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The Renner Monoids and Cell Decompositions of the Classical Algebraic Monoids

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The Renner Monoids and Cell Decompositions of the Classical Algebraic Monoids

by

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ABSTRACT

The Renner monoids, cross section lattices and cell decompositions of the classical algebraic monoids are studied.

The Renner monoid is extremely important in the theory of reductive algebraic monoids. It is well know that the Renner monoid $\mathcal{R}_n$ of $\mathbf{M}_n(K)$ is the monoid of all zero-one matrices which have at most one entry equal to one in each row and column, i.e., $\mathcal{R}_n$ consists of injective partial maps on a set of n elements. We obtain that the Renner monoids of the symplectic algebraic monoids and special orthogonal algebraic monoids turn out to be submonoids of $\mathcal{R}_n$ consisting of symplectic and special orthogonal 1-1 partial maps, respectively. The cardinalities of the Renner monoids are obtained, as well.

The cross section lattice is another very important concept in the theory of irreducible algebraic monoids. The cross section lattices of the symplectic and special orthogonal algebraic monoids are explicitly characterized.

The cell decompositions of symplectic algebraic monoids and special orthogonal monoids are explicitly determined. Each cell here turns out to be an intersection of the monoid with some cell of $\mathbf{M}_n(K)$.

KEYWORDS: Renner monoid, reductive monoid, cross section lattice, cell decomposition, symplectic algebraic monoid, special orthogonal algebraic monoid, the Weyl group, Borel subgroup.

DEDICATION

I would like to dedicate this thesis to my grandmother Yongxiu Guo, my father Shaoyuan Li, my mother Huaping Li and my wife Xiuzhen Lu. They have given me a great deal of support and love in my life.

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CHAPTER I

INTRODUCTION

The objective of this thesis is to determine the Renner monoids and cell decompositions of the symplectic monoids and special orthogonal monoids.

The two pioneers in the development of the theory of linear algebraic monoids are M. Putcha and L. Renner. They originated this area independently around 1980. Over the last two decades the Putcha-Renner theory of linear algebraic monoids has made possible significant progress in a number of areas: algebraic groups, Lie theory, abstract semigroups, algebraic combinatorics, Hecke algebras, etc. [see 5, 9 11–22].

1. The Renner Monoids

The Renner monoid is an extremely important concept in the theory of reductive monoids. It generalizes the Weyl group from algebraic groups to monoids. Actually, the unit group of the Renner monoid is a Weyl group. Let M be a reductive algebraic monoid, $T \subseteq G$ a maximal torus of the unit group G, $B \subseteq G$ a Borel subgroup with $T \subseteq B$, N the normalizer of T in G, $\overline{N}$ the Zariski cloure of N in M. Then $\overline{N}$ is a unit regular inverse monoid which normalizes T, so $\mathcal{R} = \overline{N}/T$ is a monoid. Thus

$$\mathcal{R} = \overline{N}/T \supseteq N/T = W, \text{ the Weyl group.}$$

Not only did Renner [17] define this concept, now called the Renner monoid, but he also found an analogue of the Bruhat decomposition for reductive algebraic monoids and obtained a monoid version of the Tits System which is now a central idea in the structure theory of linear algebraic monoids.

If $M = \mathbf{M}_n$, then the Renner monoid $\mathcal{R}_n$ of M may be identified with the monoid of all zero-one matrices which have at most one entry equal to one in each row and column, i.e., $\mathcal{R}_n$ consists of all injective, partial maps on a set of n elements. The cardinality of $\mathcal{R}_n$ is $|\mathcal{R}_n| = \sum_{r=0}^{n} \binom{n}{r}^2 r!$. The unit group of $\mathcal{R}_n$ is the group P_n of permutation matrices. Let G_0 be the symplectic algebraic group $\mathbf{Sp}_n$ or the special orthogonal algebraic group $\mathbf{SO}_n$ (char $K \neq 2$, if $G_0 = \mathbf{SO}_n$, see Humphreys [7]). Let $G = K^* G_0 \subset \mathbf{GL}_n$. Then G is a connected reductive group and $M = \overline{G}$, Zariski closure of G in $\mathbf{M}_n$, is a reductive algebraic monoid called symplectic or special orthogonal depending on whether $G_0 = \mathbf{Sp}_n$ or $G_0 = \mathbf{SO}_n$. In this thesis we study the Renner monoids of the symplectic monoids and the special orthogonal monoids. They turn out to be submonoids of $\mathcal{R}_n$. Their unit groups are the Weyl groups of the symplectic algebraic groups and the special orthogonal algebraic groups respectively. We established their cardinalities as well.

2. Cross Section Lattices

The cross section lattice is another key concept in the theory of irreducible algebraic monoids. It was first introduced by M. Putcha [13]. If M is reductive, then the cross section lattice is defined as follows

$$\Lambda = \Lambda(B) = \{e \in E(\overline{T}) | Be = eBe\}.$$

We discuss the cross section lattices of the symplectic and the special orthogonal algebraic monoids in detail.

3. Cell Decompositions

The most commonly studied cell decmopositions in algebraic geometry are the ones obtained by the method of Bialynicki-Birula: Let K be an algebraically closed field. If $S = K^*$ acts on a smooth complete variety X with finite fixed point set $F \subseteq X$, then $X = \bigsqcup_{\alpha \in F} X_\alpha$, where $X_\alpha = \{x \in X \mid \lim_{t \to 0} tx = \alpha\}$. Furthermore, X_α is isomorphic to an affine space. If, further, a semisimple group G acts on X extending the action of S, we may assume (replacing S if necessary) that each X_α is stable under the action of some Borel subgroup B of G with $S \subseteq B$. In case X is a complete homogeneous space for G, each cell X_α turns out to consist of exactly one B-orbit.

In case $X = M$, a reductive algebraic monoid with the two-sided G-action on it, we encounter the following challenging difficulties:

(i) Each [BB]-cell is an intriguing finite union of $B \times B$-orbits, yet there is no explicit algorithm for deciding how each cell is made up from the $B \times B$-orbits. On the other hand, the set of $B \times B$-orbits has been calculated explicitly [17, 18].

(ii) One hopes to find a good "cell" decomposition for any reductive monoid. However, the $[BB]$-procedure has not been developed to yield discriminating results in the presence of singularities. What we need in this situation is a more direct definition of cells; initially for reductive algebraic monid M where $M \setminus \{0\}$ has exactly one minimal nonzero $G \times G$-orbit J (eg: $M = \mathbf{M}_n(K)$ and $J = \{x \in M \mid \text{rank}(x) = 1\}$).

The more direct definition of cells is as follows (due to L. Renner). Let $B \subseteq M$ be a Borel subgroup with $T \subseteq B$ a maximal torus and let $r \in \mathcal{R}(1)$, rank one elements in the Renner monoid. Then there exist unique $e, f \in E_1(\overline{T})$, the set of rank one idempotents in $\overline{T}$, such that $r = erf$. Define $C_r = \{y \in M \mid eBy = eBey \subseteq rB\}$.

L. Renner has already obtained the following results:

(i) $M \setminus \{0\} = \bigsqcup_{r \in \mathcal{R}(1)} C_r$.

(ii) Any [BB]-decomposition of $(M \setminus \{0\})/K^*$ with finite fixed point set has exactly $|\mathcal{R}(1)|$ cells.

(iii) For $M = \mathbf{M}_n(K)$ let $r = E_{ij}$, the matrix unit, where $i, j = 1, ..., n$. Then

$$C_{ij}(K) = C_{E_{ij}} = \left\{ (a_{pq}) \in \mathbf{M}_n(K) \;\middle|\; \begin{array}{l} a_{ij} \neq 0;\ a_{pq} = 0, \text{ if } i < p \leq n, \\ \text{or if } p = i \text{ and } 1 \leq q \leq j-1 \end{array} \right\}.$$

We explicitly determine cell decompositions of the symplectic algebraic monoids and the special orthogonal algebraic monoids. Each cell turns out to be an intersection of the monoid with a cell $C_{ij}(K)$ of $\mathbf{M}_n(K)$.

CHAPTER II

FACTS ABOUT THE MATRIX MONOIDS

Let K be an algebraically closed field. Let $\mathbf{M}_n = \mathbf{M}_n(K)$ denote the set of all $n \times n$ matrices over K. Then $\mathbf{M}_n$ is an algebraic monoid with the general linear group $\mathbf{GL}_n = \mathbf{GL}_n(K)$ as its unit group, and $\overline{\mathbf{GL}}_n = \mathbf{M}_n$, the Zariski closure of $\mathbf{GL}_n$ in $\mathbf{M}_n$. Let

$$\mathbf{B}_n = \mathbf{B}_n(K) = \{(a_{ij}) \in \mathbf{M}_n(K) |\ a_{ij} = 0, \text{ if } \ i > j\}$$

be a Borel subgroup of $\mathbf{GL}_n$. The monoid $\mathbf{D}_n = \mathbf{D}_n(K)$ consists of diagonal matrices in $\mathbf{M}_n$. The subgroup $\mathbf{T}_n = \mathbf{T}_n(K)$ of $\mathbf{D}_n$ consisting of all invertible diagonal matrices is a maximal torus of $\mathbf{GL}_n$, and $\overline{\mathbf{T}}_n = \mathbf{D}_n$ is the Zariski closure of $\mathbf{T}_n$ in $\mathbf{M}_n$. We use $\mathcal{R}_n = \mathcal{R}_n(K)$ to denote the Renner monoid of $\mathbf{M}_n$. Then

$$\mathcal{R}_n = \left\{ (a_{ij}) \in \mathbf{M}_n \;\middle|\; \begin{array}{l} a_{ij} \text{ is 0 or 1, and at most one non-} \\ \text{zero entry in each row and column} \end{array} \right\}.$$

The set of the idempotents of $\mathcal{R}_n$ is:

$$E(\mathcal{R}_n) = \{(a_{ij}) \in \mathbf{D}_n |\ a_{ij} = 0 \text{ or } 1, \text{ for all } i, j\}.$$

The cross section lattice of $\mathbf{M}_n$ is

$$\begin{aligned} \Lambda &= \Lambda(\mathbf{B}_n) \\ &= \{e \in E(\overline{\mathbf{T}}_n) \mid \mathbf{B}_n e = e\mathbf{B}_n e\} \\ &= \left\{ \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, (0) \right\}. \end{aligned}$$

It is well known that the unit group of $\mathcal{R}_n$ is the Weyl group of $\mathbf{GL}_n$ which is isomorphic to the symmetric group S_n on n letters. Let $P_n \subseteq \mathbf{GL}_n$ be the group of *permutation matrices*. Then S_n is isomorphic to P_n by the mapping $\pi \longmapsto \sum_{j=1}^{n} E_{\pi j,j}$ where $\pi \in S_n$ and $E_{\pi j,j}$ is a matrix unit.

CHAPTER III

SYMPLECTIC MONOIDS MSp_n

In this chapter we determine the Renner monoids and the cell decompositions of the symplectic algebraic monoids.

Let $n = 2l$ be even and $J_l = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \in \mathbf{M}_n$ be the nonsingular and skew symmetric matrix, where $J = \begin{pmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{pmatrix}$ of size l. The symplectic group is by definition

$$G_0 = \mathbf{Sp}_n = \{g \in \mathbf{GL}_n |\ g^T J_l g = J_l\}$$

which is connected and reductive.

Remark 1. The definition of $\mathbf{Sp}_n$ here is what J. Humphreys used in his book [7]. It is different from that of L. Solomon [22].

Let $T_0 = G_0 \cap \mathbf{T}_n$. Elements in T_0 have the shape

$$t = \text{diag}(t_1, ..., t_l, t_l^{-1}, ..., t_1^{-1})$$

where $t_1, ..., t_l$ are arbitrary in K^*. Thus T_0 is a maximal torus of dimension l. Let us recall some facts about the Weyl group $W(G_0, T_0)$ which will be simply denoted by W if there is no confusion in the context. If $\pi \in S_n$ let $p_\pi = \sum_{i=1}^n E_{\pi i,i} \in P_n$ be the corresponding permutation matrix. Then $p_\pi(a_{ij}) = (a_{\pi^{-1}i,j})$, and $(a_{ij})p_\pi = (a_{i,\pi j})$ where (a_{ij}) is any $n \times n$ matrix. It follows that $p_{\pi^{-1}}(a_{ij})p_\pi = (a_{\pi i,\pi j})$. Define an involution $\theta : i \longmapsto \bar{i}$ of $\{1, 2, ..., n\}$ by

$$\bar{i} = 2l + 1 - i, \qquad \text{for } 1 \leq i \leq 2l.$$

Let C denote the centralizer of θ in S_n. Then p_π normalizes T_0 if and only if $\pi \in C$. The group C is a semidirect product $C = C_1C_2$ where C_1 is a normal abelian subgroup of order 2^l generated by the transpositions $(1\bar{1}), ..., (l\bar{l})$ and $C_2 \simeq S_l$ consists of all permutations $\pi \in S_n$ which stablize $\{1, ..., l\}$ and act on the complement $\{l+1, ..., 2l\}$ in the unique manner consistent with the assertion that $\pi \in C$. Then W is isomorphic to C_1C_2 which is also isomorphic to $(Z_2)^l \rtimes S_l$.

If $n = 4$, then $\theta = (1\bar{1})(2\bar{2}) = (14)(23)$, and C_1 is a subgroup of $C_{S_4}(\theta)$ generated by (14) and (23). So

$$C_1 = \{1, (14), (23), (14)(23)\}.$$

Taking $\pi = (12)(34)$, we see that $\theta\pi = \pi\theta$ which means that $\pi \in C_{S_4}(\theta)$. It is clear that π stablizes $\{1, ..., l\} = \{1, 2\}$ and $\pi \notin C_1$. Let C_2 be a subgroup of $C_{S_4}(\theta)$ generated by π. Then

$$C_2 = \{1, (12)(34)\}.$$

Thus the Weyl group

$$W = C_1C_2 = \{1, (14), (23), (14)(23), (12)(34), (1243), (1342), (13)(24)\}.$$

The corresponding matrix form of the Weyl group is

$$W = \left\{ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix} \right\}.$$

$$= \{1, \rho, \sigma, \theta, \pi, \rho\pi, \sigma\pi, \theta\pi\}.$$

4. The Renner Monoids of the Symplectic Monoids MSp_n

We compute the Renner monoids of the symplectic algebraic monoids MSp_n in this section. Some by-products are obtained as well, such as the cardinalities of the Renner monoids.

Let $G = K^*G_0 \subseteq \mathbf{GL}_n$ where $n = 2l$. Then G is a connected reductive group with rank $r = l + 1$ and semisimple rank l. Let $T = K^*T_0$. Then T is a maximal torus of G, and the Weyl group $W(G, T)$ is isomorphic to $W(G_0, T_0)$ (see [22]). We identify them in what follows and let W denote either of these groups. The Weyl group plays an important role in identifying the Renner monoids.

4.1. Definition. *The monoid $\overline{G}$, Zariski closure of G in $\mathbf{M}_n(K)$, is called the symplectic monoid which will be denoted by MSp_n.*

In this section we compute the Renner monoid $\mathcal{R}$ of the symplectic monoid. To do so, we need the following definition (due to Solomon [22]).

4.2. Definition. *A subset $I \subseteq \{1, ..., n\}$ is called admissible if $j \in I$ implies $\bar{j} \notin I$, where $\bar{j} = \theta(j)$ as above; the empty set ϕ and the whole set $\{1, ..., n\}$ are also considered to be admissible.*

If $n = 4$, then all the admissible subsets of $\{1, 2, 3, 4\}$ are

$$\phi,\ \{1\},\ \{2\},\ \{3\},\ \{4\},\ \{1, 2\},\ \{3, 4\},\ \{1, 3\},\ \{2, 4\},\ \{1, 2, 3, 4\}.$$

An admissible subset I is referred to as standard if $I = \phi$, or there is an integer $i \in \{1, ..., l, 2l\}$ such that $I = \{1, ..., i\}$.

The standard admissible subsets of $\{1, 2, 3, 4\}$ are ϕ, $\{1\}$, $\{1, 2\}$, $\{1, 2, 3, 4\}$.

A similar discussion to [22, p336] gives the following lemma describing the relationship between admissible subsets and idempotents in $\overline{T}$. We omit those details.

4.3. Lemma.

a) W maps admissible sets to admissible sets, and $w^{-1}e_I w = e_{wI}$, for $w \in W$.

b) The map

$$I \longmapsto e_I = \sum_{j \in I} E_{jj}$$

is bijective from the admissible subsets of $\{1, ..., n\}$ to $E(\overline{T})$, where $e_I = 0$, if $I = \phi$.

c) The set $E(\overline{T})$ of idempotents in $\overline{T}$ is

$$E(\overline{T}) = \{e_I \mid I \subseteq \{1, ..., n\} \text{ is admissible}\}.$$

d) $e_{I_1} \cdot e_{I_2} = e_{I_1 \cap I_2}$, for any $e_{I_1}, e_{I_2} \in E(\overline{T})$.

Proof. For a), b) and c), see [22, p336]. By checking directly, we get d). □

If $n = 4$, the set of idempotents of MSp_4 is

$$E(\overline{T}) = \{0, 1, E_{11}, E_{22}, E_{33}, E_{44}, E_{11} + E_{22}, E_{33} + E_{44}, E_{11} + E_{33}, E_{22} + E_{44}\}.$$

Remark 2. Rank one elements in $E(\overline{T})$ are in one to one correspondence with the admissible subsets containing exactly one element of $\{1, ..., n\}$. There are no admissible subsets with size k $(l < k < 2l)$.

4.4. Proposition.

$$|E(\mathcal{R})| = |E(\overline{T})| = \sum_{i=0}^{l} \sum_{j=0}^{i} \binom{l}{j} \binom{l-j}{i-j} + 1.$$

Proof. It is true by counting the number of admissible subsets of $\{1, ..., n\}$ and [17, Proposition 3.2.1]. □

For $i = 1, ..., l$, let $E_i(\mathcal{R}) \subseteq E(\mathcal{R})$ (resp. $E_i(\overline{T}) \subseteq E(\overline{T})$) denote the set of rank i idempotent elements in $\mathcal{R}$ (resp. $\overline{T}$). Then by Lemma 4.3 and [17, Proposition 3.2.1] we have the following

4.5. Proposition.

a) $E_1(\mathcal{R}) = E_1(\overline{T}) = \{E_{ii} |\ i = 1, ..., n\}$.

b) $|E_1(\mathcal{R})| = |E_1(\overline{T})| = n$.

We now find the set of rank one elements in $\mathcal{R}$. To this end, let $\mathcal{R}(i)$ denote the set of rank i elements in the Renner monoid $\mathcal{R}$, for $i \in \{1, ..., n\}$,

4.6. Lemma. *Under the notation above, one has*

a) $\mathcal{R}(1) = \{E_{ij} \mid i, j = 1, ..., n\}$.

b) $|\mathcal{R}(1)| = n^2$.

Proof. For any $j \in \{1, ..., n\}$, let

$$w = \begin{cases} (1\bar{1}), & \text{if } j = \bar{1} \\ (1j)(\bar{1}\bar{j}), & \text{if } j \neq \bar{1} = 2l. \end{cases}$$

Then w is in the Weyl group W and $j = w(1)$. It follows that

$$E_{1j} = E_{1,w1} = E_{11}w \in E_{11}W, \text{ for } j = 1, ..., n.$$

Thus $E_{11}W = \{E_{1j} |\ j = 1, ..., n\}$.

Similarly, $E_{ii}W = \{E_{ij} \mid j = 1, ..., n\}$, for $i = 2, ..., n$. Therefore, the set of rank one elements in the Renner monoid $\mathcal{R}$ is $\{E_{ij} \mid i, j = 1, ..., n\}$ which proves a).

It is clear that b) follows from a). □

Remark 3. The lemma above shows that $\mathcal{R}(1) = \mathcal{R}_n(1)$. However, $\mathcal{R}(2) \neq \mathcal{R}_n(2)$, since $\{1, 2l\}$ is not an admissible subset of $\{1, ..., n\}$, but $E_{11} + E_{2l,2l} \in \mathcal{R}_n(2)$. For the same reason, we know $\mathcal{R}(i) \neq \mathcal{R}_n(i)$, for $i = 3, ..., n$.

4.7. Theorem. *For any admissible subset* $I \subseteq \{1, ..., n\}$ *with* $|I| = i$*, where* $i = 1, ..., l, 2l$*, there exist* $w \in W$ *and a unique standard admissible subset* $I_0 = \{1, ..., i\}$ *such that* $wI = I_0$*.*

Proof. If $I = \{1, ..., n\}$, then $I_0 = I$ and $w = 1 \in W$ and we are done. Now let I be admissible and $I \neq \{1, ..., n\}$. Then $|I| = i \in \{1, ..., l\}$. We use induction on the size i of the admissible subset I. If $i = 1$, then $I = \{j\}$ for some $j \in \{1, ..., n\}$. From Lemma 4.6 there exists $w \in W$ such that $w(I) = \{w(j)\} = \{1\}$, i.e., $wI = I_0$.

Suppose that $I \subseteq \{1, ..., n\}$ is any admissible subset with $1 < |I| \leq l$. Then $I = J \cup \{k\}$ where J is a subset of I with $|J| = i - 1$ and $k \in I \setminus J$. It follows that J is admissible. By the induction hypothesis there exist $w' \in W$ and a unique standard admissible subset $I' = \{1, ..., i-1\}$ such that $w'J = I'$. Then $w'I = I' \cup \{p\}$ where $p = w'(k)$, and hence $w'I = \{1, ..., i-1\} \cup \{p\}$ is a disjoint union. If $p = i$, then $I_0 = I' \cup \{i\}$ and $w = w'$ are what we want. If $p \neq i$, let

$$w_1 = \begin{cases} (i\bar{i}), & \text{if } p = \bar{i} \\ (ip)(\bar{i}\bar{p}), & \text{if } p \neq \bar{i} = 2l + 1 - i. \end{cases}$$

It follows that $w_1 \in W$ and $w_1(p) = i$. Note that $\bar{p} \notin w'I$, since $p \in w'I$ which is admissible. It follows that $p, i, \bar{p}, \bar{i} \notin I'$, and so $w_1(j) = j$, for $j \in I' = \{1, ..., i-1\}$. Taking $w = w_1 w'$, we obtain

$$\begin{aligned} w(I) &= w_1(w'(I)) \\ &= w_1(\{1, ..., i-1\} \cup \{p\}) \\ &= \{1, ..., i\}. \end{aligned}$$

Hence, the result is as stated. □

4.8. Proposition. *The cross section lattice*

$$\Lambda = \{e_I \in E(\overline{T}) \mid I \textit{ is a standard admissible subset of } \{1,...,n\}\}.$$

$$= \left\{ I_{2l}, \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I_{l-1} & 0 \\ 0 & 0 \end{pmatrix}, \ \cdots \ , \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}, (0) \right\}.$$

$$\simeq \{I \mid I \textit{ is a standard admissible subset of } \{1,...,n\}\}, \textit{ under the inclusion.}$$

$$\simeq \{0, 1, ..., l, l+1\}, \textit{ under the linear order.} \quad \square$$

The cross section lattice of MSp_4 is

$$\Lambda = \left\{ \begin{pmatrix} 0 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 1 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} \right\}.$$

The Hasse diagram of the partial order structure for the cross section lattice in Proposition 4.8 is given by

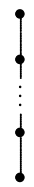

Figure 1.

