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Statistical Modelling, Optimal Strategies and Decisions in Two-Period Economies

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Abstract

Motivated by some real problems, our thesis puts forward two general two-period pricing models and explore optimal buying and selling strategies in two states of the two-period decision, when buyer/seller's decisions in the two periods are uncertain: commodity valuations may or may not be independent, may or may not follow the same distribution, be heavily or just lightly influenced by exogenous economic conditions, and so on. For both the example of buying laptops and the example of selling houses, the connections between each example and the two-envelope paradox encourage us to explore optimal strategies based on the works of McDonnell and Abbott (2009) and McDonnell et al. (2011), which proposed optimal strategies on the condition that the amount of the first envelope is known. In the case of buying laptop whether on *Black Friday*, or on *Boxing day*, We derive an optimal strategy for minimizing the expected loss in the two-period economy when a pivotal decision needs to be made during the first time-period and cannot be subsequently reversed. In the cases of selling real estate in Punta del Este, a resort town in Uruguay, real-estate property is in demand by both domestic and foreign buyers. There are several stages of selling residential units: before, during, and after the actual construction. Different pricing strategies are used at every stage. We propose a general model to derive, under various scenarios of practical relevance, optimal strategies for setting prices within two-stage selling framework, as well as to explore the optimal timing for accomplishing these tasks in order to maximize the overall seller's expected revenue. The optimal strategies in this model draw hints from the example of selling real estate proposed by Egozcue et al. (2013), where the ideas of McDonnell and Abbott (2009) and McDonnell et al. (2011) were applied in two states of the one-period decision scenario. All our optimal buying/selling strategies are illustrated with numerical and graphical examples using appropriately constructed parametric models.

Keywords: Decision theory; two-period economy; price discrimination; strategy; game theory; conditional probability; statistical modelling; behavioral economics; uncertainty; background risk model; gamma distribution.

Dedicated to my beloved Yijuan & Runmei

for their love, endless support, encouragement and sacrifices.

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Chapter 1

Introduction

Optimal strategies and decisions in two-period economies are important and popular in decision making and economic analysis. Naturally, in a two-period model, the first period is the current period (or today) and the second period represents the latter period (the future, or tomorrow). These kinds of studies cover domains of economic problems including the consumer's problems, the producer's problems, and general equilibrium models (Chiang, 1984; Daher et al., 2014; Farmer, 1993; Grossmann and Dietl, 2009; Hlouskova et al., 2017; Obstfeld and Rogoff, 1996; Rabitsch and Stepanchuk, 2014).

We are now focusing on decision-making theory frameworks to exploring optimal decision-making strategies which balance the trade-off between the benefits in the present (current period) and the benefits in the future (next period). The vast majority of these kinds of studies are of these types:

- the buying decisions which consider buying today at current prices or waiting until next period;
- the selling decisions which consider selling today at current prices or waiting until next period.

Incorporating time dimensions into decision making and economic analysis, we construct a two-period framework of optimal decision-making strategies to understand inter-temporal choices and dynamic decision-making issues. The generalization to multiple periods is straightforward: if the two-period framework is constructed, then we can consider a three-period framework as a initial stage connecting the rest stage, which is also a two-period framework; so continue, until you can consider all multistage framework into the two-period framework.

The rest of this chapter is organized as follows. In Section 1.1, we review the analyses of optimal decision strategies in two states of the one-period scenario. Among these analyses, the strategies suggested by McDonnell and Abbott (2009) and McDonnell et al. (2011) are attractive and very different. Based on the review of exploring optimal decision strategies in two states of the one-period scenario, we describe optimal decision problems in two states of the two-period scenario in Section 1.2. Section 1.3 gives an overview of the structure introduction of the whole thesis.

1.1 Optimal decision strategies in two states of the one-period scenario

Among vast amount of optimal strategies and decision theories with two states of the one-period scenario, McDonnell and Abbott (2009) proposed novel strategies and decision theories based on their solution of the famous "two-envelope" paradox [1].

Most of the solutions to the "two-envelope" paradox indicate temporal strategies when making decisions that maximize the payoff (Albers et al., 2005). McDonnell and Abbott (2009) and McDonnell et al. (2011) proposed new strategy that can enable a player to beat the game regarding increasing their payoff. The strategies could be applied to optimize gains of buying strategies of consumers economic behavior, gains of selling strategies of firms theory of economics, financial investments and so on.

Generally, in a standard two-envelope paradox, a player has to choose one of two envelopes,

[1] The "two-envelope" paradox attracts wide interest, because it impacts on various fields such as game theory, probability theory, economic theory, and decision theory (Langtree, 2004).

and therefore keep the money in it — one of them contains twice as much money as the other one. Key points in the following strategies are

- if the player opens one envelope they choose, then he/she cannot choose the other one any more;
- or he/she swaps envelopes, and therefore keeps the money in the other one.

Thus, indeed, everyone including the player knows the other envelope has either twice the money or half the money as the former one. The player needs to make a wise decision even though he/she does not know which is which in the initial period.

In a strategy without many careful considerations, since a player has a 50-50 chance of choosing either envelope, he/she has an equal chance of gaining or losing money. It doesn't matter whether he/she decides to swap or keep the original envelope. However, a little bit more mathematical calculations may indicate that it is always better to swap based on probability theory (Falk, 2008).

Furthermore, McDonnell and Abbott (2009) and McDonnell et al. (2011) developed more novel and different strategies for the "two-envelope" paradox. Their strategies followed Cover's switching function, which are,

- on the one hand, people want envelopes when the observed amount is a large sum of money;
- on the other hand, people want to swap with greater likelihood if the initial observed amount is small.

Cover's proposal is that the probability of switching should be some monotonically decreasing function of the observed amount. Based on these results, the strategy of McDonnell and Abbott (2009, 2011) discovered — whether the player should swap the envelopes or not depends on that the observed amount is large or not in the current period.

The strategies suggested by McDonnell and Abbott (2009, 2011) can be applied comprehensively in many fields. As an application in real estate, Egozcue et. al. (2013) put forward a model and then derived an optimal strategy that maximizes the expected real-estate selling price when one of the only two remaining buyers had already made an offer, but the other one had yet to make an offer. The strategies of McDonnell and Abbott (2009, 2011) provided pretty useful hints for optimal real-estate sellers strategies. When selling a house, the seller can not make sure whether the other buyer would make a lower or higher offer than the first-come (current) buyer. Indeed, the seller needs optimal strategies to decide whether to accept or to reject the first-come offer. By a connection between the motivating problem and the two-envelope problem, Egozcue et. al. (2013) derived an optimal sellers strategy, which is used to maximize the expected selling price based on the contributions of McDonnell and Abbott (2009, 2011) from the seller's perspective.

1.2 Optimal decision strategies in two states of the two-period scenario

When it comes to optimal decision strategies in two states of the two-period scenario based on the optimal decision strategies in two states of the one-period scenario, at least the following three different pieces of information have to be considered:

- the trade-off between temporal decision and the decision in latter period;
- the impact of current decisions on decisions of next period;
- the effect of time between two terms on random variables of the latter period.

In our real life, many things that people encounter in daily life related to two states of the two-period decisions would also pose an increasing practical demand for optimal strategies and decision making.

For example, here is a case from the real life. The author had an urgent need to buy a laptop in a very soon coming period, for example, in one or two months of rationality and he was looking for a good deal online. Due to the advantages of online shopping, he got much information about the price of the particular type of Lenovo laptop, such as T540, very quickly. He knew that, in coming two months, the price of T540 would fluctuate for many reasons, and two important promotions among those reasons are more attractive, *Black Friday promotion*, from November 27^{th}, 2015, to December 3^{th}, 2015, and *Boxing day promotion*, from December 26^{th}, 2015, to January 3^{th}, 2016. Based on that, his buying strategy was to get a better choice among the two promotions.

Obviously, because of the promotion, it was prudent to buy the laptop during the Black Friday promotion period, with the discounted price of 1,431.01 Canadian dollars for the laptop. However, it was not easy to justify at that moment whether he wanted to buy the laptop at that price or wait until the Boxing Day promotion. Can there be a good buying strategy for making good decisions during the Black Friday promotion period?

The above example of buying the laptop here raises a practical demand for optimal *buying strategies* in two states of the two-period decisions from the buyers' perspective. Furthermore, sometimes we can also observe the demand for optimal *selling strategies* in two states of the two-period decisions from the sellers' perspective. The real estate example below proposes a real need for optimal selling strategies when selling houses.

Namely, in Punta del Este, a resort town in Uruguay, the real-estate property is in demand by both domestic and foreigner buyers. As a recent example, the distribution of buyers for a certain high-rise building were 10% Uruguayans, 75% Argentineans, 10% Brazilians, and the remaining 5% were from the rest of the world (Chile, U.S.A., and so on). Naturally, the ratio of domestic and foreign buyers varies and depends on many factors, such as economic and financial. We also assume that the average foreign buyer is wealthier than the domestic one, and thus tends to exhibit a higher bidding price. Furthermore, it is important to note that given the diversity of buyers, the prices are usually in US dollars (USD), but some of the building costs such as salaries of workers are in Uruguayan pesos (UYU).

Suppose now that, contracted by an investor, a construction company is in the process of building a high-rise apartment building. There are various stages of selling apartments: before, during, and after the actual construction of the building. Different pricing strategies are used at every stage. For the sake of concreteness, we only deal with the case when the building has already been built, but some units are still to be sold. Suppose that, initially, the investor wishes to sell the units en masse and thus hires a real estate agent for several months. If the sale turns out to be unsuccessful during the initial selling stage, then the units would be put on sale individually, with no particular time horizon set in advance and at a possibly different (higher or lower) price. For the property investor, the task is to set a proper price for the initial selling stage, and also another price, usually different from the original price, for higher profit. Can there be a good selling strategy for making good decisions during initial period from the seller's perspective?

1.3 Overview

The rest of this thesis is organized as follows. In Chapter 2, we introduce all the fundamental notions and techniques used throughout the thesis. The starting point of the logic of this thesis is "the two-envelope paradox", which is described in Section 2.1.1. In Section 2.1.2, the varies of solutions of the two-envelope paradox are briefly concluded. Among many valuable solutions, McDonnell and Abbott (2009) proposed a wise strategy which is demonstrated in Section 2.1.3. Also, all three particular cases of the strategy of McDonnell and Abbott (2009) are illustrated in Section 2.1.4. Proposition 2.1.1 expresses clearly the strategy — it is optimal to switch and complementary intervals where no-switching is the optimal strategy.

As an application of the strategies of McDonnell and Abbott (2009) and McDonnell et al. (2011), Section 2.1.5 shows us an example of real-estate sale, which is analyzed in Egozcue et al. (2013). Egozcue et al. (2013) solved the maximization problem of "strategy risk parameter" described in the Proposition 2.1.2 as abstract of the logic of real-estate selling. And, in turn, Proposition 2.1.3 solve the maximizing strategy function which is described in Proposition

2.1.2 and Proposition 2.1.4 proposed a threshold-type strategy.

In Chapter 3, we propose an optimal strategy to minimize the expected buying price from the buyer's perspective in a two-period scenario. The story in this chapter is realistic. When one of the authors need to buy a laptop in the Fall of 2015, it was prudent, but difficult, for the author to buy during the former period — the Black Friday promotion period. The Section 3.1 describes the motivation problem from the buyer's perspective. In Section 3.2, the optimal strategy, which minimizes the expected buying price with no attempts to guess the possible price to be offered during the second time-period, is shown in Theorem 3.2.1. As a natural extended part of Theorem 3.2.1, the optimal strategy, which minimizes the expected buying price with guessed price to be offered during the second time-period, is shown in Theorem 3.2.2. When specifying the detailed information in the proposed purchase laptop problems, the behaviors of the salespersons L and H, who are defined in Section 3.2, are depicted by two independent random variables and their distribution $f_{X_L,X_H}(x,y)$. Section 3.3 models the joint distribution $f_{X_L,X_H}(x,y)$ through both the characters of the behaviors of the salespersons L and H and the influences of the company's marketing team or management. The probability that the salesperson L makes an offer during the first time-period given that the prices provided by L and H are x and y, respectively, is $p(x,y)$, which is described in Section 3.4. Because of the characters of the two-period economies, the theory of the third-degree price discrimination is applied to shape $p(x,y)$ in the monopolistic competitive market. Theorem 3.2.1 and Theorem 3.2.2 are proved in Section 3.5 as a somehow conclusion of Chapter 3.

In Chapter 4, we propose an optimal strategy from the seller's perspective which maximizes the overall seller's expected revenue. Especially we set first- and second-stage prices to maximize the overall seller's expected revenue under various scenarios of practical relevance in a two-period scenario. In Section 4.1, we show a motivation problem of selling real-estate property in Punta del Este, a resort town in Uruguay in the introduction part of Subsection 4.1.1. Following the introduction part, some basic results and findings of solving the motivation problem are concisely concluded in Subsection 4.1.2. Subsection 4.1.3 reviews some pieces of literature related to the motivation problem, and its solutions.

In Section 4.2 we present several illustrative examples that clarify certain key aspects of our general model, which will be proposed in Section 4.3. These aspects are described as sequential and simultaneous price settings, differing valuations and thus bid prices, costs associated with holding unsold property. The differences of four kinds of scenarios between setting the two prices sequentially and simultaneously are shown by numerical 4 tables, Tables 4.1 – 4.4, and 16 graphs, Figures 4.2 – 4.17 in Subsection 4.2.2 . These different cases of 4 scenarios show how many considerations influence the two prices in a two-period scenario.

When the second period, T, is pre-specified and thus deterministic, this kind of examples were discussed in Wu and Zitikis (2017). When the second period, T, is generally unknown and thus treated as a random variable, it is the general case that is considered as the general model proposed in Section 4.3.

In order to figure out the ideas of the general model, a number of other simplifying yet practically sound assumptions are made to simplify the technicalities in the general model. Section 4.4 and Section 4.5. In Section 4.4 we analyze the initial-stage selling probability. We concentrate on the forces that give rise to the amount of money that the buyer (domestic or foreign) is willing to pay for the property during the initial selling stage. Subsection 4.4.1 takes into account individual considerations detached both from the exogenous factors and various exogenous factors to describe the initial-stage selling probability. These the exogenous factors and various exogenous factors are modeled by some proper gamma distributions in Subsection 4.4.2.

The second-stage selling possibility is explored in Section 4.5. Very similar to the first stage, we take into account individual considerations detached both from the exogenous factors and various exogenous factors to describe the second-stage selling conditional probability in Subsection 4.5.1. The exogenous factors and various exogenous factors are modeled by some proper gamma distributions in Subsection 4.5.2. In Section 4.6 we discuss modeling initial- and second-stage value functions and then use them to illustrate our general model through some figures numerically (See Figures 4.18-4.21).

In Chapter 5, the conclusions of the whole theses are given in Section 5.1 and remaining open questions and future works are proposed in Section 5.2.

Chapter 2

Fundamental notions and the techniques

2.1 Two-envelope paradox and its resolutions

2.1.1 Two-envelope paradox

The two-envelope paradox, also known as the two-envelope problem, is a paradox in logic, game theory, decision theory, probability, and mathematics. Sometimes this problem goes to be very paradoxical, if you are not careful with your analysis. The problem typically is expressed by a theoretical problem of the following descriptions.

Suppose you are playing a game for money, and you are given two indistinguishable envelopes. One containing some money, say \$$x$, the other one contains twice, say \$$2x$. But you don't know which one is which. Initially, you can pick one of the two envelopes and keep the amount it contains, say \$$v$. Before opening it, you are offered a chance to swap envelopes to get more money (of course, if your guessing is wrong, then you would get less money). A question is "should you swap the envelopes or not"?

Obviously swapping the envelopes or not depends on comparison of the value without swapping envelopes and the expected returns with swapping envelopes. Suppose the player has a 50-50 chance of choosing either envelope at this moment, and therefore he/she has an equal chance of gaining or losing money. After a few steps of calculations, it is not difficult to understand

- if he/she doesn't swap, he/she get $\$v$;
- if he/she swaps, he/she get

$$\frac{1}{2}(v/2) + \frac{1}{2}(2v) \quad [= \frac{5}{4}v > v].$$

The results would mean we expect to gain $\frac{5}{4}v$ on average based on swapping the envelopes. It gives you some hints that it doesn't matter which envelope you choose initially — you should always swap.

Indeed, besides the above example, there are many other versions of the two-envelope paradox. The beginning of the two-envelope paradox started from necktie paradox (Kraitchik, 1930) and wallet game (Gardner, 1982). A more popular version of the two-envelope paradox was constructed by Zabell (1988).

There is a body of work related to the two-envelope paradox since the logic of the two-envelope paradox is widely existing in the fields of game theory, probability theory, and decision theory. What's more, when problems of random switching between two unstable states are involved in a study, the two-envelope paradox can be applied to these problems frequently in fields of physics, engineering and economics (Abbott, 2009; Allison & Abbott, 2001; Harmer & Abbott, 1999a, 1999b, 2002).

2.1.2 Exploiting for resolutions of the two-envelope paradox

There are plentiful of resolutions to the two-envelope paradox, but many researchers insist that most of them are meaningless because of the lack of consensus (Christensen & Utts, 1992; McGrew et al., 1997; Castell & Batens, 1994; Clark & Shackel, 2000; Meacham & Weisberg, 2003). As a result, the two-envelope paradox is widely considered as an opening problem (Albers et al., 2005).

Among these abundant studies of finding the solutions of the two-envelope paradox, there is one type of the two-envelope paradox — where the player observes the amount in one envelope and keeps the amount of the other one unknown for exploiting (Nalebuff, 1989; Nickerson and Falk, 2006). Following the idea that observing the amount in the first selected envelope, the player swaps to the second envelope with a probability. Mcdonnell and Abbott (2009) demonstrated a wise strategy for dealing with the two-envelope paradox. The smart strategy shows

- to swap envelopes with less likelihood when the observed amount is a large sum of money;

- to swap envelopes with more likelihood when the observed amount is small.

2.1.3 Problem formulation

Suppose there are two indistinguishable envelopes, the amounts of which are x and $2x$ respectively. Let random variable X defined on $(\Omega, \mathcal{F}, \mathbf{P})$ be the amount of the first envelope opened by the player with probabilities $\mathbf{P}(X = x) = p$ and $\mathbf{P}(X = 2x) = 1 - p$. Then the *switching probabilities* given the amount $X = x$ are $P_S(x)$ and $P_S(2x)$ respectively, where $0 \leq P_S(x) \leq 1$ for all $x > 0$. Hence, the probability that the player ends the trial with amount x is

$$P_x(x) = p\,(1 - P_S(x)) + (1 - p)P_S(2x),$$

and the probability that the player ends with amount $2x$ is

$$P_{2x}(x) = pP_S(x) + (1 - p)\,(1 - P_S(2x))\,.$$

Thus, the unconditional expected return given $X = x$ is

$$\begin{aligned}
R(x) &= x \cdot P_x(x) + 2x \cdot P_{2x}(x) \\
&= x \cdot p\,(1 - P_S(x)) + (1-p)P_S(2x) + 2x \cdot pP_S(x) + (1-p)\,(1 - P_S(2x)) \\
&= x(2-p) + p(x(P_S(x) + P_S(2x)) - xP_S(2x)),
\end{aligned}$$

where $x > 0$. Assuming the density function of X is $f_X(x)$, then the unconditional expected return is

$$\begin{aligned}
R &= \int_0^\infty f_X(x)R(x)dx \\
&= \int_0^\infty f_X(x)(x(2-p) + p(x(P_S(x) + P_S(2x)) - xP_S(2x)))dx \\
&= (2-p)\mathbf{E}[X] + \int_0^\infty f_X(x)p(x(P_S(x) + P_S(2x)) - xP_S(2x))dx. \qquad (2.1)
\end{aligned}$$

Here a natural benchmark strategy could be defined as "never switching", which is equivalent to $P_S(x) \equiv 0$ for all $x > 0$. Then accordingly, the unconditional expected return of the benchmark strategy is

$$R_B := (2-p)\mathbf{E}[X].$$

Therefore, the gain or loss due to certain switching strategy P_S is

$$\mathcal{S} := R - R_B = \int_0^\infty x f_X(x)\left[pP_S(x) - (1-p)P_S(2x)\right]dx. \qquad (2.2)$$

Clearly, to maximize the gain $\mathcal{S}$, the player should determine the switching region depending on P_S. And such a switching region could be written as $\{x \in [0, \infty) : s(x) > 0\}$, where

$$s(x) = pP_S(x) - (1-p)P_S(2x). \qquad (2.3)$$

2.1.4 McDonnell and Abbott (2009): three particular cases

It might be very difficult to obtain the optimal function P_S through maximizing $\mathcal{S}$ directly. However, if the form of function as well as certain parameters is given, it is possible to obtain the required switching region. In McDonnell and Abbott (2009), three particular cases for function P_S are given to illustrate the derivation of the switching region from P_S.

(i). *Constant switching probability*

Let the switching probability $P_S(x)$ be a constant with respect to x, namely, $P_S(x) \equiv q$ for all $x > 0$. Then by (2.2) we have

$$\mathcal{S} = q(2p - 1)\mathbf{E}[X].$$

Hence, there are three cases for decision-making:

- If $p = 0.5$, then $\mathcal{S} = 0$, which means that there are neither gain nor loss for the unconditional expected return regardless of the player's decision.
- If $p < 0.5$, then $\mathcal{S} \leq 0$ and the equality holds if and only if $q = 0$. Notice that $q = 0$ corresponds to the benchmark strategy. We may conclude that the benchmark strategy is the optimal one among all strategies under this situation, i.e. the player should not switch no matter what value of x he/she observed when openning the first envelope.
- If $p > 0.5$, then $\mathcal{S} \in [0, (2p - 1)\mathbf{E}[X]]$ and is maximized at $q = 1$. Thus, we may conclude that the player should always choose to switch no matter what value of x he/she observed when openning the first envelope.

(ii). *Switching with smoothly decreasing probability*

Suppose that $P_S(x)$ is a monotonically continuously decreasing function of $x > 0$. A particular case of this situation is that $P_S(x)$ decays exponentially, namely,

$$P_{S_1}(x) = e^{-ax}, \quad x > 0$$

for some $a > 0$.

Definition 2.1.1 The logit of a number $p \in [0, 1]$ is given by the formula

$$\begin{aligned} \text{logit}(p) &= \log(\frac{p}{1-p}) \\ &= \log(p) - \log(1-p). \end{aligned}$$

Hence, change envelope everywhere we have from (2.3) that the switching region is $\{x \geq 0 : x > c_1\}$ where

$$c_1 = \max\left\{-\frac{1}{a}\text{logit}(p), 0\right\}.$$

This result indicates that as long as $p \geq 0.5$, the player should always choose to switch no matter what value of x he/she observed when opening the first envelope. Similar cases include the switching probability

$$P_{S_2}(x) = \frac{2}{1 + e^{a_1 x}}, \quad x > 0$$

for some $a_1 > 0$. Hence we have from (2.3) that the switching region is $\{x \geq 0 : x > c_2\}$ where

$$c_2 = \frac{1}{a_1} \log\left(\frac{1 + \sqrt{1 + 4y(1-y)}}{2y}\right),$$

where $y = \min\{p/(1-p), 1\}$. This result also indicates that as long as $p \geq 0.5$, the player should always choose to switch no matter what value of x he/she observed when openning the first envelope.

Definition 2.1.2 *sech* function, also known as Hyperbolic secant function, is defined by

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$$

for $x \in \mathcal{R}$.

The last case we would like to discuss is

$$P_{S_3}(x) = \operatorname{sech}(a_2 x), \quad x > 0$$

for some $a_2 > 0$.

Definition 2.1.3 *cosh* function, also known as Hyperbolic cosine function, is defined by

$$\cosh(x) = \cos(ix)$$

for $x \in \mathcal{R}$, and i is the imaginary unit which satisfies $i^2 = 1$.

Hence we have from (2.3) that the switching region is $\{x \geq 0 : x > c_3\}$ where

$$c_3 = \frac{1}{a_2} \cosh^{-1}\left(\frac{1 + \sqrt{1 + 8z}}{4z}\right),$$

where $z = \min\{p/(1-p), 1\}$. And the same conclustion that when $p \geq 0.5$ the player should always choose to switch no matter what value of x he/she observed when opening the first envelope could be obtained again.

(iii). *Switching according to a threshold value*

For this case, the switching probability could always be written as

$$P_{S_4}(x) = \mathbf{1}_{\{x \leq b\}}, \quad x > 0$$

for some $b > 0$, where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. Then according to (2.3), we have

$$s(x) = \begin{cases} 2p - 1 & \text{if} \quad x \leq 0.5b, \\ p & \text{if} \quad 0.5b < x \leq b, \\ 0 & \text{if} \quad x > b. \end{cases}$$

Thus, the switching region should be $\{x \geq 0 : c_4 < x \leq b\}$ where $c_4 = 0.5b\mathbf{1}_{\{p<0.5\}}$. Notice that when $x > b$ there are neither gain nor loss for the unconditional expected return regardless of the player's decision. As a result, when $p \geq 0.5$, the player could choose to switch no matter what value of x he/she observed when openning the first envelope.

In McDonnell et al. (2011), the suboptimality of P_S is discussed as the 'non-blind' two-envelope problem. Notice that from (2.2) we may derive that

$$\mathcal{S} := \int_0^\infty y P_S(y) g(y) dy,$$

where

$$g(y) := p f_X(y) - \frac{1-p}{4} f_X\left(\frac{y}{2}\right).$$

Hence the following result holds.

Proposition 2.1.1 *(McDonnell et al. 2011, Theorem 2.4) The optimal switching function $P_S^*(y)$ for the 'non-blind' two-envelope problem is*

$$P_S^*(y) = \mathbf{1}_{\{g(y) \geq 0\}}.$$

This result indicates that there will be intervals of y for which it is optimal to switch and complementary intervals where no-switching is the optimal strategy.

2.1.5 Egozcue et al. (2013): applications to real-estate selling

An example of applying the two-envelope framework on other types of decision-making problems is considered in Egozcue et al. (2013). In that article, the seller of the house has to decide between a current offer and a potential next offer (the seller will not wait for the third offer for some reasons, for instance, he/she has a new position starting very soon in another city). In this situation, the seller either accepts the current offer or rejects it, which means he/she has to accept the next one no matter what price is offered. The main purpose of the article is to help the seller make the decision based on the price of the current offer.

This problem is similar to a generalized version of the standard two-envelope problem. If we consider the two offers as the two envelopes. Then the offered prices could be seen as the rewards contained in the envelopes which are denoted as X_H and X_L respectively. Moreover, X_H represents the potentially higher price compared with X_L. When the seller looks at the price of the current offer, it is similar to that the seller opens one of the envelopes and sees the reward in that envelope, however, he/she has no idea whether the amount stands for the higher one. Thus, the framework of the two-envelope problem is a suitable tool for analyzing this situation.

The problem formulation is as follows. Suppose binary random variable $\Pi_1 \in \{L, H\}$ represents the person who comes first to put an offer. As the seller sees the price, his/her decision is also a binary random variable $\Delta_1 \in \{A, R\}$. $\Delta_1 = R$ means reject the current offer while $\Delta_1 = A$ means acceptance. Then the seller's strategy function is

$$S(y) \equiv \mathbf{P}[\Delta_1 = R | X_{\Pi_1} = y]$$

based on the price of the current offer given the following two assumptions:

1. Whether the first-come offer Π_1 is made by L or H does not depend on the (random) prices X_L and X_H. This assumption allows us to denote

$$p \equiv \mathbf{P}[\Pi_1 = L] = \mathbf{P}[\Pi_1 = L | X_L, X_H]$$

and thus $\mathbf{P}[\Pi_1 = H] = 1 - p$.

