplices in $\Delta^{(J-j_n)\cup M\cup i}$. With this setup, the restriction of x to $\Delta^{(J-j_n)\cup M}$ can be lifted to an element $\bar{x} \in C^k_-(\Delta^{(J-j_n)\cup M\cup i})$ using induction hypothesis. Since $\bar{x}$ and $\hat{x}|_{\Delta^{J\cup M}} = x$ agree on $C_k(\Delta^{(J-j_n)\cup M})$, we can paste these two cochains to get an element in $C^k(\Delta^{J\cup M\cup i})$. That is our $\hat{x}$.

STEP II: If $n = 1$, then $\hat{x}$ constructed in the first step is already our $\tilde{x}$. We are going to show that this $\tilde{x}$ has the desired properties later. If $n > 1$, in this step we are going to change the value of $\hat{x}$ on some short sets to get $\tilde{x}$. The reason of doing that is that the $\hat{x}$ constructed in *STEP I* is not yet in $C^k_-(\Delta^{J\cup M\cup i})$. For example, $\hat{x}$ is not guaranteed to vanish on a long subset with $k+2$ elements containing $M \cup \{j_n, i\}$ but excluding j_{n-1}.

We define

$$\tilde{A} := \{K \subseteq (J \cup M \cup i) - \{j_{n-1}, j_n\} : M \cup i \subseteq K,\, K \cup j_{n-1} \text{ short},\, K \cup j_n \text{ long}\} \tag{7.6.15}$$

and set

$$\forall K \notin \tilde{A}, \quad \tilde{x}(K) := \hat{x}(K). \tag{7.6.16}$$

For any $K \in \tilde{A}$, values of $\tilde{x}$ on all facets of $K \cup j_n$ except K has been set in (7.6.16). We then set $\tilde{x}(K)$ such that $\tilde{x}(\partial(K \cup j_n)) = 0$, that is, for any $K \in \tilde{A}$, we set

$$\tilde{x}(K) := (-1)^{[j_n:K\cup j_n]+1} \sum_{k\in K} (-1)^{[k:K\cup j_n]} \tilde{x}((K-k) \cup j_n) \tag{7.6.17}$$

which is exactly the same formula as (7.6.11). We have completed the construction of $\tilde{x}$.

The rest of the proof is to verify that $\tilde{x}$ constructed above satisfies the desired properties. We first show that $\tilde{x}$ satisfies the condition (a), (b) and (c).

(a): For any simplex $K \subseteq J \cup M$, because of (7.6.16) and definition of $\tilde{A}$, we have $\tilde{x}(K) = \hat{x}(K)$. So $\tilde{x}(K)$ is equal to $x(K)$ because of (7.6.7). We have shown that $\tilde{x}$ satisifies condition (a).

(b): For any simplex $K \in A$, $\tilde{x}(K) = \hat{x}(K)$ because by (7.6.16) and definition of $\tilde{A}$, $\tilde{x}(K) = \hat{x}(K)$ for any short subset K containing j_n. So for any $K \in A$, $\tilde{x}(K) = a_K$ because of (7.6.8). We have shown that $\tilde{x}$ satisfies condition (b).

(c): For any simplex K not containing M, $\tilde{x}(K) = \hat{x}(K)$ because by (7.6.16) and definition of $\tilde{A}$, such K is not an element of $\tilde{A}$. Since we require (7.6.10) in the construction of $\hat{x}$, for any K not containing M, we have $\tilde{x}(K) = \hat{x}(K) = 0$. $\tilde{x}$ satisfies condition (c).

The rest of the proof is to verify that $\tilde{x} \in C^k_-(\Delta^{J\cup M\cup i})$ which is defined in (7.5.5).

We first show that $\tilde{x}$ vanishes on $C_k(\Delta_+^{J\cup i})$. For any long subset $K \subset J \cup M \cup i$, it follows from (7.6.16) and definition of $\tilde{A}$ that $\tilde{x}(K) = \hat{x}(K)$. So $\tilde{x}(K) = 0$ because of (7.6.9). We have shown that $\tilde{x}$ vanishes on $C_k(\Delta_+^{J\cup i})$.

Next we show that $\tilde{x}$ vanishes on $\partial C_{k+1}(\Delta_+^{J\cup M\cup i})$. We assume $H \subseteq J \cup M \cup i$ is a long subset containing $k + 2$ elements from now on.

We have the following observations from the nature of the collection $\tilde{A}$ and (7.6.16) in *STEP II*. Let K be a face of a long set H. We have $\tilde{x}(K) = \hat{x}(K)$ in the following situations:

- $M \cup i \not\subseteq H$: Indeed, if $M \cup i \not\subseteq H$, then $M \cup i \not\subseteq K$ and thus $K \notin \tilde{A}$.
- $M \cup \{j_{n-1}, i\} \subseteq H$: Indeed, if $j_{n-1} \in K$, then $K \notin \tilde{A}$. If $j_{n-1} \in K$, then $K \cup j_{n-1} = H$ is long and thus $K \notin \tilde{A}$.

