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Labeling of graphs, sumset of squares of units modulo n and resonance varieties of matroids

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Abstract

This thesis investigates problems in a number of different areas of graph theory and its applications in other areas of mathematics.

Motivated by the 1-2-3-Conjecture, we consider the *closed distinguishing number* of a graph G, denoted by $\mathrm{dis}[G]$. We provide new upper bounds for $\mathrm{dis}[G]$ by using the Combinatorial Nullstellensatz. We prove that it is **NP**-complete to decide for a given planar subcubic graph G, whether $\mathrm{dis}[G] = 2$. We show that for each integer t there is a bipartite graph G such that $\mathrm{dis}[G] > t$. Then some polynomial time algorithms and **NP**-hardness results for the problem of partitioning the edges of a graph into regular and/or locally irregular subgraphs are presented. We then move on to consider Johnson graphs to find resonance varieties of some classes of sparse paving matroids. The last application we consider is in number theory, where we find the number of solutions of the equation $x_1^2 + \cdots + x_k^2 = c$, where $c \in \mathbb{Z}_n$, and x_i are all units in the ring $\mathbb{Z}_n$. Our approach is combinatorial using spectral graph theory.

Keywords: Edge-partition problems, Semiregular number, Adjacency matrix, Resonance variety, Closed distinguishing labeling, Computational complexity, Combinatorial Nullstellensatz, Sparse paving matroid and Johnson graph.

Co-Authorship

This thesis incorporates material that is result of joint research, as follows:

- Chapter 2 is based on the paper

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  Authors played equal roles in the research.

- Chapter 3 is based on the paper

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  Authors played equal roles in the research.

- Chapter 4 is based on the preprint

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- Chapter 5 is based on the paper

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This thesis is lovingly dedicated to my mother.

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Chapter 1

Introduction

Graph theory is a rapidly developing field with extensive applications in other fields of modern mathematics. Many theorems in graph theory are related to different ways of coloring (labeling) graphs. For example, the famous four-color theorem states that every planar graph is four-colorable. In this thesis, at first we study the closed distinguishing labelings of a graph which is motivated by the 1-2-3-Conjecture. Then motivated by the 1-2-3-Conjecture over regular graphs, we consider the problem of partitioning the edges of a graph into regular and/or locally irregular subgraphs. We then turn our attention to the Johnson graphs to find resonance varieties of some classes of sparse paving matroids. We provide some examples of sparse paving matroids with non-trivial resonance varieties, which generalizes the previous examples. Finally, with the help of spectral graph theory and labeling of Cayley graphs, we solve a counting problem in number theory.

1.1 Labeling of graphs

A vertex coloring is an assignment of labels or colors to each vertex of a graph such that no edge connects two identically colored vertices. It is **NP**-complete to decide if a given graph admits a k-coloring for a given $k \geq 3$. The 1-2-3-Conjecture, posed in 2004 by Karoński, Łuczak, and Thomason [5], states that one may label the edges of any connected graph on at least 3 vertices from the set $\{1, 2, 3\}$ (call the label function w) so that the function $f(v) = \sum_{u \in N(v)} w(uv)$ is a proper vertex colouring.

Motivated by the 1-2-3-Conjecture, Axenovich et al. [1], introduced closed distinguishing labelings of a graph. An assignment of numbers to the vertices of a graph G is said to be *closed distinguishing* if for any two adjacent vertices v and u the sum of labels of the vertices in the closed neighborhood of the vertex v differs from the sum of labels of the vertices in the closed neighborhood of the vertex u unless they have the same closed neighborhood. The *closed distinguishing number* of a graph G, denoted by $\mathrm{dis}[G]$, is the smallest integer k such that there is a closed distinguishing labeling for G using integers from the set $\{1, 2, \ldots, k\}$. Define also $\mathrm{dis}(G)$ using $N(u)$ instead of $N[u]$ and call the corresponding coloring open distinguishing. We prove that it is **NP**-complete to decide, for a given planar subcubic graph G, whether $\mathrm{dis}[G] = 2$. We also prove the same result for a given bipartite subcubic graph. Among other results, we show that for each integer t there exists a bipartite graph G such that $\mathrm{dis}[G] > t$. This give a partial answer to a question raised by Axenovich et al. that how $\mathrm{dis}[G]$ function depends on the chromatic number of a graph. Finally, using the Combinatorial Nullstellensatz we improve the current upper bound and give various upper bounds for the closed distinguishing number of graphs.

A graph is *locally irregular* if its adjacent vertices have distinct degrees. The irregular chromatic index, denoted by $\chi'_{irr}(G)$, is the minimum number k such that the graph G can be partitioned into k locally irregular subgraphs. Let G be a regular graph. It follows immediately that $\chi'_{irr}(G) = 2$ if and only if one may label the edges of G from the set $\{1, 2\}$ so that the function $f(v) = \sum_{u \in N(v)} w(uv)$ is a proper vertex colouring. Since the status of the 1-2-3-Conjecture regarding regular graphs is still not clear, the aforementioned observation is of interest.

A graph G is *weakly semiregular* if there are two numbers a and b, such that the degree of every vertex is a or b. The *weakly semiregular number* of a graph G, denoted by $wr(G)$, is the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is a weakly semiregular graph. We present a polynomial time algorithm to determine whether the weakly semiregular number of a given tree is two. On the other hand, we show that determining whether $\mathrm{wr}(G) = 2$ for a given bipartite graph G with at most three numbers in its degree set is **NP**-complete. Among other results, for every tree T, we show that $\mathrm{wr}(T) \leq 2 \log_2 \Delta(T) + \mathcal{O}(1)$.

A graph G is a *$[d, d+s]$-graph* if the degree of every vertex of G lies in the interval $[d, d+s]$. A $[d, d+1]$-graph is said to be *semiregular*. The *semiregular number* of a graph G, denoted by $\mathrm{sr}(G)$, is the minimum number of subsets into which the edge set of G

can be partitioned so that the subgraph induced by each subset is a semiregular graph. We prove that the semiregular number of a tree T is $\lceil \frac{\Delta(T)}{2} \rceil$. On the other hand, we show that determining whether $\mathrm{sr}(G) = 2$ for a given bipartite graph G with $\Delta(G) \leq 6$ is **NP**-complete.

1.2 Resonance varieties of matroids

Let $A = (A^\bullet, d)$ be a commutative, differential graded algebra (or simply CDGA) over complex numbers. So $A = \bigoplus_{i \geq 0} A^i$ is a graded $\mathbb{Q}$-vector space, with a multiplication map $\cdot\colon A^i \otimes A^j \to A^{i+j}$ where $u \cdot v = (-1)^{ij} v \cdot u$, and a differential $d\colon A^i \to A^{i+1}$ where $d(u \cdot v) = du \cdot v + (-1)^i u \cdot dv$, for all $u \in A^i$ and $v \in A^j$.

We will assume that $A^0 = \mathbb{Q}$, and A^i is finite-dimensional, for all $i \geq 0$. So we can identify the vector space $H^1(A) = Z^1(A)/B^1(A)$ with the cocycle space $Z^1(A)$. For each element $a \in Z^1(A) \cong H^1(A)$, we have the following cochain complex,

$$(A^\bullet, \delta_a)\colon\ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots, \tag{1.1}$$

where $\delta_a^i(u) = a \cdot u + du$, for all $u \in A^i$. It is easy to see that $\delta_a^{i+1}\delta_a^i(u) = 0$.

For each integer $i \geq 0$, define the degree-i resonance variety

$$R^i(A) = \{a \in H^1(A) \mid H^i(A^\bullet, \delta_a) \neq 0\}. \tag{1.2}$$

The study of resonance varieties has led to interesting connections with other branches of mathematics. For example, generalized Cartan matrices [7], Latin squares [11] and the Bethe Ansatz [2].

The main motivation to the try to find the resonance varieties comes from the tangent cone formula which relates the degree-one resonance varieties to the characteristic varieties of G, where G is a finitely presented 1-formal group.

Let M be a matroid (or any combinatorial object like graph). By the Brieskorn-Orlik-Solomon Theorem [7], it is known that the resonance varieties associated with a matroid depend only on combinatorics of the matroid. So it is natural to try to find the resonance varieties of the matroid. For degree 1, there is a full characterization [4]. For cohomological degree greater than 1, the full characterization is at present far from being solved.

There has been some work, for example [3], but little is known. For instance, Papadima and Suciu in [8], proved that for the sum of two matroids M_1 and M_2 we have

$$R^k(M_1 \oplus M_2) = \bigcup_{p+q=k} R^p(M_1) \times R^q(M_2).$$

A *paving matroid* is a matroid in which every circuit has size at least as large as the matroid's rank. A *sparse paving* matroid is a paving matroid in which its dual is a paving matroid. It has been conjectured that almost all matroids are sparse paving matroids [6]. In chapter 4, with the help of Johnson graphs and combinatorial techniques we find the resonance varieties of some classes of sparse paving matroids.

The *Johnson graphs* are a special class of graphs defined from systems of sets. Let E be a finite set of size n, and let $0 < r < n$. The Johnson graph $J(n, r)$ is the graph with vertex set $\{X \subseteq E : \ |X| = r\}$ in which any two vertices are adjacent if and only if they have $r - 1$ elements in common. It is known that $B \subseteq \{X \subseteq E : \ |X| = r\}$ is the collection bases of a sparse paving matroid if and only if $\{X \subseteq E : \ |X| = r\} - B$ is an independent set in $J(n, k)$.

Let M be a sparse paving matroid of rank r. Let $a = \sum_{i=1}^{n} \alpha_i e_i$ and $f_a : \overline{I}^{r-1} \to \overline{I}^{r}$ be defined by left multiplication by a. We show that $a \in R^{r-2}(M)$ if and only if f_a is not injective. Using the structure of Johnson graphs we show that if the intersection of all of the minimum circuits of M is non-empty, then $R^{r-2}(M)$ is trivial. Also, we find $R^{r-2}(M)$, if the intersection of all of the minimum circuits of M except one of them is non-empty. Among other results, we show that if the rank of M is large enough in comparison to the number of minimum circuits, then $R^{r-2}(M)$ is trivial.

1.3 On the addition of squares of units modulo n

Another application of graph theory we consider is in number theory. The problem of finding explicit formulas for the number of representations of a natural number n as the sum of k squares is one of the interesting and classical problems in number theory. For example, if $k = 4$, then Jacobi's four-square theorem states that this number is $8 \sum_{m|c} m$ if c is odd and 24 times the sum of the odd divisors of c if c is even.

Let $a_1, \dots, a_k$ be arbitrary elements in the ring $\mathbb{Z}_n$. Recently, Tóth [9] found formulas for the number of solutions of the equation $a_1x_1^2 + \cdots + a_kx_k^2 = c$, where $c \in \mathbb{Z}_n$, and x_i all belong to $\mathbb{Z}_n$.

Now, consider the equation

$$x_1^2 + \cdots + x_k^2 = c,$$

where $c \in \mathbb{Z}_n$, and x_i are all units in the ring $\mathbb{Z}_n$. We denote the number of solutions of this equation by $\mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$. Yang and Tang [10] obtained a formula for $\mathscr{S}_{sq}(\mathbb{Z}_n, c, 2)$. Here we provide an explicit formula for $\mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$, for an arbitrary k.

The idea may be sketched as follows: first, it is easy to show that if m, n are coprime numbers, then $\mathscr{S}_{sq}(\mathbb{Z}_{mn}, c, k) = \mathscr{S}_{sq}(\mathbb{Z}_m, c, k)\mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$. So it is enough to find a formula for $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$ where p is a prime number. Let $\mathbb{Z}_n^{\times 2} = \{x^2; x \in \mathbb{Z}_n^\times\}$. Let p be an odd prime number. There is a natural map between solutions of the above equation and $(0, c)$-walks in the directed Cayley graph $Cay(\mathbb{Z}_{p^\alpha}, \mathbb{Z}_{p^\alpha}^{\times 2})$, defined by sending $(\pm x_1, \dots, \pm x_k)$ to the walk $0, x_1^2, x_1^2 + x_2^2, \dots, x_1^2 + \cdots + x_k^2$. Thus, enumerating the number of solutions amounts to 2^k times enumerating these walks. By exploiting the structure of this graph, one can reduce this calculation to the case that $\alpha = 1$. The number of walks can then be identified as a particular entry in the kth power of the adjacency matrix of this graph; in this case the adjacency matrix can be described explicitly, and hence one can obtain an exact formula. An exact formula for $\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$ can be found by direct counting.

Bibliography

[1] M. Axenovich, J. Przybylo, R. Sotak, M. Voigt, J. Weidlich, *A note on adjacent vertex distinguishing colorings number of graphs*, Discrete Appl. Math. 205 (2016) 1–7.

[2] D. Cohen and A. Varchenko, *Resonant local systems and representations of* sl_2. Geom. Dedicata, 101:217–233, 2003.

[3] G. Denham, *Higher resonance varieties of matroids,* in: Configurations Spaces, 41–66, Springer INdAM series, vol. 14, Springer, Cham, 2016.

[4] M. Falk and S. Yuzvinsky, Multinets, *resonance varieties, and pencils of plane curves*, Compos. Math. 143 (2007), no. 4, 1069–1088.

[5] M. Karoński, T. Łuczak, A. Thomason, *Edge weights and vertex colours*, J. Combin. Theory Ser. B 91 (1) (2004) 151–157.

[6] D. Mayhew, M. Newman, D. Welsh and G. Whittle: *On the asymptotic proportion of connected matroids*, European J. Combin. 32 (2011), 882–890.

[7] P. Orlik and L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. 56 (1980), no. 2, 167–189.

[8] S. Papadima, A.I. Suciu, *Bieri–Neumann–Strebel–Renz invariants and homology jumping loci*, Proc. London Math. Soc. **100** (2010), no. 3, 795–834.

[9] L. Tóth, *Counting solutions of quadratic congruences in several variables revisited*, J. Integer Seq. 17 (2014), Article 14.11.6.

[10] Q. Yang, M. Tang, *On the addition of squares of units and nonunits modulo n,* J. Number Theory., 155 (2015) 1–12.

[11] S. Yuzvinsky, *Realization of finite abelian groups by nets in* $\mathbb{P}^2$. Compos. Math., 140(6):1614–1624, 2004

Chapter 2

On the algorithmic complexity of adjacent vertex closed distinguishing number of graphs

2.1 Introduction

In 2004, Karoński et al. in [19] introduced a new coloring of a graph which is generated via edge labeling. Let $f : E(G) \to \mathbb{N}$ be a labeling of the edges of a graph G by positive integers such that for every two adjacent vertices v and u, $S(v) \neq S(u)$, where $S(v)$ denotes the sum of labels of all edges incident with v. It was conjectured that three integer labels $\{1, 2, 3\}$ are sufficient for every connected graph, except K_2 [19] (1-2-3 Conjecture). Currently the best bound that was proved by Kalkowski et al. is five [18]. For more information we refer the reader to a survey on the 1-2-3 Conjecture and related problems by Seamone [27] (also see [4, 6, 8, 10, 12, 24, 30]). Different variations of distinguishing labelings of graphs have also been considered, see [5, 7, 17, 20–22, 25, 26, 28].

On the other hand, there are different types of labelings which consider the closed neighborhoods of vertices. In 2010, Esperet et al. in [13] introduced the notion of locally identifying coloring of a graph. A proper vertex-coloring of a graph G is said to be *locally identifying* if for any pair u, v of adjacent vertices with distinct closed neighborhoods, the sets of colors in the closed neighborhoods of u and v are different. In 2014, Aïder et al. [1] introduced the notion of relaxed locally identifying coloring of graphs. A vertex-coloring of a graph G (not necessary proper) is said to be *relaxed locally identifying* if for any pair u, v of adjacent vertices with distinct closed neighborhoods, the sets of colors in the closed neighborhoods of u and v are different. Note that a relaxed locally identifying coloring of a graph that is similar to locally identifying coloring for which the coloring is not necessary proper. For more information see [14, 16, 27].

Motivated by the 1-2-3 Conjecture and the relaxed locally identifying coloring, the closed distinguishing labeling as a vertex version of the 1-2-3 Conjecture was introduced by Axenovich et al. [3]. For every vertex v of G, let $N[v]$ denote the closed neighborhood of v. An assignment of numbers to the vertices of a graph G is *closed distinguishing* if for any two adjacent vertices v and u the sum of labels of the vertices in the closed neighborhood of the vertex v differs from the sum of labels of the vertices in the closed neighborhood of the vertex u unless $N[u] = N[v]$ (i.e. they have the same closed neighborhood). The *closed distinguishing number* of a graph G, denoted by $\mathrm{dis}[G]$, is the smallest integer k such that there is a closed distinguishing assignment for G using integers from the set $\{1, 2, \ldots, k\}$. For each vertex $v \in V(G)$, let $L(v)$ denote a list of natural numbers available at v. A *list closed distinguishing labeling* is a closed distinguishing labeling f such that $f(v) \in L(v)$ for each $v \in V(G)$. A graph G is said to be *closed distinguishing k-choosable* if every k-list assignment of natural numbers to the vertices of G permits a list closed distinguishing labeling of G. The *closed distinguishing choice number* of G, $\mathrm{dis}_\ell[G]$, is the minimum natural number k such that G is closed distinguishing k-choosable. In this work we study closed distinguishing number and closed distinguishing choice number of graphs.

In this work, we also consider another parameter, the minimum number of integers required in a closed distinguishing labeling. For a given graph G, the minimum number of integers required in a closed distinguishing labeling is called its *strong closed distinguishing number* $\mathrm{dis}_s[G]$. Note that a vertex-coloring of a graph G (not necessary proper) is said to be strong closed distinguishing labeling if for any pair u, v of adjacent vertices with distinct closed neighborhoods, the multisets of colors in the closed neighborhoods of u and v are different.

2.2 Closed distinguishing labeling

In this section we study the closed distinguishing number and the closed distinguishing choice number of graphs. We prove theorems in Section 2.5.

2.2.1 The difference between $\mathrm{dis}[G]$ and $\mathrm{dis}_\ell[G]$

It was shown [3] that for every graph G with $\Delta \geq 2$, $\mathrm{dis}[G] \leq \mathrm{dis}_\ell[G] \leq \Delta^2 - \Delta + 1$. Also, there are infinitely many values of Δ for which G might be chosen so that $\mathrm{dis}[G] = \Delta^2 - \Delta + 1$ [3]. We prove that the difference between $\mathrm{dis}[G]$ and $\mathrm{dis}_\ell[G]$ can be arbitrary large and show that for every number t there is a graph G such that $\mathrm{dis}_\ell[G] - \mathrm{dis}[G] \geq t$.

Theorem 2.2.1. *For every positive integer t there is a graph G such that* $\mathrm{dis}_\ell[G] - \mathrm{dis}[G] \geq t$.

2.2.2 The complexity of determining $\mathrm{dis}[G]$

Let $T \neq K_2$ be a tree. It was shown [3] that $\mathrm{dis}_\ell[T] \leq 3$ and $\mathrm{dis}[T] \leq 2$. Here, we investigate the computational complexity of determining $\mathrm{dis}[G]$ for planar subcubic graphs and bipartite subcubic graphs.

Theorem 2.2.2. *Let G be a planar subcubic graph G. It is* **NP***-complete to decide whether* $\mathrm{dis}[G] = 2$.

Although for a given tree T, we can compute $\mathrm{dis}[T]$ in polynomial time [3], but the problem of determining the closed distinguishing number is hard for bipartite graphs.

Theorem 2.2.3. *Let G be bipartite subcubic graph G. It is* **NP***-complete to decide whether* $\mathrm{dis}[G] = 2$.

Note that in the proof of Theorem 2.2.3, which follows in Section 2.5, we reduced Not-All-Equal to our problem and the planar version of Not-All-Equal is in **P** [23], so the computational complexity of deciding whether $\mathrm{dis}[G] = 2$ for planar bipartite graphs remains unsolved.

Theorem 2.2.4. *For every integer $t \geq 3$, it is* **NP***-complete to decide whether* $\mathrm{dis}[G] = t$ *for a given graph G.*

Note that Theorems 2.2.2,2.2.3 and 2.2.3 are in **NP**.

2.2.3 Upper bounds for $\mathrm{dis}_\ell[G]$ and $\mathrm{dis}[G]$

It was shown that for every graph G with $\Delta \geq 2$, $\mathrm{dis}[G] \leq \mathrm{dis}_\ell[G] \leq \Delta^2 - \Delta + 1$ [3]. Here, we improve the previous bound.

Theorem 2.2.5. *Let G be a simple graph on n vertices with degree sequence $\Delta = d_1 \geq d_2 \geq \ldots \geq d_n = \delta$ and $\Delta \neq 1$. Define $s := d_1 + \cdots + d_\Delta - \Delta$.*
(i) $\mathrm{dis}_\ell[G] \leq s + 1 \leq \Delta^2 - \Delta + 1$.
(ii) $\mathrm{dis}_\ell[G] \leq m$, *where m is the number of edges.*
(iii) If there are exactly t vertices with degree Δ, then

$$\mathrm{dis}_\ell[G] \leq \min\{\Delta^2 - 2\Delta + t + 1, \Delta^2 - \Delta + 1\}.$$

(iv) If there is a unique vertex with degree Δ, then $\mathrm{dis}_\ell[G] \leq \Delta^2 - 3\Delta + 4$.
(v) If G is a strongly regular graph with parameters (n, k, λ, μ), then

$$\mathrm{dis}_\ell[G] \leq k(k - \lambda - 1) + 1.$$

(vi) $\mathrm{dis}_\ell[G] \leq (\frac{n-1}{2})^2 + 1$.

2.2.4 Lower bound for $\mathrm{dis}[G]$

Let G be a bipartite graph with partite sets A and B which is not a star. Let, for $X \in \{A, B\}$; $\Delta_X = \max_{x \in X} d(x)$ and $\delta_{X,2} = \min_{x \in X, d(x) \geq 2} d(x)$. It was shown [3] that

$$\mathrm{dis}[G] \leq min\{c\sqrt{|E(G)|}, \lfloor \frac{\Delta_A - 1}{\delta_{B,2} - 1} \rfloor + 1, \lfloor \frac{\Delta_B - 1}{\delta_{A,2} - 1} \rfloor + 1\},$$

where c is some constant. Thus, for a given bipartite graph G, $\mathrm{dis}[G] = \mathcal{O}(\Delta)$ [3]. Regarding $\mathrm{dis}[G]$ as a function, Axenovich et al. [3] said: "One of the challenging problems in the area is to determine how $\mathrm{dis}[G]$ depends on the chromatic number of a graph. The situation is far from being understood even for bipartite graphs." We give a negative answer to this problem and show that for each t there is a bipartite graph G such that $\mathrm{dis}[G] > t$.

Theorem 2.2.6. *For each integer t, there is a bipartite graph G such that* $\mathrm{dis}[G] > t$.

2.2.5 Split graphs

A *split graph* is a graph whose vertex set may be partitioned into a clique K and an independent set S. It is well-known that split graphs can be recognized in polynomial time, and that finding a canonical partition of a split graph can also be found in polynomial time. We prove the following result.

Theorem 2.2.7. *If G is a split graph, then* $\mathrm{dis}[G] \leq (\omega(G))^2$.

2.3 Strong closed distinguishing number

In this section, we focus on the strong closed distinguishing number of graphs. For any graph G, we have the following.

$$\mathrm{dis}_s[G] \leq \mathrm{dis}[G] \leq \mathrm{dis}_\ell[G] \tag{2.1}$$

For a given connected bipartite graph $G = [X, Y]$, except K_2, define $f : V(G) \to \{1, \Delta\}$ such that:

$$f(v) = \begin{cases} 1, & \text{if } v \in X \\ \Delta, & \text{if } v \in Y \end{cases}$$

Let $x \in X$ and $y \in Y$. If $\sum_{v \in N[x]} f(v) = \sum_{v \in N[x]} f(v)$, then $\Delta \deg x + 1 = \Delta + \deg y$. Hence $\Delta(\deg x - 1) = \deg y - 1$. Thus, $\mathrm{dis}_s[G] \leq 2$. So, by Theorem 2.2.6, the difference between $\mathrm{dis}[G]$ and $\mathrm{dis}_s[G]$ can be arbitrary large. Here we increase the gap.

Theorem 2.3.1. *For each n, there is a graph G with n vertices such that* $\mathrm{dis}[G] - dis_s[G] = \Omega(n^{\frac{1}{3}})$.

Let G be an r-regular graph and $f : V(G) \to \{a, b\}$ be a closed distinguishing labeling. Define:

$$g(v) = \begin{cases} a', & \text{if } \ f(v) = a, \\ b', & \text{if } \ f(v) = b. \end{cases}$$

It is easy to check that if $a' \neq b'$, then $g : V(G) \to \{a', b'\}$ is a closed distinguishing labeling. Thus, for an r-regular graph G, $\mathrm{dis}_s[G] = 2$ if and only if $\mathrm{dis}[G] = 2$.

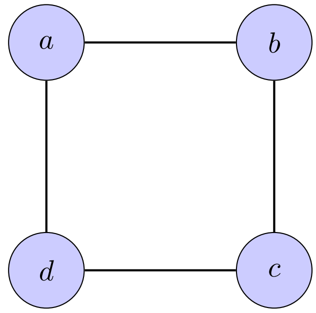


FIGURE 2.1: $G = C_4$

Let a and b be two numbers and $a \neq b$, we show that for a given 4-regular graph G, it is **NP**-complete to decide whether there is a closed distinguishing labeling from $\{a, b\}$.

Theorem 2.3.2. *For a given 4-regular graph G, it is* **NP***-complete to decide whether $dis_s[G] = 2$.*

2.4 Notation and Tools

All graphs considered in this chapter are finite, undirected, with no loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Also, $\Delta(G)$ denotes the maximum degree of G and simply denoted by Δ. For every $v \in V(G)$, $d_G(v)$ and $N_G(v)$ denote the degree of v and the set of neighbors of v, respectively. Also $N[v] = N(v) \cup \{v\}$. For a given graph G, we use $u \sim v$ if two vertices u and v are adjacent in G.

Let G be a graph and K be a non-empty set. A proper vertex coloring of G is a function $c : V(G) \to K$, such that if $u, v \in V(G)$ are adjacent, then $c(u) \neq c(v)$. A *proper vertex k-coloring* is a proper vertex coloring with $|K| = k$. The smallest number of colors needed to color the vertices of G for obtaining a proper vertex coloring is called the *chromatic number* of G and denoted by $\chi(G)$. Let G be the graph of Figure 2.1. Let $\psi : V(G) \to \{1, 2\}$ be defined by $\psi(a) = \psi(c) = 1$ and $\psi(b) = \psi(d) = 2$. Then ψ is a proper vertex coloring and it is easy to see that $\chi(G) = 2$. Also, one can verify that the labeling ψ is closed distinguishing for G and so $\mathrm{dis}[G] = 2$.

A *k-regular* graph is a graph whose each vertex has degree k. A regular graph G with n vertices and degree k is said to be *strongly regular* if there are integers λ and μ such that every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors and is denoted by $SRG(n, k, \lambda, \mu)$.

The *Cartesian product* $H \square G$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ where vertices (g, h) and (g', h') are adjacent if and only if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.

We say that a set of vertices is *independent* if there is no edge between these vertices. The *independence number*, $\alpha(G)$, of a graph G is the size of a largest independent set of G. Also, a *clique* in a graph G is a subset of its vertices such that every two vertices in the subset are connected by an edge. The *clique number* $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. A *split graph* is a graph whose vertex set may be partitioned into a clique K and an independent set S. We suppose, without loss of generality, that K is maximal, that is no vertex in S is adjacent to all vertices in K. The pair (K, S) is then called a canonical partition of G. For such a partition, we have $\omega(G) = |K|$.

We use the notation $f(x) = \Theta(g(x))$, if for sufficiently large values of x, we have $ag(x) \leq f(x) \leq bg(x)$, for some positive a and b values. The notation $f(x) = \mathcal{O}(g(x))$ is used, if for sufficiently large values of x, we have $|f(x)| \leq a|g(x)|$, for some positive value a. The notation $f(x) = \Omega(g(x))$ is used, if for sufficiently large values of x, we have $|f(x)| \geq a|g(x)|$, for some positive value a.

Consider a formula $\Phi = (X, C)$, where the two sets $X = \{x_1, \ldots, x_n\}$ and $C = \{c_1, \ldots, c_m\}$ are the sets of variables and clauses of Φ, respectively. We say that a formula Φ is in *conjunctive normal form (CNF)* if it is a conjunction of clauses, where a clause is a disjunction of literals. For example, $(x \vee \neg y \vee z) \wedge (x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z)$ is a formula in conjunctive normal form with the the set of variables $\{x, y, z\}$ and the set of clauses $\{(x \vee \neg y \vee z), (x \vee y \vee z), (x \vee \neg y \vee \neg z)\}$. Also, a *literal* is either a variable or the negation of a variable. For instance, the clause $(x \vee \neg y \vee z)$ contains three literals $x, \neg y, z$. Throughout the work, when we consider a formula, we mean a formula in conjunctive normal form.

The following problems are NP-complete.

Cubic Monotone NAE (2,3)-Sat.

INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $\mid c \mid \in \{2, 3\}$, every variable appears in exactly three clauses and there is no negation

in the formula.
QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal and at least one false literal?

Monotone Not-All-Equal 3-Sat.
INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $\mid c \mid = 3$ and there is no negation in the formula.
QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal and at least one false literal?

In computational complexity theory, P, is a complexity class. It contains all decision problems that can be solved by a deterministic Turing machine using a polynomial amount of computation time, or polynomial time. NP is the set of decision problems solvable in polynomial time by a non-deterministic Turing machine. NP-hardness in computational complexity theory, is the defining property of a class of problems that are, informally, "at least as hard as the hardest problems in NP". More precisely, a problem H is NP-hard when every problem L in NP can be reduced in polynomial time to H. A decision problem is NP-complete when it is both in NP and NP-hard.

We follow [29] for terminology and notation where they are not defined here. The main tool we use in the proof of Theorem 2.2.4 is the Combinatorial Nullstellensatz.

Proposition 2.4.1. *(Combinatorial Nullstellensatz [2]) Let F be a field, let $d_1, \ldots, d_n \geq 0$ be integers, and let $P \in F[x_1, \ldots, x_n]$ be a polynomial of degree $d_1 + \cdots + d_n$ with a non-zero $x_1^{d_1} \ldots x_n^{d_n}$ coefficient. Then P cannot vanish on any set of the form $E_1 \times \ldots \times E_n$ with $E_1, \ldots, E_n \subset F$ and $|E_i| > d_i$ for $i = 1, \ldots, n$.*

2.5 Proofs

Here we prove that the difference between $\mathrm{dis}[G]$ and $\mathrm{dis}_\ell[G]$ can be arbitrary large.

Proof of Theorem 2.2.1

For every integer t, $t \geq 4$, we construct a graph G such that $\mathrm{dis}_\ell[G] - \mathrm{dis}[G] \geq t$. Our construction consists of four steps.

Step 1. Consider $2t-1$ copies of the complete graph K_{2t} and call them $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_{2t-1}$. For every i, $1 \leq i \leq 2t-1$, let $\{v_1^i, v_2^i, \ldots, v_t^i, u_1^i, u_2^i, \ldots, u_t^i\}$ be the set of vertices of the complete graph $\mathcal{K}_i$.

Step 2. For each (i,j,k), where $1 \leq i < j \leq t$ and $1 \leq k \leq 2t-1$, put two new vertices $x_{i,j}^k$ and $y_{i,j}^k$, and put the edges $x_{i,j}^k y_{i,j}^k$, $x_{i,j}^k v_i^k$ and $y_{i,j}^k v_j^k$. Similarly, for every (i,j,k), where $1 \leq i < j \leq t$ and $1 \leq k \leq 2t-1$, put two new vertices $a_{i,j}^k$ and $b_{i,j}^k$, and put the edges $a_{i,j}^k b_{i,j}^k$, $a_{i,j}^k u_i^k$ and $b_{i,j}^k u_j^k$.

Step 3. For every (i,i',k), where $1 \leq i \leq t$, $1 \leq i' \leq t$ and $1 \leq k \leq 2t-1$, put two new vertices $g_{i,i'}^k$ and $h_{i,i'}^k$, and put the edges $g_{i,i'}^k h_{i,i'}^k$, $g_{i,i'}^k v_i^k$ and $h_{i,i'}^k u_{i'}^k$.

Step 4. Finally, put a new vertex p and join the vertex p to each vertex in $\{g_{i,i'}^k : 1 \leq i \leq t, 1 \leq i' \leq t, 1 \leq k \leq 2t-1\}$. Call the resulting graph G.

Next, we discuss the basic properties of the graph G. Let f be a closed distinguishing labeling for G.

