6-27-2018 2:00 PM

Optimization Studies and Applications: in Retail Gasoline Market

Daero Kim, *The University of Western Ontario*

Supervisor: Davison, Matt, *The University of Western Ontario*
Co-Supervisor: Odegaard, Fredrik, *The University of Western Ontario*
A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Applied Mathematics

Abstract

The study of the retail gasoline market is of great interest in financial economics, since it allows many theories about price formation, oligopolistic markets, and consumer search to be tested. In addition, the risk management of gasoline prices is an important instance of the management of any consumable commodity cost. For the retailer, the tool of dynamic pricing may be found to be useful.

This thesis contributes to the study of retail gasoline markets in three main ways, each in its own paper. The first paper tests various economic models to confirm earlier results about pricing behavior in retail gasoline markets and the setting of optimal pricing strategies. The second paper presents optimal refueling strategies, analysis of optimal swap contracts. It also proposes and analyzes a price guarantee "gasoline option" based on loyalty programs and financial swap structures. The final paper presents optimal refueling strategies using empirical data, and compares the value obtained using a simulated approach with the value using an empirical approach.

Keywords: retail gasoline; oligopoly; game theory; dynamic programming; optimization; optimal pricing; pricing and searching behavior; asymmetric response; floating-to-fixed swap; loyalty program; guaranteed price; optimal control; optimal refueling strategies.

The Co-Authorship Statement

The materials covered in chapter 3 have been written with Matt Davison and Fredrik Ødegaard. I, Daero Kim, am the first author. The material briefly introduced in chapter 2 have been written with Matt Davison and Harald Keller, Matt Davison was the first author. Other forthcoming papers drawn from chapter 4, 5, and 6 will be written with Matt Davison and Fredrik Ødegaard.

Acknowledgments

I would like to thank all the staff at the University of Western Ontario for their support. I am deeply grateful to my supervisors, Dr. Matt Davison and Dr. Fredrik Ødegaard, they have provided endless support, inspiring advice, and guidance throughout my studies. They have contributed many hours in regular meetings and encouraged me while completing my thesis in part-time while working full-time, and I am forever in debt and grateful. I would also like to thank my wife, Gina, and my parents, Janghwan and Myungsoon for supporting me through the good times and the bad times. Finally, I would like to thank all my friends for their generous support and feedback.

Table of Contents

List of Tables

List of Figures

List of Appendices

Chapter 1

Introduction

1.1 Retail Gasoline Market

Retail gasoline is the end product of a very long supply chain. The sale at local retail stations is the last step of that long process, including refining, retailing and distribution, and marketing. Due to the long supply chain, it is not clear what goes into the price of a gallon of gasoline, what drives its prices up and down, and what drives price differentials between cities, towns, and streets. Thus, in order to understand retail gasoline markets, not only do we need to examine market size, the level of competition, price variation between communities, but we also need to study how gasoline is produced, distributed, priced and sold.

In Canada, during 2014, the total net sale of gasoline was 40.9 billion liters, while the total population was estimated at 35.5 million, and the total number of road motor vehicle registrations was 23.5 million (StatCan (2015) and StatCan (2014)). We can estimate the average fuel consumption rate, 1,740 liters per car per year, by dividing the total net sale of gasoline by the total number of registered road motor vehicles. Then, by simple calculation, the average annual cost of the fuel consumption is about \$1,740 assuming \$1/L and about \$1,722.6 assuming \$0.99/L, thus, every cent difference at the pump is worth about \$20 per year for the typical consumer.

The dollar amount may look small from the consumers point of view, but from the retailers point of view, every cent at the pumps is worth about \$410 million per year for the industry as a whole, as the total annual quantity of gasoline sold in Canada is about 41 billion liters. The sale of motor fuel is also a major economic activity in most countries, especially in poorly planned and designed cities where the residential area is far from business, commercial or industrial areas, as the cost of motor fuel is a major cost of transportation for many commuters.

The share of the retail gasoline price are the cost of crude oil (about 50%), followed by taxes (about 20% depending on your home province), refiner's cost and margin (about 20%), and retailer's operational cost including distribution, marketing, storage and gross margin (about 10%). Therefore, even before the gasoline is delivered to the local retail stations, over 90% of the cost of gasoline is already determined.

The biggest factor of the retail gasoline price is the cost of crude oil, which varies over time and across regions. The crude oil price is strongly driven by the world market and Organization of Petroleum Exporting Countries (OPEC), who controls about half of the world's oil production and the world's crude oil reserves. Other factors can also affect the price of crude oil. For instance, a disruption in global oil supplies due to natural disasters or conflicts in the petroleum exporting countries can lead the price of crude oil to rise sharply. Prices may also fall in response to reduced demand due to global recession or as inventories rise (Kilian and Murphy (2014)).

There are two main benchmarks for pricing crude oil, namely West Texas Intermediate (WTI) and Brent crude oil as shown in Figure 1.1. WTI is a crude produced in Texas and southern Oklahoma and Brent is produced in the North Sea region, both are light, sweet crude oils although WTI is generally sweeter (low in sulphur) and lighter (higher petroleum density) than Brent. The crude oil price is volatile, for instance, their price decreased about 66% between 2008 and 2009, but increased about 100% between 2009 and 2010, as depicted in Figure 1.1. The crack spread measures

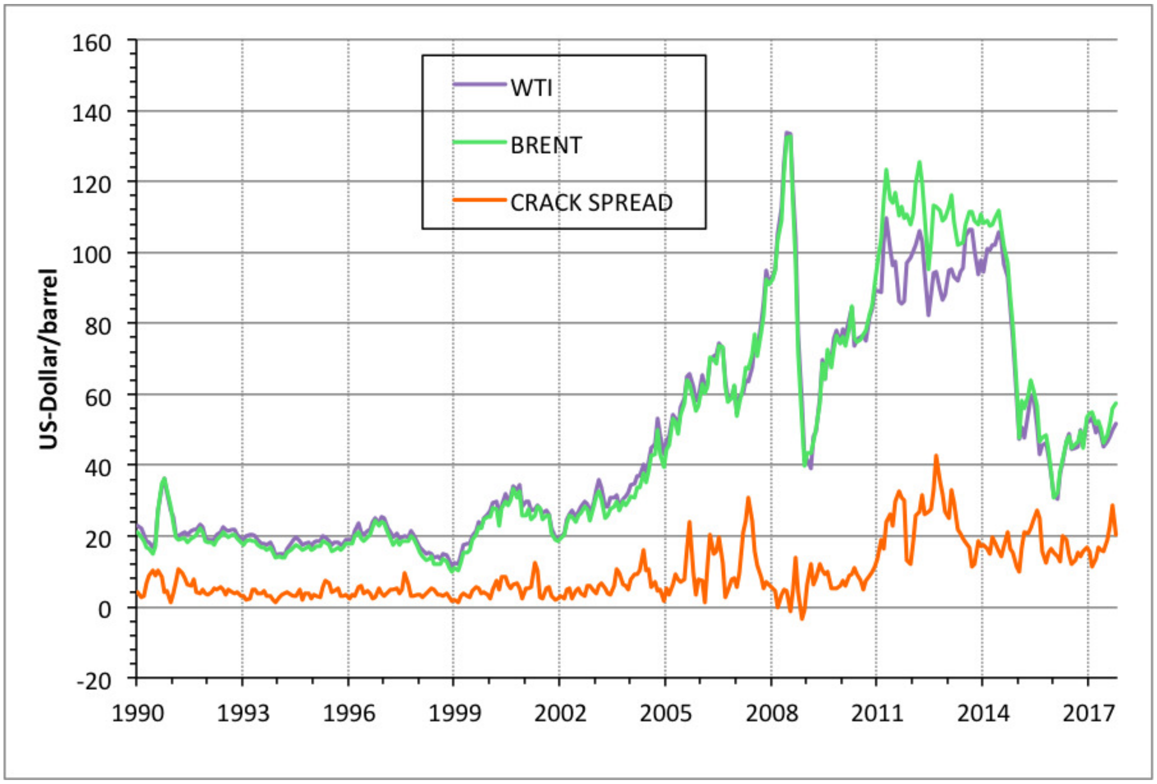


Figure 1.1: *Historical spot prices of West Texas Intermediate (WTI) and Brent, the current market rates at which the commodity can be purchased "on the spot", and crack spread measures the differential between the price of WTI and New York Harbor Conventional Gasoline Regular Spot Price (Source: U.S. Energy Information Administration (Oct 2017)).*

the differential between the price of WTI or Brent and gasoline or other products. Thus, the spread approximates the profit margin an oil refinery can expect to earn by refining crude oil; as of October 2017, the crack spread is about 20 US dollar per barrel, see Figure 1.1 (EIA (2017)).

The volatility of crude oil productions feeds into the volatility of petroleum products because crude oil is refined to produce a wide array of petroleum products, including gasoline for automobiles, heating oil, diesel and jet fuels, or other chemical products. Thus, a change in world crude oil productions and imbalance in supply and demand of crude oil based products can also lead to a change in wholesale gasoline price and downstream retail gasoline price.

1.2 Infrastructure of Retail Gasoline in Canada

Over the years, the structure of the retail gasoline market has changed significantly in part because of trends in market de-regulation. Today's retail gasoline market is much more complex and fragmented in many countries including Canada (NRC (2014)). Only a small number of gas stations are now directly owned and operated by vertically integrated refiners or by national oil companies albeit many still display national brands. Many national brands are run either by retail operators who own the site inventory or by independent owners who have total control over the price. Thus, today's retail gasoline market is a highly competitive arena in which major branded companies fight for their market share nation wide, while local small independent owners compete against each other in niche markets. The price competition between fuel retailers leads to price differentials across cities, towns, and streets (McCaffrey et al. (2011)).

In Canada, total sales of refined petroleum products increased at a slow but steady pace during most of the 1990s and early 2000s, stimulated by economic and population growth. Transportation is largely responsible for the growth in total sales of refined petroleum products since 1990, accounting for more than two-thirds the sales of refined petroleum products, primarily motor gasoline, low-sulphur diesel fuel and aviation fuel. According to Natural Resources Canada, the sales of refined petroleum products including gasoline in Canada, the fifth largest oil producer in the world, totalled 89.1 billion litres in 2014, about 46% was motor gasoline (includes ethanol), 20% was distillate fuel (heating oil and diesel fuel), and 8% was jet fuel. On average, Ontario and Quebec account for about 60% of the gasoline consumed in Canada, the Western provinces account for about 32%, and the remaining 8% of gasoline is consumed in the Atlantic provinces and the Territories (NRC (2014), Statista (2014)).

Depending on classification and based on the way crude oil is distilled, petroleum products can be separated into three categories: light distillates (separated components from distillation, such as, liquefied petroleum gas, gasoline, naphtha), middle distillates (kerosene, diesel), other distillates (heavy fuel oil, lubricating oils, wax, asphalt). The unbalanced demand for gasoline and for other distillates can also create challenges for refiners. This is because a refinery has a limited range of flexibility in setting the gasoline to other distillates production ratio. Beyond a certain point, the production of other distillates can only be increased by also increasing gasoline production. Therefore, refiners can have an excess supply of gasoline compared to other distillates or vice versa depending on the regional demand for gasoline and other distillates (NRC (2014)).

A recent survey study, entitled the National Retail Petroleum Site Census 2014, identified a total of 11,811 retail gasoline stations operating in Canada as of December 31, 2014. The number of retail gasoline stations in Canada has decreased since 1989, when over 20,000 retail outlets existed. The study determined that fifteen percent of all 11,811 gas stations are under the price control of one of the three big oil companies, and nineteen percent of all gas stations are under the price control of the refiner-marketers. While most people are familiar with the big brands like Suncor, Esso, Shell, and Petro Canada, many of them are independently owned and operated by individual business people who set their own price (MJ Ervin & Associates (2015)). The survey report illustrated the diversity of gasoline brands, with over 94 different brand names in Canada. However, most of these are brands originate from fourteen refineries, operated by eight refining companies. There are 66 companies involved in the retail management of these brands. The provision of goods or services, such as the type of pump service (full, self or split), convenience store size, car washes, fast food, automotive service, and to what degree diesel fuel is offered for sale, other than gasoline is also vital part that differentiate retail gasoline outlets, since the gross margin on gasoline itself is generally very small (MJ Ervin & Associates (2015)).

With the help of other services, margin plays a critical role for the profits of retail gasoline station as discussed in Chapter 3. Due to the high degree of competition, fuel retailers must optimally set their prices to maintain gross profit in response to volatile costs and frequent price changes across competitors. We test different pricing strategies and project the expected sales using a model developed based on the historical data; retailer's own price, sales, competitor's price, and wholesale price.

In Chapter 2, we introduce key ideas of dynamic programming, basic optimality equation, and the principle of optimality. Then, we show dynamic programming application examples in radiation therapy, and discuss the analogy between radiation therapy and a retail gasoline market.

In Chapter 3, we study dynamic competition between nearby retailers in the retail gasoline market for multiple periods. We first model this dynamic retail gasoline competition as a repeated single-period game where the retailers set their price simultaneously, then, extend the model to the sequential game where the retailers set their price sequentially. Using backward induction, we find the optimal prices for each retailers in the sequential game.

In Chapter 4, in contrast to the previous chapter, which is written from the perspective of a retailer, we change our attention to the end customers and introduce loyalty program which provide customers with a guaranteed price to hedge against the fluctuation of price. We also study the expected outcomes of the guaranteed price and optimal quantity of gasoline to be hedged by retailers in order to provide this guaranteed price.

In Chapter 5, we extend dynamic programming to determine an optimal solution when a dynamic state evolves stochastically. We analytically prove existence of the optimal decision boundary and examine the optimal refueling time and quantity without and with the loyalty program. We also numerically compute the optimal decision boundary and optimal policies, then, we find the value of loyalty program.

In Chapter 6, we compare the value using a simulation approach with the value using an empirical approach. The results show that it might be worth it to freely offer the guaranteed price to valued customers as a loyalty program.

The thesis concludes in Chapter 7 which summarizes and provides next steps.

Bibliography

Short-Term Energy Outlook January (2016). U.S. Energy Information Administration, Department of Energy, Retrieved from: http://www.eia.gov/forecasts/steo/pdf/uncertainty.pdf Date retrieved: 27 Jan 2016.

Spot Prices for Crude Oil and Petroleum Products October (2017). U.S. Energy Information Administration, Department of Energy, Retrieved from: http://www.eia.gov/dnav/pet/pet-pri-spt-s1-m.htm Date retrieved: 6 December 2016.

Kilian, L., and Murphy, D. P. (2014). The role of inventories and speculative trading in the global market for crude oil, *Journal of Applied Econometrics*, Vol. 29, No.3, pp.454-478.

McCaffrey, D., Jenkins, B., and Liptrot, T. (2011). Optimization of forecourt fuel pricing, *IMA Journal of Management Mathematics*, Vol. 23, No.3, pp.265-281.

MJ Ervin and Associates (2015), National Retail Petroleum Site Census 2014, Retrieved from: http://www.kentgroupltd.com/wp-content/uploads/2014/06/Executive-Summary-2014-National-Retail-Petroleum-Site-Census.pdf Date retrieved: 15 Aug 2015.

Natural Resources Canada, Refinery Economics. Crude Oil Input, Crude Slates, Refinery Configuration, Product Slates, Refinery Utilization, Environmental Initiatives, Retrieved from: http://www.nrcan.gc.ca/energy/crude-petroleum/4561 Date retrieved: 2 June 2015.

Refined petroleum products consumption in Canada from 1986 to 2016, Retrieved from: https://www.statista.com/statistics/494977/canadian-refined-petroleum-consumption-daily-basis/ Date retrieved: 20 Dec 2017.

Statistics Canada (2015), CANSIM, Statistics Canada Catalogue no. 62-001-X. Version updated July 2015. Ottawa, Retrieved from: http://www.statcan.gc.ca/

tables-tableaux/sum-som/l01/cst01/trade37a-eng.htm Date retrieved: 7 Nov 2015.

Statistics Canada (2014), Table 23-10-0067-01 Road motor vehicle registrations, by type of vehicle, Retrieved from: https://www150.statcan.gc.ca/t1/tbl1/en/tv.action?pid=2310006701 Date retrieved: 7 Nov 2015.

Chapter 2

Mathematical Foundations for Dynamic Programming

2.1 Optimization in High Dimension and the Need for Dynamic Programming

One can easily find an optimal solution by the brute force method of checking all possible candidate solutions for a simple 1-dimensional optimization problem when the number of possible solutions is small. It gets harder to find an optimal solution by the brute force method, as the dimension space gets bigger and the number of feasible solutions grows. For example, to optimize N functions taking inputs from a d-dimensional space, one must check N^d potential solutions. For example, using $N = 100$ and $d = 10$, one must check 100^{10} spots to find the optimum. That motivates dynamic programming, which converts a d-dimensional problem into d 1-dimensional problems. For the same example above now, one needs to solve for 10 problems, each of checking for 100 spots, for a total of 1000 or 10^4 function evaluations, which is much easier.

2.2 Dynamic Programming

Dynamic programming, proposed by Bellman (Bellman (1954)), is a powerful numerical method for solving optimization problems efficiently. In addition to the simplification described in Section 2.1, the main advantage of dynamic programming over other methods is the guaranteed global optimality of the solution.

One can use the dynamic programming technique to decompose an N-decision problem into a sequence of N separate, but interrelated, single-decision sub-problems, and then combine the solutions of the smaller problems to obtain the solution of the entire larger model.

First, we decompose a big systemic problem into small sub-problems each with only one decision variable.
The basic optimality equation for an one-stage problem is

$$V(i) = \max_{\{a \in A\}} R(i,a), \tag{2.1}$$

which can easily be extended to n-stage problems of the form $V_n(i)$ via the relation:

$$V_n(i) = \max_{\{a \in A\}} \left[R(i,a) + \sum_j \mathbf{P}_{ij}(a) V_{n-1}(j) \right]. \tag{2.2}$$

Let

$$\mathbf{P}_{ij} = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise.} \end{cases}$$

and where A is the finite set of all possible actions or decisions, a. If action a is chosen, then the reward earned is $R(i,a)$ and the next state will be j with probability

$P_{ij}(a)$, where i is the current state. In practice, it often takes a great effort to solve (2.2) analytically, thus numerical methods are one of the alternative options. The most naive approach to solving such problems is to solve for all possible combination of actions, selecting the series of actions that gives the biggest value of $V_n(i)$, but, as described in Section 2.1, this is computationally expensive. In contrast, a general technique for recursively solving for $V_n(i)$ follows

1. First obtain $V_N(i)$ given the final condition, i.e. a=max(A).

2. Now solve for $V_n(i)$ when $n = N - 1$,

3. and so on, up to $V_1(i)$ for all $i = 0, 1, \cdots, M$.

The optimal policy is a path of actions a that maximizes the $V_n(i)$ for $n = 1 \cdots N$. When the state variables at stage i and the decision variables a to use are allowed to take on only a finite set of discrete values, the result is a discrete dynamic programming problem.

The solution of the discrete dynamic programming problem will always be a global maximum (or minimum) regardless of the concavity, convexity, or even the continuity of the functions $R(i, a)$ (Bertsekas (1995)).

In this thesis, we consider discrete time finite horizon dynamic programming problems. The dynamic programming applications in retail gasoline market are;

1. Optimal pricing where firms sequentially compete on price,

2. Optimal refueling time and amount without a guaranteed price,

3. Optimal refueling time and amount with a guaranteed price.

2.3 Simple Dynamic Programming Example

The reader who is familiar with dynamic programming is encouraged to skip this section. Although in this thesis we mainly focus on retail gasoline markets, dynamic programming is applied in many other areas. For instance, in radiotherapy, radiation, which is high energy X or gamma rays, is directed to damage a tumor while avoiding surrounding healthy normal tissue. Cancer cells are naturally more susceptible to radiation due to their faster growth rate and oxygen saturation level than that of normal tissues. It is very important to minimize dose to a normal tissue and to ensure that sensitive biological structures near the tumor are not damaged more than a certain amount (this corresponds to a tank level constraint in the gasoline problem considered in Chapter 5) so that we minimize adverse reactions to sensitive organs. This leads an optimization problem for optimal dose sequences and fractionation schedules (optimal refueling amount and strategy in Chapter 5 gasoline problem), whether to deliver a large number of relatively low doses (fill only minimum in Chapter 5 gasoline problem) or deliver a small number of high doses (fill-up in gasoline problem in Chapter 5).

The linear quadratic (LQ) dose response model is a popular model to study the response of biological tissues to the radiation doses. For a tumor tissue, this is given by;

$$T(d_k) = \alpha_T d_k + \beta_T d_k^2 \tag{2.3}$$

While for sensitive normal tissue, it is given by;

$$S(d_k) = \alpha_S \omega d_k + \beta_S (\omega d_k)^2 \tag{2.4}$$

Here $0 < \omega < 1$ is the 'sparing factor', which depends on the level of dose, d_k measured in Gray (Gy), α_T and β_T are the tumor specific response coefficients, while α_S and β_S are the sensitive tissue response coefficients that are assumed to be constant for a patient over the duration of radiation treatments (Davison, Kim, and Keller (2011) and Keller et al. (2013)).

The constrained optimization problem to maximize the damage to the tumor is

$$\max_{\{d_1,d_2,\cdots,d_N\}} \sum_{k=1}^{N} T(\{d_k\}) \tag{2.5}$$

$$\text{s.t.} \qquad \textstyle\sum_{k=1}^{N} S(\{d_k\}) = \Omega \qquad \text{for } k = 1 \text{ to } N.$$

The DP optimality equation for the above optimization problem is

$$V_n(i) = \max_{\{a \in A_n\}} \left[R(i,a) + \sum_j \mathbf{P}_{ij}(a) V_{n-1}(j) \right]. \tag{2.6}$$

with a sensitive tissue constraint below,

$$\sum_{k=1}^{N} S(d_k) \leq \Omega \qquad \text{and} \qquad d_k \geq 0 \quad for \quad k = 1, \cdots, N$$

$$\text{where} \qquad R(i,a) = T(d_k) \qquad a = d_k$$

At maximum the sensitivity tissue constraint will bind, and we have

$$V_n(i) = \max_{\{d_k \in S^{-1}\left(\Omega - \sum_{k=1}^{n} S(d_k)\right)\}} \left[T(d_k) + \sum_j \mathbf{V}_{n-1}(j) \right]. \tag{2.7}$$

We have the final condition that all the remaining dose must be applied by the end of decision periods with terminal condition;

$V_T(s) = \alpha_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-1} S(d_k)\big)\Big] + \beta_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-1} S(d_k)\big)\Big]^2$

Proposition 0:

- if $T(d_t) = \alpha_T d_M + \beta_T d_M^2 \geq N(\alpha_T d_e + \beta_T d_e^2)$ $(\frac{\alpha_T}{\beta_T} \geq \frac{\alpha_S}{\omega\beta_S})$
  $V_t(s) = \alpha_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{t-1} S(d_k)\big)\Big] + \beta_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{t-1} S(d_k)\big)\Big]^2$
  and $d_t^* = d_M = S^{-1}\big(\Omega - \sum_{k=1}^{t-1} S(d_k)\big)$ for $t \leq T$.

- if $T(d_t) = \alpha_T d_M + \beta_T d_M^2 < N(\alpha_T d_e + \beta_T d_e^2)$ $(\frac{\alpha_T}{\beta_T} < \frac{\alpha_S}{\omega\beta_S})$
  $V_t(s) = \alpha_T d_e + \beta_T d_e^2 + \alpha_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{t} S(d_e)\big)\Big] + \beta_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{t} S(d_e)\big)\Big]^2$
  and $d_t^* = d_e$ for $t \leq T$.

Proof by induction:

for $t = T-1$ (for one stage-to-go)

$$\begin{aligned} V_{T-1}(s) &= \alpha_T d_t^* + \beta_T d_t^{*2} + V_T(s') \\ &= \alpha_T d_t^* + \beta_T d_t^{*2} + \alpha_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-1} S(d_k)\big)\Big] + \beta_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-1} S(d_k)\big)\Big]^2 \end{aligned}$$

where

$V_{T-1}(s) = \alpha_T d_t^* + \beta_T d_t^{*2} + \alpha_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-1} S(d_k)\big)\Big] + \beta_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-1} S(d_k)\big)\Big]^2$

Thus, if $\alpha_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)\Big] + \beta_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)\Big]^2 \geq 2\Big[\alpha_T\big[\frac{S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)}{2}\big] + \beta_T\big[\frac{S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)}{2}\big]^2\Big]$, we maximize d_t^*, and $d_t^* = d_M = S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)$, and if $\alpha_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)\Big] + \beta_T\Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)\Big]^2 < 2\Big[\alpha_T\big[\frac{S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)}{2}\big] + \beta_T\big[\frac{S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)}{2}\big]^2\Big]$, we minimize d_t^*, and $d_t^* = d_e = \frac{S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)}{2}$.

The Proposition 0 holds for $t = T-1$.

If $d_t^* = d_e$,

$V_{T-1}(s) = \alpha_T d_e + \beta_T d_e^2 + \alpha_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-1} S(d_e)\big)\Big] + \beta_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-1} S(d_e)\big)\Big]^2$

and if $d_t^* = S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)$,
$V_{T-1}(s) = \alpha_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)\Big] + \beta_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{T-2} S(d_k)\big)\Big]^2$

Assume the proposition 0 holds for all $t = j+1, j+2, \cdots, T-1$.

Prove the proposition 0 holds for $t = j$ (for $T - t$ stages-to-go)

$$\begin{aligned} V_j(s) &= \alpha_T d_j^* + \beta_T d_j^{*2} + V_{j+1}(s') \\ &= \alpha_T d_j^* + \beta_T d_j^{*2} + \alpha_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{j} S(d_k)\big)\Big] + \beta_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{j} S(d_k)\big)\Big]^2 \end{aligned}$$

where
$V_j(s) = \alpha_T d_j^* + \beta_T d_j^{*2} + \alpha_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{j} S(d_k)\big)\Big] + \beta_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{j} S(d_k)\big)\Big]^2$

Thus, if $\alpha_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{j-1} S(d_k)\big)\Big] + \beta_T \Big[S^{-1}\big(\Omega - \sum_{k=1}^{j-1} S(d_k)\big)\Big]^2 \geq \Big[T - (j - 1)\Big] \Big[\alpha_T \big[\frac{S^{-1}\left(\Omega - \sum_{k=1}^{j-1} S(d_k)\right)}{T-(j-1)}\big] + \beta_T \big[\frac{S^{-1}\left(\Omega - \sum_{k=1}^{j-1} S(d_k)\right)}{T-(j-1)}\big]^2\Big]$, we maximize d_j^*, and $d_j^* = S^{-1}\big(\Omega - \sum_{k=1}^{j-1} S(d_k)\big)$, and $d_1^* = S^{-1}(\Omega) = d_M$,
and else if, we minimize d_j^*, and $d_j^* = d_e = \frac{S^{-1}\left(\Omega - \sum_{k=1}^{j-1} S(d_k)\right)}{T-(j-1)}$.

By dynamic programming, we have shown there are only two extreme solutions to this problem, an equal dose (d_e) per fraction and a megadose (d_M) treatments.

$$d_e = S^{-1}\Big(\frac{\Omega}{N}\Big)$$

in which $d_1 = d_2 = \cdots = d_N = d_e$ and $N * S(d_e) = \Omega$

$$d_M = S^{-1}(\Omega)$$

in which $S(d_M) = \Omega$

If α_s, β_s, Ω are all known and constant, the equal dose and the megadose are;

$$d_e = \frac{\sqrt{\alpha_s^2 + 4\beta_s(\Omega/N)} - \alpha_s}{2\beta_s\omega}$$

$$d_M = \frac{\sqrt{\alpha_s^2 + 4\beta_s(\Omega)} - \alpha_s}{2\beta_s\omega}$$

The optimal dose and the number of fractions depend on whether $\alpha_T d_M + \beta_T d_M^2 \geq N(\alpha_T d_e + \beta_T d_e^2)$ or $\alpha_T d_M + \beta_T d_M^2 < N(\alpha_T d_e + \beta_T d_e^2)$. By simple math, the above conditions can be rewritten as, $\frac{\alpha_T}{\beta_T} \geq \frac{\alpha_S}{\omega\beta_S}$ or $\frac{\alpha_T}{\beta_T} < \frac{\alpha_S}{\omega\beta_S}$.

With these retail gasoline market preliminary (in Chapter 1) and Dynamic Programming preliminary (in Chapter 2) out of the way, we now proceed to some modeling and analysis of gasoline markets. We begin in Chapter 3 with the discussion of price setting behaviour by gasoline retailers.

Bibliography

Bellman, R. (1954). The theory of dynamic programming, (No. RAND-P-550), RAND Corp Santa Monica CA.

Bertsekas, D. P. (1995). Dynamic programming and optimal control, Vol. 1, No. 2, Belmont, MA: Athena Scientific.

Davison, M., Kim, D., and Keller, H. (2011). Radiotherapy Dose Fractionation under Parameter Uncertainty, *Advances In Mathematical And Computational Methods: Addressing Modern Challenges of Science, Technology, and Society*, Vol. 1368, No. 1, pp.179-182. AIP Publishing.

Keller, H., Hope, A., Meier, G., and Davison, M. (2013). A novel dose volume metric for optimizing therapeutic ratio through fractionation: Retrospective analysis of lung cancer treatments. *Medical physics*, 40(8), 084101.

Puterman, M. L. (2014). Markov decision processes: discrete stochastic dynamic programming. John Wiley & Sons.

Chapter 3

Optimal Prices in the Case of the Retail Gasoline Market?

3.1 Introduction

Economists have long been fascinated with retail gasoline markets because of the light they shed on several microeconomic questions. Their work, as discussed in more detail in the following literature review section, includes price formation, consumer search, and oligopolistic competition. This study analyzes a rich new European dataset to validate or partially validate some of the existing economic conclusions on gasoline pricing, all of which were based on North American data. The economic models we examine tend to have a descriptive focus (i.e., how do consumers and retailers act?), but in addition to validating these models, we also take a slightly prescriptive approach by following up with this question: Given the economic insights gleaned, how should economic agents react? Our main new contribution in this regard is to add time series-based optimal strategies to the setting of gasoline prices.