4.9. Theorem. *With the notation above, the Renner monoid of the symplectic monoid MSp_{2l} is as follows*

$$\mathcal{R} = \left\{ \sum_{i \in I, w \in W} E_{i,wi} \in \mathcal{R}_{2l} \mid I \subseteq \{1,...,2l\} \textit{ is admissible} \right\}.$$

Proof. Since $\mathcal{R} = E(\overline{T})W$ by [17, Proposition 3.2.1], it suffices to compute $e_I w$ for every $e_I \in E(\overline{T}), w \in W$, where I is admissible. From Lemma 4.3 b) we know $e_I = \sum_{i \in I} E_{ii}$. Thus $e_I w = \sum_{i \in I} E_{ii} w = \sum_{i \in I} E_{i,wi}$, and so the Theorem. $\square$

4.10. Corollary.

$$\mathcal{R} = \left\{ \sum_{i\in I, w\in W} E_{wi,i} \in \mathcal{R}_{2l} \mid I \subseteq \{1, ..., 2l\} \textit{ is admissible} \right\}.$$

Proof. This result comes from the fact that $\mathcal{R} = WE(\overline{T})$ and

$$w^{-1}e_{_I} = \sum_{i\in I} w^{-1}E_{ii} = \sum_{i\in I} E_{wi,i}. \quad \square$$

Now we think of $\mathcal{R}_{2l}$, the Renner monoid of $\mathbf{M}_n(K)$, as the set of partial 1-1 maps from $\{1, ..., 2l\}$ to itself. In other words, $x \in R_{2l}$ is a 1-1 map from a subset of $\{1, ..., 2l\}$ to $\{1, ..., 2l\}$.

4.11. Theorem.

$$\mathcal{R} \setminus W = \left\{ x \in \mathcal{R}_{2l} \,\middle|\, \begin{array}{l} x \textit{ is singular; both } D(x) \\ \textit{and } R(x) \textit{ are admissible} \end{array} \right\}$$

where $D(x)$ is the domain of x, and $R(x)$ is the range of x.

Proof. Let $\mathcal{R}'$ denote the set of the right hand side in the Theorem. It follows from Theorem 4.9 that $\mathcal{R} \setminus W \subseteq \mathcal{R}'$, since W maps admissible sets to admissible sets.

We now prove the other inclusion. For any $x \in \mathcal{R}'$, one knows $|D(x)| = |R(x)|$ which will be denoted by i. Then $i \leq l$, since x is singular and both $D(x)$ and $R(x)$ are admissible. It follows from Theorem 4.7 that there exist a unique standard admissible set $I_0 = \{1, ..., i\}(i \leq l)$ and $w_1, w_2 \in W$ such that

$$w_1 D(x) = w_2 R(x) = I_0.$$

Thus $w_1^{-1} x w_2 = e_{I_0} \in \Lambda \subseteq \mathcal{R}$, and hence $x = w_1 e_{I_0} w_2^{-1} \in \mathcal{R}$, since $\mathcal{R} = WE(\overline{T}) = E(\overline{T})W$. But $x \notin W$, because $x \in \mathcal{R}_{2l}$ is singular. So, $\mathcal{R}' \subseteq \mathcal{R} \setminus W$.

Therefore, $\mathcal{R} \setminus W = \mathcal{R}'$, i.e., the Theorem is true. $\square$

Remark 4. In the proof above we obtain $\mathcal{R} = W\Lambda W$ as well.

Now let us consider some examples. If $n = 2$, then all the admissible subsets of $\{1,2\}$ are $\phi, \{1\}, \{2\}, \{1,2\}$. Note that they are all standard except $\{2\}$. So, the cross section lattice of MSp_2 by Proposition 4.8 is

$$\Lambda = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

From Theorem 4.11, the Renner monoid of MSp_2 is

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

On the other hand, note that $\mathbf{Sp}_2 = \mathbf{SL}_2$ by checking the definitions directly. Then $MSp_2 = \mathbf{M}_2$. The Renner monoid of $\mathbf{M}_2$ was obtained first by L. Renner in 1986 (see [17, Proposition 4.1.1]). Our result here for MSp_2 matches that of L. Renner for $\mathbf{M}_2$.

The Renner monoid of MSp_4 is given by

$$\begin{aligned}
\mathcal{R} = \{ & 0,\ E_{11},\ E_{12},\ E_{13},\ E_{14},\ E_{21},\ E_{22},\ E_{23},\ E_{24},\ E_{31},\ E_{32},\ E_{33},\ E_{34}, \\
& E_{41},\ E_{42},\ E_{43},\ E_{44},\ E_{11}+E_{22},\ E_{22}+E_{41},\ E_{11}+E_{32},\ E_{32}+E_{41}, \\
& E_{34}+E_{42},\ E_{12}+E_{31},\ E_{21}+E_{42},\ E_{12}+E_{21},\ E_{33}+E_{44},\ E_{14}+E_{33}, \\
& E_{23}+E_{44},\ E_{14}+E_{23},\ E_{13}+E_{24},\ E_{24}+E_{43},\ E_{13}+E_{34},\ E_{34}+E_{43}, \\
& E_{11}+E_{33},\ E_{33}+E_{41},\ E_{11}+E_{23},\ E_{23}+E_{41},\ E_{13}+E_{31},\ E_{31}+E_{43}, \\
& E_{13}+E_{21},\ E_{21}+E_{43},\ E_{22}+E_{44},\ E_{14}+E_{22},\ E_{32}+E_{44},\ E_{14}+E_{32}, \\
& E_{24}+E_{42},\ E_{12}+E_{24},\ E_{34}+E_{42},\ E_{12}+E_{34}, E_{11}+E_{22}+E_{33}+E_{44}, \\
& E_{14}+E_{22}+E_{33}+E_{41}, E_{11}+E_{23}+E_{32}+E_{44}, E_{14}+E_{23}+E_{32}+E_{41}, \\
& E_{13}+E_{24}+E_{31}+E_{42}, E_{12}+E_{24}+E_{31}+E_{43}, E_{13}+E_{21}+E_{34}+E_{42}, \\
& E_{12}+E_{21}+E_{34}+E_{43} \}.
\end{aligned}$$

We now express the Renner monoid of MSp_4 in terms of matrices.

$$
\begin{aligned}
\mathcal{R} = \Bigg\{ 0, \; & \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \\
& \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \\
& \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix},
$$

$$
\begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&1&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&1&0 \end{pmatrix},
$$

$$
\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&1\\ 0&1&0&0\\ 0&0&1&0\\ 1&0&0&0 \end{pmatrix},
\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix},
$$

$$
\left.\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix},
\begin{pmatrix} 0&1&0&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&0&1&0 \end{pmatrix},
\begin{pmatrix} 0&0&1&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix},
\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}\right\}.
$$

The following result is an analogue of Proposition 7.3 of [17].

4.12. Proposition. *For any $e_I \in \Lambda$ with $|I| = i$, where $i = 0, 1, ..., l$,*

$$
\begin{aligned}
We_IW &= \{x \in \mathcal{R} \mid \ \textit{rank}(x) = i\} \\
&= \{x \in \mathcal{R} \mid x \textit{ has } i \textit{ nonzero rows}\} \\
&= \{x \in \mathcal{R}_{2l} \mid D(x) \textit{ and } R(x) \textit{ are admissible with } |D(x)| = |R(x)| = i \ \},
\end{aligned}
$$

where $D(x)$ is the domain of x and $R(x)$ the range of x. Furthermore,

$$
|We_IW| = \left[\sum_{j=0}^{i} \binom{l}{j}\binom{l-j}{i-j}\right]^2 i!.
$$

Proof. Observe that $Ge_IG = \bigsqcup_{x\in We_IW} BxB$ consists of $n \times n$ matrices of rank i in MSp_{2l} where $i = |I| = 0, 1, ..., l$. We obtain the first part of the Proposition.

Now there are $\sum_{j=0}^{i} \binom{l}{j}\binom{l-j}{i-j}$ ways to choose i of the n rows such that $D(x)$ is admissible. There are the same number of ways to choose i of the n columns making

$R(x)$ admissible. For each of these choices there are $i!$ elements of $\mathcal{R}$, of rank i, with a nonzero entry in each of the i rows and each of the i columns chosen. Thus there are $\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!$ possibilities. $\square$

4.13. Corollary. $|\mathcal{R}| = \sum_{i=0}^{l}\left(\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!\right) + 2^l l!,$ *for* $l \geq 1$.

For example, the Renner monoid of symplectic monoid MSp_4 has 57 elements.

5. Cell Decompositions of the Symplectic Monoids MSp_n

The main purpose of this section is to determine the cell decompositions of the symplectic algebraic monoids MSp_n. Each cell turns out to be an intersection of MSp_n with a cell $C_{ij}(K)$ of $\mathbf{M}_n(K)$.

Let $B_0 = \mathbf{B}_n \cap \mathbf{Sp}_n$. Then B_0 is a Borel subgroup of $\mathbf{Sp}_n$, and $B = K^*B_0$ is a Borel subgroup of $G = K^*\mathbf{Sp}_n$. A simple calculation tells us that

$$B_0 = \left\{ \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in \mathbf{Sp}_{2l} \;\middle|\; \begin{matrix} b_1, b_3 \in \mathbf{M}_l(K) \text{ are upper triangular,} \\ b_3^T J b_1 = J, \quad b_3^T J b_2 = b_2^T J b_3 \end{matrix} \right\}.$$

More concretely,

$$B_0 = \left\{ \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in \mathbf{Sp}_{2l} \;\middle|\; \begin{matrix} b_1 \in \mathbf{M}_l(K) \text{ is upper triangular,} \quad b_2 = b_1 a J \text{ for} \\ \text{some } a \in \mathbf{M}_l(K) \text{ symmetric, } b_3 = J(b_1^{-1})^T J \end{matrix} \right\}.$$

We can now find a relationship among the Borel subgroup B, idempotents and rank one elements in $E(\overline{T})$.

5.1. Theorem. *Let* $T = K^*T_0 \subseteq B$ *be a maximal torus in* G. *Then for every non-zero idempotent* $e_I \in E(\overline{T})$, *there exists a unique* $e_i = E_{ii} \in E_1(\overline{T})$ *such that* $e_i B e_I = e_i B e_i$, *where* i *is the maximal number in* I.

Proof. Let $e_{_I} \in E(\overline{T})$ where $I = \{i_1, i_2, ..., i_m\} \subseteq \{1, ..., n\}$ is admissible with $i_1 < i_2 < ... < i_m$. For any upper triangular matrix $b = (b_{jk}) \in B \subseteq G$, the matrix $be_{_I}$ is an upper triangular matrix whose k-th column is exactly the k-th column of $b = (b_{jk})$ for $k = i_1, ..., i_m$, and the other columns of $be_{_I}$ are all zero. Let $i = i_m$ which is maximal in I. Taking $e_i = E_{ii}$, we get $e_i be_{_I} = E_{ii} be_{_I}$ is a matrix whose $(i,i)-$entry is b_{ii} and the other entries are all zero. It follows that $e_i be_{_I} = e_i be_i$ by calculating directly. Therefore, $e_i Be_{_I} = e_i Be_i$. From the procedure above we also see the uniqueness of such $e_i = E_{ii} \in E_1(\overline{T})$. □

5.2. Definition. *For any $e_i = E_{ii} \in E_1(\overline{T})$, define*

$$\mathcal{R}(e_i) = \{x \in \mathcal{R} \mid e_i Bx = e_i Be_i x \neq 0\}.$$

5.3. Corollary. *The set of non-zero elements of the Renner monoid has a decomposition*

$$\mathcal{R}^\times = \bigsqcup_{e_i \in E_1(\overline{T})} \mathcal{R}(e_i) = \bigsqcup_{i=1}^{2l} \mathcal{R}(E_{ii}).$$

Applying Theorem 5.1, we can now get a surjective map τ from the set $E(\overline{T})$ of idempotents in $\overline{T}$ onto the set $E_1(\overline{T})$ of rank one elements. This map can also be extended to $\mathcal{R}^\times$ to $E_1(\overline{T})$.

5.4. Theorem.

a) There is a surjective map τ from $E(\overline{T})$ onto $E_1(\overline{T})$ by

$$e_{_I} \longmapsto \tau(e_{_I}) = e_i, \ \text{if } e_i Be_{_I} = e_i Be_i \neq 0.$$

b) The map τ extends to $\mathcal{R}^\times = \mathcal{R} \setminus \{0\}$ by, for every $x \in \mathcal{R}^\times$, defining,

$$\tau(x) = e_i, \ \text{if } x \in e_{_I} W \text{ and } \tau(e_I) = e_i,$$

where $I \neq \phi$ is admissible and i is maximal in I.

Proof. a) is clear. To prove b), note that for any $x \in \mathcal{R}^{\times}$, there exist $w \in W$ and a unique $e_{_I} \in E(\overline{T})$ such that $x = e_{_I} w$. It follows that there is a unique $e_i = E_{ii} \in E_1(\overline{T}) = E_1(\mathcal{R})$ such that $e_i B e_I = e_i B e_i \neq 0$. Then we obtain the map from $\mathcal{R}^{\times}$ to $E_1(\mathcal{R}) = E_1(\overline{T})$ by $\tau(x) = e_i$, as required. □

For $i = 1, ..., 2l$, denote by $I(i)$ the set of all the admissible sets $I \subseteq \{1, ..., 2l\}$ such that the i is maximal in I. Then we have the following proposition

5.5. Proposition.

a) $\mathcal{R}(e_i) = \tau^{-1}(e_i) = \bigsqcup_{I \in I(i)} e_{_I} W$*, for* $i = 1, ..., 2l$*.*

b) $\mathcal{R}^{\times} = \bigsqcup_{i=1}^{2l} \tau^{-1}(e_i)$*, a disjoint union.*

Proof. It is straightforward. □

For MSp_4, $E_1(\overline{T}) = \{e_1 = E_{11}, e_2 = E_{22}, e_3 = E_{33}, e_4 = E_{44}\}$, and

$$\begin{aligned}
\mathcal{R}(e_1) &= \bigsqcup_{I \in I(1)} e_I W \\
&= E_{11} W \\
&= \left\{ \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \right\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}(e_2) &= \textstyle\bigsqcup_{I \in I(2)} e_I W \\
&= E_{22} W \cup (E_{11} + E_{22}) W \\
&= \left\{ \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \right.
\end{aligned}$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\left.\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\},$$

$$\mathcal{R}(e_3) = \bigsqcup_{I\in I(3)} e_I W$$

$$= E_{33}W \cup (E_{11}+E_{33})W$$

$$= \left\{\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\right.$$

$$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},$$

$$\left.\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\right\},$$

$$\mathcal{R}(e_4) = \bigsqcup_{I\in I(4)} e_I W$$

$$= E_{44}W \cup (E_{33}+E_{44})W \cup (E_{22}+E_{44})W \cup W$$

$$= \left\{\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\right.$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\1&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$
$$\left.\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right\}.$$

To get our cell decomposition of MSp_{2l}, we first determine the cell decomposition of the symplectic Renner monoid. Note that for any $r = E_{ij} \in \mathcal{R}(1)$, there exist unique $e_r = E_{ii}$ and $f_r = E_{jj}$ in $E_1(\overline{T})$ such that $r = e_r r f_r$.

5.6. Definition. *For any $r = e_r r f_r \in \mathcal{R}(1)$, call $\mathcal{C}_r = \{x \in \mathcal{R}(e_r) \mid e_r x f_r = r\}$ a cell of the Renner monoid $\mathcal{R}$ corresponding to the rank one element r.*

Remark 5. L. Renner has shown that

$$\mathcal{C}_r = \{x \in \mathcal{R}(e_r) \mid e_r x f_r \neq 0\}$$

$$= \{x \in \mathcal{R}(e_r) \mid e_r x = r\}.$$

5.7. Proposition. *For any $e \in E_1(\overline{T})$ and $x \in \mathcal{R}(e)$, there is a unique $r = ex \in \mathcal{R}(1)$ such that $x \in \mathcal{C}_r$.*

Proof. Let $x \in \mathcal{R}(e)$ where $e \in E_1(\overline{T})$. Then $eBx = eBex \neq 0$ where B is the Borel subgroup of $G \subseteq MSp_n$. Since $r = ex$, $r \in \mathcal{R}(1)$ and $r = erf_r$ for the unique $e, f_r \in E_1(\overline{T})$. For if $r = e_r r f_r$ and $e_r \neq e$, then $r = ex = e(ex) = er = e(e_r)rf_r = (ee_r)rf_r = 0$, since $ee_r = 0$. Thus $x \in \mathcal{R}(e)$ and $exf_r = rf_r = (erf_r)f_r = erf_r = r$, i.e., $x \in \mathcal{C}_r$.

Suppose there is another $r' \in \mathcal{R}(1)$ such that $x \in \mathcal{C}_{r'} = \{x \in \mathcal{R}(e) \mid exf' = r'\}$, where $r' = er'f'$ for the unique $e, f' \in E_1(\overline{T})$. If $f' \neq f_r$ then $r' = exf' = (ex)f' = rf' = (exf_r)f' = ex(f_rf') = 0$, since $f_rf' = 0 \in E(\overline{T})$, which is a contradiction. Therefore, the uniqueness. $\square$

By Corollary 5.3 and the above proposition we get the following

5.8. Corollary. *a)* $\mathcal{R}(e) = \bigsqcup_{\substack{r\in\mathcal{R}(1)\\ er=r}} \mathcal{C}_r$, *where* $e \in E_1(\overline{T})$.

b) $\mathcal{R}^\times = \bigsqcup_{r\in\mathcal{R}(1)} \mathcal{C}_r$.

Now, we can establish a surjective map φ from $\mathcal{R}^\times$ to the set $\mathcal{R}(1)$ consisting of rank one elements in $\mathcal{R}$ by declaring $\varphi(x) = r$ if $x \in \mathcal{C}_r$ where $x \in \mathcal{R}^\times$ and $r \in \mathcal{R}(1)$. It is an extension of τ. Furthermore, $\varphi^{-1}(r) = \mathcal{C}_r$ for $r \in \mathcal{R}(1)$.

5.9. Theorem. *The above surjective map* φ *from* $\mathcal{R}^\times$ *to* $\mathcal{R}(1)$ *satisfies* $\varphi(x) = e_iw$ *if* $x = e_{_I}w \in \mathcal{R}^\times$ *and* $\tau(e_{_I}) = e_i$, *where* $e_{_I} \in E(\overline{T})$ *and* $w \in W$.

Proof. Since $\mathcal{R}^\times = \bigsqcup_{e_i\in E_1(\overline{T})} \mathcal{R}(e_i)$ where $\mathcal{R}(e_i) = \bigsqcup_{I\in I(i)} e_{_I}W$, there is a unique $e_i \in E_1(\overline{T})$ such that $x \in \mathcal{R}(e_i)$. It follows that if $x = e_{_I}w \in \mathcal{R}^\times$ and $\tau(e_{_I}) = e_i$, then $I \in I(i)$. Thus $\varphi(x) = e_ix = e_i(e_{_I}w) = (e_ie_{_I})w = e_iw$, the required result. $\square$

5.10. Theorem. *For any* $r = E_{ij} \in \mathcal{R}(1), i, j = 1, ..., 2l$,

$$\mathcal{C}_r = \mathcal{C}_{E_{ij}} = \{(x_{pq}) \in \mathcal{R} \mid x_{ij} = 1; x_{pq} = 0, \text{ if } i < p \leq 2l, 1 \leq q \leq 2l\}.$$

Proof. If $x = (x_{pq}) \in \mathcal{R}$ is an $n \times n$ matrix, then $E_{ii}x = E_{ij}$ if and only if $x_{iq} = \delta_{qj}$, for $i, q, j = 1, ..., n$. Then

$$\begin{aligned}\mathcal{C}_r = \mathcal{C}_{E_{ij}} &= \{(x_{pq}) \in \mathcal{R}(e_i) \mid E_{ii}(x_{pq}) = E_{ij}\}\\ &= \{(x_{pq}) \in \mathcal{R}(e_i) \mid x_{ij} = 1\}\\ &= \bigsqcup_{I \in I(i)} \{(x_{pq}) \in e_I W \mid x_{ij} = 1\}\\ &= \{(x_{pq}) \in \mathcal{R} \mid x_{ij} = 1; x_{pq} = 0, \text{ if } i < p \leq 2l, 1 \leq q \leq 2l\},\end{aligned}$$

which proves the Theorem. □

In the sequel, the cells in Theorem 5.10 will be simply denoted by $\mathcal{C}_{ij}$.

If $n = 4$, the cells of the Renner monoid of MSp_4 are:

$$\mathcal{C}_{11} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathcal{C}_{12} = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{13} = \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathcal{C}_{14} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{21} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{22} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{23} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{24} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{31} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{32} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{33} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{34} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{41} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{42} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{43} = \left\{ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&1&0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{44} = \left\{ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} \right\}.$$

Notice the fact that the cells $\mathcal{C}_{ij}(K)$ of the Renner monoid $\mathcal{R}_{2l}$ of $\mathbf{M}_{2l}(K)$ are

$$\mathcal{C}_{ij}(K) = \{(x_{pq}) \in \mathcal{R}_{2l} \mid x_{ij} = 1, x_{pq} = 0, \text{ if } i < p \leq 2l, 1 \leq q \leq 2l\},$$

where $i, j = 1, ..., 2l$. We can now get

5.11. Theorem. *$\mathcal{C}_{ij} = \mathcal{C}_{ij}(K) \cap \mathcal{R}$, where $i, j = 1, ..., 2l$.*

We begin to describe the cell decomposition of the symplectic algebraic monoid MSp_n using the following definition.

5.12. Definition. *The sets $C_{ij} = B\mathcal{C}_{ij}B$ for $i, j = 1, ..., n$ are called the cells for the symplectic monoid with respect to the Borel subgroup B.*

5.13. Theorem. *The cells of the symplectic monoid are*

$$C_{ij} = \left\{ (a_{pq}) \in MSp_n \;\middle|\; \begin{array}{l} a_{ij} \neq 0; \quad a_{iq} = 0 \text{ if } 1 \leq q < j; \\ a_{pq} = 0, \text{ if } i < p \leq 2l \text{ and } 1 \leq q \leq 2l \end{array} \right\}$$

where $i, j = 1, ..., 2l$.