2. The probability of rejecting the current offer depends only on the amount of the current offer.

Similar to the discussion in McDonnell and Abbott (2009), a benchmark expected price is defined as the unconditional expected value of the price at deal when the seller always accepts the current offer, i.e. $S(y) \equiv 0$. Hence the benchmark expected price is

$$\mathrm{BEP} = p\mathbf{E}[X_L] + (1-p)[X_H],$$

which is equivalent to R_B in McDonnell and Abbott (2009). Hence if the unconditional expected price given that some strategy $S(y)$ is adopted is denote as μ_X, the following result is given by Egozcue et al. (2013).

Proposition 2.1.2 *(Theorem 2.1, Egozcue et al., 2013) The difference of μ_X and* BEP *is*

$$\mu_X - \mathrm{BEP} = p\mathbf{E}[S(X_L)\{\mathbf{E}[X_H|X_L] - X_L\}] + (1-p)\mathbf{E}[S(X_H)\{\mathbf{E}[X_L|X_H] - X_H\}].$$

And this difference is called "strategy risk parameter" and denoted as SRP(S) *given the strategy S.*

Proposition 2.1.2 reduces the main purpose to the following maximization problem:

$$\max_S \mathrm{SRP}(S) \tag{2.4}$$

subject to $0 \leq S(y) \leq 1$ for all $y \in [0, \infty)$. This maximization problem could be solved through the following result.

Proposition 2.1.3 *(Corollary 3.1, Egozcue et al., 2013) Assume that X_L and X_H have densities f_L and f_H, respectively. Then the maximizing strategy function $\mathcal{S}_{\mathrm{MAX}}(y)$ is the indicator $\mathbf{1}_A(y)$*

of the set $A = \{x \in [0, \infty) : \mathcal{H}_{\text{MAX}}(x) > 0\}$*, where*

$$\mathcal{H}_{\text{MAX}}(x) = p\left\{\mathbf{E}\left[X_H | X_L = x\right] - x\right\} f_L(x) + (1-p)\left\{\mathbf{E}\left[X_L | X_H = x\right] - x\right\} f_H(x).$$

According to the appendix of McDonnell et al. (2011), Proposition 2.1.3 is expected to be proved in at least three ways. However, as we could write $\text{SRP}(S)$ as

$$\text{SRP}(S) = \int S(x)\mathcal{H}_{\text{MAX}}(x)dx,$$

it is clear that if $\text{SRP}(S)$ is supposed to be maximized $S(x)$ has to be equal to 1 when $\mathcal{H}_{\text{MAX}}(x) > 0$ while $S(x)$ has to be equal to 0 when $\mathcal{H}_{\text{MAX}}(x) \leq 0$ intuitively.

Egozcue et al. (2013) claims that the ultimate maximizing strategy $\mathcal{S}_{\text{MAX}}(y)$ is often a threshold-type strategy, namely, $S_b(y) = \mathbf{1}_{[0,b)}(y)$, where $b \geq 0$ is a "threshold". Two situations are discussed in that article to illustrate this intuition, of which the first situation is when random variables X_L and X_H are independent. In this situation, the function $\mathcal{H}_{\text{MAX}}(x)$ reduces to

$$\mathcal{H}_{\text{MAX}}(x) = p\left\{\mathbf{E}\left[X_H\right] - x\right\} f_L(x) + (1-p)\left\{\mathbf{E}\left[X_L\right] - x\right\} f_H(x).$$

Moreover, it is assumed that X_H is greater than or equal to X_L in the likelihood ratio sense, namely,

$$w(x) = \frac{f_H(x)}{f_L(x)}$$

is a non-decreasing function with respect to x (this is the reason why subscript H is used to stand for potentially "higher" price while subscript L stands for potentially "lower" price).

Notice that

$$\begin{aligned}\mu_H &:= \mathbf{E}[X_H] \\ &= \mathbf{E}[X_L w(X_L)] \\ &= \mathbf{Cov}(X_L, w(X_L)) + \mathbf{E}[X_L],\end{aligned}$$

it is easy to verify that $\mu_H \geq \mu_L := \mathbf{E}[X_L]$ due to that $w(x)$ is non-decreasing. Thus, if X_L and X_H have the same support (x_1, x_2) for some $0 \leq x_1 < x_2 \leq \infty$, the explicit solution of $\mathcal{S}_{\text{MAX}}(y)$ could be obtained as the following result.

Proposition 2.1.4 *(Theorem 3.1, Egozcue et al., 2013) The optimal strategy function* $\mathcal{S}_{\text{MAX}}(y)$ *is*

$$\mathcal{S}_{\text{MAX}}(y) = \mathbf{1}_{(x_1,b)}(y)$$

with the threshold $b := \sup\{x > \mu_L : v(x) > w(x)\}$*, where*

$$v(x) := \frac{p(\mu_H - x)}{(1-p)(x - \mu_L)}.$$

In particular, when

$$\mu_L = \mu_H (\equiv \mu),$$

then

$$\mathcal{S}_{\text{MAX}}(y) = \mathbf{1}_{(x_1,\mu)}(y).$$

The other situation discussed in Egozcue et al. (2013) is that

$$X_H = \alpha X_L$$

for a constant $\alpha > 1$. This is equivalent to

$$f_H(x) = (1/\alpha) f_L(x/\alpha)$$

and thus

$$\mathrm{SRP}(S) = (\alpha - 1) \int S(x)x \left[p f_L(x) - (1-p)\frac{1}{\alpha^2} f_L\left(\frac{x}{\alpha}\right)\right] dx.$$

Hence function $\mathcal{H}_{\mathrm{MAX}}(x)$ could be chosen as

$$\mathcal{H}_{\mathrm{MAX}}(x) = p f_L(x) - (1-p)\frac{1}{\alpha^2} f_L\left(\frac{x}{\alpha}\right).$$

The readers are referred to the integrand of (2.2) of McDonnell and Abbott (2009) as a particular case $\alpha = 2$. As the result of $\mathcal{S}_{\mathrm{MAX}}(y)$ depends on $f_L(x)$, the rest of the discussion in Egozcue et al. (2013) is based on three particular distributions for X_L.

1. When X_L is uniform on $[A, B]$ for some $0 \le A < B \le \infty$, then

$$\mathcal{S}_{\mathrm{MAX}}(y) = \begin{cases} \mathbf{1}_{[A,\alpha A)}(y) & \text{when} \quad \alpha A \le B \text{ and } p \le \frac{1}{1+\alpha^2}, \\ \mathbf{1}_{[A,B]} & \text{otherwise.} \end{cases}$$

   To decipher this result in terms of the decision-making language, consider the doublet $(\alpha, p) \in (1,\infty) \times [0,1]$. If we define a subset:

$$\Delta := \left\{(\alpha, p) \in (1,\infty) \times [0,1] : \alpha \le \frac{B}{A}, p \le \frac{1}{1+\alpha^2}\right\},$$

   then the decision rule is:

   - when $(\alpha, p) \notin \Delta$, then the current offer should always be rejected, irrespectively of the price y;
   - when $(\alpha, p) \in \Delta$, then the current offer should be rejected if the price $y < \alpha A$. Otherwise, the current offer should be accepted.

2. When X_L is a log-normal random variable having density

$$f_L(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log(x) - \mu)^2}{2\sigma^2}\right\} \mathbf{1}_{(0,\infty)}(x)$$

for some parameters $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Then

$$\mathcal{S}_{\text{MAX}}(y) = \mathbf{1}_{(0,b)}(y)$$

with the threshold

$$b = \sqrt{\alpha} e^{\mu} \left(\frac{p\alpha}{1-p} \right)^{\sigma^2 / \log \alpha}.$$

3. When X_L is a Pareto random variable having density

$$f_L(x) = \frac{\theta}{x_0} \left(\frac{x_0}{x} \right)^{\theta+1} \mathbf{1}_{[x_0,\infty)}(x)$$

for some parameters $x_0 > 0$ and $\theta > 1$. Then

$$\mathcal{S}_{\text{MAX}}(y) = \begin{cases} \mathbf{1}_{[x_0,\alpha x_0)}(y) & \text{when} \quad p \le \frac{1}{1+\alpha^{1-\theta}}, \\ \mathbf{1}_{[x_0,\infty]} & \text{otherwise.} \end{cases}$$

To decipher this result in terms of the decision-making language, consider the doublet $(\alpha, p) \in (1, \infty) \times [0, 1]$. If we define a subset:

$$\Xi := \left\{ (\alpha, p) \in (1, \infty) \times [0, 1] : p \le \frac{1}{1 + \alpha^{1-\theta}} \right\},$$

then the decision rule is:

- when $(\alpha, p) \notin \Xi$, then the current offer should always be rejected, irrespectively of the price y,
- when $(\alpha, p) \in \Xi$, then the current offer should be rejected if the price $y < \alpha x_0$. Otherwise, the current offer should be accepted.

2.2 Price discrimination

Price discrimination describes pricing behaviors and pricing preferences of monopoly enterprises in a monopolistic competitive market. Price discrimination happens when units of the same product are sold at different prices, either to the same consumers or different consumers.

Definition 2.2.1 Price discrimination is a microeconomic pricing strategy where identical or largely similar goods or services are transacted at different prices by the providers in different markets/consumers.

According to the classification of Pigou (1920), price discrimination may take three forms: *first-degree price discrimination*, *second-degree price discrimination* and *third-degree price discrimination*.

Definition 2.2.2 First-degree price discrimination is a pricing strategy where each consumer is charged the maximum amount they are willing to pay for each unit.

First-degree price discrimination means that the producer can charge whatever the market will bear and thus captures the entire consumer surplus. It involves the seller charging a different price for each unit of the commodity in such a way that the amount charged for each unit is equal to the maximum willingness-to-pay for that unit. It is also known as perfect price discrimination.

Definition 2.2.3 Second-degree price discrimination is a pricing strategy where price varies according to quantity demanded. Larger quantities of the same product are sold at a lower unit price; Smaller quantities of the same product are sold at a higher unit price.

Second-degree price discrimination involves selling larger quantities of the same product at a lower unit price. It occurs when price differs depending on the number of units that the good bought, but not across consumers. This phenomenon is also known as nonlinear pricing. Each consumer faces the same price schedule, but the schedule involves different price for the different amount of the good purchased. Quantity discounts or premia are the prominent examples.

Definition 2.2.4 Third-degree price discrimination is a pricing strategy where producers charge a different price to different consumer groups.

Third-degree price discrimination takes place when different prices are charged for the same product to different consumers. It means that different purchasers are charged different prices, but each buyer pays a constant amount for each unit of the good bought. It is perhaps the most common form of price discrimination; examples are student discounts or charging different prices on different days of the week.

2.2.1 Willingness-to-pay and marginal willing-to-pay of price discrimination

To understand three forms of price discrimination in an economic environment, we introduce a model of willingness-to-pay (Varian, 1992). Suppose there are many potential consumers whose utility functions are

$$u_i(x) + y,$$

for $i \in 1, 2, \ldots$, and x represents the interested good, y represents all other goods. Denote $r(i)$ is the maximum willingness-to-pay for some consumption level x such that it is the solution to the equation

$$u_i(0) + y = u_i - r_i(x) + y. \tag{2.5}$$

We define $u_i(x) = 0$ for simlicity, and get naturally

$$r_i(x) \equiv u_i(x). \tag{2.6}$$

Furthermore if the consumer faces a per-unit price $p(x)$ of x, budget limit of m, and chooses the optimal level of consumption, then equivalently it is the maximization problem

$$\max_{x,y} \quad u_i(x) + y$$
$$\text{subject to} \quad p(x)x + y = m.$$

The first order condition for this problem is

$$p(x) = u_i'(x). \tag{2.7}$$

The price necessary to induce consumer i to choose consumption level x, i.e., the marginal willing-to-pay, is

$$p(x) = p_i(x) \equiv u_i'(x).$$

Definition 2.2.5 Suppose that the maximum willingness-to-pay for the good by consumer k, always exceeds the maximum willingness-to-pay by consumer j,

$$u_k(x) > u_j(x), \tag{2.8}$$

for all x. For the marginal willingness-to-pay, we suppose

$$u_k'(x) > u_j'(x), \tag{2.9}$$

which means the marginal willingness-to-pay for the good by consumer k exceeds the marginal willingness-to-pay by consumer j. Then the consumer k is regarded as the *high demand consumer* and the consumer j is regarded as the *low demand consumer*.

2.2.2 First-degree price discrimination

First-degree price discrimination is also known as perfect price discrimination. In term of first-degree price discrimination, the consumer surplus is 0, because it is reaped by the producer. Selling an extra unit does not require lowering the price of the previously sold units, since each one may be sold at a different price. Demand and marginal revenue curves are therefore confounded, and the profit maximizing condition (marginal revenue equals marginal cost) yields the same output as a competitive market. If a monopoly can practice perfect discrimination, the outcome will thus be Pareto efficient, which is a quite counter-intuitive result. Transaction and research costs involved in finding the willingness-to-pay of each consumer make this case extremely unlikely in the real world, i.e., it is an ideal case.

Definition 2.2.6 *Consumer surplus* is defined as the difference between the total amount that consumers are willing and able to pay for a good or service (indicated by the demand curve) and the total amount that they do pay (i.e. the market price).

Definition 2.2.7 *Producer surplus* is an economic measure of the difference between the amount a producer of a good received and the minimum amount the producer is willing to accept for a good. The difference, or surplus value, is the benefit the producer receives for selling the commodity in the market.

Theoretically first-degree price discrimination requires the monopoly seller of a good or service to know the absolute maximum price (or reservation price) that every consumer is willing to pay. By knowing the reservation price, the seller is able to sell the good or service to each consumer at the maximum price he is willing to pay, and thus transform the consumer surplus into revenues. So the profit is equal to the sum of consumer surplus and producer surplus (see Figure 2.1). The marginal consumer is the one whose reservation price equals to the marginal cost of the product. Suppose a monopolist offers goods to one agent in the market

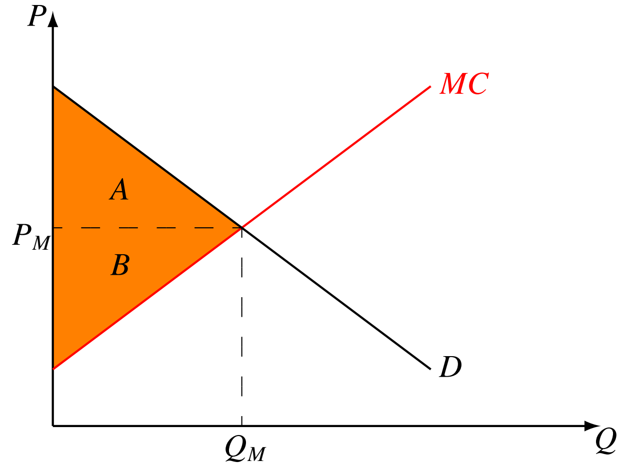


Figure 2.1: The first-degree price discrimination. The vertical axis represents the price level of a commodity. The horizontal axis represents the quantity of goods. Curve MC is marginal cost curve. Curve D is demand curve, also known as marginal revenue curve here. The profit maximizing condition (marginal revenue equals marginal cost $MR = MC$) indicates the pairs of the quantity and the price, (Q_M, P_M). The price P_M is a market clearing price. Q_M is a market clearing quantity, which matchs the market clearing price. $A + B$ means the profit, which is equal to the sum of consumer surplus and producer surplus here.

at this moment. The profit maximization problem of the monopolist is

$$\begin{aligned} \max_{x,r} \quad & r - cx \\ \text{subject to} \quad & u(x) \geq r \end{aligned}$$

where c is a constant marginal cost of per unit of the monopolist. The first order condition for this problem is

$$u'(x^*) = c.$$

The corresponded willing-to-pay price is

$$r = r(x^*) = u(x^*).$$

Now let's imagine next if the monopolist sells each unit of output to the consumer at a different price. Suppose, for example, that the firm breaks up the output into n pieces of size Δx, so that $x = n\Delta x$. Considering the willing-to-pay price,

$$\begin{aligned} u(0) &= u(\Delta) - p_1, \\ u(\Delta) &= u(2\Delta) - p_2, \\ &\vdots \\ u((n-1)\Delta) &= u(x) - p_n. \end{aligned}$$

we have

$$\sum_{n}^{i=1} p_n = u(x),$$

by adding up these n equations. That is the sum of the marginal willingnesses-to-pay, which must equal the total willingness-to-pay.

2.2.3 Second-degree price discrimination

Second-degree price discrimination is also known as nonlinear pricing: prices vary across units, but not across people. The usual example of second-degree price discrimination is quantity discounts, when larger quantities of the same product are sold at a lower unit price. Producers use this technique when they know different consumer groups exist, but have no way of observing them. The complete extraction of consumer surplus is not possible as it was in the first-degree price discrimination, but the producer can still design some price scheme to force consumers to reveal their type. Second-degree price discrimination is a more realistic case of price discrimination. It is often practiced by public utilities, as the price per unit of water and electricity often depends on how much is consumed. Consumers self-select themselves into consumption categories. The description for Figure 2.2 are as follows. Suppose there are two groups of people with different numbers of people, a smaller number Q_1 for group 1, and a larger number Q_2 for group 2. In terms of the second-degree price discrimination, the producers offer a higher price

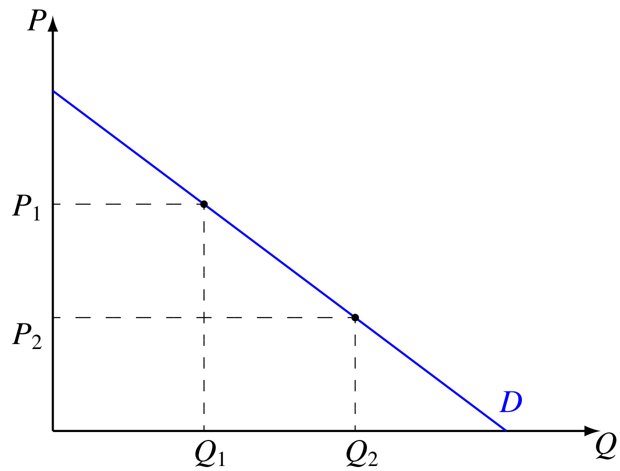


Figure 2.2: The second-degree price discrimination. The vertical axis represents the price level of a commodity. The horizontal axis represents the quantity of goods. Curve D is demand curve here. The point (Q_1, P_1) on the curve D means the producers offer price P_1 to the group 1 with the consumer group number who has the group number Q_1. The point (Q_2, P_2) on the curve D means the producers offer price P_2 to the consumer group 2 who has the group number Q_1.

P_1 of goods for group 1, and a lower price P_2 of goods for group 2, i.e. $Q_1 < Q_2$ and $P_2 < P_1$. Figure 2.2 shows a kind of price discrimination which indicates price varies according to the quantity demanded.

We suppose there are two consumers, and each consumer want to consume the amount x_i and be willing to pay the price r_i,

$$u_i(x_i) - r_i \geq 0,$$

for $i = 1, 2$. Also we suppose each consumer prefer his consumption to the consumption of the other consumer

$$\begin{cases} u_1(x_1) - r_1 & \geq u_1(x_2) - r_2, \\ u_2(x_2) - r_2 & \geq u_2(x_1) - r_1. \end{cases}$$

Because of the equations of 2.8 and 2.9, we can continue to calculate and get

$$\begin{cases} r_2 &= u_2(x_2) - u_2(x_1) + r_1, \\ r_1 &= u_1(x_1). \end{cases}$$

Considering the profit function of the monopolist is

$$\pi = (r_1 - cx_1) + (r_2 - cx_2),$$

we replace r_1 and r_2 and reach equivalently,

$$\pi = (u_1(x_1) - cx_1) + (u_2(x_2) - u_2(x_1) + r_1 - cx_2).$$

Differentiating it with respect to x_l and x_2 to get a maximized expression

$$\begin{cases} u_1'(x_1) &= c + (u_2'(x_1) - u_1'(x_1)), \\ u_2'(x_2) &= c. \end{cases} \tag{2.10}$$

Equation (2.10) says that at the optimal nonlinear prices, the *high demand consumer* has a marginal willingness-to-pay which is equal to marginal cost; and the *low demand consumer* has a (marginal) value for the good that exceeds marginal cost. Hence he consumes an inefficiently small amount of the good.

2.2.4 Third-degree price discrimination

Third-degree price discrimination consists of distinguishing several groups of consumers with different demands to charge them different prices. Producers discriminate between groups of people, but not across units, as consumers are charged different prices, but each one faces a constant price for all units of output purchased. It is used to take advantage of the fact that different groups of people have different demand functions. The usual example of third-degree price discrimination is student discounts at the movies or sports events. Third-degree price discrimination is also practiced by pharmaceutical firms who sell the same drugs in different countries at different prices.

Definition 2.2.8 Price elasticity of demand is a measure used in economics to show the relationship between a change in the quantity demanded of a good or service and a change in its price. It gives the percentage change in quantity demanded in response to a one percent change in price.

$$\epsilon_{(p)} = \frac{dQ/Q}{dP/P}. \tag{2.11}$$

where Q is the quantity demanded of a good or service, and P are the price of the good or the service.

Figure 2.3 gives us a brief view about the third-degree price discrimination. Suppose there are two markets/groups, markets/groups 1, and markets/groups 2, with different price elasticities of demand. Assume group 1 is relatively lack of elasticity of demand, and group 2 is relatively elastic. In terms of the third-degree price discrimination, the producers offer a higher price p_1 of goods for group 1, and a lower price p_2 of goods for group 2, i.e. $p_2 < p_1$. Figure 2.3 shows a kind of price discrimination which indicates producers charge a different price to different consumers of markets/groups.

The model description of the third-degree price discrimination is following. Indeed we suppose there are two groups of consumers in two separate markets/groups, and define p_i, for $i = 1, 2$, as the inverse demand function for group i of consumers. The monopolist's profit maximization problem is

$$\max_{x_1, x_2} \quad p_1(x_1)x_1 + p_2(x_2)x_2 - cx_1 - cx_2. \tag{2.12}$$

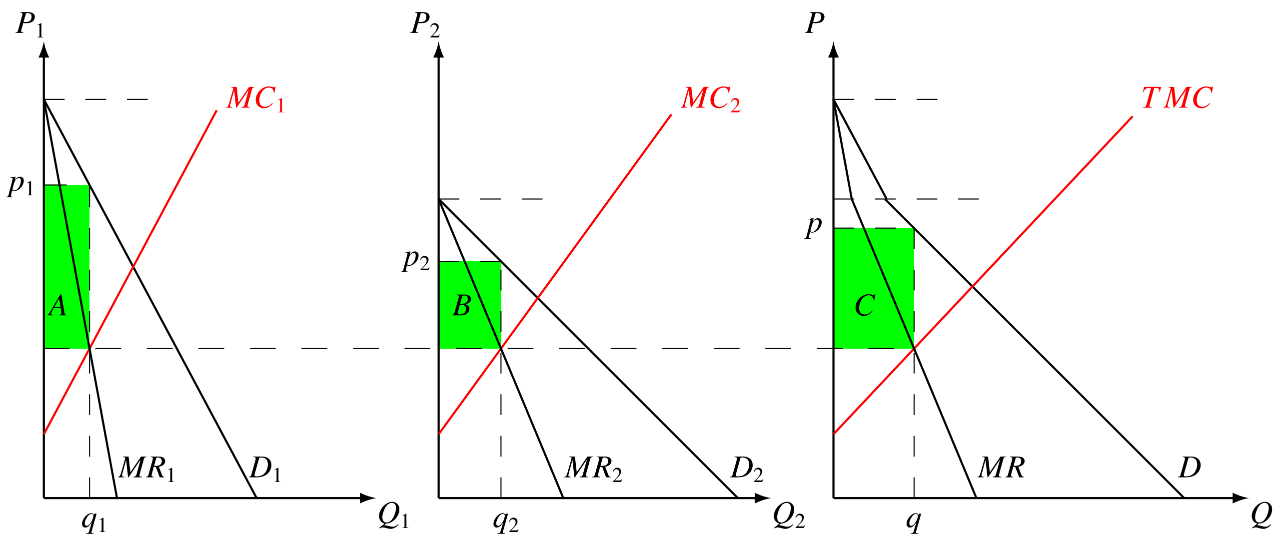


Figure 2.3: The third-degree price discrimination. The vertical axis represents the price level of a commodity. The horizontal axis represents the quantity of goods. Curve MC is marginal cost curve. Also, curve MC is supply curve here. Curve D is demand curve, also known as average revenue curve here. The demand curves D_1 and D_2 are different, which means the price elasticities of demand in two markets/groups are different. Curve MR is marginal revenue curve. The profit maximizing condition (marginal revenue equals marginal cost $MR = MC$) indicates the pairs of the quantities and the prices, (q_1, p_1), (q_2, p_2) and (q, p) respectively. A means the profit from the market/group 1. B means the profit from market/group 2. $A + B = C$. C means the total profits from both two markets/groups.

Differentiating the equation (2.12) with respect to x_l and x_2 to get a maximized expression

$$\begin{cases} p_1(x_1) + p_1'(x_1)x_1 &= c, \\ p_2(x_2) + p_2'(x_2)x_2 &= c. \end{cases} \tag{2.13}$$

We rewrite the equations (2.13) by

$$\begin{cases} p_1(x_1)(1 - \frac{1}{\epsilon_1}) &= c, \\ p_2(x_2)(1 - \frac{1}{\epsilon_2}) &= c. \end{cases} \tag{2.14}$$

Proposition 2.2.1 *If we have the equations* (2.14)*, then*

$$p_1(x_1) > p_2(x_2)$$

if and only if

$$|\epsilon_1| < |\epsilon_2|.$$

As one of the behaviours of monopoly enterprises in a monopolistic competitive market, the third-degree price discrimination means a selling strategy that the same provider charges different prices for identical or just similar goods or services in different consumer groups with different demand curves, and demand elasticities as well (cf., Aguirre, 2010; Holmes, 1989; Pindyck, and Rubinfeld, 2001; Schmalensee, 1981; Schwartz, 1990; Yoshida, 2000; Varian, 1992). The product differentiation is one of the crucial reasons why firms have some degree of control over the price. The more successful it is at differentiating its product from other firms selling similar products, the more monopoly power the form has. Product differences are due to quality, functional features, design, and so on, and so there is imperfect substitution between commodities even when they are in the same category (cf., Krugman, 1980; Head, and Ries, 2001; Helpman, 1981; Nguyen, 2014; Varian, 1992).

In our cases, the Lenovo laptop gets differentiated from other laptops because of its brand, quality, reputation, and so on, and thus gets imperfect substitution between different kinds of laptops. Consider that most consumers who buy laptops during the Black Friday promotion period have stronger laptop buying intention than most of the consumers who buy laptops during the Boxing Day promotion period. That is, the demand curves during these two periods have different demand elasticities of price: during the *Black Friday* promotion period, the demand elasticities of price is smaller than that during the *Boxing Day* promotion period. Considering this difference between buying intentions during two periods, the buyers during the two periods can be regarded as two separate markets with different demand curves and different demand elasticities, and the monopolist firm can tend to enforce the division of the selling prices during the two periods; this is the third-degree price discrimination.

2.3 Parametrical models

2.3.1 Beta distribution

Naturally, distributions supported on a bounded domain are considered for modelling the discounted prices of certain goods since the given prices are bounded above by the original prices while bounded below by 0 (in general, a seller will not sell a product if the price is below certain levels $x_{\min} > 0$). One of the very basic distributions with bounded supports is the Beta distribution, which will be employed to model the prices that vary within a fixed interval throughout this thesis. A formall definition of the *Beta distribution* is provided below from the perspective of the seller (the exact range of the discounted price is known to the seller).