Lemma 2.5.1. *We have:*
$d(v_i^k) = d(u_i^k) = 4t-2$, *for each* i,k, $1 \leq i \leq t$ *and* $1 \leq k \leq 2t-1$,
$d(x_{i,j}^k) = d(y_{i,j}^k) = d(a_{i,j}^k) = d(b_{i,j}^k) = 2$, *for each* i,j,k, $1 \leq i < j \leq t$ *and* $1 \leq k \leq 2t-1$,
$d(g_{i,i'}^k) = 3$, $d(h_{i,i'}^k) = 2$, *for each* i,i',k, $1 \leq i \leq t$, $1 \leq i' \leq t$ *and* $1 \leq k \leq 2t-1$.

Lemma 2.5.2. *Let* $M = \{x_{i,j}^k, y_{i,j}^k, a_{i,j}^k, b_{i,j}^k : 1 \leq i < j \leq t, 1 \leq k \leq 2t-1\}$. *There is a function* $f' : M \to \{1,2,\ldots,t\}$, *such that for each* k,

$$\sum_{l \in N[v_1^k] \cap M} f'(l), \ldots, \sum_{l \in N[v_t^k] \cap M} f'(l), \sum_{l \in N[u_1^k] \cap M} f'(l), \ldots, \sum_{l \in N[u_t^k] \cap M} f'(l)$$

are $2t$ *distinct integers.*

Proof. Let k be a fixed number and $f' : M \to \{1,2,\ldots,t\}$ be an arbitrary labeling. For each i, $1 \leq i \leq t$ we have:

$$|\{l : l \in N[v_i^k] \cap M\}| = |\{l : l \in N[u_i^k] \cap M\}| = t-1.$$

Thus,

$$t-1 \leq \sum_{l \in N[v_i^k] \cap M} f'(l) \leq t(t-1).$$

On the other hand, for each i, j, $i \neq j$, we have:

$$(N[v_i^k] \cap M) \cap (N[v_j^k] \cap M) = \emptyset.$$

Also, for each i, i', $1 \leq i, i' \leq t$, we have:

$$(N[v_i^k] \cap M) \cap (N[u_{i'}^k] \cap M) = \emptyset.$$

Since $N[v_i^k] \cap M$ and $N[v_j^k] \cap M$ are disjoint and $t \geq 4$, one can define $f' : M \rightarrow \{1, 2, \ldots, t\}$ such that for each i, $1 \leq i \leq t$,

$$\sum_{l \in N[v_i^k] \cap M} f'(l) = t - 1 + i - 1$$

and

$$\sum_{l \in N[u_i^k] \cap M} f'(l) = 2t - 1 + i - 1.$$

This completes the proof of Lemma. $\square$

Lemma 2.5.3. *For each (i, j, k), where $1 \leq i < j \leq t$ and $1 \leq k \leq 2t-1$, $f(v_i^k) \neq f(v_j^k)$ and $f(u_i^k) \neq f(u_j^k)$.*

Proof. Consider the two adjacent vertices $x_{i,j}^k$ and $y_{i,j}^k$. Since f is a closed distinguishing labeling for G, we have,

$$\sum_{l \in N[x_{i,j}^k]} f(l) \neq \sum_{l \in N[y_{i,j}^k]} f(l).$$

Thus,

$$f(v_i^k) + f(x_{i,j}^k) + f(y_{i,j}^k) \neq f(v_j^k) + f(x_{i,j}^k) + f(y_{i,j}^k).$$

Therefore, $f(v_i^k) \neq f(v_j^k)$. Similarly, by considering the two adjacent vertices $a_{i,j}^k$ and $b_{i,j}^k$, we have $f(u_i^k) \neq f(u_j^k)$. $\square$

By Lemma 2.5.3, $f(v_1^1), f(v_2^1), \ldots, f(v_t^1)$ are t distinct integers. So dis$[G] \geq t$. Now, we show that dis$[G] \leq t$. Let f' be a labeling that has the conditions of Lemma 2.5.2 and consider the following labeling for G:

$f : V(G) \to \{1, 2, \ldots, t\}$,
$f(v_i^k) = f(u_i^k) = i, \quad$ for each i, k, $1 \le i \le t$ and $1 \le k \le 2t-1$,
$f(g_{i,i'}^k) = f(h_{i,i'}^k) = 1, \quad$ for each i, i', k, $1 \le i \le t$, $1 \le i' \le t$ and $1 \le k \le 2t-1$,
$f(p) = t$, $t \ge 4$,
$f(l) = f'(l), \quad$ for each $l \in M$.

Now, we show that f is a closed distinguishing labeling for G. We have:

$\sum_{l \in N[p]} f(l) = t^2(2t-1) + t \ge 4t$, $\sum_{l \in N[g_{i,i'}^k]} f(l) = t + 2 + i \le 3t$, $\sum_{l \in N[h_{i,i'}^k]} f(l) = 2 + i' \le 3t$,
$\sum_{l \in N[x_{i,j}^k]} f(l) = f'(x_{i,j}^k) + f'(y_{i,j}^k) + i \le 3t$, $\sum_{l \in N[y_{i,j}^k]} f(l) = f'(x_{i,j}^k) + f'(y_{i,j}^k) + j \le 3t$,
$\sum_{l \in N[a_{i,j}^k]} f(l) = f'(a_{i,j}^k) + f'(b_{i,j}^k) + i \le 3t$, $\sum_{l \in N[b_{i,j}^k]} f(l) = f'(a_{i,j}^k) + f'(b_{i,j}^k) + j \le 3t$,
Since $d(v_i^k) = d(u_i^k) = 4t-2$ and $f(v_i^k) = f(u_i^k) = i$, we have $\sum_{l \in N[v_i^k]} f(l) \ge 4t$ and $\sum_{l \in N[u_i^k]} f(l) \ge 4t$.

For every two adjacent vertices v_i^k and u_j^k, we have

$$\sum_{l \in N[v_i^k] \setminus M} f(l) = \sum_{l \in N[u_j^k] \setminus M} f(l).$$

Thus, by Lemma 2.5.2, the sum of labels of the vertices in the closed neighborhood of the vertex v_i^k differs from the sum of labels of the vertices in the closed neighborhood of the vertex u_j^k. We have a similar result for every two adjacent vertices v_i^k and v_j^k. For other pairs of adjacent vertices, from the values shown above it is clear that for every two adjacent vertices z, s, the sum of labels of the vertices in the closed neighborhood of the vertex z differs from the sum of labels of the vertices in the closed neighborhood of the vertex s. So, f is a closed distinguishing labeling for G. Thus $\mathrm{dis}[G] = t$.

Next, we show that $\mathrm{dis}_\ell[G] \ge 2t$. To the contrary assume that $\mathrm{dis}_\ell[G] \le 2t-1$ and let $N = \{u_i^k : 1 \le i \le t, 1 \le k \le 2t-1\}$. Consider the following lists for the vertices of the graph G:

$L(u_i^k) = \{1 + k, 2 + k, 3 + k, \ldots, 2t - 1 + k\}$,

$L(l) = \{1, 2, 3, \ldots, 2t-1\}$, for every $l \in V(G) \setminus N$.

Assume that f is a closed distinguishing labeling for G from the lists that shown above (i.e. for each vertex v, $f(v) \in L(v)$). Without loss of generality assume that $f(p) = w$. Consider the set of vertices $v_1^w, v_2^w, \ldots, v_t^w, u_1^w, u_2^w, \ldots, u_t^w$. We have:

$L(u_i^w) = \{1+w, 2+w, 3+w, \ldots, 2t-1+w\}$,

$L(v_i^w) = \{1, 2, 3, \ldots, 2t-1\}$.

Consider the following covering for the set of numbers $L(u_i^w) \cup L(v_i^w)$,

$$\{1+w, 1\}, \{2+w, 2\}, \ldots, \{2t-1+w, 2t-1\}.$$

By the pigeonhole principle and Lemma 2.5.3, there are indices r, i and j such that $f(v_i^w), f(u_j^w) \in \{r+w, r\}$, so $f(v_i^w) = r$ and $f(u_j^w) = r+w$. Therefore,

$$\sum_{l \in N[g_{i,j}^w]} f(l) = \sum_{l \in N[h_{i,j}^w]} f(l).$$

This is a contradiction, so $\mathrm{dis}_\ell[G] \geq 2t$. Here, we investigate the computational complexity of determining $\mathrm{dis}[G]$ for planar subcubic graphs. We show that for a given planar subcubic graph G, it is **NP**-complete to determine whether $\mathrm{dis}[G] = 2$.

Proof of Theorem 2.2.2

It is clear that this problem is in NP. Let Φ be a 3SAT formula with clauses $C = \{c_1, \ldots, c_\gamma\}$ and variables $X = \{x_1, \ldots, x_n\}$. Let $G(\Phi)$ be a graph with the vertices $C \cup X \cup (\neg X)$, where $\neg X = \{\neg x_1, \ldots, \neg x_n\}$, such that for each clause $c = (y \vee z \vee w)$, c is adjacent to y, z and w, also every $x \in X$ is adjacent to $\neg x$. Φ is called a strongly planar formula if $G(\Phi)$ is a planar graph. It was shown that the problem of satisfiability for strongly planar formulas is **NP**-complete [11] (for more information about strongly planar formulas see [9]). We reduce the following problem to our problem.

Problem: *Strongly planar* 3*SAT.*
INPUT: A strongly planar formula Φ.
QUESTION: Is there a truth assignment for Φ that satisfies all the clauses?

Consider a strongly planar formula Φ with the variables X and the clauses C. We transform this into a planar subcubic graph $\mathcal{G}$ such that $\text{dis}[\mathcal{G}] = 2$ if and only if Φ is satisfiable. For every $x \in X$ consider a cycle $C_{24\gamma}$, where γ is the number of clauses in Φ (call that cycle $\mathcal{C}_x$). Suppose that $\mathcal{C}_x = v_1 v_2 \dots v_{24\gamma} v_1$ and color the vertices of $\mathcal{C}_x$ by function ℓ.

$$\ell(v_i) = \begin{cases} \text{red}, & \text{if } 1 \le i \le 12\gamma \text{ and } i \equiv 1 \pmod 6 \\ \text{black}, & \text{if } 1 \le i \le 12\gamma \text{ and } i \equiv 4 \pmod 6 \\ \text{black}, & \text{if } 1 + 12\gamma \le i \le 24\gamma \text{ and } i \equiv 1 \pmod 6 \\ \text{blue}, & \text{if } 1 + 12\gamma \le i \le 24\gamma \text{ and } i \equiv 4 \pmod 6 \\ \text{white}, & \text{otherwise.} \end{cases}$$

For every $c \in C$ consider a path P_8 with the vertices $u_1, u_2, \dots, u_8$, in that order. Next put two new isolated vertices u' and u'', and join the vertex u_3 to the vertex u' and join the vertex u_6 to the vertex u''. Call that resultant graph $\mathcal{P}_c$. Next, for every $c \in C$, without loss of generality assume that $c = (a \vee b \vee w)$, where $a, b, w \in X \cup (\neg X)$. If $a \in X$ ($a \in \neg X$) then join the vertex u_1, $u_1 \in \mathcal{P}_c$ to one of the red (blue) vertices with degree two of $\mathcal{C}_a$. Similarly, if $b \in X$ ($b \in \neg X$) then join the vertex u_1, $u_1 \in \mathcal{P}_c$ to one of the red (blue) vertices with degree two of $\mathcal{C}_b$. Furthermore, if $w \in X$ ($w \in \neg X$) then join the vertex u_4, $u_4 \in \mathcal{P}_c$ to one of the red (blue) vertices of degree two of $\mathcal{C}_w$; also, join the vertex u_8, $u_8 \in \mathcal{P}_c$ to one of the red (blue) vertices of degree two of $\mathcal{C}_w$. In the resulting graph for every red or blue vertex l with degree two, put a new isolated vertex l' and join the vertex l to the vertex l'. Also, for every black vertex l, put a new isolated vertex l' and join the vertex l to the vertex l'. So in the final graph the degree of every blue, red or black vertex is three. Call the resultant subcubic graph $\mathcal{G}$. Note that since Φ is strongly planar ($G(\Phi)$ is planar), we can construct $\mathcal{G}$ such that it is a planar graph.

Assume that $f : V(\mathcal{G}) \to \{1, 2\}$ is a closed distinguishing labeling for $\mathcal{G}$. We have the following lemmas:

Lemma 2.5.4. *For every $x \in X$, we have:*

▷ *for every $z \in V(\mathcal{C}_x)$, if $\ell(z) =$red then $f(z) = 2$ and if $\ell(z) =$blue then $f(z) = 1$,*

or

▷ *for every $z \in V(\mathcal{C}_x)$, if $\ell(z) =$red then $f(z) = 1$ and if $\ell(z) =$blue then $f(z) = 2$.*

Proof. Let $h_1, h_2, h_3, h_4, h_5, h_6, h_7 \in V(\mathcal{C}_x)$, and $h_1h_2h_3h_4h_5h_6h_7$ be a path of length six in $\mathcal{C}_x$ with $\ell(h_2) = \ell(h_3) = \ell(h_5) = \ell(h_6) =$white. Since f is a a closed distinguishing labeling, we have:

$$\sum_{g\in N[h_2]} f(g) \neq \sum_{g\in N[h_3]} f(g).$$

Thus, $f(h_1) \neq f(h_4)$. Similarly, $f(h_4) \neq f(h_7)$. Hence $f(h_1) = f(h_7)$. Therefore, the labels of red vertices are the same. Also, the labels of blue vertices are the same. In $\mathcal{C}_x$, we have $\ell(v_1) =$red, $\ell(v_{24\gamma-2}) =$blue and $\ell(v_{24\gamma-1}) = \ell(v_{24\gamma}) =$white. Thus, $f(v_{24\gamma-2}) \neq f(v_1)$. This completes the proof. $\square$

Define $f' : X \cup (\neg X) \rightarrow \{1, 2\}$ such that for every $a \in X$ ($a \in \neg X$), $f'(a) = 2$ if and only if the values of function f for the red (blue) vertices in $\mathcal{C}_a$ are two.

Lemma 2.5.5. *Let c be an arbitrary clause and $c = (a\vee b\vee w)$, where $a, b, w \in X\cup(\neg X)$. We have $2 \in \{f'(a), f'(b), f'(w)\}$.*

Proof. To the contrary assume that $f'(a) = f'(b) = f'(w) = 1$. Since $\sum_{l\in N[u_1]} f(l) \neq \sum_{l\in N[u_2]} f(l)$, we have $f(u_3) = 1$. Also since $\sum_{l\in N[u_4]} f(l) \neq \sum_{l\in N[u_5]} f(l)$, we have $f(u_6) = 1$. Finally, since $\sum_{l\in N[u_7]} f(l) \neq \sum_{l\in N[u_8]} f(l)$, we have $f'(w) = 2$. But this is a contradiction. $\square$

Let $\Gamma : X \rightarrow \{true, false\}$ be a function such that $\Gamma(x) = true$ if and only if $f'(x) = 2$. By Lemma 2.5.5, Γ is a satisfying assignment for Φ.

Next, suppose that Φ is satisfiable with the satisfying assignment Γ. For every $x \in X$ if $\Gamma(x) = true$ then for $\mathcal{C}_x$ define:

$$f(v_i) = \begin{cases} 2, & \text{if } \ell(v_i) = \text{red} \\ 1, & \text{if } \ell(v_i) = \text{blue} \\ 2, & \text{if } \ell(v_i) = \text{white} \\ 1, & \text{if } \ell(v_i) = \text{black and } 1 \leq i \leq 12\gamma \\ 2, & \text{if } \ell(v_i) = \text{black and } 1 + 12\gamma \leq i \leq 24\gamma, \end{cases}$$

and if $\Gamma(x) = false$ then for $\mathcal{C}_x$ define:

$$f(v_i) = \begin{cases} 1, & \text{if } \ \ell(v_i) = \text{red} \\ 2, & \text{if } \ \ell(v_i) = \text{blue} \\ 2, & \text{if } \ \ell(v_i) = \text{white} \\ 2, & \text{if } \ \ell(v_i) = \text{black} \ \text{ and } 1 \leq i \leq 12\gamma \\ 1, & \text{if } \ \ell(v_i) = \text{black} \ \text{ and } 1 + 12\gamma \leq i \leq 24\gamma. \end{cases}$$

Next, for every $c = (a \vee b \vee w)$, if $\Gamma(w) = true$ then for $\mathcal{P}_c$, define:

$$f(v_i) = \begin{cases} 2, & \text{if } \ v_i = u'' \\ 1, & \text{otherwise }, \end{cases}$$

otherwise, if $\Gamma(w) = false$ then for $\mathcal{P}_c$, define:

$$f(v_i) = \begin{cases} 2, & \text{if } \ v_i \in \{u_3, u_6, u', u''\} \\ 1, & \text{otherwise }. \end{cases}$$

Finally, label remaining vertices by number 2. One can check that f is a closed distinguishing labeling for $\mathcal{G}$. $\diamondsuit$

Next, we show that it is **NP**-complete to determine whether $\text{dis}[G] = 2$, for a given bipartite subcubic graph G.

Proof of Theorem 2.2.3

We reduce *Monotone Not-All-Equal 3Sat* to our problem in polynomial time. It was shown that the following problem is **NP**-complete [15].

Consider an instance Φ with the set of variables X and the set of clauses C. We transform this into a bipartite graph G, such that Φ has a Not-All-Equal satisfying assignment if and only if there is a closed distinguishing labeling $f : V(G) \rightarrow \{1, 2\}$. For every $x \in X$ consider a cycle $C_{12\gamma}$, where γ is the number of clauses in Φ (call that cycle $\mathcal{C}_x$). Suppose that $\mathcal{C}_x = v_1 v_2 \ldots v_{12\gamma} v_1$ and color the vertices of $\mathcal{C}_x$ by function ℓ.

$$\ell(v_i) = \begin{cases} \text{red}, & \text{if } i \equiv 1 \pmod 6 \\ \text{blue}, & \text{if } i \equiv 4 \pmod 6 \\ \text{white} & \text{otherwise.} \end{cases}$$

For every $c = (x \vee y \vee z)$, $c \in C$, do the following three steps:

Step 1. Put two paths $\mathcal{P}_c = c_1^1 c_2^1 c_3^1 c_4^1 c_5^1$ and $\mathcal{P}_c^{'} = c_1^2 c_2^2 c_3^2 c_4^2 c_5^2$. Also, put two isolated vertices c', c'' and add the edges $c' c_3^1$, $c'' c_3^2$.
Step 2. Let $\{v_i, v_j, v_k\}$ be a set of vertices such that each of them has degree two, the value of function ℓ for each of them is red, $v_i \in V(\mathcal{C}_x)$, $v_j \in V(\mathcal{C}_y)$ and $v_k \in V(\mathcal{C}_z)$. Add the edges $c_1^1 v_i, c_1^1 v_j, c_5^1 v_k$.
Step 3. Let $\{v_{i'}, v_{j'}, v_{k'}\}$ be a set of vertices such that each of them has degree two, the value of function ℓ for each of them is blue, $v_{i'} \in V(\mathcal{C}_x)$, $v_{j'} \in V(\mathcal{C}_y)$ and $v_{k'} \in V(\mathcal{C}_z)$. Add the edges $c_1^2 v_{i'}, c_1^2 v_{j'}, c_5^2 v_{k'}$.

Next, in the resulting graph for every red or blue vertex u with degree two, put a new isolated vertex u' and join the vertex u to the vertex u'. This graph has no cycle of odd order. Call the resultant bipartite subcubic graph G.

First, assume that f is a closed distinguishing labeling for G. For every $x \in X$, we have:

- $\diamond$ for every $u \in V(\mathcal{C}_x)$, if $\ell(u)$ =red then $f(u) = 2$, and if $\ell(u)$ =blue then $f(u) = 1$,

or

- $\diamond$ for every $u \in V(\mathcal{C}_x)$, if $\ell(u)$ =red then $f(u) = 1$, and if $\ell(u)$ =blue then $f(u) = 2$,

(see the proof of Lemma 2.5.4). Define $\Gamma : X \rightarrow \{true, false\}$ such that for every $x \in X$, $\Gamma(x) = true$ if and only if the values of function f for the red vertices in $\mathcal{C}_x$ are two. By the structure of clause gadgets, for every clause $c = (x \vee y \vee z)$,

$$\sum_{u \in N[c_1^1]} f(u) \neq \sum_{u \in N[c_2^1]} f(u) \text{ and } \sum_{u \in N[c_4^1]} f(u) \neq \sum_{u \in N[c_5^1]} f(u).$$

So, $true \in \{\Gamma(x), \Gamma(y), \Gamma(z)\}$. On the other hand,

$$\sum_{u \in N[c_1^2]} f(u) \neq \sum_{u \in N[c_2^2]} f(u) \text{ and } \sum_{u \in N[c_4^2]} f(u) \neq \sum_{u \in N[c_5^2]} f(u).$$

Thus, $false \in \{\Gamma(x), \Gamma(y), \Gamma(z)\}$. Therefore, Γ is a Not-All-Equal assignment.

Next, suppose that Φ has a Not-All-Equal assignment Γ. For every $x \in X$ if $\Gamma(x) = true$ then:

$\diamond$ for every $u \in V(\mathcal{C}_x)$, if $\ell(u)$ =red then put $f(u) = 2$ and if $\ell(u)$ =blue then put $f(u) = 1$,

and if $\Gamma(x) = false$ then:

$\diamond$ for every $u \in V(\mathcal{C}_x)$, if $\ell(u)$ =red then put $f(u) = 1$ and if $\ell(u)$ =blue then put $f(u) = 2$.

For every white vertex l, put $f(l) = 2$. Also, for every clause $c = (x \vee y \vee z)$, $c \in C$, put:

$$f(c_1^1) = f(c_2^1) = f(c_4^1) = f(c_5^1) = f(c_1^2) = f(c_2^2) = f(c_4^2) = f(c_5^2) = 1.$$

Also, put $f(c_3^1) = 1$ and $f(c_3^2) = 2$ if and only if $\Gamma(z) = false$. Finally, label all remaining vertices by number 2. One can see that the resulting labeling is a closed distinguishing labeling for G. $\diamondsuit$

Here, we prove that for every integer $t \geq 3$, it is **NP**-complete to determine whether $\text{dis}[G] = t$ for a given graph G.

Proof of Theorem 2.2.4

In order to prove the theorem, we reduce t-Colorability to our problem for each $t \geq 3$. It was shown [15] that for each t, $t \geq 3$, the following problem is **NP**-complete.

Problem: *t-Colorability.*
INPUT: A graph G.
QUESTION: Is $\chi(G) \leq t$?

Let G be a given graph and t be a fixed number. We construct a graph G^* in polynomial time such that $\chi(G) \leq t$ if and only if G^* has a closed distinguishing labeling from $\{1, 2, \ldots, t\}$. Our construction consists of two steps.

Step 1. Consider a copy of the graph G. For every vertex $v \in V(G)$ put $\Delta(G) - d_G(v) + 1$ new isolated vertices $u_1^v, \ldots, u_{\Delta(G)-d_G(v)}^v, z^v$ and join them to the vertex v. Call the resulting graph G'. In the resulting graph the degree of each vertex is $\Delta(G) + 1$ or 1.

Step 2. Let $|V(G)| = n$, $|V(G')| = n' + n$ and $\alpha = (n' + 1)(t - 1) + 2$. Consider a copy of the complete graph K_α with the set of vertices $\{x_1, x_2, \ldots, x_\alpha\}$. For each i, $1 \le i < \alpha$, put $n'+1$ new isolated vertices and join them to the vertex x_i. Finally, for each $v \in V(G)$ join the vertex x_α to the vertices $v, u_1^v, \ldots, u_{\Delta(G)-d_G(v)}^v$. Call the resulting graph G^*. In the final graph for each i, $1 \le i < \alpha$, $d_{G^*}(x_i) = \alpha + n'$ and $d_{G^*}(x_\alpha) = \alpha + n' - 1$.

Let $f : V(G^*) \to \{1, 2, \ldots, t\}$ be a closed distinguishing labeling. We have the following lemmas:

Lemma 2.5.6. *For every vertex $v \in V(G)$, $f(v) = f(u_1^v) = \cdots = f(u_{\Delta(G)-d_G(v)}^v) = 1$.*

Proof. Consider the set of vertices $V(K_\alpha) = \{x_1, x_2, \ldots, x_\alpha\}$. Since f is a a closed distinguishing labeling,

$$\sum_{l \in N[x_1]} f(l), \sum_{l \in N[x_2]} f(l), \ldots, \sum_{l \in N[x_\alpha]} f(l),$$

are α distinct numbers. Thus,

$$\sum_{\substack{l \in N[x_1] \\ l \notin V(K_\alpha)}} f(l), \sum_{\substack{l \in N[x_2] \\ l \notin V(K_\alpha)}} f(l), \ldots, \sum_{\substack{l \in N[x_\alpha] \\ l \notin V(K_\alpha)}} f(l),$$

are α distinct numbers. For each i, $1 \le i < \alpha$,

$$n' + 1 \le \sum_{\substack{l \in N[x_i] \\ l \notin V(K_\alpha)}} f(l) \le (n' + 1)t.$$

Since there are exactly $\alpha - 1$ values in this range, we have

$$\{ \sum_{\substack{l \in N[x_i] \\ l \notin V(K_\alpha)}} f(l) : 1 \le i < \alpha\} = \{n' + 1, n' + 2, \ldots, (n' + 1)t\}.$$

Thus

$$\sum_{\substack{l \in N[x_\alpha] \\ l \notin V(K_\alpha)}} f(l) \le n'.$$

On the other hand,

$$\sum_{\substack{l\in N[x_\alpha]\\ l\notin V(K_\alpha)}} f(l) \geq |\{l : l \in N[x_\alpha], l \notin V(K_\alpha)\}| = n'.$$

Therefore

$$\sum_{\substack{l\in N[x_\alpha]\\ l\notin V(K_\alpha)}} f(l) = n'$$

and for every vertex $v \in V(G^*)$, $f(v) = f(u_1^v) = \cdots = f(u^v_{\Delta(G)-d_G(v)}) = 1$. This completes the proof of Lemma. $\square$

Lemma 2.5.7. *Let v and v' be two adjacent vertices in G. We have $f(z^v) \neq f(z^{v'})$.*

Proof. For two adjacent vertices v and v' in G we have $\sum_{l\in N_{G^*}[v]} f(l) \neq \sum_{l\in N_{G^*}[v']} f(l)$. By Lemma 2.5.6 and since $d_{G^*}(v) = d_{G^*}(v')$, we have $f(z^v) \neq f(z^{v'})$. $\square$

By Lemma 2.5.7, the following function is proper vertex t-coloring for G:

$$c : V(G) \to \{1, 2, \ldots, t\} \text{ such that } c(v) = f(z^v)$$

On the other hand, if G is t-colorable (and c is a proper vertex t-coloring for the graph G), define:

$f(v) = f(u_1^v) = \cdots = f(u^v_{\Delta(G)-d_G(v)}) = 1$ and $f(z^v) = c(v)$, for every vertex $v \in V(G)$, $f(l) = t$, for every vertex $l \in T$ (note that $T = \{x_1, x_2, \ldots, x_\alpha\}$).

Also, for every vertex x_i, $1 \leq i < \alpha$, label the set of vertices $\{l : l \in N[x_i], l \notin V(K_\alpha)\}$, such that in the final labeling

$$\{\sum_{\substack{l\in N[x_i]\\ l\notin V(K_\alpha)}} f(l) : 1 \leq i < \alpha\} = \{n'+1, n'+2, \ldots, (n'+1)t\}.$$

One can check that f is a closed distinguishing labeling. $\diamondsuit$

In the next theorem we give some upper bounds by using the Combinatorial Nullstellensatz.

Proof of Theorem 2.2.5

Let $V(G) = \{x_1, \ldots, x_n\}$ and $S_i := \sum\limits_{x_j \in N[x_i]} x_j$. Define the following polynomial:

$$f(x_1, \ldots, x_n) = \prod_{\substack{x_t \sim x_s \\ t<s}}^{N[x_t] \neq N[x_s]} (S_t - S_s).$$

One can check that $f(a_1, \ldots, a_n) \neq 0$ if and only if $(a_1, \ldots, a_n)$ is a closed distinguishing coloring.

(i) Let x_v be a vertex in G. The term x_v appears in S_t if $x_v \in N[x_t]$. Hence the term x_v appears in $(S_t - S_s)$ if $x_v \in N[x_t] \cup N[x_s]$ and $x_v \notin N[x_t] \cap N[x_s]$. Thus x_v appears in $f(x_1, \ldots, x_n)$ at most $\sum_{x_j \in N(x_v)} (d(x_j) - 1)$ times, which is less than or equal to $(d_1 - 1) + \cdots + (d_\Delta - 1)$. Hence for each monomial in $f(x_1, \ldots, x_n)$ like $x_1^{a_1} \ldots x_n^{a_n}$, we have $a_i \leq s$. Let $E_i = L(v_i)$, such that $|L(v_i)| \geq s + 1$ for $1 \leq i \leq n$. Then by the Combinatorial Nullstellensatz $f(x_1, \ldots, x_n)$ cannot vanish on $E_1 \times \ldots \times E_n$. Then there exists $(a_1, \ldots, a_n) \in E_1 \times \ldots \times E_n$ such that $f(a_1, \ldots, a_n) \neq 0$ which completes the proof.

(ii) The degree of $f(x_1, \ldots, x_n)$ is m. Also, $f(x_1, \ldots, x_n)$ is not the zero polynomial. So for each monomial like $x_1^{a_1} \ldots x_n^{a_n}$ we have $a_i \leq m$. It is easy to check that there exists a monomial such that $a_i < m$ for $1 \leq i \leq n$. Therefore, the Combinatorial Nullstellensatz finishes the proof.

(iii) Since $\Delta - 1 \geq d_{t+1} \geq d_{t+2} \ldots \geq d_\Delta$, it follows that $s \leq \Delta^2 - 2\Delta + t$. Also for $1 \leq i \leq n$, $d_i \leq \Delta$. Hence $s \leq \Delta^2 - \Delta$.

(iv) Let x_v be the only vertex such that $d(x_v) = \Delta$. Assign 1 as the label for the vertex x_v. For every i, the term x_i appears at most $(d_1 - 1) + \cdots + (d_{\Delta-1} - 1)$ times. We have:

$$d_1 + \cdots + d_{\Delta-1} - (\Delta - 1) \leq \Delta + (\Delta - 2)(\Delta - 1) - (\Delta - 1) = \Delta^2 - 3\Delta + 3.$$

(v) Let x_v be a vertex in G. Let $N(x_v) = \{x_{a_1}, \ldots, x_{a_k}\}$. The term x_v doesn't appear in $(S_{a_i} - S_b)$, where $x_v \in N(x_{a_i}) \cap N(x_b)$. Then for every $1 \leq i \leq n$, the term x_i appears at most $k(k - \lambda - 1)$ times in $f(x_1, \ldots, x_n)$. Then the Combinatorial

Nullstellensatz completes the proof.

(vi) Let x_v be a vertex in G such that $d(x_v) = \alpha$. Assume that $N(x_v) = \{x_{a_1}, \ldots, x_{a_\alpha}\}$. Let $V(G) - N[x_v] = \{x_{b_1}, \ldots, x_{b_{n-\alpha-1}}\}$. Let $x_r, x_t \in V(G)$ be adjacent. Then x_v appears in $S_r - S_t$ if and only if exactly one of x_r, x_t belongs to $N(x_v)$ and another one belongs to $V(G) - N[x_v]$. So in f, the term x_v appears at most $\alpha(n - \alpha - 1)$, which is less than or equal to $[\frac{n-1}{2}]^2$. Now the proof is complete by the Combinatorial Nullstellensatz.$\diamondsuit$

Here, we show that for each positive integer t there is a bipartite graph G such that $\text{dis}[G] > t$.

Proof of Theorem 2.2.6

Let t be a fixed number. We construct a bipartite graph G such that $\text{dis}[G] > t$. Let $\alpha = t^2$. Define:

$V(G) = X \cup Y \cup Z$,
$X = \{x_1, x_2, \ldots, x_\alpha\}$,
$Y = \{y_1, y_2, \ldots, y_\alpha\}$,
$Z = \{z_{A,B}, z'_{A,B} : A, B \subseteq \{1, 2, 3, \ldots, \alpha\}, A \neq \emptyset, B \neq \emptyset\}$.

$$E(G) = \bigcup_{A,B} \{z_{A,B}z'_{A,B}, z_{A,B}x_i, z'_{A,B}y_j : i \in A, j \in B\}.$$

The graph G is bipartite with parts $X \cup Y$ and Z. To the contrary assume that $f : V(G) \to \{1, 2, 3, \ldots, t\}$ is a closed distinguishing labeling. For every two vertices x_i and y_j, we have:

$$x_i z_{\{i\},\{j\}}, z_{\{i\},\{j\}} z'_{\{i\},\{j\}}, y_j z'_{\{i\},\{j\}} \in E(G),$$

so

$$\sum_{l \in N[z_{\{i\},\{j\}}]} f(l) \neq \sum_{l \in N[z'_{\{i\},\{j\}}]} f(l),$$

thus $f(x_i) \neq f(y_j)$. Let S_1 and S_2 be two subsets of $\{1, 2, 3, \ldots, t\}$ such that $S_1 \cap S_2 = \emptyset$. Without loss of generality, we can assume that for each i, $1 \leq i \leq \alpha$, $f(x_i) \in S_1$ and for every j, $1 \leq j \leq \alpha$, $f(y_j) \in S_2$.

