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Annuity Product Valuation and Risk Measurement under Correlated Financial and Longevity Risks

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Abstract

Longevity risk is a non-diversifiable risk and regarded as a pressing socio-economic challenge of the century. Its accurate assessment and quantification is therefore critical to enable pension-fund companies provide sustainable old-age security and maintain a resilient global insurance market. Fluctuations and a decreasing trend in mortality rates, which give rise to longevity risk, as well as the uncertainty in interest-rate dynamics constitute the two fundamental determinants in pricing and risk management of longevity-dependent products. We also note that historical data reveal some evidence of strong correlation between mortality and interest rates and must be taken into account when modelling their joint dynamics. In this thesis, we model and examine the impact of nonlinearity and correlation on an annuity product. A regime-switching approach to address nonlinearity is embedded both in the Lee-Carter model for mortality rate modelling and prediction, and in the Vasiček model for capturing interest-rate movements. In the valuation and computation of risk measures for an annuity that are being carried out to satisfy regulatory requirements, the correlation structure between mortality and financial risks is explicitly modelled. Our proposed modelling framework is implemented on simulated data as well as actual data covering the South Korean population and Korean bond yields for the period 1980-2015. Our results demonstrate the significant effect of correlation on annuity and risk-metric values. Finally, we found that the use of regime-switching techniques for both mortality and interest rate modelling creates a greater latitude in obtaining accurate prices, based on models' parameter estimates, and in setting capital adequacy that avoids substantial over-reserving or under-reserving.

Keywords: Markovian regime-switching models, Lee-Carter model, Vasiček model, correlation, mortality risk, interest rate, insurance product valuation

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Chapter 1

Introduction

The United Nations, in one of its technical reports [55], declared that aging populations poses a threat to the stability of social security and consequently can burden regional and global economy. Although medical advances are positive developments, increased life expectancy over the last few decades in both the developed and developing countries brought financial consequences that are becoming a major concern for the public and private sectors; see Roser [48]. A growing elderly population implies more pension funds are required to be disbursed. Undoubtedly, as the aging population gets bigger on a global scale, reliable methods on life expectancy modelling and prediction are hugely important. Moreover, crucial attention must be paid to the management of the associated risks for the various aspects of the pension and insurance business.

The key concept akin to a longer life expectancy is longevity risk, which refers to the risk arising from the situation where the annuitants' actual life expectancy is greater than what was predicted. Equivalently, it is the risk whereby the actual future mortality rate is lower than what was expected. Kontis, et al. [30] identified elements such as medical progress and improved health technologies that collectively contribute to mortality rate reduction, and longevity risk is then gradually realised. Historically, according to the International Monetary Fund (IMF) [28], the actual life expectancy consistently exceeded the forecasted value; and this finding is not only limited to a certain demographic group but rather on a global scale since the 1990's.

The severity of longevity risk can now be felt more clearly than ever before. Consequently, this has started to strain the capabilities of governments as well as the private pension/annuity providers in fulfilling their financial obligations to the retirees. To calculate the value of additional pension payments, it is essential to estimate the mortality rate at the time of death along with the interest rate that must be employed in discounting future pension payments to present value. An IMF's study [28] concluded that extra resources valued around 40 % to 80 % of the GDP are required for advanced countries to sustain their social security systems if the life expectancy is increased by 3 years from the presently expected value. The dire impact of longevity risk combined with the underestimation of life expectancy necessitates the improvement of existing models and the methods to measure the associated risks and costs.

As emphasised above, the primary factors that need to be estimated in dealing with longevity risk and the associated costs are the mortality and interest rates. The most widely used models for the evolution of these variables are linear time series models such as the autoregressive (AR), moving average (MA), and AR integrated MA (ARIMA) models. These models have the advantage that they are easy to understand and apply. However, they lack the feature of capturing nonlinear dynamic patterns; for example, these time series models are limited in their ability to capture very well asymmetry and volatility clustering, as described in Mamon and Elliott [39]. Such a limitation is a main source of error in the long-term estimation of mortality and interest rates given that both factors exhibit randomness amidst nonlinear patterns. Nonlinear time series models are therefore more suitable for modeling mortality and interest rates.

A Markov regime-switching methodology, introduced by Hamilton [21], is simple yet powerful in modelling nonlinearities in time series data. In the context of this thesis, a regime-switching approach utilises multiple time series models having different parameter values to characterise different regimes. Our approach then allows the model to switch amongst multiple equations in order to capture more complex nonlinear behaviour. Regime-switching model specifications are benchmarked vis-a-vis a one-state model (i.e., no switching and typically describing the simplest dynamics) throughout this thesis.

In current practice, the modelling of mortality and interest rates is still conducted separately when valuing annuity products. This means that the correlation between these rates are ignored, and they are simply assumed independent of each other. But, there is robust evidence particularly in developed countries from historical joint series of mortality and interest rates of their factual correlation [42]. Although the exact cause of this correlation is difficult to pinpoint, a general cause-and-effect argument of the two risk factors can be described. A decrease in mortality rate imposes financial stress on the government and national savings. This in turn negatively affects the local economy that ultimately contributes to the rate of returns and interest rate. On the other hand, a high interest rate directly impacts both the economy and individuals with debts and mortgages. Increased interest rates reduce one's capacity to afford medical care and quality food and/or nourishment that directly impact the longevity of individuals. It is therefore necessary to include mortality and interest rates' correlation, especially when it is highly non-zero; and it must be incorporated in the model that supports the pricing and risk management of insurance products.

1.1 Research scope and objectives

The main objective of this thesis is to improve traditional methodologies in the valuation and risk measurement of insurance products. We shall develop a modelling framework via a regime-switching approach that incorporates correlation to capture nonlinear dynamics of the underlying variables and their dependence. More specifically, our aims are as follows:

(i) To analyse the impact of the Markovian regime-switching method in the modelling of mortality and interest rates by developing one-, two- and three-state models. The performance of these models (e.g., fitting, out-of-sample forecasting ability, etc) will be compared and validated, against a set of benchmarks, using simulated and observed data.

(ii) To project correlated mortality and interest rates for the next 45 years under the three models in (i) for pricing and risk management.

(iii) To find out the impact of correlation and regime-switches on the price and risk metrics, we shall also perform systematic sensitivity analyses. This will give us insights on what parameters to monitor more closely.

1.2 Structure of the thesis

This thesis is organised in the following manner. Chapter 2 presents an overview of the key concepts and the methodologies that will be used throughout this research exposition. The development of the mortality rate model as well as the projection of mortality rates for a 45-year period is presented in Chapter 3. The Markov-modulated regime-switching method is employed where one-, two- and three-state models are formulated and compared under some information-based criteria and forecasting metrics. Similar to Chapter 3, the development of the interest rate model, also with regime-switching feature, is discussed in Chapter 4 along with the simulation implementation over a 45-year period; in this chapter, we also investigate how correlation influences the joint evolution of mortality and interest rates . The valuation of annuity and measurement of longevity-risk impact based on the models presented in Chapters 3 and 4 are demonstrated in Chapter 5. Finally, Chapter 6 gives some concluding remarks.

Chapter 2

Modeling methodologies

This chapter provides an overview of the modelling methodologies that are widely used to model and predict mortality and interest rates. Relative to this current literature, we shall propose modelling approaches for both mortality and interest rate dynamics.

2.1 Mortality rate prediction

2.1.1 Survey of relevant existing models

Models for mortality rate modelling and projections are classified into two main categories. The first type estimates the present mortality rate based on available data. The primary issue is the way mortality rates are calculated for old-age groups. Here, the size of the population data for these particular groups is too small to be deemed reliable. Therefore, data from the age groups with a significant population size are used to extrapolate mortality rates for old ages. Examples of models that fall into this category include Gompertz [19]; Gompertz and Makeham [36]; Himes, et al. [23]; Coale and Kisker [9]; and Brass [6]. As this class of models is designed chiefly to describe *present* mortality rates, it is not well-suited for the pricing of insurance products that must depend on *future* mortality rates.

The second type of models was developed to forecast and simulate future mortality rates on the basis of past and present mortality rate trends. The models of this type include Lee and Carter [32]; Lee and Miller [33]; Booth, et al. [4]; Li, et al. [34]; Renshaw and Haberman [44]; and Cairns, et al. [7], amongst others. Under this category, the Lee-Carter (LC) model is a widely used model for mortality and gave rise to the emergence of other models. That is, other models in this category are extended versions of the original LC model and are often collectively referred to as the extended Lee-Carter models. In spite of copious alternatives from these extended models, the LC model remains the most popular. It is a model that is extensively applied owing to its mortality rates' age- and time-dependence specification while keeping the simplicity of its modelling structure.

One critical frailty of the LC model is its assumption, in an effort to achieve simplicity, that mortality improvement at each age remains invariant with time. This assumption, however, is not in agreement with the data collected from several countries over the past decades [3]. The invariance of mortality rate improvement poses inaccuracy to mortality rate projections and could produce implausible results in the long term. Extended models, as mentioned above, attempt to mitigate such drawback by augmenting the number of parameters to the LC model in order to capture certain characteristics exhibited by the time series of mortality data.

In this thesis, the aforementioned failing of the LC model — the assumption that the mortality rate improvement at each age remains invariant over time — is addressed via a Markovian regime- switching approach, which is outlined in section 2.3. In other words, the original LC model is enriched by our approach in an effort to obtain dependable mortality forecasts. The technical aspects of the LC model is detailed in the next section.

2.1.2 The LC model

This model was based on the works of Lee and Carter in 1990's on mortality rate estimation [32]. It forecasts mortality based on the persistent long-term historical patterns and trends, instead of relying on certain behavioural, medical, or social influences.

The LC model states that the logarithm of the mortality $m_{x,t}$ satisfies

$$\ln(m_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t}, \tag{2.1}$$

or

$$m_{x,t} = e^{a_x + b_x k_t + \varepsilon_{x,t}}. \tag{2.2}$$

The subscripts x and t represent age and time, respectively. The coefficients a_x and b_x are age-specific constants, k_t is a time-varying index and $\varepsilon_{x,t}$ is an error term. The parameters in equations (2.1)–(2.2) are interpreted as follows:

- a_x describes the general shape of $\ln(m_{x,t})$ across age,
- k_t depicts the variation in the mortality level with time t,
- b_x refers to the sensitivity of the logarithm of the mortality at age x to variations in the time index k_t, and
- $\varepsilon_{x,t}$ reflects the particular age-time variation of historical influences not captured by the other components of the model.

Mortality forecasting in the LC model is performed in two steps. In the first step, the parameters a_x, b_x and k_t are estimated based on historical mortality data. However, the model expressed by equation (2.1) or (2.2) cannot be fitted by the usual ordinary least squares (OLS) method. This is because a_x, b_x and k_t are not observable variables. To rectify this problem, Lee and Carter [32] proposed a singular value decomposition (SVD) method [1] under two imposed constraints (i.e., $\sum_t k_t = 0$ and $\sum_x b_x = 1$) to determine the parameters of the model. The values of a_x and b_x are time-independent, and thus, their values found in this step are sufficient in forecasting future mortality rates. However, the obtained value of the time-varying k_t is solely based on the historical data, and an extrapolation for k_t is required for the purpose of pricing.

[1] In linear algebra, the singular value decomposition of a complex matrix $\mathbf{A}$ is the factorisation of $\mathbf{A}$ into the product of three matrices $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ where the columns of $\mathbf{U}$ and $\mathbf{V}$ are orthonormal and the matrix $\mathbf{D}$ is diagonal with positive real entries. Applications abound in signal processing, statistics and other allied fields.

In the second step, the fitted values of k_t are extrapolated, as indicated above, using the autoregressive integrated moving average (ARIMA) method pioneered by Box and Jenkins [5]. ARIMA encompasses a general class of models that are fitted to time series data to forecast future data values. There are many types of ARIMA modelling specifications, but the model used in the original LC paper, and is also adopted in this thesis, expresses k_t as

$$k_t = k_{t-1} + c + \varepsilon_t. \tag{2.3}$$

The model in equation (2.3) is also referred to as a random walk with drift. The terms c and ε_t are the drift term and a white noise components, respectively. The drift is the average change of k_t from time $t-1$ to time t, and the white noise is also called an innovation of k_t with $\varepsilon_t \sim N(0, \sigma^2)$.

2.2 Interest-rate modelling

2.2.1 Pertinent existing literature

As Chan et al. [8] argued, the interest rate is perhaps the most important economic variable that drives the financial markets. Modelling endeaovours, to be able to predict its trends and dynamics, are expended more than any other modelling efforts for other economic indicators. Many of the interest-rate models in both academic research and practice are built to capture the dynamic behaviour of the short-term rate process. As pointed out in Fabbozzi [15], there are two classes of models for describing the interest-rate dynamics: the no-arbitrage and equilibrium models .

The no-arbitrage models use the current term structure of interest rates[2] as an input to achieve an exact fit to longer-term bond prices and other interest rate derivatives. This means that no-arbitrage models employ current market prices of interest-related products (e.g., bond and derivatives) and adjust the model parameters to fit the prices exactly. Consequently, this

[2]The term structure of interest rates, also called the yield curve, illustrates the relationship between the interest rates and different terms/investment horizons with an expectation about the interest rate changes in the future.

type of models inherently avoids the occurrence of arbitrage, which is why it is called "no-arbitrage" or "arbitrage-free" models. Models developed by Ho and Lee [24], Hull and White [26] and Heath et al. [22] fall into this classification.

The equilibrium models, on the other hand, aim to capture the time-evolving trends of the term structure. In contrast to the no-arbitrage models, equilibrium models do not use a term structure at a particular point in time to avoid arbitrage. Instead, equilibrium models makes use of a statistical approach where the term structure is estimated based on past market prices. As this model type is based on the average interest rate over time (mean reversion), the projected interest rate converges the historic average over time, which is where the term "equilibrium" comes from. The equilibrium models are more commonly used for the risk management of financial institutions due to their future interest-rate exposures on various investment positions. The representative models that fall into the equilibrium model category are the Vasiček model [56] and the Cox-Ingersoll-Ross (CIR) model [11]. For further discussion of equilibrium and no-arbitrage models, see Hull [25].

In general, the change in the interest, in the continuous-time setting, for both the no-arbitrage and equilibrium models can be expressed as

$$dr_t = \mu(r,t)dt + \sigma(r,t)dW_t, \tag{2.4}$$

where dr_t is the change in the interest rate over the time interval dt. The change in the interest rate is decomposed into two parts: drift ($\mu(r,t)$) and volatility ($\sigma(r,t)$). The drift $\mu(r,t)$ specifies the expected or average interest rate change (thus the term "drift") at each instance. The component $\mu(r,t)$ is also the term in which certain characteristics of interest rates (e.g., mean reversion) can be incorporated. The second term on the right-hand side of equation (2.4), $\sigma(r,t)dW_t$, consists of volatility, $\sigma(r,t)$, and the increment of a standard Brownian motion, dW_t. The quantity dW_t is normally distributed with a mean of zero and the time interval dt is its variance.

Most, if not all, of the interest-rate models share the form of equation (2.4) in which it has the drift and the volatility representations. The differences in the models arise in how the drift and volatility terms are given as shown in Table 2.1. The key distinction is that the no-arbitrage models set the drift or volatility terms to be time-dependent in order to match today's term structure, whilst the equilibrium models describe the two terms with a small number of statistically estimated constant parameters drawn from historical market data.