Proof. Since $C_{ij} = \bigcup_{x\in\mathcal{C}_{ij}} (K^*B_0)x(K^*B_0) = K^*\left(\bigcup_{x\in\mathcal{C}_{ij}} B_0xB_0\right)$, for $i, j = 1, ..., 2l$, we need only to consider elements in B_0xB_0, where $x \in \mathcal{C}_{ij}$ and

$$B_0 = \left\{ \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in \mathbf{Sp}_{2l} \middle| \begin{array}{l} b_1, b_3 \in \mathbf{M}_l(K) \text{ are upper triangular,} \\ b_3^T J b_1 = J, \quad b_3^T J b_2 = b_2^T J b_3 \end{array} \right\}.$$

Now for every element $b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in B_0$, suppose that $b = (b_{pq})_{2l\times 2l}$ which is upper triangular. Then $b_{pp} \neq 0$, for $p = 1, ..., 2l$. For any $x = (x_{pq}) \in \mathcal{C}_{ij} \subseteq \mathcal{R}(E_{ii}) \subseteq \mathcal{R}$, let $I = \{i_1, ..., i_{m-1}, i_m\}$ denote the index set of non-zero rows of x where $i_1 < ... < i_{m-1} < i_m$ and $i_m = i$. Let $J = \{j_1, ..., j_{m-1}, j_m\}$ denote the index set of non-zero columns such that $j_m = j$ and $x_{i_k j_k} = 1$, for $k = 1, ..., m$. Generally, we do not have $j_1 < ... < j_{m-1} < j_m$.

Thus bx is a matrix whose j_k-th column is the i_k-th of b where $k = 1, ..., m$, and all rows under row i are zero. The shape of bx is

$$i\text{-th row} \longleftarrow \begin{pmatrix} * & \cdots & * & b_{1i} & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & b_{i-1i} & * & \cdots & * \\ 0 & \cdots & 0 & b_{ii} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$\downarrow$$

$$j\text{-th column}$$

where $b_{ii} \neq 0$ is the (i, j)-entry of bx. Taking any $b' = (b'_{pq})_{2l\times 2l} \in B_0$, one obtains the shape of bxb' is

$$i\text{-th row} \longleftarrow \begin{pmatrix} * & \dots & * & * & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \dots & * & * & * & \dots & * \\ 0 & \dots & 0 & b_{ii}b'_{jj} & b_{ii}b'_{jj+1} & \dots & b_{ii}b'_{j2l} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\downarrow$$

$$j\text{-th column}$$

where $b_{ii}b'_{jj} \neq 0$ is the (i,j)-entry of bxb'. From the arbitrariness of $b, b' \in B_0$ and $x \in \mathcal{C}_{ij}$, we get

$$C_{ij} = \left\{ (a_{pq}) \in MSp_n \;\middle|\; \begin{array}{l} a_{ij} \neq 0; \quad a_{iq} = 0 \text{ if } 1 \leq q < j; \\ a_{pq} = 0 \text{ if } i < p \leq 2l \text{ and } 1 \leq q \leq 2l \end{array} \right\}$$

where $i, j = 1, ..., 2l$, since $B = K^* B_0$. $\square$

It follows from the Bruhat-Renner decomposition [17, Corollary 5.8] of MSp_{2l} and Corollary 5.8 that

5.14. Corollary. *Keeping the notation above, we have*

$$MSp_{2l} \setminus \{0\} = \bigsqcup_{i,j=1}^{2l} C_{ij}.$$

From the shapes of elements in the cells $C_{ij}(K)$ of $\mathbf{M}_n(K)$ we obtain the following

5.15. Theorem. $C_{ij} = C_{ij}(K) \cap MSp_n$, *for* $i, j = 1, ..., 2l$.

6. Submonoids of the Symplectic Algebraic Monoids MSp_n

The main purpose of this section is to establish some properties of the submonoid $(MSp_n)_e = \{y \in MSp_n |\ ye = ey = e\}$ of MSp_n where $e \in E(MSp_n)$ and $n = 2l$. We simply denote by M_e the submonoid $(MSp_n)_e$. Let $G_e = M_e \cap G$ where $G = K^*\mathbf{Sp}_n$ is the unit group of MSp_n. Then by [15, Theorem 6.11] one has $M_e = \overline{G}_e$. Thus, saying something about G_e is necessary.

6.1. Lemma. *Let* $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{2l \times 2l} \in MSp_n$. *Then* $y \in \mathbf{Sp}_{2l}$ *if and only if* $y_1 \in \mathbf{Sp}_{2l-2}$.

Proof. Recall $J_l = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \in \mathbf{M}_{2l}(K)$ be the nonsingular and skew symmetric matrix, where $J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size l. Rewrite J_l to be $J_l = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J_{l-1} & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} y \in \mathbf{Sp}_{2l} &\Longleftrightarrow y^T J_l y = J_l \\ &\Longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & y_1^T J_{l-1} y_1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J_{l-1} & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &\Longleftrightarrow y_1^T J_{l-1} y_1 = J_{l-1} \\ &\Longleftrightarrow y_1 \in \mathbf{Sp}_{2l-2}. \quad \square \end{aligned}$$

6.2. Theorem. *Let* $e_1 = E_{11} \in \Lambda$ *and* $G = K^*\mathbf{Sp}_{2l}$. *Then* G_{e_1} *is isomorphic to* $K^*\mathbf{Sp}_{2l-2}$. *Furthermore,* M_{e_1} *is isomorphic to* MSp_{2l-2}.

Proof. Suppose that $y = tx \in G$ with $x = (x_{ij})_{i,j=1}^{2l} \in \mathbf{Sp}_{2l}$ and $t \in K^*$. Then $ye_1 = e_1 y = e_1$ is equivalent to $xe_1 = e_1 x = (1/t)e_1$. So

$$x = \begin{pmatrix} 1/t & 0 \\ 0 & x_1 \end{pmatrix} \in \mathbf{Sp}_{2l},$$

where $x_1 = (x_{ij})_{i,j=2}^{2l}$ is a $2l-1 \times 2l-1$ matrix. Let $A = (0, ..., 1)_{1\times 2l-1}$ and rewrite $J_l = \begin{pmatrix} 0 & A \\ -A^T & J' \end{pmatrix}$. Notice that

$$x^T J_l x = \begin{pmatrix} 0 & (1/t)Ax_1 \\ (-1/t)x_1^T A^T & x_1^T J' x_1 \end{pmatrix}.$$

Thus $x^T J_l x = J_l$ gives us $(1/t)Ax_1 = A$, $(-1/t)x_1^T A^T = -A^T$ and $x_1^T J' x_1 = J'$. It follows that $x_{2l,2} = ... = x_{2l,2l-1} = 0$ and $x_{2l,2l} = t$, which shows that x_1 has the shape

$$x_1 = \begin{pmatrix} x_2 & X \\ 0 & t \end{pmatrix}_{2l-1\times 2l-1},$$

where $x_2 = (x_{ij})_{i,j=2}^{2l-1}$ is a $2l-2 \times 2l-2$ matrix and $X = (x_{2,2l}, ..., x_{2l-1,2l})^T$. Since $J' = \begin{pmatrix} J_{l-1} & 0 \\ 0 & 0 \end{pmatrix}$ and $x_1^T J' x_1 = \begin{pmatrix} x_2^T J_{l-1} x_2 & x_2^T J_{l-1} X \\ X^T J_{l-1} x_2 & X^T J_{l-1} X \end{pmatrix}$, it follows from $x_1^T J' x_1 = J'$ that $x_2^T J_{l-1} x_2 = J_{l-1}$, $x_2^T J_{l-1} X = 0$, $X^T J_{l-1} x_2 = 0$ and $X^T J_{l-1} X = 0$. Thus $X = 0$ and $x_1 = \begin{pmatrix} x_2 & 0 \\ 0 & t \end{pmatrix}$, where $x_2 \in \mathbf{Sp}_{2l-2}$. Therefore,

$$x = \begin{pmatrix} 1/t & & \\ & x_2 & \\ & & t \end{pmatrix} \in \mathbf{Sp}_{2l},$$

where $t \in K^*$ and $x_2 \in \mathbf{Sp}_{2l-2}$. It follows easily that

$$G_e = \left\{ t \cdot \begin{pmatrix} 1/t & & \\ & x_2 & \\ & & t \end{pmatrix} \middle| \; t \in K^*, x_2 \in \mathbf{Sp}_{2l-2} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & & \\ & tx_2 & \\ & & t^2 \end{pmatrix} \middle| \; t \in K^*, x_2 \in \mathbf{Sp}_{2l-2} \right\}.$$

Define a mapping f from G_{e_1} to $K^*\mathbf{Sp}_{2l-2}$ by

$$y = \begin{pmatrix} 1 & & \\ & tx_2 & \\ & & t^2 \end{pmatrix} \longmapsto tx_2 \in K^*\mathbf{Sp}_{2l-2}$$

Then f is an algebraic group isomorphism from G_{e_1} to $K^*\mathbf{Sp}_{2l-2}$. Hence, $\overline{G}_{e_1}$ is isomorphic to $\overline{K^*\mathbf{Sp}_{2l-2}}$ which is MSp_{2l-2}. But it follows from [15, Theorem 6.11] that $M_{e_1} = \overline{G}_{e_1}$. Therefore, M_{e_1} is isomorphic to MSp_{2l-2}. This proves the Theorem. □

6.3. Corollary.

a) For any $e \in E_1(\overline{T})$, M_e *is isomorphic to* MSp_{2l-2}.

b) For any $e \in E_1(MSp_{2l})$, *the rank one elemenets in* $E(MSp_{2l})$, M_e *is isomorphic to* MSp_{2l-2}.

Proof. To prove a), note that $E_1(\overline{T}) = \{w^{-1}e_1w \mid w \in W\}$, where $e_1 = E_{11}$. Then for any $e \in E_1(\overline{T})$, there exists $w \in W$ such that $e = w^{-1}e_1w$. Since $ye = ey = e$ is equivalent to $(wyw^{-1})e_1 = e_1(wyw^{-1}) = e_1$, it follows that M_e is isomorphic to M_{e_1} by the mapping $y \longmapsto wyw^{-1}$. From Theorem 6.2 one obtains that M_e is isomorphic to MSp_{2l-2}. Similar recipes of a) apply to b) by using $E_1(MSp_{2l}) = \{g^{-1}e_1g \mid g \in G\}$. □

A similar discussion to that of Theorem 6.2 gives the following

6.4. Theorem. *Let* $G = K^*\mathbf{Sp}_{2l}$ *and* $e_I \in \Lambda$ *with* $I = \{1, ..., i\}$ *standard admissible, where* $e_I = \sum_{j\in I} E_{jj} \in \Lambda$ *and* $i = 1, ..., l$. *Then* G_{e_I} *is isomorphic to* $K^*\mathbf{Sp}_{2l-2i}$. *Furthermore,* M_{e_I} *is isomorphic to* MSp_{2l-2i}.

6.5. Corollary. *Keeping the same notations in Theorem 5.4, we have*

a) For every $e_J \in E_i(\overline{T})$ *with* J *admissible and* $|J| = i$, *for* $i = 1, ..., l$; M_{e_J} *is isomorphic to* MSp_{2l-2i}.

b) For every $e \in E_i(MSp_{2l})$, *the rank* i *elements in* $E(MSp_{2l})$, M_e *is isomorphic to* MSp_{2l-2i}, *for* $i = 1, ..., l$.

Proof. For a), note that the W acts transitively on the set $E_i(\overline{T})$ of rank i ($i = 1, ..., l$) idempotents in $E(\overline{T})$. Then for every $e_J \in E_i(\overline{T})$, there exist $w \in W$ and $e_I \in \Lambda$ such that $e_J = we_I w^{-1}$. It follows that $ye_J = e_J y = e_J$ is equivalent to $(w^{-1}yw)e_I = e_I(w^{-1}yw) = e_I$, and hence M_{e_J} is isomorphic to M_{e_I} by the mapping $y \longmapsto w^{-1}yw$. But, by Theorem 6.4, M_{e_I} is isomorphic to MSp_{2l-2i}. Applying a) and [17, Corollary 6.10 (ii)] one gets b) easily. $\square$

CHAPTER IV

SPECIAL ORTHOGONAL MONOIDS MSO_{2l}

In this chapter we discuss the Renner monoids and the cell decompositions of the special orthogonal algebraic monoids, even case. Throughout this chapter we assume that the characteristic of K is not 2.

Let $n = 2l$ be even and $J_l = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \in \mathbf{M}_n$ be the symmetric matrix, where $J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{pmatrix}$ is an $l \times l$ matrix. The special orthogonal group is by definition

$$G_0 = \mathbf{SO}_n = \{g \in \mathbf{SL}_n |\ g^T J_l g = J_l\}$$

which is connected and reductive.

Remark 6. The definition of $\mathbf{SO}_n$ here is from [7, pp.52-53].

Let $T_0 = G_0 \cap \mathbf{T}_n$. Elements in T_0 have the shape

$$t = \mathrm{diag}(t_1, ..., t_l, t_l^{-1}, ..., t_1^{-1})$$

where $t_1, ..., t_l$ are arbitrary in K^*. Thus T_0 is a maximal torus of dimension l. Let us recall some facts about the Weyl group $W(G_0, T_0)$ which will be simply denoted by W if there is no confusion in the context. If $\pi \in S_n$, let $p_\pi = \sum_{i=1}^n E_{\pi i,i} \in P_n$ be the corresponding permutation matrix. Then $p_\pi(a_{ij}) = (a_{\pi^{-1}i,j})$, and $(a_{ij})p_\pi = (a_{i,\pi j})$ where (a_{ij}) is any $n \times n$ matrix. It follows that $p_\pi^{-1}(a_{ij})p_\pi = p_{\pi^{-1}}(a_{ij})p_\pi = (a_{\pi i,\pi j})$. Define an involution $\theta : i \longmapsto \bar{i}$ of $\{1, 2, ..., n\}$ by

$$\bar{i} = 2l + 1 - i, \qquad \text{for } 1 \le i \le 2l.$$

Let C denote the centralizer of θ in S_n. Then p_π normalizes T_0 if and only if $\pi \in C$. The group C is a semidirect product $C = C_1 C_2$ where C_1 is a normal abelian subgroup of order 2^l generated by the transpositions $(1\bar{1}), ..., (l\bar{l})$ and $C_2 \simeq S_l$ consists of all permutations $\pi \in S_n$ which stablize $\{1, ..., l\}$ and act on the complement $\{l+1, ..., 2l\}$ in the unique manner consistent with the assertion that $\pi \in C$. Note that permutation matrices in C need not be in $\mathbf{SO}_{2l}$. So, let C_1' be a subgroup of C_1 generated by $(1\bar{1})(2\bar{2}), (2\bar{2})(3\bar{3}), ..., (l-1\,\overline{l-1})(l\bar{l})$. Then C_1' consists of even permutations in C_1. Let $C_2' = C_2$ and $C' = C_1'C_2'$. It follows that $C' \subset \mathbf{SO}_{2l}$ and $|C'| = 2^{l-1}l!$. But $\omega_1 T_0 = \omega_2 T_0$ if and only if $\omega_1 = \omega_2$ for any $\omega_1, \omega_2 \in C'$. Thus W is isomorphic to $C' \subset S_n$. Also, W is isomorphic to $(Z_2)^{l-1} \rtimes S_l$.

If $n = 4$, then $\theta = (1\bar{1})(2\bar{2}) = (14)(23)$, and C_1' is a subgroup of $C_{S_4}(\theta)$ generated by $(14)(23)$. So

$$C_1' = \{1, (14)(23)\}.$$

Taking $\pi = (12)(34)$, we see that $\theta\pi = \pi\theta$ which means that $\pi \in C_{S_4}(\theta)$. It is clear that π stablizes $\{1, ..., l\} = \{1, 2\}$ and $\pi \notin C_1$. Let C_2' be a subgroup of $C_{S_4}(\theta)$ generated by π. Then

$$C_2' = \{1, (12)(34)\}.$$

Thus the Weyl group $W = C_1'C_2' = \{1, (14)(23), (12)(34), (13)(24)\}$. The corresponding matrix form of the Weyl group is

$$W = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

$$= \{1, \theta, \pi, \theta\pi = \pi\theta\}.$$

7. The Renner Monoids of the Special Orthogonal Monoids MSO_{2l}

The main purpose of this section is to determine the Renner monoids of the special orthogonal algebraic monoids, even case. We get some by-products as well, such as the cardinalities of the Renner monoids.

Let $G = K^*G_0 \in \mathbf{GL}_n$ where $n = 2l$. Then G is a connected reductive group with rank $r = l + 1$ and semisimple rank l.

7.1. Definition. *The monoid $\overline{G}$, the Zariski closure of G in $\mathbf{M}_n(K)$, is called the special orthogonal monoid which will be denoted by MSO_n, where $n = 2l$.*

7.2. Definition. *A subset $I \subseteq \{1, ..., n\}$ is called admissible if $j \in I$ implies $\bar{j} \notin I$, where $\bar{j} = \theta(j)$ as above; the empty set ϕ and $\{1, ..., n\}$ are also considered to be admissible.*

If $n = 4$, then the admissible subsets of $\{1, 2, 3, 4\}$ are

$$\phi,\ \{1\},\ \{2\},\ \{3\},\ \{4\},\ \{1, 2\},\ \{3, 4\},\ \{1, 3\},\ \{2, 4\},\ \{1, 2, 3, 4\}.$$

An admissible subset I is referred to as standard if there is an integer $i \in \{1, ..., l, 2l\}$ such that $I = \{1, ..., i\}$; the empty set and the set $\{1, ..., l-1, l+1\}$ are also considered to be standard. For example, the standard admissible subsets of $\{1, 2, 3, 4\}$ are ϕ, $\{1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 2, 3, 4\}$.

Remark 7. The standard admissible subsets here are a little different from that of the symplectic situation.

A similar discussion to [22, p336] gives the following lemma describing the relationship between admissible subsets and idempotents in $\overline{T}$ with $T = K^*T_0$.

7.3. Lemma.

a) W maps admissible sets to admissible sets, and $w^{-1}e_I w = e_{wI}$ for any $w \in W$.

b) The map

$$I \longmapsto e_I = \sum_{j \in I} E_{jj}$$

is bijective from the admissible subsets of $\{1, ..., n\}$ to $E(\overline{T})$, where $e_I = 0$, if $I = \phi$.

c) The set $E(\overline{T})$ of idempotents in $\overline{T}$ is

$$E(\overline{T}) = \{e_I \mid I \text{ is admissible}\}.$$

d) $e_{I_1} \cdot e_{I_2} = e_{I_1 \cap I_2}$ for any $e_{I_1}, e_{I_2} \in E(\overline{T})$.

Proof. For a), b) and c), see [22, p336]. By checking directly, we get d). □

If $n = 4$, the set of idempotents of MSO_4 is

$$E(\overline{T}) = \{0, 1, E_{11}, E_{22}, E_{33}, E_{44}, E_{11} + E_{22}, E_{33} + E_{44}, E_{11} + E_{33}, E_{22} + E_{44}\}.$$

Remark 8. There are no admissible subsets with size k $(l < k < 2l)$, and the rank one elements in $E(\overline{T})$ are in one to one correspondence with the admissible subsets containing exactly one element of $\{1, ..., n\}$.

7.4. Proposition.

$$|E(\mathcal{R})| = |E(\overline{T})| = \sum_{i=0}^{l} \sum_{j=0}^{i} \binom{l}{j} \binom{l-j}{i-j} + 1.$$

Proof. By counting the number of admissible subsets of $\{1, ..., n\}$ and applying [17, Proposition 3.2.1], we know the result is true. □

Let $E_1(\mathcal{R}) \subseteq E(\mathcal{R})$ (resp. $E_1(\overline{T}) \subseteq E(\overline{T})$) denote the set of rank one idempotent elements in $\mathcal{R}$ (resp. $\overline{T}$). Then by Lemma 7.3 and [17, Proposition 3.2.1] we have the following

7.5. Proposition.

a) $E_1(\mathcal{R}) = E_1(\overline{T}) = \{E_{ii}|\ i = 1, ..., n\}$.

b) $|E_1(\mathcal{R})| = |E_1(\overline{T})| = n$.

We now find the set of rank one elements in $\mathcal{R}$. To this end, let $\mathcal{R}(i)$ denote the set of rank i elements in the Renner monoid $\mathcal{R}$, for $i \in \{1, ..., n\}$.

7.6. Lemma. $\mathcal{R}(1) = \{E_{ij}|\ i, j = 1, ..., n\}$, *and* $|\mathcal{R}(1)| = n^2$.

Proof. It suffices to show that $\{E_{ij} \mid i, j = 1, ..., n\} \subseteq \mathcal{R}(1)$.

Firstly, we prove that $\{E_{1j} \mid j = 1, ..., n\} \subseteq \mathcal{R}(1)$. There are three cases,

a) If $j \in \{1, ..., l\}$, let $w = (1j)(\bar{1}\bar{j})$. Then w stablizes $\{1, ..., l\}$ and $w\theta = \theta w$. It follows that $w \in W_2 \subseteq W$ and $w(j) = 1$.

b) If $j = \bar{1} = 2l \in \{l+1, ..., 2l\}$, let $w = (1j)(2\bar{2}) = (1\bar{1})(2\bar{2})$. Then $w \in W_1 \subseteq W$ and $w(j) = 1$.

c) If $j \in \{l+1, ..., 2l\}$ but $j \neq \bar{1} = 2l$, let $w_2 = (j\bar{1})(\bar{j}1)$. Then w_2 stablizes $\{1, ..., l\}$ and $w_2\theta = \theta w_2$. So $w_2 \in W_2 \subseteq W$ and $w_2(j) = \bar{1}$. Let $w_1 = (1\bar{1})(2\bar{2}) \in W_1$ and $w = w_1 w_2$. Then $w \in W$ and $w(j) = w_1(w_2(j)) = w_1(\bar{1}) = 1$.

So, $E_{1j} = E_{1,w1} = E_{11}w \in E_{11}W$ for $j = 1, ..., n$. Thus $\{E_{1j} \mid j = 1, ..., n\} = E_{11}W$ which is a subset of $\mathcal{R}(1)$.

Similarly, $\{E_{ij} \mid j = 1, ..., n\} = E_{ii}W \subseteq \mathcal{R}(1)$ for $i = 2, ..., n$.

Therefore, $\mathcal{R}(1) = \{E_{ij} \mid i, j = 1, ..., n\}$ with size n^2. □

Remark 9. The lemma above shows that $\mathcal{R}(1) = \mathcal{R}_n(1)$. However, $\mathcal{R}(2) \neq \mathcal{R}_n(2)$, since $\{1, 2l\}$ is not an admissible subset of $\{1, ..., n\}$, and so $E_{11}+E_{2l,2l} \notin \mathcal{R}(2)$, but $E_{11} + E_{2l,2l} \in \mathcal{R}_n(2)$. For the same reason, we know $\mathcal{R}(i) \neq \mathcal{R}_n(i)$, for $i = 3, ..., n$.

7.7. Theorem. *For any admissible subset* $I \subseteq \{1, ..., n\}$ *with* $|I| = i$, *where* $i = 1, ..., l-1, 2l$, *there exist* $w \in W$ *and a unique standard admissible subset* $I_0 = \{1, ..., i\}$ *such that* $wI = I_0$.

Proof. If $I = \{1, ..., n\}$, then $I_0 = I$ and $w = 1 \in W$ and we are done. Now let I be admissible and $I \neq \{1, ..., n\}$. Then $|I| = i \in \{1, ..., l-1\}$. Use induction on the size i of the admissible subset I. If $i = 1$, then $I = \{j\}$ for some $j \in \{1, ..., n\}$ and $I_0 = \{1\}$. By Lemma 7.6 we know there exists $w \in W$ such that $w(I) = I_0$.

