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Properties of k-isotropic functions

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Abstract

The focus of this work is a family of maps from the space of $n \times n$ symmetric matrices, S^n, into the space $S^{\binom{n}{k}}$ for any $k = 1, \dots, n$, invariant under the conjugate action of the orthogonal group O^n. This family, called generated k-isotropic functions, generalizes known types of maps with similar invariance property, such as the spectral, primary matrix, isotropic functions, multiplicative compound, and additive compound matrices on S^n.

The notion of operator monotonicity dates back to a work by Löwner in 1934. A map $F : S^n \to S^m$ is called *operator monotone*, if $A \succeq B$ implies $F(A) \succeq F(B)$. (Here, '$\succeq$' denotes the semidefinite partial order in S^n.) Often, the function F is defined in terms of an underlying simpler function f. Of main interest is to find the properties of f that characterize operator monotonicity of F. In that case, it is said that f is also operator monotone. Classical examples are the Löwner's operators and the spectral (scalar-valued isotropic) functions. Operator monotonicity for these two classes of functions is characterized in seemingly very different ways.

The work in Chapter 1 extends the notion of operator monotonicity to symmetric functions f on k arguments. The latter is used to define *(generated) k-isotropic functions* $F : S^n \to S^{\binom{n}{k}}$ for any $n \geq k$. Necessary and sufficient conditions are given for f to characterize an operator monotone k-isotropic map F. Then, in Chapter 2, we give necessary and sufficient conditions for the analyticity of (generated) k-isotropic functions.

When $k = 1$, the k-isotropic map becomes a Löwner's operator and when $k = n$ it becomes a spectral functions. This allows us to reconcile and explain the differences between the conditions for monotonicity and analyticity for the Löwner's operators and the spectral functions.

We say that a function $F : S^n \to S^{n^k}$ is *k-tensor isotropic*, if it satisfies

$$F(UAU^\top) = (U^{\otimes k})F(A)(U^{\otimes k})^\top$$

for all $U \in O^n$ and all A in the domain of F. Here, '$\otimes k$' denotes the k-th tensor power. The

goal of Chapter 3 is to investigate the internal structure of the k-tensor isotropic functions and formulate a canonical representation of F in terms of simpler functions on $\mathbb{R}^n$. We achieve this goal in the case when $k = 2$ for any natural n. In the process, we characterize the structure of the matrices in the centralizer of certain orthogonal subgroups of O^{n^k}.

An orthogonally invariant class of operator functions, F^H is studied in Chapter 4. Then, we connect F^H to (generated) k-isotropic functions, when it is restricted to block diagonal matrices. This connection allows us to establish various smoothness properties of F^H.

Keywords: Spectral function, primary matrix function, Löwner's operator, isotropic function, k-isotropic function, symmetric function, analyticity, multiplicative compound matrix, additive compound matrix, tensor product, anti-symmetric tensor product, operator monotone function, Pick function, Bernstein function, positive map, k-tensor isotropic function, centralizer, orthogonal group, differentiability, orthogonally invariant function

Co-Authorship Statement

I hereby declare that this thesis incorporates materials that are direct results of my main efforts.

Chapter 2 is the paper co-authored with Dr. H. Sendov, submitted to *Linear Algebra and its Applications*.

Chapter 3 is the paper co-authored with Dr. S. Mousavi, and H. Sendov, submitted to *Linear Algebra and its Applications*.

Chapter 4 is the paper co-authored with Dr. H. Sendov, submitted to *Electronic Journal of Linear Algebra*.

Chapter 5 is the paper co-authored with Dr. H. Sendov, submitted to *Operators and Matrices*.

This thesis employed an integrated-article format follows Western's thesis guidelines. Each chapter is self-contained and can be read independently.

Dedicated to my wife and parents

for their support, love, and encouragement.

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Chapter 1

Introduction

Let $\mathbb{N}_n := \{1, \ldots, n\}$. Denote by $\mathbb{R}^{n\times n}$ the space of all $n \times n$ real matrices and denote by S^n the space of all $n\times n$ symmetric matrices with inner product $\langle A, B\rangle := \mathrm{Tr}\,(AB)$. Let O^n be the group of $n\times n$ orthogonal matrices. Denote by $\mathbb{R}^n_{\geq}$ the convex cone in $\mathbb{R}^n$ consisting of all vectors with coordinates non-increasingly ordered. For any $A \in S^n$, let $\lambda(A) \in \mathbb{R}^n_{\geq}$ be the vector of ordered eigenvalues of A. Denote by $\mathrm{Diag}\, x$ the $n \times n$ matrix with $x \in \mathbb{R}^n$ on the main diagonal. Denote by P^n the collection of all $n \times n$ permutation matrices.

For any $A \in S^n$, we use the notation $A \succeq 0$, whenever A is positive semidefinite. Denote by S^n_+ the closed convex cone consisting of all positive semidefinite matrices in S^n. The cone S^n_+ defines a partial order on S^n in the following way. For any $A, B \in S^n$, we use the notation $A \succeq B$, whenever $A - B \succeq 0$.

Definition 1.0.1 *A map $F : S^n \to S^m$ is called operator monotone, if*

$$A \succeq B \text{ implies } F(A) \succeq F(B)$$

for any A and B in the domain of F.

A characterization of operator monotonicity is as follows.

Proposition 1.0.2 *Let $F : S^n \to S^m$ be a C^1 map defined on a convex domain with non-empty* *interior. Then, F is operator monotone, if and only if $\nabla F(A)[H] \geq 0$ for all $H \in S^n_+$ and all A in the domain of F.*

The focus of this thesis is to show the operator monotonicity and analyticity of a class of orthogonally invariant matrix-valued functions and study the connection to other types of orthogonally invariant matrix-valued functions. This class of functions captures three previously investigated classes of orthogonally invariant matrix-valued functions. We start with the three special cases.

Definition 1.0.3 *A real-valued function $F : S^n \to \mathbb{R}$ is called a spectral function, if*

$$F(UAU^\top) = F(A)$$

holds for all $U \in O^n$ and all $A \in S^n$ in the domain of F.

Spectral functions are also known as *scalar-valued isotropic functions*. They have been extensively studied and applied in various areas, for example, engineering, see [25], material science, see [21], and optimization and variational analysis, see [14].

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *symmetric*, if $f(Px) = f(x)$ holds for any $x \in \mathbb{R}^n$ and any permutation matrix $P \in P^n$. The representation theorem of spectral functions is as follows and can be found in [6] and [22].

Theorem 1.0.4 *A real-valued function $F : S^n \to \mathbb{R}$ is a spectral function, if and only if there exists a unique symmetric function $f : \mathbb{R}^n \to \mathbb{R}$ such that $F(A) = (f \circ \lambda)(A)$ for all $A \in S^n$.*

Properties of a spectral function and its corresponding symmetric function f are closely connected. For example, a spectral function F is differentiable at A, if and only if f is differentiable at $\lambda(A)$. The evolution of research in this area can be found in [13], [15], [19], [20], and [23]. The analyticity of spectral functions has been proven in [24]: F is is analytic at A, if and only if f is analytic at $\lambda(A)$.