Gasoline is a product that many consumers must buy on a regular basis. Despite the best efforts of retailers to differentiate their brand from others, gasoline is essentially a commodity, the wholesale price of which is set on financial markets and fluctuates from day to day. The retail price of gasoline is easy for consumers to monitor, both because it is posted in large numerals outside every gasoline station and because several websites (e.g., www.gasbuddy.com, www.gaspricewatch.com) now post aggregated data about each individual station's prices. The market for retail gasoline provides an interesting laboratory through which to study several microeconomic questions. Gasoline markets are used as a study setting to examine the ways that retailers can pass along price changes to consumers.

Gasoline is sold by small independent retailers but mostly by large chains of retailers and, as such, it provides a useful field-based situation for studying the way retailers may collude in setting prices when input prices change rapidly. Gasoline stations occupy a variety of locations; some are farther from and some closer to consumers. The impact of this spatial variation allows for an investigation of some interesting questions on consumer search behaviours. Despite the large amount of data about gasoline prices, even relatively simple tasks like estimating the price elasticity of consumers to gasoline prices are remarkably hard to settle; these parameters may, in fact, vary with time.

The profit of the individual retail gasoline station depends on both the direction of the upstream cost change and the demand change. The upstream cost change is something that cannot be controlled by the individual retailers, but the demand change can be indirectly regulated by their pricing decisions. As a consequence, demand can be modeled as a function of the individual retailers' prices and their competitors' prices. Dynamic competition occurs when firms compete in the market for multiple periods. We model this dynamic retail gasoline competition as a repeated single-period game.

This paper combines theoretical and empirical studies of retail gasoline pricing behaviors. Section 3.2 describes the dataset, followed by a statistical analysis. Section 3.3 presents several economic models, including the classic Bertrand model, the differentiated Bertrand model, the repeated single-period game model, and a sequential game model to derive the optimal pricing strategies in an oligopoly setting. Section 3.4 provides a quantitative case study based on a linear regression model for site-specific gasoline prices and sales volumes with a self-selected competitor's price. Section 3.4 also provides empirical results based on models from Section 3.3 and shows the profit for using optimal pricing strategy for each game. Also, we examine empirical proof for price asymmetry using daily data from cities in a western European country, and we support the idea that implicit collusion may constitute a possible source of asymmetry. A qualitative analysis follows, exploring how imperfect information on upstream cost affects consumers' search behaviors and retailers' pricing decisions.

3.1.1 Literature Review

The retail gasoline pricing literature includes research that studies the price differentials across individual stations using station-level empirical data and the response of aggregated gasoline prices to wholesale gasoline prices. Some studies report asymmetric gasoline price response to upstream costs and Edgeworth price cycles in gasoline markets in which there are two phases: an undercutting phase and a relenting phase.

Slade (1987) investigated the Vancouver retail gasoline market by collecting daily prices, sales volume, and cost data for three types of gasoline sold at 13 service stations during a price-war period in the summer of 1983. In order to estimate response matrices, Slade focused on the behavior of the subject firms during highly competitive periods rather than during cooperative periods. Bertrand-Nash, best-reply, and monopoly prices were calculated using the estimated demand and price change equations and were then compared to the prices observed both during and after the price war.

Noel (2007) examined three different pricing phenomena, namely cost-based pricing, sticky pricing, and sharp asymmetric pricing, observing all three in several Canadian retail gasoline markets with a panel set of 19 cities over 574 weeks (January 1989 to December 1999). Using Markov-switching regression, Noel found cycling activity in 43% of the sample, sticky pricing in 30%, and cost-based pricing in 27%. Cycling activity becomes prevalent when there are more small firms, but sticky pricing dominates with few small firms.

Verlinda (2008) found that retail gasoline prices respond asymmetrically to the cost of wholesale gasoline. Prices rise at a much faster rate with cost increases compared to the rate at which they recede with cost declines. In a study of weekly gas station prices in Southern California from September 2002 to May 2003, Verlinda illustrated the estimated asymmetry graphically through differences in Cumulative Response Functions (CRF). CRF posits a single cost shock at time t and describes the path of subsequent cumulative price changes until the price settles. Verlinda also found that market power and collusion are related to price-response asymmetry.

Estimating the price elasticity of consumer demand to gasoline prices is hard to settle, as discussed in Espey (1996), who finds that the average price elasticity of demand for gasoline is -0.26, while in the long-run, the price elasticity of demand rises to -0.58. Similarly, Hanly et al. (2002) also find that the price elasticity of demand is -0.25 in the short-run, while in the long-run, the price elasticity of demand rises to -0.64. However, Hughes (2008) showed that the price and income elasticities of gasoline demand have changed considerably over the past several decades. Hughes compared two periods, from 1975 to 1980 and from 2001 to 2006, and found that short-run price elasticities differed in range from -0.21 to -0.34 against -0.034 to -0.077, respectively.

Lewis and Noel (2011) studied how the Edgeworth price cycle affects the speed with which price responds to cost changes. As soon as falling prices become as low as wholesale costs, prices jump back up again and a new cycle begins. Using a latent regime Markov-switching regression framework, Lewis and Noel (2011) built a model specific to cycling markets that incorporated the Edgeworth cycle price dynamics and studied broad-panel data of daily retail and wholesale gasoline prices from 90 cities during 2004 and 2005.

Janssen et al. (2011) examined the properties of a perfect Bayesian equilibrium, satisfying a reservation price property (PBERP) where consumers buy if the observed price is below the reservation price and search for a better price otherwise. In this framework, these authors analyzed a sequential consumer search model in the context of cost uncertainty and showed that, when the cost is uncertain, the average price and the expected lowest price are higher and consumer welfare is relatively lower.

Chandra and Tappata (2011) tested temporal price dispersion using a panel data set from the U.S. retail gasoline market and found that price dispersion increases with the number of firms and search costs and decreases with the production cost. These results imply that there are fewer gains for consumers to search at such times.

3.2 Description of the Data

Most previous studies have used either daily station-level short-panel data, such as Slade (1987), Chandra and Tappata (2011), and Lewis and Noel (2011); or weekly station-level long panel data, like Verlinda (2008); or daily city-level cross-sections, like Lewis [2012] and Noel [2012]. A few studies used daily station-level long-panel data, collecting sales data from individual gas stations (like that used in this study) instead of using aggregated local demand.

The dataset used in this study is unique and broad in its temporal and spatial dimensions. There are 109 different retail sites (stations index $i, i = 1, 2, ..., 109$), each with anonymized site ID in the data set. Each site is independent of the others, and all are located in western European cities. The data collected per site contains a date index t ($t = 1, ..., 380$)[1]; a weekday index from 1 (Sunday) to 7 (Saturday); and station-specific daily retail price (p_t^i, in euros per liter) data; the replacement cost of regular grade gasoline (C_t^i, in euros per liter) on the date specified, including delivery from the wholesale terminal to the retail site; and the daily total regular gasoline grade fuel sales volume (in liters). Each set of station-specific data also contains self-selected competitors' daily retail prices ($p_{k,t}^i, k = 1, ..., N_i$, in euros per liter), where N_i is the number of competitors to station i, with five to eight competitors assigned to each site. We define the static average competitor price to station i on date t as $p_{c,t}^i = \sum_{k=1}^{N_i} \frac{p_{k,t}^i}{N_i}$. The data were collected every morning by retailers who contracted with a leading global pricing and solution company for the period from January 2009 to December 2009. Unlike previous studies, most of the data is complete, and there are no missing observations during the sample period. Due to a confidentiality agreement, no further detail description can be provided.

We use the total daily sales volume for regular-grade gasoline as a proxy for demand. We apply 21% VAT on the replacement cost (i.e., the cost of the gasoline that will replace the inventory is being sold) as a proxy for marginal cost to retailers, using the raw data to show the behavior of the retail gasoline market. For each retail gasoline station i, the static average price of each self-selected competitor (five to eight competitors per retailer) is considered for the linear regression model, and thus, the regression results are site-specific. We site-specifically regress quantity demanded(q_t^i) against the subject retailer's own price(p_t^i) and the static average competitor's price($p_{c,t}^i$) over time, given as:

[1]Slightly more than 1 year.

Model 0: $q_t^i \sim p_t^i + p_{c,t}^i$.

In order to capture a general characteristic and improve model fit, we also try to smooth our data when fitting a model and estimating parameters. Let q_t be the original demand series. To smooth this, now define $\overline{q_t}$ as a seven-day moving average; as defined by:

$$\overline{q_t} = \frac{(q_{t-3} + q_{t-2} + q_{t-1} + q_t + q_{t+1} + q_{t+2} + q_{t+3})}{7}. \tag{3.1}$$

Similarly, we define $\overline{p_t}$ and $\overline{p_{c,t}}$ as a seven-day moving average retailer's own price and static average competitors' price, respectively. The smoothed data makes it easier to observe the relationship between the demand and the price. We use this symmetric averaging to determine elasticities that are not regularly updated in our dynamic models. We tried other models, such as quadratic models, and we also tried additional variables, but none of these more complicated models significantly improved the model fit for the complexity. We regressed the moving average of demand on the moving averages of the retailer's own price and static average competitor's price, given as:

Model 1: $\overline{q_t^i} \sim \overline{p_t^i} + \overline{p_{c,t}^i}$.

The adjusted R-squared improves from 0.14 (Model 0) to 0.41 (Model 1); see Table 3.1 for an example of a particular station.

Table 3.1: *Estimation of quantity demanded with own price and a static average competitor's price ($q = \beta_0 + \beta_1 p + \beta_2 p_c$) and estimation of the moving average of demand with moving average of own price and of static average competitor's price ($\overline{q} = \beta_0 + \beta_1 \overline{p} + \beta_2 \overline{p_c}$), where over bar denotes moving average and variables starred **, and *** indicate significance at the 1%, and 0.1% level, respectively. Standard errors are in parentheses.*

Variable	Model 0	Variable	Model 1
Constant	β_0 = 14128*** (1956)	Constant	β_0 = 13378.1*** (957.2)
p	β_1 = -170866*** (24735)	$\overline{p}$	β_1 = -217990.8*** (15076.5)
p_c	β_2 = 167407*** (25098)	$\overline{p_c}$	β_2 = 215560.5** (15274.1)
Adjusted R-squared	0.14	Adjusted R-squared	0.41
Degrees of Freedom	361	Degrees of Freedom	355

3.3 A Duopoly Model Against a Static Player

First, we consider a modified Bertrand model, which describes a situation with only two firms that reach perfect competition with prices set at the marginal cost (Maskin and Tirole 1988) and (Ledvina and Sircar 2011). While the classical Bertrand model is well suited to firms that compete on prices, the model assumes that the products are undifferentiated, that the firms do not cooperate, that the firms compete by setting prices, and that consumers buy everything from the firm with the lowest price (Maskin and Tirole 1988). In this framework, firms will simply lower their prices until they reach the marginal cost in order to gain any demand.

Although it is true that, with only two firms, the model reaches perfect competition, (Maskin and Tirole 1988), this scenario is not directly relevant to the real retail gasoline market, in which there are nearly always more than two competitors in each geographic area. Although the gasoline that is sold at different stations may be fairly homogeneous, it is nonetheless differentiated by several factors, including brand, facility, service, and location. Because of these factors, stations with higher-than-average prices may retain some non-zero demand. Thus, we consider a modified formulation in which we assume that the products are differentiated, and in the modified Bertrand model, we assume a functional form for how the quantity demanded for retail gasoline reacts to the static average competitor's price change as well as to the individual retailer's own price change. We investigate the optimal pricing strategy for an individual gas station against a static player (i.e. average competitor). We denote the price in a station by p and the static average competitor's price by p_c. We denote the demand in a station by q and assume a linear relation between prices and demand, where β_0, β_1, β_2 are nonnegative regression coefficients. We have explicitly accounted for the fact that β_1 is negative.

$$q = \beta_0 - \beta_1 * p + \beta_2 * p_c \tag{3.2}$$

The quantity demanded has a positive relationship with static competitor's price but a negative relationship with the retailer's own price, as shown by Table 3.1. The coefficients of the function were derived from empirical data using OLS regression. We assume the marginal cost is nearly the same for the stations, and we denote it by C, and finally denote profit by π. The profit function is then simply the product of quantity and margin:

$$\pi = q(p - C) = (\beta_0 - \beta_1 p + \beta_2 p_c)(p - C) \tag{3.3}$$

The price at a station, p, is bounded by the maximum price (the price that drives demand to zero) and the minimum price (the marginal cost) so that the profit always takes a non-negative value. We look for solutions in which $\frac{d\pi}{dp} = 0$. Simple algebra then shows that a firm's best response to a static competitor's price is:

$$p^* = \frac{(\beta_0 + \beta_1 C + \beta_2 p_c)}{2\beta_1} \tag{3.4}$$

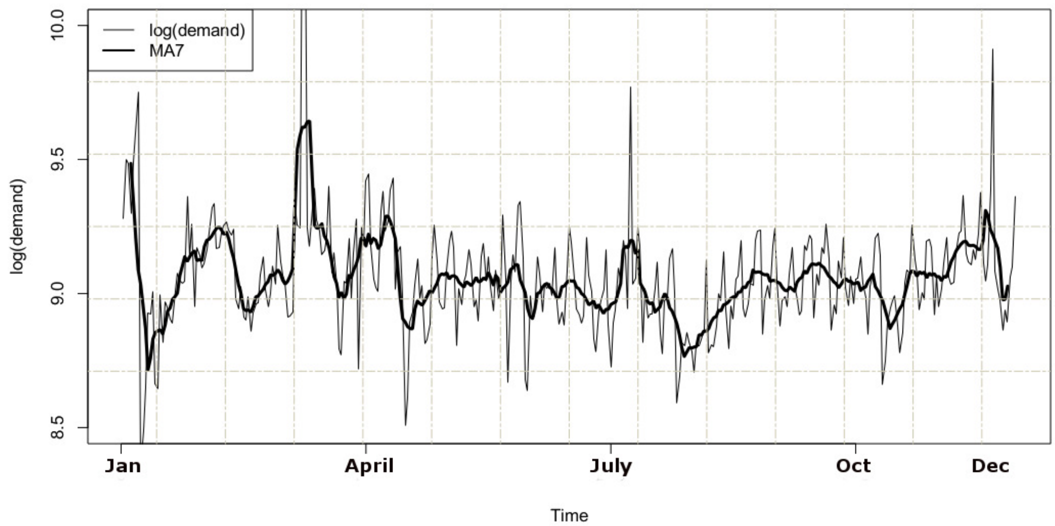


Figure 3.1: *The logged sales volume (demand) data for a specific site from a Western European country is shown in solid line, and the 7 days moving average is shown in thicker solid line. The vertical axis displays the log demand which is scaled logarithmically and ranges from 4914.8 to 22026.4 Liters, and the horizontal axis displays time (Jan-Dec 2009).*

Here, p^* is the optimal price when only one firm plays the optimal strategy, and there are no equilibrium price pairs; $p \neq p_c$, where p_c is simply the observed static average competitor's price at t assuming the retailer sets their price myopically, believing that their competitor keep their prices static. The resulting profit function is:

$$\pi(p^*, p_c) = (\beta_0 - \beta_1 p^* + \beta_2 p_c)(p^* - C) \tag{3.5}$$

3.4 A Case Study in Retail Gasoline Markets

In order to further investigate the retailer's pricing behavior and sales volume, we focus our discussion on a case study of one specific retailer from the dataset. Assuming that nearby retail gasoline stations have similar marginal costs and taxes, the retail margin would be the only factor that differentiates retail gasoline prices across stations. Each individual gasoline station is unique, however, in that their market power and consumers' preferences may differ. Brands and the proximity of competitors and consumers affect their pricing strategies. For example, national brands located near rental car locations are able to set prices higher than other stations in the same metropolitan area (Jaureguiberry 2010). In fact, the evidence suggests that, for gasoline stations, brand and location are the most important factors for garnering local market power (Verlinda 2008) and (Jaureguiberry 2010).

We have only limited information about the competitors' market data. Thus, it remains unclear how individual retailers set their prices and to what extent they can alter their prices relative to a competitor's price. Using the limited information available to us, we attempt to determine the pricing strategies used in a specific retail gasoline station. Then, based on our model described in Section 3.3, we attempt to optimize the price.

3.4.1 Descriptive Statistics on Demand and Price

The trend of the quantity demanded (sales volume) for gasoline in a specific site is shown in Figure 3.1. The demand shows periodic spikes due to the seasonal (longer scale) and weekly (shorter scale) trends of the retail price, as well as a few random spikes, but without a long-run trend. The moving average of demand still fluctuates over the course of one year, but the short-term periodic cycles are mostly removed, as depicted by the thicker solid line in Figure 3.1. Thus, the sales volume of this site is non-stationary without deterministic trends.

To examine the trend of the quantity demanded for gasoline for each day of the week, we plotted the quantity demanded for gasoline in each site against the relevant day of the week. Figure 3.2 shows a weekly cycle in gasoline sales for a specific site, with the demand for gasoline gradually rising as the end of the week approaches, with a

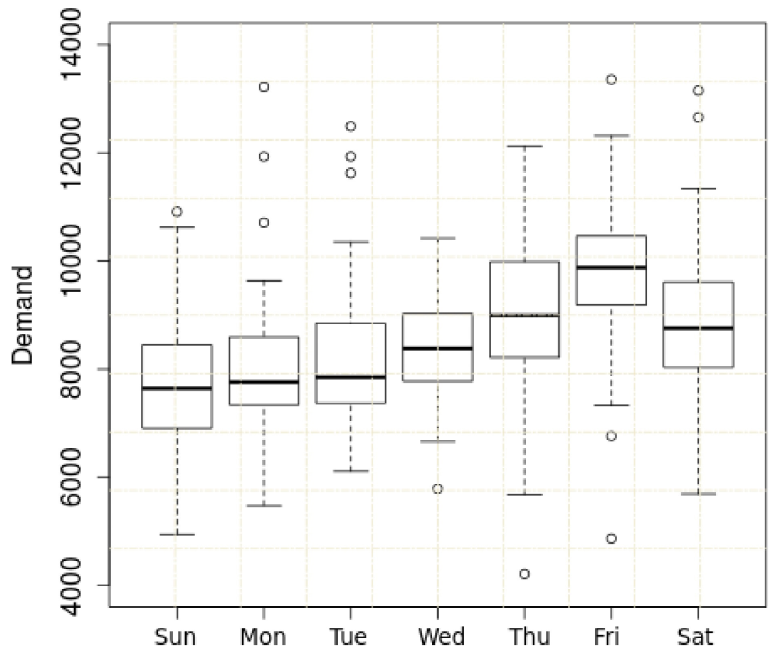


Figure 3.2: *The weekly trend of gasoline sales volume (demand) for a specific site from a Western European country is shown with error bars. The vertical axis displays sales (or demand) in Liters, and the horizontal axis displays the day of week labeled as Weekday.*

peak on Friday.

Although the demand and the margin still represent the two most important key factors for retail gasoline profits, we observe that the demand is fairly stable against the margin. If the upstream price information was also given to consumers, we would expect to see the demand fall as the margin rises and rise as the margin falls. Figure 3.3 plots the pairs of observed retailers' own prices and the static average competitors' prices as a scatter plot. The pairs of observed prices form a small band along the diagonal line of the plot, and the actual prices are not observed in the region below the small band, where the retailer's own price is much less than the static average competitor's price, nor in the region above the band, where the retailer's own price is much greater than the static average competitor's price. This result implies that

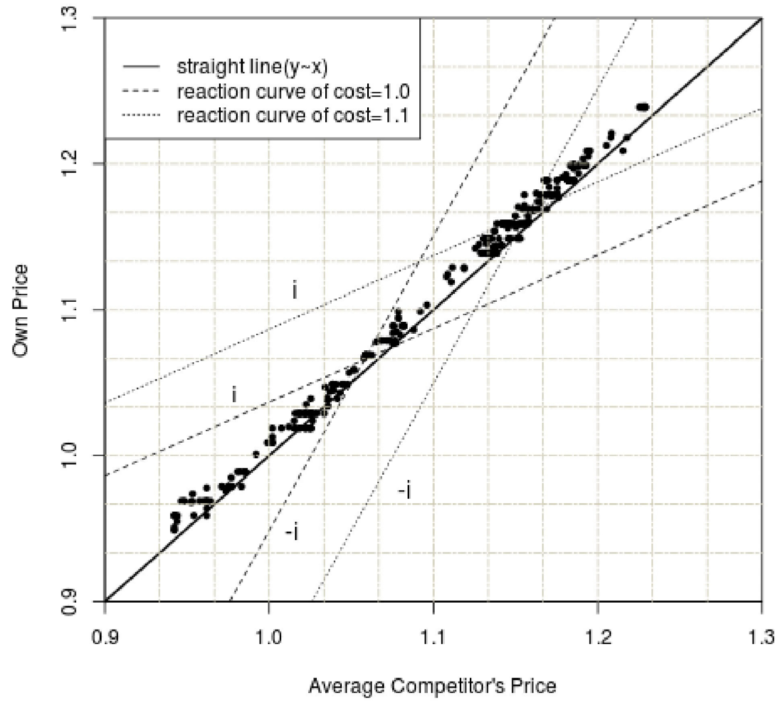


Figure 3.3: *The scatter plot of static average competitor's price and own price for a specific site from a Western European country, 364 observations plotted as points. The vertical axis displays own price, and horizontal axis displays static average competitor's price in Euros. Reaction curve, given by Eq. (3.4), lines for fixed cost (1.0, 1.1) and straight line (y=x) are added to the plot.*

the observed price differences between the retailers' own prices and the static average competitor's prices are bounded within a small window of about 2 to 3 cents. Perfect competition suggests that prices would, on average, converge to the same level for all retailers. If this were true, the straight line ($y = x$) would be at the center of the scatter plot. However, the situation depicted is still far from one of perfect competition, with the retailers' own prices nearly always sitting a few cents higher than the static average competitors' prices. This is consistent with the proposed modified Bertrand competition model and is backed up by the other dashed lines' reaction curves.

From the first-order condition, Eq. (3.4), for each firm's profit function, we can compute the reaction curve for each firm, which crosses at the equilibrium price pair

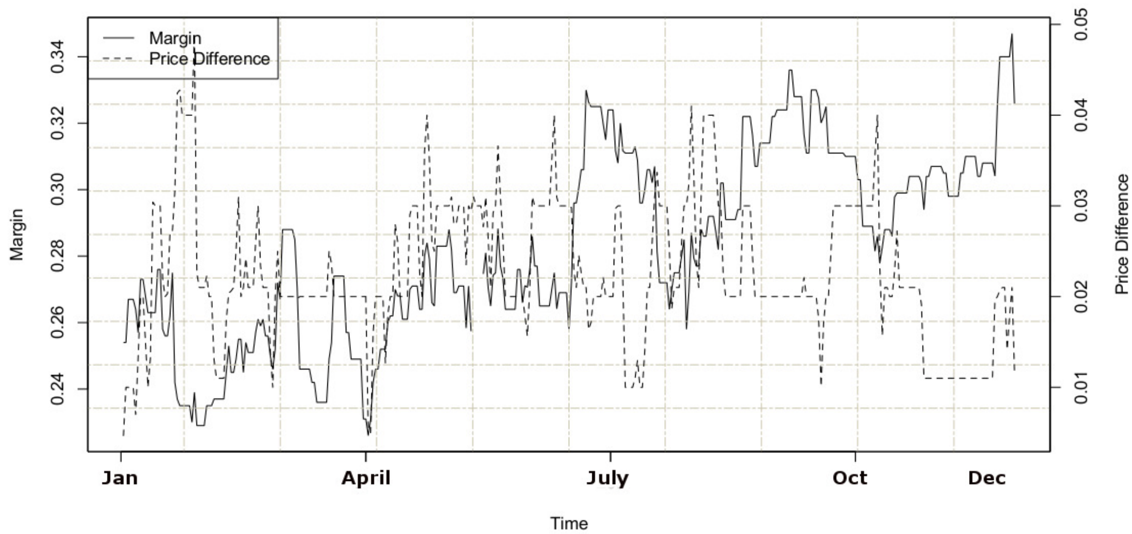


Figure 3.4: *The solid line shows margin over time, the dotted line shows the price difference, own price - minimum competitor's price, for gasoline from a Western European country over time. The left hand vertical axis displays margin in Euros, and the right hand vertical axis displays the price difference in Euros, and the horizontal axis displays time (days).*

of marginal cost for a homogeneous Bertrand model.

The competitor's demand is estimated based on the same linear demand function. The dashed line in Figure 3.3 shows the reaction curves for the different costs of 1.0 and 1.1. The lines that cross the vertical axis are the reaction curve for a firm labeled as i, and the lines that cross the horizontal axis are the reaction curve for the static average competitor of the firm labeled as $-i$. The reaction curves for a firm and its static average competitor cross each other at the equilibrium price pair. All equilibrium price pairs for a range of different costs lie between the marginal cost and the monopoly price, implying a moderate competition in the retail gasoline market, which is consistent with the modified Bertrand model.

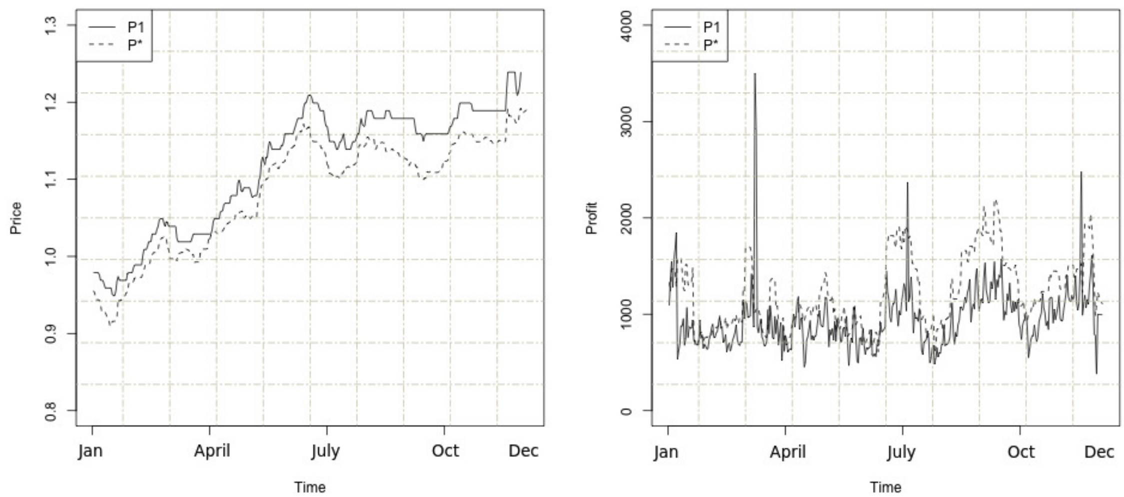
In order to study how retail gasoline prices are set and how they change over time, we plot the margin and the price difference between a given station's price and the minimum competitor's price against time. From Figure 3.4, it can be speculated that stations do not use constant margin pricing because we observe that the margin is never stationary but fluctuates within the range of 6 $\sim$ 16 euro cent over time. On the other hand, the price difference fluctuates within a relatively small interval (5 euro cents) and is always positive. It is also observed that, the price difference stays at a certain level for quite a long period of time, at 2 cents over 50 days. Based on these pieces of evidence, it can be inferred that stations may maintain their relative position to the minimum competitor prices in recognition of the competitor's greater market power. It is also possible that, in an inverse manner, the minimum competitor may have set its price a few cents lower than the subject station's own price. In the following subsection of our paper, we develop a pricing strategy that may beat the existing strategy.

3.4.2 Prescriptive Pricing

We adopt a price range given as $p \in (p_{\min}, p_{\max})$, where $p_{\min}$ is the minimum price giving zero profit, and $p_{\max}$ is the maximum price that drives demand to zero. The optimal price and profit can be computed for the given cost and the competitor's price using Eqs. (3.2) and (3.3). For a sensitivity test, we vary the static competitor's price while fixing the cost. The result shows that as the static competitor's price drops, $p_{\max}$ decreases, and vice versa. In fact, $p_{\max}$ directly depends on the static competitor's price, mathematically given as:

$$p_{max} = \frac{(\beta_0 + \beta_2 p_c)}{\beta_1} \tag{3.6}$$

The optimal prices and profits for the repeated single period game are computed for floating time points using Eq. (3.4). Figure 3.5(a) shows optimal prices for a single player as p^*. We observe that the trend of p^* is always a few cents lower than the



(a) **(b)**

Figure 3.5: *(a): p_1 is the trend of observed gasoline price over the course of year from a Western European country, p^* is the optimal price for a single firm. The vertical axis displays gasoline price in Euros, and the horizontal axis displays time. (b): p_1 indicates the trend of profit realized by actual price over the course of year from a Western European country, p^* indicates the optimal profit realized by optimal price for a single firm. The vertical axis displays the profit in Euros, and the horizontal axis displays time.*

actual price trend.

Figure 3.5(b) shows the optimal profit for the same observed price and the optimal prices as in Figure 3.5(a). A higher profit is observed for p^* than the the observed price. Also, we observe an asymmetry in which, when the actual price rises, the optimal price is closer to the observed price than when the actual price falls. This is because retailers set their margins low when the upstream cost rises, forcing the possible price range $(p_{\min}, p_{\max})$ to shrink; and they set their margins high when the upstream cost falls, forcing the possible price range $(p_{\min}, p_{\max})$ to expand. We will discuss empirical evidence of this price response asymmetry in Section 3.4.4.

3.4.3 Asymmetric Pricing and Searching Behavior

In this section of the paper, we conduct a more qualitative analysis to study retailers' pricing decisions and consumers' search behaviors. Firms earn high profit margins by setting their prices high, relative to the upstream cost, when the upstream cost falls. This action is possible partially because the demand does not fall much during high-margin times. The inelastic demand response to the margin implies that the consumers have only limited upstream cost information. This may further imply that consumer's purchasing decision is based mainly on their expected price, which is estimated based on previous period's prices and current competitor's prices (Chandra and Tappata 2011). Therefore, consumers with limited access to upstream cost information find it difficult to maximize their welfare or utility (Janssen et al. 2011).

