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Non-Linear Time Series Modelling with Applications to Equity and Fixed Income Markets

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Abstract

My thesis focuses on theoretical and empirical aspects of modelling time series during different financial and economic conditions. It consists of three separate chapters in which the properties of Threshold Vector Autoregressive Model (TVAR) models are addressed with subsequent applications to equity and fixed income markets.

In the first chapter, which is a joint work with my supervisor Lars Stentoft, we examine the steady state properties of the TVAR model. Assuming the trigger variable is exogenous and the regime process follows a Bernoulli distribution, we derive the necessary and sufficient conditions for existence of a stationary distribution. The derived stationarity conditions for the TVAR model could help to validate existing and future empirical studies, which are using this type of framework. We analyze a situation related to so called locally explosive models, where the stationary distribution exists though the model is explosive in one regime. Using simulation methods we show that locally explosive models can generate some of the key properties of financial and economic data, usually implied by the literature on bubble formation. Thus, having closed form solutions for the stability properties, which describe locally explosive models, could be potentially useful for the studies of bubbles in a multivariate setting. We also demonstrate that assessing the stationarity of threshold models based on simulations might well lead to wrong conclusions, which highlights the challenges when making inference in non-linear threshold models.

In the second chapter, I study the stock market liquidity and volatility relation over the period of 2000 - 2015 in an empirical TVAR model with two regimes, which are defined endogenously by the past level of stock market liquidity. I find supporting evidence that the link between liquidity and volatility is non-linear and this result is robust for all the 4 stocks in my sample. My results demonstrate that the relationship between market liquidity and

volatility is stronger when market liquidity is low. I demonstrate that a shock to the market liquidity and volatility can lead to vicious cycles when liquidity remains low forever, which is related to the liquidity and volatility spirals described in the literature. The estimated level of liquidity threshold could serve as informative indicator for market makers about destabilizing liquidity conditions in the equity market. On the other hand, I find supporting evidence that a single negative shock to volatility and liquidity is not enough to create the explosive series when the model evolves between regimes.

In the third paper, I model the distribution of the Canadian swap rates during normal times and during the Low Interest Rate (LIR) period. I examine the properties of the interest rates around LIR periods and show that the whole distribution changes. To capture this effect, I propose to use a mixture of t-scaled and Gaussian distributions with time-varying weights. The estimated mixture of distributions model defines two different distributions with the sharp transition between them at around 1.0% level of the short interest rate. My model can generate the leptokurtic pattern of interest rates during normal interest rate regime, as well as very low and possibly negative interest rates during LIR regime. I show that the resulting methodology leads to more accurate empirical performance when compared to the standard (one-regime) models used in the literature. The proposed mixture of distribution model could improve on the models for interest rates and other risk factors, like exchanges rates, used for derivative pricing and risk management in the LIR or low volatility environment.

keywords: volatility, threshold models, interest rate, market liquidity, zero lower bound

Co-Authorship Statement

This thesis contains Chapter 1 co-authored with Lars Stentoft.

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To My Family

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1.1 Introduction

Correct theoretical and empirical modelling of financial time series remains challenging. First of all, the usual linear framework often falls short of properly describing the data which instead exhibit important non-linear features. Secondly, economic theory regularly results in models with multiple equilibria and asymmetries which the time series model should be able to accommodate. Finally, data is often interconnected and hence simple univariate models generally fall short of appropriately describing the complex nature of the data. The Global Financial Crisis of 2008 demonstrated this very clearly and reinforced the need to use a multivariate non-linear framework in economic models, in general, and in empirical finance, in particular.

Among the many possible candidate non-linear models, threshold models are particularly interesting and they have been extensively used in the existing empirical literature. These models are straightforward generalizations of linear models. For example, the simple two regime Threshold Autoregressive (TAR) model specifies a different autoregressive structure for each of the regimes and a threshold variable that determines which regime is active. These models are therefore relatively simple to estimate, and since at time t the regime state is known they are more suitable for forecasting than other non-linear models, in particular hidden Markov models. Finally, TAR models allow for reasonably simple tests of the non-linear structure against linear alternatives and to test the number of regimes. The multivariate generalization of the TAR model instead uses vector autoregressive (VAR) structures in the regimes and is therefore naturally referred to as the TVAR model (Tsay, 1998, Hubrich and Tersvirta, 2013).

Empirical studies have used threshold models to explore the asymmetry of shocks and non-linear relationship between variables in financial markets and data from the real and monetary economy. For instance, TVAR models are widely used to study the asymmetric effect of fiscal and monetary policies in different credit, interest and inflation rate regimes (Fazzari et al., 2015, Balke, 2000, Shen and Chiang, 1999). For example, Balke (2000) studies the propagation of shocks to output growth, Fed funds rate, inflation and measures of credit conditions during "tight" and "normal" credit market conditions using a TVAR framework with two regimes. The results suggest that shocks have a larger effect on output in "tight" credit regimes and that contractionary monetary shocks are more effective than expansionary ones. A similar approach is followed by Calza and Souza (2005) to study the transmission of monetary shocks across two credit regimes in the EU area and by Li and St-Amant (2010) to evaluate the effect of financial stress conditions on monetary policy effectiveness in Canada.

Another important application of threshold models has been to study the business cycle. For example, Altissimo and Violante (2001) study the joint dynamics of US output and unemployment using a bivariate TVAR model for recessions and expansions. Here the lagged feedback variable, which measures the depth of the recession, defines the regime. The resulting model is a VAR model with a fixed number of lags when the economy is in expansion and a time varying lag order when the economy is in recession. The authors find that nonlinearities are statistically significant only for unemployment, but it transmits to output through cross-correlation. Further evidence on the usefulness of threshold models for analysing the business cycle can be found in Koop and Potter (1999), Koop et al. (1996), Peel and Speight (1998), and Potter (1995), amongst others.

Threshold models are also popular when it comes to exploring the asymmetric relation between different varibales in financial markets. In particular, a common application of TAR models includes determining the threshold effect in price movements related to transaction cost (Yadav et al., 1994). The threshold ARCH class of models has been applied to study the non-linear effect in volatility processes (Rabemananjara and Zakoian, 1993). Finally, multivariate threshold models have been extensively used in studying the dynamics in stock prices, returns, volatilities, inflation and economic activity (Barnes, 1999, Griffin et al., 2007, Huang et al., 2005, Li et al., 2015). For example, Griffin et al. (2007) study the joint dynamics of stock market turnover, returns and volatility in 46 countries using a TVAR model with two regimes which are separated by the sign of the past return. The authors conclude that small negative return shocks, rather than large ones, are the drivers for the decrease in turnover after a decreasing returns. Li et al. (2015) study the interaction between the Shanghai and Shenzhen stock markets in a bivariate three regime TVAR model where the threshold variable is the average difference of the log returns between the two markets. The results suggest that the strength of interaction between markets is regime dependent. In particular, the Shanghai market leads most of the time, except in the third regime, where both markets interact simultaneously. A detailed review of the application of threshold models in empirical economics can be found in Hansen (2011).

One challenge with non-linear time series models in general and by extension also with threshold models is to assess model stationarity. Establishing stationarity is important as it is a fundamental assumption in most theoretical research. Indeed, the asymptotic properties of estimators in threshold models are generally established under a set of standard regularity conditions, which include the existence of finite higher order moments and the strict stationarity of the data generating process (Tsay, 1998). Moreover, existing inference approaches assume stationarity of the data generating process (Tsay, 1998, Hansen, 1996, 2000) and violation of this assumption might lead to spurious non-linearities (Calza and Souza, 2005) and could invalidate the use of Hansen (1996) simulated p-values for inference.

While significant progress has been made to establish conditions which ensure stationarity of the univariate threshold case (Chan and Tong, 1985, Brockwell et al., 1992, Petruccelli and Woolford, 1984, Knight and Satchell, 2011, Chen et al., 2011) to the best our knowledge very little is known about the multivariate extension. Please see Chen et al. (2011) for an extensive review about recent findings regarding the stationarity of TAR models. If one was to use the general approach from this literature to establish the stationarity of TVAR models it would require proving the convergence of an infinite sum of products of random matrices. This is clearly difficult and likely explains the absence of theoretical results for TVAR model.

In this paper we fill this gap in the existing literature and analyse the properties of the

TVAR model in detail. To achieve this, we assume that the trigger variable is exogenous and that the regime process follows a Bernoulli distribution. First, we derive necessary and sufficient conditions for second order stationarity, which are not available in the existing literature, when the variance-covariance matrices of the random vector and the error process are assumed to have full rank. Next, we characterize the joint conditional distribution of the data generating process when the error vector follows a multivariate normal distribution. Finally, we derive the unconditional distribution for a special case of the TVAR model, and we demonstrate that in this case, the distribution of the threshold model is an infinite mixture of normals. This shows that TVAR models are very general and can accommodate many of the stylized features of financial data.

As an application of our results, we consider the special case where the elements of the random vector are positively correlated and we describe a model which is explosive in one regime, but still allows for the existence of steady state distribution. A similar idea was introduced in Knight et al. (2014) in the univariate case as a so-called "locally explosive model". In particular, they study the univariate threshold autoregressive model with exogenous trigger and its application to bubble formation. We extend the notion of locally explosive models to the bivariate TVAR model. The derived conditions for the existence of the stationary distribution have simple economic intuition and are easy to interpret. In particular, our results show that in the stationary model there is a trade-off between autoregressive dependence in the regime and the probability of the regime.

Next, we conduct an empirical analysis of the locally explosive models. In the absence of explicit theoretical conditions which guarantee stationarity of the model, the previous literature suggested to establish stationarity indirectly by demonstrating, using a simulation study, that the estimated model does not contradict the stability assumption. To assess this procedure, we simulate the bivariate locally explosive TVAR model for different distributions of the regimes. Our results show that a simulation study aimed at verifying stability of a particular model might give inconclusive or even wrong results. Specifically, we show that the simulation exercise may very well fail to reject stability of non-stationary TVAR models when the probability of the explosive regime is low.

Finally, we empirically document that the locally explosive TVAR model can be associated with bubble formation processes. In fact, our simulated locally explosive models appear to possess explosive and unit root behaviour while overall remaining stationary. These properties are implied by the definition of bubbles prevailing in the current literature and formally described by Evans (1991) and Phillips and Yu (2011). Our results should encourage further research into threshold models and their use to study the formation of and existence of bubbles in financial data.

The structure of the paper is as follows: In Section 1.2 we derive the necessary and sufficient conditions for second order stationarity and for the existence of a stationary distribution for the TVAR model. This section also derives closed form solutions for the stationary distribution. In Section 1.3, we consider the so-called locally explosive models, in which the TVAR model is explosive in one regime, while overall remaining stationary. This section also presents some interesting special cases and reports the results from a simulation study. Finally, Section 1.4 concludes. Appendix A contains proofs and additional figures.

1.2 The Threshold Vector Autoregressive model

Throughout this paper we consider the threshold vector autoregressive model given by

$$Y_t = \Phi^1 I(X_{t-1} \in R_1)Y_{t-1} + \Phi^2 I(X_{t-1} \in R_2)Y_{t-1} + \epsilon_t, \tag{1.1}$$

where Y_t is a $(n \times 1)$ random vector, Φ^1 and Φ^2 are $(n \times n)$ parameter matrices, where n is the number of time series, $I()$ is the indicator function, X_t is a random variable, which determines the regime, and ϵ_t is a sequence of independent multivariate random vectors, such that $E(\epsilon_t) = 0$ and $Var(\epsilon_t) = \Sigma$, $\forall t$, where Σ is positive definite with full rank. We assume that $E(\epsilon_t|X_s) = 0$ for all $s \leq t$ and that the sequence (ϵ_t, X_t), $t \geq 1$, is *iid*.

The regime process is defined as $S_t = I(X_t \in R_2)$, $\forall t$, where $Prob(X_t \in R_2) = \pi$ and $Prob(X_t \in R_1) = 1 - \pi$, with $R_1 \cup R_2 = R$ and $R_1 \cap R_2 = \emptyset$. From this it follows that S_t is a Bernoulli variable with $S_t = 0$ with probability $1 - \pi$ and $S_t = 1$ with probability π. Using S_t, (1.1) can be rewritten as $Y_t = (\Phi^1 + S_{t-1}\Phi^0)Y_{t-1} + \epsilon_t$, where $\Phi^0 = \Phi^2 - \Phi^1$. If we further denote by $B_t = S_t\Phi^0 - \pi\Phi^0$, where $E(B_t) = 0$, $\forall t$, the model in (1.1) can be rewritten as a Random Coefficient Model (RCM) (see Nicholls and Quinn (1982)) given by

$$Y_t = (\Phi + B_{t-1})Y_{t-1} + \epsilon_t, \tag{1.2}$$

where $\Phi = \Phi^1 + \pi\Phi^0 = (1 - \pi)\Phi^1 + \pi\Phi^2$.

In the following sections we examine in detail the TVAR model specified above. First we provide the necessary and sufficient conditions under which the TVAR model is second order stationary. We also derive expressions for the moments and the stationary solution to the model given in (1.1). Secondly, we derive the distribution associated with this data generating process. For simplicity, we assume only two regimes in Equation (1.1). However, the theoretical results obtained here can easily be generalized to multiple regimes.

1.2.1 Stationarity of the TVAR model

Theorem 1.1 provides conditions under which the TVAR model above is second order stationary, i.e. that $E(Y_t)$ is constant and $Cov(Y_t, Y_{t+h})$ depends only on the lag length h.

Theorem 1.1 *The process Y_t, $t = 0, 1, 2, ...$ defined in* (1.1) *is second order stationary with positive definite covariance matrix $V = Var(Y_0)$ if and only if:*

1. $\mu = 0$*, where μ is a mean of the initial vector, $\mu = E(Y_0)$,*
2. *the covariance matrix, V, solves $V - \Phi V \Phi' - E(B_{t-1} V B'_{t-1}) = \Sigma$, and*
3. $|\lambda| < 1$*, where λ is the maximum eigenvalue of the matrix* $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$.

Proof. See Appendix A.

Condition 2 of Theorem 1.1 provides an expression for calculating the covariance matrix of the second order stationary process Y_t. Notice that after vectorization of this expression we can obtain a closed form formula for this. Remark 1 provides this formula.

Remark 1 *From vectorization of the expression $V - \Phi V \Phi' - E(B_{t-1} V B'_{t-1}) = A$ the equation for the variance of Y_t can be obtained from*

$$vecV = (I - \Phi' \otimes \Phi' - \pi(1-\pi)\Phi^0 \otimes \Phi^0)^{-1} vec\Sigma. \tag{1.3}$$

We note that our sufficient conditions for the existence of moments are special cases of the conditions for the stationarity of RCMs derived by Nicholls and Quinn (1981) and

Feigin and Tweedie (1985). Theorem 1.1, however, generalizes these results and provides necessary conditions, which have been missing in the literature, for the stationarity of the TVAR model in (1.1).

Theorem 1.2 provides an expression for the stationary solution to the model in (1.1) and the corresponding conditions for the existence of this solution. Theorem 1.2 also shows that this solution is unique and strictly stationary.

Theorem 1.2 *Assume that V is positive definite with full rank. Then the TVAR model in* (1.1) *has a unique stationary solution given by*

$$Y_t = \epsilon_t + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \Phi + B_{t-k} \right) \epsilon_{t-n}, \tag{1.4}$$

if and only if $\mid \lambda \mid < 1$, *where λ is the maximum eigenvalue of the matrix* $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$.

Proof. See Appendix A.

In Remark 2 we provide the restriction on the eigenvalues of the matrix Φ, which is necessary for the stationary model (1.1) and follows from Theorem 1.1 and 1.2. This condition is more tractable, and it is used in Section 3 to simplify the analysis of the stationary TVAR model with one explosive regime.

Remark 2 *Let the process Y_t, $t = 0, 1, 2, \ldots$ defined in* (1.1) *be stationary with positive definite covariance matrix V. Then the maximum eigenvalue of the matrix Φ is less than* 1.

Proof. See Appendix A.

The results of Theorem 1.1 and 1.2 can be extended to TVAR models with more than one lag. Corollary 3 presents the conditions for the stationarity of the TVAR model, which contains more than one lag.

Corollary 3 *Consider the following two-regime TVAR model with p lags in each regime*

$$Y_t = I(X_{t-1} \in R_1) \sum_{j=1}^{p} \Phi^{1j} Y_{t-j} + I(X_{t-1} \in R_2) \sum_{j=1}^{p} \Phi^{2j} Y_{t-j} + \epsilon_t, \tag{1.5}$$

where the properties of X_t and ϵ_t are those following Equation (1.1)*. This model has a unique stationary solution given by*

$$Z_t = \eta_t + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} A + D_{t-k} \right) \eta_{t-n}, \tag{1.6}$$

if $|\ \lambda\ | < 1$*, where* λ *is the maximum eigenvalue of the matrix* $(1-\pi)A^1 \otimes A^1 + \pi A^2 \otimes A^2$*, and only if* $|\ \lambda_1\ | < 1$*, where* λ_1 *is the maximum eigenvalue of the matrix* $A = (1-\pi)A^1 + \pi A^2$*, where* A^i*,* $i = 1, 2$*, is defined as* $A^i =$

$$\begin{pmatrix} \Phi^{i1} & \Phi^{i2} & \Phi^{i3} & ... & \Phi^{i(p-1)} & \Phi^{ip} \\ I_n & 0 & 0 & ... & 0 & 0 \\ 0 & I_n & 0 & ... & 0 & 0 \\ 0 & 0 & I_n & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & I_n & 0 \end{pmatrix}.$$

Z_t *and* η_t *are* $np \times 1$ *vectors given by* $Z_t' = [Y_t', Y_{t-1}', Y_{t-2}', ...Y_{t-(p-1)}']$ *and* $\eta_t = [\epsilon_t', 0, 0, .., 0]$*, respectively, and* $D_t = (S_t - \pi)A^2 + (\pi - S_t)A^1$.

Proof. See Appendix A.

The distinctive feature of the TVAR model is that it is a linear Vector Autoregresive Model (VAR) in each of the regimes and an interesting question therefore is how the stability of each regime contributes to the stationarity of the whole TVAR model. Knight and Satchell (2011) investigate this question in detail for the univariate TAR model and Niglio et al. (2012) provide evidence that, when the univariate TAR model is stationary in both regimes, the whole TAR model cannot explode. The most interesting situation however occurs when the model in (1.1) is explosive in one of the regimes.

The results of Theorem 1.1 and 1.2 can be used to analyse this particular situation, one in which the TVAR model in (1.1) is explosive in one of the regimes. For example, the following example shows that the TVAR model can still be stationary in that case provided the probability to be in the explosive regime is not too large. See also Section 3 for further analysis.

Example. Consider the model in (1.1), where $\Phi^1 = \begin{pmatrix} 0.70 & 0.21 \\ 0.31 & 0.80 \end{pmatrix}$, $\Phi^2 = \begin{pmatrix} 0.20 & 0.32 \\ 0.10 & 0.25 \end{pmatrix}$ and $\pi = Prob(X_t \in R_2) = 0.3$. Since one of the eigenvalues of Φ^1 is equal to 1.01, the model is not stationary in regime one. On the other hand $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2 = \begin{pmatrix} 0.36 & 0.12 & 0.12 & 0.0 \\ 0.16 & 0.41 & 0.06 & 0.14 \\ 0.16 & 0.06 & 0.41 & 0.14 \\ 0.07 & 0.18 & 0.18 & 0.47 \end{pmatrix}$, and its maximum eigenvalue $\lambda = 0.78$. Thus, overall the model is stationary.

1.2.2 The stationary distribution

In this section we describe the stationary distribution associated with the model in (1.1). Throughout, we assume that $\epsilon_t \sim N(0, \Sigma)$ are independent random vectors. Let Y_t be defined by (1.4) and let $S_n(t) = \prod_{k=1}^{n}(\Phi + B_{t-k})$, $n \geq 1$, with $S_0(t) = 1$. It follows that

$$Y_t = \epsilon_t + \sum_{n=1}^{\infty} S_n(t)\epsilon_{t-n}. \tag{1.7}$$

From this we have that

$$Y_t|S_n(t) \sim N(0, \Sigma + \sum_{n=1}^{\infty} S_n(t)\Sigma S_n'(t)), \tag{1.8}$$

and from the definition of $S_n(t)$ we notice that the stationary distribution of Y_t is a complicated mixture of Normal distribution. Since, it is difficult to establish the distribution of Y_t in general, we will derive it under the assumption that $\Phi^1 = 0$.

From (1.8) we see that the characteristic function of Y_t conditioned on $S_n(t)$ is given by

$$\phi(t, Y_t|S_n(t)) = exp\left(-\frac{1}{2}t\Sigma t' - \frac{1}{2}t\sum_{n=1}^{\infty} S_n(t)\Sigma S_n'(t)t'\right). \tag{1.9}$$

Notice that when $\Phi^1 = 0$ and $\Phi^2 = \Psi$ then $B_t = (S_t - \pi)\Psi$ and $\Phi = \pi\Psi$, and hence $S_n(t) = \prod_{k=1}^{n} S_{t-k}\Psi$. Note also that $\prod_{k=1}^{n} S_{t-k}\Psi\Sigma \prod_{k=1}^{n} S_{t-k}\Psi = \prod_{k=1}^{n} S_{t-k}\Psi\Sigma \prod_{k=1}^{n} \Psi'$. The conditional characteristic function (1.9) therefore becomes

$$\phi(t, Y_t|S_n(t)) = exp\left(-\frac{1}{2}t\Sigma t' - \frac{1}{2}t\sum_{n=1}^{\infty}\prod_{k=1}^{n} S_{t-k}\Phi^2\Sigma\prod_{k=1}^{n}\Phi^{2'}t'\right). \tag{1.10}$$

Given the conditional characteristic function and the distribution of $S_n(t)$ we can obtain the unconditional characteristic function and the marginal stationary distribution of Y_t. The results are presented in Theorem 1.3.

Theorem 1.3 *The stationary distribution of the TVAR process with* $\Phi^1 = 0$ *and* $\Phi^2 = \Psi$

has the following characteristic function

$$\phi(t, Y_t) = (1 - \pi) \sum_{K=0}^{\infty} \pi^K exp\left(-\frac{1}{2}t \sum_{n=0}^{K} \Psi^n \Sigma \Psi'^n t\right). \tag{1.11}$$

Moreover, the probability distribution function is given by

$$f(Y_t) = (1 - \pi) \sum_{K=0}^{\infty} \pi^K N\left(0, \sum_{n=0}^{K} \Psi^n \Sigma \Psi'^n\right), \tag{1.12}$$

where $N(A, B)$ is the multivariate normal distribution function with mean A and covariance matrix B.

Proof. See Appendix A.

Theorem 1.3 is a generalization of a result for the univariate threshold autoregressive process developed in Knight and Satchell (2011) and shows that when $\Phi^1 = 0$ the distribution function of Y_t given in (1.12) is an infinite mixture of multivariate Normals. This type of distribution can generate excess kurtosis. Such distributional characteristics are interesting when it comes to analysing financial markets and economic problems, since it can support the special features of this type of data. For instance, the distributions of equity returns and typical measures of realized volatility are characterized by large kurtosis. Thus, the theorem shows that TVAR models can be used to study these processes.

1.3 Locally explosive TVAR models

Threshold autoregressive models where one regime is non-stationary are related to the so-called locally explosive models.[1] Knight et al. (2014) defines the locally explosive model as a model in which some regimes may be explosive, but the whole model has a stationary distribution. They study univariate threshold models and apply the idea of locally explosive models to investigate the formation of bubbles. In this section, we generalize the notion of locally explosive models to the bivariate setting. In order to do so, we need to link the stationarity of the whole model in (1.1) provided in Theorem 1.1 and 1.2 to the stability of the model in each particular regime.

The derived conditions for the existence of a stationary solution are simple conditions on the matrix $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$, and it is not possible to relate the eigenvalues of this matrix to the eigenvalues of the parameter matrices Φ^1 and Φ^2 without adding extra structure. In the following section we therefore consider a bivariate TVAR model, where the parameter matrices Φ^1 and Φ^2 have either positive entries only or are upper triangular. We first obtain the conditions on the parameter matrices under which the locally explosive TVAR model remains stationary. We next provide a simulation study to examine the characteristics of this model and show that graphically it is very difficult to assess model stationarity using simulated data.

[1]Notice that the locally explosive models considered in this paper are models, which are state explosive. When $X_t = t$ instead the TVAR model is related to the models derived in Phillips and Yu (2009) and Phillips et al. (2011) where the explosive behaviour is defined in the time series context.