Table 2.1: Interest model comparison.

Type	Model	Dynamics	Drift	Volatility
no-arbitrage	Ho-Lee [24]	$dr_t = \theta(t)dt + \sigma dW_t$	time-dependent	constant
no-arbitrage	Hull-White [26]	$dr_t = (\theta(t) - \alpha r_t)dt + \sigma(t)dW_t$	time-dependent	time-dependent
equilibrium	Vasiček [56]	$dr_t = a(b - r_t)dt + \sigma dW_t$	mean-reverting	constant
equilibrium	CIR [12]	$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$	mean-reverting	proportional

The two types of models have different focus (and thus, strengths). For emphasis again, the no-arbitrage models focus on fitting the price of interest-dependent products exactly based on present term-structure to avoid arbitrage, whilst the equilibrium model focuses on projecting future interest rate movement based on the mean reversion. The equilibrium model is a more suitable type for the research work explored in this thesis because our main objective entails the projection of interest rates into the future as far as measuring the longevity risk goes.

Both the Vasiček and CIR models have withstood the test of time for practical implementation because they both have easy interpretations and analytic bond pricing solutions, which are excellent for risk management and model calibration considerations. The Vasiček and CIR models's drifts and volatilities are shown in Table 2.1. They satisfy a corresponding system of ordinary differential equations, for each model, whose solutions yield a closed-form bond pricing expression.

As noted in Copeland, et al. [10], both the Vasiček and CIR models have the drift term $(a(b - r_t))$ that depends linearly on time. This becomes the main source of the misspecification for these models. This misspecification problem can be rectified by using a Markovian regime-switching approach detailed in Section 2.3. The difference between the two models comes from how the volatility term is set forth; it is constant in the Vasiček model and is proportional to the square root of the interest rate in the CIR model. The square root term implies that the CIR model assumes that r_t takes only non-negative values. For the CIR model, the choice of the parameters can be controlled in such a way that only positive rates are generated; refer to Mamon [37], for instance.

Of these two interest-rate models, the Vasiček model is employed in this thesis for simulating future interest rates because of the nature of interest rates in South Korea in the next 30 years. In particular, South Korean rates have the positive probability of hitting negative values (i.e., likely to occur) during some intervals based on historical trends whereby (i) the current interest rate is below 2 % and (ii) the interest rate has been continuously declining for the past 20 years. We explain the development and features of the Vasiček model in the next subsection.

2.2.2 The Vasiček model

In continuous-time finance, Vasiček is regarded as the first stochastic model for interest rates. The model assumes that the short-term interest rate, r_t, is governed by the stochastic differential equation (SDE)

$$dr_t = a(b - r_t)dt + \sigma dW_t, \tag{2.5}$$

where a, b and σ are constants. The respective parameters b, a and σ are the r_t's long-term mean, speed of adjustment in its attempt to reach b, and volatility. As noted in Mamon [38], the interest rate, r_t, follows a Markov process and is memoryless (i.e., the future development of r_t is independent of its past). The short-term rate r_t will be pulled back towards b when it deviates away from it.

The main purpose of estimating the future interest rates is to determine the appropriate discount factor and the pricing of bonds. The mathematical framework, under the Vasiček set up, for bond valuation is discussed in the sequel following Mamon's formulation and development [38].

The underlying background for the stochastic modelling of interest rates is a probability space, which is a triplet $(\Omega, \mathcal{F}, P)$ endowed with a standard filtration $\{\mathcal{F}_t, t \geq 0\}$. Here, Ω is a sample space, $\mathcal{F}$ is the event space and P is a probability function equipped with a standard filtration $\{\mathcal{F}_t\}$ ($P : \mathcal{F} \to [0,1]$, where $0 \leq P(A) \leq 1$ is the probability that the event $A \in \mathcal{F}$ occurs). The solution of equation (2.5) is

$$r_t = e^{-at}\Big[r_0 + \int_o^t abe^{au}du + \sigma \int_o^t e^{au}dW_u\Big], \tag{2.6}$$

which can be verified using Itô's formula[3] Equation (2.6) can be re-written as

$$\begin{aligned} r_t &= e^{-at}\Big[r_0 + \int_o^t abe^{au}du + \sigma \int_o^t e^{au}dW_u\Big] \\ &= e^{-at}\left[r_0 + b(e^{at} - 1) + \sigma \int_o^t e^{au}dW_u\right] \\ &= e^{-at}\left[r_0 + b(e^{at} - 1)\right] + e^{-at}\sigma \int_o^t e^{au}dW_u \\ &= \mu_t + \sigma \int_o^t e^{a(u-t)}dW_u, \end{aligned} \tag{2.7}$$

where μ_t is a deterministic function, which also gives the expectation value of r_t, i.e., $E[r_t]$.

Assume that there is a risk-neutral measure Q equivalent to P. With E^Q denoting the expectation under Q, the price of a zero-coupon bond with maturity T at time t, $B(t,T)$, is expressed as

$$B(t,T) = E^Q\left[\exp\left(-\int_t^T r_u du\right)\Big| F_t\right]. \tag{2.8}$$

Suppose r_t has the same specification given in equation (2.5) but under Q instead. It is demonstrated in Mamon [37], employing three different ways (probabilistic, partial differential equation (PDE) and forward-measure-based technique), that

$$B(t,T,r_t) = \exp\Big(-A(t,T)\,r_t + D(t,T)\Big), \tag{2.9}$$

[3] This is a well-known result in Stochastic Calculus and has become a mathematical tool in deriving the dynamics of a function of a stochastic process, see Shreve [51].

where

$$A(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{2.10}$$

and

$$D(t,T) = \left(b - \frac{\sigma^2}{2a^2}\right)\Big[A(t,T) - (T-t)\Big] - \frac{\sigma^2 A(t,T)^2}{4a}. \tag{2.11}$$

Therefore, given the paramters a, b and σ, the yield rate, $y(t,T)$ of zero-coupon bond, is

$$\begin{aligned} y(t,T) &= -\frac{\log B(t,T,r_t)}{T-t} \\ &= -\frac{1}{T-t}\Big(-A(t,T)\,r_t + D(t,T)\Big) \end{aligned} \tag{2.12}$$

and the yield curve of the bond is completely determined.

2.3 The Markovian regime-switching method

Linear time series models are widely used for analysing the dynamic behaviour of economic and financial variables. However, these variables could exhibit non-linear patterns. The simplest method to capture them is to introduce multiple states so that a mixture of states could reproduce those non-linear patterns. A (latent) factor driving a variable can assume only one state at any given time, and in the next time step it can either stay on that state or switch to another state with corresponding probability values. The method of assigning multiple states to a factor or variable to capture the non-linearity in time series is the basic principle of a regime-switching approach. The 'force' behind that dictates the switching is a Markov chain in either discrete or continuous-time setting. Such a Markov chain assumes that the current state depends only on the most recent previous state.

In the context of this work, the relevant non-linear time series model employing a Markovian regime-switching framework is Gao, et al. [18]. The classic work that promotes the Markov regime-switching method was due to Hamilton [21]. Since then, various models have used the Markovian regime-switching approach to explain the characteristics of interest rates and the term structure. Some examples include Bansal and Zhou [2], Smith [52], and Kalimipalli and Susmel [29], Erlwein and Mamon [13], Zhou and Mamon [60], Xi and Mamon ([57] and [58]), and Grimm, et al. [20], amongst many research studies.

The success of Markov-modulated regime-switching methodology in finance has permeated into actuarial research. Milidonis, et al. [41] showcased the power of this method in modelling the mortality dynamics. In recent years, Gao, et al. [40] and Gao, et al. [18] priced and provided risk measurement analysis for guaranteed annuity options.

In this thesis, the Markov regime-switching approach is applied to determine the time-varying index, k_t, of the LC model and the interest rate, r_t, of the Vasiček model. This approach will rectify the limitation the linear time series models, where k_t in the LC model and r_t in the Vasiček model does not have the adequate flexibility to reflect the dynamic changes exhibited by the stochastic behaviour of the process observed over time.

Following Zhou and Mamon [60], two- and three-state Markovian regime-switching methods are assumed throughout this thesis. For the two-state model, the regime in the next time step, $s_{t+1} \in \{1, 2\}$, in relation to the regime of current time step, $s_t \in \{1, 2\}$, is governed by a 2×2 transition probability matrix

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1 - p_{12} & p_{12} \\ p_{21} & 1 - p_{21} \end{bmatrix}, \tag{2.13}$$

where p_{ij} $(i, j = 1, 2)$ is the transition probability from state i at time t to state j at time $t + 1$. As $p_{i1} + p_{i2} = 1$ for $i = 1, 2$, p_{12} and p_{21} are the only two parameters that need to be estimated.

Under a two-state Markov regime-switching model for r_t, there are five parameters that must be determined: p_{12} and p_{21} in equation (2.13), and a, b, and σ in equation (2.5). These parameters are determined by maximising the log-likelihood function, $l(\theta)$, where θ is a set containing the parameters p_{12}, p_{21} a, b, and σ. The function $l(\theta)$ is defined as

$$l(\theta) = \sum_{t=1}^{n} \log f(r_{t+1} | \mathcal{F}_t; \theta), \tag{2.14}$$

where f is the probability density function of r_t, $\mathcal{F}_t$ denotes the filtration generated by r_t, and n is the total number of observed interest rates. This modelling formulation leads to the distribution of r_t being a mixture of two Gaussian distributions considering that for each regime, one Brownian motion (which is normally distributed) drives the r_t process.

The probability density function f of r_{t+1} is then expressed as

$$f(r_{t+1}|\mathcal{F}_t;\theta) = \boldsymbol{\pi}^{\top} \cdot \boldsymbol{\eta}\,, \tag{2.15}$$

where $\boldsymbol{\pi}$ is the unconditional probabilities in the underlying Markov process , $\boldsymbol{\eta}$ is the vector of probability density functions, and $\cdot$ is the dot product of two vectors. The vector $\boldsymbol{\pi}$ is given by

$$\boldsymbol{\pi} = \begin{bmatrix} p_{21}\,/\,(p_{21}+p_{12}) \\ p_{12}\,/\,(p_{21}+p_{12}) \end{bmatrix} \tag{2.16}$$

associated with the transition probability matrix in equation (2.13) whilst $\boldsymbol{\eta}$ has the representation

$$\begin{aligned} \boldsymbol{\eta} &= \begin{bmatrix} f(r_{t+1}|\,\mathbf{s}_t = 1, \mathcal{F}_t;\theta) \\ f(r_{t+1}|\,\mathbf{s}_t = 2, \mathcal{F}_t;\theta) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2\pi\sigma_1}} \exp -\frac{(r_{t+1}-r_t-a_1(b_1-r_t))^2}{2\sigma_1^2} \\ \frac{1}{\sqrt{2\pi\sigma_2}} \exp -\frac{(r_{t+1}-r_t-a_2(b_2-r_t))^2}{2\sigma_2^2} \end{bmatrix}. \end{aligned} \tag{2.17}$$

The two-state Markovian regime-switching model for k_t is formulated in a similar manner.

2.4 Correlation between mortality and interest rate

Mortality and interest rates are assumed independent of each other for simplicity and calculation tractability. However, the presence of correlation between the two risk factors was discussed in Liu et al. [35] and Gao [16]. More specifically, mortality risk can affect the overall economy, which in turn affects interest rate. The decrease in mortality rate, for instance, creates stress on governments and private pension/annuity providers. This directly affects both local and global economy, which in turn impacts the interest rate. The interest rate, on the other hand, also directly affects the economy that influences daily lives of ordinary individuals. The change in economic conditions affects the longevity of individuals without much financial security who may no longer be able to afford even basic sustenance, and more importantly, needed medical care.

Moving onwards, inclusion of correlation between the mortality and interest rates is therefore the proper route to modelling that supports pricing and risk management of annuity and insurance products.

To tackle the dependence of mortality and interest rates, a joint filtration is necessary to support the two processes ; see Liu, et al. [35] for details. Suppose the respective dynamics of the interest rate process r_t and the time-varying k_t are given by

$$dk_t = c dt + \xi dZ_t^1 \tag{2.18}$$

and

$$dr_t = a(b - r_t)\, dt + \sigma dW_t, \tag{2.19}$$

where Z_t^1 and W_t are correlated Brownian motions, c is the average change of k_t over the period dt, and ξ is the volatility. The process k_t in equation (2.18) is the difference equation in (2.3).

Furthermore, the expectation of the product of the correlated Brownian motions is

$$\mathrm{E}[Z_t^1\, W_t] = \rho t, \tag{2.20}$$

where ρ is a correlation coefficient[4] defined as

$$\rho_t = \frac{\mathrm{Cov}[k_t,\ r_t]}{\xi\,\sigma}. \tag{2.21}$$

The correlation structure covering the two Brownian motions, as an offshoot of Levy's Theorem, is

$$dW_t = \rho dZ_t^1 + \sqrt{1-\rho^2}dZ_t^2, \tag{2.22}$$

where Z_t^2 is another standard Brownian motion and is independent of Z_t^1 in equation (2.18). Equation (2.22) shows that W_t in equation (2.19) is also a Brownian motion correlated with Z_t^1. Both the initial values r_0 and k_0 are assumed known at time 0. To summarise, equations (2.18)–(2.19) state that our modelling set up consists of the Vasiček interest rate model, and the LC mortality model with an embedded time-dependent k_t.

2.5 Mortality and interest-rate datasets

South Korean mortality and interest rate data are used in our analysis. Both data were compiled by the Korean Statistical Information Service (KOSIS) [31], which provides official statistics sourced out from over 120 statistical agencies covering more than 500 subject matters.

The mortality rate data for the LC model are available for ages 0 to 99, and they are provided for each age spanning the period 1970–2015 for males, females and both genders. We note that the data prior to 1983 have less credibility due to various omissions and frequent delays of death reports [43]. The data starting from 1983 were collected using improved mechanics and this why we chose them for our implementation. In addition, the female mortality rate in each age group is used as an input for the Lee-Carter model because it shows a more stable trend and an emergence of longer life expectancy.

[4] In sections 4.1 and 6.1, ρ is estimated as 0.857. This is based on a given synchronised windows of data for k_t and r_t. We are aware of cointegration and presence of spurious correlation; thus, additional appropriate statistical/econometric tests may be employed to validate further the strong correlation between these two time series.

The interest rate data is based on the values of the monetary stability bond (MSB)[5]instead of the Korean treasury bond yield data. Although the paired time-series yield values of the MSB ad Korean treasury bond are almost identical (below one-percent average margin of error), MSB values are available for a much longer period (1980 – 2016 for MSB versus 2000 – 2016 for Korean treasury bond). The MSB yield values are available for daily, monthly quarterly and yearly frequency. In our numerical work, the yearly time-series values are chosen in order to be consistent with the mortality data, which are presented as time series of yearly values.