Examples of operator monotone spectral functions are shown as follow: $\det A$, $-\det A^{-1}$ for $A \succ 0$, and $\operatorname{Tr} A$. The next theorem shows the characterization of operator monotonicity of spectral functions.

Theorem 1.0.5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be symmetric function with corresponding spectral function $F : S^n \to \mathbb{R}$. Then, F is operator monotone, if and only if f is non-decreasing in each argument.*

We now introduce another class of orthogonally invariant functions.

Definition 1.0.6 *A function $F : S^n \to S^n$ is called a primary matrix function, if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$F(A) = U(\operatorname{Diag}(f(\lambda_1(A)), \ldots, f(\lambda_n(A))))U^\top, \tag{1.1}$$

where $U \in O^n$ is such that $A = U(\operatorname{Diag} \lambda(A))U^\top$.

Primary matrix functions are also known as *Löwner's operator* functions. One can see that primary matrix functions are well-defined, since the right-hand side of (1.1) does not depend on the choice of the diagonalizing matrix U of A.

Several properties, for example, derivatives, operator monotonicity, and operator convexity of primary matrix functions are studied and are characterized in terms of the underlying function f, see for example [4, Chapter V] and [10, Chapter 6].

The primary matrix function F is a continuously differentiable at A, if and only if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable at each $\lambda_i(A)$ for $i \in \mathbb{N}_n$, see [4, Theorem V.3.3]. The differential of F is expressed by the first divided differences of f, see [4, Theorem V.3.3]. For any $x \in \mathbb{R}^n$ in the domain of f, define the $n \times n$ divided difference matrix

$$(f^{[1]}(x))_{ij} := \begin{cases} f'(x_i) & \text{if } x_i = x_j, \\ \dfrac{f(x_i) - f(x_j)}{x_i - x_j} & \text{if } x_i \neq x_j, \end{cases} \tag{1.2}$$

for $i, j \in \mathbb{N}_n$. Then, we have

$$\nabla F(A)[H] = U(f^{[1]}(\lambda(A)) \circ (U^\top HU))U^\top$$

for any A in the domain of F and $U \in O^n$ such that $A = U(\operatorname{Diag} \lambda(A))U^\top$. Here, '$\circ$' denotes the Hadamard product between two matrices.

If f is analytic, then (1.1) becomes

$$F(A) = \oint_\Gamma f(z)(zI - A)^{-1}\, dz,$$

where Γ is a Jordan curve in the complex plane enclosing the eigenvalues of A. Thus, the primary matrix function F is analytic, if and only if f is analytic, see [12, Chapter 7].

Definition 1.0.7 *A function $f : \mathbb{R} \to \mathbb{R}$ is called operator monotone of order n, if the corresponding primary matrix function $F : S^n \to S^n$ is operator monotone. A function $f : \mathbb{R} \to \mathbb{R}$ is called operator monotone, if the corresponding primary matrix function $F : S^n \to S^n$ is operator monotone for all n.*

The following examples are collected in [4, Chapter V]. The functions x^r for $x \geq 0$ and $r \in [0, 1]$, $-1/x$ for $x > 0$, and $x/(1 + x)$ for $x > 0$ are operator monotone. See [7] for more examples. Operator monotonicity of order n is characterized as follows.

Theorem 1.0.8 *Let I be an interval in $\mathbb{R}$. A continuously differentiable function $f : I \to \mathbb{R}$ is operator monotone of order n, if and only if $f^{[1]}(x)$ given by* (1.2) *is a positive semidefinite matrix for every $x \in \mathbb{R}^n$ with coordinates in I.*

Note that in Theorem 1.0.8, the matrix dimension n was implicitly fixed. An operator monotone function f defined on an interval I can be characterized by a *Pick function*, see [4, Theorem V.4.7] and Nevanlinna's theorem [4, Theorem V.4.11].

Theorem 1.0.9 *A function $f : I \to \mathbb{R}$ is an operator monotone function, if and only if*

$$f(x) = a + bx + \int_{-\infty}^{+\infty} \Big(\frac{1}{\lambda - x} - \frac{\lambda}{\lambda^2 + 1}\Big)\, d\mu(\lambda),$$

for some $a \in \mathbb{R}, b \geq 0$, and μ a positive Borel measure on $\mathbb{R}$ with zero mass on I, such that

$$\int_{-\infty}^{+\infty} \frac{1}{\lambda^2 + 1}\, d\mu(\lambda) < \infty.$$

In [18, Chapter 6], the connection between operator monotone functions and *complete Bernstein functions* is explained. That is, a function $f : (0, \infty) \to [0, \infty)$ is operator monotone, if and only if it is a complete Bernstein function.

The class of primary matrix functions is a special case of the following class of functions.

Definition 1.0.10 *A function $F : S^n \to S^n$ is called a tensor-valued isotropic function, if*

$$F(UAU^\top) = UF(A)U^\top,$$

for all $U \in O^n$ and all $A \in S^n$ in the domain of F.

An example of tensor-valued isotropic functions found in [21] is the Piola-Kirchhoff stress function in an isotropic solid.

We say a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *symmetric*, if $f(Px) = Pf(x)$ holds for any $x \in \mathbb{R}^n$ and any permutation matrix $P \in P^n$. The following representation theorem for tensor-valued isotropic functions can be found in [21] and [22].

Theorem 1.0.11 *A function $F : S^n \to S^n$ is a tensor-valued isotropic function, if and only if there exists a symmetric function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$F(A) = U(\mathrm{Diag}\, f(\lambda(A)))U^\top,$$

where $U \in O^n$ is such that $A = U(\mathrm{Diag}\, \lambda(A))U^\top$.

Note that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that $f(x) = (g(x_1), \dots, g(x_n))$ for some $g : \mathbb{R} \to \mathbb{R}$, then the tensor-valued isotropic function becomes primary matrix function (1.1).

We now introduce the class of functions that capture the spectral, the primary matrix functions, and tensor-valued isotropic functions.

Let $\mathbb{N}_{n,k}$ be the set of all subsets of $\mathbb{N}_n$ of size $k \in \mathbb{N}_n$ with elements ordered non-decreasingly. Order the elements in $\mathbb{N}_{n,k}$ lexicographically so that we use them to index the coordinates of vectors in $\mathbb{R}^{\binom{n}{k}}$ and the entries of matrices in $\mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$. For any $\mathbf{x} \in \mathbb{R}^{\binom{n}{k}}$ and any $\rho \in \mathbb{N}_{n,k}$, denote by $\mathbf{x}_\rho$ the ρ-th element in vector $\mathbf{x}$. For any $\mathbf{A} \in \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$ and any $\rho, \tau \in \mathbb{N}_{n,k}$, denote by $\mathbf{A}_{\rho,\tau}$ the element in the ρ-th row and τ-th column of $\mathbf{A}$.

