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Local Higher Category Theory

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Abstract

The purpose of this thesis is to give presheaf-theoretic versions of three of the main extant models of higher category theory: the Joyal, Rezk and Bergner model structures. The construction of these model structures takes up Chapters 2, 3 and 4 of the thesis, respectively. In each of the model structures, the weak equivalences are local or stalkwise weak equivalences. In addition, it is shown that certain Quillen equivalences between the aforementioned models of higher category theory extend to Quillen equivalences between the various models of local higher category theory.

Throughout, a number of features of local higher category theory are explored. For instance, at the end of Chapter 3, descent for the local Joyal model structure is given a simple characterization in terms of descent for the Jardine model structure. Interestingly enough, this characterization is a consequence of the Quillen equivalence between the local complete Segal model structure and the local Joyal model structure.

The right properness of the local Bergner model structures means that one has an attractive theory of cocycles and torsors. This theory can be interpreted as defining non-abelian H^1 with coefficients in an ∞ -groupoid, generalizing the results found in [15, Chapter 9]. This is explored in the last section of Chapter 4.

Keywords: Homotopy Theory, Grothendieck Topoi, Quasi-Categories, Complete Segal Spaces, Descent, Simplicial Categories, Non-Abelian Cohomology

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Notational Conventions

For each $n \in \mathbb{N}$, write $[\mathbf{n}]$ for the ordinal number category with $n + 1$ objects $\{0, 1, \dots, n - 1, n\}$. We write Set , sSet for the categories of sets and simplicial sets, respectively.

We will write $\text{hom}_C(x, y)$ for the set of morphisms between two objects x, y of a category C . Oftentimes, we will omit the subscript, because the category is obvious from the context. Given a small category C , we will write $\text{Iso}(C)$ for the subcategory of C whose objects are the objects of C and whose morphisms are the isomorphisms of C . Given a small category C , we write $\text{Mor}(C)$ and $\text{Ob}(C)$ for the set of morphisms and objects of C , respectively. In the internal description of the category, the source, target, identity and composition maps are respectively denoted s, t, ident, c .

Given a category C enriched over simplicial sets, we will write $\text{Ob}(C)$ and $\text{Mor}(C)$ for the set of objects and the simplicial sets of morphisms, respectively. The source, target, identity and composition maps are respectively denoted s, t, ident, c in the internal description of the simplicially enriched category.

Given a small category C , write $B(C)$ for the nerve of C . Given two simplicial sets K, Y , write Y^K for the simplicial set whose n -simplices are maps $\text{hom}(K \times \Delta^n, Y)$.

Bisimplicial sets are functors $X : \Delta^{op} \times \Delta^{op} \rightarrow \text{Set}$. We write $X_{m,n}$ for $X(m, n)$. We write $s^2\text{Set}$ for the category of bisimplicial sets. Given two simplicial sets K, L , write $K \tilde{\times} L$ for the bisimplicial set whose (m, n) bisimplices are $K_m \times L_n$. We refer to $X_{m,n}$ as the (m, n) bisimplices of X . Given two bisimplicial sets K and X , there is a simplicial set $\mathbf{hom}(K, X)$, whose n -simplices are elements of $\text{hom}((\Delta^0 \tilde{\times} \Delta^n) \times K, X)$. We write X^K for internal hom in bisimplicial sets.

Throughout the document, we will fix a small Grothendieck site \mathcal{C} . Thus, $s\mathbf{Pre}(\mathcal{C})$ and $s^2\mathbf{Pre}(\mathcal{C})$ are the simplicial and bisimplicial presheaves on \mathcal{C} , respectively. Similarly, $s\mathbf{Sh}(\mathcal{C})$ and $s^2\mathbf{Sh}(\mathcal{C})$ are the simplicial and bisimplicial sheaves on \mathcal{C} , respectively.

Central to this thesis is the use of the **injective model structure** on $s\mathbf{Pre}(\mathcal{C})$, in which the weak equivalences are ‘stalkwise weak equivalences’ and cofibrations are the monomorphisms. We will call its weak equivalences **local weak equivalences** and its fibrations **injective fibrations**. The model structure is constructed in [15, Chapters 4 and 5]. Given a simplicial set K , we will also denote by K the constant simplicial presheaf defined by $U \mapsto K$ for all $U \in \text{Ob}(\mathcal{C})$. Moreover, given two simplicial presheaves, X, Y , write $\mathbf{hom}(X, Y)$ for the simplicial set defined by $\mathbf{hom}(X, Y)_n = \text{hom}(X \times \Delta^n, Y)$.

Chapter 1

Introduction

Introduction

Local homotopy theory is the study of model structures on simplicial presheaves and their applications. The term ‘local’ refers to the fact that the weak equivalences are defined stalkwise rather than sectionwise. The prototypical example of a local model structure is the Jardine model structure on simplicial presheaves in which the cofibrations are monomorphisms and the weak equivalences are stalkwise standard weak equivalences of simplicial sets. The Jardine model structure has found numerous applications in algebraic geometry, abstract homotopy theory and theoretical computer science. For instance, classical sheaf cohomology can be viewed as a ‘derived’ construction of the Jardine model structure; it is represented in the homotopy category by certain objects called Eilenberg-Mac Lane objects. Giraud’s theory of non-abelian cohomology can be reworked in homotopy-theoretic terms using Jardine’s theory (see [15, Chapter 9]). Finally, following Simpson (see [32]), we can use this model structure to give a homotopy-theoretic definition of higher stacks.

The purpose of this thesis is to establish rigorous foundations of local higher category theory, in which one considers presheaf-theoretic extensions of the various models of higher category theory. The following three chapters will each focus on one model of local higher category theory, based on the Joyal, Rezk and Bergner model structures, respectively. One of the central themes of this thesis is that these model structures can be connected by a zig-zag of Quillen equivalences, which extend the Quillen equivalences linking various models of higher category theory.

The first model of local higher category theory was found in Hirschowitz and Simpson’s paper ‘Descent Pour Les N-Champs’ ([10, Theorem 3.2]); the weak equivalences are stalkwise weak equivalences in the injective model structure on Segal precategories. This model structure was originally invented to serve as a setting for $(\infty, 1)$ -stack theory; an $(\infty, 1)$ -stack is a presheaf of Segal categories which is sectionwise weakly equivalent to its fibrant replacement in the model structure of [10, Theorem 3.2]. More generally, [10] establishes a theory of (∞, n) -stacks for all $n \geq 1$. The theory of (∞, n) -stacks is built up inductively using $(\infty, 0)$ -stacks (simplicial presheaves satisfying descent with respect to the Jardine model structure) as a base case.

Thus, one of the major reasons for studying local higher category theory is as a setting for higher stack theory. The Quillen equivalences between the various models of higher category theory extend to the presheaf level and thus we can use, say, the local Joyal or local complete Segal model structures to study higher stacks in the sense of Simpson. These model structures have some technical advantages over Simpson’s original presentation which will be discussed further in the chapter introductions.

Another motivation for studying local higher category is that it potentially allows one to define new variants of cohomology theory as ‘derived constructions’ of the various models of local higher category theory.

A derived construction of the local Joyal model structure may be particularly useful in computer science applications. Sheaf cohomology has found its way into applications such as the study of coding networks and electric circuits (see [7], [31]). However, the disadvantage with ordinary sheaf cohomology is that, as a derived construction on the homotopy category of the classical Jardine model structure (see [15, Chapter 8]), it doesn’t always detect directionality to the extent that one would like; directionality is inherently very important in these applica-

tions. For instance, the path category is not an invariant of the classical Kan model structure. However, it is an invariant of the Joyal model structure. Thus, a derived construction of the local Joyal model structure could potentially provide a refined cohomology theory for computer science applications.

In Section 4.5 of the thesis, we provide the first example of a variant of cohomology theory defined using local higher category theory. Following Jardine’s approach to non-abelian cohomology found in [15, Chapter 9], we can define non-abelian H^1 with coefficients in an ∞ -groupoid. An ∞ -groupoid can be interpreted as a higher category in which morphisms are only invertible up to homotopy.

The thesis is arranged into an introduction and three chapters, corresponding to the three models of local higher category theory.

Chapter 2 discusses the local Joyal model structure. The first section of this chapter reviews properties of the Joyal model structure that are relevant to the proof of the existence of the local Joyal model structure. For instance, the section includes a characterization of fibrations in the Joyal model structure whose target is a quasi-category (2.1.27) and a characterization of Joyal equivalences in terms of standard weak equivalences (2.1.25). The latter allows one to exploit the Jardine model structure and the technique of Boolean localization in the proof of the existence of the local Joyal model structure.

The relevant facts about the Jardine model structure and Boolean localization are reviewed in the second section of the chapter. The local Joyal model structure is constructed in the final section.

Chapter 3 concerns the local analogue of the complete Segal model structure. We define this as a left Bousfield localization of the Jardine model structure on an appropriately defined site. The advantage of this is that there is a good description of localizations of the Jardine model structure (see [15, Section 7]). This makes the fibrant objects and the phenomena of descent easier to describe. It is also possible that one can define a model of local (∞, n) -category theory more generally using this localization technique, as Bergner and Rezk define a model of (∞, n) -category theory using localization theory in [3]. The main technical difficulty in this chapter is showing that the model structure obtained is Quillen equivalent to the local Joyal model structure; this takes up most of the chapter. This ultimately shows that one can recover the usual definition of local weak equivalences in terms of Boolean localization (see 3.3.12).

The final section of this chapter is devoted to proving a characterization of descent with respect to the local Joyal model structure, in terms of descent with respect to the Jardine model structure. The proof of this statement depends on the Quillen equivalence with the complete Segal model structure and exploits the localization theory. This simple characterization is highly useful, and will be exploited extensively in a future work ([26]).

The final chapter concerns the local Bergner model structure. The primary advantage of a local analogue of the Bergner model structure is that, unlike other models of local higher category theory, it is right proper. This means that one can describe the homotopy category in terms of cocycles. The use of cocycle categories is the essential ingredient to establishing the variant of non-abelian cohomology theory defined at the end of the chapter. Cocycle categories were also essential to Jardine’s approach to non-abelian cohomology.

The local Bergner model structure is the most technical of the model structures to establish. The weak equivalences are defined to be maps $f : X \rightarrow Y$ of presheaves of simplicial categories

such that

1. The following diagram is homotopy cartesian for the Jardine model structure

$$\begin{array}{ccc}
 \text{Mor}(X) & \longrightarrow & \text{Mor}(Y) \\
 \downarrow & & \downarrow (s,t) \\
 \text{Ob}(X) \times \text{Ob}(X) & \longrightarrow & \text{Ob}(Y) \times \text{Ob}(Y)
 \end{array}$$

2. $\pi_0(X) \rightarrow \pi_0(Y)$ is a local equivalence of presheaves of categories (4.2.2), where π_0 is the path component functor for simplicial categories applied sectionwise.

A complicating factor is that cofibrations in the Bergner model structure are not easy to describe; it is difficult to show that they behave appropriately with Boolean localization. It is because of this that one has to take the cofibrations to be ‘projective cofibrations.’

A major theme of this chapter is that the Quillen equivalence

$$\mathcal{C} : \text{sSet} \rightleftarrows \text{sCat} : \mathcal{B}$$

between the Bergner and Joyal model structures extends to the local theory. This Quillen equivalence for ordinary higher category theory is discussed extensively in the first section of the chapter. The treatment follows that of [19].

In a forthcoming paper ([26]), the local Joyal model structure will be applied to construct various interesting examples of higher stacks. In particular, the higher stack of unbounded chain complexes will be constructed. Informally, this allows one to study how unbounded chain complexes are glued together along quasi-isomorphisms. Other examples of higher stacks constructed include the higher stack of higher stacks and the higher stack of complexes of quasi-coherent sheaves (or vector bundles). Many of the results are analogues of results found in Hirschowitz and Simpson ([10]). Although most of the general ideas come from [10], the quasi-categorical machinery employed leads to significantly simplified proofs and in some cases generalizations.

In conclusion, the models of local higher category theory discussed allow one to extend many of the classical applications of local homotopy theory. However, we have only scratched the surface in understanding the ways in which local higher category theory can be used to study stack theory and cohomology theory. Nonetheless, the results obtained offer a tantalizing glimpse of the possibilities. These model structures should also lead to novel computer science applications.

Chapter 2

The Local Joyal Model Structure

Introduction

The purpose of this chapter is to develop an analogue of the injective (Jardine) model structure on simplicial presheaves in which, rather than having the weak equivalences be ‘local Kan equivalences’, the weak equivalences are ‘local Joyal equivalences’. This model structure is called the **local Joyal model structure**. The results of this chapter appear in [24].

It can be shown ([15, Theorem 8.25]) that for a sheaf of groups A on a space X , there is a natural isomorphism:

$$H^n(X, A) \cong [*, K(n, A)]_{s\text{Pre}(X)},$$

where $H^n(X, A)$ is sheaf cohomology, $[*,]_{s\text{Pre}(X)}$ denotes the maps in the homotopy category of the injective model structure and $K(n, A)$ is the **Eilenberg Mac Lane object** associated to A and to n . More generally, given a simplicial presheaf X and a sheaf of groups A , the n th cohomology of X with coefficients in A can be defined as:

$$H^n(X, A) = [X, K(n, A)]_{s\text{Pre}(\mathcal{C})}.$$

Hence, it seems plausible that by considering maps

$$[X, T]_{s\text{Pre}(\mathcal{C})}$$

in the homotopy category of the local Joyal model structure for presheaves of quasi-categories X and T , we may obtain an interesting variant of cohomology theory.

It is possible that this may be useful in applications where directionality is important, such as in theoretical computer science (see [7], [31]). Indeed, the Joyal model structure seems much more sensitive to directionality than the usual Kan model structure on simplicial sets. In particular, Joyal equivalences induce equivalences of path categories (see 2.1.5).

Traditionally, one says that a sheaf of groups is a stack if and only if it satisfies an effective descent condition (see [15, pg. 276-277]). It can be shown that a sheaf of groups G satisfies effective descent if and only if $B(G)$ is sectionwise equivalent to its fibrant replacement in the injective model structure.

More generally, one can define a presheaf of Kan complexes to be a higher stack if and only if it is sectionwise equivalent to its fibrant replacement in the Jardine model structure. This concept of higher stack was studied extensively by Carlos Simpson (see [32]).

One advantage of descent is that, given a presheaf of Kan complexes X on a site \mathcal{C} that satisfies descent, one can produce a descent spectral sequence

$$E_2^{p,q} = H^p(\mathcal{C}, L^2(\pi_s(X))) \implies \pi_{t-s}(X(S))$$

which computes the homotopy groups of the space of global sections $X(S)$. Such descent spectral sequences are of fundamental interest in algebraic K-theory, which is where the study of homotopy-theoretic descent originated. For instance, the Quillen-Lichtenbaum conjecture can be viewed as a statement about the convergence of a descent spectral sequence (see [17] for more details).

Given the many applications of injective descent, and given the close analogy between the local Joyal and injective model structures (see 2.3.7), it seems reasonable to investigate

the analogous concept of descent for the local Joyal model structure. That is, a presheaf of quasi-categories X is said to satisfy quasi-injective descent if and only if it is sectionwise Joyal equivalent to its fibrant replacement in the local Joyal model structure. A preliminary characterization of quasi-injective descent will be given at the end of Chapter 3.

One interesting perspective on quasi-injective descent seems to be suggested by [4, Proposition 8.1]. This theorem states that a map $f : X \rightarrow Y$ of quasi-categories is a Joyal equivalence if and only if

1. $\pi_0 J(f)$ is surjective.
2. For each $x, y \in X_0$, $\text{Map}_X(x, y) \rightarrow \text{Map}_X(f(x), f(y))$ is a weak equivalence.

Here J is Joyal's core functor and $\text{Map}_X(x, y)$ refers to one of the models of mapping spaces in a quasi-category found in [22, pg. 27].

This is very similar to the description of weak equivalences in the model structure for Segal precategories of [10]. Thus, it seems reasonable that one can prove a quasi-categorical analogue of [10, Theorem 10.2]. This would say that a presheaf of quasi-categories X satisfies descent if and only if

1. For each object U of the underlying site and $x, y \in X(U)$, $\text{Map}_{X|_U}(x, y)$ satisfies descent.
2. X satisfies an effective descent condition.

This will be explored in a future paper ([26]).

The first section of this chapter reviews some facts about quasi-categories and the Joyal model structure that will be used to prove the main theorem of the chapter. In particular, the path category and core of a quasi-category are described. Joyal equivalences are characterized in a manner compatible with Boolean localization (2.1.25). In 2.1.27, we characterize fibrations of quasi-categories in the Joyal model structure as pseudo-fibrations.

The second section is devoted to reviewing the technique of Boolean localization, which is essential to proving the existence of the injective model structure, as well as the local Joyal model structure. Boolean localization states that every Grothendieck topos has a cover by the topos of sheaves on a complete Boolean algebra. This theorem is proven in both [23, Chapter IX] and [15, Chapter 3]. The book [15] gives a comprehensive exposition of the construction of the injective model structure using Boolean localization.

The third section is devoted to proving the existence of the local Joyal model structure, as well as its analogue for simplicial sheaves. This is the heart of the chapter.

2.1 Preliminaries on Quasi-Categories

Definition 2.1.1 An **inner fibration** is a map of simplicial sets which has the right lifting property with respect to all inner horn inclusions $\Lambda_i^n \subset \Delta^n$, $0 < i < n$. Say that a simplicial set X is a **quasi-category** if the map $X \rightarrow *$ is an inner fibration.

Suppose that S is a collection of monomorphisms in $s\mathbf{Pre}(\mathcal{C})$ and $0, 1 : * \rightarrow I$ are two global sections of a simplicial presheaf I . Let $\square^n = I^{\times n}$. For $0 \leq i \leq n$, $\epsilon = 0, 1$, let

$$d^{(i,\epsilon)} : \square^{n-1} \rightarrow \square^n$$

be the map

$$I^{n-1} \cong I^i \times * \times I^{n-i-1} \xrightarrow{(id_i, \epsilon, id_{I^{n-i-1}})} I \times I \times I^{n-i-1}$$

Let $\partial \square^n$ denote the complex generated by the face inclusions $d^{(i, \epsilon)} : \square^{n-1} \subseteq \square^n, 0 \leq i \leq n, \epsilon = 0, 1$. Let $\Pi_{(j, \gamma)}^n$ be the complex generated by the face inclusions $d^{(i, \epsilon)} : \square^{n-1} \subseteq \square^n, (i, \epsilon) \neq (j, \gamma)$.

The **anodyne (I, S)-cofibrations** are the maps which are in the saturation of the maps

$$(A \times \square^n) \cup (B \times \partial \square^n) \rightarrow B \times \square^n$$

for $n \in \mathbb{N}, A \rightarrow B$ a morphism of S and the maps

$$(C \times \square^n) \cup (D \times \Pi_{(i, \epsilon)}^n) \rightarrow D \times \square^n$$

for $0 \leq i \leq n, n > 0$ and $C \rightarrow D$ a generating cofibration of $s\mathbf{Pre}(\mathcal{C})$.

Definition 2.1.2 We call a map $f \in s\mathbf{Pre}(\mathcal{C})$ an **injective map** if and only if it has the right lifting property with respect to all anodyne (I, S) -cofibrations. We call an object X **injective** if and only if $X \rightarrow *$ is an injective map.

We write $\pi_I(X, Z)$ for the I -homotopy classes of maps between X, Z . We say that a map $f : X \rightarrow Y$ is an **(I, S)-equivalence** if and only if $\pi_I(Y, Z) \rightarrow \pi_I(X, Z)$ is a bijection for each injective object Z .

The following theorem is due to Cisinski. However, the exposition in this thesis follows that in [12].

Theorem 2.1.3 *There is a model structure on $s\mathbf{Pre}(\mathcal{C})$ in which the cofibrations are the monomorphisms and the weak equivalences are the (I, S) -equivalences defined above. The fibrant objects are the injective objects.*

We call this the **(I, S) model structure**. This class of model structures are also called Cisinski model structures.

It is important to note that the simplicial presheaves on the trivial site $*$ can be identified with simplicial sets. Thus, we can use 2.1.3 to produce model structures on simplicial sets.

Definition 2.1.4 We write $P : s\mathbf{Set} \rightarrow \mathbf{Cat}$ for left adjoint to the nerve functor. For X a simplicial set, write $\pi(X)$ for the groupoid completion of $P(X)$. We call P the **path category** of X and $\pi(X)$ the **fundamental groupoid** of X .

Note that P preserves finite products.

The existence of the **Joyal model structure** on simplicial sets is asserted in [22, Theorem 2.2.5.1] and [20, Theorem 6.12]. The fibrant objects of this model structure are the quasi-categories. The cofibrations are the monomorphisms. The weak equivalence are called **Joyal equivalences** and can be described as follows. Given two simplicial sets K and X , $\tau_0(K, X)$ will denote **Joyal's set**, which is defined to be the set of isomorphism classes of objects in $P(X^K)$.

The Joyal equivalences are defined to be maps $f : A \rightarrow B$ such that, for each quasi-category X , the map

$$\tau_0(B, X) \rightarrow \tau_0(A, X)$$

is a bijection. The fibrations of this model structure are called **quasi-fibrations**. The trivial fibrations are the trivial Kan fibrations. Moreover, the Joyal model structure is an (I, S) -model structure in the sense of 2.1.3 (see [20, Theorem 6.12] and [16, pg. 19-21]), with $I = B\pi(\Delta^1)$ and S the set of inner horn inclusions. That is, it is a Cisinski model structure.

Let \mathcal{S}_{Joyal} denote the functorial fibrant replacement for the Joyal model structure, obtained via the small object argument with respect to the inner horn inclusions (see [12, Corollary 4.14]).

Lemma 2.1.5 *A Joyal equivalence induces an equivalence of path categories.*

Proof Let $f : X \rightarrow Y$ be a Joyal equivalence of quasi-categories. Then $X \times I$ is a cylinder object for X in the Joyal model structure. There exists a map $g : Y \rightarrow X$ and homotopies $f \circ g \sim id_Y, g \circ f \sim id_X$. Since $P(I) = \pi(\Delta^1)$ and P preserves finite products, $P(f)$ is an equivalence of categories.

If $f : X \rightarrow Y$ is a Joyal equivalence, form the diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \mathcal{S}_{Joyal}(X) \\ \downarrow f & & \downarrow \mathcal{S}_{Joyal}(f) \\ Y & \xrightarrow{i_Y} & \mathcal{S}_{Joyal}(Y) \end{array}$$

where the horizontal maps are the natural fibrant replacements for the Joyal model structure (i.e. constructed by taking transfinite composites of pushouts of inner horn inclusions). By [20, Lemma 1.6], the maps $P(\Lambda_i^n) \rightarrow P(\Delta^n)$ induced by inner horn inclusions are isomorphisms. Thus, since P commutes with colimits, $P(i_X)$ and $P(i_Y)$ are isomorphisms. The first paragraph implies that $P(X) \rightarrow P(Y)$ is an equivalence of categories, as required.

Definition 2.1.6 Suppose that X is a quasi-category. Say that 1-simplices $\alpha, \beta : x \rightarrow y$ of X are **right homotopic**, written by $\alpha \Rightarrow_R \beta$, if and only if there exists a 2-simplex with boundary

$$\begin{array}{ccc} & y & \\ \alpha \nearrow & & \searrow s_0(y) \\ x & \xrightarrow{\beta} & y \end{array}$$

Similarly, say that β, α are **left homotopic** (written $\beta \Rightarrow_L \alpha$) if and only if there exists a 2-simplex with boundary

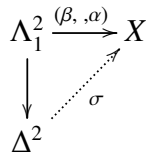
$$\begin{array}{ccc} & x & \\ s_0(x) \nearrow & & \searrow \alpha \\ x & \xrightarrow{\beta} & y \end{array}$$

Lemma 2.1.7 *If α and β are 1-simplices in a quasi-category, then*

$$\alpha \Rightarrow_R \beta \text{ (i)} \iff \beta \Rightarrow_R \alpha \text{ (ii)} \iff \alpha \Rightarrow_L \beta \text{ (iii)} \iff \beta \Rightarrow_L \alpha \text{ (iv)}.$$

*If any of the preceding relations are true, then we say that α and β are **homotopic**. Moreover, homotopy in this sense is an equivalence relation.*

Example 2.1.8 In the case X is a quasi-category, $P(X) = \text{ho}(X)$, where $\text{ho}(X)$ has the following description. It is the category which has objects the vertices of X and morphisms the homotopy classes of 1-simplices $[\alpha] : x \rightarrow y$ in X . Composition is defined for classes $[\alpha] : x \rightarrow y$, $[\beta] : y \rightarrow z$, by $[d_1(\sigma)] = [\beta] \circ [\alpha]$, where σ is the 2-simplex depicted in the following diagram:



In this category, $s_0(x) = [id_x]$.

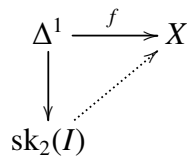
The following theorem is the main result of [18].

Theorem 2.1.9 *A quasi-category X is a Kan complex if and only if its path category $P(X)$ is a groupoid.*

Definition 2.1.10 There is a functor $J : \text{Quasi} \rightarrow \text{Kan}$ which associates to each quasi-category its maximal Kan subcomplex. Here, Quasi and Kan are, respectively, the full subcategories of sSet of quasi-categories and Kan complexes.

The functoriality of J follows from the fact that the simplices of $J(X)$ are precisely the simplices of X whose edges are invertible in $P(X)$.

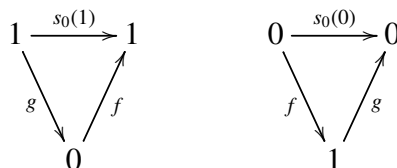
Lemma 2.1.11 *Let $I = B\pi(\Delta^1)$ and X be a quasi-category. A 1-simplex f is invertible in $P(X)$ if and only if there exists a lift in the diagram*



The same statement is true if we replace $\text{sk}_2(I)$ with I .

Proof We prove the first statement. Necessity follows from 2.1.8.

For the converse, note that it suffices to produce 2-simplices with boundaries



By symmetry, it suffices to produce a 2-simplex with boundary as depicted on the left.

Suppose that f has a left inverse in $P(X)$, g , so that $f \circ g$ is right-homotopic to the identity. Consider a map $\sigma : \Lambda_2^3 \rightarrow X$, so that σ_{012} expresses $f \circ g$ as a composite of f, g and σ_{123} , σ_{023} are, respectively, the 2-simplices with boundaries depicted below:

$$\begin{array}{ccc} 1 & \xrightarrow{g} & 0 \\ & \searrow g & \nearrow s_0(0) \\ & 0 & \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{s_0(0)} & 0 \\ & \searrow f \circ g & \nearrow s_0(0) \\ & 0 & \end{array}$$

Note that σ_{023} expresses the right homotopy between $f \circ g$ and $s_0(0)$. Extending σ to a 3-simplex σ' , $d_2(\sigma')$ gives the required 2-simplex.

We now prove the second statement. Suppose that $f : \Delta^1 \rightarrow X$ is invertible in the path category. By the definition of J , finding a lift

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{f} & X \\ \downarrow & \nearrow & \\ I & & \end{array}$$

is equivalent to finding a lift

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{f} & J(X) \\ \downarrow & \nearrow & \\ I & & \end{array}$$

and the latter exists since $\Delta^1 \rightarrow I$ is a trivial cofibration for the Kan model structure.

Conversely, if we can find an extension of f to I , then we can find an extension of f to $sk_2(I)$. Thus, f represents an invertible element of $P(X)$ by the first part of the lemma.

Definition 2.1.12 Suppose that Ω^* is a cosimplicial object of a category C . Then there is a pair of adjoint functors

$$|\!|_{\Omega^*} : \mathbf{sSet} \rightleftarrows C : \mathbf{Sing}_{\Omega^*}.$$

The left adjoint is given by

$$|\!|_{\Omega^*} S = \lim_{\substack{\longrightarrow \\ \Delta^n \rightarrow S}} \Omega^n.$$

The right adjoint is given by $\mathbf{Sing}_{\Omega^*}(S)_n = \mathbf{hom}(\Omega^n, S)$, and is known as a **singular functor associated to Ω^*** .