2.6 Analysis tools and programme

The simulations in this thesis are performed using the statistical software R, which is the most widely used language and environment for statistical computing. The software R has an extensive collection of packages that extend its functionality in a wide range of modern statistics; see [54] for further description. Moreover, two packages are utilised extensively for this work: demography [27] and MSwM [49]. The demography package includes functions for demographic analysis and univariate time series forecasts such as ARIMA modelling. The MSwM package is used for the implementation of the Markovian regime-switching approach under the linear and generalised models in conjunction with the Expectation–Maximisation (EM) algorithm[6].

[5]MSB stands for monetary stability bond, which helps absorb liquidity. This is issued by the Bank of Korea (BOK) – Korea's central bank that mandates the monetary policy.

[6]This is an iterative method to find the maximum likelihood or maximum a posteriori (MAP) estimates of parameters in statistical models, where the model depends on unobserved or latent variables.

Chapter 3

Mortality rate modeling and forecasting

It was asserted in the first chapter that an accurate prediction of mortality rates is imperative for pricing insurance products and for measuring insurance risk. The two main objectives of this chapter are:

(i) to develop a mortality rate model by fitting the Lee-Carter model to the historical data and incorporating a regime-switching approach as an extension, and

(ii) to forecast future mortality rates using the proposed extended model in (i).

Under the regime-switching framework, one-, two- and three-state models are constructed and their corresponding performances are assessed. The assessment involves the goodness-of-fit benchmark via the root-mean-square error (RMSE) values The mortality rates for a 45-year period (2016–2060) are simulated using the three models for valuation's intent.

3.1 Empirical analysis of mortality data

The mortality rates' behaviour for the Korean female population from 1983 to 2015 is shown in Figure 3.1. We see that the mortality rate is strongly dependent on both time and age, which makes it necessary to consider time and age variables simultaneously for modelling the mortality rate.

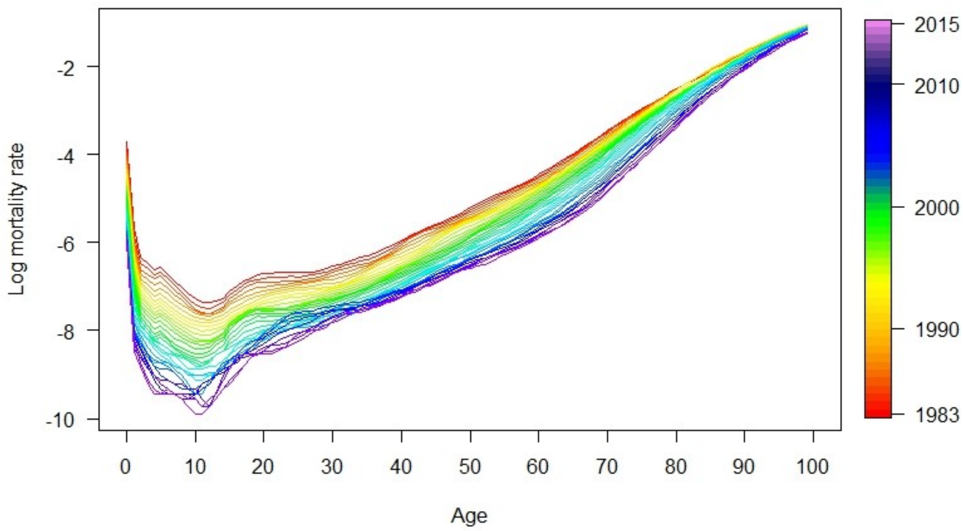

Figure 3.1: Log mortality rate for the Korean female population from 1983 to 2015, published by the Korean Statistical Information Service (KOSIS) [31].

In addition to time and age variables, another important aspect to consider is the mortality-improvement rate, which is defined as

$$\text{Mortality improvement rate (\%)} = \frac{m_{x,t+1} - m_{x,t}}{m_{x,t+1}} \times 100, \tag{3.1}$$

where m is the mortality rate and the subscripts x and t represent age and time, respectively. The mortality improvement rates for Korean female at ages 10, 30, 50 and 70 from 1983 to 2015 are shown in Figure 3.2. This shows that the mortality improvement rate does not remain steady; the improvement rate is changing abruptly with time. It is this non-stationary mortality improvement rate that makes the mortality rate modelling a challenging mathematical exercise.

The non-stationarity of the mortality improvement rate influences the accuracy of the projected life expectancy. Its impact can be seen further in Figure 3.3, which shows the actual and projected life expectancy for Korean females from 1970 to 2065. The life expectancies has increased continuously since 1970, and it is expected to be higher than 90 years in 2030 with a 57% probability, which is the highest worldwide [30]. More importantly, all of the projected life expectancies published at any given time show a significant discrepancy with respect to the actual life expectancy. Such a discrepancy is attributed to the change in the mortality improve-

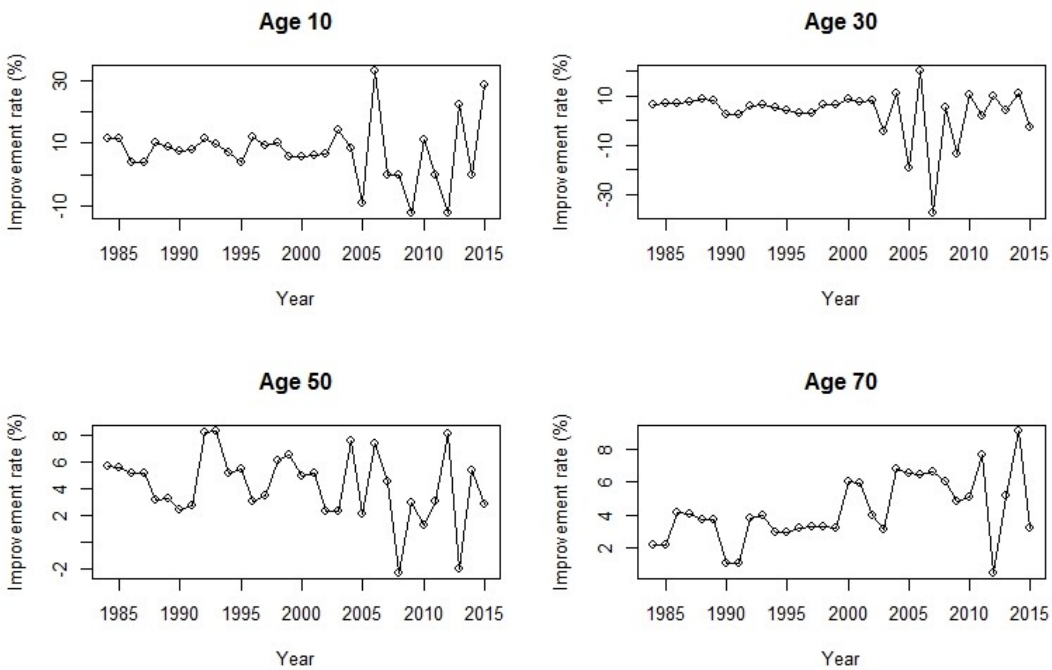

Figure 3.2: Age-specific mortality improvement rate for Korean females, 1983–2015.

ment rate, which makes it necessary to reflect the fluctuations of mortality improvement rate in forecasting the future mortality rates.

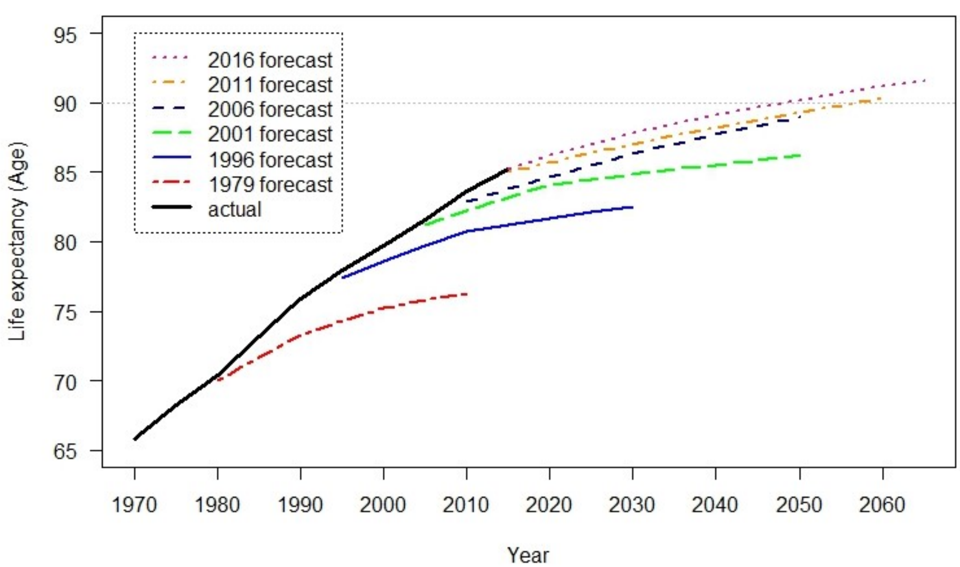

Figure 3.3: Actual and projected life expectancy for Korean females, 1970–2065, published by Statistics Korea [53]

3.2 Determining the parameters for mortality prediction

The mortality rates for Korean females are modeled using the LC model, whose parameters are calculated using the data for the past 33 years (1983–2015). The calculation steps for determining the parameters associated with the mortality rate prediction are discussed in this section. In the first step, the age-specific constants of the LC model, a_x and b_x, and a time-varying index, k_t, are estimated by fitting the LC model to historical data. The LC model alone cannot be used to forecast the future mortality rates since the future k_t values must be estimated by regression. In order to reflect the fluctuations of mortality improvement rates shown in Figure 3.2, k_t is modelled following the ARIMA model with a regime-switching method incorporated in the subsequent step.

3.2.1 Estimation of parameters in the LC model

After fitting the LC model to historical data, the parameters a_x, b_x and k_t are obtained and their plots are depicted in Figures 3.4, 3.5 and 3.6. The numerical values of all three parameters are also depicted in Appendix A.

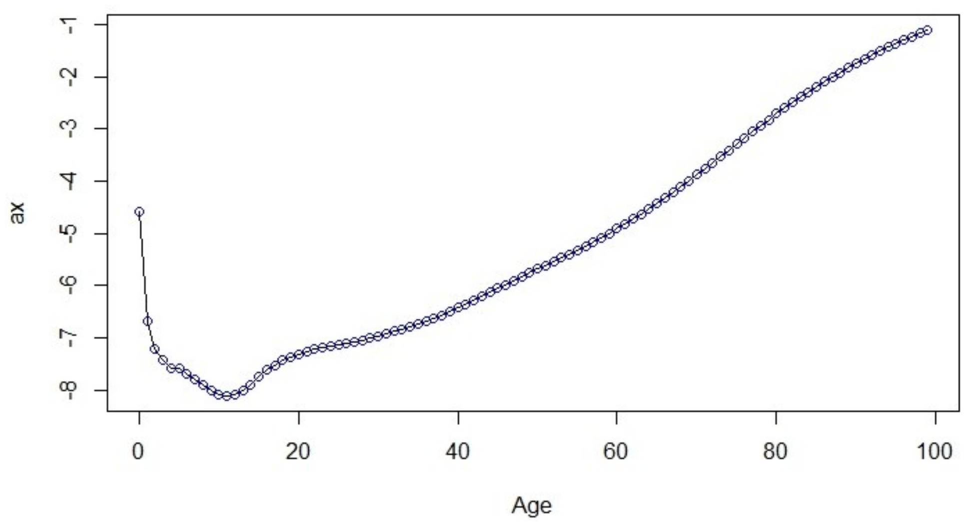

Figure 3.4: a_x for ages from 0 to 99.

The first parameter, a_x, shown in Figure 3.4 is an age-specific constant that explains the general shape of the logarithm of mortality rates across age. The pattern shows that the mortality rates decrease sharply from age 0 to age 13. The mortality rates then increases steadily from age 13 onwards. The pattern of a_x is very similar to the general mortality rate pattern for all ages shown in Figure 3.1.

Figure 3.5 illustrates another age-specific constant, b_x, of the LC model. Unlike a_x, the parameter b_x measures the sensitivity of the $\ln m_{x,t}$ at age x to variations in the time index k_t. Parameter b_x has its highest value around age 10, and then it declines as age increases. An interesting feature of b_x is that the rate of decline plateaus in the ages between 40 and 60, but then continues again to decrease steadily for the ages older than 60. A time-varying index k_t in Figure 3.6 shows that the mortality rate level decreases as a function of time.

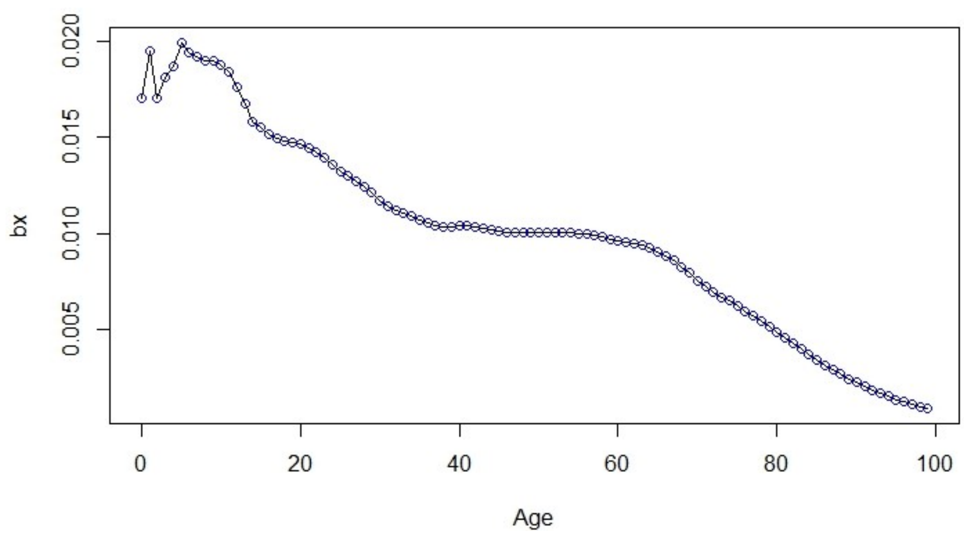

Figure 3.5: b_x for ages from 0 to 99.