- $M \cup i \subseteq H$ and $\{j_{n-1}, j_n\} \cap M = \emptyset$: Indeed, if $M \cup i \nsubseteq K$, then $K \notin \tilde{A}$. If $M \cup i \subseteq K$, then $K \cup j_{n-1} = (H - j) \cup i$ for some $j \in J - \{j_{n-1}, j_n\}$. Since $j_{n-1} > j$, the set $K \cup j_{n-1}$ is long and thus $K \nsubseteq \tilde{A}$.

Now we show that $\tilde{x}$ vanishes on $\partial C_{k+1}(\Delta_+^{J \cup M \cup i})$ by looking at the following cases:

(1) If $M \cup i \nsubseteq H$, then $\tilde{x}(\partial H) = 0$ because of the first observation above and $\hat{x}(\partial H) = x(\partial H) = 0$.

(2) If $M \cup i \subseteq H$ and $\{j_{n-1}, j_n\} \cap H = \emptyset$, then from the third observation above we have $\tilde{x}(\partial H) = \hat{x}(\partial H)$. Since any face of H does not contain j_n, we get $\tilde{x}(\partial H) = \bar{x}(\partial H) = 0$ from (7.6.13) and the fact that $\bar{x} \in C_-^k(\Delta^{(J - j_n) \cup M \cup i})$.

(3) If H contains $M \cup \{j_{n-1}, i\}$ but not j_n, then it follows from the second observation above that $\tilde{x}(\partial H) = \hat{x}(\partial H)$. By (7.6.13) we have $\hat{x}(K) = \bar{x}(K)$ for all $K \subseteq H$. Since $\bar{x} \in C_-^k(\Delta^{(J - j_n) \cup M \cup i})$, we have $\partial \tilde{H} = \bar{x}(\partial H) = 0$.

(4) If $M \cup \{j_{n-1}, j_n, i\} \subseteq H$, then from the second observation above we have $\tilde{x}(\partial H) = \hat{x}(\partial H)$. Let us look at $\hat{x}(\partial H)$ now. If $H - j_n$ is long, then by Remark 7.6.1, all facets of H are long and thus $\hat{x}(\partial H) = 0$ by (7.6.9). If $H - j_n$ is short, then by (7.6.11), $\hat{x}(H - j_n)$ is set such that $\hat{x}(\partial H) = \hat{x}(\partial((H - j_n) \cup j_n)) = 0$. In both cases, $\hat{x}(\partial H) = 0$ and thus $\tilde{x}(\partial H) = 0$.

Note that we have shown in (1), (3) and (4) that for any H containing j_{n-1}, we have $\tilde{x}(\partial H) = 0$.

(5) If H contains $M \cup \{j_n, i\}$ but not j_{n-1}, then $\tilde{x}(\partial H)$ may not be equal to $\hat{x}(\partial H)$ anymore because they may differ on the simplex $H - j_n$ afterr *STEP II*. Instead of focusing on $\tilde{x}(H)$, let us look at the $(k+3)$-simplex $H \cup j_{n-1}$. If at least one facet of $H \cup j_{n-1}$ is short, then by Remark 7.6.1, $(H \cup j_{n-1}) - j_n = (H - j_n) \cup j_{n-1}$ is short. Since $(H - j_n) \cup j_n = H$ is long, by (7.6.15), $H \in \tilde{A}$ and thus $\tilde{x}(\partial H) = 0$ by (7.6.17) in this case.

   If all facets of $H \cup j_{n-1}$ are long, then we have shown that $\tilde{x}$ vanishes on the boundary of all facets of $H \cup j_{n-1}$ except H because all facets of $H \cup j_{n-1}$ except H contains j_{n-1}.

We can conclude that $\tilde{x}(H) = 0$ from this observation because $\tilde{x}(\partial\partial(H \cup j_{n-1}))$ is an alternating sum of the value of $\tilde{x}$ on boundary of all facets of $H \cup j_{n-1}$ and is always 0 because $\partial\partial(H \cup j_{n-1}) = 0$.

We have completed the proof. □

7.7 Proof of Lemma 7.3.1 and Theorem 7.1.4

Now we prove Lemma 7.3.1. In the end of this section we will complete the proof of Theorem 7.1.4.

With Corollary 7.4.3, we are only interested in the exactness of the cochain complex $P_k^*(F_J)$ for any $0 \le k \le r-1$ and $J \subseteq [r]$. Furthermore, with commutativity and horizontal isomorphism in (7.5.6), for any integer k, we can rewrite the cochain complex $P_k^*(F_J)$ as

$$P_k^i(F_J) = \bigoplus_{\substack{M \subseteq [r]-J \\ |M| = r-i-|J|}} C_-^{k-1}(\Delta^{J \cup M}) \tag{7.7.1}$$

where $C_-^*(\Delta^J)$ was defined in (7.5.5).

For any $\sigma \in P_k^i(F_J)$ and any set $M \supseteq [r] - J$ such that $|M| = r - i - |J|$, we will denote its component on $C_-^{k-1}(\Delta^{J \cup M})$ as σ_M. We also denote the differential of this cochain complex as d_J to specify that the differential depends on the subset J. This d_J on each direct summand of $P_k^i(F_J)$ in (7.7.1) has the following formula:

$$d_J(\sigma_M) = \sum_{\substack{N \subseteq M \\ |N| = r-i-1-|J|}} (-1)^{[M-N,M]} \iota_{M,N}^*(\sigma_M) \tag{7.7.2}$$

where $i_{M,N}^* : C_-^{k-1}(\Delta^{J \cup M}) \to C_-^{k-1}(\Delta^{J \cup N})$ is the cochain map induced by inclusion $\Delta^{J \cup N} \hookrightarrow \Delta^{J \cup M}$.