Now suppose that $I \subseteq \{1, ..., n\}$ is any admissible subset with $1 < |I| = i \leq l-1$. Then $I = J \cup \{k\}$ where J is a subset of I with $|J| = i-1$ and $k \in I \setminus J$. It follows that J is admissible. By the induction hypothesis there exist $w' \in W$ and a unique standard admissible subset $I' = \{1, ..., i-1\}$ such that $w'J = I'$. Then $w'I = I' \cup \{p\}$ where $p = w'(k) \notin I'$. There are four cases for p,

1) If $p = i$, then $I_0 = I' \cup \{i\}$ and $w = w'$ are what we want.

2) If $p \in \{1, ..., l\}$, and $p \neq i$, let $w_1 = (pi)(\bar{p}\bar{i})$. Then $w_1\theta = \theta w_1$ and w_1 stablizes $\{1, ..., l\}$. Thus, $w_1 \in W_2 \subseteq W$. Note that $w_1(j) = j$, for $j \in I'$. Taking $w = w_1 w'$, we obtain that $w \in W$ and $w(I) = w_1(I' \cup \{p\}) = I' \cup \{w_1(p)\} = \{1, ..., i\} = I_0$.

3) If $p = \bar{i} = 2l + 1 - i \in \{l+1, ..., 2l\}$, let $w_1 = (i\bar{i})(l\bar{l})$ with $i \leq l-1$. Then $w_1 \in W_1 \subseteq W$ and $w_1|_{I'} = 1$. Let $w = w_1 w'$. We obtain that $w \in W$ and $w(I) = w_1(I' \cup \{p\}) = w_1(I') \cup \{w_1(p)\} = I_0$.

4) If $p \in \{l+1, ..., 2l\}$ but $p \neq \bar{i} = 2l + 1 - i$. Let $w_1 = (p\bar{i})(\bar{p}i)$. Then $w_1 \in W_2 \subseteq W$ and $w_1(j) = j$ for $j \in I'$. Taking $w = (i\bar{i})(l\bar{l})w_1 w'$, we get $w \in W$ and $w(I) = (i\bar{i})(l\bar{l})w_1(I' \cup \{p\}) = (i\bar{i})(l\bar{l})(I' \cup \{\bar{i}\}) = I' \cup \{i\} = I_0$.

This proves the Theorem. $\square$

7.8. Corollary. *The Weyl group W acts on $E_i(\overline{T})$, by $w^{-1}e_I w = e_{wI}$, transitively, for $i = 1, ..., l-1$.*

7.9. Theorem. *Let $I \subseteq \{1, ..., 2l\}$ be admissible with $|I| = l$. Then there is a $w \in W$ such that either $w(I) = \{1, ..., l\}$ or $w(I) = \{1, ..., l-1, l+1\}$.*

Proof. Since $I \subseteq \{1, ..., n\}$ is an admissible subset with $|I| = l$. Then $I = J \cup \{k\}$ where J is a subset of I with $|J| = l - 1$ and $k \in I \setminus J$. It follows that J is admissible. By Theorem 4.7 there exist $w \in W$ and a unique standard admissible subset $I' = \{1, ..., l-1\}$ such that $wJ = I'$. Then $wI = I' \cup \{p\}$ where $p = w(k) \notin I'$. We claim that $p = l$ or $l + 1$. Otherwise, $p \in \{l+2, l+3, ..., 2l\}$. It follows that $\theta(p) = \bar{p} = 2l+1-p \in I' \subseteq w'(I)$, i.e., p and $\bar{p}$ are both in $w'(I)$, which is impossible since $w'(I)$ is admissible. This proves the Theorem. $\square$

7.10. Corollary. *Under the action, by conjugation, of W on $E_l(\overline{T})$, there are exactly two orbits. One is $We_{J_1}W$, and the other is $We_{J_2}W$, where $J_1 = \{1, ..., l\}$ and $J_2 = \{1, ..., l-1, l+1\}$.*

We will use the following definition soon.

7.11. Definition. *An admissible subset I of size l is called* **type I** *if there exists w in W such that $wI = J_1 = \{1, ..., l-1, l\}$;* **type II** *if $wI = J_2 = \{1, ..., l-1, l+1\}$.*

7.12. Proposition. *The cross section lattice of MSO_{2l} is*

$$\Lambda = \{e_I \in E(\overline{T}) |\ I \textit{ is a standard admissible subset of } \{1, ..., n\}\}.$$

$$= \left\{ I_{2l}, \begin{pmatrix} I_l & \\ & 0 \end{pmatrix}, \begin{pmatrix} I_{l-1} & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \begin{pmatrix} I_{l-1} & \\ & 0 \end{pmatrix}, \ \cdots \ , \begin{pmatrix} I_1 & \\ & 0 \end{pmatrix}, 0 \right\}.$$

The cross section lattice of MSO_4 is

$$\Lambda = \left\{ (0), \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

The Hasse diagram of the cross section lattice of MSO_{2l} is as follows

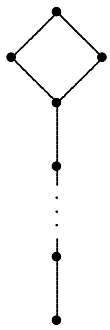

Figure 2.

7.13. Theorem. *With the notation above, the Renner monoid of the special orthogonal monoid MSO_{2l} is as follows*

$$\mathcal{R} = \left\{ \sum_{\substack{i \in I \\ w \in W}} E_{i,wi} \in \mathcal{R}_{2l} \mid I \subseteq \{1, ..., 2l\} \textit{ is admissible} \right\}.$$

Proof. Since $\mathcal{R} = E(\overline{T})W$ by [17, Proposition 3.2.1], it suffices to compute $e_I w$ for every $e_I \in E(\overline{T}), w \in W$, where I is admissible. From Lemma 7.3 b) we know $e_I = \sum_{i \in I} E_{ii}$. Thus $e_I w = \sum_{i \in I} E_{ii} w = \sum_{i \in I} E_{i,wi}$, and so the Theorem. $\square$

7.14. Corollary.

$$\mathcal{R} = \left\{ \sum_{\substack{i \in I \\ w \in W}} E_{wi,i} \in \mathcal{R}_{2l} \mid I \subseteq \{1, ..., 2l\} \textit{ is admissible} \right\}.$$

Proof. This result comes from the fact that $\mathcal{R} = WE(\overline{T})$, and

$$w^{-1} e_I = \sum_{i \in I} E_{wi,i}. \quad \square$$

7.15. Theorem.

$$\mathcal{R}\setminus W = \left\{ x\in\mathcal{R}_{2l} \,\middle|\, \begin{array}{l} x \textit{ is singular; } D(x) \textit{ and } R(x) \textit{ are admissible,} \\ \textit{and of the same type if } |D(x)|=|R(x)|=l \end{array} \right\}$$

where $D(x)$ is the domain of x, and $R(x)$ is the range of x.

Proof. Let $\mathcal{R}'$ denote the set of the right hand side in the Theorem. It follows from Theorem 7.12 that $\mathcal{R}\setminus W\subseteq\mathcal{R}'$, since W maps admissible sets I to admissible sets and both wI and I are of the same type if $|I|=l$.

We now prove the other inclusion. For any $x\in\mathcal{R}'$, one knows $|D(x)|=|R(x)|$ which will be denoted by i. Then $i\le l$, since x is singular and both $D(x)$ and $R(x)$ are admissible.

a) If $i\ne l$, it follows from Theorem 7.7 that there exist $w_1,w_2\in W$ and a unique standard admissible set $I_0=\{1,...,i\}(i\le l-1)$ such that

$$w_1D(x)=w_2R(x)=I_0.$$

Thus $w_1^{-1}xw_2=e_{I_0}\in\Lambda\subseteq\mathcal{R}$, and hence $x=w_1e_{I_0}w_2^{-1}\in\mathcal{R}\setminus W$, since $\mathcal{R}=WE(\overline{T})=E(\overline{T})W$ and x is singular.

b) If $i=l$, then $D(x)$ and $R(x)$ are of the same type because of $x\in\mathcal{R}'$. There are w_1 and w_2 in W such that

$$w_1D(x)=w_2R(x)=\begin{cases} J_1, & \text{if } D(x) \text{ and } R(x) \text{ are of type I} \\ J_2, & \text{if } D(x) \text{ and } R(x) \text{ are of type II.} \end{cases}$$

where $J_1=\{1,...,l-1,l\}$ and $J_2=\{1,...,l-1,l+1\}$. It follows that $w_1^{-1}xw_2=e_{J_1}\in\Lambda\subseteq E(\overline{T})$ or $w_1^{-1}xw_2=e_{J_2}\in\Lambda\subseteq E(\overline{T})$. That is, $x=w_1e_{J_1}w_2^{-1}$ or $x=w_1e_{J_2}w_2^{-1}$. Hence, $x\in\mathcal{R}\setminus W$, since $\mathcal{R}=WE(\overline{T})=E(\overline{T})W$ and x is singular.

Therefore, $\mathcal{R}\setminus W=\mathcal{R}'$, i.e., the Theorem is true. $\square$

Remark 10. In the proof above we obtain $\mathcal{R} = W\Lambda W$ as well.

We now consider some examples. If $n = 2$, then all the admissible subsets of $\{1, 2\}$ are

$$\phi, \{1\}, \{2\}, \{1, 2\}.$$

They are all standard. So, the cross section lattice of MSO_2 by Proposition 7.12 is

$$\Lambda = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

From Theorem 7.15, the Renner monoid of MSO_2 is

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let us now consider the Renner monoid of MSO_4. The idempotent set $E(\mathcal{R}) = E(\overline{T})$ of MSO_4 is a union of sets of rank i idempotent elements in $\overline{T}$, for $i = 0, 1, 2, 4$.

$$E(\overline{T}) = E_0(\overline{T}) \cup E_1(\overline{T}) \cup E_2(\overline{T}) \cup E_4(\overline{T}).$$

where $E_0(\overline{T}) = \{0\}$, $E_1(\overline{T}) = \{E_{11}, E_{22}, E_{33}, E_{44}\}$, $E_2(\overline{T}) = \{E_{11} + E_{22}, E_{33} + E_{44}, E_{11} + E_{33}, E_{22} + E_{44}\}$, $E_4(\overline{T}) = \{E_{11} + E_{22} + E_{33} + E_{44}\}$.

Since $\mathcal{R} = E_0(\overline{T})W \cup E_1(\overline{T})W \cup E_2(\overline{T})W \cup E_4(\overline{T})W$, we get

$$\begin{aligned} \mathcal{R} = \{ & 0,\ E_{11},\ E_{12},\ E_{13},\ E_{14},\ E_{21},\ E_{22},\ E_{23},\ E_{24},\ E_{31},\ E_{32},\ E_{33},\ E_{34}, \\ & E_{41},\ E_{42},\ E_{43},\ E_{44},\ E_{11}+E_{22},\ E_{14}+E_{23},\ E_{12}+E_{21},\ E_{13}+E_{24}, \\ & E_{33}+E_{44},\ E_{32}+E_{41},\ E_{34}+E_{43},\ E_{31}+E_{42},\ E_{11}+E_{33},\ E_{14}+E_{32}, \\ & E_{12}+E_{34},\ E_{13}+E_{31},\ E_{22}+E_{44},\ E_{23}+E_{41},\ E_{21}+E_{43},\ E_{24}+E_{42}, \\ & E_{11}+E_{22}+E_{33}+E_{44}, E_{14}+E_{23}+E_{32}+E_{41}, E_{12}+E_{21}+E_{34}+E_{43}, \\ & E_{13}+E_{24}+E_{31}+E_{42}\}. \end{aligned}$$

The matrix expression of the Renner monoid $\mathcal{R}$ of MSO_4 is

$$
\begin{aligned}
\mathcal{R} = \Bigg\{ & \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \\
& \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}, \\
& \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \\
& \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix} \Bigg\}.
\end{aligned}
$$

The following result is an analogue of Proposition 7.3 of [17].

7.16. Proposition. *For any $e_I \in \Lambda$ with $|I| = i$, where $i = 0, 1, ..., l-1$,*

$$\begin{aligned} We_IW &= \{x \in \mathcal{R} |\ rank(x) = i\} \\ &= \{x \in \mathcal{R} |\ x \text{ has } i \text{ nonzero rows}\} \\ &= \{x \in \mathcal{R}_{2l} |\ D(x) \text{ and } R(x) \text{ are admissible with } |D(x)| = |R(x)| = i\ \}. \end{aligned}$$

Furthermore,

$$|We_IW| = \left[\sum_{j=0}^{i} \binom{l}{j}\binom{l-j}{i-j}\right]^2 i!,$$

where $D(x)$ is the domain of x and $R(x)$ the range of x.

Proof. Observe that $Ge_IG = \bigsqcup_{x \in We_IW} BxB$ consists of $n \times n$ matrices of rank i in MSO_{2l} where $i = |I| = 0, 1, ..., l-1$. One gets the first part of the Proposition.

Now there are $\sum_{j=0}^{i} \binom{l}{j}\binom{l-j}{i-j}$ ways to choose i of the n rows making $D(x)$ admissible. There are the same number of ways to choose i of the n columns such that $R(x)$ is admissible. For each pair of the choices of the rows and columns there are $i!$ elements of $\mathcal{R}$, of rank i, with a nonzero entry in each of the i rows and each of the i columns chosen. Thus, there are $\left[\sum_{j=0}^{i} \binom{l}{j}\binom{l-j}{i-j}\right]^2 i!$ possibilities. □

Similarly, we get the following

7.17 Proposition. *Let $J_1 = \{1, ..., l\}$, and $J_2 = \{1, ..., l-1, l+1\}$. Then*

$$\begin{aligned} We_{J_1}W \cup We_{J_2}W &= \{x \in \mathcal{R} |\ rank(x) = l\} \\ &= \{x \in \mathcal{R} |\ x \text{ has } l \text{ nonzero rows}\} \\ &= \left\{ x \in \mathcal{R}_{2l} \,\middle|\, \begin{array}{l} D(x) \text{ and } R(x) \text{ are admissible and of} \\ \text{the same type with } |D(x)| = |R(x)| = l \end{array} \right\}. \end{aligned}$$

Furthermore,

$$|We_{J_1}W \cup We_{J_2}W| = \frac{1}{2}\left[\sum_{j=0}^{l}\binom{l}{j}\right]^2 l!,$$

where $D(x)$ is the domain of x and $R(x)$ the range of x.

Proof. The first part follows from Theorem 7.15 above.

Now there are $\frac{1}{2}\sum_{j=0}^{l}\binom{l}{j}$ ways to choose l of the n rows such that the resulting subsets are of type I (resp. II). There are the same number of ways to choose l of the n columns. For each pair of the choices of the rows and columns there are $l!$ elements of $\mathcal{R}$ of rank l with a nonzero entry in each of the l rows and each of the l columns chosen. Thus there are $\frac{1}{4}\left[\sum_{j=0}^{l}\binom{l}{j}\right]^2 l!$ possibilities for elements on $We_{J_1}W$ (resp. $We_{J_2}W$). Hence, the number of elements in $We_{J_1}W \cup We_{J_2}W$ is as stated. □

7.18. Corollary. $|\mathcal{R}| = \sum_{i=0}^{l-1}\left(\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!\right) + (2^l+1)2^{l-1}l!,$ *for $l \geq 1$.*

Proof. It is clear that

$$\begin{aligned}
|\mathcal{R}| &= \sum_{i=0}^{l-1}\left(\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!\right) + |We_{I_1}W \cup We_{I_2}W| + |W| \\
&= \sum_{i=0}^{l-1}\left(\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!\right) + \frac{1}{2}\left[\sum_{j=0}^{l}\binom{l}{j}\right]^2 l! + 2^{l-1}l! \\
&= \sum_{i=0}^{l-1}\left(\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!\right) + \frac{1}{2}2^{2l}l! + 2^{l-1}l! \\
&= \sum_{i=0}^{l-1}\left(\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!\right) + (2^l+1)2^{l-1}l!. \quad \square
\end{aligned}$$

For instance, the Renner monoid of MSO_4 has 37 elements.

8. Cell Decompositions of the Special Orthogonal Monoids MSO_{2l}

The main purpose of this section is to determine the cell decompositions of the special orthogonal monoids, even case.

Let $B_0 = \mathbf{B}_n \cap \mathbf{SO}_n$. Then B_0 is a Borel subgroup of $\mathbf{SO}_n$, and $B = K^* B_0$ is a Borel subgroup of $G = K^* \mathbf{SO}_n$. A simple calculation tells us that

$$B_0 = \left\{ \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in \mathbf{SO}_{2l} \middle| \begin{array}{l} b_1, b_3 \in \mathbf{M}_l(K) \text{ are upper triangular,} \\ b_3^T J b_1 = J, \quad b_3^T J b_2 = -b_2^T J b_3 \end{array} \right\}.$$

The following theorem states a relationship among the Borel subgroup B, idempotents and rank one elements in $E(\overline{T})$.

8.1. Theorem. *Let $T = K^* T_0 \subseteq B$ be a maximal torus in G. Then for every $e_{_I} \in E(\overline{T})$, there exists a unique $e_i = E_{ii} \in E_1(\overline{T})$ such that $e_i B e_{_I} = e_i B e_i$, where i is the maximal number in I.*

Proof. Let $e_{_I} \in E(\overline{T})$ where $I = \{i_1, i_2, ..., i_m\} \subseteq \{1, ..., n\}$ is admissible with $i_1 < i_2 < ... < i_m$. For any upper triangular matrix $b = (b_{jk}) \in B \subseteq G$, the matrix $be_{_I}$ is an upper triangular matrix whose k-th column is exactly the k-th column of $b = (b_{jk})$, for $k = i_1, ..., i_m$, and the other columns of $be_{_I}$ are all zero. Let $i = i_m$ which is maximal in I. Taking $e_i = E_{ii}$, we get $e_i b e_{_I} = E_{ii} b e_{_I}$, a matrix whose $(i,i)-$entry is b_{ii} and the other entries are all zero. It follows that $e_i b e_{_I} = e_i b e_i$ by calculating directly. Therefore, $e_i B e_{_I} = e_i B e_i$. From the procedure above we also see the uniqueness of such $e_i = E_{ii} \in E_1(\overline{T})$. $\square$

8.2. Definition. *For any $e_i = E_{ii} \in E_1(\overline{T})$, define*

$$\mathcal{R}(e_i) = \{x \in \mathcal{R} |\ e_i B x = e_i B e_i x \neq 0\}.$$

8.3. Corollary. *The set of non-zero elements of the Renner monoid has a decomposition*

$$\mathcal{R}^{\times} = \bigsqcup_{e_i \in E_1(\overline{T})} \mathcal{R}(e_i) = \bigsqcup_{i=1}^{2l} \mathcal{R}(E_{ii}).$$

It follows from Theorem 8.1 that there is a surjective map τ from the set $E(\overline{T})$ of idempotents in $\overline{T}$ onto $E_1(\overline{T})$ of rank one elements. This map can also be extended to a map of $\mathcal{R}^{\times}$ to $E_1(\overline{T})$.

8.4. Theorem.

a) There is a surjective map τ from $E(\overline{T})$ onto $E_1(\overline{T})$ by

$$e_{_I} \longmapsto \tau(e_{_I}) = e_i, \ \textit{if } e_i B e_{_I} = e_i B e_i \neq 0$$

b) The map τ extends to $\mathcal{R}^{\times} = \mathcal{R} \setminus \{0\}$ by, for every $x \in \mathcal{R}^{\times}$, defining,

$$\tau(x) = e_i, \ \textit{if } x \in e_{_I} W \ \textit{and } \tau(e_I) = e_i,$$

where $I \neq \phi$ admissible and i is maximal in I.

Proof. a) is clear. To prove b), note that for any $x \in \mathcal{R}^{\times}$, there exist $w \in W$ and a unique $e_{_I} \in E(\overline{T})$ such that $x = e_{_I} w$. It follows that there is a unique $e_i = E_{ii} \in E_1(\overline{T}) = E_1(\mathcal{R})$ such that $e_i B e_I = e_i B e_i \neq 0$. Then we obtain the map from $\mathcal{R}^{\times}$ to $E_1(\mathcal{R}) = E_1(\overline{T})$ by $\tau(x) = e_i$, as required. □

8.5. Proposition. *Let $I(i) = \{I \subseteq \{1, ..., 2l\} | \ I$ admissible with i =max(I)$\}$, where $i = 1, ..., 2l$. Then*

a) $\mathcal{R}(e_i) = \tau^{-1}(e_i) = \bigsqcup_{I \in I(i)} e_{_I} W$, for $i = 1, ..., 2l$.

b) $\mathcal{R}^{\times} = \bigsqcup_{i=1}^{2l} \tau^{-1}(e_i)$, a disjoint union.

Proof. It is straightforward. □

For MSO_4, $E_1(\overline{T}) = \{e_1 = E_{11}, e_2 = E_{22}, e_3 = E_{33}, e_4 = E_{44}\}$, and

$$\begin{aligned}
\mathcal{R}(e_1) &= \bigsqcup_{I \in I(1)} e_I W \\
&= E_{11} W \\
&= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}(e_2) &= \textstyle\bigsqcup_{I \in I(2)} e_I W \\
&= E_{22} W \cup (E_{11} + E_{22}) W \\
&= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}(e_3) &= \textstyle\bigsqcup_{I \in I(3)} e_I W \\
&= E_{33} W \cup (E_{11} + E_{33}) W \\
&= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}(e_4) &= \textstyle\bigsqcup_{I \in I(4)} e_I W \\
&= E_{44} W \cup (E_{33} + E_{44}) W \cup (E_{22} + E_{44}) W \cup W
\end{aligned}$$

$$= \left\{ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix} \right\}$$

Our task now is to find the cell decomposition of the special orthogonal Renner monoid. Note that for any $r = E_{ij} \in \mathcal{R}(1)$, there exists a unique $e_r = E_{ii}$ and $f_r = E_{jj}$ in $E_1(\overline{T})$ such that $r = e_r r f_r$.

8.6. Definition. *For any $r = e_r r f_r \in \mathcal{R}(1)$, call $\mathcal{C}_r = \{x \in \mathcal{R}(e_r) \mid e_r x f_r = r\}$ a cell of the Renner monoid $\mathcal{R}$ of MSO_{2l} corresponding to the rank one element r.*

8.7. Proposition. *For any $e \in E_1(\overline{T})$ and $x \in \mathcal{R}(e)$, there is a unique $r \in \mathcal{R}(1)$ such that $x \in \mathcal{C}_r$ where $r = ex$.*

Proof. Let $x \in \mathcal{R}(e)$ where $e \in E_1(\overline{T})$. Then $eBx = eBex \neq 0$ where B is the Borel subgroup of $G \subseteq MSO_n$. Let $r = ex$. Then $r \in \mathcal{R}(1)$ and $r = erf_r$ for the unique $e, f_r \in E_1(\overline{T})$. For if $r = e_r r f_r$ and $e_r \neq e$, then $r = ex = e(ex) = er = e(e_r)rf_r = (ee_r)rf_r = 0$, since $ee_r = 0$. Thus $x \in \mathcal{R}(e)$ and $exf_r = rf_r = (erf_r)f_r = erf_r = r$, i.e., $x \in \mathcal{C}_r$.