1.3.1 Special cases of the TVAR model

We consider the special case where Y_t in (1.1) is bivariate and the parameter matrices have positive entries. We introduce the following additional notation for $\Phi^1 = \begin{pmatrix} \phi^1_{11} & \phi^1_{12} \\ \phi^1_{21} & \phi^1_{22} \end{pmatrix}$ and $\Phi^2 = \begin{pmatrix} \phi^2_{11} & \phi^2_{12} \\ \phi^2_{21} & \phi^2_{22} \end{pmatrix}$. The following corollary to Theorem 1.1 and 1.2 provides conditions in terms of the individual ϕ's above under which the TVAR model is second order stationary. These conditions do not rule out the possibility of an explosive regime, and if we assume that one regime is explosive, we derive the conditions on the coefficient matrix of the stationary regime.

Corollary 4 *Let the matrices* Φ^1 *and* Φ^2 *have positive entries. If* $(1-\pi)(\phi^1_{j1} + \phi^1_{j2})^2 + \pi(\phi^2_{i1} + \phi^2_{i2})^2 < 1$, $\forall i, j = 1, 2$, *then the model in* (1.1) *is stationary. Moreover, if the model in* (1.1) *is explosive in one of the regimes* $x \in \{1, 2\}$, *then* $(\phi^{-x}_{i1} + \phi^{-x}_{i2}) < 1$, $\forall i = 1, 2$, *where* $-x \in \{1, 2\} \setminus \{x\}$.

Proof. See Appendix A.

Corollary 4 shows that if the model in (1.1) is explosive in one regime, the persistence of the variables in this regime is restricted by the probability of the regime and the persistence of the variables in the other regime.[2] In other words, Corollary 4 states that there is a trade-off between how persistent a given regime can be and the probability of this particular regime. In addition, when the conditions of Corollary 4 hold and one of the regimes is explosive, the sum of the coefficients of the other regime's matrix is naturally bounded by one.

[2]The persistence of the variables is defined as the column sum of the coefficients of Φ^1 and Φ^2.

Corollary 4 provide sufficient conditions for stationarity of the model, even when the underlying relationship is explosive in one of the regimes. We believe that the above finding might be useful for a number of financial and macroeconomic models. In fact, the assumption of positive entries only in Φ^1 and Φ^2 is not very restrictive for economic research and there are a variety of well documented cases with positive relationships between variables and their lags. For example, it is shown to be the case for asset returns and asset market illiquidity, consumption and GDP, volatility and trading volume and inflation and stock volatility, among many other pairs (Amihud and Mendelson, 1986, Engle and Rangel, 2005, Jagannathan et al., 2000, Wang and Yau, 2000).

When we add slightly more structure and assume that Φ^1 and Φ^2 are triangular matrices with nonnegative diagonal entries, we can derive the necessary conditions directly in terms of the eigenvalues of Φ^1 and Φ^2. Corollary 5 summarizes these findings.

Corollary 5 *Let the process Y_t, $t = 0, 1, 2, ...$ defined in* (1.1) *be stationary. Then the following conditions hold*

1. $\lambda_1^2 \lambda_2^2 \leq \frac{1}{\pi}$,
2. $\lambda_1^1 \lambda_2^1 \leq \frac{1}{(1-\pi)}$,
3. $\lambda_1^1 \lambda_2^2 \leq \sqrt{\frac{1}{(1-\pi)\pi}}$, *and*
4. $\lambda_1^2 \lambda_2^1 \leq \sqrt{\frac{1}{(1-\pi)\pi}}$,

where λ_1^i and λ_2^i are the eigenvalues of the matrix Φ^i, $i = 1, 2$.

Proof. See Appendix A.

Since the eigenvalues of a triangular matrix is its diagonal entries, Corollary 5 could equivalently be stated as follows.

Corollary 6 *Let the process* Y_t*,* $t = 0, 1, 2, ...$ *defined in* (1.1) *be stationary. Then the following conditions hold*

1. $\phi_{11}^{2}\phi_{22}^{2} \leq \frac{1}{\pi}$,

2. $\phi_{11}^{1}\phi_{22}^{1} \leq \frac{1}{(1-\pi)}$,

3. $\phi_{11}^{1}\phi_{22}^{2} \leq \sqrt{\frac{1}{(1-\pi)\pi}}$, *and*

4. $\phi_{11}^{2}\phi_{22}^{1} \leq \sqrt{\frac{1}{(1-\pi)\pi}}$.

Corollary 5 and 6 illustrate explicitly that there is a trade-off between how persistent a regime in the TVAR model can be and the probability of that regime while ensuring the overall stationarity of the process. Again, it is noteworthy that the stationarity of the TVAR model does not rule out the possibility of an explosive regime, but it restricts the value of the own autoregressive coefficients.

1.3.2 Simulation

Second order stationarity implies that means, variances and covariances are time-invariant and finite. If stationarity is not satisfied, however, it could be that shocks to the data generating process could lead to a time series that have unbounded moments. Previously, and in the absence of explicit stationarity conditions such as the ones derived in our paper, the literature instead suggested to verify that the estimated model does not contradict stability assumptions by use of simulation studies (Hubrich and Tersvirta, 2013, Franses and Dijk, 2000). Specifically, the literature proposed to switch off the noise and simulate the estimated model for different histories. If the generated series converge to the same point, the natural conclusion would be that the simulated model is stationary. In contrast, finding at least one starting point that leads to an explosive time series would be sufficient to invalidate the stationarity assumption.

In this section we follow the above procedure and perform a graphical analysis to "test" the stationarity of the TVAR model as suggested in the existing literature for different locally explosive TVAR models. Our result show that this rough-and-ready approach does not allow us to draw the correct conclusion and the outcome of it is affected by the distribution of the explosive regime and the persistence of this regime. To be specific, we simulate the bivariate TVAR model in (1.1) with different parameter values. We generate time series from the model of length equal to $n = 250$, which is equivalent to one year of daily observations. The number of simulations is equal to $m = 200$. The initial values of the time series, Y_0, are equally distributed over the interval $[-0.15, 0.24]$ for the first series and equally distributed over the interval over $[-0.17, 0.23]$ for the second series. In Appendix A, we report additional results when $n = 2000$ to check the robustness of our result.

The parameter values used in the simulation study are shown in Table 1.1. As the table shows, regime 2 is by construction always explosive and we vary the value of π, the probability of this regime, such that the overall TVAR model can be stationary or non-stationary. This is indicated by the maximum eigenvalue of the matrix $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$, which is reported in column six labelled λ_{max}. In particular, we define 3 groups of models, such that models within each group have the same coefficient matrices, but the probability to be in the explosive regime 2, π, varies.

Models 1-6 are stronger related to lags in the explosive regime 2 than in regime 1. We contrast our models such that the persistence of the models in the second regime is stronger in group 2 than group 1. When the second regime is mildly explosive, like the models

Table 1.1: Parameter values used in simulating the bivariate TVAR model

Group	**Model**	**Regime 1,** Φ^1	**Regime 2,** Φ^2	**Probability of regime 2,** π	λ_{max}
1	1	0.2 0.3 0.3 0.4	0.3 0.8 0.5 0.8	0.3	0.72
	2	0.2 0.3 0.3 0.4	0.3 0.8 0.5 0.8	0.5	0.95
	3	0.2 0.3 0.3 0.4	0.3 0.8 0.5 0.8	0.7	1.18
2	4	0.2 0.3 0.3 0.4	1.1 1.2 1.2 1.05	0.1	0.84
	5	0.2 0.3 0.3 0.4	1.1 1.2 1.2 1.05	0.3	1.8
	6	0.2 0.3 0.3 0.4	1.1 1.2 1.2 1.05	0.5	2.8
3	7	0.9 0.05 0.7 0.3	0.3 0.8 0.2 0.8	0.1	0.95
	8	0.9 0.05 0.7 0.3	0.3 0.8 0.2 0.8	0.3	0.99
	9	0.9 0.05 0.7 0.3	0.3 0.8 0.2 0.8	0.5	1.03

Notes: This table shows the parameter values used in the simulated TVAR models. The distribution of the regimes is Bernoulli with probability to be in regime 2 equal to π. Notice that regime 2 is not stable in any of the models. In the right hand column we report the maximum eigenvalue of the matrix $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$, λ_{max}.

from group 1, this regime has to occur very frequently, in order to make the whole TVAR model non-stationary. In contrast, model 6 is unstable even when the probability to be in the explosive regime is as low as 0.3. Thus, when one regime is not stable, the distribution of the regimes is crucial for the stationarity of the whole TVAR model.

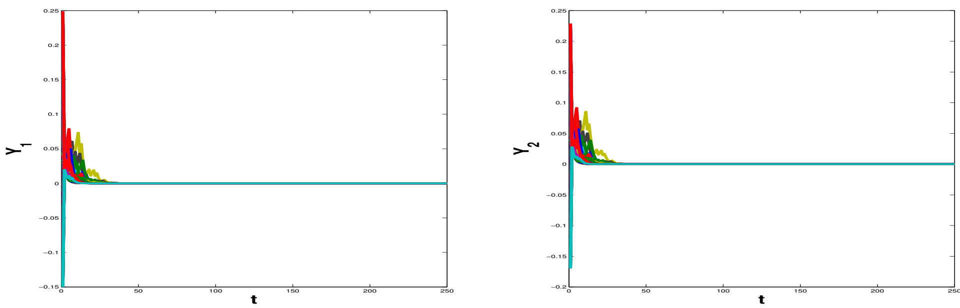
Figure 1.1 shows the simulated paths from models 1-3. When π is fairly low (Panel a) the time series appear stationary. When π gets higher and λ is closer to 1, the simulated model looks like a unit root (Panel b). While model 1 and 2 generate spikes, the simulated series return to the initial level all the time, a characteristic of stationary processes. When $\pi = 0.7$ (Panel c), the series is no longer stable and this is also evident from the figure. This conclusion is also valid when $n = 2000$ (see Figure A.1 in Appendix A).

Figure 1.2 shows the simulated paths from models 4-6. These models are very persistent in regime 2 and they can generate huge spikes even when the probability of this regime is low (Panel a and b). Both models 4 and 5 look like unit root models, which explode, though they return to the initial level afterwards. Thus, the simulation exercise cannot reject stability of model 5, even though it is non-stationary by construction. The simulation study though does reject stability of model 6, when the probability to be in the explosive regime increases to 50% (Panel c). Thus, the results of the simulation might be misleading about non-stationary TVAR model with low probability of the explosive regime.[3] The result of the simulation of models 4-6 prevails when $n = 2000$ (see Figure A.2 in Appendix A).

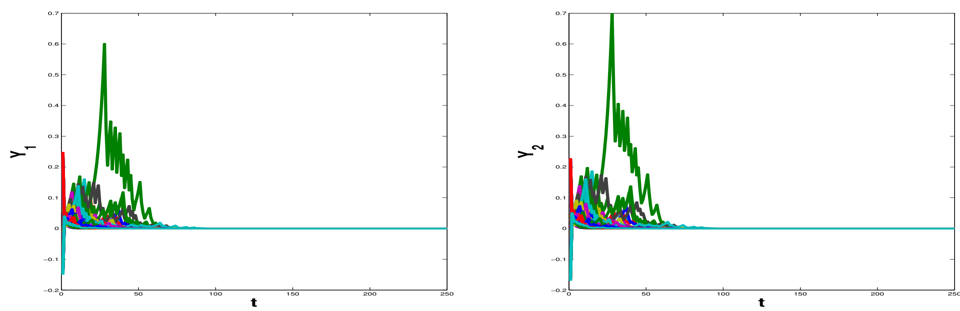
Models 7-9 describe a type of relationship, where a particular time series is stronger related with its own lag in regime 1 and with the other time series in regime 2. These models are quite persistent in regime 1, but still remain stationary in this regime. Figure 1.3 shows the simulated paths from these models. The explosive performance of model 9 is evident from Panel c. The conclusion however is not clear about model 8. This model is quite persistent in both regimes, thus it can generate growing series, like those shown in Panel b, and simulation of model 8 may in fact lead to rejecting the stability of a stationary model. However, we cannot reject stability of the model when $n = 2000$ (Figure A.3 in Appendix A, Panel b). In fact, when the length of the simulated time series is increased to $n = 2000$ the series from model 8 grows first but then returns to the initial level later

[3] It is of course impossible to check all starting points in a simulation study and we might not be lucky enough to have a starting point that allows rejecting stability of the model.

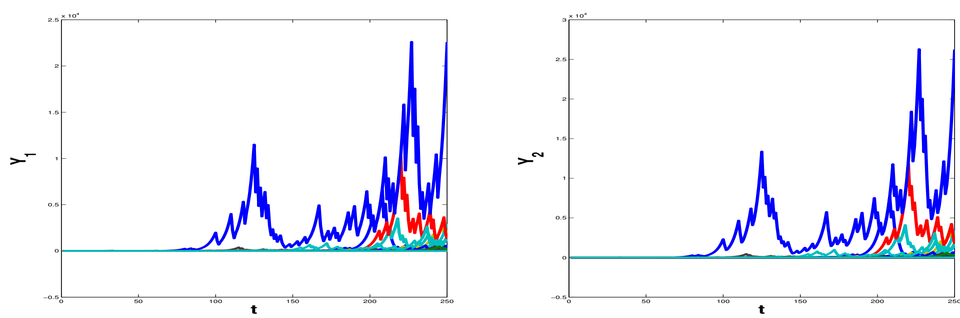
Figure 1.1: Simulated paths from models 1-3



(a) Model 1, $\pi = 0.3$



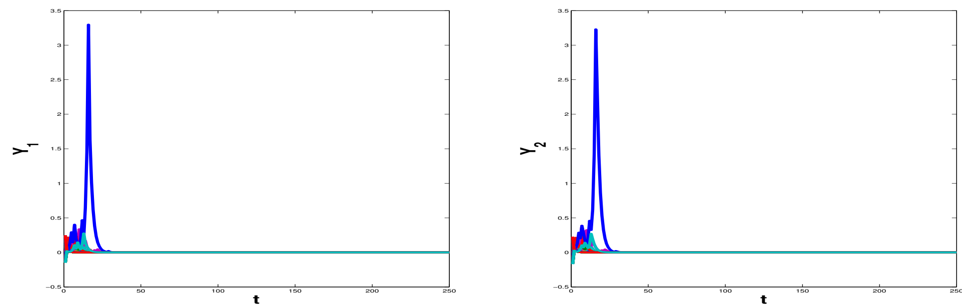
(b) Model 2, $\pi = 0.5$



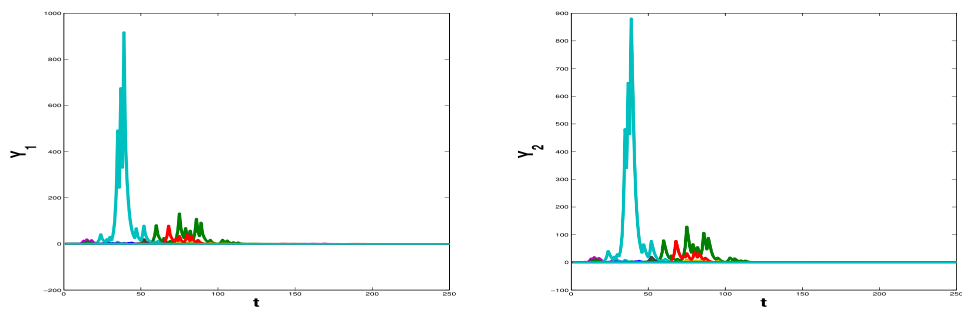
(c) Model 3, $\pi = 0.7$

Notes: This figure shows the simulated paths from models 1-3 for different set of histories over a $n = 250$ period using $m = 200$ simulated paths. The parameters are those from Table 1.1 and the probability to be in the explosive regime 2 is equal to π.

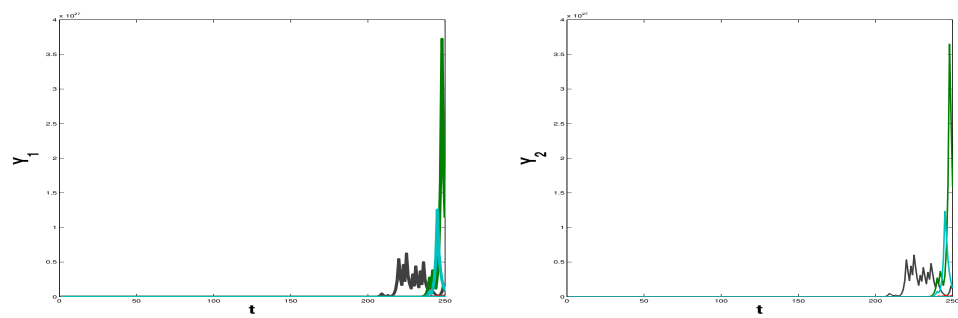
Figure 1.2: Simulated paths from models 4-6



(a) Model 4, $\pi = 0.1$



(b) Model 5, $\pi = 0.3$



(c) Model 6, $\pi = 0.5$

Notes: This figure shows the simulated paths from models 4-6 for different set of histories over a $n = 250$ period using $m = 200$ simulated paths. The parameters are those from Table 1.1 and the probability to be in the explosive regime 2 is equal to π.

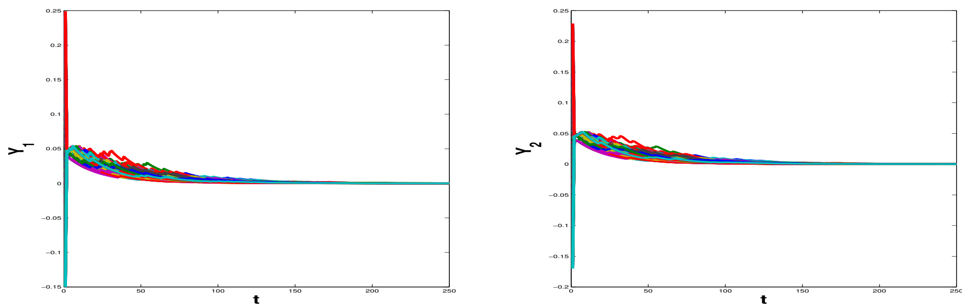
on. Thus, the results from simulating model 8 show that the conclusion from this type of simulation study may also be sensitive to the sample size used in the simulation.

Figure 1.2 shows the simulated paths from models 4-6. These models are very persistent in regime 2 and they can generate huge spikes even when the probability of this regime is low (Panel a and b). Both models 4 and 5 look like unit root models, which explode, though they return to the initial level afterwards. Thus, the simulation exercise cannot reject stability of model 5, even though it is non-stationary by construction. The simulation study though does reject stability of model 6, when the probability to be in the explosive regime increases to 50% (Panel c). Thus, the results of the simulation might be misleading about non-stationary TVAR model with low probability of the explosive regime. The result of the simulation of models 4-6 prevails when $n = 2000$ (see Figure A.2 in Appendix A).

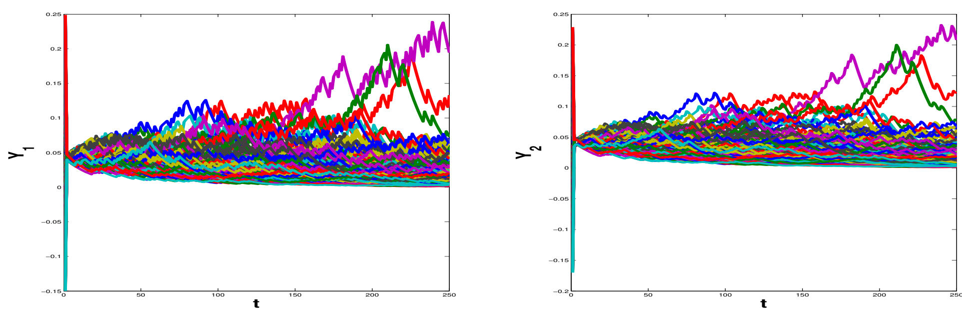
Figure 1.4 shows the simulated Y_{1t} from TVAR models 4 and 5 specified in the Table (1.1). We end this section by noting that the simulated series of Y_t could be associated with data generating processes of financial or economic bubbles. Evans (1991) defines periodically collapsing explosive processes of bubbles such that the explosive behaviour of this process prevails through the whole sample, with non zero probability to collapse when it faces some threshold level. Phillips and Yu (2011) suggest a locally explosive process of bubbles, where asset prices transit from a unit root regime to an explosive regime and claim that this approach is consistent with other propagation mechanisms in financial markets like rational bubbles, exuberant responses to economic fundamentals and herd behaviour. Our simulation exercise shows that a simple bivariate locally explosive yet globally stationary TVAR model can generate unit root or explosive behaviour, which is consistent with these existing definitions of bubbles.[4]

[4] An open question in the literature relates to how one can test for bubbles. In a recent paper, Ahmed and Satchell (2016) examine the performance of the Generalized Sup Augmented Dickey Fuller test proposed by

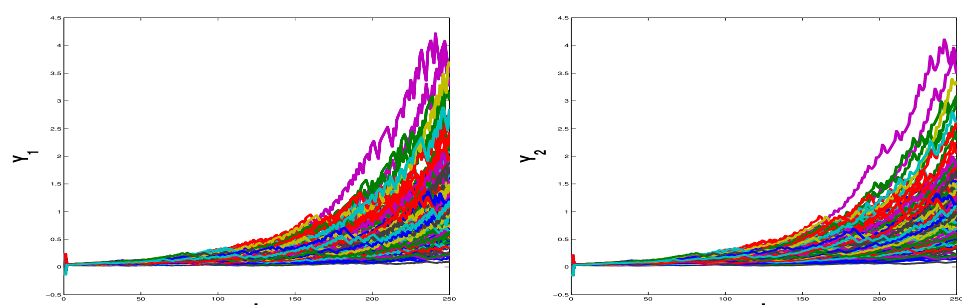
Figure 1.3: Simulated paths from models 7-9



(a) Model 7, $\pi = 0.1$



(b) Model 8, $\pi = 0.3$

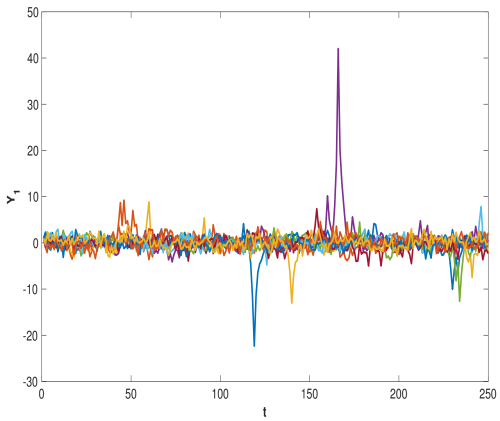


(c) Model 9, $\pi = 0.5$

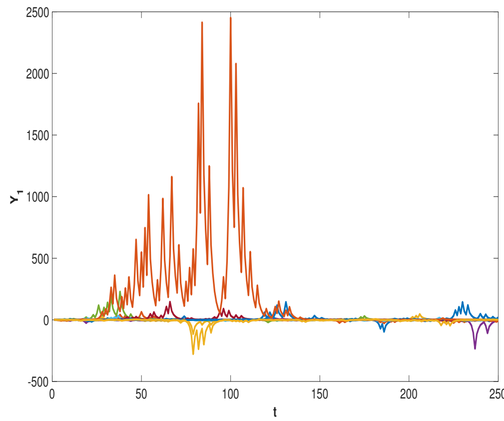
Notes: This figure shows the simulated paths from models 7-9 for different set of histories over a $n = 250$ period using $m = 200$ simulated paths. The parameters are those from Table 1.1 and the probability to be in the explosive regime 2 is equal to π.

Phillips et al. (2013) for the detection of explosive roots in univariate TAR models. They show that the power of the test drops considerably even though locally explosive regimes continue to be present when the process has a stationary distribution. We conjecture that this conclusion generalizes to the multivariate setting used in our paper.

Figure 1.4: Simulated Y_{1t}



(a) Model 4, $\pi = 0.1$



(b) Model 5, $\pi = 0.3$

Notes: This figure shows the simulated Y_{1t} of model in (1.1) with parameters from models 4 and 5 for over $n = 250$ period, where $\epsilon_t \in N\{0, 1\}$ The parameters are those from Table 1.1 and the probability to be in the explosive regime 2 is equal to π.

1.4 Conclusion

This paper derives the necessary and sufficient conditions for the existence of a stationary distribution of the TVAR model with two regimes, when the regime process follows a Bernoulli distribution. These results are to the best of our knowledge unavailable in the existing literature. We further derive a closed form solution for the stationary distribution in the special case when there is no autoregressive structure in one of the regimes.

When the variables of interest are positively related we describe a bivariate TVAR model, which is explosive in one regime, but allows for a stationary distribution along with finite moments. These results are related to so-called locally explosive models and our results extend the notion of locally explosive univariate processes to the bivariate case. We show that such models may remain stationary and to ensure this there is a trade-off between the persistence in a given regime and the probability of this regime.

In an empirical application we simulate from various bivariate TVAR models, which are explosive in one of the regimes. We show how these models can capture the unit root and explosive behaviour, usually implied by the literature on bubble formation. We also demonstrate that a simulation study may fail to reject the stability of non-stationary TVAR models, when the probability of the explosive regime is low.