Then, for any $A \in \mathbb{R}^{n\times n}$ and any $\rho, \tau \in \mathbb{N}_{n,k}$, denote by $A_{\rho\tau}$ the $k \times k$ minor of A with indexes in the intersection of ρ_1-th , . . . , ρ_k-th rows and τ_1-th , . . . , τ_k-th columns.

For $A \in \mathbb{R}^{n\times n}$, the *k-th multiplicative compound matrix of* A, $A^{(k)} \in \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$, is defined by

$$(A^{(k)})_{\rho,\tau} := \det(A_{\rho\tau}) \quad \text{for any } \rho, \tau \in \mathbb{N}_{n,k}.$$

Definition 1.0.12 *A function $F : S^n \to S^{\binom{n}{k}}$ is called k-isotropic, if*

$$F(UAU^\top) = U^{(k)}F(A)(U^{(k)})^\top$$

for all $U \in O^n$ and $A \in S^n$ in the domain of F.

Definition 1.0.13 *A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is called symmetric, if the equation*

$$\operatorname{Diag} \mathbf{f}(Px) = P^{(k)}(\operatorname{Diag} \mathbf{f}(x))(P^{(k)})^\top$$

holds for all $x \in \mathbb{R}^n$ and all permutation matrices $P \in P^n$.

The representation theorem of k-isotropic functions in [16] is shown as follows.

Theorem 1.0.14 *A function $F : S^n \to S^{\binom{n}{k}}$ is k-isotropic, if and only if there is a unique symmetric function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ such that*

$$F(A) = U^{(k)}(\operatorname{Diag}\mathbf{f}(\lambda(A)))(U^{(k)})^{\top}, \tag{1.3}$$

for all $A \in S^n$ and $U \in O^n$ such that $A = U(\operatorname{Diag}\lambda(A))U^{\top}$.

For any $x \in \mathbb{R}^n$ and any $\rho \in \mathbb{N}_{n,k}$, let

$$x_\rho := (x_{\rho_1}, \ldots, x_{\rho_k}) \in \mathbb{R}^k.$$

Any symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ defines a symmetric function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$, see [16], by

$$\mathbf{f}_\rho(x) := f(x_\rho) \text{ for all } x \in \mathbb{R}^n \text{ and all } \rho \in \mathbb{N}_{n,k}.$$

In this case, (1.3) is called a *(generated) k-isotropic function*.

A (generated) k-isotropic function becomes a spectral function, if we take $k = n$. In that case, we have $U^{(k)} = \det(U) = \pm 1$, since U is orthogonal and the set $\mathbb{N}_{n,n}$ contains only one element $\{1, 2, \ldots, n\}$. Thus,

$$F(A) = f(\lambda_1(A), \ldots, \lambda_n(A)).$$

A (generated) k-isotropic function becomes a primary matrix function, if we take $k = 1$. In that case, we have $U^{(1)} = U$ and the set $\mathbb{N}_{n,1}$ contains n elements $\{1\}, \ldots, \{n\}$. Thus,

$$F(A) = U(\operatorname{Diag}(f(\lambda_1(A)), \ldots, f(\lambda_n(A))))U^T.$$

A k-isotropic function becomes a tensor-valued isotropic function, if we take $k = 1$. Similar to the case of primary matrix functions, we have $U^{(1)} = U$ and the set $\mathbb{N}_{n,1}$ contains n elements $\{1\}, \ldots, \{n\}$. Thus,

$$F(A) = U(\operatorname{Diag}\mathbf{f}(\lambda(A)))U^T.$$

Some other examples are shown as follows. If we take $f(x_1, \ldots, x_k) := x_1 + \cdots + x_k$, a (generated) k-isotropic function becomes the k-th additive compound matrix, see [11, page 19]. If we take $f(x_1, \ldots, x_k) := x_1 \cdots x_k$, a (generated) k-isotropic function becomes the k-th multiplicative compound matrix, see [17].

The main result shown as follows generalizes the results in [3], [20], and [23], when $k = n$, and generalizes the result in [5], when $k = 1$, see [16].

Theorem 1.0.15 *Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is symmetric and $F : S^n \to S^{\binom{n}{k}}$ is its corresponding k-isotropic function. Then, F is C^r if and only if $\mathbf{f}$ is C^r for any $r = 1, \ldots, \infty$.*

The technique in [16] cannot show the analyticity of k-isotropic functions. Thus, a different technique is used in Chapter 2. We lift a (generated) k-isotropic function to a map from S^n to S^{n^k}. By proving the analyticity of such lifted map, we obtain that a (generated) k-isotropic function is analytic at A, if and only if the underlying symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ is analytic at $\lambda(A)$.

The main goal in Chapter 3 is to characterize operator monotonicity of the (generated) k-isotropic functions in terms of the underlying symmetric function $f : \mathbb{R}^k \to \mathbb{R}$. The main result is shown as follows. Let $I^k := I \times \cdots \times I$ for k times. A symmetric, C^1 function $f : I^k \to \mathbb{R}$ is operator monotone (of order n), if and only if the function $f(\cdot, \dot{x})$ is operator monotone on I (of order $n - k + 1$) for all $\dot{x} \in \mathbb{R}^{k-1}$.

This result explains the apparent difference between Theorem 1.0.8 and Theorem 1.0.5, if we increase k from 1 to n.

The class of orthogonally invariant functions studied in works [1], [2], [8], [9], and [26] generalizes primary matrix functions to a class of functions on several operator arguments.

We now give the construction. For any fixed $n_1, \ldots, n_k$, k-tuples in $\mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$ are ordered lexicographically. Any function $f : \mathbb{R}^k \to \mathbb{R}$ defines an operator map $F^H : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$ by

$$F^H(A_1, \ldots, A_k) := (\otimes_{i=1}^k U_i)(\mathrm{Diag}_l\, f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k)))(\otimes_{i=1}^k U_i)^\top \tag{1.4}$$

for $l = (l_1, \ldots, l_k) \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$, where $U_i \in O^{n_i}$ are such that $A_i = U_i(\operatorname{Diag} \lambda(A_i))U_i^\top$ for $i \in \mathbb{N}_k$. Here, Diag_l denotes a diagonal matrix with the values $f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$ on the main diagonal ordered lexicographically. Function (1.4) is well-defined, since the right-hand side of the function does not depend on the choice of the diagonalizing matrices U_i for $i \in \mathbb{N}_k$. One can see that the map (1.4) becomes a primary matrix function (1.1), when $k = 1$.

The map defined by (1.4) is orthogonally invariant, that is,

$$F^H(U_1 A_1 U_1^\top, \ldots, U_k A_k U_k^\top) = (\otimes_{i=1}^k U_i) F^H(A_1, \ldots, A_k) (\otimes_{i=1}^k U_i)^\top \tag{1.5}$$

for any $U_i \in O^{n_i}, i \in \mathbb{N}_k$.