Remark 2.1.13 Note that a map of cosimplicial objects $\Omega^* \rightarrow \Xi^*$ in C induces natural maps $|\!|_{\Omega^*} \rightarrow |\!|_{\Xi^*}$ and $\mathbf{Sing}_{\Xi^*} \rightarrow \mathbf{Sing}_{\Omega^*}$. In simplicial degree n , the latter is the map $\mathbf{hom}(\Xi^n, C) \rightarrow \mathbf{hom}(\Omega^n, C)$ induced by $\Omega^n \rightarrow \Xi^n$.

Definition 2.1.14 We write

$$k_! : \mathbf{sSet} \rightleftarrows \mathbf{sSet} : k^!$$

for the pair of adjoint functors associated to the cosimplicial object $\Omega : \Delta \rightarrow \mathbf{sSet}$, defined by $\Omega([n]) = B(\pi\Delta^n)$, with face and degeneracy maps induced by those of Δ^n .

Remark 2.1.15 Let Δ^* be the cosimplicial object defined by $\Delta^n = B([\mathbf{n}])$ (the standard n -simplex) with the obvious face and degeneracy maps. Then one has a natural isomorphism $|\mathcal{S}|_{\Delta} \cong S$.

The inclusions $\Delta^n \rightarrow B\pi(\Delta^n)$ yield a map of cosimplicial objects $\Delta^* \rightarrow \Omega^*$, where Ω^* is as in 2.1.14. Thus, we have natural maps $X \rightarrow k_!(X)$ and $k^!(X) \rightarrow X$.

Lemma 2.1.16 *Filtered colimits of weak equivalences are weak equivalences. Filtered colimits of Joyal equivalences are Joyal equivalences.*

Proof We prove the first statement.

Note that filtered colimits of trivial Kan fibrations of Kan complexes are trivial Kan fibrations of Kan complexes. Thus, by Ken Brown's lemma ([9, Corollary 7.7.2]), filtered colimits preserve weak equivalences of Kan complexes. But the Ex^∞ functor of [8, pg. 188], which is a natural fibrant replacement for the standard model structure, commutes with filtered colimits. Thus, filtered colimits preserve arbitrary weak equivalences.

The second statement has a proof which is identical to the first; note that the functorial fibrant replacement functor for the Joyal model structure commutes with filtered colimits and the trivial fibrations are the trivial Kan fibrations.

Lemma 2.1.17 *The functor $k_!$ preserves monomorphisms, and the natural map $X \rightarrow k_!(X)$ is a weak equivalence for simplicial sets. In particular, $k_!$ preserves weak equivalences.*

Proof The fundamental groupoid functor takes pullbacks

$$\begin{array}{ccc} \Delta^{n-2} & \longrightarrow & \Delta^{n-1} \\ \downarrow & & \downarrow d_i \\ \Delta^{n-1} & \xrightarrow{d_j} & \Delta^n \end{array}$$

to pullbacks. All maps $B\pi(\Delta^{n-1}) \rightarrow B\pi(\Delta^n)$ are monomorphisms and there is a coequalizer diagram

$$\coprod_{i < j} B\pi(\Delta^{n-2}) \rightrightarrows \coprod_{0 \leq i \leq n} B\pi(\Delta^{n-1}) \rightarrow C,$$

where C is the union of the images in $B\pi(\Delta^n)$. The functor $k_!$ preserves coequalizers so that $C \cong k_!(\partial\Delta^n)$ and the induced map

$$k_!(\partial\Delta^n) \rightarrow k_!(\Delta^n) = B\pi(\Delta^n)$$

is a monomorphism. The monomorphisms are the saturation of the inclusions $\partial\Delta^n \subset \Delta^n$. Since $k_!$ preserves colimits, it follows that $k_!$ preserves monomorphisms.

We show by induction on skeleta that $X \rightarrow k_!(X)$ is a weak equivalence for all n -skeletal finite simplicial sets X . In the case $n = 0$, this is trivial. In general, we can obtain X as a pushout

$$\begin{array}{ccc} \coprod \partial\Delta^n & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \coprod \Delta^n & \longrightarrow & X \end{array}$$

where $Y \rightarrow k_!(Y)$ is a weak equivalence. By the inductive hypothesis, $\partial\Delta^n \rightarrow k_!(\partial\Delta^n)$ is a weak equivalence. Furthermore, $\Delta^n \rightarrow k_!(\Delta^n) = B\pi(\Delta^n)$ is a weak equivalence. Thus, by the gluing lemma ([8, Lemma 2.8.8]), we conclude that $X \rightarrow k_!(X)$ is a weak equivalence.

Let X be an infinite simplicial set. Let $M(X)$ be the set of finite subcomplexes of X . We have a commutative diagram

$$\begin{array}{ccc} \lim_{\substack{\longrightarrow \\ K \in M(X)}} K & \longrightarrow & \lim_{\substack{\longrightarrow \\ K \in M(X)}} k_!(K) \\ \cong \downarrow & & \cong \downarrow \\ X & \longrightarrow & k_!(X) \end{array}$$

where the top horizontal map is a filtered colimit of weak equivalences. Since weak equivalences are preserved by filtered colimits by 2.1.16, the map $X \rightarrow k_!(X)$ is a weak equivalence in general.

Lemma 2.1.18 *If X is a Kan complex, the canonical map $k^!(X) \rightarrow X$ is a trivial Kan fibration of simplicial sets. If X is a quasi-category, the induced map $k^!(X) \rightarrow J(X)$ is a trivial fibration.*

Proof The lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & k^!(X) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^n & \longrightarrow & X \end{array}$$

is equivalent to a lifting problem

$$\begin{array}{ccc} k_!(\partial\Delta^n) \cup_{\partial\Delta^n} \times\Delta^n & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \\ k_!(\Delta^n) & & \end{array}$$

The diagram of monomorphisms

$$\begin{array}{ccccc} \partial\Delta^n & \xrightarrow{w.e.} & k_!(\partial\Delta^n) & & \\ \downarrow & & \downarrow & \searrow & \\ \Delta^n & \xrightarrow{w.e.} & k_!(\partial\Delta^n) \cup_{\partial\Delta^n} \times\Delta^n & & \\ & \searrow & & \searrow & \\ & & & & k_!(\Delta^n) \end{array}$$

$w.e.$

shows that $k_!(\partial\Delta^n) \cup_{\partial\Delta^n} \times\Delta^n \rightarrow k_!(\Delta^n)$ is a trivial cofibration. Therefore, since X is a Kan complex, the required lift exists.

For the second statement, note that every map $B\pi(\Delta^n) \rightarrow X$ factors through $J(X)$ by 2.1.9. Thus, $k^!J(X) \rightarrow k^!(X)$ is an isomorphism. The induced map is the diagonal in the diagram

$$\begin{array}{ccc} k^!(J(X)) & \xrightarrow{\cong} & J(X) \\ \downarrow & \nearrow & \downarrow \\ k^!(X) & \longrightarrow & X \end{array}$$

where $k^!J(X) \rightarrow J(X)$ is a trivial fibration by the first statement.

Lemma 2.1.19 *The functor J takes quasi-fibrations to Kan fibrations.*

Proof If $f : X \rightarrow Y$ is a quasi-fibration, then it is an inner fibration and the non-trivial part is finding lifts of the form

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & J(X) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & J(Y) \end{array}$$

with $i = 0, n$. If $n = 1$, then the lift is equivalent to finding a lift

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & X \\ d^i \downarrow & \nearrow & \downarrow \\ B\pi(\Delta^1) & \longrightarrow & Y \end{array}$$

which exists since $\Delta^0 \xrightarrow{d^i} B\pi(\Delta^1)$ is a trivial cofibration for the Joyal model structure.

In the case $i = 0, n > 1$, the image of the initial edge $0 \rightarrow 1$ represents an isomorphism in $P(X)$ by the definition of J , so that the lift exists by the path lifting property for inner fibrations ([20, Theorem 4.13]). Dualizing the argument, we find the lift for $i = n, n > 1$.

Lemma 2.1.20 *Suppose that $q : X \rightarrow Y$ is a quasi-fibration and Y is a quasi-category. Then $k^1(q)$ is a Kan fibration.*

Proof All horn inclusions $\Lambda_k^n \rightarrow \Delta^n$ induce trivial cofibrations by 2.1.17. Every diagram

$$\begin{array}{ccc} k_!(\Lambda_k^n) & \longrightarrow & X \\ i_* \downarrow & & \downarrow \\ k_!(\Delta^n) & \longrightarrow & Y \end{array}$$

can be refined to a diagram

$$\begin{array}{ccccc} k_!(\Lambda_k^n) & \longrightarrow & J(X) & \longrightarrow & X \\ i_* \downarrow & \nearrow & \downarrow J(q) & & \downarrow q \\ k_!(\Delta^n) & \longrightarrow & J(Y) & \longrightarrow & Y \end{array}$$

The map i_* is a trivial cofibration by 2.1.17. Thus, to show that the lifting exists, it suffices to note that $J(q)$ is a Kan fibration by 2.1.19.

Lemma 2.1.21 *The functor $k_!$ takes weak equivalences to Joyal equivalences.*

Proof Suppose Z is a quasi-category. Then $k^1(Z)$ is a Kan complex by 2.1.20. The functor k^1 preserves trivial fibrations, and takes quasi-fibrations between quasi-categories to Kan fibrations.

Suppose that

$$\begin{array}{ccc} & & Z^! \\ & \nearrow & \downarrow \\ Z & \xrightarrow{\Delta} & Z \times Z \end{array}$$

is a path object for the Joyal model structure. Then the induced diagram

$$\begin{array}{ccc} & & k^!(Z') \\ & \nearrow & \downarrow (p_{0*}, p_{1*}) \\ k^!(Z) & \xrightarrow{\Delta} & k^!(Z) \times k^!(Z) \end{array}$$

is a path object for the standard model structure. It follows that there are natural bijections

$$[X, k^!(Z)] \cong \pi(X, k^!(Z)) \cong \pi(k_!(X), Z) \cong [k_!(X), Z],$$

where $\pi(K, Y)$ is the set of right homotopy classes for the respective path objects constructed above. Therefore, $k_!$ takes weak equivalences to Joyal equivalences.

Corollary 2.1.22 *The adjoint pair*

$$k_! : s\mathbf{Set} \rightleftarrows s\mathbf{Set} : k^!$$

is a Quillen adjunction between the standard model structure on simplicial sets and the Joyal model structure.

Proof Follows from 2.1.17 and 2.1.21.

We call a map $f : X \rightarrow Y$ of quasi-categories a **pseudo-fibration** if and only if it is an inner fibration and there exists a lift in each diagram of the form

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & J(X) \\ d^0 \downarrow & \nearrow & \downarrow \\ \Delta^1 & \longrightarrow & J(Y) \end{array}$$

Theorem 2.1.23 *Suppose that $f : X \rightarrow Y$ is a pseudo-fibration. For any monomorphism $i : L \rightarrow K$,*

$$X^K \rightarrow X^L \times_{Y^L} Y^K$$

is a pseudo-fibration. In particular, if X is a quasi-category and K is a simplicial set, then X^K is a quasi-category.

Proof The first statement is [20, Theorem 5.13]. The second is obtained from the first by taking f to be $X \rightarrow *$ and $i = id_K$.

Lemma 2.1.24 *Suppose that $f : X \rightarrow Y$ is a Joyal equivalence of quasi-categories and K is a simplicial set. Then $X^K \rightarrow Y^K$ is a Joyal equivalence.*

Proof f has a factorization $g \circ h$ where g is a trivial fibration and h is a section of trivial fibration. The functor $(-)^K$ preserves trivial fibrations, and thus preserves Joyal equivalences of quasi-categories.

Lemma 2.1.25 *A map $f : X \rightarrow Y$ is a Joyal equivalence of quasi-categories if and only if*

$$\pi(J(X^K)) \rightarrow \pi(J(Y^K))$$

is an equivalence of categories for all finite simplicial sets K .

Proof Note that f is a Joyal equivalence if and only if $[K, X] \rightarrow [K, Y]$ is a bijection for each finite simplicial set K . However, $[K, X]$ can be naturally identified with $\tau_0(K, X)$ by [22, Proposition 2.2.5.7]. Note that X^K is a quasi-category. Thus, $\tau_0(K, X)$ is in bijection with $\pi_0 J(X^K)$ since the edges of $J(X^K)$ are precisely those representing isomorphisms in $P(X^K)$. Thus, f is a Joyal equivalence if and only if

$$\pi_0(J(X^K)) \rightarrow \pi_0(J(Y^K))$$

is a bijection for each finite simplicial set K .

Suppose that $f : X \rightarrow Y$ is a Joyal equivalence of quasi-categories. By 2.1.18 and 2.1.22, J sends Joyal equivalences of quasi-categories to weak equivalences of Kan complexes. Thus, since $(-)^K$ preserves Joyal equivalences of quasi-categories, $J(X^K) \rightarrow J(Y^K)$ is a weak equivalence for all finite simplicial sets K . Thus, $\pi J(X^K) \rightarrow \pi J(Y^K)$ is an equivalence of categories.

The required result follows by combining the observations of the preceding paragraphs.

The proof of the preceding lemma also shows the following is true.

Corollary 2.1.26 *A map $f : X \rightarrow Y$ is a Joyal equivalence of quasi-categories if and only if*

$$J(X^K) \rightarrow J(Y^K)$$

is a weak equivalence for all finite simplicial sets K .

Lemma 2.1.27 (see [20, Theorem 5.22]). *Let $f : X \rightarrow Y$ be a map of quasi-categories. Then f is a quasi-fibration if and only if it is a pseudo-fibration.*

Proof Recall that the Joyal model structure is an (I, S) model structure in the sense of 2.1.3 (see [20, Theorem 6.12] and [16, pg. 19-21]), with $I = B\pi\Delta^1$ and S the set of inner horn inclusions. Let $\square^n = I^{\times n}$, $\partial\square^n$ denote the complex generated by the face inclusions $d^{(i,\epsilon)} : \square^{n-1} \subseteq \square^n$, $0 \leq i \leq n$, and $\Pi_{(i,\epsilon)}^n$ be the complex generated by the face inclusions $d^{(j,\gamma)} : \square^{n-1} \subseteq \square^n$, $(j,\gamma) \neq (i,\epsilon)$.

The fact that every quasi-fibration is a pseudo-fibration is trivial. We will show that pseudo-fibrations of quasi-categories are quasi-fibrations.

It follows from [12, Lemma 4.13] that a map whose target is a quasi-category is a quasi-fibration if it is **injective** in the sense of 2.1.2. That is, it has the right lifting property with respect to the maps

$$(\Delta_i^k \times \square^n) \cup (\Delta^k \times \partial\square^n) \rightarrow \Delta^k \times \square^n$$

for $0 < i < k, k > 0, n \in \mathbb{N}$ and the maps

$$(\partial\Delta^m \times \square^n) \cup (\Delta^m \times \Pi_{(i,\epsilon)}^n) \rightarrow \Delta^m \times \square^n$$

for $0 \leq i \leq n, \epsilon \in \{0, 1\}, m \in \mathbb{N}, n > 0$.

By adjunction, the first lifting problem is equivalent to showing that

$$X^{\square^n} \rightarrow X^{\partial \square^n} \times_{Y^{\partial \square^n}} Y^{\square^n}$$

is an inner fibration, which is true by 2.1.23.

By 2.1.23 and adjunction, the second lifting problem is equivalent to showing that $\Pi_{(i,\epsilon)}^n \rightarrow \square^n$ has the left lifting property with respect to all pseudo-fibrations $f : X \rightarrow Y$. We proceed by induction on n . The case $n = 1$ is trivial. Assume the result is true for all $n < n'$. Note that for $i < n'$, $\Pi_{(i,\epsilon)}^{n'} \rightarrow \square^{n'}$ is naturally isomorphic to

$$(\Pi_{(i,\epsilon)}^{n'-1} \times \square^1) \cup (\square^{n'-1} \times \partial \square^1) \subseteq \square^{n'-1} \times \square^1$$

(see [12, pg. 92]). A lifting problem

$$\begin{array}{ccc} (\Pi_{(i,\epsilon)}^{n'-1} \times \square^1) \cup (\square^{n'-1} \times \partial \square^1) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \square^{n'-1} \times \square^1 & \longrightarrow & Y \end{array}$$

for a pseudo-fibration f is equivalent to finding a lift

$$\begin{array}{ccc} \Pi_{(i,\epsilon)}^{n'-1} & \longrightarrow & X^{\square^1} \\ \downarrow & \nearrow & \downarrow \\ \square^{n'-1} & \longrightarrow & Y^{\square^1} \times_{Y^{\partial \square^1}} X^{\partial \square^1} \end{array}$$

by adjunction. The lift can be found by the inductive hypothesis and 2.1.23. Similarly, $\Pi_{(n',\epsilon)}^{n'} \rightarrow \square^{n'}$ is naturally isomorphic to

$$(\partial \square^1 \times \square^{n'-1}) \cup (\square^1 \times \Pi_{(n'-1,\epsilon)}^{n'-1}) \rightarrow \square^1 \times \square^{n'-1},$$

and we can use a similar argument to that above to solve all lifting problems

$$\begin{array}{ccc} \Pi_{(n',\epsilon)}^{n'} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \square^{n'} & \longrightarrow & Y \end{array}$$

for a pseudo-fibration f .

2.2 Preliminaries on Boolean Localization

Suppose that K is a simplicial set and X is a simplicial presheaf. Write $\text{hom}(K, X)$ for the simplicial presheaf $U \mapsto \text{hom}(K, X(U))$. Write X^K for the simplicial presheaf $U \mapsto X(U)^K$. Note that $\text{hom}(K, X) = X_0^K$.

Definition 2.2.1 Let \mathcal{L}, \mathcal{M} be Grothendieck topoi. A **geometric morphism** $p : \mathcal{L} \rightarrow \mathcal{M}$ is a pair of functors $p^* : \mathcal{M} \rightarrow \mathcal{L}$, $p_* : \mathcal{L} \rightarrow \mathcal{M}$ so that p^* preserves finite limits and is left adjoint to p_* . Call a geometric morphism **surjective** if and only if it satisfies the following equivalent properties:

1. p^* is faithful.
2. p^* reflects isomorphisms.
3. p^* reflects monomorphisms.
4. p^* reflects epimorphisms.

The following theorem is proven in [23, pg. 515], as well as in [15, Section 3.5].

Theorem 2.2.2 (Barr) *Let \mathcal{L} be any Grothendieck topos. Then there exists a surjective geometric morphism $p = (p^*, p_*) : \mathbf{Sh}(\mathcal{B}) \rightarrow \mathcal{L}$ such that \mathcal{B} is a complete Boolean algebra.*

Such a surjective geometric morphism (from $\mathbf{Sh}(\mathcal{B})$) is called a **Boolean localization** of \mathcal{L} .

Definition 2.2.3 Suppose that $i : K \subseteq L$ is an inclusion of finite simplicial sets and that $f : X \rightarrow Y$ is a map of simplicial presheaves. Say that f has the **local right lifting property** with respect to i if for every commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & X(U) \\ \downarrow & & \downarrow \\ L & \longrightarrow & Y(U) \end{array}$$

there is some covering sieve $R \subseteq \text{hom}(-, U)$, $U \in \text{Ob}(\mathcal{C})$, such that the lift exists in the diagram

$$\begin{array}{ccccc} K & \longrightarrow & X(U) & \xrightarrow{X(\phi)} & X(V) \\ \downarrow & & & \nearrow \text{dotted} & \downarrow \\ L & \longrightarrow & Y(U) & \xrightarrow{Y(\phi)} & Y(V) \end{array}$$

for each $\phi \in R$. Similarly, say that f has the **sectionwise right lifting property** with respect to i if and only if there exists a lifting

$$\begin{array}{ccc} K & \longrightarrow & X(U) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ L & \longrightarrow & Y(U) \end{array}$$

for each $U \in \text{Ob}(\mathcal{C})$.

Definition 2.2.4 Say that a map of simplicial presheaves is a **local inner fibration** (respectively **local Kan fibration**) if and only if it has the local right lifting property with respect to the inner horn inclusions $\Lambda_i^n \rightarrow \Delta^n$, $0 < i < n$ (respectively the horn inclusions $\Lambda_i^n \rightarrow \Delta^n$, $0 \leq i \leq n$).

If X is a simplicial presheaf so that the map to the terminal sheaf $X \rightarrow *$ has the local right lifting property with respect to the inner horn inclusions, we say that X is **local Joyal fibrant**. Similarly, there is a notion of **locally Kan fibrant** simplicial presheaves.

Call a map $f : X \rightarrow Y$ of simplicial presheaves a **sectionwise Kan fibration** if and only if for each $U \in \text{Ob}(\mathcal{C})$, $X(U) \rightarrow Y(U)$ is a Kan fibration. There are analogous definitions of **sectionwise trivial fibrations** and **sectionwise quasi-fibrations**.

Note that $X \rightarrow *$ has the sectionwise right lifting property with respect to $\Lambda_i^n \rightarrow \Delta^n$, $0 < i < n$ (respectively $\Lambda_i^n \rightarrow \Delta^n$, $0 \leq i \leq n$) if and only if X is a presheaf of quasi-categories (respectively a presheaf of Kan complexes).

Lemma 2.2.5 ([15, Lemma 4.8]) *A map of simplicial presheaves $f : X \rightarrow Y$ has the local right lifting property with respect to a finite inclusion of simplicial sets $i : K \rightarrow L$ if and only if*

$$X^L \xrightarrow{(i^*, f_*)} X^K \times_{Y^K} Y^L$$

is a local epimorphism in simplicial degree 0.

Throughout the rest of the thesis, we fix a Boolean localization $p : \mathbf{Sh}(\mathcal{B}) \rightarrow \mathbf{Sh}(\mathcal{C})$ of the small site \mathcal{C} . It is important to note that the Boolean localization is chosen for sheaves, rather than presheaves, since a Boolean localization is a geometric morphism of topoi. We will write

$$p^* : s\mathbf{Sh}(\mathcal{C}) \rightleftarrows s\mathbf{Sh}(\mathcal{B}) : p_*$$

for the adjoint pair obtained by applying the left and right adjoint parts of p sectionwise to a simplicial sheaf. We write L^2 for the sheafification functor.

Lemma 2.2.6 *Let K be a finite simplicial set and X be a simplicial presheaf. Then there are natural isomorphisms:*

1. $p^* \text{hom}(K, L^2(X)) \cong \text{hom}(K, p^* L^2(X))$.
2. $p^* L^2(X^K) \cong (p^* L^2(X))^K$.
3. $L^2 \text{hom}(K, X) \cong \text{hom}(K, L^2(X))$.
4. $L^2(X^K) \cong (L^2(X))^K$.

Proof 1 and 3 follow from the facts that p^* and L^2 preserve finite limits and a simplicial set is a colimit of its non-degenerate simplices. The implications $1 \implies 2$, $3 \implies 4$ are obvious.

Lemma 2.2.7 *Let $f : X \rightarrow Y$ be a map of simplicial sheaves on a complete Boolean algebra. Then f has the local right lifting property with respect to inclusion $i : L \rightarrow K$ of finite simplicial sets if and only if it has the sectionwise right lifting property with respect to i .*

Proof Follows from the axiom of choice for $s\mathbf{Sh}(\mathcal{B})$ ([15, Lemma 3.30]), 2.2.5 and 2.2.6.

Lemma 2.2.8 ([15, Lemma 4.9 and Lemma 4.10]) *The functors p^* and L^2 both reflect and preserve the property of having the local right lifting property with respect to an inclusion of finite simplicial sets.*

Proof Follows from 2.2.5 and 2.2.6.

We write $\mathrm{Ex}^\infty : s\mathbf{Pre}(\mathcal{C}) \rightarrow s\mathbf{Pre}(\mathcal{C})$ for the functor obtained by applying Kan's Ex^∞ functor (see [8, pg. 188]) sectionwise to a simplicial presheaf.

Definition 2.2.9 A map f of simplicial presheaves is a **local weak equivalence** if and only if $p^*L^2\mathrm{Ex}^\infty(f)$ is a sectionwise weak equivalence.

The intuition behind Boolean localization is that it can be regarded as giving a ‘fat’ point for a site (for more details see [11, Section 1]). Thus, the definition of local weak equivalence generalizes the idea of stalkwise weak equivalence in the case of a topos with enough points. This definition of weak equivalence is independent of the choice of Boolean localization by [15, Theorem 4.5].

Remark 2.2.10 By 2.2.6, there is a natural isomorphism $p^*L^2\mathrm{Ex}^\infty \cong L^2\mathrm{Ex}^\infty p^*L^2$. Thus, $X \rightarrow L^2(X)$ is a local weak equivalence and L^2 preserves and reflects local weak equivalences.

The fact that the definition of weak equivalence is independent of the choice of Boolean localization means that if \mathcal{C} is a Boolean site, the choice of Boolean localization can be taken to be the identity. It follows that p^* preserves and reflects local weak equivalences.

Lemma 2.2.11 ([15, Lemma 4.23]; also [28, Corollary 10.9]). *Let $f : X \rightarrow Y$ be a map of sheaves of Kan complexes on \mathcal{B} . Then f is a local weak equivalence if and only if it is a sectionwise weak equivalence.*

Definition 2.2.12 A map of simplicial presheaves $f : X \rightarrow Y$ is said to be a **local equivalence of fundamental groupoids** if and only if $B\pi(f)$ is a local weak equivalence. There is an analogous notion of **sectionwise weak equivalences of fundamental groupoids**.

Lemma 2.2.13 *Suppose that $f : X \rightarrow Y$ is a local equivalence of groupoids of sheaves of Kan complexes on \mathcal{B} . Then $\pi(f)$ is an equivalence of categories in each section.*

Proof Note that by the functorial factorization in the standard model structure for simplicial sets (see 2.3.17 and 2.3.18), we can assume that f is a sectionwise Kan fibration.

Note that since f is a sectionwise Kan fibration, if x and y lie in the same path component of $Y(U)$, then x is in the image of f if and only if y is. Similarly, if σ_1, σ_2 are 1-simplices representing the same morphism in $\pi X(U)$, then σ_1 is in the image of f if and only if σ_2 is. The map $\pi_0(f)$ is a local epimorphism and

$$\mathrm{Mor}(\pi(X)) \rightarrow \mathrm{Mor}(\pi(Y)) \times_{(\mathrm{Ob}(\pi(Y)) \times \mathrm{Ob}(\pi(Y)))} (\mathrm{Ob}(\pi(X)) \times \mathrm{Ob}(\pi(X)))$$

is a local epimorphism. Thus, f has the local, and hence sectionwise right lifting property with respect to $\emptyset \rightarrow \Delta^0$ and $\partial\Delta^1 \rightarrow \Delta^1$. We conclude that $\pi(f)$ is full and essentially surjective in sections.

Finally, note that every presheaf on a complete Boolean algebra is separated (see [28, Proposition 10.4], [15, Lemma 3.13]); the natural map $X \rightarrow L^2(X)$ is a monomorphism. Thus, $\pi(f)$ is faithful in sections if $L^2\pi(f)$ is. But $BL^2\pi(f) \cong L^2B\pi(f)$ is a sectionwise weak equivalence, so that $L^2\pi(f)$ is faithful in sections.

Corollary 2.2.14 *A map $f : X \rightarrow Y$ of simplicial presheaves of Kan complexes is a local equivalence of fundamental groupoids if and only if $\pi p^*L^2(f)$ is an equivalence of categories in each section.*

Proof By 2.2.11, $p^*L^2(B\pi(f))$ is a sectionwise weak equivalence if and only if $B\pi(f)$ is a local weak equivalence. We have natural isomorphisms

$$p^*L^2B\pi(X) \cong p^*BL^2\pi(X) \simeq p^*BL^2\pi(L^2X) \cong BL^2\pi p^*L^2X.$$

The result now follows from 2.2.13.