3.2.2 A regime-switching approach for the estimation of k_t

In order to estimate or simulate the future values for k_t, we examine the historical values of k_t shown in Figure 3.6, which are first fitted to a one-state model

$$dk_t = c_1 dt + \xi_1 dZ_t^1, \tag{3.2}$$

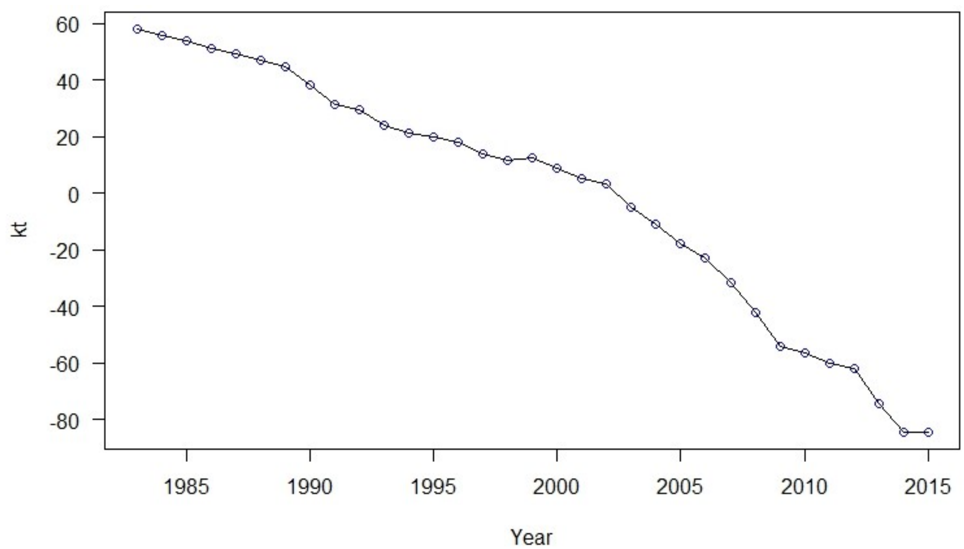

Figure 3.6: k_t for years 1983 to 2015.

where c_1 is a drift term corresponding to an average change of k_t between 1983 and 2015, ξ_1 is a volatility and Z_t^1 is a Brownian motion. In multi-state regime-switching models, each regime has its own set of c_i's and ξ_i's. The values of c_i's and ξ_i's for one-, two- and three-state models are shown in Table 3.1.

Table 3.1: Estimated parameter values for 1-, 2- and 3-state models of k_t fitted to 1983–2015 data. Numbers enclosed in parentheses are standard errors.

Parameters	1-state	2-state	3-state
$\widehat{c_1}$	−4.46 (0.10)	−8.00 (0.37)	−10.91 (0.29)
$\widehat{c_2}$	–	−2.26 (0.06)	−6.66 (0.10)
$\widehat{c_3}$	–	–	−2.30 (0.06)
$\widehat{\xi_1}$	3.44 (0.24)	2.71 (0.51)	1.29 (0.31)
$\widehat{\xi_2}$	–	1.20 (0.05)	1.02 (0.05)
$\widehat{\xi_3}$	–	–	1.23 (0.03)
$\widehat{p_{12}}$ (%)	–	30.2 (0.01)	15.5 (0.04)
$\widehat{p_{13}}$ (%)	–	–	29.6 (0.03)
$\widehat{p_{21}}$ (%)	–	18.3 (0.01)	1.2 (0.03)
$\widehat{p_{23}}$ (%)	–	–	38.7 (0.04)
$\widehat{p_{31}}$ (%)	–	–	9.7 (0.02)
$\widehat{p_{32}}$ (%)	–	–	9.6 (0.03)

As multi-state models describe the change in k_t with more states, the drift terms in the multi-state models take on values that are both larger and smaller than the drift term in the 1-state model. For example, $c_1 < c_2 < c_3$ in the 3-state model and the c_1, but the value of c_1 in a one-state model could be in between $[c_1, c_3]$ in the 3-state model.

The volatility terms (i.e., ξ_1, ξ_2 and ξ_3) of the multi-state models become smaller when the number of states is increased because the estimation error decreases with a larger number of states in the model. The transition-probability estimates from regime i at time t to regime j at time $t + 1$, p_{ij}, are also shown in Table 3.1. The total transition probability from any given regime to all of the possible regimes is always equal to 1 (e.g., $p_{11}+p_{12}+p_{13} = 1$ in a three-state model). For the case of the 2–state model, both p_{12} and p_{21} are below 50%, which implies that the transitions are less likely to occur regardless of the original regime. In addition, the lower value of p_{21} compared to the value of p_{12} indicates that under the 2-state model, the process has a higher probability of being in regime 2 than in regime 1. The lower p_{21} also indicates that c_2 and ξ_2 contribute more to k_t than c_1 and ξ_1 do.

For the case of the three-state model, $p_{33} = 1 - p_{31} - p_{32} = 80.7$ %; this indicates that the model has a very high probability of remaining in regime 3 at time $t+1$ if it is in regime 3 at time t. For the other two regimes, the probabilities are also high of staying in the original regimes (i.e., $p_{11} = 54.9$ %, $p_{22} = 60.1$ %), which means that according to the model the mortality rate process is not likely to switch regimes. The higher value of p_{33} compared to the values of p_{11} and p_{22} shows that the process mostly will be staying in regime 3. Furthermore, the situation $p_{13} > p_{12}$ and $p_{23} > p_{21}$ also tell us that the process is most likely to switch to regime 3 when it transitions out of both regime 1 and regime 2. Consequently, c_3 and ξ_3 contribute the most to k_t.

The probability of the process, under the two- and three-state models, to be in each regime at any given point in time is shown in Figures 3.7 and 3.8, respectively[1]. For example, it is more likely for the process to be in regime 1 in year 1993 because the probability of being in regime

[1]Filtered probability is the probability that the unobserved Markov chain under a Markovian regime-switching model is in a particular regime *at time t*, conditional on observing sample information up to time t. Smoothed probability, on the other hand, is the probability that the unobserved Markov chain is in a particular regime *in period t* (i.e., at time prior to t), conditional on observing all sample information

1 is 0.6 whilst the probability of being in regime 2 is 0.4. It can be seen clearly in Figures 3.7 and 3.8 that the switches of regimes do occur over the data period of 33 years; a regime having a dominant probability changes over time. In Figure 3.7, the overall probability of being in regime 2 is higher than the probability of being regime 1, which is consistent with the fact that p_{12} is higher than p_{21}. Similarly, the overall probability of being regime 3 in Figure 3.8 is the highest of all three; this also agrees with the aforementioned discussion where p_{33} is larger than both p_{11} and p_{22}.

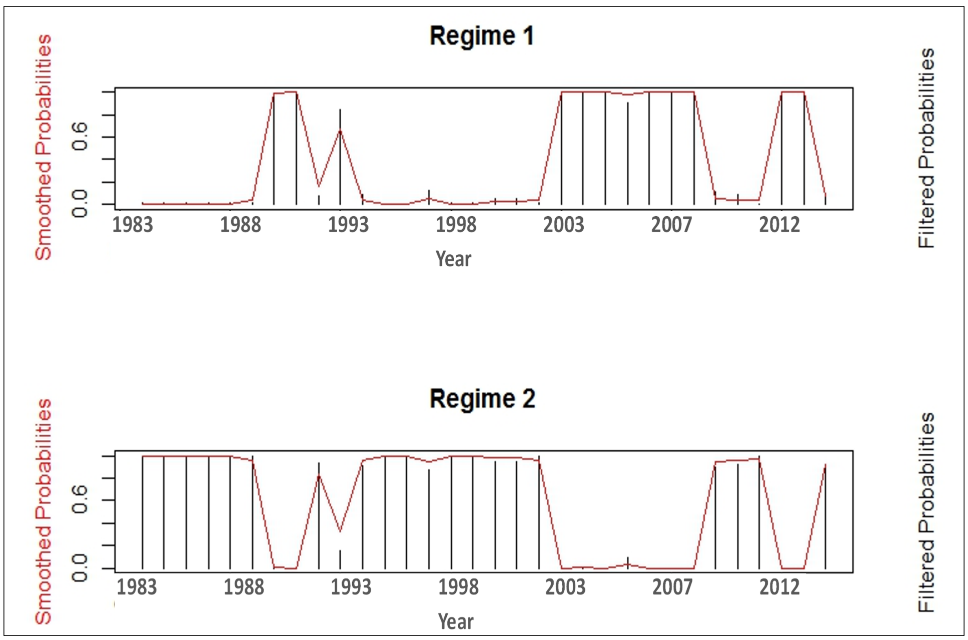

Figure 3.7: Filtered and smoothed probabilities to be in regime 1 or 2 under the 2-state model.

3.2.3 Evaluation of the goodness of fit

The goodness of fit of the three models is evaluated using three criteria: maximum log-likelihood value (MLL), Akaike information criterion (AIC) [1] and Bayesian information criterion (BIC) [50]. The MLL is defined as

$$\mathrm{MLL} = \arg\max_{\theta} \sum_{i=1}^{n} \log f(x_i \,|\, \theta), \tag{3.3}$$

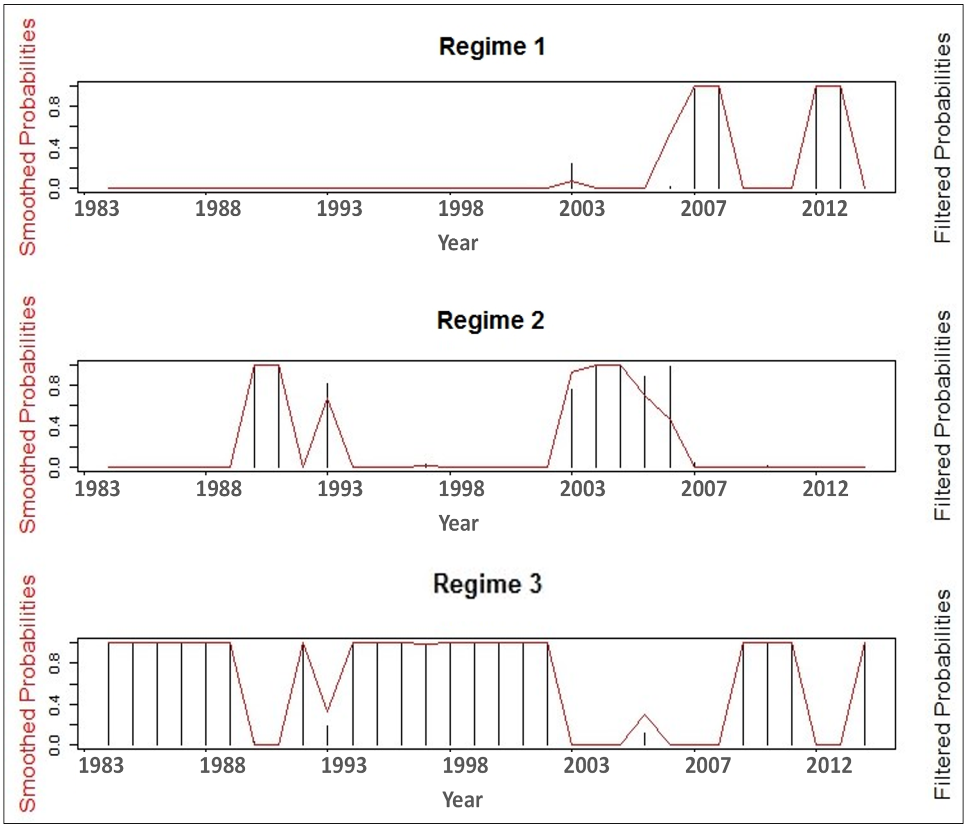

Figure 3.8: Filtered and smoothed probabilities to be in regime 1, 2 or 3 under the 3-state model
.

where θ is the collection of parameters (i.e., c and ξ), x_i is the i^{th} observation and n is the number of observations. The AIC is derived from the MLL and given by

$$\text{AIC} = -2\text{MLL} + 2p, \tag{3.4}$$

where p is the number of parameters. Similarly, the BIC is defined as

$$\text{BIC} = -2\text{MLL} + p \log n. \tag{3.5}$$

The AIC and BIC gives merit to the increase in log-likelihood, but penalises the introduction of additional parameters (p) and observations (n). Based on the specifications of the criteria in equations (3.3), (3.4) and (3.5), a higher value of MLL and a smaller value of AIC and

BIC indicate a better model to choose. Higher likelihood values mean better fitting whilst the penalty for more parameters is a disinclination towards complexity.

The estimated AIC, BIC and MLL values of the models are shown in Table 3.2. Based on the MLL and AIC, the three-state model is the best, although the differences between the corresponding MLL and AIC values are small for the two-state model vis-à-vis three-state model. The BIC value is the highest for the two-state model, which indicates that the two-state model is the best after taking into account the balance between goodness of fit and penalty for both the number of parameters and observations to be included when performing calculations in the model.

Table 3.2: Selection-criteria results for choosing the best model for k_t. The estimated criterion value that gives the best-performing model is marked in bold.

	1-state	2-state	3-state
MLL	−84.4	−74.9	**−72.9**
AIC	172.9	153.9	**151.8**
BIC	175.8	**163.7**	166.6

3.3 Model validation for k_t

Model validation is a necessary diagnostic because a model that yields better fit does not always guarantee better prediction performance. Therefore, the modelling approach discussed in Section 3.2 is validated by using the available data for past 33 years (1983–2015). More specifically, the first 23 years (1983–2005) of data are used for estimating parameters and the remaining data for the last 10 years (2006–2015) are used for the out-of-sample period in evaluating model's prediction ability.

The model parameter estimates obtained by employing the data from 1983 to 2005 are shown in Table 3.3. The values shown in Table 3.3 differ from those shown in Table 3.1 due to different ranges of data in the parameter estimation.

Based on the parameter values in Table 3.3, the mortality rate, $m_{x,t}$, for the out-of-sample

Table 3.3: Estimated parameter values, under the 1-, 2- and 3-state models, of k_t fitted to the 1983–2005 data.

	1-state	2-state	3-state
$\widehat{c}_1$	−4.86 (0.11)	−9.85 (0.39)	−8.77 (0.22)
$\widehat{c}_2$	–	−3.26 (0.07)	−5.52 (0.12)
$\widehat{c}_3$	–	–	−2.87 (0.06)
$\widehat{\xi}_1$	3.37 (0.26)	1.56 (0.52)	0.66 (0.19)
$\widehat{\xi}_2$	–	1.74 (0.04)	3.86 (0.10)
$\widehat{\xi}_3$	–	–	0.10 (0.06)
$\widehat{p}_{12}$ (%)	–	44.7 (0.01)	100.0 (0.06)
$\widehat{p}_{13}$ (%)	–	–	0.0 (0.03)
$\widehat{p}_{21}$ (%)	–	8.3 (0.01)	25.4 (0.03)
$\widehat{p}_{23}$ (%)	–	–	37.7 (0.04)
$\widehat{p}_{31}$ (%)	–	–	0.0 (0.04)
$\widehat{p}_{32}$ (%)	–	–	33.7 (0.05)

period (2005–2015) are predicted using the following procedure:

1. The time-varying index, k_t, of the out-of-sample period is projected using an ARIMA model based on $dk_t = cdt + \xi dZ_t^1$ specified by equation (2.18).

2. Repeat the first step 1,000 times to obtain 1,000 values of k_t.

3. Compute $(m_{x,t})$ using the LC model in equation (2.1) making use of the 1,000 projected k_t and the age-specific constants, a_x and b_x.

The k_t values (steps 1 and 2 above) are shown in Figure 3.9. The colours red, blue and green represent results for the one-, two- and three-state models, respectively. The solid lines represent the average of 1,000 simulations and the region between the two dotted lines represents the 95 % confidence interval. It can be observed from Figure 3.9 that the predicted k_t's using the two- and three-state models are very similar both in terms of the average value and the 95 % confidence interval. On the other hand, the predicted k_t using the one-state model exhibits a steeper decrease. Also, the lower values of k_t signify that the mortality rate improves more significantly. Here, the one-state model estimates appears to be giving a rosier prognosis of the mortality-rate improvement than what the two- and three-state models are showing.