Suppose that there is another $r' \in \mathcal{R}(1)$ such that $x \in \mathcal{C}_{r'} = \{x \in \mathcal{R}(e) \mid exf' = r'\}$ where $r' = er'f'$ for the unique $e, f' \in E_1(\overline{T})$. If $f' \neq f_r$ then $r' = exf' = (ex)f' = rf' = (exf_r)f' = ex(f_rf') = 0$, since $f_rf' = 0 \in E(\overline{T})$, which is a contradiction. Therefore, the uniqueness. □

8.8. Corollary.

a) $\mathcal{R}(e) = \bigsqcup_{\substack{r\in\mathcal{R}(1)\\ er=r}} \mathcal{C}_r$, *where* $e \in E_1(\overline{T})$.

b) $\mathcal{R}^\times = \bigsqcup_{r\in\mathcal{R}(1)} \mathcal{C}_r$.

Proof. It is straightforward by Corollary 8.3 and the above proposition □

Now, we can establish a surjective map φ from $\mathcal{R}^\times$ to the set $\mathcal{R}(1)$ consisting of rank one elements in $\mathcal{R}$ by declaring $\varphi(x) = r$ if $x \in \mathcal{C}_r$ where $x \in \mathcal{R}^\times$ and $r \in \mathcal{R}(1)$. It is an extension of τ. Furthermore, $\varphi^{-1}(r) = \mathcal{C}_r$ for $r \in \mathcal{R}(1)$.

8.9. Theorem. *The above surjective map* φ *from* $\mathcal{R}^\times$ *to* $\mathcal{R}(1)$ *satisfies* $\varphi(x) = e_iw$ *if* $x = e_{_I}w \in \mathcal{R}^\times$ *and* $\tau(e_{_I}) = e_i$, *where* $e_{_I} \in E(\overline{T})$ *and* $w \in W$.

Proof. Since $\mathcal{R}^\times = \bigsqcup_{e_i\in E_1(\overline{T})} \mathcal{R}(e_i)$ where $\mathcal{R}(e_i) = \bigsqcup_{I\in I(i)} e_{_I}W$, there is a unique $e_i \in E_1(\overline{T})$ such that $x \in \mathcal{R}(e_i)$. It follows that if $x = e_{_I}w \in \mathcal{R}^\times$ and $\tau(e_{_I}) = e_i$, then $I \in I(i)$. Thus $\varphi(x) = e_ix = e_i(e_{_I}w) = (e_ie_{_I})w = e_iw$, the required result. □

8.10. Theorem. *For any* $r = E_{ij} \in \mathcal{R}(1), i, j = 1, ..., 2l$,

$$\mathcal{C}_r = \mathcal{C}_{E_{ij}} = \{(x_{pq}) \in \mathcal{R} \mid x_{ij} = 1; x_{pq} = 0, \textit{ if } i < p \leq 2l, 1 \leq q \leq 2l\}.$$

Proof. If $x = (x_{pq}) \in \mathcal{R}$ is an $n\times n$ matrix, then $E_{ii}x = E_{ij}$ if and only if $x_{iq} = \delta_{qj}$, for $i, q, j = 1, ..., n$. Then

$$\begin{aligned}
\mathcal{C}_r = \mathcal{C}_{E_{ij}} &= \{(x_{pq}) \in \mathcal{R}(e_i) \mid E_{ii}(x_{pq}) = E_{ij}\} \\
&= \{(x_{pq}) \in \mathcal{R}(e_i) \mid x_{ij} = 1\} \\
&= \bigsqcup_{I \in I(i)} \{(x_{pq}) \in e_I W \mid x_{ij} = 1\} \\
&= \{(x_{pq}) \in \mathcal{R} \mid x_{ij} = 1; x_{pq} = 0, \text{ if } i < p \leq 2l, 1 \leq q \leq 2l\}.
\end{aligned}$$

which proves the Theorem. $\square$

In the sequel, the cells in Theorem 7.10 will be simply denoted by $\mathcal{C}_{ij}$.

If $n = 4$, the cells of the Renner monoid of MSO_4 are:

$$\mathcal{C}_{11} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathcal{C}_{12} = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{13} = \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathcal{C}_{14} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{21} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{22} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{23} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{24} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{31} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{32} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{33} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{34} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{41} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{42} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{43} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{44} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Notice the fact that the cells $\mathcal{C}_{ij}(K)$ of the Renner monoid $\mathcal{R}_{2l}$ of $\mathbf{M}_n(K)$ are

$$\mathcal{C}_{ij}(K) = \{(x_{pq}) \in \mathcal{R}_{2l} \mid x_{ij} = 1, x_{pq} = 0, \text{ if } i < p \leq 2l, 1 \leq q \leq 2l\}.$$

where $i, j = 1, ..., 2l$. We can now get

8.11. Theorem. $\mathcal{C}_{ij} = \mathcal{C}_{ij}(K) \cap \mathcal{R}$, *where* $i, j = 1, ..., 2l$.

We begin to describe the cell decomposition of the special orthogonal algebraic monoid MSO_n using the following definition.

8.12. Definition. *The sets* $C_{ij} = B\mathcal{C}_{ij}B$ *for* $i, j = 1, ..., n$ *are called cells for special orthogonal monoids* MSO_{2l} *with respect to the Borel subgroup* B.

8.13. Theorem. *The cells of the special orthogonal monoid are*

$$C_{ij} = \left\{ (a_{pq}) \in MSO_n \;\middle|\; \begin{array}{l} a_{ij} \neq 0; \quad a_{iq} = 0 \textit{ if } 1 \leq q < j; \\ a_{pq} = 0, \textit{ if } i < p \leq 2l \textit{ and } 1 \leq q \leq 2l \end{array} \right\}$$

where $i, j = 1, ..., 2l$.

Proof. Since $C_{ij} = \cup_{x \in \mathcal{C}_{ij}} BxB = \cup_{x \in \mathcal{C}_{ij}} (K^* B_0) x (K^* B_0) = K^* \left(\cup_{x \in \mathcal{C}_{ij}} B_0 x B_0 \right)$, for $i, j = 1, ..., 2l$, we need only to consider elements in $B_0 x B_0$, where $x \in \mathcal{C}_{ij}$ and

$$B_0 = \left\{ \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in \mathbf{SO}_{2l} \;\middle|\; \begin{array}{l} b_1, b_3 \in \mathbf{M}_l(K) \text{ are upper triangular,} \\ b_3^T J b_1 = J, \quad b_3^T J b_2 = -b_2^T J b_3 \end{array} \right\}.$$

Now for every element $b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in B_0$, suppose that $b = (b_{pq})_{2l \times 2l}$ which is upper triangular. Then $b_{pp} \neq 0$, for $p = 1, ..., 2l$. For any $x = (x_{pq}) \in \mathcal{C}_{ij} \subseteq \mathcal{R}(E_{ii}) \subseteq \mathcal{R}$, let $I = \{i_1, ..., i_{m-1}, i_m\}$ denote the index set of non-zero rows of x where $i_1 < ... < i_{m-1} < i_m$ and $i_m = i$. Let $J = \{j_1, ..., j_{m-1}, j_m\}$ denote the index set of non-zero columns such that $j_m = j$ and $x_{i_k j_k} = 1$, for $k = 1, ..., m$. Generally, we do not have $j_1 < ... < j_{m-1} < j_m$.

Thus bx is a matrix whose j_k-th column is the i_k-th of b where $k = 1, ..., m$, and all rows under row i are zero. The shape of bx is

$$i\text{-th row} \longleftarrow \begin{pmatrix} * & \dots & * & b_{1i} & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \dots & * & b_{i-1i} & * & \dots & * \\ 0 & \dots & 0 & b_{ii} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\downarrow$$

$$j\text{-th column}$$

where $b_{ii} \neq 0$ is the (i,j)-entry of bx. Taking any $b' = (b'_{pq})_{2l\times 2l} \in B_0$, one obtains the shape of bxb' is

$$i\text{-th row} \longleftarrow \begin{pmatrix} * & \dots & * & * & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \dots & * & * & * & \dots & * \\ 0 & \dots & 0 & b_{ii}b'_{jj} & b_{ii}b'_{jj+1} & \dots & b_{ii}b'_{j2l} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\downarrow$$

$$j\text{-th column}$$

where $b_{ii}b'_{jj} \neq 0$ is the (i,j)-entry of bxb'. From the arbitrariness of $b, b' \in B_0$ and $x \in \mathcal{C}_{ij}$, we get

$$C_{ij} = \left\{ (a_{pq}) \in MSO_n \;\middle|\; \begin{array}{l} a_{ij} \neq 0; \quad a_{iq} = 0 \text{ if } 1 \leq q < j; \\ a_{pq} = 0 \text{ if } i < p \leq 2l \text{ and } 1 \leq q \leq 2l \end{array} \right\}$$

where $i, j = 1, ..., 2l$, since $B = K^* B_0$. $\square$

8.14. Corollary. *Keeping the notation above, we have*

$$MSO_{2l} \setminus \{0\} = \bigsqcup_{i,j=1}^{2l} C_{ij}.$$

Proof. The statement is true by Corollary 7.8 and the Bruhat-Renner decomposition of MSp_{2l}. See [17, Corollary 5.8]. $\square$

8.15. Theorem. $C_{ij} = C_{ij}(K) \cap MSO_n$, *for* $i, j = 1, ..., 2l$.

Proof. From Theorem 8.13 and the shapes of elements in the cells $C_{ij}(K)$ of $\mathbf{M}_n(K)$ we know the result is as stated. $\square$

9. Submonoids of the Special Orthogonal Monoids MSO_{2l}

The main purpose of this section is to establish some properties of the submonoid $(MSO_n)_e = \{y \in MSO_n \mid ye = ey = e\}$ of MSO_n, where $e \in E(MSO_n)$ and $n = 2l$. We simply denote by M_e the submonoid $(MSO_n)_e$. Let $G_e = M_e \cap G$ where $G = K^*\mathbf{SO}_n$ is the unit group of MSO_n. Then by [15, Theorem 6.11] one has $M_e = \overline{G}_e$. Thus, unveiling some properties of G_e is necessary.

9.1. Lemma. *Let* $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{2l \times 2l} \in MSO_n$. *Then* $y \in \mathbf{SO}_{2l}$ *if and only if* $y_1 \in \mathbf{SO}_{2l-2}$.

Proof. Recall that $J_l = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \in \mathbf{M}_{2l}(K)$ is a symmetric matrix, where $J =$

$\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size l. Rewrite J_l to be $J_l = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J_{l-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} y \in Sp_{2l} &\Longleftrightarrow y^T J_l y = J_l \\ &\Longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & y_1^T J_{l-1} y_1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J_{l-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &\Longleftrightarrow y_1^T J_{l-1} y_1 = J_{l-1} \\ &\Longleftrightarrow y_1 \in \mathbf{SO}_{2l-2}. \quad \square \end{aligned}$$

9.2. Theorem. *Let $e_1 = E_{11} \in \Lambda$ and $G = K^*\mathbf{SO}_{2l}$. Then G_{e_1} is isomorphic to $K^*\mathbf{SO}_{2l-2}$. Furthermore, M_{e_1} is isomorphic to MSO_{2l-2}.*

Proof. Suppose that $y = tx \in G$ with $x = (x_{ij})_{i,j=1}^{2l} \in \mathbf{SO}_{2l}$ and $t \in K^*$. Then $ye_1 = e_1 y = e_1$ is equivalent to $xe_1 = e_1 x = (1/t)e_1$. So

$$x = \begin{pmatrix} 1/t & 0 \\ 0 & x_1 \end{pmatrix} \in \mathbf{SO}_{2l},$$

where $x_1 = (x_{ij})_{i,j=2}^{2l}$ is a $2l-1 \times 2l-1$ matrix. Let $A = (0, ..., 1)_{1\times 2l-1}$ and rewrite $J_l = \begin{pmatrix} 0 & A \\ A^T & J' \end{pmatrix}$. Notice that

$$x^T J_l x = \begin{pmatrix} 0 & (1/t)Ax_1 \\ (1/t)x_1^T A^T & x_1^T J' x_1 \end{pmatrix}.$$

Thus $x^T J_l x = J_l$ gives us $(1/t)Ax_1 = A$, $(1/t)x_1^T A^T = A^T$ and $x_1^T J' x_1 = J'$. It follows that $x_{2l,2} = ... = x_{2l,2l-1} = 0$ and $x_{2l,2l} = t$, which shows that x_1 has the shape

$$x_1 = \begin{pmatrix} x_2 & X \\ 0 & t \end{pmatrix}_{2l-1\times 2l-1},$$

where $x_2 = (x_{ij})_{i,j=2}^{2l-1}$ is a $2l-2 \times 2l-2$ matrix and $X = (x_{2,2l}, ..., x_{2l-1,2l})^T$. Since $J' = \begin{pmatrix} J_{l-1} & 0 \\ 0 & 0 \end{pmatrix}$ and $x_1^T J' x_1 = \begin{pmatrix} x_2^T J_{l-1} x_2 & x_2^T J_{l-1} X \\ X^T J_{l-1} x_2 & X^T J_{l-1} X \end{pmatrix}$, it follows from

$x_1^T J' x_1 = J'$ that $x_2^T J_{l-1} x_2 = J_{l-1}$, $x_2^T J_{l-1} X = 0$, $X^T J_{l-1} x_2 = 0$ and $X^T J_{l-1} X = 0$. Thus $X = 0$ and $x_1 = \begin{pmatrix} x_2 & 0 \\ 0 & t \end{pmatrix}$, where $x_2 \in \mathbf{SO}_{2l-2}$. Therefore,

$$x = \begin{pmatrix} 1/t & & \\ & x_2 & \\ & & t \end{pmatrix} \in \mathbf{SO}_{2l},$$

where $t \in K^*$ and $x_2 \in \mathbf{SO}_{2l-2}$. It follows easily that

$$\begin{aligned} G_e &= \left\{ t \cdot \begin{pmatrix} 1/t & & \\ & x_2 & \\ & & t \end{pmatrix} \middle| \; t \in K^*, x_2 \in \mathbf{SO}_{2l-2} \right\} \\ &= \left\{ \begin{pmatrix} 1 & & \\ & tx_2 & \\ & & t^2 \end{pmatrix} \middle| \; t \in K^*, x_2 \in \mathbf{SO}_{2l-2} \right\}. \end{aligned}$$

Define a mapping f from G_{e_1} to $K^*\mathbf{SO}_{2l-2}$ by

$$y = \begin{pmatrix} 1 & & \\ & tx_2 & \\ & & t^2 \end{pmatrix} \longmapsto tx_2 \in K^*\mathbf{SO}_{2l-2}$$

Then f is an algebraic group isomorphism from G_{e_1} to $K^*\mathbf{SO}_{2l-2}$. Hence, $\overline{G}_{e_1}$ is isomorphic to $\overline{K^*\mathbf{SO}}_{2l-2}$ which is MSO_{2l-2}. But it follows from [15, Theorem 6.11] that $M_{e_1} = \overline{G}_{e_1}$. Therefore, M_{e_1} is isomorphic to MSO_{2l-2}. This proves the Theorem. □

9.3. Corollary.

a) For any $e \in E_1(\overline{T})$, M_e is isomorphic to MSO_{2l-2}.

c) For any $e \in E_1(MSO_{2l})$, the rank one elements in $E(MSO_{2l})$, M_e is isomorphic to MSO_{2l-2}.

Proof. For a), note that $E_1(\overline{T}) = \{w^{-1}e_1 w \mid w \in W\}$. Then for any $e \in E_1(\overline{T})$, there exists $w \in W$ such that $e = w^{-1}e_1 w$, where $e_1 = E_{11}$. Since $ye = ey = e$

is equivalent to $(wyw^{-1})e_1 = e_1(wyw^{-1}) = e_1$, it follows that M_e is isomorphic to M_{e_1} by the mapping $y \longmapsto wyw^{-1}$. From Theorem 9.2 one obtains that M_e is isomorphic to MSO_{2l-2}. Similar recipes of a) apply to b) by using $E_1(MSO_{2l}) = \{g^{-1}e_1 g \mid g \in G\}$. □

9.4. Theorem. *Let $G = K^*\mathbf{SO}_{2l}$ and $e_I \in \Lambda$ with I standard admissible, where $e_I = \sum_{j\in I} E_{jj} \in \Lambda$ and $i = 1, ..., l$. Then G_{e_I} is isomorphic to $K^*\mathbf{SO}_{2l-2i}$. Furthermore, M_{e_I} is isomorphic to MSO_{2l-2i}.*

Proof. It is similar to that of Theorem 9.2. □

9.5. Corollary. *Keeping the same notations in Theorem 8.4, we have*

a) For every $e_J \in E_i(\overline{T})$ with J admissible and $|J| = i$, for $i = 1, ..., l$, M_{e_J} is isomorphic to MSO_{2l-2i}.

b) For every $e \in E_i(MSO_{2l})$, the rank i elements in $E(MSO_{2l})$, M_e is isomorphic to MSO_{2l-2i}, for $i = 1, ..., l$.

Proof. For a), note that for every $e_J \in E_i(\overline{T})$, the set of rank i ($i = 1, ..., l$) idempotents of $E(\overline{T})$, there exist unique $e_I \in \Lambda$ and $w \in W$ such that $e_J = we_Iw^{-1}$. But $ye_J = e_Jy = e_J$ is equivalent to $(w^{-1}yw)e_I = e_I(w^{-1}yw) = e_I$. Hence M_{e_J} is isomorphic to M_{e_I} by the mapping $y \longmapsto w^{-1}yw$. It follows from Theorem 9.4 that M_{e_J} is isomorphic to MSO_{2l-2i}. Applying a) and [17, Corollary 6.10 (ii)] one gets b) easily. □

CHAPTER V

SPECIAL ORTHOGONAL MONOIDS MSO_{2l+1}

In this chapter we consider the Renner monoids and the cell decompositions of the special orthogonal monoid, odd case. Throughout this section the characteristic of the base field K is not two, unless otherwise stated.

Let $n = 2l+1$ and $J_l = \begin{pmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{pmatrix} \in \mathbf{M}_n$ be the symmetric matrix, where $J = \begin{pmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{pmatrix}$ of dimension $l \times l$. The special orthogonal group is by definition

$$G_0 = \mathbf{SO}_n = \{g \in \mathbf{SL}_n | \; g^T J_l g = J_l\}$$

which is a closed, connected and reductive subgroup of the linear algebraic group.

Remark 11. The definition of $\mathbf{SO}_n$ here is different than what J. Humphreys used in his book [7].

From now on we assume $n = 2l+1$. Let $T_0 = G_0 \cap \mathbf{T}_n$. Let $t = \mathrm{diag}(t_1, ..., t_{2l+1})$ be an element in T_0. Then by $t^T J_l t = J_l$ one obtains that

$$t = \mathrm{diag}(t_1, ..., t_l, \pm 1, t_l^{-1}, ..., t_1^{-1}).$$

Since $\mathbf{SO}_n \subset \mathbf{SL}_n$, elements in T_0 have the shape

$$t = \mathrm{diag}(t_1, ..., t_l, 1, t_l^{-1}, ..., t_1^{-1}).$$

where $t_1, ..., t_l$ are arbitrary in K^*. Thus T_0 is a maximal torus of dimension l. Let us recall some facts about the Weyl group $W(G_0, T_0)$. If $\pi \in S_n$, let $p_\pi = \sum_{i=1}^{2l+1} E_{\pi i,i} \in P_n$ be the corresponding permutation matrix, where E_{ij} are the matrix unit. Then $p_\pi(a_{ij}) = (a_{\pi^{-1}i,j})$, and $(a_{ij})p_\pi = (a_{i,\pi j})$, where (a_{ij}) is any $n \times n$ matrix. It follows that $p_\pi^{-1}(a_{ij})p_\pi = p_{\pi^{-1}}(a_{ij})p_\pi = (a_{\pi i,\pi j})$.

Define an involution $\theta : i \longmapsto \bar{i}$ of $\{1, 2, ..., l, l+1, l+2, ..., 2l+1\}$ by

$$\bar{i} = 2l + 2 - i, \quad \text{for } 1 \leq i \leq 2l+1,$$

Since $p_\pi^{-1}\text{diag }(t_1, ..., t_{2l+1})p_\pi = \text{diag}(t_{\pi(1)}, ..., t_{\pi(2l+1)})$, p_π normalizes T_0 if and only if $\pi\theta(i) = \theta\pi(i)$, for $i = 1, ..., 2l+1$. Taking $i = l+1$, one finds that $\pi(l+1) = l+1$. Let C denote the centralizer of θ in S_n. Then p_π normalizes T_0 if and only if $\pi \in C$. The group C is a semidirect product $C = C_1C_2$ where C_1 is a normal abelian subgroup of order 2^l generated by the transpositions $(1\bar{1}), ..., (l\bar{l})$ and $C_2 \simeq S_l$ consists of all permutations $\pi \in S_n$ which fix $l+1$, stablize $\{1, ..., l\}$ and act on $\{l+2, ..., 2l+1\}$ in the unique manner consistent with the assertion that $\pi \in C$. Then $W(G_0, T_0) \simeq C_1C_2$. Also, $W \simeq (Z_2)^l \rtimes S_l$.

If $n = 5$, then $\theta = (1\bar{1})(2\bar{2}) = (15)(24) \in S_5$, and C_1 is a subgroup of $C_{S_5}(\theta)$ generated by (15) and (24). So

$$C_1 = \{1, (15), (24), (15)(24)\}.$$

Taking $\pi = (12)(45) \in S_5$, we see that $\theta\pi = \pi\theta$ which means that $\pi \in C_{S_5}(\theta)$. It is clear that $\pi(3) = 3$ and π stabilizes $\{1, ..., l\} = \{1, 2\}$. Let C_2 be a subgroup of $C_{S_5}(\theta)$ generated by π. Then

$$C_2 = \{1, (12)(45)\}.$$

Thus the Weyl group

$$W(G_0, T_0) = C_1C_2 = \{1, (15), (24), (15)(24), (1254), (1452), (14)(25), (12)(45)\}.$$

The corresponding matrix form of the Weyl group is

$$W(G_0,T_0)=\left\{\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix},\right.$$

$$\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&0&0&1&0\end{pmatrix},$$

$$\left.\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}\right\}.$$

$$=\{1,\rho,\sigma,\theta,\theta\pi,\rho\pi,\sigma\pi,\pi\}.$$

We will use the Weyl group to describe the Renner monoid of the special orthogonal monoid in the following section.

10. The Renner Monoids of the Special Orthogonal Monoids MSO_{2l+1}

We now compute the Renner monoid $\mathcal{R}$ of the special orthogonal monoid, odd case.

Let $G = K^*G_0 \in \mathbf{GL}_n$, where $n = 2l+1$. Then G is a connected reductive group with rank $r = l+1$ and semisimple rank l. Let $T = K^*T_0$. Then T is a maximal torus of G. The Weyl group $W(G,T)$ is isomorphic to $W(G_0,T_0)$ (see [22]). We identify them in what follows and let W denote either of these groups.