Table 3.2: *Estimation of margin with price difference and cost change ($m = \alpha_0 + \alpha_1 \Delta p + \alpha_2 \Delta C$). Variables starred *, **, and *** indicate significance at the 5, 1, and 0.1 % level, respectively. Standard errors are in parentheses.*

Variable	Model 2
Intercept	α_0 = 0.122938 ***
	(0.003803)
Δp	α_1 = -0.603528 ***
	(0.165248)
ΔC	α_2 = -2.510867***
	(0.346661)
Adjusted R-squared	0.2004
Degrees of Freedom	355

We run a regression using site-specific price difference ($\Delta p_t = p_t - \min(p_{c,t})$), where $\min(p_{c,t})$ is a minimum competitor's price at t; and cost change ($\Delta C_t = C_t - C_{t-1}$) to examine the relationship of these two variables on margins (m_t), given as:

Model 2: $m_t \sim \Delta p_t + \Delta C_t$

The retailer's gain is offset by the consumer's gain, and the relationship between the two parties is more like a competition for available utility than it is a partnership. Therefore, one might argue that consumers should be able to save more money by searching harder when retailers enjoy big profit margins (i.e., when cost falls) than when retailers enjoy small profit margins (i.e., when cost rises) if retailers do not collude to keep their own price high when cost falls. However, exploring the regression results using Model 2, this argument receives little empirical support, as we observe that price dispersion is negatively correlated to the margin (see Table 3.2). Therefore, firms enjoy higher margin when the dispersion between their own price and the minimum competitor's price is small, and vice versa. Also, firms indeed enjoy higher margin when cost falls compared to when cost rises, as indicated by the negative regression coefficient of cost change in Table 3.2.

This observation supports the result of Borenstein et al. (1997), which states that retailers collude (at least, implicitly) to set high margins when upstream cost falls. Since the retailers are colluding at those times, the price dispersion is smaller, as seen in Figure 3.4. These observations support Borenstein et al. (1997) theory providing smaller incentives for consumers to search when the upstream cost falls partially due to the small price dispersion, and partially due to the lower price than their expected price. This explains why other studies describe that consumers search less for cheaper prices when prices fall than when prices rise.

Recall that the raw data for the quantity demanded displayed weekly periodicity, as shown in Figure 3.2. In our data, the weekly trend of the quantity demanded for gasoline gradually rises as the week approaches the weekend.

3.4.4 Evidence for Price Response Asymmetry in the European Retail Gasoline Market

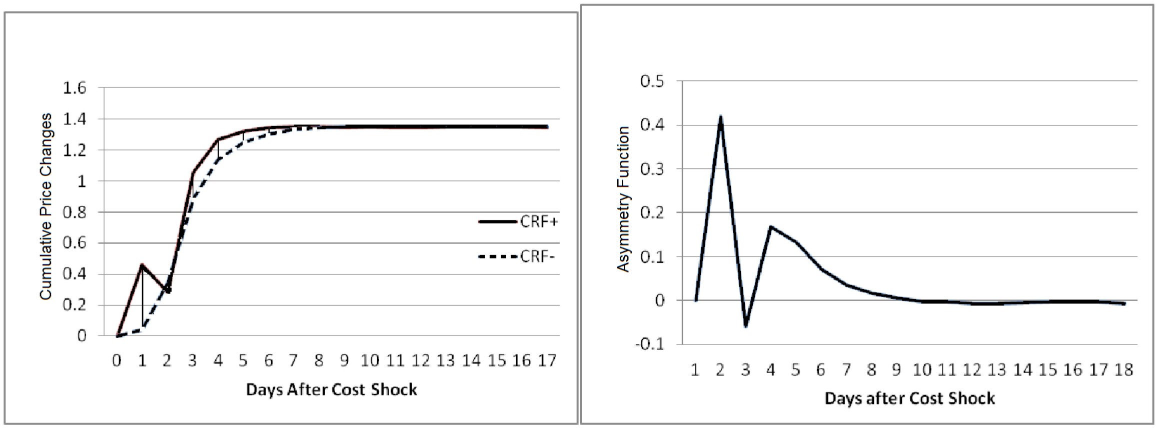
Cumulative Response Functions (CRF) have theoretical and practical importance in examining the response asymmetry between retail prices and costs. The price response asymmetry describes a gasoline price behavior in which prices rise quickly following an increase in the cost, but fall slowly following a decrease in the cost. For more details about CRF, see Borenstein et al. (1997) and Verlinda (2008). We present a response asymmetry, which is an error correction model with separate coefficients for positive and negative changes in wholesale costs and retail prices, given as:

$$\Delta p_t = \sum_l (b_l^+ \Delta C_{t-l}^+ + b_l^- \Delta C_{t-l}^-) + \sum_m (\gamma_m^+ \Delta p_{t-m}^+ + \gamma_m^- \Delta p_{t-m}^-) + \lambda e_t + \varepsilon_t \tag{3.7}$$

where $e_t = p_{t-1} - \alpha - \theta C_{t-1}$ for $l = 0, 1, ..., L$ and $m = 0, 1, ..., M$

Here Δp_t^+ is the positive price change; $\Delta p_{t-m}^+ = \max(p_{t-m} - p_{t-m-1}, 0)$, Δp_t^- is the negative price change; $\Delta p_{t-m}^- = \min(p_{t-m} - p_{t-m-1}, 0)$. ΔC_{t-l}^+ is the positive cost change; $\Delta C_{t-l}^+ = \max(C_{t-l} - C_{t-l-1}, 0)$, ΔC_{t-l}^- is the negative cost change; $\Delta C_{t-l}^- = \min(C_{t-l} - C_{t-l-1}, 0)$, p_{t-1} is the previous period's price, and C_{t-1} denotes the previous period's wholesale cost. λ measures the rate of reversion to the long-run relationship; in the long-run, the gasoline price should be proportional to the cost, specified by e_t. The long-run response of expected prices to a cost is denoted by θ, and α is the intercept for the long-run expected margin. The parameters $b_l^+, b_l^-, \gamma_m^+, \gamma_m^-$ are the regression coefficients, and ε_t is the noise term.

We estimate the three lagged cost change(L=3) coefficients (b_l^+, b_l^-) separately for wholesale cost increases and decreases, and we also estimate the two lagged price change(M=2) coefficients (γ_m^+, γ_m^-) separately for each of the cases of price moving up



(a) **(b)**

Figure 3.6: *(a): The figure shows CRF, the path of cumulative price change after positive and negative cost shock at day 1. The unit of the vertical axis is Euros labeled as cumulative price change, and the time unit in the horizontal axis is days. (b): The asymmetry function(A_t) is the difference between the path of cumulative price change after positive and negative cost shock at day 1. The unit of the vertical axis is Euros labeled as asymmetry function, and the time unit in the horizontal axis is days.*

or moving down. The resulting coefficient estimates are used to construct a CRF as in Lewis and Noel (2011) and Verlinda (2008). The subsequent price change, Δp_t, describes the CRF at each period t, which is the previous period's predicted price plus the predicted change in prices in the current period:

$$CRF_t = p_{t-1} + \Delta p_t \tag{3.8}$$

where p_{t-1} is the previous period's predicted price, and Δp_t is the predicted change in prices from Eq.(11) that describes how retail prices pass through an upstream cost change.

Figure 3.6(a) plots the CRF (i.e., the path of cumulative price changes) until the price settles at the equilibrium value after a one-unit change in cost at day 1. The path of cumulative price change for negative cost shock is also plotted on the same positive domain.

Lastly, we use an asymmetry function $A_t = CRF_t^+ - CRF_t^-$, as defined in Verlinda

(2008), where CRF_t^+ represents the path of cumulative price changes after positive single shock in costs, and CRF_t^- represents that after negative single shock in costs. Figure 3.6(b) plots the A_t, and the observed deviation from 0 indicates a price asymmetry toward the direction of the deviation.

Positive values for the cumulative response asymmetry are observed, implying that the price response to negative cost shocks is much slower than that of positive cost shocks. Cost increases are initially passed through more quickly than cost decreases. However, prices fully respond to cost changes for both positive and negative cost shocks over the course of a week following a cost change. This rate of asymmetric response is faster than the findings of previous studies (three to six weeks), as in Lewis and Noel (2011). The rate of asymmetric response may depend on the type of gasoline market or pricing policy.

3.5 Conclusion

In this study, we found optimal prices and profits in a stylistic repeated single-period game in the retail gasoline market. We also showed that retailers do not fully optimize their prices, possibly because they do not know how demand responds to their prices in a quantitative scale, so they instead respond to their competitor's prices. Individual gasoline retailers have no control over external factors like upstream cost changes and static average competitor's prices. Given the upstream cost and static average competitor's price, we showed that retailers may optimize their profit using our strategy when the strategy is practiced on a small scale. Obviously, when a given retailer is the only one or is part of a small group of retailers who set prices at the optimal price, that retailer will earn a higher profit than the actual profit. But when all retailers use the same price-setting strategy, depending on price conditions, the subject retailer might earn higher or lower profit, assuming similar market power.

Although all models are incomplete representations of reality, the stylistic repeated sequential leader and follower game is very realistic in the following sense: Once a retailer has garnered big market power with a big brand name and a great location, that retailer will remain the leader until a new, stronger competitor enters the market. Similarly, once a retailer becomes a follower, that retailer will remain the follower for a long period of time because the location and the brand name are fixed external factors, and market power cannot shift quickly.

This evidence of price asymmetry (or collusive behavior) is presented with daily data using the same approach as that used in other recent studies. However, the entire market does not show the price leadership characteristics. According to Noel (2007), about 40% of the market shows an Edgeworth price cycle where retailers pass through the slow undercutting phase and short relenting phase. Thus, we need more than one model to capture multiple market characteristics, and we may be able to model this price cycle behavior by alternating the leader's and follower's responses, and by raising one of their prices when near the marginal cost.

Our main goal for this study was to investigate how an individual retailer might maximize its profit by optimally setting the retail gasoline price under modified Bertrand model. The economic models we proposed do not incorporate market phenomena like collusion and random demand spikes. A comparison of model predictions with empirical data may reveal those phenomena. While these models do have some limitations, we have obtained much insight from empirically testing the modified Bertrand model and its derivatives. The empirical data shows competitive market behavior followed by somewhat cooperative market behavior thereafter. This finding suggests that the retail gasoline market may change its behavior over time.

For future study, our study can be extended to more complicated scenarios, where every retailer responds to everyone's price instead of just to the static average competitor's price as well as to price cycle behavior using alternating sequential game models. There are still many elements that can push the envelope of our understanding of the retail gasoline market and the effects of its imperfect information on both consumers and retailers. Thus, as a setting for future research, the retail gasoline market holds much potential.

In the next Chapter, we change the focus to fluctuating retail gasoline prices from consumers' perspective, and we introduce guaranteed retail gasoline price using swap structure.

Bibliography

Baye, Michael R. (2010), Managerial Economics and Business Strategy, *The McGraw-Hill Series Economics. 7th ed., McGraw-Hill/Irwin*, New York

Borenstein, Severin and Shephard, Andrea (1996), Dynamic Pricing in Retail Gasoline Markets, RAND *Journal of Economics*, Vol. 27, No. 3, pp.429-51.

Borenstein, Severin, Cameron A. Colin, and Gilbert, Richard (1997), Do gasoline prices respond asymmetrically to crude oil price changes?, *Quarterly Journal of Economics*, Vol. 112, No. 1, pp.305-339.

Chandra, A. and Tappata, M. (2011), Consumer search and dynamic price dispersion: an application to gasoline markets, *RAND Journal of Economics*, Vol. 42, No. 4, pp.681-704.

Cleveland, W.S., E. Grosse, and W.M. Shyu (1992), Local regression models, Chapter 8 of Statistical Models, *Wadsworth and Brook/Cole*.

Deltas, G. (2008), Retail Gasoline Price Dynamics and Local Market Power, *The Journal of Industrial Economics*, Vol. 56, No. 3, pp.613-628.

Espey, Molly (1996), Explaining the Variation in Elasticity Estimates of Gasoline Demand in the United States: A Meta-Analysis, *The Energy Journal, International Association for Energy Economics*, Vol. 0, No. 3, pp.49-60.

Hanly, M., Dargay J., and Goodwin, P. (2002), Review of Income and Price Elasticities in the Demand for Road Traffic, *Department for Transport*, London

Havranek, T.,Irsova, Z., and Janda, K. (2012), Demand for gasoline is more price-inelastic than commonly thought, *Energy Economics, Elsevier*, Vol. 34, No. 1, pp.201-207.

Hughes, Jonathan E., Knittel, Christopher R. and Sperling , Daniel (2008), Evidence of a shift in the short-run price elasticity of gasoline demand, *The Energy Journal*, International Association for Energy Economics, Vol. 29, No. 1, pp.113-134.

Janssen, M.,Pichler, P. and Weidenholzer, S. (2011), Oligopolistic markets with sequential search and production cost uncertainty. *RAND Journal of Economics*, Vol. 42, No. 3, pp.444-470.

Jaureguiberry, Florencia. (2010), An Analysis of Strategic Price Setting in Retail Gasoline Markets, PhD dissertation, Public Policy Analysis, *The Pardee RAND Graduate School*, Santa Monica.

Kedem, Benjamin and Fokianos, Konstantinos (2002), Regression analysis, Time-series analysis, *Wiley series in probability and statistics*

Ledvina, Andrew and Sircar, Ronnie (2011), Dynamic Bertrand Oligopoly, Applied Mathematics and Optimization, *Springer-Verlag*, Vol. 63, No. 1, pp.11-44.

Lewis, Matthew S. (2011), Asymmetric price adjustment and consumer search: an examination of the retail gasoline market, *Journal of Economics Management Strategy*, Vol. 20, No. 2, pp.409-449.

Lewis, Matthew S. and Noel, Michael D. (2011), The speed of gasoline price response in markets with and without Edgeworth cycles, *Review of Economics and Statistics*, Vol. 92, No. 3, pp.672-682.

Mas-Colell, A., Whinston, M., and Green, J.R. (1995), Microeconomic Theory, New York.

Maskin, Eric and Tirole, Jean (1988), A Theory of Dynamic Oligopoly, I: Overview and Quantity Competition with Large Fixed Costs, *Econometrica, Econometric Society*, Vol. 56, No. 3, pp.549-69.

Maskin, Eric and Tirole, Jean (1988), A Theory of Dynamic Oligopoly, II: Price Competition, Kinked Demand Curves, and Edgeworth Cycles, *Econometrica, Econometric Society*, Vol. 56, No. 3, pp.571-99.

Montgomery, Douglas C., Jennings, Cheryl L., Kulahci, Murat (2008), Time-series analysis, Forecasting, *Wiley series in probability and statistics*

Natural Resources Canada (2013), Understanding Gasoline Markets in Canada and Economic Drivers Influencing Prices, Vol. 11, No. 8.

Noel, Michael D. (2007), Edgeworth price cycles, cost-based pricing, and sticky pricing in retail gasoline markets, *Review of Economics and Statistics*,Vol. 89, No. 2, pp.324-334.

Radchenko, Stanislav (2005), Oil price volatility and the asymmetric response of gasoline prices to oil price increases and decreases, *Energy Economics, Elsevier*, Vol. 27, No. 5, pp.708-730.

Slade, Margaret E. (1987), Interfirm rivalry in a repeated Game: An empirical test of tacit collusion, *The Journal of Industrial Economics*, Vol. 35, No. 4, pp.499-516.

Verlinda, Jeremy A. (2008), Do rockets rise faster and feathers fall slower in an atmosphere of local market power? Evidence from the retail gasoline market, *The Journal of Industrial Economics*, Vol. 56, No. 3, pp.581-612.

Chapter 4

Retail Gasoline Price Guarantees with Swap Structures

4.1 Introduction: a general background to the problem and overview of idea

As a result of retailer competition and changes in key input costs, retail gasoline prices and margins fluctuate over time (Deltas 2008). Fluctuating retail gasoline prices are problematic for both retailers and consumers in that the fluctuating retail gasoline prices lead to uncertainty in sales and competitive strategies from retailer's perspective, and to uncertainty in cost from consumers' perspective. When a commodity price suddenly rises and so retail gasoline prices also rise, customers would observe higher prices and search for lower prices. Thus, with price competition between retailers, margins may drop. When a commodity price suddenly falls, the retail gasoline prices also falls, customers would observe lower prices, and feel less concern about prices.

Some companies (e.g., http://www.mygallons.com, www.firstfuelbank.com) offer customers the opportunity to buy a quantity of gasoline at current prices and then cash in the quantity when prices rise. However, they still expose customers to the risk of taking the wrong side of price movements, and involve both hidden fees and a large premium to pay. For example, MyGallons charges customers a 6% fee upon pre-purchasing, with an additional charge of six cents per gallon as a pre-purchased quantity is cashed in, and a $30 annual membership fee. First Fuel Bank sells a price-locked gallon of gasoline almost a dollar above the market price. While these companies offer a way to protect against a big surge in the future, they are profit-oriented and charge a premium for their services. Thus, customers would benefit from these services only if gasoline prices rise very high.

In this study, gasoline markets are used as an example to examine another way customers can hedge against price changes regardless of their directions and magnitude. The fluctuations of gasoline prices make searching for an optimal refueling strategy (time and quantity) for consumers an interesting problem to solve. Likewise, retailers may want to hedge against the fluctuation of wholesale gasoline price to stabilize their overall sales and profits.

Large fuel consuming companies, such as airlines, use commodity swaps or options to reduce their exposure to volatile and potentially rising fuel costs. Hedging the fuel costs by locking in the future fuel prices can protect against sudden losses from rising fuel prices, but also prevents sudden gains from decreasing fuel prices. The reason that airlines still choose to hedge fuel prices is not to improve profits but to stabilize overall costs, cash flows, and profits (Morrell and Swan (2006)).

As mentioned above, commodity swaps, futures, and options are widely used to hedge commodity prices by producers of commodities, such as precious and base metals, energy stores (natural gas and crude oil) and food (including corn, wheat, pork bellies and cattle). The swap is a contractual agreement between two parties who agree to exchange their cash flows; one of which is fixed and the other is fluctuating prices, for a defined period of time and quantity without physical delivery (Dubofsky and Thomas (2003)).

The simplest examples of hedging are for a portfolio manager using a call option, a right to buy a underlying stock at strike price at maturity, to hedge against rising stock price; or using put option, a right to sell an underlying stock at strike price at maturity, to hedge against falling stock price. Farmers can also use a forward contract to sell their crops at predetermined prices protecting against sudden losses from falling crop prices. Airlines can use a commodity swap to reduce their exposure to volatile and potentially rising fuel costs. Using the floating-to-fixed rate cash flow swaps, producers can lock in fixed cash flows which meet their budget estimates, and hedge against falling prices (Davies et al. (2004)).

In what follows, we introduce a new loyalty program which provides customers with a guaranteed price. The floating-to-fixed swap may also be used to establish a fixed cost when prices surge, so that retailers can provide their consumers with the guaranteed price. We primarily focus on the end customers in this study, but gasoline retailers can also use this product format to drive loyalty and boost repeated sales. Using a case study, we quantitatively investigate the expected outcomes of the guaranteed price and optimal quantity to hedge under price uncertainty.

4.1.1 Literature Review

Much of the fuel hedging related literature has focused on airlines. The hedging portion of sales vary between different airlines, but generally airlines only partially hedge their fuel requirements for a certain period (Morrell and Swan (2006)). The main topic of the literature includes research that studies the benefits of hedging using options, swaps, futures and forwards to manage risks.

Morrell and Swan (2006) reach the conclusion that there is no reason to contradict the economic fundamentals of hedging; and a policy of permanent hedging of fuel costs should leave expected long-run profits unchanged. Depending on the stage of the economic cycle, this hedging may or may not damp out profit volatility. Data suggests it may only damp out volatility when oil price and air travel demand are negatively correlated. However, when oil prices and air travel demand cycles are positively correlated, then hedging is less effective in reducing volatility. So oil prices can be observed to either increase or decrease airline profit cycles, depending on the time period sampled. The economic cycle can be made worse by hedging, when oil price moves are demand-driven rather than supply-driven.

Vieira et al. (2014) examine a case study based on American Airlines' quarterly financial reports collected from Reuters Knowledge Database from 1989 to 2010, the quarterly jet fuel price calculated from daily spot jet fuel price for kerosene-type jet fuel data from the US EIA, and share prices collected from Yahoo Finance. Using time series regression coefficients as a measure of risk sensitivity, Vieira found it makes economic sense for American Airlines to have some hedge for fuel price risk, but suggest that the company has not been able to optimize the level of its hedging strategy.

Jin and Jorion (2006) investigate the hedging activities of 119 U.S. oil and gas producers from 1998 to 2001, and test for a difference in firm value between firms with and without hedging strategy. Using Tobin's Q ratio, which is defined as the ratio of the market value to the replacement value of assets, Jin and Jorion do not find a significant relation between the firms value and its hedging activities. Instead, they suggest the oil and gas context is closer to a perfect capital market, where there are no information asymmetries, taxes, or transaction costs, and hedging should not add value to the firm since it can be easily done by individual investors.

Westbrooks (2005) discusses hedging solutions available to airlines. The most desirable way for airlines to hedge against an increase in their fuel cost to directly pass it on to their customers. When reasons such as competitive pressure make that impossible, the next best alternative is to hedge the fuel cost using financial tools such as forwards, futures, and options contracts. Because of the volatility of the price of jet fuel, airline financial managers are enticed to use hedging as a risk management tool to insulate their companies from price increases, and it has been shown that hedging jet fuel price exposure has a positive effect on an airline stock values (12-13% value added for the extent of hedging).

Acharya and Lochstoer (2013) investigate the hedging activities using a theoretical model, and empirical proxy based on data of spot and futures prices for heating oil, crude oil, gasoline, and natural gas over the period 1979 to 2010, pairing these data with crude oil and natural gas producing firms' reported hedging policies from their financial accounting standards 133 disclosures from 2000 to 2010. They find that, when firm-specific default risk is high, these firms are more likely to hedge: an increase in measures of the aggregate default risk of producers of commodities with the volatility of the commodity prices forecasts a significant increase in the excess return on short term futures, and limits to arbitrage in the financial market generate limits to hedging for firms in the real economy. Consequently, corporate hedging policy changes affect asset prices; such market generated limits to hedging have predictive power for commodity futures and spot prices, inventory.

In summary, previous studies have mixed support for hedging, and suggests that investors should only value a company higher if its hedging strategy yields benefits that offset higher costs. But there are some distinctions in the retail gasoline market, in which price is driven not primarily by demand but by the supply and hence by wholesale prices. Also, retail gasoline demand would be less sensitive to its prices than air travel demand to its prices as air travel is more expensive and not a daily necessity. For instance, people need to drive to work regardless of the gasoline price, but they can choose not to travel to Cuba when air tickets are expensive. Thus, gasoline retailers may use hedging as risk management and marketing tools against increase in fuel cost without passing it on to the consumer. Hedging commodities like gasoline is possible when relatively large amounts of fuel are being traded and the upstream of the retail gasoline market is fully understood.

4.2 Design of the Swap Structure

In the previous section we have introduced that hedging means, reduced risks using financial tools, such as options, forwards, swaps, etc, designed to share the risks with speculators, financial institutions, or insurance companies. We also discussed how other industries use hedging instruments to reduce their exposure to volatile price and potentially rising costs.

In this study, we focus our attention to design a swap structure for retailers and customers to hedge against fluctuating gasoline prices. In the proposed swap structure, a retailer sells the gasoline to a swap provider at an average floating price set by retail gasoline market for a defined period of time and quantity, e.g. $\$g$/litre for 10,000 litres a month. In return, the swap provider sells it back to the retailer at a guaranteed price set by swap provider, e.g. $\$G$/litre for 10,000 litres a month. Therefore, the retailer gets the floating cash flow which will cover the floating cost, and may offer guaranteed prices to their customers.

This guaranteed price feature would give customers the possibility to hedge against increasing price movements. This would eliminate the possibility of having to pay higher price when market price rises above the guaranteed price, while still enjoying favourable price movements when market price falls below the guaranteed price, i.e., to buy gasoline at the minimum of market price and the guaranteed price. In other words, customers may buy gasoline at whichever offers them a lower price between market price (g), which is random, and guaranteed price (G), which is deterministic at a cost of the option premium; see the Figure 4.1 and Figure 4.2. In this way, not only customers but also retailers can reduce the risk of sales and profit fluctuations due to the price fluctuations.

Swap providers may enter into multiple swap transactions with multiple retailers and/or speculators with different preferences for particular prices, quantities, and terms. The swap provider may pay floating price in one transaction (to retailers) and pay guaranteed price in the other (to speculators) (Davies et al. (2004)). For these swap transactions to be feasible there must be price spreads between the two fixed prices for the swap provider's interest.

4.2.1 Retail Gasoline Price Guarantees

To enjoy the price guarantee benefits, customers first need to register a loyalty program at a participating gas station. With the loyalty program, the loyal customers

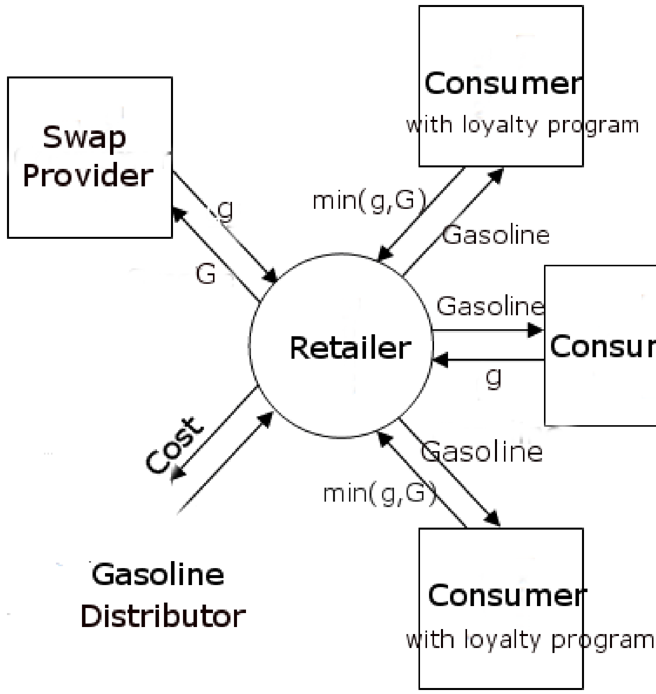


Figure 4.1: *Schematic diagram shows the cash flow between swap provider, retailer, and consumers with and without guaranteed price (G). Between the retailer and swap provider, there is only cash flow exchange, floating price (g) and guaranteed price (G), not the actual gasoline.*

can enter into a option contract with the retailer who gives them a right to buy gasoline at a guaranteed price (G) for a given period of time and for a desired quantity (e.g. 200 L for 1 month). Only loyal customers can enter into the contract, and pay a premium (P) upfront at the beginning of each month for an anticipated quantity of gasoline, e.g. 200 L for 1 month. The loyal customers can enjoy the guaranteed price (G) with the peace of mind which comes from hedging against rising price.

4.2.2 Outcomes of the Guaranteed Price

How would this swap impact the consumers' payoff, if the floating price were 20% higher or lower during the next month than the guaranteed price of \$1.3/L? Let's assume that a retailer anticipates to sell 500,000 L during the coming month, and wants to hedge against the fluctuating sales entering into the floating-to-fixed swap. Also,

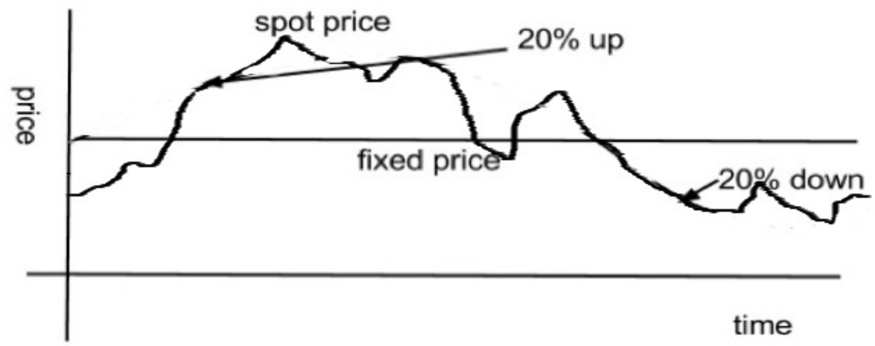


Figure 4.2: *Schematic diagram of floating gasoline price, which change up and down with time, and guaranteed price.*

assume that the premium for each consumer to enter into this contract is \$0.13/L for a desired quantity 200 L, which is aggregated to 500,000 L for 2500 consumers.

If the floating price falls below the guaranteed price by 20% during the next month, loyal customers would buy gasoline at floating prices; and the loss for the consumers on the floating-to-fixed swap is limited to just the premium paid, \$65,000 at aggregated-level for 500,000 L (200 L(-\$0.13/L) = \$26 at individual-level).
On the other hand, if the market price rises above the guaranteed price by 20% during the next month, the loyal customers will buy the gasoline at the guaranteed price; and the gain for the consumers on the floating-to-fixed swap is \$65,000 at aggregated-level for 500,000 L (200 L(\$1.56/L - \$1.3/L - \$0.13/L) = \$26 at individual-level). In this study, we mainly focus on the consumers' payoff.

4.3 Optimal Strategy under Uncertainty

Economic theory suggests that rational consumers would continue to search for a better price until the marginal cost of searching exceeds the marginal benefit. However, consumers are generally not informed of wholesale price, and tend to search more when retail price rises and less when price falls (Chandra and Tappata 2011). Since consumers search more when retail price rises, this lead retailers to lower their margins, and their prices would be less dispersed. On the other hand, when retail price falls retailers have little incentives to lower their margins, and their prices would be more dispersed. This feature is described as the gasoline price rising fast like rockets and falling slow like feathers in an atmosphere of local market power (Verlinda 2008). Retailers might decide to cooperate for high margin, or decide to compete for high sales as they would want to sell as much as possible when margin is high. When the retailers decides to compete for high sales, their sales would highly fluctuate depending on their relative prices against their competitors' prices. They also need to make sure they do not run out of fuel inventory. While the demand of an individual purchase is limited by the capacity of the gas tank, the supply is also limited with much larger underground tanks in the gas stations. For a very low-traffic and small gas stations which do not get deliveries as frequently as big companies, it might take days or a week to refill the underground tanks. Thus, while conserving the minimum level of tank, retailers need to sell as much as possible when the margin is high.

4.3.1 Optimal Quantity to Hedge

The guaranteed retail gasoline price could potentially stabilize fuel sales across stations and time. Furthermore, it could enable retailers to foresee the desired quantity from customers and manage fuel inventory better. In order to sell the contracted quantity of gasoline at a guaranteed price, retailers need to hedge some portion of their underground tank against rising price because they could just sell the unhedged portion when the price falls and sell the hedged portion when the price rises.