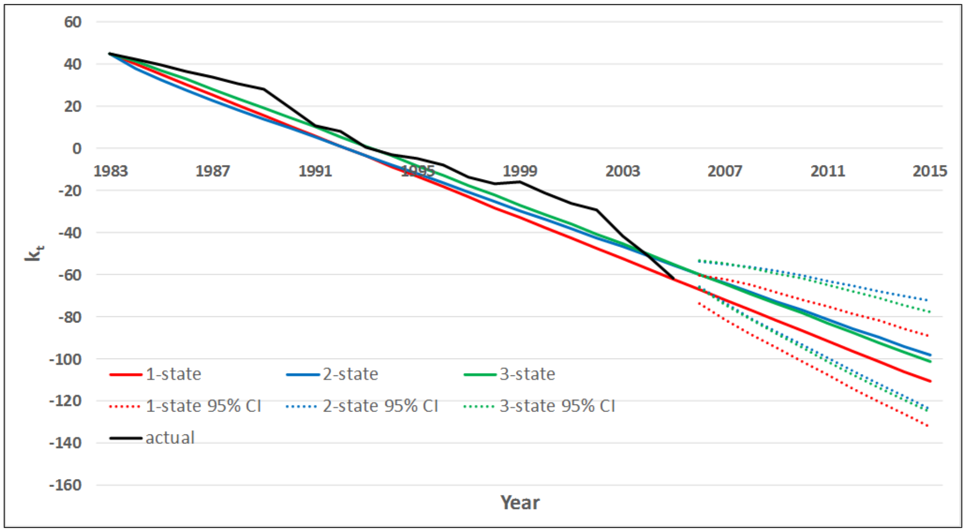

Figure 3.9: Estimated and forecasted k_t. Models were fitted to 1983–2005 data and projections obtained for 2006–2015.

The projected values of k_t are used to predict the log mortality rate, $\log m_{x,t}$, using the LC model. The performance of the prediction is evaluated using the root-mean-square error (RMSE), which is defined as

$$\text{RMSE of } \log m_x = \sqrt{\frac{\sum_{t=1}^{N}(\log \widehat{m}_{x,t} - \log m_{x,t})^2}{N}}, \tag{3.6}$$

where $\widehat{m}_{x,t}$ and $m_{x,t}$ are the predicted and actual mortality rates at time t for an individual aged x, respectively; and N is the total number of predicted values $\widehat{m}_{x,t}$. The RMSE results for $m_{x,t}$ under the three models are depicted in Table 3.4 for ages 20, 40, 60 and 80. The projected and actual mortality rates for age 60 are shown in Figure 3.10.

Table 3.4: RMSE results for $\log m_x$ at selected ages.

Age	1-state	2-state	3-state
20	0.495	0.339	0.380
40	0.429	0.310	0.331
60	0.073	0.037	0.021
80	0.186	0.228	0.218

A key observation regarding Table 3.4 is that the RMSEs under the two- and three-state

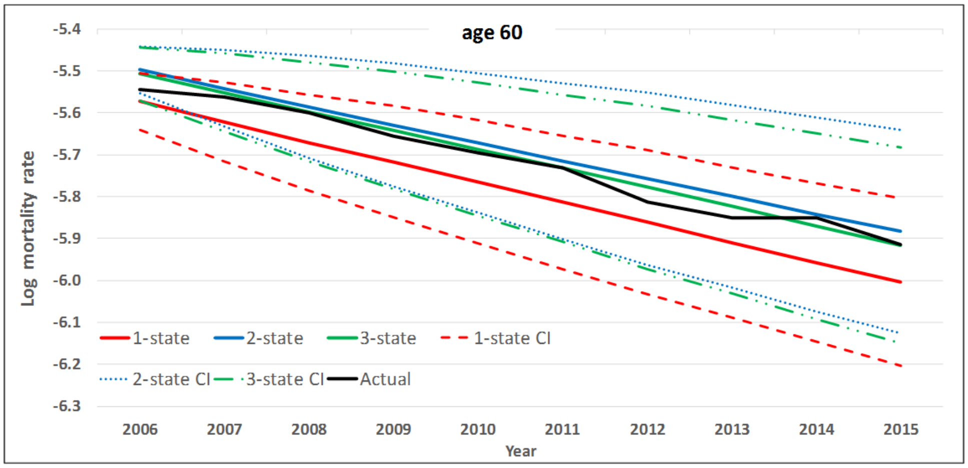

Figure 3.10: Actual and projected mortality rate for age 60 for 2006–2015.

models are close to each other for all ages. The RMSE under the one-state model, on the other hand, is considerably larger for all ages except at age 80. For all models, the prediction at age 60 shows the best performance, with significantly lower RMSE values, for all three model settings. As noted above, it is under the one-state model where we can have the most mortality-rate improvement. Thus, from the last row of Table 3.4, the RMSE being the lowest in the one-state model at age 80 signals that the mortality rate is improved more significantly at age 80 compared with the other age groups. Overall, the two- and three-state model outperforms the one-state model.

3.4 Forecasting future mortality rate

Following the procedure detailed in Section 3.3, future mortality rates over the 45-year horizon (2016–2060) are forecasted/simulated based on the estimated parameters reported in Table 3.1. Figure 3.11 shows the time-varying index k_t from 1983 to 2060. The black line represents the k_t generated by fitting the LC model to historical data. The red-, blue- and green-coloured lines denote the k_t obtained under the one-, two- and three-state ARIMA models, respectively. For the ARIMA modelling forecasts, the solid lines correspond to the average values, whilst the region between the two dashed lines correspond to the 95 % confidence interval. The average values almost coincide for all the three models, which make it difficult to distinguish the three lines in Figure 3.11. The width of the 95 % confidence interval is significantly narrower for the one-state model compared to those of the two- and three-state models, which are almost identical with each other .

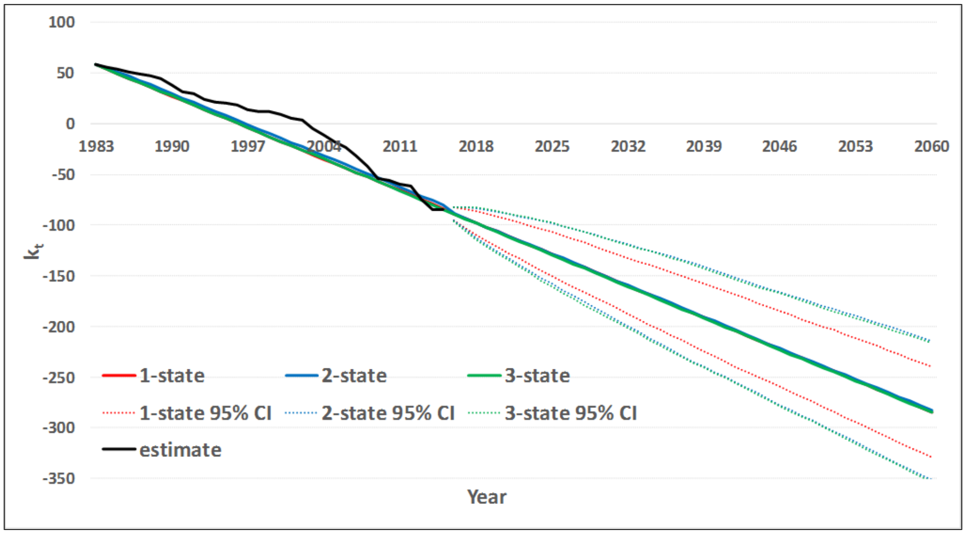

Figure 3.11: Forecasted k_t; fitted on 1983–2015 data along with forecasts covering 2016–2060.

The forecasted age-specific mortality rates are shown in Figure 3.12 at ages 20, 40, 60 and 80. The actual age-specific mortality rates for the past 33 years are shown by the black lines. The average mortality rates for 1,000 simulations under the one-, two- and three-state models are shown in red, blue and green lines, respectively. Similar to the k_t patterns in Figure 3.11, the average values of the mortality rates are almost identical under all the three models. However, the k_t under the 2- and 3-state model has a wider 95% confidence interval at age 60. Although the confidence interval is shown only at age 60 in Figure 3.12 to get a a clearer illustration, the same pattern holds at all other ages being examined. The three models' forecasting performance seems similar based on their average mortality rates. However, the differences in the confidence intervals will have implications in the pricing and computation of related risks of insurance products, which will be analysed in Chapter 5.

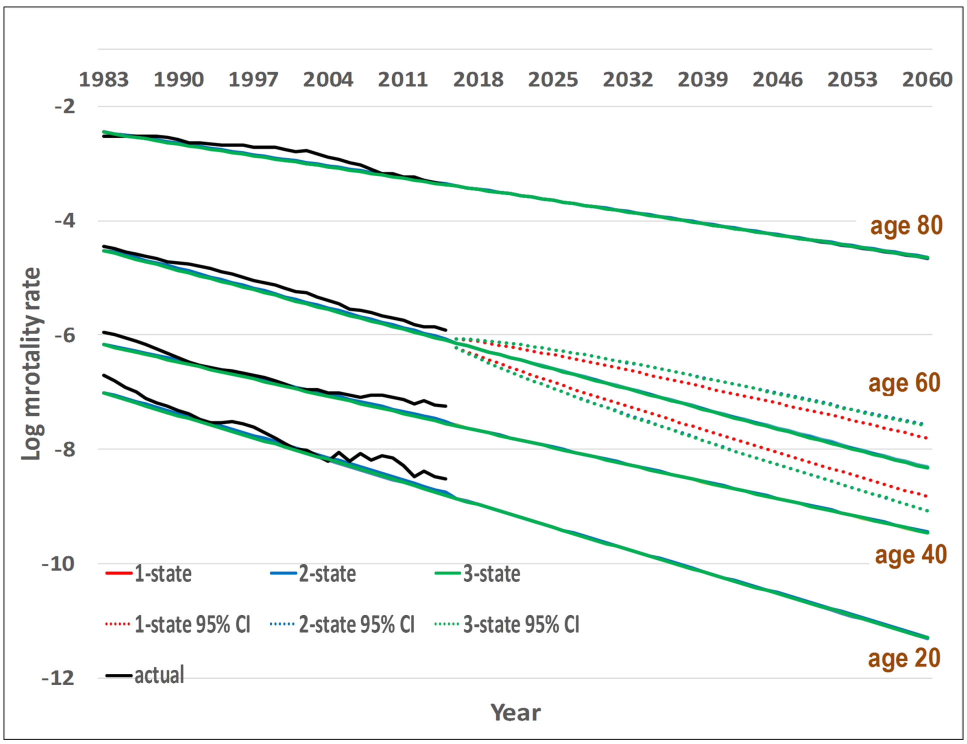

Figure 3.12: Forecasted mortality rates for selected specific ages; models fitted on 1983–2015 data and forecasts generated for 2016–2060.

Chapter 4

Interest-rate modeling and forecasting

The time value of money is reflected in the interest rates involved in lending and borrowing transactions. This principle bespeaks that such a time value is an essential component in the valuation of contingent claims at a future-time horizon. These claims encompass financial and insurance products, and an appropriate interest rate, serving as a discount factor, is needed to determine their present values. One factor that distinguishes the pricing of insurance products from other types of contingent claims is the probability of death or survival of the policy holder. Hence, as already seen in previous chapters, the mortality rate in addition to the interest rate is a principal factor in determing the value of the benefit payout at the outset of the contract.

The format of this chapter closely resembles with that of Chapter 3 considering that the essence of modelling is almost the same. In particular, an interest rate model is developed by fitting the historical data to the Vasiček model and future interest rates are simulated/forecasted using a regime-switching approach. Nonetheless, there are some key differences and these are:

1. The Vasiček model does not require an ARIMA time-series modelling input to project the rates into the future (cf the LC model in the previous chapter);
2. The interest rates are generated with a specified correlation with the mortality rate; and
3. The impact of incorporating correlation influencing the interest rate model is investigated.

Interest rate models under the one-, two- and three-state frameworks are formulated to simulate interest rates for 45 years (2016–2060).

4.1 Empirical analysis of interest data

We use the data on yields from the Korean Monetary Stability Bond (MSB) from 1980 to 2015 as a proxy for short-term interest rate r_t. The plot of the yields data, constituting our interest rates, is shown in Figure 4.1. The rates have a general decreasing trends, with fluctuations, from 25.0 % in 1980 to 1.70 % in 2015. Interest rate is affected by many economic factors such as inflation and GDP. The Korean economy went through a high-growth, high-price period in the 1980's, which contributed to a double-digit interest rate. However, as the growth rate and the inflation rate slowed down gradually in the mid-1990's, the downward pressure on interest rates intensified. The interest rate trend in Figure 4.1 exhibits characteristics of mean reversion (e.g., interest rate in 1985–1998 tend towards the average price over 1980–1983). The wide range of interest rate values and different levels of fluctuations delineated in Figure 4.1 suggest strongly that a regime switching model with multiple states is needed to describe the interest-rate movement accurately.

It can also be seen in Figure 4.1 that the interest rate and the calculated time-varying index of the the LC model, k_t show a similar pattern. Such a similar pattern hints that the interest rate and the mortality rates are correlated. The correlation coefficient, ρ, computed using equation (2.21) is 0.857, which indicates that the interest rate, r_t, and k_t are strongly correlated indeed. Therefore, it is important to consider this correlation when pricing insurance products (e.g., annuity, life insurance, endowment, etc.).

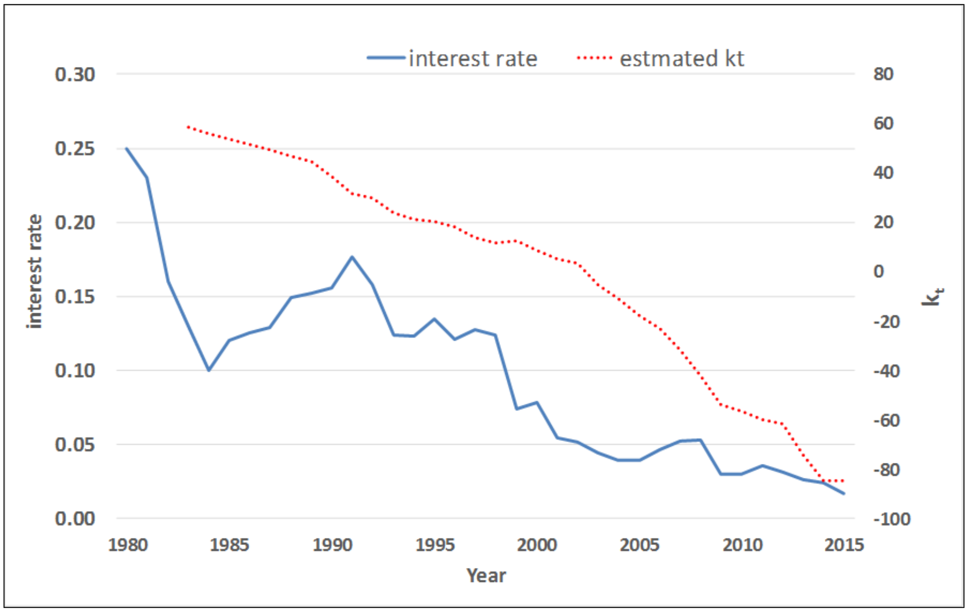

Figure 4.1: Evolution of the MSB yields (1980 – 2015) published by the Korean Statistical Information Service (KOSIS) [31], as well as the time-varying index k_t for the Korean female population (1983 – 2015).