10.1. Definition. *The monoid $\overline{G}$, Zariski closure of G in $\mathbf{M}_n(K)$, is called the special orthogonal monoid which will be denoted by MSO_n with $n = 2l+1$.*

10.2. Definition. *A subset $I \subseteq \{1, ..., 2l+1\}$ is called admissible if $j \in I$ implies $\bar{j} \notin I$, where $\bar{j} = \theta(j)$ as above; the empty set ϕ and the whole set $\{1, ..., 2l+1\}$ are also considered to admissible.*

An admissible subset I is referred to as standard if $I = \phi$, or there is an integer $i \in \{1, ..., l, 2l+1\}$ such that $I = \{1, ..., i\}$.

If $n = 5$, then the admissible subsets of $\{1, 2, 3, 4, 5\}$ are

$$\phi, \{1\}, \{2\}, \{4\}, \{5\}, \{1,2\}, \{4,5\}, \{1,4\}, \{2,5\}, \{1,2,3,4,5\},$$

The standard admissible subsets of $\{1, 2, 3, 4, 5\}$ are $\phi, \{1\}, \{1,2\}, \{1,2,3,4,5\}$.

A similar discussion to [22, p336] gives the following lemma stating the relationship between admissible subsets and idempotents in $\overline{T}$. We omit those simple details.

10.3. Lemma.

a) W maps admissible sets to admissible sets, and $w^{-1} e_I w = e_{wI}$ for any $w \in W$.

b) The map

$$I \longmapsto e_I = \sum_{j \in I} E_{jj}$$

is bijective from the admissible subsets of $\{1, ..., 2l+1\}$ to $E(\overline{T})$, where $e_I = 0$, if $I = \phi$.

c) The set $E(\overline{T})$ of idempotents in $\overline{T}$ is

$$E(\overline{T}) = \{e_I \mid I \text{ is admissible}\}.$$

d) $e_{I_1} \cdot e_{I_2} = e_{I_1 \cap I_2}$ for any $e_{I_1}, e_{I_2} \in E(\overline{T})$.

Proof. For a), b) and c), see [22, p336]. By checking directly, we get d). □

If $n = 5$, the set of idempotents in $\overline{T}$ of MSO_5 is

$$E(\overline{T}) = \{\,0,\ \ E_{11},\ \ E_{22},\ \ E_{44},\ \ E_{55},\ \ E_{11}+E_{22},\ E_{44}+E_{55},$$

$$E_{11}+E_{44},\ E_{22}+E_{55},\ E_{11}+E_{22}+E_{33}+E_{44}+E_{55}\}.$$

Remark 12. The rank one elements in $E(\overline{T})$ are in one to one correspondence with the admissible subsets containing exactly one element of $\{1, ..., 2l+1\}$. There are no admissible subsets with size k $(l < k \leq 2l)$.

10.4. Corollary.

$$|E(\mathcal{R})| = |E(\overline{T})| = 1 + \sum_{i=0}^{l}\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}.$$

Proof. Thanks to [17, Proposition 3.2.1], it is straightforward by counting the number of admissible subsets of $\{1, ..., 2l+1\}$. $\square$

For $i = 1, ..., l, 2l+1$, let $E_i(\mathcal{R}) \subseteq E(\mathcal{R})$ (resp. $E_i(\overline{T}) \subseteq E(\overline{T})$) denote the set of rank i idempotent elements in $\mathcal{R}$ (resp. $\overline{T}$). Then by Lemma 10.3 and [17, Proposition 3.2.1] we have the following

10.5. Proposition.

a) $E_1(\mathcal{R}) = E_1(\overline{T}) = \big\{E_{ii} \mid i \in \{1, ..., 2l+1\} \setminus \{l+1\}\big\}$.

b) $|E_1(\mathcal{R})| = |E_1(\overline{T})| = 2l$.

We now find the set of rank one elements in $\mathcal{R}$. Let $\mathcal{R}(i)$ denote the set of rank i elements in the Renner monoid $\mathcal{R}$, for $i \in \{1, ..., 2l+1\}$

10.6. Lemma. $\mathcal{R}(1) = \big\{E_{ij} \mid i, j \in \{1, ..., 2l+1\} \setminus \{l+1\}\big\}$, *and* $|\mathcal{R}(1)| = 4l^2$.

Proof. For any $j \in \{1, ..., 2l+1\} \setminus \{l+1\}$, let

$$w = \begin{cases} (1\bar{1}), & \text{if } j = \bar{1} \\ (1j)(\bar{1}\bar{j}), & \text{if } j \neq \bar{1}. \end{cases}$$

Then $w \in C_{S_n}(\theta)$. In other words, w is in the Weyl group W. It follows from $j = w(1)$ that

$$E_{1j} = E_{1,w1} = E_{11}w \in E_{11}W \text{ for } j \in \{1, ..., 2l+1\} \setminus \{l+1\}\}.$$

Thus, $E_{11}W = \{E_{1j} \mid j \in \{1, ..., 2l+1\} \setminus \{l+1\}\}$.

Similarly, if $i \neq l+1$, then $E_{ii}W = \{E_{ij} \mid j \in \{1, ..., 2l+1\} \setminus \{l+1\}\}$. Therefore, the set of rank one elements in the Renner monoid $\mathcal{R}$ is $\mathcal{R}(1) = \{E_{ij} \mid i, j \in \{1, ..., 2l+1\} \setminus \{l+1\}\}$. $\square$

Remark 13. The lemma above shows that there is a bijection from $\mathcal{R}(1)$ to $\mathcal{R}_{2l}(1)$. But not true for $\mathcal{R}(2)$ and $\mathcal{R}_{2l}(2)$, since $\{1, 2l+1\}$ is not an admissible subset of $\{1, ..., 2l+1\}$, and so not in $\mathcal{R}(2)$, but $E_{11} + E_{2l+1,2l+1} \in \mathcal{R}_{2l}(2)$ up to an isomorphism. For the same reason, we know $\mathcal{R}(i) \neq \mathcal{R}_{2l}(i)$ for $i = 3, ..., 2l$.

10.7. Theorem. *For any admissible subset $I \subseteq \{1, ..., 2l+1\}$ with $|I| = i$, where $i = 1, ..., l, 2l+1$, there exist a unique standard admissible subset $I_0 = \{1, ..., i\}$ and $w \in W$ such that $wI = I_0$.*

Proof. If $I = \{1, ..., 2l+1\}$, then $I_0 = I$ and $w = 1 \in W$ and we are done. Now let I be admissible and $I \neq \{1, ..., 2l+1\}$. Then $|I| = i \in \{1, ..., l\}$. We use induction on i. If $i = 1$, then $I = \{j\}$ for some $j \in \{1, ..., 2l+1\} \setminus \{l+1\}$. From Lemma 10.6 there exists $w \in W$ such that $w(I) = \{w(j)\} = \{1\}$, i.e., $wI = I_0$.

Suppose that $I \subseteq \{1, ..., 2l+1\}$ is any admissible subset with $1 < |I| \leq l$. Then $I = J \cup \{k\}$ where J is a subset of I with $|J| = i-1$ and $k \in I \setminus J$. It follows that J is admissible. By the induction hypothesis there exist a $w' \in W$ and a unique standard admissible subset $I' = \{1, ..., i-1\}$ such that $w'J = I'$. Then $w'I = I' \cup \{p\}$ where $p = w'(k)$, and hence $w'I = \{1, ..., i-1\} \cup \{p\}$ is a disjoint union. If $p = i$, then $I_0 = I' \cup \{i\}$ and $w = w'$ are what we want. If $p \neq i$, let

$$w_1 = \begin{cases} (i\bar{i}), & \text{if } p = \bar{i} \\ (ip)(\bar{i}\bar{p}), & \text{if } p \neq \bar{i}. \end{cases}$$

It follows that $w_1 \in W$. Note that $\bar{p} \notin w'I$, since $p \in w'I$ which is admissible. Then $p, i, \bar{p}, \bar{i} \notin I'$, and so $w_1(j) = j$, for $j \in I' = \{1, ..., i-1\}$. Taking $w = w_1 w'$, we obtain

$$w(I) = w_1(\{1, ..., i-1\} \cup \{p\})$$

$$= \{1, ..., i\}$$

which proves the Theorem. $\square$

10.8. Proposition. *The cross section lattice of* MSO_{2l+1}

$$\Lambda = \{e_I \in E(\overline{T}) \mid I \text{ is a standard admissible subset of } \{1, ..., 2l+1\}\}$$

$$= \left\{ I_{2l+1}, \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I_{l-1} & 0 \\ 0 & 0 \end{pmatrix}, \ \dots \ , \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}, (0) \right\}$$

$$\simeq \{I \mid I \text{ is a standard admissible subset of } \{1, ..., 2l+1\}\}$$

$$\simeq \{0, 1, ..., l, l+1\}, \text{ under the linear order.}$$

The cross section lattice of MSO_5 is

$$\Lambda = \left\{ (0), \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\}.$$

The Hasse diagram of the cross section lattice of MSO_{2l+1} is given by

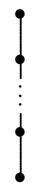

Figure 3.

10.9. Theorem. *With the notation above, the Renner monoid of the special orthogonal monoid MSO_n is as follows*

$$\mathcal{R} = \left\{ \sum_{\substack{i \in I \\ w \in W}} E_{i,wi} \in \mathcal{R}_n \mid I \subseteq \{1, ..., 2l+1\} \textit{ is admissible} \right\}.$$

Proof. By [17, Proposition 3.2.1] $\mathcal{R} = E(\overline{T})W$. It suffices to compute $e_{_I} w$ for every $e_{_I} \in E(\overline{T}), w \in W$, where I is admissible. From Lemma 10.3 b) we know $e_{_I} = \sum_{i \in I} E_{ii}$. Thus $e_{_I} w = \sum_{i \in I} E_{ii} w = \sum_{i \in I} E_{i,wi}$, and so the Theorem. □

10.10. Corollary.

$$\mathcal{R} = \left\{ \sum_{\substack{i \in I \\ w \in W}} E_{wi,i} \in \mathcal{R}_n \mid I \subseteq \{1, ..., 2l+1\} \textit{ is admissible} \right\}.$$

Proof. This result follows from the fact that $\mathcal{R} = WE(\overline{T})$ and

$$w^{-1} e_{_I} = \sum_{i \in I} w^{-1} E_{ii} = \sum_{i \in I} E_{wi,i}. \quad \square$$

10.11. Theorem.

$$\mathcal{R} \setminus W = \left\{ x \in \mathcal{R}_{2l+1} \;\middle|\; \begin{array}{l} x \textit{ is singular; both } D(x) \\ \textit{and } R(x) \textit{ are admissible} \end{array} \right\}$$

where $D(x)$ is the domain of x, and $R(x)$ is the range of x.

Proof. Let $\mathcal{R}'$ denote the set of the right hand side in the Theorem. It follows from Theorem 10.9 that $\mathcal{R} \setminus W \subseteq \mathcal{R}'$, since W maps admissible sets to admissible sets.

We now prove the other inclusion. For any $x \in \mathcal{R}'$, one knows $|D(x)| = |R(x)|$ which will be denoted by i. Then $i \leq l$, since x is singular and both $D(x)$ and $R(x)$ are admissible. It follows from Theorem 4.7 that there exist a unique standard admissible set $I_0 = \{1, ..., i\}(i \leq l)$ and $w_1, w_2 \in W$ such that

$$w_1 D(x) = w_2 R(x) = I_0.$$

Thus $w_1^{-1} x w_2 = e_{I_0} \in \Lambda \subseteq \mathcal{R}$, and hence $x = w_1 e_{I_0} w_2^{-1} \in \mathcal{R}$, since $\mathcal{R} = WE(\overline{T}) = E(\overline{T})W$. But $x \notin W$, because $x \in \mathcal{R}_{2l}$ is singular. So, $\mathcal{R}' \subseteq \mathcal{R} \setminus W$.

Therefore, $\mathcal{R} \setminus W = \mathcal{R}'$, i.e., the Theorem is true. $\square$

Remark 14. In the proof above we obtain $\mathcal{R} = W\Lambda W$ as well.

Now let us consider some examples. If $n = 3$, then all the admissible subsets of $\{1, 2, 3\}$ are $\phi, \{1\}, \{3\}, \{1, 2, 3\}$. Note that they are all standard except $\{3\}$. So, the cross section lattice of MSO_3 by Proposition 10.8 is

$$\Lambda = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

From Theorem 10.11, the Renner monoid of MSO_2 is

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

The Renner monoid $\mathcal{R}$ of MSO_5 is as follows

$$\begin{aligned}\mathcal{R} = \{&0,\ E_{11},\ E_{12},\ E_{15},\ E_{14},\ E_{21},\ E_{22},\ E_{25},\ E_{24},\ E_{41},\ E_{42},\\ &E_{44},\ E_{45},\ E_{51},\ E_{52},\ E_{54},\ E_{55},\ E_{11}+E_{22},\ E_{22}+E_{51},\\ &E_{11}+E_{42},\ E_{42}+E_{51},\ E_{45}+E_{52},\ E_{12}+E_{41},\ E_{21}+E_{52},\\ &E_{12}+E_{21},\ E_{44}+E_{55},\ E_{15}+E_{44},\ E_{24}+E_{55},\ E_{15}+E_{24},\\ &E_{14}+E_{25},\ E_{25}+E_{54},\ E_{14}+E_{45},\ E_{45}+E_{54},\ E_{11}+E_{44},\\ &E_{44}+E_{51},\ E_{11}+E_{24},\ E_{24}+E_{51},\ E_{14}+E_{41},\ E_{41}+E_{54},\\ &E_{14}+E_{21},\ E_{21}+E_{54},\ E_{22}+E_{55},\ E_{15}+E_{22},\ E_{42}+E_{55},\\ &E_{15}+E_{42},\ E_{25}+E_{52},\ E_{12}+E_{25},\ E_{45}+E_{52},\ E_{12}+E_{45},\\ &E_{11}+E_{22}+E_{33}+E_{55}+E_{44},\quad E_{15}+E_{22}+E_{33}+E_{51}+E_{44},\\ &E_{11}+E_{24}+E_{33}+E_{55}+E_{42},\quad E_{15}+E_{24}+E_{33}+E_{51}+E_{42},\\ &E_{14}+E_{25}+E_{33}+E_{52}+E_{41},\quad E_{12}+E_{25}+E_{33}+E_{54}+E_{41},\\ &E_{14}+E_{21}+E_{33}+E_{52}+E_{45},\quad E_{12}+E_{21}+E_{33}+E_{54}+E_{45}\}.\end{aligned}$$

We now express the Renner monoid of MSO_5 in term of matrices.

$$\mathcal{R} = \{\ 0,$$

$$\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$
\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix},
$$
$$
\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0 \end{pmatrix},
\begin{pmatrix} 1&0&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0 \end{pmatrix},
$$
$$
\left.\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{pmatrix},
\begin{pmatrix} 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix}\right\}
$$

The following result is an analogue of Proposition 7.3 of [17].

10.12. Proposition. *For any $e_I \in \Lambda$ with $|I| = i$, where $i = 0, 1, ..., l$,*

$$
\begin{aligned}
We_IW &= \{x \in \mathcal{R} \mid rank(x) = i\} \\
&= \{x \in \mathcal{R} \mid x \text{ has } i \text{ nonzero rows } \} \\
&= \{x \in \mathcal{R}_n \mid \ D(x) \text{ and } R(x) \text{ are admissible with } |D(x)| = |R(x)| = i\ \}.
\end{aligned}
$$

Furthermore,

$$|We_IW| = \left[\sum_{j=0}^{i} \binom{l}{j}\binom{l-j}{i-j}\right]^2 i!,$$

where $D(x)$ is the domain of x and $R(x)$ the range of x.

Proof. Observe that $Ge_IG = \bigsqcup_{x\in We_IW} BxB$ consists of $n \times n$ matrices of rank i in MSO_{2l+1}. Thus the first part of the Proposition is true.

Now there are $\sum_{j=0}^{i} \binom{l}{j}\binom{l-j}{i-j}$ ways to choose i of the n rows such that $D(x)$ is admissible. There are the same ways to choose i of the n columns making $R(x)$ admissible. For each of these choices there are $i!$ elements of $\mathcal{R}$, of rank i, with a nonzero entry in each of the i rows and each of the i columns chosen. Hence, there are $\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!$ posibilities. $\square$

10.13. Corollary. $|\mathcal{R}| = \sum_{i=0}^{l}\left(\left[\sum_{j=0}^{i}\binom{l}{j}\binom{l-j}{i-j}\right]^2 i!\right) + 2^l l!,$ *for* $l \geq 1$.

11. Cell Decompositions of the Special Orthogonal Monoids MSO_{2l+1}

The main purpose of this section is to determine the cell decompositions of the special orthogonal algebraic monoids, odd case.

We first figure out what the Borel subgroup is by using the cross section lattice of MSO_{2l+1}. From Proposition 10.8, the cross section lattice Λ of MSO_{2l+1} is

$$\Lambda = \left\{ I_{2l+1}, \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I_{l-1} & 0 \\ 0 & 0 \end{pmatrix}, \ \ldots \ , \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}, (0) \right\}.$$

Then $B = \{g \in G \mid ge = ege \text{ for all } e \in \Lambda\}$ is a Borel subgroup, where $G = K^*\mathbf{SO}_{2l+1}$ is the unit group of MSO_{2l+1}.

Let $y \in B$. Then $y = tb$, for some $t \in K^*, b = (b_{ij})_{i,j=1}^{2l+1} \in \mathbf{SO}_{2l+1}$. Thus $be = ebe$, for all $e \in \Lambda$. So, b has the shape

$$b = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \ldots & b_{1l} & b_{1,l+1} & b_{1,l+2} & \ldots & b_{1,2l+1} \\ 0 & b_{22} & b_{23} & \ldots & b_{2l} & b_{2,l+1} & b_{2,l+2} & \ldots & b_{2,2l+1} \\ 0 & 0 & b_{33} & \ldots & b_{3l} & b_{3,l+1} & c_{3,l+2} & \ldots & b_{3,2l+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & b_{ll} & b_{l,l+1} & b_{l,l+2} & \ldots & b_{l,2l+1} \\ 0 & 0 & 0 & \ldots & 0 & b_{l+1,l+1} & b_{l+1,l+2} & \ldots & b_{l+1,2l+1} \\ 0 & 0 & 0 & \ldots & 0 & b_{l+2,l+1} & b_{l+2,l+2} & \ldots & b_{l+2,2l+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & b_{2l+1,l+1} & b_{2l+1,l+2} & \ldots & b_{2l+1,2l+1} \end{pmatrix}.$$

Let $X = (b_{1,l+1}, ..., b_{l,l+1})^T, Y = (b_{l+1,l+2}, ..., b_{l+1,2l+1}), Z = (b_{l+2,l+1}, ..., b_{2l+1,l+1})^T$.

Then

$$b = \begin{pmatrix} b_1 & X & b_2 \\ 0 & b_{l+1,l+1} & Y \\ 0 & Z & b_3 \end{pmatrix},$$

where $b_1 = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1l} \\ 0 & b_{22} & \dots & b_{2l+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{ll} \end{pmatrix}$, $b_2 = \begin{pmatrix} b_{1,l+2} & \dots & b_{1,2l+1} \\ \vdots & & \vdots \\ b_{l,l+2} & \dots & b_{l,2l+1} \end{pmatrix}$ and

$$b_3 = \begin{pmatrix} b_{l+2,l+2} & \dots & b_{l+2,2l+1} \\ \vdots & & \vdots \\ b_{2l+1,l+2} & \dots & b_{2l+1,2l+1} \end{pmatrix}.$$

By $x^T J_l x = J_l$, one has

$$\begin{pmatrix} 0 & b_1^T JZ & b_1^T Jb_3 \\ 0 & b_{l+1,l+1}^2 + X^T JZ & b_{l+1,l+1} Y + X^T Jb_3 \\ b_3^T Jb_1 & b_3^T JX + Y^T b_{l+1,l+1} + b_2^T JZ & Y^T Y + b_3^T Jb_2 + b_2^T Jb_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{pmatrix}.$$

It follows that

$$\begin{cases} b_1^T Jb_3 = J & b_1^T JZ = 0 \\ b_{l+1,l+1}^2 + X^T JZ = 1 & b_{l+1,l+1} Y + X^T Jb_3 = 0 \\ b_3^T JX + Y^T b_{l+1,l+1} + b_2^T JZ = 0 & Y^T Y + b_3^T Jb_2 + b_2^T Jb_3 = 0, \end{cases}$$

and hence b_1 and b_3 are upper triangular and invertible, $Z = 0$ and $b_{l+1,l+1} = 1$. Thus the Borel subgroup B_0 of $\mathbf{SO}_{2l+1}$ is

$$B_0 = \left\{ \begin{pmatrix} b_1 & X & b_2 \\ 0 & 1 & Y \\ 0 & 0 & b_3 \end{pmatrix} \middle| \begin{array}{l} b_1, b_3 \in \mathbf{M}_l(K) \text{ are upper triangular,} \\ b_1^T Jb_3 = J, Y^T Y + b_3^T Jb_2 + b_2^T Jb_3 = 0, \\ Y + X^T Jb_3 = 0 \end{array} \right\}.$$

Therefore, $B = K^* B_0 = \{tb \mid b \in B_0, t \in K^*\}$ is a Borel subgroup of G and $B = G \cap \mathbf{B}_n$.

We can now find the important relationship involving the Borel subgroup B, idempotents and rank one elements in $E(\overline{T})$.

11.1. Theorem. *Let $T = K^* T_0 \subseteq B$ be a maximal torus in G. Then for every $e_I \in E(\overline{T})$, there exists a unique $e_i = E_{ii} \in E_1(\overline{T})$ such that $e_i B e_I = e_i B e_i$, where i is the maximal number in I.*

Proof. Let $e_I \in E(\overline{T})$ where $I = \{i_1, i_2, ..., i_m\} \subseteq \{1, ..., 2l+1\}$ is admissible with $i_1 < i_2 < ... < i_m$. For any upper triangular matrix $b = (b_{jk}) \in B \subseteq G$, the matrix be_I is an upper triangular matrix whose k-th column is exactly the k-th column of $b = (b_{jk})$ for $k = i_1, ..., i_m$, and the other columns of be_I are all zero. Let $i = i_m$ which is maximal in I. Taking $e_i = E_{ii}$, we get $e_i be_I = E_{ii} be_I$ is a matrix whose $(i,i)-$entry is b_{ii} and the other entries are all zero. It follows that $e_i be_I = e_i be_i$ by calculating directly. Therefore, $e_i Be_I = e_i Be_i$. From the procedure above we also see the uniqueness of such $e_i = E_{ii} \in E_1(\overline{T})$. □

11.2. Definition. *For any $e_i = E_{ii} \in E_1(\overline{T})$, define*

$$\mathcal{R}(e_i) = \{x \in \mathcal{R} \mid e_i Bx = e_i Be_i x \neq 0\}.$$

11.3. Corollary. *The set of non-zero elements of the Renner monoid has a decomposition*

$$\mathcal{R}^\times = \bigsqcup_{e_i \in E_1(\overline{T})} \mathcal{R}(e_i) = \bigsqcup_{i=1}^{2l+1} \mathcal{R}(E_{ii}).$$

Applying Theorem 11.1, we can now get a surjective map τ from the set $E(\overline{T})$ of idempotents in $\overline{T}$ onto the set $E_1(\overline{T})$ of rank one elements. This map can also be extended to a map of $\mathcal{R}^\times$ to $E_1(\overline{T})$.