We want a representation theorem of functions $F^H : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$ that satisfy (1.5) in the same pattern as we have for Theorems 1.0.11 and 1.0.14. In Chapter 4, we work under a particular case, when $n_1 = \cdots = n_k =: n$ and $A_1 = \cdots = A_k =: A \in S^n$ to formulate a representation theorem for maps $F : S^n \to S^{n^k}$, called *k-tensor isotropic functions*, satisfying

$$F(UAU^\top) = (\otimes_{i=1}^k U) F(A) (\otimes_{i=1}^k U)^\top$$

for all $U \in O^n$ and all A in the domain of F. We solve the problem fully in the case of $k = 2$.

In Chapter 5, we study the connection between the class of (generated) k-isotropic functions and (1.4), when the underlying function is symmetric. It allows us to characterize the differentiability of (1.4) by applying Theorem 1.0.15. Characterization of the analyticity of F^H in terms of f is obtained, where f is not necessarily symmetric.

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Chapter 2

A unified approach to operator monotone functions

2.1 Introduction

2.1.1 Connecting operator monotone and spectral functions

Denote by $\mathbb{R}^{n\times n}$ the space of all $n \times n$ real matrices. Denote by $S^n \subset \mathbb{R}^{n\times n}$ the Euclidean space of symmetric matrices with $\langle A, B\rangle := \mathrm{Tr}\,(AB)$ and Frobenius norm $\|A\| := \sqrt{\mathrm{Tr}\,(AA)}$. Let O^n be the group of $n \times n$ orthogonal matrices. Denote by $\mathbb{R}^n_{\geq}$ the convex cone in $\mathbb{R}^n$ of all vectors with non-increasing coordinates. For any $x, y \in \mathbb{R}^n$, we write that $x \geq y$ when $x_i \geq y_i$ for all $i = 1, \ldots, n$. For any $A \in S^n$, let $\lambda(A) \in \mathbb{R}^n_{\geq}$ be the vector of eigenvalues of A, ordered non-increasingly: that is, $\lambda_i(A)$ is the i-th largest eigenvalue. For any vector $x \in \mathbb{R}^n$, let $\mathrm{Diag}\, x \in S^n$ be the matrix with x on the main diagonal and zeros elsewhere. For any $A \in \mathbb{R}^{n\times n}$, let $\mathrm{diag}\, A \in \mathbb{R}^n$ be the diagonal of A.

For $A \in S^n$, we write $A \succeq 0$ when A is positive semidefinite matrix. The set of all positive semidefinite matrices in S^n is a closed convex cone, denoted by S^n_+. This cone defines a partial order on S^n as follows. For any $A, B \in S^n$, we write $A \succeq B$, whenever $A - B \succeq 0$. The focus of this work is the monotonicity of functions with respect to this partial order.

Definition 2.1.1 *A map $F : S^n \to S^m$ is called operator monotone, if*

$$A \geq B \text{ implies } F(A) \geq F(B)$$

for any A and B in the domain of F.

A characterization of operator monotonicity is easy to obtain as we now recall.

Proposition 2.1.2 *Let $F : S^n \to S^m$ be a C^1 map defined on a convex domain with non-empty interior. Then, F is operator monotone, if and only if $\nabla F(A)[H] \geq 0$ for all $H \in S^n_+$ and all A in the domain of F.*

Proof Suppose that F is operator monotone. Let $A \in S^n$ be in the interior of the domain of F and let $H \in S^n_+$. Then, we have $F(A + tH) - F(A) \geq 0$ for all small enough positive t. This implies that $\nabla F(A)[H]$ is positive semidefinite. (Taking limit, the conclusion holds for all A in the domain of F.)

For the other direction, suppose that $\nabla F(A)[H] \geq 0$ for all $H \in S^n_+$ and all A in the domain of F. For any $A, B \in S^n$ with $B \geq A$, let $A(t) := (1 - t)A + tB$ for $t \in [0, 1]$. Then, one can see that

$$F(B) - F(A) = \int_0^1 \nabla F(A(t))[A'(t)]\, dt \geq 0,$$

since $A'(t) = B - A \geq 0$ for all $t \in [0, 1]$.

Often, a function F on a domain of S^n, is defined in terms of an underlying simpler function f. In that case, of main interest is to find what properties of f characterize operator monotonicity of F. Classical examples are the primary matrix functions and the spectral functions that we now describe.

Definition 2.1.3 *A real-valued function $F : S^n \to \mathbb{R}$ is called a spectral function, if*

$$F(UAU^{\top}) = F(A)$$

holds for all $U \in O^n$ and all A in the domain of F.

Spectral functions are also known as *scalar-valued isotropic functions*. They have numerous applications in the fields of optimization, engineering, and material science, see for example [13], [23], and [27].

The following representation theorem of spectral functions is easy to deduce, see [5] or [24]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *symmetric*, if $f(Px) = f(x)$ holds for any $n \times n$ permutation matrix P and any $x \in \mathbb{R}^n$ in the domain of f.

Theorem 2.1.4 *A real-valued function $F : S^n \to \mathbb{R}$ is a spectral function, if and only if there exists a unique symmetric function $f : \mathbb{R}^n \to \mathbb{R}$ such that $F(A) = (f \circ \lambda)(A)$ for all A in the domain of F.*

The spectral functions are in one-to-one correspondence with the symmetric functions. Efforts have been focused on identifying the properties of the spectral functions that are inherited from the corresponding symmetric functions. For example, F is convex, if and only if the corresponding symmetric function f is, see [3]. Even though the eigenvalue map $\lambda : S^n \to \mathbb{R}^n$ is not differentiable everywhere, F is differentiable at A, if and only if f is differentiable at $\lambda(A)$, see [12], [14], [21], [22], and [25]. Further, in [26], the authors show that F is analytic at A if and only if f is analytic at $\lambda(A)$. The list of such transferable properties is quite long, but not every property of f is inherited directly by F, for example Gâteaux differentiability, see [12, page 587].

Examples of operator monotone spectral functions include $\det A$; $-\det A^{-1}$ for $A \succ 0$; and $\operatorname{Tr} A$. It is easy to see when a spectral function is operator monotone as the next result shows. Its proof is easy and well-known, we include it for completeness.

Theorem 2.1.5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be symmetric function with corresponding spectral function $F : S^n \to \mathbb{R}$. Then, F is operator monotone, if and only if f is non-decreasing in each argument.*

Proof Suppose that F is operator monotone. For any $x, y \in \mathbb{R}^n$ with $x \geq y$, we have $f(x) = F(\operatorname{Diag} x) \geq F(\operatorname{Diag} y) = f(y)$.

For the other direction, suppose that the symmetric function f is non-decreasing in each argument. For any $A, B \in S^n$ with $A \geq B$, by Weyl's monotonicity theorem, see [2, Corollary III.2.3], we have $\lambda(A) \geq \lambda(B)$. In that case, $F(A) = f(\lambda(A)) \geq f(\lambda(B)) = F(B)$.

We proceed with describing the primary matrix functions.