Before we state and prove the main proposition, we want to introduce a filtration on $P_k^*(F_J)$. Let $i_1 > i_2 > \cdots > i_n$ where $n = r - |J|$ be all the elements in $[r] - J$. There is a filtration on

$P_k^*(F_J)$:

$$F^n P_k^*(F_J) \subseteq F^{n-1} P_k^*(F_J) \subseteq \cdots F^1 P_k^*(F_J) \subseteq F^0 P_k^*(F_J) = P_k^*(F_J) \tag{7.7.3}$$

where

$$F^s P_k^*(F_J) = \bigoplus_{M \not\ni i_1, \ldots, i_s} C_-^{k-1}(\Delta^{J \cup M}) \tag{7.7.4}$$

and the direct sum is over $M \subseteq [r] - J$ such that $|M| = r - i - |J|$. The first several terms of the filtration look like the following:

$$\begin{array}{rcccccc}
F^0 P_k^*(F_J): \cdots \to & \bigoplus\limits_{i,j \in [r]-J} C_-^{k-1}(\Delta^{J \cup \{i,j\}}) & \to & \bigoplus\limits_{i \in [r]-J} C_-^{k-1}(\Delta^{J \cup i}) & \to & C_-^{k-1}(\Delta^J) \\
& \uparrow & & \uparrow & & \uparrow {\scriptstyle =} \\
F^1 P_k^*(F_J): \cdots \to & \bigoplus\limits_{\substack{i,j \in [r]-J \\ i,j \neq i_1}} C_-^{k-1}(\Delta^{J \cup \{i,j\}}) & \to & \bigoplus\limits_{\substack{i \in [r]-J \\ i \neq i_1}} C_-^{k-1}(\Delta^{J \cup i}) & \to & C_-^{k-1}(\Delta^J) \\
& \uparrow & & \uparrow & & \uparrow {\scriptstyle =} \\
F^2 P_k^*(F_J): \cdots \to & \bigoplus\limits_{\substack{i,j \in [r]-J \\ i,j \neq i_1, i_2}} C_-^{k-1}(\Delta^{J \cup \{i,j\}}) & \to & \bigoplus\limits_{\substack{i \in [r]-J \\ i \neq i_1, i_2}} C_-^{k-1}(\Delta^{J \cup i}) & \to & C_-^{k-1}(\Delta^J).
\end{array} \tag{7.7.5}$$

Remark 7.7.1. (1) Every $F^i P_k^*(F_J)$ is indeed a cochain subcomplex with differential inherited from d_J.

(2) The left end of the filtration, that is, $F_k^n(F_J)$ is the following cochain complex centered at degree $r - |J|$:

$$0 \to C_-^{k-1}(\Delta^J) \to 0. \tag{7.7.6}$$

We can prove the following proposition now.

Proposition 7.7.2. *For any $k \geq 0$, any $J \subseteq [r]$, if $\mu(l) \geq p \geq 2$, then for any $q > \max(r - |J| - p + 1, 0)$ we have*

(1) The map between cohomology induced by the cochain map $F^s P_k^*(F_J) \to F^{s-1} P_k^*(F_J)$

$$H^q(F^s P_k^*(F_J)) \to H^q(F^{s-1} P_k^*(F_J)) \tag{7.7.7}$$

is surjective for $1 \le s \le q$.

(2) The cochain complex $F^s P_k^*(F_J)$ *is exact at degree* q *for* $0 \le s \le q-1$.

Proof. We prove the proposition by induction on p. We start with the base case $\boldsymbol{p = 2}$.

If $J = [r]$, then $r - |J| - p + 1 = -1 < 0$. In this case, the cochain complex $P_k^*(F_J)$ has only one nonzero term $C_-^{k-1}(\Delta^{[r]})$ at degree 0. In other words, $P_k^*(F_J)$ is

$$0 \to C_-^{k-1}(\Delta^{[r]}) \to 0. \tag{7.7.8}$$

So $P_k^*(F_J)$ is trivially exact at degrees greater than 0 and thus (1) of the proposition is true in this case. In this case, since $[r] - J = \varnothing$, the filtration in (7.7.3) is trivial and thus (2) of the proposition is trivially true in this case.

If $J \ne [r]$, then $[r] - J$ is not empty. Since $r - 1 - |J| \ge 0$ when $J \ne [r]$, the exactness of $F^s P_k^*(F_J)$ in degrees $q > \max(r - |J| - p + 1, 0) = r - 1 - |J|$ is reduced to the exactness at degree $q = r - |J|$ which is the right end of $F^s P_k^*(F_J)$ (diagram in (7.7.5) gives a clearer picture). For $s \le q - 1$, the summand of $F^s P_k^*(F_J)$ in degree $r - 1 - |J|$ is not empty, that is, the second rightmost term in $F^s P_k^*(F_J)$ in (7.7.5) is not a trivial sum. Part (2) of the proposition then follows from Corollary 7.6.3. Part (1) of the proposition is trivially true because all the vertical maps at the right end of (7.7.5) are identity maps. That completes the proof of the base case.