Definition 2.2.15 Let

$$\mathcal{S}_{Joyal} : s\mathbf{Pre}(\mathcal{C}) \rightarrow s\mathbf{Pre}(\mathcal{C})$$

be the functor which applies the usual fibrant replacement functor (i.e. constructed via the small object argument with respect to inner horn inclusions) for the Joyal model structure sectionwise to a simplicial presheaf. Let $s\mathbf{Pre}(\mathcal{C})^{\text{Quasi}}$, $s\mathbf{Pre}(\mathcal{C})^{\text{Kan}}$ denote the full subcategories of $s\mathbf{Pre}(\mathcal{C})$ consisting of presheaves of quasi-categories and presheaves of Kan complexes, respectively. Sectionwise application of the functor J (as defined in 2.1.10) defines a functor

$$J : s\mathbf{Pre}(\mathcal{C})^{\text{Quasi}} \rightarrow s\mathbf{Pre}(\mathcal{C})^{\text{Kan}}.$$

Definition 2.2.16 For a simplicial set X , the cardinality of X is defined to be $|X| = \sup_{n \in \mathbb{N}} (|X_n|)$. For each simplicial presheaf X , and infinite cardinal α , say that X is α -bounded if

$$\sup_{U \in \text{Ob}(\mathcal{C})} (|X(U)|) < \alpha.$$

Say that a monomorphism $A \rightarrow B$ is α -bounded if B is α -bounded.

Lemma 2.2.17 *The functor \mathcal{S}_{Joyal} has the following properties:*

1. \mathcal{S}_{Joyal} preserves filtered colimits.
2. \mathcal{S}_{Joyal} preserves cofibrations.

3. Suppose that γ is a regular cardinal so that $\gamma > |\text{Mor}(\mathcal{C})|$. For a simplicial presheaf X , let $\mathcal{F}_\gamma(X)$ denote the filtered system of subobjects of X which has cardinality $< \gamma$. The natural map

$$\lim_{\substack{\longrightarrow \\ Y \in \mathcal{F}_\gamma(X)}} \mathcal{S}_{\text{Joyal}}(Y) \rightarrow \mathcal{S}_{\text{Joyal}}(X)$$

is an isomorphism.

4. Let $\lambda > |\text{Mor}(\mathcal{C})|$ be a regular cardinal. If $|X| \leq 2^\lambda$, then $|\mathcal{S}_{\text{Joyal}}(X)| \leq 2^\lambda$.
5. $\mathcal{S}_{\text{Joyal}}$ preserves pullbacks.

Proof By arguing sectionwise, this is the same argument as [12, Theorem 4.8].

2.3 Existence of the Model Structure

Definition 2.3.1 Define a map $f : X \rightarrow Y$ of simplicial presheaves on \mathcal{C} to be a **sectionwise Joyal equivalence** if and only if $X(U) \rightarrow Y(U)$ is a Joyal equivalence for each $U \in \text{Ob}(\mathcal{C})$. Define f to be a **local Joyal equivalence** if and only if $L^2\mathcal{S}_{\text{Joyal}}p^*L^2(f)$ is a sectionwise Joyal equivalence. Note that local Joyal equivalences automatically satisfy the 2 out of 3 property. A **quasi-injective fibration** is a map that has the right lifting property with respect to maps which are both monomorphisms and local Joyal equivalences.

Corollary 2.3.2 *The map $X \rightarrow L^2(X)$ is a local Joyal equivalence.*

The following theorem is the main theorem of this chapter; the remainder of 2.3 is devoted to its proof.

Theorem 2.3.3 *There exists a left proper model structure on $s\text{Pre}(\mathcal{C})$, with the weak equivalences the local Joyal equivalences, the cofibrations monomorphisms and the fibrations the quasi-injective fibrations.*

Lemma 2.3.4 *Let $\Gamma^* : s\text{Set} \rightarrow s\text{Sh}(\mathcal{C})$ be the composite of the constant simplicial presheaf functor and sheafification. The functors*

$$p^*(- \times \Gamma^*(C)), p^*(-) \times \Gamma^*(C) : s\text{Sh}(\mathcal{C}) \rightarrow s\text{Sh}(\mathcal{B})$$

are naturally isomorphic for each simplicial set C .

Proof Follows easily by the exactness of p^*L^2 .

Lemma 2.3.5 *There is a natural isomorphism $p^*L^2\mathcal{S}_{\text{Joyal}} \cong L^2\mathcal{S}_{\text{Joyal}}p^*L^2$. In particular, f is a local Joyal equivalence if and only if $p^*L^2\mathcal{S}_{\text{Joyal}}(f)$ is a sectionwise Joyal equivalence.*

Proof Since p^* and L^2 commute with colimits, by the construction of \mathcal{S}_{Joyal} , it suffices to show that $p^*L^2E_1(X) \cong L^2E_1p^*L^2(X)$ naturally, where E_1 is the pushout of simplicial presheaves

$$\begin{array}{ccc} \coprod_{\Lambda_k^n \subset \Delta^n} (\text{hom}(\Lambda_k^n, X) \times \Lambda_k^n) & \xrightarrow{ev} & X \\ \downarrow & & \downarrow \\ \coprod_{\Lambda_k^n \subset \Delta^n} (\text{hom}(\Lambda_k^n, X) \times \Delta^n) & \longrightarrow & E_1 \end{array}$$

where the coproducts are indexed over the set of all inner horn inclusions $\Lambda_k^n \subset \Delta^n$ and ev is the evaluation map. Thus, by 2.2.6, 2.3.4 and the fact that sheafification commutes with finite limits, $p^*L^2(E_1)$ is naturally isomorphic to the sheaf pushout

$$\begin{array}{ccc} \coprod_{\Lambda_k^n \subset \Delta^n} (\text{hom}(\Lambda_k^n, p^*L^2X) \times \Gamma^*(\Lambda_k^n)) & \xrightarrow{ev} & p^*L^2X \\ \downarrow & & \downarrow \\ \coprod_{\Lambda_k^n \subset \Delta^n} (\text{hom}(C, p^*L^2X) \times \Gamma^*(\Delta^n)) & \longrightarrow & S \end{array}$$

which is naturally isomorphic to $L^2E_1p^*L^2(X)$, as required.

Lemma 2.3.6 1. *If X is a sheaf of quasi-categories on \mathcal{B} , then the natural map $J(X) \rightarrow L^2J(X)$ is an isomorphism.*

2. *Let X be a presheaf of quasi-categories on \mathcal{C} . For $n \in \mathbb{N}$, let E_n denote the set of edges $\Delta^1 \rightarrow \Delta^n$. For each $e \in E_n$, form the pullback*

$$\begin{array}{ccc} P_n^e & \xrightarrow{\phi_e} & \text{hom}(\Delta^n, X) \\ \downarrow & & \downarrow e^* \\ \text{hom}(\text{sk}_2(I), X) & \xrightarrow{i^*} & \text{hom}(\Delta^1, X) \end{array}$$

where e^*, i^* are induced by inclusion. The n -simplices of $J(X)$ are equal to the presheaf-theoretic image of $\bigcap_{e \in E_n} (P_n^e) \xrightarrow{\phi} \text{hom}(\Delta^n, X)$ induced by the ϕ_e 's.

3. *Let X be a presheaf of quasi-categories on \mathcal{C} . Then there is a natural isomorphism $Jp^*L^2(X) \cong p^*L^2J(X)$.*

Proof First, suppose that X is a sheaf on a Boolean site. Then $L^2J(X)$ is a locally Kan simplicial sheaf, and hence is sectionwise Kan by 2.2.7. Furthermore, sheafification preserves monomorphisms, so there is a diagram

$$\begin{array}{ccc} J(X) & \longrightarrow & X \\ \downarrow & & \parallel \\ L^2J(X) & \longrightarrow & L^2(X) \end{array}$$

Thus, $J(X) \rightarrow L^2J(X)$ is an inclusion of sub-presheaves of X . But $J(X)$ is the maximal sectionwise Kan subcomplex of X , so we conclude that $J(X) = L^2(J(X))$.

Statement 2 follows immediately from 2.1.9 and 2.1.11.

For the final statement, it is clear that the P_n^e 's are preserved under p^*L^2 since this composite preserves finite limits. Thus, $p^*L^2(J(X))_n$ is isomorphic to the sheaf theoretic image of

$$\bigcap_{e \in E_n} P_n^e \xrightarrow{\phi} \text{hom}(\Delta^n, p^*L^2(X)),$$

which is $L^2J(p^*L^2(X))_n$ by 2. But there is an isomorphism $L^2J(p^*L^2(X))_n \cong Jp^*L^2(X)_n$ by 1.

Theorem 2.3.7 *A map $X \xrightarrow{\phi} Y$ is a local Joyal equivalence if and only if the map*

$$\pi J((\mathcal{S}_{\text{Joyal}}(X))^K) \rightarrow \pi J((\mathcal{S}_{\text{Joyal}}(Y))^K)$$

is a local weak equivalence of fundamental groupoids for each finite simplicial set K .

Proof The results 2.2.6 and 2.3.6 imply that for each finite simplicial set K there are isomorphisms

$$p^*L^2J((\mathcal{S}_{\text{Joyal}}(X))^K) \cong Jp^*L^2((\mathcal{S}_{\text{Joyal}}(X))^K) \cong J((p^*L^2\mathcal{S}_{\text{Joyal}}(X))^K).$$

Thus, by 2.2.14, the assertion that $J((\mathcal{S}_{\text{Joyal}}(\phi))^K)$ is a local equivalence of fundamental groupoids is equivalent to

$$\pi J((p^*L^2\mathcal{S}_{\text{Joyal}}(X))^K)(b) \rightarrow \pi J((p^*L^2\mathcal{S}_{\text{Joyal}}(Y))^K)(b)$$

being an equivalence of groupoids for all $b \in \text{Ob}(\mathcal{B})$.

The result now follows from 2.1.25.

The proof of this theorem, along with 2.1.26, shows that the following weaker result is also true.

Lemma 2.3.8 *A map $X \xrightarrow{\phi} Y$ is a local Joyal equivalence if and only if the map*

$$J((\mathcal{S}_{\text{Joyal}}(X))^K) \rightarrow J((\mathcal{S}_{\text{Joyal}}(Y))^K)$$

is a local weak equivalence for each finite simplicial set K .

Corollary 2.3.9 *A sectionwise Joyal equivalence $X \rightarrow Y$ of simplicial presheaves is a local Joyal equivalence.*

Corollary 2.3.10 *A local Joyal equivalence between sheaves of quasi-categories on a Boolean site is a sectionwise Joyal equivalence.*

Corollary 2.3.11 *A map $f : X \rightarrow Y$ of presheaves of quasi-categories is a local Joyal equivalence if and only if $p^*L^2(f)$ is a sectionwise Joyal equivalence.*

Proof Suppose that f is a local Joyal equivalence. The map $p^*L^2(X) \rightarrow \mathcal{S}_{\text{Joyal}}p^*L^2(X) \rightarrow L^2\mathcal{S}_{\text{Joyal}}p^*L^2(X)$ is a local Joyal equivalence in $s\mathbf{Pre}(\mathcal{B})$ by 2.3.2 and 2.3.9. Furthermore,

$L^2\mathcal{S}_{\text{Joyal}}p^*L^2(f)$ is a sectionwise, and hence local Joyal equivalence. Thus, the commutative diagram

$$\begin{array}{ccc} p^*L^2(X) & \longrightarrow & L^2\mathcal{S}_{\text{Joyal}}p^*L^2(X) \\ \downarrow & & \downarrow L^2\mathcal{S}_{\text{Joyal}}p^*L^2(f) \\ p^*L^2(Y) & \longrightarrow & L^2\mathcal{S}_{\text{Joyal}}p^*L^2(Y) \end{array}$$

and the 2 out of 3 property imply that $p^*L^2(f)$ is a local Joyal equivalence in $\mathbf{sPre}(\mathcal{B})$. But a local Joyal equivalence between sheaves of quasi-categories on \mathcal{B} is a sectionwise Joyal equivalence.

The converse is similar, but easier.

Corollary 2.3.12 *The natural map $X \rightarrow \mathcal{S}_{\text{Joyal}}(X)$ is a local Joyal equivalence.*

Corollary 2.3.13 *A map f of simplicial presheaves is a local Joyal equivalence if and only if $\mathcal{S}_{\text{Joyal}}(f)$ is a local Joyal equivalence.*

Corollary 2.3.14 *Let $X_\alpha \rightarrow Y_\alpha$ be natural transformations consisting of local Joyal equivalences of presheaves of quasi-categories, indexed by some filtered category M . Then the induced map*

$$\lim_{\alpha \in M} X_\alpha \rightarrow \lim_{\alpha \in M} Y_\alpha$$

is a local Joyal equivalence.

Lemma 2.3.15 *The functor p^*L^2 preserves the property of being a sectionwise quasi-fibration of presheaves of quasi-categories.*

Proof By 2.1.27, it suffices to show that p^*L^2 preserves and reflects the property of being a sectionwise pseudo-fibration. But this follows from 2.2.7, 2.2.8 and 2.3.6.

A **local trivial fibration** is a map which has the local right lifting property with respect to the boundary inclusions $\partial\Delta^n \subseteq \Delta^n$ for all $n \geq 0$.

Lemma 2.3.16 *A local trivial fibration is a local Joyal equivalence. Suppose that $f : X \rightarrow Y$ is a sectionwise quasi-fibration of presheaves of quasi-categories. Then f is a local Joyal equivalence if and only if it is a local trivial fibration.*

Proof If f is a local trivial fibration, then $p^*L^2(f)$ is a sectionwise trivial fibration, and hence a sectionwise Joyal equivalence. Thus it is a local Joyal equivalence. But p^*L^2 reflects local Joyal equivalences, so that f is a local Joyal equivalence.

Now, suppose that f is a local Joyal equivalence of presheaves of quasi-categories and a sectionwise quasi-fibration. By 2.3.15, $p^*L^2(X) \rightarrow p^*L^2(Y)$ is a sectionwise quasi-fibration. By 2.3.11, it is also a sectionwise Joyal equivalence so $p^*L^2(X) \rightarrow p^*L^2(Y)$ is a sectionwise trivial fibration. The result follows from 2.2.8.

Example 2.3.17 This example gives the construction of the quasi-fibration replacement for a map $f : X \rightarrow Y$ of quasi-categories. This construction is standard in a category of fibrant objects. Form the diagram of quasi-categories

$$\begin{array}{ccccc} X \times_Y Y^I & \xrightarrow{f_*} & Y^I & \xrightarrow{d_1} & Y \\ d_{0*} \downarrow & \nearrow sf & \downarrow d_0 & & \\ X & \xrightarrow{f} & Y & & \end{array}$$

Note that Y^I is a path object for a quasi-category Y in the Joyal model structure. Since Y is a quasi-category, d_0 is a sectionwise trivial fibration. Thus, d_{0*} is a sectionwise trivial fibration. The section s of d_0 induces a section s_* of d_{0*} , and

$$(d_1 f_*) s_* = d_1 s f = f.$$

Finally, there is a pullback diagram of quasi-categories

$$\begin{array}{ccc} X \times_Y Y^I & \xrightarrow{f_*} & Y^I \\ (d_{0*}, d_1 f_*) \downarrow & & \downarrow (d_0, d_1) \\ X \times Y & \xrightarrow{f \times 1} & Y \times Y \end{array}$$

and the projection map $pr_R : X \times Y \rightarrow Y$ is a quasi-fibration since X is a presheaf of quasi-categories. Thus, $pr_R(d_{0*}, d_1 f_*) = d_1 f_*$ is a sectionwise quasi-fibration. Write $Z_f = X \times_Y Y^I$ and $\pi = d_1 f_*$. Then π is a functorial replacement of f by a quasi-fibration, and there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{s_*} & Z_f & \xrightarrow{(d_0)_*} & X \\ & \searrow f & \downarrow \pi & & \\ & & Y & & \end{array}$$

where $(d_0)_*$ is a trivial fibration and $(d_0)_* \circ s_* = id_X$.

Remark 2.3.18 An analogous construction to that of 2.3.17 produces the sectionwise Kan fibration replacement of a map of presheaves of Kan complexes. Taking pullbacks gives a functorial Kan fibration replacement for all simplicial presheaf maps. However, this technique does not work for the Joyal model structure since the Joyal model structure is not right proper.

Lemma 2.3.19 *Let α be a regular cardinal so that $\alpha > |\text{Mor}(\mathcal{C})|$. Let $C \subseteq A$ be an inclusion of simplicial presheaves such that C is α -bounded and A is a presheaf of quasi-categories. Then there exists an α -bounded presheaf of quasi-categories B so that $C \subseteq B \subseteq A$.*

Proof The set of lifting problems

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & C(U) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

for $U \in \text{Ob}(\mathcal{C})$ is α -bounded and can be solved over A . Furthermore, since A is a colimit of its α -bounded subobjects, there is a subobject B_1 of A so that $C \subseteq B_1$, all of the preceding lifting problems can be solved over B_1 and B_1 is α -bounded. Repeating this procedure countably many times produces an ascending sequence

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots .$$

Set $B = \bigcup_{i=1}^{\infty} B_i$.

Lemma 2.3.20 (see [15, Theorem 5.2]) *Suppose that α is a regular cardinal such that $\alpha > |\text{Mor}(\mathcal{C})|$. Suppose that there is a diagram of monomorphisms of simplicial presheaves of quasi-categories*

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

where A is α -bounded and $X \rightarrow Y$ is a local Joyal equivalence. Then there exists an α -bounded presheaf of quasi-categories B such that $A \subseteq B \subseteq Y$ and $B \cap X \rightarrow B$ is a local Joyal equivalence.

Proof If $B \subseteq Y$ is a presheaf of quasi-categories, write $\pi_B : Z_B \rightarrow B$ for the natural quasi-fibration replacement of $B \cap X \rightarrow B$ (as explained in 2.3.17). By 2.3.16, $B \cap X \rightarrow B$ is a local Joyal equivalence if and only if π_B is a local trivial fibration. Now, suppose there is a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Z_A(U) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^n & \longrightarrow & A(U) \end{array}$$

Then this lifting problem can be solved locally over some covering $\{U_i \rightarrow U\}$ having at most α elements. There is an identification

$$\lim_{\substack{\longrightarrow \\ |B| < \alpha}} Z_B = Z_Y$$

since Y is a filtered colimit of its α -bounded subobjects. Thus, it follows from the regularity assumption on α that there exists an α -bounded $A' \subseteq Y$, $A \subseteq A'$, over which the preceding lifting problem can be solved. The set of all such lifting problems is α -bounded. Thus, there is an α -bounded presheaf of quasi-categories $B_1 \subseteq Y$ such that each lifting problem can be solved over B_1 by 2.3.19. Repeating this procedure countably many times produces an ascending sequence of presheaves of quasi-categories

$$B_1 \subseteq B_2 \cdots \subseteq B_n \cdots$$

such that all lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Z_{B_i}(U) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^n & \longrightarrow & B_i(U) \end{array}$$

can be solved locally over B_{i+1} . Put $B = \cup B_i$. Then B is α -bounded by the regularity of α . Furthermore, B is a presheaf of quasi-categories. Since the construction of Z_B commutes with filtered colimits, $Z_B \rightarrow B$ is a local trivial fibration, as required.

Lemma 2.3.21 *Let $\beta > |\text{Mor}(\mathcal{C})|$ be a cardinal. Put $\alpha = 2^\beta + 1$, so that α is a regular cardinal since it is a successor. Suppose that there is a diagram of monomorphisms of simplicial presheaves*

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

where A is α -bounded and $X \rightarrow Y$ is a local Joyal equivalence. Then there exists α -bounded simplicial presheaves A' and B' such that

1. $\mathcal{S}_{\text{Joyal}}(A) \subseteq A' \subseteq \mathcal{S}_{\text{Joyal}}(Y)$, $\mathcal{S}_{\text{Joyal}}(X) \cap A' \rightarrow A'$ is a local Joyal equivalence.
2. $A \subseteq B'$, $A' \subseteq \mathcal{S}_{\text{Joyal}}(B')$

Proof Since $\mathcal{S}_{\text{Joyal}}$ preserves monomorphisms, 2.2.17 implies that there is a diagram of α -bounded monomorphisms

$$\begin{array}{ccc} & & \mathcal{S}_{\text{Joyal}}(X) \\ & & \downarrow \\ \mathcal{S}_{\text{Joyal}}(A) & \longrightarrow & \mathcal{S}_{\text{Joyal}}(Y) \end{array}$$

Thus, there is an A' with the desired properties by 2.3.20. Now, note that by 2.2.17

$$\lim_{\substack{\longrightarrow \\ Y \in \mathcal{F}_\alpha(X)}} \mathcal{S}_{\text{Joyal}}(Y) \cong \mathcal{S}_{\text{Joyal}}(X).$$

Furthermore, the set of elements

$$\{(x, U) : x \in A'(U) - \mathcal{S}_{\text{Joyal}}(A)(U), U \in \text{Ob}(\mathcal{C})\}$$

is α -bounded, so there exists an α -bounded object B' with the desired properties.

Theorem 2.3.22 *Let α be as in 2.3.21. Suppose that there is diagram of monomorphisms of simplicial presheaves*

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

where A is α -bounded and $X \rightarrow Y$ is a local Joyal equivalence. Then there exists an α -bounded subobject B , $A \subseteq B \subseteq Y$, so that $B \cap X \rightarrow B$ is a local Joyal equivalence.

Proof For each $n \in \mathbb{N}$, define α -bounded objects A_n and B_n inductively, so that the following properties hold:

1. $\mathcal{S}_{Joyal}(B_{n'}) \subseteq A_n \subseteq \mathcal{S}_{Joyal}(Y)$ for all $n' < n$ and $\mathcal{S}_{Joyal}(X) \cap A_n \rightarrow A_n$ is a local Joyal equivalence.
2. $A \subseteq B_n \subseteq Y$ and $A_n \subseteq \mathcal{S}_{Joyal}(B_n)$.

Start the induction by setting $A_0 = B_0 = A$. In general, having defined $A_{n'}$ and $B_{n'}$ for $n' < n$, apply 2.3.21 to the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ B_{n-1} & \longrightarrow & Y \end{array}$$

to produce A_n and B_n . Let

$$B = \lim_{\substack{\longrightarrow \\ n \in \mathbb{N}}} B_n.$$

B is α -bounded by the regularity of α . Now, note that by 2.2.17, for X' a subobject of Y , there are natural isomorphisms

$$\mathcal{S}_{Joyal}(B \cap X') \cong \lim_{\substack{\longrightarrow \\ n \in \mathbb{N}}} \mathcal{S}_{Joyal}(B_n \cap X') \cong \lim_{\substack{\longrightarrow \\ n \in \mathbb{N}}} \mathcal{S}_{Joyal}(B_n) \cap \mathcal{S}_{Joyal}(X') \cong \lim_{\substack{\longrightarrow \\ n \in \mathbb{N}}} (A_n \cap \mathcal{S}_{Joyal}(X')),$$

so that $\mathcal{S}_{Joyal}(B \cap X) \rightarrow \mathcal{S}_{Joyal}(B)$ is a local Joyal equivalence by 2.3.14. Thus, the map $B \cap X \rightarrow X$ is a local Joyal equivalence by 2.3.13, as required.

Lemma 2.3.23 *Let α be a cardinal as in 2.3.21 and 2.3.22. Then a map f has the right lifting property with respect to all maps which are cofibrations (respectively local Joyal equivalences and cofibrations) if and only if it has the right lifting property with respect to all α -bounded cofibrations (respectively α -bounded local Joyal equivalences and cofibrations).*

Proof For cofibrations, this is just [15, Theorem 5.6]. For cofibrations that are local Joyal equivalences, use 2.3.22 and the method of [15, Lemma 5.4].

Lemma 2.3.24 *A map $f : X \rightarrow Y$ of simplicial presheaves which has the right lifting property with respect to all cofibrations is a local Joyal equivalence and a quasi-injective fibration.*

Proof The map f is a quasi-injective fibration by definition. The map is also a sectionwise trivial fibration, and hence a local trivial fibration. Conclude using 2.3.16.

Lemma 2.3.25 *Consider a pushout diagram of simplicial presheaves*

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow b & & \downarrow b' \\ C & \xrightarrow{a'} & B \cup_A C \end{array}$$

where a is a cofibration. Then b' is a local Joyal equivalence if b is. If a is a local Joyal equivalence, then so is a' .

Proof In the case that A, B and C are sheaves of quasi-categories on the Boolean algebra \mathcal{B} , this is immediate from the left properness of the Joyal model structure and 2.3.10.

Now, suppose A, B, C and D are arbitrary simplicial presheaves. In the following diagram each of the vertical maps are sectionwise Joyal equivalences

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}_{Joyal}(B) & \longleftarrow & \mathcal{S}_{Joyal}(A) & \longrightarrow & \mathcal{S}_{Joyal}(C) \end{array}$$

The gluing lemma ([8, Lemma II.8.8]) implies that the induced map

$$B \cup_A C \rightarrow \mathcal{S}_{Joyal}(B) \cup_{\mathcal{S}_{Joyal}(A)} \mathcal{S}_{Joyal}(C)$$

is a sectionwise and hence local Joyal equivalence. Thus, the diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ b \downarrow & & \downarrow b' \\ C & \xrightarrow{a'} & B \cup_A C \end{array}$$

is local Joyal equivalent to

$$\begin{array}{ccc} \mathcal{S}_{Joyal}(A) & \xrightarrow{\mathcal{S}_{Joyal}(a)} & \mathcal{S}_{Joyal}(B) \\ \mathcal{S}_{Joyal}(b) \downarrow & & \downarrow s \\ \mathcal{S}_{Joyal}(C) & \xrightarrow{s'} & \mathcal{S}_{Joyal}(B) \cup_{\mathcal{S}_{Joyal}(A)} \mathcal{S}_{Joyal}(C) \end{array}$$

Since p^*L^2 preserves pushouts and cofibrations, the case of sheaves of quasi-categories on \mathcal{B} implies that $p^*L^2(s)$ is a local Joyal equivalence. Thus, so is s , since local Joyal equivalences are reflected by Boolean localization.

Lemma 2.3.26 *Let $f : X \rightarrow Y$ be a map of simplicial presheaves. Then it can be factored as*

$$\begin{array}{ccc} & Z & \\ i \nearrow & & \searrow p \\ X & \xrightarrow{f} & Y \\ q \searrow & & \nearrow j \\ & W & \end{array}$$

where

1. i is a local Joyal equivalence and a cofibration and p is a quasi-injective fibration.
2. j is a cofibration and q is a quasi-injective fibration and local Joyal equivalence.

Proof For the first factorization choose a cardinal $\lambda > 2^\alpha$, where α is chosen as in 2.3.23, and do a small object argument of size λ to solve all lifting problems

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

where i is an α -bounded trivial cofibration. The result follows from the fact that cofibrations which are local Joyal equivalences are preserved under pushout, which is 2.3.25.

The second statement is proven in a similar manner, using 2.3.24.

Proof of Theorem 1.3.3 CM1 and CM2 are trivial to verify. CM3 follows from the definition of local Joyal equivalences. CM5 is 2.3.26.

Let $f : X \rightarrow Y$ be a map which is a local Joyal equivalence and a quasi-injective fibration. The next paragraph will show that f has the right lifting property with respect to cofibrations.

By 2.3.26, the map has a factorization

$$\begin{array}{ccc} X & \xrightarrow{g} & W \\ & \searrow f & \downarrow h \\ & & Y \end{array}$$

where h has the right lifting property with respect to all cofibrations, and is therefore a local Joyal equivalence, and g is a cofibration. Hence by the 2 out of 3 property, g is a local Joyal equivalence. It is also a cofibration. Thus, there is a lifting in the diagram

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ g \downarrow & \nearrow n & \downarrow f \\ W & \xrightarrow{h} & Y \end{array}$$

Finally, the diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & W & \xrightarrow{n} & X \\ f \downarrow & & \downarrow h & & \downarrow f \\ Y & \xrightarrow{id} & Y & \xrightarrow{id} & Y \end{array}$$

shows that f is a retract of h . Hence, f has the right lifting property with respect to all cofibrations (since right lifting property is preserved under retracts), as required. This argument is standard; it is found, for instance, in [15, Theorem 5.8].