Definition 2.1.6 *A function $F : S^n \to S^n$ is called a primary matrix function, if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$F(A) = U(\operatorname{Diag}(f(\lambda_1(A)), \ldots, f(\lambda_n(A))))U^\top,$$

where $U \in O^n$ is such that $A = U(\operatorname{Diag} \lambda(A))U^\top$.

Primary matrix functions are also known as *Löwner's operator* functions. It is easy to see that they are well-defined, meaning that the value of $F(A)$ does not depend on the choice of the orthogonal matrix U diagonalizing A.

Characterizing when a primary matrix function F is operator monotone in terms of its corresponding function f has been the topic of extensive research in the past. One can specialize Proposition 2.1.2 and for that we need a description of the differential of F. It is known that F is a a continuously differentiable at A, if and only if $f : \mathbb{R} \to \mathbb{R}$ is such at each $\lambda_i(A)$ for $i = 1, \ldots, n$, see [2, Theorem V.3.3]. The differential of F is described in terms of the first divided differences of f as follows. For any $x \in \mathbb{R}^n$, such that all $x_1, \ldots, x_n$ are in the domain of f, define the $n \times n$ divided difference matrix

$$(f^{[1]}(x))_{ij} := \begin{cases} f'(x_i) & \text{if } x_i = x_j, \\ \dfrac{f(x_i) - f(x_j)}{x_i - x_j} & \text{if } x_i \neq x_j, \end{cases} \tag{2.1}$$

for $i, j = 1, \ldots, n$. Then, see for example [2, Theorem V.3.3], we have

$$\nabla F(A)[H] = U(f^{[1]}(\lambda(A)) \circ (U^\top H U))U^\top$$

for any A in the domain of F and $U \in O^n$ such that $A = U(\operatorname{Diag}\lambda(A))U^\top$. Here, '$\circ$' denotes the Hadamard product between two matrices.

Definition 2.1.7 *A function $f : \mathbb{R} \to \mathbb{R}$ is called operator monotone of order n, if the corresponding primary matrix function $F : S^n \to S^n$ is operator monotone. A function $f : \mathbb{R} \to \mathbb{R}$ is called operator monotone, if the corresponding primary matrix function $F : S^n \to S^n$ is operator monotone for all n.*

For example, x^r, for $x \geq 0$ is operator monotone for $r \in [0, 1]$; as well as $-1/x$ for $x > 0$ and $x/(1 + x)$ for $x > 0$ see for example [2, Chapter V]. See [6], among other places, for more examples. Operator monotone functions of order n are characterized as follows.

Theorem 2.1.8 *Let I be an interval in $\mathbb{R}$. A continuously differentiable function $f : I \to \mathbb{R}$ is operator monotone of order n, if and only if $f^{[1]}(x)$ is a positive semidefinite matrix for every $x \in \mathbb{R}^n$ with coordinates in I.*

Operator monotone function f defined on an interval I can be characterized by *Pick functions* that take real values on I, see [2, Theorem V.4.7] and Nevanlinna's theorem [2, Theorem V.4.11].

Theorem 2.1.9 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is an operator monotone function, if and only if*

$$f(x) = a + bx + \int_{-\infty}^{+\infty} \Big(\frac{1}{\lambda - x} - \frac{\lambda}{\lambda^2 + 1}\Big)\, d\mu(\lambda), \tag{2.2}$$

for some $a \in \mathbb{R}, b \geq 0$, and μ a positive Borel measure on $\mathbb{R}$ with zero mass on I, such that

$$\int_{-\infty}^{+\infty} \frac{1}{\lambda^2 + 1}\, d\mu(\lambda) < \infty. \tag{2.3}$$

There is a connection between operator monotone functions and *complete Bernstein functions*, as explained in [20, Chapter 6]. The class of complete Bernstein functions is equal to the class of non-negative Pick functions on $(0, \infty)$. Hence, $f : (0, \infty) \to [0, \infty)$ is operator monotone, if and only if it is a complete Bernstein function.

We now introduce the class of functions that unite the spectral and the primary matrix functions. They will explain the apparent different nature of Theorems 2.1.5 and 2.1.8. That is one of the main goals of this work.

Let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ be defined by

$$\mathbf{f}_\rho(x) := f(x_{\rho_1}, \ldots, x_{\rho_k})$$

for all $x \in \mathbb{R}^n$ and all $\rho := (\rho_1, \ldots, \rho_k)$ that satisfy $1 \leq \rho_1 < \cdots < \rho_k \leq n$. For any $n \times n$ matrix U, denote by $U^{(k)}$ its k-th multiplicative compound matrix, where $1 \leq k \leq n$. Section 2.2 gives the precise definition and properties, but at the moment recall that $U^{(k)}$ is a $\binom{n}{k} \times \binom{n}{k}$ matrix that is orthogonal, if U is.

Definition 2.1.10 *A function $F : S^n \to S^{\binom{n}{k}}$ is called (generated) k-isotropic, if*

$$F(A) := U^{(k)}(\operatorname{Diag} \mathbf{f}(\lambda(A)))(U^{(k)})^\top, \tag{2.4}$$

where $U \in O^n$ is such that $A = U(\operatorname{Diag} \lambda(A))U^\top$.

It can be shown that the right-hand side of (2.4) does not depend on the choice of the diagonalizing matrix U, see [1] or [17]. The (generated) k-isotropic functions form a "bridge" between the spectral and the primary matrix functions. Indeed, when $k = n$, one has $U^{(n)} = \det(U) = \pm 1$ and $\mathbb{N}_{n,n} := \{\{1, \ldots, n\}\}$, thus (2.4) becomes a spectral function. When $k = 1$, one has $U^{(1)} = U$ and $\mathbb{N}_{n,1} := \{\{1\}, \ldots, \{n\}\}$, thus (2.4) becomes a primary matrix function.

A natural extension of Definition 2.1.7 is the following.

Definition 2.1.11 *A symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ is called operator monotone of order n, if the corresponding (generated) k-isotropic function $F : S^n \to S^{\binom{n}{k}}$ is operator monotone. A symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ is called operator monotone, if the corresponding (generated) k-isotropic function $F : S^n \to S^{\binom{n}{k}}$ is operator monotone for every $n \geq k$.*

A main goal of this work is to characterize operator monotonicity of the (generated) k-isotropic functions in terms of the corresponding symmetric function $f : \mathbb{R}^k \to \mathbb{R}$. This is achieved in Corollary 2.3.5. The comments after it describe how Theorem 2.1.8 connects to Theorem 2.1.5 as k increases from 1 to n. Another connection between these theorems is the second main result in this work. Its proof is at the end of Section 2.3.

Theorem 2.1.12 *Let I be an interval in $\mathbb{R}$ and $I^k := I \times \cdots \times I$, k times.*

A symmetric, C^1 function $f : I^k \to \mathbb{R}$ is operator monotone (of order n), if and only if the function $f(\cdot, \dot{x})$ is operator monotone on I (of order $n - k + 1$) for all $\dot{x} \in \mathbb{R}^{k-1}$.