11.4. Theorem.

a) There is a surjective map τ from $E(\overline{T})$ onto $E_1(\overline{T})$ by

$$e_I \longmapsto \tau(e_I) = e_i, \text{ if } e_i Be_I = e_i Be_i \neq 0$$

b) The map τ extends to $\mathcal{R}^\times = \mathcal{R} \setminus \{0\}$ by, for every $x \in \mathcal{R}^\times$, defining,

$$\tau(x) = e_i, \text{ if } x \in e_I W \text{ and } \tau(e_I) = e_i,$$

where $I \neq \phi$ admissible and i is maximal in I.

Proof. a) is clear. To prove b), note that for any $x \in \mathcal{R}^\times$, there is a $w \in W$ and a unique $e_{_I} \in E(\overline{T})$ such that $x = e_{_I}w$. It follows that there is a unique $e_i = E_{ii} \in E_1(\overline{T}) = E_1(\mathcal{R})$ such that $e_iBe_{_I} = e_iBe_i \neq 0$. Then we obtain the map from $\mathcal{R}^\times$ to $E_1(\mathcal{R}) = E_1(\overline{T})$ by $\tau(x) = e_i$, as required. $\square$

11.5. Proposition.

Let $I(i) = \{I \subseteq \{1,...,2l+1\} \mid I$ *are admissible with* $i =$*max(I)*$\}$, *where* $i = 1,...,2l+1$. *Then*

a) $\mathcal{R}(e_i) = \tau^{-1}(e_i) = \bigsqcup_{I\in I(i)} e_{_I}W$, *for* $i = 1,...,2l+1$.

b) $\mathcal{R}^\times = \bigsqcup_{i=1}^{2l+1} \tau^{-1}(e_i)$, *a disjoint union.*

Proof. It is straightforward. $\square$

For MSO_5, $E_1(\overline{T}) = \{e_1 = E_{11}, e_2 = E_{22}, e_4 = E_{44}, e_5 = E_{55}\}$, and

$\mathcal{R}(e_1) = \bigsqcup_{I\in I(1)} e_{_I}W = E_{11}W =$

$$\left\{\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\right\}$$

$\mathcal{R}(e_2) = \bigsqcup_{I\in I(2)} e_{_I}W = E_{22}W \cup (E_{11} + E_{22})W =$

$$\left\{\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\right.$$

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\Bigg\}$$

$\mathcal{R}(e_4) = \bigsqcup_{I\in I(4)} e_{I} W = E_{44}W \cup (E_{22}+E_{44})W =$

$$\Bigg\{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix}\Bigg\}$$

$\mathcal{R}(e_5) = \bigsqcup_{I\in I(5)} e_{I} W = E_{55}W \cup (E_{44}+E_{55})W \cup (E_{11}+E_{55})W \cup W =$

$$\Bigg\{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&1&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix},$$

$$\left.\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}\right\}$$

Our task now is to find the cell decomposition of the special orthogonal Renner monoid. Note that for any $r = E_{ij} \in \mathcal{R}(1)$, there exists a unique $e_r = E_{ii}$ and $f_r = E_{jj}$ in $E_1(\overline{T})$ such that $r = e_r r f_r$.

11.6. Definition. *For any $r = e_r r f_r \in \mathcal{R}(1)$, call $\mathcal{C}_r = \{x \in \mathcal{R}(e_r) \mid e_r x f_r = r\}$ a cell of the Renner monoid $\mathcal{R}$ of MSO_{2l+1} corresponding to the rank one element r.*

11.7. Proposition. *For any $e \in E_1(\overline{T})$ and $x \in \mathcal{R}(e)$, there is a unique $r \in \mathcal{R}(1)$ such that $x \in \mathcal{C}_r$, where $r = ex$.*

Proof. Let $x \in \mathcal{R}(e)$ where $e \in E_1(\overline{T})$. Then $eBx = eBex \neq 0$ where B is the Borel subgroup of $G \subseteq MSO_n$. Let $r = ex$. Then $r \in \mathcal{R}(1)$ and $r = erf_r$ for the unique $e, f_r \in E_1(\overline{T})$. For if $r = e_r r f_r$ and $e_r \neq e$, then $r = ex = e(ex) = er = e(e_r)rf_r = (ee_r)rf_r = 0$, since $ee_r = 0$. Thus $x \in \mathcal{R}(e)$ and $exf_r = rf_r = (erf_r)f_r = erf_r = r$, i.e., $x \in \mathcal{C}_r$.

Suppose that there is another $r' \in \mathcal{R}(1)$ such that $x \in \mathcal{C}_{r'} = \{x \in \mathcal{R}(e) \mid exf' = r'\}$ where $r' = er'f'$ for the unique $e, f' \in E_1(\overline{T})$. If $f' \neq f_r$ then $r' = exf' = (ex)f' = rf' = (exf_r)f' = ex(f_rf') = 0$, since $f_rf' = 0 \in E(\overline{T})$, which is a contradiction. Therefore, the uniqueness. $\square$

By Corollary 11.3 and the above proposition we get the following

11.8. Corollary.

a) $\mathcal{R}(e) = \bigsqcup_{\substack{r \in \mathcal{R}(1) \\ er=r}} \mathcal{C}_r$, *where* $e \in E_1(\overline{T})$.

b) $\mathcal{R}^\times = \bigsqcup_{r \in \mathcal{R}(1)} \mathcal{C}_r$.

Now, we can establish a surjective map φ from $\mathcal{R}^\times$ to the set $\mathcal{R}(1)$ consisting of rank one elements in $\mathcal{R}$ by declaring $\varphi(x) = r$ if $x \in \mathcal{C}_r$ where $x \in \mathcal{R}^\times$ and $r \in \mathcal{R}(1)$. It is an extension of τ. Furthermore, $\varphi^{-1}(r) = \mathcal{C}_r$ for $r \in \mathcal{R}(1)$.

11.9. Theorem. *The above surjective map φ from $\mathcal{R}^\times$ to $\mathcal{R}(1)$ satisfies*

$$\varphi(x) = e_i w, \text{ if } x = e_I w \in \mathcal{R}^\times \text{ and } \tau(e_I) = e_i,$$

where $e_I \in E(\overline{T})$ and $w \in W$.

Proof. Since $\mathcal{R}^\times = \bigsqcup_{e_i \in E_1(\overline{T})} \mathcal{R}(e_i)$ where $\mathcal{R}(e_i) = \bigsqcup_{I \in I(i)} e_I W$, there is a unique $e_i \in E_1(\overline{T})$ such that $x \in \mathcal{R}(e_i)$. It follows that if $x = e_I w \in \mathcal{R}^\times$ and $\tau(e_I) = e_i$, then $I \in I(i)$. Thus $\varphi(x) = e_i x = e_i(e_I w) = (e_i e_I)w = e_i w$, the required result. $\square$

11.10. Theorem. *For any $r = E_{ij} \in \mathcal{R}(1), i, j = 1, ..., 2l+1$, but $i, j \neq l+1$,*

$$\mathcal{C}_r = \mathcal{C}_{E_{ij}} = \{(x_{pq}) \in \mathcal{R} \mid x_{ij} = 1; x_{pq} = 0 \text{ if } i < p \leq 2l+1, 1 \leq q \leq 2l+1\}.$$

Proof. If $x = (x_{pq}) \in \mathcal{R}$ is an $n \times n$ matrix, then $E_{ii}x = E_{ij}$ if and only if $x_{iq} = \delta_{qj}$, for $i, q, j = 1, ..., 2l + 1$. Then

$$\mathcal{C}_r = \mathcal{C}_{E_{ij}} = \{(x_{pq}) \in \mathcal{R}(e_i) \mid E_{ii}(x_{pq}) = E_{ij}\}$$

$$= \{(x_{pq}) \in \mathcal{R}(e_i) \mid x_{ij} = 1\}$$

$$= \{(x_{pq}) \in \mathcal{R} \mid x_{ij} = 1; x_{pq} = 0 \text{ if } i < p \leq 2l + 1, 1 \leq q \leq 2l + 1\}.$$

which proves the Theorem. □

In the sequel, the cells in Theorem 11.10 will be simply denoted by $\mathcal{C}_{ij}$.

If $n = 5$, the cells of the Renner monoid of MSO_5 are:

$$\mathcal{C}_{11} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathcal{C}_{12} = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{14} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \mathcal{C}_{15} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{21} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{22} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{24} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{25} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{41} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{42} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{44} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{45} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}_{51} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\mathcal{C}_{52}=\left\{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&1&0&0&0\end{pmatrix}\right\}$$

$$\mathcal{C}_{54}=\left\{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}\right\}$$

$$\mathcal{C}_{55}=\left\{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}\right\}$$

Notice the fact that the cells $\mathcal{C}_{ij}(K)$ of the Renner monoid $\mathcal{R}_n$ of $\mathbf{M}_n(K)$ are

$$\mathcal{C}_{ij}(K)=\{(x_{pq})\in\mathcal{R}_n|\ x_{ij}=1, x_{pq}=2l+1 \text{ if } i<p\le n, 1\le q\le n\}.$$

where $i,j=1,...,n$. We can now get

11.11. Theorem. *$\mathcal{C}_{ij}=\mathcal{C}_{ij}(K)\cap\mathcal{R}$, where $i,j\in\{1,...,2l+1\}\setminus\{l+1\}$.*

We begin to describe the cell decomposition of the special orthogonal monoid MSO_n using the following definition.

11.12. Definition. *The sets* $C_{ij} = B\mathcal{C}_{ij}B$ *for* $i, j \in \{1, ..., 2l+1\} \setminus \{l+1\}$ *are called the cells for the special orthogonal monoid with respect to the Borel subgroup* B.

To find the relationship between the cells of the special orthogonal monoid and those of $\mathbf{M}_n$, the following theorem about the structure of the cells of MSO_{2l+1} is necessary.

11.13. Theorem. *The cells of the special orthogonal monoid are*

$$C_{ij} = \left\{ (c_{pq})_{p,q=1}^{2l+1} \in MSO_n \;\middle|\; \begin{array}{l} c_{ij} \neq 0; \quad c_{iq} = 0 \quad \text{for } 1 \leq q < j; \\ c_{pq} = 0, \text{ for } i < p \leq 2l+1, \; 1 \leq q \leq 2l+1 \end{array} \right\}.$$

where $i, j \in \{1, ..., 2l+1\} \setminus \{l+1\}$.

Proof. Since $C_{ij} = \cup_{x \in \mathcal{C}_{ij}} BxB = \cup_{x \in \mathcal{C}_{ij}} (K^* B_0) x (K^* B_0) = K^* \left(\cup_{x \in \mathcal{C}_{ij}} B_0 x B_0 \right)$, for $i, j \in \{1, ..., 2l+1\} \setminus \{l+1\}$, we need only to consider elements in $B_0 x B_0$, where $x \in \mathcal{C}_{ij}$ and

$$B_0 = \left\{ \begin{pmatrix} b_1 & X & b_2 \\ 0 & 1 & Y \\ 0 & 0 & b_3 \end{pmatrix} \;\middle|\; \begin{array}{l} b_1, b_3 \in \mathbf{M}_l(K) \text{ are upper triangular,} \\ b_1^T J b_3 = J, Y^T Y + b_3^T J b_2 + b_2^T J b_3 = 0, \\ Y + X^T J b_3 = 0 \end{array} \right\}.$$

Now let $b = (b_{pq})_{p,q=1}^{2l+1} = \begin{pmatrix} b_1 & X & b_2 \\ 0 & 1 & Y \\ 0 & 0 & b_3 \end{pmatrix} \in B_0$. Then $b_{00} = 1$ and $b_{pp} \neq 0$, for $p = 1, ..., 2l+1$. For any $x = (x_{pq}) \in \mathcal{C}_{ij} \subseteq \mathcal{R}(E_{ii}) \subseteq \mathcal{R}$, let $I = \{i_1, ..., i_{m-1}, i_m\}$ denote the index set of non-zero rows of x where $i_1 < ... < i_{m-1} < i_m$ and $i_m = i$. Let $J = \{j_1, ..., j_{m-1}, j_m\}$ denote the index set of non-zero columns such that $j_m = j$ and $x_{i_k j_k} = 1$, for $k = 1, ..., m$. Generally, we don't have $j_1 < ... < j_{m-1} < j_m$.

Thus bx is a matrix whose j_k-th column is the i_k-th of b where $k = 1, ..., m$, and all rows under row i are zero. The shape of bx is

$$
i\text{-th row} \longleftarrow \begin{pmatrix} * & \dots & * & b_{1i} & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \dots & * & b_{i-1i} & * & \dots & * \\ 0 & \dots & 0 & b_{ii} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix},
$$

$$\downarrow$$

j-th column

where $b_{ii} \neq 0$ is the (i, j)-entry of bx and neither i and j are $l + 1$. Taking any $b' = (b'_{pq})_{n\times n} \in b_0$, one obtains the shape of bxb' is

$$
i\text{-th row} \longleftarrow \begin{pmatrix} * & \dots & * & * & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \dots & * & * & * & \dots & * \\ 0 & \dots & 0 & b_{ii}b'_{jj} & b_{ii}b'_{jj+1} & \dots & b_{ii}b'_{j2l+1} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix},
$$

$$\downarrow$$

j-th column

where $b_{ii}b'_{jj} \neq 0$ is the (i, j)-entry of bxb'. From the arbitrariness of $b, b' \in B_0$ and $x \in \mathcal{C}_{ij}$, observing that $B = K^*B_0$, we get, for $i, j = 1, ..., 2l + 1$.

$$
C_{ij} = \left\{ (c_{pq})_{p,q=1}^{2l+1} \in MSO_n \,\middle|\, \begin{array}{l} c_{ij} \neq 0; \quad c_{iq} = 0, \quad \text{for } 1 \leq q < j; \\ c_{pq} = 0, \text{ for } i < p \leq 2l+1,\ 1 \leq q \leq 2l+1 \end{array} \right\}. \qquad \square
$$

It follows from the Bruhat-Renner decomposition [17, Corollary 5.8] of MSO_n and Corollary 11.8 that

11.14. Corollary. *Keeping the notation above, we have*

$$MSO_n \setminus \{0\} = \bigsqcup_{\substack{i,j=1 \\ i,j\neq l+1}}^{2l+1} C_{ij}.$$

From the shapes of elements in the cells $C_{ij}(K)$ of $\mathbf{M}_n(K)$, where $i, j = 1..., 2l+1$, we obtain the following

11.15. Theorem. $C_{ij} = C_{ij}(K) \cap MSO_n$*, for* $i, j \in \{1, ..., 2l+1\} \setminus \{l+1\}$. $\square$

12. Submonoids of the Special Orthogonal Algebraic Monoids MSO_{2l+1}

The main purpose of this section is to establish some properties of the submonoid $(MSO_n)_e = \{y \in MSO_n \mid ye = ey = e\}$ of MSO_n where $e \in E(MSO_n)$ and $n = 2l+1$ is odd. We simply denote by M_e the submonoid $(MSO_n)_e$. Let $G_e = M_e \cap G$ where $G = K^*\mathbf{SO}_n$ is the unit group of MSO_n. Then by [15, Theorem 6.11] one has $M_e = \overline{G_e}$. Thus, searching for some properties about G_e is necessary.

12.1. Lemma. *Let* $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{n\times n} \in MSO_n$*. Then* $y \in \mathbf{SO}_n$ *if and only if* $y_1 \in \mathbf{SO}_{n-2}$.

Proof. Recall $J_l = \begin{pmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{pmatrix} \in \mathbf{M}_n(K)$ be the symmetric matrix, where $J =$

$\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size $l \times l$. Rewrite J_l to be $J_l = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J_{l-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} y \in \mathbf{SO}_n &\iff y^T J_l y = J_l \\ &\iff \begin{pmatrix} 0 & 0 & 1 \\ 0 & y_1^T J_{l-1} y_1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J_{l-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &\iff y_1^T J_{l-1} y_1 = J_{l-1} \\ &\iff y_1 \in \mathbf{SO}_{n-2}. \quad \square \end{aligned}$$

12.2. Theorem. *Let $e_1 = E_{11} \in \Lambda$ and $G = K^*\mathbf{SO}_n$ $(n = 2l+1)$. Then G_{e_1} is isomorphic to $K^*\mathbf{SO}_{n-2}$. Furthermore, M_{e_1} is isomorphic to MSO_{n-2}.*

Proof. Suppose that $y = tx \in G$ with $x = (x_{ij})_{i,j=1}^{2l+1} \in \mathbf{SO}_n$ and $t \in K^*$. Then $ye_1 = e_1 y = e_1$ is equivalent to $xe_1 = e_1 x = (1/t)e_1$. Thus

$$x = \begin{pmatrix} 1/t & 0 \\ 0 & x_1 \end{pmatrix} \in \mathbf{SO}_n,$$

where $x_1 = (x_{ij})_{i,j=2}^{n}$ is a $2l \times 2l$ matrix, $A = (0, ..., 1)_{1\times 2l}$ and rewrite

$$J_l = \begin{pmatrix} 0 & A \\ A^T & J' \end{pmatrix}$$

where $J' = \begin{pmatrix} J_{l-1} & 0 \\ 0 & 0 \end{pmatrix}_{2l\times 2l}$. Notice that

$$x^T J_l x = \begin{pmatrix} 0 & (1/t)Ax_1 \\ (1/t)x_1^T A^T & x_1^T J' x_1 \end{pmatrix}.$$

Thus $x^T J_l x = J_l$ gives us $(1/t)Ax_1 = A$, $(1/t)x_1^T A^T = A^T$ and $x_1^T J' x_1 = J'$. It follows that $x_{2l+1,2} = ... = x_{2l+1,2l} = 0$ and $x_{2l+1,2l+1} = t$, which shows that x_1 has the shape

$$x_1 = \begin{pmatrix} x_2 & X \\ 0 & t \end{pmatrix}_{2l\times 2l},$$

where $x_2 = (x_{ij})_{i,j=2}^{2l}$ is a $2l-1 \times 2l-1$ matrix and $X = (x_{2,2l+1}, ..., x_{2l,2l+1})^T$. Since $J' = \begin{pmatrix} J_{l-1} & 0 \\ 0 & 0 \end{pmatrix}$ and $x_1^T J' x_1 = \begin{pmatrix} x_2^T J_{l-1} x_2 & x_2^T J_{l-1} X \\ X^T J_{l-1} x_2 & X^T J_{l-1} X \end{pmatrix}$, it follows from $x_1^T J' x_1 = J'$ that $x_2^T J_{l-1} x_2 = J_{l-1}$, $x_2^T J_{l-1} X = 0$, $X^T J_{l-1} x_2 = 0$ and $X^T J_{l-1} X = 0$. Thus $X = 0$ and $x_1 = \begin{pmatrix} x_2 & 0 \\ 0 & t \end{pmatrix}$, where $x_2 \in \mathbf{SO}_{n-2}$. Therefore,

$$x = \begin{pmatrix} 1/t & & \\ & x_2 & \\ & & t \end{pmatrix} \in \mathbf{SO}_n,$$

where $t \in K^*$ and $x_2 \in \mathbf{SO}_{n-2}$. It follows easily that

$$\begin{aligned} G_e &= \left\{ t \cdot \begin{pmatrix} 1/t & & \\ & x_2 & \\ & & t \end{pmatrix} \middle| \; t \in K^*, x_2 \in \mathbf{SO}_{n-2} \right\} \\ &= \left\{ \begin{pmatrix} 1 & & \\ & tx_2 & \\ & & t^2 \end{pmatrix} \middle| \; t \in K^*, x_2 \in \mathbf{SO}_{n-2} \right\}. \end{aligned}$$

Define a mapping f from G_{e_1} to $K^*\mathbf{SO}_{n-2}$ by

$$y = \begin{pmatrix} 1 & & \\ & tx_2 & \\ & & t^2 \end{pmatrix} \longmapsto tx_2 \in K^*\mathbf{SO}_{n-2}$$

Then f is an algebraic group isomorphism from G_{e_1} to $K^*\mathbf{SO}_{n-2}$ Hence, $\overline{G}_{e_1}$ is isomorphic to $\overline{K^*\mathbf{SO}}_{n-2}$ which is MSO_{n-2}. But it follows from [15, Theorem 6.11] that $M_{e_1} = \overline{G}_{e_1}$. Therefore, M_{e_1} is isomorphic to MSO_{n-2}.

This proves the Theorem. □

12.3. Corollary.

a) For any $e \in E_1(\overline{T})$, M_e is isomorphic to MSO_{n-2}.

c) For any $e \in E_1(MSO_n)$, the rank one elemenets in $E(MSO_n)$, M_e is isomorphic to MSO_{n-2}.

Proof. For a), note that $E_1(\overline{T}) = \{w^{-1}e_1w \mid w \in W\}$. Then for any $e \in E_1(\overline{T})$, there exists $w \in W$ such that $e = w^{-1}e_1w$, where $e_1 = E_{11}$. Since $ye = ey = e$ is equivalent to $(wyw^{-1})e_1 = e_1(wyw^{-1}) = e_1$, it follows that M_e is isomorphic to M_{e_1} by the mapping $y \longmapsto wyw^{-1}$. From Theorem 12.2 one obtains that M_e is isomorphic to MSO_{n-2}. Similar recipes apply to b) by using $E_1(MSO_{2l}) = \{g^{-1}e_1g \mid g \in G\}$. $\square$

12.4. Theorem. *Let $G = K^*\mathbf{SO}_n$ and $e_{_I} \in \Lambda$ with I standard admissible, where $e_{_I} = \sum_{j\in I} E_{jj} \in \Lambda$ and $|I| = 1, ..., l$. Then $G_{e_{_I}}$ is isomorphic to $K^*\mathbf{SO}_{n-2|I|}$. Furthermore, $M_{e_{_I}}$ is isomorphic to $MSO_{n-2|I|}$.*

Proof. It is similar to that of Theorem 12.2. $\square$

12.5. Corollary. *Keeping the same notations in Theorem 11.4, we have*

a) For every $e_{_J} \in E_i(\overline{T})$ with J admissible and $i = 1, ..., l$, $M_{e_{_J}}$ is isomorphic to MSO_{n-2i}.

b) For every $e \in E_i(MSO_n)$, the rank i elements in $E(MSO_n)$, M_e is isomorphic to MSO_{n-2i}, for $i = 1, ..., l$.