Theorem 2.3.27 *The category $s\mathbf{Sh}(\mathcal{C})$ along with the class of local Joyal equivalences, monomorphisms, and quasi-injective fibration forms a left proper model structure. Furthermore, there is a Quillen equivalence*

$$L^2 : s\mathbf{Pre}(\mathcal{C}) \rightleftarrows s\mathbf{Sh}(\mathcal{C}) : i,$$

where i is the inclusion of sheaves into presheaves and L^2 is sheafification.

Proof The associated sheaf functor preserves and reflects local Joyal equivalences and it also preserves cofibrations. Hence, the inclusion functor preserves quasi-injective fibrations. Thus, the functors form a Quillen pair. The unit map of the adjunction $X \rightarrow L^2(X)$ is a local Joyal equivalence, and the counit map is the identity. Thus, the second statement follows from the first, and it suffices to prove the first statement.

Axiom CM1 follows from completeness and cocompleteness of the sheaf category. Axioms CM2-CM4 follow from the corresponding statements for simplicial presheaves. Let α be a cardinal as in 2.3.23. Then choose a regular cardinal β such that $L^2(f)$ is β -bounded for each α -bounded trivial cofibration. Then a map f is a quasi-injective fibration if and only if it has the right lifting property with respect to all β -bounded trivial cofibrations. Doing a small object argument of size 2^β as in 2.3.26 gives one half of CM5. The other half has an analogous proof.

Left properness comes from the corresponding statement for simplicial presheaves, as well as the fact that $X \rightarrow L^2(X)$ is a local Joyal equivalence.

Chapter 3

Local Complete Segal Spaces

Introduction

The purpose of this chapter is to develop a model structure on bisimplicial presheaves in which the weak equivalences are stalkwise equivalences in the complete Segal model structure on bisimplicial sets, and show that it is Quillen equivalent to the local Joyal model structure on simplicial presheaves of 2.3.3. The technique of Boolean localization is used extensively to develop this model structure. The contents of this chapter were developed originally in [27].

Given model category M , one can construct a bisimplicial set $N(M)$ such that

$$N(M)_{n,*} = B(\text{we}(M^{[n]})),$$

where $\text{we}(M^{[n]})$ is the subcategory of the diagram category whose morphisms are the object-wise weak equivalences. By taking Reedy fibrant replacement, one produces a complete Segal space $N^f(M)$. If the model category M is a simplicial model category, $N^f(M)$ is a good approximation to M . Indeed, [29, Theorem 8.3] says that

1. The homotopy category of the complete Segal space $N^f(M)$ ([29, 5.5]) can be identified with the homotopy category $\text{Ho}(M)$ of M .
2. For each $x, y \in \text{Ob}(M)$, there is a weak equivalence of simplicial sets $\mathbf{Map}_{N^f(M)}(x, y) \rightarrow \mathbf{hom}_M(x, y)$, where $\mathbf{Map}_{N^f(M)}(x, y)$ is the mapping space defined in 4.1.20.

Thus, one motivation for studying the local complete Segal model structure is to study sheaves of homotopy theories M as in the Hirschowitz-Simpson paper by replacing them with the construction $N^f(M)$. Specifically, one would be interested in whether objects in M could be glued along weak equivalences. That is, whether the presheaf $N^f(M)$ satisfies an appropriate descent condition (as in 3.4.1). This could potentially give a simpler exposition of the results of [10]; the object $N^f(M)$ is much simpler to describe than the Segal category $\mathcal{S}_{\text{Seg}}(LM)$ used to approximate M in [10]. This potential application was suggested in [29, 1.3].

In the first section of this chapter, we review some properties of the complete Segal model structure, as well as describe a Quillen equivalence between the complete Segal model structure and Joyal model structure. These results are necessary for establishing the main results of the chapter.

In Section 3.2, we define the local complete Segal model structure as the left Bousfield localization of the local Reedy model structure for bisimplicial presheaves (see 3.2.5) along the constant bisimplicial presheaf maps $G(n) \subset F(n)$, $F(0) \subset \tilde{I}$. Using the technique of fibred sites (see 3.2.1), we can identify the local Reedy model structure for bisimplicial presheaves with the injective model structure on simplicial presheaves $s\mathbf{Pre}(\mathcal{C}/\Delta)$, where \mathcal{C}/Δ is the site defined in 3.2.3. Thus, we can use the localization theory of simplicial presheaves of [15, Chapter 7] to construct the local complete Segal model structure.

More generally, one could define a local analogue of the n -fold complete Segal space structure (found in [3]) as a left Bousfield localization of $s\mathbf{Pre}(\mathcal{C}/\Delta^{\times n})$ along the constant maps of multisimplicial sets described in [3, Theorem 5.6]. Here, $\Delta^{\times n}$ is the constant presheaf of categories with value the n -fold product of the ordinal number category. This would give an appropriate model of local (∞, n) -category theory. However, we do not pursue this in this thesis.

In Section 3.3, we establish the main result of this chapter: the Quillen equivalence between the local complete Segal model structure and the local Joyal model structure.

In Section 3.4, we establish a result which relates descent in the local Joyal model structure to descent in the injective model structure. Interestingly, this result is proven using the Quillen equivalence established in Section 3.3.

3.1 Complete Segal Spaces

Given simplicial sets K and L , we can define a bisimplicial set with $K \tilde{\times} L$ so that $(K \tilde{\times} L)_{m,n} = K_m \times L_n$. We write $\Delta^{p,q}$ for $\Delta^p \tilde{\times} \Delta^q$. Recall that for a simplicial set K , $\pi(K)$ denotes its fundamental groupoid, and for a category C , $B(C)$ denotes its nerve.

Definition 3.1.1 Given a category C , its **discrete nerve**, $\text{Disc}(C)$, is defined to be the bisimplicial set $B(C) \tilde{\times} \Delta^0$. We write $\tilde{I} = \text{Disc}(\pi(\Delta^1))$. Write $F(n)$ for $\text{Disc}([n]) = \Delta^n \tilde{\times} \Delta^0$.

Note that there is a map $F(0) \rightarrow \tilde{I}$ induced by the inclusion of the initial vertex $0 \in \pi(\Delta^1)$.

Remark 3.1.2 Throughout this chapter, we will identify sSet with a subcategory of s^2Set via the embedding $K \mapsto \Delta^0 \tilde{\times} K$.

If X is a bisimplicial set, then in the expression $X_{m,n}$ we call m the **horizontal** coordinate and n the **vertical** coordinate.

Definition 3.1.3 Let $G(n)$ be the glued together string of 1-simplices $1 \leq 2 \leq \dots \leq n$ inside Δ^n regarded as a vertically discrete bisimplicial set. Thus, there are natural inclusions $G(n) \subset F(n)$.

Remark 3.1.4 Note that for a bisimplicial set X , $\mathbf{hom}(F(k), X) \cong X_{k,*}$, the vertical simplicial set in horizontal degree k since

$$\begin{aligned} (\mathbf{hom}(\Delta^k \tilde{\times} \Delta^0, X))_n &\cong \mathbf{hom}(\Delta^k \tilde{\times} (\Delta^0 \times \Delta^n), X) \\ &= \mathbf{hom}(\Delta^{k,n}, X) \\ &\cong X_{k,n}. \end{aligned}$$

Note that this implies that $\mathbf{hom}(F(n), X) \rightarrow \mathbf{hom}(G(n), X)$ can be identified with the map

$$X_{n,*} \rightarrow X_{1,*} \times_{X_{0,*}} X_{1,*} \cdots \times_{X_{0,*}} X_{1,*},$$

where the right hand side is the limit of the diagram

$$X_{1,*} \xrightarrow{d_1} X_{0,*} \xleftarrow{d_0} X_{1,*} \cdots$$

constructed from n copies of $X_{1,*}$.

Example 3.1.5 The **Reedy model structure** on $s^2\text{Set}$ has cofibrations which are horizontal levelwise monomorphisms and weak equivalences which are horizontal levelwise weak equivalences. The generating cofibrations for the Reedy model structure are of the form

$$\partial(\Delta^n \tilde{\times} \Delta^k) = (\partial\Delta^n \tilde{\times} \Delta^k) \cup (\Delta^n \tilde{\times} \partial\Delta^k) \subset \Delta^n \tilde{\times} \Delta^k$$

for $k, n \in \mathbb{N}$. The generating trivial cofibrations are of the form

$$(\Delta^k \tilde{\times} \Delta^n) \cup (\Delta^k \tilde{\times} \Lambda_r^n) \subset \Delta^k \tilde{\times} \Delta^n,$$

where $0 \leq r \leq n$.

The Reedy model structure is a simplicial model category, with simplicial hom given by the usual simplicial hom for bisimplicial sets.

Definition 3.1.6 The **complete Segal model structure** is the left Bousfield localization of the Reedy model structure on $s^2\text{Set}$ along the set of maps $G(n) \subset F(n)$, $n \in \mathbb{N}$, and the natural inclusion $F(0) \rightarrow \tilde{I}$, where \tilde{I} is as in 3.1.1. The fibrant objects of this model category are called **complete Segal spaces**.

The complete Segal model structure first appeared in [29].

Example 3.1.7 If S is some set of maps in a simplicial model category, we say that X is **S-local** if and only if X is fibrant and for each $g \in S$, $g^* : \mathbf{hom}(D, X) \rightarrow \mathbf{hom}(C, X)$ is a weak equivalence. By [9, Theorem 4.1.1], an object of X is fibrant for the model structure of 3.1.6 if and only if it is fibrant in the Reedy model structure and it is S -local, where S is the set of maps in 3.1.6.

Let $t_! : s^2\text{Set} \rightarrow s\text{Set}$ be the colimit-preserving functor defined by

$$t_!(\Delta^n \tilde{\times} \Delta^m) = \Delta^n \times B\pi(\Delta^m)$$

and let $t^!$ be its right adjoint. The following theorem is proven using 2.1.22 (see [21, Sections 2-4]).

Theorem 3.1.8 *There is a Quillen equivalence*

$$t_! : s^2\mathbf{Set} \rightleftarrows s\mathbf{Set} : t^!$$

between the complete Segal space model structure and the Joyal model structure.

Example 3.1.9 Recall from 2.1.14 the definitions of the functors $k_!, k^! : s\text{Set} \rightarrow s\text{Set}$. Observe that

$$t^!(Y)_{m,n} \cong \mathbf{hom}(\Delta^m \times B\pi(\Delta^n), Y) \cong \mathbf{hom}(B\pi(\Delta^n), \mathbf{hom}(\Delta^m, Y)),$$

so that

$$t^!(Y)_{m,*} = k^!(\mathbf{hom}(\Delta^m, Y)). \quad (3.1)$$

A bisimplicial set map $f : X \rightarrow t^!(Y)$ consists of maps

$$f : k_!(X_{m,*}) \times \Delta^m \rightarrow Y,$$

so that the diagrams

$$\begin{array}{ccc} k_!(X_{n,*}) \times \Delta^m & \xrightarrow{1 \times \theta} & k_!(X_{n,*}) \times \Delta^n \\ \theta^* \times 1 \downarrow & & \downarrow \\ k_!(X_{m,*}) \times \Delta^m & \xrightarrow{f} & Y \end{array}$$

commute for all ordinal number maps $\theta : [\mathbf{m}] \rightarrow [\mathbf{n}]$. It follows that

$$t_!(X) \cong d(k_!(X)). \quad (3.2)$$

Lemma 3.1.10 *Let K be a finite bisimplicial set (i.e. having finitely many nondegenerate bisimplices) and $X \in s^2\mathbf{Pre}(\mathcal{C})$. Then we have isomorphisms, natural in K and X :*

1. $p^*\mathrm{hom}(K, X) \cong \mathrm{hom}(K, p^*(X))$ if X is a bisimplicial sheaf.
2. $p^*(X^K) \cong p^*(X)^K$ if X is a bisimplicial sheaf.
3. $L^2\mathrm{hom}(K, X) \cong \mathrm{hom}(K, L^2(X))$.
4. $L^2(X^K) \cong L^2(X)^K$.

where L^2 denotes sheafification and p is our choice of Boolean localization.

Example 3.1.11 *Suppose that X is a simplicial sheaf and K is a simplicial set. Let $p : s\mathbf{Sh}(\mathcal{B}) \rightarrow \mathbf{Sh}(\mathcal{C})$ be a Boolean localization. We have isomorphisms*

$$p_*\mathrm{hom}(K, X) \cong \lim_{\substack{\leftarrow \\ \Delta^n \rightarrow K}} p_*(X_n) \cong \mathrm{hom}(K, p_*(X)).$$

Thus, there is a natural isomorphism of sheaves

$$p_*k^!(X) \cong k^!p_*(X).$$

Thus, by adjunction

$$p^*L^2k_! \cong L^2k_!p^*L^2. \quad (3.3)$$

3.2 Fibred Sites and Localization Theory For Simplicial Presheaves

The following construction is a Grothendieck construction for a presheaf of categories A on the site \mathcal{C} .

Definition 3.2.1 There is a site \mathcal{C}/A whose objects are all pairs (U, x) , where U is an object of \mathcal{C} and $x \in \text{Ob}(A)(U)$. A morphism $(\alpha, f) : (V, y) \rightarrow (U, x)$ in the category \mathcal{C}/A is a pair consisting of a morphism $\alpha : V \rightarrow U$ of \mathcal{C} , along with a morphism $f : x \rightarrow \alpha^*(y)$ of $A(U)$. Given another morphism (γ, g) , the composite $(\alpha, f) \circ (\gamma, g)$ is defined by

$$(\alpha, f) \circ (\gamma, g) = (\alpha\gamma, g \cdot \gamma^*(f)).$$

There is a forgetful functor $c : \mathcal{C}/A \rightarrow \mathcal{C}$ which is defined by $(U, x) \mapsto U$. The covering sieves for \mathcal{C}/A are the sieves which contain a sieve of the form $c^{-1}(S)$ for S is a covering sieve of \mathcal{C} .

Definition 3.2.2 Denote $s, t : \text{Mor}(A) \rightarrow \text{Ob}(A)$ the source and target maps. We will regard $\text{Mor}(A)$ as a simplicial presheaf and $\text{Ob}(A)$ as a discrete simplicial presheaf. An **A-diagram** is a simplicial presheaf map $\pi_X : X \rightarrow \text{Ob}(A)$, together with an ‘action diagram’

$$\begin{array}{ccc} X \times_s \text{Mor}(A) & \xrightarrow{m} & X \\ \text{pr} \downarrow & & \downarrow \pi_X \\ \text{Mor}(A) & \xrightarrow{t} & \text{Ob}(A) \end{array}$$

One further requires that m respects compositions and identities. We denote by $s\text{Pre}(\mathcal{C})^A$ the category of A-diagrams whose morphisms are natural transformations

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \searrow & & \swarrow \pi_Y \\ & \text{Ob}(A) & \end{array}$$

that respect compositions and identities.

Example 3.2.3 Note if A is a constant simplicial presheaf and \mathcal{C} is a site, then we have an isomorphism

$$\mathcal{C}/A \cong \mathcal{C} \times A.$$

In particular,

$$\text{Pre}(*/\Delta) \cong s\text{Pre}(*).$$

is the category of simplicial sets. Consequently, we have an identification

$$s\text{Pre}(\mathcal{C}/\Delta) \cong s\text{Pre}(\mathcal{C} \times \Delta) \cong s^2\text{Pre}(\mathcal{C}).$$

Theorem 3.2.4 ([14, pg. 817-819]). *Let A be a presheaf of categories on \mathcal{C} . There is an equivalence of categories between $s\text{Pre}(\mathcal{C}/A)$ and $s\text{Pre}(\mathcal{C})^{A^{op}}$. This equivalence induces a model structure on $s\text{Pre}(\mathcal{C})^{A^{op}}$ defined as follows.*

1. A weak equivalence (respectively a cofibration)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \swarrow \\ & \text{Ob}(A) & \end{array}$$

of A^{op} -diagrams is a map such that the simplicial presheaf map $f : X \rightarrow Y$ is a local weak equivalence (respectively monomorphism).

2. A fibration of A^{op} -diagrams is a map which has the right lifting property with respect to all trivial cofibrations.

Remark 3.2.5 3.2.3 and 3.2.4 imply that there is a Quillen equivalence

$$s\mathbf{Pre}(\mathcal{C}/\Delta) \rightleftarrows s^2\mathbf{Pre}(\mathcal{C}),$$

where the latter is equipped with a model structure in which a map $f : X \rightarrow Y$ is a weak equivalence (respectively cofibration) if and only if $X_{n,*} \rightarrow Y_{n,*}$ is a local weak equivalence (respectively monomorphism).

We call this model structure on bisimplicial presheaves the **local Reedy model structure** and its weak equivalence **local Reedy equivalences**.

Suppose we choose a set S of monomorphisms in $s\mathbf{Pre}(\mathcal{C})$. By the results of [15, Section 5.1], the injective model structure on $s\mathbf{Pre}(\mathcal{C})$ is cofibrantly generated. We can form a smallest saturated set of monomorphisms \mathcal{F} , with $S \subseteq \mathcal{F}$, subject to the following conditions:

1. The class \mathcal{F} contains a family of generating trivial cofibrations j for the injective model structure and all elements of S .
2. If $C \rightarrow D$ is an α -bounded cofibration, and $A \rightarrow B$ is an element of \mathcal{F} , then $(A \times D) \cup (B \times C) \rightarrow B \times D$ is an element of \mathcal{F} .

The following theorem is [15, Theorem 7.18]:

Theorem 3.2.6 *Let \mathcal{F} be the set of cofibrations defined above. We call a object X of $s\mathbf{Pre}(\mathcal{C})$ **\mathcal{F} -injective** if the map $X \rightarrow *$ has the right lifting property with respect to each map in \mathcal{F} . We call a map an **\mathcal{F} -local equivalence** if and only if $\mathbf{hom}(f, Z)$ is a weak equivalence of simplicial sets for each \mathcal{F} -injective object Z (here, $\mathbf{hom}(\cdot, \cdot)$ is the usual simplicial hom for the injective model structure). There is a model structure on $s\mathbf{Pre}(\mathcal{C})$, called the **\mathcal{F} -local model structure**, in which the weak equivalences are the \mathcal{F} -equivalences and cofibrations are monomorphisms.*

Note that local weak equivalences are \mathcal{F} -equivalences.

Lemma 3.2.7 *An \mathcal{F} -equivalence between two \mathcal{F} -injective objects of $s\mathbf{Pre}(\mathcal{C})$ is a sectionwise weak equivalence.*

Proof The \mathcal{F} -injective objects are the fibrant objects ([15, Corollary 7.12]), and a weak equivalence of fibrant objects is a simplicial homotopy equivalence.

Definition 3.2.8 Recall that we can identify bisimplicial sets with constant bisimplicial presheaves. Under this identification, let

$$S = \{G(n) \subset F(n) : n \in \mathbb{N}\} \cup \{F(0) \subset I\}$$

Let \mathcal{F} be the smallest saturated set containing S as in 3.2.6. Then the identification of 3.2.5 and 3.2.6 applied to the family \mathcal{F} give a model structure on $s^2\mathbf{Pre}(\mathcal{C})$ called the **local complete Segal model structure**. We call its weak equivalences **local complete Segal equivalences**. We call its fibrations **Segal-injective** fibrations.

Let $U \in \text{Ob}(\mathcal{C})$. Then there exists a functor $L_U : s^2\mathbf{Set} \rightarrow s^2\mathbf{Pre}(\mathcal{C})$ defined by $L_U(K) = \text{hom}(-, U) \times K$.

Remark 3.2.9 Note that if X is a fibrant object for the local complete Segal model structure, then it is a presheaf of complete Segal spaces.

Indeed, X has the right lifting property with respect to $L_U(i)$ where i is one of the generating cofibrations for the Reedy model structure in 3.1.5. Thus, X is sectionwise Reedy fibrant.

Let $U \in \text{Ob}(\mathcal{C})$. Let $j_n : G(n) \rightarrow F(n)$ be the inclusion. By basic localization theory, $\mathbf{hom}(L_U(j_n), X)$ is a weak equivalence for $n \in \mathbb{N}$. But this can be identified with $\mathbf{hom}(F(n), X(U)) \rightarrow \mathbf{hom}(G(n), X(U))$ (note that under the identification of 3.2.3, the constant simplicial presheaf Δ^n gets identified with the constant bisimplicial presheaf $\Delta^0 \tilde{\times} \Delta^n$).

3.3 Equivalence with the Local Joyal Model Structure

Let

$$\mathcal{S}_{CSeg} : s^2\mathbf{Pre}(\mathcal{C}) \rightarrow s^2\mathbf{Pre}(\mathcal{C})$$

denote the functor obtained by applying the complete Segal fibrant replacement functor sectionwise. Let \mathcal{L}_{CSeg} , \mathcal{L}_{Joyal} and \mathcal{L}_{inj} denote the fibrant replacement functor for the local complete Segal, local Joyal and injective model structures, respectively.

We define functors $t_! : s^2\mathbf{Pre}(\mathcal{C}) \rightarrow s\mathbf{Pre}(\mathcal{C})$ and $t^! : s\mathbf{Pre}(\mathcal{C}) \rightarrow s^2\mathbf{Pre}(\mathcal{C})$ by composition with $t_!$ and $t^!$ respectively. We also have functors $k_! : s\mathbf{Pre}(\mathcal{C}) \rightarrow s\mathbf{Pre}(\mathcal{C})$ and $k^! : s\mathbf{Pre}(\mathcal{C}) \rightarrow s\mathbf{Pre}(\mathcal{C})$.

Lemma 3.3.1 *There is a natural isomorphism $L^2 t_! p^* L^2 \cong p^* L^2 t_!$.*

Proof This follows from equation 3.2 of 3.1.9 and equation 3.3 of 3.1.11.

Lemma 3.3.2 *Let $f : X \rightarrow Y$ be a local weak equivalence. Then $k_!(f)$ is a local Joyal equivalence.*

Proof Consider the natural sectionwise fibrant replacement map $\phi_X : X \rightarrow \text{Ex}^\infty(X)$. The map $k_!(\phi_X)$ is a sectionwise, and hence local Joyal equivalence by 2.3.9. Thus, the diagram

$$\begin{array}{ccc} k_!(X) & \longrightarrow & k_!\text{Ex}^\infty(X) \\ k_!(f) \downarrow & & \downarrow k_!\text{Ex}^\infty(f) \\ k_!(Y) & \longrightarrow & k_!\text{Ex}^\infty(Y) \end{array}$$

and the 2 out of 3 property imply that we may assume that f is a map of presheaves of Kan complexes. The fact that $p^* L^2$ preserves local weak equivalences, along with 2.2.11, imply that $p^* L^2(f)$ is a sectionwise weak equivalence. Consider the diagram

$$\begin{array}{ccc} k_! p^* L^2(X) & \longrightarrow & L^2 k_! p^* L^2(X) \\ k_! p^* L^2(f) \downarrow & & \downarrow L^2 k_! p^* L^2(f) \\ k_! p^* L^2(Y) & \longrightarrow & L^2 k_! p^* L^2(Y) \end{array}$$

The left vertical map is a sectionwise, and hence local Joyal equivalence. By 2.3.2, the horizontal maps are local Joyal equivalences. Thus, $L^2k_1p^*L^2(f) \cong p^*L^2k_1(f)$ is a local Joyal equivalence. But p^*L^2 reflects local Joyal equivalences.

Lemma 3.3.3 *Let $f : A \rightarrow B$ be a local Joyal equivalence and $g : C \rightarrow D$ be a cofibration. Then $h : A \times C \rightarrow B \times C$ and $u : (A \times D) \cup_{A \times C} (B \times C) \rightarrow B \times D$ are local Joyal equivalences.*

Proof The second statement follows from left properness and the first statement. We prove the first statement.

If $A \rightarrow B, C \rightarrow D$ are Joyal equivalences, then so is $A \times C \rightarrow B \times D$ (see [22, Corollary 2.2.5.4] or [16, Lemma 30]). Thus, it suffices to prove the statement for A, B and C presheaves of quasi-categories.

By 2.3.11, $p^*L^2(f)$ is a sectionwise Joyal equivalence. Thus, since p^*L^2 preserves finite limits, $p^*L^2(h)$ is isomorphic to

$$p^*L^2(A) \times p^*L^2(C) \rightarrow p^*L^2(B) \times p^*L^2(C),$$

which is a sectionwise Joyal equivalence. Thus, h is a local Joyal equivalence, as required.

Example 3.3.4 By a matching space argument, the generating trivial cofibrations for the local Reedy model structure on $s^2\mathbf{Pre}(\mathcal{C})$ are of the form $f = (\Delta^k \tilde{\times} X) \cup (\partial \Delta^k \tilde{\times} Y) \rightarrow \Delta^k \tilde{\times} Y$, where $X \rightarrow Y$ is an α -bounded trivial cofibration.

Thus, since $t_!$ preserves colimits, we have

$$t_!(f) = (\Delta^k \times k_!(X)) \cup (\partial \Delta^k \times k_!(Y)) \rightarrow \Delta^k \times k_!(Y).$$

The map $k_!(X) \rightarrow k_!(Y)$ is a local Joyal equivalence by 3.3.2. Thus, the map $t_!(f)$ is a local Joyal equivalence by 3.3.3.

Lemma 3.3.5 *The natural map $t_!(X) \rightarrow t_!(\mathcal{L}_{CSeg}(X))$ is a local Joyal equivalence.*

Proof Let \mathcal{F} be the family defined in 3.2.8. The fibrant objects of the local complete Segal model structure are the \mathcal{F} -injective objects by [15, Corollary 7.12]. Thus, \mathcal{L}_{CSeg} is obtained by taking iterated pushouts along maps in a set \mathcal{H} generating \mathcal{F} (see [15, Lemma 10.21]). The functor $t_!$ commutes with colimits, and filtered colimits of local Joyal equivalences are local Joyal equivalences. Thus, it suffices to show that $t_!(\phi)$ is a local Joyal equivalence, where ϕ is in the diagram

$$\begin{array}{ccc} \coprod_{\mathcal{H}} Q \times \mathrm{hom}(Q, X) & \longrightarrow & X \\ \downarrow & & \downarrow \phi \\ \coprod_{\mathcal{H}} R \times \mathrm{hom}(Q, X) & \longrightarrow & E_1(X) \end{array}$$

and $Q \rightarrow R$ is an element of \mathcal{H} . We can take \mathcal{H} to be the set of maps $A \times D \cup B \times C \rightarrow B \times D$, where $C \rightarrow D$ is a α -bounded cofibration and $A \rightarrow B$ is one of:

1. $G(n) \subset F(n)$

2. $F(0) \rightarrow \tilde{I}$
3. A generating trivial cofibration for the local Reedy model structure.

Let X be a complete Segal space. Then $\mathbf{hom}(\tilde{I} \times D, X) \rightarrow \mathbf{hom}(F(0) \times D, X)$ is naturally isomorphic to $\mathbf{hom}(\tilde{I}, X^D) \rightarrow \mathbf{hom}(F(0), X^D)$. By basic localization theory, X^D is a complete Segal space. Since $F(0) \rightarrow \tilde{I}$ is a complete Segal equivalence, $\mathbf{hom}(\tilde{I}, X^D) \rightarrow \mathbf{hom}(F(0), X^D)$ is a weak equivalence. It follows that $F(0) \times D \subset \tilde{I} \times D$ is a complete Segal equivalence. Similarly, we can show that $G(n) \times D \subset F(n) \times D$ is a complete Segal equivalence.

The functor $t_!$ takes sectionwise complete Segal equivalences to sectionwise Joyal equivalences by [21, Theorem 4.12]. The maps $t_!(F(0) \times D) \subset t_!(\tilde{I} \times D)$ and $t_!(F(0) \times D) \subset t_!(\tilde{I} \times D)$ are sectionwise Joyal equivalences. If f is a generating trivial cofibration for the local Reedy model structure, then $t_!(f \times id_D)$ is a local Joyal equivalence by 3.3.4 and 3.3.3. Thus, for $g \in \mathcal{H}$, $t_!(g)$ can be written as

$$(t_!(A \times D)) \cup (t_!(B \times C)) \rightarrow t_!(B \times D).$$

The maps $t_!(A \times D) \rightarrow t_!(B \times D)$ and $t_!(A \times C) \rightarrow t_!(B \times C)$ are local Joyal trivial cofibrations by 3.3.3 and the paragraph above. Thus, the map $t_!(g)$ is a local Joyal trivial cofibration. In conclusion, $t_!(\phi)$ is a pushout of a trivial cofibration for the local Joyal model structure, and is thus a trivial cofibration.