The (generated) k-isotropic functions are maps on a single matrix argument. But since one can vary n freely, as long as $n \geq k$, one can obtain a function on several symmetric matrix arguments, simply by

$$\mathcal{F}(A_1, \ldots, A_m) := F(\mathrm{Diag}\,(A_1, \ldots, A_m)),$$

where $A_i \in S^{n_i}$ for all $i = 1, \ldots, m$ with $n = n_1 + \cdots + n_m$.

2.2 Main definition and notation

Denote by

$$\mathbb{N}_n := \{1, \ldots, n\}$$

the set of the first n natural numbers and denote by $\mathbb{N}_{n,k}$ the set of all subsets of $\mathbb{N}_n$ of size k with elements ordered increasingly, where $k = 1, \ldots, n$. The elements of the set $\mathbb{N}_{n,k}$ are ordered lexicographically. For any $\mathbf{x} \in \mathbb{R}^{\binom{n}{k}}$, $\mathbf{x}_\rho$ denotes the ρ-th element in $\mathbf{x}$, where $\rho \in \mathbb{N}_{n,k}$. For any $\mathbf{A} \in \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$, $\mathbf{A}_{\rho,\tau}$ denotes the element in the ρ-th row and τ-th column, where $\rho, \tau \in \mathbb{N}_{n,k}$. For any $x \in \mathbb{R}^n$ and any $\rho \in \mathbb{N}_{n,k}$, define $x_\rho := (x_{\rho_1}, \ldots, x_{\rho_k}) \in \mathbb{R}^k$. Finally, for any $A \in \mathbb{R}^{n\times n}$ and any $\rho, \tau \in \mathbb{N}_{n,k}$, denote by $A_{\rho\tau}$ the $k \times k$ minor of A with elements at the intersections of rows $\rho_1, \ldots, \rho_k$ and columns $\tau_1, \ldots, \tau_k$.

For any $A \in \mathbb{R}^{n\times n}$, the *k-th multiplicative compound matrix* of A, denoted $A^{(k)} \in \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$, is defined by

$$\left(A^{(k)}\right)_{\rho,\tau} := \det(A_{\rho\tau}), \text{ for any } \rho, \tau \in \mathbb{N}_{n,k}.$$

The following properties of the k-th multiplicative compound are well-known, see for example [4]:

$$A^{(k)}B^{(k)} = (AB)^{(k)} \quad \text{and} \quad \left(A^{(k)}\right)^\top = (A^\top)^{(k)},$$

for any matrices $A, B \in \mathbb{R}^{n\times n}$, and if A is invertible,

$$(A^{(k)})^{-1} = (A^{-1})^{(k)}.$$

Denote by $\{e^1, \ldots, e^n\}$ the standard orthonormal basis in $\mathbb{R}^n$ and by $\{\mathbf{e}^\rho : \rho \in \mathbb{N}_{n,k}\}$ the standard orthonormal basis in $\mathbb{R}^{\binom{n}{k}}$. For any permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, there exists a permutation matrix P such that

$$Px = \left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)^\top$$

for all $x \in \mathbb{R}^n$ or equivalently $Pe^{\sigma(i)} = e^i$ for all $i = 1, \ldots, n$. Every permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ defines a permutation $\sigma^{(k)}$ on $\mathbb{N}_{n,k}$ by

$$\sigma^{(k)}(\rho) := \text{the increasing rearrangement of } \{\sigma(\rho_1), \ldots, \sigma(\rho_k)\} \tag{2.5}$$

for all $\rho \in \mathbb{N}_{n,k}$. The corresponding permutation matrix $\mathbf{P}$ of $\sigma^{(k)}$ is defined by

$$\mathbf{P}\mathbf{e}^{\sigma^k(\rho)} := \mathbf{e}^{\rho} \text{ for all } \rho \in \mathbb{N}_{n,k}.$$

Let $\epsilon_{\sigma,\rho}$ be $+1$, if the permutation ordering $(\sigma(\rho_1), \ldots, \sigma(\rho_k))$ increasingly is even and be -1, if it is odd. The relationship between $P^{(k)}$ and $\mathbf{P}$ can be shown to be:

$$P^{(k)}\mathbf{e}^{\sigma^{(k)}(\rho)} = \epsilon_{\sigma,\rho}\mathbf{P}\mathbf{e}^{\sigma^{(k)}(\rho)}.$$

Any symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ defines a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ by

$$\mathbf{f}_\rho(x) := f(x_\rho) \text{ for all } x \in \mathbb{R}^n \text{ and all } \rho \in \mathbb{N}_{n,k}. \tag{2.6}$$

Such $\mathbf{f}$ is symmetric, in the sense that

$$\mathbf{f}(Px) = \mathbf{P}\mathbf{f}(x)$$

for all $x \in \mathbb{R}^n$ and all $n \times n$ permutation matrices P, see [1].

Definition 2.2.1 *A function $F : S^n \to S^{\binom{n}{k}}$ is called (generated) k-isotropic, if*

$$F(A) := U^{(k)}(\operatorname{Diag}\mathbf{f}(\lambda(A)))(U^{(k)})^\top, \tag{2.7}$$

where $U \in O^n$ is such that $A = U(\operatorname{Diag}\lambda(A))U^\top$; and we say F is generated by f.

For example, when $k = n$, one has $U^{(n)} = \det(U) = \pm 1$ and $\mathbb{N}_{n,n} := \{\{1, \ldots, n\}\}$, thus (2.7) becomes a spectral function. When $k = 1$, one has $U^{(1)} = U$ and $\mathbb{N}_{n,1} := \{\{1\}, \ldots, \{n\}\}$, thus (2.7) becomes a primary matrix function.

In addition, if $f : \mathbb{R}^k \to \mathbb{R}$ taken to be $f(x_1, \ldots, x_k) := x_1 \cdots x_k$, then (2.7) turns into the k-th multiplicative compound matrix of $A \in S^n$, that is $F(A) = A^{(k)}$.

And if $f : \mathbb{R}^k \to \mathbb{R}$ is taken to be $f(x_1, \ldots, x_k) := x_1 + \cdots + x_k$, then (2.7) turns into the k-th additive compound matrix of $A \in S^n$, denoted henceforth by $\Delta_k(A)$.