Definition 2.3.1 A random variable X defined on $[x_{\min}, x_{\max}]$ is said to have Beta distribution if the density function $f_X(x)$ exists and has the following location-scale form:

$$f_X(x) = \begin{cases} \dfrac{(x - x_{\min})^{\alpha-1}(x_{\max} - x)^{\beta-1}}{B(\alpha,\beta)\rho^{\alpha+\beta-1}} & \text{if} \quad x_{\min} \le x \le x_{\max}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.15}$$

where $\alpha > 0$ and $\beta > 0$ are the shape parameters, $x_{\min}$ ($x_{\min} < x_{\max}$) is the location parameter, $\rho = x_{\max} - x_{\min}$ is the scale parameter, and $B(\alpha, \beta)$ is the Beta function of α and β.

Without loss of generality, we set $x_{\min} = 0$ and $x_{\max} = 1$, such that $\rho = 1$. We have graphed the probability density functions and cumulative distribution functions of Beta distribution with different values of α and β, in Figure 2.4 and in Figure 2.5.

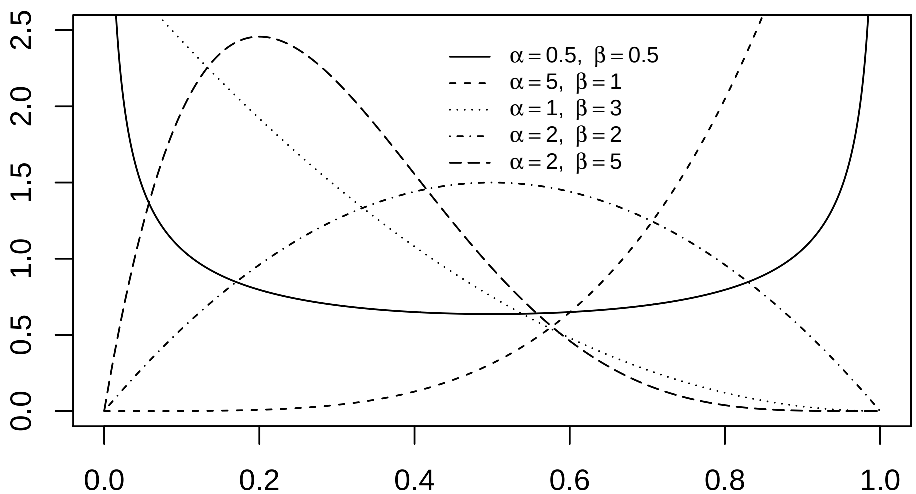


Figure 2.4: The probability density functions of Beta distribution with different parameter values of α and β. The simulated random variable is defined on [0, 1] and the selected parameter combinations of Beta distribution are $B(0.5, 0.5)$, $B(5, 1)$, $B(1, 3)$, $B(2, 2)$ and $B(2, 5)$.

Some other distributional properties cannot be expressed in terms of elementary functions, hence for the computational reason we introduce the *hypergeometric functions* defined below.

Definition 2.3.2 The *hypergeometric function* ${}_2F_1$ is defined by the power series

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

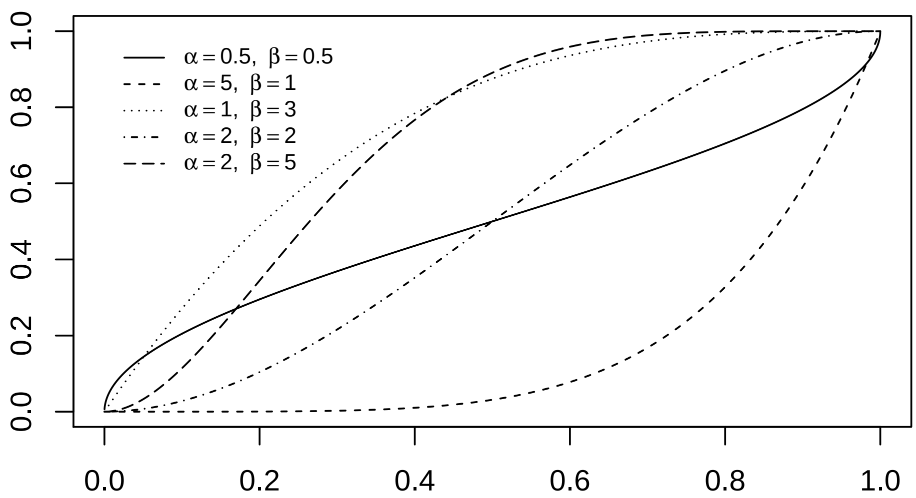


Figure 2.5: The cumulative distribution functions of Beta distribution with different parameter values of α and β. The simulated random variable is defined on $[0, 1]$ and the selected parameter combinations of Beta distribution are $B(0.5, 0.5)$, $B(5, 1)$, $B(1, 3)$, $B(2, 2)$ and $B(2, 5)$.

for $|z| < 1$, where $(q)_n$ is the *Pochhammer symbol* defined by

$$(q)_n = \begin{cases} \prod_{i=0}^{n-1}(q+i) & \text{if} \quad n > 1, \\ 1 & \text{if} \quad n = 0. \end{cases}$$

A famous integral representation of the hypergeometric function ${}_2F_1(a, b; c; z)$ is given below, which reveals the potential relation between ${}_2F_1(a, b; c; z)$ and a Beta distribution.

Proposition 2.3.1 *For $\Re(c) > \Re(a) > 0$ and $z \notin \{x \in \mathbb{R} : x \geq 1\}$, the hypergeometric function ${}_2F_1(a, b; c; z)$ could be written as*

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt.$$

Hence some other distributional properties of the Beta distribution could be expressed in terms of the hypergeometric function ${}_2F_1(a,b;c;z)$ or Pochhammer symbol.

Proposition 2.3.2 *If random variable X follows Beta distribution having density function* (2.15)*, then*

1. *The cumulative distribution function of X could be written as*

$$F_X(x) = \begin{cases} \dfrac{(x - x_{\min})/\rho}{\alpha B(\alpha,\beta)} {}_2F_1(\alpha, 1-\beta; \alpha+1; (x - x_{\min})/\rho) & \text{if} \quad x_{\min} \le x \le x_{\max}, \\ 0 & \text{if} \quad x < x_{\min}, \\ 1 & \text{if} \quad x > x_{\max}. \end{cases}$$

2. *The rth moment of $X - x_{\min}$ is*

$$\mathbf{E}\left[(X - x_{\min})^r\right] = \frac{\rho^r (\alpha)_r}{(\alpha+\beta)_r}.$$

   In particular,

$$\mathbf{E}[X] = x_{\min} + \frac{\rho\alpha}{\alpha+\beta}$$

   and

$$\mathbf{Var}(X) = \frac{\rho^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

3. *The characteristic function of X could be written as*

$$\phi_X(t) = {}_1F_1(\alpha; \alpha+\beta; i\rho t) e^{-it\rho x_{\min}},$$

   where $i = \sqrt{-1}$ and ${}_1F_1(a;b;z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}.$$

In later chapters, the two-period choice problem as an analog of the two-envelope paradox will be discussed, in which the dependence between the prices given in different period plays an important role. Hence the density function of the product of two independent random variable following Beta distributions is provided below in terms of the hypergeometric function ${}_2F_1(a, b; c; z)$.

Proposition 2.3.3 *(Corallary 3.1.2, Nagar and Zarrazola, 2004) If X and Y are two independent random variables having the standard Beta distribution ($x_{\min} = 0$ and $\rho = 1$) with the shape parameters (α_X, β_X) and (α_Y, β_Y) respectively, then the density function of their product could be expressed in terms of the Gauss hypergeometric function ${}_2F_1$, namely,*

$$f_{XY}(z) = \frac{\Gamma(\alpha_X + \beta_X)\Gamma(\alpha_Y + \beta_Y)}{\Gamma(\beta_X + \beta_Y)\Gamma(\alpha_X)\Gamma(\alpha_Y)} z^{\alpha_X - 1}(1 - z)^{\beta_X + \beta_Y - 1}\, {}_2F_1(\beta_Y, \alpha_X + \beta_X - \alpha_Y; \beta_X + \beta_Y; 1 - z)$$

for all $z \in [0, 1]$, and $f_{XY}(z) = 0$ otherwise.

Corollary 2.3.4 *Under the assumptions provided in Proposition 2.3.3, if $\alpha_Y = \alpha_X + \beta_X$, then XY follows a Beta distribution with shape parameters α_X and $\beta_X + \beta_Y$.*

These distributional results will help illustrate how the optimal choice of purchase should be made based on the theoretical framework provided in this thesis.

2.3.2 Gamma distribution

Unlike the discounted prices of goods, bidding prices do not seem to have an obvious upper bound. As a result, Beta distribution may not be appropriate for modelling the bidding prices. Instead, distributions with unbounded support such as the Gamma distribution could be a better choice. Throughout this thesis, bidding prices are assumed to have Gamma distributions (denoted as $Ga(\alpha, \beta)$) with the following density function

$$f_{\alpha,\beta}(t) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \leq 0. \end{cases} \tag{2.16}$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the rate parameter.

Without loss of generality, we have graphed the probability density functions and cumulative distribution functions of Gamma distribution with different parameter values of α and β, in Figure 2.6 and in Figure 2.7.

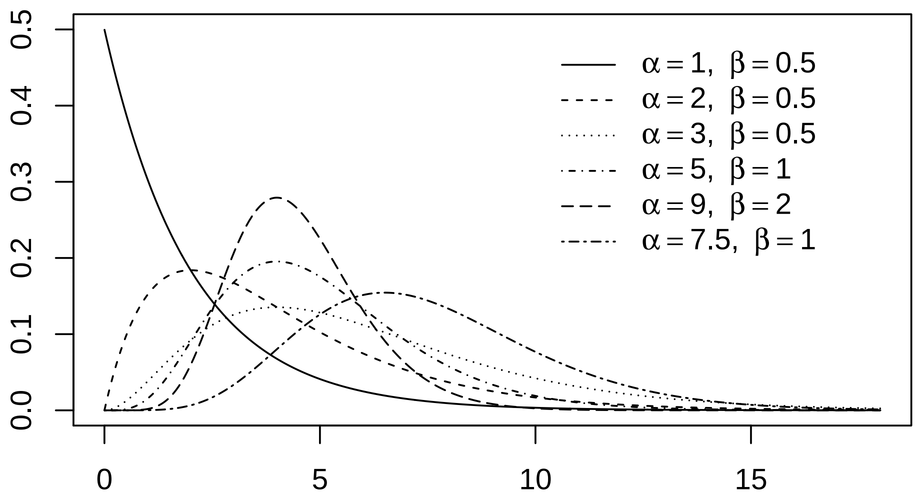


Figure 2.6: The probability density functions of Gamma distribution with different parameter values of α and β. The simulated random variable is defined on $[0, \infty)$ and the selected parameter combinations of Gamma distribution are $G_a(1, 0.5)$, $G_a(2, 0.5)$, $G_a(3, 0.5)$, $G_a(5, 1)$, $G_a(9, 2)$ and $G_a(7.5, 1)$.

Gamma distributions are extensively used in the literature of modeling prices (see, e.g., Hong & Shum, 2006; Pratt, Wise & Zeckhauser, 1979). In particular, Hong and Shum (2006) apply the gamma distribution to model search costs, including time, energy and money spent on researching products, or services, for purchasing. There are numerous cases of using the gamma distribution when modeling insurance losses (e.g., Alai et al., 2013; Furman & Landsman, 2005; Hürlimann, 2001).

Distributional properties pertaining to the Gamma distribution parameterized as (2.16) include:

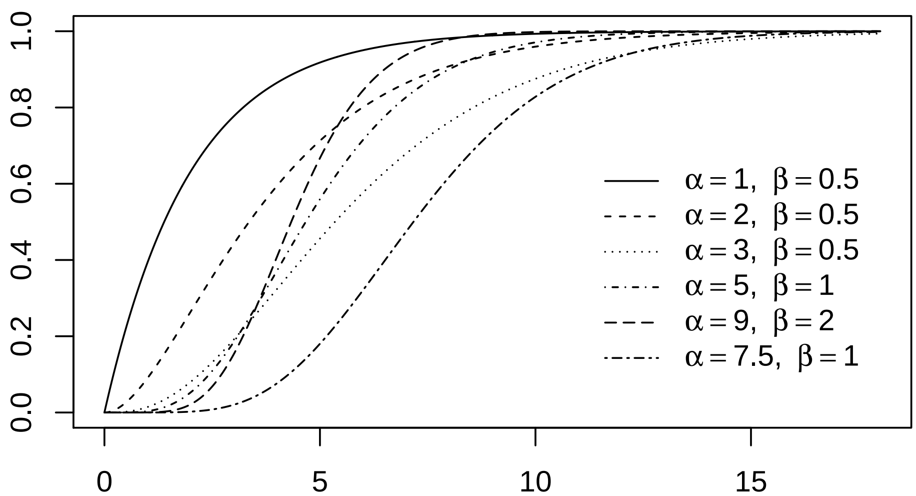


Figure 2.7: The cumulative distribution functions of Gamma distribution with different parameter values of α and β. The simulated random variable is defined on $[0, \infty)$ and the selected parameter combinations of Gamma distribution are $G_a(1, 0.5)$, $G_a(2, 0.5)$, $G_a(3, 0.5)$, $G_a(5, 1)$, $G_a(9, 2)$ and $G_a(7.5, 1)$.

1. The cumulative distribution function corresponding to $f_{\alpha,\beta}$ is

$$F_{\alpha,\beta}(x) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}, \quad x > 0,$$

   where $\gamma(\cdot, \cdot)$ is the *lower incomplete gamma function*.

2. The rth moments of the gamma distribution are given by

$$\mathbf{E}\left[X^r\right] = \frac{\Gamma(\alpha + r)}{\beta^r \Gamma(\alpha)}.$$

   In particular, the mean of $Ga(\alpha, \beta)$ is $\mathbf{E}[X] = \frac{\alpha}{\beta}$ and the variance is $\mathbf{Var}(X) = \frac{\alpha}{\beta^2}$.

3. The characteristic function of $Ga(\alpha, \beta)$ is

$$\phi_X(t) = \left(\frac{\beta}{\beta - it}\right)^{\alpha},$$

where $i = \sqrt{-1}$.

Similar to the Beta distributions, the distribution functions product of independent Gamma random variables cannot be expressed in terms of elementary functions as well. Hence we shall introduce the Meijer G-function defined as follows to help develop the representation of these distribution functions.

Definition 2.3.3 In general, the Meijer G-function is defined by

$$G^{m,n}_{p,q}\left(z\middle|\begin{matrix}a_1, \cdots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds, \quad z \neq 0$$

for $0 \leq m \leq q$ and $0 \leq n \leq p$ where $m, n, p, q \in \mathbb{N}$. Moreover, $a_k - b_j \neq 1, 2, 3, \ldots$ for $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The integral path L has three possible choices.

1. L runs from $-i\infty$ to $i\infty$ such that all poles of $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$ are located on the right of the path while all poles of $\Gamma(1 - a_k + s)$, $k = 1, 2, \ldots, n$ are located on the right of the path. The integral then converges on $\{z \in \mathbb{C} : |\arg z| < \delta\pi\}$, where

$$\delta = m + n - \frac{1}{2}(p + q)$$

and $\delta > 0$.

2. L is a loop beginning and ending at $+\infty$, encircling all poles of $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$ exactly once in the negative direction, but not encircling any of poles of $\Gamma(1 - a_k + s)$, $k = 1, 2, \ldots, n$. Then the integral converges for all z if $q > p \geq 0$; it also converges for $q = p > 0$ on $\{z \in \mathbb{C} : |z| > 1\}$.

3. L is a loop beginning and ending at $-\infty$, encircling all poles of $\Gamma(1-a_k+s)$, $k = 1, 2, \ldots, n$ exactly once in the positive direction, but not encircling any of poles of $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$. Then the integral converges for all z if $p > q \geq 0$; it also converges for $p = q > 0$ on $\{z \in \mathbb{C} : |z| > 1\}$.

Now the distribution functions as well as the Laplace transform of independent Gamma random variables could be expressed in terms of the Meijer G-functions as shown by Nadarajah (2011).

Proposition 2.3.5 *(Lemma 1, Nadarajah, 2011) Suppose X and Y are independent random variables following $Ga(\alpha_X, \beta_X)$ and $Ga(\alpha_Y, \beta_Y)$ respectively. Then the Lapace transform of XY are*

$$\mathbf{E}\left[e^{-sXY}\right] = \frac{1}{\Gamma(\alpha_X)\Gamma(\alpha_Y)} G_{m,1}^{1,m}\left(\frac{s}{\beta_X \beta_Y} \middle| \begin{matrix} 1-\alpha_X, \alpha_Y \\ 0 \end{matrix} \right)$$

for $s \in \{z \in \mathbb{C} : \Re(z) \geq 0\}$.

Proposition 2.3.6 *(Lemma 2, Nadarajah, 2011) Suppose X and Y are independent random variables following $Ga(\alpha_X, \beta_X)$ and $Ga(\alpha_Y, \beta_Y)$ respectively. Then the density function and the cumulative distribution function of XY are*

$$f_{XY}(z) = \frac{1}{\Gamma(\alpha_X)\Gamma(\alpha_Y) z} G_{0,2}^{2,0}\left(\beta_X \beta_Y z \middle| \begin{matrix} - \\ \alpha_X, \alpha_Y \end{matrix} \right)$$

and

$$F_{XY}(z) = \frac{1}{\Gamma(\alpha_X)\Gamma(\alpha_Y)} G_{1,3}^{2,1}\left(\beta_X \beta_Y z \middle| \begin{matrix} 1 \\ \alpha_X, \alpha_Y, 0 \end{matrix} \right)$$

respectively, for $z > 0$.

2.3.3 Geometric Brownian motion

When foreign investors take part in the bidding for certain asset, the foreign currency exchange rates matter. Particularly for the two-period decision problem, the decision-making procedure in the first period must take into account the projection of these exchagne rates for the second period . A simplest model for describing the behavior of the exchange rates is the geometric Brownian motion, which was used by Biger and Hull (1983) for valuing the currency options. As this thesis is not focusing on modelling the exchange rates, we adopt the assumption that exchange rates could be described by geometric Brownian motions. A formal definition of geometric Brownian motion is given below.

Definition 2.3.4 Suppose that $\{W_t\}_{t\geq 0}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ and that $\mu_B \in \mathbb{R}$ and $\sigma_B > 0$. Let

$$X_t = x \exp\left\{\left(\mu_B - \frac{\sigma_B^2}{2}\right) t + \sigma_B W_t\right\}, \quad t \in [0, \infty). \tag{2.17}$$

Then the stochastic process $\{X_t\}_{t\geq 0}$ is a geometric Brownian motion starting from x. $GBM(\mu_B, \sigma_B)$ is a geometric Brownian motion with drift μ_B and volatility σ_B.

Without loss of generality, we have graphed the simulated geometric Brownian motion with $x = 1$, and different drift parameters and volatility parameters of μ_B and σ_B, $GBM(1, 0.5)$ and $GBM(0.5, 1)$, in Figure 2.8.

For any fixed $t \geq 0$, the distributional properties of the random variable X_t are listed below:

1. X_t follows a log-normal distribution with density function

$$f_t(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi t}\sigma_B x} \exp\left\{-\dfrac{\left[\log(x) - \left(\mu_B - \sigma_B^2/2\right) t\right]^2}{2\sigma^2 t}\right\} & \text{if} \quad x \in (0, \infty), \\ 0 & \text{if} \quad x \leq 0. \end{cases}$$

2. The cumulative distribution function of X_t is given by

$$F_t(x) = \begin{cases} \Phi\left\{\dfrac{\log(x) - \left(\mu_B - \sigma_B^2/2\right) t}{\sigma_B \sqrt{t}}\right\} & \text{if} \quad x \in (0, \infty), \\ 0 & \text{if} \quad x \leq 0, \end{cases}$$

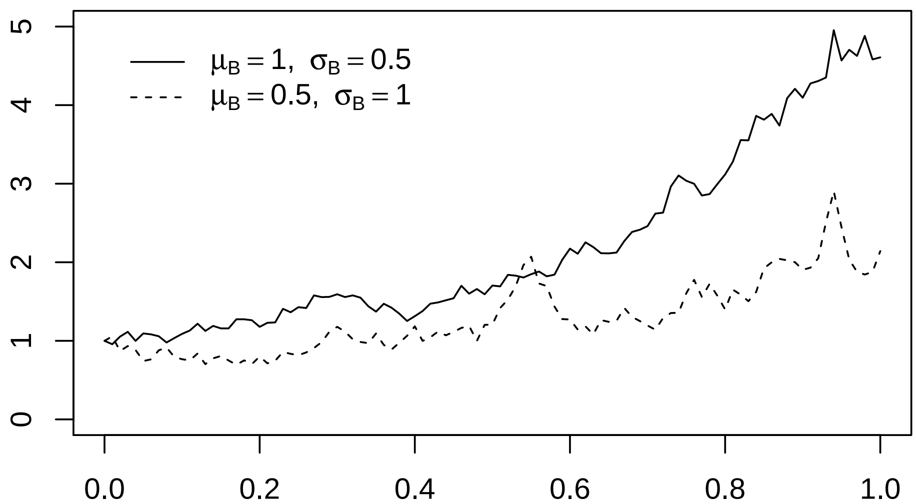


Figure 2.8: The simulated geometric Brownian motion with $x = 1$, and different drift parameters and volatility parameters of μ and σ. The simulated random variable is defined on $[0, 1]$ and the selected parameter combinations of geometric Brownian motion are $GBM(1, 0.5)$ and $GBM(0.5, 1)$.

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

3. The p-quantile function of X_t is given by

$$F_t^{-1}(p) = \exp\left\{\left(\mu_B - \sigma_B^2/2\right)t + \sigma_B\sqrt{t}\Phi^{-1}(p)\right\}, \quad p \in (0, 1),$$

where $\Phi^{-1}(\cdot)$ is the standard normal quantile function.

4. For $n \in \mathbb{N}$, the n-th moment

$$\mathbf{E}\left[X_t^n\right] = e^{n\mu_B t}.$$

In particular, we have

$$\mathbf{E}\left[X_t\right] = e^{\mu_B t}$$

and

$$\mathbf{Var}\,(X_t) = \left[\exp\left(\sigma_B^2 t\right) - 1\right]\exp(\mu_B t).$$

Besides, geometric Brownian motion satisfies centain stochastic differential equation.

Proposition 2.3.7 *The geometric Brownian motion* $\{X_t\}_{t\geq 0}$ *defined by* (2.17) *satisfies the stochastic differential equation*

$$dX_t = \mu_B X_t dt + \sigma_B X_t dW_t, \quad t \geq 0.$$

Chapter 3

Two-stage optimal decisions from the buyer's perspective

3.1 Motivation

In need of a laptop in the Fall of 2015, I was looking for a good on-line deal. Benefiting from the advantages of on-line shopping, he acquired considerable information on the Lenovo T540 laptop price. Two forthcoming time-periods were of immediate interest: the *Black Friday* promotion from November 27th to December 3rd, 2015, and the *Boxing Day* promotion from December 26th, 2015, to January 3rd, 2016.

Obviously, it was prudent to wait until the Black Friday promotion period, which revealed the discounted price of 1,431.01 Canadian dollars for the laptop, but it was not obvious at that moment whether he wanted to buy the laptop at that price or wait until the Boxing Day promotion. (The price of Lenovo T540 laptop during the Boxing Day promotion period turned out to be 1,461.60 Canadian dollars, which was not, of course, known during the Black Friday promotion period.) Can there be a strategy for making good decisions during the Black Friday promotion period?

The present chapter aims at answering this question by deriving an optimal strategy that minimizes the expected buying price. The idea for tackling this problem stems from the pioneering work of McDonnell and Abbott (2009) on the two-envelope paradox, with further far-reaching considerations by McDonnell et al. (2011). It also relies on some of the techniques put forward by Egozcue et al. (2013) who have extended the aforementioned works beyond the two-envelope paradox.

The rest of this chapter is organized as follows. In Section 3.2 we lay out the necessary mathematical background and derive two optimal strategies: one when there is no guessing of the second time-period price, and another one when such guessing takes place. In Sections 3.3 and 3.4 we discuss price modeling and parameter specifications of practical relevance. Section 3.5 contains proofs.

3.2 Main results

We start out by carefully describing the decision-making process, and also introduce the necessary notation. First, the prospective buyer contacts a sales representative during the first time-period. To somewhat simplify the problem, we assume that there are two kinds of sales representatives:

i) those, call them L, who tend to offer larger discounts and thus *lower* prices X_L;

ii) others, say H, who tend to offer smaller discounts and thus *higher* prices X_H.

Both X_L and X_H are random variables. We denote their joint cumulative distribution function (cdf) by $F_{X_L,X_H}(x,y)$ and assume that it is absolutely continuous, that is, has a density $f_{X_L,X_H}(x,y)$, which is a natural assumption in the current context.

Let Π_1 denote the random variable that takes the values L and H depending on which (kind of) salesperson takes the prospective buyer's call during the first time-period. Hence, when $\Pi_1 = L$, then the price is X_L, and when $\Pi_1 = H$, then X_H.

Once the prospective buyer learns the price during the first time-period (i.e., Black Friday), he has two options: to either accept the offer or reject it and then inevitably wait till the second time-period (i.e., Boxing Day). If the buyer thinks that the first time-period offer is good enough, he accepts it and the purchasing process ends. However, if the buyer rejects the offer, then he has to wait until the second time-period and then inevitably accept whatever offer is made to him at that time. He has to do so, because he needs a laptop and the regular price is less attractive than any of the discounted ones.

Let Δ_1 denote the random variable that represents the prospective buyer's decision during the first time-period: $\Delta_1 = A$ if the buyer accepts the first-period offer and $\Delta_1 = R$ if he rejects it. The aim of the present article is to offer an optimal strategy that minimizes the average $\mathbf{E}[X]$ of the buying price X, which could be either X_L or X_H depending the the buyer's decision during the first time-period.

We assume that the decision to accept or reject the price offered during the first time-period does not depend on who, L or H, makes the offer – it depends only on the price being offered. That is, we assume that the equation

$$\mathbf{P}[\Delta_1 = \delta \mid X_{\Pi_1} = x, X_{\Pi_2} = y, \Pi_1 = \pi] = \mathbf{P}[\Delta_1 = \delta \mid X_{\Pi_1} = x, X_{\Pi_2} = y] \tag{3.1}$$

holds for every decision $\delta \in \{A, R\}$ (i.e., accept or reject) and for every salesperson $\pi \in \{L, H\}$, where Π_2 ($\neq \Pi_1$) denotes the salesperson who makes the offer during the second time-period.

Theorem 3.2.1 *When the price of the first time-period is v and there is no attempt to guess the possible price to be offered during the second time-period, then the strategy that minimizes the expected buying price is to accept the offer when $R(v) \leq 0$ and to reject it when $R(v) > 0$, where the "no guessing strategy" function $R(v)$ is*

$$R(v) = \int (v - w)p(v, w)f_{X_L,X_H}(v, w)dw + \int (v - w)(1 - p(w, v))f_{X_L,X_H}(w, v)dw \tag{3.2}$$

with the notation

$$p(x, y) = \mathbf{P}(\Pi_1 = L \mid X_L = x, X_H = y), \tag{3.3}$$

which is the probability that the salesperson L makes an offer during the first time-period given that the prices offered by L and H are x and y, respectively.