Proof. For a), note that for every $e_{_J} \in E_i(\overline{T})$, the set of rank i $(i = 1, ..., l)$ idempotents of $E(\overline{T})$, there exist unique $e_{_I} \in \Lambda$ and $w \in W$ such that $e_{_J} = we_{_I}w^{-1}$. But $ye_{_J} = e_{_J}y = e_{_J}$ is equivalent to $(w^{-1}yw)e_{_I} = e_{_I}(w^{-1}yw) = e_{_I}$. Hence $M_{e_{_J}}$ is isomorphic to $M_{e_{_I}}$ by the mapping $y \longmapsto w^{-1}yw$. It follows from Theorem 12.4 that $M_{e_{_J}}$ is isomorphic to MSO_{n-2i}. Applying a) and [17, Corollary 6.10 (ii)] one gets b) easily. $\square$

CHAPTER VI

G_m ACTIONS AND CELL DECOMPOSITIONS

The most interesting cell decompositions in algebraic geometry are the BB-cells obtained in [1]. If $T = K^*$ acts on a smooth complete variety X with finite fixed point set $F \subseteq X$, then $X = \bigsqcup_{\alpha \in F} X_\alpha$, where $X_\alpha = \{x \in X \mid \lim_{t \to 0} tx = \alpha\}$. Furthermore, X_α is isomorphic to an affine space. If, further, a semisimple group G acts on X extending the action of T, we may assume (replacing T if necessary) that each X_α is stable under the action of some Borel subgroup B of G with $T \subseteq B$. In case X is a complete homogeneous space for G, each cell X_α turns out to consist of exactly one B-orbit.

However, we can get BB-"cells" even if X is not smooth. They may not be affine spaces, but cones instead.

13 Basic Definitions and Preliminary Results

The results of this section are the unpublished work of L. Renner. We begin this section with the following definition

13.1 Definition. *Let X be an algebraic variety and $K^* \times X \longrightarrow X$ be an action of K^* on the variety X. Then for any $x \in X$, there is a morphism $\lambda : K^* \to X$ defined by $\lambda(t) = tx$. We say*

$$\lim_{t \to 0} tx = x_0,$$

where $x, x_0 \in X$, $t \in K^*$, if, there exists an extension $\bar{\lambda}$: $K \to X$ with $\bar{\lambda}(0) = x_0$. Thus, $\lim_{t \to 0} tx$ is unique, if it exists.

Let G_0 be one of the classical algebraic groups and $G = G_0 \times K^*$. Then $\overline{G}$, the Zariski closure of G in $\mathbf{M}_n$, is an irreducible algebraic monoid which will be denoted by M, i.e., $M = \overline{G}$. Let $X = (M \setminus \{0\})/K^*$. Then $X = \bigsqcup_{\alpha \in F} X_\alpha$, where $X_\alpha = \{x \in X \mid \lim_{t \to 0} tx = \alpha\}$ is referred to as the BB-cells of X (due to A. Bialynicki-Birula).

13.2 BB-Cells of $\mathbf{M}_n$

In case $M = \mathbf{M}_n$, with the two sided $\mathbf{GL}_n(K)$-action on it, $X = (\mathbf{M}_n \setminus \{0\})/K^*$. Let $\mathbf{T}_n$ be a maximal torus contained in a Borel subgroup $\mathbf{B}_n$ of $\mathbf{GL}_n$. Suppose $S = \mathbf{T}_n \times \mathbf{T}_n$. Then, for any $s = (\lambda, \mu) \in S$ and $x = \overline{(x_{ij})} \in X$ with $(x_{ij}) \in \mathbf{M}_n \setminus \{0\}$, where $\lambda =$ diag $(\lambda_1, ..., \lambda_n)$ and $\mu =$ diag $(\mu_1, \mu_2, ..., \mu_n)$, define an action of S on X by $sx = \overline{\lambda(x_{ij})\mu} = \overline{(\lambda_i x_{ij} \mu_j)}$. The fixed point set $X^S = \left\{\overline{E}_{ij} \mid E_{ij} \in \mathbf{M}_n \setminus \{0\}\right\}$.

Let $\lambda(t) = \text{diag}(t^{(n-1)n}, ..., t^n, t^0)$ and $\mu(t) = \text{diag}(t^1, t^2, ..., t^n)$, where $t \in K^*$. Then $\lambda(t), \mu(t) \in \mathbf{T}_n$. So the action of G_m on X is given by

$$tx = \overline{\lambda(t)(x_{ij})\mu(t)}.$$

Notice that

$$\lambda(t)(x_{ij})\mu(t) = \begin{pmatrix} t^{(n-1)n+1}x_{11} & t^{(n-1)n+2}x_{12} & \dots & t^{n^2}x_{1n} \\ t^{(n-2)n+1}x_{21} & t^{(n-2)n+2}x_{22} & \dots & t^{n^2-n}x_{2n} \\ \vdots & \vdots & \dots & \vdots \\ t^{2n+1}x_{n-2,1} & t^{2n+2}x_{n-2,2} & \dots & t^{3n}x_{n-2,n} \\ t^{n+1}x_{n-1,1} & t^{n+2}x_{n-1,2} & \dots & t^{2n}x_{n-1,n} \\ tx_{n1} & t^2x_{n2} & \dots & t^nx_{nn} \end{pmatrix},$$

where all the exponents of t's in the above matrix are distinct.

Let $a_{ij} = (n-i)n + j$, where $i, j = 1, 2, ..., n$. Then

$$\lambda(t)(x_{ij})\mu(t) = (t^{a_{ij}}x_{ij})_{n \times n},$$

and hence

$$
\begin{aligned}
tx = x &\Longleftrightarrow \overline{\lambda(t)(x_{ij})\mu(t)} = x \\
&\Longleftrightarrow \lambda(t)(x_{ij})\mu(t) = \alpha(x_{ij}), \text{ for some } \alpha \in K^* \\
&\Longleftrightarrow (t^{a_{ij}} x_{ij}) = (\alpha x_{ij}).
\end{aligned}
$$

So there is at most one element $x_{ij} \neq 0$. By the fact that $(x_{ij})_{n\times n} \in \mathbf{M}_n \setminus \{0\}$, it follows that $(x_{ij}) = x_{ij}E_{ij}, x_{ij} \in K^*$ and $x = \overline{E}_{ij}$ for some $i, j \in \{1, 2, , , , n\}$. Therefore, $X^T = \{\overline{E}_{ij} \mid E_{ij} \in \mathbf{M}_n \setminus \{0\}\}$, i.e., $X^T = X^S$.

Let $X_{ij} = \{x \in X \mid \lim_{t\to 0} tx = \overline{E}_{ij}\}$. Then we have the following

13.3 Theorem. *The BB-cells X_{ij} of X are*

$$
X_{ij} = \left\{ \overline{(x_{pq})} \in X \,\middle|\, \begin{matrix} (x_{pq}) \in \mathbf{M}_n \setminus \{0\}, x_{ij} \neq 0, \ but \\ x_{pq} = 0, \ if\ p > i, \ or\ p = i \ and\ q < j \end{matrix} \right\}.
$$

$$
= \left\{ \overline{\begin{pmatrix} * & \dots & * & * & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \dots & * & * & * & \dots & * \\ 0 & \dots & 0 & x_{ij} & * & \dots & * \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}} \in X \mid x_{ij} \in K^*, * \in K \right\}.
$$

Proof. To establish concretely the BB-cells $X_{ij} = \{x \in X \mid \lim_{t\to 0} tx = \overline{E}_{ij}\}$, we need the following calculation

$$
\lambda(t)(x_{ij})\mu(t) = t^{a_{ij}} \begin{pmatrix} t^{a_{11}-a_{ij}} x_{11} & \dots & t^{a_{1j}-a_{ij}} x_{1j} & \dots & t^{a_{1n}-a_{ij}} x_{1n} \\ t^{a_{21}-a_{ij}} x_{21} & \dots & t^{a_{2j}-a_{ij}} x_{2j} & \dots & t^{a_{2n}-a_{ij}} x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t^{a_{i1}-a_{ij}} x_{i1} & \dots & x_{ij} & \dots & t^{a_{in}-a_{ij}} x_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t^{a_{n1}-a_{ij}} x_{n1} & \dots & t^{a_{nj}-a_{ij}} x_{nj} & \dots & t^{a_{nn}-a_{ij}} x_{nn} \end{pmatrix},
$$

where $a_{pq} - a_{ij} > 0$, if $p < i$, or $p = i$ and $q > j$, and $a_{pq} - a_{ij} < 0$, if $p > i$, or $p = i$ and $q < j$.

Consider elements $x = \overline{(x_{pq})_{n\times n}} \in X$ where $(x_{pq})_{n\times n}$ in $\mathbf{M}_n$ with $x_{pq} = 0$ for $p > i$, or $p = i$ and $q < j$, but $x_{ij} \neq 0$. It follows easily that $\lim_{t\to 0} tx = \overline{E}_{ij}$, so $x \in X_{ij}$. Let

$$X'_{ij} = \left\{ \overline{(x_{pq})} \in X \middle| \begin{array}{l} x_{ij} \neq 0, \text{ but } x_{pq} = 0, \text{ if} \\ p > i \text{ or } p = i \text{ and } q < j \end{array} \right\}.$$

Then $X'_{ij} \subseteq X_{ij}$ and $X = \bigsqcup_{i,j=1}^{n} X'_{ij}$. On the other hand, $X = \bigsqcup_{i,j=1}^{n} X_{ij}$, but then $X'_{ij} = X_{ij}$, for $i, j = 1, ..., n$.

This proves the theorem. $\square$

Corollary 13.4. *Let π be the canonical map of $\mathbf{M}_n$ to $X = (\mathbf{M}_n \setminus \{0\})/K^*$. Then we have the following cell decomposition*

$$\mathbf{M}_n \setminus \{0\} = \bigsqcup_{i,j=1}^{2l} \pi^{-1}(X_{ij}),$$

where $t \in K^*$ and $\pi^{-1}(X_{ij})$ coincide with the cells $C_{ij}(K)$ of $\mathbf{M}_n \setminus \{0\}$, for $i, j = 1, ..., n$.

14 G_m Action and BB-Cells for MSp_n

In case $M = MSp_n(K)$, with the two sided $K^*Sp_n(K)$-action on it, then $X = (MSp_n \setminus \{0\})/K^*$, where $n = 2l$. Let T be a maximal torus contained in a Borel subgroup B of Sp_n as before. Let $S = T \times T$. Then, for any $s = (\lambda, \mu)$ and

$x = \overline{(x_{ij})} \in X$, we define an action of S on X as follows

$$sx = \overline{\lambda(x_{ij})\mu}$$

$$= \overline{(\lambda_i x_{ij} \mu_j)} \in X.$$

where $\lambda = \text{diag}\ (\lambda_1, ..., \lambda_n)$, $\mu = \text{diag}\ (\mu_1, \mu_2, ..., \mu_n)$ and $(x_{ij}) \in MSp_n \setminus \{0\}$. Let X^S denote the fixed point set. It follows that $X^S = \{\overline{E}_{ij} \mid E_{ij} \in MSp_n \setminus \{0\}\}$.

Let

$$\lambda(t) = \text{diag}(t^{ln^2}, t^{(l-1)n^2}, ..., t^{n^2}, t^{-n^2}, ..., t^{-(l-1)n^2}, t^{-ln^2})$$

$$\mu(t) = \text{diag}(t^{-l}, t^{-(l-1)}, ..., t^{-1}, t, ..., t^{l-1}, t^{l}),$$

where $t \in K^*$. Then $\lambda(t), \mu(t) \in T$. Define a G_m action on X by $tx = \overline{\lambda(t)(x_{ij})\mu(t)}$.

Notice that $\lambda(t)(x_{ij})\mu(t) =$

$$\begin{pmatrix} t^{ln^2-l}x_{11} & \cdots & t^{ln^2-1}x_{1l} & t^{ln^2+1}x_{1,l+1} & \cdots & t^{ln^2+l}x_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t^{2n^2-l}x_{l-1,1} & \cdots & t^{2n^2-1}x_{l-1,l} & t^{2n^2+1}x_{l-1,l+1} & \cdots & t^{2n^2+l}x_{l-1,n} \\ t^{n^2-l}x_{l1} & \cdots & t^{n^2-1}x_{ll} & t^{n^2+1}x_{l,l+1} & \cdots & t^{n^2+l}x_{ln} \\ t^{-n^2-l}x_{l+1,1} & \cdots & t^{-n^2-1}x_{l+1,l} & t^{-n^2+1}x_{l+1,l+1} & \cdots & t^{-n^2+l}x_{l+1,n} \\ t^{-2n^2-l}x_{l+2,1} & \cdots & t^{-2n^2-1}x_{l+2,l} & t^{-2n^2+1}x_{l+2,l+1} & \cdots & t^{-2n^2+l}x_{l+2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t^{-ln^2-l}x_{n1} & \cdots & t^{-ln^2-1}x_{nl} & t^{-ln^2+1}x_{n,l+1} & \cdots & t^{-ln^2+l}x_{nn} \end{pmatrix},$$

where all the exponents of t's in the above matrix are distinct.

Let

$$a_{ij} = \begin{cases} (l-i+1)n^2 - (l-j+1), & \text{for } 1 \le i \le l, 1 \le j \le l \\ (l-i+1)n^2 - (l-j), & \text{for } 1 \le i \le l, l+1 \le j \le n \\ (l-i)n^2 - (l-j+1), & \text{for } l+1 \le i \le n, 1 \le j \le l \\ (l-i)n^2 - (l-j), & \text{for } l+1 \le i \le n, l+1 \le j \le n. \end{cases}$$

Then

$$\lambda(t)(x_{ij})\mu(t) = (t^{a_{ij}}x_{ij})_{n\times n},$$

and hence

$$\begin{aligned} tx = x &\Longleftrightarrow \overline{(\lambda(t)x\mu(t))} = x \\ &\Longleftrightarrow \lambda(t)(x_{ij})\mu(t) = \alpha(x_{ij}),\ \text{for some } \alpha \in K^* \\ &\Longleftrightarrow (t^{a_{ij}}x_{ij}) = (\alpha x_{ij}). \end{aligned}$$

So there is at most one element $x_{ij} \neq 0$. By the fact that $(x_{ij}) \in MSp_n \setminus \{0\}$, it follows that $(x_{ij}) = x_{ij}E_{ij}, x_{ij} \in K^*$ and $x = \overline{E}_{ij}$. Therefore, $X^T = \{\overline{E}_{ij} \mid E_{ij} \in MSp_n \setminus 0\}$, i.e., $X^T = X^S$.

Theorem 14.1. *The BB-cells X_{ij} of X are*

$$X_{ij} = \left\{ \overline{(x_{pq})} \in X \middle| \begin{array}{l} x_{ij} \neq 0,\ but\ x_{pq} = 0,\ if \\ p > i\ or\ p = i\ and\ q < j \end{array} \right\},$$

$$= \left\{ \overline{\begin{pmatrix} * & \dots & * & * & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \dots & * & * & * & \dots & * \\ 0 & \dots & 0 & x_{ij} & * & \dots & * \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}} \in X \mid x_{ij} \in K^*, * \in K \right\}.$$

Proof. Notice that

$$\lambda(t)(x_{ij})\mu(t) = t^{a_{ij}} \begin{pmatrix} t^{a_{11}-a_{ij}}x_{11} & \dots & t^{a_{1j}-a_{ij}}x_{1j} & \dots & t^{a_{1n}-a_{ij}}x_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ t^{a_{i1}-a_{ij}}x_{i1} & \dots & x_{ij} & \dots & t^{a_{in}-a_{ij}}x_{in} \\ \dots & \dots & \dots & \dots & \dots \\ t^{a_{n1}-a_{ij}}x_{n1} & \dots & t^{a_{nj}-a_{ij}}x_{nj} & \dots & t^{a_{nn}-a_{ij}}x_{nn} \end{pmatrix},$$

where $a_{pq} - a_{ij} > 0$, if $p < i$, or $p = i$ and $q > j$, and $a_{pq} - a_{ij} < 0$, if $p > i$, or $p = i$ and $q < j$.

Consider elements $x = \overline{(x_{pq})_{n\times n}} \in X$ where $(x_{pq})_{n\times n}$ in MSp_n with $x_{pq} = 0$ for $p > i$, or $p = i$ and $q < j$, but $x_{ij} \neq 0$. It follows easily that $\lim_{t\to 0} tx = \overline{E}_{ij}$. In other words, $x \in X_{ij}$. Let

$$X'_{ij} = \left\{ \overline{(x_{pq})} \in X \middle| \begin{array}{l} x_{ij} \neq 0, \text{ but } x_{pq} = 0, \text{ if} \\ p > i \text{ or } p = i \text{ and } q < j \end{array} \right\}.$$

Then $X'_{ij} \subseteq X_{ij}$ and $X = \bigsqcup_{i,j=1}^{n} X'_{ij}$. On the other hand, $X = \bigsqcup_{i,j=1}^{n} X_{ij}$, but then $X'_{ij} = X_{ij}$, for $i, j = 1, ..., n$. $\square$

Corollary 14.2. *Let π be the canonical map of MSp_{2l} to $X = (MSp_{2l} \setminus \{0\})/K^*$. Then we have the following cell decomposition*

$$MSp_{2l} \setminus \{0\} = \bigsqcup_{i,j=1}^{2l} \pi^{-1}(X_{ij}),$$

where $t \in K^*$ and $\pi^{-1}(X_{ij})$ coincide with the cells C_{ij} of MSp_{2l} (see Definition 5.12), for $i, j = 1, ..., 2l$.

Remark 15. The same $\lambda(t), \mu(t)$ works for MSO_{2l}.

Remark 16. For the case MSO_{2l+1}, we need the following

$$\lambda(t) = \text{diag}(t^{ln^2}, t^{(l-1)n^2}, ..., t^{n^2}, 1, t^{-n^2}, ..., t^{-(l-1)n^2}, t^{-ln^2}),$$

$$\mu(t) = \text{diag}(t^{-l}, t^{-(l-1)}, ..., t^{-1}, 1, t, ..., t^{l-1}, t^{l}), \text{ where } t \in K^*.$$

There is much more information that should be possible concerning these cells, even though they may not be affine spaces. Are they irreducible? What are they topologically? Unfortunately, these problems are for future work.

BIBLIOGRAPHY

1. A. Bialynicki-Birula, *Some Theorems on Actions of Algebraic Groups*, Ann. of Math. **98** (1973), 480-497.
2. A. Borel, *Linear Algebraic Groups*, Springer Verlag, New York, 1991.
3. R. Carter, *Finite Groups of Lie Type; Conjugacy Classes and Complex Characters*, John Wiley and sons, New York, 1985.
4. C. DeConcini and C. Procesi, *Complete Symmetric Varieties*, Lecture Notes in Mathematics **996** (1973), Springer, 1-44.
5. W. Huang, *Algebraic Monoids with Approaches to Linear Associative Algebras*, Dissertation, The University of Western Ontario, 1995.
6. J.M. Howie, *An Introduction to Semigroup Theory*, L.M.S. Monograph 8, Academic Press, 1972.
7. J. E. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics, vol 21, Springer Verlag, New York, 1981.
8. J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.
9. M. E. Hull, *Centralizer of a Semisimple Element on a Reductice Algebraic Monoid*, Dissertation, The University of Western Ontario, 1994.
10. R. Kane, *Reflection Groups and Invariant Theory*, Canadian Mathematical Society Book Series, Springer Verlag, 2001.
11. Z. Li, *Orbit Structure of Finite and Reductive Monoids*, Dissertation, The University of Western Ontario, 1997.
12. E. A. Pennell, M. S. Putcha and L. E. Renner, *Analogue of the Bruhat-*

Chevalley Order for Reductive Monoids, J. Algebra **196** (1997), 339-368.

13. M. S. Putcha, *Idempotent Cross-Section of $\mathcal{J}$-Classes*, Semigroup Forum **26** (1983), 103-109.

14. M. S. Putcha, *A Semigroup Approach to Linear Algebraic Groups*, J. Algebra **80** (1983), 164-185.

15. M. S. Putcha, *Linear Algebraic Monoids*, London Mathematical Society Lecture Notes Series 133, Cambridge University Press, 1988.

16. M. S. Putcha and L. Renner, *The System of Idempotents and the Lattice of $\mathcal{J}$-Classes of Reductive Algebraic Monoids*, J. Algebra **116, No. 2** (1988), 385-399.

17. L. Renner, *Analogue of the Bruhat Decomposition for Algebraic Monoids*, J. Algebra **101, No. 2** (1986), 303-338.

18. L. Renner, *Analogue of the Bruhat Decomposition for Algebraic Monoids II. The length function and the trichotomy*, J. Algebra **175** (1995), 697-714.

19. L. Renner, *Classification of Semisimple Algebraic Monoids*, Transactions of the American Mathematical Society **292, No. 1** (1985), 193-223.

20. L. Renner, *Classification of Semisimple Varieties*, J. Algebra **122** (1989), 275-287.

21. L. Renner, *An Explicit Cell Decomposition of the Wonderful Compactification of a Semisimple Algebraic Group*, to appear.

22. L. Solomon, *An Introduction to Reductive Monoids*, in: Semigroups, Formal Languages and Groups (ed. J. Fountain), Kluwer (1995), 295-352.

23. T. A. Springer, *Linear Algebraic Groups*, Boston, Birkhauser, 1998.

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a) Articles Published in Refereed Journals

1. Zhenheng Li (with Yidong Jin and Ruipu Bai), *Relationships Between Reflections Determined by Imaginary Roots and the Weyl Group of a GKM Algebra*, J. of Math. Research and Exposition **19(3)** (1999), 501-507.
2. Zhenheng Li (with Yidong Jin and Zhixue Zhang), *Notes on Imaginary Roots and Weights for Generalized Kac-Moody Algebras*, Chinese Science Bulletin **42(1)** (1997).
3. Zhenheng Li, *On the q-Analogue of the Algebra of Differential Operators and Its Derivations*, J. of Hebei University (in Chinese) **16(4)** (1996).
4. Zhenheng Li (with Yidong Jin and Kaiyuan Xin), *Generalized Kac-Moody Algebras and Their Representation Properties*, J. of North China Institute of Electric Power (in Chinese) **19(1)** (1995).
5. Zhenheng Li and Zhuo Li, *Some Properties of Irreducible Modules over Kac-Moody Algebras*, J. of Hebei University (in Chinese) **15(4)** (1995).
6. Zhenheng Li, *A Note on Imaginary Roots of Kac-Moody Algebras*, J. of Hebei University (in Chinese) **14(2)** (1994).
7. Zhenheng Li, *Some Properties of Modules over Generalized Kac-Moody Algebras*, Chinese Science Bulletin **39(1)** (1994), 9-13.
8. Zhenheng Li, *Complete Lie Superalgebras*, J. of Math. Research and Exposition **14(2)** (1994), 159-164.
9. Zhenheng Li, *Uniqueness of the Decomposition of Complete Lie Superalgebras*, Northeastern Math. J. **9(3)** (1993), 403-405.

10. Zhenheng Li, *Structures of a Class of Lie Algebras with Chain Conditions*, J. of Hebei University (in Chinese) **13(3)** (1993).
11. Zhenheng Li, *Structures and Eigenvalues of the Adjoint Representation of* $k(A)$, J. of Hebei University (in Chinese) **9(5)** (1989), 83-86.
12. Zhenheng Li, *Structures of Lie Algebras with Chain Conditions*, J. of Hebei University (in Chinese) **9(2)** (1989), 1-5.

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