We first assume that loyal customers use a simple refueling strategy, and buy gasoline evenly over time regardless of price changes, and consider for instance a North American driver who drives about 13,000 miles evenly over a year (e.g. 15 gallon every 10 days, see Appendix B).

To minimize the total cost of hedging, retailers must decide how much gasoline to hedge. We find the optimal quantity of gasoline to hedge for retailers with simple models and assumptions. For simplicity, we only consider the portion of gasoline to be sold to customers participating in a loyalty program, not to regular customers.

Quantity to hedge: $q = \{0, 1, \cdots, Q\}$,

Reward: $r(q) = \frac{q}{T}\sum_{t=1}^{T}(\min(g_t, G) - G + P) + \frac{Q-q}{T}\sum_{t=1}^{T}(\min(g_t, G) - g_t + P)$

rewrite as

$$r(q) = \frac{Q}{T}\sum_{t=1}^{T}(\min(g_t, G) - g_t + P) - \frac{q}{T}\sum_{t=1}^{T}(G - g_t)$$
$$r^*(q) = \max_q(\frac{Q}{T}\sum_{t=1}^{T}(\min(g_t, G) - g_t + P) - \frac{q}{T}\sum_{t=1}^{T}(G - g_t))$$

where

Q: an average anticipated quantity of gasoline to be sold to loyal customers over a month

q: a quantity of gasoline for retailers to hedge against rising price over a month

t: current time

T: a maturity time of the contract

P: a premium for the loyalty program.

g_t: floating gasoline price

G: guaranteed gasoline price

To find the optimal quantity (q) to hedge, assume this decision has to be made every first day of month, taking the first order partial derivative with respect to q, we get:

$\frac{dr(q)}{dq} = \frac{1}{T}\sum_{t=1}^{T}(g_t - G)$

Optimal quantity to hedge:

if $\frac{1}{T}\sum_{t=1}^{T}(g_t - G) \geq 0$ $\qquad q^* = \arg\max_q r(q) = Q$

if $\frac{1}{T}\sum_{t=1}^{T}(g_t - G) < 0$ $\qquad q^* = \arg\max_q r(q) = 0$

Then, the optimal quantity for retailers to hedge would be;

$q^* = P(\frac{1}{T}\sum_{t=1}^{T}(g_t - G) \geq 0)Q = P(E[g_t] \geq G)Q$

For instance, with equal chance of $E[g_t] \geq G$ or $E[g_t] < G$

$q^* = \frac{1}{2}Q$

Next, we assume that loyal customers use a quasi-optimal refueling strategy, and buy minimum (10% of full tank) only if the tank is low ($k = 1$), when $g \geq G$ and fill up when $g < G$.

Optimal quantity to hedge:

if $\frac{1}{T}\sum_{t=1}^{T}(g_t - G) \geq 0$ $\qquad q^* = \arg\max_q r(q) = Q$

if $\frac{1}{T}\sum_{t=1}^{T}(g_t - G) < 0$ $\qquad q^* = \arg\max_q r(q) = 0$

Then, the optimal quantity for retailers to hedge would be;

$q^* = 0.1P(k = 1|E[g_t] \geq G)Q$ (10% of full tank)

Thus, under the quasi-optimal refueling strategy, the retailers should hedge a smaller quantity than under the simple strategy where gasoline is purchased evenly over time.

Another approach to solving the above problem is by minimizing the variance of the hedged position as below;

$r(q) = \frac{Q}{T}\sum_{t=1}^{T}(\min(g_t, G) - g_t + P) - \frac{q}{T}\sum_{t=1}^{T}(G - g_t)$

let $r_A = \frac{1}{T}\sum_{t=1}^{T}(\min(g_t, G) - g_t + P)$ and $r_B = \frac{1}{T}\sum_{t=1}^{T}(G - g_t)$, rewrite as

$r(q) = Qr_A - qr_B$

r_A is the average return rate without hedging and r_B is the average return rate from the floating-to-fixed rate swap. We define the variance of r_A as σ_A^2; the variance of r_B as σ_B^2; and their correlation coefficient as ρ.

As a consequence, the variance of $r(q)$ is: $\mathrm{var}(Qr_A - qr_B) = Q^2\sigma_A^2 - 2\rho Qq\sigma_A\sigma_B + q^2\sigma_B^2$

In order to minimize the hedging cost we find the optimal quantity(q^*) to hedge, taking the first order partial derivative of the variance of $r(q)$ with respect to q, we get the same results as above:

$$\frac{\delta r(q)}{\delta q} = -2\rho Q\sigma_A\sigma_B + 2q\sigma_B^2 = 0$$

$$q* = \frac{Q\rho\sigma_A}{\sigma_B}$$

For special cases:

if $g_t < G$ for all t $\quad q^* = \frac{Q\rho\sigma_A}{\sigma_B} = 0$ when $\sigma_A = 0$ and $\rho = 0$.

if $g_t \geq G$ for all t $\quad q^* = Q\rho = Q$ when $\sigma_A = \sigma_B$ and $\rho = 1$.

For general cases, $q^* = \frac{Q\rho\sigma_A}{\sigma_B}$ when $\sigma_A \neq \sigma_B > 0$ and $0 < \rho < 1$.

In this chapter, we introduced how a retail gasoline call option-like guaranteed price can be created using the swap structure. We also briefly discussed the expected outcomes of the guaranteed price and optimal strategy under price uncertainty. In the following chapter, we will discuss optimal exercise of the guaranteed price by retail consumers and examine the optimal refueling time and quantity without and with the loyalty program.

Bibliography

Acharya, Viral V. and Lochstoer, Lars A. and Ramadorai, Tarun (2013), Limits to Arbitrage and Hedging: Evidence from Commodity Markets, *Journal of Financial Economics*, Vol. 109, No. 2, pp.441-465.

Chandra, A. and Tappata, M. (2011), Consumer search and dynamic price dispersion: an application to gasoline markets, *RAND Journal of Economics*, Vol. 42, No. 4, pp.681-704.

Deltas, G. (2008), Retail Gasoline Price Dynamics and Local Market Power, *The Journal of Industrial Economics*, Vol. 56, No. 3, pp.613-628 .

Dubofsky, David A., and Thomas W. Miller Jr. (2003), *Derivatives : valuation and risk management*. Oxford University Press, New York ; Oxford

Jin, Y. and Jorion, P (2006), Firm Value and Hedging: Evidence from U.S. Oil and Gas Producers, *The Journal of Finance*, Vol. 61, No. 2, pp.893-919.

Karlin, S., and Taylor, H. M. (1975). A rst course in stochastic processes. Academic Press, New York.

Morrell, Peter and Swan, William (2006), Airline Jet Fuel Hedging: Theory and practice, *Transport Reviews*, Vol. 26, No. 6, pp.713-730.

Natural Resources Canada (2013), Understanding Gasoline Markets in Canada and Economic Drivers Influencing Prices, Vol. 11, No. 8, pp.1-7.

Davies, R. L and Stephen A. Spitz (2004), Interest Rate Swaps: Application to Tax Exempt Financing, Orrick, Herrington & Sutcliffe. Retrieved from:https://www.orrick.com/Generic-Articles/∼/media/A5B8684FD24347E58AF86201F6FEFEFF .ashx Date retrieved: 5 October 2014.

Verlinda, Jeremy A. (2008), Do rockets rise faster and feathers fall slower in an atmosphere of local market power? Evidence from the retail gasoline market, *The Journal of Industrial Economics*, Vol. 56, No. 3, pp.581-612.

Vieira, B. L. S., Alberto, J. G. C., Oliveira, S. L., Bonaldo, E., Ribeiro, J. L. S., and Moreira, A. (2014). Does Hedging Make Economic Sense for American Airlines? Paper presented at XXXVIII Enconro Da ANPAD, 13-17 September 2014. 1-13. Retrieved from: http://www.anpad.org.br/admin/pdf/2014-EnANPAD-FIN681.pdf. Date retrieved: 15 October 2014.

Westbrooks, C. Lester (2005), Airline Fuel Hedging: An Overview of Hedging Solutions Available to Airlines, *Journal of Aviation/Aerospace Education & Research*, Vol. 14, No. 2, pp.19-25.

Chapter 5

Optimal Refueling Strategy

5.1 Introduction

In this chapter we quantify the value of the loyalty program proposed in Chapter 4. We begin by deriving and solving the optimal refueling strategy without loyalty program in Section 5.2. Next we prove that there exists a non-trivial price threshold and characterize the optimal control for the refueling cost with loyalty program in Section 5.3. In Section 5.4, we present numerical methodology that was used to find the price threshold and the value of the loyalty program. In Section 5.5, we present the numerical results using simulated prices, following the optimal policy (refueling only when the floating price is below the optimal decision boundary, or when running out of fuel is imminent, otherwise waiting), and compare the price thresholds and the value of loyalty program for different parameters and states.

5.2 Optimal Control without Loyalty Program

The purpose of this section is to study the consumers optimal refueling strategy when there is no price guarantee. Based on simple, yet general, price models and some natural assumptions about the type of strategies available to retail gasoline purchasers, we solve the consumer's optimization problem using dynamic programming.

We first assume consumers refuel at most once a day, the opportunity cost of having to visit the station at most once a day is negligible, and that the quantity used on each day is constant. Based on normal day to day living (i.e. not while on a road trip), the assumption of refueling at most once a day is reasonable. While quantities of gas consumed each day are, for most drivers, somewhat variable, for a commuter traveling say 50 km to and another 50 km from work, the variability around the daily average may be rather small. Let the amount of fuel used each day be one unit and the price also be quoted in terms of that single unit, e.g., a commuter using 10 liters daily; 1 unit equals to 10 liters.

Although the gas tank can hold any continuous volume, we consider a discrete tank of K units of fuel. The amount of fuel that can be purchased at each time is an integer multiple of the fuel units. The tank level cannot go below 1, the minimum tank reserve level to avoid engine damage, or above K, the maximum tank capacity.

We formulate the refueling problem as a Markov Decision Process as follows;

Time Horizon: we consider a finite time horizon with T discrete periods (i.e. $T = 30$ for a month)

Decision Stages: $t = \{1, 2, \cdots, T\}$. We assume decision to refuel happens at the end of each day, and $T - t$ is stages-to-go (or number of days-to-go).

State: $s = (k, g_t)$, where k is the discrete tank level at the end of day, $k \in \{1, \cdots, K\}$, the maximum tank level is K, the minimum tank reserve level is 1, and g_t is the gasoline price at the end of day t, $g_t \in R^+, t = 1, 2, \cdots, T$.

Actions: set of permissible decision variable on how much gasoline to fill up at each stage

$$A = \begin{cases} \{0, 1, \cdots, K-k\} \text{ if } k>1, \\ \{1, \cdots, K-1\} \text{ if } k = 1. \end{cases}$$

Refueling Cost: $r_t(s, a) = a \times g_t$ cost to fill a quantity of gasoline without the loyalty program, $a \in A$

Terminal Cost: $r_T(s) = (K - k) \times g_T$, assume at the end of period, must fill up. This corresponds to the requirement that the driver must fill the tank at the end of the period.

Transitional Probability Function: The transition of the tank level state variable is deterministic and defined by $k_{t+1} = k_t - 1 + a$. The gasoline price state variable is stochastic and follows a transitional probability function, $f(g_{t+1})$, where $g_{t+1} \in R^+$ is the next day's gasoline prices. Typically this probability will be conditional on today's price, $f(g_{t+1}|g_t)$.

Remark 5.2.1. *Given that we are implicitly considering a short time horizon like a daily time increment, we will ignore the discount factor.*

The consumer's objective is to minimize the total expected cost of refueling. The optimality equation for the optimization problem follows for $t < T$:

$$\begin{aligned} V_t(s) &= \inf_{a \in A}\{r(s,a) + \int_0^\infty V_{t+1}(s')f(g_{t+1})dg_{t+1}\} \\ &= \inf_{a \in A}\{g_t a + \int_0^\infty V_{t+1}(s')f(g_{t+1})dg_{t+1}\} \end{aligned}$$

with terminal condition: $V_T(s) = g_T(K - k_T)$ and $s' = (k_t - 1 + a, g_{t+1})$

This can be written as,

$$V_t(s) = g_t a^* + \int_0^\infty V_{t+1}(s')f(g_{t+1})dg_{t+1} \tag{5.1}$$

where $a^* = \arg\min_{a \in A}\{r(s,a) + \int_0^\infty V_{t+1}(s')f(g_{t+1})dg_{t+1}\}$

We now provide a result for a special case in which gasoline price moves are expected to be constant.

Proposition 5.2.2. *Let the expected gasoline price change be constant and non-negative,* $E[g_{t+1} - g_t|g_t] = \Delta \geq 0, t = 1, 2, \cdots, T$.

The optimal refueling is to fill up daily, i.e. $a^* = K - k_t$ *for* $t \leq T$. *This results in the following optimality equation;*

$$V_t(s) = g_t(K - k_t + (T - t)) + \frac{(T-t)(T-t+1)}{2}E[g_{t+1} - g_t|g_t] \tag{5.2}$$

Proof. Proposition 5.2.2

From the optimality equation (5.1), we have

for $t = T-1$ (for one stage-to-go),

$$\begin{aligned}
V_{T-1}(s) &= g_t a^* + \int_0^\infty V_T(s' = (k_t - 1 + a^*, g_{t+1})) f(g_{t+1}) dg_{t+1} \\
&= g_t a^* + \int_0^\infty (g_{t+1})(K - k_t + 1 - a^*) f(g_{t+1}) dg_{t+1} \\
&= g_t a^* + g_t(K - k_t + 1 - a^*) + (K - k_t + 1 - a^*) \int_0^\infty (g_{t+1} - g_t) f(g_{t+1}) dg_{t+1} \\
&= g_t a^* + g_t(K - k_t + 1 - a^*) + (K - k_t + 1 - a^*) E[g_{t+1} - g_t | g_t]
\end{aligned}$$

$$V_{T-1}(s) = -a^* E[g_{t+1} - g_t | g_t] + (K - k_t + 1)(g_t + E[g_{t+1} - g_t | g_t]) \tag{5.3}$$

If $E[g_{t+1} - g_t | g_t] \geq 0$, the optimality equation to be minimized is a linear function of a^* with a negative slope. Hence, the minimum is achieved at $a^* = K - k_t$.

Substituting $a^* = K - k_t$ into equation (5.3), we have

$$\begin{aligned}
V_{T-1}(s) &= -(K - k_t) E[g_{t+1} - g_t | g_t] + (K - k_t + 1)(g_t + E[g_{t+1} - g_t | g_t]) \\
&= g_t(K - k_t + 1) + E[g_{t+1} - g_t | g_t] \\
V_{T-1}(s) &= g_t(K - k_t + (T - t)) + \frac{(T-t)(T-t+1)}{2} E[g_{t+1} - g_t | g_t]
\end{aligned}$$

Thus, the solution of equation (5.3) satisfies (5.2) and so Proposition 5.2.2 is true for $t = T - 1$.

Now assume that proposition 5.2.2 holds for all $t = j+1, j+2, \cdots, T-1$.

for $t = j$, from the optimality equation (5.1), we have

$$\begin{aligned}
V_j(s) &= \inf_{a \in A} \left\{ g_j a + \int_0^\infty V_{j+1}(s') f(g_{j+1}|g_j) dg_{j+1} \right\} \\
V_j(s) &= g_j a^* + \int_0^\infty \Big((g_{j+1})(K - k_j + 1 - a^*) \\
&\quad + (T - j - 1)(g_{j+1}) + \frac{(T-j-1)(T-j)}{2} E[g_{j+1} - g_j|g_j] \Big) f(g_{j+1}|g_j) dg_{j+1} \\
&= g_j a^* + (K - k_j + 1 - a^*)(g_j + E[g_{j+1} - g_j|g_j]) + (T - j - 1)(g_j + E[g_{j+1} - g_j|g_j]) \\
&\quad + \frac{(T-j-1)(T-j)}{2} E[g_{j+1} - g_j|g_j] \\
&= -a^* E[g_{j+1} - g_j|g_j] + (K - k_j + 1)(g_j + E[g_{j+1} - g_j|g_j]) \\
&\quad + (T - j - 1)(g_j + E[g_{j+1} - g_j|g_j]) + \frac{(T-j-1)(T-j)}{2} E[g_{j+1} - g_j|g_j]
\end{aligned}$$

$$\begin{aligned}
V_j(s) &= -a^* E[g_{j+1} - g_j|g_j] + (K - k_j + 1)(g_j + E[g_{j+1} - g_j|g_j]) \\
&\quad + (T - j - 1)(g_j + E[g_{j+1} - g_j|g_j]) + \frac{(T-j-1)(T-j)}{2} E[g_{j+1} - g_j|g_j]
\end{aligned} \tag{5.4}$$

If $E[g_{j+1} - g_j|g_j] \geq 0$, it is linear function of a^* with a negative slope. Hence, the minimum is achieved at $a^* = K - k_t$.

Substituting $a^* = K - k_t$ into equation (5.4), we have

$$\begin{aligned}
V_j(s) &= -(K - k_j) E[g_{j+1} - g_j|g_j] + (K - k_j + 1)(g + E[g_{j+1} - g_j|g_j]) \\
&\quad + (T - j - 1)(g_j + E[g_{j+1} - g_j|g_j]) + \frac{(T-j-1)(T-j)}{2} E[g_{j+1} - g_j|g_j] \\
&= g_j(K - k_j) + g_j + E[g_{j+1} - g_j|g_j] + (T - j - 1)(g_j + E[g_{j+1} - g_j|g_j]) \\
&\quad + \frac{(T-j-1)(T-j)}{2} E[g_{j+1} - g_j|g_j] \\
V_j(s) &= g_j(K - k_j + (T - j)) + \frac{(T-j)(T-j+1)}{2} E[g_{j+1} - g_j|g_j]
\end{aligned}$$

Thus, it has been proved that the Proposition 5.2.2 is true for all t. $\square$

Proposition 5.2.3. *Let the expected gasoline price change be constant and negative,* $E[g_{t+1} - g_t|g_t] = \Delta < 0, t = 1, 2, \cdots, T$.

- *If* $k_t > 1$, *then* $a^* = 0$ *for* $t < T$, *and*

$$V_t(s) = (K - k_t + (T - t))(g_t + (T - t)E[g_{t+1} - g_t|g_t]) \tag{5.5}$$

- *If* $k_t = 1$, *then* $a^* = 1$ *for* $t < T$, *and*

$$V_t(s) = -(T-t)E[g_{t+1}-g_t|g_t]+(K-k_t+(T-t))(g_t+(T-t)E[g_{t+1}-g_t|g_t]) \tag{5.6}$$

Proof. Proposition 5.2.3

From equation (5.3), we have for $t = T - 1$, $V_t(s) = -a^*E[g_{t+1} - g_t|g_t] + (K - k_t + 1)(g_t + E[g_{t+1} - g_t|g_t])$. If $E[g_{t+1} - g_t|g_t] < 0$, the optimality equation to be minimized is linear function of a^* with a positive slope. Hence, the minimum is achieved at $a^* = 0$ or $a^* = 1$.

If $E[g_{t+1} - g_t|g_t] < 0$ and $k_t > 1$, $a^* = 0$ for $t < T$, substituting $a^* = 0$ into (5.3) we have, $V_{T-1}(s) = (K - k_t + 1)(g_t + E[g_{t+1} - g_t|g_t])$. If $E[g_{t+1} - g_t|g_t] < 0$ and $k_t = 1$, $a^* = 1$ for $t < T$, substituting $a^* = 1$ into (5.3) we have $V_{T-1}(s) = -E[g_{t+1} - g_t|g_t] + (K - k_t + 1)(g_t + E[g_{t+1} - g_t|g_t])$. Thus, the above solutions of equation (5.3) satisfy equation (5.5) and (5.6) for $t = T - 1$, and Proposition 5.2.3 is true for $t = T - 1$.

Now assume that proposition 5.2.3 holds for all $t = j + 1, j + 2, \cdots, T - 1$. For $t = j$, from the optimality equation (5.1), we have that

$V_j(s) = \inf_{a\in A} \left\{ g_j a + \int_0^\infty V_{j+1}(s')f(g_{j+1}|g_j)dg_{j+1} \right\}$

If $E[g_{t+1} - g_t|g_t] < 0$ and $k_t > 1$, using the above results, we get

$$\begin{aligned}
V_j(s) &= g_j a^* + \int_0^\infty \Big(K - k_j + (T-j) - a^*)(g_{j+1} + (T-j-1)E[g_{j+1} - g_j|g_j]\Big) f(g_{j+1}|g_j) dg_{j+1} \\
&= g_j a^* + (K - k_j + (T-j) - a^*)(g_j + (T-j-1)E[g_{j+1} - g_j|g_j]) \\
&+ (K - k_j + (T-j) - a^*) \int_0^\infty (g_{j+1} - g_j) f(g_{j+1}|g_j) dg_{j+1} \\
&= -a^*(T-j)E[g_{j+1} - g_j|g_j] + (K - k_j + (T-j))(g_j + (T-j)E[g_{j+1} - g_j|g_j])
\end{aligned}$$

$$V_j(s) = -a^*(T-j)E[g_{j+1} - g_j|g_j] + (K - k_j + (T-j))(g_j + (T-j)E[g_{j+1} - g_j|g_j]) \quad (5.7)$$

If $E[g_{t+1} - g_t|g_t] < 0$ and $k_t > 1$, it is a linear function of a^* with a positive slope. This is minimized when a^* is as small as possible, hence $a^* = 0$ for $t = j$.

$$V_j(s) = (K - k_j + (T-j))(g_j + (T-j)E[g_{j+1} - g_j|g_j])$$

Similarly, if $E[g_{t+1} - g_t|g_t] < 0$ and $k_t = 1$, the minimum is achieved at $a^* = 1$ for $t = j$.

$$V_j(s) = -(T-j)E[g_{j+1} - g_j|g_j] + (K - k_j + (T-j))(g_j + (T-j)E[g_{j+1} - g_j|g_j])$$

Thus, it has been proved that the proposition 5.2.3 is true for all t.

□

Illustration 5.2.4. *Arithmetic Brownian Motion (ABM)*

The dynamic price changes can be modeled using different price processes, each with advantages and disadvantages. The simpler processes like Brownian motion may miss out some desired characteristic, but are easier to interpret and calibrate from market prices. More complicated models like Ornstein Uhlenbeck Process or jump diffusions process may better characterise the behaviour of certain commodities, but require more parameters to be estimated with higher probability of model errors. Since retail gasoline price does not experience extreme price changes, it is reasonable to illustrate the price process using Arithmetic Brownian Motion (ABM).

ABM price process is g_t such that $dg_t = \mu dt + \sigma dZ_t$ where both μ and σ are constant. Let $\Delta_t = g_t - g_0$ denote the total price increment, then $\Delta_t = \mu t + \sigma Z_t$. Since $Z_t \sim N(0, t)$, and $\Delta_t \sim N(\mu t, \sigma^2 t)$, the normal density function $N(\Delta_t; \mu t, \sigma^2 t)$ is given as;

$$N(\Delta_t; \mu t, \sigma^2 t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{-\left(\Delta_t - \mu t\right)^2}{2\sigma^2 t}}$$

where the expected value and variance given by $E[\Delta_t] = \mu t$ and $Var(\Delta_t) = \sigma^2 t$, respectively.

Note that, using ABM, technically prices can go negative for, but would be very unlikely for a sufficiently large positive g_0 and μ, and for a relatively small t and σ.

Let Δ be the conditional price increment, given g_t, written as;

$$\Delta = E[g_{t+1} - g_t|g_t] = E[g_{t+1}|g_t] - E[g_t|g_t] = \mu$$

From (5.4), for $E[g_{t+1} - g_t|g_t] = \mu \geq 0$

$V_t(s) = -a^* E[g_{t+1} - g_t|g_t] + (K - k_t + 1)(g_t + E[g_{t+1} - g_t|g_t]) + (T - t - 1)(g_t + E[g_{t+1} - g_t|g_t]) + \frac{(T-t-1)(T-t)}{2} E[g_{t+1} - g_t|g_t]$

From (5.7), for $E[g_{t+1} - g_t | g_t] = \mu < 0$,

$V_t(s) = -a^*(T-t)E[g_{t+1} - g_t|g_t] + (K - k_t + (T-t))(g_t + (T-t)E[g_{t+1} - g_t|g_t])$

We have shown the optimality equations have a negative relationship with $a^*(T-t)E[g_{t+1} - g_t|g_t]$. As a consequence, the optimal solution is to make a as large as possible; $a^* = K - k_t$, if $E[g_{t+1} - g_t|g_t] \geq 0$. On the other hand, if $E[g_{t+1} - g_t|g_t] < 0$ then make a as small as possible; $a^* = 0$, if $k_t > 1$, $a^* = 1$ if $k_t = 1$.

With the expected price increment of $E[g_{t+1} - g_t|g_t] = \mu$, for $t = 1, 2, \cdots, T-1$

1) If $\mu \geq 0$, then $a^* = K - k_t$

2) If $\mu < 0$, then $a^* = 0$ if $k_t > 1$

3) If $\mu < 0$, then $a^* = 1$ if $k_t = 1$

The illustrations with ABM price process imply that people should fill up their car daily when price is expected to rise and wait when price is expected to fall. The reason we might not see this in real life, instead seeing people just fill up their car once it is empty is because there is a hidden cost to fill up (time, convenience, etc). We could potentially include an additional fixed cost for each time you fill up, but for simplicity we ignore this cost.

Although this example is useful to build insights, the model is flawed as an ABM model for gas prices allows the gas price to take on negative values. As a result, in what follows we use more special case in which g_{t+1} is a non-negative i.i.d. r.v.s.

5.3 Optimal Control with Loyalty Program

We now consider the optimal refueling strategy with the loyalty program described in Chapter 4. We first discuss the expected price overrun and the premium associated with the current period's gasoline price, then we show that there exist a non-trivial $\hat{g}_t$ at which the expected price overrun is zero and the current period's gasoline price g_t is equal to the expected future price (above which the current period's gasoline price g_t is higher than the expected future fueling price; vice versa below $\hat{g}_t$ the current gasoline price g_t is lower than the expected future fueling price) for $t = \{1, 2, \cdots, T\}$. Finally, we solve the optimization problem based on some assumptions.

5.3.1 Existence of Price Threshold

Definition 5.3.1. *We define the expected price overrun, $c_t(g_t)$, as the current price per unit volume minus the expected price of refueling a unit of gasoline in the future. Mathematically, we denote this by,*

$c_t(g_t) = g_t - N_t(g_t)$

the expected future fueling price per unit volume, $N_t(g_t)$, is given recursively by;

$N_t(g_t) = \int_0^\infty \min(g_{t+1}, N_{t+1}(g_{t+1}))f(g_{t+1})dg_{t+1}$

the expected fueling price per unit volume at the final time horizon is given by;

$N_T(g) = \min(g, G)$ *where G is a guaranteed price*

Remark 5.3.2. *Assumptions are needed to show $c_t(g_t)$ has a non-trivial root, $\hat{g}_t$.*

1) Gasoline price can rise or fall with nonzero probability except at $g_t = 0$; $P(g_{t+1} > g_t) > 0$ and $P(g_{t+1} < g_t) > 0$, $g_t \in (0, \infty), \forall t$, this is reasonable assumption for various price processes, and some price processes require that $g_{t+1} = 0$ when $g_t = 0$.

2) If $g_1 > g_2$, then $N_t(g_1) \geq N_t(g_2)$. This is a reasonable assumption and, for differentiable functions $N_t(g)$, corresponds to the assumption that $\frac{dN_t}{dg} > 0$.

Lemma 5.3.3. *The expected refueling price per unit volume in the future has the following properties;*

1) $N_t(g) \leq G$ *and* $N_{t+1}(g) \leq G \quad t = T-1, \forall g$

2) $N_t(g) \leq N_{t+1}(g) \leq G \quad t < T-1, \forall g$

where $N_t(g) = \int_0^\infty \min(g', N_{t+1}(g')) f(g'|g) dg' = E[\min(g', N_{t+1}(g'))|g]$

Proof. for t=T-1 and for all g

$$\begin{aligned}
G - N_{T-1}(g) &= G - E[\min(g', N_T(g'))|g] \quad \textbf{(since } N_{T-1}(g) = E[\min(g', N_T(g'))|g] \textbf{)} \\
&= G - E[\min(g', \min(g', G))|g] \quad \textbf{(since } N_T(g) = \min(g, G) \textbf{ thus } N_T(g) \leq G \textbf{)} \\
&= G - E[\min(g', G)|g] \quad \textbf{(since } \min(g', \min(g', G)) = \min(g', G) \textbf{)} \\
&= G - E[\min(g', G)|g] \quad \textbf{(if } g \geq G\textbf{, } \min(g, G) = G\textbf{)} \\
&= -E[\min(g' - G, 0)|g] \quad \textbf{(since } \min(x, 0) \textbf{ is less than equal to zero)} \\
&\geq 0
\end{aligned}$$

Thus, $G \geq N_{T-1}(g)$ and $G \geq N_T(g)$ for $t = T-1, \forall g$

for t=T-2 and for all g, note that $N_t(g) = E[\min(g', N_{t+1}(g'))|g]$,

$$\begin{aligned}
N_{T-1}(g) - N_{T-2}(g) &= E[\min(g', N_T(g'))|g] - E[\min(g', N_{T-1}(g'))|g] \\
&= E[\min(g', N_T(g')) - \min(g', N_{T-1}(g'))|g] \\
&= E[\min(g', G) - \min(g', N_{T-1}(g'))|g] \\
&\geq 0 \quad (\text{if } G \geq N_{T-1}(g') \text{ for all } g)
\end{aligned}$$

$$\begin{aligned}
\big(\min(g', G) - \min(g', N_{T-1}(g')) &= g' - g' = 0 \quad \text{if} \quad G \geq N_{T-1}(g') > g' \\
\min(g', G) - \min(g', N_{T-1}(g')) &= g' - N_{T-1}(g') > 0 \quad \text{if} \quad G \geq g' > N_{T-1}(g') \\
\min(g', G) - \min(g', N_{T-1}(g')) &= g' - N_{T-1}(g') \geq 0 \quad \text{if} \quad g' > G \geq N_{T-1}(g')\big)
\end{aligned}$$

Thus, $E[\min(g', G) - \min(g', N_{T-1}(g'))] \geq 0$ if $G \geq N_{T-1}(g')$, and $E[\min(g', N_T(g'))|g] \geq E[\min(g', N_{T-1}(g'))|g]$ and $N_{T-1}(g) \geq N_{T-2}(g) \quad \forall g$

The desired result, $N_t(g) \leq N_{t+1}(g)$, holds true for $t = T-1$ and $t = T-2$.