It is natural to think of the joint density $f_{X_L,X_H}(v,w)$ as a continuous function with compact support $[x_{\min}, x_{\max}] \times [x_{\min}, x_{\max}]$, where $x_{\min}$ is the reservation price from the supply side (i.e., computer technology company) and $x_{\max}$ is the reservation price from the demand side (i.e., consumer). Hence, $f_{X_L,X_H}(w,v) = 0$ when v or w, or both, are outside the interval $(x_{\min}, x_{\max})$.

We also expect that under normal circumstances there should be a point $v_0 \in (x_{\min}, x_{\max})$ such that $R(v) < 0$ (accept the offer) for all $v \in (x_{\min}, v_0)$ and $R(v) > 0$ (reject the offer) for all $v \in (v_0, x_{\max})$, with $R(v_0) = 0$. We indeed see this pattern in our illustrative Figure 3.1, where and elsewhere when graphing in this chapter we set the reservation prices to be

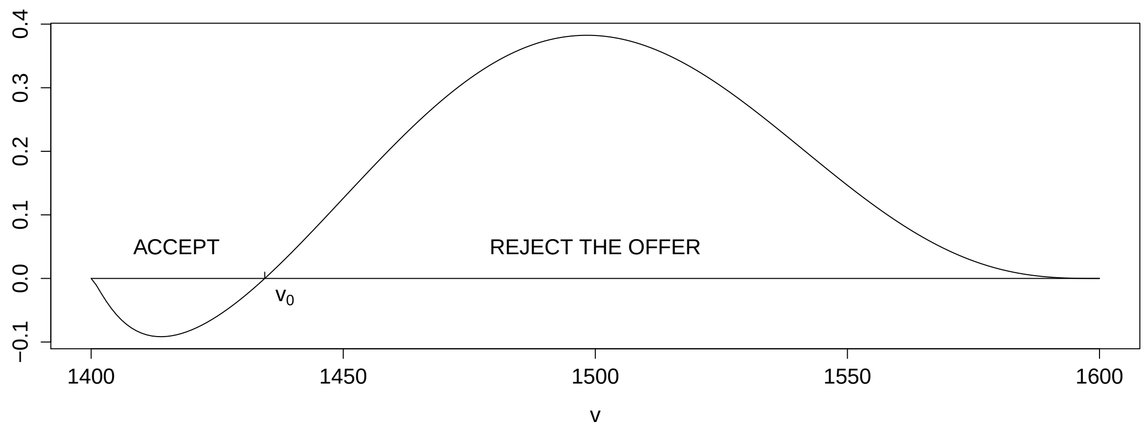


Figure 3.1: The no-guessing-strategy function $R(v)$. v_0 [$\approx$ 1,434.43] is the point where the function R(v) crosses the horizontal axis and $R(v_0) = 0$. The region where $R(v) < 0$ for all $v \in (x_{\min}, v_0)$ is the accept region (accept the offer) and the region where $R(v) > 0$ for all $v \in (v_0, x_{\max})$ is the reject region (reject the offer).

$x_{\min} = 1,400$ and $x_{\max} = 1,600$ Canadian dollars. For other specifications, including modelling of the functions $f_{X_L,X_H}(v,w)$ and $p(v,w)$, we refer to Sections 3.3 and 3.4.

We note that under the specifications, the point where the function $R(v)$ crosses the horizontal axis is $v_0 \approx 1,434.43$, which delineates the acceptance (to the left) and rejection (to the right) regions. It should also be noted that the inclusion of the point v_0 into the acceptance region is arbitrary: whenever $v \in (x_{\min}, x_{\max})$ is such that $R(v) = 0$, we could very well flip a coin to decide whether to accept the offer or reject it and wait until the next promotion period.

Theorem 3.2.2 *When the price of the first time-period is v and the guessed price to be offered during the second time-period is w, then the strategy that minimizes the expected buying price is to accept the offer when $R(v,w) \le 0$ and to reject it when $R(v,w) > 0$, where the "guessing strategy" surface is*

$$R(v,w) = (v-w)p(v,w)f_{X_L,X_H}(v,w) + (v-w)(1-p(w,v))f_{X_L,X_H}(w,v) \tag{3.4}$$

with $p(v,w)$ defined in equation (3.3).

Note that $R(v,w) = 0$ when $v = w$, which is natural. It is also natural to expect that $R(v,w) < 0$ (i.e., accept the first time-period price v) when $v < w$, and $R(v,w) > 0$ (i.e., reject the first time-period price v) when $v > w$. We indeed see this pattern in Figure 3.2 and Figure 3.3, where we have depicted the surface $R(v,w)$ and its contours.

We conclude this section with a brief note on the possible shapes of $p(v,w)$, with more details to be provided in Section 3.4 below. Namely, upon recalling that $p(v,w)$ is the probability that L makes an offer during the first time-period, given that the offers of L and H are x and y respectively, it is natural to model $p(v,w)$ as $F(v-w)$ with some cdf F such that $F(0) = 1/2$. In Section 3.4, for example, we shall use the beta cdf with the same shape parameters, in which case we shall have the equation $F(x) = 1 - F(-x)$ for all x and thus, in particular, the requirement $F(0) = 1/2$. The following corollary to Theorems 3.2.1 and 3.2.2 deals with this special case.

Corollary 3.2.3 *Let $p(v,w) = 1 - p(w,v)$ for all v and w. Then the guessing-strategy surface is*

$$R(v,w) = (v-w)p(v,w)\big(f_{X_L,X_H}(v,w) + f_{X_L,X_H}(w,v)\big),$$

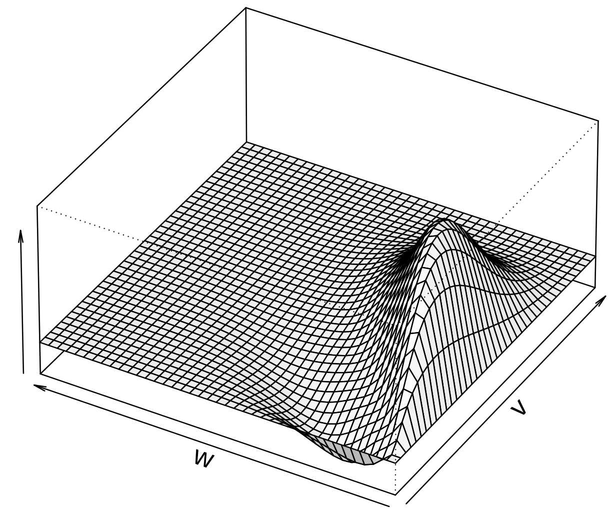


Figure 3.2: The guessing-strategy surface $R(v, w)$. This figure is a $3D$ - visualization of the function $R(v, w)$. The lower left part of the figure indicates the area where $R(v, w) < 0$ when $v < w$, i.e. accept the first time-period price v. The top right part of the figure indicates the area where $R(v, w) > 0$ when $v > w$, i.e., reject the first time-period price v. The area where $R(v, w) = 0$ is fuzzy and uncertain in this figure, and we refer to the Figure 3.3.

and the no-guessing-strategy function is

$$R(v) = \int R(v, w)dw.$$

The earlier drawn Figures 3.1, 3.2 and 3.3 are based on this corollary, with practically relevant modelling of $f_{X_L,X_H}(v, w)$ and $p(v, w)$ to be discussed in the next two sections.

3.3 Modelling $f_{X_L,X_H}(x, y)$

If the salespersons L and H were in total isolation, their offered prices would be outcomes of two independent random variables, which we denote by X_L^0 and X_H^0, both taking values in the interval $[x_{\min}, x_{\max}]$. It is natural to assume that, for example, X_L^0 and X_H^0 are beta distributed on

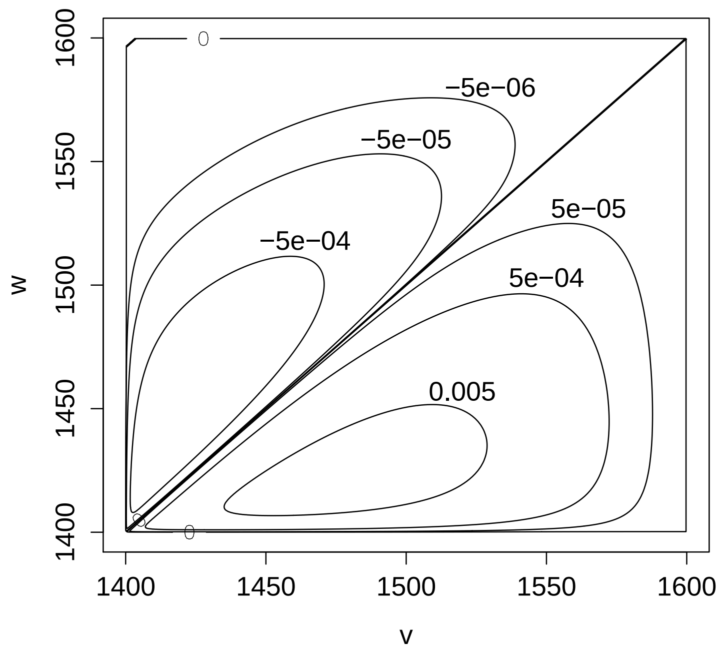


Figure 3.3: The contours of the guessing-strategy surface $R(v, w)$. This figure is the contour of the Figure 3.2. The diagonal here, from lower left to top right, is the line of $R(v, w) = 0$ when $v = w$. The top left area, where $R(v, w) < 0$ when $v < w$, is the accept region (accept the first time-period price v) and the lower right area, where $R(v, w) > 0$ when $v > w$, is the reject region (reject the first time-period price v).

$[x_{\min}, x_{\max}]$ with positive shape parameters (α_L, β_L) and (α_H, β_H), respectively, that is,

$$f_{X_L^0}(x) = \frac{(x - x_{\min})^{\alpha_L - 1}(x_{\max} - x)^{\beta_L - 1}}{B(\alpha_L, \beta_L)\rho^{\alpha_L + \beta_L - 1}}, \quad x_{\min} < x < x_{\max},$$

and

$$f_{X_H^0}(x) = \frac{(x - x_{\min})^{\alpha_H - 1}(x_{\max} - x)^{\beta_H - 1}}{B(\alpha_H, \beta_H)\rho^{\alpha_H + \beta_H - 1}}, \quad x_{\min} < x < x_{\max},$$

where

$$\rho = x_{\max} - x_{\min}[> 0]$$

is the range of possible prices. Given the earlier noted numerical values of $x_{\min}$ and $x_{\max}$, we have $\rho = 200$. When graphing, we set the parameter values to $(\alpha_L, \beta_L) = (2.5, 4.5)$ and

$(\alpha_H, \beta_H) = (4.5, 1.5)$ throughout this chapter.

The (observable) prices offered by L and H are, however, not X_L^0 and X_H^0 but those that have been influenced by, e.g., the company's marketing team or management. This naturally leads us to the background model, which we choose to be multiplicative (cf., e.g., Franke et al., 2006, 2011; Asimit et al., 2016; and references therein). Namely, suppose that $Y \in [x_{\min}, x_{\max}]$ is the (random) price that the company's marketing team, or management, would think appropriate, and which therefore influences the actual decisions of L and H. The multiplicative background model would suggest that the observable prices X_L and X_H would be of the form $X_L Y$ and $X_H Y$, but when defined in this way they are outside the natural price-range $[x_{\min}, x_{\max}]$. To rectify the situation, we first standardize the prices X_L^0, X_H^0, and Y using the equations

$$Z_L^0 = (X_L^0 - x_{\min})/\rho,$$

$$Z_H^0 = (X_H^0 - x_{\min})/\rho,$$

and

$$Z = (Y - x_{\min})/\rho,$$

and then model the observable prices as

$$X_L = Z_L^0 Z\rho + x_{\min}$$

and

$$X_H = Z_H^0 Z\rho + x_{\min}.$$

Throughout the rest of this section, we assume that X_L^0, X_H^0 and Y are independent and thus, in turn, their standardized versions Z_L^0, Z_H^0 and Z are independent as well. We shall soon find this assumption convenient; in fact, it is a natural assumption.

Let, for example, Y follow the beta distribution on the interval $[x_{\min}, x_{\max}]$ with some (positive) shape parameters (α_0,β_0). Then the pdf $f_Z(t)$ of Z is equal to

$$f_{\alpha_0,\beta_0}(t) := \frac{t^{\alpha_0-1}(1-t)^{\beta_0-1}}{B(\alpha_0,\beta_0)}, \quad 0<t<1.$$

The marginal pdf's of X_L and X_H are

$$f_{X_L}(x) = \frac{1}{\rho}\int_{t_x}^{1} f_{\alpha_L,\beta_L}\Big(\frac{x-x_{\min}}{t\rho}\Big)\frac{1}{t} f_{\alpha_0,\beta_0}(t)dt \tag{3.5}$$

and

$$f_{X_H}(x) = \frac{1}{\rho}\int_{t_x}^{1} f_{\alpha_H,\beta_H}\Big(\frac{x-x_{\min}}{t\rho}\Big)\frac{1}{t} f_{\alpha_0,\beta_0}(t)dt, \tag{3.6}$$

respectively, where $t_x = (x-x_{\min})/\rho$. We have depicted the pdf's in Figure 3.4 and 3.5

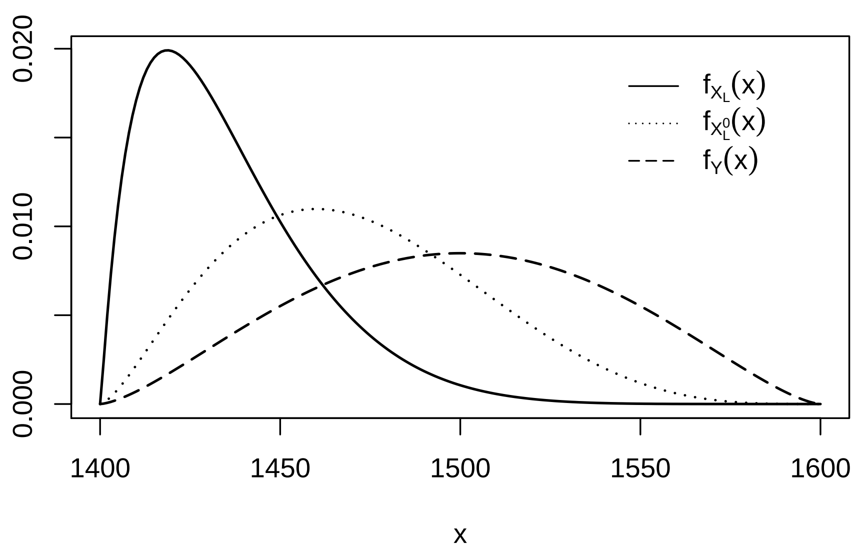


Figure 3.4: The influence of Y (dashed pdf in both panels) on X_L^0.

using $(\alpha_0,\beta_0) = (2.5, 2.5)$, and the earlier noted parameter choices for (α_L,β_L) and (α_H,β_H). Similarly to equations (3.5) and (3.6), we derive the joint pdf

$$f_{X_L,X_H}(x,y) = \frac{1}{\rho^2}\int_{t_{x,y}}^{1} f_{\alpha_L,\beta_L}\Big(\frac{x-x_{\min}}{t\rho}\Big) f_{\alpha_H,\beta_H}\Big(\frac{y-x_{\min}}{t\rho}\Big)\frac{1}{t^2} f_Z(t)dt,$$

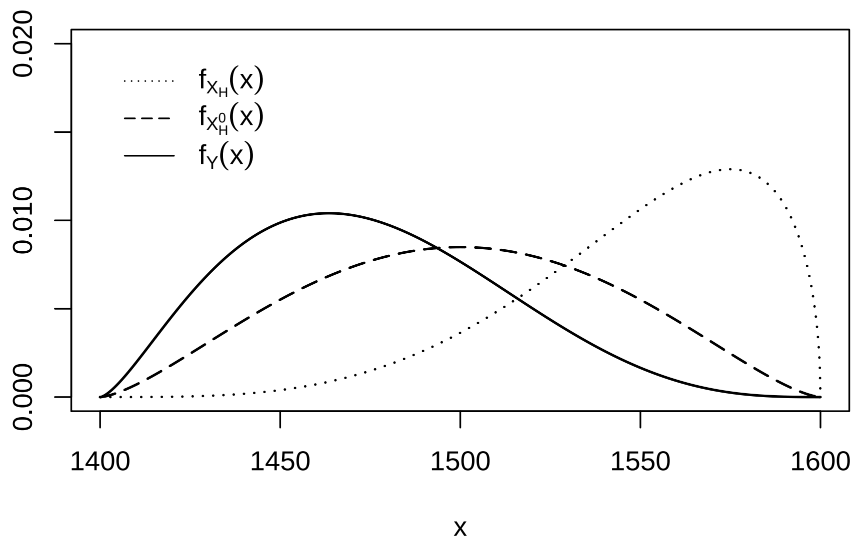


Figure 3.5: The influence of Y (dashed pdf in both panels) on X_H^0.

where $t_{x,y} = (\max\{x, y\} - x_{\min})/\rho$. We have depicted this pdf in Figures 3.6 and 3.7.

3.4 Modelling $p(x, y)$

To begin with, we view our problem within the context of two-period (also known as two-stage) economy. Indeed, it is reasonable to assume that many consumers who buy laptops during the first time-period have stronger buying intention than most of the consumers who buy laptops during the second time-period. That is, the demand curves during the two periods have different demand elasticities of price: during the first time-period, the demand elasticity of price is smaller than that during the second time-period. Considering this difference, the buyers during the two periods can be viewed as two separate markets with different demand curves and different demand elasticities, and monopolistic firms would tend to enforce the division of selling prices during the two periods.

This leads us to the topic of third-degree price discrimination (e.g., Aguirre et al., 2010; Schwartz, 1990), which in the monopolistic competitive market means that the same provider

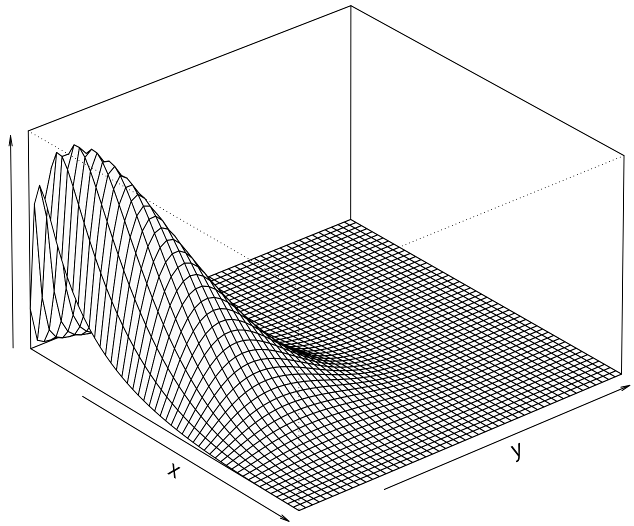


Figure 3.6: The joint pdf of X_L and X_H. This figure is a $3-D$ visualization of the joint distribution $f_{X_L,X_H}(x, y)$. The arrows in the figure indicate the larger trend of the values of x, y and $f_{X_L,X_H}(x, y)$.

would charge different prices for similar goods or services in different consumer groups having different demand curves and demand elasticities, such as those who buy laptops during the first time-period and those who buy during the second period. In general, product differentiation is one of the key factors why firms have some degree of control over the prices, and the more successful a firm is at differentiating its products from other firms selling similar products, the more monopoly power the firm has. Product differences arise due to quality, functional features, design, and so on, and so there is imperfect substitution between products even when they are in the same category (e.g., Head & Ries, 2001; Krugman, 1980).

Hence, in the case of our problem concerning laptops, there is arguably a tendency to offer higher prices during the first time-period because the demand curve in the first period is a relatively inelastic demand curve, due to stronger buying intention. Hence, it is natural that firms would take advantage of this buying intention and offer higher prices during the first

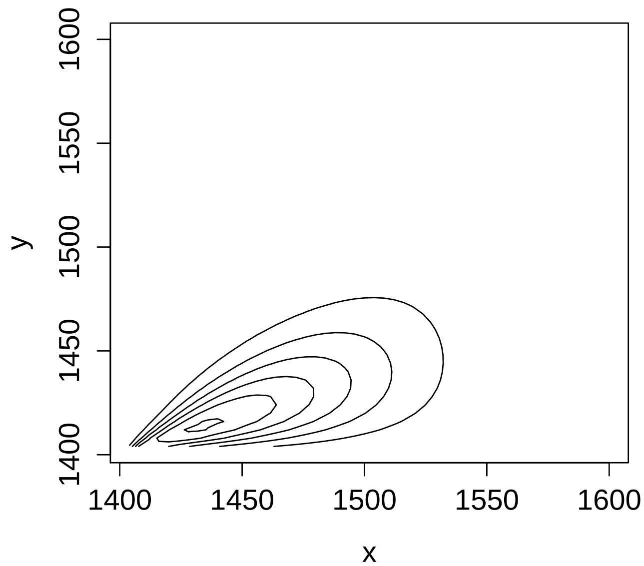


Figure 3.7: The contours of the joint pdf of X_L and X_H. This figure is the contour of the Figure 3.6.

time-period. Consequently, it would seem that the higher the price is offered by L, the higher the probability that the consumer will receive an offer from L during the first time-period. The higher the laptop price is offered by H, the lower the probability that an offer will come from L during the first time-period. When L and H offer laptops at the same or similar price, then the probability of getting an offer from L would be more or less the same as the probability of getting an offer from H. In view of these arguments, the following properties seem natural:

1) $p(x, y)$ is non-decreasing in x, for every fixed $y \in (x_{\min}, x_{\max})$;

2) $p(x, y)$ is non-increasing in y, for every fixed $x \in (x_{\min}, x_{\max})$;

3) $p(x, y) = 1/2$ whenever $x = y$.

We next suggest an example of $p(x, y)$ by setting it to be $F(x - y)$, where F can be any cdf such that $F(0) = 1/2$. For example, let F be the beta cdf on the interval $[-\rho, \rho]$ with

$\rho = x_{\max} - x_{\min}$ and equal shape parameters, say $\gamma > 0$. That is,

$$p(x, y) = \frac{1}{B(\gamma, \gamma)(2\rho)^{2\gamma-1}} \int_{-\rho}^{x-y} (\rho + t)^{\gamma-1}(\rho - t)^{\gamma-1} dt,$$

depicted in Figure 3.8 and 3.9 with the parameter $\gamma = 10$, which we always use when graphing.

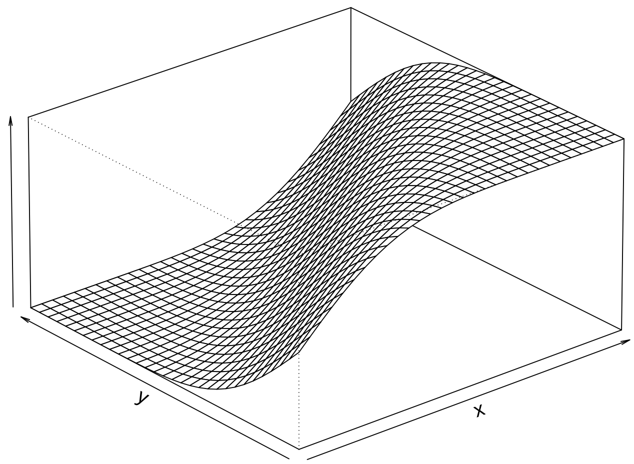


Figure 3.8: The probability surface $p(x, y)$. This figure is a $3 - D$ visualization of the joint distribution $p(x, y)$. The arrows in the figure indicate the larger trend of the values of x, y and p(x,y).

3.5 Proofs of Theorems 3.2.1 and 3.2.2

Our goal is to derive a strategy that leads to the minimal expected value $\mathbf{E}[X]$ of the buying price X, which could be either X_L or X_H depending on the outcomes of the random variables $\Pi_1 \in \{L, H\}$ and $\Delta_1 \in \{A, R\}$. We start with the equation

$$\mathbf{E}[X] = \iint \mathbf{E}[X \mid X_L = x, X_H = y] dF_{X_L, X_H}(x, y) \tag{3.7}$$

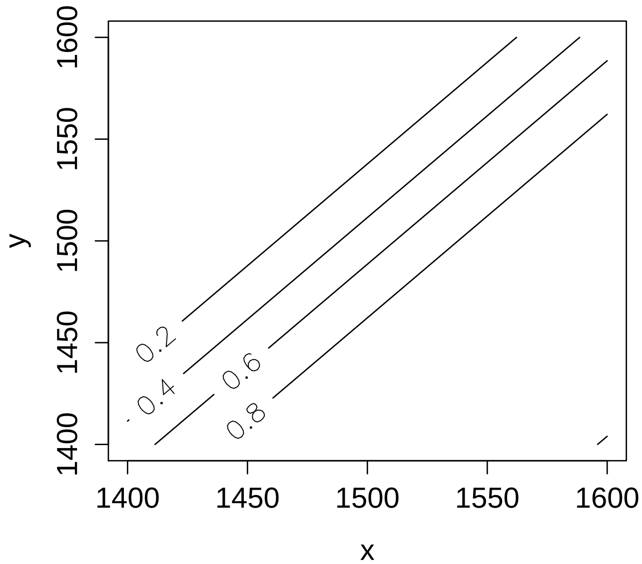


Figure 3.9: The contours of the probability surface $p(x, y)$. This figure is the contour of the Figure 3.8.

and then work with the conditional expectation inside the integral.

Given $X_L = x$ and $X_H = y$, the random variable X can take only the values x or y. Consequently, we have the equation

$$\mathbf{E}[X \mid X_L = x, X_H = y] = x\mathbf{P}[X = x \mid X_L = x, X_H = y] + y\mathbf{P}[X = y \mid X_L = x, X_H = y]. \quad (3.8)$$

We next calculate the two probabilities on the right-hand of equation (3.8) based on which of the two salespersons, L or H, is making offers during the first time-period, and also on the consumer behavior during this period, who can either accept or reject the first-come offer.

To begin with, we employ the random variable $\Pi_1 \in \{L, H\}$ and have

$$\begin{aligned}
&\mathbf{P}[X = x \mid X_L = x, X_H = y] \\
&= \mathbf{P}[X = x \mid X_L = x, X_H = y, \Pi_1 = L]\mathbf{P}[\Pi_1 = L \mid X_L = x, X_H = y] \\
&\quad + \mathbf{P}[X = x \mid X_L = x, X_H = y, \Pi_1 = H]\mathbf{P}[\Pi_1 = H \mid X_L = x, X_H = y]. \quad (3.9)
\end{aligned}$$

We next tackle the four probabilities on the right-hand side of equation (3.9), starting with the first probability.