Chapter 2

Stock Market Liquidity and Volatility: A Non-Linear Approach

2.1 Introduction

Market liquidity is related to the ability to sell and buy a large quantity of an asset without affecting its price. In illiquid market the asking price tends to be above its fundamental value, while the bidding price tends to fall below. The resulting transaction cost increases, making market less liquid. In extreme situation, liquidity can dry up, eliminating investor's opportunity to enter or exist current positions. It may cause a significant fall in asset prices, amplified by possible fire sales and leverage effect to meet margin calls (Brunnermeier, 2009). Thus, liquidity is crucial for the trading process and stability of the financial system (Pedersen, 2009).

Liquidity changes over time. The recent financial crisis reinforced the importance of market liquidity, since it was characterized by low liquidity for most financial assets and a stronger link between liquidity and other financial fundamentals (Acharya et al., 2013, Rosch and Kaserer, 2014). The goal of this paper is to explore the potential non-linearities between stock market liquidity and volatility in the time series dimension. In particular, I ask the following questions: Are market liquidity and volatility non-linearly related? How do shocks to liquidity or volatility affect subsequent stock market liquidity and volatility? What is considered to be a sustainable level of market liquidity? And what constitutes a critically low liquidity level? In this paper I build a model of stock market liquidity and volatility, such that the relationship between these variables varies and depends on the specific level of past liquidity, which I determine endogenously within the model. There are several stylized facts and recent empirical findings that motivate me to follow the described methodology.

Liquidity and volatility are related. Both of these variables share similar time series properties, like time variation, long memory, clustering and countercyclical behaviour. Chen et al. (2016) demonstrate high correlation between various aggregate liquidity measures and volatility of assets traded on NYSE. When volatility goes high, the probability of mispricing of asset is higher, thus bid-ask spread becomes wider. As a result the cost of trading increases, and market liquidity declines. A number of theoretical models link trading more volatile assets to higher inventory risks, which decrease asset liquidity (Brunnermeier and Pedersen, 2009, Grossman and Miller, 1988, Stoll, 1978). Comerton-Forde et al. (2010) empirically document this prediction, and also show that the impact of inventory on market liquidity is larger when trading results are poor. Thus, the liquidity differential between assets of low and high volatility increases, known as ”flight to quality” phenomenon when highly volatile assets become especially illiquid. Other empirical findings suggest that market liquidity declines when fundamental volatility increases (Benston and Hagerman, 1974, Amihud and Mendelson, 1989).

Liquidity affects volatility through a variety of channels. Since liqudity varies over time, investors, who face uncertainty about future transactions costs may require higher compensation for possible liquidity risk. Amihud and Mendelson (1986) demonstrate how the expected return is related to the transaction cost. Acharya and Pedersen (2005) present the liquidity adjusted CAPM model, where liquidity risk is priced in the cross section of stock returns. Other studies numerically establish that illiquid assets have higher expected returns (Chordia et al., 2009, Amihud, 2002, Hasbrouck, 2002). Thus, changes in liquidity may impact investor's expectations and contribute to price fluctuations and asset volatility. Finally, when there are many buyers and sellers who want and can trade easily, then price movements will be smoother since any shock will be incorporated into the price quickly based on market consensus about their significance. Otherwise, when it is harder to trade, in other words market liquidity is low, shocks might generate additional price movements and increase volatility.

The interation between volatility and market liquidity is described in the theoretical model of Brunnermeier and Pedersen (2009). This model links market liquidity and volatility to margin requirements and availability of capital to support trade. The main result of the paper predicts that there could be two equilibria in market liquidity, volatility and capital requirements relationship. When financiers believe that price movements are due to fundamental shocks, then negative liquidity shock increases price volatility, which rises the expectation about future volatility, then capital requirements increase, which again decrease the possibility to complete the trade, in other words worsen market liquidity and so on. The subsequent liquidity spirals arise, volatility and capital requirements increases. Otherwise, when financiers belive that price movements are due to temporary shocks, the above described feedback effect might not exist. Finally, the authors claim that the link between capital constraints and market liquidity is stronger during financial downturn, when availability of capital is already low. I test the predictions of the Brunnermeier and Pedersen (2009) model by considering a two regime model of market liquidity and volatility. However, I do not propose a pure test of the theoretical predictions of Brunnermeier (2009) in this paper, which would require capital requirements data.

The non-linearity of liquidity shocks is empirically documented in the literature, which usually aims to explain the time variation of liquidity and generally relates it to the state of the financial market. Acharya and Pedersen (2005) note a greater impact of market liquidity shocks on asset prices fluctuations in times of financial distress, which is usually associated with high volatility and illiquidity. Billio et al. (2012) study hedge fund risks using a Markov regime switching model. Their fundings suggest that liquidity shocks are highly episodic and are associated with large and negative return fluctuations. Acharya et al. (2013) study the link between liquidity risk and corporate bond returns using Markov regime switching model. They suggest that the effect of liquidity risk on the corporate bond prices is regime dependent, and is much more vivid during adverse macroeconomic and financial market conditions. Acharya et al. (2013) find the regime dependent liquidity impact on stocks, where the same factors which define regimes in the bond market, are relevant for stock market as well. Degiannakis et al. (2013) also suggest to explore the effect of return dispersion on the dynamics of market liquidity in the state dependent framework.

My paper adds to the literature that explores the non-linearity of market liquidity within regime-switching framework. First I model the joint dynamic of market liquidity and volatility. Second, in contrary to Acharya et al. (2013) and Billio et al. (2012) where regimes are defined in the statistical setting, I identify regimes according to threshold level of the past market liquidity, which allows me to shed light on the critically ”bad” level of liquidity.

Engle et al. (2012) study the potential non-linearity in the joint dynamics of volatility and liquidity of the US Treasury bonds using the multiplicative error model. The authors consider different periods related to the recent financial crisis, specifically focusing on the the dates of important economic announcement and flight to safety episodes. They find that liquidity negatively influences volatility, but the inverse effect varies, depending on the considered price tier. Specifically, the authors find a negative relationship between volatility and liquidity of bonds, which goes in both directions for the first price tier, and intensifies during financial crisis. The authors relate the feedback effect between volatility and liquidity to observed liquidity spirals and high volatility in the crash periods.

My goal in this paper is to explore the dynamics of market liquidity and volatility of stocks during, similarly to Engle et al. (2012), the recent financial crises as well. I consider a longer time period, which includes several episodes of financial downturns, and use a level of past liquidity, estimated in the model, as an indicator to distinguish between favourable and bad market conditions.

I estimate the link between market liquidity and volatility in a two regime Threshold Vector Autoregresive Model (TVAR), formally described by Tsay (1998). I assume that a regime is defined by past unknown level of market liquidity, which I call a trigger or threshold variable. I consider two regimes of volatility and liquidity, which is suggested by the previous literature and is naturally translated into good and bad regime. I estimate the TVAR model for Bank of America (BAC), Kimko Realty Corporation(KIM), Dow Chemical Corporation (DOW), and Ford Motor Company(F) stocks traded on NYSE, AMEX and NASDAQ for the period from January of 2000 to November of 2015. Such long time span allows me to capture the time-variation of liquidity and volatility with a special focus on the recent financial crisis. I consider stocks which come from different industries in order to verify whether my findings are robust to stock specific characteristics.

My results confirm the non-linear link between volatility and market liquidity. I find that a TVAR model of market liquidity and volatility is strongly statistically preferred over the linear alternative based on Hansen (1996) and Hansen (1997) inference procedures and this result is robust across all the stocks in my sample. I estimate the unknown threshold level of market liquidity, which separates two regimes of liquidity and volatility along with its confidence interval. My model identifies the ”bad” or low liquidity regime, which coincides with major financial market declines like the 9/11 attack, WorldCom bankruptcy in September 2002, 2007-2009 financial crisis and Eurozone downturn with a peak in Greek government-debt crisis in May, 2011. My results suggest that the link between stock market liquidity and volatility is stronger and generally more persistent during ”bad” liquidity regimes. The impact of liquidity on future volatility and liquidity is substantially bigger in low liquidity regime. I find that volatility shocks are stock specific in low liquidity regime. However, when volatility affects stock market liquidity in both regimes, it has a bigger impact when liquidity is low. I estimate the responce to shocks over time when the model stay in a particular regime forever. I find that illiquidity and volatility shocks generate exploding volatility and illiquidity in low liquidity regime for all stocks but DOW. These findings might be related to episodes of evaporating liquidty observed in the data and the ’liquidity spiral” described in Brunnermeier and Pedersen (2009).

I estimate non-linear IRF of liquidity and volatility shocks, which happens in a specific regime, when the model can move between regimes, which may lead to size and sign sensitivity of shocks. My findings suggest that a positive shock to illiquidity or volatility increases future illiquidity and volatility. I show that the volatility and illiquidity response to illiquidity shocks is greater, when a shock happens in low liquidity regime. My results demonstrate the sign and size asymmetry of liquidity shocks, such that a large drop in liquidity contribute more to the change in the volatility and liquidity, than the improvement in liquidity to the same extend. Large liquidity drops may signify major adverse events to market players, who usually dislike these and as a result may have a larger impact on price movements. Finally, my analysis shows that even though the explosive reaction of volatility and liquidity to shocks does exist when market liquidity is low, a single shock in the bad regime is not enough to generate liquidity spirals.

In summary, my threshold model of market liquidity and volatility adds to the large body of recent literature that stress the non-linearity of market liquidity in relation to other assets characteristics (Brunnermeier and Pedersen, 2009, Acharya et al., 2013, Acharya and Pedersen, 2005, Billio et al., 2012, Engle et al., 2012, Christoffersen et al., 2014). The results in my paper are consistent with some of the theoretical predictions or related to other empirical findings:

1. There are different regimes in illiquidity and volatility relationship; The link between these variables is stronger during scarce financial conditions.

2. Liquidity and volatility may reinforce each other when liquidity is low, leading to liquidity spirals and flight to quality/liquidity.

3. The impact of volatility on market liquidity varies within the same class of assets.

4. Liquidity shocks are asymmetric.

I believe that the proposed methodology gives useful insights into the necessary features of future models of asset pricing.

The rest of the paper is organized as following. In section 2.2 I specify TVAR model and the estimation approach. Section 2.3 describes data construction and provides preliminary data analyses. Estimation results are presented in Section 2.4. Section 2.5 concludes. Additional figures, description of estimation procedure and the results are presented in the Appendix B.

2.2 Empirical Model

I model the volatility and market liquidity using a TVAR, a multivariate extension of Threshold Autoregressive Model (TAR), which is described in Tsay (1998) and Hubrich and Tersvirta (2013). For example, the simple two regime TVAR model specifies a different autoregressive structure for each of the regimes, and there is a threshold variable that determines which regime is active. The TVAR specification has several advantages. First, it allows capturing the joint dynamics between volatility and market liquidity. Second, it characterizes in relatively simple way potential non-linearities such as regime switching, asymmetric reaction to shocks and existence of multiple equilibria implied either by the theoretical model or empirical observations. Third, it is relatively simple to estimate, and, since at time t the regime state is known, it is relatively easy to use for forecasting, in comparison for instance to other non-linear models, in particular hidden Markov models. Lastly, the trigger variable that governs regimes can itself be an endogenous variable included into the TVAR framework, as it is in this paper, which allows to generate the distribution of the regimes after the shock happened.

I consider the following TVAR model given by:

$$Y_t = (A^1 + \Phi^{1,p_1} Y_{t-1}^{p_1}) I(X_{t-d} \leq \gamma) + (A^2 + \Phi^{2,p_2} Y_{t-1}^{p_2}) I(X_{t-d} > \gamma) + \epsilon_t, \tag{2.1}$$

where Y_t is a (2×1) vector of liquidity and volatility, A^1 and A^2 are (2×1) constant coefficient vectors, Φ^1 and Φ^2 are $(2 \times 2p_1)$ and $(2 \times 2p_2)$ parameter matrices, where p_1 and p_2 are the number of lags in regime 1 and 2 respectively, $I()$ is the indicator function, X_t is a random variable, which determines the regime, i.e. threshold variable, d is a lag parameter, γ is unknown level of threshold variable, which separates two regimes, and ϵ_t is a sequence of independent multivariate random vectors, such that $E(\epsilon_t) = 0$ and $Var(\epsilon_t) = \Sigma$, $\forall t$. I assume that Σ is positive definite and has a full rank, that $E(\epsilon_t | X_s) = 0$ for all $s \leq t$ and that the sequence (ϵ_t, X_t), $t \geq 1$, is *iid* and Normal.

I consider the TVAR model in (2.1) with two regimes. In general, it is possible to define a model with multiple regimes. This framework requires a large number of parameters to be estimated, that would be computational burdensome. In addition, a two-regime model makes it easy to understand the economic meaning behind each regime, i.e. one regime would be associted with favorable conditions, the other regime with bad state. Lastly, the two-regime model can accomodate the two equilibria model proposed by Brunnermeier and Pedersen (2009).

The previous literature associates the non-linearity between volatility and liquidity with the state of the financial market, which implies that there are several candidates for the choice of threshold variable (Engle et al., 2012, Brunnermeier and Pedersen, 2009, Acharya et al., 2013). I consider liquidity, volatility, returns and VIX as potential candidates for the trigger variable. All variables, except for liquidity result in a poorly behaved log-likelihood function or suggest a corner solution as an optimal point for the threshold level, which makes optimization in (2.2) somewhat ambiguous. Other extensions of the model are possible in the form of weighted linear combinations of the trigger variables and levels, but this approach would only produce statistical gains, making economic interpretation complicated. Thus, I use liquidity as the only choice for the trigger variable in this paper. Liquidity co-moves with the market, such that financial downturns are associated with evaporating liquidity, which was observed during the last financial crisis, in particular Thus, a regime, where liquidity is lower than the threshold γ would be naturally related to the "bad" illiquid state, and the situation when liquidity is above the threshold γ would be described as "good " liquid regime. Also, liquidity is known to have strong asymmetric effects, affecting financial markets more when illiquidity is high. Consequently, choosing liquidity as a threshold variable would enable my model to evaluate the effect of liquidity shocks when it is the most needed.

The estimator $\hat{\theta} = [\hat{\gamma}, \hat{d}, \hat{p}_1, \hat{p}_2, \hat{\Phi}, \hat{\Sigma}]$, where $\hat{\Phi} = [\hat{\Phi}^1, \hat{\Phi}^2, \hat{A}^1, \hat{A}^2]$ jointly maximizes the log likelihood function, i.e.:

$$\hat{\theta} = \underset{\gamma \in \Gamma, d \in D, p_1, p_2 \in P, \Phi, \Sigma}{\arg\max} LLF(\gamma, d, p_1, p_2, \Phi, \Sigma), \tag{2.2}$$

where LLF is log-likelihood function of TVAR model in (2.1). Since ϵ_t are assumed to be *iid* and Gaussian, I estimate the model in (2.1) equation by equation using the least squares approach. I perform optimization in (2.2) using a grid search over possible values of treshold level $\gamma \in \Gamma$. Here I construct Γ by trimming the top and bottom 10% of the original distribution of trigger variable X_{t-d} in order to ensure that each regime contains a sufficient number of observations for estimation, i.e. $min\{\gamma : |\gamma \in \Gamma\} = X_{t-d}^{10}$ and $max\{\gamma : |\gamma \in \Gamma\} = X_{t-d}^{90}$, where X_{t-d}^{10} and X_{t-d}^{90} are 10^{th} and 90^{th} quantiles of the distribution of X_{t-d}

rspectively. [5]. In addition, I consider every 1% quantile of the trimmed X_{t-d} distribution in Γ sample to speed up the maximization process, i.e. numver of observation in Γ is equal to 100. D is a set of delay parameters. I consider the maximum of 10 lags of the trigger variable. I chose p_1 and p_2 among $P = 20$ possible lags in each regime for each considered value of and τ and d. Sin and White (1971) show that this method is consistent in the sense that the correct lag orders will be selected with probability one asymptotically. Optimal models are selected based on the Schwartz information, which gives larger penalties for bigger sample.

In summary, I use the following algorithm to the maximize LLF function:

1. Number of lags in each regime, p_1 and p_2 are estimated by minimizing the Schwarz information criterion for every possible threshold value $\gamma \in \Gamma$ and delay parameter $d \in D$.

2. The model in (2.1) is linear in Φ and Σ for each γ and delay lag d. Maximization of LLF conditional on delay parameter uniquely yields $\hat{\gamma}$:

$$\hat{\gamma} = \arg\max_{\gamma \in \Gamma} LLF(\gamma, d|d), \tag{2.3}$$

   where maximization is performed by grid search over Γ.

3. Optimal delay parameter $\hat{d}$ maximizes the following LLF defined at optimal threshold level $\hat{\gamma}$:

$$\hat{d} = \arg\max_{d \in D} LLF(\hat{\gamma}, d). \tag{2.4}$$

[5]The minimum number of observations in each regimes is required for asymptotic results (Hansen, 1999). Although, it is unclear how to trim trigger variable sample, Hansen (1999) recommends to use 10% quantile.

4. Estimates of the matrices of parameters $\hat{\Phi}$ and variance-covariance matrix $\hat{\Sigma}$ maximizes LLF defined at optimal value of $\hat{\gamma}$ and $\hat{d}$:

$$[\hat{\Phi}, \hat{\Sigma}] = \arg\max_{\Phi,\Sigma} LLF(\hat{\gamma}, \hat{d}, \Phi, \Sigma). \tag{2.5}$$

I follow the Hansen (1996, 1997) approach to estimate the conference interval of γ. Hansen (1997) and Chan (1993) illustrate that $\hat{\gamma}$ is a consistent estimate of the true parameter. Hansen (1997) derives the limiting distribution of the *LR* statistic to make inference for threshold parameter and illustrates a convenient way to build a confidence interval for the threshold level. Hansen (1997) recommends to build the asymptotic confidence interval by inverting $LR(q)$ statistic. Thus, I construct a $LR(q)$ statistic testing a hypothesis $H_0 : \gamma = q$ as follows:

$$LR(q) = 2(LLF_1(\hat{\gamma}) - LLF_1(q))), \tag{2.6}$$

where $LLF_1(\hat{\gamma})$ is log-likelihood of model in (2.1) estimated at $\hat{\gamma}$ and $LLF_1(q)$ is estimated at each value of trigger variable q. Then, a confidence interval given significance level α is defined as:

$$\Gamma = \{q : LR(q) > c(\alpha)\} \qquad where \qquad c(\alpha) = -2ln(1 - \sqrt{1-\alpha}). \tag{2.7}$$

2.3 Data

I consider a datasample from January, 1, 2000 to November, 30, 2015, which includes 4002 trading days. This time period includes various states of financial market and allows to investigate possible non-linearities in the time series of market liquidity and volatility.

I investigate Bank of America(BAC), Kimco Realty Corporation (KIM), Dow Chemical Corporation (DOW) and Ford Motor Company (F) stocks traded on NYSE, AMEX and NASDAQ stock markets in this study. This data is available from the Center for Research in Security Prices. I include securities which come from different economic sectors to check how robust my findings are to the stock's specific characteristics.

I compute daily market illiquidity as a relative bid-ask spread calculated as a ratio between difference of daily highest ask and lowest bid and their midpoint:[6]

$$ILLIQ_t = \frac{Ask_t - Bid_t}{(Bid_t + Ask_t)/2}. \tag{2.8}$$

This measure of liquidity is related to the market tightness, since it measures the ability to buy or sell securities at about the same price and time. It is related to the transaction cost that is necessary inccured when completing a trade. The smaller the bid-ask spread is the more liquid the market is. This liquidity measure captures the cost per dollar traded and is generally considered to be a good measure of liquidity (Goyenko et al. (2009), Ait-Sahalia and Yu (2013)). It is the most widely used definition in the literature and it allows to compare liquidity measures across different stocks. Another advantage of computing illiquidity following formula in (2.8) is that it estimates liquidity even when no trade is completed for that day.

I recover the (latent) daily volatility from a GARCH(1,1) model defined as the following:

$$r_t = \sigma_t z_t, \tag{2.9}$$

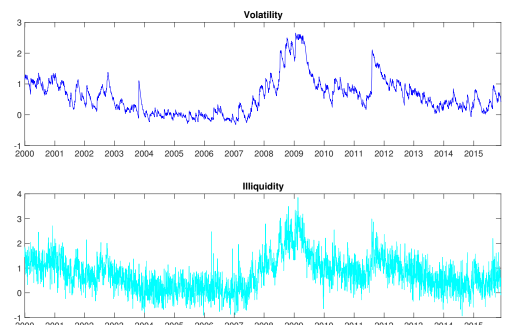
$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2, \tag{2.10}$$

[6]The convention to measure illiquidity instead of liquidity comes from the literature. Thus I will refer to illiquidity whenever I talk about market liquidity throughout the rest of the paper.

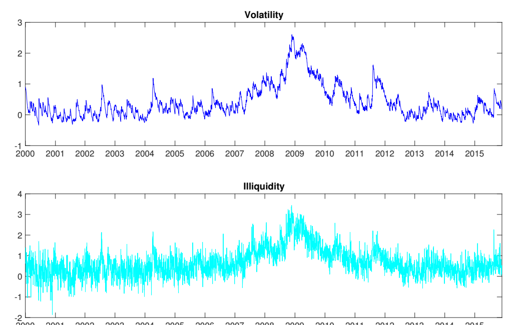
where $z_t \sim IIN(0,1)$ is the shock to returns and σ_t is the conditional volatility of return. Finally, to mitigate the impact of extremes on the estimation and for the sake of better distribution properties I consider the log-transformed liquidity and volatility series, thus $ILLIQ_t$ denotes *log* of (2.8) and σ_t is *log* of volatility series obtained from the GARCH model in (2.10) throughout this paper.

Figure 2.1: Illiquidity and Volatility

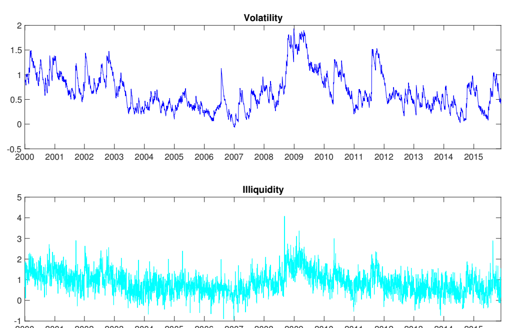
(a) Bank of America



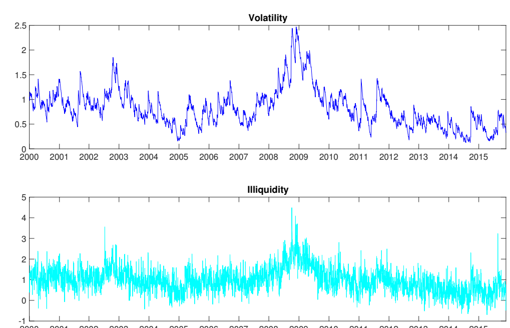
(b) Kimco Realty Corporation



(c) Dow Chemical Corporation



(d) Ford Motor Company



Notes: This figure shows the historical paths of illiquidity and volatility of BAC, KIM, DOW and F stocks during Jan, 2000 - Nov, 2015.

Figure 2.1 shows the time series dynamics of volatility and liquidity for stocks of Bank of America, Kimco Realty Corporation, Dow Chemcal Corporation and Ford Motor Company. Several observations are immidiate from this graph. First, stock market liquidity and volatility shows a pattern of time variation and clustering, though the persistence is more vivid for volatility. Second, markets are more liquid when stock returns are less volatile. Third, there is clear time series dependency between volatility and liquidity, and both series co-moves with the market. Illiquidity and volatility are getting worse around important economics events and financial downturns such as 9/11 attack, WorldCom bankruptcy in September 2002, 2007 - 2009 financial crisis, Eurozone downturn with a peak in Greek government-debt crisis in May, 2011. Liquidity evaporated during 2007 - 2009 financial crises, which could be associated with capital availability problems faced by the market participants, as is suggested by the model of Brunnermeier and Pedersen (2009) and described as liquidity spirals.

Table 2.1 presents the summary statistics of $ILLIQ_t$ and σ_t. Both series are relatively symmetric, where liquidity skewness is a bit closer to zero and varies from 0.51 to 0.63, and volatility skewness varies from 0. 85 to 1.56. Liquidity and volatility kurtosis are not far away from 3, and ranges from 3.59 to 5.40. The null hypotheses of presence of unit root are rejected for all series based on Dickey-Fuller test, with 5% significance level.