To finish the proof by induction, we show that $N_j(g) \geq N_{j-1}(g)$ if $N_{j+1}(g) \geq N_j(g)$. for all $j < T-1$ and for all g

$$\begin{aligned}
N_j(g) - N_{j-1}(g) &= E[\min(g', N_{j+1}(g'))|g] - E[\min(g', N_j(g'))|g] \\
&= E[\min(g', N_{j+1}(g')) - \min(g', N_j(g'))|g] \\
&\text{(But, } N_{j+1}(g') \geq N_j(g') \text{ for all } g' \text{ so this can never be negative)} \\
&\geq 0
\end{aligned}$$

Thus, the relation holds true for all t. We have proved Lemma 5.3.3, and $N_t(g) \leq N_{t+1}(g) \leq G \quad \forall t < T-1, g$.

$\square$

This can be explained intuitively, as there is no state of nature in which we would ever pay more than G for a unit of gas, and many in which we would pay less. Next,

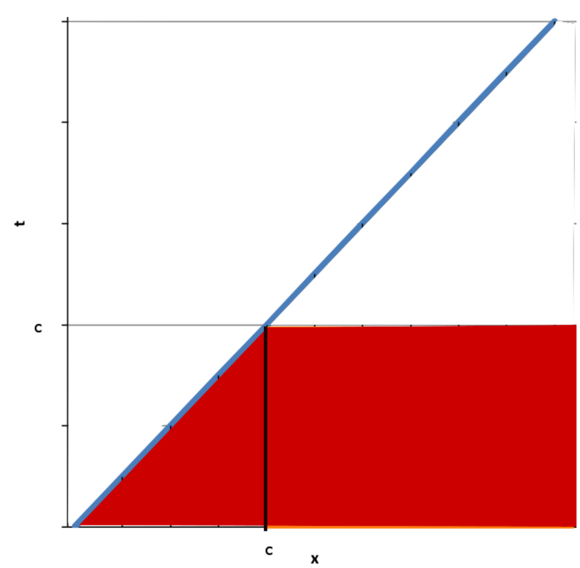


Figure 5.1: *Illustrative Diagram of the Region of Integration for the Order Change*

we will prove that there exist a non-trivial price threshold, which is the main result of this Chapter. To prove that we require the following Lemma 5.3.4.

Lemma 5.3.4. *From Karlin and Taylor (1975); Let X be a non-negative random variable with cumulative distribution function $F(x) = \Pr(X \leq x)$, $x \geq 0$, and $X_c = \min(X, c)$ where c is a given constant, then, $E[X_c] = \int_0^c [1 - F(x)]dx$.*

Proof. Lemma 5.3.4: We have

$$\int_0^c \Big[1 - F(t)\Big]dt = \int_0^c \Big[\Pr(X > t)\Big]dt = \int_0^c \Big(\int_t^\infty f(x)dx\Big)dt$$

(the region of integration is coloured in red, see Figure 5.1)

$t \leq x \leq \infty, 0 \leq t \leq c$

(since x can be split into $0 \leq x \leq c$ and $c \leq x \leq \infty$)

$0 \leq t \leq x \quad \text{for} \quad 0 \leq x \leq c, \quad \text{and} \quad 0 \leq t \leq c \quad \text{for} \quad c \leq x \leq \infty$

The integral with the order changed is then,

$$\begin{aligned}
\int_0^c \Big[1 - F(t)\Big]dt &= \int_0^c \Big(\int_0^x dt\Big)f(x)dx + \int_c^\infty \Big(\int_0^c dt\Big)f(x)dx \\
&= \int_0^c xf(x)dx + \int_c^\infty cf(x)dx \\
&= \int_0^\infty \min(x, c)f(x)dx = E[X_c]
\end{aligned}$$

□

Note this proof can easily be modified to apply to cumulative conditional distribution functions simply by writing $F(x|y) = \Pr(X \leq x|y)$ throughout.

Lemma 5.3.5. *Let $c_t(g_t) = g_t - N_t(g_t)$ from definition 5.3.1, assume $c_t(g_t)$ and $N_t(g_t)$ are continuous and twice differentiable, then $\forall t < T$, there exists a $\hat{g}_t$, such that 1) $c_t(\hat{g}_t) = 0$; 2) $c_t(g_t) < 0 \quad \forall g_t < \hat{g}_t$; 3) $c_t(g_t) > 0 \quad \forall g_t > \hat{g}_t$.*

Remark 5.3.6. *Lemma 5.3.5 implies that there exists a non-trivial 'price threshold', $\hat{g}_t$, below which it is optimal to buy gas, and above which it is optimal to wait.*

Proof. Lemma 5.3.5, there are two cases 1) $c_t(0) < 0$ and $c_t(G) > 0$, and 2) $c_t(0) = 0$ and $c_t(G) > 0$ for all t. In the first case, from the intermediate Value Theorem, it is sufficient to show that $c_t(0) < 0$ and $c_t(G) > 0$, then there exist non-trivial value $\hat{g}_t$ for which $c_t(\hat{g}_t) = 0$. In the second case, $c_t(0) = 0$ and $c_t(G) > 0$, we also need to show $\frac{dc_t(0)}{dg} \leq 0$ for all g and t, then there is a non-trivial root $\hat{g}_t$ by basic calculus and continuity of $c_t(g_t)$.

For above two cases, from the definition 5.3.1, for $t = T - 1$, we have

$$\begin{aligned} c_t(g_t) &= g_t - N_t(g_t) \\ &= g_t - \int_0^\infty \min(g_{t+1}, G) f(g_{t+1}) dg_{t+1} \quad \textbf{(from the definition 5.3.1)} \\ &= g_t - \int_0^G [1 - F(g_{t+1})] dg_{t+1} \quad \textbf{(applying Lemma 5.3.4 to i.i.d. r.v.s.)} \\ &= g_t - \int_0^G dg_{t+1} + \int_0^G F(g_{t+1}) dg_{t+1} \end{aligned}$$

Thus, $c_t(g_t) = g_t - G + \int_0^G F(g_{t+1}) dg_{t+1}$,

$c_t(0) = 0 - G + \int_0^G F(g_{t+1}) dg_{t+1} \leq 0$, this is clearly true since $\int_0^G F(g_{t+1}) dg_{t+1} \leq G$, and $c_t(0) < 0$ if $\int_0^G F(g_{t+1}) dg_{t+1} < G$ with some probability $g_{t+1} > G$ when $g_t = 0$.

$c_t(G) = G - G + \int_0^G F(g_{t+1}) dg_{t+1} = \int_0^G F(g_{t+1}) dg_{t+1} > 0$, this is true with some probability $g_{t+1} \leq G$ when $g_t = G$, and this is very reasonable as any random price process, with non-zero probability, can rise or fall as discussed in Remark 5.3.6.

Thus, for $t = T - 1$ $c_t(0) \leq 0$, and $c_t(G) > 0$. From the Intermediate value theorem, if $c_t(0) < 0$, $c_t(G) > 0$, then we are guaranteed an interior point root, but if $c_t(0) = 0$ and $c_t(G) > 0$ for many classes of probability density function, require the condition of $\frac{dc_t(0)}{dg} \leq 0$ for an interior point root, see Remark 5.3.6 for the proof.

Assume the above statement is true for all $t = j+1, j+2, \cdots, T-2$.

We prove that $c_t(0) \leq 0$, and $c_t(G) > 0$ for $t = j$.

From the definition 5.3.1, for $t = j$, we have

$$
\begin{aligned}
c_t(g_t) &= g_t - N_t(g_t) \\
&= g_t - \int_0^{\hat{g}_{t+1}} g_{t+1} f(g_{t+1}) dg_{t+1} - \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1}
\end{aligned}
$$

for $\quad g_t = 0$

$$c_t(0) = 0 - \int_0^{\hat{g}_{t+1}} g_{t+1} f(g_{t+1}) dg_{t+1} - \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1}$$

$$c_t(0) \leq 0$$

for $\quad g_t = G$

$$
\begin{aligned}
c_t(G) &= G - \int_0^{\hat{g}_{t+1}} g_{t+1} f(g_{t+1}) dg_{t+1} - \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1} \\
&> G - \int_0^{\hat{g}_{t+1}} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1} - \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1}
\end{aligned}
$$

Because $g_{t+1} < N_{t+1}(g_{t+1})$ for $g_{t+1} \in [0, \hat{g}_{t+1})$

$$
\begin{aligned}
c_t(G) &> G - \int_0^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1} \\
&> G - \int_0^{\infty} G f(g_{t+1}) dg_{t+1} \\
&\text{(because } N_{t+1}(g_{t+1}) \leq G \text{ for all } t \text{ from Lemma 5.3.3)} \\
&> 0
\end{aligned}
$$

Therefore, $c_t(0) \leq 0$ and $c_t(G) > 0$ for all t.

We now have shown that there are two possible cases, $c_t(0) < 0$ and $c_t(G) > 0$, or $c_t(0) = 0$ and $c_t(G) > 0$, and we further prove the two cases in the following.

Case 1: When $c_t(0) < 0$ and $c_t(G) > 0$, given $c_t(g_t)$ is a continuous function, then we are guaranteed an interior point root from the intermediate value theorem, thus, there exists a non-trivial root $\hat{g}_t$.

Case 2: When $c_t(0) = 0$ and $c_t(G) > 0$, we require a condition of $\frac{dc_t(0)}{dg_t} \leq 0$ to have a non-trivial root, $\hat{g}_t$.

$c_t(g_t)$ and $N_t(g_t)$ are continuous and twice differentiable from Lemma 5.3.5,

$c_t(g_t) = g_t - N_t(g_t)$,

$\frac{dc_t}{dg_t} = 1 - \frac{dN_t}{dg_t}$,

$\frac{d^2c_t}{dg_t^2} = -\frac{d^2N_t}{dg_t^2}$

We also know that, $c_t(g_t)$ is a convex function with positive curvature because $N_t(g_t) = \int_0^\infty \min(g_{t+1}, N_{t+1}(g_{t+1}))f(g_{t+1})dg_{t+1}$, and $N_t(g_t) \leq G$ for all g_t and t from Lemma 5.3.3, thus, $\frac{d^2c_t}{dg_t^2} \geq 0$ and this guarantees that $\frac{d^2N_t}{dg_t^2} \leq 0 \; \forall t, g_t$. N_t is twice-differentiable and $\frac{d^2N_t}{dg_t^2} \leq 0$, then N_t is concave.

If N_t is concave, then it is bounded above by its first-order Taylor approximation (Varian (1992));

$$\frac{N_t(\hat{g}_t) - N_t(0)}{(\hat{g}_t - 0)} \leq \frac{dN_t(0)}{dg_t}$$

$$(N_t(\hat{g}_t) = \hat{g}_t \text{ from Lemma 5.3.5 and } N_t(0) = 0 \text{ since } c_t(0) = 0 \text{ from Remark 5.3.6})$$

$$\frac{\hat{g}_t}{\hat{g}_t} \leq \frac{dN_t(0)}{dg_t}$$

$$1 \leq \frac{dN_t(0)}{dg_t}$$

Thus, $\frac{dc_t(0)}{dg_t} = 1 - \frac{dN_t(0)}{dg_t} \leq 0$. Therefore, for both case 1 and 2, there exists a non-trivial root $\hat{g}_t$ such that $c_t(\hat{g}_t) = 0$, $c_t(g_t) \leq 0 \quad \forall g_t < \hat{g}_t$, and $c_t(g_t) > 0 \quad \forall g_t > \hat{g}_t$, $\forall t$.

□

Calculations illustrating the price condition for a binomial tree approach are included in Appendix B.

Remark 5.3.7. *From Lemma 5.3.3, $N_t(g)$ is monotonically increasing with time;*
(i) $c_t(g)$ is decreasing with time; $c_t(g) \geq c_{t+1}(g)$, because $c_t(g) = g - N_t(g)$ and $c_{t+1}(g) = g - N_{t+1}(g) \quad \forall t < T - 1$ and $\forall g$
(ii) $\hat{g_t}$ is increasing with time; $0 < \hat{g_t} \leq \hat{g_{t+1}} < G \; \forall t < T - 1$, because $c_t(\hat{g_{t+1}}) \geq c_{t+1}(\hat{g_{t+1}})$ from (i), $N_t(\hat{g_{t+1}}) \leq N_{t+1}(\hat{g_{t+1}})$ from Lemma 5.3.3, and $\hat{g_t} = N_t(\hat{g_t})$, $\hat{g_{t+1}} > N_t(\hat{g_{t+1}})$, $\hat{g_{t+1}} = N_{t+1}(\hat{g_{t+1}})$ from Lemma 5.3.5, thus, $\hat{g_t} = N_t(\hat{g_t}) \leq N_t(\hat{g_{t+1}}) < N_{t+1}(\hat{g_{t+1}}) = \hat{g_{t+1}}$

As we have proved Lemma 5.3.5, there exists a price threshold, $\hat{g_t}$, which also grows with time from Remark 5.3.7, below which it is optimal to fill up the tank, and above which it is optimal to wait for the price to fall. Note that we have not excluded the possibility that the price threshold depends on the tank level. Indeed later numerical work will suggest that it does, at least where there is insufficient gas in the tank to make it to the end of the decision period without refueling. See Section 5.5.

5.3.2 Optimal Refueling Strategy with Loyalty Program

Up until now, we discussed the expected price overrun and proved that there exists a price threshold. In the remainder of the Chapter 5, we present general forms of the optimality equation for the refueling strategy problem with the loyalty program, and solve the optimality equation.

As before, we assume the time horizon T is known and sufficiently short to ignore discounting. At the end of each day, loyal customer can buy gas at minimum of market price (g_t) and guaranteed price (G); or wait until the next day depending on her or his tank level. Again, we assume one unit of gas is used each period, and the customers must fill up the tank at the end of the time horizon.

Decision Stages: T is the known final stage, and $t = \{1, 2, \cdots, T\}$, is a discrete periods of days. We assume decision to refuel happens at the end of each day, and $T - t$ denotes stages-to-go.

State: $s = (k, g_t)$, where k is the discrete tank level at the end of day, $k \in \{1, \cdots, K\}$, the maximum tank level is K, the minimum tank reserve level is 1 (i.e, avoid damaging the engine), and the tank level at the next period is k_{t+1}, which depends on the purchasing quantity at the end of each period t, a_t, and a fuel consumption rate, that we normalize to 1. g_t is the current gasoline price at the end of day t, $g_t \in R^+$.

Actions: permissible decision variable on how much gasoline to fill up at each stage

$$A = \begin{cases} \{0, 1, \cdots, K-k\} \text{ if } k > 1, \\ \{1, \cdots, K-k\} \text{ if } k = 1. \end{cases}$$

The refueling cost with a loyalty program at the end of each period is given by: $r_t(s, a_t) = a_t \min(g_t, G)$, where g_t is a market gasoline price and G is a guaranteed price

Terminal Cost: $r_T(s) = (K - k_T) \min(g_T, G)$, assume at the end of period, must fill up

Transitional Probability Function: The transition of the tank level state variable is deterministic and defined by $k_{t+1} = k_t - 1 + a$. The gasoline price state variable is stochastic and follows $f(g_{t+1})$, where $g_{t+1} \in R^+$ is the next day's gasoline prices, which is a non-negative i.i.d. random variable.

The optimality equation for $t < T$ is given below;

$$V_t(s) = \inf_{a_t}\{a_t \min(g_t, G) + \int_0^\infty V_{t+1}(s' = (k_{t+1}, g_{t+1}))f(g_{t+1})dg_{t+1}\} \tag{5.8}$$

where $k_{t+1} = k_t + a_t - 1$, a_t is the control variable, with 1 unit fuel consumption rate

Theorem 5.3.8. *When $g_t \geq G$ for $t < T$, the optimal refueling strategy is trivial, and should wait ($a_t^* = 0$ is dominant) until the price falls below the guaranteed price unless the tank is at the minimum level.*

Proof. Theorem 5.3.8 is financially obvious, as there is no state of nature in which we would ever pay more than G for a unit of gas, and many in which we would pay less. If $g_t \geq G$ for $t < T$, and if the tank level permits, it's weakly optimal to wait for the price to fall; you cannot do worse than waiting as the price will never exceed G. Thus, $a_t^* = 0$. □

Theorem 5.3.9. *When $g_t < G$ for $t < T$, there exists an optimal policy a_t^* such that:*
(i) if $g_t \leq \hat{g}_t$, then $a_t^* = K - k_t$ (i.e. fill up),
(ii) if $g_t > \hat{g}_t$ and $k_t > 1$, then $a_t^* = 0$ (i.e. don't fill), and
(iii) if $g_t > \hat{g}_t$ and $k_t = 1$, then $a_t^* = 1$ (i.e. fill only 1 unit).

Proof. If $g_t < G$ for $t < T$, the problem becomes more interesting, and we proved in Lemma 5.3.5 that there exists a price threshold, $\hat{g}_t$, below which it is optimal to fill up the tank, and above which it is optimal to wait for the price to fall if the tank level permits. □

We now prove Proposition 5.3.10, the solutions of the optimality equation (5.8).

Proposition 5.3.10. *Provided $g_t < G$, the optimality equation (5.8) results in the following optimality equations;*

- *if $g_t \leq \hat{g}_t$, then $a_t^* = K - k_t$ for $t \leq T - 1$, and*

$$V_t(s) = (K - k_t)g_t + N_t(g_t) + R_t(g_t) \tag{5.9}$$

- *if* $g_t > \hat{g}_t$ *and* $k_t > 1$*, then* $a_t^* = 0$ *for* $t \leq T - 1$*, and*

$$V_t(s) = (K - k_t + 1)N_t(g_t) + R_t(g_t) \tag{5.10}$$

- *if* $g_t > \hat{g}_t$ *and* $k_t = 1$*, then* $a_t^* = 1$ *for* $t \leq T - 1$*, and*

$$V_t(s) = g_t + (K - k_t)N_t(g_t) + R_t(g_t) \tag{5.11}$$

The expected prices are divided into two terms, namely $N_t(g_t)$ and $R_t(g_t)$. This enables us to simplify the expressions which follow. $N_t(g_t)$ is the expected price per unit volume in the future, and $R_t(g_t)$ is the expected remainder cost due to the one unit consumption each day after filling up the tank at t (the remainder amount, $T - t - 1$). $c_t(g_t)$ is the expected price overrun as defined in 5.3.1.

$$c_t(g_t) = g_t - \int_0^{\hat{g}_{t+1}} g_{t+1} f(g_{t+1}) dg_{t+1} - \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1} \tag{5.12}$$

$$N_t(g_t) = \int_0^{\hat{g}_{t+1}} g_{t+1} f(g_{t+1}) dg_{t+1} + \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1} \geq 0 \tag{5.13}$$

$$R_t(g_t) = \int_0^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1} + \int_0^{\infty} R_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1} \geq 0 \tag{5.14}$$

Proof. (5.9) and (5.10) in Proposition 5.3.10 by induction:

for $t = T-1$

$$\begin{aligned}
V_t(s) &= \inf_{a_t}\{a_t g_t + \int_0^\infty V_{t+1}(s')f(g_{t+1})dg_{t+1}\} \\
&= \inf_{a_t}\{a_t g_t + (K-k_t+1-a_t)(\int_0^\infty \min(g_{t+1}, G)f(g_{t+1})dg_{t+1})\} \\
&= \inf_{a_t}\{a_t g_t + (K-k_t+1-a_t)(\int_0^G g_{t+1}f(g_{t+1})dg_{t+1} + \int_G^\infty Gf(g_{t+1})dg_{t+1})\} \\
&= \inf_{a_t}\{a_t(g_t - \int_0^G g_{t+1}f(g_{t+1})dg_{t+1} - \int_G^\infty Gf(g_{t+1})dg_{t+1})\} \\
&+ (K-k_t+1)(\int_0^G g_{t+1}f(g_{t+1})dg_{t+1} + \int_G^\infty Gf(g_{t+1})dg_{t+1})
\end{aligned}$$

$$\text{Let}\quad N_t(g_t) = \int_0^G g_{t+1}f(g_{t+1})dg_{t+1} + \int_G^\infty Gf(g_{t+1})dg_{t+1} < G \tag{5.15}$$

$$c_t(g_t) = g_t - \int_0^G g_{t+1}f(g_{t+1})dg_{t+1} - \int_G^\infty Gf(g_{t+1})dg_{t+1} \tag{5.16}$$

$$= g_t - N_t(g_t) \tag{5.17}$$

$$\text{Thus,}\quad V_t(s) = \inf_{a_t}\{a_t c_t(g_t)\} + (K-k_t+1)N_t(g_t) \tag{5.18}$$

Since $V_t(s)$ is linear in a_t, the optimal solution lies at an extreme point of the set of feasible control variables.

If $g_t \le \hat{g}_t$, then $c_t(g_t) \le 0$ and $g_t \le N_t(g_t)$ from Lemma 5.3.5. In other words, if the current price is less than or equal to the expected price in the future, the optimality function has a negative linear relationship with a_t. Thus, it is minimized when a_t^* is as large as possible, and $a_t^* = K - k_t$.

Using expressions (5.17), (5.18), we get;

$$V_{T-1}(s) = (K-k_t)g_t + N_t(g_t) + R_t(g_t) \quad \text{if} \quad g_t \le \hat{g}_t$$

If $g_t > \hat{g}_t$, then $c_t(g_t) > 0$ and $g_t > N_t(g_t)$ from Lemma 5.3.5. the current price is greater than the expected price in the future, and the optimality function has a positive linear relationship with a_t. Thus, it is minimized when a_t^* is as small as possible, hence $a_t^* = 0$ if $k_t > 1$ and $a_t^* = 1$ if $k_t = 1$ (minimum constraint on tank level).

$$V_{T-1}(s) = (K - k_t + 1)N_t(g_t) + R_t(g_t) \quad \text{if} \quad g_t > \hat{g}_t \text{ and } k_t > 1$$
$$V_{T-1}(s) = g_t + (K - k_t)N_t(g_t) + R_t(g_t) \quad \text{if} \quad g_t > \hat{g}_t \text{ and } k_t = 1$$

(5.9) and (5.10) in Proposition 5.3.10 holds for $t = T - 1$, but since there is only one period left there is no remainder expected cost due to the one unit consumption each day after filling up at t, $T - t - 1 = T - (T - 1) - 1 = 0$, thus, $R_t(g_t) = 0$.

Assume (5.9) and (5.10) in Proposition 5.3.10 hold for all $t = j + 1, j + 2, \cdots, T - 2$, using expressions (5.12), (5.13), and (5.14) the following proof can be derived: $a_t^* = K - k_t$ if $g_t \le \hat{g}_t$ and $a_t^* = 0$ if $g_t > \hat{g}_t$ and $k_t > 1$ for $t = j$.

$$\begin{aligned}
V_t(s) &= \inf_{a_t}\{a_t g_t + \int_0^\infty V_{t+1}(s') f(g_{t+1}) dg_{t+1}\} \\
&= \inf_{a_t}\{a_t g_t + \int_0^{\hat{g}_{t+1}} \big((K - k_t + 1 - a_t)\ g_{t+1} + N_{t+1}(g_{t+1}) + R_{t+1}(g_{t+1})\big) \\
&\quad f(g_{t+1}) dg_{t+1} + \int_{\hat{g}_{t+1}}^\infty \big((K - k_t + 2 - a_t) N_{t+1}(g_{t+1}) + R_{t+1}(g_{t+1})\big) f(g_{t+1}) dg_{t+1}\}
\end{aligned}$$

$$V_t(s) = \inf_{a_t}\{a_t\big(g_t - \int_0^{\hat{g}_{t+1}} g_{t+1}f(g_{t+1})dg_{t+1} - \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1})f(g_{t+1})dg_{t+1}\big)\}$$
$$+ (K - k_t + 1)\big(\int_0^{\hat{g}_{t+1}} g_{t+1}f(g_{t+1})dg_{t+1} + \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1})f(g_{t+1})dg_{t+1}\big)$$
$$+ \int_0^{\infty} (N_{t+1}(g_{t+1}) + R_{t+1}(g_{t+1}))f(g_{t+1})dg_{t+1}$$

$$\text{Thus,} \quad V_t(s) = \inf_{a_t}\{a_t c_t(g_t)\} + (K - k_t + 1)N_t(g_t) + R_t(g_t) \tag{5.19}$$

$a^*_t = K - k_t$ if $g_t \leq \hat{g}_t$, and $a^*_t = 0$ if $g_t > \hat{g}_t$ and $k_t > 1$, and using expression (5.17), $c_t(g_t) = g_t - N_t(g_t)$ and plugging in a^*_t we get;

$$V_t(s) = (K - k_t)g_t + N_t(g_t) + R_t(g_t) \quad \text{if} \quad g_t \leq \hat{g}_t$$
$$V_t(s) = (K - k_t + 1)N_t(g_t) + R_t(g_t) \quad \text{if} \quad g_t > \hat{g}_t \quad \text{and} \quad 1 < k_t < K$$

Thus, (5.9) and (5.10) in Proposition 5.3.10 hold for all $t = j$.

Similarly, $a^*_t = K - k_t$ if $g_t \leq \hat{g}_t$ and $a^*_t = 1$ if $g_t > \hat{g}_t$ and $k_t = 1$ for $t = j$.

$$V_t(s) = \inf_{a_t}\{a_t g_t + \int_0^{\infty} V_{t+1}(s')f(g_{t+1})dg_{t+1}\}$$
$$= \inf_{a_t}\{a_t g_t + \int_0^{\hat{g}_{t+1}} \big((K - k_t + 1 - a_t)\ g_{t+1} + N_{t+1}(g_{t+1}) + R_{t+1}(g_{t+1})\big)$$
$$f(g_{t+1})dg_{t+1} + \int_{\hat{g}_{t+1}}^{\infty} \big(\min(g_{t+1}, G) + (K - k_t + 1 - a_t)N_{t+1}(g_{t+1}) + R_{t+1}(g_{t+1})\big)$$
$$f(g_{t+1})dg_{t+1}\}$$

$$
\begin{aligned}
V_t(s) = \inf_{a_t}\{a_t\big(g_t - &\int_0^{\hat{g}_{t+1}} g_{t+1} f(g_{t+1}) dg_{t+1} - \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1}\big)\} \\
&+ (K - k_t + 1)\Big(\int_0^{\hat{g}_{t+1}} g_{t+1} f(g_{t+1}) dg_{t+1} + \int_{\hat{g}_{t+1}}^{\infty} N_{t+1}(g_{t+1}) f(g_{t+1}) dg_{t+1}\Big) \\
&+ \int_0^{\hat{g}_{t+1}} (N_{t+1}(g_{t+1}) + R_{t+1}(g_{t+1})) f(g_{t+1}) dg_{t+1} \\
&+ \int_{\hat{g}_{t+1}}^{\infty} (\min(g_{t+1}, G) + R_{t+1}(g_{t+1})) f(g_{t+1}) dg_{t+1}
\end{aligned}
$$

Since $a_t^* = 1$ if $g_t > \hat{g}_t$ and $k_t = 1$, notice that $R_t(g_t)$ is slightly different.

$$
\begin{aligned}
V_t(s) &= g_t + (K - k_t) N_t(g_t) + \int_0^{\hat{g}_{t+1}} (N_{t+1}(g_{t+1}) + R_{t+1}(g_{t+1})) f(g_{t+1}) dg_{t+1} \\
&\quad + \int_{\hat{g}_{t+1}}^{\infty} (\min(g_{t+1}, G) + R_{t+1}(g_{t+1})) f(g_{t+1}) dg_{t+1} \\
V_t(s) &= g_t + (K - k_t) N_t(g_t) + R_t(g_t) \quad \text{if} \quad g_t > \hat{g}_t \quad \text{and} \quad k_t = 1
\end{aligned}
$$

Thus, (5.11) in Proposition 5.3.10 holds for all $t = j$. □

Therefore, we have proved all (5.9), (5.10), and (5.11) in Proposition 5.3.10, including the optimality equation (5.8) and Theorem 5.3.9. We also have proved the existence of price thresholds, $\hat{g}_t$, and that it is optimal to buy $K - k_t$ if $g_t \leq \hat{g}_t$, and buy nothing if $k_t > 1$ and if $g_t > \hat{g}_t$ and buy one unit if $k_t = 1$ and if $g_t > \hat{g}_t$.

Remark 5.3.11. *Assuming $g < G$ and an optimal solution exists at either extreme bounds, fill-up or fill minimum or nothing, we define numerical price threshold as the point where expected cost is indifferent between the extreme bounds.*
If $k > 1$, then price threshold is g, such that $g = \frac{E[V_{t+1}(k-1,g')|g] - E[V_{t+1}(K-1,g')|g]}{K-k}$
If $k = 1$, then price threshold is g, such that $g = \frac{E[V_{t+1}(1,g')|g] - E[V_{t+1}(K-1,g')|g]}{K-2}$
where $E[V_{t+1}(k, g')|g]$ is the expected cost-to-go for the random price g' and a tank-level k, as will be seen in Section 5.5, the price threshold for the smaller tank-level are higher than that for the larger tank-level than the number of days-to-go.

5.4 Numerical Methodology

In previous Section, we have analytically studied the existence of a non-negative threshold on the optimal decision below which it is optimal to fill-up and above which it is optimal to wait or fill one day's worth of travel minimum depending on the time-to-go and tank level. We now numerically compute the optimal decision boundary, $\hat{g}_k$, and show how the threshold value changes with different parameters.

We start by looking at the optimal policy as we hold the days-to-go $(1, 2, \cdots, 6, 7)$ and the tank level $(1, 2, \cdots, 9, 10)$ at one value, and allow two parameters of gasoline price process, μ and σ, to vary at a time. We found that the price thresholds are more sensitive to σ than to μ. This might be because the numerical value of μ is up to 10 times smaller than the values for σ used in this study.
Figure 5.2 shows the optimal policy separated by the price thresholds for a specific tank level, days-to-go, and μ, while varying σ from 0.002 to 0.01. Price thresholds changes from about 3.12 to 3.08.