Using the random variable $\Delta_1 \in \{A, R\}$, we have the equation

$$\begin{aligned}
&\mathbf{P}[X = x \mid X_L = x, X_H = y, \Pi_1 = L] \\
&= \mathbf{P}[X = x \mid X_L = x, X_H = y, \Pi_1 = L, \Delta_1 = A]\mathbf{P}[\Delta_1 = A \mid X_L = x, X_H = y, \Pi_1 = L] \\
&\quad + \mathbf{P}[X = x \mid X_L = x, X_H = y, \Pi_1 = L, \Delta_1 = R]\mathbf{P}[\Delta_1 = R \mid X_L = x, X_H = y, \Pi_1 = L]. \quad (3.10)
\end{aligned}$$

The first probability on the right-hand side of equation (3.10) is equal to 1, and the third probability is equal to 0. Hence, equation (3.10) simplifies to

$$\mathbf{P}[X = x \mid X_L = x, X_H = y, \Pi_1 = L] = \mathbf{P}[\Delta_1 = A \mid X_L = x, X_H = y, \Pi_1 = L]. \quad (3.11)$$

Similarly, we obtain the expression

$$\mathbf{P}[X = x \mid X_L = x, X_H = y, \Pi_1 = H] = \mathbf{P}[\Delta_1 = R \mid X_L = x, X_H = y, \Pi_1 = H] \quad (3.12)$$

for the third probability on the right-hand side of equation (3.9). Using equations (3.11) and (3.12) on the right-hand of equation (3.9), we have

$$\begin{aligned}
&\mathbf{P}[X = x \mid X_L = x, X_H = y] \\
&= \mathbf{P}[\Delta_1 = A \mid X_L = x, X_H = y, \Pi_1 = L]\mathbf{P}[\Pi_1 = L \mid X_L = x, X_H = y] \\
&\quad + \mathbf{P}[\Delta_1 = R \mid X_L = x, X_H = y, \Pi_1 = H]\mathbf{P}[\Pi_1 = H \mid X_L = x, X_H = y] \\
&= \mathbf{P}[\Delta_1 = A \mid X_{\Pi_1} = x, X_{\Pi_2} = y, \Pi_1 = L]\mathbf{P}[\Pi_1 = L \mid X_L = x, X_H = y] \\
&\quad + \mathbf{P}[\Delta_1 = R \mid X_{\Pi_2} = x, X_{\Pi_1} = y, \Pi_1 = H]\mathbf{P}[\Pi_1 = H \mid X_L = x, X_H = y], \quad (3.13)
\end{aligned}$$

where Π_2 ($\neq \Pi_1$) is the salesperson who offers prices during the second time-period. We now recall assumption (3.1) that tells us that the decision to accept or reject the first-come offer does not depend on who makes the offer – the decision depends only on the size of the offer. Hence, we have the equations

$$\mathbf{P}[\Delta_1 = A \mid X_{\Pi_1} = x, X_{\Pi_2} = y, \Pi_1 = L] = \mathbf{P}[\Delta_1 = A \mid X_{\Pi_1} = x, X_{\Pi_2} = y] \tag{3.14}$$

and

$$\mathbf{P}[\Delta_1 = R \mid X_{\Pi_2} = x, X_{\Pi_1} = y, \Pi_1 = H] = 1 - \mathbf{P}[\Delta_1 = A \mid X_{\Pi_1} = y, X_{\Pi_2} = x]. \tag{3.15}$$

Given the description of our problem, we might naturally think that the decision variable Δ_1 is independent of the hypothetical/speculative future value of X_{Π_2}, and thus the right-hand sides of equations (3.14) and (3.15) simplify by leaving out the second conditions associated with X_{Π_2}. We shall indeed consider this situation later, but at the moment we admit the possibility (cf. Theorem 3.2.2) that some clues to the possible price offerings during the second time-period might be available to the consumer.

With the notation

$$\mathcal{S}_w(v) = \mathbf{P}[\Delta_1 = A \mid X_{\Pi_1} = v, X_{\Pi_2} = w],$$

the right-hand side of equation (3.14) is equal to $\mathcal{S}_y(x)$ and the right-hand side of equation (3.15) is equal to

$$1 - \mathcal{S}_x(y).$$

Consequently, equation (3.13) turns into the following one

$$\begin{aligned}\mathbf{P}[X = x \mid X_L = x, X_H = y]& \\ &= \mathcal{S}_y(x)\mathbf{P}[\Pi_1 = L \mid X_L = x, X_H = y] + (1 - \mathcal{S}_x(y))\mathbf{P}[\Pi_1 = H \mid X_L = x, X_H = y].\end{aligned} \tag{3.16}$$

This is a desired expression for the first probability on the right-hand side of equation (3.8). As to the second probability, analogous considerations lead to the equations

$$\begin{aligned}
&\mathbf{P}[X = y \mid X_L = x, X_H = y]\\
&= \mathbf{P}[\Delta_1 = A \mid X_L = x, X_H = y, \Pi_1 = H]\mathbf{P}[\Pi_1 = H \mid X_L = x, X_H = y]\\
&\quad + \mathbf{P}[\Delta_1 = R \mid X_L = x, X_H = y, \Pi_1 = L]\mathbf{P}[\Pi_1 = L \mid X_L = x, X_H = y]\\
&= \mathcal{S}_x(y)\mathbf{P}[\Pi_1 = H \mid X_L = x, X_H = y] + (1 - \mathcal{S}_y(x))\mathbf{P}[\Pi_1 = L \mid X_L = x, X_H = y]. \qquad (3.17)
\end{aligned}$$

Applying equations (3.16) and (3.17) on the right-hand side of equation (3.8), we have

$$\begin{aligned}
&\mathbf{E}[X \mid X_L = x, X_H = y]\\
&= x\Big(\mathcal{S}_y(x)\mathbf{P}[\Pi_1 = L \mid X_L = x, X_H = y] + (1 - \mathcal{S}_x(y))\mathbf{P}[\Pi_1 = H \mid X_L = x, X_H = y]\Big)\\
&\quad + y\Big(\mathcal{S}_x(y)\mathbf{P}[\Pi_1 = H \mid X_L = x, X_H = y] + (1 - \mathcal{S}_y(x))\mathbf{P}[\Pi_1 = L \mid X_L = x, X_H = y]\Big)
\end{aligned}$$

which, with $p(x, y)$ defined in equation (3.3), becomes

$$\begin{aligned}
\mathbf{E}[X \mid X_L = x, X_H = y] = x&\Big(\mathcal{S}_y(x)p(x, y) + (1 - \mathcal{S}_x(y))(1 - p(x, y))\Big)\\
&+ y\Big(\mathcal{S}_x(y)(1 - p(x, y)) + (1 - \mathcal{S}_y(x))p(x, y)\Big). \qquad (3.18)
\end{aligned}$$

We next rearrange the terms on the right-hand side of equation (3.18) by separating the strategy-free and strategy-dependent terms:

$$\begin{aligned}
\mathbf{E}[X \mid X_L = x, X_H = y] &= x(1 - p(x, y)) + yp(x, y)\\
&+ \mathcal{S}_y(x)\Big(xp(x, y) - yp(x, y)\Big) + \mathcal{S}_x(y)\Big(y(1 - p(x, y)) - x(1 - p(x, y))\Big). \qquad (3.19)
\end{aligned}$$

Combining equations (3.19) and (3.7), we obtain the following decomposition

$$\mathbf{E}[X] = \mu_0 + \mu_1(\mathcal{S}), \qquad (3.20)$$

where the strategy-free term is

$$\begin{aligned}\mu_0 &= \iint \Big(x(1-p(x,y)) + yp(x,y)\Big) dF_{X_L,X_H}(x,y)\\ &= \mathbf{E}[X_L(1-p(X_L,X_H))] + \mathbf{E}[X_H p(X_L,X_H)]\end{aligned}$$

and the strategy-dependent term is

$$\begin{aligned}\mu_1(\mathcal{S}) = \iint \mathcal{S}_y(x)\Big(xp(x,y) - yp(x,y)\Big) dF_{X_L,X_H}(x,y)&\\ + \iint \mathcal{S}_x(y)\Big(y(1-p(x,y)) - x(1-p(x,y))\Big) dF_{X_L,X_H}(x,y).&\end{aligned}$$

Rewriting the above equation in terms of the joint density $f_{X_L,X_H}(x,y)$, and also slightly changing some notation to avoid potential confusion, we arrive at the equation

$$\mu_1(\mathcal{S}) = \int\int \mathcal{S}_w(v) R(v,w) dv dw, \tag{3.21}$$

where $R(v,w)$ is defined by equation (3.4). Since $\mathcal{S}_w(v)$ is a probability and can thus take values only in the unit interval $[0,1]$, integral (3.21) achieves its minimal value when

$$\mathcal{S}_w(v) = \begin{cases} 1 & \text{if} \quad R(v,w) \le 0, \\ 0 & \text{if} \quad R(v,w) > 0. \end{cases}$$

This completes the proof of Theorem 3.2.2.

We next deal with the case (cf. Theorem 3.2.1) when the decision random variable Δ_1 is independent of X_{Π_2} and thus $\mathcal{S}_w(v)$ does not depend on the price w offered during the second time-period. Hence, instead of $\mathcal{S}_w(v)$, we now deal with the strategy function

$$\mathcal{S}(v) = \mathbf{P}[\Delta_1 = A \mid X_{\Pi_1} = v].$$

Consequently, the above defined $\mu_1(\mathcal{S})$ reduces to the integral

$$\mu_1(\mathcal{S}) = \int \mathcal{S}(v)R(v)dv, \tag{3.22}$$

where $R(v)$ is defined by equation (3.2). Integral (3.22) achieves its minimal value when

$$\mathcal{S}(v) = \begin{cases} 1 & \text{if} \quad R(v) \le 0, \\ 0 & \text{if} \quad R(v) > 0. \end{cases}$$

This completes the proof of Theorem 3.2.1.

Chapter 4

Optimal two-stage pricing strategies from the seller's perspective

4.1 Introduction

Commodity pricing has been a prominent topic in the literature, with various models and strategies suggested and explored. In this chapter, motivated by a problem described next, we put forward and investigate (both theoretically and numerically) a general model for pricing within the two-period framework that naturally arises in the context of the motivating problem.

4.1.1 Motivating problem

In Punta del Este, a resort town in Uruguay, real-estate property is in demand by both domestic and foreign buyers. As a recent example, the frequency distribution of buyers for certain high-rise buildings was approximately as follows: 10% Uruguayans, 75% Argentineans, 10% Brazilians, and the remaining 5% from the rest of the world (Chile, U.S.A., and so on). A few immediate observations follow. First, the ratio of domestic and foreign buyers varies depending on a number of factors, including economic, financial, and political. Second, it has been observed that the average foreign buyer is wealthier than the average domestic one, and thus tends to exhibit higher bidding prices. Furthermore, given the diversity of buyers, the prices are usually in the US dollars (USD), but some of the building costs such as salaries of workers are in the Uruguayan pesos (UYU).

To properly understand our problem, we need to describe the property development and selling processes. Namely, contracted by an investor, a construction company starts building, say, a residential tower. There are several stages of selling residential units: before, during, and after the actual construction of the tower. Different pricing strategies are used at every stage. It is frequently the case that, at least initially, the investor wishes to sell the units en masse and thus hires a real-estate agent for several months. If the sale is not successful during this initial stage, then the units are put on sale individually, with no particular time horizon set in advance, and at a possibly different price, which could be higher or lower than the original price.

The goal that we set out in this chapter is to derive, under various scenarios of practical relevance, optimal strategies for setting first- and second-stage prices, as well as to propose the optimal timing for accomplishing these tasks, in order to maximize the overall seller's expected revenue. In the next subsection, we give a brief appraisal of what we have accomplished in the current chapter, with a related though brief literature review given in the following subsection.

4.1.2 Results and findings – an appraisal

First, in this chapter we put forward a highly encompassing, yet tractable, model and explore optimal pricing strategies from the seller's perspective when buyer's real-estate valuations and decisions in the two stages are uncertain: they can be independent or dependent, identically distributed, or stochastically dominate each other, be influenced by exogenous factors at various degrees, and so on. In particular, we shall see from our considerations and examples in the next section that the simultaneous pricing strategies yield higher expected revenues than those under the sequential pricing strategy.

Second, we study the case when real estate costs are possibly denominated in different currencies, as is the case in our motivating problem and, in general, is an important and very common factor in developing countries where large fractions of building costs are denominated in foreign currencies. Hence, currency exchange-rate movements become important in that they influence optimal pricing determination.

Third, our model provides conditions under which second-stage prices could be higher or lower than the first-stage prices. This might, initially, be surprising because it is a common intuitive assumption that if a property is not sold during the first stage, then the property price should be reduced before commencing the second stage. As we shall see from our following considerations, however, the relationship between the two stage prices is much more complex: higher holding costs, currency exchange movements, or some type of dominance between the first- and second-stage price distributions, could very much influence the determination of the second-stage price, thus possibly making it larger than that of the first stage, assuming of course that the property was not sold during the first stage.

Finally, our general model accommodates sellers with different shapes of their utility functions, such as those arising in Behavioral Economics (see, e.g., Dhami, 2016). In general, while working on this project, we were considerably influenced by, and benefited from, research contributions by many authors, and the following brief literature snapshot highlights some of those that we have found particularly related to the present chapter.

4.1.3 Related literature

House pricing from the seller's and buyer's perspectives has been studied by many authors. For instance, Quan and Quigley (1991), and Biswas and McHardy (2007) adopt the seller's viewpoint in their research. Furthermore, Stigler (1962), Rothschild (1974), Gastwirth (1976), Quan and Quigley (1991), Bruss (2003), and Egozcue et al. (2013) explore the problem from the buyer's perspective. Pricing under different seller's risk attitudes has been studied in the real estate literature as well. For instance, seller's risk neutral behavior has been researched by Arnold (1992, 1999), and Deng et al. (2012). Biswas and McHardy (2007) analyze optimal pricing for risk averse sellers. In addition, Genesove and Mayer (2001), Anenberg (2011), and

Bokhari and Geltner (2011) study house price determination for sellers whose risk behavior follows the teachings of Prospect Theory (Kahneman and Tversky, 1979).

Bruss (1998; 2003), Egozcue et al. (2013), Egozcue and Fuentes García (2015), and Wu and Zitikis (2017) apply a two-period model to determine optimal commodity (e.g., real estate, computer, etc.) prices that maximize the expected revenue, or minimize the expected loss. Some of the aforementioned works have been influenced by the two-envelope problem, and in particular by the viewpoint put forward by McDonnell and Abbott (2009), and McDonnell et al. (2011). Furthermore, Titman (1985) considers a two-period model to analyze the optimal land prices when the condominium unit prices are uncertain. We also refer to Lazear (1986), Nocke and Peitz (2007), Heidhues and Koszegi (2014), and reference therein, for additional two-period pricing models for real estate.

4.2 Sequential vs simultaneous price setting

In this section we discuss scenarios that clarify various aspects of the problem at hand. In particular, we shall see the difference between setting the two prices sequentially and simultaneously. We shall also see how the two prices are influenced by considerations such as seeking certain gross or net profits, taking into account possibly different treatments of domestic and foreign buyers, and so on.

We work with a discrete-time two-period economy: $t = 0$ and $t = 1$. Let X_0 and X_1 denote the amounts (i.e., bidding prices) that the buyer is willing to pay for the property during the initial (i.e., $t = 0$) and subsequent (i.e., $t = 1$) selling stages, respectively. Both X_0 and X_1 are random variables from the seller's perspective, and thus we also view them in this way. For the seller, the task is to set an appropriate price x_0 for the initial selling stage, and also an appropriate price x_1 (which is usually different from x_0) for the following selling stage.

It is natural to think that the seller would tend to first set x_0 that would result in a desired outcome such as the maximal expected profit during the initial selling stage, and then, if the sale fails, the seller would set x_1 that would maximize the expected profit during the following selling stage. As we shall illustrate below, the two prices set in this sequential manner may not maximize the expected overall profit, and thus a sensible strategy for the seller who is not in a rush would be to set both x_0 and x_1 before commencing the initial selling stage. The above caveat 'who is not in a rush' is important because rushed decisions usually give rise to very different forces at play, such as willingness to set the price x_0 low enough to ensure a very high probability of selling the property during the initial selling stage. There are of course many other scenarios of practical interest, but in this chapter we concentrate on maximizing the expected (gross or net) profit.

The rest of the section consists of two subsections: the first one contains preliminary facts such as sequential and simultaneous pricing, and the second subsection discusses four scenarios that clarify (and justify) the complexity of our general model that we start developing in Section 4.3.

4.2.1 Preliminaries

Sequential price setting

Suppose that the seller decides to set the prices x_0 and x_1 sequentially: x_0 before commencing the initial selling stage and x_1 just before the subsequent selling stage. In this case, the maximal expected seller's gross profit during the initial selling stage is the maximal value of the function

$$R_0(x_0) = \mathbf{P}[X_0 \geq x_0]x_0, \tag{4.1}$$

which is achieved at the price

$$x_{0,\max} = \arg\max_{x_0} R_0(x_0). \tag{4.2}$$

Given the sequential manner of setting the prices, the maximal expected seller's gross profit during the second selling stage is the maximal value of the function

$$R_1(x_1) = \mathbf{P}[X_0 < x_{0,\max}, X_1 \geq x_1]x_1, \tag{4.3}$$

which is achieved at the price

$$x_{1,\max} = \arg\max_{x_1} R_1(x_1). \tag{4.4}$$

Simultaneous price setting

The seller may decide to set the two prices x_0 and x_1 simultaneously, before commencing the initial selling stage. In this case, the two expected-profit maximizing prices are

$$(x_0^{\max}, x_1^{\max}) = \arg\max_{x_0,x_1} R(x_0, x_1), \tag{4.5}$$

where

$$R(x_0, x_1) = \mathbf{P}[X_0 \geq x_0]x_0 + \mathbf{P}[X_0 < x_0, X_1 \geq x_1]x_1. \tag{4.6}$$

Since $R_0(x_{0,\max}) + R_1(x_{1,\max})$ is equal to $R(x_{0,\max}, x_{1,\max})$, which cannot exceed $R(x_0^{\max}, x_1^{\max})$ by the very definition of $(x_0^{\max}, x_1^{\max})$, the seller cannot be worse off by simultaneously setting the prices before commencing the initial selling stage.

Note 4.2.1 The simultaneous setting of prices can be viewed as a strategic decision, whereas setting the prices sequentially just before commencing the respective selling stages are tactical choices, which in view of the above arguments cannot outperform the strategic (i.e., simultaneous) one. Deciding on which of these alternatives, and when to make them, has been a prominent topic in the literature, particularly in enterprise risk management (e.g., Fraser & Simkins, 2010; Louisot & Ketcham, 2014; Segal, 2011).

Gamma distributed bidding prices

To illustrate the above arguments numerically, and to also highlight certain aspects of the general model to be developed later in this chapter, in the following subsection we consider four scenarios based on dependent or independent random variables of the form

$$X_0 = a_0 + G_0 \quad \text{and} \quad X_1 = a_0 + G_1,$$

where a_0, which we set to 200 thousands of dollars in our numerical explorations henceforth, is the seller's reservation price during the initial selling stage (i.e., $t = 0$), which is the smallest amount that the seller could possibly ask given the building costs and other expenses, and G_0 and G_1 are two (dependent or independent) gamma distributed random variables.

Although our general model is not limited to any specific price distribution, in our numerical illustrative considerations, we assume that the prices follow the gamma distribution, which is a very reasonable assumption, extensively used in the literature (see, e.g.,Hong & Shum, 2006; Pratt, Wise & Zeckhauser, 1979; Quan & Quigley, 1991). In particular, Quan and Quigley (1991) characterize the density function of the reservation price of a group of self-selected buyers using this distribution. Hong and Shum (2006) apply the gamma distribution to model search costs, including time, energy and money spent on researching products, or services, for purchasing. There are numerous cases of using the gamma distribution when modeling insurance losses (e.g., Alai et al., 2013; Hürlimann, 2001; Furman & Landsman, 2005;).

Since different parameterizations of the gamma distribution have appeared in the literature, we note that throughout this chapter we work with the one, defined by $Ga(\alpha,\beta)$, whose probability density function (pdf) is[1]

$$f_{\alpha,\beta}(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}, \quad t > 0. \tag{4.7}$$

[1]The mean of this gamma distribution is α/β and the variance is α/β^2.

We denote the corresponding cumulative distribution function (cdf) by $F_{\alpha,\beta}$, which for numerical purposes can conveniently be expressed in terms of the lower incomplete gamma function $\gamma(\cdot,\cdot)$ by the formula

$$F_{\alpha,\beta}(x) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}. \tag{4.8}$$

We also recall that the mean of this gamma distribution is α/β. In Figure 4.1 we have depicted

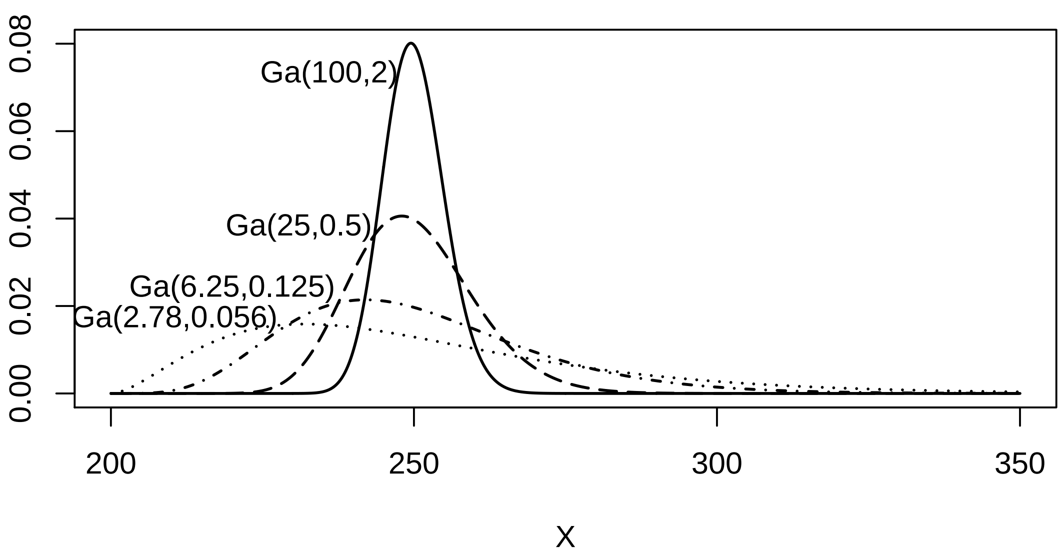


Figure 4.1: The pdf of the price $X = 200 + G$ depicted in solid $G \sim Ga(100, 2)$, dashed $G \sim Ga(25, 0.5)$, dot-dashed $G \sim Ga(6.25, 0.125)$, and dotted $G \sim Ga(2.78, 0.056)$ lines.

its pdf for several parameter choices that we use in our numerical explorations later in this chapter. The choices are such that we always have the same mean $\mu_G = \alpha/\beta = 50$ of G but varying standard deviations σ_G: equal to 5 (solid), 10 (dashed), 20 (dot-dashed), and 30 (dotted).

4.2.2 Scenarios

Scenario A: identical X_0 and X_1

Consider the case when the buyer decides on the same bidding price irrespective of the seller's perspective. This bidding price is random, and we denote it by X. In other words, the earlier introduced two random variables X_0 and X_1 are identical, that is, both are equal to a random variable X, which we set to be

$$X = a_0 + G \tag{4.9}$$

with the earlier defined a_0 and the gamma random variable $G \sim Ga(\alpha,\beta)$. Naturally, if the property is not sold during the initial stage, then under Scenario A, in order to at least hope to be successful during the subsequent stage, the seller has no alternative but to reduce the price, and we shall see this clearly from our following mathematical considerations. We note at the outset, however, that other scenarios to be discussed below will show the possibilities of increasing second-stage prices and still be able to successfully sell the property.

Hence, under Scenario A, and with $x_{0,\max}$ defined by equation (4.2) via the function $R_0(x_0)$ given by equation (4.1) with $X_0 = X$, the function $R_1(x_1)$ is given by the formula

$$R_1(x_1) = \mathbf{P}[x_1 \le X_1 < x_{0,\max}]x_1. \tag{4.10}$$

Note 4.2.2 As $\mathbf{P}[X_0 \ge x_0]x_0$ is a continuous function of x_0 and $\mathbf{P}[X_0 \ge x_0]x_0 \le \mathbf{E}[X_0]$, the value of $x_{0,\max}$ exists.

Note 4.2.3 As $\mathbf{P}[x_1 \le X_1 < x_{0,\max}]x_1$ is a continuous function of x_1 and $\mathbf{P}[x_1 \le X_1 < x_{0,\max}]x_1 \le \mathbf{P}[x_1 \le X_1]x_1 < \mathbf{E}[X_1]$, the value of $x_{1,\max}$ exists.

The (simultaneous) expected gross profit $R(x_0, x_1)$, which is defined by equation (4.6) with $X_0 = X_1 = X$, becomes

$$R(x_0, x_1) = \mathbf{P}[X \ge x_0]x_0 + \mathbf{P}[x_1 \le X < x_0]x_1. \tag{4.11}$$

We now use specification (4.9) to reduce the above formulas to more computationally tractable ones. First, we calculate $x_{0,\max}$, which is the point where the function

$$R_0(x_0) = \left(1 - \frac{\gamma(\alpha, \beta(x_0 - a_0))}{\Gamma(\alpha)}\right)x_0 \tag{4.12}$$

achieves its maximum. Next, we calculate $x_{1,\max}$, which is the point where the function

$$R_1(x_1) = \frac{\gamma(\alpha, \beta(x_{0,\max} - a_0)) - \gamma(\alpha, \beta(x_1 - a_0))}{\Gamma(\alpha)}\mathbf{1}\{x_1 \leq x_{0,\max}\}x_1 \tag{4.13}$$

achieves its maximum, where the indicator $\mathbf{1}\{x_1 \leq x_{0,\max}\}$ is equal to 1 when $x_1 \leq x_{0,\max}$ and 0 otherwise. Finally, we calculate the pair $(x_0^{\max}, x_1^{\max})$ that maximizes the function

$$R(x_0, x_1) = \left(1 - \frac{\gamma(\alpha, \beta(x_0 - a_0))}{\Gamma(\alpha)}\right)x_0 + \frac{\gamma(\alpha, \beta(x_0 - a_0)) - \gamma(\alpha, \beta(x_1 - a_0))}{\Gamma(\alpha)}\mathbf{1}\{x_1 \leq x_0\}x_1. \tag{4.14}$$

Note 4.2.4 As $\mathbf{P}[X_0 \geq x_0]\, x_0 \leq \mathbf{E}[X_0]$ and $\mathbf{P}[x_1 \leq X_1 < x_0]\, x_1 \leq \mathbf{P}[X_1 \geq x_1]\, x_1 \leq \mathbf{E}[X_1]$, we have

$$R(x_0, x_1) \leq \mathbf{E}[X_0] + \mathbf{E}[X_1].$$

Besides, since

$$\lim_{x_0\to\infty}\lim_{x_1\to\infty} R(x_0, x_1) = \lim_{x_0\to\infty} \mathbf{P}[X_0 \geq x_0]\, x_0 + \lim_{x_0\to\infty}\left(\lim_{x_1\to\infty} \mathbf{P}[x_1 \leq X_1 < x_0]\, x_1\right) = 0,$$

due to Note 4.2.2 and the fact that $\mathbf{P}[x_1 \leq X_1 < x_0]\, x_1 \equiv 0$ whenever $x_1 > x_0$, while

$$\lim_{x_1\to\infty}\lim_{x_0\to\infty} R(x_0, x_1) = \lim_{x_1\to\infty}\left(\lim_{x_0\to\infty} \mathbf{P}[X_0 \geq x_0]\, x_0 + \mathbf{P}[X_1 \geq x_1]\, x_1\right) = 0$$

due to the continuity of $R(x_0, x_1)$ with respect to x_0 and x_1 as well as Notes 4.2.2 and 4.2.3. Thus, for $(x_0, x_1) \to (\infty, \infty)$ from any path we have that $R(x_0, x_1)$ vanishes. Therefore, there exists a pair of prices $\left(x_0^{\max}, x_1^{\max}\right)$ maximizing $R(x_0, x_1)$ simultaneously.