Let us prove the proposition for p assuming that it holds for $p - 1$. If J is a subset such that $r - |J| - p + 1 < 0$, then we have $\max(r - |J| - p + 1, 0) = \max(r - |J| - p + 2, 0)$ and we are done by induction hypothesis. So we can assume that J is a subset such that $r - |J| - p + 1 \ge 0$. By induction hypothesis, we only have to show that (1) and (2) are true for $q = r - |J| - p + 2$.

Let us show (1) first. We fix an $s \le q$. The following short exact sequence

$$0 \to F^s P_k^*(F_J) \to F^{s-1} P_k^*(F_J) \to {}^{F^{s-1}P_k^*(F_J)}/_{F^s P_k^*(F_J)} \to 0 \tag{7.7.9}$$

induces a long exact sequence

$$\cdots \to H^q(F^sP_k^*(F_J)) \to H^q(F^{s-1}P_k^*(F_J)) \to H^q({}^{F^{s-1}P_k^*(F_J)}/_{F^sP_k^*(F_J)}) \to \cdots . \tag{7.7.10}$$

Let us show that $H^q({}^{F^{s-1}P_k^*(F_J)}/_{F^sP_k^*(F_J)}) = 0$ and that directly implies (1).

Let S be the set $\{i_1, \ldots, i_s\}$. The quotient cochain complex ${}^{F^{s-1}P_k^*(F_J)}/_{F^sP_k^*(F_J)}$ has the following form in degree i:

$$\left({}^{F^{s-1}P_k^*(F_J)}/_{F^sP_k^*(F_J)}\right)^i = \bigoplus_{\substack{(S-i_s)\cap M=\varnothing \\ i_s\in M}} C_-^{k-1}(\Delta^{J\cup M}) \tag{7.7.11}$$

where the direct sum is over $M \subseteq [r] - J$ such that $|M| = r - i - |J|$. Differential on this cochain complex is induced by d_J.

A closer look at this quotient cochain complex reveals that we have

$$ {}^{F^{s-1}P_k^i(F_J)}/_{F^sP_k^i(F_J)} \simeq F^{s-1}P_k^{i+1}(F_{J\cup i_s}) \tag{7.7.12}$$

as vector spaces, that is, the degree-i term of it is isomorphic to the degree-$(i + 1)$ term of $F^{s-1}P_k^*(F_{J\cup i_s})$.

Furthermore, we claim that the differential $d_{J\cup i_s}$ of $F^{s-1}P_k^*(F_{J\cup i_s})$ coincides with the differential d_J of the quotient cochain complex. Indeed, the differential $d_{J\cup i_s}$ on $\sigma \in F^{s-1}P_k^{i+1}(F_{J\cup i_s})$ is given by the following formula on the component σ_M where $\{i_1, \ldots, i_{s-1}\} \cap M = \varnothing$ and $i_s \in M$:

$$d_{J\cup i_s}(\sigma_M) = \sum_{\substack{N\subseteq M \\ i_s\in N}} (-1)^{[M-N,M-i_s]} \iota_{M,N}^*(\sigma_M) \tag{7.7.13}$$

where the sum is over N with $|N| = r-i-1-|J|$ as in (7.7.2). Since i_s is the largest element in M and N contains i_s, the sign $(-1)^{[M-N,M-i_s]}$ in front of $\iota_{M,N}^*(\sigma_M)$ in (7.7.13) is exactly $(-1)^{[M-N,M]}$. Comparing with (7.7.2) shows our claim.

We have shown the following isomorphism of cochain complexes:

$$F^{s-1}P_k^*(F_J)/F^sP_k^*(F_J) \simeq F^{s-1}P_k^*(F_{J\cup i_s})[-1] \tag{7.7.14}$$

where the degree-i term of $F^{s-1}P_k^*(F_{J\cup i_s})[-1]$ is $F^{s-1}P_k^{i+1}(F_{J\cup i_s})$.

Now it follows from the induction hypothesis that the cochain complex $F^{s-1}P_k^*(F_{J\cup i_s})$ is exact at degree $r-|J\cup i_s|-p+3$ for $s-1\leq r-|J\cup i_s|-p+2$, that is, degree $q+1$ for $s\leq q$. So the quotient complex in (7.7.11) is exact at degree q for any $s\leq q$ and thus we have

$$H^q(F^{s-1}P_k^*(F_J)/F^sP_k^*(F_J)) = 0. \tag{7.7.15}$$

Then (1) follows from the exact sequence (7.7.10).

Let us prove (2) now. By (1) we only have to show that $F^sP_k^*(F_J)$ is exact at $q=r-|J|-p+2$ for $s=q-1$. We look at the following part of the filtration in (7.7.3):

$$\begin{array}{llcl} F^sP_k^*(F_J): & C_-^{k-1}(\Delta^{[r]-S}) & \to & \bigoplus_{\substack{S\cap M=\emptyset\\ |M|=p-2}} C_-^{k-1}(\Delta^{J\cup M}) \to \cdots \\ & \uparrow & & \uparrow \\ F^{s+1}P_k^*(F_J): & 0 & \longrightarrow & C_-^{k-1}(\Delta^{[r]-(S\cup i_{s+1})}) \to \cdots \end{array} \tag{7.7.16}$$

where again, $S=\{i_1,\dots,i_s\}$. The two columns of the above diagram in both rows are of degree $q-1$ and q. It follows from a cardinality argument that we have only one term on each entry of the main diagonal in the above diagram.