Lemma 3.3.6 *J preserves both trivial Kan fibrations and Kan fibrations of quasi-categories.*

Proof Let $f : X \rightarrow Y$ be a Kan fibration. The map f creates equivalences (i.e. 1-simplices that represent isomorphisms in the path category), since $\Delta^1 \rightarrow B\pi\Delta^1$ is a weak equivalence (both of these spaces are weakly contractible) and a monomorphism. Thus, one has a pullback

$$\begin{array}{ccc} J(X) & \longrightarrow & J(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

The same proof applies to trivial fibrations.

Lemma 3.3.7 *J preserves local trivial fibrations between presheaves of quasi-categories.*

Proof Let f be a local trivial fibration. Then $p^*L^2(f)$ is a sectionwise trivial fibration, so that $Jp^*L^2(f)$ is a sectionwise trivial fibration by 3.3.6. But 2.3.6 implies that $Jp^*L^2(f) \cong p^*L^2J(f)$. Thus, $J(f)$ is a local trivial fibration by 2.2.7.

Lemma 3.3.8 *Let $f : X \rightarrow Y$ be a local Joyal equivalence of presheaves of quasi-categories. Then $t^!(f)$ is a local Reedy equivalence.*

Proof By functorial factorization 2.3.17, we can assume that f is a sectionwise quasi-fibration (note that $t^!$ preserves Joyal equivalences of quasi-categories). Thus, f is a local trivial fibration by 2.3.16. Thus, so are the maps f^{Δ^n} . By 3.3.7, each $J(f^{\Delta^n})$ is a local trivial fibration. But $J(f^{\Delta^n})$ is sectionwise Joyal equivalent to $t^!(f)_{n,*} = k^!(f^{\Delta^n})$ by 2.1.18.

Theorem 3.3.9 *There is a Quillen equivalence*

$$t_! : s^2\mathbf{Pre}(\mathcal{C}) \rightleftarrows s\mathbf{Pre}(\mathcal{C}) : t^!$$

from the local complete Segal model structure to the local Joyal model structure.

Proof Let X be a fibrant object of the local Joyal model structure. Then X is a presheaf of quasi-categories, and $t_!t^!(X) \rightarrow X$ is a sectionwise Joyal equivalence by [21, Theorem 4.12] (note that every object is cofibrant in the model structures involved).

We want to show that the natural map $X \rightarrow t^!\mathcal{L}_{Joyal}t_!(X)$ is a local complete Segal equivalence. There is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & t^!\mathcal{L}_{Joyal}t_!(X) \\ \downarrow & & \downarrow \\ \mathcal{L}_{CSeg}(X) & \longrightarrow & t^!\mathcal{L}_{Joyal}t_!\mathcal{L}_{CSeg}(X) \end{array}$$

The map $\mathcal{L}_{Joyal}t_!(X) \rightarrow \mathcal{L}_{Joyal}t_!\mathcal{L}_{CSeg}(X)$ is a local Joyal equivalence of presheaves of quasi-injective objects by 3.3.5. Thus, it is a sectionwise Joyal equivalence. It follows from [21, Theorem 4.12] that the right vertical map is a sectionwise complete Segal equivalence of presheaves of complete Segal spaces. In particular, it is a sectionwise Reedy, and hence local complete Segal equivalence. The left vertical map is a local complete Segal equivalence by definition. Thus, we may assume that X is a presheaf of complete Segal spaces.

The map $\mathcal{S}_{Joyal}t_!(X) \rightarrow \mathcal{L}_{Joyal}t_!(X)$ is a local Joyal equivalence of presheaves of quasi-categories. Thus, $t^!\mathcal{S}_{Joyal}t_!(X) \rightarrow t^!\mathcal{L}_{Joyal}t_!(X)$ is a local complete Segal equivalence by 3.3.8. By [21, Theorem 4.12], the map $X \rightarrow t^!\mathcal{S}_{Joyal}t_!(X)$ is a sectionwise complete Segal equivalence. It is also a sectionwise Reedy equivalence (since it is also a map of presheaves of complete Segal spaces), and hence a local complete Segal equivalence. It follows that the map

$$X \rightarrow t^!t_!(X) \rightarrow t^!\mathcal{S}_{Joyal}t_!(X) \rightarrow t^!\mathcal{L}_{Joyal}t_!(X)$$

is a local complete Segal equivalence, as required.

Lemma 3.3.10 *$t^!$ preserves and reflects local Joyal equivalences of presheaves of quasi-categories.*

Proof Consider the diagram

$$\begin{array}{ccccc} X & \longrightarrow & t_!t^!(X) & \xrightarrow{a} & t_!\mathcal{S}_{Joyal}t^!(X) \\ \downarrow f & & \downarrow t_!t^!(f) & & \downarrow \\ Y & \longrightarrow & t_!t^!(Y) & \xrightarrow{b} & t_!\mathcal{S}_{Joyal}t^!(Y) \end{array}$$

The horizontal composites, a and b are all local Joyal equivalences. Thus, by 2 out of 3, the left horizontal maps are local Joyal equivalences. We conclude that f is a local Joyal equivalence if and only if $t_!t^!(f)$ is a local Joyal equivalence. But $t_!$ preserves and reflects local complete Segal equivalences; it is the left adjoint of a Quillen equivalence between model categories in which every object is cofibrant.

Corollary 3.3.11 *A sectionwise complete Segal equivalence is a local complete Segal equivalence.*

Proof Let f be a sectionwise complete Segal equivalence. Then $t_!(f)$ is a sectionwise Joyal equivalence, and hence a local Joyal equivalence. But $t_!$ reflects weak equivalences between cofibrant objects of the local complete Segal model structure, as required.

Corollary 3.3.12 *p^*, L^2 both preserve and reflect local complete Segal equivalences.*

Proof We prove that p^*L^2 preserves local complete Segal equivalences. The proof that it reflects them is similar.

Suppose that f is a local complete Segal equivalence. Then since $t_!$ is the left adjoint of a Quillen adjunction, it preserves weak equivalences of cofibrant objects. But everything is cofibrant for the local complete Segal model structure, so that $t_!(f)$ is a local Joyal equivalence. Thus, so is $p^*L^2t_!(f) \cong L^2t_!p^*L^2$ (this isomorphism follows from equations 3.2 and 3.3). It follows that $t_!p^*L^2(f)$ is a local Joyal equivalence, and hence that $p^*L^2(f)$ is a local complete Segal equivalence, since $t_!$ reflects local complete Segal equivalences of cofibrant objects.

We call a fibration for the local complete Segal model structure a **Segal-injective fibration**.

Theorem 3.3.13 *The category $s^2\mathbf{Sh}(\mathcal{C})$, along with the class of local complete Segal equivalences, monomorphisms and Segal-injective fibrations, forms a left proper model structure. Let i denote the inclusion of bisimplicial sheaves into bisimplicial presheaves. There is a Quillen equivalence*

$$L^2 : s^2\mathbf{Pre}(\mathcal{C}) \rightleftarrows s^2\mathbf{Sh}(\mathcal{C}) : i.$$

from the local complete Segal model structure on bisimplicial presheaves to the local complete Segal model structure on bisimplicial sheaves.

Proof The associated sheaf functor preserves and reflects local complete Segal equivalences, and it preserves cofibrations. Hence, the inclusion functor preserves Segal-injective fibrations. Thus, the functors form a Quillen pair. The unit map of the adjunction $X \rightarrow L^2(X)$ is a local Reedy, and hence local complete Segal equivalence, and the counit map is the identity. Thus, if we prove the first statement, we have the second.

Axiom CM1 follows from completeness and cocompleteness of the sheaf category. Axioms CM2-CM4 follow from the corresponding statements for local complete Segal model structure on $s^2\mathbf{Pre}(\mathcal{C})$. By [15, Theorem 7.5], there exists a regular cardinal α , so that a map is a fibration in the complete Segal model structure if and only if it has the right lifting property with respect to α -bounded trivial cofibrations. Choose a regular cardinal β such that $L^2(f)$ is β -bounded for each α -bounded f . Then a sheaf map f is a Segal-injective fibration if and only if it has the right lifting property with respect to all β -bounded trivial cofibrations. Doing a small object argument of size 2^β , as in [15, Lemma 5.7], gives CM5.

Theorem 3.3.14 *There is a Quillen equivalence*

$$L^2t_! : s^2\mathbf{Sh}(\mathcal{C}) \rightleftarrows s\mathbf{Sh}(\mathcal{C}) : t^!$$

from the local complete Segal model structure to the local Joyal model structure.

Proof Immediate from 3.3.9, and the fact that $t^!$ commutes with sheafification by equation 3.1.

3.4 Descent Results

Definition 3.4.1 One says that a simplicial presheaf (respectively bisimplicial presheaf, respectively simplicial presheaf) X **satisfies descent** for the injective (respectively local complete Segal, local Joyal) model structure if and only if $X \rightarrow \mathcal{L}_{inj}(X)$ (respectively $X \rightarrow \mathcal{L}_{CSeg}(X)$, $X \rightarrow \mathcal{L}_{Joyal}(X)$) is a sectionwise weak equivalence (respectively sectionwise complete Segal equivalence, sectionwise Joyal equivalence).

Lemma 3.4.2 *Let S be a simplicial set. $(-)^S$ preserves quasi-injective fibrations.*

Proof The statement follows from 3.3.3 since $(-)^S$ is right adjoint to $- \times S$.

Lemma 3.4.3 *Let X be a fibrant object in the local Reedy model structure on $s^2\mathbf{Pre}(\mathcal{C})$ (see 3.2.3). Then $X_{n,*}$ is a fibrant object in the injective model structure.*

Proof $X \rightarrow *$ must have the right lifting property with respect to all maps $\Delta^n \tilde{\times} A \rightarrow \Delta^n \tilde{\times} B$, where $A \rightarrow B$ is a trivial cofibration in the injective model structure.

Lemma 3.4.4 *If X is a presheaf of complete Segal spaces, then its local Reedy fibrant replacement (i.e. injective fibrant replacement under the identification of 3.2.3) $\mathcal{L}_{inj}(X)$ is Segal-injective fibrant. In particular, X satisfies descent for the injective model structure if and only if it satisfies descent for the local complete Segal model structure.*

Proof Consider the presheaf maps

$$\begin{array}{ccc} X^{G(n)} & \longrightarrow & \mathcal{L}_{inj}(X)^{G(n)} \\ \downarrow & & \downarrow \\ X^{F(n)} & \longrightarrow & \mathcal{L}_{inj}(X)^{F(n)} \\ \\ X^{\tilde{I}} & \longrightarrow & \mathcal{L}_{inj}(X)^{\tilde{I}} \\ \downarrow & & \downarrow \\ X^{F(0)} & \longrightarrow & \mathcal{L}_{inj}(X)^{F(0)} \end{array}$$

To show that $\mathcal{L}_{inj}(X)$ is Segal-injective fibrant, it suffices to show that the right vertical maps in the above diagram are local weak equivalences. The left vertical maps are sectionwise Reedy equivalences. The map $X \rightarrow \mathcal{L}_{inj}(X)$ is a local weak equivalence of presheaves of Kan complexes in each simplicial degree. Since $(-)^A$ preserves local trivial fibrations, it preserves local weak equivalences of presheaves of Kan complexes by the functorial factorization of 2.3.18. Thus, the horizontal maps in the above diagram are all local Reedy equivalences. Thus, by 2 out of 3, the right vertical maps are local weak equivalences, as required.

Lemma 3.4.5 *Let X and Y be presheaves of quasi-categories. A map $f : X \rightarrow Y$ is a local Joyal equivalence if and only if for all $n \in \mathbb{N}$*

$$J(X^{\Delta^n}) \rightarrow J(Y^{\Delta^n})$$

is a local weak equivalence.

Proof If X is a presheaf of quasi-categories, then so is each X^{Δ^n} . Also, there is a sectionwise weak equivalence

$$k^!(X^{\Delta^n}) \rightarrow J(X^{\Delta^n})$$

by 2.1.18. Thus, the condition is equivalent to saying that $t^!(f)$ is a local Reedy equivalence. The result follows from 3.3.8 and 3.3.10.

Lemma 3.4.6 *Let X be a presheaf of quasi-categories. Then X satisfies descent with respect to the local Joyal model structure if and only if $t^!(X)$ satisfies descent with respect to the local complete Segal model structure.*

Proof The map $t^!(X) \rightarrow t^!\mathcal{L}_{\text{Joyal}}(X)$ is a local complete Segal equivalence, and $t^!\mathcal{L}_{\text{Joyal}}(X)$ is fibrant for the local complete Segal model structure. In particular, $t^!\mathcal{L}_{\text{Joyal}}(X)$ is a fibrant model of $t^!(X)$ in the local complete Segal model structure. The result follows from the fact that $t^!$ preserves and reflects sectionwise equivalences of sectionwise fibrant objects.

Theorem 3.4.7 *Let X be a presheaf of quasi-categories. Then X satisfies descent in the local Joyal model structure if and only if each $J(X^{\Delta^n})$ satisfies descent with respect to the injective model structure.*

Proof If each $J(X^{\Delta^n})$ satisfies descent, then each $k^!(X^{\Delta^n})$ satisfies descent, because of the sectionwise weak equivalence $k^! \rightarrow J$ of 2.1.18. By 3.4.3, for $n \in \mathbb{N}$,

$$k^!(X^{\Delta^n}) = t^!(X)_{n,*} \rightarrow \mathcal{L}_{\text{CSeg}}(t^!(X))_{n,*}$$

is an injective fibrant replacement (and a sectionwise weak equivalence). Therefore, $t^!(X)$ satisfies descent for the injective model structure. Conclude using 3.4.4 and 3.4.6.

The proof of the converse is similar.

Lemma 3.4.8 *If C is a category, then $JB(C) \cong B(\text{Iso}(C))$.*

Proof By construction, the n -simplices of $JB(C)$ are precisely the strings $a_1 \rightarrow \cdots \rightarrow a_n$ of invertible arrows in $PB(C) \cong C$.

Corollary 3.4.9 *Let C be a presheaf of categories. Then $B(C)$ satisfies descent for the local Joyal model structure if and only if for each $n \in \mathbb{N}$, $\text{Iso}(C)^{[n]}$ is a stack.*

Proof This follows from the preceding two results and the natural isomorphism $B(C)^{\Delta^n} = B(C)^{B([n])} \cong B(C^{[n]})$.

Theorem 3.4.10 *Let X be a presheaf of quasi-categories. Then one has a bijection $[\ast, J(X)] = [\ast, X]_q$. Here, $[\ast, \]_q$ denotes maps in the local Joyal homotopy category and $[\ast, \]$ denotes maps in the ordinary homotopy category on simplicial presheaves.*

Proof The constant simplicial presheaf $I = B\pi(\Delta^1)$ is an interval object for the local Joyal model structure. Furthermore, every map $I \rightarrow X$ factors through $J(X)$ by 2.1.11. Since $\mathcal{L}_{Joyal}(X)$ satisfies descent, we have

$$[* , X]_q \cong [* , \mathcal{L}_{Joyal}(X)] \cong \pi_I(* , \mathcal{L}_{Joyal}(X)) \cong \pi_I(* , J\mathcal{L}_{Joyal}(X)),$$

where $\pi_I(A, B)$ denotes the I -homotopy classes of maps. The constant simplicial presheaf map $\Delta^1 \rightarrow I$ is a trivial cofibration in the injective model structure, so we have

$$\pi_I(* , J\mathcal{L}_{Joyal}(X)) \cong \pi_{\Delta^1}(* , J\mathcal{L}_{Joyal}(X)) \cong [* , J(X)]$$

by 3.4.5.

Example 3.4.11 If A is a presheaf of categories, one has an identification $[* , BA]_q = [* , B(\text{Iso}(A))]$. In particular, [15, Corollary 9.15] implies that $[* , BA]_q$ is a non-abelian H^1 invariant.

Chapter 4

Cocycles in Local Higher Category Theory

Introduction

In [15, Theorem 6.5], it is shown that given a right proper model category M , whose weak equivalences are closed under finite products, the maps $[X, Y]_M$ in the homotopy category of M can be described as the path components of a **cocycle category** $h(X, Y)_M$. Its objects are diagrams

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

in M with f a weak equivalence and its morphisms are commutative diagrams

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow g \\ X & & Y \\ f' \swarrow & & \searrow g' \\ & A' & \end{array}$$

If G is a sheaf of groups on a site \mathcal{C} , then a **G -torsor** is traditionally defined to be a sheaf F with a principal and transitive G -action. This is equivalent to the fact that for the classical Borel construction $EG \times_G F$ the unique map

$$EG \times_G F \rightarrow *$$

is a local weak equivalence. Thus, every G -torsor determines a cocycle

$$* \leftarrow EG \times_G F \rightarrow BG.$$

This map induces a bijection

$$\pi_0(\mathbf{Tors}_G) \rightarrow \pi_0 h(*, B(G))_{s\text{Pre}(\mathcal{C})}$$

between path components of the category of G -torsors and path components of the cocycle category $h(*, B(G))_{s\text{Pre}(\mathcal{C})}$, leading to the homotopy classification of torsors (i.e. the homotopy theoretic interpretation of non-abelian H^1).

The technique of cocycles has numerous other applications, which are described in [15, Chapter 9]. These include the homotopy classification of gerbes (non-abelian H^2) and an explicit model for stack completion.

The ultimate purpose of this chapter is to show that cocycle-theoretic techniques apply to local higher category theory. However, the problem is that neither the Joyal or the complete Segal model structures (and by extension their local analogues) are right proper. In fact, the only known right proper model of higher categories is Bergner's model structure on simplicial categories. A major goal of this chapter is to prove the existence of a local version of Bergner's model structure on simplicial categories and show that it is Quillen equivalent to the local Joyal model structure. The contents of this chapter appear in [25].

As an application of the theory of cocycles, we will prove a generalization of the homotopy classification of torsors. In particular, given an arbitrary presheaf of Kan complexes, we will describe a bijection between the path components of a category of torsors and the maps

$$[* , X]_{s\text{Pre}(\mathcal{C})}$$

in the homotopy category of the injective model structure (4.5.17).

In the first section of this chapter, we review facts about the Bergner model structure. We also give a precise description of the Quillen equivalence between the Joyal model structure and the Bergner model structure. This is important because we want to show that this Quillen equivalence extends to the presheaf level.

In the second section, we also define an appropriate local analogue of weak equivalences for the Bergner model structure (i.e. DK-equivalences). In the third section, we prove some auxiliary results related to Boolean localization and local fibrations. In the fourth section, we prove the existence of a local analogue of the Bergner model structure, and show that it is Quillen equivalent to the local Joyal model structure.

In the final section, the technique of cocycles is applied to describe the maps $[\ast, X]$ in the homotopy category of the local Bergner model structure, in the case that X is a presheaf of ∞ -groupoids (see 4.5.1), as a non-abelian H^1 invariant. The proof involves a number of substantial results, such as the generalized Eilenberg-Zilber theorem of [33].

4.1 The Bergner Model Structure

We call a category enriched in simplicial sets a **simplicial category**. We will denote the category of small simplicial categories by \mathbf{sCat} .

Given a simplicial category C , one can construct a category $\pi_0(C)$, whose objects are the objects of C and satisfying $\mathrm{hom}_{\pi_0(C)}(x, y) = \pi_0 \mathrm{hom}_C(x, y)$. A map $f \in \mathrm{hom}_C(x, y)_0$ is called an **equivalence** if and only if it is an isomorphism in $\pi_0(C)$.

In [1], Bergner constructs a model category on the category of simplicial categories, with the following properties:

1. The weak equivalences (**sCat-equivalences**) are those maps $f : C \rightarrow D$ such that:
 - (a) $\mathrm{hom}_C(x, y) \rightarrow \mathrm{hom}_D(f(x), f(y))$ are weak equivalences for all $x, y \in C$.
 - (b) $\pi_0(C) \rightarrow \pi_0(D)$ is an equivalence of categories.
2. The fibrations (**sCat-fibrations**) are maps $f : C \rightarrow D$ such that:
 - (a) $\mathrm{hom}_C(x, y) \rightarrow \mathrm{hom}_D(f(x), f(y))$ are Kan fibrations for all $x, y \in C$.
 - (b) Any equivalence $f(x) \rightarrow y$ in D lifts to an equivalence $x \rightarrow z$ in C .
3. The cofibrations are those maps which have the left lifting property with respect to maps which are both fibrations and weak equivalences.

The \mathbf{sCat} -equivalences are referred to as DK-equivalences in [1].

Lemma 4.1.1 *The Bergner model structure is right proper.*

Proof Suppose that we have a pullback diagram

$$\begin{array}{ccc}
 D = B \times_A C & \xrightarrow{h} & C \\
 \downarrow & & \downarrow g \\
 B & \xrightarrow{f} & A
 \end{array}$$

where f is an sCat-equivalence and g is an sCat-fibration. We want to show that h is an sCat-equivalence.

First, note that for $a, b \in D$, $\text{hom}_D(a, b) \rightarrow \text{hom}_D(h(a), h(b))$ is a weak equivalence by the right properness of the Kan model structure.

Now, we want to show that $\pi_0(h)$ is an equivalence of categories. By applying π_0 to the result of the previous paragraph, $\pi_0(h)$ is fully faithful. Next, we show that it is essentially surjective.

Choose an object $y \in C$. Then since f is an sCat-equivalence, we can choose an object $x \in B$ and an equivalence $a : g(y) \rightarrow f(x)$. One then chooses an equivalence $y \rightarrow z$ which lifts a , i.e. $g(z) = f(x)$. Because D is a pullback, $(x, z) \in D$, and there is an equivalence $y \rightarrow h(x, z)$, as required.

Definition 4.1.2 There is a functor $\mathcal{U} : \text{sSet} \rightarrow \text{sCat}$ such that $\mathcal{U}(S)$ is a simplicial category which has two objects x, y , $\text{hom}_{\mathcal{U}(S)}(x, y) = S$, and $\mathcal{U}(S)$ has no other non-identity morphisms. There is a functor $\text{Disc} : \text{Set} \rightarrow \text{sCat}$ which takes S to the simplicial category with objects S and no non-identity morphisms.

Example 4.1.3 The Bergner model structure is cofibrantly generated with generating cofibrations:

1. $\mathcal{U}(\partial\Delta^n) \rightarrow \mathcal{U}(\Delta^n)$ for $n \in \mathbb{N}$.
2. $\emptyset \rightarrow *$.

and generating trivial cofibrations:

1. The inclusions $\mathcal{U}(\Lambda_i^n) \rightarrow \mathcal{U}(\Delta^n)$.
2. The inclusion maps $* \rightarrow \mathcal{H}$, where \mathcal{H} runs over a set of representatives of isomorphism classes of simplicial categories with the following properties:
 - (a) $\text{Ob}(\mathcal{H}) = \{x, y\}$.
 - (b) The simplicial sets $\text{hom}_{\mathcal{H}}(x, y)$, $\text{hom}_{\mathcal{H}}(x, x)$, $\text{hom}_{\mathcal{H}}(y, y)$ and $\text{hom}_{\mathcal{H}}(y, x)$ are weakly contractible and have countably many non-degenerate simplices.
 - (c) $* \xrightarrow{x} \mathcal{H}$ is a cofibration.

As such, we have a functorial fibrant replacement functor for the Bergner model structure, which we denote $\mathcal{S}_{\text{Berg}}$.

Remark 4.1.4 Note that given a diagram of simplicial categories

$$\begin{array}{ccc} * & \xrightarrow{\phi} & X \\ \downarrow & \nearrow \phi' & \downarrow \\ \mathcal{H} & \longrightarrow & * \end{array}$$

one can automatically find a lift by letting ϕ' be the composite $\mathcal{H} \rightarrow * \xrightarrow{\phi} X$. Thus, the fibrant objects in the Bergner model structure are precisely those whose simplicial mapping spaces are Kan complexes. We call these **fibrant simplicial categories**.

Definition 4.1.5 Given a simplicial category C , write $\text{Ex}^\infty(C)$ for the usual Ex^∞ functor applied to the internal description of C (i.e. $\text{Mor}(C)$, $\text{Ob}(C)$, composition operation, etc.).

Note that there is a natural map $C \rightarrow \text{Ex}^\infty(C)$ which is an sCat -equivalence.

Remark 4.1.6 Note that Ex^∞ preserves sCat -fibrations. Indeed, the usual Ex^∞ for simplicial sets preserves Kan fibrations, so that condition (a) in the definition of sCat -fibration on pg. 50 is preserved by Ex^∞ . For condition (b), note that there are bijections $\pi_0(X) \rightarrow \pi_0\text{Ex}^\infty(X)$ and $X_0 \rightarrow \text{Ex}^\infty(X)_0$, so condition (b) is preserved and reflected by Ex^∞ .

Recall that a functor $f : C \rightarrow D$ between ordinary categories is called an **isofibration** if and only if each isomorphism $f(c) \rightarrow d$ in D lifts to an isomorphism $c \rightarrow d'$ in C .

Lemma 4.1.7 *Let $f : C \rightarrow D$ be a map of simplicial categories. Then f is an sCat -fibration if and only if*

1. $\text{hom}_C(x, y) \rightarrow \text{hom}_D(f(x), f(y))$ are Kan fibrations for all $x, y \in C$.
2. $\pi_0(f)$ is an isofibration.

Proof Suppose that f satisfies the hypotheses of the lemma. Note that condition (b) in the definition of sCat -fibrations on pg. 50 is preserved and reflected by Ex^∞ , so it suffices to show that $\text{Ex}^\infty(f)$ satisfies condition (b) of pg. 50. In particular, since $\text{Ex}^\infty(f)$ satisfies the hypotheses of the lemma, we may assume X and Y are both fibrant.

Given an equivalence $w : f(c) \rightarrow d$ in Y , we can choose a 1-simplex $\sigma \in \text{hom}(f(c), d)$, so that $d_0(\sigma) = w$ and $d_1(\sigma) = f(v)$ for some equivalence v in X . But since $\text{hom}(c, d') \rightarrow \text{hom}(f(c), f(d'))$ is a Kan fibration, we can find a lift

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & \text{hom}(c, d') \\ d^1 \downarrow & \nearrow q & \downarrow \\ \Delta^1 & \xrightarrow{\sigma} & \text{hom}(f(c), f(d')) \end{array}$$

and $d_0(q)$ is an equivalence lifting w .

The other direction is trivial.

Lemma 4.1.8 *Let C and D be categories. A functor $f : C \rightarrow D$ is injective on objects and faithful if and only if $B(f)$ is a monomorphism of simplicial sets.*

Proof If $B(f)$ is a monomorphism, then $\text{Ob}(C) \rightarrow \text{Ob}(D)$ is a monomorphism (vertices) and $\text{Mor}(C) \rightarrow \text{Mor}(D)$ is a monomorphism (1-simplices).

On the other hand, suppose that f is injective on objects and faithful, and $\sigma_1, \sigma_2 : [\mathbf{n}] \rightarrow C$ are n -simplices which have the same image in D . Then $\sigma_1(i) = \sigma_2(i)$ for $0 \leq i \leq n$ since f is injective on objects. Each 1-simplex $i \rightarrow i+1$ has the same image under σ_1 and σ_2 since f is faithful. Thus, $\sigma_1 = \sigma_2$. Thus, $BC \rightarrow BD$ is a monomorphism.

Lemma 4.1.9 1. Consider the pushout diagram

$$\begin{array}{ccc} \{0, 1\} & \longrightarrow & C \\ (x,y) \downarrow & & \downarrow g \\ [I] & \longrightarrow & D \end{array}$$

of categories. Then the map g is a monomorphism.