Let $T_X = \{f(x_1), \ldots, f(x_\alpha)\}$ and $T_Y = \{f(y_1), \ldots, f(y_\alpha)\}$. We know that $f(x_i) \in S_1$ and $f(y_j) \in S_2$. Let $|S_1| = \mu$ and $|S_2| \leq t - \mu$. By the pigeonhole principle there exists an element, say r, in S_1 such that r appears at least $\dfrac{\alpha}{\mu}$ times in T_X. Similarly, there exists an element, say p, in S_2 such that p appears at least $\dfrac{\alpha}{t-\mu}$ times in T_Y. Since $\alpha = t^2$, we have:

$$\frac{\alpha}{\mu} = \frac{t^2}{\mu} \geq \frac{t^2}{t} \geq p.$$

Also,

$$\frac{\alpha}{t-\mu} = \frac{t^2}{t-\mu} \geq \frac{t^2}{t} \geq r.$$

Thus in T_X there exists r at least p times, and in T_Y there exists p at least r times. Consequently, one can find two sets $A, B \subseteq \{1, 2, 3, \ldots, \alpha\}$ such that $|A| = p$, $|B| = r$, for each $i \in A$, $f(x_i) = r$ and for each $j \in B$, $f(y_j) = p$. Thus,

$$\sum_{l \in N[z_{A,B}]} f(l) = \sum_{l \in N[z'_{A,B}]} f(l).$$

But this is a contradiction. Therefore, $\mathrm{dis}[G] > t$. □

Note that in the graph G which was constructed in the previous theorem, we have:

$$|V(G)| = 2\alpha + 2(2^\alpha - 1)^2 = 2t^2 + 2(2^{t^2} - 1)^2.$$

It is interesting to find a bipartite graph G such that $V(G) = \mathcal{O}(t^c)$ and $\mathrm{dis}[G] > t$, where c is a constant number. Next, we show that if G is a split graph, then $\mathrm{dis}[G] \leq (\omega(G))^2$.

Proof of Theorem 2.2.7

Let (K, S) be a canonical partition of G and assume that $S = \{v_1, v_2, \ldots, v_{|S|}\}$. For each i, $1 \leq i \leq |S|$, let G_i be the induced subgraph on the set of vertices $K \bigcup_{j=1}^{i} v_j$. Let G_0 be the induced graph on the set of vertices K and $f_0 : V(G_0) \to \{1\}$ be a closed distinguishing labeling of G_0 such that for every vertex $u \in V(G_0)$, $f_0(u) = 1$. For $i = 1$ to $i = |S|$ do the following procedure:

▷ For each j, $1 \leq j \leq (\omega(G))^2$, define

$$g^i_j(x) = \begin{cases} f_{i-1}(x) & x \in V(G_{i-1}), \\ j & x \in V(G_i) \setminus V(G_{i-1}). \end{cases}$$

For a fixed number j if there are two vertices $x, y \in V(G_0)$, such that $N_{G_i}(x) \neq N_{G_i}(y)$ and

$$\sum_{l \in N_{G_i}[x]} g^i_j(l) = \sum_{l \in N_{G_i}[y]} g^i_j(l),$$

then $g^i_j(x)$ is not a closed distinguishing labeling for G_i. The graph G_0 has $\binom{\omega(G)}{2}$ edges, so there are at most $\binom{\omega(G)}{2}$ restrictions. Thus, there is an index j such that for every two vertices $x, y \in V(G_0)$, if $N_{G_i}(x) \neq N_{G_i}(y)$, then

$$\sum_{l \in N_{G_i}[x]} g^i_j(l) \neq \sum_{l \in N_{G_i}[y]} g^i_j(l).$$

For that j, put $f_i \leftarrow g^i_j$. (End of procedure.)

When the procedure terminates, the function $f_{|S|}$ is a closed distinguishing labeling for G. This completes the proof.

Here, we show that for each n, there is a graph G with n vertices such that $\text{dis}[G] - \text{dis}_s[G] = \Omega(n^{\frac{1}{3}})$.

Proof of Theorem 2.3.1

Let $t = 10k$ and consider a copy of the complete graph K_{t^2} with the set of vertices $\{v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq t,\}$. For each (i, j), $1 \leq i \leq t, 1 \leq j \leq t$, put $i + j$ new vertices $x^{i,j}_1, x^{i,j}_2, \ldots, x^{i,j}_i, y^{i,j}_1, y^{i,j}_2, \ldots, y^{i,j}_j$ and join them to the vertex $v_{i,j}$. Call the resultant graph G. Note that for each (i, j), $1 \leq i \leq t, 1 \leq j \leq t$, $d(v_{i,j}) = t^2 + i + j - 1$.

First, we show that $\text{dis}_s[G] = 2$. Define:

$f : V(G) \to \{1, \Delta(G) + 1\}$,

$f(v_{i,j}) = f(x^{i,j}_1) = f(x^{i,j}_2) = \cdots = f(x^{i,j}_i) = \Delta(G) + 1$, for every i and j,

$f(y^{i,j}_1) = f(y^{i,j}_2) = \cdots = f(y^{i,j}_j) = 1$, for every i and j.

It is easy to check that f is a closed distinguishing labeling for G. Next, assume that $f : V(G) \to \text{dis}[G]$ is a closed distinguishing labeling for G. Consider the set of vertices $R = \{v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq t\}$. The function f is a closed distinguishing labeling therefore,

$$\{ \sum_{l \in N[v_{i,j}]} f(l) : 1 \leq i \leq t, 1 \leq j \leq t\},$$

are t^2 distinct numbers. Thus,

$$\{ \sum_{\substack{l \in N[v_{i,j}] \\ l \notin R}} f(l) : 1 \leq i \leq t, 1 \leq j \leq t\},$$

are t^2 distinct numbers. For each (i, j), $1 \leq i \leq t, 1 \leq j \leq t$,

$$2 \leq i + j \leq \sum_{\substack{l \in N[v_{i,j}] \\ l \notin R}} f(l) \leq (i + j)\text{dis}[G] \leq (2t) \times \text{dis}[G].$$

So $2t \times \text{dis}[G] - 2 + 1 \geq t^2$. Thus $\text{dis}[G] \geq 5k$. On the other hand,

$$|V(G)| = t^2 + \sum_{i=1}^{t} \sum_{j=1}^{t} (i + j) \leq t^2 + \sum_{i=1}^{t} \sum_{j=1}^{t} (2t) = \mathcal{O}(t^3) = \mathcal{O}(k^3).$$

This completes the proof.

Here, we show that for a given 4-regular graph G, it is **NP**-complete to determine whether $\text{dis}_s[G] = 2$.

Proof of Theorem 2.3.2

Clearly, the problem is in **NP**. We prove the **NP**-hardness by a reduction from the following well-known **NP**-complete problem [15].

3SAT.
INSTANCE: A 3CNF formula $\Psi = (X, C)$.
QUESTION: Is there a truth assignment for X?

Let $\Psi = (X, C)$ be an instance of *3SAT* and also assume that α and β are two numbers such that $\alpha \neq \beta$. We convert Ψ into a 4-regular graph G such that Ψ has a satisfying

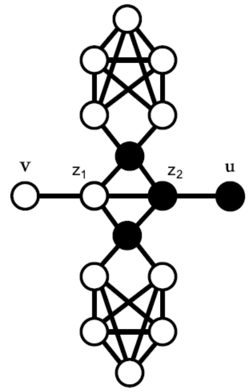


FIGURE 2.2: The gadget $I(v, u)$. Let G be a 4-regular graph and $f : V(G) \to \{\alpha, \beta\}$ be a closed distinguishing labeling. If $I(v, u)$ is a subgraph of G, then $f(v) \neq f(u)$.

assignment if and only if G has a closed distinguishing labeling from $\{\alpha, \beta\}$. First, we introduce a useful gadget.

Construction of the gadget $\mathcal{T}_k$.

Consider a copy of the bipartite graph $P_2 \square C_{2k}$ and let $\ell : V(P_2 \square C_{2k}) \to \{1, 2\}$ be a proper vertex 2-coloring. Call the set of vertices $V(P_2 \square C_{2k})$, *the main vertices*. Construct the gadget $\mathcal{T}_k$ by replacing every edge vu of $P_2 \square C_{2k}$ with a copy of the gadget $I(v, u)$ which is shown in Fig. 2.2.

Note that the gadget $\mathcal{T}_k$ has $4k$ main vertices and the degree of each main vertex is three. Also, in $\mathcal{T}_k$ the degree of each vertex that is not a main vertex is four.

For each variable $x \in X$ assume that the variable x appears in exactly $\mu(x)$ clauses (positive or negative) and suppose that $|C| = \lambda$. Next, we present the construction of the main graph.

Construction of the graph G.

Put a copy of $\mathcal{T}_{3\lambda}$ and call it F. Also, for every variable $x \in X$, put a copy of the gadget $\mathcal{T}_{\mu(x)}$ and call it D_x. Furthermore, for every clause $c \in C$, put a copy of the path $P_2 = c_1c_2$. For every $x \in X$, define:

$$S_x^2 = \{v \in V(D_x) : v \text{ is a main vertex and } \ell(v) = 2\},$$

$$S_x^1 = \{v \in V(D_x) : v \text{ is a main vertex and } \ell(v) = 1\}.$$

Also, define:

$$R^2 = \{v \in V(F) : v \text{ is a main vertex and } \ell(v) = 2\},$$

$$R^1 = \{v \in V(F) : v \text{ is a main vertex and } \ell(v) = 1\}.$$

Next, for every $c \in C$, without loss of generality assume that $c = (a \vee b \vee s)$, where $a, b, s \in X \cup (\neg X)$. If $a \in X$ ($a \in \neg X$) then join the vertex c_1, to a vertex $v \in S_a^2$ ($v \in S_{\neg a}^1$) of degree three. Also, if $b \in X$ ($b \in \neg X$) then join the vertex c_1, to a vertex $v \in S_b^2$ ($v \in S_{\neg b}^1$) of degree three. Similarly, if $s \in X$ ($s \in \neg X$) then join the vertex c_1, to a vertex $v \in S_s^2$ ($v \in S_{\neg s}^1$) of degree three.
Furthermore, join the vertex c_2 to three vertices $v, u, z \in R^1$ of degree three. Call the resultant graph G'. Note that the degree of every vertex in G' is three or four.

Now, consider two copies of the graph G'. For each vertex h with degree three in G', call its corresponding vertex in the first copy of G', h', and call its corresponding vertex in the second copy of G', h''. Next, connect the vertices h' and h'' through a copy of the gadget $I(h', h'')$. Call the resulting 4-regular graph G. In the next, we just focus on the vertices in the first copy of G' and talk about their properties.

First, assume that $f : V(G) \to \{\alpha, \beta\}$ is a closed distinguishing labeling. We have the following lemmas about the vertices in the first copy of G'.

Lemma 2.5.8. *For each $x \in X$, for every two vertices $h, g \in S_x^2$, $f(h) = f(g)$ and for every two vertices $h, g \in S_x^1$, $f(h) = f(g)$. Also, for each two vertices $h \in S_x^2$ and $g \in S_x^1$, $f(h) \neq f(g)$.*

Proof. Let G be a 4-regular graph and $f : V(G) \to \{\alpha, \beta\}$ be a closed distinguishing labeling for G. Assume that $I(v, u)$ is a subgraph of G. For two adjacent vertices z_1 and z_2 in $I(v, u)$ we have:

$$\sum_{l \in N[z_1]} f(l) \neq \sum_{l \in N[z_2]} f(l).$$

Thus, $f(v) \neq f(u)$. Consequently, in each copy of the gadget $I(v, u)$, we have $f(v) \neq f(u)$. So, in the gadget D_x, for every two main vertices l_1 and l_2 that are connected through a copy of gadget $I(l_1, l_2)$, we have $f(l_1) \neq f(l_2)$. On the other hand, the gadget D_x is constructed from a bipartite graph by replacing each edge with a copy of the gadget $I(v, u)$. The main vertices of D_x can be partitioned into two sets, based on the function ℓ which is a proper vertex 2-coloring for the base bipartite graph. In each part, the values of function f for the main vertices in that part are the same. So, for each $x \in X$, for every two vertices $h, g \in S_x^2$, $f(h) = f(g)$ and for every two vertices $h, g \in S_x^1$, $f(h) = f(g)$. Also, for each two vertices $h \in S_x^2$ and $g \in S_x^1$, $f(h) \neq f(g)$. $\diamondsuit$ □

Lemma 2.5.9. *For every two vertices $g, h \in R^2$, $f(g) = f(h)$ and for every two vertices $g, h \in R^1$, $f(g) = f(h)$. Also, for each two vertices $v \in R^2$ and $u \in R^1$, $f(g) \neq f(h)$.*

Proof. The proof is similar to the proof of Lemma 2.5.8. □

Note that if $f : V(G) \to \{\alpha, \beta\}$ is a closed distinguishing labeling, then

$$f'(v) = \begin{cases} \beta, & \text{if } f(v) = \alpha, \\ \alpha, & \text{if } f(v) = \beta. \end{cases}$$

is a closed distinguishing labeling for G. Now, without loss of generality assume that the values of function f for the set of vertices R^2 are β. Define $\Gamma : X \to \{true, false\}$ such that for every $x \in X$, $\Gamma(x) = true$ if and only if the values of function f for the set of vertices S_x^2 are β. By Lemma 2.5.8, Lemma 2.5.9 and structure of G, it is easy to check that Γ is a satisfying assignment.

Next, suppose that Ψ is satisfiable with the satisfying assignment Γ. Define the following values for the function f for the vertices in the first copy of G':

▷ For every $v \in S_x^2$, if $\Gamma(x) = true$ then put $f(v) = \beta$ and if $\Gamma(x) = false$ then put $f(v) = \alpha$.

▷ For every $v \in S_x^1$, if $\Gamma(x) = true$ then put $f(v) = \alpha$ and if $\Gamma(x) = false$ then put $f(v) = \beta$.

▷ For every $v \in R^2$, put $f(v) = \beta$ and for every $v \in R^1$, put $f(v) = \alpha$.

▷ For every $c \in C$, put $f(c_1) = \beta$ and $f(c_2) = \alpha$.

Also, for every vertex l, $l \in S_x^2 \cup S_x^1 \cup R^2 \cup R^1 \cup \{c_1, c_2 : c \in C\}$ in the second copy of G', put $f(l) = 1$ if and only if the value of function f for the vertex l in the first copy of G' is two. Finally, for each subgraph $I(v, u)$, without loss of generality assume that

$f(v) = \alpha$ and $f(u) = \beta$. Label the vertices of $V(I(v, u)) \setminus \{v, u\}$ such that the labels of black vertices are β and the labels of white vertices are α (see Fig. 2.2). It is easy to check that this labeling is a closed distinguishing labeling for G. This completes the proof.

2.6 Concluding remarks and future work

In this chapter, we worked on the closed distinguishing labeling which is very similar to the concept of relaxed locally identifying coloring. A vertex-coloring of a graph G (not necessary proper) is said to be relaxed locally identifying if for any pair u, v of adjacent vertices with distinct closed neighborhoods, the sets of colors in the closed neighborhoods of u and v are different and an assignment of numbers to the vertices of graph G is *closed distinguishing* if for any two adjacent vertices v and u the sum of labels of the vertices in the closed neighborhood of the vertex v differs from the sum of labels of the vertices in the closed neighborhood of the vertex u unless they have the same closed neighborhood.

2.6.1 The computational complexity

We proved that for a given bipartite subcubic graph G, it is **NP**-complete to decide whether $dis[G] = 2$. On the other hand, it was shown that for every tree T, $\text{dis}_\ell[T] \leq 3$ [3]. Here, we ask the following question.

Problem 2.6.1. *For a given tree T, for every vertex $v \in V(T)$, let $L(v)$ be a list of size two of natural numbers. Determine the computational complexity of deciding whether there is a closed distinguishing labeling f such that for each $v \in V(T)$, $f(v) \in L(v)$.*

It was shown in [3] that for every tree T, $\text{dis}[T] \leq 2$. On the other hand, we proved that for a given bipartite subcubic graph G it is **NP**-complete to decide whether $dis[G] = 2$. In the proof of Theorem 2.2.3, we reduced Not-All-Equal to our problem and the planar version of Not-All-Equal is in **P** [23], so the computational complexity of deciding whether $\text{dis}[G] = 2$ for planar bipartite graphs remains unsolved.

Problem 2.6.2. *For a given planar bipartite graph G, determine the computational complexity of deciding whether* $\text{dis}[G] = 2$.

Let G be an r-regular graph. If $\text{dis}[G] = 2$ then for every two numbers a, b ($a \neq b$), G has a closed distinguishing labeling from $\{a, b\}$. We proved that for a given 4-regular graph G, it is **NP**-complete to decide whether $\text{dis}[G] = 2$. Determining the computational complexity of deciding whether $\text{dis}[G] = 2$ for 3-regular graphs can be interesting.

Problem 2.6.3. *For a given 3-regular graph G, determine the computational complexity of deciding whether* $\text{dis}[G] = 2$.

Summary of results and open problems in the complexity of determining whether $\text{dis}[G] = 2$ is shown in Table 1.

TABLE 2.1: Recent results and open problems

	$\text{dis}[G] = 2$?	Refrence
Tree	**P**	see [3]
Planar bipartite	Open	Problem 2.6.2
Bipartite subcubic	**NP**-c	Theorem 2.2.3
Planar subcubic	**NP**-c	Theorem 2.2.2
3-regular	Open	Problem 2.6.3
4-regular	**NP**-c	Theorem 2.3.2

2.6.2 Bipartite graphs

Let G be a bipartite graph with partite sets A and B which is not a star. Let, for $X \in \{A, B\}$; $\Delta_X = \max\limits_{x \in X} d(x)$ and $\delta_{X,2} = \min\limits_{x \in X, d(x) \geq 2} d(x)$. It was shown [3] that

$$dis[G] \leq min\{c\sqrt{|E(G)|}, \lfloor \frac{\Delta_A - 1}{\delta_{B,2} - 1} \rfloor + 1, \lfloor \frac{\Delta_B - 1}{\delta_{A,2} - 1} \rfloor + 1\},$$

where c is some constant. Thus, for a given bipartite graph G, $\text{dis}[G] = \mathcal{O}(\Delta)$ [3]. On the other hand, we proved that for each integer t, there is a bipartite graph G such that $\text{dis}[G] > t$ (to see an example see Fig. 2.3). Here, we ask the following:

Problem 2.6.4. *For each positive integer t, is there a bipartite graph G such that $V(G) = \mathcal{O}(t^c)$ and* $\text{dis}[G] > t$*, where c is a constant number.*

What can we say about the upper bound in bipartite graphs? Perhaps one of the most intriguing open question in this scope is the case of bipartite graphs.

Problem 2.6.5. *Let G be a bipartite graph, is* $\text{dis}[G] \leq \mathcal{O}(\sqrt{\Delta(G)})$*?*

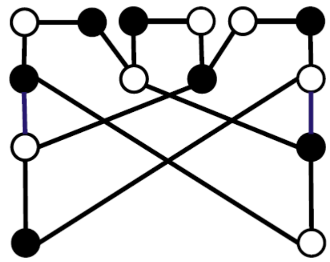


FIGURE 2.3: A bipartite graph G with the closed distinguishing number greater than two.

We proved that the difference between $\text{dis}[G]$ and $\text{dis}_\ell[G]$ can be arbitrary large. What can we say about the difference in bipartite graphs?

Problem 2.6.6. *For any positive integer t, is there any bipartite graph G such that $\text{dis}_\ell[G] - \text{dis}[G] \geq t$?*

For a given bipartite graph $G = [X, Y]$, define $f : V(G) \to \{1, \Delta\}$ such that:

$$f(v) = \begin{cases} 1, & \text{if } v \in X \\ \Delta, & \text{if } v \in Y \end{cases}$$

It is easy to see that f is a closed distinguishing labeling for G. Thus, for a bipartite graph G, $\text{dis}_s[G] \leq 2$. On the other hand, for a general graph G, the best upper bound we know is $\text{dis}_s[G] \leq |V(G)|$.

Problem 2.6.7. *Is this true "for any graph G, $\text{dis}_s[G] \leq \chi(G)$?"*

For each positive integer n, we proved that there is a graph G with n vertices such that $\text{dis}[G] - \text{dis}_s[G] = \Omega(n^{\frac{1}{3}})$. It would be desirable to increase the gap into $\Omega(\sqrt{n})$.

Problem 2.6.8. *Is this true? "For each positive integer n, there is a graph G with n vertices such that $\text{dis}[G] - \text{dis}_s[G] = \Omega(\sqrt{n})$".*

2.6.3 Split graphs

It is well-known that split graphs can be recognized in polynomial time, and that finding a canonical partition of a split graph can also be found in polynomial time. In this work, we proved that if G is a split graph, then $\operatorname{dis}[G] \leq (\omega(G))^2$. Let G be a split graph and (K, S) be a canonical partition of G. Assume that $S = \{v_1, v_2, \ldots, v_{|S|}\}$. Define:

$$f(u) = \begin{cases} 1, & \text{if } \ u \in V(K) \\ (\Delta+1)^{i-1}, & \text{if } \ u = v_i \text{ and } 1 \leq i \leq |S| \end{cases}$$

It is easy to check that f is a closed distinguishing labeling for G. Thus, $\operatorname{dis}_s[G] \leq \alpha(G)$. However, one further step does not seem trivial.

Problem 2.6.9. *Is it true that if G is a split graph, then* $\operatorname{dis}[G] = \mathcal{O}(\omega(G))$*?*

Problem 2.6.10. *Can one decide in polynomial time whether* $\operatorname{dis}[G] \leq \omega(G)$ *for every split graph G?*

Bibliography

[1] M. Aïder, S. Gravier, S. Slimani, *Relaxed locally identifying coloring of graphs*, Graphs Combin. 32 (2016) 1651. http://dx.doi.org/10.1007/s00373-016- 1677-z.

[2] N. Alon, M. Tarsi, *Colorings and orientations of graphs*, Combinatorica 12 (2) (1992) 125–134.

[3] M. Axenovich, J. Przybylo, R. Sotak, M. Voigt, J. Weidlich, *A note on adjacent vertex distinguishing colorings number of graphs*, Discrete Appl. Math. 205 (2016) 1–7.

[4] T. Bartnicki, J. Grytczuk, S. Niwczyk, *Weight choosability of graphs*, J. Graph Theory 60 (3) (2009) 242–256.

[5] O. Baudon, J. Bensmail, É. Sopena, *An oriented version of the 1–2–3 conjecture*, Discuss. Math. Graph Theory 35 (1) (2015) 141–156.

[6] P. Bennett, A. Dudek, A. Frieze, L. Helenius, *Weak and strong versions of the 1–2–3 conjecture for uniform hypergraphs*, Electron. J. Combin. 23 (2) (2016) 21. Paper 2.46.

[7] J. Bensmail, Partitions and decompositions of graphs (Ph.D. thesis), Université de Bordeaux, 2014.

[8] A. Davoodi, B. Omoomi, *On the 1–2–3-conjecture*, Discrete Math. Theor. Comput. Sci. 17 (1) (2015) 67–78.

[9] A. Dehghan, *On strongly planar not-all-equal 3sat*, J. Comb. Optim. 32 (3) (2016) 721–724.

[10] A. Dehghan, M.-R. Sadeghi, A. Ahadi, *Algorithmic complexity of proper labeling problems*, Theoret. Comput. Sci. 495 (2013) 25–36.

[11] D.-Z. Du, K.-K. Ko, J. Wang, Introduction to Computational Complexity, Higher Education Press, 2002.

[12] A. Dudek, D. Wajc, *On the complexity of vertex-coloring edge-weightings*, Discrete Math. Theor. Comput. Sci. 13 (3) (2011) 45–50.

[13] L. Esperet, S. Gravier, M. Montassier, P. Ochem, A. Parreau, *Locally identifying coloring of graphs*, Electron. J. Combin. 19 (2) (2012) 21. Paper 40.

[14] F. Foucaud, I. Honkala, T. Laihonen, A. Parreau, G. Perarnau, *Locally identifying colourings for graphs with given maximum degree*, Discrete Math. 312 (10) (2012) 1832–1837.

[15] M.R. Garey, D.S. Johnson, A guide to the theory of NP-completeness, in: Computers and Intractability, in: A Series of Books in the Mathematical Sciences, W. H. Freeman and Co, San Francisco, Calif, 1979.

[16] D. Gonçalves, A. Parreau, A. Pinlou, *Locally identifying coloring in bounded expansion classes of graphs*, Discrete Appl. Math. 161 (18) (2013) 2946–2951.

[17] E. Győri, C. Palmer, *A new type of edge-derived vertex coloring*, Discrete Math. 309 (22) (2009) 6344–6352.

[18] M. Kalkowski, M. Karoński, F. Pfender, *Vertex-coloring edge-weightings: towards the 1–2–3-conjecture*, J. Combin. Theory Ser. B 100 (3) (2010) 347–349.

[19] M. Karoński, T. Łuczak, A. Thomason, *Edge weights and vertex colours*, J. Combin. Theory Ser. B 91 (1) (2004) 151–157.

[20] M. Khatirinejad, R. Naserasr, M. Newman, B. Seamone, B. Stevens, *Digraphs are 2-weight choosable*, Electron. J. Combin. 18 (1) (2011) 4. Paper 21.

[21] M. Khatirinejad, R. Naserasr, M. Newman, B. Seamone, B. Stevens, *Vertex-colouring edge-weightings with two edge weights*, Discrete Math. Theor. Comput. Sci. 14 (1) (2012) 1–20.

[22] H. Lu, Q. Yu, C.-Q. Zhang, *Vertex-coloring 2-edge-weighting of graphs*, European J. Combin. 32 (1) (2011) 21–27.

[23] B.M. Moret, *Planar NAE3SAT is in P*, SIGACT News 19 (2) (1988) 51–54.

[24] J. Przybyło, M. Woźniak, *On a 1, 2 conjecture*, Discrete Math. Theor. Comput. Sci. 12 (1) (2010) 101–108.

[25] B. Seamone, Derived colourings of graphs (Ph.D. thesis), Carleton University, Ottawa, 2012.

[26] B. Seamone, *Bounding the monomial index and (1, l)-weight choosability of a graph*, Discrete Math. Theor. Comput. Sci. 16 (3) (2014) 173–187.

[27] B. Seamone, *The 1–2–3 conjecture and related problems: a survey*, arXiv:1211.5122.

[28] B. Seamone, B. Stevens, *Sequence variations of the 1–2–3 conjecture and irregularity strength*, Discrete Math. Theor. Comput. Sci. 15 (1) (2013) 15–28.

[29] D.B. West, Introduction to Graph Theory, Prentice Hall Inc., Upper Saddle River, NJ, 1996.

[30] T.-L. Wong, J. Wu, X. Zhu, *Total weight choosability of cartesian product of graphs*, European J. Combin. 33 (8) (2012) 1725–1738.

Chapter 3

Algorithmic complexity of weakly semiregular partitioning, and the representation number of graphs

3.1 Introduction

This chapter consists of two parts. In the first part, we consider the problem of partitioning the edges of a graph into regular and/or locally irregular subgraphs. In this part, we present some polynomial time algorithms and **NP**-hardness results. In the second part of the work, we focus on the representation number of graphs. It was conjectured that the determination of rep(G) for an arbitrary graph G is a difficult problem [38]. In this part, we confirm this conjecture and show that if $\mathbf{NP} \neq \mathbf{P}$, then for any $\epsilon > 0$, there is no polynomial time $(1-\epsilon)\frac{n}{2}$-approximation algorithm for the computation of representation number of regular graphs with n vertices.

3.2 Partitioning the edges of graphs

In 1981, Holyer [31] focused on the computational complexity of edge partitioning problems and proved that for each t, $t \geq 3$, it is **NP**-complete to decide whether a given graph can be edge-partitioned into subgraphs isomorphic to the complete graph K_t. Afterwards, the complexity of edge partitioning problems have been studied extensively by several authors, for instance see [21–23]. Nowadays, the computational complexity of edge partitioning problems is a well-studied area of graph theory and computer science. For more information we refer the reader to a survey on graph factors and factorization by Plummer [40].

If we consider the Holyer problem for a family $\mathcal{G}$ of graphs instead of a fixed graph then, we can discover interesting problems. For a family $\mathcal{G}$ of graphs, a $\mathcal{G}$-decomposition of a graph G is a partition of the edge set of G into subgraphs isomorphic to members of $\mathcal{G}$. Problems of $\mathcal{G}$-decomposition of graphs have received a considerable attention, for example, Holyer proved that it is **NP**-hard to edge-partition a graph into the minimum number of complete subgraphs [31]. To see more examples of $\mathcal{G}$-decomposition of graphs see [15, 19, 33].

3.2.1 Related works and motivations

We say that a graph is *locally irregular* if its adjacent vertices have distinct degrees and a graph is *regular* if each vertex of the graph has the same degree. In 2001, Kulli *et al.* introduced an interesting parameter for the partitioning of the edges of a graph [34]. The *regular number* of a graph G, denoted by $\mathrm{reg}(G)$, is the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is regular. The *edge chromatic number* of a graph, denoted by $\chi'(G)$, is the minimum size of a partition of the edge set into 1-regular subgraphs and also, by Vizing's theorem [45], the edge chromatic number of a graph G is equal to either $\Delta(G)$ or $\Delta(G)+1$, therefore the regular number problem is a generalization for the edge chromatic number and we have the following bound: $\mathrm{reg}(G) \leq \chi'(G) \leq \Delta(G)+1$. It was asked [27] to determine whether $\mathrm{reg}(G) \leq \Delta(G)$ holds for all connected graphs.

Conjecture 1. *[27, The degree bound]* For any connected graph G, $\mathrm{reg}(G) \leq \Delta(G)$.

It was shown [4] that not only there exists a counterexample for the above-mentioned bound but also for a given connected graph G decide whether $\mathrm{reg}(G) = \Delta(G)+1$ is **NP**-complete. Also, it was shown that the computation of the regular number for a given connected bipartite graph G is **NP**-hard [4]. Furthermore, it was proved that determining whether $\mathrm{reg}(G) = 2$ for a given connected 3-colorable graph G is **NP**-complete [4].

On the other hand, Baudon *et al.* introduced the notion of edge partitioning into locally irregular subgraphs [12]. In this case, we want to partition the edges of the graph G

into locally irregular subgraphs, where by a partitioning of the graph G into k locally irregular subgraphs we refer to a partition $E_1, \ldots, E_k$ of $E(G)$ such that the graph $G[E_i]$ is locally irregular for every i, $i = 1, \ldots, k$. The *irregular chromatic index* of G, denoted by χ'_{irr}, is the minimum number k such that the graph G can be partitioned into k locally irregular subgraphs. Baudon *et al.* characterized all graphs which cannot be partitioned into locally irregular subgraphs and call them exceptions [12]. Motivated by the 1-2-3-Conjecture, they conjectured that apart from these exceptions all other connected graphs can be partitioned into three locally irregular subgraphs [12]. For more information about the 1-2-3-Conjecture and its variations, we refer the reader to a survey on the 1-2-3-Conjecture and related problems by Seamone [43] (see also [1, 2, 11, 14, 20, 42, 44]).

Conjecture 2. *[12]* For every non-exception graph G, we have $\chi'_{irr}(G) \leq 3$.

Regarding the above-mentioned conjecture, Bensmail *et al.* [16] proved that every bipartite graph G which is not an odd length path satisfies $\chi'_{irr}(G) \leq 10$. Also, they proved that if G admits a partitioning into locally irregular subgraphs, then $\chi'_{irr}(G) \leq 328$. Recently, Lužar *et al.* improved the previous bound for bipartite graphs and general graphs to 7 and 220, respectively [36]. For more information about this conjecture see [41].

Regarding the complexity of edge partitioning into locally irregular subgraphs, Baudon *et al.* [13] proved that the problem of determining the irregular chromatic index of a graph can be handled in linear time when restricted to trees. Furthermore, in [13], Baudon *et al.* proved that determining whether a given planar graph G can be partitioned into two locally irregular subgraphs is **NP**-complete.