Table 2.1: Summary Statistics

	$ILLIQ_t$				σ_t			
	BAC	KIM	DOW	F	BAC	KIM	DOW	F
Mean	0.76	0.66	0.86	0.98	0.72	0.41	0.67	0.82
St Dev	0.69	0.67	0.55	0.60	0.59	0.53	0.38	0.39
Skewness	0.62	0.83	0.51	0.62	1.09	1.56	0.85	1.07
Kurtosis	3.59	4.17	3.61	4.22	4.31	5.40	3.62	4.80
D-F Test	0.00	0.00	0.00	0.00	0.02	0.00	0.03	0.04
L-B Test Test 10 lags	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
L-B Test Test 20 lags	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
L-B Test Test 30 lags	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Normality J-B Test	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
ARCH Test	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Notes: p-value is reported for D-F, L-B, Normality J-B and ATCH Test

2.4 Estimation Results

2.4.1 Estimation of TVAR Model of Liquidity and Volatility

Table 2.2 shows the estimation results for the TVAR model in (2.1). The estimated quantile of $\hat{\gamma}$, shown in Column 4, varies between 58 - 78 across the stocks. It is the smallest for BAC, indicating that liquidity and volatility dynamics of BAC are most sensitive to liquidity levels. In other words, a smaller shock of liquidity will make the model in (2.1) to be in the low liquidity regime for BAC. Since BAC belong to banking industry, this result might suggest that financial institutions should monitor the level of liquidity closely, and it is a part of Basel III requirements now.

The confidence interval of $\hat{\gamma}$ (Column 3) is sufficiently tight, which translates into precise estimate of threshold parameter and provides additional evidence in favour of the TVAR model. For further evidence refer to Figure B.1 in the Appendix B, which shows the estimated threshold level and its confidence interval.

Table 2.2: TVAR: Estimation Results

Stock	$\hat{\gamma}$	95% Confidence Interval of $\hat{\gamma}$*	Quantile of $\hat{\gamma}$	$\hat{d}$	$\hat{p}_1$	$\hat{p}_2$
BAC	0.83	$[0.81, 0.86]$	58	1	1	2
KIM	1.04	$[1.04, 1.04]$	77	1	2	2
DOW	1.24	$[1.24, 1.24]$	78	1	2	2
F	1.22	$[1.10, 1.40]$	69	1	1	2

Notes: this table shows the estimated parameters of model (2.1): $\hat{\gamma}$ is threshold level of trigger variable $ILLIQ$; $\hat{d}$ is a lag value of trigger variable $ILLIQ$; $\hat{p}_1$ and $\hat{p}_2$ is a number of lags in regime 1 and 2, respectively estimated with BIC statistics;** Confidence interval is build using Hansen (1997) approach.

The estimated value of the delay parameter $\hat{d}$ for the trigger variable $ILLIQ_{t-d}$ is equal to 1 for all stocks (Column 5 of Table 2.2). Column 6 and 7 present the number of lags within each regime, which minimizes the Schwarz information criterion. The number of lags in high liquidity regime varies from 1 to 2, whereas the low liquidity regime is estimated to have 2 lags.

Table 3.1 illustrates the summary statistics for illiquidity and volatility in high and low liquidity regimes. The high liquidity regime has on average higher liquidity and lower volatility. Iliquidity and volatility at least doubled in low liquidity regime, and for some stocks the effect is even bigger. Thus, high liquidity regime could be considered as the ”good” state of the market, and low liquidity regime as the ”bad” state.

Table 2.3: Summary Statistics Across Regimes of Liquidly

	High Liquidity		**Low Liquidity**	
	Mean	St Dev	Mean	St Dev
BAC				
$ILLIQ_t$	0.40	0.48	1.25	0.53
σ_t	0.28	0.34	1.06	0.57
KIM				
$ILLIQ_t$	0.43	0.47	1.40	0.68
σ_t	0.20	0.28	1.09	0.59
DOW				
$ILLIQ_t$	0.70	0.47	1.36	0.51
σ_t	0.55	0.29	1.08	0.38
F				
$ILLIQ_t$	0.78	0.48	1.43	0.59
σ_t	0.68	0.28	1.15	0.41

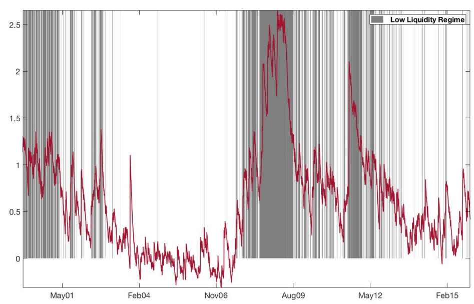
Figure 2.2 shows the time-series of volatility of different stocks through high and low liquidity regimes. The low liquidity regime coincides with 9/11 attack, WorldCom bankruptcy in September 2002, 2007 - 2009 financial crisis and Eurozone downturn with a peak in Greek government-debt crisis in May, 2011. The estimated low liliquidity regime

is longer during recent financial crisis for BAC and KIM stocks, which is expected since these stocks represent financial and real estate sectors, which lost the most during this period. Regime clustering is more vivid for BAC and KIM. Since the regime is defined by the past liquidity level, regime clustering would be related to highly persistent liquidity series. Regime clustering gives a rationale to investigate how shocks to liquidity/volatility affect liquidity or volatility when systems remains in a specific regime forever. Section 4.4 refer to this question in more detail, where I estimate IRF of model in (2.1) within each specific regime.

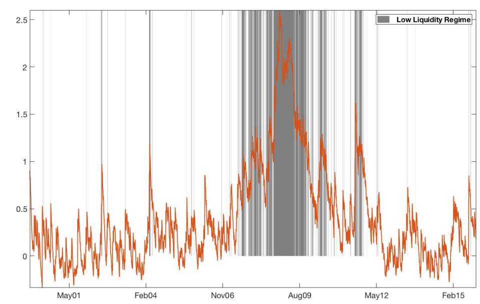
Short lived episodes of low liquidity during generally favorable financial conditions are also present, especially for DOW and F stocks. This observation is related to empirically documented sudden dry-ups of market liquidity. Brunnermeier and Pedersen (2009) explain temporary drops in liquidity by strong beliefs about temporality of price shocks and good availability of funding, such that a shock to liquidity does not lead to liquidity spiral.

Figure 2.2: Volatility and Regime Changes

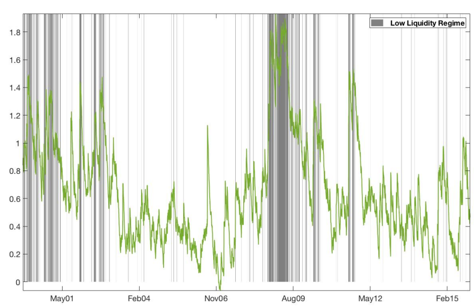
(a) Bank of America



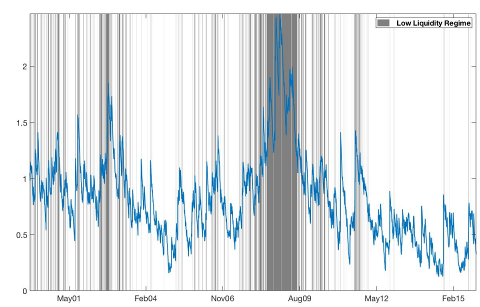
(b) Kimco Realty Corporation



(c) Dow Chemical Corporation



(d) Ford Motor Company



Notes: This figure shows the volatility series of BAC, KIM, DOW, F stocks trough low and high liquidity regimes. The shaded area represents the estimated low liquidity regime, or when $ILLIQ_{t-d} > \hat{\gamma}$.

The results of the estimation of TVAR model (1) is presented in the Tables B.1 - B.4 in Appendix B. Most of the estimated coefficients are positive and significant, and negative coefficients are either small and fairly insignificant. This gives reason to conclude that there is generally a positive time series dependency between volatility and liquidity, which is consistent with the existing theoretical and empirical literature (Benston and Hagerman, 1974, Amihud and Mendelson, 1989, Brunnermeier and Pedersen, 2009, Engle et al., 2012).

Past values of illiquidity have stronger impact on illiquidity when liquidity is already low and this result is stable for all stocks (Column 2 of Tables B.1 - B.4). Iliquidity significantly increases with volatility when liquidity is high. Volatility does not have a significant impact on market liquidity in the low liquidity regime for BAC and DOW. Since volatility affects liquidity through the expectation of the future volatility, risk of mispricing and capital requirements to support trade, in the low liquidity regime these indicators are already high, such that the marginal volatility change might not significantly impact liquidity. When volatility affects stock market liquidity in both regimes, its impact is bigger in the low liquidity regime (KIM and F stocks). Generally, I conclude that volatility impact on market liquidity is stock specific in low liquidity regime, and it is potentially interesting to consider a bigger stock sample in the future to explore this result in more details, or to investigate other channels through which volatility impacts liquidity.

Column 4 of Tables B.1 - B.4 shows the estimated coefficients of model (2.1) for the volatility dynamics. The results show that illiquidity significantly and positively affects volatility, and its impact is greater in the low liquidity regime for all stocks. My findings therefore support the hypothesis that liquidity is more important for transaction smoothing and price fluctuations when liquidity is already low.

Before moving to the analysis of the impact of shocks, I would like like to address one of the most important issues in the threshold literature, specifically testing linear model against TVAR alternative, that would help to justify my approach.

2.4.2 Testing for Non-Linearity

An important question is whether the TVAR model is statistically preferred over the linear VAR. Hence, before moving further to the discussion of the dynamics generated by the estimated TVAR model, I would like to address this question in order to justify my approach. The null hypothesis of the linear VAR model can be formulated as follows $H_0 : A^1 = A^2$ and $\Phi^1 = \Phi^2$ against the alternative that at least one of the matrices are not equal. The inference procedure would be straightforward if the threshold level is known. However, in my setting the threshold value is not known, but must be estimated. This means that the inference is complicated by the fact that the threshold value γ is not identified under the null hypothesis of no threshold (Davies, 1977, 1987), thus the corresponding assymptotic distribution of the test statistc is not chi-square. Hansen (1996, 1997) demonstrates that the asymptotic distribution can be approximated by a bootstrap procedure. I follow the non-standard inference procedure proposed by Hansen (1996, 1997) with an extension to the multivariate case. I calculate *LR* statistics for the testing of no difference between regimes for each possible threshold value. Then test statistics is defined as *supLR*, which is maximum over all possible threshold levels:

$$LR = sup_{\gamma\in\Gamma} 2(\widehat{LLF}_1(\gamma) - \widehat{LLF}_0), \tag{2.11}$$

where $\widehat{LLF_1}(\gamma)$ and $\widehat{LLF}_0$ are log-likelihood functions estimated under the null and alternative hypothesis for each value of γ. Hansen (1996) suggest to compute the empirical distribution of *LR* from the following bootstrap algorithm:

1. Estimate *LR* statistics;
2. Form bootstrap empirical distribution:
   (a) Draw residuals jointly with replacement estimated under the null hypothesis (from VAR model);
   (b) Generate new bivariate random variable Y_t^* under the null;
   (c) Estimate LR^* statistics using generated data Y_t^*;
   (d) Repeat steps a)-c) to create bootstrap sample of LR^* statistics.
3. Calculate the p-value as a share of $LR^* > LR$ in total bootstrap sample.

I performed the described procedure with 500 bootstrap replications. The estimated p-value for null hypothesis $H_0 : A^1 = A^2$ and $\Phi^1 = \Phi^2$ is 0.00 for all stocks, thus I reject the linear VAR model of liquidity and volatility against TVAR model in (2.1) with 1% level of significance.[7]

2.4.3 Regime Dependent Impulse Response

The impulse response function (IRF) describes the response of one variable to the shock to an another variable in a system of multiple variables. IRFs are a useful tool in analysing the dynamic properties of autoregressive models. Since my model has variable coefficients, the general response to the shocks might not longer be linear to size and sign of the shock, and it might depend on the history of the regimes. I start the analysis by considering a specific

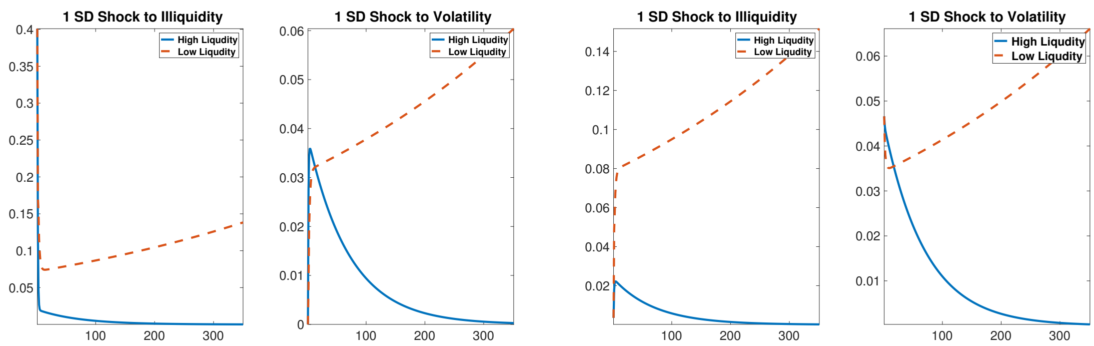
[7] I used 500 bootstrap replications. A small bootstrap sample may result in the extremely low p-value.

regime scenario, when the model starts and remains in a particular regime forever. Such regime dependent IRFs will allow to better demonstrate the difference between regimes in the dynamic perspective. Also, the persistence of the trigger variable, $ILLIQ_t$ and the resulting regime clustering gives additional rational to explore response of liquidity and volatility to shocks within each regime. This IRF would be fully described by the coefficients of a correspondent regime, and can be easily calculated as a IRF of standard VAR model. I orthogonalize shocks using the Cholesky decomposition of the estimated covariance matrix of innovations. In general it is hard to address the ordering of market liquidity and volatility. Here, I consider the market liquidity first, though the alternative ordering produces similar results.

Figures 2.3 - 2.6 show the estimated response of a one standard deviation shock to liquidity or volatility when the system stays in a particular regime forever. The striking conclusion from Figures 2.3 - 2.6 is that any shock to liquidity or volatility makes these variables to explode in low liquidity regime for all stocks but DOW. The impact of liquidity shocks is always bigger when liquidity is low. In contrast, volatility shocks are more pronounced in the good regime during the first week after the shock, but they escalate very fast in the low liquidity regime.

The impact of a liquidity shock on liquidity is close to a 0.4 increase in illiquidity which is roughly equal to 60% - 70% of a standard deviation of illiquidity, and it declines rapidly on the next day for both regimes (Left of Panel (a) in Figures 2.3 - 2.6). The impact of the shock in the high liquidity regime disappear over time. In contrast, liquidity start growing in the low liquidity regime and the effect of the shock is around 35% of the initial impact on illiquidity 350 days after the shock for BAC and KIM, and 64% of initial impact on illiquidity for F. This translates into an increase in $ILLIQ_t$ of around 20% of standard deviation for BAC and KIM and by 45% of standard deviation for F 350 days after the shock. The liquidity shock has a longer lasting effect on illiquidity in bad regimes than in good regimes for DOW, however the impact of the shocks have almost disappeared 150 days after the shock in both regimes.

Figure 2.3: Regime Dependent Impulse Response. BAC.



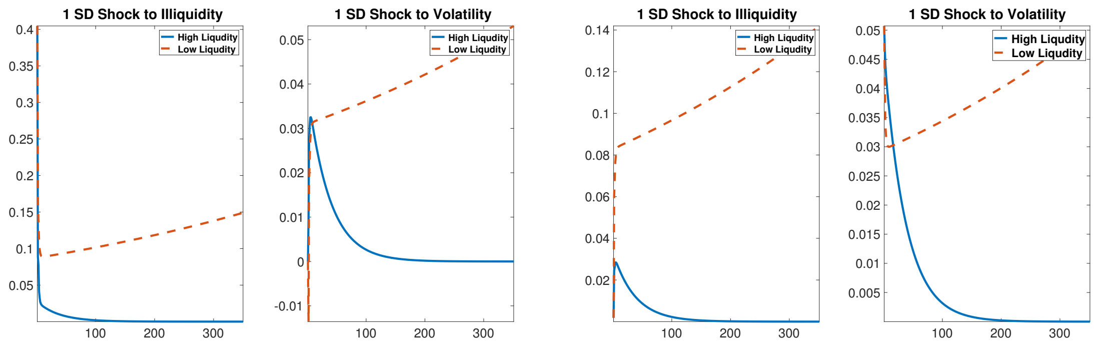
(a) Illiquidity Response (b) Volatility Response

Notes: This figure shows the response of stock market illiquidity and volatility of BAC stock to one standard deviation shock (1 SD) of liquidity and volatility. Blue line shows the response when start and remain in high liquidity regime. Red dashed line shows the response, when model start and remain in low liquidity regime.

A one standard deviation of volatility shock increases future illiquidity by 0.03 - 0.04 for BAC, KIM and F during the first week after the shock, which translates into an increase of 5% of standard deviation (Right of Panel (a) in Figures 2.3, 2.4 and 2.6). Then it takes 55 - 110 days for illiquidity to return to 20% value of initial increase in illiquidity, when the model stays in the high liquidity regime. The impact of volatility shocks on illiquidity grows rapidly over time in the low liquidity regime. The effect of the shock almost doubles for BAC and KIM, and triples for F 350 days after of the shock. This increase is equal to around 9% of a standard deviation for BAC and KIM and 16% of a standard deviation of illiquidity for F.

This dynamic is different for DOW stock, where a one standard deviation volatility

Figure 2.4: Regime Dependent Impulse Response. KIM.



(a) Illiquidity Response (b) Volatility Response

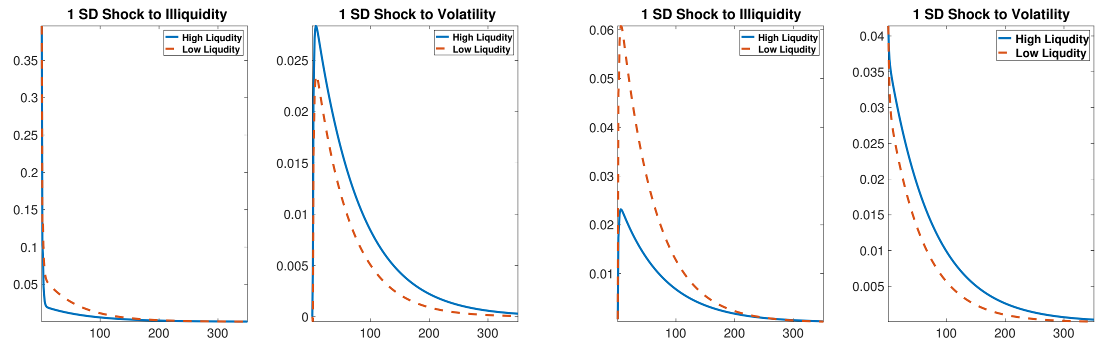
Notes: This figure shows the response of stock market illiquidity and volatility of KIM stock to one standard deviation shock (1 SD) of liquidity and volatility. Blue line shows the response when start and remain in high liquidity regime. Red dashed line shows the response, when model start and remain in low liquidity regime.

shock rises stock market illiquidity by 0.03 in the high liquidity regime and by 0.02 in the low liquidity regime, which is equal to to 5% and 4% of a standard deviation of illiquidity respectively (Right of Panel (a) in Figure 2.5). The persistence of the shock is similar in both regimes, such that it takes 116 days and 91 days for the impact of the shock to return to its 20% values in good and bad regimes, respectively.

A one standard deviation shock to illiquidity rises future volatility by approximately 0.02 and 0.08 during the first week after the shock, which amounts to 4% - 5% and 13% -16% of a standard deviation and in high and low regimes, respectively (Left of Panel (b) in Figures 2.3 - 2.6). The effect of the shock decays fast in the high liquidity regime, and converge to its 20% of the initial value in 55 - 110 days after the shock for BAC, KIM and F, respectively. Volatility grows rapidly after liquidity shock in low liquidity regimes for these stocks. The impact of the shock almost doubles in 350 days post shock for BAC and KIM and triple for F, such that volatility increases by 26% - 61% of a standard deviation.

A one standard deviation shock to volatility rises future stock volatility by 0.04 - 0.06,

Figure 2.5: Regime Dependent Impulse Response. DOW.

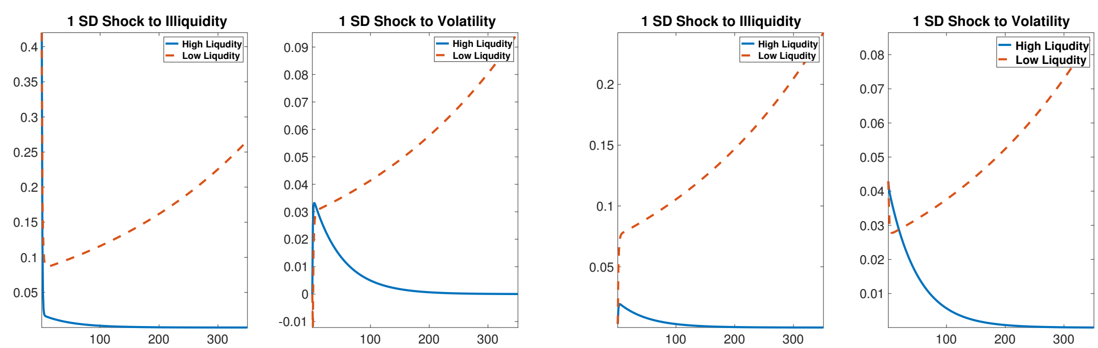


(a) Illiquidity Response

(b) Volatility Response

Notes: This figure shows the response of stock market illiquidity and volatility of DOW stock to one standard deviation shock (1 SD) of liquidity and volatility. Blue line shows the response when start and remain in high liquidity regime. Red dashed line shows the response, when model start and remain in low liquidity regime.

Figure 2.6: Regime Dependent Impulse Response. F.



(a) Illiquidity Response

(b) Volatility Response

Notes: This figure shows the response of stock market illiquidity and volatility of F stock to one standard deviation shock (1 SD) of liquidity and volatility. Blue line shows the response when start and remain in high liquidity regime. Red dashed line shows the response, when model start and remain in low liquidity regime.

or by 8% - 11% of a standard deviations (Right of Panel (a) in Figures 2.3 - 2.6). The response of volatility decays, when liquidity is high, and grows when liquidity is low. The effect of the shock increases by 100% - 200% in the low liquidity regime 350 days after the shock, which corresponds to an increase of 10% - 22% of a standard deviation of volatility for BAC, KIM and F. The impact of the shock decays to its 20% value in 55 - 110 days in high liquidity regimes for these stocks.

The dynamics of volatility response to shocks is different for DOW (Panel (b) in Figure 2.5). The effect of a liquidity shock is bigger in the low liquidity regime, whereas the effect of volatility shocks are the same. A one standard deviation of illiquidity shock will increase volatility by 6% and 16% of standard deviation in high and low regimes respectively. A volatility shock increases volatility by approximately 0.04, which is equal to 11% of a standard deviation of volatility in both regimes. The volatility response to shocks declines over time, and it disappears faster in the low liquidity regime. The effect of a liquidity shock reaches its 20% value 116 and 89 days after the shock in high and low regimes, respectively. The volatility shock converges to its 20% value in 112 and 76 days in high and low regimes, respectively.

In summary, my analysis of the impact of shocks when the model stays in one regime forever demonstrates the non-linearity in volatility and liquidity interactions. Figures 2.3 - 2.6 show that volatility and liquidity are more sensitive to their own shocks in the low liquidity regime for BAC, KIM and F. The effect of the shocks declines quickly in the high liquidity regime. Shocks lead to evaporating liquidity and growing volatility in the low liquidity regime. Liquidity/volatility response to the shocks is among the fastest growing in low liquidity regime, when the impact of the shock doubles or even triples past 350 period after the shock. Thus, the reinforcing effect between liquidity and volatility in the low liquidity regime, generated by the model, could be related to liquidity dry ups observed in the data and liquidity spirals described in the literature. The feedback effect between volatility and liqudity that leads to liquidty spirals and growing volatility is well described in the model of Brunnermeier and Pedersen (2009), and is associated with the increase in capital requirements as liquidity worsen. Brunnermeier and Pedersen (2009) also explains that a similar spiral effect might not exist, when market participants consider liquidity movements as temporary. Thus, margin requirements do not increase due to lower liquidity, which might explain decaying shocks for DOW, and in high regimes for other stocks. Lastly, I notice that this spiral effect is the biggest for F, which is the most illiquid and volatile of the stocks. This observations could be related to the theoretical predictions of Brunnermeier and Pedersen (2009), where liquidity shocks are stronger for volatile and illiquid securities.