2. A transfinite composite of monomorphisms of categories is a monomorphism.

3. A cofibration of simplicial categories is faithful and injective on objects.

Proof For the first statement, consider the pushout

$$\begin{array}{ccc} \partial\Delta^1 & \longrightarrow & BC \\ \downarrow & & \downarrow h \\ \Delta^1 & \longrightarrow & X \end{array}$$

Since P preserves pushouts, $PBC \rightarrow PX$ is naturally isomorphic to $C \rightarrow D$. Note that $BC \rightarrow X$ is a monomorphism, so that $C \rightarrow D$ is a monomorphism on objects. X is obtained by adjoining a 1-simplex α to BC . Now, the morphism of $P(X)$ can be represented by strings of 1-simplices modulo an equivalence relation. Note that if α appears in a string

$$a_0 \rightarrow \cdots \rightarrow a_i \rightarrow x \xrightarrow{\alpha} y \rightarrow \cdots \rightarrow a_n,$$

then no composition relation of C can remove it. Thus, if y_1, y_2 are morphisms in C and $y_1 \simeq y_2$, then they must be equivalent by some composition laws in C , so that $y_1 = y_2$.

For the second statement, note that if we have a transfinite composite

$$A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow \cdots \rightarrow A_\alpha \rightarrow A_{\alpha+1} \rightarrow \cdots$$

indexed by an ordinal λ then its colimit A is the category such that $\text{Ob}(A) = \cup_{\alpha < \lambda} \text{Ob}(A_\alpha)$ and $\text{Mor}(A) = \cup_{\alpha < \lambda} \text{Mor}(A_\alpha)$ with the obvious composition laws. The result follows since a filtered colimit of monomorphisms of simplicial sets is a monomorphism.

For the third statement, note that a pushout of $i_n : \mathcal{U}(\partial\Delta^n) \rightarrow \mathcal{U}(\Delta^n)$ is obtained in each simplicial degree by adding morphisms, so a pushout of i_n is a monomorphism by the first part. a pushout of $\emptyset \rightarrow *$ is simply obtained by adding a single object. Thus, the result follows from part 2 above.

If \mathcal{O} is a set, let $\text{sCat}_{\mathcal{O}}$ denote the subcategory of sCat consisting of simplicial categories whose object set is \mathcal{O} and whose morphisms are the identity on objects.

Theorem 4.1.10 (see [6, Propositions 7.2 and 7.3]). Let \mathcal{O} be a set. Then there is a proper model category on $\text{sCat}_{\mathcal{O}}$ whose fibrations are those maps which induce Kan fibrations of simplicial homs and whose weak equivalences are maps that induce weak equivalences of simplicial homs.

We call the model structure of 4.1.10 the **Dwyer-Kan model structure**. Recall that inclusion of simplicial sets $S \rightarrow T$ induces a map $\mathcal{U}(S) \rightarrow \mathcal{U}(T)$. Let $\text{Disc}\{x, y\} \rightarrow \text{Disc}(\mathcal{O})$ be the inclusion for $x, y \in \mathcal{O}$. The generating trivial cofibrations for the Dwyer-Kan model structure are of the form

$$\mathcal{U}(\Lambda_i^n) \coprod_{\text{Disc}\{x,y\}} \text{Disc}(\mathcal{O}) \rightarrow \mathcal{U}(\Delta^n) \coprod_{\text{Disc}\{x,y\}} \text{Disc}(\mathcal{O}),$$

where $0 \leq i \leq n$, $x, y \in \mathcal{O}$. The generating cofibrations are

$$\mathcal{U}(\partial\Delta^n) \coprod_{\text{Disc}\{x,y\}} \text{Disc}(\mathcal{O}) \rightarrow \mathcal{U}(\Delta^n) \coprod_{\text{Disc}\{x,y\}} \text{Disc}(\mathcal{O}),$$

where $0 \leq i \leq n$, $x, y \in \mathcal{O}$.

Definition 4.1.11 If C is a category, then there exists a **free category** $F(C)$ with one generator for each non-identity map in C . There are functors

$$\phi : FC \rightarrow C, \psi : FC \rightarrow F^2C$$

defined by $Fc \mapsto c$, $Fc \mapsto F(Fc)$, respectively. Given a category C , there is a **simplicial resolution** of C , F_*C , so that $F_n(C) = F^n(C)$. The face and degeneracy maps

$$d_i : F_n(C) \rightarrow F_{n-1}(C), s_i : F_n(C) \rightarrow F_{n+1}(C)$$

are given by $F^i\phi F^{n-i}$ and $F^i\psi F^{n-i}$, respectively. Here, F^n means the result of applying F n times. Given a simplicial category C , write $\text{DK}(C)$ for the simplicial category defined by $\text{DK}(C)_n = d(F_*(C_*))$.

Lemma 4.1.12 *Given a simplicial category C , $\text{DK}(C)$ is cofibrant.*

Proof If C is a simplicial category, then $\text{DK}(C)_n$ is a free category. Furthermore, by the definition of $\text{DK}(C)$, each degeneracy of a generator of $\text{DK}(C)_{n-1}$ is a generator of $\text{DK}(C)_n$. Thus, by [6, 7.6], $\text{DK}(C)$ is cofibrant in the Dwyer-Kan model structure on $\text{sCat}_{\text{Ob}(C)}$. Since the generating cofibrations of $\text{sCat}_{\text{Ob}(C)}$ are cofibrations for the Bergner model structure and pushouts (resp. transfinite composites) in $\text{sCat}_{\text{Ob}(C)}$ are pushouts (resp. transfinite composites) in sCat , it follows that the natural inclusion $\text{Disc}(\text{Ob}(C)) \rightarrow \text{DK}(C)$ is a cofibration in the Bergner model structure. Finally, the definition of generating cofibrations in the Bergner model structure implies that $\text{Disc}(\text{Ob}(C))$ is cofibrant, from which the result follows.

Corollary 4.1.13 *The natural map $\text{DK}(C) \rightarrow C$ is a cofibrant replacement for the Bergner model structure.*

Proof If C is a discrete simplicial category, then the function $a \mapsto F(a)$ forms a contracting homotopy of $\text{hom}_{\text{DK}(C)}(x, y)$ onto $\text{hom}_C(x, y)$. If C is a simplicial category, then $\text{hom}_{\text{DK}(C)}(x, y) = d\text{hom}_{\text{DK}(C_*)}(x, y)$. The map $d\text{hom}_{\text{DK}(C_*)}(x, y) \rightarrow \text{hom}_C(x, y)$ is a weak equivalence by [8, Proposition IV.1.7] and the case C is discrete.

Example 4.1.14 (see [22, Definition 1.1.5.1]). Given a set Q , let $\mathcal{P}(Q)$ denote the poset of subsets of Q . For $n \in \mathbb{N}$, let Φ^n be the simplicial category whose objects are the elements of $\{0, 1, \dots, n-1, n\}$ and whose simplicial set of morphisms

$$\mathrm{hom}_{\Phi^n}(i, j)$$

is the nerve of the poset $\mathcal{P}_n[i, j]$ of subsets of the interval $[i, j]$ which contains the endpoints. Then

$$\mathcal{P}_n[i, j] \cong \mathcal{P}[i+1, j-1].$$

The latter set has $j-i-1$ elements, and we have an isomorphism

$$\mathrm{hom}_{\Phi^n}(i, j) \cong (\Delta^1)^{j-i-1}.$$

The composition law

$$\mathrm{hom}_{\Phi^n}(i, j) \times \mathrm{hom}_{\Phi^n}(j, k) \rightarrow \mathrm{hom}_{\Phi^n}(i, k)$$

is induced by a poset morphism

$$\mathcal{P}_n[i, j] \times \mathcal{P}_n[j, k] \rightarrow \mathcal{P}_n[i, k]$$

given by

$$(A, B) \mapsto A \cup B.$$

Given an ordinal number map $\theta : \mathbf{m} \rightarrow \mathbf{n}$ and $A \in \mathcal{P}_m[i, j]$, the image $\theta(A)$ is contained in $\mathcal{P}_n[\theta(i), \theta(j)]$. Thus, θ induces a functor

$$\Phi(\theta) : \Phi^m \rightarrow \Phi^n.$$

Together, the $\Phi(\theta)$'s determine a cosimplicial object Φ^* in sCat .

Recall from 2.1.12 that, given a cosimplicial object Ω in a category C , there is a singular functor $\mathrm{Sing}_\Omega : C \rightarrow \mathrm{sSet}$ associated to it. We call the singular functor associated to the cosimplicial object Φ of 4.1.14 the **homotopy coherent nerve**. We write \mathfrak{B} for the homotopy coherent nerve and \mathfrak{C} for its left adjoint. Note that

$$\mathfrak{B}(X)_n = \mathrm{hom}(\Phi^n, X).$$

The following is [22, Theorem 2.2.5.1] (it is also proven in [19]):

Theorem 4.1.15 *The adjoint pair*

$$\mathfrak{C} : \mathrm{sSet} \rightleftarrows \mathrm{sCat} : \mathfrak{B}$$

gives a Quillen equivalence between the Joyal model structure and the Bergner model structure.

We give an outline of the proof of 4.1.15 found in [19].

Lemma 4.1.16 *The functor \mathfrak{C} preserves cofibrations.*

Proof It suffices to show that \mathfrak{C} preserves the generating cofibrations $\partial\Delta^n \subset \Delta^n$. In the case $n = 0$, this gets taken to $\emptyset \subseteq *$, which is a cofibration. On the other hand, if $n > 0$, $\mathfrak{C}(\partial\Delta^n)$ has the following description:

1. The objects are the same as the objects of $\mathfrak{C}(\Delta^n)$.
2. For each i, j , $\text{hom}_{\mathfrak{C}(\partial\Delta^n)}(i, j) = \text{hom}_{\mathfrak{C}(\Delta^n)}(i, j)$ except in the case $i = 0, j = n$.
3. $\text{hom}_{\mathfrak{C}(\Delta^n)}(0, n)$ is the cone on $\text{hom}_{\mathfrak{C}(\partial\Delta^n)}(0, n)$ with cone point the string of maximal length.

We want to solve lifting problems

$$\begin{array}{ccc} \mathfrak{C}(\partial\Delta^n) & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathfrak{C}(\Delta^n) & \longrightarrow & Y \end{array}$$

where $X \rightarrow Y$ is a trivial fibration. By the above description of $\mathfrak{C}(\partial\Delta^n)$, the lifting problem is equivalent to solving a lifting problem involving

$$\mathcal{U}(\text{hom}_{\mathfrak{C}(\partial\Delta^n)}(0, n)) \subseteq \mathcal{U}((\Delta^1)^{n-1}),$$

which is a cofibration.

Lemma 4.1.17 *For $0 < i < n$, the map*

$$\mathfrak{C}(\Lambda_i^n) \rightarrow \mathfrak{C}(\Delta^n)$$

is an sCat-equivalence.

Proof More generally, we will prove that if S is a proper subset of the $n - 1$ simplices of Δ^n containing d^0 and d^n , then for the subcomplex $\langle S \rangle$ of Δ^n generated by the elements of S , $\mathfrak{C}(\langle S \rangle) \rightarrow \mathfrak{C}(\Delta^n)$ is an sCat-equivalence. We will proceed by increasing induction on $|S|$ and n .

In the case that $|S| = 2$, we can express $\mathfrak{C}(\langle S \rangle)$ as a pushout

$$\begin{array}{ccc} \mathfrak{C}(\Delta^{n-2}) & \xrightarrow{d^0} & \mathfrak{C}(\Delta^{n-1}) \\ d^{n-1} \downarrow & & \downarrow \\ \mathfrak{C}(\Delta^{n-1}) & \longrightarrow & \mathfrak{C}(\langle S \rangle) \end{array}$$

Note that the maps $\mathfrak{C}(\Delta^n) \rightarrow [\mathbf{n}]$, natural in n , are weak equivalences in the Bergner model structure. The pushout diagram

$$\begin{array}{ccc} [\mathbf{n-2}] & \xrightarrow{d^{n-1}} & [\mathbf{n-1}] \\ d^0 \downarrow & & \downarrow d^0 \\ [\mathbf{n-1}] & \xrightarrow{d^n} & [\mathbf{n}] \end{array}$$

induces a map $\mathfrak{C}(\langle S \rangle) \rightarrow [\mathbf{n}]$, which is an sCat-equivalence by the glueing lemma [8, Lemma II.8.8]. We can apply the glueing lemma because everything in the preceding two displays

is cofibrant for the Bergner model structure and the left vertical maps in both displays are cofibrations by 4.1.16. But the map $\mathfrak{C}(\langle S \rangle) \rightarrow [\mathbf{n}]$ factors as

$$\mathfrak{C}(\langle S \rangle) \rightarrow \mathfrak{C}(\Delta^n) \rightarrow [\mathbf{n}].$$

Thus, by 2 out of 3, $\mathfrak{C}(\langle S \rangle) \rightarrow \mathfrak{C}(\Delta^n)$ is an sCat-equivalence.

In general, suppose that we have proven the statement for all S such that either $S \subseteq (\Delta^m)_{m-1}$, $m < n$ or $|S| < i$ and $S \subseteq (\Delta^n)_{n-1}$. Suppose that $|S| = i$ and $S \subseteq (\Delta^n)_{n-1}$. Suppose that S is obtained from S' by adding a simplex d^j , $0 < j < n$. The intersection $d^j \cap \langle S' \rangle$ is the subcomplex of $\Delta^{n-1} = \langle d^j \rangle$ generated by $d^j d^a$, $d^j d^b$, with $a < j, b > j$. In particular, it contains $d^j d^0$ and $d^j d^n$. Thus, the inductive hypothesis implies that $\mathfrak{C}(\langle S'' \rangle) \rightarrow \mathfrak{C}(\Delta^{n-1})$ is an sCat-equivalence.

We have a pushout

$$\begin{array}{ccc} \mathfrak{C}(\langle S'' \rangle) & \longrightarrow & \mathfrak{C}(\langle S' \rangle) \\ \downarrow & & \downarrow \\ \mathfrak{C}(\Delta^{n-1}) & \longrightarrow & \mathfrak{C}(\langle S \rangle) \end{array}$$

The left vertical map is a trivial cofibration. Hence, so is the right vertical map. The fact that $\mathfrak{C}(\langle S \rangle) \rightarrow \mathfrak{C}(\Delta^n)$ is an sCat-equivalence follows from the inductive hypothesis.

Lemma 4.1.18 *Suppose that X is a simplicial set. Then $\pi_0 \mathfrak{C}(X) \cong P(X)$.*

Proof First, note that $\pi_0(\mathfrak{C}(\Delta^n)) \cong [\mathbf{n}]$. The functor π_0 is the left adjoint of the functor $f : \text{Cat} \rightarrow \text{sCat}$ which regards a category as a discrete simplicial category. Thus, it preserves colimits. We thus have natural isomorphisms

$$\pi_0 \mathfrak{C}(X) \cong \pi_0 \mathfrak{C}(\lim_{\Delta^n \rightarrow X} \Delta^n) \cong \lim_{\Delta^n \rightarrow X} (\pi_0 \mathfrak{C}(\Delta^n)) \cong \lim_{\Delta^n \rightarrow X} ([\mathbf{n}]).$$

On the other hand, P is a left adjoint and we have natural isomorphisms

$$P(X) \cong P(\lim_{\Delta^n \rightarrow X} \Delta^n) \cong \lim_{\Delta^n \rightarrow X} (PB([\mathbf{n}])) \cong \lim_{\Delta^n \rightarrow X} ([\mathbf{n}]).$$

Lemma 4.1.19 *The adjoint pair of 4.1.15 is a Quillen adjunction.*

Proof We can prove that the above is a Quillen adjunction as follows.

We will first show that \mathfrak{B} preserves fibrations of fibrant simplicial categories. By 2.1.27, we want to prove:

1. $\mathfrak{B}(f)$ has the right lifting property with respect to the inner horn inclusions $\Lambda_i^n \subseteq \Delta^n$.
2. $\mathcal{J}\mathfrak{B}(f)$ has the right lifting property with respect to $\Delta^0 \xrightarrow{d^0} \Delta^1$.

The first statement follows from 4.1.17 by adjunction.

For the second part, we want to find a lift in the diagram

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & J\mathfrak{B}(X) \\ d^0 \downarrow & \nearrow & \downarrow \mathfrak{B}(f) \\ \Delta^1 & \xrightarrow{g} & J\mathfrak{B}(Y) \end{array}$$

By adjunction, this is equivalent to finding a lift

$$\begin{array}{ccc} \mathfrak{C}(\Delta^0) & \longrightarrow & X \\ d^0 \downarrow & \nearrow & \downarrow f \\ \mathfrak{C}(\Delta^1) & \xrightarrow{h} & Y \end{array}$$

The map g extends to a map $B\pi\Delta^1 \xrightarrow{g'} \mathfrak{B}(Y)$. Since $\pi_0\mathfrak{C}(B\pi\Delta^1) \cong P(B\pi\Delta^1) \cong \pi\Delta^1$ is a groupoid, the vertex of $\text{hom}_{\mathfrak{C}(\Delta^1)}(0, 1)$ corresponding to g is mapped under h to an equivalence, so that one can find the required lift by the definition of sCat-fibrations.

Now, we will prove that \mathfrak{B} preserves arbitrary fibrations. Suppose that $f : X \rightarrow Y$ is a fibration of simplicial categories. Form the diagram

$$\begin{array}{ccc} X & \xrightarrow{a} & V \\ f \downarrow & & \downarrow q \\ Y & \xrightarrow{j} & \mathcal{S}_{\text{Berg}}(Y) \end{array}$$

where a is a trivial cofibration and q is a fibration. Form the pullback $q_* : Y \times_{\mathcal{S}_{\text{Berg}}(Y)} V \rightarrow Y$. Then $\mathfrak{B}(q_*)$ is a fibration since it is a pullback of fibration by the preceding paragraph. The map $j_* : Y \times_{\mathcal{S}_{\text{Berg}}(Y)} V \rightarrow V$ is an sCat-equivalence since the Bergner model structure is right proper. The induced map $\theta : X \rightarrow Y \times_{\mathcal{S}_{\text{Berg}}(Y)} V$ is thus an sCat-equivalence. Factorize it as

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & \searrow \theta & \downarrow \pi \\ & & Y \times_{\mathcal{S}_{\text{Berg}}(Y)} V \end{array}$$

where i is a trivial cofibration and π is a trivial fibration. Since \mathfrak{C} preserves cofibrations, \mathfrak{B} preserves trivial fibrations. Then $\mathfrak{B}(q_*\pi)$ is a fibration, and the existence of the lift

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ \downarrow & \nearrow f & \downarrow \\ Z & \xrightarrow{q_*\pi} & Y \end{array}$$

says that f is a retract of $q_*\pi$. It follows that $\mathfrak{B}(f)$ is a fibration.

We will write SeCat for the category of bisimplicial sets X such that $X_{*,0}$ is a discrete simplicial set. We call the objects of SeCat **Segal precategories**. Let $G(n)$ denote the glued together 1-simplices inside Δ^n , regarded as a vertically discrete bisimplicial set. Write $F(n) = \Delta^n \tilde{\times} \Delta^0$. We call a Segal precategory a **Segal category** if and only if the map

$$\mathbf{hom}(F(n), X) \rightarrow \mathbf{hom}(G(n), X)$$

induced by the inclusion $G(n) \subseteq F(n)$ is a weak equivalence of simplicial sets (compare with 3.1.7).

Definition 4.1.20 We write $\mathcal{S}_{\text{Seg}}(X)$ for the canonical replacement of a Segal precategory by a Segal category, obtained by doing a small object argument to solve all lifting problems of the form

$$\begin{array}{ccc} (\Delta_i^n \times F(m)) \cup (\Delta^n \times G(m)) & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n \times F(m) & & \end{array}$$

where $m, n \in \mathbb{N}, 0 \leq i \leq n$.

Definition 4.1.21 Given a Segal category X , we write $\text{Ob}(X) = X_{0,0}$ and write $\mathbf{Map}_X(x, y)$ for the pullback

$$\begin{array}{ccc} \mathbf{Map}_X(x, y) & \longrightarrow & X_{1,*} \\ \downarrow & & \downarrow (d_0, d_1) \\ * & \xrightarrow{(x,y)} & X_{0,*} \times X_{0,*} \end{array}$$

Definition 4.1.22 We call a map $f : X \rightarrow Y$ of Segal categories a **DK-equivalence** if and only if

1. For each $x, y \in \text{Ob}(X)$, $\mathbf{Map}_Y(x, y) \rightarrow \mathbf{Map}_X(x, y)$ is a weak equivalence of simplicial sets.
2. $\text{Ho}(f) : \text{Ho}(X) \rightarrow \text{Ho}(Y)$ is an equivalence of categories. Here $\text{Ho}(X)$ denotes the homotopy category of a Segal category, defined in [29, 5.5].

The **injective model structure** for Segal precategories ([2, Theorem 5.1]) is a model structure in which

1. The cofibrations are monomorphisms.
2. Weak equivalences are maps f so that $\mathcal{S}_{\text{Seg}}(f)$ is a DK-equivalences.

Given a simplicial category C and $n \in \mathbb{N}$, we can construct a simplicial category $C^{(n)}$ such that its objects are objects of C and $\text{hom}_{C^{(n)}}(x, y) = \text{hom}_C(x, y)^{\Delta^n}$. Given simplicial categories C and D , we can define a simplicial set $\text{Hom}(C, D)$ such that $\text{Hom}(C, D)_n = \text{Hom}(C, D^{(n)})$. This defines a simplicial enrichment of sCat .

Example 4.1.23 Let Φ be the cosimplicial object defined in 4.1.14. There is a pair of adjoint functors

$$K_! : \text{SeCat} \rightleftarrows \text{sCat} : K^!,$$

where $K_{n,*}^!(Y) = \text{Hom}(\Phi^n, Y)$. Moreover, by [21, Theorem 5.6], there is Quillen equivalence

$$q^* : \text{sSet} \rightleftarrows \text{SeCat} : j^*$$

between the Joyal model structure and the injective model structure on SeCat , where $j^*(X) = X_{*,0}$. We have a commutative diagram

$$\begin{array}{ccc} \text{sCat} & \xrightarrow{\quad} & \text{SeCat} \\ & \searrow^{K^!} & \downarrow j^* \\ & \mathfrak{B} & \text{sSet} \end{array}$$

Thus, since q^* preserves and reflects weak equivalences (it is the left adjoint of a Quillen equivalence between model structures in which everything is cofibrant), to show that $K_!, K^!$ form a Quillen pair, it suffices to show that $K_!$ preserves cofibrations. This is done in [19, 1.23-1.26]).

For $m \geq 1, n \geq 0$, define $P_{m,n}$ to be the pushout

$$\begin{array}{ccc} \text{sk}_0(\Delta^n) \tilde{\times} \partial \Delta^m & \longrightarrow & \Delta^n \tilde{\times} \partial \Delta^m \\ \downarrow & & \downarrow \\ \text{sk}_0(\Delta^n) \tilde{\times} \Delta^0 & \longrightarrow & P_{m,n} \end{array}$$

and let $Q_{m,n}$ be the pushout

$$\begin{array}{ccc} \text{sk}_0(\Delta^n) \tilde{\times} \Delta^m & \longrightarrow & \Delta^n \tilde{\times} \Delta^m \\ \downarrow & & \downarrow \\ \text{sk}_0(\Delta^n) \tilde{\times} \Delta^0 & \longrightarrow & P_{m,n} \end{array}$$

The **projective model structure** for Segal precategories ([2, Theorem 7.1]) is a model structure in which

1. The cofibrations are the saturation of the induced maps $i_{m,n} : P_{m,n} \rightarrow Q_{m,n}$.
2. Weak equivalences are maps f so that $\mathcal{S}_{\text{Seg}}(f)$ is a DK-equivalences.

Outline of Proof of 4.1.15 By 4.1.23 and the 2 out of 3 property for Quillen equivalences, it suffices to show that $K_!, K^!$ form a Quillen equivalence between the injective model structure for Segal categories and the Bergner model structure.

The identity functor induces a Quillen equivalence between the projective and injective model structures on SeCat ([2, Theorem 7.5]). By the two out of three property for Quillen equivalences, it suffices to show that

$$K_! : \text{SeCat} \rightleftarrows \text{sCat} : K^!$$

is a Quillen equivalence between the projective model structure on SeCat and the Bergner model structure.

By [2, Section 8], there is a Quillen equivalence

$$G_! : \text{SeCat} \rightleftarrows \text{sCat} : G^!$$

between the projective model structure on SeCat and the Bergner model structure, where

$$G^!(C) = \text{Hom}([\mathbf{n}], C)$$

i.e. $G^!(C)_{*,n} = B(C_n)$.

The augmentation map $\Phi^n \rightarrow [\mathbf{n}]$ induces a map $K_! \rightarrow G_!$. To show that $K_!, K^!$ form a Quillen equivalence, it suffices to prove that for projective cofibrant Segal precategories C , $K_!(C) \rightarrow G_!(C)$ is an sCat -equivalence. This is the main objective of [19, Section 2].

Example 4.1.24 Given a map $f : X \rightarrow Y$ of fibrant simplicial categories, we want to construct a functorial fibration replacement of f in the Bergner model structure. Write $Y^!$ for the fibrant simplicial category $Y^{(1)}$ defined above 4.1.23. For $i = 0, 1$, let $d_i : Y^! \rightarrow Y$ be the map such that it is the identity on objects and for $x, y \in Y^!$, $\text{hom}_{Y^!}(x, y) \rightarrow \text{hom}_Y(x, y)$ is the map

$$\text{hom}_Y(x, y)^{\Delta^1} \rightarrow \text{hom}_Y(x, y)$$

induced by $d^i : \Delta^0 \rightarrow \Delta^1$. This map is a trivial fibration since Y is fibrant. Thus, each d_i is trivial sCat -fibration. Moreover, the d_i 's have a common section s , and one can apply the argument of 2.3.17 verbatim to construct a functorial sCat -fibration replacement

$$\begin{array}{ccc} X & \xrightarrow{s_*} & Z_f \\ & \searrow f & \downarrow \pi \\ & & Y \end{array}$$

such that s_* is the section of a trivial sCat -fibration.

Example 4.1.25 We can use the right properness of the Bergner model structure to construct a functorial fibration replacement for arbitrary maps of simplicial categories $f : X \rightarrow Y$. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & \text{Ex}^\infty(X) & & \\ & \searrow \theta_f & \downarrow \text{Ex}^\infty(f) & \searrow s_* & \\ & & \tilde{Z}_f & \xrightarrow{j_*} & Z_{\text{Ex}^\infty(f)} \\ & \swarrow \pi_f & & \swarrow \pi & \\ Y & \xrightarrow{j} & \text{Ex}^\infty(Y) & & \end{array}$$

in which $\pi \circ s_*$ is the functorial factorization constructed in 4.1.24 and the front face is a pullback. By the right properness of the Bergner model structure, j_* is an sCat -equivalence. Thus, so is θ_f . Finally, π_f is an sCat -fibration, and the map π_f is a fibration replacement of f .

This construction is functorial and commutes with filtered colimits.

4.2 Local sCat-Equivalences

Let $f : C \rightarrow D$ be a functor. Consider the pullback of categories $\text{Iso}(D)^{[1]} \times_D C$, whose objects are isomorphisms $f(c) \rightarrow d$ in D and whose morphisms are commutative squares

$$\begin{array}{ccc} f(c) & \longrightarrow & d \\ \downarrow & & \downarrow \\ f(c') & \longrightarrow & d' \end{array}$$

in D . We can define a map $\phi_f : \text{Iso}(D)^{[1]} \times_D C \rightarrow D$ by $(f(c) \rightarrow d) \mapsto d$.

Lemma 4.2.1 *A functor $f : C \rightarrow D$ is an equivalence of categories if and only if*

1. f is fully faithful.
2. $\phi_f : \text{Iso}(D)^{[1]} \times_D C \rightarrow D$ has the right lifting property with respect to $\emptyset \rightarrow *$.

Proof It suffices to show that the second condition above is equivalent to essential surjectivity. The stated lifting property is equivalent to

$$\phi : \text{Iso}(D)^{[1]} \times_D C \rightarrow D$$

being surjective on objects. This means that, for each $d \in \text{Ob}(D)$, there exists $c \in \text{Ob}(C)$ and an isomorphism $t : f(c) \rightarrow d$, which is equivalent to essential surjectivity.