In 2015, Bensmail and Stevens considered the problem of partitioning the edges of graph into some subgraphs, such that in each subgraph every component is either regular or locally irregular [18]. The *regular-irregular chromatic index* of graph G, denoted by $\chi'_{reg-irr}(G)$, is the minimum number k such that G can be partitioned into k subgraphs, such that each component of every subgraph is locally irregular or regular [18]. They conjectured that the edges of every graph can be partitioned into at most two subgraphs, such that each component of every subgraph is regular or locally irregular [17, 18].

Conjecture 3. *[17, 18]* For every graph G, we have $\chi'_{reg-irr}(G) \leq 2$.

Recently, motivated by Conjecture 2 and Conjecture 3, Ahadi *et al.* in [5] presented the following conjecture.

Conjecture 4. *[5] Every graph can be partitioned into 3 subgraphs, such that each subgraph is locally irregular or regular.*

Note that in Conjecture 4, each subgraph (instead of each component of every subgraph) should be locally irregular or regular. Also, note that it was shown that deciding whether a given planar bipartite graph G with maximum degree three can be partitioned into at most two subgraphs such that each subgraph is regular or locally irregular is **NP**-complete [5].

In [5], Ahadi *et al.* considered the problem of partitioning the edges into locally regular subgraphs. We say that a graph G is *locally regular* if each component of G is regular (note that a regular graph is locally regular but the converse does not hold). The *regular chromatic index* of a graph G denoted by χ'_{reg} is the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is locally regular. From the definitions of locally regular and regular graphs we have the following bound: $\chi'_{reg}(G) \leq reg(G) \leq \Delta(G) + 1$. It was shown that every graph G can be partitioned into $\Delta(G)$ subgraphs such that each subgraph is locally regular and this bound is sharp for trees [5].

Lemma 3.2.1. [5] *Every graph G can be partitioned into $\Delta(G)$ subgraphs such that each subgraph is locally regular and this bound is sharp for trees.*

In conclusion, we can say that the problem of partitioning the edges of graph into regular and/or locally irregular subgraphs is an active area in graph theory and computer science. What can we say about the edge decomposition problem if we require that each subgraph (instead of each component of every subgraph) should be a graph with at most k numbers in its degree set. With this motivation in mind, we investigate the problem of partitioning the edges of graphs into subgraphs such that each subgraph has at most two numbers in its degree set. In this work, we consider partitioning into weakly semiregular and semiregular subgraphs.

3.2.2 Weakly semiregular graphs

A graph G is *weakly semiregular* if there are two numbers a, b, such that the degree of every vertex is a or b. The *weakly semiregular number* of a graph G, denoted by $\mathrm{wr}(G)$, is the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is weakly semiregular. This parameter is well-defined for any graph G since one can always partition the edges into 1-regular subgraphs. Throughout the paper, we say that a graph G is (a, b)-graph if the degree of every vertex is a or b (in other words, if the degree set of the graph G is $\{a, b\}$).

Remark 1. There are infinitely many values of Δ for which the graph G might be chosen so that $\mathrm{wr}(G) \geq \log_3 \Delta(G)$. Assume that G is a graph (G can be a tree) such that for each i, $1 \leq i \leq \Delta$, there is a vertex with degree i in that graph. Also, let $E_1, E_2, \ldots, E_{wr(G)}$ be a weakly semiregular partitioning for the edges of that graph. The degree set of the subgraph $G_i = (V, E_i)$ has at most three elements. By adding $wr(G)$ such degree sets, one corresponding to each subset E_i, we get a degree set that contains at most $3^{wr(G)}$ elements. Hence, the degree set of the graph G contains at most $3^{wr(G)}$ elements. This completes the proof.

In this work, we focus on the algorithmic aspects of weakly semiregular number. We present a polynomial time algorithm to determine whether the weakly semiregular number of a given tree is two. We prove the following theorem in Section 3.5.

Theorem 3.2.2. *(i) There is an $\mathcal{O}(n^2)$ time algorithm to determine whether the weakly semiregular number of a given tree is two, where n is the number of vertices in the tree.*
(ii) Let c be a constant, there is a polynomial time algorithm to determine whether the weakly semiregular number of a given tree is at most c.
(iii) For every tree T, $\mathrm{wr}(T) \leq 2\log_2 \Delta(T) + \mathcal{O}(1)$.

Remark 2. If G is a graph with $\Delta(G) \leq 4$, then $\mathrm{wr}(G) \leq 2$. If the graph G is not regular, then consider two copies of the graph G and for each vertex v with degree less than 4, join the vertex v in the first copy of G to the vertex v in the second copy of the graph G. By repeating this procedure we can obtain a 4-regular graph G'. A subgraph F of a graph H is called a factor of H if F is a spanning subgraph of H. If a factor F has all of its degrees equal to k, it is called a k-factor. A k-factorization for a graph H is a partition of the edges into disjoint k-factors. For $k \geq 1$, every $2k$-regular graph admits a 2-factorization [39], thus the graph G' can be partitioned into two 2-regular graphs G'_1 and G'_2. Let $f : E(G') \to \{1, 2\}$ be a function such that $f(e) = 1$ if and only if $e \in E(G'_1)$. One can see that the function f can partition the edges of the graph G into two (1,2)-graphs. Therefore, $\mathrm{wr}(G) \leq 2$. This completes the proof.

If G is a graph with at most two numbers in its degree set, then its weakly semiregular number is one. On the other hand, if $\Delta \leq 4$ by Remark 2, the weakly semiregular number of the graph is at most two. We show that determining whether $\mathrm{wr}(G) = 2$ for a given bipartite graph G with $\Delta(G) = 6$ and at most three numbers in its degree set, is **NP**-complete. The proof is in section 3.6.

Theorem 3.2.3. *Determining whether* $\mathrm{wr}(G) = 2$ *for a given bipartite graph* G *with* $\Delta(G) = 6$ *and at most three numbers in its degree set, is* **NP**-*complete.*

3.2.3 Semiregular graphs

A graph G is a $[d, d+s]$-*graph* if the degree of every vertex of G lies in the interval $[d, d+s]$. A $[d, d+1]$-graph is said to be *semiregular*. Semiregular graphs are an important family of graphs and their properties have been studied extensively, see for instance [8, 9]. The *semiregular number* of a graph G, denoted by $\mathrm{sr}(G)$, is the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is semiregular. We prove that the semiregular number of a tree T is $\lceil \frac{\Delta(T)}{2} \rceil$. On the other hand if $\Delta \leq 4$ by Remark 2, the semiregular number of a graph is at most two. We show that determining whether $\mathrm{sr}(G) = 2$ for a given bipartite graph G with $\Delta(G) \leq 6$, is **NP**-complete. The proof is in section 3.7.

Theorem 3.2.4. *(i) Let* T *be a tree, then* $\mathrm{sr}(T) = \lceil \frac{\Delta(T)}{2} \rceil$.
(ii) Let G *be a graph, then* $\mathrm{sr}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.
(iii) Determining whether $\mathrm{sr}(G) = 2$ *for a given bipartite graph* G *with* $\Delta(G) \leq 6$, *is* **NP**-*complete.*

Every semiregular graph is a weakly semiregular graph, thus by the above-mentioned theorem, we have the following bound:

$$\mathrm{wr}(G) \leq \mathrm{sr}(G) \leq \lceil \frac{\Delta(G) + 1}{2} \rceil. \tag{3.1}$$

3.2.4 Partitioning into locally irregular and weakly semiregular subgraphs

Bensmail and Stevens [18] considered the outcomes on Conjecture 2 of allowing components isomorphic to the complete graph K_2, or more generally regular components. In fact their investigations are motivated by the following question: "How easier can

Conjecture 2 be tackled if we allow a locally irregular partitioning to induce connected components isomorphic to the complete graph K_2?" They conjectured that the edges of every graph can be partitioned into at most two subgraphs, such that each component of every subgraph is regular or locally irregular [18]. Motivated by this conjecture we pose the following conjecture. Note that in Conjecture 5, each subgraph (instead of each component of every subgraph) should be locally irregular or weakly semiregular.

Conjecture 5. Every graph can be partitioned into 3 subgraphs, such that each subgraph is locally irregular or weakly semiregular.

Note that if Conjecture 2 or Conjecture 4 is true, then Conjecture 5 is true. Also, if every graph can be partitioned into 2 subgraphs such that each component of every subgraph is a locally irregular graph or K_2, then Conjecture 5 is true. We conclude this section by the following hardness result. We provide the proof in section 3.8.

Theorem 3.2.5. *Determining whether a given graph G, can be partitioned into 2 subgraphs, such that each subgraph is locally irregular or weakly semiregular is* **NP***-complete.*

3.2.5 Summary of results

A summary of results and open problems on edge-partition problems are shown in Table 3.1.

Table 3.1: Recent results on edge partitioning of graphs into subgraphs

	= 2 (for trees)	= 2	Upper bound
Regular subgraphs	**P** (see [27])	**NP**-c (see [4])	$\Delta + 1$ (see [27])
Locally regular subgraphs	**P** (see [5])	**NP**-c (see [5])	Δ (see [5])
Weakly semiregular subgraphs	**P** (Th. 3.2.2)	**NP**-c (Th. 3.2.3)	$\lceil \frac{\Delta+1}{2} \rceil$ (Th. 3.2.4)
Semiregular subgraphs	**P** (Th. 3.2.4)	**NP**-c (Th. 2.2.3)	$\lceil \frac{\Delta+1}{2} \rceil$ (Th. 3.2.4)
Locally irregular subgraphs	**P** (see [13])	**NP**-c (see [13])	3 (Conj. 2)
regular-irregular subgraphs	Open (see [5])	**NP**-c (see [5])	3 (Conj. 4)
regular-irregular components	**P** (see [18])	P (Conj. 3)	2 (Conj. 3)

3.3 Representation number

A finite graph G is said to be *representable modulo* r, if there exists an injective map $\ell : V(G) \to \{0, 1, \ldots, r-1\}$ such that vertices v and u of the graph G are adjacent if and only if $|\ell(u) - \ell(v)|$ is relatively prime to r. The *representation number* of G, denoted by $rep(G)$, is the smallest positive integer r such that the graph G has a representation modulo r. In 1989, Erdős and Evans introduced representation numbers and showed that every finite graph can be represented modulo some positive integer [24]. They used representation numbers to give a simpler proof of a result of Lindner et al. [35] that, any finite graph can be realized as an orthogonal Latin square graph (an orthogonal Latin square graph is one whose vertices can be labeled with Latin squares of the same order and same symbols such that two vertices are adjacent if and only if the corresponding Latin squares are orthogonal). The existence proof of Erdős and Evans gives an unnecessarily large upper bound for the representation number [24]. During the recent years, representation numbers have received considerable attention and have been studied for various classes of graphs, see [6, 25, 26, 32, 37].

Narayan and Urick conjectured that the determination of $\mathrm{rep}(G)$ for an arbitrary graph G is a difficult problem [38]. In the following theorem we discuss about the computational complexity of $\mathrm{rep}(G)$ for regular graphs. The proof for this theorem is in Section 3.9

Theorem 3.3.1. *(i) If* $\mathbf{NP} \neq \mathbf{P}$*, then for any* $\epsilon > 0$*, there is no polynomial time* $(1-\epsilon)\frac{n}{2}$*-approximation algorithm for the representation number of regular graphs with* n *vertices. (ii) For every* $\epsilon > 0$ *there is a polynomial time* $((1+\epsilon)\dfrac{en}{2})$*-approximation algorithm for computing* $\mathrm{rep}(G)$ *where* G^c *is a triangle-free* r*-regular graph.*

3.4 Notation and tools

All graphs considered in this chapter are finite and undirected. If G is a graph, then $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Also, $\Delta(G)$ denotes the maximum degree of G and simply denoted by Δ. For every $v \in V(G)$, $d_G(v)$ and $N_G(v)$ denote the degree of v and the set of neighbors of v, respectively. Also, $N[v] = N(v) \cup \{v\}$. For a given graph G, we use $u \sim v$ if two vertices u and v are adjacent in G.

The *degree sequence* of a graph is the sequence of non-negative integers listing the degrees of the vertices of G. For example, the complete bipartite graph $K_{1,3}$ has degree sequence $(1, 1, 1, 3)$, which contains two distinct elements: 1 and 3. The *degree set* D of a graph G is the set of distinct degrees of the vertices of G. For $k \in \mathbb{N}$, a *proper edge* k*-coloring* of G is a function $c : E(G) \longrightarrow \{1, \ldots, k\}$, such that if $e, e' \in E(G)$ share a common endpoint, then $c(e)$ and $c(e')$ are different. The smallest integer k such that G has a proper edge k-coloring is called the *edge chromatic number* of G and denoted by $\chi'(G)$. By Vizing's theorem [45], the edge chromatic number of a graph G is equal to either $\Delta(G)$ or $\Delta(G)+1$. Those graphs G for which $\chi'(G) = \Delta(G)$ are said to belong to *Class* 1, and the other to *Class* 2.

Let G be a graph and f be a non-negative integer-valued function on $V(G)$. Then a spanning subgraph H of G is called an f-factor of G if $d_H(v) = f(v)$, for all $v \in V(G)$. Let G be a graph and let f, g be mappings of $V(G)$ into the non-negative integers. A (g, f)-factor of G is a spanning subgraph F such that $g(v) \leq d_F(v) \leq f(v)$ for all $v \in V(G)$. In 1985, Anstee gave a polynomial time algorithm for the (g, f)-factor problem and his algorithm either returns one of the factors in question or shows that none exists, in $\mathcal{O}(n^3)$ time [7]. Note that this complexity bound is independent of the functions g and f. We will use this algorithm in our proof. We follow [46] for terminology and notation where they are not defined here.

3.5 Proof of Theorem 3.2.2

Here we prove Theorem 3.2.2.

(i) Let T be an arbitrary tree. Any subgraph of a tree is a forest, so if T can be partitioned into two weakly semiregular forests T_1 and T_2, then there are two numbers α, β (not necessary distinct) such that T_1 is a $(1, \alpha)$-forest and T_2 is a $(1, \beta)$-forest (note that a forest T is a (a, b)-forest if the degree of every vertex is a or b). Without loss of generality, we can assume that $1 \leq \alpha \leq \beta \leq \Delta(T) \leq n$. Let D be the degree set of T, we have $D \subseteq \{1, 2, \alpha, \alpha + 1, \beta, \beta + 1, \alpha + \beta\}$. So if $|D| \geq 8$, then the tree T cannot be partitioned into two weakly semiregular forests. On the other hand, one can see that if $|D| \leq 7$, then the number of possible cases for (α, β) is $\mathcal{O}(1)$.

In Algorithm 1, we present an $\mathcal{O}(n^2)$ time algorithm to check whether T can be partitioned into two weakly semiregular forests T_1 and T_2, such that the forest T_1 is $(1, \alpha)$-forest and the forest T_2 is $(1, \beta)$-forest. If the algorithm returns NO, it means that T cannot be partitioned and if it returns YES, it means that T can be partitioned.

Here, let us to introduce some notation and state a few properties of Algorithm 1. Suppose that $|V(T)| = n$ and choose an arbitrary vertex $v \in V(T)$ to be its root. Perform a breadth-first search algorithm from the vertex v. This defines a partition $L_0, L_1, \ldots, L_h$ of the vertices of T where each part L_i contains the vertices of T which are at depth i (at distance exactly i from v). Let $p(x)$ denote the neighbor of the vertex x on the xv-path, i.e. its parent. Also, let $\{v_1 = v, v_2, \ldots, v_n\}$ be a list of the vertices according to their distance from the root. We use form this list of vertices in the algorithm. See Algorithm 1.

Algorithm 1

Input: The tree T and two numbers α, β.
Output: Can T be partitioned into two weakly semiregular forests T_1 and T_2, such that T_1 is $(1, \alpha)$-forest and T_2 is $(1, \beta)$-forest.
Let $g : E(T) \to \{red, blue, free\}$ and put $g(e) \leftarrow free$ for all edges
Let $f : E(T) \to \{red, blue, free\}$ and put $f(e) \leftarrow free$ for all edges
while there is an edge e such that $g(e) = free$ **do**
    For any edge e, put $f(e) \leftarrow free$
    $s \leftarrow$YES
    for $i = 1$ to $i = n$ **do**
        if there is no labeling like h for the set of edges $S_i = \{v_i v_j : j > i\}$ with the colors red and blue such that $|\{e : e \in S_i, h(e) = red\} \cup \{v_i p(v_i) : f(v_i p(v_i)) = red, i > 1\}| \in \{0, 1, \alpha\}$, also $|\{e : e \in S_i, h(e) = blue\} \cup \{v_i p(v_i) : f(v_i) = blue, i > 1\}| \in \{0, 1, \beta\}$ and for each edge $e \in S_i$, if $g(e) \neq free$, then $g(e) = h(e)$ **then**
            if $g(v_i p(v_i)) \neq free$ **then**
                return NO
            end if
            if $f(v_i p(v_i)) = blue$ **then**
                $g(v_i p(v_i)) \leftarrow red$
                $s \leftarrow$NO
                break the for loop
            end if
            if $f(v_i p(v_i)) = red$ **then**
                $g(v_i p(v_i)) \leftarrow blue$
                $s \leftarrow$NO
                break the for loop
            end if
        end if
        if $s =$YES **then**
            Label the set of edges $S_i = \{v_i v_j : j > i\}$ with the colors red and blue such that $|\{e : e \in S_i, h(e) = red\} \cup \{v_i p(v_i) : \text{ if } f(v_i) = red\}| \in \{0, 1, \alpha\}$, also $|\{e : e \in S_i, h(e) = blue\} \cup \{v_i p(v_i) : \text{ if } f(v_i) = blue\}| \in \{0, 1, \beta\}$ and for each edge $e \in S_i$, if $g(e) \neq free$, then $g(e) = h(e)$
            For each edge e in $\{v_i v_j : j > i\}$ assign the label of e to the variable $f(e)$
        end if
    end for
    if $s =$YES **then**
        return YES
    end if
end while
return NO

Sketch of Algorithm 1

The algorithm starts from the vertex v_1 and labels the set of edges $S_1 = \{v_1 v_j : j > 1\}$

with labels *red* and *blue* such that the number of edges in S_1 with label *red* is 0 or 1 or α and the number of edges in S_1 with label *blue* is 0 or 1 or β. The algorithm saves the partial labeling in f. Next, at step i, $i > 1$ of the *for loop*, the algorithm labels the set of edges $S_i = \{v_i v_j : j > i\}$ with labels *red* and *blue* such that the number of edges in $S_i \cup \{v_i p(v_i)\}$ with label *red* is 0 or 1 or α and the number of edges in $S_i \cup \{v_i p(v_i)\}$ with label *blue* is 0 or 1 or β. If the algorithm runs the *for loop* completely, then we are sure that the tree can be partitioned and if at step i, there is no labeling for S_i, then it shows that the label of edge $v_i p(v_i)$ should not be $f(v_i p(v_i))$. So, the algorithm saves the correct label of $v_i p(v_i)$ in g, erases the labels of edges, breaks the *for loop* and starts the *for loop* from the beginning. In the next iteration of the *for loop*, the algorithm has some restrictions in its labeling (if the label of an edge e in $g(e)$ is not *free*, then its label must be equal to $g(e)$). Therefore, at step i of the *for loop*, the algorithm should find a labeling like h for the set of edges S_i such that $|\{e : e \in S_i, h(e) = red\} \cup \{v_i p(v_i) : f(v_i p(v_i)) = red\}| \in \{0, 1, \alpha\}$, also $|\{e : e \in S_i, h(e) = blue\} \cup \{v_i p(v_i) : f(v_i) = blue\}| \in \{0, 1, \beta\}$ and for each edge $e \in S_i$, if $g(e) \neq free$, then $g(e) = h(e)$. If the algorithm runs the *for loop* completely, then we are sure that the tree can be partitioned and if at step i, there is no labeling for S_i, then it shows that the label of the edge $v_i p(v_i)$ should not be $f(v_i p(v_i))$. Now, if $g(v_i p(v_i)) = f(v_i p(v_i))$, it shows that T cannot be partitioned into two weakly semiregular forests and if $g(v_i p(v_i)) = free$, the algorithm saves the correct label of $v_i p(v_i)$ in g, erases the labels, breaks the *for loop* and starts the *for loop* from the beginning. Note that if the algorithm does not run the *for loop* completely, then the label of one edge in the function g will be changed from $free$ into *red* or *blue*. Thus, the *while loop* will be run at most $|E(T)|$ times. So, finally the algorithm finds a labeling or terminates and returns NO.

Complexity of Algorithm 1

In Algorithm 1, if the *for loop* in Line 8, was completely run (if it was not *broken* in Line 16, 21 or 11), then the algorithm will return YES in line 30. Otherwise, the label of one edge in function g will be changed from $free$ into *red* or *blue*. Thus, the *while loop* will be run at most $|E(T)|$ times. On the other hand, the *for loop* in Line 8, takes at most $\mathcal{O}(n)$ times. Consequently, the running time of Algorithm 1 is $\mathcal{O}(n^2)$, hence we can determine whether T can be partitioned into two weakly semiregular forests T_1 and T_2 in $\mathcal{O}(n^2)$.

(ii) Let T be an arbitrary tree and c be a constant number. Any subgraph of a tree is a forest, so if T can be partitioned into c weakly semiregular forests $T_1, T_2, \ldots, T_c$, then there are c numbers $\alpha_1, \alpha_2, \ldots, \alpha_c$ (not necessary distinct) such that T_i is $(1, \alpha_i)$-forest.

Without loss of generality, we can assume that $1 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_c \leq \Delta(T) \leq n$. For each possible candidate for $(\alpha_1, \alpha_2, \ldots, \alpha_c)$ we run Algorithm 2. Since $1 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_c \leq \Delta(T) \leq n$, the number of candidates is a polynomial in terms of the number of vertices (in terms of n).

Here, let us to introduce some notation and state a few properties of Algorithm 2. Suppose that $|V(T)| = n$ and choose an arbitrary vertex $v \in V(T)$ to be its root. Perform a breadth-first search algorithm from the vertex v. This defines a partition $L_0, L_1, \ldots, L_h$ of the vertices of T where each part L_i contains the vertices of T which are at depth i (at distance exactly i from v). Let $p(x)$ denote the neighbor of x on the xv-path, i.e. its parent. Also, let $\{v_1 = v, v_2, \ldots, v_n\}$ be a list of the vertices according to their distance from the root. If Algorithm 2 returns NO, it means that T cannot be partitioned and if it returns YES, it means T can be decomposed. See Algorithm 2.

In Algorithm 2, at each step the variable ℓ for each edge shows the set of forbidden colors for that edge. In other words, at any time, for each edge e, $\ell(e)$ is a subset of $\{1, 2, \ldots, c\}$.

Assume that we want to find a labeling like h with the labels $\{1, 2, \ldots, c\}$ for the set of edges incident with a vertex u such that for each k, $1 \leq k \leq c$, $|\{e : e \ni u, h(e) = k\}| \in \{\gamma_k\}$ and for each edge e and color t, if $t \in \ell(e)$, then $h(e) \neq t$. We claim that this problem can be solved in polynomial time. Construct the bipartite graph H with the vertex set $V(H) = \{a_1, a_2, \ldots, a_c\} \cup \{e : e \ni u\}$ such that $a_t e \in E(H)$ if and only if $t \notin h(e)$. In this graph we want to find an F-factor such that for each i, $F(a_i) = \gamma_k$ and for each edge e, $F(e) = 1$. If the graph H has an F-factor then there is a labeling like h for the the set of edges incident with the vertex u with the specified properties. In 1985, Anstee gave a polynomial time algorithm for the F-factor problem and his algorithm either returns one of the factors in question or shows that none exists, in $\mathcal{O}(n^3)$ time [7]. Thus, Line 9 and Line 20 can be performed in polynomial time (**Fact 1**).

Complexity of Algorithm 2

In Algorithm 2, if the *for loop* in Line 8, was completely run (it was not *broken* in Line 16, 21 or 11), then Algorithm will return YES in line 30. Otherwise, a color will be added to the set of forbidden colors of an edge. Thus, the *while loop* will be run at most $c|E(T)|$ times. On the other hand, by Fact 1, the *for loop* in Line 8, takes a polynomial time to run. Consequently, the running time of Algorithm 1 is a polynomial in terms of n.

Sketch of Algorithm 2

Algorithm 2

Input: The tree T and c numbers $\alpha_1, \alpha_2, \ldots, \alpha_c$.
Output: Can T be partitioned into c weakly semiregular forests $T_1, T_2, \ldots, T_c$, such that T_i is $(1, \alpha_i)$-forest.
Let $f : E(T) \to \{1, 2, \ldots, c, free\}$ and put $f(e) \leftarrow free$ for all edges
Let $\ell : E(T) \to \{2^{\{1,2,\ldots,c\}}\}$ and put $\ell(e) \leftarrow \emptyset$ for all edges
while there is an edge e such that $\ell(e) \neq \{1, 2, \ldots, c\}$ **do**
  For any edge e, put $f(e) \leftarrow free$
  $s \leftarrow$YES
  for $i = 1$ to $i = n$ **do**
    if there is no labeling like h for the set of edges $S_i = \{v_i v_j : j > i\}$ with the colors $\{1, 2, \ldots, c\}$ such that for each k, $1 \leq k \leq c$, $|\{e : e \in S_i, h(e) = k\} \cup \{v_i p(v_i) : f(v_i p(v_i)) = k, i > 1\}| \in \{0, 1, \alpha_k\}$ and for each edge $e \in S_i$ and color t, if $t \in \ell(e)$, then $h(e) \neq t$ **then**
      if $f(v_i p(v_i)) \in \ell(v_i p(v_i))$ **then**
        return NO
      end if
      if $f(v_i p(v_i)) \notin \ell(v_i p(v_i))$ **then**
        $\ell(v_i p(v_i)) \leftarrow \ell(v_i p(v_i)) \cup \{f(v_i p(v_i))\}$
        $s \leftarrow$NO
        break the for loop
      end if
    end if
    if $s =$YES **then**
      Label the set of edges $S_i = \{v_i v_j : j > i\}$ with the colors $\{1, 2, \ldots, c\}$ such that for each k, $1 \leq k \leq c$, $|\{e : e \in S_i, h(e) = k\} \cup \{v_i p(v_i) : f(v_i p(v_i)) = k, i > 1\}| \in \{0, 1, \alpha_k\}$ and for each edge $e \in S_i$ and color t, if $t \in \ell(e)$, then $h(e) \neq t$
      For each edge e in $\{v_i v_j : j > i\}$ assign the label of e to the variable $f(e)$
    end if
  end for
  if $s =$YES **then**
    return YES
  end if
end while
return NO

Performance of Algorithm 2 is similar to Algorithm 1, except that in Algorithm 2, at each step the variable ℓ for each edge shows the set of forbidden colors for that edge. So at any time, for each edge e, $\ell(e)$ is a subset of $\{1, 2, 3, \ldots, c\}$. This completes the proof.

(iii) Suppose that $|V(T)| = n$ and choose an arbitrary vertex $v \in V(T)$ to be its root. Perform a breadth-first search algorithm from the vertex v. This defines a partition $L_0, L_1, \ldots, L_h$ of the vertices of T where each part L_i contains the vertices of T which are at depth i (at distance exactly i from v). Let $\{v_1 = v, v_2 \ldots, v_n\}$ be a list of the vertices according to their distance from the root. Do Algorithm 3 for the vertices of T according to their indices.

Algorithm 3

Input: The tree T.
Output: A decomposition of T into $2\log_2 \Delta(T) + \mathcal{O}(1)$ weakly semiregular subgraphs.
for $i = 1$ to $i = n$ **do**
  If $i = 1$, let $a[0]a[1]\cdots a[\lfloor \log_2 \Delta \rfloor]$ be the binary representation of the number $d(v_1)$ (note that for each r, $a[r]$ is either a 1 or a 0) and label the set of edges $\{v_1v_j : j > 1\}$, such that for each t, $0 \leq t \leq \lfloor \log_2 \Delta \rfloor$, if $a[t] = 1$, then the number of edges incident with v_1 with label t is 2^t.
  If $i > 1$, $v_i \in L_k$ and $k = 0 (\mod 2)$. Let $a[0]a[1]\cdots a[\lfloor \log_2 \Delta \rfloor]$ be the binary representation of the number $d(v_i) - 1$ and label the set of edges $\{v_iv_j : j > i\}$, such that for each t, $0 \leq t \leq \lfloor \log_2 \Delta \rfloor$, if $a[t] = 1$, then the number of edges in $\{v_iv_j : j > i\}$ with label $t + \lfloor \log_2 \Delta \rfloor + 1$ is 2^t.
  If $i > 1$, $v_i \in L_k$ and $k = 1 (\mod 2)$. Let $a[0]a[1]\cdots a[\lfloor \log_2 \Delta \rfloor]$ be the binary representation of the number $d(v_i) - 1$ and label the set of edges $\{v_iv_j : j > i\}$, such that for each t, $0 \leq t \leq \lfloor \log_2 \Delta \rfloor$, if $a[t] = 1$, then the number of edges in $\{v_iv_j : j > i\}$ with label t is 2^t.
end for

In Algorithm 3, for each t, $0 \leq t \leq \lfloor \log_2 \Delta \rfloor$, the set of edges with label t forms a $(1, 2^t)$-graph. Also, the set of edges with label $t + \lfloor \log_2 \Delta \rfloor + 1$ forms a $(1, 2^t)$-graph. Thus, one can see that the above-mentioned labeling is partitioning of edges into $2\log_2 \Delta(T) + \mathcal{O}(1)$ weakly semiregular subgraphs. This completes the proof.

3.6 Proof of Theorem 3.2.3

It was shown [4] that the following version of *Not-All-Equal (NAE) satisfying assignment problem* is **NP**-complete.

Cubic Monotone NAE (2,3)-Sat.
INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $\mid c \mid \in \{2, 3\}$, every variable appears in exactly three clauses and there is no negation in the formula.
QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal and at least one false literal?

We reduce *Cubic Monotone NAE (2,3)-Sat* to our problem in polynomial time. Consider an instance Φ, we transform this into a bipartite graph G in polynomial time such that $wr(G) = 2$ if and only if Φ has an NAE truth assignment. We use three auxiliary gadgets $\mathcal{H}_c$, $\mathcal{I}_c$ and $\mathcal{B}$ which are shown in Figure 3.1 and Figure 3.2.

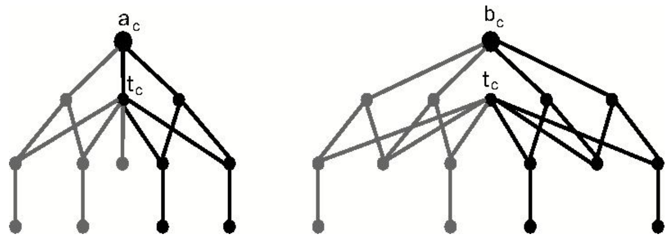


FIGURE 3.1: The two gadgets $\mathcal{H}_c$ and $\mathcal{I}_c$. The graph $\mathcal{I}_c$ is on the right hand side of the figure.

Our construction consists of three steps.

Step 1. Put a copy of the graph $\mathcal{B}$, a copy of the complete bipartite graph $K_{1,6}$ and a copy of the complete bipartite graph $K_{3,6}$.

Step 2. For each clause $c \in C$ with $\mid c \mid= 3$, put a copy of the gadget $\mathcal{H}_c$ and for each clause $c \in C$ with $\mid c \mid= 2$, put a copy of the gadget $\mathcal{I}_c$.

Step 3. For each variable $x \in X$, put a vertex x and for each clause $c = y \vee z \vee w$, where $y, z, w \in X$ add the edges $a_c y$, $a_c z$ and $a_c w$. Also, for each clause $c = y \vee z$, where $y, z \in X$ add the edges $b_c y$ and $b_c z$.

Call the resultant graph G. The degree of every vertex in the graph G is 1 or 3 or 6 and the resultant graph is bipartite. Assume that the graph G can be partitioned into two weakly semiregular graphs G_1 and G_2, we have the following lemmas.