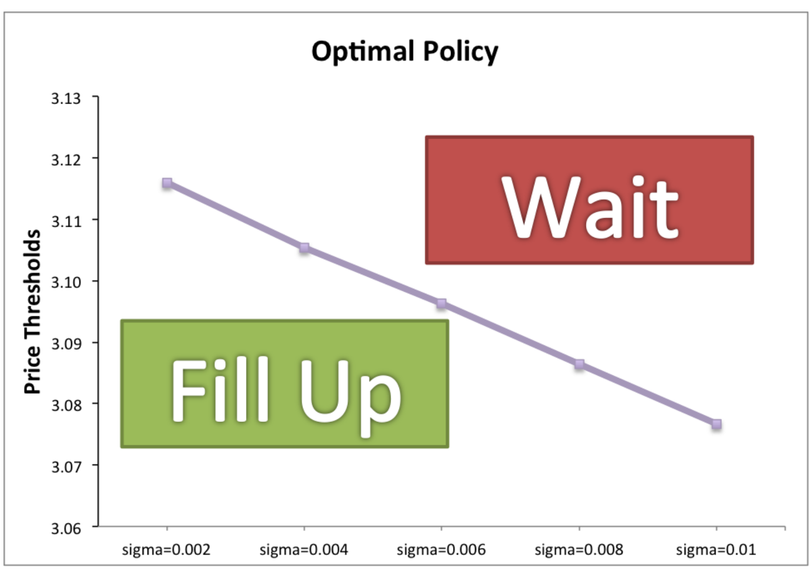


Figure 5.2: *$\hat{g}$ against σ, varied from 0.002 to 0.01, while fixing $G = 3.13$, tank level=4 and days-to-go=6, and $\mu = 0.0002$*

Figure 5.3 shows the optimal policy separated by the price thresholds for specific tank level, days-to-go, and σ, while varying μ from 0.0001 to 0.0005. Price thresholds do not change much.

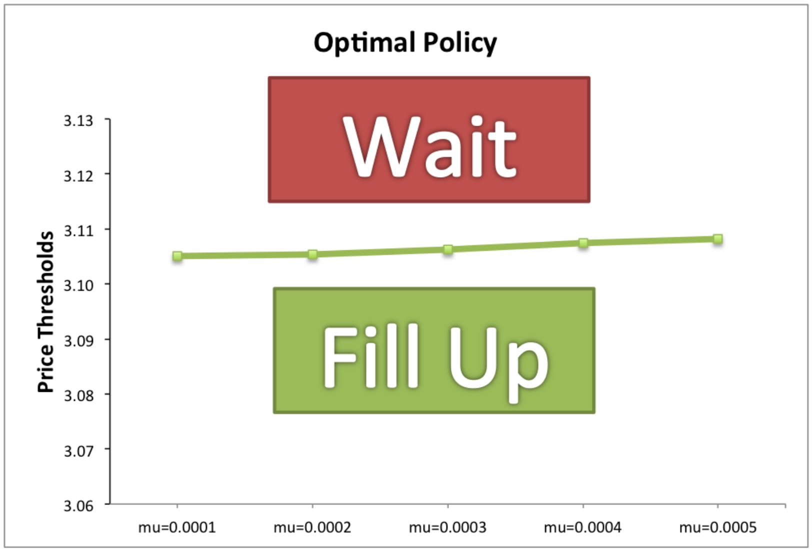


Figure 5.3: *$\hat{g}$ against μ, varied from 0.0001 to 0.0005, while fixing $G = 3.13$, tank level=4 and days-to-go=6, and $\sigma = 0.004$*

5.4.1 Numerical Algorithm

Problem: Given a finite tank (10 unit tank-level) and a number of days-to-go (7 days-to-go), minimize the refueling cost with a constraint of filling up the tank by the 7^{th} day with a guaranteed price G.

Inputs: price grid points (g_k), $0 \leq g_k \leq G$, a list of daily drift rate (μ) varied from 0.0001 to 0.001, a list of daily volatility (σ) varied from 0.002 to 0.01, and the guaranteed price $G = 3.13$.

Outputs: We simulate n possible prices for time $k+1$ at each price grid (g_k) and at each days-to-go, given drift rates and volatilities. We compute the Cost-to-Go following optimal policy both with the guaranteed price and without the guaranteed price at each state. Then, we find the value of loyalty program as the cost-to-go difference between with the guaranteed price and without the guaranteed price following optimal policies at each state. We also find $\hat{g}_t$, which is the maximum price that satisfies the optimal policy to re-fill at each state.

Dynamic Programming Algorithm:

I) One-Period Random Price Movement

Generate a $M \times N$ matrix with one-period price transitions following a non-negative price process for a specific μ and σ. Each row is a discrete price-level, g_k, with interval of $\frac{G}{M-1}$, i.e. $0, \frac{G}{M-1}, \frac{2G}{M-1}, \cdots, G$, and each column is a simulated instance of the price one-period later given an initial price g_k (i.e. $N = 1000$ and $M = 10$).

For each possible tank-level $1, 2, 3, \cdots, 10$, do the following;

II a) Expected cost-to-go which follows optimal policy for period 6

Generate a $M \times 2$ matrix with each row corresponding to the current price level, g_k. In the first column, simply take the closed form solution for the cost of filling up tank (last period) at the simulated price, g_{k+1}, and then take average of these costs.

with guaranteed price

$E[$ refuel cost at price g_{k+1} in period $7|g_k] = ($ 10 - current tank level $)\frac{\sum \min(g_{k+1}, G)}{N}$

In the second column, you calculate the expected cost-to-go, given by;

min(refuel cost at price g_k in period 6 $+E[$ refuel cost at price g_{k+1} in period 7 $]$)

In general, the expected cost-to-go is the argmin of the previous column, which follows the optimal policy, given as;

$$\text{cost-to-go}(i, j, g_k) = \min_a (a \times \min(g_k, G) + E[\text{cost-to-go}(i + a - 1, j - 1, g_{k+1})])$$

Where g_{k+1} is the simulated price one-period later and g_k is the current period price level, i is the current tank-level, and j is the current number of days-to-go.

II b) Expected Cost-to-go following Optimal policy for period 5, 4, 3, 2, and 1

Generate $M \times 5$ matrix with each row corresponding to the current price level. Start with period 5 and move back to period 1.

In the following columns of each matrix we calculate the expected cost-to-go;

min(refuel cost at price g_k in period $t + E$[refuel cost at price g_{k+1} in period $t + 1$])

Where,

E[refuel cost at price g_{k+1} in period $t + 1] = \frac{1}{N} \sum$(interpolated cost-to-go in $t + 1$)

Since the majority of simulated prices, g_{k+1}, do not fall in exactly one of the discrete price grid points $g_k = 0, \frac{G}{M-1}, \frac{2G}{M-1}, \cdots, \frac{(M-3)G}{M-1}, \frac{(M-2)G}{M-1}, G$, but fall in between the price grid points, we interpolate the cost-to-go from the nearest discrete price buckets, above and below the simulated price. We then take the average of all cost-to-go.

$$[0, \frac{G}{M-1}], [\frac{G}{M-1}, \frac{2G}{M-1}], \cdots, [\frac{(M-3)G}{M-1}, \frac{(M-2)G}{M-1}], [\frac{(M-2)G}{M-1}, G]$$

To find the optimal decision boundary or "price thresholds", and the value of the guaranteed price, we use dynamic programming (DP) and the recursive value function or "cost-to-go" function, which follows the optimal policy. The optimal policy resulting from this cost-to-go function is described by the value of the guaranteed price matrices for each of the 10 tank-levels, of 7 days-to-go, and of g_k. The price thresholds are, then, mapped from the optimal policy and the value of the guaranteed price; as the maximum price such that it is optimal to fill-up and has the value greater than 0.005.

The price grid points are constructed by assuming that all simulated prices, g_{k+1}, will fall on or between the discrete price grid points. The dynamics of the price are only depend on μ and σ, and for reasonable μ and σ, g_{k+1} will be between the price grid points, but in the case of $g_k = G$, the simulated price will not be between the price grid points, and we do not have the cost-to-go for $g_{k+1} > G$. To overcome this issue, we use one additional price-level as we approach the final period. i.e. the discrete price-levels increases $\frac{MG}{M-1}, \frac{(M+1)G}{M-1}, \cdots, \frac{(M+5)G}{M-1}$ as the number of days-to-go decreases.

5.4.2 Price Simulation Approach

Using the algorithm we presented in Section 5.4.1, we can numerically find the value of the loyalty program and the optimal decision boundary. For the numerical values, we need particular price process, and we now impose Geometric Brownian motion (GBM) on the price-movement because of its links with quantitative finance, stationary normal increments, and non-negativity. As discussed, more complicated processes can be used and may better characterize the behaviour of gas prices, but may be more difficult to understand and requires more parameters to be estimated. For GBM, we just need only to estimate parameter values μ and σ as described in Appendix D from the logged daily gasoline price changes.

Let g_{t+1} be the next period price and g_t be the current period price, then, we have $g_{t+1} = g_t e^{\mu - \frac{\sigma^2}{2} + \sigma Z_t}$ where μ is the daily drift rate and σ is the daily volatility. For example, the drift rate of 0.0002 means logged gasoline price will move up .02%, each day, on average and volatility of 0.004 means some day logged price will move up 0.4%, others down 0.4%, and most of the time in the middle.

5.5 Simulation Results

5.5.1 Price Thresholds

In what follows, we present the numerical results using simulated prices for different parameters, and compare the price thresholds and the value of the loyalty program that follows the optimal refueling strategy. The simulated prices at time $k+1$ at each number of days-to-go are based on the price grid points g_k, and the full price paths are unknown. Assuming the full price paths and price cycles are unknown, if g_k is greater than G, then the (less interesting) optimal strategy is simply to wait or to refill the minimum unit of gas at G. Under the same assumption, if g_k is less than or equal to G, then, the optimal strategy is to compare today's price with price thresholds and fill-up if today's price is below the appropriate price thresholds, otherwise wait unless the tank is near empty.

We learned that the optimal decision boundaries or price thresholds are time-varying as well as state-dependent while fixing parameters to parameters appropriate for historical proxy ($\mu = 0.0002, \sigma = 0.004$). The lower the tank level and the fewer the days-to-go, the higher the price thresholds with some exceptions. In fact, the price thresholds is monotonically increases as approach to the final stage because there is lower the chance of price changes.

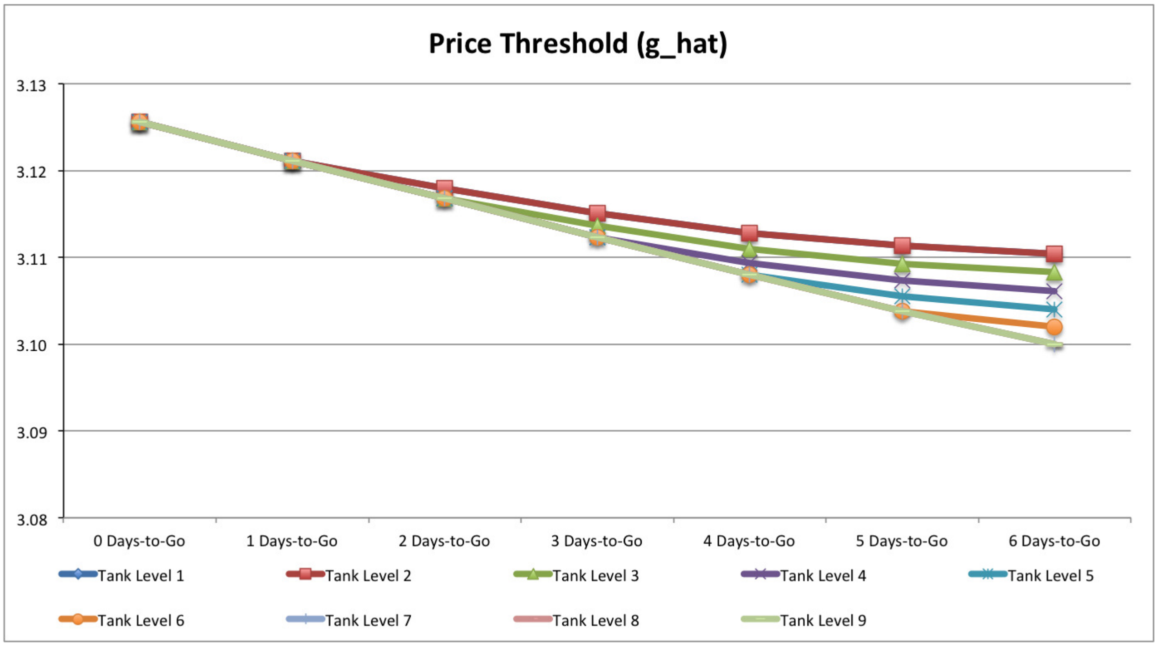


Figure 5.4: *Price Thresholds, $\hat{g}$ (y-axis), against the number of days-to-go (x-axis), while fixing* $\mu = 0.0002, \sigma = 0.004,$ *and* $G = 3.13$.

The price thresholds are the same for all tank levels at the final stage, also the same for the minimum tank-level ($k = 1$) and tank-level ($k = 2$), and same for tank-levels ($k = 7, 8, 9$), while the difference between different tank levels grows as the number of days-to-go increases for other tank levels. This is intuitively reasonable because of the constraint of filling up the tank at the final stage for all tank levels, the constraint of filling up at least 1 unit for the minimum tank-level ($k = 1$), and never forced to refill except the final stage for tank-level ($k = 7, 8, 9$), make their expected cost-to-go equivalent between, the above mentioned, tank levels.

Figure 5.4 shows price thresholds for specific μ and σ, price thresholds monotonically decrease as time-to-go increase, also decreases as tank level increases. This makes perfect sense both financially and psychologically when you are less desperate about re-filling your tank either because you have more time to make your decision or have more reserve in your tank, you can get more picky about price.

5.5.2 Value of a Loyalty Program

A call option is in the money if the price of the underlying asset is higher than the strike price, it is out of the money if the price of the underlying asset is lower than the strike price, and it is at the money if the price of the underlying asset is the same as the strike price. Similarly, loyalty program is in the money if the gas price is higher than the guaranteed price, it is out of the money if the gas price is lower than the guaranteed price, and it is at the money if the gas price is the same as the guaranteed price.

The difference between the gas price and the guaranteed price is instant cost-saving, and the possibility that it will move into the money is potential cost-saving. Thus, the loyalty program would only be valuable when the current price is near or greater than the guaranteed price. In fact, from Section 5.5.1, for reasonable parameters the price thresholds exist near the guaranteed price, thus, g_k near G or near at-the-money is expected to be more interesting and variable than for deep out-of-the-money ($g_k << G$) or deep in-the-money ($g_k >> G$). The value of the loyalty program is expected to be very small when $g_k << G$, relatively bigger when g_k is near G, and very big when $g_k >> G$. As introduced in Section 5.4.1, using dynamic programming and the recursive cost-to-go functions, we find the expected cost-to-go both with and without guaranteed price that follows the optimal policy at each state of the 10 tank-levels, of 7 days-to-go, and of g_k. The expected cost-to-go with guaranteed price is given as;

$$\text{cost-to-go}(i, j, g_k) = \min(a * \min(g_k, G) + E[\text{cost-to-go}(i + a - 1, j - 1, g_{k+1})])$$

The expected cost-$\bar{\text{t}}$o-go without guaranteed price is given as;

$$\text{cost-}\bar{\text{t}}\text{o-go}(i, j, g_k) = \min(a * g_k + E[\text{cost-}\bar{\text{t}}\text{o-go}(i + a - 1, j - 1, g_{k+1})])$$

The value of the loyalty program is then calculated as the expected cost-to-go difference between with the guaranteed price and without the guaranteed price following optimal policies at each state of the 10 tank-levels, of 7 days-to-go, and of g_k.

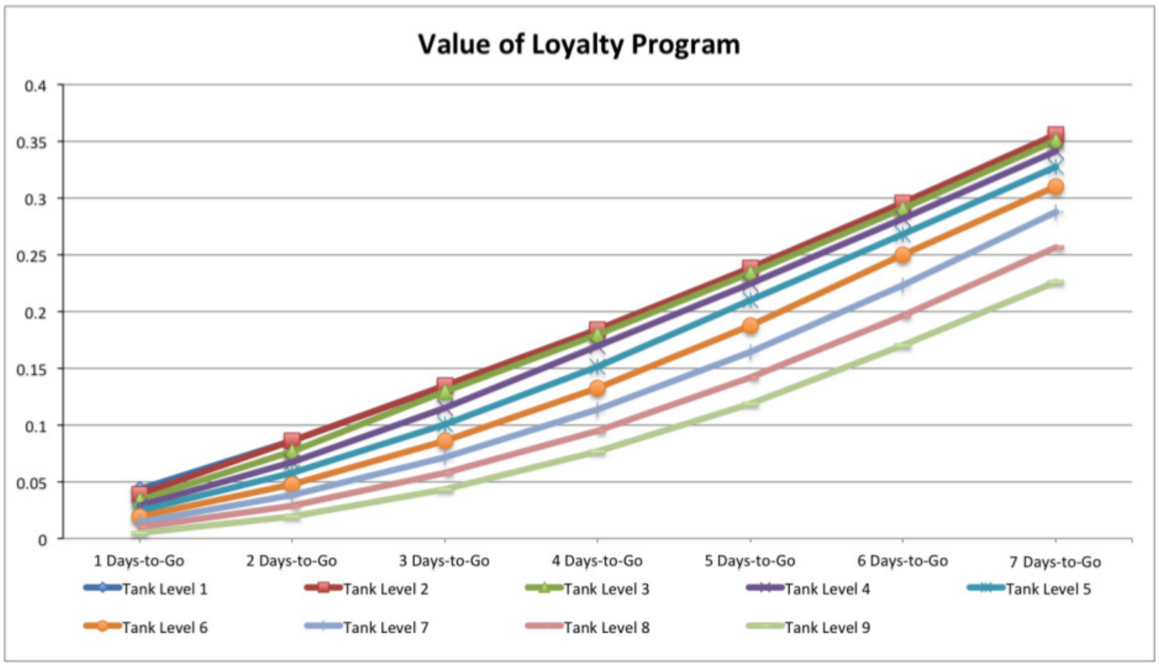


Figure 5.5: *The value of loyalty program (y-axis) against the number of days-to-go (x-axis). while fixing* $\mu = 0.0002, \sigma = 0.004$*, and* $g_k = G = 3.13$

Figure 5.5 shows the value of loyalty program for $g_k = G$ with specific μ, and σ, the value of loyalty program increases almost linearly as the number of days-to-go increases and as the tank level gets lower; the value as well as the variations of the value shrinks as the number of days-to-go decreases and as tank level increases. This implies that the value of loyalty program is determined by both tangible and intangible factors; the more gas to fill and the longer the time to go, the higher the value gets.

This is more evident when we plot the same value against the tank level for different number of days-to-go, see Figure 5.6. The values are clearly higher for more days-to-go and for lower tank levels. More interestingly, the value increases linearly as tank level gets lower at 1 day-to-go. This is because of the constraint of filling up the tank by the 7th day, and there is no potential cost-saving, just the instant cost-saving at 1 day-to-go. For longer days-to-go, the potential cost-saving is also added to the instant cost-saving, and the potential cost-saving grows slowly, approximately logarithmically with smaller tank level, especially when that tank level is not large enough to wait and be forced to refill, see Figure 5.6.

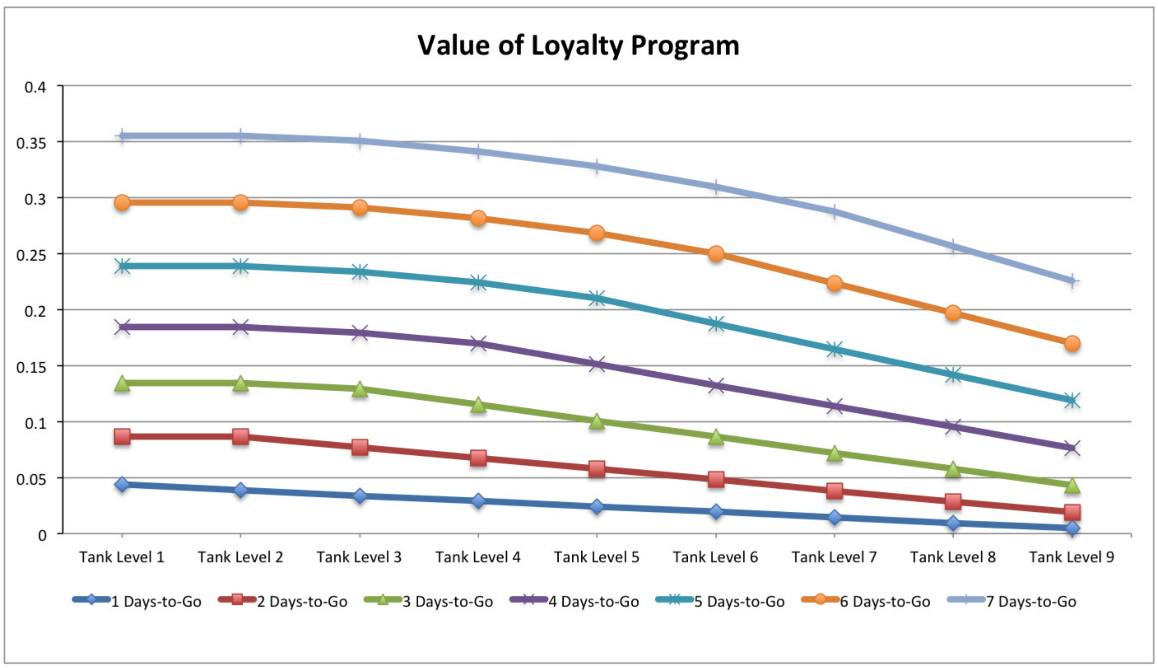


Figure 5.6: *The value of loyalty program (y-axis) against the tank level (x-axis), while fixing* $\mu = 0.0002, \sigma = 0.004,$ *and* $g_k = G = 3.13$

We now fix the tank level and days-to-go at one value, and vary two of μ, σ, g_k. As expected, value gets higher as g_k gets near G. The value shows the most variations at g_k near G, and less variations away from the G, also, more sensitive to σ than to μ. This is because the changes in volatility parameter, σ, is greater than the drift parameter, $\mu - \frac{\sigma^2}{2}$, due to the relative magnitude difference between the two parameters, μ and σ.

Figure 5.7 shows the value of loyalty program for fixed tank level, days-to-go, and μ, while varying g_k for different σ. The value shows the most variation in g_k near G.

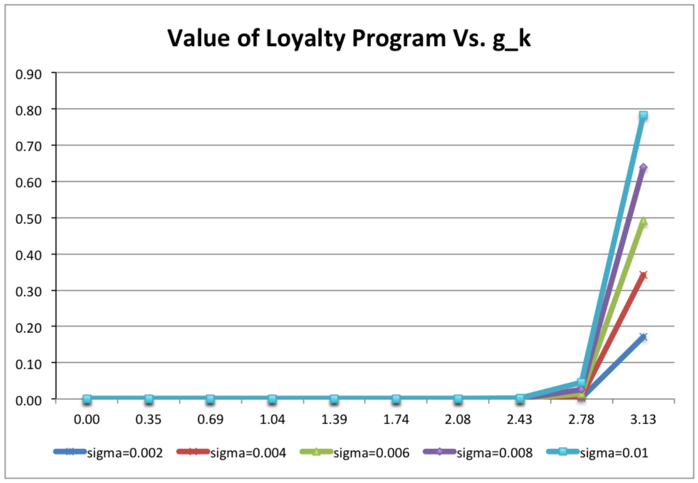


Figure 5.7: *The value (y-axis) against g_k (x-axis), while fixing tank level=4 and days-to-go=6, $G = 3.13$, and $\mu = 0.0002$*

Figure 5.8 shows the value of loyalty program for fixed tank level, days-to-go, and σ, while varying g_k for different μ. Not much variations are observed for all g_k.

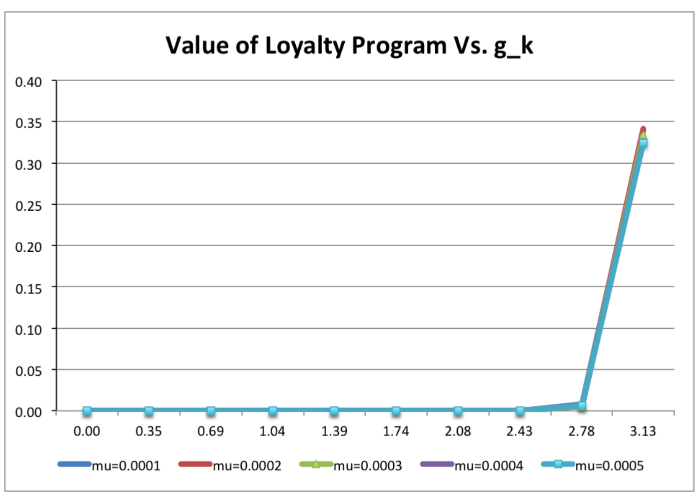


Figure 5.8: *The value (y-axis) against g_k (x-axis), while fixing tank level=4 and days-to-go=6, $G = 3.13$, and $\sigma = 0.004$*

For coarse grid points, in Figure 5.7 and 5.8, we observed the progression of g_k moving from out of the money to in the money. Thus, we study the figure in more detail using finer grid points, for gas price values near G, in the following. Figure 5.9 shows the value of loyalty program for specific tank level, days-to-go, and μ, while varying g_k and σ. The value of σ matters most at near the money, 3.11, while deep out of the money, σ does not matter much.

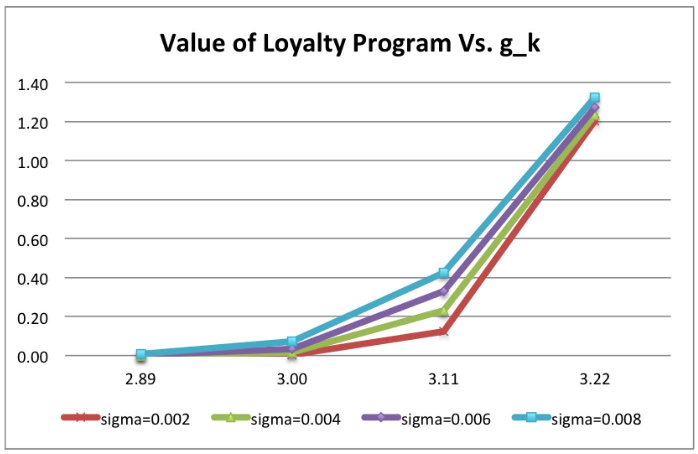


Figure 5.9: *The value (y-axis) against g_k (x-axis), while fixing tank level=4 and days-to-go=6, $G = 3.13$, and $\mu = 0.0002$*

Figure 5.10 shows the value of loyalty program for specific tank level, days-to-go, and σ, while varying g_k and μ. The value of μ does not matter as much as the value of g_k and σ.

As discussed, the results show that the value of loyalty program is fairly independent of μ (as long as $\mu > 0$). We further study the value of guaranteed price for $\mu < 0$. Figure 5.11 shows the value of loyalty program for $g_k = G$ with $\mu = -0.004$, and $\sigma = 0.004$. For $\mu = -0.004 < 0$, the value of loyalty program is still small positive numbers, increase as the number of days-to-go increase and tank levels get lower.

Figure 5.12 shows the value of loyalty program for $g_k = G$ with $\mu = -0.016$, and

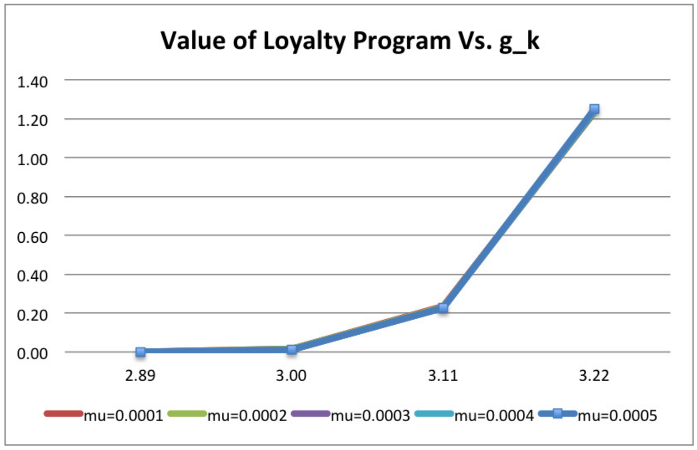


Figure 5.10: *The value (y-axis) against g_k (x-axis), while fixing tank level=4 and days-to-go=6, $G = 3.13$, and $\sigma = 0.004$*

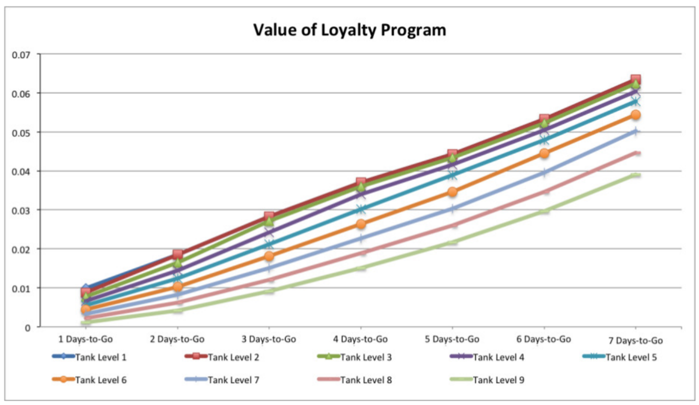


Figure 5.11: *The value of loyalty program (y-axis) against the number of days-to-go (x-axis) while fixing $\mu = -0.004, \sigma = 0.004$, and $g_k = G = 3.13$*

$\sigma = 0.004$, the value of loyalty program now becomes zero for all states.

In order to drive the value of loyalty program to zero, the μ has to be big enough, need condition of $\mu - \frac{\sigma^2}{2} + \sigma \times Z < 0$, and $\mu < \frac{\sigma^2}{2} - \sigma \times Z$, i.e. $\mu < \frac{0.004^2}{2} - 0.004 \times Z, \mu \lesssim -0.015$ because the percentage changes over one day for GBM are independent log-normal random variables.