2.4.4 Non-Linear Impulse Response

The IRF of non-linear models is much more complicated than in linear models. Since shocks can lead to switches between regimes, it is not possible to construct the Wold decomposition to easily compute the IRF. Consequently, IRFs of non-linear models do not preserve the convenient feature of linearity to the sign and the size of the shock, and they might depend on the initial condition Ξ_{t-1} as shown in Potter (1994) and Koop et al. (1996). I follow the approach of Koop et al. (1996) to compute the non-linear IRF, which is formally defined as

$$IRF_i = E(Y_t|\Xi_{t-1}, e_t^i) - E(Y_t|\Xi_{t-1}), \tag{2.12}$$

where Ξ is the information set at time $t-1$, e_t^i is a realization of an exogenous shock to a particular variable i. [8] The IRF defined in (2.12) depends on initial conditions and the size

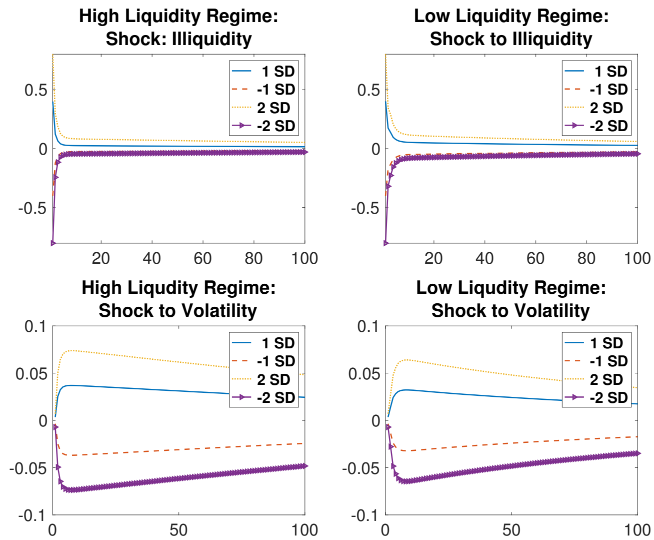
[8] Koop et al. (1996) refers to the non-linear IRF specified in (2.12) as the Generalized Impulse Response Function.

of the shock. Thus, to fully describe the dynamic of the model, the IRF in (2.12) should be estimated for different values of the shock, as well as specific initial conditions. Also, the IRF of non-linear models is sensitive to the specific ordering of the shocks in the same way as the IRF of standard linear VAR model. I follow the same ordering I use in the section 2.4.4. for the regime dependent IRF, i.e. keeping market liquidity first.

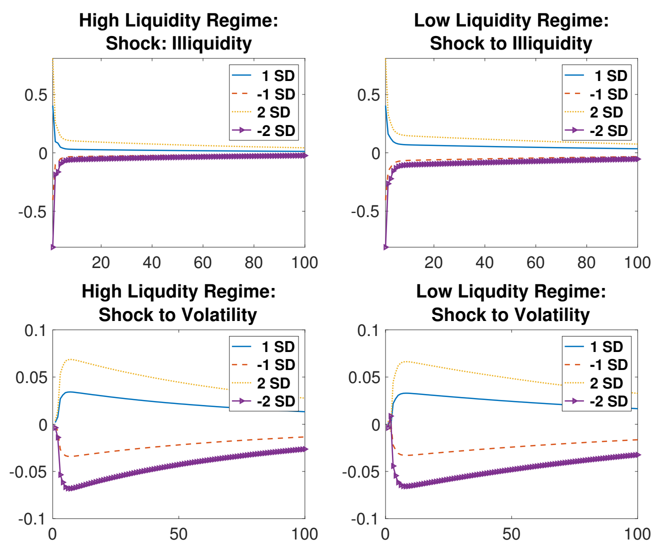
I estimate the non-linear IRF assuming the model starts in low or high liquidity regimes, but it moves between regimes afterwards. First, I randomly pick a vector of initial conditions $\xi_b, b = 1, .., 500$ from a set of possible histories $\Xi_{t-1} \in \{high, low\}$. Then, for each initial condition I randomly draw a series of residuals $u_{t+h}, h = 0, .., l$, with replacement and compute the baseline path of $Y_{t+h}, h = 0, 1, .., l$ of the estimated model. To preserve the joint distribution of the residuals I draw residuals for $ILLIQ_t$ and Vol_t simultaneously. Assuming there is a shock at time t to the variable $i, i \in \{1, 2\}$, I replace the residual u_t^i with the value of shock e_t^i. In order to account for possible asymmetries, I assume that the shocks can take values of ± 1 and ± 2 of a standard deviation of the residuals. I repeat this procedure for each history, i.e. 500 times to estimate the conditional expectations in (2.12). Then I average calculated IRFs over initial conditions to get an non-linear IRF for each regime. A detailed procedure for how the non-linear IRF is calculated is described in Appendix B.

Figure 2.7 - 2.10 show the response of illiquiidty and volatility to shocks, when the model evolves between regimes after a shock occurs. I notice that there is a positive link between volatility and illiquidity such that any positive shock to illiquidity make volatility go up, and vise versa for all stocks, which is consistent with spells of liquidity disappearing and volatility rising during crisis periods. Liquidity shocks on liquidity and volatility are bigger and longer when liquidity is in high demand. A single liquidity shock happening in

Figure 2.7: Liquidity Response, BAC and KIM



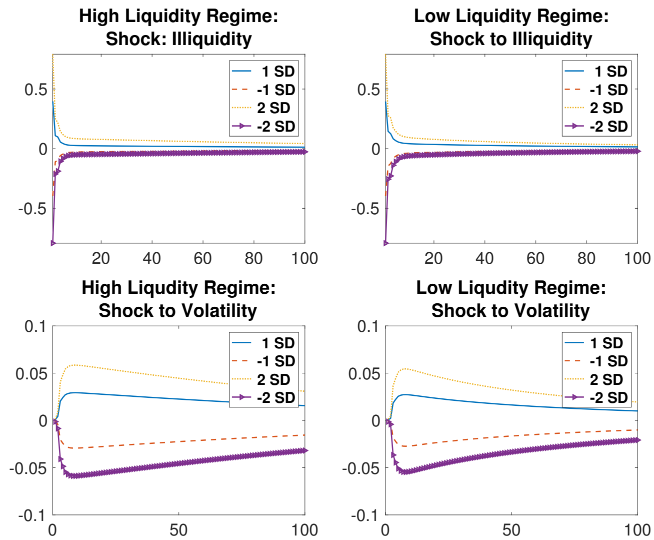
(a) Bank of America



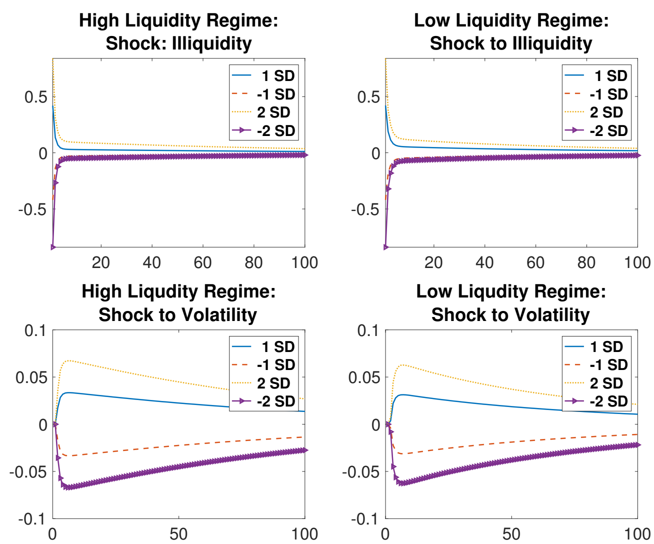
(b) Kimko Realty Corporation

Notes: This figure shows the response of stock market illiquidity to shocks of liquidity and volatility, when model start in low or high liquidity regime. 1 SD shows the response to one standard deviation shock. 2 SD shows the response to two standard deviation shock. -1 SD shows the response to minus one standard deviation shock. -2 SD shows the response to minus two standard deviation shock.

Figure 2.8: Liquidity Response, DOW and F



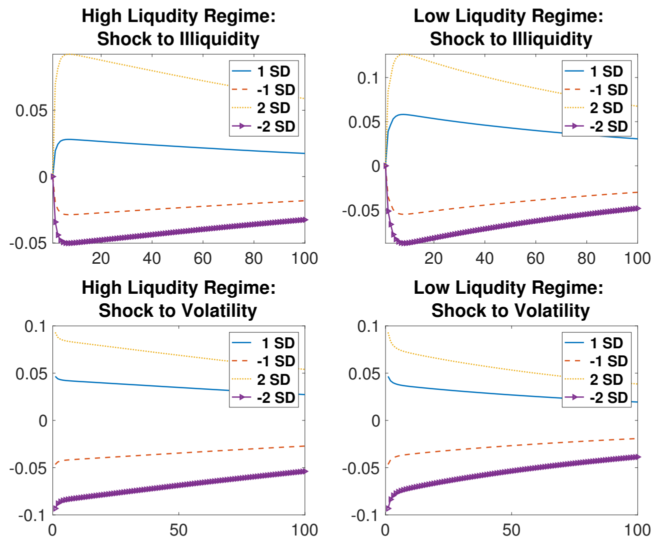
(a) Dow Chemical Corporation



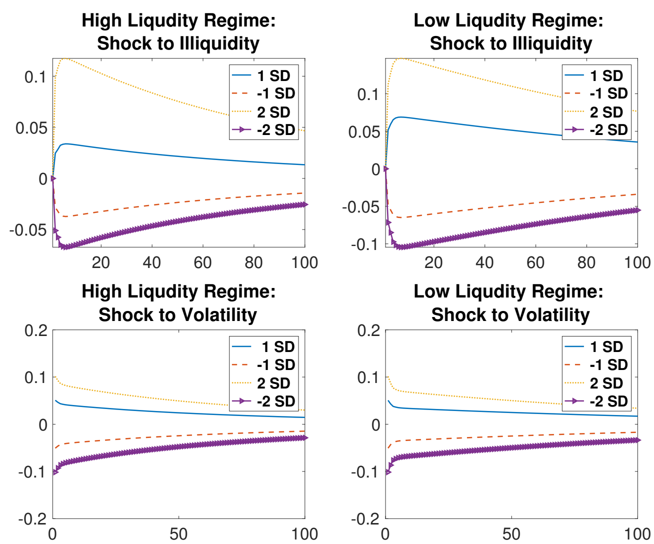
(b) Ford Motor Company

Notes: This figure shows the response of stock market illiquidity to shocks of liquidity and volatility, when model start in low or high liquidity regime. 1 SD shows the response to one standard deviation shock. 2 SD shows the response to two standard deviation shock. -1 SD shows the response to minus one standard deviation shock. -2 SD shows the response to minus two standard deviation shock.

Figure 2.9: Volatility Response, BAC and KIM



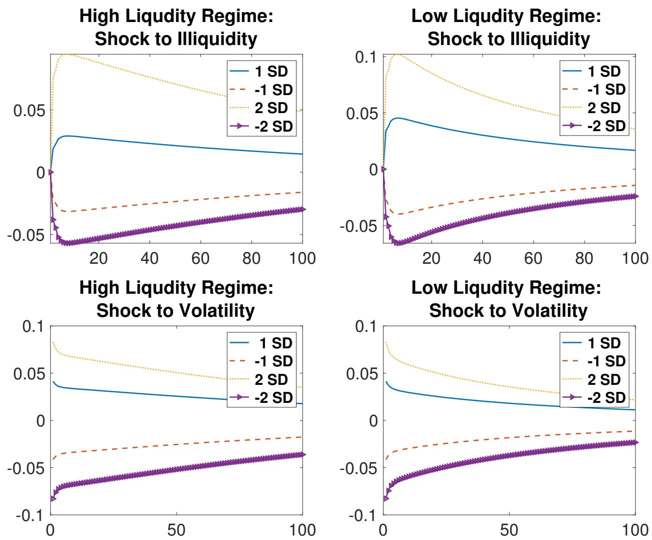
(a) Bank of America



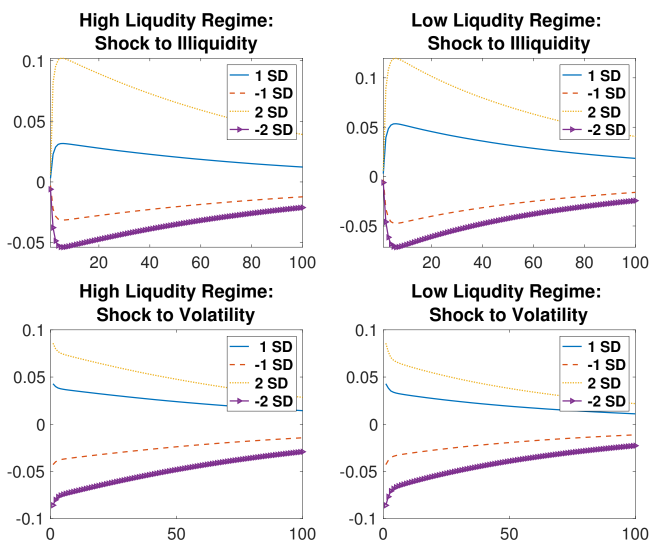
(b) Kimko Realty Corporation

Notes: This figure shows the response of volatility to shocks of liquidity and volatility, when model start in low or high liquidity regime. 1 SD shows the response to one standard deviation shock. 2 SD shows the response to two standard deviation shock. -1 SD shows the response to minus one standard deviation shock. -2 SD shows the response to minus two standard deviation shock.

Figure 2.10: Volatility Response, DOW and F



(a) Dow Chemical Corporation



(b) Ford Motor Company

Notes: This figure shows the response of volatility to shocks of liquidity and volatility, when model start in low or high liquidity regime. 1 SD shows the response to one standard deviation shock. 2 SD shows the response to two standard deviation shock. -1 SD shows the response to minus one standard deviation shock. -2 SD shows the response to minus two standard deviation shock.

high or low liquidity regimes is enough to create a different dynamics thereafter. The same is not true for volatility shocks, where the results suggest even longer impact of volatility shocks on liquidity in high liquidity regimes. This finding demonstrates the investor's fear about the future liquidity during sudden liquidity dry ups observed in the data.

A striking conclusion from Figures 2.7 - 2.10 is that the response to the impulse decays over time in each regime. In contrast to the situation where the model remains in low liquidity forever after the shock, shown on Figures 2.3, 2.4 and 2.6 (panel b), this specific case scenario cannot generate the evaporating liquidity and growing volatility. For that purpose a different shock scenario might be suggested, for instance when there are repetitive liquidity shocks which will keep the model in the low liquidity regime for a longer time. Indeed this finding is related to asset pricing literature with a special focus on events that generates crisis. For instance, Ornthanalai (2014) proposed to use a model with frequent moderate shocks to generate financial crisis instead of adding infrequent jumps of large magnitude to the benchmark diffusion model. In essence, his continuous time modelling framework is a similar to the regime switching model developed in the discrete time set-up, where jumps indicate a different regime.

My analysis of non-linear responses also shows that there is size and sign asymmetry of liquidity shocks. In particular, liquidity and volatility are highly sensitive to large drops in liquidity, but responds less to an improvement of liquidity of the same extend. A large two standard deviation drop in liquidity has a bigger impact than an improvement of liquidity of the same size, but the exact asymmetry response varies among stocks. Large drops in liquidity may signify a major market distress and have a bigger impact on investor's expectation through the fear of market instability.

To sum up, I estimate the IRF of volatility and liquidity, when the model starts in a particular regime, but is allowed to move between regimes thereafter. I demonstrate that illiquidity and volatility are positively related to each other's shocks. The effect of liquidity shocks on liquidity and volatility are bigger when liquidity is in high demand. The impact of shocks does not lead to explosive dynamics in liquidity and volatility in either of the regimes.

2.5 Conclusion

This paper investigates the non-linear relationship between volatility and market liquidity for different stocks. Using a two regime TVAR model of market liquidity and volatility when liquidity determines the regime, I find supportive evidence of non-linearity between liquidity and volatility and this result is robust for my sample of stocks. My model identifies two different regimes — low and high liquidity, respectively, in the liquidity and volatility interactions, where the low liquidity regime coincides with major financial downturns. My results suggest that threshold level of liquidity which identifies low liquidity regime varies within the same class of assents an it is the lowest for the banking industry.

My empirical findings establish that the link between volatility and liquidity is stronger in a ”bad” liquidity regime. The results suggest that shocks to volatility and illiquidity make these variables grow when the model remains in the low liquidity regime forever. This feedback effect between liquidity and volatility can be related to liquidity dry ups and growing volatility, which were present during the financial crisis, and are described as liquidity spirals in the literature. Thus, the estimated level of liquidity which separates two regimes could serve as a barometer that identifies the state of liquidity and volatility crashes, and should be monitored, especially by financial institutions where it is estimated to be the lowest.

Next, my findings show that liquidity shocks are asymmetric in both regimes. Specifically, I show that liquidity and volatility are more sensitive to large drops in liquidity than to positive liquidity dynamics, which might reflect investors fear of liquidity crashes.

Finally I demonstrate that a single shock does not generate evaporating liquidity and growing volatility, when the model is allowed to move between regimes. This result is related to the crisis creations literature where repetitive moderate shocks are more important for generation of crisis than single large shock. A different case scenario with potentially multiple subsequent shocks could be studied in future work to explore the mechanism behind the spiral creation.

Chapter 3

Modelling the Distribution of Interest Rates

3.1 Introduction

Modelling the dynamics of interest rates is an essential question for the risk and portfolio management, derivative pricing, macroeconomics and monetary policy. Low and persistent government yields, which has been recently observed in many countries, created an additional methodological challenges for financial models, since early adopted practices do not work in the new environment. Specifically, the majority of the early models restrict interest rate from being negative (Cox et al., 1985, Black and Karasinski, 1991) since investors were not supposed to pay a fee to lend their money (i.e. interest rate is negative). On the other hand, models that allow negative rates (Vasicek, 1977, Hull and White, 1990), and Gaussian affine models (Brigo and Mercurio, 2001), cannot explain why short rates did not rebound quickly, but had been stalled for along time. Thus, the rapid decline of the interest rate around the world during the financial crises in 2007 - 2009 created a new environment – called Zero Lower Bound (ZLB) – where interest rate has been either very small for a long time (USA, Canada) or become even negative (Euro Zone, Switzerland, Denmark). To stress the ability of interest rates to become negative I will refer to this regime as a low interest rate (LIR) regime rather than ZLB, although the latter is more widely used in the literature.

In this paper I propose to model this new environment by incorporating LIR as an different regime in the distribution of the interest rate, since it is characterized by the distinct features compared to the period of high interest rates. The difference between regimes is driven not only by the statistical properties, but also by the economic theory, in the sense that a "liquidity trap" prevents Central Bank's intervention to change its key policy rate (Keynes, 1936, Krugman et al., 1998). Indeed, once short term interest rate hits zero lower bound, the Central Bank cannot decrease it further. By contrast, Central Banks has higher flexibility to adjust its key policy rates during normal times, which leads to fast mean reversion of the interest rate.

I distinguish between the above mentioned regimes by modelling the distribution of interest rates as a mixture of two distributions, corresponding to the normal state (high interest rates) and the LIR regime, respectively. In the first regime, I assume that interest rates have heteroscedastic volatility with fat tails innovations, which is in line with classical models of short interest rate (Cox et al., 1985). By contrast, interest rates follow a Gaussian distribution in the LIR regime with constant volatility. The changes in regimes are driven by a state-dependent weighing function and depend on the level of interest rate. Thus, the resulting conditional distribution is flexible enough to accommodate distinctive features of the interest rate dynamics during periods of high interest rates and during the LIR regime.

I estimate the distribution of short Canadian Swap rates from December, 1994 till January, 2017. Since short interest rates are not directly observable I filter the from weekly government yields using the dynamic Nielson-Siegel (DNS) approach of Diebold and Li (2006). The DNS model is very popular among financial market practitioners because of its accurate performance in fitting the term structure of interest rates and forecasting its dynamics. First, I analize the statistical properties and stylized facts of the yield curve, and conclude that these are indeed captured by the DNS model. I demonstrate that the yield curve changes its dynamics in the LIR state, and the shift is more pronounced for the short term yields. Similar results are reported in Christensen (2015), Christensen and Rudebusch (2015), and Meucci and Loregian (2016) when considering the term structure of U.S. Treasury bills.

To motivate my model further, I investigate the statistical properties of the estimated short interest rate and conclude that there is indeed a shift of the distribution of the interest rate around LIR regime. In particular, the interest rate distribution is leptokurtic during normal states, but remains close to Gaussian distributed, when the interest rate is low. Based on this observation, I model the distribution of interest rate by a mixture of a t-Scaled distribution and a Gaussian distribution with time varying weights.

My findings identify periods of normal regimes and LIR states. In particular the estimated weighting function, suggests that when the interest rate is above 0.87%, the probability of the financial stability regime is higher than the probability of the LIR regime. It approaches 100% very fast, i.e. it already reaches 95% when the interest rate is above 1.44% . Thus, my model estimates periods of leptokurtic interest rate increments during during Jan, 1994 - Feb, 2009 and , Sep, 2010 - Jan, 2017. On the other hand, when the interest rate is below 0.87%, the probability of the LIR regime is greater, and it is above

85% from June, 2009 till May, 2010, when Canadian short interest rate was close to zero.

Next, I compare the empirical performance of my model to one regime benchmarks, which assume either a Normal or t-Scaled distribution of interest rate changes. I find that my proposed mixture of distributions model clearly outperforms these mentioned competitors based on Likelihood and BIC statistics.

The analysis provided in this paper is related to the vast literature on modelling the yield curve in the low interest rate enviroment and papers related to assessing the macroeconomic effect of leaving LIR regimes (Meucci and Loregian, 2016, Bauer and Rudebusch, 2014, Kim and Singleton, 2012, Wu and Xia, 2014). This area of research takes into account the existence of LIR regimes, and handles it either by using regime switching models or Gaussian shadow rates models proposed by Black (1995). Unlike the majority of these papers aiming to model the drift and diffusion of the interest rate process, I model the whole distribution of the interest rates, which is more applicable for option pricing and portfolio risk management, for example.

The rest of the paper is organized as follows. In section 3.2 define the mixture of distributions model for the interest rate increments with time varying weights. In section 3.3 I describe the data, explain how to construct the short term interest rate and provide a preliminary analysis regarding its distributional properties. Section 3.4 contains the empirical findings and discusses its implications. Section 3.5 concludes.

3.2 Model

In this section I discuss the proposed model for the distribution of interest rate increments, which combines both a fat tail distribution and a Gaussian distribution. This modelling framework is based on the stylized facts observed in the data, which I document in more details in the next section. Thus, my modelling approach is different from popular affine type models, see Piazzesi (2010) for general overview, in the sense that I do not define state vectors that drive dynamics of the yields. The major benefit of the affine type models is that they allow obtaining tractable closed or semi-closed form expressions of the bond yields. Meanwhile, researchers have to assume restrictive dynamics of the state vectors to get these tractable solutions. Moreover, affine models cannot explain a zero lower bound regime, which motivates the recent research to extend standard Gaussian affine model. In particular, Christensen (2015) used a regime switching affine model, Monfort et al. (2017) considered non-negative affine processes, and Christensen and Rudebusch (2015) incorporated the shadow rate approach into the affine Nelson-Siegel framework. Although these models could explain zero lower bound regimes, they can not produce negative rates, observed in the Euro Zone, Switzerland and Denmark.

In my paper, I take a very different approach: I build my model based on the empirical stylized facts, including the existence of low and high interest rates regime. Furthermore, the transition probability between two regimes is time varying and state dependent.

I assume that the short interest rate dynamics in the high interest rate regime is described:[9]

$$r_{t+1} = r_t + \epsilon_{t+1} r_t, \tag{3.1}$$

$$\epsilon_{t+1} \overset{iid}{\sim} t(\nu, \mu_1, \sigma_1). \tag{3.2}$$

[9] The extension of the model to incorporate mean reversion in the dynamics of the interest rates is straightforward. However, the current time series are very close to unit root given weekly frequency of the data, which justifies the simplification and focus on differences of returns.

In this regime, interest rate increments, $\Delta r_{t+1} = r_{t+1} - r_t$, have heteroscedastic volatility, driven by the previous value of the interest rate level, r_t, and follow a t-scale distribution.

The conditional variance of returns increments is proportional to the square of the interest rate, which is consistent with the lognormal specification of the short rate process, see Dothan (1978) and Black and Karasinski (1991). While different models assumed different functional specifications, Conley et al. (1997) empirically estimated the variance elasticity of Federal funds interest rate sampled at daily frequency to be between three and four. My model has lower variance elasticity (two) due to the ability to generate fat tails using t-scaled distribution, while continuous time models are primarily constrained to Gaussian processes.

The dynamics of the rates in the LIR regime is defined as follows:

$$r_{t+1} = r_t + \omega_{t+1}, \tag{3.3}$$

$$\omega_{t+1} \overset{iid}{\sim} N(\mu_2, \sigma_2). \tag{3.4}$$

The interest rates increments have homoscedastic volatility σ_2 and follows Normal distribution, which allows to generate negative rates in the LIR regime.