2.2.1 A note about domains

Denote by $\operatorname{dom} f \subseteq \mathbb{R}^k$ the domain of the symmetric function $f : \mathbb{R}^k \to \mathbb{R}$. Naturally, $\operatorname{dom} f$ is a symmetric set: $Px \in \operatorname{dom} f$ for all $x \in \operatorname{dom} f$ and all $k \times k$ permutation matrices P. Define the set

$$\operatorname{dom}_n f := \{x \in \mathbb{R}^n : x_\rho \in \operatorname{dom} f \text{ for all } \rho \in \mathbb{N}_{n,k}\}.$$

It is easy to see that $\operatorname{dom}_n f$ is a symmetric set. Indeed, choose any $n \times n$ permutation matrix P, with corresponding permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$. Then, for any $x \in \operatorname{dom}_n f$ and any $\rho \in \mathbb{N}_{n,k}$, one sees by (2.5) that vector $(Px)_\rho = (x_{\sigma(\rho_1)}, \ldots, x_{\sigma(\rho_k)})$ is a permutation of $x_{\sigma^{(k)}(\rho)}$. Since $x_{\sigma^{(k)}(\rho)} \in \operatorname{dom} f$ and the latter set is symmetric, we get $(Px)_\rho \in \operatorname{dom} f$. That is, $Px \in \operatorname{dom}_n f$.

To avoid pathological situations, we

$$\text{assume throughout that the set } \operatorname{dom} f \subseteq \mathbb{R}^k \text{ is convex.}$$

It is easy to see that this implies that $\operatorname{dom}_n f \subseteq \mathbb{R}^n$ is convex.

Then, the domain of the k-isotropic function $F : S^n \to S^{\binom{n}{k}}$ generated by $f : \mathbb{R}^k \to \mathbb{R}$ is

$$\begin{aligned}\operatorname{dom} F &:= \{A \in S^n : \lambda(A) \in \operatorname{dom}_n f\} \\ &= \{A \in S^n : \lambda_\rho(A) \in \operatorname{dom} f \text{ for all } \rho \in \mathbb{N}_{n,k}\}.\end{aligned}$$

Since $\operatorname{dom}_n f$ is convex and symmetric, Theorem 7 in [15], asserts that $\operatorname{dom} F$ is a convex set as well.

2.2.2 The differential of a (generated) k-isotropic function

A well-known connection between the k-th multiplicative compound and the k-th additive compound matrices is

$$\Delta_k(A) = \frac{d}{dt}(I + tA)^{(k)}\Big|_{t=0}, \tag{2.8}$$

see for example [18].

We now state the main result from [1], which is also re-derived in a more general context in [17].

Theorem 2.2.2 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a symmetric function with corresponding (generated) k-isotropic function $F : S^n \to S^{\binom{n}{k}}$. Then, F is C^1 at A, if and only if f is C^1 at $\lambda_\rho(A)$ for all $\rho \in \mathbb{N}_{n,k}$. In that case, the differential of F, at* Diag x *in the direction $H \in S^n$, is given by*

$$(\nabla F(\operatorname{Diag} x)[H])_{\rho,\tau} = \begin{cases} \displaystyle\sum_{i=1}^{k} \frac{\partial f}{\partial x_{\rho_i}}(x_\rho)H_{\rho_i\rho_i} & \text{if } \rho = \tau, \\ (-1)^{i+j}\dfrac{f(x_\rho) - f(x_\tau)}{x_{\rho_i} - x_{\tau_i}}H_{\rho_i\tau_j} & \text{if } |\rho \cap \tau| = k-1 \text{ and } x_{\rho_i} \neq x_{\tau_j}, \\ (-1)^{i+j}\dfrac{\partial f}{\partial x_{\rho_i}}(x_\rho)H_{\rho_i\tau_j} & \text{if } |\rho \cap \tau| = k-1 \text{ and } x_{\rho_i} = x_{\tau_j}, \\ 0 & \text{if } |\rho \cap \tau| < k-1, \end{cases} \tag{2.9}$$

where in the second and the third cases, the indexes $i, j \in \mathbb{N}_k$ are such that $\rho_i \in \rho \setminus \tau$ and $\tau_j \in \tau \setminus \rho$. (Here, $|\rho \cap \tau|$ denotes the number of common elements in ρ and τ.) For arbitrary $A \in S^n$, we have

$$\nabla F(A)[H] = U^{(k)}(\nabla F(\operatorname{Diag} \lambda(A))[U^\top HU])(U^{(k)})^\top \tag{2.10}$$

for all $U \in O^n$ such that $A = U(\operatorname{Diag} \lambda(A))U^\top$.

Using formula (2.9), one can obtain an explicit expressions for the entries of the k-th additive compound matrix (2.8). Indeed,

$$\Delta_k(H) = \lim_{t\to 0} \frac{(I+tH)^{(k)} - I^{(k)}}{t} = \nabla F(I)[H],$$

where F is generated by $f(x_1, \ldots, x_k) := x_1 \cdots x_k$. Thus, we have

$$(\Delta_k(H))_{\rho,\tau} = \begin{cases} \sum_{i=1}^{k} H_{\rho_i\rho_i} & \text{if } \rho = \tau, \\ (-1)^{i+j} H_{\rho_i\tau_j} & \text{if } |\rho \cap \tau| = k-1 \text{ and } \rho_i \in \rho\backslash\tau, \tau_j \in \tau\backslash\rho, \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

One may also refer to [1, Corollary 2.1] for a direct derivation of formula (2.11).

2.3 Characterization of operator monotone k-isotropic functions

For any $\dot\rho \in \mathbb{N}_{n,k-1}$ let

$$\dot\rho^c := \mathbb{N}_n \backslash \dot\rho$$

be the complement of $\dot\rho$ in $\mathbb{N}_n$. Note that $\dot\rho^c \in \mathbb{N}_{n,n-k+1}$.

While the j-th element of $\dot\rho \in \mathbb{N}_{n,k-1}$ is denoted by $\dot\rho_j$, to keep the notation lighter, the j-th element of $\dot\rho^c$ is denoted by $\dot\rho^j$, that is

$$\dot\rho^j := (\dot\rho^c)_j.$$

For any $s \in \dot\rho^c$, let $\alpha(s)$ be the position of s in $\dot\rho^c$. That is,

$$\alpha : \dot\rho^c \to \mathbb{N}_{n-k+1} \text{ and } s = \dot\rho^{\alpha(s)}. \tag{2.12}$$

We say that $\rho \in \mathbb{N}_{n,k}$ *is an extension of* $\dot{\rho} \in \mathbb{N}_{n,k-1}$ *by* s, if $s \in \dot{\rho}^c$ and $\rho = \dot{\rho} \cup \{s\}$. Such extensions are denoted by $\dot{\rho} + s$.

For any $s \in \dot{\rho}^c$, let $\beta(s)$ be the position of s in $\dot{\rho} + s$. That is,

$$\beta : \dot{\rho}^c \to \mathbb{N}_k \text{ and } s = (\dot{\rho} + s)_{\beta(s)}. \tag{2.13}$$

Finally, for any $i \in \mathbb{N}_{n-k+1}$, let $\gamma(i)$ is the position of $\dot{\rho}^i$ in the extension $\dot{\rho} + \dot{\rho}^i$. That is,

$$\gamma : \mathbb{N}_{n-k+1} \to \mathbb{N}_k \text{ and } \dot{\rho}^i = (\dot{\rho} + \dot{\rho}^i)_{\gamma(i)}. \tag{2.14}$$

Replacing s by $\dot{\rho}^i$ in (2.13), we see that $\gamma(i) = \beta(\dot{\rho}^i)$. Conversely, replacing i by $\alpha(s)$ in (2.14), and using (2.12), one obtains

$$\gamma(\alpha(s)) = \beta(\dot{\rho}^{\alpha(s)}) = \beta(s)$$

for any $s \in \dot{\rho}^c$. That is, the following diagram commutes:

$$\begin{tikzcd} \dot{\rho}^c \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathbb{N}_{n-k+1} \arrow[d, "\gamma"] \\ & \mathbb{N}_k \end{tikzcd}$$

The maps α, β, and γ depend on $\dot{\rho}$ but we suppress that for the sake of simplicity.