4.2 Determining the parameters for interest rate prediction

Interest rates are modeled using the Vasiček model specified in equation (2.19), i.e.,

$$dr_t = a(b - r_t)dt + \sigma dW_t,$$

which has three parameters a, b and σ. The values of three parameters are estimated using the interest rate data of past 33 years (1983–2015) and the goodness of fit corresponding to the one-, two- and three-state models are compared.

Remarks:

(i) The valuation of contingent claims is done under a risk-neutral measure to obtain a no-arbitrage price, and this works well for short-term (6 months or less) financial instruments. However, for long-term contracts such as annuity and insurance, this presents a difficulty. As Gao et al.[17] argued, when modelling guaranteed maturity benefits, cur-

rent market statistics, which are used to back out risk-neutral measure, may not provide sufficient market information. This is because the guaranteed maturity benefits often have longer maturities than the traded options. They vary with term to maturity so that it is hard to assert that current market conditions can provide an appropriate assumption when analysing future cash flows.

(ii) Invoking the same rationale from (i) for our case where we price a 30-year annuity contract, the risk-neutral parameters here are not used but rather the parameters calculated under the objective measure using past data. Thus, despite the risk-neutral valuation framework in this work, we do not, therefore, concern ourselves with model calibration such as those given in Rodrigo and Mamon [47] for term-structure modelling; and the parallel issues in option valuation addressed in Xi, et al. [59] and Rodrigo and Mamon ([46] and [45]).

4.2.1 Estimation of parameters in the Vasiček model

The estimated parameters are listed in Table 4.1. The parameter b corresponds to a long-term mean of r_t, whilst a corresponds to the speed of adjustment of r_t to revert back to b. The parameter σ is the volatility term. For multi-state models, one must set or estimate various levels for the long-term mean, speed of adjustment and volatility values associated with each regime.

In the case of the one-state model, the b value is 0.046 and r_t converges to 0.046 in the long term. For the two- and three-state models, on the other hand, there are different b's for each regime, which means that r_t converges to a value incorporating the probability of belonging to each regime. As the parameter a corresponds to the speed of adjustment of r_t to revert to the mean, a regime with a negative a implies that r_t continues to move away from the long-term mean without reverting to the mean. However, a negative a only occurs for one of the regimes for the multi-state models. Therefore, r_t converges to a long-term average b under the combined influence of a "pull" from the regimes with positive a values and a "push" from the

Table 4.1: Estimated parameter values under the 1-, 2- and 3-state models for r_t fitted on the 1980–2015 MSB data. Numbers enclosed in parentheses are standard errors.

	1-state	2-state	3-state
$\widehat{a}_1$	0.124 (0.01)	0.181 (0.04)	0.098 (0.01)
$\widehat{a}_2$	–	−0.098 (0.02)	0.311 (0.02)
$\widehat{a}_3$	–	–	−0.118 (0.37)
$\widehat{b}_1$	0.046 (0.00)	0.015 (0.00)	0.061 (0.01)
$\widehat{b}_2$	–	0.045 (0.01)	0.013 (0.01)
$\widehat{b}_3$	–	–	0.040 (0.01)
$\widehat{\sigma}_1$	0.018 (0.00)	0.017 (0.00)	0.010 (0.00)
$\widehat{\sigma}_2$	–	0.007 (0.00)	0.009 (0.00)
$\widehat{\sigma}_3$	–	–	0.006 (0.00)
$\widehat{p}_{12}$ (%)	–	33.2 (0.01)	55.4 (0.02)
$\widehat{p}_{13}$ (%)	–	–	45.6 (0.03)
$\widehat{p}_{21}$ (%)	–	31.4 (0.01)	100.0 (0.02)
$\widehat{p}_{23}$ (%)	–	–	0.0 (0.04)
$\widehat{p}_{31}$ (%)	–	–	80.4 (0.04)
$\widehat{p}_{32}$ (%)	–	–	0.0 (0.02)

regime with a negative a value only more especially when the regime with a negative a is not dominant. Even when the regime with a negative a is non-dominant, the model will revert to the long-term average very slowly if the contributions of "pull" and "push" are similar.

For the case of a two-state model, regime 2 has a negative a whilst the probability of the process to be in either regime is very similar as shown in Figure 4.2. The similar values of the two probabilities suggest that the model will either (i) fail to converge to a long-term average or (ii) converge to the long-term average very slowly. In the three-state model, only regime 3 has a negative a whilst its probability is much smaller than the probabilities of regime 1 and regime 2 combined, as shown in Figure 4.3. As the regime with a positive a (regime 3) is not dominant, the model will converge to a long-term average.

The evaluation of the goodness of fit is carried out via the MLL, AIC and BIC and their values are listed in Table 4.2. The three-state model has the best fit to the actual values based on MLL and AIC, whilst the one-state model has the best fit based on BIC. Our comparison shows that the three-state model fits more effectively to the actual data, but the one-state model

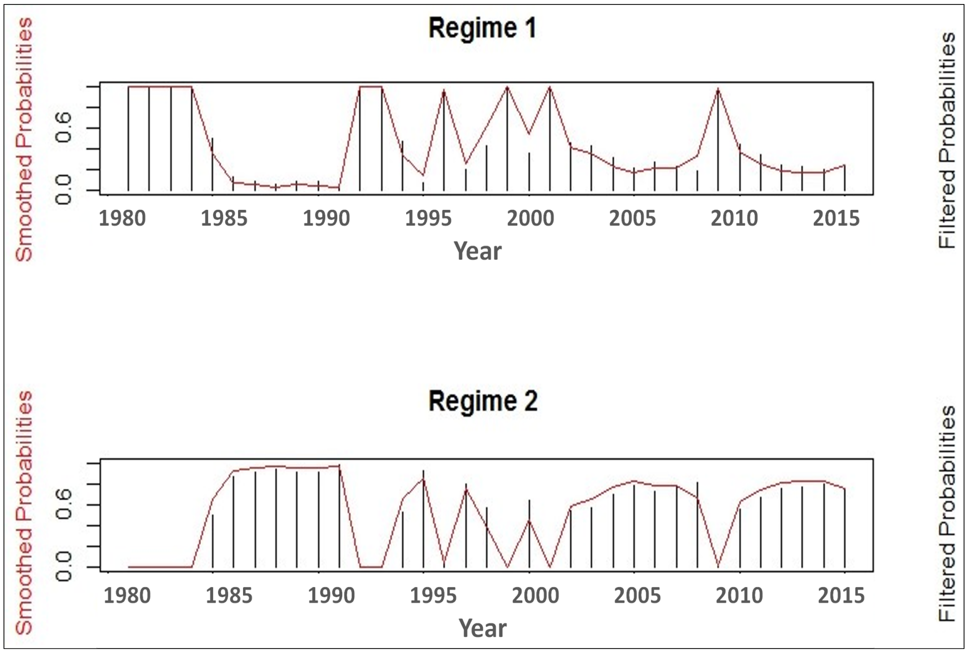

Figure 4.2: Filtered and smoothed probabilities under the 2-state model to be in regime 1 or 2.

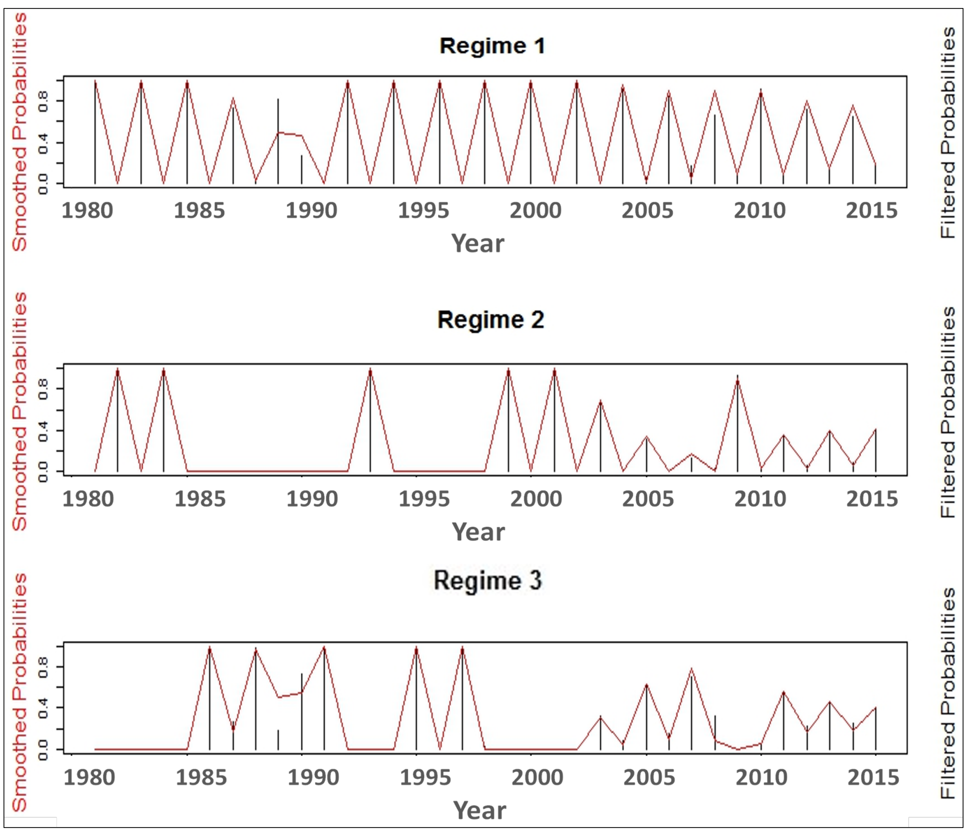

Figure 4.3: Filtered and smoothed probabilities under the 3-state model to be in regime 1, 2 or 3.

minimally outperforms the multi-state models if the penalty for the number of parameters and the size of the observation are considered. Apparently, if the number of observations is even slightly reduced, the multi-state models will beat the one-state model.

Table 4.2: Goodness of fit of the various model setting for r_t. The best-performing model is marked in bold.

	1-state	2-state	3-state
MLL	91.8	100.3	**107.6**
AIC	−177.6	−192.5	**−203.2**
BIC	**−172.9**	−172.1	−172.5

4.3 Model validation

The interest-rate model is validated using the first 26 years of the 36 years of data as in-sample period, whilst the last 10 years of data are used as out-of-sample period to determine the RMSE of the model. The values of estimated parameters of Vasiček model based on in-sample period (1980–2005) are listed in Table 4.3.

The values in Table 4.3 differ from the values in Table 4.1 due to the change in the data range of the new parameter estimation, but the overall trend remains unchanged. One fundamental difference is that the values of long-term mean level, b, in Table 4.3 is higher than the values in Table 4.1 because the omitted data (2006–2015) have the lowest interest rates in the entire data set (1980–2015). Another important difference is that the regime 2 of the two-state model has a negative a. Since regime 2 is not dominant, r_t under the two-state model is expected to revert to the long-term mean in a relatively short period of time.

A simulation of the interest rate, r_t, for out-of-sample period (2006-2015) is performed 1,000 times. The averages and their 95 % confidence intervals under various model settings are shown in Figure 4.4. The actual interest rate (black line) decreases from 4.7 % in 2006 to 1.7 % in 2015 with some fluctuations. The rate r_t, under the one-state model, approaches 7.6 % according to its long-term mean value b_1, where r_t reaches 7.2 % in 2015. Rates under the two-

Table 4.3: Estimated parameter values under the 1-, 2- and 3-state models of r_t fitted on the 1980–2005 data

	1-state	2-state	3-state
$\widehat{a}_1$	0.170 (0.01)	0.215 (0.02)	0.136 (0.03)
$\widehat{a}_2$	–	−0.125 (0.05)	0.316 (0.07)
$\widehat{a}_3$	–	–	−0.161 (0.10)
$\widehat{b}_1$	0.076 (0.01)	0.051 (0.01)	0.088 (0.05)
$\widehat{b}_2$	–	0.067 (0.01)	0.016 (0.01)
$\widehat{b}_3$	–	–	0.067 (0.01)
$\widehat{\sigma}_1$	0.021 (0.00)	0.019 (0.00)	0.011 (0.00)
$\widehat{\sigma}_2$	–	0.008 (0.00)	0.009 (0.00)
$\widehat{\sigma}_3$	–	–	0.007 (0.00)
$\widehat{p}_{12}$ (%)	–	18.7 (0.01)	55.6 (0.02)
$\widehat{p}_{13}$ (%)	–	–	44.4 (0.03)
$\widehat{p}_{21}$ (%)	–	32.3 (0.01)	100.0 (0.05)
$\widehat{p}_{23}$ (%)	–	–	0.0 (0.04)
$\widehat{p}_{31}$ (%)	–	–	74.4 (0.08)
$\widehat{p}_{32}$ (%)	–	–	0.0 (0.05)

and three-state models do not change much from the original 3.97 % in 2005 and attain 4.00 % in 2015. In addition, the confidence intervals narrow down as the number of states in the model increases. This is because r_t, under the one-state model, has a large volatility, σ, given large fluctuations from 23% in 1980 to 3.97 % in 2005. The volatility values decrease as more regimes are introduced, and there is further ability to capture these large fluctuations and ranges of r_t. The actual r_t falls within the confidence interval of all three models, which is partly due to the large confidence interval especially for the one-state model. Although the average values of simulated r_t's under the two- and three-state models are very close, the tighter confidence interval of the three-state model makes it superior.

The interest rates simulated with and without the correlation with the mortality rate (or with k_t to be more specific) are shown in Figure 4.5. Although the fitting for the r_t is done using the actual data from 1980 to 2005, the mortality rate data is only available from 1983 onwards. Consequently, the correlation is only included from 1983. The solid and dotted lines in Figure 4.5 correspond to the average of the 1,000 simulated values of r_t with and without the

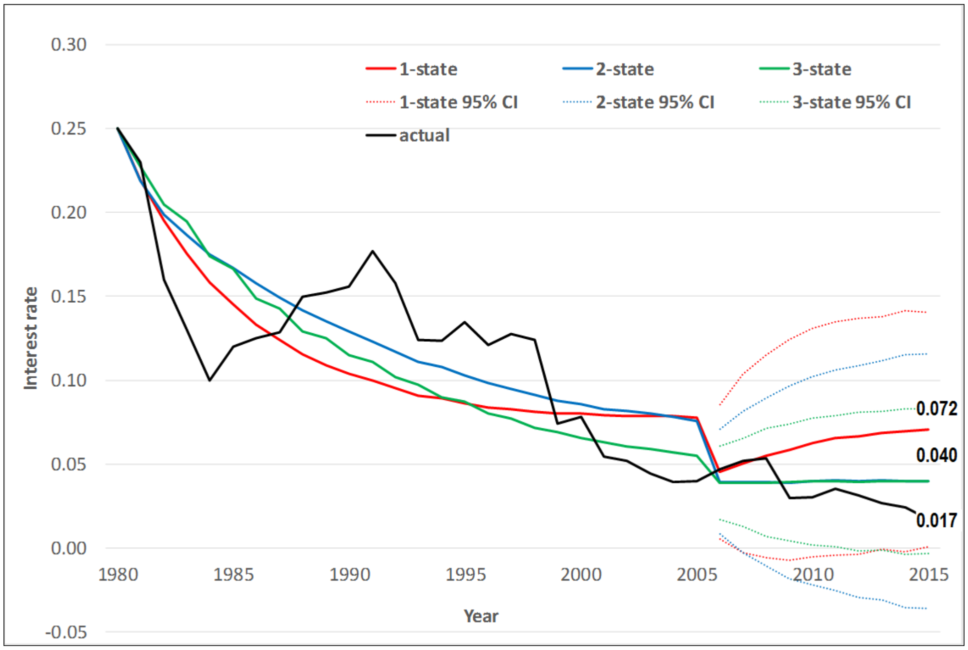

Figure 4.4: Actual and projected r_t accompanied by confidence interval; fitted on 1980–2005 and projected for 2006–2015.

correlation, respectively. The impact of the correlation is not entirely visible in Figure 4.5. We note that the inclusion of correlation is through the Brownian motions describing r_t and k_t. In other words, the random number multiplied to the volatility term is the only link between the two, which makes the mortality rate and interest rate 'swing' in the same direction. Hence, both the average value and the confidence interval, if interest rate is considered not correlated with mortality, exhibit similar behaviour with those obtained with the assumption of correlation. However, we shall see that the correlation affects the annuity pricing and risk measurement.