Lemma 3.6.1. *The graphs G_1 and G_2 are (1,3)-graph.*

Proof. Without loss of generality, assume that the graph G_1 is (α_1, α_2)-graph and the graph G_2 is (β_1, β_2)-graph. Since $\Delta(G) = 6$, by attention to the structure of the graph $\mathcal{B}$, with respect to the symmetry, the following cases for $(\alpha_1\alpha_2, \beta_1\beta_2)$ can be considered: $(16, 12), (15, 12), (24, 12), (14, 12), (13, 13)$. The graph G contains a copy of the complete bipartite graph $K_{1,6}$, so the case (24,12) is not possible, also, the graph G contains a copy of complete bipartite graph $K_{3,6}$, so the cases $(16, 12), (15, 12), (14, 12)$ are not possible. Thus, the graphs G_1 and G_2 are (1,3)-graph. □

Lemma 3.6.2. *For every vertex v with degree three all edges incident with the vertex v are in one part.*

Proof. Since the graphs G_1 and G_2 are (1,3)-graph, the proof is clear. □

Now, we present an NAE truth assignment for the formula Φ. For every $x \in X$, if all edges incident with the vertex x are in the graph G_1, put $\Gamma(x) = true$ and if all edges incident with the vertex x are in graph G_2, put $\Gamma(x) = false$. Let $c = y \vee z \vee w$ be an arbitrary clause, if all edges $a_c y, a_c z, a_c w$ are in the graph G_1 (G_2, respectively), then by the construction of the gadget $\mathcal{H}_c$, the degree of the vertex t_c in the graph G_2 (G_1, respectively) is at least 5 (5, respectively). This is a contradiction. Similarly, let $c = y \vee z$ be an arbitrary clause, if all edges $b_c y, b_c z$ are in the graph G_1 (G_2, respectively), then by the construction of $\mathcal{I}_c$, the degree of the vertex t_c in the graph G_2 (G_1, respectively) is 6. This is a contradiction. Hence, Γ is an NAE satisfying assignment. On the other hand, suppose that Φ is NAE satisfiable with the satisfying assignment $\Gamma : X \to \{true, false\}$. For every variable $x \in X$, put all edges incident with the the vertex x in G_1 if and only if $\Gamma(x) = true$. By this method, one can show that the graph G can be partitioned into two weakly semiregular subgraphs. This completes the proof.

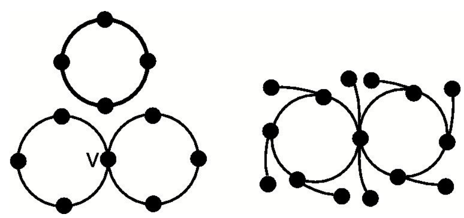


FIGURE 3.2: The two gadgets $\mathcal{P}$ and $\mathcal{B}$. The graph $\mathcal{B}$ is on the right hand side of the figure.

3.7 Proof of Theorem 3.2.4

(i) Any subgraph of a tree is a forest, so in every decomposition of a tree T into some semiregular subgraphs, each subgraph is a (1,2)-forest. Thus, $sr(T) \geq \lceil \frac{\Delta(T)}{2} \rceil$. For every bipartite graph H, we have $\chi'(H) = \Delta(H)$ (see for example [46]). Assume that $f : E(T) \to \{1, \ldots, \Delta(T)\}$ is a proper edge coloring for T. The following partition is a decomposition of T into $\lceil \frac{\Delta(T)}{2} \rceil$ semiregular subgraphs.

$$E(T) = \bigcup_{i=1}^{\lceil \frac{\Delta(T)}{2} \rceil} \{e : f(e) = i \text{ or } f(e) = i + \lceil \frac{\Delta(T)}{2} \rceil\}.$$

This completes the proof.

(ii) Let G be an arbitrary graph. By Vizing's theorem [45], the edge chromatic number of a graph G is equal to either $\Delta(G)$ or $\Delta(G)+1$. The following partition is a decomposition of G into $\lceil \frac{\chi'(G)}{2} \rceil$ semiregular subgraphs.

$$E(G) = \bigcup_{i=1}^{\lceil \frac{\chi'(G)}{2} \rceil} \{e : f(e) = i \text{ or } f(e) = i + \lceil \frac{\chi'(G)}{2} \rceil\}.$$

So the graph G can be partitioned into $\lceil \frac{\Delta(G)+1}{2} \rceil$ semiregular subgraphs.

(iii) We use a reduction from the following **NP**-complete problem [4].

Cubic Monotone NAE (2,3)-Sat.

INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $\mid c \mid \in \{2, 3\}$, every variable appears in exactly three clauses and there is no negation in the formula.
QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal and at least one false literal?

We reduce *Cubic Monotone NAE (2,3)-Sat* to our problem in polynomial time. Consider an instance Φ, we transform this into a bipartite graph G with $\Delta(G) \leq 6$ in polynomial time such that $sr(G) = 2$ if and only if Φ has an NAE truth assignment. We use three auxiliary gadgets $\mathcal{D}_c$, $\mathcal{F}_c$ and $\mathcal{P}$ which are shown in Figure 3.2 and Figure 3.3.

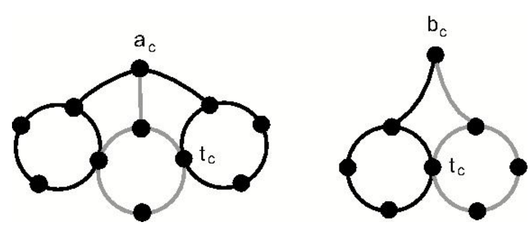


FIGURE 3.3: The two auxiliary gadgets $\mathcal{F}_c$ and $\mathcal{D}_c$. $\mathcal{D}_c$ is on the right hand side of the figure.

Our construction consists of three steps.
Step 1. Put a copy of the graph $\mathcal{P}$.
Step 2. For each clause $c \in C$ with $\mid c \mid= 3$, put a copy of the gadget $\mathcal{F}_c$ and for each clause $c \in C$ with $\mid c \mid= 2$, put a copy of the gadget $\mathcal{D}_c$.
Step 3. For each variable $x \in X$, put a vertex x and for each clause $c = y \vee z \vee w$,

where $y, z, w \in X$ add the edges $a_c y$, $a_c z$ and $a_c w$. Also, for each clause $c = y \vee z$, where $y, z \in X$ add the edges $b_c y$ and $b_c z$.

Call the resultant graph G. The degree set of the graph G is $\{2, 3, 4, 6\}$ and the graph is bipartite. Assume that G can be partitioned into two semiregular graphs G_1 and G_2, we have the following lemmas.

Lemma 3.7.1. *The graphs G_1 and G_2 are (2,3)-graph.*

Proof. Without loss of generality assume that G_1 is $(\alpha-1, \alpha)$-graph such that $\Delta(G_1) = \alpha$ and G_2 is $(\beta - 1, \beta)$-graph such that $\Delta(G_2) = \beta$. In the graph G any vertex of degree six has a neighbor of degree three, Thus, $\alpha \neq 6$ and $\beta \neq 6$. Also, there is no vertex of degree five, and any neighbor of each vertex of degree six has degree three, so we can assume that $\alpha \neq 5$ and $\beta \neq 5$. In the graph $\mathcal{P}$, the degree of the vertex v is four and the degree of each of its neighbor is two (note that the graph G contains a copy of the graph $\mathcal{P}$). Thus, by the structure of $\mathcal{P}$ and since the graph G contains a vertex of degree six, we have $\alpha \neq 4$ and $\beta \neq 4$. On the other hand, since $\Delta(G) = 6$, we have $\alpha = \beta = 3$. Hence, the graphs G_1 and G_2 are (2,3)-graph. □

Lemma 3.7.2. *For every vertex z with degree three or two all edges incident with the vertex z are in one part.*

Proof. Since the graphs G_1 and G_2 are (2,3)-graph, the proof is clear. □

Now, we present an NAE truth assignment for the formula Φ. For every $x \in X$, if all edges incident with the vertex x are in G_1, put $\Gamma(x) = true$ and if all edges incident with the vertex x are in G_2, put $\Gamma(x) = false$. Let $c = y \vee z \vee w$ be an arbitrary clause, if all edges $a_c y, a_c z, a_c w$ are in G_1 (G_2, respectively), then by the construction of the gadget $\mathcal{F}_c$ and Lemma 3.7.2, the degree of the vertex t_c in the graph G_2 (G_1, respectively) is 4 (4, respectively). This is a contradiction. Similarly, let $c = y \vee z$ be an arbitrary clause, if all edges $b_c y, b_c z$ are in G_1 (G_2, respectively), then by the construction of the gadget $\mathcal{D}_c$, the degree of the vertex t_c in the graph G_2 (G_1, respectively) is 4. This is a contradiction. Hence, Γ is an NAE satisfying assignment. On the other hand, suppose that Φ is NAE satisfiable with the satisfying assignment $\Gamma : X \to \{true, false\}$. For every variable $x \in X$, put all edges incident with the the vertex x in G_1 if and only if $\Gamma(x) = true$. By this method, it is easy to show that G can be partitioned into two semiregular subgraphs. $\diamondsuit$

3.8 Proof of Theorem 3.2.5

Let G be a graph. We say that an edge-labeling $\ell : E(G) \to \mathbb{N}$ is an additive vertex-colorings if and only if for each edge uv, the sum of labels of the edges incident to u is different from the sum of labels of the edges incident to v. It was shown that determining whether a given 3-regular graph G has an edge-labeling which is an additive vertex-coloring from $\{1, 2\}$ is **NP**-complete [3]. For a given 3-regular graph G, it is easy to see that G has an edge-labeling which is an additive vertex-coloring from $\{1, 2\}$ if and only if the edge set of G can be partitioned into at most two locally irregular subgraphs. Thus, determining whether a given 3-regular graph G can be decomposed into two locally irregular subgraphs is **NP**-complete [3]. We will reduce this problem to our problem. Let G be a 3-regular graph. We construct a graph G' such that G can be partitioned into two locally irregular subgraphs if and only if G' can be partitioned into 2 subgraphs, such that each subgraph is locally irregular or weakly semiregular. Let $G' = G \cup C_4 \cup P_5 \cup K_{9,9} \bigcup_{i=4}^{8} K_{1,i}$.

The degree set of G' is $D = \{j : 1 \leq j \leq 9\}$, so $|D| \geq 9$, thus G' cannot be partitioned into two weakly semiregular subgraphs. Now, assume that G' can be partitioned into two subgraphs $\mathcal{I}$ and $\mathcal{R}$ such that $\mathcal{I}$ is locally irregular and $\mathcal{R}$ is (α, β)-graph. The graph G' contains a copy of C_4, thus $2 \in \{\alpha, \beta\}$. Also, G' contains a copy of P_5, thus $1 \in \{\alpha, \beta\}$. Hence $\mathcal{R}$ is a $(1, 2)$-graph. Note that $K_{9,9}$ cannot be partitioned into two subgraphs $\mathcal{I}$ and $\mathcal{R}$ such that $\mathcal{I}$ is locally irregular and $\mathcal{R}$ is $(1, 2)$-graph. Thus, G' cannot be partitioned into two subgraphs $\mathcal{I}$ and $\mathcal{R}$ such that $\mathcal{I}$ is locally irregular and $\mathcal{R}$ is weakly semiregular. On the other hand, it is easy to see that the graph $C_4 \cup P_5 \cup K_{9,9} \bigcup_{i=4}^{8} K_{1,i}$ can be partitioned into two locally irregular subgraphs. Therefore, G can be partitioned into two locally irregular subgraphs if and only if G' can be partitioned into 2 subgraphs, such that each subgraph is locally irregular or weakly semiregular. $\diamondsuit$

3.9 Proof of Theorem 3.3.1

(i) Let $\epsilon > 0$ be a fixed number and G be a 3-regular graph with sufficiently large number of vertices in terms of ϵ. Construct the graph H from the graph G by replacing every edge ab of G by a copy of the gadget $I(a, b)$ which is shown in Fig. 3.4. It was shown that it is **NP**-complete to determine whether the edge chromatic number of a cubic graph is three [30]. Assume that the number of vertices in the graph H is n. We show that if $\chi'(G) = 3$

then $rep(H^c) \leq (1+\epsilon)(\frac{n}{2})^3$ and if $\chi'(G) > 3$ then $rep(H^c) \geq (\frac{n}{2})^4$, consequently, there is no polynomial time θ-approximation algorithm for computing $rep(A^c)$ when

$$\frac{(\frac{n}{2})^4}{(1+\epsilon)(\frac{n}{2})^3} = \frac{n}{2(1+\epsilon)} > (1-\epsilon)\frac{n}{2} = \theta.$$

By the structure of the gadget $I(a,b)$, the graph H is 3-regular and triangle-free, also by the structure of H, $\chi'(G) = \chi'(H)$. Let $a = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$, set $rad(a) := p_1 \ldots p_k$. Let H be a triangle-free graph and $rep(H^c) = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$, and let $f(i)$ be the label for the vertex i. For each i, let $g(i) := rad(f(i))$, and let $n = rad(p_1^{\alpha_1} \ldots p_d^{\alpha_d})$. One can check that function g is a representation labeling. Then $rep(H^c)$ is square-free. Let $rep(H^c) = \prod_{i=1}^{d} p_i$, where for each i, $i = 1, \ldots, d$, p_i is a prime number. Assume that $r : V(H^c) \longrightarrow \mathbb{Z}_{rep(H^c)}$ is a representation for H^c. For each vertex v of H^c define a d-triple $(r_v^1, \ldots, r_v^d) \in \prod_{i=1}^{d} \mathbb{Z}_{p_i}$ such that $r_v^i = (r(v) \mod p_i)$. By the definition of the function r, vw is an edge in H if and only if there exists an index i such that $r_v^i = r_w^i$. For each edge $e = xy$ of H, define $S(e) = \{i : r_x^i = r_y^i\}$. So for each edge e, $S(e)$ is non-empty and since the graph H is triangle-free, for every two incident edges e and e' we have $S(e) \cap S(e') = \emptyset$. Let $c : E(H) \longrightarrow \{1, \ldots, d\}$ be a function such that $c(e) = \min S(e)$. It is clear that c is a proper edge coloring for the graph H. So

$$d \geq \chi'(H) \tag{3.2}$$

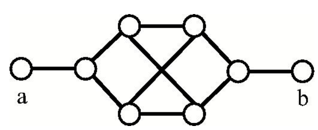


FIGURE 3.4: The auxiliary graph $I(a,b)$.

Define $M_i = \{e \in E(H), i \in S(e)\}$ for every i, $1 \leq i \leq d$. The set M_i contains all edges of the graph H like $e = vu$ such that v and u have a same i-th component. Since the graph H is triangle-free, it follows that the set of edges M_i is a matching. Also, since H^c is a triangle-free graph, each $z \in \mathbb{Z}_{p_i}$ appears at most 2 times as the i-th component of vertices in the graph H. Also, each vertex of H which is not adjacent to any vertex of M_i, has a unique i-th component. For each i denote the number of edges of M_i by m_i (note that $|V(H)| = |V(H^c)| = n$). We have:

$$p_i \geq m_i + (n - 2m_i) = n - m_i. \tag{3.3}$$

Also, since every matching has at most $\frac{n}{2}$ edges, it follows that

$$p_i \geq \frac{n}{2}. \tag{3.4}$$

Now, let $\chi'(H) = 3$ and $f : E(H) \longrightarrow \{1, 2, 3\}$ be a proper edge coloring of H. The edges of H can be partitioned into three perfect matchings f_1, f_2 and f_3, where $f_i = \{e : f(e) = i\}$. For each i, $i = 1, 2, 3$, label $f_i = \{e_i^1, \ldots, e_i^{\frac{n}{2}}\}$.

It follows from the prime number theorem that for any real $\alpha > 0$ there is a $n_0 > 0$ such that for all $n' > n_0$ there is a prime p such that $n' < p < (1 + \alpha)n'$ (see [29] page 494). Thus for a sufficiently large number n, there are three prime numbers p_1, p_2, p_3 such that $\frac{n}{2} \leq p_1 < p_2 < p_3 < \frac{n}{2}(1+\epsilon)^{\frac{1}{3}}$. For every vertex v of the graph H, call the edges incident with the vertex v, e_1^{α}, e_2^{β} and e_3^{γ} and let $\psi : V(H^c) \longrightarrow \mathbb{Z}_{p_1 p_2 p_3}$ be a function such that $\psi(v) = (\alpha, \beta, \gamma)$. Clearly, this is a representation, so $rep(H^c) \leq p_1 p_2 p_3 < (1 + \epsilon)(\frac{n}{2})^3$. On the other side, assume that $\chi'(G) > 3$, so $\chi'(H) > 3$. Thus, we have:

$$\begin{aligned} rep(H^c) &= \prod_{i=1}^{d} p_i \\ &\geq \prod_{i=1}^{4} p_i && \text{By Equation (3.2),} \\ &\geq (\frac{n}{2})^4 && \text{By Equation (3.4),} \end{aligned}$$

This completes the proof.

(ii) Let G^c be a triangle-free r-regular graph. By Vizing's theorem [45], the chromatic index of G^c is equal to either $\Delta(G^c)$ or $\Delta(G^c) + 1$. Thus, for every r-regular graph G^c, $r \leq \chi'(G^c) \leq r + 1$. Therefore, the set of edges of G^c can be partitioned into $r + 1$ matchings $M_1, \ldots, M_{r+1}$. By an argument similar to the proof of part (i), we have:

$$\begin{aligned} \text{rep}(G) &\leq (1 + \epsilon) \prod_{i=1}^{r+1} (n - |M_i|) \\ &\leq (1 + \epsilon)(n - \frac{rn}{2(r+1)})^{r+1} && \text{By Equation (3.3),} \\ &\leq (1 + \epsilon)e(\frac{n}{2})^{r+1} && \text{By inequality } (1 + \frac{1}{x})^x < e. \end{aligned}$$

On the other hand:

$$\text{rep}(G) \geq (\frac{n}{2})^r.$$

Therefore we have a polynomial time $(1+\epsilon)\frac{e}{2}n$ approximation algorithm for computing $rep(G)$. $\diamondsuit$

3.10 Concluding remarks and future work

3.10.1 Trees

We proved that for every tree T, $\mathrm{wr}(T) \leq 2\log_2 \Delta(T) + \mathcal{O}(1)$. On the other hand, there are infinitely many values of Δ for which the tree T might be chosen so that $\mathrm{wr}(T) \geq \log_3 \Delta(T)$. Finding the best upper bound for trees can be interesting.

Problem 3.10.1. *Find the best upper bound for the weakly semiregular numbers of trees in terms of the maximum degree.*

We proved that there is an $\mathcal{O}(n^2)$ time algorithm to determine whether the weakly semiregular number of a given tree is two. Also, if c is a constant number, then there is a polynomial time algorithm to determine whether the weakly semiregular number of a given tree is at most c. However, one further step does not seem trivial. Is there any polynomial time algorithm to determine the weakly semiregular number of trees?

Problem 3.10.2. *Is there any polynomial time algorithm to determine the weakly semiregular number of trees?*

In this work we present an algorithm with running time $\mathcal{O}(n^2)$ to determine whether the weakly semiregular number of a given tree is at most two. Is there any algorithm with running time $\mathcal{O}(n \log n)$ for this problem?

Problem 3.10.3. *Is there any algorithm with running time $o(n^2)$ for determining whether the weakly semiregular number of a given tree is at most two?*

3.10.2 Planar graphs

Balogh *et al.* proved that a planar graph can be partitioned into three forests so that one of the forests has maximum degree at most 8 [10]. On the other hand, we proved that for every tree T, $\mathrm{wr}(T) \leq 2\log_2 \Delta(T) + \mathcal{O}(1)$. Thus, for every planar graph G, we have $\mathrm{wr}(G) \leq 4\log_2 \Delta(G) + \mathcal{O}(1)$. Also, it was shown that every planar graph with girth

$g \geq 6$ has an edge partition into two forests, one having maximum degree at most 4 [28]. Thus, for every planar graph G with girth $g \geq 6$, we have $\mathrm{wr}(G) \leq 2\log_2 \Delta(G) + \mathcal{O}(1)$. Finding a good upper bound for all planar graphs can be interesting.

Problem 3.10.4. *Is this true "For every planar graph G, we have $\mathrm{wr}(G) \leq 2\log_2 \Delta(G) + \mathcal{O}(1)$"?*

3.10.3 Representation Number

In this work, we proved that if $\mathbf{NP} \neq \mathbf{P}$, then for any $\epsilon > 0$, there is no polynomial time $(1-\epsilon)\frac{n}{2}$-approximation algorithm for the computation of representation number of regular graphs with n vertices. In 2000 it was shown by Evans, Isaak and Narayan [25] that if $n, m \geq 2$, then $rep(nK_m) = p_i p_{i+1} \dots p_{i+m-1}$ where p_i is the smallest prime number greater than or equal to m if and only if there exists a set of $n-1$ mutually orthogonal Latin squares of order m. It is interesting to investigate what our result implies about the Orthogonal Latin Square Conjecture (there exists $n-1$ mutually orthogonal Latin squares of order n if and only if n is a prime power). That is, can our reduction be extended from regular graphs to just nK_m?

Bibliography

[1] A. Ahadi, A. Dehghan, *The inapproximability for the (0,1)-additive number*, Discrete Math. Theor. Comput. Sci. 17(3) (2016) 217–226.

[2] A. Ahadi, A. Dehghan, M. Kazemi, E. Mollaahmadi, *Computation of lucky number of planar graphs is NP-hard*, Inform. Process. Lett. 112(4) (2012) 109–112.

[3] A. Ahadi, A. Dehghan, M.-R. Sadeghi, *Algorithmic complexity of proper labeling problems*, Theoret. Comput. Sci. 495 (2013) 25–36.

[4] A. Ahadi, A. Dehghan, M.-R. Sadeghi, *On the complexity of deciding whether the regular number is at most two*, Graphs Combin. 31(5) (2015) 1359–1365.

[5] A. Ahadi, A. Dehghan, M.-R. Sadeghi, B. Stevens, *Partitioning the edges of graph into regular, locally regular and/or locally irregular subgraphs is hard no matter which you choose*, 2015, submitted for publication.

[6] R. Akhtar, A.B. Evans, D. Pritikin, *Representation numbers of complete multipartite graphs*, Discrete Math. 312(6) (2012) 1158–1165.

[7] R.P. Anstee, *An algorithmic proof of Tutte's f-factor theorem*, J. Algorithms 6(1) (1985) 112–131.

[8] C. Balbuena, D. González-Moreno, X. Marcote, *On the connectivity of semiregular cages*, Networks 56(1) (2010) 81–88.

[9] C. Balbuena, X. Marcote, D. González-Moreno, *Some properties of semiregular cages*, Discrete Math. Theor. Comput. Sci. 12(5) (2010) 125–138.

[10] J. Balogh, M. Kochol, A. Pluhár, X. Yu, *Covering planar graphs with forests*, J. Combin. Theory Ser. B 94(1) (2005) 147–158.

[11] E. Barme, J. Bensmail, J. Przybyło, M. Woźniak, *On a directed variation of the 1–2–3 and 1–2 Conjectures*, Discrete Appl. Math. 217(part 2) (2017) 123–131.

[12] O. Baudon, J. Bensmail, J. Przybyło, M. Woźniak, *On decomposing regular graphs into locally irregular subgraphs*, European J. Combin. 49 (2015) 90–104.

[13] O. Baudon, J. Bensmail, E. Sopena, *On the complexity of determining the irregular chromatic index of a graph*, J. Discrete Algorithms 30 (2015) 113–127.

[14] P. Bennett, A. Dudek, A. Frieze, L. Helenius, *Weak and strong versions of the 1–2–3 conjecture for uniform hypergraphs*, Electron. J. Combin. 23(2) (2016), Paper 2.46, 21.

[15] J. Bensmail, Partitions and Decompositions of Graphs, PhD thesis, Université de Bordeaux, 2014.

[16] J. Bensmail, M. Merker, C. Thomassen, *Decomposing graphs into a constant number of locally irregular subgraphs*, European J. Combin. 60 (2017) 124–134.

[17] J. Bensmail, B. Stevens, Decomposing Graphs into Locally Irregular Subgraphs: Allowing k2's Helps a Lot, Bordeaux Graph Workshop, France, 2014, p.103.

[18] J. Bensmail, B. Stevens, *Edge-partitioning graphs into regular and locally irregular components*, Discrete Math. Theor. Comput. Sci. 17(3) (2016) 43–58.

[19] F.R.K. Chung, *On the decomposition of graphs*, SIAM J. Algebr. Discrete Methods 2(1) (1981) 1–12.

[20] A. Dehghan, M. Mollahajiaghaei, *On the algorithmic complexity of adjacent vertex closed distinguishing colorings number of graphs*, Discrete Appl. Math. 218 (2017) 82–97.

[21] A. Dehghan, M.-R. Sadeghi, Colorful *edge decomposition of graphs: some polynomial cases*, Discrete Appl. Math. (2016), http://dx.doi.org/10.1016/j.dam.2016.10.019.

[22] A.A. Diwan, J.E. Dion, D.J. Mendell, M.J. Plantholt, S.K. Tipnis, *The complexity of P4-decomposition of regular graphs and multigraphs*, Discrete Math. Theor. Comput. Sci. 17(2) (2015) 63–75.

[23] Dorit Dor, Michael Tarsi, *Graph decomposition is NP-complete: a complete proof of Holyer's conjecture*, SIAM J. Comput. 26(4) (1997) 1166–1187.

[24] P. Erdős, A.B. Evans, *Representations of graphs and orthogonal Latin square graphs*, J. Graph Theory 13(5) (1989) 593–595.

[25] A.B. Evans, G.H. Fricke, C.C. Maneri, T.A. McKee, M. Perkel, *Representations of graphs modulo n*, J. Graph Theory 18(8) (1994) 801–815.

[26] A.B. Evans, G. Isaak, D.A. Narayan, *Representations of graphs modulo n*, Discrete Math. 223(1–3) (2000) 109–123.

[27] A. Ganesan, R.R. Iyer, *The regular number of a graph*, J. Discrete Math. Sci. Cryptogr. 15(2–3) (2012) 149–157.

[28] D. Gonçalves, *Covering planar graphs with forests, one having bounded maximum degree*, J. Combin. Theory Ser. B 99(2) (2009) 314–322.

[29] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, fifth edition, The Clarendon Press, Oxford University Press, New York, 1979.

[30] I. Holyer, *The NP-completeness of edge-coloring*, SIAM J. Comput. 10(4) (1981) 718–720.

[31] I. Holyer, *The NP-completeness of some edge-partition problems*, SIAM J. Comput. 10(4) (1981) 713–717.

[32] D. Kiani, M. Mollahajiaghaei, *On the unitary Cayley graphs of matrix algebras*, Linear Algebra Appl. 466 (2015) 421–428.

[33] D.G. Kirkpatrick, P. Hell, *On the complexity of general graph factor problems*, SIAM J. Comput. 12(3) (1983) 601–609.

[34] V.R. Kulli, B. Janakiram, R.R. Iyer, *Regular number of a graph*, J. Discrete Math. Sci. Cryptogr. 4(1) (2001) 57–64.

[35] Charles C. Lindner, E. Mendelsohn, N.S. Mendelsohn, Barry Wolk, *Orthogonal Latin square graphs*, J. Graph Theory 3(4) (1979) 325–338.

[36] B. Lužar, J. Przybyło, R. Soták, *New bounds for locally irregular chromatic index of bipartite and subcubic graphs*, arXiv preprint arXiv:1611.02341, 2016.

[37] D. Narayan, *An upper bound for the representation number of graphs with fixed order*, Integers 3(4) (2003) A12.

[38] D. Narayan, J. Urick, *Representations of split graphs, their complements, stars, and hypercubes*, Integers 7(13) (2007) A9.

[39] J. Petersen, *Die Theorie der regulären Graphs*, Acta Math. 15(1) (1891) 193–220.

[40] M.D. Plummer, *Graph factors and factorization: 1985–2003: a survey,* Discrete Math. 307(7–8) (2007) 791–821.

[41] J. Przybyło, *On decomposing graphs of large minimum degree into locally irregular subgraphs*, Electron. J. Combin. 23(2) (2016), Paper 2.31, 13.

[42] J. Przybyło, M. Woźniak, *On a 1, 2 conjecture*, Discrete Math. Theor. Comput. Sci. 12(1) (2010) 101–108.

[43] B. Seamone, *The 1–2–3 conjecture and related problems: a survey*, arXiv:1211.5122.

[44] B. Seamone, B. Stevens, *Sequence variations of the 1–2–3 conjecture and irregularity strength*, Discrete Math. Theor. Comput. Sci. 15(1) (2013) 15–28.

[45] V.G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz No. 3 (1964) 25–30.

[46] D.B. West, Introduction to Graph Theory, Prentice Hall Inc., Upper Saddle River, NJ, 1996.

Chapter 4

Resonance varieties of sparse paving matroids

4.1 Introduction

Let $A = (A^{\bullet}, d)$ be a commutative, differential graded algebra (or simply CDGA) over the complex numbers. So $A = \bigoplus_{i \geq 0} A^i$ is a graded $\mathbb{Q}$-vector space, with a multiplication map $\cdot\colon A^i \otimes A^j \to A^{i+j}$ where $u \cdot v = (-1)^{ij} v \cdot u$, and a differential $d\colon A^i \to A^{i+1}$ where $d(u \cdot v) = du \cdot v + (-1)^i u \cdot dv$, for all $u \in A^i$ and $v \in A^j$.

We will assume that $A^0 = \mathbb{Q}$, and A^i is finite-dimensional, for all $i \geq 0$. So we can identify the vector space $H^1(A) = Z^1(A)/B^1(A)$ with the cocycle space $Z^1(A)$. For each element $a \in Z^1(A) \cong H^1(A)$, we have the following cochain complex,

$$(A^{\bullet}, \delta_a)\colon \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots , \tag{4.1}$$

where $\delta_a^i(u) = a \cdot u + du$, for all $u \in A^i$. It is easy to see that $\delta_a^{i+1} \delta_a^i(u) = 0$.

For each integer $i \geq 0$, we can define the degree-i resonance variety as:

$$R^i(A) = \{a \in H^1(A) \mid H^i(A^{\bullet}, \delta_a) \neq 0\}. \tag{4.2}$$

By [8], we have:

$$R^i(A \otimes A') \subseteq \bigcup_{p+q=i} R^p(A) \times R^q(A'). \tag{4.3}$$

Moreover, if the differentials of both A and A' are zero, we have equality in the above product formula.

The study of resonance varieties has led to interesting connections with other branches of mathematics. For example, generalized Cartan matrices [6], Latin squares [10] and the Bethe Ansatz [1].

The main motivation to the try to find the resonance varieties comes from the tangent cone formula which relates the degree-one resonance varieties to the characteristic varieties of G, where G is a finitely presented 1-formal group.

Let M be a matroid (or any combinatorial object like graph). It is of interest and an interesting research topic to find the resonance varieties of M in terms of the combinatorial data coming from the M. Falk and Yuzvinsky [2], have given a characterization of $R^1(M)$. In degree greater than 1, there has been some works, for example [4], but little is known. For instance, Papadima and Suciu in [8], proved that for the sum of two matroids M_1 and M_2 we have

$$R^k(M_1 \oplus M_2) = \bigcup_{p+q=k} R^p(M_1) \times R^q(M_2).$$

This chapter is organized as follows. Section 2 presents some preliminaries and some basic properties. In this section we show that for a given $a \in A^1$ and a sparse paving matroid M of rank r, a belongs to $R^{r-2}(M)$, if and only if the map f_a is not injective. In Section 3, we proceed with the study of f_a. We correspond a matrix to this map. We find basic properties of this matrix. In the last section we will look more closely at the matrix expression of f_a. We drive some interesting results. For example, we show that if the rank of M is large enough in comparison to the number of minimum circuits, then $R^{r-2}(M)$ is trivial. We have a number of reduction theorems. Finally, we provide some examples, which generalizes of current known examples.

4.2 Background and basic properties

A matroid M is a pair (E, I), where E is a finite set (called the *ground set*) and I is a family of subsets of E (called the *independent sets*) with the following properties:

- The empty set is independent.

- Every subset of an independent set is independent.
- If A and B are two independent sets and $|A| > |B|$, then there exists $x \in B - A$ such that $B \cup \{x\}$ is in I.

A subset of the ground set E that is not independent is called *dependent*. A maximal independent set is called a *basis*. A *circuit* in a matroid M is a minimal dependent subset. It is known that, any two bases of a matroid of M have the same number of elements. This number is called the *rank* of M.

Let M be a matroid on the ground set $[n] := \{1, 2, \ldots, n\}$, and let $V = \mathbb{Q}^n$, a vector space with a basis $e_1, \ldots, e_n$. Let ∂ be the derivation on $E := \Lambda(V)$ defined by $\partial(e_i) = 1$ for all $1 \leq i \leq n$. Let I be generated by

$$\{\partial(e_C) \colon \text{circuits } C \subseteq [n] \text{ of } M\}, \tag{4.4}$$

where $e_C := \prod_{i \in C} e_i$, with indices are in increasing order. The Orlik-Solomon algebra $A := A(M)$ is the quotient of an exterior algebra E by an ideal $I = I(M)$ generated by homogeneous relations indexed by circuits in M. The algebra E is graded. Let $\overline{I}^i = \overline{I} \cap E_i$, where $\overline{I}$ has the same generators as I. The projective Orlik-Solomon algebra $\overline{A}$ is defined as follows. Let $\overline{E}$ be the subset of E generated by all differences $e_i - e_j$. Then we set $\overline{A} := \dfrac{\overline{E}}{\overline{I}}$.