Now let us prove that $F^sP_k^*(F_J)$ is exact at degree q. For a cochain $\sigma\in F^sP_k^q(F_J)$ in $\ker d_J$, it follows from (2) that it is cohomologous to an element $\tilde{\sigma}\in F^{s+1}P_k^q(F_J)$ in $\ker d_J$ which is just one single term $C_-^{k-1}(\Delta^{[r]-(S\cup i_{s+1})})$ by (7.7.16). Then we are in a situation where we can apply Lemma 7.6.2. Since $\tilde{\sigma}$ is in $\ker d_J$, it satisfies the condition (7.6.2). It follows from Lemma 7.6.2 that there is $\tau\in C_-^{k-1}(\Delta^{[r]-S}) = F^sP_k^{q-1}(F_J)$ such that the following property is satisfied:

$$d_J(\tau) = \tilde{\sigma}. \tag{7.7.17}$$

Indeed, property(c) in Lemma 7.6.2 implies that components of $d_J(\tau)$ on $C_-^{k-1}(\Delta^{J\cup M})$ are zeros except when $J \cup M = [r] - (S \cup i_{s+1})$. Property(a) in Lemma 7.6.2 then implies (7.7.17).

We have completed our proof. □

Proof of Lemma 7.3.1. By Corollary 7.4.3 it suffices to show that

$$H^i(P_k^*(F_J)) = 0 \tag{7.7.18}$$

for all $J \subseteq [r]$, $0 \leq k \leq r-1$ and $i > \max(r - |J| - \mu(l) + 1, 0)$. This is the result of Proposition 7.7.2(2) for $s = 0$. □

Corollary 7.7.3. *Let $a \geq 2$, $b, r \geq 1$, then for any generic length vector $l \in \mathbb{R}^r$ we have*

$$\operatorname{syzord} H_T^*(X_{a,b}(l)) = \mu(l) - 1. \tag{7.7.19}$$

Proof. The case when the fixed point set is empty was proved in Lemma 7.2.1. The case when the fixed point set is nonempty was proved in Lemma 7.3.1. □

Now we can complete the proof of Theorem 7.1.4.

Proof of Theorem 7.1.4. The proof follows from Corollary 7.7.3 and Lemma 4.2.2. □

Chapter 8

Applications

In this section, we prove several applications of the main theorem we proved. Our main theorem says that we can compute the syzygy order of the equivariant cohomology of a big polygon space $X_{a,b}(l)$ directly from the combinatorial properties of the length vector l. However, two big polygon spaces having the same syzygy order may not have the same equivariant diffeomorphism type. In this section we will see that in certain cases, we can infer the equivariant diffeomorphism type of a big polygon space from the syzygy order of its equivariant cohomology.

The significance of our application lies in Remark 4.1.2. According to Remark 4.1.2, the number of equivariant diffeomorphism types grows fast when r increases. So it will be good if we can rule out many equivariant diffeomorphism types using syzygy order. Our application shows that it is possible if we consider syzygy orders that are high enough.

To make it more convenient to read, we will indicate number of elements in a set whenever is necessary by writing a bracket with number over it.

One immediate application is a new proof of [16, Theorem 1.2].

Corollary 8.1 ([16, Theorem 1.2])**.** *Let $a, b \geq 1, m \geq 0$, and $l \in \mathbb{R}^r$ be generic with $0 \leq l_1 \leq \cdots \leq l_r$.*

(1) Assume $r = 2m + 1$, then $\operatorname{syzord} H_T^*(X_{a,b}(l)) = m$ if and only if $X_{a,b}(l)$ is equivariantly

diffeomorphic to $X_{a,b}(\overbrace{1,\ldots,1}^{2m+1})$.

(2) *Assume* $r = 2m+2$*, then* $\operatorname{syzord} H_T^*(X_{a,b}(l)) = m$ *if and only if* $X_{a,b}(l)$ *is equivariantly diffeomorphic to* $X_{a,b}(0,\overbrace{1,\ldots,1}^{2m+1})$.

Let us first prove two lemmas that will be used a lot.

Lemma 8.2. *If* $l \in \mathbb{R}^r$ *is generic and* $\mu(l) \geq m$*, then all the subsets of* $[r]$ *with less than* m *elements are short.*

Proof. If $M \subseteq [r]$ has less than m elements and is long, then all the subsets of M have to be long. Indeed, if the collection of long subsets of M is not empty, then we can take a minimal such subset because $\varnothing$ is always short. On the other hand, a minimal long subset $N \subseteq M$ satisfies:

$$\sigma_l(N) < m. \tag{8.1}$$

That contradicts the assumption $\mu(l) \geq m$ because by Definition 7.1.1 we should have $\mu(l) \leq \sigma_l(N)$. □

Lemma 8.3. *Assume* $l \in \mathbb{R}^r$ *is generic,* $l_1 \leq \cdots \leq l_r$ *and* $\mu(l) \geq m \geq 2$*. Let* M *be a long subset of* $[r]$ *and* N *consists of* n *largest elements in* $[r] - M$ *with* $n \leq m-1$*. Then for any subset* $N' \subseteq M$ *with* n *elements, the following set*

$$(M - N') \cup N \tag{8.2}$$

is long.