The predicted and actual mortality rates for the out-of-sample period (2006-2015) are compared in terms of the RMSE values, which are shown in Table 4.4. The RMSE value of the one-state model is significantly larger than both of those in the two- and three-state models; this indicates the poor performance of the one-state model. The RMSE values of two- and three-state models are very similar. However, the smaller confidence interval of the three-state model makes it a better model.

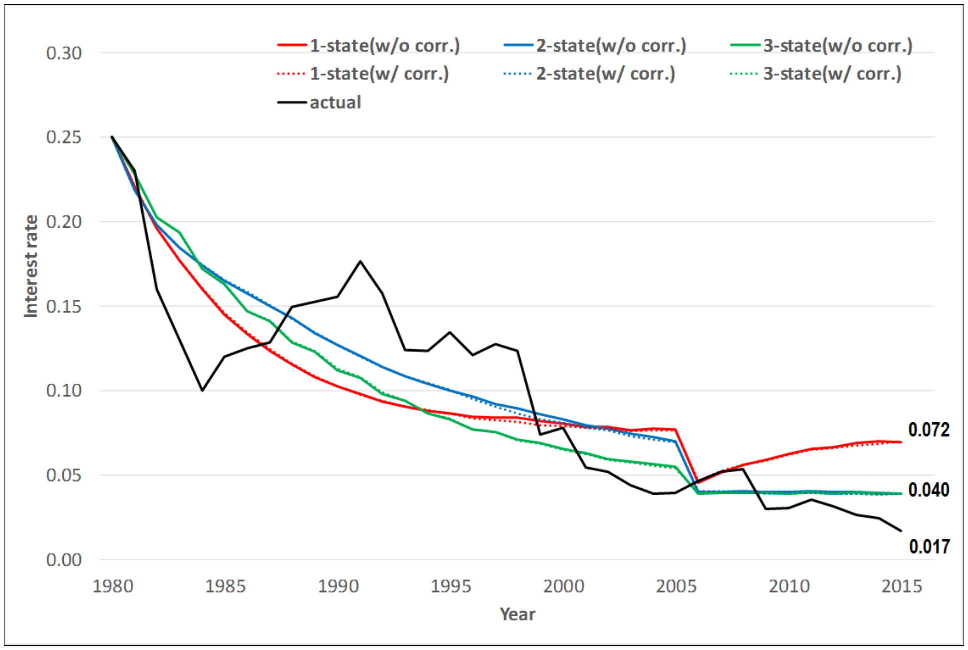

Figure 4.5: Actual and projected r_t; fitted on 1980–2005 and projected for 2006–2015. r_t is simulated with and without the correlation with k_t.

Table 4.4: RMSE for the one-, two- and three-state models of r_t, with and without correlation

	1-state	2-state	3-state
without correlation	0.0324	0.0124	0.0123
with correlation	0.0320	0.0121	0.0121

4.4 Forecasting future interest rate

Future interest rates for next 45 years (2016–2060) are forecasted/projected using the same method described in Section 4.3 using the parameter values obtained by fitting the interest rate data from 1980 to 2015; see Table 4.1. The projected interest rates, both the average and 95 % confidence interval of the 1,000 simulations, are shown in Figure 4.6. The impact of correlation is not included in the analysis considering that correlation does not have much impact in interest rate simulation alone as discussed above. But, again the correlation will make a difference to the valuation and risk measurement of annuity contracts in the next chapter.

When we consider *individual simulated sample paths*, mean-reversion in the interest rates is certainly evident. For the averaged process of the simulated paths, the key feature of Figure 4.6

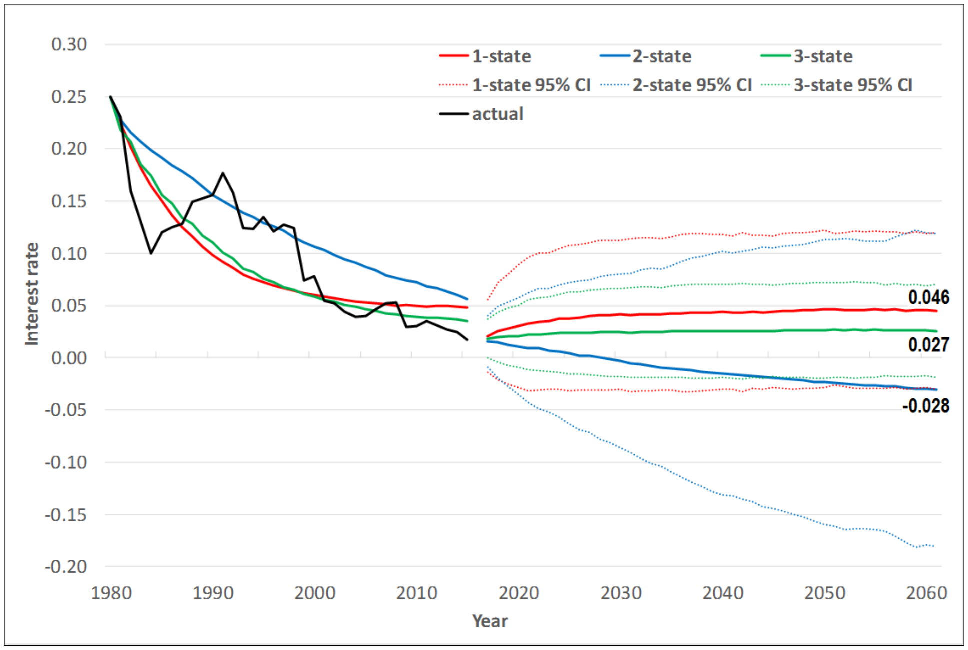

Figure 4.6: Projected r_t for 2016-2060; model parameters obtained by fitting the models on 1980–2015 data.

that differs from the results shown in the validation step (Figure 4.4) is that, under the two-state model there does not appear to be a sign of mean-reversion. This difference stems from the presence of higher probability being in state 2 corresponding to a negative a; this means having a slow mean reversion or a lack of it. Consequently, the average interest rate under the two-state model continues to decrease and reaches -2.8 %. The confidence interval under the two-state model keeps on increasing with time as the model fails to revert to the long-term mean. The one- and three-state models, on the other hand, converges to the long-term mean values of 4.6 % and 2.7 %, respectively. Furthermore, the confidence intervals of the one- and three-state models converge with time as the interest rates reach the long-term mean. The three-state model has a significantly smaller confidence interval than that of the one-state model because its flexibility enable it to capture large fluctuations of interest rates and the volatility values being not too large also helps.

Chapter 5

Pricing annuity and measuring the effect of longevity risk

Pricing the insurance products require accurate estimates of both interest and mortality rates. Here, the pricing of an annuity product is implemented using the results of Chapter 3 for mortality and Chapter 4 for interest rate. The prices of the annuity product varies depending on the age of the policy holder and thus, only the case of Korean female with age 60 is chosen for discussion in this chapter.

The three main objectives of this chapter are:

1. to determine the annuity price and risk, under the one-, two- and three-state models, arising from the time-varying index, k_t,

2. to determine the annuity price and risk, under the one-, two- and three-state models, arising from the interest rate, r_t, and

3. to determine the impact of correlation between k_t and r_t on the annuity price and the associated risk measure.

In order to analyse the impact of different aspects of the modeling approaches, the analysis are conducted in three steps. In the first step, the annuity price and longevity risk are determined without considering the discount factor in order to understand the impact of using the regime-switching method in determining k_t. The discount factor is then included without the correlation with the mortality rate to understand the impact of the interest rate. Lastly, the correlation between the interest rate and the mortality rate is included to understand the impact of the correlation on the annuity price and the longevity risk.

5.1 Assumptions in pricing and longevity-risk measurement methodology

In order to price annuity and measure the longevity risk, the product type, pricing structure and the method for measuring the longevity risk must to be defined. These are elaborated in the following subsection.

5.1.1 Product assumption

- Product type: whole life annuity of \$1 per year payable on the condition that the policyholder remains alive at the end of the payment year. The maximum survival age is assumed to be 90[1].

- Sample policyholder: Korean female of age 60 at the end of 2015. The policyholder can receive the annuity until she reaches age 90 in year 2045.

[1] The maximum age is limited to 90 because the small set of survival rate data for the age above 90 makes the model and its predictions unreliable.

5.1.2 Pricing structure of an annuity product

The price, $a(x,t)$, of a life annuity issued to a policyholder aged x in year t, paying \$1 at the end of each year until her death is given by

$$a(x,t) = \sum_{n=1}^{\omega-x} v_n(t)\,{}_np_x(t), \tag{5.1}$$

where ω is the maximum survivable age, ${}_np_x(t)$ is the probability of a person aged x in year t will survive for n years, and $v_n(t)$ is the n-year discounting factor for an annuity payment n years from year t. The discount factor, $v_n(t)$, is

$$\begin{aligned} v_n(t) &= \exp\left(-\int_t^{t+n} r(u)du\right) \\ &\approx \exp\left(-\sum_{u=1}^{n} r(t+u)\right), \end{aligned} \tag{5.2}$$

where $r(u)$ is the instantaneous interest rate at time u. Since the interest rate is calculated for each year (i.e., r is a discrete function of u), the integral of equation (5.2) becomes a summation where forecasted interest rates of each year from t to $t+u$ are added.

The probability of an individual aged x in year t surviving for the next n years, ${}_np_x(t)$, is expressed as

$$\begin{aligned} {}_np_x(t) &= p_x(t)p_{x+1}(t+1)\cdots p_{x+n-1}(t+n-1) \\ &= (1-q_x(t))\,(1-q_{x+1}(t+1))\cdots(1-q_{x+n-1}(t+n-1)) \\ &= \prod_{j=1}^{n-1}\left(1-q_{x+j}(t+j)\right), \end{aligned} \tag{5.3}$$

where $p_x(t)$ is the probability of an individual aged x in year t survives for one year. The notation $q_x(t)$, on the other hand, is the probability that an individual aged x in year t to die within a year (i.e., $p_x(t)+q_x(t)=1$). It is assumed that the mortality rate, $m_{x,t}$, in the LC model is the same as $q_x(t)$ in equation (5.3).

5.1.3 Method for measuring longevity risk

The longevity risk is calculated based on a 95 % Value at Risk (VaR). To be more specific, the longevity risk is measured as the difference between the top 5 percentile of the simulation results and the average, or

$$\text{Longevity risk} = 95\%\text{ percentile of } a(x,t) - \text{average of } a(x,t). \tag{5.4}$$

Such a method of calculating the longevity risk is based on the standard method put forward in the Solvency II document for the European Insurance and Occupational Pensions Authority (EIOPA) [14]. In addition, most companies use this VaR-based method for their internal model in calculating longevity risks.

5.2 Annuity price and longevity risk with fixed discount factor

In this implementation, the annuity price for a policyholder assumed to have joined/invested in the pension product at the end of 2015 is calculated using equation (5.1). To analyse the risk due to mortality only, $v_n(t)$ in equation (5.2) is set to 1 for calculating the annuity price. There are 1,000 annuity prices are obtained with 1,000 mortality-rate simulated paths in Section 3.4.

The probability that the policyholder survives for the next n years, ${}_np_{60}$, is shown in Figure 5.1. The maximum n, which is also the maximum duration of payments, is capped at 30 because the policyholder reaches the assumed maximum survival age of 90 with this cap. The black line is ${}_np_{60}$ calculated from the mortality rates of ages between 60 to 90 in year 2015. In other words, the black line does not assume any future mortality-rate improvement and is based solely on the 2015 data. Red-, blue- and green-coloured lines correspond to ${}_np_{60}$ calculated under the one-, two and three-states models of k_t, respectively. The solid coloured lines represent the average of the 1,000 simulations whilst the region between the dotted lines represent a 95

% confidence interval.

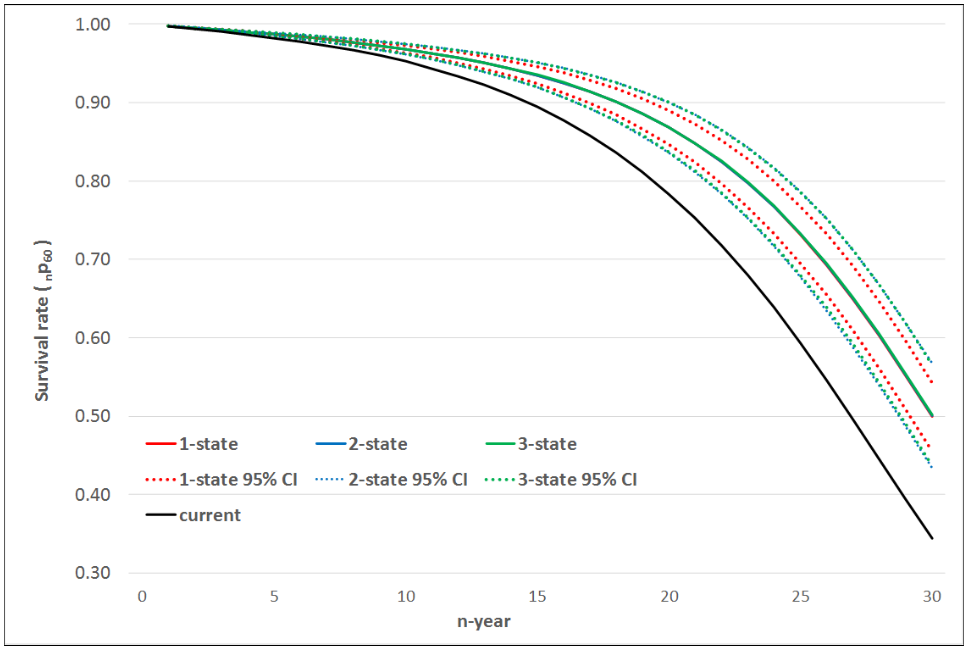

Figure 5.1: Current and forecasted $_n p_{60}$ for a Korean female population.