Since A is a quotient of an exterior algebra, multiplication by an element $a \in E^1$ gives a degree one differential on A, yielding a cochain complex:

$$A^0 \longrightarrow A^1 \longrightarrow \cdots \longrightarrow A^r \longrightarrow \cdots$$

We denote this complex by (A, a), and its cohomology by $H^*(A, a)$. Let $p \geq 0$. The degree-p resonance variety of A is the set

$$R^p(A) = \{a \in A^1 \mid H^p(A, a) \neq 0\}.$$

Clearly, $H^*(A, a) = H^*(A, ca)$, for all $c \in \mathbb{Q}^\times$. Thus, each resonance variety $R^p(A)$ is homogeneous. Also, it is known that (see [3]), if M is a matroid of rank r, then

$$\{0\} \subseteq R^0(M) \subseteq \cdots \subseteq R^r(M) \subseteq \overline{V}, \tag{4.5}$$

where $\overline{V} = \{\sum_{i=1}^{n} \alpha_i e_i : \sum_{i=1}^{n} \alpha_i = 0\}$.

A *paving matroid* is a matroid in which every circuit has size at least as large as the matroid's rank. A *sparse paving* matroid is a paving matroid in which its dual is a paving matroid. It has been conjectured ([7]) that almost all matroids are sparse paving matroids, i.e. that $\lim_{n\to\infty} \frac{s_n}{m_n} = 1$, where m_n denotes the number of matroids on n elements, and s_n the number of sparse paving matroids. Pendavingh and van der Pol [9] proved that $\lim_{n\to\infty} \frac{\log s_n}{\log m_n} = 1$. So it seems that any property of sparse paving matroid is a property of almost all matroids.

In view of 4.5, the first thing we show is that $R^1(M) = \cdots = R^{r-3}(M) = \{0\}$ and we find $R^{r-2}(M)$.

Proposition 4.2.1. *Let M be a paving matroid of rank r. Then $R^1(M) = \cdots = R^{r-3}(M) = \{0\}$ and $R^{r-2}(M) = \{a \mid f_a$ is not injective$\}$, where $f_a : \overline{I}^{r-1} \to \overline{I}^r$ be defined by left multiplication by a.*

Proof. Since the rank of M is r and every circuit has size at least r, we have $R^1(M) = \cdots = R^{r-3}(M) = \{0\}$.

For any non-zero $a \in E^1$, there is a short exact sequence of complexes of E-modules,

$$0 \longrightarrow (I, a) \longrightarrow (E, a) \longrightarrow (A, a) \longrightarrow 0.$$

This gives a long exact sequence in cohomology and $H^p(A, a) \cong H^{p+1}(I, a)$ for all p, since $H^\bullet(E, v) = 0$. Similarly, since $\overline{A} = \frac{\overline{E}}{\overline{I}}$ is the projective Orlik-Solomon algebra, we have $H^p(\overline{A}, a) \cong H^{p+1}(\overline{I}, a)$ so we have $H^{r-2}(\overline{A}, a) \cong H^{r-1}(\overline{I}, a) = \{u \in \overline{I}^{r-1} \mid ua = 0\}$. Then $R^{r-2}(M) = \{a \mid \overline{I}^{r-1} \to \overline{I}^r$ is not injective$\}$. Then $a \in R^{r-2}(M)$ if and only if f_a is not injective. ∎

Let E be a finite set, and let $0 < r < |E|$. The *Johnson graph* $J(E, r)$ is the graph with vertex set

$$\binom{E}{r} := \{X \subseteq E : \ |X| = r\},$$

in which any two vertices are adjacent if and only if they have $r-1$ elements in common; equivalently, the vertices X and Y are adjacent whenever $|X \triangle Y| = 2$.

Proposition 4.2.2. [5] The Johnson graph $J(n, r)$ has $\binom{n}{r}$ vertices, and

(i) is regular of valency $r(n-r)$,

(ii) with eigenvalues $(n-i)(n-r-i)-i$ with multiplicity $\binom{n}{i}-\binom{n}{i-1}$ $(0 \leq i \leq r)$. In particular, the smallest eigenvalue of $J(n,r)$ is $-r$.

(iii) $J(n,r) \cong J(n,n-r)$.

Given a simple graph $G=(V,E)$ a subset of vertices $S \subset V$ is an independent set if and only if there is no edge in E between any two vertices in S.

The next well-known theorem, establishes the relation between Johnson graphs and sparse paving matroids.

Theorem 4.2.3. [9] Let $B \subseteq \binom{E}{r}$ is the collection of bases of a sparse paving matroid if and only if $\binom{E}{r}-B$ is an independent set in $J(n,k)$.

Let B be a subset of $[n]$. Write $sgn_B^i := (-1)^{|\{b \in B; i<b\}|}$. For simplicity of notation, we write $\partial(A)$ or $\partial(e_A)$ instead of $\partial(\prod_{i\in A} e_i)$. Let M be a paving matroid of rank r, thus $R^1(M)=\cdots=R^{r-3}(M)=\{0\}$, so here when we say resonance variety we mean $R^{r-2}(M)$. We say $R^{r-2}(M)$ is trivial if $R^{r-2}(M)$ has no non-local component.

4.3 Matrix expression

In this section, we write the map f_a (Proposition 4.2.1) as a matrix. Then we find basic properties of this matrix.

Proposition 4.3.1. *Let M be a paving matroid on the ground set $[n]$ of rank r. Then $\overline{I}^{r-1}$ has a generating set indexed by minimum circuits. If M is a sparse paving matroid, then $\dim \overline{I}^r = \binom{n-1}{r}$ and $\mathcal{A}=\{\partial(A); |A|=r+1 \text{ and } 1 \in A\}$ is a basis of $\overline{I}^r$.*

Proof. Let $\{C_1,\ldots,C_t\}$ be the set of minimum circuits of M. First we prove that $\mathcal{A}$ is an independent set. Let $\sum c_i\partial(C_i)=0$. Each $\partial(C_i)$ has exactly one term without e_1. For $1 \leq i < j \leq \binom{n-1}{r}$, the term without e_1 in $\partial(C_i)$ is different with the term without e_1 in $\partial(C_j)$. Then $c_1=\cdots=c_{\binom{n-1}{r}}=0$.

Now, we show that $\mathcal{A}$ is a generating set for $\overline{I}^r$. Let A be a subset of $[n]$ of cardinality $r+1$, such that $1 \notin A$. We know that $\partial(\partial(\prod_{i\in A\cup\{1\}} e_i))=0$. Then

$$\partial(\sum_{j\in A\cup\{1\}} sgn^j_{A\cup\{1\}} \prod_{i \in A\cup\{1\}-\{j\}} e_i)=0.$$

Therefore, $\partial(A)$ can be written as a linear combination of elements of $\mathcal{A}$.

Now, let M be a sparse paving matroid. It is enough to show that $\partial(C_1), \ldots, \partial(C_t)$ are independent. Let $\sum_{i=1}^{t} c_i \partial(C_i) = 0$. Without loss of generality, we can assume that $c_1 \neq 0$. Let $a \in C_1$. Then there exists $1 \neq i$ and $b \in C_i$ such that $C_1 - \{a\} \neq C_i - \{b\}$ which contradicts the fact that $|C_1 \cap C_i| \leq r-2$. □

Theorem 4.3.2. *Let M be a sparse paving matroid of rank r with minimum circuits $\{A_1, \ldots, A_t\}$ such that for $1 \leq i < j \leq t$, $|A_i \cap A_j| < r-2$. Then $R^{r-2}(M)$ is trivial.*

Proof. Let $a = \sum_{i=1}^{n} \alpha_i e_i \in R^{r-2}(M)$. Then there exists $c_1, \ldots, c_t$ such that $a \cdot \sum_{i=1}^{t} c_i \partial(A_i) = 0$. Since for $1 \leq i < j \leq t$, $|A_i \cap A_j| < k-2$, it follows that there is no common term in $c_i a \cdot \partial(A_i)$ and $c_j a \cdot \partial(A_j)$. Hence $c_1 a \cdot \partial(A_1) = \cdots = c_t a \cdot \partial(A_t) = 0$. □

Let $a = \sum_{i=1}^{n} \alpha_i e_i$ and $A \in \binom{E}{r}$. The next proposition, computes the expression of $a \cdot \partial(A)$ as a linear combination of $\mathcal{A}$, where $1 \in A$.

Proposition 4.3.3. *Let $a = \sum_{i=1}^{n} \alpha_i e_i$ and $A \in \binom{E}{r}$ such that $1 \notin A$. Then*

$$a \cdot \partial(A) = \sum_{i \notin A} (-1)^r sgn_A^i \alpha_i \partial(A \cup \{i\}). \tag{4.6}$$

Proof. Let $A = \{i_1 < i_2 < \cdots < i_r\}$. Then we have

$$a \cdot \partial(A) = (\sum_{i \in A} \alpha_i e_i) \cdot \partial(A) + (\sum_{i \notin A} \alpha_i e_i) \cdot \partial(A).$$

Hence

$$\begin{aligned} a \cdot \partial(A) &= (-1)^{r-1} (\sum_{i \in A} \alpha_i)(\prod_{i \in A} e_i) + (\sum_{i \notin A} \alpha_i e_i)(\sum_{j \in A} sgn_A^j \prod_{l \in A, l \neq j} e_l) \\ &= (-1)^r (\sum_{i \notin A} \alpha_i)(\prod_{i \in A} e_i) + (\sum_{i \notin A} \alpha_i e_i)(\sum_{j \in A} sgn_A^j \prod_{l \in A, l \neq j} e_l) \\ &= (-1)^r (\sum_{i \notin A} \alpha_i)(\prod_{i \in A} e_i) + (\sum_{i \notin A} \sum_{j \in A} \alpha_i sgn_A^j e_i \prod_{l \in A, l \neq j} e_l). \end{aligned}$$

Thus $a \cdot \partial(A)$ equals to

$$(-1)^r (\sum_{i \notin A} \alpha_i)(\prod_{i \in A} e_i) + (\sum_{i \notin A} \sum_{j \in A} \alpha_i sgn_A^j (-1)^{r-1} sgn_{A-\{j\}}^i \prod_{l \in A \cup \{i\}, l \neq j} e_l). \tag{4.7}$$

A standard computation shows that

$$(-1)^r sgn^i_A sgn^j_{A\cup\{i\}} = (-1)^{r-1} sgn^j_A sgn^i_{A-\{j\}} \tag{4.8}$$

Thus by (4.7) and (4.8)

$$a\cdot\partial(A) = (-1)^r(\sum_{i\notin A}\alpha_i)(\prod_{i\in A}e_i) + (\sum_{i\notin A}\sum_{j\in A}\alpha_i(-1)^r sgn^i_A sgn^j_{A\cup\{i\}}\prod_{l\in A\cup\{i\},l\neq j}e_l).$$

Then

$$a\cdot\partial(A) = \sum_{i\notin A}(-1)^r sgn^i_A\alpha_i\partial(A\cup\{i\}). \tag{4.9}$$

Assume $1\in A$. In this case, Equation (4.6) is an expression of $a\cdot\partial(A)$ in $\mathcal{A}$. □

The next proposition, computes the expression of $a\cdot\partial(A)$ in $\mathcal{A}$, where $1\notin A$.

Proposition 4.3.4. *Let $a=\sum_{i=1}^n\alpha_ie_i$ and $A\in\binom{E}{r}$ such that $1\notin A$. Then*

$$a\cdot\partial(A) = \alpha_1\partial(A\cup\{1\}) + \sum_{i\notin A,i\neq 1} sgn^i_A\alpha_i\sum_{j\in A\cup\{i\}} sgn^j_{A\cup\{1,i\}}\partial(A\cup\{1,i\}-\{j\}). \tag{4.10}$$

Proof. By Equation (4.6), we have

$$a\cdot\partial(A) = (-1)^r sgn^1_A\alpha_1\partial(A\cup\{1\}) + \sum_{i\notin A,i\neq 1}(-1)^r sgn^i_A\alpha_i\partial(A\cup\{i\}). \tag{4.11}$$

Since $\mathcal{A}$ is a basis for $\overline{I}^r$, we try to write $\partial(A\cup\{i\})$ as a linear combination of elements in $\mathcal{A}$. We have

$$\partial(A\cup\{i\}) = \sum_{j\in A\cup\{i\}} sgn^j_{A\cup\{i\}}\prod_{l\in(A\cup\{i\})-\{j\}}e_l.$$

We know that

$$\partial(\partial(A\cup\{1,i\})) = 0.$$

Then

$$\partial(\sum_{j\in A\cup\{1,i\}} sgn^j_{A\cup\{1,i\}}\prod_{l\in(A\cup\{1,i\})-\{j\}}e_l) = 0.$$

Hence

$$\partial((-1)^{r+1}\prod_{l\in A\cup\{i\}}e_l) + \partial(\sum_{j\in A\cup\{i\}} sgn^j_{A\cup\{1,i\}}\prod_{l\in(A\cup\{1,i\})-\{j\}}e_l) = 0.$$

This means that

$$\partial(A \cup \{i\}) = (-1)^r \partial\Big(\sum_{j \in A\cup\{i\}} sgn^j_{A\cup\{1,i\}} \prod_{l \in (A\cup\{1,i\})-\{j\}} e_l\Big).$$

Thus

$$\partial(A \cup \{i\}) = (-1)^r \sum_{j \in A\cup\{i\}} sgn^j_{A\cup\{1,i\}} \partial\Big(\prod_{l \in (A\cup\{1,i\})-\{j\}} e_l\Big).$$

Finally we have

$$\partial(A \cup \{i\}) = (-1)^r \sum_{j \in A\cup\{i\}} sgn^j_{A\cup\{1,i\}} \partial(A \cup \{1,i\} - \{j\}).$$

Then by Equation (4.11), we have

$$a \cdot \partial(A) = \alpha_1 \partial(A \cup \{1\}) + \sum_{i \notin A, i \neq 1} sgn^i_A \alpha_i \sum_{j \in A\cup\{i\}} sgn^j_{A\cup\{1,i\}} \partial(A \cup \{1,i\} - \{j\}). \quad (4.12)$$

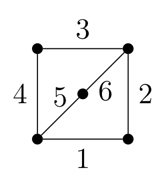
$\square$

Remark 3. Let $1 \notin A$. For the column A, the array in the row $\partial(A \cup \{1\})$ is

$$\alpha_1 + \sum_{i \notin A, i \neq 1} sgn^i_A sgn^i_{A\cup\{1,i\}} \alpha_i = \alpha_1 + \sum_{i \notin A, i\neq 1} \alpha_i = \sum_{i \notin A} \alpha_i.$$

The rest of the entries in the column A are monomials or zero.

Example 1. Let M be the sparse paving matroid on the ground set $\{1, \ldots, 6\}$ with minimum circuits $\{1234, 1256, 3456\}$. It is easy to see that this matroid is the circuit matroid of the graph



Let $a = \sum_{i=1}^{6} \alpha_i e_i$, where $\sum_{i=1}^{6} \alpha_i = 0$. Then

$$A = \begin{pmatrix} \alpha_5 & 0 & \alpha_2 \\ \alpha_6 & 0 & -\alpha_2 \\ 0 & \alpha_3 & \alpha_2 \\ 0 & \alpha_4 & -\alpha_2 \\ 0 & 0 & \alpha_1 + \alpha_2 \end{pmatrix}$$

Hence

$$A^T A = \begin{pmatrix} \alpha_5^2 + \alpha_6^2 & 0 & \alpha_2\alpha_5 - \alpha_2\alpha_6 \\ 0 & \alpha_3^2 + \alpha_4^2 & \alpha_2\alpha_3 - \alpha_2\alpha_4 \\ \alpha_2\alpha_5 - \alpha_2\alpha_6 & \alpha_2\alpha_3 - \alpha_2\alpha_4 & 4\alpha_2^2 + (\alpha_1 + \alpha_2)^2 \end{pmatrix}$$

Then

$$\det(A^T A) = (\alpha_5^2+\alpha_6^2)(\alpha_3^2+\alpha_4^2)(\alpha_1+\alpha_2)^2+\alpha_2^2(\alpha_3^2+\alpha_4^2)(\alpha_5+\alpha_6)^2+\alpha_2^2(\alpha_5^2+\alpha_6^2)(\alpha_3+\alpha_4)^2$$

Therefore $a \in R^2(M)$ if and only if

$$\begin{cases} \alpha_5 = \alpha_6 = 0 \text{ or} \\ \alpha_3 = \alpha_4 = 0 \text{ or} \\ \alpha_1 = \alpha_2 = 0 \text{ or} \\ \alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = \alpha_5 + \alpha_6 = 0. \end{cases}$$

Then the only non-trivial component of $R^2(M)$ is $\{a(e_1 - e_2) + b(e_3 - e_4) + c(e_5 - e_6) \mid a, b, c \in \mathbb{R}\}$. Also one can get the same result by just comparing the columns of A.

Definition 1. *Let A and B be r-subsets of $[n]$, such that $1 \in A - B$. We say there is a two-way step from A to B by (a, b, c) if $A \cup \{a\} = B \cup \{b, 1\} - \{c\}$.*
If $1 \notin A \cup B$, we say there is a special two-way step from A to B by (a, b, c, d) if $A \cup \{1, a\} - \{d\} = B \cup \{b, 1\} - \{c\}$.

Remark 4. If $1 \in A - B$ and $|A \cap B| < r - 2$, then there is no two-way step from A to B. If $|A \cap B| = r - 2$, then there exist exactly 2, two-way step from A to B by (b_1, a, b_2) and (b_2, a, b_1), where $A - B = \{1, a\}$ and $B - A = \{b_1, b_2\}$.

If $1 \notin A \cup B$ and $|A \cap B| < r - 2$, then there is no special two-way step from A to B. If $|A \cap B| = r - 2$, then there exist exactly 4, special two-way step from A to B by (b_1, a_2, b_2, a_1), (b_1, a_1, b_2, a_2), (b_2, a_2, b_1, a_1) and (b_2, a_1, b_1, a_2), where $A - B = \{a_1, a_2\}$ and $B - A = \{b_1, b_2\}$.

Remark 5. Let M be the sparse paving matroid on the ground set [8] with minimum circuits $A_1, \ldots, A_t$ such that $1 \in (\cap_{i=1}^{k} A_i - \cap_{i=k+1}^{t} A_i)$. Let $B := A^T A$. Then B is the following block matrix:

$$\begin{array}{cc} & \begin{array}{cc} A_1, \ldots, A_k & A_{k+1}, \ldots, A_t \end{array} \\ \begin{array}{c} A_1, \ldots, A_k \\ A_{k+1}, \ldots, A_t \end{array} & \left(\begin{array}{c|c} D & E \\ E^t & F \end{array} \right) \end{array}$$

Here D is the diagonal $k \times k$ matrix, $(D)_{ii} = \sum_{j \notin A_i} \alpha_j^2$, and $(E)_{ij}$ is:

$$\begin{cases} \alpha_a(\alpha_b \pm \alpha_c) & \text{if there is a two-way step from } A_i \text{ to } A_j \text{ by } (a, b, c) \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq i \leq k$ and $k+1 \leq j \leq t$. Also, F is a $(t-k) \times (t-k)$ matrix, where

$$(F)_{ij} = \begin{cases} (\alpha_a \pm \alpha_b)(\alpha_c \pm \alpha_d) & \begin{array}{c}\text{if } i \neq j \text{ and there is a special two-way step} \\ \text{from } A_i \text{ to } A_j \text{ by } (a, b, c, d)\end{array} \\ (\sum_{l \in A^c} \alpha_l)^2 + (r-1) \sum_{l \in A^c - \{1\}} \alpha_l^2 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

for $k+1 \leq i, j \leq t$.

Remark 6. If $|C \cap D| < r - 2$ for any distinct C and D, then the matrix $A^T A$ is a diagonal matrix.

Here are some elementary properties of this matrix.

Proposition 4.3.5. *Let M be a sparse paving matroid with minimum circuits $A_1, \ldots, A_t$ of rank r. Then*

(i) If A is a minimum circuit such that $1 \in A$. Then in the column A, there are exactly r non-zero elements.

(ii) If A is a minimum circuit such that $1 \notin A$. Then in the column A, there are exactly $r(n-r-1)+1$ non-zero elements.

(iii) If there exists only one minimum circuit, say C, such that $1 \notin C$, then in each row there exists at most two non-zero elements, which one of them must be from the column C.

(iv) *Let C be a minimum circuit such that $1 \notin C$. Let $1 \neq i \notin C$. Then $\pm\alpha_i$ appears in exactly r places in the column C. To be exact $\pm\alpha_i$ appears in rows $\partial(C \cup \{1, i\} - \{j\})$ for $j \in C$.*

Proof. It follows immediately from the matrix expression of M. □

Proposition 4.3.6. *Let M be a sparse paving matroid of rank r. Let A and B be two minimum circuits of M. Then*

(i) *If $1 \in A \cap B$, then in the matrix expression of M the columns A and B have no intersection.*

(ii) *Let $1 \in A$ and $1 \notin B$. Let $|A \cap B| = r - 2$, $A - B = \{1, a\}$ and $B - A = \{b_1, b_2\}$. Then in the matrix expression of M the columns A and B have two intersections in rows $A \cup \{b_1\} = B \cup \{1, a\} - \{b_2\}$ and $A \cup \{c_2\} = B \cup \{1, a\} - \{b_1\}$.*

$$\begin{array}{r c l}
 & \begin{array}{ccccc} \cdots & A & \cdots & B & \cdots \end{array} & \\
\begin{array}{r} \\ A \cup \{b_1\} \\ \\ A \cup \{b_2\} \\ \\ \end{array} & \left(\begin{array}{ccccc} & \vdots & & \vdots & \\ & \pm\alpha_{b_1} & \cdots & \pm\alpha_a & \\ & \vdots & & \vdots & \\ & \pm\alpha_{b_2} & \cdots & \pm\alpha_a & \\ & \vdots & & \vdots & \end{array}\right) & \begin{array}{l} \\ B \cup \{1, a\} - \{b_2\} \\ \\ B \cup \{1, a\} - \{b_1\} \\ \\ \end{array}
\end{array}$$

(iii) *Let $1 \in A$ and $1 \notin B$. If $|A \cap B| \neq r - 2$, then in the matrix expression of M the columns A and B have no intersection.*

(iv) *Let $1 \notin A \cup B$. Let $|A \cap B| = r - 2$, $A - B = \{a_1, a_2\}$ and $B - A = \{b_1, b_2\}$. Then in the matrix expression of M the columns A and B have four intersections in rows $A \cup \{1, b_1\} - \{a_1\} = B \cup \{1, a_2\} - \{b_2\}$, $A \cup \{1, b_1\} - \{a_2\} = B \cup \{1, a_1\} - \{b_2\}$, $A \cup \{1, b_2\} - \{a_1\} = B \cup \{1, a_2\} - \{b_1\}$ and $A \cup \{1, b_2\} - \{a_2\} = B \cup \{1, a_1\} - \{b_1\}$.*

$$\begin{array}{rcl}
 & \begin{array}{ccccc} \cdots & A & \cdots & B & \cdots \end{array} & \\
\begin{array}{r} \\ A\cup\{1,b_1\}-\{a_1\} \\ \\ A\cup\{1,b_1\}-\{a_2\} \\ \\ A\cup\{1,b_2\}-\{a_1\} \\ \\ A\cup\{1,b_2\}-\{a_2\} \\ \\ \end{array} &
\begin{pmatrix} & \vdots & & \vdots & \\ & \pm\alpha_{b_1} & \cdots & \pm\alpha_{a_2} & \\ & \vdots & & \vdots & \\ & \pm\alpha_{b_1} & \cdots & \pm\alpha_{a_1} & \\ & \vdots & & \vdots & \\ & \pm\alpha_{b_2} & \cdots & \pm\alpha_{a_2} & \\ & \vdots & & \vdots & \\ & \pm\alpha_{b_2} & \cdots & \pm\alpha_{a_1} & \\ & \vdots & & \vdots & \end{pmatrix} &
\begin{array}{l} \\ B\cup\{1,a_2\}-\{b_2\} \\ \\ B\cup\{1,a_1\}-\{b_2\} \\ \\ B\cup\{1,a_2\}-\{b_1\} \\ \\ B\cup\{1,a_1\}-\{b_1\} \\ \\ \end{array}
\end{array}$$

(v) *Let $1 \notin A\cup B$. If $|A\cap B| \neq r-2$, then in the matrix expression of M the columns A and B have no intersection.*

Proof. It follows immediately from the matrix expression of M. □

4.4 Resonance varieties

In this section, we restrict our attention to the matrix expression of f_a to find the resonance varieties of sparse paving matroids. In this section, we show that if the intersection of all of the minimum circuits of M is non-empty, then $R^{r-2}(M)$ is trivial. Also, we find $R^{r-2}(M)$, if the intersection of all of the minimum circuits of M except one of them is non-empty. Among other results, we show that if the rank of M is large enough in comparison to the number of minimum circuits, then $R^{r-2}(M)$ is trivial. We have a number of reduction theorems. For example, we show that if the rank of the incidence matrix of minimum circuits of a matroid is n, then $R^{r-2}(M) = \cup_{i=1}^{t} R^{r-2}(M - A_i)$. Also, we provide some examples which is a generalization of the current known examples.

We first prove that if the intersection of all minimum circuits of M is non-empty, then all resonance varieties of M are trivial.

Theorem 4.4.1. *Let M be a sparse paving matroid with minimum circuits $A_1, \dots, A_t$ of rank r. If $\cap_{i=1}^{t} A_i \neq \emptyset$, Then $R^{r-2}(M)$ is trivial.*

Proof. There is no loss of generality in assuming $1 \in \cap_{i=1}^{t} A_i$. If we prove that $a \cdot \partial(A_1), \ldots, a \cdot \partial(A_t)$ are linearly independent, the assertion follows. Let $1 \leq i < j \leq t$. Since $|A_i \cap A_j| \leq r-2$, it follows that if $u \notin A_i$ and $v \notin A_j$, then $A_i \cup \{u\} \neq A_j \cup \{v\}$. Now Equation (4.6) shows that in the matrix expression of $a \cdot \partial(A_1), \ldots, a \cdot \partial(A_t)$ in $\mathcal{A}$, there is at most one element in each row, which completes the proof. □

Definition 2. *Let $A_1, \ldots, A_t$ be subsets of $[n]$. Set $m_j := |\{A_i; j \in A_i\}|$.*

If there exists an integer j such that $m_j = t$, then by Theorem 4.4.1, the resonance is trivial. We have $\sum_{j=1}^{n} m_j = rt$. Then $\max\{m_j\} \geq rt/n$. We can assume that $m_1 = \max\{m_j\}$. Let $M - A_i$ be the sparse paving matroid on the ground set $[n]$ with minimum circuits $\{A_1, \ldots, A_t\} - \{A_i\}$.

Theorem 4.4.2. *Let M be a sparse paving matroid with minimum circuits $A_1, \ldots, A_t$ of rank r. If for $1 \leq i \leq t$ we have $|A_1 \cap A_i| \neq r-2$, then $R^{r-2}(M) = R^{r-2}(M - A_1)$.*

Proof. It follows immediately from the matrix expression of M. □

Example 2. Let M be the sparse paving matroid on the ground set $\{1, \ldots, 6\}$ with minimum circuits $\{126, 145, 235, 346\}$. The set $\mathcal{A} = \{\partial(1234), \partial(1235), \partial(1236), \partial(1245), \partial(1246), \partial(1256), \partial(1345), \partial(1346), \partial(1356), \partial(1456)\}$ is a basis. Then

$$A = \begin{array}{c} \begin{array}{cccc} 126 & 145 & 235 & 346 \end{array} \\ \begin{pmatrix} 0 & 0 & -\alpha_4 & -\alpha_2 \\ 0 & 0 & \alpha_1+\alpha_4+\alpha_6 & 0 \\ \alpha_3 & 0 & -\alpha_6 & \alpha_2 \\ 0 & -\alpha_2 & -\alpha_4 & 0 \\ \alpha_4 & 0 & 0 & -\alpha_2 \\ \alpha_5 & 0 & \alpha_6 & 0 \\ 0 & -\alpha_3 & \alpha_4 & -\alpha_5 \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_5 \\ 0 & 0 & -\alpha_6 & -\alpha_5 \\ 0 & -\alpha_6 & 0 & \alpha_5 \end{pmatrix} \end{array} \begin{array}{l} \\ \partial(1234) \\ \partial(1235) \\ \partial(1236) \\ \partial(1245) \\ \partial(1246) \\ \partial(1256) \\ \partial(1345) \\ \partial(1346) \\ \partial(1356) \\ \partial(1456) \end{array}$$

From A it is easy to see that the only non-local component of $R^1(M)$ is *span*$\{e_1 + e_3 - e_5 - e_6, e_2 + e_4 - e_5 - e_6\}$.

The next definition is a generalization of Example 1.

Definition 3. *Let M_n be the sparse paving matroid on the ground set $[2n]$ with minimum circuits $\{A_1, \dots, A_n\}$, where $A_i = [2n] - \{2i-1, 2i\}$. The rank of M_n is $2n-2$.*

Let L_M be the union of local components for the matroid M.

Theorem 4.4.3. *Let n be a natural number and $a = \sum_{i=1}^{2n} \alpha_i e_i$, where $\sum_{i=1}^{2n} \alpha_i = 0$. Then $R^{2n-4}(M_n) = L_{M_n} \cup Span_{\mathbb{Q}}\{e_1 - e_2, e_3 - e_4, \dots, e_{2n-1} - e_{2n}\}$.*

Proof. It follows immediately from the matrix expression of M_n. □

Theorem 4.4.4. *Let M be a sparse paving matroid on the ground set $[n]$ with minimum circuits $\{A_1, \dots, A_t\}$ of rank r. If $m \notin \cup_{i=1}^{t} A_i$, then in any non-local component of $R^{r-2}(M)$ we have $\alpha_m = 0$.*

Proof. We can assume that $1 \in \cap_{i=1}^{s} A_i - \cup_{i=s+1}^{t} A_i$. Consider the row $\partial(A_1 \cup \{m\})$. In this row and the column A_1, we have $\pm\alpha_m$. In this row the rest of entries are zero. Then $\alpha_m = 0$. □

If $\max\{m_j\} = t-1$, we have the following theorem.

Theorem 4.4.5. *Let M be a sparse paving matroid on the ground set $[n]$ with minimum circuits $\{A_1, \dots, A_t\}$ of rank r. Suppose $\cap_{i=1}^{t} A_i = \emptyset$ and $1 \in \cap_{i=1}^{t-1} A_i$. Then*

$$R^{r-2}(M) = \Big(\bigcup_{i=1}^{t} R^{r-2}(M - A_i)\Big) \bigcup V, \tag{4.13}$$

where $V \neq \emptyset$ if and only if M is isomorphic to M_s for some s.

Proof. By Theorems 4.4.1 and 4.4.4, we can assume that $\cup_{i=1}^{t} A_i = [n]$ and $\cup_{i=1}^{t} A_i^c = [n]$. If there exists $1 \le i \le t-1$ such that $|A_i \cap A_t| < r-2$, then $R^{r-2}(M)$ is trivial.

Then for all $1 \le i \le t-1$ we have $|A_i \cap A_t| = r-2$. Consider the column A_t. Let $x \in A_t^c$. On the rows $A_t \cup \{1, x\} - \{y\}$ for $y \in A_t$, the entry is $\pm\alpha_x$. Let $A_t - A_i = \{a_i, b_i\}$, and let $A_i - A_t = \{1, c_i\}$. The columns A_t and A_i intersect in only two rows, $A_i \cup \{a_i\} = A_t \cup \{1, c_i\} - \{b_i\}$ and $A_i \cup \{b_i\} = A_t \cup \{1, c_i\} - \{a_i\}$, where for both of rows the entry on the column A_t is $\pm\alpha_{c_i}$. This means that if x is outside of the sets A_t and $\cap(A_i - A_t)$, then $\alpha_x = 0$.

First, we show that if $x \in A_t^c \cap A_1^c$, then $\alpha_x = 0$. Since $x \notin A_1$, on the row $A_1 \cup \{x\}$ and the column A_1, the entry is α_x. On the same row and the column A_t, the entry is zero, because the intersection of two column happens at rows $A_t \cup \{1, c_1\} - \{a_1\}$ and $A_t \cup \{1, c_1\} - \{b_1\}$. Hence $\alpha_x = 0$. Now, if $x \in \cup_{i=1}^{t-1}(A_t^c \cap A_i^c)$, then $\alpha_x = 0$. It is easy to show that $\cup_{i=1}^{t-1}(A_t^c \cap A_i^c) = \left(A_t \cup (\cap_{i=1}^{t-1} A_i)\right)^c$.