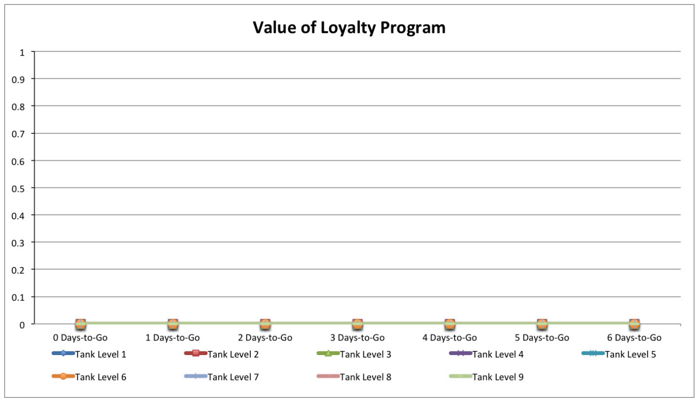


Figure 5.12: *The value of loyalty program (y-axis) against the number of days-to-go (x-axis) while fixing* $\mu = -0.016, \sigma = 0.004,$ *and* $g_k = G = 3.13$

5.6 Conclusion

In this chapter we found optimal purchasing strategies for customers, providing both theoretical results about the abstract structure of these strategies and numerical values for these policies computed with numerical parameters. Using simulated data, we found the price thresholds, estimated the value of the loyalty program, the premium to be paid, and quantitatively supported the loyalty program. The empirical study of loyalty programs was limited to potential value and benefits for customers because the particular loyalty program we proposed has never been globally introduced in retail gasoline market. Once the loyalty program is launched, with the comprehensive 'before' and 'after' data, the impact of a loyalty program can be assessed, how much it would increase market share and repeat-purchase, as well as attract new customers.

Since 1980's, trading in crude oil has shifted from a domain of buyers and sellers to a market where speculators bet on a price of a given crude oil on a specific future date to make profits from speculation (Engdahl (2008)). Studies show that unregulated speculation in oil futures, can drive prices up, which cannot be justified by the supply and demand theory (Engdahl (2008)).

Thus, to avoid such price manipulations in the retail gasoline market, the quantity any speculator buys on the retail gasoline market, needs to be regulated. But, this may be a lesser problem in the retail gasoline market because the quantity can be restricted naturally by the quantity which consumers are willing to buy and by which retailers are willing to sell. The value of petroleum product will eventually become very low, discouraging speculators as fuel powered vehicles are replaced by electric vehicles.

While further investigation is needed to fully understand the benefits and risks of bringing the guaranteed price in the world of retail gasoline market, we presented some promising theoretical results to promote the loyalty program and show that it would be a great tool for both retailers and consumers, hedging against the price and sales fluctuations. Truck and taxi companies could also manage their fleet costs more effectively using the same loyalty program with much bigger potential benefits by the volume.

In addition to the positive value of loyalty program, we postulate the guaranteed price in the loyalty program would also discourage consumers from 'price search' behaviour. Thus, sales and profit are expected to be less sensitive to the price change over the time with the guaranteed prices and brand loyalty. Thus, it might also be interesting to study the impact of the loyalty program in gasoline market on both price searching and pricing behaviour.

In the next Chapter, we investigate the performance of this on real market data and compare numerical results with empirical data.

Bibliography

Engdahl, F. William. (2008), Perhaps 60% of today's oil price is pure speculation. Retrieved from: https://www.globalresearch.ca/perhaps-60-of-today-s-oil-price-is-pure-speculation/8878 Date retrieved: 27 October 2014.

Hull, John C. (2008), Options, Futures and Other Derivatives 7th Edition. Prentice Hall.

Jaureguiberry, Florencia. (2010), An Analysis of Strategic Price Setting in Retail Gasoline Markets, PhD dissertation, Public Policy Analysis, The Pardee RAND Graduate School, Santa Monica

Karlin, S., and Taylor, H. M. (1975). A First course in stochastic processes. Academic Press, New York.

Longstaff, F. A., and Schwartz, E. S. (2001), Valuing American options by simulation: a simple least-squares approach, *Review of Financial studies*, Vol. 14, No. 1, pp.113-147.

Mas-Colell, Andreu, Michael Dennis Whinston, and Jerry R. Green. (1995), Microeconomic theory. Vol. 1. New York: Oxford university press

Maskin, Eric, and Tirole, Jean (1988), A theory of dynamic oligopoly, I: Overview and quantity competition with large fixed costs, *Journal of the Econometric Society*, Vol. 56, No. 3, pp.549-569.

Maskin, Eric, and Tirole, Jean (1988), A theory of dynamic oligopoly, II: Price competition, kinked demand curves, and Edgeworth cycles, *Journal of the Econometric Society*, Vol. 56, No. 3, pp.571-599.

Noel, Michael D. (2007), Edgeworth price cycles, cost-based pricing, and sticky pricing in retail gasoline markets, *Review of Economics and Statistics*, Vol. 89, No. 2, pp.324-334.

Varian, H.R. (1992) Microeconomic Analysis. 3rd Edition, W.W. Norton & Company, New York.

Weber, R. (2010). Optimization and control. University of Cambridge.

Chapter 6

Optimal Refueling Strategy Using Empirical Data

6.1 Comparison with Empirical Data

In this Chapter, we present empirical data (historical values), and find the optimal refueling strategy based on a theoretical model introduced in Section 5.4.1. The empirical data set used in this Chapter is unique and broad in its temporal and spatial dimensions. There are over 1000 different retail sites each with anonymized site ID in the data set. Each site is independent of the others, and all are located in North American cities.

The data collected per site contains a date index; a weekday index from 1 (Sunday) to 7 (Saturday); and station-specific daily retail price (in $ per gallon) data. The data were collected by retailers who contracted with a leading global pricing and solution company for the period from July 2012 to June 2013. Due to a confidentiality agreement, no further detail may be provided here.

In what follows, we analyze results from the empirical data, and compare the empirical results with results from the simulated data of chapter 5.

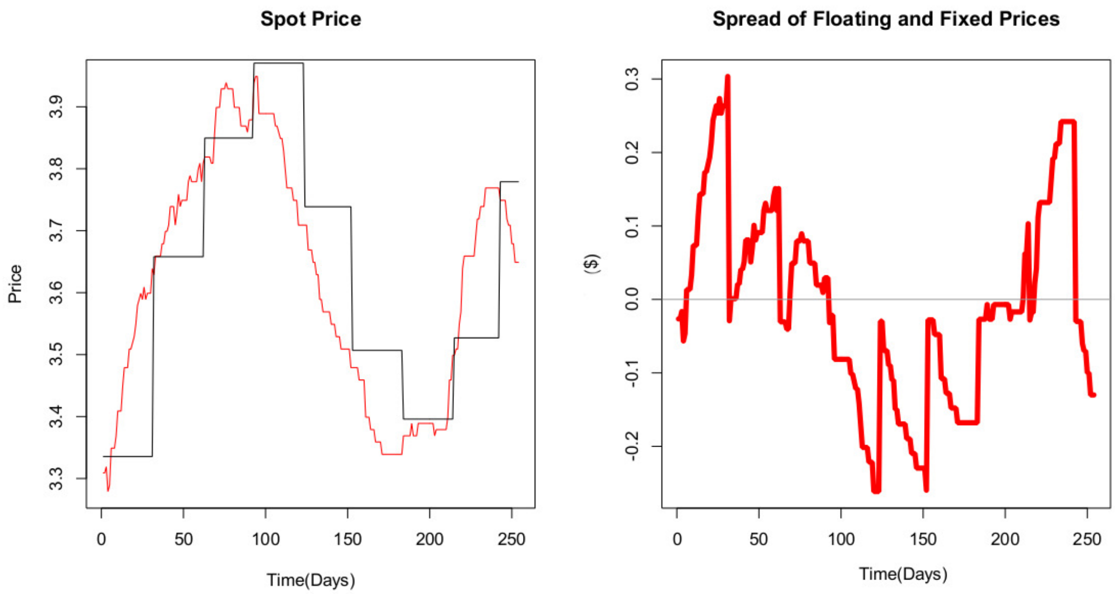


Figure 6.1: *Floating prices(g_t) and monthly constant guaranteed prices(e.g. $G_{1,2,\cdots,9} = 1.008g0_{1,2,\cdots,9}$) are plotted using gasoline price data in US (Firm G, site 2015) from July 2012 to March 2013. On the right, the spread of floating and guaranteed prices of gasoline price for the same site, which changes positively and negatively with time.*

The spread of floating and guaranteed prices, as seen in Figure 6.1, is the difference between floating and customizable guaranteed prices, and the average of which is the value of the floating-to-fixed swap, assuming negligible transaction costs and cash flows occur at nearly the same time. The spread is within \$0.3/gal, but it has variations over time with linear trends either upward or downward on a smaller scale. On a bigger scale, the spread is discontinuous, and shows cyclical patterns because we assume that the guaranteed price is updated monthly adapting to the floating price changes.

While the value of the swap is simply given as the spread between floating and guaranteed prices, we must estimate the expected value (the premium) of the loyalty program, which is designed to be paid by customers upfront at the beginning of each month. The expected value of the loyalty program, with which customers have options to purchase gasoline up to a pre-set quantity (e.g., 40 gal, about four times fill-up for 10 gal tank) at a guaranteed price (G) at any discrete time during finite time horizon $[0, T]$, where T is the option's maturity time (e.g., a month), is not directly observable and easily calculated.

The relative prices in future are uncertain, furthermore, purchasing frequency and quantity at each time would depend on the particular refueling strategy. The purchasing frequency would also depend on the individual's consumption rate and the quantity at each time is bounded by lower bound (when full, 0 gal) and upper bound (when empty, e.g.,10 gal) depending on the state of individual's tank and gasoline price.

6.1.1 Implementation

First, retailers need to set a guaranteed price, which can be matched to or a few cents lower than the guaranteed price that the customers get, with swap providers (e.g., as in forward contract $G = g_0 \exp^{\mu T}$) and set the premium for the loyalty program as discussed in Chapter 4. In order to set the premium, retailers need to know the monthly guaranteed price (G), and to determine the guaranteed price must know the monthly initial floating price (g_0), daily drift rate (μ) and volatility (σ) of retail gasoline price, and the contract time period (T).

All the information is given except the two parameters, the daily drift rate and volatility, that must be estimated. The two parameters can be estimated from historical data over the life of contract looking back. Additionally, swap providers may find speculators to hedge their exposed floating rate risk or may solely enter into the fixed-to-floating swap without speculators. Swap documentation should include the following; guaranteed price (G), the average floating gasoline price (g_t, e.g., average retail gasoline price over a month), the scheduled time for cash flow exchange (e.g., at the end of each month) between retailers and swap providers, the termination date (e.g., end of the month), and the conditions of termination.

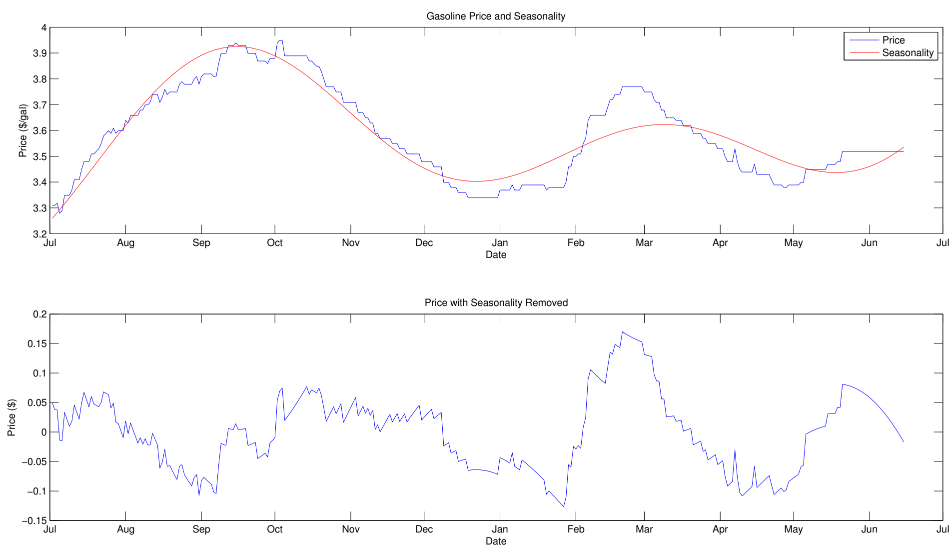


Figure 6.2: *Historical Retail Gasoline Price from July 2012 to June 2013 with (upper panel) and without (lower panel) Seasonality*

6.1.2 Empirical Price Approach

We have solved the optimality problem in retail gasoline market both theoretically and numerically in Chapter 5. In the following Sections, we aim to show that our optimal strategy derived from the optimality problem is effective in practice.

To support the proposed idea of loyalty program and a guaranteed gasoline price in the retail gasoline market, we numerically investigate how much money the guaranteed price could actually have saved individuals who follow the optimal strategy using individual realized price paths. The optimal strategy is to refuel as soon as the floating price drops below the price thresholds, otherwise wait as described in Section 5.3.2 and as obtained in Section 5.5.1.

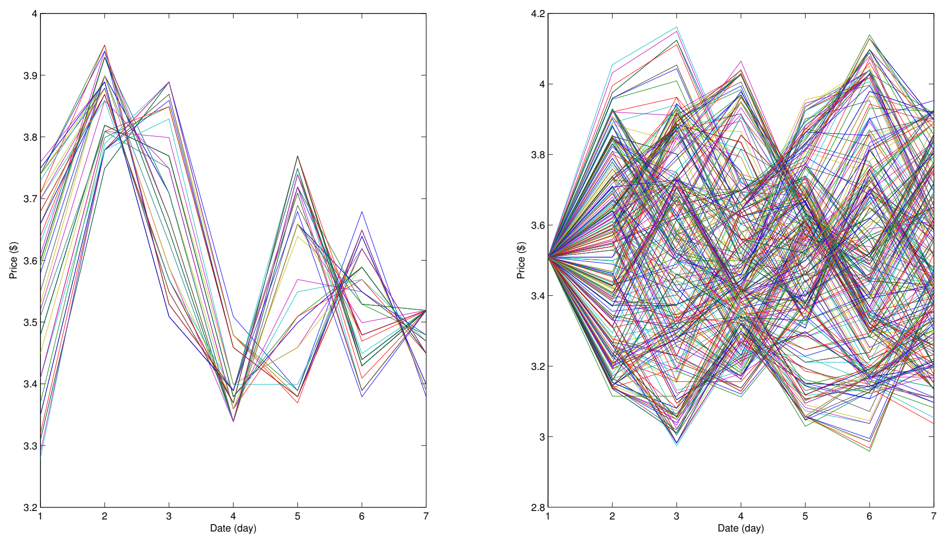


Figure 6.3: *Left panel: Illustration of one of the seven segmented realized prices from July 2012 to June 2013 by 7 days. Right panel: After shifting by one day for each day of week, the seven segmented realized prices are combined and scaled with the same initial prices.*

For the individual realized price paths, the time series of historical retail gasoline price, shown in Figure 6.2, is segmented into 7 days in order to keep using the same model framework and assumptions (7 days to go, fill-up at the end of decision periods). Next, we adjust the weekly seasonality by shifting the segmented prices, see Figure 6.3 (Left panel), by one day for each day of week, and combine them all. Then, we scale the combined prices so that all the price paths can have the same initial value

Table 6.1: *Summary of Calibrated Parameters from the Short-term (7 days) segmented realized prices.*

Summary	μ	σ
Mean	0.0002	0.0033
Max	0.0063	0.0089
Min	-0.0045	0.0000

as illustrated in Figure 6.3 (Right panel).

After this price data manipulation, we re-calibrate μ and σ for short-term (7 days), and compare these parameters with the parameters estimated from the long-term. The long-term (350 days) price changes have only one set of calibrated parameters, while the short-term (7 days) price changes have multiple sets of calibrated parameters (50 segments). Table 6.1 shows the summary statistics of 50 sets of calibrated parameters from the short-term (7 days) price changes.

The single set of calibrated parameters from the long-term (350 days) prices are daily $\mu = 0.0002$ and $\sigma = 0.0046$. The average price drifts and variations of the short-term (7 days) price changes are very close to the long-term price drifts and variations over almost a year, but the calibrated parameters from the short-term (7 days) prices vary between -0.0045 to 0.0063 for μ and between 0 to 0.0089 for σ.

In the following, we compute the total costs over time (7 days) with and without the guaranteed price, assume guaranteed prices, $G = 3.52$, following the optimal strategy. We first study the empirical realized price paths, and then, the larger number of simulated price paths given the price thresholds.

As introduced in Section 5.4.1, we find the expected cost-to-go both with and without guaranteed price that follows the optimal policy found in the Section 5.3.2 for each of the 10 tank-levels, and of 7 days-to-go. The main difference is that we now use individual price paths (where full price paths are known) instead of price distributions (where full price paths are unknown). Then, we compute the value of loyalty program using the individual price paths that a consumer would earn under the optimal strategy. The value, the expected cost-saving, is the difference between the expected cost-to-go with and without guaranteed price throughout 7 days.

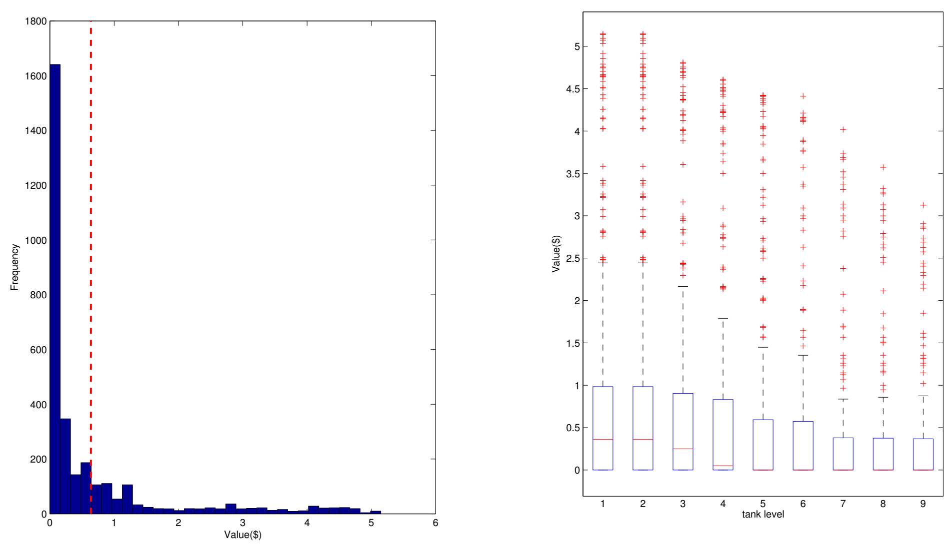


Figure 6.4: *Histogram (Left panel) and Box-plots (Right panel) of the value of loyalty program for the realized price paths for all tank level over 7 days, the mean of a distribution is overlaid with the vertical red dotted line, numerical results based on the empirical data in US (Firm G, site 2015) from July 2012 to June 2013.*

The empirical value of loyalty program under the optimal strategy has some variation changing between \$0 and \$5, see Figure 6.4 (Left panel), increases as the initial tank level gets lower, see Figure 6.4 (Right panel), and the mean expected cost-saving is about \$0.72. Assuming one unit of tank level is consumed at each day and the average tank level is 5, on average 11 units needs to be filled per week.

Thus, the value of the loyalty program is \$0.72/11 units per week which is about 7 cents/unit per week. The empirical results show that the loyalty program under the optimal strategy is beneficial in practice, in that using guaranteed price saves some money when i.e. premium $<$ 7 cents/unit, especially for companies who own fleets of vehicles.

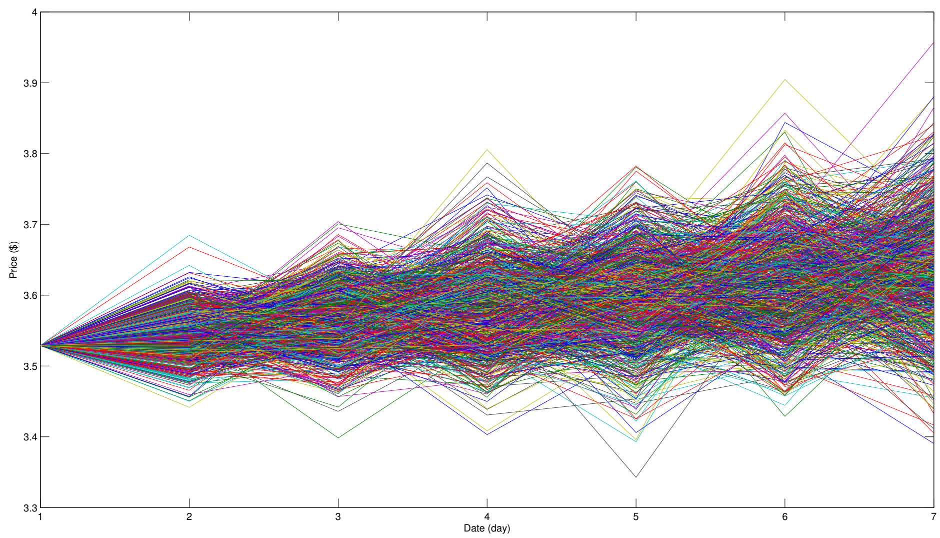


Figure 6.5: *Illustration of 1000 Simulated Individual Price Paths over 7 days*

Next, we repeat our findings with a larger number of simulated price paths. We simulate 1000 price paths using one of the calibrated parameters μ (0.005) and σ (0.0089) from the Short-term (7 days) segmented realized prices. Then, we compute the total costs over time (7 days) for each of the 10 tank-levels, with and without the guaranteed price, assume $G = 3.52$, following the optimal strategy. For 1000 simulated price paths, shown in Figure 6.5, we find the expected cost-saving, difference between the total costs over time with and without guaranteed price, as shown in Figure 6.6. The value of loyalty program using the larger number of simulated price paths is very close to the value using the realized price paths in Figure 6.4. The value lies between \$0 and \$4.5, and the mean value is \$0.78, see Figure 6.6. Thus, the value of loyalty program for the simulated prices is \$0.78/11 units per week or 7 cents/unit per week.

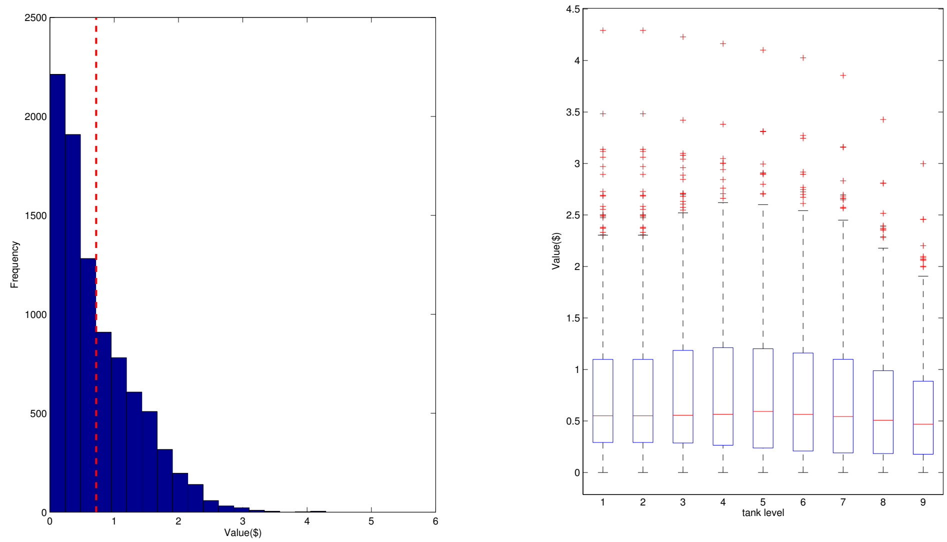
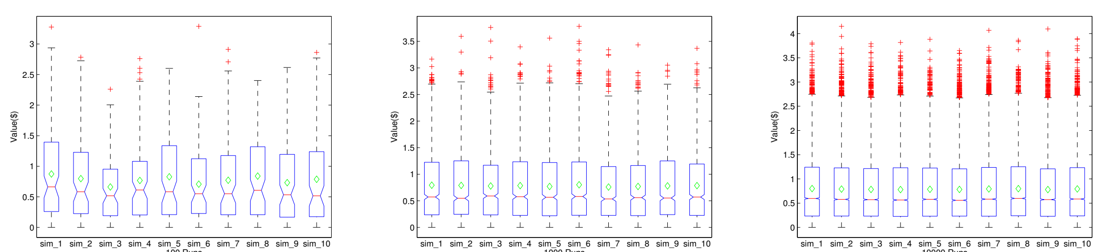


Figure 6.6: *Histogram (Left panel) and Box-plots (Right panel) of the value of loyalty program for the simulated price paths for all tank level, g0=3.52, mu=0.005, and sigma= 0.0089*

In what follows, we examine to get a sense of how variable the values of loyalty program are against the number of simulations. In Figure 6.7, the boxplots shows the variability of the values for each of 100, 1,000, and 10,000 simulated price paths. The variability of the values gets reasonably small from 1,000 samples, in that, the average values (green rhombus) vary between \$0.66 and 0.87 (standard deviation=0.06) for 100 samples, the average values vary between \$0.76 and 0.80 (standard deviation=0.01) for 1,000 samples, and the average values vary between \$0.78 and 0.80



(a) **(b)** **(c)**

Figure 6.7: *(a): Boxplots of the value of loyalty program for 100 simulated price paths (b): Boxplots of the value of loyalty program for 1,000 simulated price paths (c): Boxplots of the value of loyalty program for 10,000 simulated price paths for initial g0=3.52 and tank level of 5, and mu=0.005 and sigma= 0.0089*

(standard deviation=0.0075) for 10,000 samples.

We now compare the value estimated from individual price paths approach with the value estimated from the price distributions in the price grids in Section 5.5.2. It is worth noting that they convey slightly different stories and that their simulated prices are derived from a different stochastic process. The value from the price grids approach tells us given the current state (tank-level and days-to-go), parameters (μ and σ), and current price (g_k), what the expected value is following the optimal policy at a point in time. While, the value from the individual price paths approach tells us given the initial price (g_0), initial tank level, parameters (μ and σ), and total number of days-to-go, what the total expected value is following the optimal policy throughout the time.

Although, the values from two different approaches are not directly comparable, it is still worthwhile to compare the two values for a benchmark under the reasonable assumptions with the same parameters N(μ = 0.005, σ = 0.0089). Also, g_k near G is most interesting and variable as shown in Figure 5.9. Thus, to compare the two approaches we use $g_k = G = 3.52$ and $g_0 = G = 3.52$ as a benchmark, but the values are simply lower when $g_k < G$ or $g_0 < G$, and bigger when $g_k > G$ or $g_0 > G$.

Thus, we assume the current state (tank level=5 and 7 days-to-go), parameters (μ = 0.005 and σ = 0.0089), and current price ($g_k = G = 3.52$) for the price grids approach. Similarly, we assume the initial price ($g_0 = G = 3.52$), initial tank level (tank level=5), parameters (μ = 0.005 and σ = 0.0089), and total number of days-to-go (7 days-to-go) for the individual price paths approach. The values are computed for the price grids approach first, and then, using the same price thresholds the values are computed for the individual price paths approach; for the 1000 individual price paths, 10 runs are repeated to get standard deviation.

Table 6.2: *Estimated values for the parameters (μ = 0.005, σ = 0.0089),all starting from the same fill level of 5. Y is the estimated value for price distributions on price grids, from Y1 to Y10 are the estimated value for individual price paths(each with 10 runs). The first run(row) of Y is compared to 10 runs(10 rows) of Y1, the second run(row) of Y to 10 runs(10 rows) of Y2, $\cdots$, the 10^{th} run(row) of Y to 10 runs(10 rows) of Y10.*

Runs	Y	Y1	Y2	Y3	Y4	Y5	Y6	Y7	Y8	Y9	Y10
1	0.71	0.74	0.78	0.76	0.79	0.80	0.78	0.76	0.80	0.81	0.76
2	0.72	0.79	0.78	0.77	0.77	0.79	0.76	0.77	0.80	0.80	0.81
3	0.71	0.75	0.79	0.80	0.75	0.81	0.75	0.80	0.76	0.82	0.79
4	0.73	0.79	0.80	0.80	0.83	0.80	0.78	0.80	0.76	0.76	0.75
5	0.72	0.79	0.79	0.77	0.77	0.79	0.80	0.79	0.74	0.78	0.78
6	0.71	0.78	0.78	0.77	0.82	0.78	0.81	0.78	0.77	0.79	0.79
7	0.71	0.78	0.77	0.76	0.81	0.81	0.82	0.78	0.78	0.78	0.80
8	0.72	0.80	0.79	0.77	0.79	0.77	0.78	0.78	0.79	0.79	0.74
9	0.72	0.80	0.78	0.80	0.83	0.81	0.77	0.76	0.81	0.77	0.75
10	0.72	0.78	0.77	0.79	0.77	0.77	0.82	0.78	0.80	0.77	0.84
Mean	0.72	0.78	0.78	0.78	0.79	0.79	0.79	0.78	0.78	0.78	0.78
Deviation	0.007	0.01	0.01	0.02	0.03	0.02	0.03	0.01	0.02	0.02	0.03

Table 6.2 shows that the estimated values from the price grids approach, Y, are more consistent with smaller variations (SD = 0.007) than the estimated values from individual price paths approach, Y1-10 with slightly bigger variations (SD = 0.01-0.03). The estimated values from the price grids approach, Y, is also close to the estimated values from individual price paths approach, Y1-10. However, they are not expected to be at the same level because of the aforementioned differences in their price simulation framework (point in time vs. throughout the time) and the different story they convey. Although most of their estimated values from the price grids approach, Y, are within 2 to 3 standard deviation of the mean values from individual price paths approach, Y1-10, the variations may change for different state conditions and

different parameters.

Both numerical results based on empirical and simulated data in this Chapter, where parameters are calibrated from the short-term price variations, provide convincing evidence that it might be worthwhile to offer the guaranteed price to valued customers as a loyalty program free of charge, since the value of the loyalty program would be fairly small at individual level when it is out of the money ($g_k << G$), relatively small when it is near at the money ($g_k \lesssim G$), as long as G is set reasonably higher than g_0. The average value of loyalty program for both simulated prices and the empirical data, is about 7 cents/unit per week for the given $G = 3.52$ when it is near at the money, considering the average gross margin is about 15-20 cents/unit, the loyalty program may be freely offered or offered with small membership fee, i.e. 7 cents/unit $\times$ 15unit about \$1 per week.