More generally, given a map of presheaves of categories $f : C \rightarrow D$, we can form a pullback $\text{Iso}(D)^{[1]} \times_D C$ such that $\text{Ob}(\text{Iso}(D)^{[1]} \times_D C)(U)$ consists of isomorphisms $f(c) \rightarrow d$ in $D(U)$. We also have a map $\phi_f : \text{Iso}(D)^{[1]} \times_D C \rightarrow D$ which in each section is the map ϕ_f of 4.2.1.

Definition 4.2.2 Suppose that $f : X \rightarrow Y$ is a map of presheaves of categories. Then we say that f is a **local equivalence of presheaves of categories** if and only if

1. The sheafification of the diagram

$$\begin{array}{ccc} \text{Mor}(X) & \longrightarrow & \text{Mor}(Y) \\ \downarrow & & \downarrow \\ \text{Ob}(X) \times \text{Ob}(X) & \longrightarrow & \text{Ob}(Y) \times \text{Ob}(Y) \end{array}$$

is a pullback.

2. $\phi_f : \text{Iso}(Y)^{[1]} \times_Y X \rightarrow Y$ has the local right lifting property with respect to $\emptyset \rightarrow *$.

Lemma 4.2.3 *Let $f : X \rightarrow Y$ be a map of simplicial categories. Then f is an sCat-equivalence if and only if*

1. The diagram

$$\begin{array}{ccc}
\text{Mor}(X) & \longrightarrow & \text{Mor}(Y) \\
\downarrow & & \downarrow (s,t) \\
\text{Ob}(X) \times \text{Ob}(X) & \longrightarrow & \text{Ob}(Y) \times \text{Ob}(Y)
\end{array}$$

is homotopy cartesian.

2. $\pi_0(X) \rightarrow \pi_0(Y)$ is an equivalence of categories.

Proof Since $\text{Ob}(C) \rightarrow \text{Ob}(D)$ is a map of discrete simplicial sets, it is a Kan fibration. Thus, the pullback diagram

$$\begin{array}{ccc}
\text{Mor}(Y) \times_{(\text{Ob}(Y) \times \text{Ob}(Y))} (\text{Ob}(X) \times \text{Ob}(X)) & \longrightarrow & \text{Mor}(Y) \\
\downarrow & & \downarrow (s,t) \\
\text{Ob}(X) \times \text{Ob}(X) & \longrightarrow & \text{Ob}(Y) \times \text{Ob}(Y)
\end{array}$$

is homotopy cartesian and the diagram above is homotopy cartesian if and only if

$$\text{Mor}(X) \rightarrow \text{Mor}(Y) \times_{(\text{Ob}(Y) \times \text{Ob}(Y))} (\text{Ob}(X) \times \text{Ob}(X))$$

is a weak equivalence. But $\text{Mor}(X)$ is a disjoint union of $\text{hom}_X(x, y)$ for all $x, y \in \text{Ob}(X)$, so that this is equivalent to condition (1) in the definition of weak equivalences for the Bergner model structure.

Throughout the rest of the document write $\text{CatPre}(\mathcal{C})$ and $\text{sCatPre}(\mathcal{C})$ for the presheaves of categories on the site \mathcal{C} and the presheaves of simplicial categories on \mathcal{C} , respectively. In light of 4.2.3, one can define a local sCat-equivalence of presheaves of simplicial categories as follows.

Definition 4.2.4 We call a map $f : X \rightarrow Y$ of presheaves of simplicial categories a **local sCat-equivalence** if and only if

1. The following diagram is homotopy cartesian for the injective model structure:

$$\begin{array}{ccc}
\text{Mor}(X) & \longrightarrow & \text{Mor}(Y) \\
\downarrow & & \downarrow (s,t) \\
\text{Ob}(X) \times \text{Ob}(X) & \longrightarrow & \text{Ob}(Y) \times \text{Ob}(Y)
\end{array}$$

2. $\pi_0(X) \rightarrow \pi_0(Y)$ is a local equivalence of presheaves of categories.

The following is [15, Lemma 5.20]:

Lemma 4.2.5 Suppose we have a pullback diagram of simplicial presheaves

$$\begin{array}{ccc}
B \times_D C & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{f} & D
\end{array}$$

where f is a local Kan fibration. Then the diagram is homotopy cartesian for the injective model structure.

Remark 4.2.6 Let $f : X \rightarrow Y$ be a map of simplicial presheaves. Note that $\text{Ob}(X) \times \text{Ob}(X) \rightarrow \text{Ob}(Y) \times \text{Ob}(Y)$ is a sectionwise Kan fibration. Thus, the diagram in condition (1) of 4.2.4 is homotopy cartesian for the injective model structure if and only if

$$\text{Mor}(X) \rightarrow \text{Mor}(Y) \times_{(\text{Ob}(Y) \times \text{Ob}(Y))} (\text{Ob}(X) \times \text{Ob}(X))$$

is a local weak equivalence.

Remark 4.2.7 It is clear from 4.2.1 and 4.2.6 that a sectionwise $s\text{Cat}$ -equivalence of presheaves of simplicial categories is also a local $s\text{Cat}$ -equivalence.

Lemma 4.2.8 *Suppose that $f : C \rightarrow D$ is a map of presheaves of simplicial categories. Then f is a local $s\text{Cat}$ -equivalence if and only if*

1. *The following diagram is homotopy cartesian for the injective model structure*

$$\begin{array}{ccc} \text{Mor}(X) & \longrightarrow & \text{Mor}(Y) \\ \downarrow & & \downarrow (s,t) \\ \text{Ob}(X) \times \text{Ob}(X) & \longrightarrow & \text{Ob}(Y) \times \text{Ob}(Y) \end{array}$$

2. *The map $\phi_{\pi_0(f)} : (\text{Iso}(\pi_0 D))^{[I]} \times_{(\pi_0 D)} (\pi_0 C) \rightarrow (\pi_0 D)$ of 4.2.2 has the local right lifting property with respect to $\emptyset \rightarrow *$.*

Proof It suffices to show that these properties imply that $\pi_0(f)$ satisfies condition (1) of 4.2.2. Note that by 4.2.6, we have a bijection

$$L^2 \pi_0 \text{Mor}(X) \rightarrow L^2 \pi_0 (\text{Mor}(Y) \times_{(\text{Ob}(Y) \times \text{Ob}(Y))} (\text{Ob}(X) \times \text{Ob}(X))). \quad (4.1)$$

Since π_0 commutes with coproducts, there is a natural isomorphism $\pi_0 \text{Mor}(X) \cong \text{Mor}(\pi_0 X)$, and by definition $\text{Ob}(\pi_0 X) = \text{Ob}(X)_0 = \pi_0 \text{Ob}(X)$. It follows that the map in 4.1 is naturally isomorphic to the sheafification of

$$\text{Mor}(\pi_0 X) \rightarrow \text{Mor}(\pi_0 Y) \times_{(\text{Ob}(\pi_0 Y) \times \text{Ob}(\pi_0 Y))} (\text{Ob}(\pi_0 X) \times \text{Ob}(\pi_0 X)),$$

as required.

Lemma 4.2.9 *$X \rightarrow L^2(X)$ is a local $s\text{Cat}$ -equivalence.*

Proof The maps $\pi_0(X) \rightarrow \pi_0(L^2(X))$ and

$$\text{Mor}(X) \rightarrow (\text{Ob}(X) \times \text{Ob}(X)) \times_{\text{Ob}(L^2(X)) \times \text{Ob}(L^2(X))} \text{Mor}(L^2(X))$$

both induce isomorphisms on associated sheaves.

4.3 Boolean Localization and Local SCat-Fibrations

Definition 4.3.1 We call a map $f : X \rightarrow Y$ of presheaves of simplicial categories a **local trivial sCat-fibration** if and only if it has the local right lifting property with respect to all maps

1. $\emptyset \rightarrow *$.
2. $\mathcal{U}(\partial\Delta^n) \rightarrow \mathcal{U}(\Delta^n), n \in \mathbb{N}$.

We call a category C **finite** if and only if it has a finite number of objects and each $\text{hom}_C(x, y)$ is a finite set. We call a simplicial category C **finite** if and only if it has a finite number of objects and each $\text{hom}_C(X, Y)$ is a finite simplicial set (i.e. has finitely many non-degenerate simplices).

Lemma 4.3.2 *There are isomorphisms*

$$p^*L^2\text{hom}(C, X) \rightarrow \text{hom}(C, p^*L^2X)$$

natural in finite categories C (respectively, finite simplicial categories C) and presheaves of categories X (respectively presheaves of simplicial categories X).

Proof Suppose that C is a category and X is a presheaf of categories. Note that $\text{hom}(C, X)$ is naturally isomorphic to $\text{hom}(BC, BX) \cong \text{hom}(\text{sk}_2(BC), BX)$. The simplicial set $\text{sk}_2B(C)$ is finite. Thus, by 2.2.6, we have

$$p^*L^2\text{hom}(C, X) \cong \text{hom}(\text{sk}_2(BC), p^*L^2B(X)) \cong \text{hom}(\text{sk}_2BC, Bp^*L^2(X)) \cong \text{hom}(C, p^*L^2X).$$

If C is a finite simplicial category and X is a presheaf of simplicial categories, form bisimplicial sets C', X' so that $C'_{n,*} = \text{sk}_2B(C_n)$ and $X'_{n,*} = B(X_n)$. The constructions of C' and X' are natural, and there are natural isomorphisms

$$\text{hom}(C, X) \cong \text{hom}(C', X').$$

Note that C' is a finite bisimplicial set. Thus, we can use 3.1.10 and the argument of the case of presheaves of categories to complete the proof.

Corollary 4.3.3 *There is a natural isomorphism $p^*L^2\mathfrak{B} \cong \mathfrak{B}p^*L^2$.*

Corollary 4.3.4 *p^*L^2 preserves and reflects the property of having the local right lifting property with respect to a map of finite simplicial categories. In particular, f is a local trivial sCat-fibration if and only if $p^*L^2(f)$ is a sectionwise trivial fibration in the Bergner model structure.*

Corollary 4.3.5 *p^*L^2 preserves and reflects local equivalences of categories.*

Proof The non-trivial part is showing that p^*L^2 preserves and reflects condition (2) of 4.2.2. Given a map of presheaves of categories f , write ϕ_f for the map in condition (2) of 4.2.2. Note that Iso commutes with p^*L^2 by 3.4.8 and 2.3.6. Thus, $p^*L^2(\phi_f) \cong \phi_{p^*L^2(f)}$ by 4.3.2. But p^*L^2 also preserves and reflects the property of being an epimorphism on objects, as required.

Lemma 4.3.6 p^*L^2 preserves and reflects local $s\text{Cat}$ -equivalences.

Proof First note that π_0 is left adjoint to the functor $i : \text{CatPre}(\mathcal{C}) \rightarrow s\text{CatPre}(\mathcal{C})$ which regards a presheaf of categories as a presheaf of discrete simplicial categories. We have natural isomorphisms $p_*i \cong ip_*$, so that by adjunction, we have a natural isomorphism

$$p^*L^2\pi_0 \cong L^2\pi_0p^*L^2.$$

Now, condition (1) of 4.2.8 is preserved and reflected under Boolean localization by 4.2.6. For condition (2), note that if f is a map of presheaves of simplicial categories, $\pi_0p^*L^2(f)$ is a local equivalence of categories if and only if $L^2\pi_0p^*L^2(f) \cong p^*L^2\pi_0(f)$ is local equivalence of categories. In turn, this is true if and only if $\pi_0(f)$ is a local equivalence of categories.

Corollary 4.3.7 A local trivial $s\text{Cat}$ -fibration $f : X \rightarrow Y$ is a local $s\text{Cat}$ equivalence.

Proof $p^*L^2(f)$ is a sectionwise trivial $s\text{Cat}$ -fibration. Therefore, it is a local $s\text{Cat}$ -equivalence.

Lemma 4.3.8 Let $f : X \rightarrow Y$ be a map of presheaves of simplicial categories that is a sectionwise $s\text{Cat}$ -fibration. Then f is a local trivial $s\text{Cat}$ -fibration if and only if it is a local $s\text{Cat}$ -equivalence.

Proof If f is a local trivial $s\text{Cat}$ -fibration, then it is a local $s\text{Cat}$ -equivalence by 4.3.7.

Conversely, suppose that f is a local $s\text{Cat}$ -equivalence. Then

$$\text{Mor}(X) \rightarrow \text{Mor}(Y) \times_{(\text{Ob}(Y) \times \text{Ob}(Y))} (\text{Ob}(X) \times \text{Ob}(X))$$

is a sectionwise Kan fibration and a local weak equivalence. By [15, Theorem 4.32], it is a local trivial fibration. In particular, f has the local right lifting property with respect to $\mathcal{U}(\partial\Delta^n) \rightarrow \mathcal{U}(\Delta^n)$.

Choose an object $a \in Y(U)$. The map

$$\phi_f : \text{Ob}(\text{Iso}(\pi_0(Y))^{[1]} \times_{(\pi_0 Y)} (\pi_0 X)) \rightarrow \text{Ob}(\pi_0 Y)$$

is a local epimorphism. Thus, there exists a covering $\{U_\alpha \rightarrow U\}$ and equivalences $s_\alpha : f(b_\alpha) \rightarrow a|_{U_\alpha}$.

Since each $X(U_\alpha) \rightarrow Y(U_\alpha)$ is an $s\text{Cat}$ -fibration, there exists an equivalence $s'_\alpha \in \text{Mor}(X)(U_\alpha)$ such that $f(s'_\alpha) = s_\alpha$. This means that $f(r_\alpha) = a|_{U_\alpha}$ for some r_α . Thus, f has the local right lifting property with respect to $\emptyset \rightarrow *$.

Lemma 4.3.9 Suppose that f is a map of presheaves of fibrant simplicial categories. Then f is a local $s\text{Cat}$ -equivalence if and only if $\mathfrak{B}(f)$ is a local Joyal equivalence.

Proof We can factorize f as a sectionwise trivial cofibration for the Bergner model structure followed by a sectionwise $s\text{Cat}$ -fibration. Since \mathfrak{B} is the right adjoint of a Quillen equivalence, it preserves and reflects weak equivalences between fibrant objects, so we can assume that f is a sectionwise $s\text{Cat}$ -fibration.

If f is a local $s\text{Cat}$ -equivalence, it is a local trivial $s\text{Cat}$ -fibration by 4.3.8. But then $\mathfrak{B}p^*L^2(f) \cong p^*L^2\mathfrak{B}(f)$ is a sectionwise trivial fibration, so that $\mathfrak{B}(f)$ is a local Joyal equivalence.

Conversely, if $\mathfrak{B}(f)$ is a local Joyal equivalence, then $p^*L^2\mathfrak{B}(f) \cong \mathfrak{B}p^*L^2$ is a sectionwise trivial fibration by 2.3.16. But \mathfrak{B} reflects sectionwise weak equivalences between sectionwise fibrant objects, so that $p^*L^2(f)$ is a sectionwise $s\text{Cat}$ -equivalence.

Corollary 4.3.10 *Let $f : X \rightarrow Y$ be a map of sheaves of fibrant simplicial categories on a Boolean site \mathcal{B} . Then f is a local $s\text{Cat}$ -equivalence if and only if it is a sectionwise $s\text{Cat}$ -equivalence.*

Proof One direction is trivial.

For the other, note that $\mathfrak{B}(f)$ is a local Joyal equivalence of map of sheaves of quasi-categories by 4.3.3 and 4.3.9. It is thus a sectionwise Joyal equivalence by 2.3.10. Thus, f is a sectionwise $s\text{Cat}$ -equivalence since \mathfrak{B} reflects $s\text{Cat}$ -equivalences of fibrant simplicial categories.

4.4 The Local Bergner Model Structure

Since the Bergner model structure is cofibrantly generated, there is a **global projective model structure** on $s\text{CatPre}(\mathcal{C})$ in which the fibrations and weak equivalences are respectively sectionwise fibrations and weak equivalences in the Bergner model structure. The cofibrations are called **projective cofibrations**. The generating set of projective cofibrations consists of maps of the form $\text{hom}(-, U) \times \phi$, where ϕ is a generating cofibration for the Bergner model structure.

The objective of this section is to prove the following theorem:

Theorem 4.4.1 *There is a right proper model structure on $s\text{CatPre}(\mathcal{C})$ such that*

1. *The cofibrations are the projective cofibrations.*
2. *The weak equivalences are the local $s\text{Cat}$ -equivalences.*
3. *The fibrations are the maps which have the right lifting property with respect to maps which are both local $s\text{Cat}$ -equivalences and projective cofibrations. We call these **$s\text{Cat}$ -injective fibrations**. We call the fibrant objects **$s\text{Cat}$ -injective**.*

We call this the **local Bergner model structure**.

Recall that \mathcal{B} is a Boolean site we fixed at the beginning Chapter 2.

Lemma 4.4.2 *Suppose that $F \in \text{Sh}(\mathcal{B})$ is a discrete sheaf. Then*

1. *F is projective cofibrant.*
2. *$F \times \mathcal{U}(\partial\Delta^n) \rightarrow F \times \mathcal{U}(\Delta^n)$ is a projective cofibration.*

Proof We prove the second statement; the first is similar. Let $i : \partial\Delta^n \subseteq \Delta^n$ be the inclusion. If F is empty, the statement is trivial. Thus, we can assume $F \neq \emptyset$. Consider the poset \mathcal{Y} of subsheaves E of F such that

$$E \times \mathcal{U}(\partial\Delta^n) \rightarrow E \times \mathcal{U}(\Delta^n)$$

is a projective cofibration.

First note that this poset is nonempty. Choose an object $x \in E(b)$, and note that $E = \text{hom}(-, b) \cong *|_b \xrightarrow{x} F$ is contained in \mathcal{Y} .

By way of contradiction, suppose that the largest element of this poset is $E \subsetneq F$. Then E has a complement in F since \mathcal{B} is Boolean ([15, Lemma 3.29]). Write $E \coprod E' = F$. Choose an object $b \in \mathcal{B}$ such that $E'(b) \neq \emptyset$. Then $(*_|_b \coprod E) \times (i) \cong \text{hom}(-, b) \coprod E \times (i)$ is a projective cofibration, a contradiction.

Lemma 4.4.3 *If f is a projective cofibration, then $p^*L^2(f)$ is isomorphic to $L^2(f')$ for some projective cofibration f' .*

Proof If g is a monomorphism of simplicial sets, then

$$p^*L^2(\mathcal{U}(g) \times \text{hom}(-, U)) \cong \mathcal{U}(g) \times F$$

is a monomorphism for some sheaf F . It is therefore a projective cofibration by 4.4.2.

By the first paragraph, $p^*L^2(f)$ is in the saturation (in the category $s\text{Cat}\mathbf{Sh}(\mathcal{B})$) of maps which have the right lifting property with respect to the sectionwise trivial sCat-fibrations. Thus, $p^*L^2(f)$ has the right lifting property with respect to all sectionwise trivial sCat-fibrations of sheaves of simplicial categories.

One can factor $p^*L^2(f) = h \circ g$, where $g : X \rightarrow X'$ is a projective cofibration and $h : X' \rightarrow Y$ is a sectionwise trivial sCat-fibration. Let $i : X' \rightarrow L^2(X')$ be the natural map. One can find a lift

$$\begin{array}{ccc} X & \xrightarrow{\quad} & L^2(X') \\ p^*L^2(f) \downarrow & \nearrow i \circ g & \downarrow h \\ Y & \xrightarrow{\quad} & Y \end{array}$$

A standard retract argument now shows that f is the composite of a projective cofibration followed by sheafification.

Lemma 4.4.4 *Suppose that we have a pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{\quad} & C \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{\quad} & D \end{array}$$

where f is a local sCat-equivalence and a projective cofibration. Then so is g .

Proof Let Q be the fibrant replacement functor for the global projective model structure on $s\text{Cat}\mathbf{Pre}(\mathcal{C})$. Consider the iterated pushout diagram

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & C & \xrightarrow{\quad j} & Q(C) \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{\quad} & D & \xrightarrow{\quad} & Q(C) \cup D \end{array}$$

The map j is a sectionwise trivial cofibration for the Bergner model structure. Thus, we can assume that C is a presheaf of fibrant simplicial categories. Form the iterated pushout diagram

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & Q(A) & \xrightarrow{\quad} & C \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{\quad j'} & B \cup Q(A) & \xrightarrow{\quad} & B \cup C \end{array}$$

where the top horizontal composite is a factorization of $A \rightarrow C$. The map j' is a sectionwise cofibration and weak equivalence for the Bergner model structure. Thus, we can assume that A and C are sectionwise fibrant. Using the argument of the first paragraph, we can assume B is sectionwise fibrant as well.

Thus, we can assume that A, B and C are presheaves of fibrant simplicial categories. $p^*L^2(f)$ is a sectionwise sCat-equivalence by 4.3.10. By using the argument at the end of the proof of 4.4.3, we can show that $p^*L^2(f)$ is the composite of a sectionwise trivial cofibration for the Bergner model structure $X \rightarrow X'$ and the natural map $X' \rightarrow L^2(X')$. One concludes that the pushout of $p^*L^2(f)$ is a local sCat-equivalence, as required.

Lemma 4.4.5 *Let C be a category. Let $M \subseteq \text{Mor}(C)$ be an uncountable subset. Then there exists a subcategory C' , such that $M \subseteq \text{Mor}(C')$ and $|C'| = |M|$.*

Proof The smallest subcategory of C containing objects of M has a set of morphisms consisting of finite strings of composites of arrows in M and is of at most size

$$\sum_{i=0}^{\infty} |M|^i = |M|.$$

Corollary 4.4.6 *Suppose that $\alpha > |\text{Mor}(\mathcal{C})|$ is a regular cardinal. Suppose that $X \subseteq Y$ is an inclusion of presheaves of simplicial categories and that X is α -bounded. Suppose that we have an α -bounded simplicial presheaf $J \subseteq \text{Mor}(Y)$. Then there exists some α -bounded object Z such that*

1. $X \subseteq Z \subseteq Y$.
2. $J \subseteq \text{Mor}(Z)$.

Lemma 4.4.7 *Let $\alpha > |\text{Mor}(\mathcal{C})|$ be a regular cardinal. Suppose that we have a diagram of monomorphisms*

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

where A is α -bounded and $X \rightarrow Y$ is a local sCat-equivalence. Then there exists an α -bounded $B, A \subseteq B \subseteq Y$, such that $B \cap X \rightarrow B$ is a local sCat-equivalence.

Proof If $B \subseteq Y$ is a presheaf of simplicial categories, write $\pi_B : Z_B \rightarrow B$ for the natural sCat-fibration replacement of $B \cap X \rightarrow B$ described in 4.1.25. By 4.3.8, $B \cap X \rightarrow B$ is a local sCat-equivalence if and only if π_B is a local trivial sCat-fibration. Now, suppose there is a lifting problem

$$\begin{array}{ccc} Q & \longrightarrow & Z_A(U) \\ \downarrow & \nearrow & \downarrow \\ R & \longrightarrow & A(U) \end{array}$$

where $Q \rightarrow R$ is one of the generating cofibrations in 4.1.3. Then this lifting problem can be solved locally over some covering $\{U_i \rightarrow U\}$ having at most α elements. There is an identification

$$\lim_{\substack{\rightarrow \\ |B| < \alpha}} Z_B = Z_Y$$

since Y is a filtered colimit of its α -bounded subobjects. Thus, it follows from the regularity assumption on α there is an α -bounded $A' \subseteq Y$ containing A , over which the lifting problem can be solved. The set of all such lifting problems is α -bounded. Thus, there is a α -bounded presheaf of simplicial categories $B_1 \subseteq Y$ such that each lifting problem can be solved over B_1 . Repeating this procedure countably many times produces an ascending sequence of α -bounded presheaves of simplicial categories

$$B_1 \subseteq B_2 \cdots \subseteq B_n \cdots$$

such that all lifting problems

$$\begin{array}{ccc} Q & \longrightarrow & Z_{B_i}(U) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ R & \longrightarrow & B_i(U) \end{array}$$

with $Q \rightarrow R$ a generating cofibration from 4.1.3 can be solved locally over B_{i+1} . Put $B = \cup B_i$. Then B is α -bounded by the regularity of α . Since the construction of Z_B commutes with filtered colimits, $Z_B \rightarrow B$ is a local trivial sCat-fibration, as required.

Let $\beta > |\text{Mor}(\mathcal{C})|$. Then $\alpha = 2^\beta + 1$ is a regular cardinal. Let \mathfrak{M} be a collection of (representatives of isomorphism classes of) all α -bounded maps which are local sCat-equivalences and monomorphisms. For each $m \in \mathfrak{M}$, form the factorization

$$\begin{array}{ccc} C & \xrightarrow{j_m} & E \\ & \searrow m & \downarrow p_m \\ & & D \end{array}$$

where j_m is a projective cofibration and p_m has the right lifting property with respect to all projective cofibrations (by the same argument as in 2.2.17, we can take α sufficiently large, so that this factorization preserves α -bounded objects). Let \mathfrak{J} denote the set of all j_m above. Note that j_m is an α -bounded local sCat-equivalence.

Lemma 4.4.8 (c.f. [15, Lemma 7.3]). *Suppose that $q : X \rightarrow Y$ is a local sCat-equivalence which has the right lifting property with respect to all elements J_m of the set \mathfrak{J} . Then q has the right lifting property with respect to all projective cofibrations.*

Proof By the description of the generating projective cofibrations and the fact that $\alpha > |\text{Mor}(\mathcal{C})|$, it suffices to solve the lifting problems

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow q \\ B & \longrightarrow & Y \end{array}$$

where $A \rightarrow B$ is a α -bounded projective cofibration. The map q has a factorization

$$\begin{array}{ccc} X & \longrightarrow & Z \\ & \searrow j & \downarrow p \\ & & Y \end{array}$$

where j is a projective cofibration and p has the right lifting property with respect to all projective cofibrations. The projective cofibration j is also an sCat-equivalence. We can find a lifting in the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow j \\ & \nearrow \xi & Z \\ & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

The map j is a sectionwise monomorphism and local sCat-equivalence. Hence, by 4.4.7, we can choose an α -bounded subobject $D \supset im(\xi)$ of Z , so that $D \times_Z X \xrightarrow{m} D$ is a local sCat-equivalence. The original diagram has a factorization

$$\begin{array}{ccccc} A & \longrightarrow & D \times_Z X & \longrightarrow & X \\ \downarrow & & \downarrow m & & \downarrow q \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

where m is an α -bounded monomorphism and sCat-equivalence. One can factor m as $p_m \circ j_m$, where j_m is in the collection \mathfrak{J} . There is a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & D \times_Z X & \longrightarrow & X \\ \downarrow i & & \downarrow j_m & & \downarrow q \\ & \nearrow \omega & W & & \\ & & \downarrow p_m & & \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

The lift ω exists by hypothesis on the map q , and the lift κ exists since p_m has the right lifting property with respect to projective cofibrations.

Lemma 4.4.9 *A map f is a sCat-injective fibration if and only if it has the right lifting property with respect to all maps in the set \mathfrak{J} .*

Proof Suppose that $i : A \rightarrow B$ is an sCat-equivalence and a projective cofibration. Then i has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ & \searrow i & \downarrow p \\ & & B \end{array}$$

where j is in the saturation of \mathfrak{J} and p has the right lifting property with respect to all members of \mathfrak{J} . By 4.4.4, j is a local $s\text{Cat}$ -equivalence. Thus, so is p . It follows from 4.4.8 that p has the right lifting property with respect to all projective cofibrations and there exists a lift

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ i \downarrow & \nearrow & \downarrow p \\ B & \xrightarrow{id} & B \end{array}$$

The map i is therefore a retract of j .

Thus, if a map has the right lifting property with respect to all maps which are in \mathfrak{J} , then it has the right lifting property with respect to all maps which are projective cofibrations and $s\text{Cat}$ -equivalences. It is therefore a $s\text{Cat}$ -injective fibration.

Lemma 4.4.10 p^*L^2 preserves sectionwise $s\text{Cat}$ -fibrations of presheaves of fibrant simplicial categories.