With the above specification, the probability distribution function of interest rates increments becomes:

$$\begin{aligned} pdf(r_{t+1} - r_t | I_t) &= \pi_t f_1(\epsilon_{t+1} \cdot r_t) + (1 - \pi_t) f_2(\omega_{t+1}), \\ \pi_t(r_t) &= \frac{1}{1 + e^{-\gamma(r_t - \tau)}}, \end{aligned} \tag{3.5}$$

where $\pi_t(r_t)$ is a time varying probability function of interest rate level r_t with parameters γ and τ. Here τ is interpreted as a threshold between two regimes, such that $\pi_t(r_t) = \frac{1}{2}$, when $r_t = \tau$, and $\pi_t(r_t)$ approaches 1 as r_t goes above τ, and it reaches 0, when r_t drops below τ. The speed of convergence of $\pi_t(\cdot)$ depends on the parameter γ, which determines the smoothness of the probability function $\pi_t(r_t)$, i.e. smoothness of the transition from one regime to another. I simplify the mixture pdf in (3.5) as follows:

$$pdf(\Delta r_t | I_t) = \pi_t f_1(\epsilon_{t+1})\frac{1}{r_t} + (1-\pi_t) f_2(\omega_{t+1}). \tag{3.6}$$

Next, I specify the pdfs in each regime as follows

$$f_1(\epsilon_{t+1}) = \frac{\Gamma(\frac{v+1}{2})}{\sigma_1\sqrt{v\pi}\Gamma(\frac{v}{2})}\left[\frac{v + \left(\frac{\epsilon_{t+1}-\mu_1}{\sigma_1}\right)^2}{v}\right]^{-\left(\frac{v+1}{2}\right)}, \tag{3.7}$$

$$f_2(\omega_{t+1}) = \frac{1}{2\sigma_2\sqrt{\pi}} exp\left(-\left[\frac{\omega_{t+1}-\mu_2}{2\sigma_2}\right]^2\right), \tag{3.8}$$

and I assume that

$$E(\Delta r_t) = 0, \qquad i.e. \quad \pi\mu_1 + (1-\pi)\mu_2 = 0, \tag{3.9}$$

which involves an identification restriction for μ_1 and μ_2. To identify both conditional means of Δr_t within each mixture I assume that one mean is constant, while the other is time varying. Thus, I estimate one conditional mean while identifying the other mean from (3.9).

I introduce the t-Scaled distribution in (3.1) to capture heavy tails in high interest rate regime. The function $\Gamma(\cdot)$ in (3.7) is a Gamma function and the parameters μ_1, σ_1 and v are location, scale and shape parameters respectively of t-Scaled distribution. The $f_1(\epsilon_t)$ distribution arises from the distribution of $\mu + \sigma T$ variable, where T has Student's t-distribution, and μ and σ are location and scale parameter respectively. Small values of the shape parameter v are associated with heavy tails of the distribution, and $f_1(\epsilon_t)$ approaches the Normal distribution, when $v \rightarrow \infty$.

3.3 Data Analysis

In this section I discuss the construction of the short term interest rate. I start by analyzing the main trends of bonds yields of different maturities. Then, I estimate the yield curve and elaborate on how well it captures the major stylized facts of the data. Finally, I estimate the interest rate using yield curve factors and discuss its dynamics.

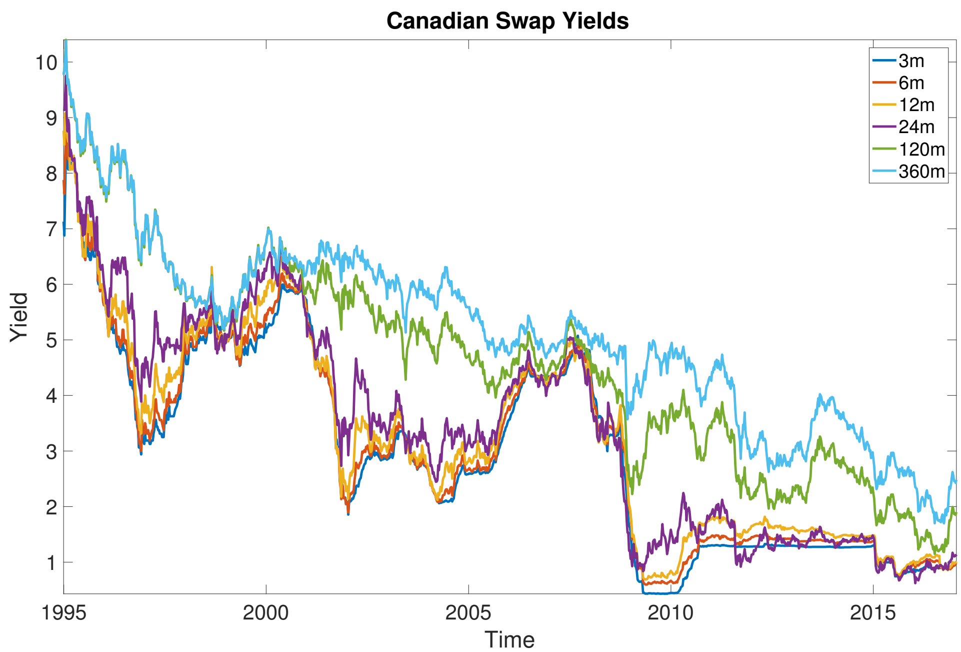
3.3.1 The Data

I consider the zero coupon yields on Canadian Swaps from December, 1994 till January, 2017. I focus on the fixed maturities of 2, 6, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120. 180, 240, 360 months. I collected data from Bloomberg and then constructed weekly observations from daily data.

Since I assess the interest rate from various yields I focus on them now in more details. Figure 3.1 shows weekly yields of Canadian Swaps from December, 1994 to January, 2017 at maturities of 3, 6, 12, 24, 120, 360 months. The time variation of the yields is apparent from the figure at every maturity, but is more prominent for yields with shorter maturity during the period of 1994 - 2009. However, long-term yields are more volatile than short term yields after the financial crisis period.

Table 1 shows descriptive statistics of the yields of different maturities, as well as em-

Figure 3.1: Canadian Swap Yields



Notes: This figure shows weekly Canadian Swap Yields from December, 1994 till January, 2017 at maturities of 3, 6, 12, 24, 120, 360 months.

pirical level, slope and curvature of yield curve, which I define as the following:

$$
\begin{aligned}
&level = y_t(240), \\
&slope = y_t(240) - y_t(3), \qquad . \\
&curvature = 2y_t(24) - y_t(3) - y_t(240).
\end{aligned}
\tag{3.10}
$$

Typically, yields rise with maturity level, which suggest upward sloping yield curve. In general long term yields varies more than short term yields. Yields are very persistent, but long tern yields are more persistent than yields at the short end of the curve. Level is the least volatile and slope is less persistent than any individual yield. Curvature is highly volatile around its mean, and is the least persistent among all the other factors.

3.3.2 Fitting The Yield Curve

I consider the following DNS model of the term structure of interest rates

$$
y(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau}\right) + \beta_{3t}\left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau}\right) + \epsilon_t, \tag{3.11}
$$

where $y(\tau)$ is the yield with maturity τ at time period t. $\beta_{1t}, \beta_{2t}, \beta_{3t}$ are three latent dynamic factors. The first factor β_{1t} is considered to be a long term factor, or level, since its loading is equal to one and is constant over maturities and time. Thus, any changes to β_{1t} affect the whole level of term structure. The second factor β_{2t} is called a short term factor, or slope, since its loading monotonically decays from 1 to 0 over time. The last factor, β_{3t} is viewed as medium-term factor or curvature, since its loading increases from 0 and then declines. From (3.11) I estimate the short term interest rate according to:

Table 3.1: Descriptive Statistics, Weekly Canadian Swap Yields

Maturities(in months)	Mean	St. Dev.	Max	Min	$\rho(1)$	$\rho(12)$	$\rho(30)$
3	3.00	1.84	8.38	0.43	1.00	0.93	0.80
6	3.10	1.82	8.69	0.60	0.99	0.92	0.80
12	3.27	1.83	9.08	0.69	0.99	0.92	0.80
24	3.43	1.96	9.75	0.62	0.99	0.92	0.83
36	3.68	1.97	9.88	0.67	0.99	0.92	0.84
48	3.89	1.97	10.02	0.71	0.99	0.92	0.84
60	4.06	1.96	10.18	0.77	0.99	0.93	0.84
72	4.22	1.96	10.26	0.87	0.99	0.93	0.84
84	4.36	1.95	10.35	0.99	0.99	0.93	0.84
96	4.49	1.93	10.37	1.06	0.99	0.93	0.84
108	4.60	1.92	10.40	1.15	0.99	0.93	0.84
120	4.60	1.92	10.40	1.15	0.99	0.93	0.84
180	5.02	1.72	10.40	1.57	0.99	0.92	0.83
240 (level)	5.13	1.67	10.38	1.70	0.99	0.92	0.82
360	5.06	1.71	10.36	1.58	0.99	0.93	0.83
slope	2.13	1.14	4.55	-0.03	0.99	0.90	0.66
curvature	-1.27	0.91	1.54	-3.28	0.98	0.78	0.65

This table presents descriptive statistics for Canadian weekly swap yields at different maturities, and for the yield curve level, slope and curvature, where I define the level as the 20-year yield, the slope as the difference between the 20-year and 3-month yields, and the curvature as the twice the 2-year yield minus the sum of the 3- month and 20-year yields. The sample period is 1994:12 - 2017:01. $\rho(1), \rho(12), \rho(30)$ are sample autocorrelations at the displacements of 1, 12, and 30 months.

$$r_t = \beta_{1t} + \beta_{2t}. \tag{3.12}$$

In this section I estimate and discuss fitting of the three - factor DNS model in the equation (3.11) in the time series and cross section. I build the short interest rate from estimated level and slope components using equation in (3.12), and then estimate model in (3.5). Thus, the proposed framework allows to incorporate evolution not only of the short term rate but also of the whole term structure of interest rates.

In general, I could estimate the model in (3.11) by Non-Linear OLS, but it has slow and not-robust performance. Instead, I use a well established and simple procedure by fixing a value of λ_t and estimating the parameters in (3.11) by ordinary least squares at each point of time. Specifically, follwoing Diebold and Li (2006), I set the value of λ_t to be equal to 0.0609, calculate factors loadings and estimate the times series factors $\{\beta_{1t}, \beta_{2t}, \beta_{3t}\}$ at each t.

Table 3.2: Descriptive Statistics of $\beta_1, \beta_2, \beta_3$

Factors	Mean	St. Dev.	Max	Min	$\rho(1)$	$\rho(12)$	$\rho(30)$
β_{1t}	5.34	1.74	10.46	1.68	0.99	0.93	0.82
β_{2t}	-2.24	1.37	0.03	-5.40	0.99	0.89	0.63
β_{3t}	-2.63	2.08	2.99	-7.19	0.98	0.79	0.67

This table presents descriptive statistics for the estimated $\beta_{1t}, \beta_{2t}, \beta_{3t}$ in the term structure model in (3.11) for Canadian swap rates from 1994:12 to 2017:01 when $\lambda_t = 0.0609$. $\rho(1), \rho(12), \rho(30)$ are sample autocorrelations at the displacements of 1, 12, and 30 months.

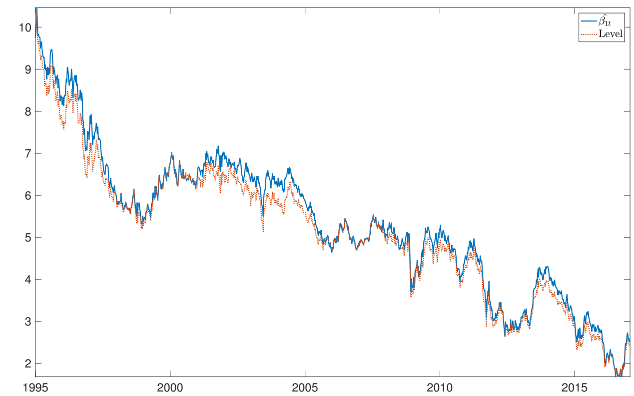
Table 3.2 shows descriptive statistics for the estimated factors $\{\beta_{1t}, \beta_{2t}, \beta_{3t}\}$. From the autocorrelation functions I notice that β_{1t} is the most persistent factor. This is consistent with calling β_{1t} a level of the yield curve, which is generally very persistent. Since, β_{1t} is the only factor in my model, which determines long rates, the model in (3.11) confirms that long rates are more persistent than short rates. β_2 is the least persistent factor. This fact gives additional rationale to call β_2 a slope, which is related to the spread of the yield curve, and is know to be less persistent than the level of the yield curve. I also notice that β_1 is more volatile than β_2. Thus my model is consistent with the observation that long rates, completely determined by β_{1t} in my setting, are generally more volatile than the short end of yield curve, which depends on β_{1t} and β_{2t}.

Figure 3.2 shows the the estimated factors $\{\beta_{1t}, \beta_{2t}, \beta_{3t}\}$ versus the empirical level, slope and curvature respectively. All three time series pairs demonstrate similar patterns. The correlation between β_{1t} and level, β_{2t} and slope and β_{3t} and curvature are 0.99, 0.99 and 0.97 respectively. This observations confirms my reasoning for referring to the respecting factors $\{\beta_{1t}, \beta_{2t}, \beta_{3t}\}$ as level, slope and curvature.

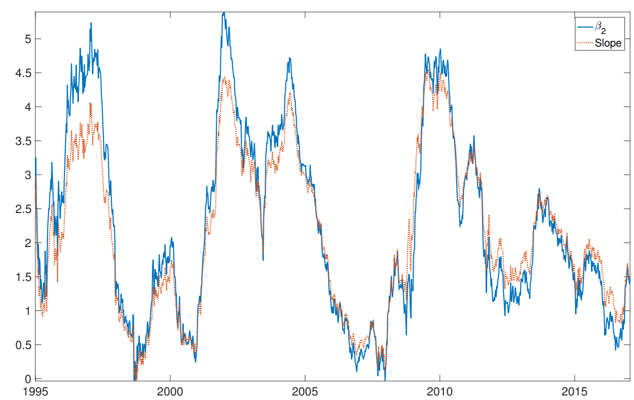
3.3.3 Interest Rate

In this section I discuss the empirical properties of the short term interest rate constructed following equation in (3.12). Figure 3.3a shows the estimated short interest rate, r_t. r_t reaches its maximum value at the beginning of 1995 and has very volatile pattern till 2008. The interest rate drops to a minimum of 0.19% during financial crisis, and the dynamics of the interest rate changes dramatically thereafter. It raises to almost 2% in 2010 and remains fairly stable around 1%-2% afterwards. At first glance, the mixture of distributions seems to a be a suitable methodology to model the distribution of interest rate. However, the dynamics of the interest rate also suggests a non-stationary series, thus it would be difficult to ensure the asymptotic properties of the estimators. Instead, I model the first difference of the interest rate, defined as $\Delta(r_t) = r_t - r_{t-1}$, which is shown on the figure 3.3b. The time series of $\Delta(r_t)$ demonstrates mean reversion and volatility clustering.

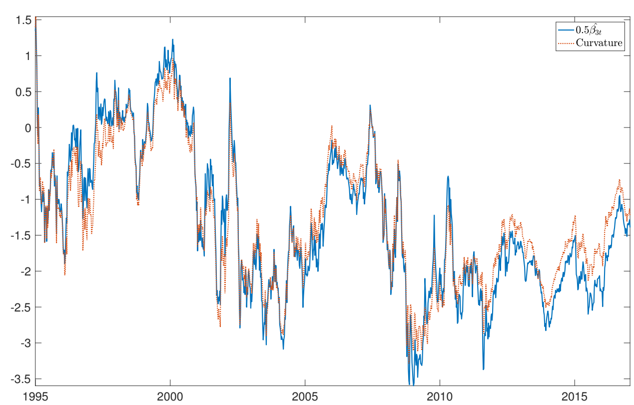
Figure 3.2: Estimated Factors and Empirical Factors



(a) Blue solid line: $\hat{\beta_1}_t$, Red dotted line: empirical level



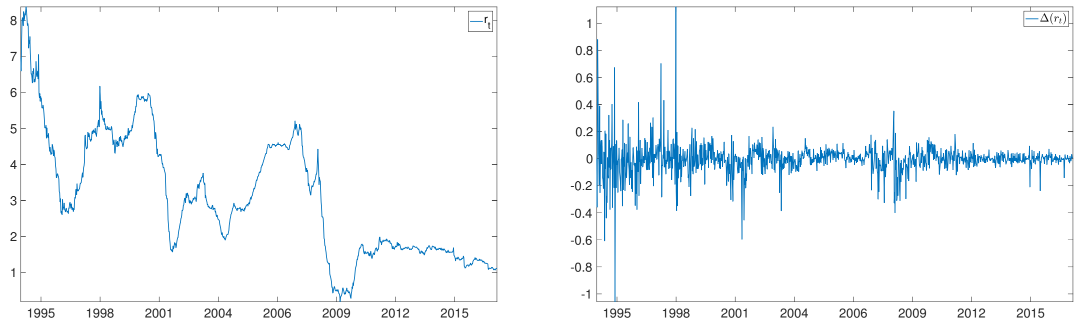
(b) Blue solid line: $-\hat{\beta}_{2t}$, Red dotted line: empirical slope



(c) Blue solid line: $0.5\hat{\beta_3}_t$, Red dotted line: empirical curvature

Notes: This figure shows the estimated $\hat{\beta_t}_1, \hat{\beta_t}_2$ and $\hat{\beta_t}_3$ vs empirical level. slope and curvature. I define the empirical level as 20-year yields, the slope as a difference between the 20-year yields and 3-month yields, and the curvature as the twice the 2-year yields minus the sum of 20-year yield and 3-month yields.

Figure 3.3: Short Interest Rate



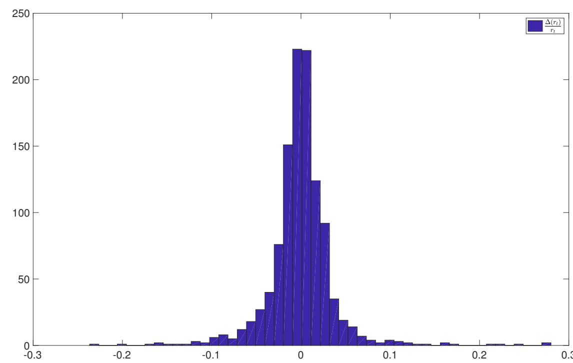
(a) $r_t = \beta_{1t} + \beta_{2t}$.   (b) $\Delta(r_t) = r_t - r_{t-1}$.

Notes: This figure shows the dynamic of the estimated instantaneous interest rate,r_t, for the Canadian Swaps from 1994:12 to 2017:01. Panel (a) shows the level of r_t, defined as $r_t = \beta_{1t} + \beta_{2t}$. Panel (b) shows the first difference of $\Delta(r_t)$, defined as $\Delta(r_t) = r_t - r_{t-1}$.

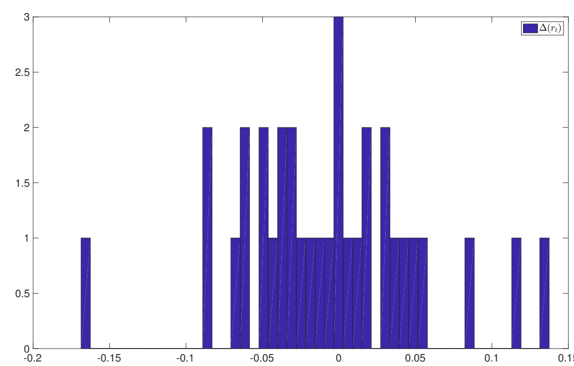
Clearly, the volatility of $\Delta(r_t)$ changes, and depends on the level of interest rate. $\Delta(r_t)$ is very volatile, when the interest rate is high, but it looks very stable, when the interest rate is close to zero. This confirms my conjecture to the model the distribution of interest rate changes as a mixture of distributions, such that the heterogeneous volatility of interest rate changes depends on the level of the interest rate only in the high interest rate regime. Interest rate models, where volatility depends on the level of interest rate are very popular in the literature, the Cox-Ingersoll-Ross model are among those. (Cox et al., 1985).

Next, I supplement my analysis investigating the distributional properties of $\Delta(r_t)$ conditional on the level of the interest rate r_t. Specifically, I fix the value of $r_t = 0.5\%$ and explore the histrogram of $\Delta(r_t)$, when r_t is below or above 0.5% (Figure 3.4). Clearly, the dynamics of $\Delta(r_t)$ is described by different distributions, when r_t is close to zero and when it is above 0.5%. The descriptive statistics of $\Delta(r_t)$, conditional on r_t confirms this finding (Table 3.3). When $r_t < 0.5\%$, the distribution of $\Delta(r_t)$ has skewness close to 0 and kurtosis close to 3, and it could be associated with a Normal distribution. When $r_t > 0.5\%$, the distribution of $\Delta(r_t)$ demonstrates leptokurtic pattern with large excess kurtosis. This observation confirms my choice of t-Scaled Distribution for the mixture of distributions model in (3.5), since it can better accommodate fat tails, which occurs when $r_t > 0.5\%$.

Figure 3.4: Distribution of Interest Rate Increments



(a) $\frac{\Delta(r_t)}{r_t}$, $r_t > 0.5$.



(b) $\Delta(r_t)$, $r_t \leq 0.5$.

Notes: This figure shows the histogram of $\Delta(r_t)$ and $\frac{\Delta(r_t)}{r_t}$ for Canadian Swaps from 1994:12 to 2017:01. Panel (a) shows $\frac{\Delta(r_t)}{r_t}$, when $r_t > 0.5$. Panel (b) shows $\Delta(r_t)$, when $r_t \leq 0.5$.

Table 3.3: Descriptive Statistics by The Level of Interest Rate.

State	Variable	Mean	St. Dev.	Skewness	Kurtossis	ADF
$r_t > 0.5$	$\Delta(r_t)$	0.00	0.12	0.38	24.41	0.00
	$\frac{\Delta(r_t)}{r_t}$	0.00	0.04	0.81	15.89	0.00
$r_t < 0.5$	$\Delta(r_t)$	-0.01	0.06	0.08	3.79	0.00
	$\frac{\Delta(r_t)}{r_t}$	0.00	0.19	1.09	6.48	0.00

This table shows the description statistics for $\frac{\Delta(r_t)}{r_t}$ and $\Delta(r_t)$ of Canadian Swaps, when $r_t < 0.5$ and $r_t > 0.5$ during 1994:122017:01. The last column shows the $p - value$ of Augmented Dickey-Fuller test.

3.4 Estimation Results

In this section I discuss the empirical performance of the model in (3.5). I conjecture that the distribution of interest rate increments is described by the mixture of fat tail distribution in normal times and Gaussian in the LIR regime. Thus, my goal is to access the importance of mixing distribution, time varying weights and economic underpinning behind the estimated threshold parameter τ. First, I compare in sample performance of my model in (3.5) to conventional models used in the literature, which assume either a Normal or a t-Scaled distribution of $\Delta(r_t)$. Table 3.4 reports the estimated parameters of the following distributions of interest rates increments:

$$\begin{array}{l} (1) - Gaussian, \\ (2) - t - Scaled \\ (3) - Mixture \quad of \quad t - Scaled \quad and \quad Gaussian \quad distribution \end{array}, \tag{3.13}$$

which are estimated by Maximum Likelihood. I note that model (3) in Table 3.4 strongly outperforms model (1) and (2) based on reported LogLikelihood statistics (L) and Schwarz

Information Criteria (BIC). Interestingly, model (1) has the worst performance according to Likelihood and BIC statistics. Model (2) outperforms model (1), since it can better captures the fat tails observed in the data.

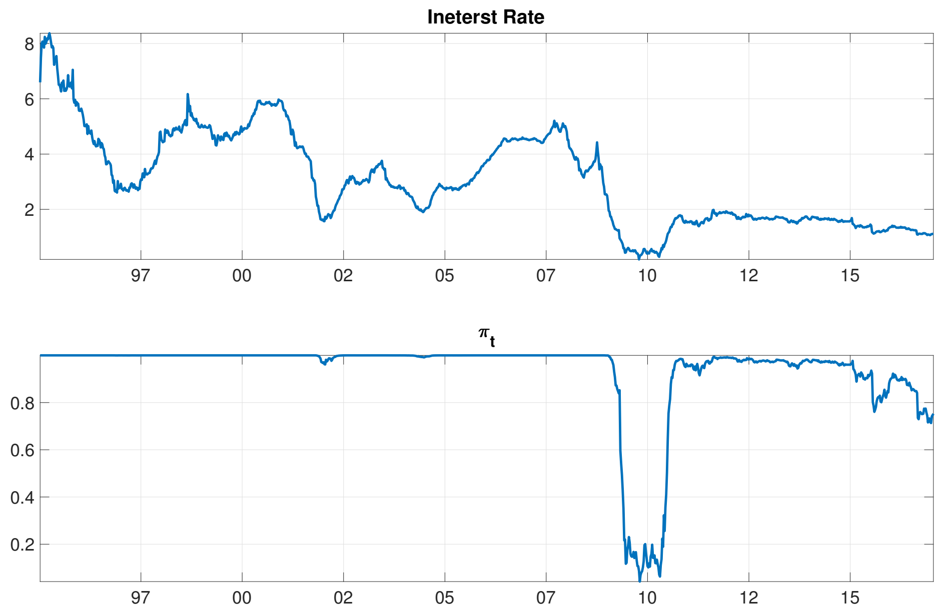
Table 3.4: Estimation Results.