We report the values of the aforementioned maximal points and the respective expected profits in Table 4.1. We note that our chosen values of α and β are such that they lead to the

α	β	σ_G	$R(x_{0,\max}, x_{1,\max})$	$x_{0,\max}$	$x_{1,\max}$	$R(x_0^{\max}, x_1^{\max})$	$x_0^{\max}$	$x_1^{\max}$
100	2	5	238.3249	238.3686	232.1596	243.6800	246.8699	236.3763
25	0.5	10	229.9880	230.1152	220.5280	238.4024	244.7218	226.8302
6.25	0.125	20	217.1578	217.3732	206.2535	229.8237	241.8907	213.1633
2.78	0.056	30	207.4594	207.5638	200.5773	223.6389	240.9744	204.4544

Table 4.1: Prices and profits when $X_0 = X_1 = X$ with $X = a_0 + G$ and $G \sim Ga(\alpha, \beta)$.

same mean $\mu_G = 250$ of G but different standard deviations σ_G $(= \sigma_X)$. We see from the table that we always have $x_{0,\max} > x_{1,\max}$ and $x_0^{\max} > x_1^{\max}$, which is natural because $X_0 = X_1$. As we already mathematically concluded (see below equation (4.6)), the numerical values in Table 4.1 confirm that $x_{0,\max} < x_0^{\max}$ and $x_{1,\max} < x_1^{\max}$, that is, setting the two selling prices simultaneously before commencing the initial selling stage proves to be more beneficial for the seller. Note also from the table that the values of all the four prices $x_{0,\max}$, $x_0^{\max}$, $x_{1,\max}$ and $x_1^{\max}$ decrease when the standard deviation σ_G $(= \sigma_X)$ increases.

When $\alpha = 25$ and $\beta = 0.5$, the functions $R_0(x_0)$ and $R_1(x_1)$ as well as the surface $R(x_0, x_1)$ are depicted in Figure 4.2 and 4.3, and in Figure 4.4 and 4.5 respectively.

Scenario B: independent X_0 and X_1

Now we assume that the bidding prices X_0 and X_1 are independent, which sets us apart from Scenario A. However, we still let the two prices follow the same distribution. Specifically,

$$X_0 =_d X \quad \text{and} \quad X_1 =_d X, \tag{4.15}$$

where $X = a_0 + G$ is the same as in equation (4.9) with $G \sim Ga(\alpha, \beta)$, and '$=_d$' denotes equality in distribution. Hence, $x_{0,\max}$ is defined by equation (4.2) via the function $R_0(x_0)$ given by equation (4.1) with $X_0 = X$, and the expected profits (4.3) and (4.6) become

$$R_1(x_1) = \mathbf{P}[X < x_{0,\max}]\mathbf{P}[X \geq x_1]x_1 \tag{4.16}$$

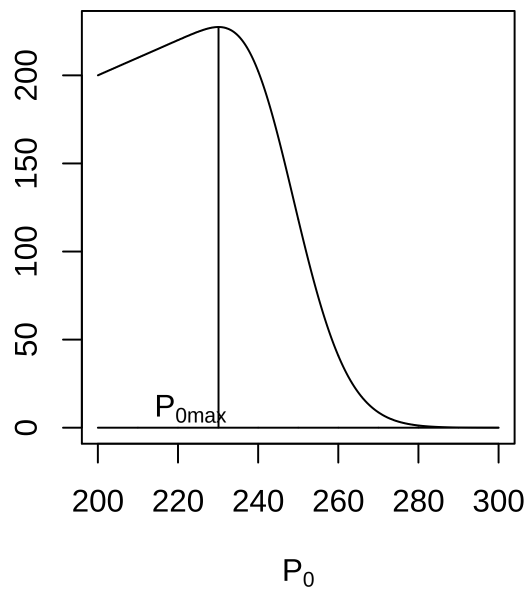


Figure 4.2: The functions $R_0(x_0)$ of price and gross profit when $X_0 = a_0 + G$ with $G \sim Ga(25, 0.5)$.

and

$$R(x_0, x_1) = \mathbf{P}[X \geq x_0]x_0 + \mathbf{P}[X < x_0]\mathbf{P}[X \geq x_1]x_1. \tag{4.17}$$

Obviously, $x_{0,\max}$ and $x_{1,\max}$ must be identical because X_0 and X_1 follow the same distribution, but there is of course no reason why $x_0^{\max}$ and $x_1^{\max}$ should be identical: the clear difference between the two will be seen from the following numerical example.

First, we see that $x_{0,\max}$ is the same as in Scenario A but $x_{1,\max}$ that maximizes the function

$$R_1(x_1) = \frac{\gamma(\alpha, \beta(x_{0,\max} - a_0))}{\Gamma(\alpha)}\left(1 - \frac{\gamma(\alpha, \beta(x_1 - a_0))}{\Gamma(\alpha)}\right)x_1 \tag{4.18}$$

is different from the corresponding one in Scenario A. We see these facts in Table 4.2 where we use the same shape α and rate β parameters as in earlier Table 4.1. In Table 4.2 we have

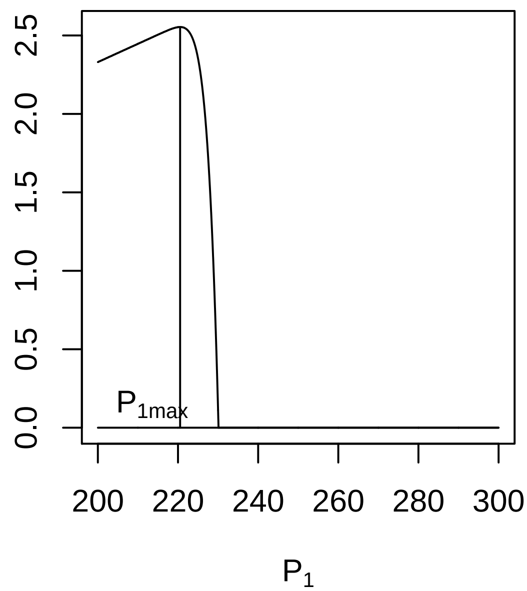


Figure 4.3: The functions $R_1(x_1)$ of price and gross profit when $X_1 = a_0 + G$ with $G \sim Ga(25, 0.5)$.

α	β	σ_G	$R(x_{0,\max}, x_{1,\max})$	$x_{0,\max}$	$x_{1,\max}$	$R(x_0^{\max}, x_1^{\max})$	$x_0^{\max}$	$x_1^{\max}$
100	2	5	238.3595	238.3686	238.3686	244.1578	247.0367	238.3651
25	0.5	10	230.0839	230.1152	230.1152	239.2700	244.9522	230.1226
6.25	0.125	20	217.3058	217.3732	217.3732	230.9968	242.0798	217.3781
2.78	0.056	30	207.5201	207.5638	207.5638	224.4556	241.0434	207.5595

Table 4.2: Prices and gross profits when X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha, \beta)$.

also reported the pairs $(x_0^{\max}, x_1^{\max})$ on which the maximum of the function

$$R(x_0, x_1) = \left(1 - \frac{\gamma(\alpha, \beta(x_0 - a_0))}{\Gamma(\alpha)}\right)x_0 + \frac{\gamma(\alpha, \beta(x_0 - a_0))}{\Gamma(\alpha)}\left(1 - \frac{\gamma(\alpha, \beta(x_1 - a_0))}{\Gamma(\alpha)}\right)x_1 \tag{4.19}$$

is achieved. Note from Table 4.1 that the values of all the four selling prices $x_{0,\max}$, $x_0^{\max}$, $x_{1,\max}$ and $x_1^{\max}$ decrease when the standard deviation σ_G $(= \sigma_X)$ increases. Note also that the bounds $x_{0,\max} < x_0^{\max}$ and $x_0^{\max} > x_1^{\max}$ hold. Furthermore, we always see the ordering $x_{0,\max} < x_0^{\max}$ in Table 4.1, but the ordering of $x_{1,\max}$ and $x_1^{\max}$ seems to depend on the value of σ_G.

In the special case $\alpha = 25$ and $\beta = 0.5$, we have depicted the functions $R_0(x_0)$ and $R_1(x_1)$ as

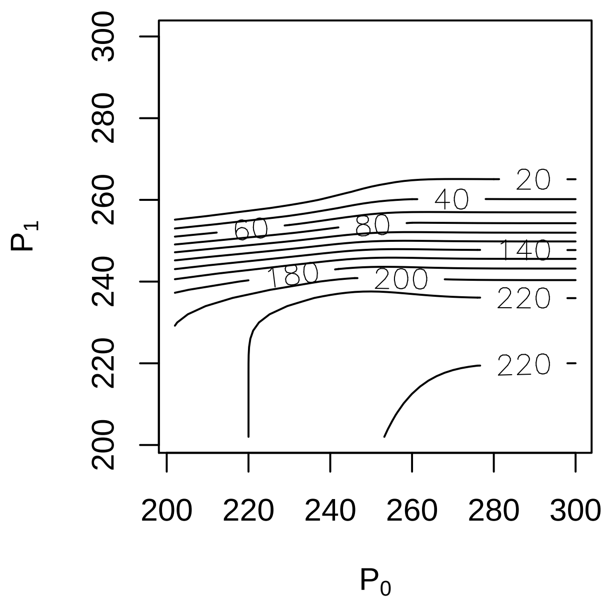


Figure 4.4: The contour of the surface $R(x_0, x_1)$ of prices and gross profits when $X_0 = X_1 = a_0 + G$ with $G \sim Ga(25, 0.5)$.

well as the surface $R(x_0, x_1)$ in Figure 4.6 and4.7, and in Figure 4.8 and 4.9.

Scenario C: X_1 stochastically dominates X_0

We see from previous two Tables 4.1 and 4.2 that neither sequential nor simultaneous second-stage selling prices are higher than the corresponding first-stage prices: we always have $x_{0,\max} \geq x_{1,\max}$ and $x_0^{\max} \geq x_1^{\max}$ in Tables 4.1 and 4.2. In practice, however, we often observe that after the failed initial sales, the sellers increase the prices and achieve successful results. There are several explanations of this phenomenon, and we shall next discuss one of them, with the other one making the contents of Scenario D below.

Namely, our first explanation is based on the assumption that, due to various reasons, buyers are often willing to pay higher prices during the second selling stage. To illustrate this situation numerically, we let

$$X_0 = a_0 + G_0 \quad \text{and} \quad X_1 = a_0 + b_1 G_1,$$

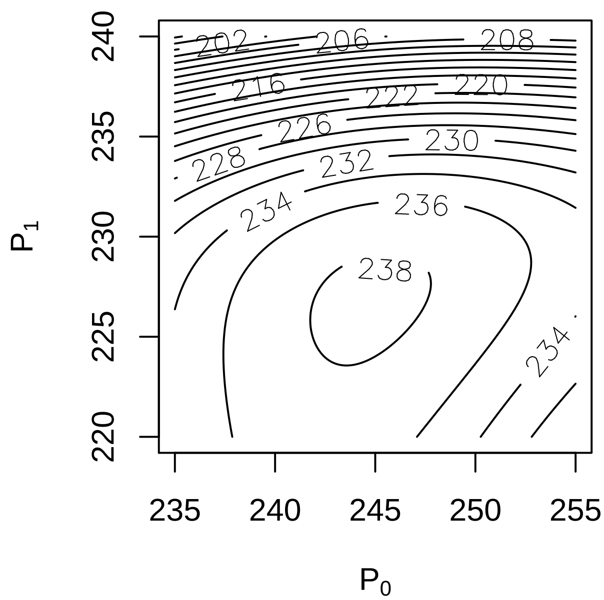


Figure 4.5: The center area of the contour of the surface $R(x_0, x_1)$ of prices and gross profits when $X_0 = X_1 = a_0 + G$ with $G \sim Ga(25, 0.5)$.

where $b_1 > 0$ is a constant, and $G_0, G_1 \sim Ga(\alpha,\beta)$ are two independent random variables. That is, the buyer is willing to change the bidding amount by $(b_1 - 1)100\%$. Note that $b_1 G_1 \sim Ga(\alpha, \beta/b_1)$, which is useful when calculating. Namely, with the same $x_{0,\max}$ as in Scenarios A and B, we now have

$$R_1(x_1) = \frac{\gamma(\alpha, \beta(x_{0,\max} - a_0))}{\Gamma(\alpha)}\left(1 - \frac{\gamma(\alpha, \beta(x_1 - a_0)/b_1)}{\Gamma(\alpha)}\right)x_1 \tag{4.20}$$

and

$$R(x_0, x_1) = \left(1 - \frac{\gamma(\alpha, \beta(x_0 - a_0))}{\Gamma(\alpha)}\right)x_0 + \frac{\gamma(\alpha, \beta(x_0 - a_0))}{\Gamma(\alpha)}\left(1 - \frac{\gamma(\alpha, \beta(x_1 - a_0)/b_1)}{\Gamma(\alpha)}\right)x_1, \tag{4.21}$$

where $x_{1,\max}$ maximizes the function $R_1(x_1)$ and the pair $(x_0^{\max}, x_1^{\max})$ maximizes the surface $R(x_0, x_1)$. In Table 4.3 we have reported the numerical values of the expected gross profits $R(x_{0,\max}, x_{1,\max})$ and $R(x_0^{\max}, x_1^{\max})$, as well as of the prices at which these maximal expected

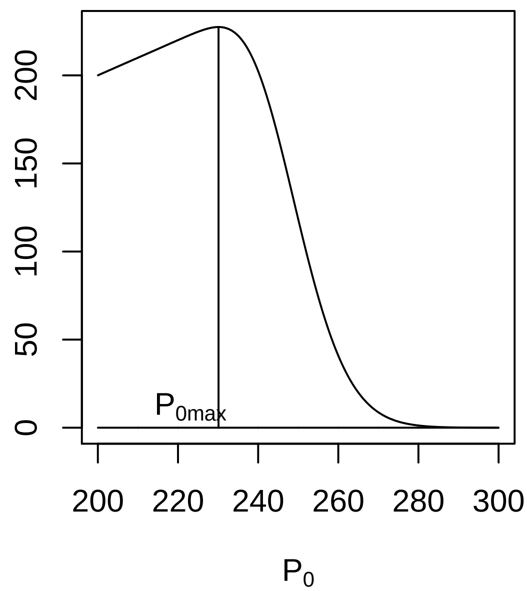


Figure 4.6: The functions $R_0(x_0)$ of price and gross profit when X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(25, 0.5)$.

	α	β	σ_G	$R(x_{0,\max}, x_{1,\max})$	$x_{0,\max}$	$x_{1,\max}$	$R(x_0^{\max}, x_1^{\max})$	$x_0^{\max}$	$x_1^{\max}$
$b_1 = 0.5$	100	2	5	238.2429	238.3686	218.6924	241.1509	243.4476	218.6442
	25	0.5	10	229.9149	230.1152	214.1065	235.7860	240.6705	214.1065
	6.25	0.125	20	217.1670	217.3732	207.1570	228.2490	238.4532	207.1570
	2.78	0.056	30	207.4754	207.5638	202.3953	223.0999	238.8913	202.3953
$b_1 = 1.1$	100	2	5	238.3830	238.3686	242.3547	241.4216	243.7144	220.6309
	25	0.5	10	230.1187	230.1152	233.4259	240.3086	246.1226	233.3253
	6.25	0.125	20	217.3363	217.3732	219.6358	231.6847	242.9417	219.6334
	2.78	0.056	30	207.5313	207.5638	208.8733	224.8072	241.2413	208.8733
$b_1 = 2.1$	100	2	5	238.6217	238.3686	282.7481	241.4216	243.7144	220.6309
	25	0.5	10	230.4780	230.1152	267.7370	250.7337	252.1044	248.5686
	6.25	0.125	20	217.6737	217.3732	245.0225	241.6766	253.5593	244.0345
	2.78	0.056	30	207.6767	207.5638	226.2926	229.9190	247.4248	225.7422

Table 4.3: Prices and gross profits when $X_0 = a_0 + G_0$ and $X_1 = a_0 + b_1 G_1$ with independent $G_0, G_1 \sim Ga(\alpha, \beta)$ and varying parameter b_1 values.

gross profits are achieved, for several values of b_1.

We see from Table 4.3 that for every noted value of b_1, the prices $x_{0,\max}$ and $x_{1,\max}$ decrease when the standard deviation σ_G (= σ_X) increases, but the pattern of $x_0^{\max}$ and $x_1^{\max}$ is unclear. Note also from the table that the ordering $x_{0,\max} < x_0^{\max}$ always holds, but various orderings hold

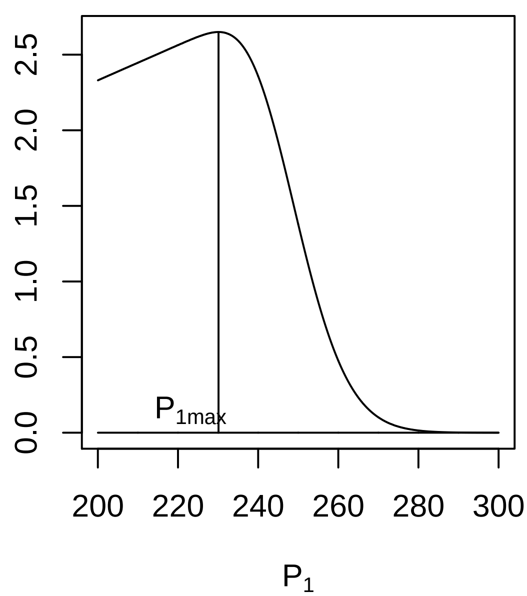


Figure 4.7: The function $R_1(x_1)$ of price and gross profit when X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(25, 0.5)$.

between the second-stage prices $x_{1,\max}$ and $x_1^{\max}$. Furthermore, we see that when $b_1 = 0.5$, we have $x_{0,\max} > x_{1,\max}$ and $x_0^{\max} > x_1^{\max}$, but when $b_1 = 1.1$ and $b_1 = 2.1$, we have $x_{0,\max} < x_{1,\max}$ and $x_0^{\max} > x_1^{\max}$.

In the special case $\alpha = 25, \beta = 0.5$ and $a = 1.1$, we have depicted the functions $R_0(x_0)$ and $R_1(x_1)$ as well as the surface $R(x_0, x_1)$ in Figures 4.10 and 4.11, and in Figures 4.12 and 4.13.

Scenario D: cost of holding the property

Based on Scenario C, when the seller guesses that the buyer might be willing to pay a large price during the second-stage selling stage, the price in the second stage can be set larger and still the maximal expected gross profit achieved.

There is also another reason why the second-stage selling price can be set larger and the seller's goals achieved, and it is based on the fact that the seller may wish to maximize, for example, the net profit instead of the gross profit. To simplify our illustration of this fact, we

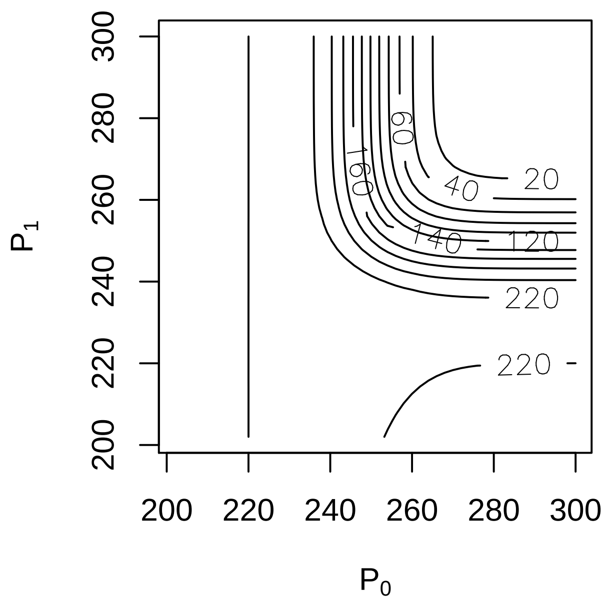


Figure 4.8: The contour of the surface $R(x_0, x_1)$ of Prices and gross profits when X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(25, 0.5)$.

take into consideration only one deductible, which is the cost c_1 of holding the property unsold, in which case the (net) profit during the second selling stage becomes $x_1 - c_1$. Furthermore, let the bidding prices X_0 and X_1 be the same as in Scenario B, that is, they are independent and follow the same distribution as $X = a_0 + G$ with $G \sim Ga(\alpha, \beta)$ (see (4.15)). Hence, $x_{0,\max}$ is the same as in Scenario B or, equivalently, as in Scenario A, that is, the selling price $x_{0,\max}$ is given by equation (4.2) via the same function $R_0(x_0)$ as in equation (4.1). The function $R_1(x_1)$ and the surface $R(x_0, x_1)$, however, need to be redefined in order to take into account the aforementioned cost c_1. Namely, we have

$$
\begin{aligned}
R_{1,c}(x_1) &= \mathbf{P}[X_0 < x_{0,\max}]\mathbf{P}[X_1 \geq x_1](x_1 - c_1) \\
&= \frac{\gamma(\alpha, \beta(x_{0,\max} - a_0))}{\Gamma(\alpha)}\left(1 - \frac{\gamma(\alpha, \beta(x_1 - a_0))}{\Gamma(\alpha)}\right)(x_1 - c_1),
\end{aligned}
$$

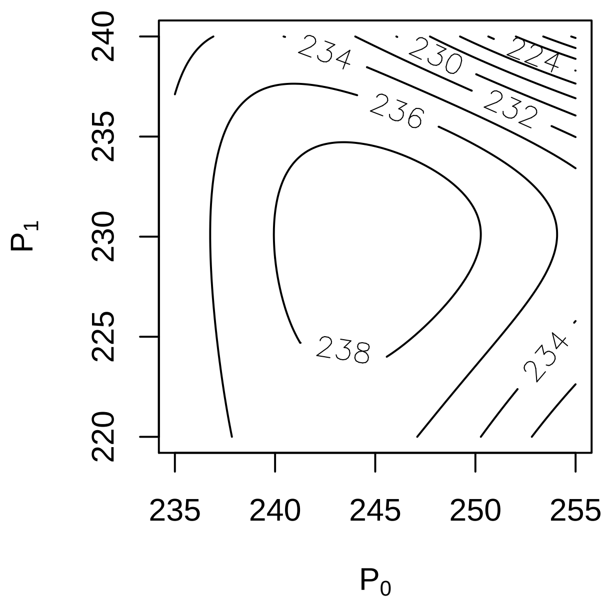


Figure 4.9: The center area of the contour of the surface $R(x_0, x_1)$ of Prices and gross profits when X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(25, 0.5)$.

with the same $x_{0,\max}$ as in Scenario B (or A), and

$$\begin{aligned} R_c(x_0, x_1) &= \mathbf{P}[X_0 \geq x_0]x_0 + \mathbf{P}[X_0 < x_{0,\max}]\mathbf{P}[X_1 \geq x_1](x_1 - c_1) \\ &= \left(1 - \frac{\gamma(\alpha, \beta(x_0 - a_0))}{\Gamma(\alpha)}\right)x_0 + \frac{\gamma(\alpha, \beta(x_0 - a_0))}{\Gamma(\alpha)}\left(1 - \frac{\gamma(\alpha, \beta(x_1 - a_0))}{\Gamma(\alpha)}\right)(x_1 - c_1). \end{aligned}$$

Thus, we have

$$x_{1,c,\max} = \arg\max_{x_1} R_{1,c}(x_1)$$

and

$$(x_{0,c}^{\max}, x_{1,c}^{\max}) = \arg\max_{x_0, x_1} R_c(x_0, x_1),$$

whose numerical values for different cost c_1 values are reported in Table 4.4.

We see from the table that for all specified values of c_1, the sequentially set selling prices follow the order $x_{0,\max} < x_{1,c,\max}$, which is the opposite of what we have seen in the previous

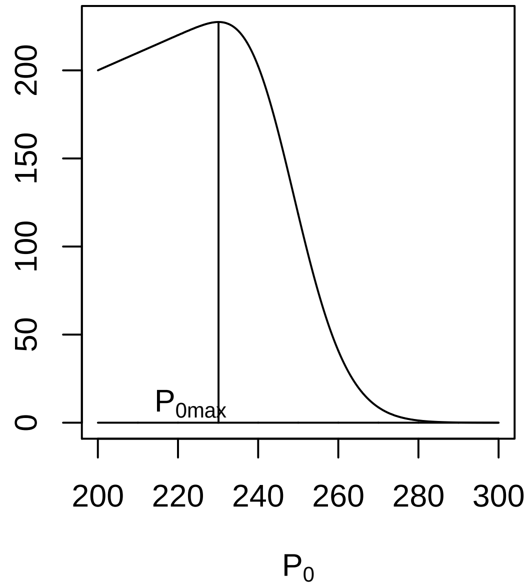


Figure 4.10: The function $R_1(x_1)$ of price and gross profit when $X_0 = a_0+G_0$ and $X_1 = a_0+b_1G_1$ with independent $G_0, G_1 \sim Ga(\alpha,\beta)$ and $b_1 = 1.1$.

	α	β	σ_G	$R(x_{0,\max}, x_{1,c,\max})$	$x_{0,\max}$	$x_{1,c,\max}$	$R(x_{0,c}^{\max}, x_{1,c}^{\max})$	$x_{0,c}^{\max}$	$x_{1,c}^{\max}$
$c_1 = 20$	100	2	5	238.2367	238.3686	238.5186	241.0633	243.3410	238.5186
	25	0.5	10	229.8538	230.1152	230.4159	234.9445	239.6038	230.4159
	6.25	0.125	20	216.9603	217.3732	217.9019	225.1370	234.1515	217.9019
	2.78	0.056	30	207.2347	207.5638	208.1217	217.2595	230.1206	208.1216
$c_1 = 100$	100	2	5	237.7470	238.3686	239.3381	238.2509	239.9507	239.3381
	25	0.5	10	228.9377	230.1152	232.1152	229.8364	233.0915	232.1152
	6.25	0.125	20	215.5916	217.3732	221.1499	216.9273	222.2088	221.1500
	2.78	0.056	30	206.1066	207.5638	212.1154	207.4254	212.7300	212.1154
$c_1 = 150$	100	2	5	237.4428	238.3686	240.2109	237.6034	239.1895	240.2109
	25	0.5	10	228.3730	230.1152	234.0368	228.6428	231.6018	234.0368
	6.25	0.125	20	214.7622	217.3732	225.3874	215.1202	219.6026	225.3874
	2.78	0.056	30	205.4343	207.5638	218.7960	205.7393	209.6705	218.7952

Table 4.4: Prices and profits for various holding cost c_1 values when the bidding prices X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha,\beta)$.

scenarios. In the case of simultaneously set prices, we have $x_{0,c}^{\max} > x_{1,c}^{\max}$ for the two smaller costs $c_1 = 20$ and $c_1 = 100$, with the opposite ordering $x_{0,c}^{\max} < x_{1,c}^{\max}$ in the case of the cost $c_1 = 150$. The sequentially set selling prices in the initial stage are always smaller than the corresponding simultaneously set prices, that is, the ordering $x_{0,\max} < x_{0,c}^{\max}$ holds throughout

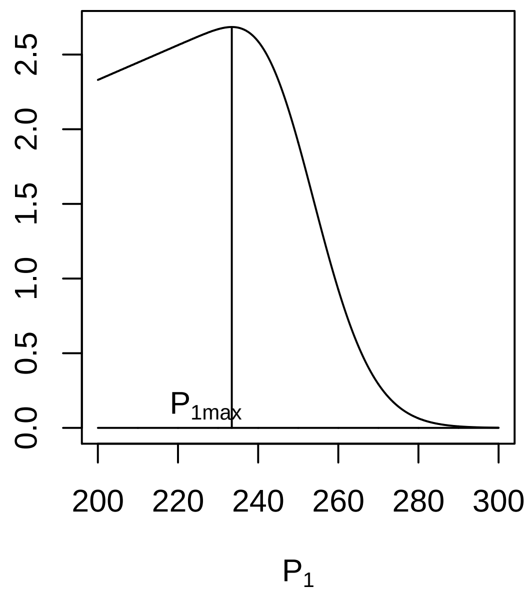


Figure 4.11: The function $R_1(x_1)$ of price and gross profit when $X_0 = a_0+G_0$ and $X_1 = a_0+b_1G_1$ with independent $G_0, G_1 \sim Ga(\alpha,\beta)$ and $b_1 = 1.1$.

the entire table. The reported in Table 4.4 numerical values of the selling prices $x_{1,c,\max}$ and $x_{1,c}^{\max}$ are very similar.