Proof Let $f : X \rightarrow Y$ be a sectionwise $s\text{Cat}$ -fibration. Condition (1) in 4.1.7 is equivalent to $\text{Mor}(X) \rightarrow \text{Mor}(Y) \times_{\text{Ob}(Y) \times \text{Ob}(Y)} (\text{Ob}(X) \times \text{Ob}(X))$ being a sectionwise Kan fibration, which is clearly preserved under p^*L^2 .

On the other hand, $\mathfrak{B}p^*L^2(f) \cong p^*L^2\mathfrak{B}(f)$ is a sectionwise quasi-fibration by 2.3.15. Thus, $P\mathfrak{B}p^*L^2(f)$ is an isofibration in each section by 2.1.11 and 2.1.27. We have natural equivalences $P\mathfrak{B}p^*L^2(f) \cong \pi_0\mathfrak{C}\mathfrak{B}p^*L^2(f) \simeq \pi_0p^*L^2(f)$ by 4.1.15 and 4.1.18, so condition (2) of 4.1.7 is verified.

We write $\text{Ex}^\infty : s\text{Cat}\mathbf{Pre}(\mathcal{C}) \rightarrow s\text{Cat}\mathbf{Pre}(\mathcal{C})$ for the functor of 4.1.5 applied sectionwise to a presheaf of simplicial categories.

Lemma 4.4.11 Consider a diagram

$$\begin{array}{ccc} C \times_B A & \longrightarrow & A \\ h \downarrow & & \downarrow s \\ C & \xrightarrow{f} & B \end{array}$$

where the $A \rightarrow B$ is a local $s\text{Cat}$ -equivalence and $C \rightarrow B$ is a sectionwise $s\text{Cat}$ -fibration. Then $C \times_B A \rightarrow C$ is a local $s\text{Cat}$ -equivalence.

Proof $p^*L^2\text{Ex}^\infty$ preserves finite limits. Moreover, $p^*L^2\text{Ex}^\infty$ preserves sectionwise $s\text{Cat}$ -fibrations by 4.4.10 and 4.1.6, and it preserves and reflects local $s\text{Cat}$ -equivalences. Thus, we are reduced to assuming that all objects are sectionwise fibrant sheaves of simplicial categories on a Boolean site. By 4.3.10, it follows that $A \rightarrow B$ is a sectionwise $s\text{Cat}$ -equivalence, so the result follows from the right properness of the Bergner model structure (4.1.1).

Proof of Theorem 4.4.1 CM1-CM3 are trivial. Given a morphism $f : X \rightarrow Y$ of presheaves of simplicial categories, 4.4.9 and 4.4.4 imply that one half of CM5 is obtained by doing a small object argument to solve lifting problems

$$\begin{array}{ccc} A & \longrightarrow & X \\ \phi \downarrow & \nearrow & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

where $\phi \in \mathfrak{J}$. The other half of CM5 is trivial.

One half of CM4 is true by definition. For the other half, let $f : X \rightarrow Y$ be an sCat-injective fibration and local sCat-equivalence. One can factor f as

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ & \searrow f & \downarrow h \\ & & Y \end{array}$$

where g is a projective cofibration and h has the right lifting property with respect to all projective cofibrations (and is thus a local sCat-equivalence). By 2 out of 3, g is also an sCat-equivalence.

Thus, we can produce a lifting

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ g \downarrow & \nearrow & \downarrow f \\ Z & \xrightarrow{h} & Y \end{array}$$

Thus, f is a retract of h , and therefore f has the right lifting property with respect to all projective cofibrations. This argument is standard (see the proof of 2.3.3).

Right properness follows from the fact that a sCat-injective fibration is a sectionwise sCat-fibration and 4.4.11.

We call a map of simplicial presheaves a **projective cofibration** if and only if it is in the saturation of

$$\partial\Delta^n \times \text{hom}(-, V) \rightarrow \Delta^n \times \text{hom}(-, V),$$

where n runs over all natural numbers and V runs over all objects of \mathcal{C} .

Theorem 4.4.12 *There is a model structure on $s\text{Pre}(\mathcal{C})$ in which*

1. *Cofibrations are projective cofibrations.*
2. *Weak equivalences are local Joyal equivalences.*
3. *The fibrations are maps which have the right lifting property with respect to maps which are both local Joyal equivalences and projective cofibrations. We call these **projective quasi-fibrations**.*

The proof of this theorem is analogous to the proof of [15, Theorem 5.41].

Proof CM1-CM3 are trivial. It is trivial to produce a factorization of a map as a cofibration followed by a trivial fibration. For the other half of CM5, factorize a map $f = g \circ h$, where g is a quasi-injective fibration (thus a projective quasi-fibration) and h is a monomorphism and local Joyal equivalence. Then factor $h = l \circ m$, where l has the right lifting property with respect to projective cofibrations (and is hence a projective quasi-fibration) and m is a projective cofibration. The required factorization is $(g \circ l) \circ m$.

One half of CM4 is trivial. For the other, suppose that f is a projective quasi-fibration and local Joyal equivalence. One can factor $f = h \circ g$, where g is a projective cofibration and h has the right lifting property with respect to projective cofibrations. Then by two out of three, g is a local Joyal equivalence, and one can show that f is a retract of h by a standard argument (see the proof of 2.3.3).

Lemma 4.4.13 *The identity map*

$$i : s\mathbf{Pre}(\mathcal{C}) \rightleftarrows s\mathbf{Pre}(\mathcal{C}) : i$$

is a Quillen equivalence from the local projective Joyal model structure to the (usual) local Joyal model structure.

Proof Trivial.

We write $\mathcal{L}_{Berg} : s\mathbf{Cat}\mathbf{Pre}(\mathcal{C}) \rightarrow s\mathbf{Cat}\mathbf{Pre}(\mathcal{C})$ for the functorial fibrant replacement for the local Bergner model structure.

Theorem 4.4.14 *There is a Quillen equivalence*

$$\mathfrak{C} : s\mathbf{Pre}(\mathcal{C}) \rightleftarrows s\mathbf{Cat}\mathbf{Pre}(\mathcal{C}) : \mathfrak{B}$$

from the local projective Joyal model structure to the local Bergner model structure.

Proof Clearly, \mathfrak{C} takes generating cofibrations to cofibrations. We want to show that \mathfrak{C} sends local Joyal equivalences $f : X \rightarrow Y$ to local sCat-equivalences. We have a diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{B}\mathcal{S}_{Berg}\mathfrak{C}(X) \\ f \downarrow & & \downarrow \mathfrak{B}\mathcal{S}_{Berg}\mathfrak{C}(f) \\ Y & \longrightarrow & \mathfrak{B}\mathcal{S}_{Berg}\mathfrak{C}(Y) \end{array}$$

The horizontal maps are sectionwise Joyal equivalence by 4.1.15. By 2 out of 3, the right vertical map is a local Joyal equivalence. Thus, $\mathcal{S}_{Berg}\mathfrak{C}(f)$ and $\mathfrak{C}(f)$ are local sCat-equivalences by 4.3.9.

Thus, the adjunction is a Quillen adjunction. Let X be a fibrant object in the local Bergner model structure. Then X is a presheaf of fibrant simplicial categories. Thus, $\mathfrak{C}\mathfrak{B}(X) \rightarrow X$ is a sectionwise sCat-equivalence by 4.1.15.

Let X be a simplicial presheaf. We have a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & \mathfrak{B}\mathfrak{C}X & \longrightarrow & \mathfrak{B}\mathcal{S}_{Berg}\mathfrak{C}(X) \\ id \downarrow & & \downarrow & & \downarrow \psi \\ X & \xrightarrow{\gamma} & \mathfrak{B}\mathcal{L}_{Berg}\mathfrak{C}(X) & \xrightarrow{\phi} & \mathfrak{B}\mathcal{S}_{Berg}\mathcal{L}_{Berg}\mathfrak{C}(X) \end{array}$$

The top horizontal composite is a sectionwise Joyal equivalence by 4.1.15. Moreover, by 4.3.9, ϕ, ψ are local Joyal equivalences. Thus, by two out of three, so is γ , as required.

Remark 4.4.15 The preceding two results show that there are Quillen equivalences relating all three models of local higher category theory.

4.5 Non-Abelian H^1 with Coefficients in an ∞ -Groupoid

Definition 4.5.1 A fibrant simplicial category C is called an ∞ -**groupoid** if and only if $\pi_0(C)$ is a groupoid.

Remark 4.5.2 If X is a fibrant simplicial category, then X is an ∞ -groupoid if and only if $\mathfrak{B}(X)$ is a Kan complex. To see this, note that there are equivalences of categories $P\mathfrak{B}(X) \simeq \pi_0(X)$ by 4.1.15 and 4.1.18. Thus, since $\mathfrak{B}(X)$ is a quasi-category, $\mathfrak{B}(X)$ is a Kan complex if and only if $P\mathfrak{B}(X)$ is a groupoid by 2.1.9, which is true if and only if X is a ∞ -groupoid.

Definition 4.5.3 Suppose that we have a model category M . For $X, Y \in \text{Ob}(M)$ there is a category $h(X, Y)_M$ in which the objects are **cocycles**, i.e. diagrams

$$X \xleftarrow{f} A \xrightarrow{g} Y,$$

where f a weak equivalence. One writes (f, g) for the cocycle depicted above. The morphisms in $h(X, Y)_M$ are commutative diagrams

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow g \\ X & & Y \\ f' \swarrow & & \searrow g' \\ & A' & \end{array}$$

The following is [15, Theorem 6.5]:

Theorem 4.5.4 Suppose that we have a model category M such that

1. Finite products preserve weak equivalences.
2. M is right proper.

Then the natural map $\pi_0 h(X, Y)_M \rightarrow [X, Y]_M$ defined by

$$(X \xleftarrow{f} A \xrightarrow{g} Y) \mapsto g \circ f^{-1}$$

is a bijection.

Lemma 4.5.5 Suppose that $f : X \rightarrow Y$ is a map of presheaves of simplicial groupoids. Then f is a local $s\text{Cat}$ -equivalence if and only if the following conditions hold:

1. The following diagram is homotopy cartesian for the injective model structure

$$\begin{array}{ccc} \text{Mor}(X) & \longrightarrow & \text{Mor}(Y) \\ \downarrow & & \downarrow (s,t) \\ \text{Ob}(X) \times \text{Ob}(X) & \longrightarrow & \text{Ob}(Y) \times \text{Ob}(Y) \end{array}$$

2. $\pi_0(X_0) \rightarrow \pi_0(Y_0)$ is a local epimorphism.

Proof We will show that Condition (2) above is equivalent to Condition (2) of 4.2.8.

We will first show that Condition (2) of 4.2.8 is equivalent to $\pi_0(\pi_0 X) \rightarrow \pi_0(\pi_0 Y)$ being a local epimorphism. Because X and Y are groupoids, condition (2) of 4.2.8 is equivalent to $\phi : \text{Ob}(\pi_0 Y^{[1]} \times_{\pi_0 X} \pi_0 Y) \rightarrow \text{Ob}(\pi_0 Y)$ being a local epimorphism. The image of ϕ consists exactly of the objects of $\pi_0(Y)$ that lie in the same path component as elements of $\text{im}(\pi_0(f))$. Thus, ϕ being a local epimorphism is equivalent to $\pi_0(\pi_0(f))$ being a local epimorphism.

Now, if G is a groupoid, then a and b lie in the same path component of $\pi_0(G)$ if and only if $\text{hom}_{\pi_0 G}(a, b) \neq \emptyset$. It follows that $\pi_0(G_n) \cong \pi_0(G_0) \cong \pi_0(\pi_0(G))$ for $n \geq 0$.

We write sGpd for the category of simplicial groupoids. We say that a map of groupoids $H \rightarrow J$ has the **path lifting property** if every map $f(a) \rightarrow b$ in J lifts to a map $a \rightarrow c$ in H .

Theorem 4.5.6 ([8, Theorem V.7.6]) *There is a cofibrantly generated, right proper model structure on sGpd in which the weak equivalences are maps $f : C \rightarrow D$ so that*

1. $\text{hom}_C(x, y) \rightarrow \text{hom}_D(f(x), f(y))$ are weak equivalences for all $x, y \in C$.
2. $\pi_0(C_0) \rightarrow \pi_0(D_0)$ is essentially surjective.

and the fibrations are maps so that

1. $\text{hom}_C(x, y) \rightarrow \text{hom}_D(f(x), f(y))$ are weak equivalences for all $x, y \in C$.
2. $C_0 \rightarrow D_0$ has the path lifting property.

Remark 4.5.7 Note that for a presheaf of simplicial groupoids the path lifting property is equivalent to condition b) in the definition of sCat -fibrations on pg. 50. Thus, the fibrations for the model structure of 4.5.6 are precisely the sCat -fibrations between simplicial groupoids. In particular, the fibrant objects are precisely those whose simplicial mapping spaces are Kan complexes.

Moreover, the isomorphism $\pi_0(G_n) \cong \pi_0(G_0) \cong \pi_0(\pi_0(G))$ for simplicial groupoids implies that the weak equivalences for the model structure of 4.5.6 are precisely the sCat -equivalences.

We will now discuss a local analogue of the model structure of 4.5.6 and its Quillen equivalence with the injective model structure. Write $\mathcal{G}(\Delta^n)_m$ for the free groupoid on the graph of $m + 1$ -simplices $\sigma : \sigma(0) \rightarrow \sigma(1)$ of Δ^n , modulo the relation that $s_0\tau$ is the identity on $\tau(0)$. If $\theta : [1, k + 1] \rightarrow [1, m + 1]$ is an ordinal number map and

$$\sigma : a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_{m+1}$$

is an $n + 1$ simplex, we let $\theta^*(\sigma) = d_0(\sigma, \theta)^{-1}d_1(\sigma, \theta)$, where (σ, θ) is the simplex

$$a_0 \rightarrow a_1 \rightarrow a_{\theta(1)} \rightarrow a_{\theta(k+1)}.$$

The maps θ^* and the $\mathcal{G}(\Delta^n)_m$ determine a simplicial groupoid $\mathcal{G}(\Delta^n)$.

The definition of simplicial groupoids $\mathcal{G}(\Delta^n)$ extends to a functor

$$\mathcal{G} : \mathbf{sSet} \rightarrow \mathbf{sGpd},$$

which is called the **loop group functor**. This functor has a right adjoint \bar{W} called the **Eilenberg-Mac Lane functor** (see [8, V.7.7]).

We write $\mathbf{sGpdPre}(\mathcal{C})$ for the presheaves of simplicial groupoids on a site \mathcal{C} . We write

$$\mathcal{G} : \mathbf{sPre}(\mathcal{C}) \rightleftarrows \mathbf{sGpdPre}(\mathcal{C}) : \bar{W}$$

for the adjoint pair obtained by applying \mathcal{G} and \bar{W} sectionwise.

Theorem 4.5.8 (see [15, Theorem 9.50, Lemma 9.52]). *There is a model structure on $\mathbf{sGpdPre}(\mathcal{C})$ defined as follows:*

1. *The weak equivalences are maps which satisfy the hypotheses of 4.5.5.*
2. *A map f is a fibration if and only if $\bar{W}(f)$ is an injective fibration.*
3. *The cofibrations are the maps which have the left lifting property with respect to the trivial fibrations.*

Moreover, the adjoint pair

$$\mathcal{G} : \mathbf{sPre}(\mathcal{C}) \rightleftarrows \mathbf{sGpdPre}(\mathcal{C}) : \bar{W}$$

gives a Quillen equivalence between the injective model structure and this model structure.

The following is the main theorem of [33] and is called the ‘generalized Eilenberg-Zilber Theorem’.

Theorem 4.5.9 *There is a natural weak equivalence $dB \rightarrow \bar{W}$.*

Suppose that X and Y are both presheaves of simplicial categories. Write $h_{\text{hyp}}(X, Y)_{\mathbf{sCat}(\mathcal{C})}$ for the full subcategory of $h(X, Y)_{\mathbf{sCat}(\mathcal{C})}$ consisting of objects (f, g) such that f is also a sectionwise fibration for the Bergner model structure. Similarly, given presheaves of simplicial groupoids X and Y , we write $h_{\text{hyp}}(X, Y)_{\mathbf{sGpd}(\mathcal{C})}$ for the full subcategory of $h(X, Y)_{\mathbf{sGpd}(\mathcal{C})}$ consisting of objects (f, g) , with f a sectionwise fibration for the model structure of 4.5.6.

Lemma 4.5.10 (c.f. [15, Lemma 6.14]) *If X and Y are presheaves of simplicial categories such that Y is sectionwise fibrant for the Bergner model structure, then the inclusion $i_1 : h_{\text{hyp}}(X, Y)_{\mathbf{sCat}(\mathcal{C})} \subseteq h(X, Y)_{\mathbf{sCat}(\mathcal{C})}$ induces a bijection on path components.*

If X and Y are presheaves of simplicial groupoids such that Y is sectionwise fibrant for the model structure of 4.5.8, then the inclusion $i_2 : h_{\text{hyp}}(X, Y)_{\mathbf{sGpd}(\mathcal{C})} \subseteq h(X, Y)_{\mathbf{sGpd}(\mathcal{C})}$ induces a bijection on path components.

Proof We prove the first statement. The second statement has an identical proof.

Objects of the cocycle category $h(X, Y)_{\text{sCat}(\mathcal{C})}$ can be identified with morphisms $(f, g) : Z \rightarrow X \times Y$ so that f is a local sCat-equivalence, and morphisms of $h(X, Y)_{\text{sCat}(\mathcal{C})}$ are commutative triangles in the obvious way. Maps of the form (f, g) have a functorial factorization

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & V \\ & \searrow j & \downarrow (p, g') \\ & & X \times Y \end{array}$$

where j is a sectionwise trivial cofibration in the Bergner model structure and (p, g') is a sectionwise sCat-fibration. It follows that the composite

$$V \xrightarrow{(p, g')} X \times Y \xrightarrow{pr} X$$

is a local sCat-equivalence. The projection map pr is a sectionwise sCat-fibration since Y is sectionwise fibrant. It follows that p is a sectionwise fibration. The assignment $(f, g) \mapsto (p, g')$ defines a functor $\psi : h(X, Y)_{\text{sCat}(\mathcal{C})} \rightarrow h_{\text{hyp}}(X, Y)_{\text{sCat}(\mathcal{C})}$.

The map j above defines natural maps $id \rightarrow \psi \circ i_1$ and $id \rightarrow i_1 \circ \psi$, from which the result follows.

Definition 4.5.11 We write $G' : \text{Cat} \rightarrow \text{Gpd}$ for the groupoid completion functor. We write $G : \text{sCat} \rightarrow \text{sGpd}$ for the functor defined by $G(C)_n = G'(C_n)$.

Theorem 4.5.12 ([6, Corollary 9.4, Proposition 9.5]). *Suppose that $f : X \rightarrow Y$ is a weak equivalence of cofibrant objects in the Dwyer-Kan model structure on sCat_O of 4.1.10. Then $G(f)$ is also an sCat-equivalence. If X is cofibrant for the Dwyer-Kan model structure and $\pi_0(X)$ is a groupoid, then $X \rightarrow GX$ is an sCat-equivalence.*

By the proof of 4.1.12, if X is cofibrant in the Bergner model structure, it is also cofibrant in the Dwyer-Kan model structure on $\text{sCat}_{\text{Ob}(X)}$.

Lemma 4.5.13 *Suppose that we have a local sCat-equivalence*

$$f : X \rightarrow Y,$$

where Y is a presheaf of groupoids, both X and Y are sectionwise fibrant and X is projective cofibrant. Then the natural map

$$k_X : X \rightarrow G(X)$$

is a local sCat-equivalence.

Proof Note that $p^*L^2(f)$ is a sectionwise sCat-equivalence. Thus, $\pi_0(p^*L^2X)$ is a presheaf of groupoids since $\pi_0(p^*L^2Y)$ is a presheaf of groupoids. Now, by 4.4.3, $p^*L^2(X) \cong L^2(X')$ for some projective cofibrant presheaf X' . Consider the diagram

$$\begin{array}{ccccc} GX' & \xleftarrow{\phi} & GDKX' & \longrightarrow & GDKp^*L^2X \\ \uparrow k_{X'} & & \uparrow & & \uparrow k_{\text{DKG}p^*L^2X} \\ X' & \longleftarrow & DKX' & \longrightarrow & DKp^*L^2X \end{array}$$

The horizontal maps in the right hand square are both local isomorphisms. By 4.5.12, the map ϕ is a sectionwise sCat-equivalence. Because $\pi_0 p^* L^2(X)$ is a presheaf of groupoids, 4.5.12 implies $k_{\text{DK}p^* L^2 GX}$ is a sectionwise sCat-equivalence. We conclude that $X' \rightarrow G(X')$ is a local sCat-equivalence. But the sheafification of this map is naturally isomorphic to $p^* L^2(k_X)$. Since $p^* L^2$ reflects local sCat-equivalences, we conclude that k_X is a local sCat-equivalence.

Theorem 4.5.14 *Let X and Y be presheaves of simplicial groupoids. Then there is a bijection*

$$[X, Y]_{sGpd\mathbf{Pre}(\mathcal{C})} \rightarrow [X, Y]_{sCat\mathbf{Pre}(\mathcal{C})}$$

between maps in the homotopy category of the model structure of 4.5.8 and maps in the homotopy category of the local Bergner model structure.

Proof Let

$$i : h_{hyp}(X, Y)_{sGpd\mathbf{Pre}(\mathcal{C})} \rightarrow h_{hyp}(X, Y)_{sCat\mathbf{Pre}(\mathcal{C})}$$

be the inclusion. This map is well-defined by 4.5.5 and 4.5.7. By 4.5.10 and 4.5.4, it suffices to show that $\pi_0(i)$ is a bijection. By 4.5.5, we can replace X and Y with their fibrant replacement in the model structure of 4.5.8. In particular, we may assume that X and Y are presheaves of fibrant simplicial groupoids.

First, note that i is a surjection. Indeed, suppose that

$$\sigma : X \xleftarrow{f} Z \xrightarrow{g} Y$$

is an element of $h_{hyp}(X, Y)_{sCat\mathbf{Pre}(\mathcal{C})}$. Let \mathcal{K} denote the cofibrant replacement functor for the global projective model structure on $sCat\mathbf{Pre}(\mathcal{C})$. Since Z is sectionwise fibrant, it follows from 4.5.13 that $G\mathcal{K}(Z) \rightarrow \mathcal{K}(Z)$ is a local sCat-equivalence. Thus,

$$\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & & Y \\ & \swarrow h & \searrow \\ & G(\mathcal{K}(Z)) & \end{array}$$

represents a map of cocycles, with the bottom cocycle an element of $h(X, Y)_{sGpd\mathbf{Pre}(\mathcal{C})}$, as required.

On the other hand, we will show that i is injective. It suffices to note that if

$$\begin{array}{ccc} & Z & \\ & \swarrow & \searrow \\ X & & Y \\ & \swarrow & \searrow \\ & W & \end{array}$$

is a map of cocycles in which everything is sectionwise fibrant, then

$$\begin{array}{ccc}
 & GK(Z) & \\
 X & \swarrow & \searrow Y \\
 & GK(W) &
 \end{array}$$

represents a map of cocycles by 4.5.13.

Given a presheaf of simplicial categories A , an **A-diagram** consists of a simplicial set map $\pi : X \rightarrow \text{Ob}(A)$ as well as an action

$$\begin{array}{ccc}
 X \times_s \text{Mor}(A) & \xrightarrow{a} & X \\
 \downarrow & & \downarrow \pi \\
 \text{Mor}(A) & \xrightarrow{t} & \text{Ob}(A)
 \end{array}$$

A map of A -diagrams is a commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\phi} & Y \\
 & \searrow \pi & \downarrow \pi \\
 & & \text{Ob}(A)
 \end{array}$$

which respects the action. We write sSet^A for the category of A -diagrams.

In sections, this is equivalent to the internal description of compatible functors $X_n : A(U)_n \rightarrow \text{Set}$. Thus, we can define a simplicial presheaf $\underline{\text{holim}}_{A_n}(X_n)$ by

$$U \mapsto \underline{\text{holim}}_{A_n(U)} X_n(U).$$

We call an A -diagram an **A-torsor** if and only if

$$\underline{\text{holim}}_{A_n}(X_n) \rightarrow *$$

is a local weak equivalence. Let \mathbf{Tors}_A denote the full subcategory of sSet^A of torsors. The above definition of torsors appeared in [13], and generalizes the classical description of a torsor (see [15, pg. 251] for a discussion).

We call a presheaf X of simplicial categories **sectionwise cofibrant** if and only if $X(U)$ is cofibrant for the Bergner model structure for all $U \in \text{Ob}(\mathcal{C})$. Note that this is not the same as being cofibrant for the local Bergner model structure.

Theorem 4.5.15 *Suppose that X is a sectionwise cofibrant presheaf of simplicial categories such that $\pi_0(X)$ is a presheaf of groupoids. Then there is a bijection*

$$\pi_0(\mathbf{Tors}_{GX}) \rightarrow [*, X]_{\text{sCatPre}(\mathcal{C})}.$$

Proof We have bijections

$$[* , X]_{sCatPre(\mathcal{C})} = [* , GX]_{sCatPre(\mathcal{C})} = [* , GX]_{sGpdPre(\mathcal{C})} = [* , \bar{W}GX]_{sPre(\mathcal{C})} = [* , dBGX]_{sPre(\mathcal{C})},$$

where $[,]_{sPre(\mathcal{C})}$ denotes homotopy classes of maps in the injective model structure. The first bijection follows from 4.5.12 and the fact that $\pi_0 X$ is a presheaf of groupoids. The second and third follow from 4.5.14 and 4.5.8, respectively. The final one comes from 4.5.9, the generalized Eilenberg-Zilber theorem.

On the other hand, [13, Theorem 24] gives a bijection

$$[* , dBGX]_{sPre(\mathcal{C})} = \pi_0(\mathbf{Tors}_{GX}).$$

Corollary 4.5.16 *Suppose that X is a presheaf of simplicial categories such that $\pi_0(X)$ is a presheaf of groupoids. Then there is a bijection*

$$\pi_0(\mathbf{Tors}_{GDK(X)}) \rightarrow [* , X]_{sCatPre(\mathcal{C})}.$$

Corollary 4.5.17 *Suppose that X is a presheaf of Kan complexes. Then we have a bijection*

$$\pi_0(\mathbf{Tors}_{G\mathcal{C}(X)}) \rightarrow [* , X]_{sPre(\mathcal{C})},$$

where $[,]_{sPre(\mathcal{C})}$ denotes homotopy classes of maps in the injective model structure.

Proof First, note that by 3.4.10 and the fact that X is a presheaf of Kan complexes, we have a bijection

$$[* , X]_{inj} = [* , X]_{LJoyal}$$

where $[* , X]_{inj}$ and $[,]_{LJoyal}$ denotes maps in the homotopy categories of the injective and local Joyal model structures, respectively. The Quillen equivalences of 4.4.13 and 4.4.14 imply that there are bijections

$$[* , X]_{LJoyal} = [* , \mathfrak{C}(X)]_{LBerg}$$

between maps in the homotopy categories of the local Joyal and local Bergner model structures, since Quillen equivalences induce equivalences of homotopy categories.

Since everything in the Joyal model structure is cofibrant, $\mathfrak{C}(X)$ is sectionwise cofibrant presheaf by the Quillen equivalence of 4.1.15. By 4.1.18, $\pi_0 \mathfrak{C}(X) \cong P(X)$. But $P(X)$ is a groupoid by 2.1.9. Thus, $\mathfrak{C}(X)$ satisfies the hypotheses of 4.5.15 and we have an identification

$$[* , \mathfrak{C}(X)]_{sCatPre(\mathcal{C})} = \pi_0(\mathbf{Tors}_{G\mathcal{C}(X)}),$$

from which the result follows.

Remark 4.5.18 In [5] and [30] an explicit description of $\mathfrak{C}(X)$ for a quasi-category X is given that may prove particularly useful for calculations (see in particular [30, Theorem 2.3]). From this description, a number of interesting properties of $\mathfrak{C}(X)$ are deduced, such as the fact that its simplicial homs are 3-coskeletal ([30, Theorem 4.1]).

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