Parameters	1	2	3
	Normal	t-Scaled	Mixture of t-Scaled and Gaussian
μ_1	-0.01	0.00	0.00
	(0.00)	(0.01)	(0.00)
σ_1	0.12	0.10	0.02
	(0.00)*	(0.00)*	(0.00)*
v		20	2.30
		(5.15)*	(0.24)*
μ_2			
σ_2			0.08
			(0.00)*
τ			0.87
			(0.09)*
γ			4.60
			(1.03)*
L	858	2549	3588
BIC	-1709	-5091	-7169

This table shows the estimated parameters of the distribution of $\Delta(r_t)$ of Canadian Swaps, during 1994:122017:01. The first column corresponds to Normal distribution, the second - to t-Scaled distribution, the last column refers to mixture of t-Scaled and Normal distribution. L refers to the estimated log-Likelihood, and BIC stands for Schwarz Information Criteria , calculated as $-2L+p*log(n)$, where p is a number of parameters estimated and n is a number of observations. Here * means that corresponding p-value is lower than 0.01

The estimated mixture of t-Scaled and Gaussian distribution defines two distinct regimes. The first regime, which is related to ”good” times, is a leptokurtic regime with relatively small shape and scale parameter, $v = 2.30$ and $\sigma = 0.02$ respectively. The estimated

Figure 3.6: Interest Rate and Weight Function, $\pi_t(r_t)$



Notes: This figure shows the dynamic of the estimated interest rate, r_t, of the Canadian Swaps from 1994:12 to 2017:01 versus the estimated probability of t-Scaled distribution $\pi_t(r_t) = \frac{1}{1+e^{-\gamma(r_t-\tau)}}$, where γ and τ are estimated in the model (3.5)

$v = 2.30$ is relatively small, indicating that the first component of the mixture of distribution in (3.5) has fat tails.

Several conclusions arise from the estimated weighted function $\pi_t(r_t)$, the dynamics of which is shown on the bottom of the figure 3.6. It fluctuates from 0.04 to 1 during 1994 - 2017, and, as expected, it is highly correlated with interest rate level. The parameters of the $\pi_t(r_t)$ have correct economic intuitions. The estimated γ is equal to 4.60, which indicates a sharp transition between two distributions. In other words, my model suggests that two different regimes in the distribution of $\Delta(r_t)$, - either t-Scaled distribution or Normal distribution exists, but rarely a combination of both.

The estimated threshold parameter is $\tau = 0.87\%$, and it describes a situation, when the distribution of Δr_t is a mixture of two distributions with equal weights. When the interest rates is above the threshold level $\tau = 0.87\%$, the probability of the high interest rate regime is greater than the probability of the LIR regime, and it is more than 0.9, when interest rate is above 1.44%. Consequently the t-Scaled distribution is related to the periods of high interest rate during Jan, 1994 - Feb, 2009, and Sep, 2010 - Jan, 2017. Accordingly, given the small value of ν, my model is able to capture the leptokurtic pattern of the interest rate during these times. On the other hand, the probability of being in the LIR regime is greater than the probability of being in the high interest regime when interest rate falls below 0.87%, and it is above 0.85 when interest rate falls below 0.60%, which happened from June, 2009 till May, 2010. The estimated path of $\pi(r_t)$ shows high probability of r_t to follow a Normal distribution, which allows for negative rates. Thus, the proposed framework could be useful for derivative pricing and risk management, where the essential question is how to model the distribution of interest rates with negative values.

While modelling the distribution of interest rate around LIR regimes, I do not make any specific assumptions on the level of interest rate which separates the LIR period. Hence I consider the proposed methodology as a general framework for modelling of the distribution of interest rates, which could be studied in the future work for other countries as well, like USA or EU, where LIR periods had different characteristics than those in Canada.

Lastly, another avenue for potential research is to apply the proposed methodology to other risk factors, e.g modelling the exchange rate dynamics of emerging countries. For instance, if a particular currency starts to rapidly depreciate, it would lead to the high volatility regime of its exchange rate. In this situation the Central Bank is likely to introduce a currency peg, which essentially implies a low volatility regime. As a result, pricing of foreign exchange options and accurate risk assessment becomes challenging for one regime models. I will leave assessing the quantitative gains of my proposed two regime model for other classes of assets for future research.

3.5 Conclusion

In this paper I model the distribution of interest rates by a mixture of a t-Scaled distribution and a Gaussian distribution. My approach expands the conventional practice in the literature to model the interest rate solely by a Gaussian distribution or a Lognormal distribution, which has several limitation like the ability to capture fat tails (essentially due to the unrealistic assumption on the volatility of interest rate increments), negative rates allowed during LIR regime, and the overall performance during long periods of time. Using the mixture of distributions model I address the issue of interest rate volatility dynamics during long periods of time. Also, including t-Scaled distribution in my mixture of distribution model, I capture the fat tails during the high interest rate regime. My model is significantly statistically preferred over standard Gaussian models or a t-Scaled distribution model. In the future, I plan to apply the proposed procedure to model the distribution of interest rate increments of Euro zone swaps, where negative yields were observable. Another application of mixing framework with two regimes (high and low volatilities) might be suitable for modelling and forecasting of the exchange rates of developing countries, which I leave for future research.

Appendix A

Chapter 1 Appendix

This appendix contains the proofs of all the theorems, remarks and corollaries in the paper, as well as provides additional figures.

A.1 Proof of Theorem 1.1

Theorem 1.1 *The process Y_t, $t = 0, 1, 2, ...$ defined in* (1.1) *is second order stationary with positive definite covariance matrix $V = Var(Y_0)$ if and only if:*

1. $\mu = 0$*, where μ is a mean of the initial vector, $\mu = E(Y_0)$,*
2. *the covariance matrix, V, solves $V - \Phi V \Phi' - E(B_{t-1} V B'_{t-1}) = \Sigma$, and*
3. $\mid \lambda \mid < 1$*, where λ is the maximum eigenvalue of the matrix* $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$.

Proof We first prove necessity. Let Y_t be second order stationary such that $E(Y_t) = \mu$. Taking expectation of (1.2), we have that $\mu = (1 - \Phi)^{-1} * 0$. From (1.2) we have that

$$(Y_t - \mu)(Y_t - \mu)' = (B_{t-1}\mu + (\Phi + B_{t-1})(Y_{t-1} - \mu) + \epsilon_t)(B_{t-1}\mu + (\Phi + B_{t-1})(Y_{t-1} - \mu) + \epsilon_t)'. \quad \text{(A.1)}$$

Taking expectations on both sides of (A.1) and noticing that the expectations of cross products are zero we have

$$Var(Y_t) = Var(B_{t-1}\mu) + Var((\Phi + B_{t-1})(Y_{t-1} - \mu)) + Var(\epsilon_t), \tag{A.2}$$

which can be rewritten as

$$Var(Y_t) = V = Var(B_{t-1}\mu) + \Phi Var(Y_{t-1})\Phi' + E(B_{t-1}Var(Y_{t-1})B'_{t-1}) + Var(\epsilon_t), \tag{A.3}$$

or

$$V = \Phi V \Phi' + E(B_{t-1}VB'_{t-1}) + \Sigma. \tag{A.4}$$

Then

$$V - \Phi V \Phi' - E(B_{t-1}VB'_{t-1}) = \Sigma. \tag{A.5}$$

By definition Σ is a positive definite matrix of full rank. Conlisk (1974) and Conlisk (1976) show there is a unique positive definite V if and only if the maximum of the moduli of $\Phi \otimes \Phi + E(B_{t-1} \otimes B_{t-1})$ is less than 1. Notice that $E(B_{t-1} \otimes B_{t-1}) = \pi(1-\pi)\Phi^0 \otimes \Phi^0$ and $\Phi \otimes \Phi + \pi(1-\pi)\Phi^0 \otimes \Phi^0 = \pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1$. Thus, the conditions used in Conlisk (1974) and Conlisk (1976) transform to $\mid \lambda \mid < 1$, where λ is the maximum eigenvalues of the matrix $\pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1$.

We now show sufficiency. Let conditions 1-3 hold. Taking expectation of equation (1.2) at $t = 1$ shows that $E(Y_1) = E(Y_0) = \mu$.[10]Iterating further, it is possible to show that $E(Y_t) = \mu$, $\forall t$. Similarly, calculating the variance of equation (1.2) at $t = 1$ shows that $Var(Y_1) = \Phi V \Phi' + E(B_{t-1}VB'_{t-1}) + A = V = Var(Y_0)$. Iterating further, it is possible to

[10]The existence of the solution of (1.2) is demonstrated by the Theorem 1.2

show that $Var(Y_t) = V, \forall t$. Since $\lambda < 1$, it follows from Conlisk (1974) and Conlisk (1976) that V is a positive definite matrix. Premultiplying (1.2) by Y_{t+h} and taking expectations, we have

$$cov(Y_t, Y_{t+h}) = (\pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1)cov(Y_t, Y_{t+h-1}). \tag{A.6}$$

Iterating further we have

$$cov(Y_t, Y_{t+h}) = (\pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1)^h cov(Y_t, Y_t) = (\pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1)^h V. \tag{A.7}$$

Thus, the process Y_t, $t = 0, 1, 2, ...$ is second-order stationary.

A.2 Proof of Theorem 1.2

Theorem 1.2. *Assume that V is positive definite with full rank. Then the TVAR model in* (1.1) *has a unique stationary solution given by*

$$Y_t = \epsilon_t + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\Phi + B_{t-k}\right)\epsilon_{t-n}, \tag{1.4}$$

if and only if $\mid \lambda \mid < 1$*, where* λ *is the maximum eigenvalue of the matrix* $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$.

Proof Let Y_t be stationary and defined by (1.4), i.e. moments of (1.4) exist and they are finite. Then, from (1.7) it follows that

$$E(Y_t Y_t') = E\left(\sum_{n=0}^{\infty} S_n(t)\epsilon_{t-n}\epsilon_{t-n}' S_n'(t)\right).$$

We may rewrite this in *vec* form as

$$
\begin{aligned}
vecE(Y_tY_t') &= vecE\left(\sum_{n=0}^{\infty} S_n(t)\epsilon_{t-n}\epsilon_{t-n}' S_n'(t)\right) \\
&= E\left(\sum_{n=0}^{\infty} S_n(t) \otimes S_n vec(\epsilon_{t-n}\epsilon_{t-n}')\right) \\
&= E\left(vec(\epsilon_t\epsilon_t') + \sum_{n=1}^{\infty}\prod_{k=1}^{n}(\Phi + B_{t-n}) \otimes (\Phi + B_{t-n}) vec(\epsilon_{t-n}\epsilon_{t-n}')\right).
\end{aligned}
$$

Since $\prod_{k=0}^{n} A_k \otimes \prod_{k=0}^{n} B_k = \prod_{k=0}^{n} A_k \otimes B_k$ for any matrices A_k and B_k whenever the matrix product exists, the later can be rewritten as

$$vecE(Y_tY_t') = \sum_{n=0}^{\infty}(\Phi \otimes \Phi + E(B_{t-n} \otimes B_{t-n}))^n vec\Sigma.$$

Then

$$vecV = \sum_{n=0}^{\infty}(\pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1)^n vec\Sigma. \tag{A.8}$$

Furthermore, since $(\pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1)vecV = \sum_{n=1}^{\infty}(\pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1)^n vec\Sigma = vecV - vec\Sigma$, we have that

$$vecV - (\pi\Phi^2 \otimes \Phi^2 + (1-\pi)\Phi^1 \otimes \Phi^1)vecV = vec\Sigma,$$

or

$$vecV - (\Phi \otimes \Phi + E(B_{t-n} \otimes B_{t-n}))vecV = vec\Sigma,$$

which is equivalent to

$$V - \Phi V \Phi - E(B_{t-n} V B_{t-n}) = \Sigma. \tag{A.9}$$

Since V and Σ are both positive definite, the maximum eigenvalue of $\Phi \otimes \Phi + E(B_{t-n} \otimes B_{t-n})$

is less than 1 (Conlisk, 1974, 1976). Thus, the maximum eigenvalue of $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$, λ, is less than 1.

We now prove sufficiency. Let all the eigenvalues of the matrix $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$ be less than 1. Following Nicholls and Quinn (1982) we consider the following:

$$W_r(t) = \epsilon_t + \sum_{n=1}^{r} \prod_{k=1}^{n} (\Phi + B_{t-k})\epsilon_{t-n} = \sum_{n=0}^{r} S_{n-1}(t)\epsilon_{t-n}. \tag{A.10}$$

Given that the eigenvalues of the matrix $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$ are less than 1, the limit $W(t)$ of $W_r(t)$ exists in mean square and thus in probability. Moreover, $W(t) = \epsilon_t + \sum_{n=1}^{\infty}(\prod_{k=1}^{n} \Phi + B_{t-k})\epsilon_{t-n}$ satisfies equation (1.2) and $W(t)$ is stationary. Now, suppose $U(t)$ is another stationary solution of (1.2) and define

$$X(t) = W(t) - U(t). \tag{A.11}$$

By definition $X(t) = (\Phi + B_{t-1})X(t-1)$, $E(X(t)) = 0$ and $X(t)$ is stationary. Then $E(X(t)X'(t)) = \Phi E(X(t-1)X'(t-1)\Phi') + E(B_{t-1}E(X(t-1)X'(t-1))B_{t-1})$. Since $X(t)$ is stationary, and $\Phi \otimes \Phi + E(B_{t-1} \otimes B_{t-1}) = (1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$, we have $(I - ((1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2)')vecE(X(t)X'(t)) = 0$. However, since the eigenvalues of $(1-\pi)\Phi^1 \otimes \Phi^1 + \pi\Phi^2 \otimes \Phi^2$ are less then 1, $E(X(t)X'(t))$=0. Thus, $W(t) = U(t)$, and $W(t)$ is the unique solution of (1.2). Since, $W(t)$ is the same for all t it is also the strictly stationary solution of (1.2).

A.3 Proof of Remark 2

Remark 2. *Let the process Y_t, $t = 0, 1, 2, ...$ in* (1.1) *be stationary with positive definite covariance matrix V. Then the maximum eigenvalue of the matrix Φ is less than 1.*

Proof Following Theorem 1.1 and 1.2, the maximum eigenvalue of $\Phi \otimes \Phi + E(B_{t-n} \otimes B_{t-n})$ is less than 1. Consider $S = \Sigma + E(B_{t-n}VB_{t-n})$ and $K = V$. S and K are positive definite, since Σ and V are positive definite. From equation (A.9) we have that

$$K - \Phi K \Phi = S.$$

From Barnett and Storey (1970) it follows that the maximum eigenvalue of Φ is less than 1.

A.4 Proof of Corollary 3

Corollary 3. *Consider the following two-regime TVAR model with p lags in each regime*

$$Y_t = I(X_{t-1} \in R_1) \sum_{j=1}^{p} \Phi^{1j} Y_{t-j} + I(X_{t-1} \in R_2) \sum_{j=1}^{p} \Phi^{2j} Y_{t-j} + \epsilon_t, \tag{A.12}$$

where the properties of X_t and ϵ_t are those following equation (1.1). This model has a unique stationary solution given by

$$Z_t = \eta_t + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} A + D_{t-k} \right) \eta_{t-n}, \tag{A.13}$$

if $| \lambda |< 1$, where λ is the maximum eigenvalue of the matrix $(1-\pi)A^1 \otimes A^1 + \pi A^2 \otimes A^2$, and only if $| \lambda_1 |< 1$, where λ_1 is the maximum eigenvalue of the matrix $A = (1 - \pi)A^1 + \pi A^2$, where A^i, $i = 1, 2$, is defined as $A^i =$

$$\begin{pmatrix} \Phi^{i1} & \Phi^{i2} & \Phi^{i3} & ... & \Phi^{i(p-1)} & \Phi^{ip} \\ I_n & 0 & 0 & ... & 0 & 0 \\ 0 & I_n & 0 & ... & 0 & 0 \\ 0 & 0 & I_n & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & I_n & 0 \end{pmatrix}.$$

Z_t and η_t are $np \times 1$ vectors given by $Z'_t = [Y'_t, Y'_{t-1}, Y'_{t-2}, ...Y'_{t-(p-1)}]$ and $\eta_t = [\epsilon'_t, 0, 0, .., 0]$, respectively, and $D_t = (S_t - \pi)A^2 + (\pi - S_t)A^1$.

Proof Given the definitions of Z_t, η_t, and $A^i, i = 1, 2$, we can rewrite model (A.12) in its companion form

$$Z_t = A^1 I(X_{t-1} \in R_1)Z_{t-1} + A^2 I(X_{t-1} \in R_2)Z_{t-1} + \eta_t. \tag{A.14}$$

Now define $A = (1 - \pi)A^1 + \pi A^2$. Then the model in (A.14) can be rewritten as a RCM as of Nicholls and Quinn (1982) given by

$$Z_t = (A + D_t)Z_{t-1} + \eta_t, \tag{A.15}$$

where $D_t = (S_t - \pi)A^2 + (\pi - S_t)A^1$, such that $ED_t = 0$. The proof of sufficient conditions are similar to the proof of Theorem 1.2 and it suffices to show necessity. Define $\Omega = varZ_t$ and assume it exists and that it is finite. Notice that $\eta(t) = (1, 0, 0, .., 0) \otimes \epsilon(t) = l \otimes \epsilon(t)$.

Define $H = ll'$. Then, following the first part of the proof of Theorem 1.2, we have

$$vec\Omega = \sum_{n=0}^{\infty}(A \otimes A + ED_{t-n} \otimes D_{t-n})^n vec(H \otimes \Sigma). \tag{A.16}$$

Following the same strategy as in the proof of Theorem 1.2, it is straightforward to show that

$$\Omega = A\Omega A + ED'_{t-n}\Omega D_{t-n} + H \otimes \Sigma. \tag{A.17}$$

Let $z' = [z'_1, ...z'_p]$ be the left eigenvector of the matrix A with corresponding eigenvalue λ and z_i are $n \times 1$ vectors. Then

$$(1 - \lambda^2)z'\Omega z = z'_1\Sigma z_1 + z'ED'_{t-n}\Omega D_{t-n}z.$$

Since Ω is positive semidefinite, $ED'_{t-n}\Omega D_{t-n}$ is positive semidefinite. Since Σ is positive definite, $\mid \lambda \mid < 1$ when $z_1 \neq 0$. Now let $z_1 = 0$. Since z' is the left eigenvector of A with eigenvalue λ we have the following system of equations

$$z'_1\Phi^i + z'_{i+1} = \lambda z'_i, \qquad \forall i = 1, .., p-1,$$

and

$$z'_1\Phi^p = \lambda z'_p.$$

Since $\lambda \neq 0$, $z_p = 0$. Thus, $z_i = 0$, $\forall i = 1, .., p-1$. However, since $z \neq 0$, this contradicts that $z_1 = 0$.

A.5 Proof of Theorem 1.3

Theorem 1.3. *The stationary distribution of the TVAR process with $\Phi^1 = 0$ and $\Phi^2 = \Psi$ has the following characteristic function*

$$\phi(t, Y_t) = (1-\pi)\sum_{K=0}^{\infty}\pi^K exp\left(-\frac{1}{2}t\sum_{n=0}^{K}\Psi^n\Sigma\Psi'^n t\right). \tag{1.11}$$

Moreover, the probability distribution function is given by

$$f(Y_t) = (1-\pi)\sum_{K=0}^{\infty}\pi^K N\left(0, \sum_{n=0}^{K}\Psi^n\Sigma\Psi'^n\right), \tag{1.12}$$

where $N(A, B)$ is the multivariate normal distribution function with mean A and covariance matrix B.

Proof The characteristic function of Y_t is

$$\phi(t, Y_t) = E_{S_n(t)}\left(exp\left(-\frac{1}{2}t\Sigma t' - \frac{1}{2}t\sum_{n=1}^{\infty}S_n(t)\Sigma S_n'(t)t'\right)\right). \tag{A.18}$$

Thus, it is defined by the distribution of $S_n(t)$. Since $\sum_{n=1}^{\infty}S_n(t) = \sum_{n=1}^{\infty}\prod_{k=1}^{n}S_{t-k}\Psi$, the probability space of $\sum_{n=1}^{\infty}S_n(t)$ is $\{0, \sum_{n=1}^{K}\Psi^n, K \geq 1\}$, and $\sum_{n=1}^{\infty}S_n(t)$ has a Geometric distribution with $P(\sum_{n=1}^{\infty}S_n(t) = 0) = (1-\pi)$ and $P(\sum_{n=1}^{\infty}S_n(t) = \sum_{n=1}^{K}\Psi^n) = (1-\pi)\pi^K$. It follows that

$$
\begin{aligned}
\phi(t, Y_t) &= E_{S_n(t)}\left(exp\left(-\frac{1}{2}t\Sigma t' - \frac{1}{2}t\sum_{n=1}^{\infty}\prod_{k=1}^{n} S_{t-k}\Psi\Sigma(\prod_{k=1}^{n} S_{t-k}\Psi)' t'\right)\right) \\
&= (1-\pi)exp\left(-\frac{1}{2}t\Sigma t'\right) + (1-\pi)\sum_{K=1}^{\infty}\pi^K exp\left(-\frac{1}{2}t\sum_{n=1}^{K}\Psi^n\Sigma\Psi'^n t'\right) \\
&= (1-\pi)\sum_{K=0}^{\infty}\pi^K exp\left(-\frac{1}{2}t\sum_{n=0}^{K}\Psi^n\Sigma\Psi'^n t'\right).
\end{aligned}
$$

Integrating the above expression, we have that the probability distribution function of Y_t is a weighted average of normal distributions

$$
f(t, Y_t) = (1-\pi)\sum_{K=0}^{\infty}\pi^K N\left(0, \sum_{n=0}^{K}\Psi^n\Sigma\Psi'^n\right)
$$

where $N(A, B)$ is the multivariate normal distribution with mean A and covariance matrix B.

A.6 Proof of Corollary 4

Corollary 4. *Let the matrices* Φ^1 *and* Φ^2 *have positive entries. If* $(1-\pi)(\phi^1_{j1} + \phi^1_{j2})^2 + \pi(\phi^2_{i1} + \phi^2_{i2})^2 < 1$, $\forall i, j = 1, 2$, *then the model in* (1.1) *is stationary. Moreover, if the model in* (1.1) *is explosive in one of the regimes* $x \in \{1, 2\}$, *then* $(\phi^{-x}_{i1} + \phi^{-x}_{i2}) < 1$, $\forall i = 1, 2$, *where* $-x \in \{1, 2\} \setminus \{x\}$.

Proof The proof of Corollary 4 uses the Perron Frobenius theorem, which states that for a matrix X with positive entries there is a unique maximum eigenvalue λ such that $\min_i \sum_j x_{ij} \le \lambda \le max_i \sum_j x_{ij}$. Let λ^2 be the maximum eigenvalue of matrix Φ^2. Then,

$1 \leq \lambda^2 \leq \max_{i\in\{1,2\}}(\phi^2_{i1}+\phi^2_{i2})$. Let λ be the maximum eigenvalue of $(1-\pi)\Phi^1\otimes\Phi^1+\pi\Phi^2\otimes\Phi^2$. Then

$$\lambda \leq \max_{i\in\{1,2\}} F_i((1-\pi)\Phi^1\otimes\Phi^1+\pi\Phi^2\otimes\Phi^2), \tag{A.19}$$

where $F_i()$ denotes the column sum for each row i. From the last equation we have that $\lambda \leq \max_{i\in\{1,2\}} F_i((1-\pi)\Phi^1\otimes\Phi^1+\pi\Phi^2\otimes\Phi^2) \leq (1-\pi)\max_{i\in\{1,2\}}(\phi^1_{i1}+\phi^1_{i2})^2+\pi\max_{i\in\{1,2\}}(\phi^2_{i1}+\phi^2_{i2})$. Since the condition of the Corollary holds for any $i, j = 1, 2$, we have that

$$(1-\pi)\max_{i\in\{1,2\}}(\phi^1_{i1}+\phi^1_{i2})^2+\pi\max_{i\in\{1,2\}}(\phi^2_{i1}+\phi^2_{i2}) < 1. \tag{A.20}$$

Thus, $\lambda < 1$ and from Theorem 1.1 and 1.2 the model in (1.1) is stationary.