Proof. Let us fix m and do induction on n. When $n = 0$, it is trivially true.

Assuming the lemma has been proved for $n-1$, let us prove it for n. Let

$$i' = \min N' \text{ and } i = \min N. \tag{8.3}$$

Since $|N - i| = n - 1$, applying the induction hypothesis to M, $N - i$ and $N' - i'$ we can get

$$M' := (M - (N' - i')) \cup (N - i) \text{ is long.} \tag{8.4}$$

If $M' - i'$ is long, then $(M - N') \cup N = (M' - i') \cup i$ is long and we are done. If $M' - i'$ is short, then since we have

$$M' = (M' - i') \cup i' \text{ is long,} \tag{8.5}$$

it follows from the equivalent definition of $\mu(l)$ in Remark 7.1.2 that there are at least m elements in $[r] - (M' - i')$ such that when one of these elements is added to $M' - i'$, the resulting subset is long.

Since N' has only $n \leq m - 1$ elements and i is the largest element in $[r] - (M' - i')$ apart from those in N', we have

$$(M' - i') \cup i = (M - N') \cup N \text{ is long.} \tag{8.6}$$

□

Remark 8.4. (1) Lemma 9.2 allows us to get some information about long and short subsets of $[r]$ directly from $\mu(l)$.
(2) Lemma 9.3 is essentially saying that we can substitute a number of elements in a long set by the same amount of the largest several elements in the complement to get another long set. That will be useful when we want to use contradiction to prove that a subset is short.

Proof of Corollary 8.1. The "if" parts of both (1) and (2) follow from the fact that equivariant cohomology is invariant under equivariant diffeomorphism and the observation that

$$\mu(\overbrace{1, \ldots, 1}^{2m+1}) = \mu(0, \overbrace{1, \ldots, 1}^{2m+1}) = m + 1. \tag{8.7}$$

By Theorem 7.1.4, we have $\operatorname{syzord} H_T^*(X_{a,b}(l)) = m$.

Let us prove the "only if" part for (1) and (2).

If $r = 2m + 1$ and syzord $H^*_T(X_{a,b}(l)) = m$, then by Theorem 7.1.4 we have

$$\mu(l) = m + 1. \tag{8.8}$$

It follows from Lemma 8.2 that for any $J \subseteq [r]$ we have

$$|J| \leq m \Rightarrow J \text{ short}. \tag{8.9}$$

Since the complement of a short set is long, the long and short subsets of $[r]$ are described by the following:

$$J \text{ is } \begin{cases} \text{long if } |J| \geq m + 1, \\ \text{short if } |J| \leq m \end{cases} \tag{8.10}$$

and that gives the same long and short subsets as the length vector $(\overbrace{1, \ldots, 1}^{2m+1})$ which implies that $l \sim (1, \ldots, 1)$. By Lemma 4.1.5(3), $X_{a,b}(l)$ is equivariantly diffeomorphic to $X_{a,b}(1, \ldots, 1)$.

If $r = 2m + 2$, the same argument as above shows that

$$J \text{ is } \begin{cases} \text{long if } |J| \geq m + 2, \\ \text{short if } |J| \leq m. \end{cases} \tag{8.11}$$

But we need some more information about subsets containing $m + 1$ elements. Let us prove that

$$|J| = m + 1 \text{ and } 1 \in J \Rightarrow J \text{ short}. \tag{8.12}$$

Indeed, let us show that the following subset with $m + 1$ elements

$$\{1, \overbrace{m + 3, m + 4..., 2m + 2}^{m}\} \tag{8.13}$$

is short and (8.12) follows from it because (8.13) is the longest subset in $[r]$ with $m+1$ elements containing 1.

Let us prove by contradiction. Assume that (8.13) is long. By Lemma 8.3, we can substitute $\{\overbrace{m+3,...,2m+2}^{m}\}$ with $\{\overbrace{3,...,m+2}^{m}\}$ and get the following long subset

$$\{1,\overbrace{3,...,m+2}^{m}\}. \tag{8.14}$$

Then the subset $\{\overbrace{2,3,...,m+2}^{m+1}\}$ is also long because we are replacing l_1 by a longer l_2. That implies that all subsets of $[r]$ with $m+1$ elements not containing 1 are long, and that shows that (8.13) is short because the complement of a long set is short. We have proved (8.12).

Since the complement of a subset with $m+1$ elements is another subset with $m+1$ elements, if J is a subset with $m+1$ elements, then we have

$$J \text{ short} \iff 1 \in J. \tag{8.15}$$

Combining (8.11) and (8.15) we get a complete description of long and short subsets of $[r]$:

$$J \text{ is } \begin{cases} \text{long if } |J| \geq m+2 \text{ or } |J| = m+1 \text{ with } 1 \notin J, \\ \text{short if } |J| \leq m \text{ or } |J| = m+1 \text{ with } 1 \in J. \end{cases} \tag{8.16}$$

It is the same as that of the length vector $(0,\overbrace{1,\ldots,1}^{2m+1})$. So $l \sim (0,1,\ldots,1)$. By Lemma 4.1.5(3) $X_{a,b}(l)$ is equivariantly diffeomorphic to $X_{a,b}(0,1,\ldots,1)$. □

Another application is to show the following new result.