Assume that $\left(A_t \cup (\cap_{i=1}^{t-1} A_i)\right)^c \neq \emptyset$. Let $x \in \left(A_t \cup (\cap_{i=1}^{t-1} A_i)\right)^c$. Then $\alpha_x = 0$. Then $x \in A_t^c$ and there exists i such that $x \notin A_i$. Since $\cup_{i=1}^{t} A_i = [n]$, it follows that there exists j such that $x \in A_j$. Thus, $x \in A_j - A_t$. So $A_j - A_t = \{1, x\}$. Now, consider rows $A_t \cup \{1, x\} - \{a_j\} = A_j \cup \{b_j\}$ and $A_t \cup \{1, x\} - \{b_j\} = A_j \cup \{a_j\}$. This means that $\alpha_{a_j} = \alpha_{b_j} = 0$. Then $V = \emptyset$.

Now, we show that if $\left(A_t \cup (\cap_{i=1}^{t-1} A_i)\right)^c = \emptyset$, then M is isomorphic to M_s for some s. Thus $A_t \cup (\cap_{i=1}^{t-1} A_i) = [n]$, and so $A_t^c \subseteq \cap_{i=1}^{t-1} A_i$. Hence $A_t^c \subseteq \cap_{i=1}^{t-1}(A_i - A_t) = \bigcap_{i=1}^{t-1}\{1, c_i\}$. Then it must be $\{1, c\}$. Then $A_t = [n] - \{1, c\}$, and so $A_i = [n] - \{a_i, b_i\}$. Obviously, $\{a_1, b_1\}, \ldots, \{a_{t-1}, b_{t-1}\}, \{1, c\}$ must be a partition for $[n]$. $\square$

Theorem 4.4.6. *Let M be the sparse paving matroid of rank r with minimum circuits $A_1, \ldots, A_t$. Let L be the incidence matrix of sets $A_1, \ldots, A_t$. If* $\operatorname{rank}(L) = n$, *then* $R^{r-2}(M) = \cup_{i=1}^{t} R^{r-2}(M - A_i)$.

Proof. Let $a = \sum_{i=1}^{n} \alpha_i e_i$. First we prove that if $a \in R^{r-2}(M) - \cup_{i=1}^{t} R^{r-2}(M - A_i)$, then $\sum_{j \in A_i} \alpha_j = 0$ for $1 \leq i \leq t$. We have two cases:

- $1 \notin A_i$. Consider the row $A_i \cup \{1\}$. In this row the only non-zero term is $\sum_{j \in A_i^c} \alpha_j$. Then $\sum_{j \in A_i} \alpha_j = 0$.
- $1 \in A_i$. Let s be an arbitrary element of A_i^c. In much the same way as in Proposition 4.3.4 and Remark 3, we can write $a \cdot \partial(A)$ as a linear combination of elements of $\mathcal{A}' = \{\partial(A); |A| = r + 1 \text{ and } s \in A\}$ and in the row $A_i \cup \{s\}$ the only non-zero term is $\sum_{j \in A_i^c} \alpha_j$. Then $\sum_{j \in A_i} \alpha_j = 0$.

Now, if $\operatorname{rank}(L) = n$, then the system of equations $\sum_{j \in A_i} \alpha_j = 0$ for any $1 \leq i \leq t$, is a non-singular system. Then $R^{r-2}(M) = \cup_{i=1}^{t} R^{r-2}(M - A_i)$. $\square$

Next theorem shows that if the rank of M is large enough in comparison to the number of minimum circuits, then $R^{r-2}(M)$ is trivial.

Theorem 4.4.7. *Let M be a sparse paving matroid on the ground set $[n]$ with minimum circuits $\{A_1, \ldots, A_t\}$ of rank r. If $r > 2(t-1)$, then $R^{r-2}(M)$ is trivial.*

Proof. It is enough to show that $R^{r-2}(M) = \cup_{i=1}^{t} R^{r-2}(M - A_i)$. On the contrary assume that $R^{r-2}(M) - \cup_{i=1}^{t} R^{r-2}(M - A_i) \neq \emptyset$. Let $a \in R^{r-2}(M) - \cup_{i=1}^{t} R^{r-2}(M - A_i)$. Let $a = \sum_{i=1}^{n} \alpha_i e_i$. If $\cap_{i=1}^{t} A_i \neq \emptyset$, then by Theorem 4.4.1, the result follows. Assume that $1 \notin A_t$. Let $1 \neq i$ be an arbitrary element of A_t^c. Thus $\pm\alpha_i$ appears r times in the column A_t, in rows $\partial(A \cup \{1, i\} - \{j\})$ with $j \in A_t$. By Proposition 4.3.6, there is $j \in A_t$ such that the only non-zero term in the row $\partial(A \cup \{1, i\} - \{j\})$ is $\pm\alpha_i$, which shows that $\alpha_i = 0$ for all $i \in A_t^c$. Then $a \in \cup_{i=1}^{t} R^{r-2}(M - A_i)$ which is a contradiction. □

Definition 4. *Let $A_1 = \{1, \ldots, t\}, \ldots, A_k = \{kt - t + 1, \ldots, kt\}$. Let $A_{i_1,\ldots,i_{k-1}} = \{i_1, \ldots, i_{k-1}\} \cup \{i_k; i_1 + \cdots + i_{k-1} + i_k \equiv 0 \mod t,$ and $i_k \in A_k\}$. Now, let $M_{k,t}$ be the sparse paving matroid with the minimal circuits $\mathbf{A} = \{A_{i_1,\ldots,i_{k-1}}; i_j \in A_j\}$. For example, the matroid $M_{3,2}$ is isomorphic to A_3. It is easy to see that $M_{3,3}$ is the Pappus arrangement. The rank of $M_{k,t}$ is k. One can see that $M_{3,t}$ corresponds to the decomposition of edges of the graph $K_{t,t,t}$ into triangles.*

The next example shows the matrix expression for $M_{3,3}$ (Pappus arrangement).

Example 3. Let M be the sparse paving matroid on the ground set [9] with minimum circuits $\{189, 239, 679, 156, 134, 246, 478, 357, 258\}$. In this matroid by multinet we know that $R^{r-2}(M)$ is spanned by $e_1 + e_2 + e_6 - (e_3 + e_7 + e_8)$ and $e_1 + e_2 + e_6 - (e_4 + e_5 + e_9)$. Here $S_{ijk} := (\sum_{t=1}^{9} \alpha_i) - (\alpha_i + \alpha_j + \alpha_k)$.

$$
\begin{array}{cc}
 & \begin{array}{ccc|cccccc} 189 & 134 & 156 & 239 & 679 & 246 & 478 & 357 & 258 \end{array} \\
\begin{array}{c}
1234 \\ 1235 \\ 1236 \\ 1237 \\ 1238 \\ 1239 \\ 1245 \\ 1246 \\ 1247 \\ 1248 \\ 1249 \\ 1256 \\ 1257 \\ 1258 \\ 1259 \\ 1267 \\ 1268 \\ 1269 \\ 1278 \\ 1279 \\ 1289 \\ 1345 \\ 1346 \\ 1347 \\ 1348 \\ 1349 \\ 1356 \\ 1357
\end{array} &
\left(\begin{array}{ccc|cccccc}
0 & -\alpha_2 & 0 & -\alpha_4 & 0 & \alpha_3 & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha_5 & 0 & 0 & 0 & -\alpha_2 & \alpha_3 \\
0 & 0 & 0 & -\alpha_6 & 0 & -\alpha_3 & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha_7 & 0 & 0 & 0 & \alpha_2 & 0 \\
0 & 0 & 0 & -\alpha_8 & 0 & 0 & 0 & 0 & -\alpha_3 \\
0 & 0 & 0 & S_{239} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha_5 & 0 & 0 & \alpha_4 \\
0 & 0 & 0 & 0 & 0 & S_{246} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha_7 & -\alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha_8 & \alpha_2 & 0 & -\alpha_4 \\
0 & 0 & 0 & -\alpha_4 & 0 & -\alpha_9 & 0 & 0 & 0 \\
0 & 0 & -\alpha_2 & 0 & 0 & -\alpha_5 & 0 & 0 & -\alpha_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 & -\alpha_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_{258} \\
0 & 0 & 0 & -\alpha_5 & 0 & 0 & 0 & 0 & -\alpha_9 \\
0 & 0 & 0 & 0 & -\alpha_2 & \alpha_7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_8 & 0 & 0 & -\alpha_6 \\
0 & 0 & 0 & -\alpha_6 & \alpha_2 & \alpha_9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 & 0 & -\alpha_7 \\
0 & 0 & 0 & -\alpha_7 & -\alpha_2 & 0 & 0 & 0 & 0 \\
-\alpha_2 & 0 & 0 & -\alpha_8 & 0 & 0 & 0 & 0 & \alpha_9 \\
0 & -\alpha_5 & 0 & 0 & 0 & 0 & 0 & \alpha_4 & 0 \\
0 & -\alpha_6 & 0 & 0 & 0 & -\alpha_3 & 0 & 0 & 0 \\
0 & -\alpha_7 & 0 & 0 & 0 & 0 & -\alpha_3 & -\alpha_4 & 0 \\
0 & -\alpha_8 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 \\
0 & -\alpha_9 & 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha_3 & 0 & 0 & 0 & 0 & -\alpha_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & S_{357} & 0
\end{array}\right)
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{ccc|cccccc} 189 & 134 & 156 & 239 & 679 & 246 & 478 & 357 & 258 \end{array} \\
\begin{array}{c} 1358 \\ 1359 \\ 1367 \\ 1368 \\ 1369 \\ 1378 \\ 1379 \\ 1389 \\ 1456 \\ 1457 \\ 1458 \\ 1459 \\ 1467 \\ 1468 \\ 1469 \\ 1478 \\ 1479 \\ 1489 \\ 1567 \\ 1568 \\ 1569 \\ 1578 \\ 1579 \\ 1589 \\ 1678 \\ 1679 \\ 1689 \\ 1789 \end{array} &
\left(\begin{array}{ccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_8 & -\alpha_3 \\
0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & -\alpha_9 & 0 \\
0 & 0 & 0 & 0 & -\alpha_3 & 0 & 0 & -\alpha_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_6 & \alpha_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\alpha_3 & \alpha_8 & 0 \\
0 & 0 & 0 & \alpha_7 & -\alpha_3 & 0 & 0 & \alpha_9 & 0 \\
-\alpha_3 & 0 & 0 & \alpha_8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha_4 & 0 & 0 & \alpha_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & -\alpha_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\alpha_5 & 0 & -\alpha_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha_4 & -\alpha_7 & \alpha_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha_8 & -\alpha_6 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_4 & -\alpha_9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & S_{478} & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha_4 & 0 & -\alpha_9 & 0 & 0 \\
-\alpha_4 & 0 & 0 & 0 & 0 & 0 & \alpha_9 & 0 & 0 \\
0 & 0 & -\alpha_7 & 0 & -\alpha_5 & 0 & 0 & \alpha_6 & 0 \\
0 & 0 & -\alpha_8 & 0 & 0 & 0 & 0 & 0 & \alpha_6 \\
0 & 0 & -\alpha_9 & 0 & \alpha_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\alpha_5 & -\alpha_8 & \alpha_7 \\
0 & 0 & 0 & 0 & -\alpha_5 & 0 & 0 & -\alpha_9 & 0 \\
-\alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_9 \\
0 & 0 & 0 & 0 & -\alpha_8 & 0 & -\alpha_6 & 0 & 0 \\
0 & 0 & 0 & 0 & S_{679} & 0 & 0 & 0 & 0 \\
-\alpha_6 & 0 & 0 & 0 & -\alpha_8 & 0 & 0 & 0 & 0 \\
-\alpha_7 & 0 & 0 & 0 & \alpha_8 & 0 & -\alpha_9 & 0 & 0
\end{array}\right)
\end{array}
$$

Theorem 4.4.8. *Let n, k be natural numbers. Then $a = a_1(e_1 + \cdots + e_t) + \cdots + a_k(e_{kt-t+1} + \cdots + e_{kt})$ belongs to $R^{k-2}(M_{k,t})$, if $a_1 + \cdots + a_k = 0$.*

Proof. Let $B_i = e_{t(i-1)+1} + \cdots + e_{ti}$. Since $(B_2 - B_3)\cdots(B_{k-1} - B_k)$ is not proportional to $(B_1 - B_2)$, it suffices to show that $(B_1 - B_2)(B_2 - B_3)\cdots(B_{k-1} - B_k) = 0$.

$$\begin{aligned}(B_1 - B_2)(B_2 - B_3)\cdots(B_{k-1} - B_k) &= \sum_{i=1}^{k}(-1)^{k-i} B_1 \cdots \widehat{B_i} \cdots B_k \\ &= \sum_{i=1}^{k}(-1)^{k-i} \sum_{t_j \in A_j, j\neq i} \prod_{j=1, j\neq i}^{k} e_{t_j} \\ &= \sum_{i_j \in A_j} \pm\partial(A_{i_1,\ldots,i_{k-1}}) \\ &= 0.\end{aligned}$$

□

Theorem 4.4.9. *Let k be a positive integer, and let L be the subspace generated by $(e_3 + e_4) - (e_1 + e_2), (e_5 + e_6) - (e_1 + e_2), \ldots, (e_{2k-1} + e_{2k}) - (e_1 + e_2)$. Then*

$$R^k(M_{k,2}) = L \cup \bigcup_{A \in \mathbf{A}} R^{k-2}(M - A).$$

Proof. Let C be a minimum circuit. We know that if $a \in R^{k-2}(M) - \cup R^{k-2}(M - A)$, then a must satisfy the equation $\sum_{i \in C} \alpha_i = 0$. Now consider circuits $\{2, 4, \ldots, 2n\}$ and $\{2, 4, \ldots, 2n-4, 2n-3, 2n-1\}$. Thus we have

$$\alpha_{2n} + \alpha_{2n-2} = \alpha_{2n-3} + \alpha_{2n-1}.$$

Now consider circuits $\{1, 4, 6, \ldots, 2n-2, 2n-1\}$ and $\{1, 4, 6, \ldots, 2n-4, 2n-3, 2n\}$. Thus we have

$$\alpha_{2n} + \alpha_{2n-2} = \alpha_{2n-3} + \alpha_{2n-1}.$$

Then $\alpha_{2n-2} = \alpha_{2n-3}$. Similarly, we have $\alpha_{2i} = \alpha_{2i-1}$, which completes the proof. □

Theorem 4.4.10. *Let M be a sparse paving matroid on the ground set $[n]$ with minimum circuits $\{A_1, \ldots, A_t\}$ of rank r. Let $\cap_{i=1}^{t} A_i = \emptyset$ and $1 \in \cap_{i=1}^{t-2} A_i - (A_{t-1} \cup A_t)$. If $r < n-4$, then*

$$R^{r-2}(M) = \bigcup_{i=1}^{t} R^{r-2}(M - A_i). \tag{4.14}$$

Proof. Let $V = R^{r-2}(M) - \bigcup_{i=1}^{t} R^{r-2}(M - A_i)$. We prove by contradiction that $V = \emptyset$. Let $a = \sum_{i=1}^{n} \alpha_i e_i \in V$. We need to consider three cases:

- There exists i such that $|A_t - A_i| = |A_{t-1} - A_i| = 2$.

  Let $A_i - A_t = \{1, b_i\}$, $A_i - A_{t-1} = \{1, b'_i\}$, $A_t - A_i = \{c_i, d_i\}$ and $A_{t-1} - A_i = \{c'_i, d'_i\}$. Since $r < n-4$, there exists $\beta \in A_i^c - \{c_i, c'_i, d_i, d'_i\}$. Then $\beta \in A_i^c$ and $\beta \notin A_{t-1} \cup A_t$. Consider the row $A_i \cup \{\beta\}$. In this row the only non-zero element is α_β. Then $\alpha_\beta = 0$. Since $\cup_{i=1}^t A_i = [n]$, it follows that there exists j such that $\beta \in A_j$. We need to consider four cases:

  - $|A_j - A_t| > 2$ and $|A_j - A_{t-1}| > 2$. In this case it is easy to see that the column A_j has no intersection with the rest of columns, which means that $a \in \cup_{i=1}^t R^{r-2}(M - A_i)$, which is a contradiction.
  - $|A_j - A_t| = 2$ and $|A_j - A_{t-1}| > 2$. Then $A_j - A_t = \{1, \beta\}$. Let $A_t - A_j = \{c_j, d_j\}$. Now consider the column A_t and row $A_t \cup \{1, \beta\} - \{c_j\}$. In this row the only non-zero entries are α_{c_j} and α_{d_j}. Then $\alpha_{c_j} = \alpha_{d_j} = 0$. Then all entries of the column A_j are zero, which means that $a \in \cup_{i=1}^t R^{r-2}(M - A_i)$, which is a contradiction.
  - $|A_j - A_t| > 2$ and $|A_j - A_{t-1}| = 2$. This case is similar to the second case.
  - $|A_j - A_t| = 2$ and $|A_j - A_{t-1}| = 2$. Then $A_j - A_{t-1} = A_j - A_t = \{1, \beta\}$. Let $A_t - A_j = \{c_j, d_j\}$ and $A_{t-1} - A_j = \{c'_j, d'_j\}$. Now consider the column A_t and rows $A_t \cup \{1, \beta\} - \{c_j\}$ and $A_t \cup \{1, \beta\} - \{d_j\}$. In these rows the only non-zero entries are α_{c_j} and α_{d_j}. Then $\alpha_{c_j} = \alpha_{d_j} = 0$, and similarly for A_{t-1} we have $\alpha_{c'_j} = \alpha_{d'_j} = 0$. Then all entries of the column A_j are zero, which means that $a \in \cup_{i=1}^t R^{r-2}(M - A_i)$, which is a contradiction.

- There exists i such that $|A_t - A_i| > 2$ and $|A_{t-1} - A_i| > 2$.

  In this case it is easy to see that the column A_i has no intersection with the rest of columns, which leads to a contradiction.

- For all $1 \leq i \leq t$, we have either

(i) $|A_t - A_i| = 2$ and $|A_{t-1} - A_i| > 2$ or

(ii) $|A_{t-1} - A_i| = 2$ and $|A_t - A_i| > 2$.

Without loss of generality we can assume that $A_1, \ldots, A_k$ belong to (i) and $A_{k+1}, \ldots, A_{t-2}$ belong to (ii). Then we have two cases:

- $|A_t \cap A_{t-1}| < r-2$. In this case two submatrices $A_1, \ldots, A_k, A_t$ and $A_{k+1}, \ldots, A_{t-2}, A_{t-1}$ has no intersection. Then

$$V \subset R^{r-2}(M - \{A_1, \ldots, A_k, A_t\}) \cap R^{r-2}(M - \{A_{k+1}, \ldots, A_{t-2}, A_{t-1}\}),$$

  which is a contradiction.

- $|A_t \cap A_{t-1}| = r-2$. Let $A_t - A_1 = \{c_1, d_1\}$. Let $\beta \in A_1^c - \{c_1, d_1\}$. Then $\beta \notin A_1$ and $\beta \notin A_t$. In the row $A_1 \cup \{\beta\}$ the only term is α_β (in the column A_1). Hence $\alpha_\beta = 0$. We have two cases:
  * $\beta \in A_{t-1}$. Then $\beta \in A_{t-1} - A_t$. Let $A_{t-1} - A_t = \{\beta, \lambda\}$ and $A_t - A_{t-1} = \{\beta', \lambda'\}$. Then by considering columns A_{t-1}, A_t, we have $a_\beta = a_{\beta'} = a_\lambda = a_{\lambda'} = 0$. It means that $V \subset R^{r-2}(M - \{A_1, \ldots, A_k, A_t\}) \cap R^{r-2}(M - \{A_{k+1}, \ldots, A_{t-2}, A_{t-1}\})$, contradicts our assumption.
  * $\beta \notin A_{t-1}$. Since $\cup_{i=1}^{t} A_i = [n]$, there exists $i \neq 1, t-1, t$ such that $\beta \in A_i$. Without loss of generality we can assume that $\beta \in A_2$. Then $A_2 - A_t = \{1, \beta\}$ and $A_t - A_2 = \{c_2, d_2\}$. The column A_2 has no intersection with other columns except the column A_t which shows that $a_{c_2} = a_{d_2} = 0$. Then all entries of the column A_j are zero, which means that $a \in \cup_{i=1}^{t} R^{r-2}(M - A_i)$, which is a contradiction.

□

Theorem 4.4.11. *Let M be a sparse paving matroid on the ground set $[n]$ with minimum circuits $\{A_1, \ldots, A_t\}$ of rank r. If $m \notin \cup_{i=1}^{t-1} A_i$, then in any non-local component of $R^{r-2}(M)$, we have $\alpha_m = 0$.*

Proof. If $m \in A_t^c$, then by Theorem 4.4.4, we have $\alpha_m = 0$. Now, let $m \in A_t$. If $1 \in A_t$, then in the row $A_1 \cup \{m\}$ the only non-zero element is α_m. Then $\alpha_m = 0$. Now assume that $1 \notin A_t$. Let $1 \in \cap_{i=1}^{k} A_i - \cup_{i=k+1}^{t} A_i$. By Theorem 4.4.5, it is easy to check the case of $t = k+1$. So $t > k+1$. Consider the column A_{t-1}. In this column α_m appears r-times, in rows $A_{t-1} \cup \{1, m\} - \{p\}$, where $p \in A_{t-1}$. In these rows the only possible intersection happen with A_t. If $r > 2$, then $\alpha_m = 0$. The case $r = 2$ is trivial. □

Bibliography

[1] D. Cohen and A. Varchenko, *Resonant local systems and representations of sl_2*. Geom. Dedicata, 101:217–233, 2003.

[2] M. Falk and S. Yuzvinsky, Multinets, *resonance varieties, and pencils of plane curves*, Compos. Math. 143 (2007), no. 4, 1069–1088.

[3] G. Denham, A. Suciu, S. Yuzvinsky, *Abelian duality and propagation of resonance*, arXiv:1512.07702.

[4] G. Denham, *Higher resonance varieties of matroids*, in: Configurations Spaces, 41–66, Springer INdAM series, vol. 14, Springer, Cham, 2016.

[5] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.

[6] A. Libgober and S. Yuzvinsky, *Cohomology of the Orlik-Solomon algebras and local systems*. Compositio mathematica, 121:337–361, 2000.

[7] D. Mayhew, M. Newman, D. Welsh and G. Whittle: *On the asymptotic proportion of connected matroids,* European J. Combin. 32 (2011), 882–890.

[8] S. Papadima, A.I. Suciu, *Bieri–Neumann–Strebel–Renz invariants and homology jumping loci*, Proc. London Math. Soc. **100** (2010), no. 3, 795–834.

[9] R. Pendavingh and J. van der Pol, *On the number of matroids compared to the number of sparse paving matroids.* Electron. J. Combin., 22(2):Paper 2.51, 17, 2015.

[10] S. Yuzvinsky, *Realization of finite abelian groups by nets in $\mathbb{P}^2$*. Compos. Math., 140(6):1614–1624, 2004.

Chapter 5

On the addition of squares of units modulo n

5.1 Introduction

Let $\mathbb{Z}_n$ be the ring of residue classes modulo n, and let $\mathbb{Z}_n^\times$ be the group of its units. Let $c \in \mathbb{Z}_n$, and let k be a positive integer. Brauer [1] gave a formula for the number of solutions of the equation $x_1 + \cdots + x_k = c$ with $x_1, \ldots, x_k \in \mathbb{Z}_n^\times$. Sander [4] found the number of representations of a fixed residue class mod n as the sum of two units in $\mathbb{Z}_n$, the sum of two non-units, and the sum of mixed pairs, respectively. Kiani and Mollahajiaghaei [3] generalized the results of Sander to an arbitrary finite commutative ring, as sum of k units and sum of k non-units, with a combinatorial approach.

The problem of finding explicit formulas for the number of representations of a natural number n as the sum of k squares is one of the most interesting problems in number theory. For example, if $k = 4$, then Jacobi's four-square theorem states that this number is $8\sum_{m|c} m$ if c is odd and 24 times the sum of the odd divisors of c if c is even. See [5] and the references given there for historical remarks.

Recently, Tóth [5] obtained formulas for the number of solutions of the equation

$$a_1x_1^2 + \cdots + a_kx_k^2 = c,$$

where $c \in \mathbb{Z}_n$, and x_i and a_i all belong to $\mathbb{Z}_n$.

Now, consider the equation

$$x_1^2 + \cdots + x_k^2 = c, \tag{5.1}$$

where $c \in \mathbb{Z}_n$, and x_i are all units in the ring $\mathbb{Z}_n$. We denote the number of solutions of this equation by $\mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$. Yang and Tang [7] obtained a formula for $\mathscr{S}_{sq}(\mathbb{Z}_n, c, 2)$. In this chapter we provide an explicit formula for $\mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$, for an arbitrary k. Our approach is combinatorial with the help of spectral graph theory.

The idea may be sketched as follows: first, it is easy to show that if m, n are coprime numbers, then $\mathscr{S}_{sq}(\mathbb{Z}_{mn}, c, k) = \mathscr{S}_{sq}(\mathbb{Z}_m, c, k)\mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$. So it is enough to find a formula for $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$ where p is a prime number. Let $\mathbb{Z}_n^{\times 2} = \{x^2; x \in \mathbb{Z}_n^\times\}$. Let p be an odd prime number. There is a natural map between solutions of the above equation and $(0, c)$-walks in the directed Cayley graph $Cay(\mathbb{Z}_{p^\alpha}, \mathbb{Z}_{p^\alpha}^{\times 2})$, defined by sending $(\pm x_1, \ldots, \pm x_k)$ to the walk $0, x_1^2, x_1^2 + x_2^2, \ldots, x_1^2 + \cdots + x_k^2$. Thus, enumerating the number of solutions amounts to 2^k times enumerating these walks. By exploiting the structure of this graph, one can reduce this calculation to the case that $\alpha = 1$. The number of walks can then be identified as a particular entry in the k-th power of the adjacency matrix of this graph; in this case the adjacency matrix can be described explicitly, and hence one can obtain an exact formula. An explicit formula for $\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$ can be found by direct counting.

5.2 Preliminaries

In this section we present some graph theoretical notions and properties used in the paper. See, e.g., the book [2]. Let G be an additive group with identity 0. For $S \subseteq G$, the *Cayley graph* $X = Cay(G, S)$ is the directed graph having vertex set $V(X) = G$ and edge set $E(X) = \{(a, b); b - a \in S\}$. Clearly, if $0 \notin S$, then there is no loop in X, and if $0 \in S$, then there is exactly one loop at each vertex. If $-S = \{-s; s \in S\} = S$, then there is an edge from a to b if and only if there is an edge from b to a.

Let $\mathbb{Z}_n^{\times 2} = \{x^2; x \in \mathbb{Z}_n^\times\}$. The *quadratic unitary Cayley graph* of $\mathbb{Z}_n$, $G_{\mathbb{Z}_n}^2 = Cay(\mathbb{Z}_n; \mathbb{Z}_n^{\times 2})$, is defined as the directed Cayley graph on the additive group of $\mathbb{Z}_n$ with respect to $\mathbb{Z}_n^{\times 2}$; that is, $G_{\mathbb{Z}_n}^2$ has vertex set $\mathbb{Z}_n$ such that there is an edge from x to y if and only if

$y - x \in \mathbb{Z}_n^{\times 2}$. Then the out-degree of each vertex is $|\mathbb{Z}_n^{\times 2}|$.

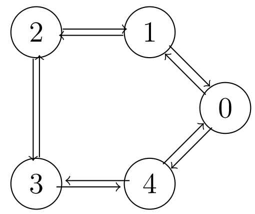


FIGURE 5.1: $G^2_{\mathbb{Z}_5}$

Let G be a directed graph without multiple edges, and let $V(G) = \{v_1, \ldots, v_n\}$. The *adjacency matrix* A_G of G is defined in a natural way. Thus, the rows and the columns of A_G are labeled by $V(G)$. For i, j, if there is an edge from v_i to v_j then $a_{v_i v_j} = 1$; otherwise $a_{v_i v_j} = 0$. We will write it simply A when no confusion can arise. For the graph $G^2_{\mathbb{Z}_n}$ the matrix A is symmetric, provided that -1 is a square mod n.

We write J_m for the $m \times m$ all 1-matrix. The identity $m \times m$ matrix will be denoted by I_m.

The complete directed graph on m vertices with a loop at each vertex is denoted by K_m^+. Thus, the adjacency matrix of K_m^+ is J_m.

A *walk* in a graph G is a sequence $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$ so that $v_i \in V(G)$ for every $0 \leq i \leq n$, and e_i is an edge from v_{i-1} to v_i, for every $1 \leq i \leq n$. We denote by $w_k(G, i, j)$ the number of walks of length k from i to j in the graph G.

One application of the adjacency matrix is to calculate the number of walks between two vertices.

Lemma 5.2.1. [2, Lemma 8.1.2] *Let G be a directed graph, and let k be a positive integer. Then the number of walks from vertex i to vertex j of length k is the entry on row i and column j of the matrix A^k, where A is the adjacency matrix.*

The next theorem provides the connection between $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$ and $w_k(G^2_{\mathbb{Z}_{p^\alpha}}, 0, c)$.

Theorem 5.2.2. *Let p be an odd prime number and α be a positive integer. Then*

$$\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k) = 2^k w_k(G^2_{\mathbb{Z}_{p^\alpha}}, 0, c).$$

Proof. Consider the graph $G^2_{\mathbb{Z}_{p^\alpha}}$. Let $(x_1, \ldots, x_k) \in (\mathbb{Z}^{\times}_{p^\alpha})^k$ such that $x_1^2 + x_2^2 + \cdots + x_k^2 = c$. Then $0, x_1^2, x_1^2 + x_2^2, \ldots, x_1^2 + x_2^2 + \cdots + x_k^2 = c$ is a walk of length k from 0 to c.

Now, let $0 = a_0, a_1, \ldots, a_k = c$ be a walk of length k. Then $a_i - a_{i-1} = y_i^2$, where $y_i \in \mathbb{Z}^{\times}_{p^\alpha}$ for $i = 1, \ldots, k$. Hence $y_1^2 + y_2^2 + \cdots + y_k^2 = c$. Then the set $\{(\epsilon_k y_1, \ldots, \epsilon_k y_k); \epsilon_i \in \{1, -1\}\}$ is a set of solutions of size 2^k, which proves the theorem. $\square$

The *tensor product* $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \otimes G_2) := V(G_1) \times V(G_2)$, with edges specified by putting (u, v) adjacent to (u', v') if and only if u is adjacent to u' in G_1 and v is adjacent to v' in G_2. It can be easily verified that the number of edges in $G_1 \otimes G_2$ is equal to the product of the number of edges in the graphs G and H.

Lemma 5.2.3. [6] *The adjacency matrix of $G \otimes H$ is the tensor product of the adjacency matrices of G and H.*

The rest of chapter is organized as follows. In Section 5.3 we reduce the case $\mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$ to the cases $\mathscr{S}_{sq}(\mathbb{Z}_p, c, k)$ and $\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$. We show that if p is an odd prime number, then $G^2_{\mathbb{Z}_{p^\alpha}} \cong G^2_{\mathbb{Z}_p} \otimes K^+_{p^{\alpha-1}}$. Section 5.4 is devoted to the study of $\mathscr{S}_{sq}(\mathbb{Z}_p, c, k)$, where $p \equiv 1 \mod 4$. In this section, we write A^k as a linear combination of matrices A, J_p and I_p, and then we obtain a formula for $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$. Similarly, we find a formula for $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$, where $p \equiv 3 \mod 4$, in Section 5.5. Last section, provides an explicit formula for $\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$ by direct counting.

5.3 General results

In this section, we reduce the case $\mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$ to the cases $\mathscr{S}_{sq}(\mathbb{Z}_p, c, k)$ and $\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$.

The next lemma shows that the function $n \to \mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$ is multiplicative.