Now suppose the model is explosive in regime 2 and let λ^2 be the maximum eigenvalue of matrix Φ^2. Then, from the Perron Frobenius theorem

$$1 \leq \lambda^2 \leq \max_{i\in\{1,2\}}(\phi^2_{i1}+\phi^2_{i2}). \tag{A.21}$$

From (A.20) and (A.21) it follows that $1 \leq (\max_{i\in\{1,2\}}(\phi^2_{i1}+\phi^2_{i2}))^2 < \frac{1}{\pi} - \frac{(1-\pi)}{\pi}\max_{i\in\{1,2\}}(\phi^1_{i1}+\phi^1_{i2})^2$ and thus $(\phi^1_{i1}+\phi^1_{i2}) < 1$ for any $i = 1, 2$.

A.7 Proof of Corollary 5

Corollary 5. *Let the process Y_t, $t = 0, 1, 2, ...$ in* (1.1) *be stationary. Then the following conditions hold*

1. $\lambda^2_1\lambda^2_2 \leq \frac{1}{\pi}$,

2. $\lambda^1_1\lambda^1_2 \leq \frac{1}{(1-\pi)}$,

3. $\lambda_1^1\lambda_2^2 \leq \sqrt{\frac{1}{(1-\pi)\pi}}$*, and*

4. $\lambda_1^2\lambda_2^1 \leq \sqrt{\frac{1}{(1-\pi)\pi}}$*,*

where λ_1^i *and* λ_2^i *are eigenvalues of the matrix* Φ^i*,* $i = 1, 2$*.*

Proof From Remark 2 we know that if the model is stationary, then the eigenvalues of the matrix Φ are less than 1. Let λ_1 and λ_2 are the eigenvalues of matrix $\Phi = (1-\pi)\Phi^1 + \pi\Phi^2$ where $\Phi^1 = \begin{pmatrix} \phi_{11}^1 & \phi_{12}^1 \\ 0 & \phi_{22}^1 \end{pmatrix}$, $\Phi^2 = \begin{pmatrix} \phi_{11}^2 & \phi_{12}^2 \\ 0 & \phi_{22}^2 \end{pmatrix}$.
Then

$$\lambda_1\lambda_2 = det\Phi = (1-\pi)^2\lambda_1^1\lambda_2^1 + \pi^2\lambda_1^2\lambda_2^2 + \pi(1-\pi)\phi_{11}^1\phi_{22}^2 + \pi(1-\pi)\phi_{22}^1\phi_{11}^2. \tag{A.22}$$

Since ϕ_{11}^i and $\phi_{22}^i, i = 1, 2$ are nonnegative,

$$\lambda_1\lambda_2 \geq (1-\pi)^2\lambda_1^1\lambda_2^1 + \pi^2\lambda_1^2\lambda_2^2. \tag{A.23}$$

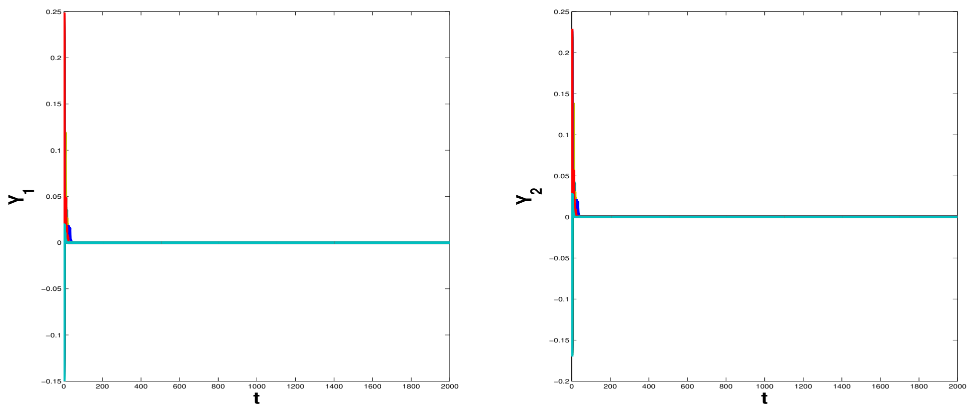
Suppose $\lambda_1^2\lambda_2^2 \geq 1$ and $\pi \geq \sqrt{\frac{1}{\lambda_1^2\lambda_1^2}}$. Since Φ^1 and Φ^2 are upper triangular matrices with nonnegative diagonal entries we have that

$$\lambda_1\lambda_2 \geq \pi^2\lambda_1^2\lambda_2^2 \geq 1. \tag{A.24}$$

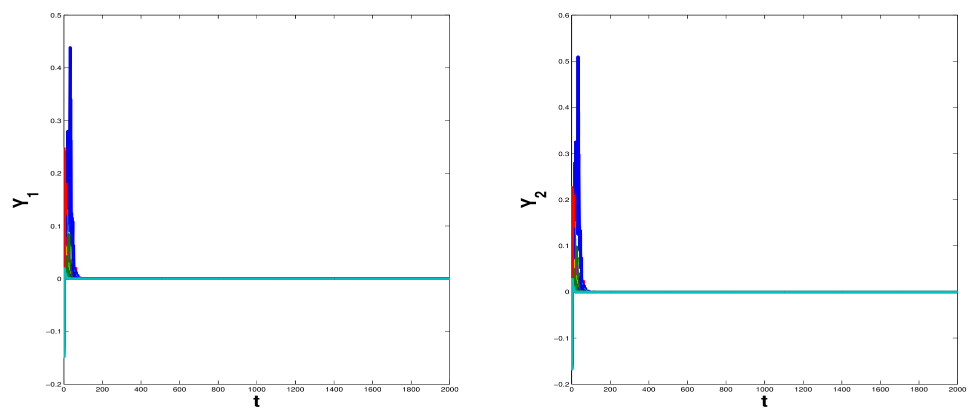
Thus, there is an eigenvalue of Φ, which is greater than 1. From Theorem 1.2 it follows that the process $Y_t, t = 0, 1, 2, ...$ is not stationary. Similarly, it can be shown that the process Y_t is not stationary when condition 2 of Corollary 5 doesn't hold.

A.8 Additional Figures

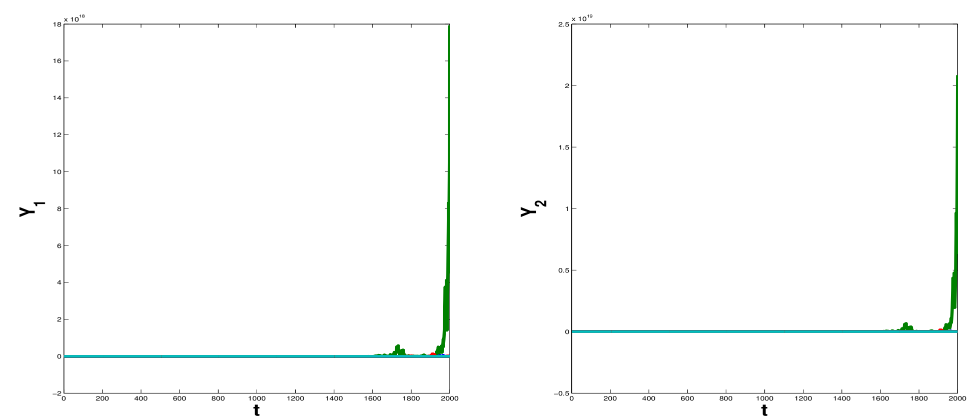
Figure A.1: Simulated paths from models 1-3



(a) Model 1, π = 0.3



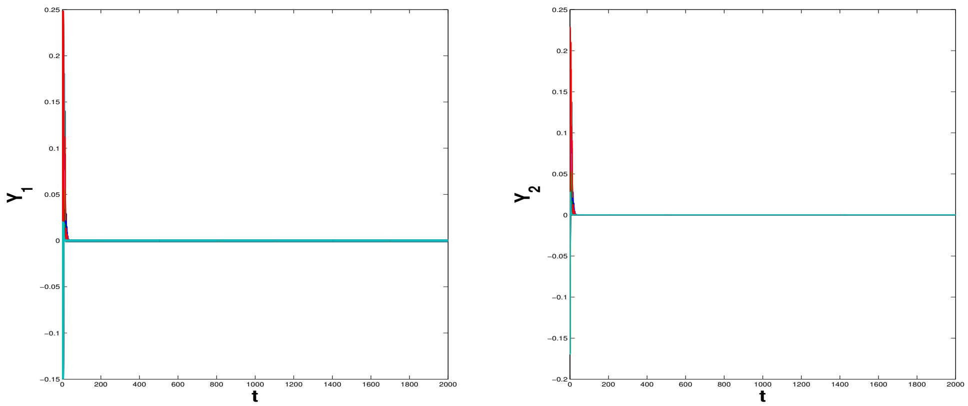
(b) Model 2, π = 0.5



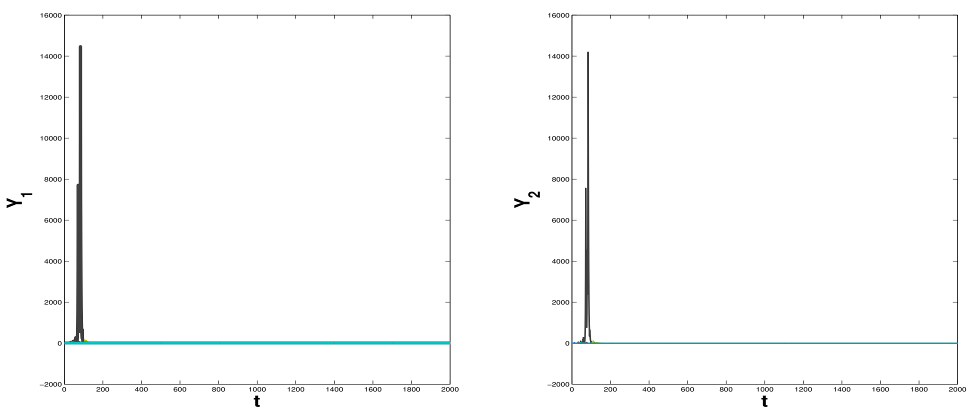
(c) Model 3, π = 0.7

Notes: This figure shows the simulated paths from models 1-3 for different set of histories over a $n = 2000$ period using $m = 200$ simulated paths. The parameters are those from Table 1.1 and the probability to be in the explosive regime 2 is equal to π.

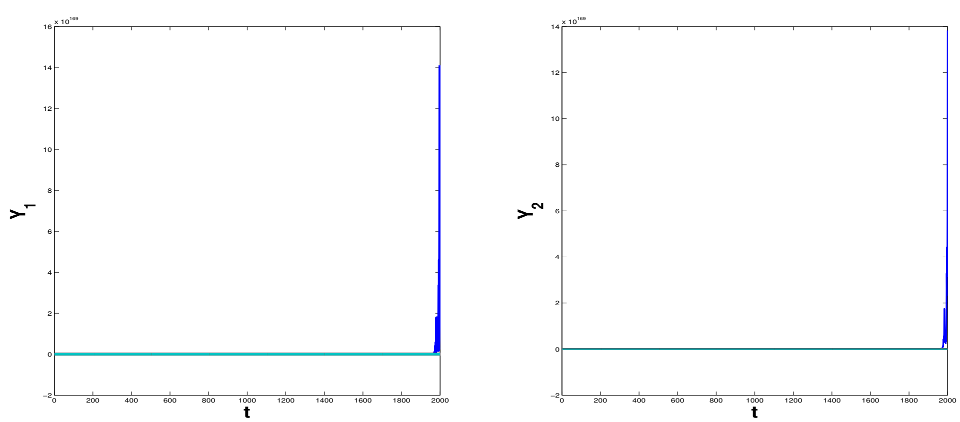
Figure A.2: Simulated paths from models 4-6



(a) Model 4, $\pi = 0.1$



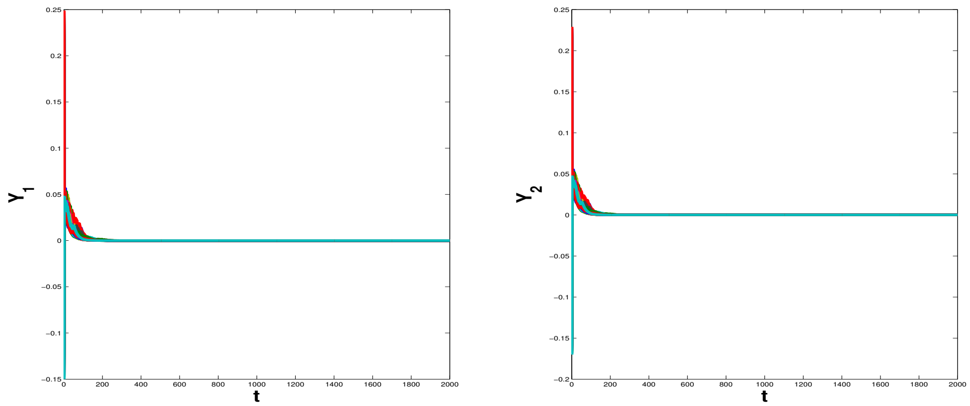
(b) Model 5, $\pi = 0.3$



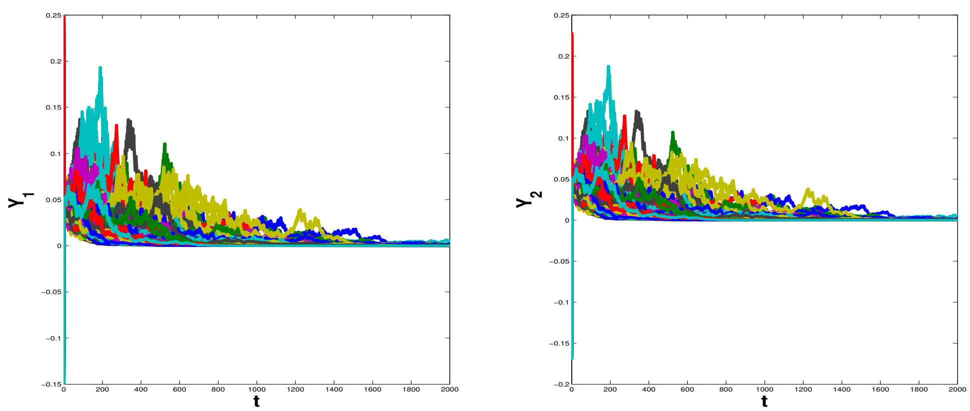
(c) Model 6, $\pi = 0.5$

Notes: This figure shows the simulated paths from models 4-6 for different set of histories over a $n = 2000$ period using $m = 200$ simulated paths. The parameters are those from Table 1.1 and the probability to be in the explosive regime 2 is equal to π.

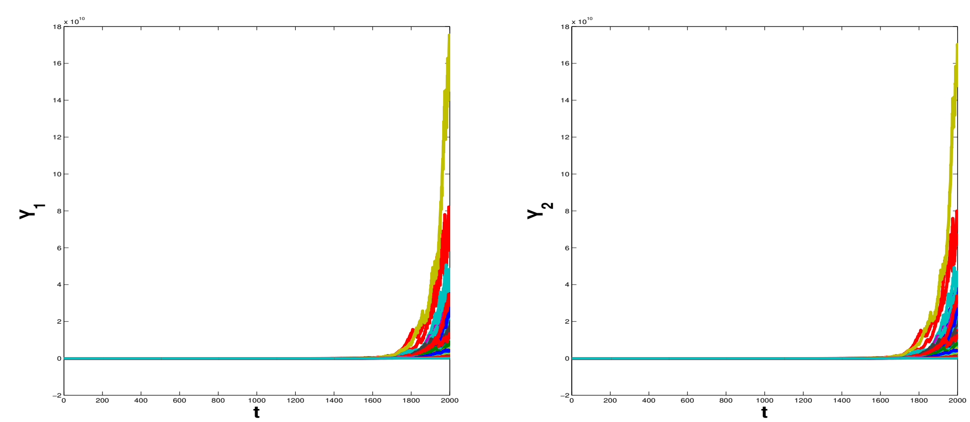
Figure A.3: Simulated paths from models 4-6



(a) Model 7, $\pi = 0.1$



(b) Model 8, $\pi = 0.3$



(c) Model 9, $\pi = 0.5$

Notes: This figure shows the simulated paths from models 7-9 for different set of histories over a $n = 2000$ period using $m = 200$ simulated paths. The parameters are those from Table 1.1 and the probability to be in the explosive regime 2 is equal to π.

Appendix B

Chapter 2 Appendix

This appendix contains the additional estimation results and estimation algorithm for IRF.

B.1 Estimation Results

Table B.1: TVAR model: Bank of America

	$ILLIQ_t$		σ_t	
	Coefficient	**(Std. Err.)**	**Coefficient**	**(Std. Err.)**
High Liquidity				
const	0.14	(0.01)*	-0.01	(0.00)*
$ILLIQ_{t-1}$	0.30	(0.03)*	0.04	(0.00)*
σ_{t-1}	0.59	(0.03)*	0.96	(0.00)*
Low Liquidity				
const	0.05	(0.03)***	-0.03	(0.01)*
$ILLIQ_{t-1}$	0.43	(0.03)*	0.11	(0.01)*
σ_{t-1}	0.17	(0.20)	0.87	(0.03)*
$ILLIQ_{t-2}$	0.18	(0.03) *	-0.01	(0.00)*
σ_{t-2}	0.19	(0.19)	0.04	(0.03)
R^2	0.66		0.99	

Notes: HAC standard errors are reported in parenthesis;
* 1% level of significance; ** 5% level of significance; *** 10% level of significance

Table B.2: TVAR model: Kimko Realty Corporation

	$ILLIQ_t$		σ_t	
	Coefficient	**(Std. Err.)**	**Coefficient**	**(Std. Err.)**
High Liquidity				
const	0.18	(0.01)*	-0.02	(0.00)*
$ILLIQ_{t-1}$	0.22	(0.02)*	0.06	(0.00)*
σ_{t-1}	0.18	(0.15)	0.93	(0.01)*
$ILLIQ_{t-2}$	0.13	(0.02)*	0.00	(0.00)
σ_{t-2}	0.33	(0.14)**	0.01	(0.00)
Low Liquidity				
const	0.04	(0.04)	-0.04	(0.01)*
$ILLIQ_{t-1}$	0.41	(0.04)*	0.15	(0.01)*
σ_{t-1}	-0.27	(0.24)	0.81	(0.04)*
$ILLIQ_{t-2}$	0.20	(0.04)*	-0.01	(0.01)
σ_{t-2}	0.68	(0.22)*	0.05	(0.04)
R^2	0.67		0.99	

Notes: HAC standard errors are reported in parenthesis;
* 1% level of significance; ** 5% level of significance; *** 10% level of significance

Table B.3: TVAR model: Dow Chemical Corporation

	$ILLIQ_t$		σ_t	
	Coefficient	**(Std. Err.)**	**Coefficient**	**(Std. Err.)**
High Liquidity				
const	0.17	(0.02)*	-0.01	(0.00)*
$ILLIQ_{t-1}$	0.26	(0.02)*	0.04	(0.00)*
σ_{t-1}	0.11	(0.18)	0.93	(0.01)*
$ILLIQ_{t-2}$	0.15	(0.02)*	0.00	(0.00)
σ_{t-2}	0.37	(0.17)**	0.03	(0.01)*
Low Liquidity				
const	0.14	(0.07)**	-0.03	(0.02)
$ILLIQ_{t-1}$	0.32	(0.05)*	0.12	(0.02)*
σ_{t-1}	-0.01	(0.28)	0.88	(0.05)*
$ILLIQ_{t-2}$	0.23	(0.04)*	-0.02	(0.01)***
σ_{t-2}	0.39	(0.26)	0.02	(0.04)
R^2	0.48		0.99	

Notes: HAC standard errors are reported in parenthesis;
* 1% level of significance; ** 5% level of significance; *** 10% level of significance

Table B.4: TVAR model: Ford Motor Comapany

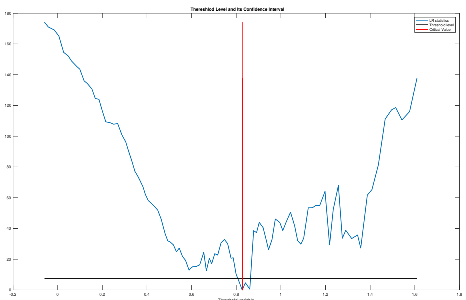
	$ILLIQ_t$		σ_t	
	Coefficient	**(Std. Err.)**	**Coefficient**	**(Std. Err.)**
High Liquidity				
const	0.17	(0.02)*	0.00	(0.00)
$ILLIQ_{t-1}$	0.31	(0.03)*	0.03	(0.00)*
σ_{t-1}	0.59	(0.03)*	0.95	(0.00)*
Low Liquidity				
const	-0.03	(0.06)	-0.02	(0.01)**
$ILLIQ_{t-1}$	0.46	(0.04)*	0.11	(0.01)*
σ_{t-1}	-0.28	(0.19)	0.84	(0.04)*
$ILLIQ_{t-2}$	0.15	(0.03)*	0.00	(0.01)
σ_{t-2}	0.72	(0.17)*	0.04	(0.03)
R^2	0.67		0.99	

Notes: HAC standard errors are reported in parenthesis;
* 1% level of significance; ** 5% level of significance; *** 10% level of significance

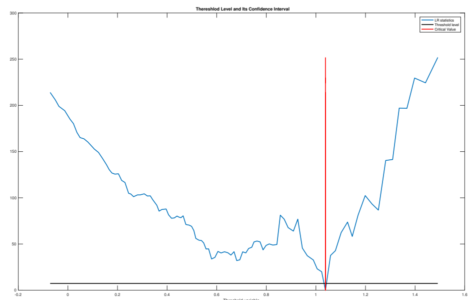
B.2 Threshold Level Estimation Results

Figure B.1: Threshold Level and Its Confidence Interval

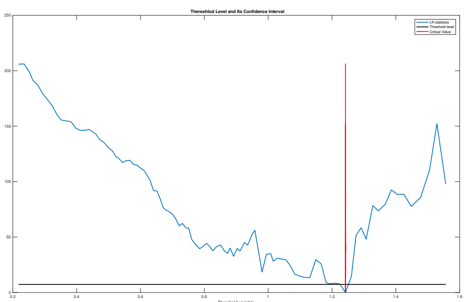
(a) Bank of America



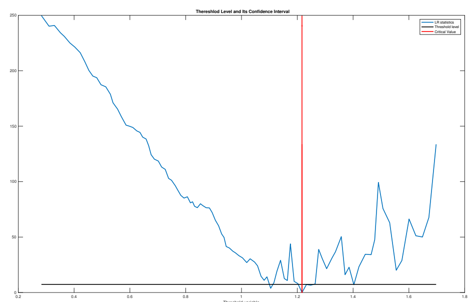
(b) Kimko Realty COrporation



(c) DOW Chemical Corporation



(d) Ford Motor Company



Notes: Blue line shows the *LR* statistic computed in (2.6) for different value of threshold level q; black line shows the estimated threshold level $\hat{\gamma}$; red line shows the estimated 95% confidence interval of $\hat{\gamma}$.

B.3 Non - Linear Impulse Response Estimation Algorithm

I estimate the following IRF

$$IRF_i = E(Y_t|\Xi_{t-1}, e_t^i) - E(Y_t|\Xi_{t-1}), \tag{2.12}$$

for initial condition being in high or low liquidity regime, $\Xi_{t-1} = \{high, low\}$, and shock to variable i takes values of $\pm 1, \pm 2$ of standard deviation. I follow the algorithm defined below:

1. First, I pick a set of histories $(\xi_1, \xi_2, ..., \xi_B)$, where $B = 500$ from a set of possible initial conditions Ξ_{t-1}, i.e. I determine the initial condition.

2. For each history ξ_b I randomly draw a set of residuals $u = (u_t, u_{t+1}, ..., u_{t+(l-1)})$ from the estimated residuals with replacement, where l denotes a forecasting horizon. I assume the joint distribution of residuals, thus residuals are simulated jointly for $ILLIQ$ and Vol.

3. I feed residuals u into the Y_t to receive a path of Y_t without shock. The regime state is defined within the model at each forecaster period.

4. I create a set of orthogonalized residuals and replace the value of the orthogonalized residual of variable i at time t by the value e_t^i. Then, by Cholesky decomposition I transform back back to the original residuals $u^* = (u_t^*, u_{t+1}, ..., u_{t+(l-1)})$.

5. I feed residuals u^* into the estimated model (**??**) to create a path of Y_t^* with shock.

6. I repeat steps 2-5 for each history ξ_b 500 times to create one observation of IRF_b in (2.12).

7. I average IRF_b over set of histories to receive a bootstaped estimation of IRF for each regime.

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