Corollary 8.5. *Let $a,b \geq 1, m \geq 0$ and $l \in \mathbb{R}^r$ be generic with $0 \leq l_1 \leq ... \leq l_r$.*

(1) If $r = 2m+1$, then $\operatorname{syzord} H_T^*(X_{a,b}(l)) = m-1$ *if and only if $X_{a,b}(l)$ is equivariantly diffeomorphic to $X_{a,b}(0,0,\overbrace{1,1,\ldots,1}^{2m-1})$.*

(2) If $r = 2m+2$, then $\operatorname{syzord} H_T^*(X_{a,b}(l)) = m-1$ *if and only if $X_{a,b}(l)$ is equivariantly diffeomorphic to* $X_{a,b}(0,0,0,\overbrace{1,1,\dots,1}^{2m-1})$ *or* $X_{a,b}(1,1,1,\overbrace{2,2,\dots,2}^{2m-1})$.

Proof. For both assertions, $\operatorname{syzord} H_T^*(X_{a,b}(l)) = m-1$ implies that $\mu(l) = m$ by Theorem 7.1.4. The "if" parts of both assertions follow from Theorem 7.1.4 and the observation that the length vectors l mentioned in both assertions satisfy $\mu(l) = m$. We only have to prove the "only if" parts.

Let us prove (1) first. Since $\mu(l) = m$, Lemma 8.2 gives the following partial description of long and short subsets of $[r]$:

$$J \text{ is } \begin{cases} \text{long if } |J| \geq m+2, \\ \text{short if } |J| \leq m-1. \end{cases} \tag{8.17}$$

Now we need to investigate subsets of $[r]$ containing m and $m+1$ elements. Let us prove that if $|J| = m$, then

$$J \text{ is short} \iff J \cap \{1,2\} \neq \emptyset. \tag{8.18}$$

We first prove "$\Leftarrow$" by contradiction. Without loss of generality, let us assume the following subset is long:

$$J = \{2, \overbrace{m+3, m+4, \dots, 2m+1}^{m-1}\} \tag{8.19}$$

because it us the longest subset containing 2 with m elements. By Lemma 8.3, we can substitute $\{\overbrace{m+3, m+4..., 2m+1}^{m-1}\}$ with $\{\overbrace{4,5,\dots,m+2}^{m-1}\}$ and get the long set

$$\{2, \overbrace{4,5,\dots,m+2}^{m-1}\} \tag{8.20}$$

with m elements. Thus the set $\{\overbrace{3,4,\dots,m+2}^{m}\}$ is long and the complement of this set is

$\{1, 2, \overbrace{m+3, m+4, \ldots, 2m+1}^{m-1}\}$ and it is short. But

$$\{1, 2, m+3, m+4, \ldots, 2m+1\} = J \cup 1 \tag{8.21}$$

and by assumption, it is long and that is a contradiction. So we have shown "$\Leftarrow$".

Let us prove "$\Rightarrow$" also by contradiction. Without loss of generality, we can assume that the subset

$$J = \{\overbrace{3, 4, \ldots, m+2}^{m}\} \tag{8.22}$$

is short. Then its complement $\{1, 2, \overbrace{m+3, m+4, \ldots, 2m+1}^{m-1}\}$ is long. It then follows from Lemma 8.3 that the set

$$\{1, 2, \overbrace{4, 5, \ldots, m+2}^{m-1}\} \tag{8.23}$$

is long. If the set

$$\{\overbrace{1, 2, 3, \ldots, m+1}^{m+1}\} \tag{8.24}$$

is also long, then all the subsets of $m+1$ elements are long because the set (8.23) is the shortest subset with $m+1$ elements. Furthermore, all subsets of m elements are short by taking complement of a subset of $m+1$ elements. But then $\mu(l) = m+1$, not m. So the set (8.24) has to be short. However this implies that the long set $\{1, \ldots, m+2\}$ with $m+2$ elements has a short subset. Since $\mu(l) = m$, the subset (8.23) has to be short, which is a contradiction.

Combining (8.17) and (8.18) we get complete knowledge on long and short subsets of $[r]$ and it coincides with that of $(0, 0, \overbrace{1, \ldots, 1}^{2m-1})$. Part (1) then follows from Lemma 4.1.5(3).

Part (2) follows from the following Lemma 8.6. □

Lemma 8.6. *Let a, b, m, l be as in Corollary 8.5(2). We have the following two situations:*

(1) If there is a long subset of $[r]$ with m elements, then $l \sim (0, 0, 0, \overbrace{1, \ldots, 1}^{2m-1})$.

(2) If there is no long subset of $[r]$ with m elements, then $l \sim (1, 1, 1, \overbrace{2, \ldots, 2}^{2m-1})$.

Proof. Since $\mu(l) = m$, we get the following long and short information of subsets of $[r]$ from Lemma 8.2:

$$J \text{ is } \begin{cases} \text{long if } |J| \geq m+3, \\ \\ \text{short if } |J| \leq m-1. \end{cases} \tag{8.25}$$

We need to investigate subsets of $[r]$ containing m, $m+1$ and $m+2$ elements.