Whether the loyalty program may be freely offered or offered with small membership fees, the potential benefits of the loyalty program are still valuable for both agents, especially for customers. The customers have the potential for rewards, but their risk is limited to the price of premium. On the other hand, retailer's direct potential rewards are limited to the price of premium, while their risk is either unlimited in case they do not get into swap contract or limited to the price of swap in case they get into swap contract, which can further be minimized when properly hedged. Instead of fully hedging, retailers can choose to hedge optimal quantity, minimizing their cost of hedging, because hedged portion of gasoline can be sold when floating price is above the guaranteed price, and unhedged portion can be sold when floating price is below the guaranteed price.

Loyalty program and some kinds of customer reward have become a norm in retail and small business such as coffee shop punch card, membership points system, or membership coupon emails due to the associated benefits that outweigh the costs of the reward. The most popular types of loyalty program are in-store rewards program and other incentive to customers who earn membership points by shopping there. Many retail businesses, from a coffee shop to ice cream store, implement these programs for marketing and sales growth. The associated benefits of rewards programs that specifically applies for retail gasoline market is that it attracts new customers and keep them coming back, and brings opportunity for the customers to shop in the convenience store attached to the gas station or to use the other services provided such as car wash and auto shop.

The most important benefits of loyalty program are that both retailers and consumers could hedge against the price and sales fluctuations respectively. As aforementioned retail gasoline prices both periodically and suddenly change, and having the guaranteed price, the sales would become less sensitive to the price change. Also, retailers have the control to make it more prestigious or more attractive to customers and build brand loyalty, by controlling the premium.

In Chapter 7, we provide a high level summary of the key findings, conclusions, recommendations, and provide next steps.

Chapter 7

Conclusion

In this dissertation, we have shown how optimization techniques can be applied to solve complex problems under various price structures; static, dynamic, and stochastic evolution of states. In all cases, the method of dynamic programming was applied in solving the complex problems using practical models in the context of economical theory.

In Chapter 2, we introduced the principle of optimality, optimality function, and how to find the optimal value of the control variable under all possible circumstances.

In Chapter 3, we found optimal prices and profits using the repeated single-period game and the repeated sequential game in the retail gasoline market. We were able to show that retailers do not fully optimize their prices, thus their profits. Given the upstream cost and average competitor's price, we showed that retailers may optimize their profit by optimally responding to their competitors' prices. Then, we showed the evidence of price asymmetry (or collusive behavior) with daily data using the same approach as found in other recent studies. The price response asymmetry has a theoretical and practical importance in examining the response asymmetry between retail prices and costs. We also learned that the market changes its behavior over time, and we need more than one model to capture multiple market characteristics.

In Chapter 4, we introduced a retail gas loyalty program which provide customers with a tool to hedge against the fluctuation of price. To the best of our knowledge, the guaranteed price has never been globally implemented either in North American retail gasoline market making it a very interesting model to study. We also investigated the expected outcomes of the guaranteed price and how optimal quantity of gasoline might be hedged by retailers.

In Chapter 5, we analytically proved existence of optimal decision boundary and studied optimal refueling time and quantity without and with the loyalty program. In contrast to the static dynamic programming studied in Chapter 2, we extended to stochastic dynamic programming where price evolve stochastically. We numerically computed the optimal decision boundary and optimal policies, and found the value of loyalty program. We learned several things from this study. First, it is often difficult to solve a dynamic programming problem analytically, and the dynamic programming equation can sometimes look very messy, especially with uncertainty and continuous transitional density functions. But, often a common pattern can be found by the nature of recursive functions from which we can simplify to find a solution. When the optimality functions are linear with a decision variable, the optimal policy follows either nothing or everything, as known as bang-bang control.

In Chapter 6, we compared the value of loyalty program using simulated approach with the value using empirical approach. We discussed that the empirical results support that the loyalty program may be a great tool for both consumers and retailers hedging against the price and the sales fluctuations respectively. We also argued that it might be worth to freely offer the guaranteed price to valued customers as a loyalty program for the following benefits it has. For instance, retailers might be able to manage the delivery effectively knowing how much volume is expected to be sold by the loyalty program, and truck and taxi companies could also manage fleet price more effectively using the loyalty program.

Future work could include investigating price competition and price cycle behavior in gasoline market using more complex models and techniques such as neural network, forecasting weekly gasoline price using both regional prices and macroeconomic variables, studying the impact of the loyalty program on retailer's own profit and on other retailers' pricing behaviour and investigating loyalty behaviour of card holders with the loyalty program setting.

Appendix A

Illustrative Example for Optimal Refueling Time and Quantity

Application of Inventory Theory

Retailers' Perspective Heuristic:

Retailers may avoid a retail out-of-stock by having a safety stock or a buffer protecting against unexpected demand shocks exceeding the average demand. Depending on the size of the station's tank and the level of traffic, a gas station's order varies from 1,500 to 9,000 gallons every week. We attempt to find an optimal quantity and time to order. As suggested by Dennis (1991) and Giordano (2009), for simplicity, assuming constant ordering costs, carrying costs, and demand rate are known, the optimal order time and quantity can be found as:

$T^* = \sqrt{\frac{2C_o}{DC_c}}$

$Q^* = \sqrt{\frac{2DC_o}{C_c}}$

where

T^* = optimal time between orders in days

Q^* = optimal delivery quantity of gasoline in gallon

C_o = the cost of placing an order per order,

C_c = carrying cost per gallon per day

D = demand rate in gallon per day

For an example, the total sales for Gulf's unleaded gasoline at a specific site is 1,079,373 gallons. Thus, the number of gallons of gasoline sold per day is 2,957.18 gallons, and the number of gallons of gasoline sold per week is 20,700.3 gallons. We assume $C_o = \$700$, and C_c is \$0.01/day.

Then, T^* is, $\sqrt{\frac{2\times700}{2{,}957.18\times0.01}} = 6.88$ days.
and Q^* is, $\sqrt{\frac{2\times2{,}957.18\times700}{0.01}} = 20,347.12$ gallons.

With a safety stock for a medium size gas station assuming $SS = 5,000$ gallons and $LT = 3$ days, an optimal reorder point would be:

$RP = D \times LT + SS$

where

D = demand rate per day

LT = the time required for a supplier to process an order and ship (assume 3 days)

SS = the desired level of safety stock

$RP = D \times LT + SS = 2,957.18$ gallons/day $\times 3$ days$+5,000$ gallons $= 13,871.54$ gallons.

Consumers' Perspective Heuristic:

$T^* = \sqrt{\frac{2C_o}{DC_c}}$

$Q^* = \sqrt{\frac{2DC_o}{C_c}}$

where

T^* = optimal time between refueling in days

Q^* = optimal refueling quantity of gasoline in gallon

C_o = fixed order cost(searching cost) per order,

C_c = holding cost per gallon per day

D = fuel consumption rate in gallon per day

For an example, an average North American mid-size car travels 21 mpg (about 11 L/100 km) city, 27 mpg (about 9 L/100 km) highway. If we assume the average North American drives 13000 miles (about 20000 km) per year, roughly consuming, 13000 miles/year/24 mpg = 541.67 gallons/year (about 2000 L/year), which is about 1.5 gallons/day (D). Assuming gasoline is selling for \$3/gallon and about 4 miles is driven for price searching on average, we estimate the cost of searching, 4 miles/24 mpg $\times$ \$3/gallon= \$0.5 (C_o). The carrying cost can be estimated to be \$0.0069/gallon because 15% more gasoline is spent for extra 400 pounds, that is, about 0.0375% gallons/pound, thus 0.0375% gallons/pound $\times$ 6.2 pounds/gallon (density of gasoline) $\times$ \$3/gallon = \$0.0069/gallon (C_c).

Then, T^* is, $\sqrt{\frac{2\times0.5}{1.5\times0.0069}} = 9.82$ days.
and Q^* is, $\sqrt{\frac{2\times1.5\times0.5}{0.0069}} = 14.74$ gallons.

With a 15 gallons fuel tank car, purchasing 14.74 gallons per visit is certainly possible. Thus, the optimal strategy for average North American who drives about 13000 miles evenly over a year is filling up every 10 days.

Appendix B

Price Threshold Conditions: Binomial Tree Approach as an Approximation to GBM

Consider the special case in which a price at time t of x becomes, at time $t+1$, a price of Ux (probability $\frac{1}{2}$) or Dx (probability $\frac{1}{2}$). in order for this to make sense we need $0 < D < U$.

Then,

$$
\begin{aligned}
F(y|x) &= 0, \quad y < Dx, \\
&= 0.5, \quad Dx \leq y < Ux, \\
&= 1, \quad y \geq Ux
\end{aligned}
$$

Thus $A(x) = \int_0^G F(y|x)dy$ is given by

$$
\begin{aligned}
A(x) &= \int_{Dx}^{Ux} 0.5dy + \int_{Ux}^{G} dy = 0.5(Ux - Dx) + G - Ux = G - 0.5(U+D)x, \quad x \leq \frac{G}{U} \\
A(x) &= \int_{Dx}^{G} 0.5dy = 0.5(G - Dx), \quad Dx < G < Ux, \quad \text{or} \quad \frac{G}{U} < x \leq \frac{G}{D} \\
A(x) &= 0, \quad x > \frac{G}{D}
\end{aligned}
$$

Now, $C_{T-1}(x) = x - G + A(x)$, so

$$
\begin{aligned}
c_{T-1}(x) &= x(1 - 0.5(U+D)), \quad x \leq \frac{G}{U} \\
c_{T-1}(x) &= x(1 - 0.5D) - 0.5G, \quad \frac{G}{U} < x \leq \frac{G}{D} \\
c_{T-1}(x) &= x - G, \quad x > \frac{G}{D}
\end{aligned}
$$

This is very instructive. The curve here is piecewise linear and continuous with $c(0) = 0$. The slopes of the three piecewise segments are increasing, so there can't be more than one root.

If $U + D > 2$ then we have that the initial slope of $c(x)$ is negative, but it then becomes positive; if $D < 2$ at the next segment, but for certain by the third segment. So in this case we have a single root.

On the other hand, if $U + D < 2$ we have that the initial slope is already positive, and only rises faster and faster. Here we have no (positive) roots except for the trivial one $c(0) = 0$.

This is a very meaningful and natural separation of models. Recall that the expected value of the gas at time $t+1$ is $\frac{U+D}{2}$.

So if $U + D > 2$ the gas price, on average, rises. As such, without a guaranteed price, you would always buy gas right away. With the guaranteed price, G, it might occasionally make sense to wait, gambling that a big move down might happen, knowing that a big move up won't hurt you much as you will never pay more than G for the gas.

If $U + D < 2$, on the other hand, the gas price falls on average, making it worthwhile to wait even without the guaranteed price to protect you. So nothing interesting happens.

If $U + D = 2$ the situation is totally pathological, with infinitely many roots.

Case 1: Let g_{k+1} be the next period price and g_k be the current period price, then, $g_{k+1} = U \times g_k$ with probability p, and $g_{k+1} = D \times g_k$ with probability $1 - p$; where $0 < D < 1 < U$.

For g_k such that, $g_k \leq \frac{G}{U}$, the slope of $C(g_k)$ is negative, and the slope of $C(g_k)$ turns positive for $g_k > \frac{G}{U}$ when $U + D > 2$, whereas the slope of $C(g_k)$ is always positive for g_k when $U + D < 2$, and the slope of $C(g_k)$ is zero for $g_k \leq \frac{G}{U}$, and the slope of $C(g_k)$ turns positive for $g_k > \frac{G}{U}$ when $U + D = 2$.

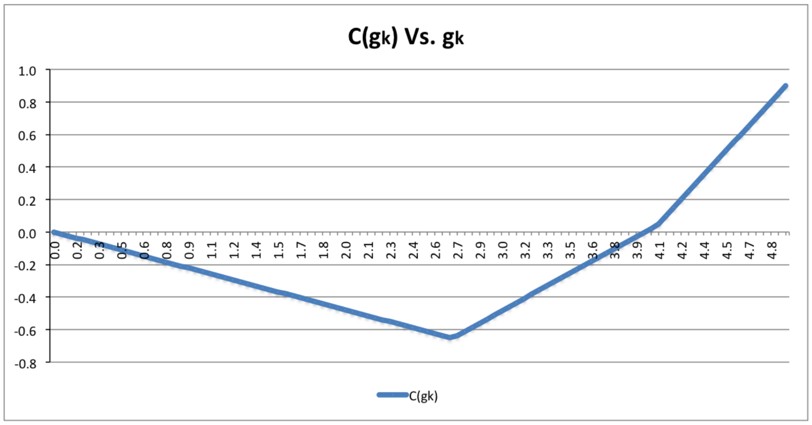


Figure B.1: *$c(g_k)$ against g_k when $U + D > 2$, where $G = 4, U = 1.5, D = 0.99$ for one period.*

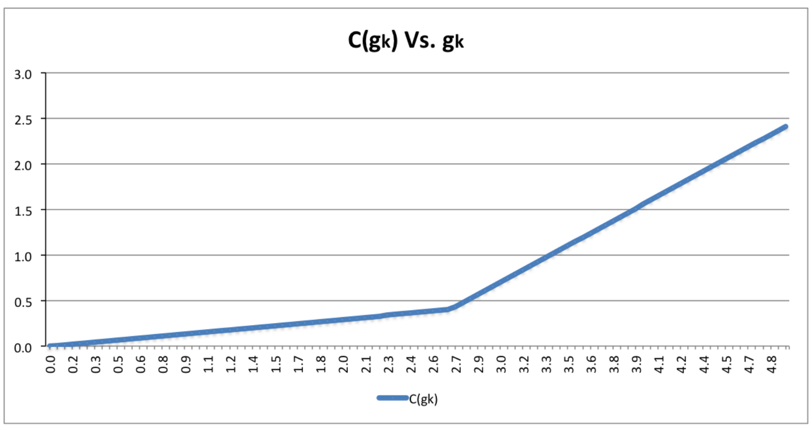


Figure B.2: *$c(g_k)$ against g_k when $U + D < 2$, where $G = 4, U = 1.5, D = 0.2$ for one period.*

Threfore, price thresholds only exist when $U + D >= 2$ and it is not sensitive to the changes in U but to the changes in D; $\hat{g}$ decreases as D decreases.

There are infinitely many roots when $U + D = 2$, and the price threshold is the biggest root.

Figures B.4 and B.5 show that $\hat{g}$ does not depend on U, but on D, with condition of $U + D \geq 2$.

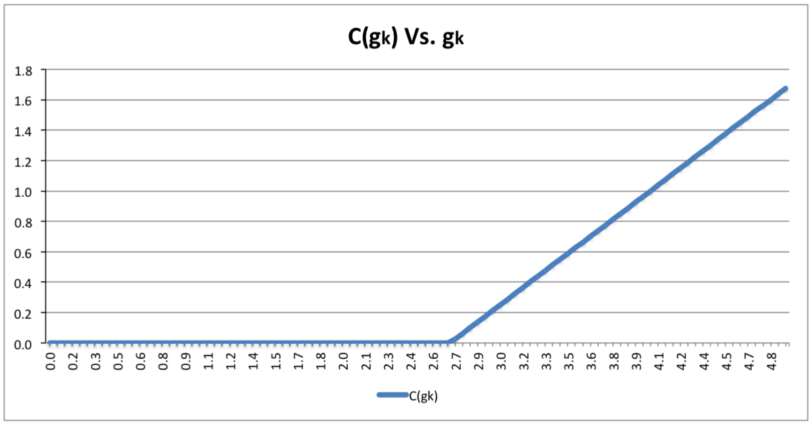


Figure B.3: *$c(g_k)$ against g_k when $U + D = 2$ where $G = 4, U = 1.5, D = 0.5$ for one period.*

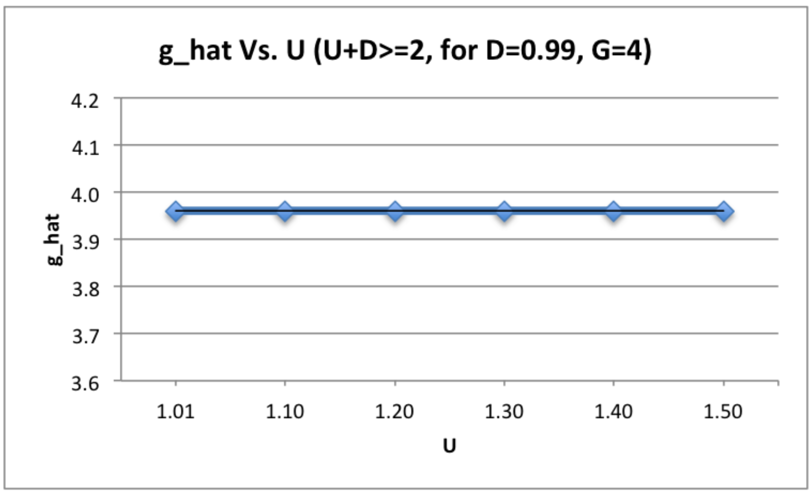


Figure B.4: *$\hat{g}$ against U when $U + D > 2$ while fixing $D = 0.5$ and $G = 4$ for one period.*

The one period model can be expanded for multiple periods-to-go. For specific U, D, and G. Figure B.6 shows $\hat{g}$ for multiple periods fixing parameters at one value, $U = 1.5, D = 0.99$ and $G = 4$.

The logged percentage change over one day, $\log(\frac{g_t}{g_{t-1}})$, is also identically distributed and has a normal distribution with mean μ and variance σ^2. Thus we can approximate geometric BM over the fixed time interval by matching the mean and variance with proper u, d, p:

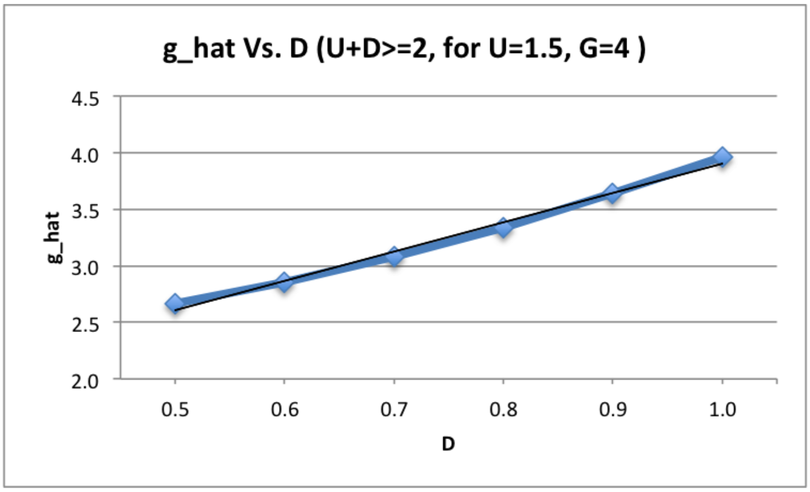


Figure B.5: *$\hat{g}$ against D when $U + D > 2$ while fixing $U = 1.5$ and $G = 4$ for one period.*

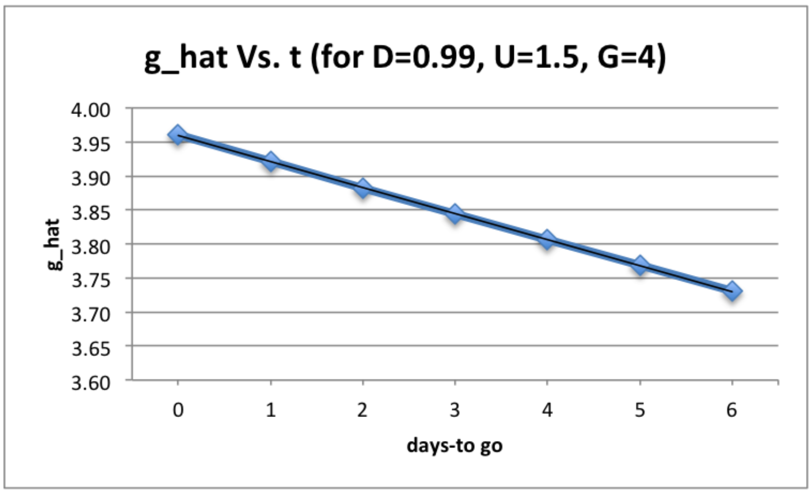


Figure B.6: *$\hat{g}$ against the number of days-to-go when $U + D > 2$, where $G = 4, U = 1.5, D = 0.99$ for multiple periods.*

$E[\frac{g_t}{g_{t-1}}] = \exp^{\mu+\frac{\sigma^2}{2}} = pu + (1-p)d$

$Var[\frac{g_t}{g_{t-1}}] = \exp^{2\mu+2\sigma^2} = pu^2 + (1-p)d^2$

Appendix C

Supplementary Results and Proofs: Proposition 5.3.10 with a normal density function

Proof by induction:

for $t = T - 1$

$$V_{T-1}(s = (k_t, g_t)) = \inf_{a_t}\{\min(g_t, G)a_t + \int_{-\infty}^{\infty} V_T(s' = (k_t - 1 + a_t, g_{t+1}))f(g_{t+1}|g_t)dg_{t+1}\}$$
$$V_{T-1}(s = (k_t, g_t)) = \inf_{a_t}\{\min(g_t, G)a_t + (K - k_t + 1 - a_t)\int_{-\infty}^{\infty} \min(g_{t+1}, G)f(g_{t+1}|g_t)dg_{t+1}\}$$

$$\begin{aligned}
V_{T-1}(s) &= \inf_{a_t}\{\min(g_t, G)a_t + (K - k_t + 1 - a_t)\int_{-\infty}^{\infty}\min(g_t + z\sigma + \mu, G)\phi(z)dz\} \\
&= \inf_{a_t}\{\min(g_t, G)a_t \\
&+ (K - k_t + 1 - a_t)(\int_{-\infty}^{\frac{G-g_t-\mu}{\sigma}}(g_t + z\sigma + \mu)\phi(z)dz + \int_{\frac{G-g_t-\mu}{\sigma}}^{\infty}G\phi(z)dz)\} \\
&= \inf_{a_t}\{a_t[\min(g_t, G) - G - (g_t + \mu - G)\Phi(\frac{G - g_t - \mu}{\sigma}) + \sigma\phi(\frac{G - g_t - \mu}{\sigma})]\} \\
&+ (K - k_t + 1)((g_t + \mu)\Phi(\frac{G - g_t - \mu}{\sigma}) - \sigma\phi(\frac{G - g_t - \mu}{\sigma}) \\
&+ G(1 - \Phi(\frac{G - g_t - \mu}{\sigma})))
\end{aligned}$$

Let $c_{T-1}(g_t) = g_t - G - (g_t + \mu - G)\Phi(\frac{G-g_t-\mu}{\sigma}) + \sigma\phi(\frac{G-g_t-\mu}{\sigma})$, which is a increasing function in g_t while $g_t < G$ and let $\hat{g}_{T-1}$ be g_t such that $c_{T-1}(g_t) = 0$. If $c_{T-1}(g_t) < 0$, it is a negative linear function with respect to a_t. Thus, it is minimized when a_t^* is as large as possible, and $a_t^* = K - k_t$. If $c_{T-1}(g_t) \geq 0$, it is a positive linear function with respect to a_t. Thus, it is minimized when a_t^* is as small as possible, hence $a_t^* = 0$.
If $g_t < \hat{g}_{T-1}$, $a_t^* = K - k_t$

$$V_{T-1}(s) = g_t(K - k_t) + G + (g_t + \mu - G)\Phi(\frac{G - g_t - \mu}{\sigma}) - \sigma\phi(\frac{G - g_t - \mu}{\sigma})$$

If $g_t \geq \hat{g}_{T-1}$, $a_t^* = 0$

$$V_{T-1}(s) = (K - k_t + 1)(G + (g_t + \mu - G)\Phi(\frac{G - g_t - \mu}{\sigma}) - \sigma\phi(\frac{G - g_t - \mu}{\sigma}))$$

Assume the proposition holds for all $t = j+1, j+2, \cdots, T-1$

Prove $a_t^* = K - k_t$ if $g_t < \hat{g}_t$ and $a_t^* = 0$ if $g_t \geq \hat{g}_t$ for $t = j$

$$V_j(s) = \inf_{a_t}\{\min(g_t, G)a_t + \int_{-\infty}^{\hat{g}_{j+1}-g_t} V_{j+1}(s')f(g_{t+1}|g_t)dg_{t+1} + \int_{\hat{g}_{j+1}-g_t}^{\infty} V_{j+1}(s')f(g_{t+1}|g_t)dg_{t+1}\}$$

$$\begin{aligned}
V_j(s) = \inf_{a_t}\{&\min(g_t, G)a_t \\
&+ \int_{-\infty}^{\hat{g}_{j+1}-g_t} \Big((g_{t+1})(K - k_t + 1 - a_t) + N_{j+1}(g_{t+1}) + R_{j+1}(g_{t+1})\Big) f(x)dx \\
&+ \int_{\hat{g}_{j+1}-g_t}^{\infty} \Big((K - k_t + 2 - a_t)N_{j+1}(g_{t+1}) + R_{j+1}(g_{t+1})\Big) f(x)dx\} \\
&= \inf_{a_t}\{a_t\Big(\min(g_t, G) - \int_{-\infty}^{\hat{g}_{j+1}-g_t} (g_{t+1})f(x)dx - \int_{\hat{g}_{j+1}-g_t}^{\infty} N_{j+1}(g_{t+1})f(x)dx\Big)\} \\
&+ (K - k_t + 1)\Big(\int_{-\infty}^{\hat{g}_{j+1}-g_t} (g_{t+1})f(x)dx + \int_{\hat{g}_{j+1}-g_t}^{\infty} N_{j+1}(g_{t+1})f(x)dx\Big) \\
&+ \int_{-\infty}^{\infty} N_{j+1}(g_{t+1})f(x)dx + \int_{-\infty}^{\infty} R_{j+1}(g_{t+1})f(x)dx
\end{aligned}$$

Where

$$R_j(g_t) = \int_{-\infty}^{\infty} N_{j+1}(g_{t+1})f(x)dx + \int_{-\infty}^{\infty} R_{j+1}(g_{t+1})f(x)dx$$

$$N_j(g_t) = \int_{-\infty}^{\hat{g}_{j+1}-g_t} (g_{t+1})f(x)dx + \int_{\hat{g}_{j+1}-g_t}^{\infty} N_{j+1}(g_{t+1})f(x)dx$$

Let $c_j(g_t) = \min(g_t, G) - \int_{-\infty}^{\hat{g}_{j+1}-g_t}(g_{t+1})f(x)dx - \int_{\hat{g}_{j+1}-g_t}^{\infty} N_{j+1}(g_{t+1})f(x)dx$, which is a locally increasing function in g_t while $g_t < G$ and let $\hat{g}_j$ be g_t such that $c_j(g_t) = 0$. If $c_j(g_t) < 0$ (or $g_t < \hat{g}_j$), it is a negative linear function with respect to a_t. Thus, it is minimized when a_t is as large as possible, and $a_t^* = K - k_t$, for $t = j$.

$$V_j(s) = g_t(K - k_t) + N_j(g_t) + R_j(g_t)$$

If $c_j(g_t) \geq 0$ (or $g_t \geq \hat{g}_j$), it is a positive linear function with respect to a_t. Thus, it is minimized when a_t is as small as possible, hence $a_t^* = 0$, for $t = j$.

$$V_j(s) = N_j(g_t)(K - k_t + 1) + R_j(g_t)$$

Appendix D

Parameter Calibration

We calibrate parameters for standard geometric Brownian motion, and the time interval, $t_i - t_{i-1}$, is fixed and there is no random time changes. The parameters were estimated from historical data. Average price was used from a sample taken from the historical data.

First, we define $X_i = log(\frac{g_{t_i}}{g_{t_{i-1}}}) \sim N(\hat{\mu}, \hat{\sigma})$, the drift and variance parameters $\hat{\mu}$ and $\hat{\sigma}$ were estimated for normal distribution.
Estimated $\hat{\mu} = 0.0001768$, $\hat{\sigma} = 0.004614$;
Since $\hat{\mu} = (\mu - \frac{\sigma^2}{2})(t_i - t_{i-1})$, $\hat{\sigma} = \sigma\sqrt{(t_i - t_{i-1})}$ and $t_i - t_{i-1} = 1$,

$$\mu = \frac{\hat{\mu}}{t_i - t_{i-1}} + \frac{\sigma^2}{2} = 0.0001874,$$
$$\sigma = \frac{\hat{\sigma}}{\sqrt{t_i - t_{i-1}}} = 0.004614$$

Daero Kim's Curriculum Vitae

Post secondary Education and Degrees:

- University of Western Ontario

  London, Ontario, Canada

  2003-2007 B.Sc. in Biophysics

- University of Western Ontario

  London, Ontario, Canada

  2007-2009 M.Sc. in Biophysics

- University of Western Ontario

  London, Ontario, Canada

  2009-2011 M.Sc. in Applied Mathematics

- University of Western Ontario

  London, Ontario, Canada

  2011-2018 Ph.D. in Applied Mathematics

Honours and Awards:

- Ontario Graduate Scholarship

  2009-2011

- CIHR Strategic Training Program in Vascular Research

  2007-2009

- Dean's Honor list, Faculty of Science

  2003-2006

Related Work Experience:

- Senior Analyst, Risk Capital and Loss Forecasting Methodology

  BMO Financial Group

  2015-Present

- Teaching Assistant in Applied Mathematics

  The University of Western Ontario

  2009-2015

- Teaching Assistant in Biophysics

  The University of Western Ontario

  2007-2009

Publications:

- Kim, Daero, Davison, Matt and Ødegaard, Fredrik. (2015) "Do Retailers Set Optimal Prices in the Case of Retail Gasoline Market?", International Journal of Revenue Management, (Vol. 8, Issue. 3-4, pp. 241-259).

- Davison, M., Kim, D., and Keller, H. (2011). 'Radiotherapy Dose Fractionation under Parameter Uncertainty', In advances in mathematical and computational methods: addressing modern challenges of science, technology, and society (Vol. 1368, No. 1, pp. 179-182). AIP Publishing.