The average values under each of the three models are almost the same. The 95 % confidence intervals under the two- and three-state models are very similar and are larger than the interval of the one-state model. This trend is consistent with the patterns observed for k_t (Figure 3.11) and m_x (Figure 3.12). In addition, the models employed for calculating $_n p_{60}$ show that around 50 % of the policyholders remain alive in 30 years whilst the 2015 data shows only around 35 % of the policyholders continue to live. Such a large increase of the survival rate indicates that the mortality improvement rate is quite substantial in the next 30 years.

With the discount factor, $v_n(t)$, set to 1, the annuity price in equation (5.1) becomes

$$\begin{aligned} a(60,t) &= \sum_{n=1}^{30} v_n(t)_n p_{60}(t) \\ &= \sum_{n=1}^{30} {}_n p_{60}(t), \end{aligned} \tag{5.5}$$

which is a summation of the $_n p_{60}$ values for each year in Figure 5.1. The calculated annuity price under each of the three models are shown in Table. 5.1. For all three models, the average annuity price is around 26.00, which is slightly less than 30 (i.e., payment of $1 for 30 years if the policyholder remains alive until the maximum age of 90). Whilst the average values are almost the same for all models, the one-state model has a significantly lower longevity risk due to a smaller confidence interval. It can also be seen that the two- and three-state models yield almost identical results.

Table 5.1: Annuity price and longevity risk calculated with a constant discount factor of 1.

No. of states for k_t	average ($)	95th percentile	longevity risk
1	26.06 (0.008)	26.48	0.42
2	26.02 (0.012)	26.65	0.63
3	26.04 (0.012)	26.68	0.64

5.3 Annuity price and longevity risk with uncorrelated discount factor

We look at the impact of interest rate on the annuity price and longevity risk. *Within the Korean data that we examined* and based on the validation results in Section 3.3 (cf Table 3.4), the *three-state model is selected because it produces the smallest RMSE value for the considered age of 60.*

The discount factors calculated using equation (5.2) under the one-, two- and three-state r_t models are shown in Figure 5.2. The discount rates under the one- and three-state models decrease from 1.000 in year 1 to 0.367 and 0.525 in year 30, respectively. For example, the discount factor of 0.367 in year 30 means that the present value of $1 in 30 years is $0.367. It is noted that the two-state model fails to revert to the long-term mean and the interest rate becomes negative (see Figure 4.6). Consequently, the discount factor for the two-state model becomes larger than 1 in year 15 and increases rapidly with time and thereafter. Furthermore, the confidence interval keeps increasing and diverges, which is quite contrary to the patterns seen for one- and three-state models where the confidence interval converges to a fixed size.

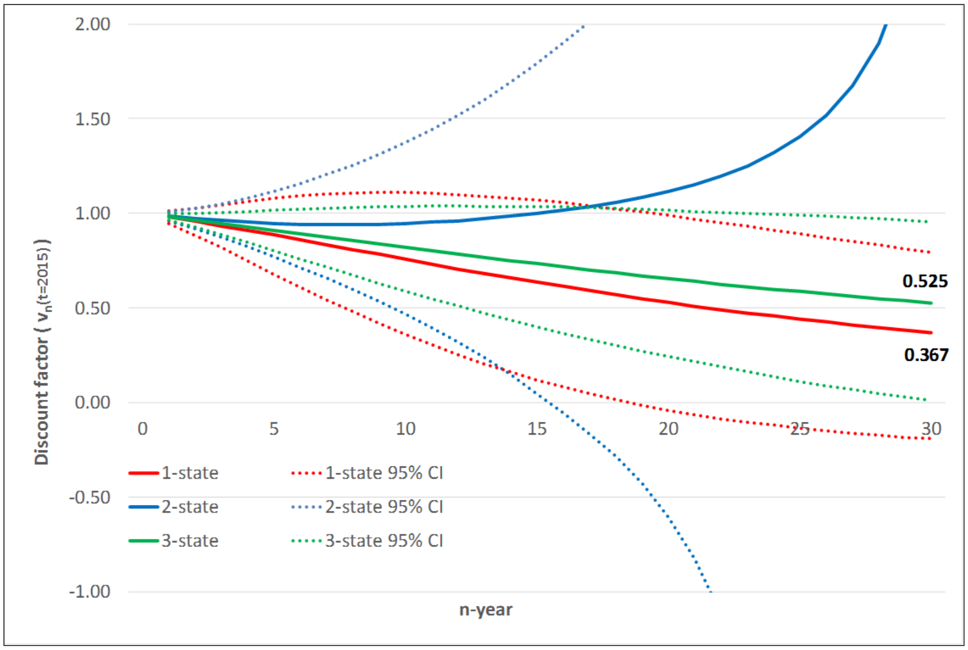

Figure 5.2: Forecasted discount factor of $v_n(2015)$

Given that the two-state model appears inappropriate for our data set, due to its shortcoming wherein r_t is unable to revert back to the long-term mean and the confidence interval diverges, further discussions will only centre on the comparison between the one- and three-state models. The annuity price and longevity risks calculated from these two models, assuming an uncorrelated discount factor, are shown in Table 5.2. The inclusion of the discount factor reduces the average price significantly by as much as 48.5%, i.e., from around \$26.00 (see Table 5.1) to \$17.55 (cf Table 5.2). The lower average price, under the one-state model, comes from a higher long-term mean interest rate, whilst the larger longevity risk under the one-state model is consistent with the associated large confidence interval (see Figure 4.6).

Table 5.2: Annuity price and longevity risk calculated with uncorrelated discount factor.

No. of states for r_t	average (\$)	95th percentile	longevity risk
1	17.55 (0.146)	26.33	8.78
3	19.72 (0.101)	25.52	5.80

5.4 Annuity price and longevity risk under the correlation assumption

As the final step in our analysis, the impact of the correlation between interest and mortality rates (more specifically, the time-varying index, k_t, of the mortality rate) on the annuity price and longevity risk are investigated.

Table 5.3 shows the average price and the longevity risk obtained with the discount factor calculated under correlation assumption. It can be seen from Table 5.3 that the average prices remain virtually unchanged when these are compared to the average prices shown in Table 5.2. This lack of change in the average prices can be attributed to the correlation structure built between r_t and k_t. This correlation is embedded in the Brownian motion which is normally distributed. The little effect of correlation on *average values* can be understood by noting that it is introduced through a stochastic process characterised by a normal distribution with zero mean. However, we see below that the impact of the correlation is on the longevity risk.

Table 5.3: Annuity price and longevity risk calculated with correlated discount factor.

No. of states for r_t	average ($)	95the percentile	longevity risk
1	17.55 (0.152)	26.58	9.02
3	19.74 (0.104)	25.88	6.14

Accounting for both the results in the one- and three-state models, the correlation increases the longevity risk ; see last column of Table 5.3. The increase of 0.30 corresponds to 3 % and 6 % increase in longevity risk for one- and three-state models, respectively. The impact of the correlation on the longevity risk can be explained as follows. The correlation between interest and mortality rates means that its shift in value due to the Brownian motion occurs in the same direction (i.e., both of them will either increase or decrease). The impact of the shift is qualitatively the same for both interest and mortality rates; an increase in rates will decrease the annuity price whilst a decrease in rates will increase the annuity price. In the cases where one of the two rates is increased whilst the other is decreased, the net effect of the two cancels each other. Introducing a correlation eliminates such a cancelling scenario, which in turn increases the longevity risk. The increase in longevity risk suggests that a disregard for correlation leads to an underestimation of longevity risk. Therefore, it is important to include the correlation between mortality and interest rates in annuity pricing and risk measurement.

Chapter 6

Conclusion

In this thesis, we developed a modelling methodology that focuses on longevity risk as well as the correlation of interest and mortality rates in the pricing and risk measurement of an annuity. This is an effort for the accurate estimation of the associated risk level and the determination of the extent certain factors and parameters contribute to prices and the setting of capital reserves. In particular, we utilised the Lee-Carter and Vasiček models to capture the stochastic behaviour of the interest and mortality dynamics. These one-state models were enriched using the Markovian regime-switching technique (up to 3 states) so that the capacity is enhanced in reproducing the observed nonlinear dynamic patterns of the data. We put forward a correlated Lee-Carter and Vasiček models, and the impact of having multiple regimes to capture the nonlinearity is assessed by comparing the one-, two- and three-state models. Finally, all of the proposed models are used to obtain the annuity prices and risk measures. Our empirical work made use of the mortality rate data for South Korean female population and South Korean interest-rate data. Such data sets were deliberately chosen since the South Korean economy underwent a rapid growth in the 1980's and has now attained reasonable stability whilst the mortality rate is improving rapidly for Korean females to the point where their life expectancy is projected to be the highest in the world [30].

6.1 Research contributions

We highlight below the contributions and findings of this thesis:

(i) Insights from the regime-switching modelling results: Both the two- and three-state models outperform the linear model (one state) in describing the time-varying index of the mortality rate. For the interest rate data that we analysed, the two-state regime-switching model did not perform better than the one-state model. There seems to be a problem for the interest rate process to revert to a long-term mean because regime 1 with a positive speed of mean reversion is not dominant in magnitude compared to regime 2 having a negative speed.

   Nonetheless, it is worth noting that the three-state model performs the best in predicting both the mortality and interest rates. The regime-switching models possess the capability to significantly improve the linear models in terms of forecasting metrics. Although, care must be taken to ensure that the chosen model behaves as intended (i.e., making the process converge to the long-term mean).

(ii) Insights from the inclusion of correlation: Again within the context of the data in our implementation, the interest rate and the time-varying index of the mortality rate are strongly correlated with an estimated correlation coefficient value of 0.857. As the correlation is reflected through the Brownian-motion terms of the two models, the impact of the correlation can only be measured when interest and mortality rates are paired with the same frequency.

   We found that the correlation does not affect much the average price of the annuity product. However, the correlation increases the longevity risk which happens when the two factors substantially deviate from their means and move in the same direction. On the other hand, when the two factors move in the opposite direction, even though there is big deviations from their their means, reduction in risk measures is attained.

6.2 Possible future research directions

This thesis put forward the utility of a regime-switching approach in a framework that also captures explicitly the correlation between the interest and mortality rates. Our simulation results adduced the huge benefit of our modelling setup. Under the appropriate circumstances backed up by observed data, such a correlation structure is a valuable mechanism in the justification when setting the “right” amount (neither under-reserving nor over-reserving) of capital requirements as a result of longevity risk reduction. The natural research direction of this work points to the following:

(i) Expansion of the sample cohort to include male and wider age groups in the pricing annuity price and measuring longevity risk;

(ii) Pricing and risk measurement of life insurance products with investment guarantees under our proposed modelling framework;

(iii) Empirical investigations of correlation between aging population and various financial instrument(e.g., stock price or bond price) in the Korean capital market.

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Appendix A

Estimated Lee-Carter model parameters

Tables A.1–A.3 display the numerical values of the three parameters of the LC model, a_x, b_x and k_t, in subsection 3.2.1. The estimation was performed using the SVD in the statistical software R applied to the Korean female mortality data (1983-2015).

Table A.1: Fitted values of a_x and b_x for the Lee-Carter model (ages 0–50)

Age	a_x	b_x	Age	a_x	b_x	Age	a_x	b_x
0	−5.05812	0.01686	17	−7.94782	0.01385	34	−7.12421	0.01138
1	−7.19304	0.02071	18	−7.85922	0.01346	35	−7.06538	0.01079
2	−7.68960	0.01686	19	−7.79353	0.01316	36	−7.00374	0.01025
3	−7.91427	0.01848	20	−7.74305	0.01251	37	−6.93727	0.00978
4	−8.08700	0.01922	21	−7.69147	0.01194	38	−6.86685	0.00957
5	−8.11405	0.02100	22	−7.64835	0.01138	39	−6.79676	0.00937
6	−8.21662	0.01997	23	−7.60909	0.01079	40	−6.72622	0.00918
7	−8.33198	0.01948	24	−7.56847	0.01025	41	−6.65348	0.00892
8	−8.43323	0.01899	25	−7.53599	0.01578	42	−6.57565	0.00867
9	−8.53394	0.01881	26	−7.50537	0.01443	43	−6.49606	0.00848
10	−8.61159	0.01838	27	−7.47130	0.01450	44	−6.41616	0.00850
11	−8.62813	0.01795	28	−7.43182	0.01408	45	−6.33589	0.00863
12	−8.57921	0.01693	29	−7.38516	0.01385	46	−6.25784	0.00882
13	−8.48148	0.01578	30	−7.33181	0.01346	47	−6.18233	0.00889
14	−8.34804	0.01443	31	−7.27988	0.01316	48	−6.10577	0.00902
15	−8.18309	0.01450	32	−7.22681	0.01251	49	−6.03028	0.00912
16	−8.05506	0.01408	37	−6.93727	0.00978	50	−5.95591	0.01051

Table A.2: Fitted values of a_x and b_x for the LC model (ages 51–99)

Age	a_x	b_x	Age	a_x	b_x	Age	a_x	b_x
51	−5.88173	0.01058	68	−4.29683	0.01079	85	−2.28215	0.00415
52	−5.80866	0.01068	69	−4.17422	0.01053	86	−2.18275	0.00376
53	−5.73689	0.01078	70	−4.04840	0.01027	87	−2.08601	0.00340
54	−5.66509	0.01089	71	−3.92469	0.00996	88	−1.99192	0.00305
55	−5.59327	0.01095	72	−3.80136	0.00965	89	−1.90049	0.00273
56	−5.51522	0.01103	73	−3.67821	0.00931	90	−1.81172	0.00243
57	−5.43219	0.01108	74	−3.55228	0.00901	91	−1.72560	0.00216
58	−5.34467	0.01108	75	−3.42658	0.00860	92	−1.64215	0.00190
59	−5.25398	0.01108	76	−3.30317	0.00815	93	−1.56135	0.00167
60	−5.16202	0.01104	77	−3.18079	0.00768	94	−1.48320	0.00146
61	−5.06602	0.01104	78	−3.06057	0.00723	95	−1.40772	0.00127
62	−4.96721	0.01106	79	−2.94140	0.00680	96	−1.33489	0.00110
63	−4.86484	0.01111	80	−2.82524	0.00637	97	−1.26472	0.00095
64	−4.75936	0.01116	81	−2.71072	0.00592	98	−1.19721	0.00083
65	−4.64870	0.01116	82	−2.59777	0.00545	99	−1.13235	0.00073
66	−4.53447	0.01111	83	−2.48847	0.00499			
67	−4.41696	0.01099	84	−2.38421	0.00456			

Table A.3: Fitted values of k_t for the LC model

Year	k_t	Year	k_t	Year	k_t
1983	58.264	1994	21.368	2005	−17.988
1984	55.974	1995	20.119	2006	−23.069
1985	53.819	1996	18.094	2007	−31.519
1986	51.429	1997	13.878	2008	−41.960
1987	49.175	1998	11.808	2009	−53.932
1988	46.872	1999	12.376	2010	−56.339
1989	44.683	2000	8.724	2011	−60.117
1990	38.346	2001	5.210	2012	−61.834
1991	31.428	2002	3.173	2013	−74.230
1992	29.677	2003	−5.082	2014	−84.564
1993	23.977	2004	−11.067	2015	−84.518

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