We first prove (1). First we show the following assertion for J with $m+1$ elements:

$$J \text{ is short } \iff |J \cap \{1,2,3\}| \geq 2. \tag{8.26}$$

Without loss of generality we will show that the set $\{2, 3, \overbrace{m+4, m+5, \ldots, 2m+2}^{m-1}\}$ is short because it is the longest subset satisfying the condition on the right hand side of (8.26). Assume it is long, then by substituting the subset $\{\overbrace{m+4, \ldots, 2m+2}^{m-1}\}$ with $\{\overbrace{5, \ldots, m+3}^{m-1}\}$, it follows from Lemma 8.3 that the set

$$\{2, 3, \overbrace{5, 6, \ldots, m+3}^{m-1}\} = \{\overbrace{2, \ldots, m+3}^{m+2}\} - \{4\} \tag{8.27}$$

is long. Since $\mu(l) = m$, it follows from the definition of $\mu(l)$ that

$$\{\overbrace{2, \ldots, m+3}^{m+2}\} \text{ has no short subsets with 1 element fewer.} \tag{8.28}$$

However, by assumption of (1), there is a long subset with m elements and thus the shortest subset with $m+2$ elements $\{1, \ldots, m+2\}$ has to be short. Then $\{2, \ldots, m+2\}$ is short and thus $\{2, \ldots, m+3\}$ has a short subset with 1 element fewer, that is a contradiction to (8.28). So we have shown (8.26).

Next we investigate sets with m elements. We are going to show that if $|J| = m$, then we have

$$J \text{ is long } \iff J \cap \{1,2,3\} = \varnothing. \tag{8.29}$$

Let us first observe by contradiction that if $|J| = m$, then we have

$$3 \in J \Rightarrow J \text{ is short.} \tag{8.30}$$

Without loss of generality we assume that $\{3, \overbrace{m+4, m+5, \dots, 2m+2}^{m-1}\}$ is long. Then by substituting $\{\overbrace{m+4, \dots, 2m+2}^{m-1}\}$ with $\{\overbrace{5, ..., m+3}^{m-1}\}$ and applying Lemma 8.3, the resulting set is $\{3, \overbrace{5, 6, \dots, m+3}^{m-1}\}$ and it is long. Thus $\{4, 5, \dots, m+3\}$ is long. It implies that the complement $\{1, 2, 3, \overbrace{m+4, \dots, 2m+2}^{m-1}\}$ is short. However, $\{1, 2, 3, m+4, \dots, 2m+2\}$ is longer than $\{3, \overbrace{5, 6, \dots, m+3}^{m-1}\}$ which is already long. That leads to a contradiction. We have shown the "$\Rightarrow$" part of (8.29).

Next we want to show the "$\Leftarrow$" part of (8.29). Since there is a long set of m elements by assumption, without loss of generality we can assume $\{\overbrace{m+3, \dots, 2m+2}^{m}\}$ is long. Then by Lemma 8.3 the set $\{\overbrace{4, \dots, m+3}^{m}\}$ is long. So the "$\Leftarrow$" part of (8.29) follows.

With (8.29) we also get a description of subsets with $m+2$ by taking complement. If $|J| = m+2$, then

$$J \text{ is long} \iff \{1, 2, 3\} \not\subseteq J. \tag{8.31}$$

Combining (8.31) with (8.29) and (8.26), we get a complete description of long and short subsets of $[r]$ and it coincides with that of $(0, 0, 0, \overbrace{1, \dots, 1}^{2m-1})$. We have shown (1).

Now let us prove (2). Again we first prove (8.26) under the assumption of (2). We go through the same process as before and try to show that the following subset

$$\{2, 3, \overbrace{m+4, m+5, \dots, 2m+2}^{m-1}\} \tag{8.32}$$

is short by contradicting (8.28).

Now we assume that (8.28) is true. Then the set $\{\overbrace{2, \dots, m+2}^{m+1}\}$ is long. Since the complement of $\{2, \dots, m+2\}$ is the longest subset with $m+1$ elements containing 1, it implies the following statement for $|J| = m + 1$:

$$J \text{ is long} \iff 1 \notin J. \tag{8.33}$$

Furthermore, it follows from the assumption of (2) that

$$J \text{ is } \begin{cases} \text{long if } |J| \geq m + 2, \\ \\ \text{short if } |J| \leq m. \end{cases} \tag{8.34}$$

Combined with (8.33) we can conclude that $\mu(l) = m + 1$ just by definition of $\mu(l)$ and looking at long and short sets of $[r]$. That is a contradiction. So (8.28) can't be true.

Note that we have shown (8.26) for (2) and thus the subset (8.32) is short. It then follows that if $|J| = m + 1$ and $|J \cap \{1, 2, 3\}| \geq 2$, then J is short because the subset (8.32) is the longest such subset. By taking complement of such J, it also follows that if $|J| = m + 1$ and $|J \cap \{1, 2, 3\}| \leq 1$, then J is long. So we have proved (8.26).

Combining (8.26) with (8.34), we get a complete description of long and short subsets of $[r]$ and it coincides with that of $(1, 1, 1, 2, ..., 2)$. We have shown (2). $\square$

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