Lemma 5.3.1. *Let m, n be coprime numbers. Then $\mathscr{S}_{sq}(\mathbb{Z}_{mn}, c, k) = \mathscr{S}_{sq}(\mathbb{Z}_m, c, k) \cdot \mathscr{S}_{sq}(\mathbb{Z}_n, c, k)$.*

Proof. For given representations

$$a_1^2 + \cdots + a_k^2 \equiv c \mod m,$$

$$b_1^2 + \cdots + b_k^2 \equiv c \mod n$$

with $a_1, \ldots, a_k \in \mathbb{Z}_m^\times$ and $b_1, \ldots, b_k \in \mathbb{Z}_n^\times$ the Chinese remainder theorem guarantees the unique existence of $c_i \mod mn$ $(i = 1, \ldots, k)$ such that

$$c_i \equiv a_i \mod m$$

and

$$c_i \equiv b_i \mod n.$$

Moreover we have:

$$c_1^2 + \cdots + c_k^2 \equiv c \mod m$$

and

$$c_1^2 + \cdots + c_k^2 \equiv c \mod n.$$

Then

$$c_1^2 + \cdots + c_k^2 \equiv c \mod mn.$$

Conversely each representation $c_1^2 + \cdots + c_k^2 \mod mn$ yields representations $a_1^2 + \cdots + a_k^2 \mod m$ and $b_1^2 + \cdots + b_k^2 \mod n$ by setting $a_i :\equiv c_i \mod m$ and $b_i :\equiv c_i \mod m$, which completes the proof. $\square$

Lemma 5.3.2. *Let p be an odd prime number, and let $\mathfrak{m}$ be the ideal generated by p in the ring $\mathbb{Z}_{p^\alpha}$. Let $u \in \mathbb{Z}_{p^\alpha}^{\times 2}$ and $r \in \mathfrak{m}$. Then $u + r \in \mathbb{Z}_{p^\alpha}^{\times 2}$.*

Proof. For this to happen, it is enough to show that $1+r$ belongs to $\mathbb{Z}_{p^\alpha}^{\times 2}$. We know that r is a nilpotent element of $\mathbb{Z}_{p^\alpha}$. Let λ be a sufficiently large integer. Then $(1+r)^{p^\lambda} = 1$. Hence, $(1+r)^{p^\lambda + 1} = 1 + r$. $\square$

Theorem 5.3.3. *Let p be an odd prime number, and let α be a positive integer. Then $G^2_{\mathbb{Z}_{p^\alpha}} \cong G^2_{\mathbb{Z}_p} \otimes K^+_{p^{\alpha-1}}$.*

Proof. Let m be the ideal generated by p, and $\mathbb{Z}_{p^\alpha} = \bigcup_{i=1}^{p}(m + r_i)$, where $m + r_i$ is a coset of the maximal ideal m in $\mathbb{Z}_{p^\alpha}$. The ring $\mathbb{Z}_{p^\alpha}/m$ is isomorphic to the field $\mathbb{Z}_p$. Then for each $r \in \mathbb{Z}_{p^\alpha}$ there is a unique i and $n_r \in m$ such that $r = r_i + n_r$. Let $\psi : G^2_{\mathbb{Z}_{p^\alpha}} \longrightarrow G^2_{\mathbb{Z}_p} \otimes K^+_{p^{\alpha-1}}$ be defined by $\psi(r) := (r_i + m, n_r)$. Obviously, this map is a bijection. Now, let (r, r') be a directed edge in $G^2_{\mathbb{Z}_{p^\alpha}}$. We show that $(\psi(r), \psi(r'))$ is also a directed edge in $G^2_{\mathbb{Z}_p} \otimes K^+_{p^{\alpha-1}}$. By definition, $\psi(r) = (r_i + m, n_r)$ and $\psi(r') = (r_j + m, n_{r'})$. We have $r' - r \in \mathbb{Z}_{p^\alpha}^{\times 2}$. Thus, $r_j - r_i + n_{r'} - n_r \in \mathbb{Z}_{p^\alpha}^{\times 2}$. Hence by Lemma 5.3.2, $r_j - r_i \in \mathbb{Z}_{p^\alpha}^{\times 2}$.

Then $r_j - r_i + m \in (\mathbb{Z}_{p^\alpha}/m)^{\times 2}$. Since the number of edges of $G^2_{\mathbb{Z}_{p^\alpha}}$ and $G^2_{\mathbb{Z}_p} \otimes K^+_{p^{\alpha-1}}$ are the same, the proof is complete. $\square$

By the aforementioned theorem, we see

$$\begin{aligned} A^k_{G^2_{\mathbb{Z}_{p^\alpha}}} &= A^k_{G^2_{\mathbb{Z}_p}} \otimes A^k_{K^+_{p^{\alpha-1}}} \\ &= A^k_{G^2_{\mathbb{Z}_p}} \otimes J^k_{p^{\alpha-1}}. \end{aligned}$$

5.4 $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$ where $p \equiv 1 \mod 4$

In this section, we find $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$, where p is a prime number with $p \equiv 1 \mod 4$.

Recall that an *strongly regular graph* with parameters (n, k, λ, μ) is a simple graph with n vertices that is regular of valency k and has the following properties:

- For any two adjacent vertices x, y, there are exactly λ vertices adjacent to both x and y.

- For any two non-adjacent vertices x, y, there are exactly μ vertices adjacent to both x and y.

Let p be a fixed prime number with $p \equiv 1 \mod 4$. The Paley graph P_p is defined by taking the field $\mathbb{Z}_p$ as vertex set, with two vertices x and y joined by an edge if and only if $x - y$ is a nonzero square in $\mathbb{Z}_p$. For example, P_5 is isomorphic to C_5.

As is well known (see e.g., [2, P. 221]), the Paley graph is strongly regular with parameters $(p, \frac{p-1}{2}, \frac{p-5}{4}, \frac{p-1}{4})$. The fact that Paley graph is strongly regular shows that A^2 can be written as a linear combination of matrices A, J_p and I_p.

Lemma 5.4.1. *[2, P. 219] Let p be a prime number such that $p \equiv 1 \mod 4$. Then the adjacency matrix of the Paley graph P_p satisfies*

$$A^2_{P_p} = -A_{P_p} + (\frac{p-1}{4})J_p + (\frac{p-1}{4})I_p. \tag{5.2}$$

Although the graph $G^2_{\mathbb{Z}_p}$ is a directed graph and P_p is a simple graph, they share the same adjacency matrix. Then $A^n_{G^2_{\mathbb{Z}_p}}$ can be written as a linear combination of $A_{G^2_{\mathbb{Z}_p}}$, I_p

and J_p.

Let

$$A^{n+1} = a_{n,p}A + b_{n,p}J_p + c_{n,p}I_p. \tag{5.3}$$

Then

$$A^{n+2} = a_{n,p}A^2 + \frac{p-1}{2}b_{n,p}J_p + c_{n,p}A.$$

Now, by Equation (5.2), we have

$$A^{n+2} = (a_{n,p}a_{1,p} + c_{n,p})A + (\frac{p-1}{2}b_{n,p} + a_{n,p}b_{1,p})J_p + (a_{n,p}c_{1,p})I_p.$$

Then we see that

$$\begin{cases} a_{n+1,p} = a_{n,p}a_{1,p} + c_{n,p}, & a_{1,p} = -1, a_{2,p} = \frac{p+3}{4}; \\ b_{n+1,p} = \frac{p-1}{2}b_{n,p} + a_{n,p}b_{1,p}, & b_{1,p} = \frac{p-1}{4}, b_{2,p} = (\frac{p-1}{4})(\frac{p-3}{2}); \\ c_{n+1,p} = a_{n,p}c_{1,p}, & c_{1,p} = \frac{p-1}{4}, c_{2,p} = -\frac{p-1}{4}. \end{cases}$$

From the first and last equations, we have the following homogeneous linear recurrence relation

$$a_{n,p} = \frac{p-1}{4}a_{n-2,p} - a_{n-1,p}.$$

Let $\tau = \dfrac{-1+\sqrt{p}}{2}$. Since $a_1 = -1$ and $a_2 = \frac{p+3}{4}$, we deduce

$$a_{n,p} = \Big(\frac{1}{\sqrt{p}}\Big)\Big(\tau^{n+1} + (-1)^n(\tau+1)^{n+1}\Big). \tag{i}$$

Now, we have the following for all $n \geq 1$,

$$c_{n,p} = \Big(\frac{\tau(\tau+1)}{\sqrt{p}}\Big)\Big(\tau^n + (-1)^{n-1}(\tau+1)^n\Big). \tag{ii}$$

Also, for all $n \geq 1$ we have

$$b_{n,p} = \frac{(p-5)(\tau(\tau+1))^{n-1}}{8} + \Big(\frac{\tau(\tau+1)}{\sqrt{p}}\Big)\Big((\tau+1)\tau^{n-2} + (-1)^{n-2}\tau(\tau+1)^{n-2}\Big). \tag{iii}$$

We can now find $\mathscr{S}_{sq}(\mathbb{Z}_p, c, k)$.

$$\mathscr{S}_{sq}(\mathbb{Z}_p, c, k) = \begin{cases} 2^k(b_{k-1,p} + c_{k-1,p}), & \text{if } c = 0; \\ 2^k(a_{k-1,p} + b_{k-1,p}), & \text{if } c = x^2, \text{ for some } x \in \mathbb{Z}_p^{\times}; \\ 2^k b_{k-1,p}, & \text{otherwise.} \end{cases} \tag{5.4}$$

The last theorem of this section provides a formula for $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$.

Theorem 5.4.2. *Let p be a prime number such that $p \equiv 1 \mod 4$. Let k and α be positive integer and $k > 1$. Then*

$$\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k) = \begin{cases} p^{(\alpha-1)(k-1)}2^k(b_{k-1,p} + c_{k-1,p}), & \text{if } c \equiv 0 \mod p; \\ p^{(\alpha-1)(k-1)}2^k(a_{k-1,p} + b_{k-1,p}), & \text{if } c = x^2, \text{ for some } x \in \mathbb{Z}_{p^\alpha}^\times; \\ p^{(\alpha-1)(k-1)}2^k b_{k-1,p}, & \text{otherwise,} \end{cases}$$

where $a_{k-1,p}$, $c_{k-1,p}$ and $b_{k-1,p}$ are defined by equations (i), (ii) and (iii), respectively, (putting $n = k-1$).

Proof. By Theorem 5.3.3 and Lemma 5.2.3, $A_{G^2_{\mathbb{Z}_{p^\alpha}}} = A_{G^2_{\mathbb{Z}_p}} \otimes A_{K^+_{p^{\alpha-1}}}$. Then

$$\begin{aligned} A^k_{G^2_{\mathbb{Z}_{p^\alpha}}} &= A^k_{G^2_{\mathbb{Z}_p}} \otimes J^k_{p^{\alpha-1}} \\ &= A^k_{G^2_{\mathbb{Z}_p}} \otimes p^{(\alpha-1)(k-1)} J_{p^{\alpha-1}}. \end{aligned}$$

Then Equation (5.4) and Lemma 5.2.1 complete the proof. □

5.5 $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$ where $p \equiv 3 \mod 4$

In this section, we find $\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$, where p is a prime number with $p \equiv 3 \mod 4$. The main idea is similar to that used in the previous section. We try to write $A^2_{G^2_{\mathbb{Z}_p}}$ as a linear combination of matrices $A_{G^2_{\mathbb{Z}_p}}$, I_p and J_p.

The field $\mathbb{Z}_p$, has no square root of -1. Then for each pair of (x, y) of distinct elements of $\mathbb{Z}_p$, either $x - y$ or $y - x$, but not both, is a square of a nonzero element. Hence in the graph $G^2_{\mathbb{Z}_p}$, each pair of distinct vertices is linked by an arc in one and only one direction. Therefore, $A_{G^2_{\mathbb{Z}_p}} + A^T_{G^2_{\mathbb{Z}_p}} = J_p - I_p$. The entry on row a and column b of the matrix $A^2_{G^2_{\mathbb{Z}_p}}$ equals to the size of the set $(a + \mathbb{Z}_p^{\times 2}) \cap (b - \mathbb{Z}_p^{\times 2})$. The goal of following lemmas is to find $|(a + \mathbb{Z}_p^{\times 2}) \cap (b - \mathbb{Z}_p^{\times 2})|$.

Lemma 5.5.1. *Let a and b be elements of $\mathbb{Z}_p$. Then $|(a + \mathbb{Z}_p^{\times 2}) \cap (b - \mathbb{Z}_p^{\times 2})| = |(a - b + \mathbb{Z}_p^{\times 2}) \cap -\mathbb{Z}_p^{\times 2}|$.*

Proof. Let $\psi : (a+\mathbb{Z}_p^{\times 2})\cap(b-\mathbb{Z}_p^{\times 2}) \longrightarrow (a-b+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2}$ be defined by $\psi(r) = r-b$. Obviously, ψ is well-defined and injective. Now, let $c \in (a-b+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2}$, so there exists $s \in \mathbb{Z}_p^{\times 2}$ such that $c = a-b+s$. Then $\psi(c+b) = c$, which completes the proof. □

Lemma 5.5.2. *Let a be a non-zero element of $\mathbb{Z}_p$. Then $|(a^2+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2}| = |(1+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2}|$ and $|(-a^2+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2}| = |(-1+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2}|$.*

Proof. Let $\psi : (a^2+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2} \longrightarrow (1+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2}$ be defined by $\psi(r) = ra^{-2}$. Obviously, ψ is well-defined and injective. Now, let $c \in (1+\mathbb{Z}_p^{\times 2})\cap -\mathbb{Z}_p^{\times 2}$. Thus, there exists $s \in \mathbb{Z}_p^{\times}$ such that $c = 1+s^2$. Then $\psi(ca^2) = c$, which completes the proof.

The proof for the second part is similar. □

Then by Lemmas 5.5.1 and 5.5.2, one can see that A^2 is a linear combination of matrices A, J_p and I_p. We show this in Lemma 5.5.5.

Lemma 5.5.3. $|(1+\mathbb{Z}_p^{\times 2})\cap(-\mathbb{Z}_p^{\times 2})| = \frac{p+1}{4}$.

Proof. We know that $\Big((1+\mathbb{Z}_p^{\times 2})\cap(-\mathbb{Z}_p^{\times 2})\Big)\cup\Big((1+\mathbb{Z}_p^{\times 2})\cap(\mathbb{Z}_p^{\times 2})\Big) = 1+\mathbb{Z}_p^{\times 2}$, and $\Big((1+\mathbb{Z}_p^{\times 2})\cap(-\mathbb{Z}_p^{\times 2})\Big)\cap\Big((1+\mathbb{Z}_p^{\times 2})\cap(\mathbb{Z}_p^{\times 2})\Big) = \emptyset$. Then $|(1+\mathbb{Z}_p^{\times 2})\cap(-\mathbb{Z}_p^{\times 2})| = \frac{p-1}{2} - |(1+\mathbb{Z}_p^{\times 2})\cap(\mathbb{Z}_p^{\times 2})|$. Now, $a \in (1+\mathbb{Z}_p^{\times 2})\cap(\mathbb{Z}_p^{\times 2})$ if and only there exist $b, c \in \mathbb{Z}_p^{\times}$ such that $a = 1+b^2 = c^2$. Thus, $(c-b)(c+b) = 1$. Hence $c = \frac{u+u^{-1}}{2}$ and $b = \frac{u-u^{-1}}{2}$, for $u \in \mathbb{Z}_p^{\times} - \{1,-1\}$. Then $(1+\mathbb{Z}_p^{\times 2})\cap(\mathbb{Z}_p^{\times 2}) = \{(\frac{u+u^{-1}}{2})^2; u \in \mathbb{Z}_p^{\times}\} - \{1\}$.

If $(\frac{u+u^{-1}}{2})^2 = (\frac{v+v^{-1}}{2})^2$, then we have two cases:

(i) $\frac{u+u^{-1}}{2} = \frac{v+v^{-1}}{2}$. A trivial verification shows that $u = v$ or $u = v^{-1}$.

(ii) $\frac{u+u^{-1}}{2} = -\frac{v+v^{-1}}{2}$. Then $u = -v$ or $u = -v^{-1}$.

Then $|(1+\mathbb{Z}_p^{\times 2})\cap(\mathbb{Z}_p^{\times 2})| = \frac{p-1-2}{4}$, and the lemma follows. □

The following lemma may be proved in much the same way as Lemma 5.5.3.

Lemma 5.5.4. $|(-1+\mathbb{Z}_p^{\times 2})\cap(-\mathbb{Z}_p^{\times 2})| = \frac{p-3}{4}$.

Lemma 5.5.5. *Let p be a prime number with $p \equiv 3 \mod 4$. Let A be the adjacency matrix of the graph $G^2_{\mathbb{Z}_p}$. Then*

$$A^2 = -A + (\frac{p+1}{4})J_p - (\frac{p+1}{4})I_p. \tag{5.5}$$

Proof. Let $a, b \in \mathbb{Z}_p$. By Lemma 5.5.1,

$$(A)_{ab} = |(a + \mathbb{Z}_p^{\times 2}) \cap (b - \mathbb{Z}_p^{\times 2})| = |(a - b + \mathbb{Z}_p^{\times 2}) \cap (-\mathbb{Z}_p^{\times 2})|.$$

If there is an edge from a to b, then by Lemmas 5.5.2 and 5.5.4,

$$(A)_{ab} = |(-1 + \mathbb{Z}_p^{\times 2}) \cap (-\mathbb{Z}_p^{\times 2})| = \frac{p-3}{4}.$$

If $a \neq b$ and there is no edge from a to b, then by a similar argument, we have $(A)_{ab} = \frac{p+1}{4}$. If $a = b$, then by Lemma 5.5.1,

$$(A)_{ab} = |(a + \mathbb{Z}_p^{\times 2}) \cap (b - \mathbb{Z}_p^{\times 2})| = |(\mathbb{Z}_p^{\times 2}) \cap (-\mathbb{Z}_p^{\times 2})| = 0,$$

which establishes Equation (5.5). ∎

Let

$$A^{n+1} = a_{n,p}A + b_{n,p}J_p + c_{n,p}I_p.$$

Hence

$$A^{n+1} = a_{n,p}A^2 + b_{n,p}\frac{p-1}{2}J_p + c_{n,p}A.$$

Then

$$A^{n+1} = (c_{n+1,p} - a_{n,p})A + (a_{n,p}\frac{p+1}{4} + b_{n+1,p}\frac{p-1}{2})J_p + (-a_{n,p}\frac{p+1}{4})I_p.$$

Thus, we have

$$\begin{cases} a_{n+1,p} = c_{n,p} - a_{n,p}, & a_{1,p} = -1, a_{2,p} = \frac{3-p}{4}; \\ b_{n+1,p} = \frac{p-1}{2}b_{n,p} + a_{n,p}\frac{p+1}{4}, & b_{1,p} = \frac{p+1}{4}, b_{2,p} = \frac{p+1}{4}(\frac{p-1}{2} - 1); \\ c_{n+1,p} = -a_{n,p}\frac{p+1}{4}, & c_{1,p} = -\frac{p+1}{4}, c_{2,p} = \frac{p+1}{4}. \end{cases}$$

From the first and last equations, we have the following homogeneous linear recurrence relation

$$a_{n+1,p} + a_{n,p} + \frac{p+1}{4}a_{n-1,p} = 0.$$

Let $\zeta = \dfrac{-1+i\sqrt{p}}{2}$. Since $a_{1,p} = -1$ and $a_{2,p} = \frac{3-p}{4}$, we deduce

$$a_{n,p} = \frac{i}{\sqrt{p}}(\overline{\zeta}^{n+1} - \zeta^{n+1}), \tag{i'}$$

where $i = \sqrt{-1}$. Then

$$c_{n,p} = \frac{i}{\sqrt{p}}\Big(\zeta^{n+1}\overline{\zeta} - \overline{\zeta}^{n+1}\zeta\Big). \tag{ii'}$$

Thus, for $b_{n,p}$ we have the following non-homogeneous linear recurrence relation

$$b_{n,p} = \frac{p-1}{2}b_{n-1,p} - \frac{i}{\sqrt{p}}\Big(\zeta^{n+1}\overline{\zeta} - \overline{\zeta}^{n+1}\zeta\Big).$$

Then by the usual methods we have,

$$b_{n,p} = \frac{1}{p}\Big((\frac{p-1}{2})^{n+1} + \zeta^{n+2} + \overline{\zeta}^{n+2}\Big). \tag{iii'}$$

Then the number of solutions of Equation (5.1) is

$$\mathscr{S}_{sq}(\mathbb{Z}_p, c, k) = \begin{cases} 2^k(b_{k-1,p} + c_{k-1,p}), & \text{if } c = 0; \\ 2^k(a_{k-1.p} + b_{k-1,p}), & \text{if } c = x^2, \text{ for some } x \in \mathbb{Z}_p^{\times}; \\ 2^k b_{k-1,p}, & \text{otherwise.} \end{cases}$$

Let $F_{p,c}(t) = \sum_{k=0}^{\infty} \mathscr{S}_{sq}(\mathbb{Z}_p, c, k)t^k$ be the ordinary generating function of $\mathscr{S}_{sq}(\mathbb{Z}_p, c, k)$. Then we have

$$F_{p,c}(t) = \begin{cases} \frac{1}{p}(\frac{1}{1-(p-1)t} - \frac{1-p+(-p-1)t}{1+2t+(p+1)t^2}), & \text{if } c = 0; \\ \frac{1}{p}(\frac{1}{1-(p-1)t} - \frac{1-p+(p-1)t}{1+2t+(p+1)t^2}), & \text{if } c = x^2, \text{ for some } x \in \mathbb{Z}_p^{\times}; \\ \frac{1}{p}(\frac{1}{1-(p-1)t} - \frac{1+(p-1)t}{1+2t+(p+1)t^2}), & \text{otherwise.} \end{cases}$$

Theorem 5.5.6. *Let p be a prime number such that $p \equiv 3 \mod 4$. Let α be a positive integer. Then*

$$\mathscr{S}_{sq}(\mathbb{Z}_{p^\alpha}, c, k) = \begin{cases} p^{(\alpha-1)(k-1)}2^k(b_{k-1,p} + c_{k-1,p}), & \textit{if } c \equiv 0 \mod p; \\ p^{(\alpha-1)(k-1)}2^k(a_{k-1,p} + b_{k-1,p}), & \textit{if } c = x^2, \textit{ for some } x \in \mathbb{Z}_{p^\alpha}^{\times}; \\ p^{(\alpha-1)(k-1)}2^k b_{k-1,p}, & \textit{otherwise,} \end{cases}$$

where $a_{k-1,p}$, $c_{k-1,p}$ and $b_{k-1,p}$ are defined by equations (i'), (ii') and (iii'), respectively, (putting $n = k-1$).

Proof. The proof is similar to that of Theorem 5.4.2. □

5.6 $\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$

In this section we find $\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$. For $\alpha = 1$ and $\alpha = 2$, this number is easy to find.

Lemma 5.6.1. *Let $n = 2^\alpha$ such that $\alpha > 2$. Then $\mathbb{Z}_n^{\times 2} = \Big\{8k + 1; k \in \{0, \dots, \frac{n}{8} - 1\}\Big\}$.*

Proof. Obviously, $\Big\{8k + 1; k \in \{0, \dots, \frac{n}{8} - 1\}\Big\} \supseteq \mathbb{Z}_n^{\times 2}$. It suffices to show that the set $\mathbb{Z}_n^{\times 2}$ has exactly $n/8$ elements. Define the equivalence relation between odd elements of $\mathbb{Z}_n$ as follows. We say $a \sim b$ if and only if $a^2 \equiv b^2 \mod 2^\alpha$. It is easy to check that each equivalence class has exactly 4 elements. Hence the number of equivalence classes is $n/8$, which is equal to the size of $\mathbb{Z}_n^{\times 2}$. □

Now, we are able to find $\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$.

Theorem 5.6.2. *Let $n = 2^\alpha$, $c \in \mathbb{Z}_n$ and $k \geq 1$. Then*

$$\mathscr{S}_{sq}(\mathbb{Z}_{2^\alpha}, c, k) = \begin{cases} 1, & \text{if } \alpha = 1 \text{ and } c \equiv k \mod 2; \\ 2^k, & \text{if } \alpha = 2 \text{ and } c \equiv k \mod 4; \\ 2^{2k+(\alpha-3)(k-1)}, & \text{if } \alpha > 2 \text{ and } c \equiv k \mod 8; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. $\alpha \leq 2$ is trivial. Let $\alpha > 2$. Let $A = \{(y_1, \dots, y_k); 8\sum_{i=1}^k y_i = c - k\}$ and $B = \{(x_1, \dots, x_k); \sum_{i=1}^k x_i^2 = c\}$. Then by Lemma 5.6.1, and since each equivalent class has 4 elements, there exists a 4^k to 1 and onto map from B to A. By Lemma 5.6.1, one can check that, if $c \equiv k \mod 8$, then $|A| = (2^{\alpha-3})^{k-1}$, which establishes the formula. □

Remark 7. Let $n = p_1^{\alpha_1} \dots p_t^{\alpha_t}$. Then by Lemma 5.3.1, we conclude that

$$\mathscr{S}_{sq}(\mathbb{Z}_n, c, k) = \prod_{i=1}^{t} \mathscr{S}_{sq}(\mathbb{Z}_{p_i^{\alpha_i}}, c, k),$$

which can be computed easily by Theorems 5.4.2, 5.5.6 and 5.6.2.

Bibliography

[1] A. Brauer, *Lösung der Aufgabe 30, Jahresber*. Dtsch. Math.-Ver. 35 (1926) 92–94.

[2] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.

[3] D. Kiani, M. Mollahajiaghaei, *On the addition of units and non-units in finite commutative rings*, Rocky Mountain J. Math. 45 (6) (2015), 1887–1896.

[4] J. W. Sander, *On the addition of units and nonunits mod m*, J. Number Theory., 129 (2009), 2260–2266.

[5] L. Tóth, *Counting solutions of quadratic congruences in several variables revisited*, J. Integer Seq. 17 (2014), Article 14.11.6.

[6] P. M. Weichsel, *The Kronecker product of graphs*. Proc. Am. Math. Soc. 13 (1962), 47–52.

[7] Q. Yang, M. Tang, *On the addition of squares of units and nonunits modulo n*, J. Number Theory., 155 (2015) 1–12.

Chapter 6

Conclusion

6.1 On the algorithmic complexity of adjacent vertex closed distinguishing number of graphs

In Chapter 2, we proved that for each integer t, there is a bipartite graph G such that $\mathrm{dis}[G] > t$. The size of the graph G is exponential. So we ask the following question:

Problem 6.1.1. *For each positive integer t, is there a bipartite graph G such that $V(G) = \mathcal{O}(t^c)$ and $\mathrm{dis}[G] > t$, where c is a constant number.*

What can we say about the upper bound in bipartite graphs? Perhaps one of the most intriguing open question in this scope is the case of bipartite graphs.

Problem 6.1.2. *Let G be a bipartite graph, is $\mathrm{dis}[G] \leq \mathcal{O}(\sqrt{\Delta(G)})$?*

The polynomial method is a relatively new and powerful method in combinatorics and graph theory. We provided some number of upper bounds by using a beautifully simple application of the Combinatorial Nullstellensatz. One may ask can we find lower bounds using the polynomial or algebraic method?

6.2 Algorithmic complexity of weakly semiregular partitioning, and the representation number of graphs

In Chapter 3, we proved that for every tree T, $\mathrm{wr}(T) \leq 2\log_2 \Delta(T) + \mathcal{O}(1)$. On the other hand, there are infinitely many values of Δ for which the tree T might be chosen so that $\mathrm{wr}(T) \geq \log_3 \Delta(T)$. Finding the best upper bound for trees can be interesting. Also, it would be desirable to generalize the upper bound to an arbitrary simple graph.

We proved that there is a polynomial time algorithm to determine whether the weakly semiregular number of a given tree is at most c. Is there any polynomial time algorithm to determine the weakly semiregular number of trees?

6.3 Resonance varieties of sparse paving matroids

In Chapter 4, we provided some theorems about the resonance varieties of sparse paving matroid. We found $R^{r-2}(M)$, if the intersection of all of the minimum circuits of M except one of them is non-empty. Also, we proved that if the rank of M is large enough in comparison to the number of minimum circuits, then $R^{r-2}(M)$ is trivial.

It would be desirable to generalize these results to an arbitrary matroid.

Also, we expressed the map f_a as a matrix. What is still lacking is an explicit description of the matrix.

6.4 On the addition of squares of units modulo n

In Chapter 5, we found an explicit formula for the number of representation of an element in the ring $\mathbb{Z}_n$ as the sum of k invertible squares. It would be interesting to generalize this formula to an arbitrary ring. This question is at present far from being solved.

Let $(t_1, \ldots, t_k) \in \mathbb{N}^k$. Consider the following equation

$$x_1^{t_1} + \cdots + x_k^{t_k} = c, \tag{6.1}$$

where $c \in \mathbb{Z}_n$, and x_i are all units in the ring $\mathbb{Z}_n$. It would be desirable to find an explicit formula for the number of solutions of this equation.

Appendix A

Curriculum Vitae

Mohsen Mollahajiaghaei

Education

2013–2017 **PhD, Pure Mathematics**, *University of Western Ontario*, *London, Canada*.

2009–2011 **MS, Pure Mathematics-Algebra**, *Amirkabir University of Technology (Tehran Polytechnic)*, *Tehran, Iran*.

2005–2009 **BS, Pure Mathematics**, *Amirkabir University of Technology*, *Tehran, Iran*.

Publications

2011 D. Kiani, M. Mollahajiaghaei, Y. Meemark and B. Suntornpoch, Energy of unitary Cayley graphs and gcd-graphs, Linear Algebra Appl, 435 (2011) 1336-1343.

2012 D. Kiani and M. Mollahajiaghaei, On the unitary cayley graph, Electronic Journal of Combinatorics, 19(2) (2012), P10.

2012 M. Mollahajiaghaei, The eigenvalues and energy of integral circulant graphs, Transactions on Combinatorics, Volume 1, Number 3, 2012, 47-56.

2015 D. Kiani and M. Mollahajiaghaei, On the unitary Cayley graph on $M_n(F)$, Linear Algebra Appl, 466 (2015), 421-428.

2015 D. Kiani and M. Mollahajiaghaei, On the addition of units and non-units in finite commutative rings, Rocky Mountain J. Math., Volume 45, Number 6 (2015), 1887–1896.

2017 M. Mollahajiaghaei, On the addition of squares of units modulo n, Journal of Number Theory 170 (2017) 35-45.

2017 A. Dehghan and M. Mollahajiaghaei, Adjacent vertex closed distinguishing colorings number of graphs, Discrete Applied Mathematics. Volume 218, 82–97.

2016 M. Mollahajiaghaei, Properties of the dot product graph of a commutative ring, submitted.

2017 A. Ahadi, A. Dehghan and M. Mollahajiaghaei, Algorithmic complexity of weakly semiregular partitioning and the representation number, Theor Comput Sci, 674 (2017), 60–72.

2017 M. Mollahajiaghaei, Resonance varieties of Sparse Paving Matroids, Preprint.

Awards, Honors and Scholarships

July 2009	16^{th} International mathematics competition for university students (IMC-2009)	Budapest, Hungary.	**Second Prize**
July 2008	15^{th} International mathematics competition for university students (IMC-2008)	Blagoevgrad, Bulgaria.	**Third Prize**
May 2008	32^{nd} National mathematical competition for university students (IMS-2008)	Tehran, Iran.	**Gold Medal**

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May 2007	31^{st} National mathematical competition for university students (IMS-2007)	Mashhad, Iran.	**Bronze Medal**
July 2009	10^{th} International scientific Olympiad (Mathematics)	Tehran, Iran.	7^{th} **Place**

March 2008 and March 2007 — Awarded as Outstanding Student by **university president**.

Experience

Research

2008-2009 — **B.Sc. Thesis**, *Amirkabir University of Technology*.
- Rainbow Connection in Graphs
- Supervised by D. Kiani

2009-2011 — **M.Sc. Thesis**, *Amirkabir University of Technology*.
- On Unitary Cayley Graphs Over Rings
- Supervised by D. Kiani

2011-2013 — **Research Assistant**, *Institute for Research in Fundamental Sciences (IPM)*.
-Supervised by D. Kiani

2013-present — **Research Assistant**, *Univesity of Western Ontario*.
-Supervised by G. Denham

Teaching

Calculus 1000A — **Mathematics department**, *University of Western Ontario*.
Fall 2016

Teaching Assistant

Amirkabir University of Technology — **Mathematics department**, *Algebra 1 (Fall 2006, Spring 2007 and Spring 2008), Algebra 2 (Spring 2011), Calculus 1 (Fall 2009 and 2010), Discrete Mathematics (Fall 2008 and Spring 2009), Graph Theory (Spring 2010), Algebraic Graph Theory (Graduate course) (Fall 2011)*.

Amirkabir University of Technology — **Electrical engineering department**, *Calculus 2 (Spring 2010)*.

Western University — **Mathematics department**, *Calculus 0110, 1000 and 1301, Math 1228, 1229, 3152 (Fall 2013-Now)*, Spectral Graph Theory (Graduate course) (Summer 2015).

Talks

June. 1-4. 2015 — Independence and domination number of simplicial rook graphs
5th biennial Canadian Discrete and Algorithmic Mathematics Conference (CanaDAM), University of Saskatoon, Canada

May. 21–24. 2015 — Representation numbers of graphs
Discrete Math Days and Ontario Combinatorics Workshop, University of Ottawa, Canada

June. 02. 2011 — On the Integral Circulant Graphs
Workshop On Graphs and Algorithms. IPM. Tehran. Iran

Nov. 03. 2010 — On Unitary Cayley graphs
Weekly Seminars on Combinatorics and Computing IPM. Tehran. Iran