In the special case $\alpha = 25, \beta = 0.5$ and $c_1 = 20$, we have depicted the functions $R_0(x_0)$ and $R_1(x_1)$ as well as the surface $R(x_0, x_1)$ in Figures 4.14 and 4.15, and in Figures 4.16 and 4.17.

4.3 The general model

We need to further elaborate on the motivating problem, and to also introduce additional notation. Hence, during the initial selling stage, which we have agreed to collapse into only one instance $t = 0$, the seller keeps the property on sale. Let X_0 be the price, viewed as a random variable, that the buyer is willing to pay for the property during the initial selling stage. Let x_0 be the price set by the seller, who wishes it to be such that certain (economic, financial, etc.) goals would be achieved. Hence, unlike X_0, the price x_0 is not random – the seller chooses it

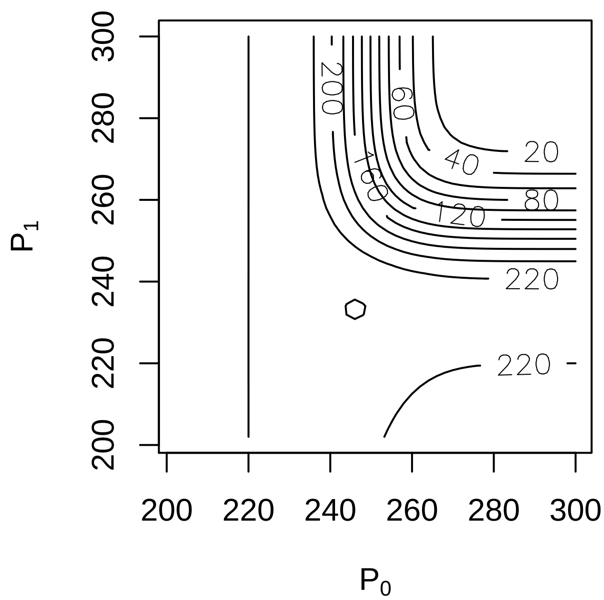


Figure 4.12: The surface $R(x_0, x_1)$ of prices and gross profits when $X_0 = a_0 + G_0$ and $X_1 = a_0 + b_1 G_1$ with independent $G_0, G_1 \sim Ga(\alpha, \beta)$ and $b_1 = 1.1$.

based on the available information and the goals to be achieved. When $X_0 \geq x_0$, the property is sold and the seller's profit is $v_0(x_0)$, where v_0 is a function, usually such that $v_0(p) \leq p$ for all $p \geq 0$. For example,

$$v_0(x_0) = (x_0 - c_0)_+, \tag{4.22}$$

where c_0 is, e.g., the property development cost evaluated during the initial selling stage. (By definition, $x_+ = x$ when $x \geq 0$, and $x_+ = 0$ when $x < 0$.) If, however, $X_0 < x_0$, then the buyer rejects the offer and makes the second (and final) attempt to buy the property at a later time, which is generally unknown and thus treated as a random variable, which we denote by T.

Note 4.3.1 There are of course situations when T is pre-specified and thus deterministic, say $T = 1$. For example, Wu and Zitikis (2017) consider a two-period economy with $t = 0$ standing for the Black Friday promotion period and $t = 1$ for the Boxing Day promotion period.

Let X_T be the amount of money that the buyer is willing to pay at time $T > 0$ during

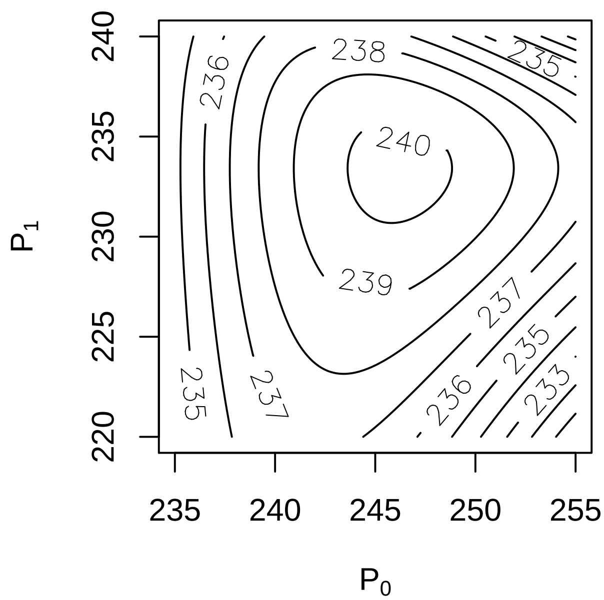


Figure 4.13: The center area of the surface $R(x_0, x_1)$ of prices and gross profits when $X_0 = a_0 + G_0$ and $X_1 = a_0 + b_1 G_1$ with independent $G_0, G_1 \sim Ga(\alpha, \beta)$ and $b_1 = 1.1$.

the second selling stage. Conditionally on T, the price X_T is a random variable from the seller's perspective. Let x_1 be the price set by the seller some time prior to commencing the second selling stage (the price can be set as early as the time of setting the initial price x_0). Analogously to the initial decision making, if $X_T \geq x_1$, then the property is sold and the seller's profit is $v_T(p)$, where v_T is a value (or utility) function, perhaps different from v_0, but usually such that $v_T(p) \leq p$ for all $p \geq 0$. For example,

$$v_T(p) = (p - c_T)_+,$$

where c_T is, e.g., the costs of property development and holding it unsold at time T. We shall provide specific details on the structure of c_T later in this chapter.

For the sake of concreteness, throughout the rest of the chapter we assume that the seller

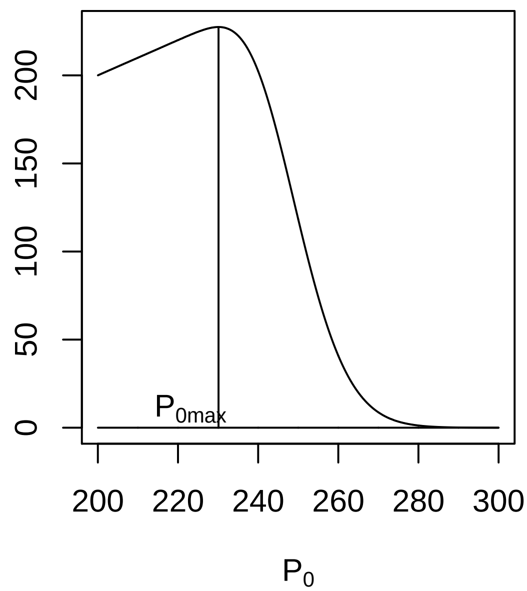


Figure 4.14: The function $R_0(x_0)$ of profit and price when the cost is $c_1 = 20$ and the bidding prices X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha, \beta)$.

wishes to determine x_0 and x_1 such that the overall two-stage expected profit

$$\begin{aligned} R(x_0, x_1) &= \mathbf{P}[X_0 \geq x_0]v_0(x_0) + \int_0^\infty \mathbf{P}[X_t \geq x_1, X_0 < x_0]v_t(x_1)\mathrm{d}F_T(t) \\ &= \mathbf{P}[X_0 \geq x_0]v_0(x_0) + (1 - \mathbf{P}[X_0 \geq x_0]) \int_0^\infty \mathbf{P}[X_t \geq x_1 \mid X_0 < x_0]v_t(x_1)\mathrm{d}F_T(t) \end{aligned} \tag{4.23}$$

would be maximal, where F_T is the cdf of T. The seller may have various goals to achieve, and our following considerations can be adjusted accordingly. When deriving equation (4.23), which involves conditioning on T, we have assumed that the events $X_t \geq x_1$ and $T = t$ are independent and in this way obtained the probability

$$\mathbf{P}[X_t \geq x_1 \mid X_0 < x_0].$$

Even though the simplifying independence assumption is natural, it can be relaxed if a ne-

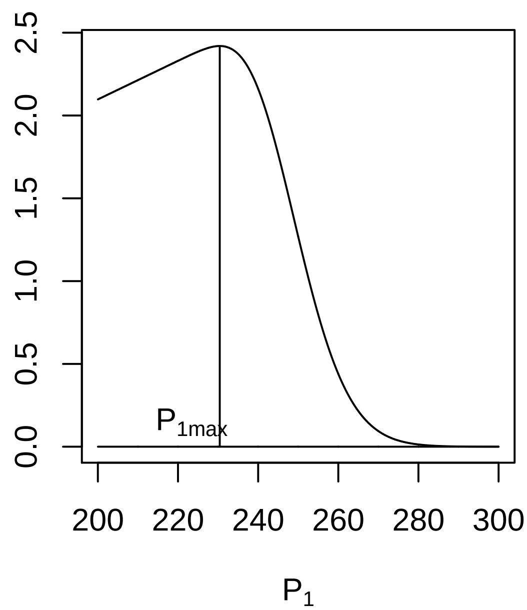


Figure 4.15: The function $R_1(x_1)$ of profit and price when the cost is $c_1 = 20$ and the bidding prices X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha, \beta)$.

cessity arises, but there are also situations when this assumption is automatically satisfied. For example, this happens in the static two-stage scenario when T always takes the same constant value, say $T = 1$. We note in this regard that the chosen value 1 is just a symbolic representation of the second selling stage, such as the Boxing Day promotion period that follows the initial (i.e., $t = 0$) Black Friday promotion period (e.g., Wu & Zitikis, 2017). In this case formula (4.23) reduces to

$$R(x_0, x_1) = \mathbf{P}[X_0 \geq x_0]v_0(x_0) + \mathbf{P}[X_1 \geq x_1, X_0 < x_0]v_1(x_1), \tag{4.24}$$

where

$$v_1(x_1) = (x_1 - c_1)_+. \tag{4.25}$$

Henceforth, we shall make a number of other simplifying yet practically sound assumptions, so that the technicalities would not be too complex.

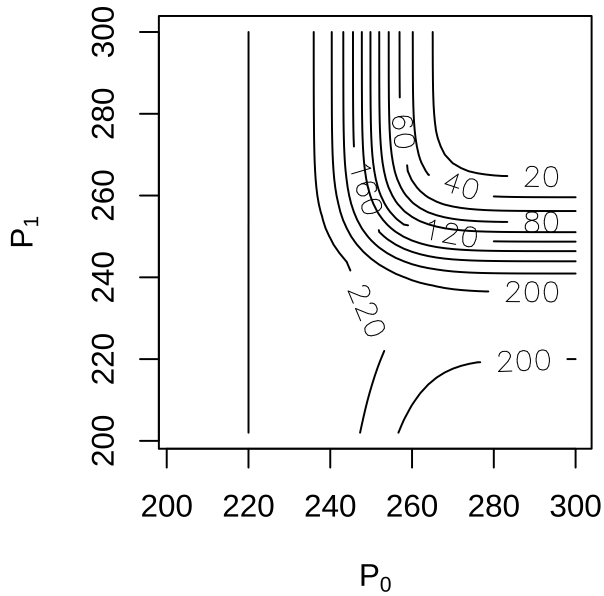


Figure 4.16: The surface $R(x_0, x_1)$ of profits and prices when the cost is $c_1 = 20$ and the bidding prices X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha, \beta)$.

4.4 The initial-stage selling probability

To assess the probabilities $\mathbf{P}[X_0 \geq x_0]$ and $\mathbf{P}[X_t \geq x_1 \mid X_0 < x_0]$ on the right-hand side of equation (4.23), we need to specify appropriate models for the random variables X_t, $t \geq 0$. Their distributions may involve population heterogeneity, as our motivating example shows, which we take into consideration. Specifically, we assume that the population of potential buyers consists of two groups: domestic buyers (D) permanently residing in Uruguay and foreigners (A) wishing to make investments.

Note 4.4.1 We have reserved F for denoting cdf's, as is usually the case in the literature, and so use A to denote foreign buyers. This notation also reflects the fact that most of the foreign property buyers in Punta del Este are Argentineans.

Since economic and financial considerations of the two types of buyers are usually different, the structures of the corresponding random variables are also different. In this section we

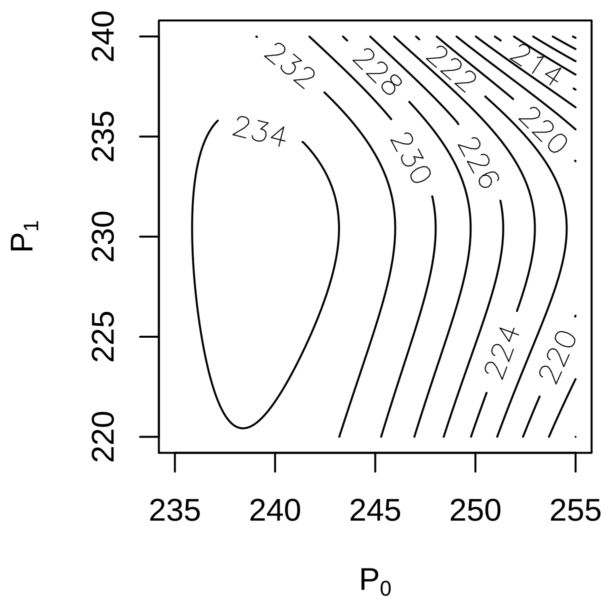


Figure 4.17: The center area of the surface $R(x_0, x_1)$ of profits and prices when the cost is $c_1 = 20$ and the bidding prices X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha, \beta)$.

concentrate on the probability $\mathbf{P}[X_0 \geq x_0]$ and thus specify the structure of X_0. For this, we first note the forces that give rise to the amount of money X_0 that the buyer (domestic or foreign) is willing to pay for the property during the initial selling stage.

In this section and throughout the rest of this chapter, background risk models will play an important role. There are two major classes of such models: additive and multiplicative. For applications and discussions of additive models in Economic Theory, we refer to Gollier and Pratt (1996) and references therein, and to problems in Actuarial Science, we refer to Furman and Landsman (2005, 2010), Tsanakas (2008), and references therein. Our current research in subsection 4.4 is essentially based on the multiplicative model, which has been extensively explored and utilized in the literature (see, e.g., Asimit et al., 2016; Franke et al., 2006, 2011; Tsetlin & Winkler, 2005; references therein). It is worth noting that a number of important parametric multiplicative models incorporate elements of both Pareto and gamma distributions, and we refer to Asimit et al. (2016), Su (2016), and Su and Furman (2017) for details and further references.

4.4.1 General considerations

Consider first the population of domestic buyers. Suppose that, initially, their buying decisions are based on individual considerations detached from all the exogenous factors, such as the overall economic situation. Let Y_{0D} be the amount of money (i.e., valuation) that the buyer thinks is affordable and worthy to pay, based on the aforementioned personal considerations. We call Y_{0D} the endogenous domestic valuation.

Naturally, the valuation Y_{0D} is subsequently revised into a more sophisticated and realistic one, which we denote by X_{0D}, taking into account various exogenous factors. We collectively model these factors with a random variable Z_0, that we call the exogenous valuation adjustment. Let h_0 be the function that couples Y_{0D} with Z_0 and gives rise to the aforementioned price X_{0D}, that is,

$$X_{0D} = h_0(Y_{0D}, Z_0). \tag{4.26}$$

This is the amount of money (i.e., valuation) that the domestic buyer can afford, and is willing, to pay for the property during the initial selling stage.

Likewise, we arrive at

$$X_{0A} = h_0(Y_{0A}, Z_0), \tag{4.27}$$

which is the amount that the foreign buyer is willing to pay during the initial selling stage, where Y_{0A} is the corresponding endogenous valuation.

Note 4.4.2 Throughout this chapter we assume that the random variables Y_{0D}, Y_{0A}, and Z_0 are independent, which is a reasonable assumption as we argue next. Indeed, suppose that Y_{0D} and Y_{0A} are dependent. This would suggest that we have not properly separated the exogenous information from the individual valuations of the domestic and foreign buyers, thus contradicting the above description of the endogenous valuations Y_{0D} and Y_{0A}.

Hence, with X_{0D} representing the amount that the domestic buyer is willing to pay during the initial selling stage, and with X_{0A} representing the corresponding amount of the foreign buyer, the valuation X_0 can be expressed by the formula

$$X_0 = \xi_0 X_{0D} + (1 - \xi_0) X_{0A}, \tag{4.28}$$

where ξ_0 is a binary random variable taking values 1 and 0, with the event $\xi_0 = 1$ meaning 'domestic buyer.' The proportion of domestic buyers depends on the value of the exogenous valuation adjustment Z_0, which naturally gives rise to the function

$$q_0(z) = \mathbf{P}(\xi_0 = 1 \mid Z_0 = z),$$

that plays a pivotal role in our subsequent considerations.

Namely, when calculating the probability $\mathbf{P}[X_0 \geq x_0]$, we first condition on Z_0, whose cdf we denote by F_{Z_0}, and then separate X_{0D} from X_{0A} by conditioning on ξ_0. We obtain the equations

$$\begin{aligned}\mathbf{P}[X_0 \geq x_0] &= \int \mathbf{P}[X_0 \geq x_0 \mid Z_0 = z] \mathrm{d}F_{Z_0}(z) \\ &= \int \Big(q_0(z) \mathbf{P}[X_0 \geq x_0 \mid \xi_0 = 1, Z_0 = z] \\ &\qquad + (1 - q_0(z)) \mathbf{P}[X_0 \geq x_0 \mid \xi_0 = 0, Z_0 = z] \Big) \mathrm{d}F_{Z_0}(z). \end{aligned} \tag{4.29}$$

Using representation (4.28) and expressions (4.26) and (4.27) on the right-hand side of equation

(4.29), we obtain

$$\begin{aligned}
\mathbf{P}[X_0 \geq x_0] &= \int \Big(q_0(z)\mathbf{P}[X_{0D} \geq x_0 \mid \xi_0 = 1, Z_0 = z] \\
&\qquad + (1 - q_0(z))\mathbf{P}[X_{0A} \geq x_0 \mid \xi_0 = 0, Z_0 = z]\Big) \mathrm{d}F_{Z_0}(z) \\
&= \int \Big(q_0(z)\mathbf{P}[h_0(Y_{0D}, z) \geq x_0 \mid \xi_0 = 1, Z_0 = z] \\
&\qquad + (1 - q_0(z))\mathbf{P}[h_0(Y_{0A}, z) \geq x_0 \mid \xi_0 = 0, Z_0 = z]\Big) \mathrm{d}F_{Z_0}(z). \qquad (4.30)
\end{aligned}$$

We find it reasonable to simplify the right-hand side of equation (4.30) by first recalling that the endogenous domestic and foreign valuations Y_{0D} and Y_{0A} are independent of the exogenous valuation adjustment Z_0, and then we additionally assume that the valuations Y_{0D} and Y_{0A} do not depend on ξ_0. All of these are justifiable assumptions from the practical point of view. Hence, equation (4.30) simplifies into

$$\mathbf{P}[X_0 \geq x_0] = \int \Big(q_0(z)\mathbf{P}[h_0(Y_{0D}, z) \geq x_0] + (1 - q_0(z))\mathbf{P}[h_0(Y_{0A}, z) \geq x_0]\Big) \mathrm{d}F_{Z_0}(z). \qquad (4.31)$$

In the next subsection, we specialize formula (4.31) into a practically sound scenario based on the gamma distribution, under which we subsequently explore the expected profit $R(x_0, x_1)$ numerically and graphically (Section 4.6 below).

4.4.2 Specific modelling

The gamma distribution provides a good way to model Y_{0D}, Y_{0A}, and Z_0. In particular, we model the endogenous domestic price Y_{0D} using the shifted gamma distribution supported on the intervals $[a_0, \infty)$, with a_0 denoting the seller's reservation price, that is, we have the equation

$$Y_{0D} = a_0 + G_{0D},$$

where $G_{0D} \sim Ga(\alpha_{0D}, \beta_{0D})$. Assuming that the exogenous valuation adjustment Z_0 is an independent gamma random variable $G_0 \sim Ga(\alpha_0, \beta_0)$, the valuation X_{0D} can then be modelled as follows

$$X_{0D} = a_0 + G_{0D}G_0 = h_0(Y_{0D}, Z_0),$$

with the coupling function

$$h_0(y, z) = a_0 + (y - a_0)z. \tag{4.32}$$

Analogously, starting with

$$Y_{0A} = a_0 + (1 + \varphi_0)G_{0A},$$

where $G_{0A} \sim Ga(\alpha_{0A}, \beta_{0A})$ is an independent gamma random variable, with the factor $1 + \varphi_0$ referring to the $(1 + \varphi_0)100\%$ price change (e.g., increase when $\varphi_0 > 0$) that the foreign buyers additionally face when compared to the domestic ones, we arrive at the representation

$$X_{0A} = a_0 + (1 + \varphi_0)G_{0A}G_0 = h_0(Y_{0A}, Z_0),$$

with the same coupling function as in equation (4.32). We have used the same G_0 as in the 'domestic case'.

Note 4.4.3 To be in line with our earlier made assumption that foreign buyers generally offer higher endogenous valuations than the domestic ones, in our numerical explorations we choose the gamma parameters so that the average of $G_{0D} \sim Ga(\alpha_{0D}, \beta_{0D})$ does not exceed the average of $G_{0A} \sim Ga(\alpha_{0A}, \beta_{0A})$, which is equivalent to bound

$$\frac{\alpha_{0D}}{\beta_{0D}} \leq \frac{\alpha_{0A}}{\beta_{0A}}. \tag{4.33}$$

Bound (4.33) is satisfied for the parameter choices that we shall specify in Note 4.6.2 below.

Since the random variables G_{0D}, G_{0A}, and G_0 are independent, formula (4.31) reduces to the following one:

$$\mathbf{P}[X_0 \geq x_0] = 1 - \int_0^\infty \Big\{ q_0(z) F_{\alpha_{0D},\beta_{0D}}\Big(\frac{x_0 - a_0}{z}\Big) + (1 - q_0(z)) F_{\alpha_{0A},\beta_{0A}}\Big(\frac{x_0 - a_0}{(1+\varphi_0)z}\Big)\Big\} f_{\alpha_0,\beta_0}(z)\mathrm{d}z. \quad (4.34)$$

It is natural to view the function $q_0(z)$ as decreasing, and such that $q_0(0) = 1$ and $q_0(\infty) = 0$. Thus, for example, we can model $q_0(z)$ as a survival function (i.e., 1 minus a cdf) on the interval $[0, \infty)$. The gamma distributions serves a good model, and we thus set

$$q_0(z) = 1 - F_{\gamma_0,\delta_0}(z) \quad (4.35)$$

in our numerical research later in the chapter 4, with appropriately chosen shape $\gamma_0 > 0$ and rate $\delta_0 > 0$ parameters.

4.5 The second-stage selling probability

In this section, we express the probability $\mathbf{P}[X_t \geq x_1 \mid X_0 < x_0]$ in terms of underlying quantities at every time instance $t > 0$. We accomplish this task in a similar way to that for $\mathbf{P}[X_0 \geq x_0]$ in the previous section.

4.5.1 General considerations

We start with additional notations, mimicking the earlier ones. Firstly, we assume that the endogenous valuations Y_{tD} and Y_{tA} as well as the exogenous valuation adjustment Z_t are independent random variables. The definition of the coupling function h_t follows that in equation (4.32) but now with a_t instead of a_0, that is,

$$h_t(y, z) = a_t + (y - a_t)z.$$

Hence, with

$$X_{tD} = h_t(Y_{tD}, Z_t) \quad \text{and} \quad X_{tA} = h_t(Y_{tA}, Z_t),$$

we have

$$X_t = \xi_t X_{tD} + (1 - \xi_t) X_{tA}. \tag{4.36}$$

Analogously to equation (4.30), we obtain

$$\begin{aligned}\mathbf{P}[X_t \geq x_1 \mid X_0 < x_0] = \int \Big(& q_t(x_0, z)\mathbf{P}[h_t(Y_{tD}, z) \geq x_1 \mid X_0 < x_0, \xi_t = 1, Z_t = z] \\ & + (1 - q_t(x_0, z))\mathbf{P}[h_t(Y_{tA}, z) \geq x_1 \mid X_0 < x_0, \xi_t = 0, Z_t = z]\Big) \mathrm{d}F_{Z_t}(z),\end{aligned} \tag{4.37}$$

where

$$q_t(x_0, z) = \mathbf{P}(\xi_t = 1 \mid X_0 < x_0, Z_t = z)$$

is the proportion of domestic buyers at time t who did not buy during the initial selling stage (i.e, $X_0 < x_0$).

To make our following considerations simpler, we assume that the endogenous domestic and foreign valuations Y_{tD} and Y_{tA} are based solely on personal considerations at time $t > 0$, that is, they do not depend on any past or current exogenous factors, nor on the past endogenous factors Y_{0D} and Y_{0A}. In other words, we assume that the random variables Y_{tD} and Y_{tA} are independent of X_0, ξ_t and Z_t. This simplifies equation (4.37) into the following one:

$$\begin{aligned}&\mathbf{P}[X_t \geq x_1 \mid X_0 < x_0] \\ &\quad = \int \Big(q_t(x_0, z)\mathbf{P}[h_t(Y_{tD}, z) \geq x_1] + (1 - q_t(x_0, z))\mathbf{P}[h_t(Y_{tA}, z) \geq x_1]\Big) \mathrm{d}F_{Z_t}(z).\end{aligned} \tag{4.38}$$

4.5.2 Specific modelling

Analogously to the initial selling stage, we set

$$Y_{tD} = a_t + G_{tD} \quad \text{and} \quad Y_{tA} = a_t + (1 + \varphi_t)G_{tA},$$

where $G_{tD} \sim Ga(\alpha_{tD},\beta_{tD})$ and $G_{tA} \sim Ga(\alpha_{tA},\beta_{tA})$ with the factor $1 + \varphi_t$ referring to the $(1 + \varphi_t)100\%$ additional amount at time t that the foreign buyer needs to pay when compared to the domestic buyer. The exogenous valuation adjustment is

$$Z_t = G_t \sim Ga(\alpha_t,\beta_t).$$

We assume that the three gamma random variables G_{tD}, G_{tA}, and G_t are independent, in which case equation (4.38) reduces to

$$\mathbf{P}[X_t \geq x_1 \mid X_0 < x_0] = 1 - \int_0^\infty \Big\{ q_t(x_0, z)F_{\alpha_{tD},\beta_{tD}}\Big(\frac{x_1 - a_t}{z}\Big) + (1 - q_t(x_0, z))F_{\alpha_{tA},\beta_{tA}}\Big(\frac{x_1 - a_t}{(1 + \varphi_t)z}\Big)\Big\} f_{\alpha_t,\beta_t}(z)\mathrm{d}z. \quad (4.39)$$

It is reasonable to assume that the seller's reservation price a_t may change over time. For example, it may grow at the inflation rate. Hence, in our numerical explorations we assume that there is a constant ρ such that

$$a_t = (1 + \rho t)a_0,$$

for all $t \geq 0$. This assumption reduces equation (4.39) to

$$\mathbf{P}[X_t \geq x_1 \mid X_0 < x_0] = 1 - \int_0^\infty \Big\{ q_t(x_0, z)F_{\alpha_{tD},\beta_{tD}}\Big(\frac{x_1 - (1 + \rho t)a_0}{z}\Big) + (1 - q_t(x_0, z))F_{\alpha_{tA},\beta_{tA}}\Big(\frac{x_1 - (1 + \rho t)a_0}{(1 + \varphi_t)z}\Big)\Big\} f_{\alpha_0,\beta_0}(z)\mathrm{d}z, \quad (4.40)$$

where, for the sake of simplicity, we have assumed that the distribution of the exogenous valuation adjustment Z_t does not change with time t, that is, $Z_t \sim Ga(\alpha_0, \beta_0)$ for all $t \geq 0$.

Finally, we introduce an appropriate model for $q_t(x_0, z)$, which is more complex than that for $q_0(z)$. We start with a few observations:

1. When $x_0 = a_0$, it is reasonable to assume that there is not anyone wishing to wait until the second selling stage, and thus $q_t(a_0, z) = 0$ for every exogenous valuation adjustment z.

2. When $x_0 = +\infty$, no one wishes to buy during the initial selling stage, and thus $q_t(+\infty, z)$ should look like $q_0(z)$. Hence, we let $q_t(+\infty, z)$ be the survival function $1 - H_t(z)$ for a cdf $H_t(z)$ on the interval $[0, \infty)$. Just like in the case of $t = 0$, a good model for the cdf H_t is the gamma cdf F_{γ_t, δ_t} with shape $\gamma_t > 0$ and rate $\delta_t > 0$ parameters, which may depend on t.

3. It is reasonable to assume that $q_t(x_0, z)$ is an increasing function of x_0, because larger prices during the initial selling stage would suggest that more domestic buyers are deferring their purchases until the second selling stage.

In summary, we have arrived at the model

$$q_t(x_0, z) = Q_t(x_0 - a_0)(1 - H_t(z)), \tag{4.41}$$

where Q_t is a non-negatively supported cdf. In Section 4.6 below, we work with the gamma cdf, that is, we set

$$\begin{aligned} q_t(x_0, z) &= F_{\eta_t, \theta_t}(x_0 - a_0)(1 - F_{\gamma_t, \delta_t}(z)) \\ &= \frac{\gamma(\eta_t, x_0 - a_0)}{\Gamma(\eta_t)}\left(1 - \frac{\gamma(\gamma_t, \delta_t z)}{\Gamma(\gamma_t)}\right). \end{aligned} \tag{4.42}$$

4.6 Value functions and a numerical exploration

To make formula (4.23) actionable, in addition to the already discussed probabilities $\mathbf{P}[X_0 \geq x_0]$ and $\mathbf{P}[X_t \geq x_1 \mid X_0 < x_0]$, we need to specify appropriate models for the value functions $v_0(x_0)$ and $v_t(x_1)$.

Value function $v_0(x_0)$

We already have a model for $v_0(x_0)$ given by equation (4.22), but in view of our motivating example, an adjustment to this function needs to be made. Namely, property prices in Punta del Este, Uruguay, are predominantly in the US dollars, while property development costs are partially in the Uruguayan pesos and partially in the US dollars. In general, the costs are mainly due to land, design and development, materials, labor costs and subcontracts. Those that are in the Uruguayan pesos are labor costs (i.e., salaries of Uruguayan workers) and they can, for example, be around 30% of the structure's costs, that is, of the total cost minus the land cost. Therefore, we can say that, for some $\nu \in (0, 1)$, the percentage $\nu 100\%$ of the total cost is in the Uruguayan pesos and the rest $(1 - \nu)100\%$ is in the US dollars.

To express these costs into one currency, we convert the Uruguayan pesos into the US dollars – because the prices x_0 and x_1 are in the latter currency – using the exchange rate (US dollars per one Uruguayan peso) at an appropriate time instance. Namely, let ε_0 be the exchange rate during the initial selling stage (i.e., $t = 0$). Then equation (4.22) turns into the following one

$$v_0(x_0) = (x_0 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 - (1 - \nu)c_{0,\mathrm{USD}})_+. \tag{4.43}$$

Strictly speaking, the exchange rates are unknown in advance, and thus predicted values need to be used. It is very likely, however, that the prices x_0 and x_1 are set just before commencing the initial selling stage, and thus the value of ε_0 can be reasonably assumed known, and thus v_0 defined in equation (4.43) becomes deterministic and fully specified.

Value function $v_t(x_1)$

The exchange rate ε_t at time $t > 0$ cannot be known beforehand, that is, at time $t = 0$, and we thus treat it as a random variable. For this reason, we define v_t analogously as v_0, but now with the averaging over the distribution of ε_t, that is, we let

$$\begin{aligned} v_t(x_1) &= \mathbf{E}[(x_1 - \nu c_{0,\mathrm{UYU}}\varepsilon_t - (1-\nu)c_{0,\mathrm{USD}})_+] \\ &= \mathbf{E}[(x_1 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 r_t - (1-\nu)c_{0,\mathrm{USD}})_+], \end{aligned} \tag{4.44}$$

where $r_t = \varepsilon_t/\varepsilon_0$. In our numerical explorations, we let r_t follows the geometric Brownian motion, that is,

$$r_t = \exp\{\mu t + \sigma W_t\},$$

where W_t is the standard Wiener process (i.e., Brownian motion). This simple model has been a popular example in financial engineering. Equation (4.44) becomes

$$v_t(x_1) = \mathbf{E}[(x_1 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 \exp\{\mu t + \sigma\sqrt{t}\, N_{0,1}\} - (1-\nu)c_{0,\mathrm{USD}})_+], \tag{4.45}$$

where $N_{0,1}$ denotes the standard normal random variable.

We conclude this section with a note that arguments of Behavioural Economics may suggest using the more general value functions

$$v_0(x_0) = u(x_0 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 - (1-\nu)c_{0,\mathrm{USD}})$$

and

$$v_t(x_1) = \mathbf{E}[u(x_1 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 r_t - (1-\nu)c_{0,\mathrm{USD}})]$$

with some function u. Note that we have so far used $u(t) = t_+$, which is a very simple member in the class of so-called S-shaped functions: concave for $t \geq 0$ and convex for $t < 0$. Reverse S-shaped functions, which are convex for $t \geq 0$ and concave for $t < 0$, have also been extensively employed by researchers. We also find many studies where even more complexly shaped functions have been justified. For related discussions, we refer to, for example, Pennings and Smidts (2003), Gillen and Markowitz (2009), Dhami (2016), and references therein.

A numerical illustration

Using formulas (4.34), (4.40), (4.42), (4.43) and (4.45) on the right-hand side of equation (4.23), and with the parameter choices specified below, we obtain an expression for the expected profit $R(x_0, x_1)$ whose maximum with respect to x_0 and x_1 we want to find. Alongside the surface $R(x_0, x_1)$ and the point $(p_0^{\max}, p_1^{\max})$ where it achieves its maximum, in Figure 4.18 and 4.19, and in Figure 4.20 and 4.21 we have also depicted the profit functions $R_0(x_0)$ and $R_1(x_1)$.

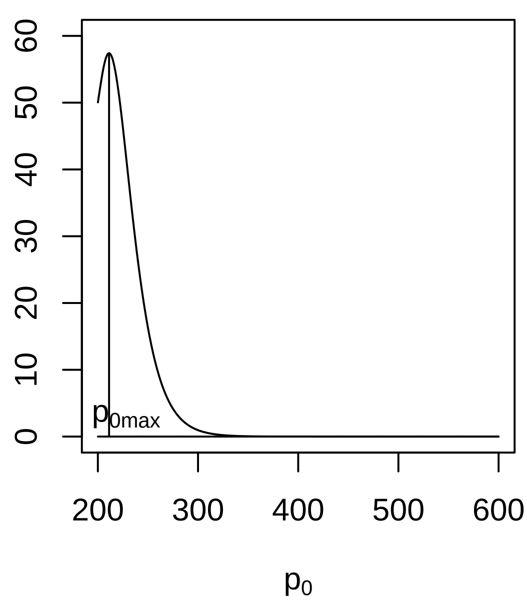


Figure 4.18: The function $R_0(x_0)$ of profit and price under the parameter specifications in Notes 4.6.1–4.6.4.

Next are the parameter choices that we have used in our numerical and graphical explorations, summarized in the four panels of Figure 4.18 and 4.21, and subsequently detailed in

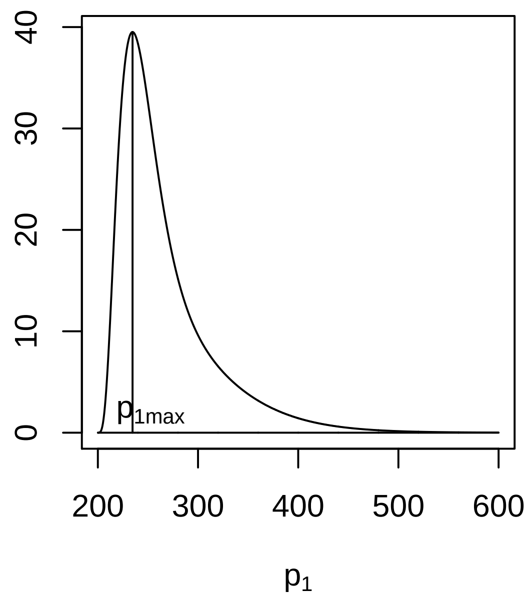


Figure 4.19: The function $R_1(x_1)$ of profit and price under the parameter specifications in Notes 4.6.1–4.6.4.

Figures from Figure 4.22 to Figure 4.24. We note that the parameter choices have arisen from our statistical analyses of (proprietary) data sets, as well as from our Economic Theory based considerations.

Note 4.6.1 We assume $T \sim Ga(\alpha_*, \beta_*)$ and set the following parameter values:

- $\alpha_* = 4$ and $\beta_* = 4$

Note 4.6.2 These are the specific parameter choices pertaining to the model of Section 4.4.2:

- $a_0 = 200$
- $\alpha_{0D} = 20$ and $\beta_{0D} = 0.6$
- $\alpha_{0A} = 30$ and $\beta_{0A} = 0.4$
- $\alpha_0 = \beta_0 = 4$

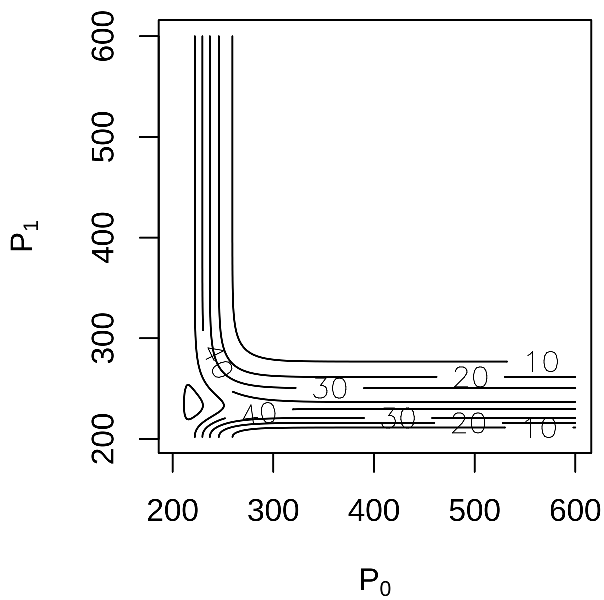


Figure 4.20: The surface $R(x_0, x_1)$ of profits and prices under the parameter specifications in Notes 4.6.1–4.6.4.

- $\varphi_0 = 0.2$
- $\gamma_0 = 10$ and $\delta_0 = 0.1$

Note 4.6.3 These are the specific parameter choices pertaining to the model of Section 4.5.2:

- $a_t = 200$ $(= a_0)$
- $\varphi_t = 0.2$
- $\rho = 0.1$
- $\alpha_{tD} = 20$ $(= \alpha_{0D})$ and $\beta_{tD} = 0.6$ $(= \beta_{0D})$
- $\alpha_{tA} = 30$ $(= \alpha_{0A})$ and $\beta_{tA} = 0.4$ $(= \beta_{0A})$
- $\alpha_t = \beta_t = 4$ $(= \alpha_0 = \beta_0)$

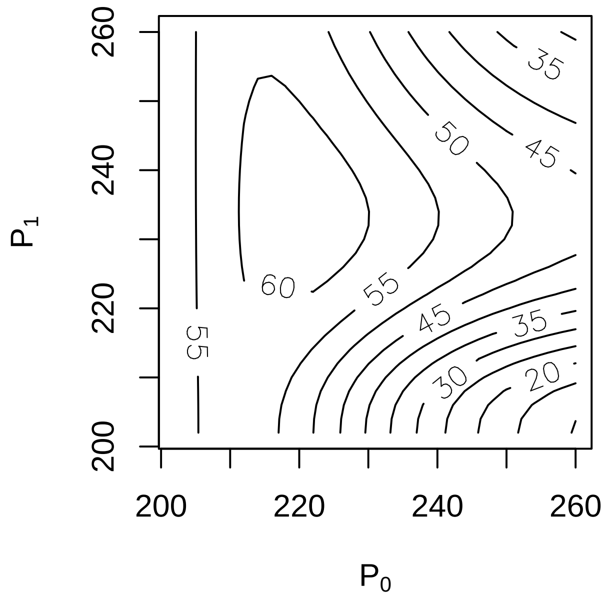


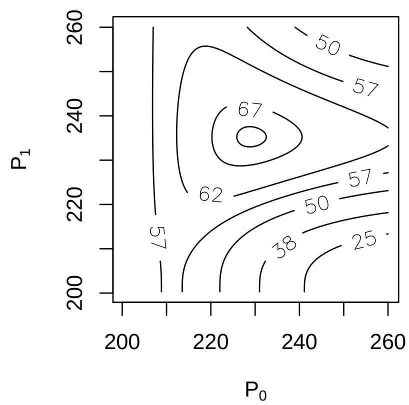
Figure 4.21: The center area of the surface $R(x_0, x_1)$ of profits and prices under the parameter specifications in Notes 4.6.1–4.6.4.

- $\eta_t = 8$ and $\theta_t = 1$
- $\gamma_t = 10$ and $\delta_t = 0.1$

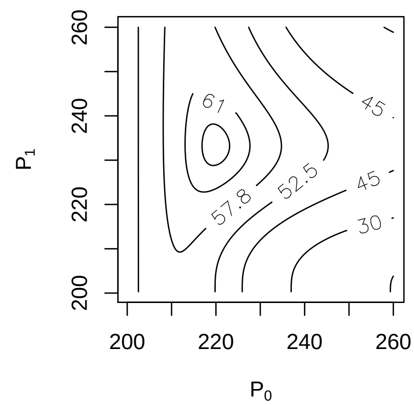
Note 4.6.4 These are the specific parameter choices pertaining to the value function $v_t(x_1)$ discussed in Section 4.6:

- $\nu = 0.3$
- $c_{0,\text{UYU}}\varepsilon_0 = 150$ and $c_{0,\text{USD}} = 150$
- $\mu = 0$ and $\sigma = 1$

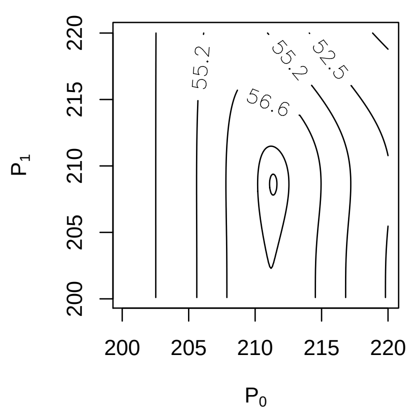
The proposed model has been developed to facilitate well-informed decisions, and the real-life example has guided us in every step of the model development. The model has, inevitably, turned out to be complex. Hence, at this initial stage of our exploration, we have prioritized certain aspects of the research according to their relevance in terms of policy implications, in order to keep considerations within reasonable space limits. The timing of price setting has perhaps been the most significant aspect that is affecting all the other ones. The dependence between the two-stage pricing decisions and the influence of the systematic (or background) risk has been among the other important aspects. The exchange rate fluctuations, though very important, have nevertheless been given a lesser attention in the present chapter, due to a justifiable reason. Namely, a detailed exploration of this aspect with due mathematical care of its various issues such as change points, heteroscedasticity, and other non-linear structures manifesting naturally in financial stochastic models would require considerable space. Our use of the simple geometric Browning motion, instead of a more complex and realistic process, has also been influenced by space considerations. Nevertheless, to give an initial idea about the influence of the mean μ and the volatility σ, we have produced a set of graphs in Figures from Figure 4.23 to Figure 4.24.



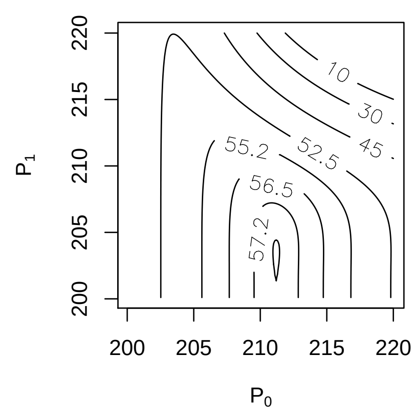
(a) $\mu = 0$ and $\sigma = 0.5$.



(b) $\mu = 0$ and $\sigma = 1$.

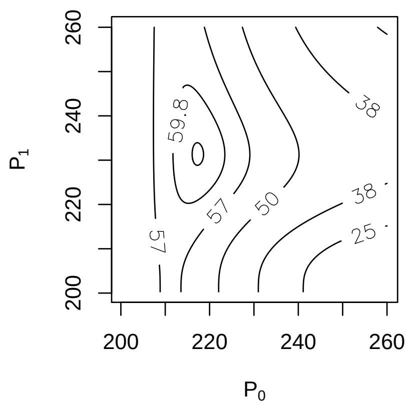


(c) $\mu = 0$ and $\sigma = 2$.

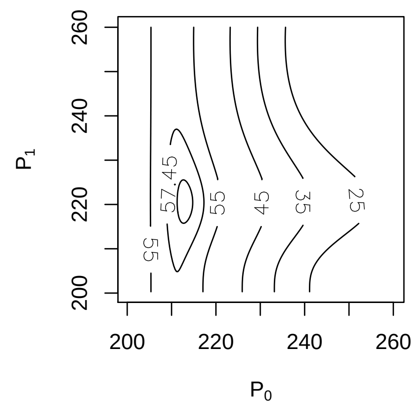


(d) $\mu = 0$ and $\sigma = 3$.

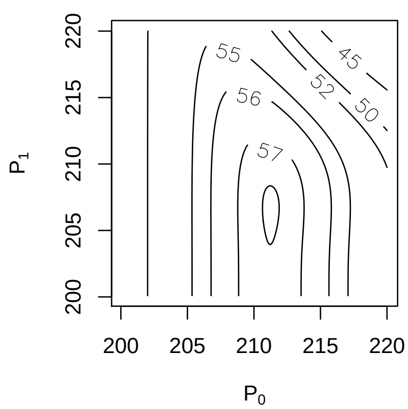
Figure 4.22: $\Pi(p_0, p_1)$ for $\mu = 0$ and various values of σ.



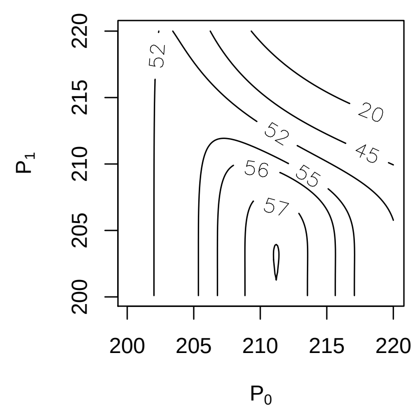
(a) μ = 0.5 and σ = 0.5.



(b) μ = 0.5 and σ = 1.

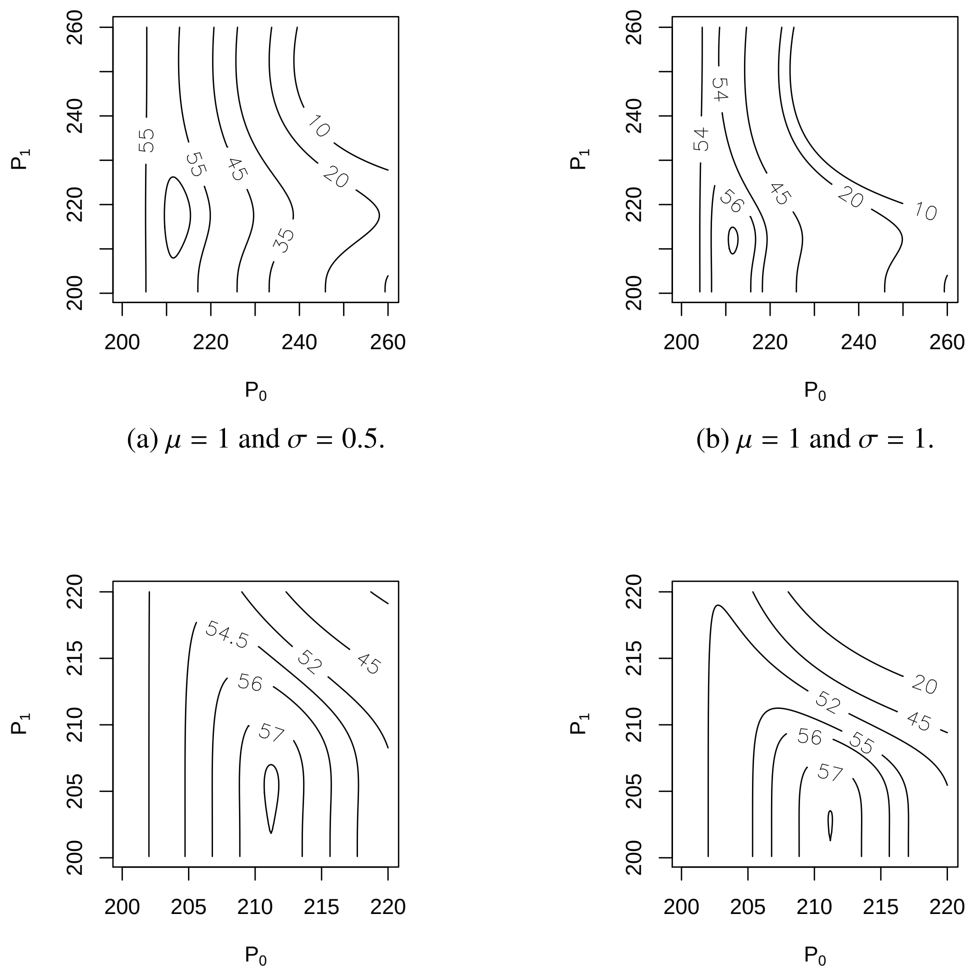


(c) μ = 0.5 and σ = 2.



(d) μ = 0.5 and σ = 3.

Figure 4.23: $\Pi(p_0, p_1)$ for $\mu = 0.5$ and various values of σ.



(a) $\mu = 1$ and $\sigma = 0.5$.

(b) $\mu = 1$ and $\sigma = 1$.

(c) $\mu = 1$ and $\sigma = 2$.

(d) $\mu = 1$ and $\sigma = 3$.

Figure 4.24: $\Pi(p_0, p_1)$ for $\mu = 1$ and various values of σ.

Chapter 5

Conclusions and future works

5.1 Conclusions

The thesis discusses optimal strategies and decisions in a two-period economy scenario, and we propose general frameworks to involve a trade-off between the the present (current period) benefits and the benefits in the future (next period). In this thesis, following the logic of the two-envelope problem and inspired by the pioneer works of McDonnell and Abbott (2009), McDonnell et al. (2011) and Egozcue et. al. (2013), we construct the two-period time horizon economic models of optimal strategies and decisions to deal with the realistic problems.

In the case of buying laptop whether in *Black Friday*, or *Boxing day*, we have proposed the general two-period optimal strategies to make decisions which minimize the expected buying price from the buyers perspective. In this model, we have tackled this topic in the setup that concerns shoppers facing two price-discount periods and needing a strategy for making beneficial (for them) decisions. Incorporating the market structure theory of economics, we have applied the third-degree price discrimination theory to describe the distribution of price variables in two stages respectively. Guided by these economic theories and also, some rigorous probabilistic considerations, we have developed a practically sound model for making such decisions. In particular, we have shown how to derive, analyze, and use strategy functions, which not only delineate the acceptance and rejection regions for the first-come offers but also tell us how confident we can be when making such decisions.

In the cases of selling real estate, we have proposed the general two-period pricing models and explored various pricing strategies from the seller's perspective. The model takes into account such practical considerations as the facts that the buyer's valuations, which are random from the seller's perspective, in the two periods may or may not be independent, may or may not follow the same distribution, and so on. We have seen in particular that the seller's simultaneous-pricing strategies yield higher expected revenues than the sequential-pricing strategies. Our general model allows for the possibility of commodity costs being denominated in different currencies, and thus being impacted by currency exchange-rate movements. The model also takes into accounts various endogenous and exogenous factors, such as seller's and buyer's considerations, general economic conditions, different seller's utility or value functions. We have illustrated our theoretical findings both numerically and graphically, using appropriately constructed multiplicative background models that quickly take into account various specific elements of the motivating problem.

5.2 Future works

This thesis provides some insights related to the application of the two-envelope paradox framework on the general decision-making process from the perspectives of both purchasing and selling. However, the distributional assumptions of the prices are straightforward. Hence one possible future work is to generalize these assumptions. For instance, when considering a discounted price (having bounded support), we could assume that the price offered by the seller has a generalized Beta distribution of the second kind (GB2). More details about the GB2 distribution could be found in textbooks such as Section 6.1 of Kleiber and Kotz (2003). As a distribution widely used in modeling risks and prices, the GB2 distribution is more flexible than Beta distribution and thus may provide more accurate results if the parameters are estimated from real data. Besides, the truncated gamma distribution used by Quan and Quigley (1991) might be another choice for modeling the discounted price. Since this distribution is less related to the Beta distribution than the GB2 distribution, if similar results hold in this case then our method will seem to be useful for a broader range of distributions that could be employed for modeling prices.

Similarly, when considering the offered prices of underlying assets such as real estates, the distribution of the offered prices could be generalized in several different ways. For instance, the offered prices could be assumed to follow the generalized gamma distribution (Kleiber and Kotz, 2003, Section 5.1), which are usually used for modeling the incomes. Other choices may include heavy tail distributions such as log-normal distribution or Pareto distribution if there is a booming market for assets like real estate.

After studying the decision-making problems with specific distributions based on our generalized two-envelope paradox framework, a natural question is to identify certain classes of distributions which may still lead to the similar results. For prices with a bounded-support, we might be interested in the difference between the results by modeling the prices using a "flatter" distribution (by flatter we mean the distribution has huge variance and similar to the uniform distribution) or a "steeper" distribution. For prices without an upper bound, we might be interested in the difference between the results by modeling the prices using a heavy-tailed distribution and a light-tailed distribution, which correspond to a booming and a falling market of certain assets, respectively.

Besides of generalizing the distributional assumptions of prices, another important potential work is to apply our generalized two-envelope framework on multi-period decision processes such as the American option. Assume an investor holds an American option, he/she has to decide whether to exercise the option at the end of each period. If the option is not exercised, the investor will keep making decisions until he/she decides to use it or the maturity date of the option. The assumptions that required by the framework may include the martingale assumption (the return of the underlying asset is a martingale), AR(1) assumptions (the return of the underlying asset is an AR(1) time series), or other more complex hypotheses.

Furthermore, in our analysis there is only one decision-maker who is considered and a right decision is made depending on the curves we obtained. In fact, this piece of information is much more crucial to the counterparty. If the seller knows the rules of decision-making of the buyer, there might be a strategic adjustment on how to determine the discounted prices on traditional holidays. Similarly, if the buyers of real estates know the rule of decision-making of the seller, they will adjust their offered prices accordingly. Therefore, we may expect a significant gaming problem between buyers and sellers under our generalized two-envelope framework.

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