For any fixed $\dot{\rho} \in \mathbb{N}_{n,k-1}$, we introduce two linear maps:

$$l_{\dot{\rho}} : S^{n-k+1} \to S^n \text{ and } L_{\dot{\rho}} : S^{n-k+1} \to S^{\binom{n}{k}}.$$

For any $A \in S^{n-k+1}$ let

$$\left(l_{\dot{\rho}}(A)\right)_{ij} := \begin{cases} A_{\alpha(i)\alpha(j)} & \text{if } i, j \in \dot{\rho}^c, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, A is the principal minor of $l_{\dot{\rho}}(A)$ at the intersection of the rows and columns with indexes in $\dot{\rho}^c$. The rest of the entries of $l_{\dot{\rho}}(A)$ are zero.

For any $A \in S^{n-k+1}$ let

$$(L_{\dot{\rho}}(A))_{\rho,\tau} := \begin{cases} A_{ij} & \text{if } \rho = \dot{\rho} + \dot{\rho}^i \text{ and } \tau = \dot{\rho} + \dot{\rho}^j, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, A is the principal minor of $L_{\dot{\rho}}(A)$ at the intersection of rows and columns with indexes in $\{\dot{\rho} + \dot{\rho}^1, \ldots, \dot{\rho} + \dot{\rho}^{n-k+1}\}$. If ρ or τ does not contain $\dot{\rho}$ as a subset, then $(L_{\dot{\rho}}(A))_{\rho,\tau} = 0$.

Let $E_{n-k+1} \in S^{n-k+1}$ be the all-one matrix and let

$$E^{\dot{\rho}} := l_{\dot{\rho}}(E_{n-k+1}).$$

Trivially, we have

$$(E^{\dot{\rho}})_{\dot{\rho}^c \dot{\rho}^c} = E_{n-k+1}. \tag{2.15}$$

It is easy to see that both E_{n-k+1} and $E^{\dot{\rho}}$ are positive semidefinite.

Note that when $k = 1$, we have one choice for $\dot{\rho}$, namely $\dot{\rho} = \emptyset$ and $\dot{\rho}^c = \mathbb{N}_n$. Then, $\alpha(s) = s$ and $\beta(s) \equiv 1$ for all $s \in \mathbb{N}_n$. In addition, $l_{\dot{\rho}}(A) = L_{\dot{\rho}}(A) = A$ for all $A \in S^{n-k+1}$ and $E^{\dot{\rho}} = E_n$.

Theorem 2.3.1 *Let* $\mathbf{A} \in S^{\binom{n}{k}}$ *with* $\operatorname{diag} \mathbf{A} = 0$. *Let J be an* $\binom{n}{k} \times n$ *matrix with* $J_{\rho,s} = 0$, *whenever* $s \notin \rho$. *Define a linear map* $\mathbf{T} : S^n \to S^{\binom{n}{k}}$ *by*

$$\mathbf{T}(H) := \mathbf{A} \circ \Delta_k(H) + \operatorname{Diag}(J(\operatorname{diag} H)). \tag{2.16}$$

Then, $\mathbf{T}(H) \geq 0$ *for all* $H \geq 0$, *if and only if* $\mathbf{T}(E^{\dot{\rho}}) \geq 0$ *for all* $\dot{\rho} \in \mathbb{N}_{n,k-1}$.

Proof First, we decompose $\mathbf{T}(H)$ as a sum of $\binom{n}{k-1}$ matrices of size $\binom{n}{k} \times \binom{n}{k}$. The decomposition is such that no two matrices in the sum have overlapping, non-zero, off-diagonal entries.

For any $\dot\rho \in \mathbb{N}_{n,k-1}$, define $T_{\dot\rho} \in S^{n-k+1}$ by

$$\left(T_{\dot\rho}\right)_{ij} := \begin{cases} J_{\dot\rho+\dot\rho^i,\dot\rho^i} & \text{if } i = j, \\ (-1)^{\gamma(i)+\gamma(j)}\mathbf{A}_{\dot\rho+\dot\rho^i,\dot\rho+\dot\rho^j} & \text{if } i \neq j, \end{cases} \tag{2.17}$$

for all $i, j \in \mathbb{N}_{n-k+1}$. Then, we have

$$(T_{\dot\rho} \circ H_{\dot\rho^c\dot\rho^c})_{ij} = (T_{\dot\rho})_{ij}(H_{\dot\rho^c\dot\rho^c})_{ij} = \begin{cases} J_{\dot\rho+\dot\rho^i,\dot\rho^i}H_{\dot\rho^i\dot\rho^i} & \text{if } i = j, \\ (-1)^{\gamma(i)+\gamma(j)}\mathbf{A}_{\dot\rho+\dot\rho^i,\dot\rho+\dot\rho^j}H_{\dot\rho^i\dot\rho^j} & \text{if } i \neq j, \end{cases}$$

for all $i, j \in \mathbb{N}_{n-k+1}$. Thus,

$$\begin{aligned}
(L_{\dot\rho}(T_{\dot\rho} \circ H_{\dot\rho^c\dot\rho^c}))_{\rho,\tau} &= \begin{cases} (T_{\dot\rho} \circ H_{\dot\rho^c\dot\rho^c})_{ij} & \text{if } \rho = \dot\rho + \dot\rho^i \text{ and } \tau = \dot\rho + \dot\rho^j, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} J_{\rho,\dot\rho^i}H_{\dot\rho^i\dot\rho^i} & \text{if } \rho = \dot\rho + \dot\rho^i \text{ and } \rho = \tau, \\ (-1)^{\gamma(i)+\gamma(j)}\mathbf{A}_{\rho,\tau}H_{\dot\rho^i\dot\rho^j} & \text{if } \rho = \dot\rho + \dot\rho^i,\ \tau = \dot\rho + \dot\rho^j \text{ and } i \neq j, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} J_{\rho,\rho_i}H_{\rho_i\rho_i} & \text{if } \dot\rho = \rho \setminus \{\rho_i\} \text{ and } \rho = \tau, \\ (-1)^{i+j}\mathbf{A}_{\rho,\tau}H_{\rho_i\tau_j} & \text{if } \dot\rho = \rho \setminus \{\rho_i\},\ \dot\rho = \tau \setminus \{\tau_j\} \text{ and } |\rho \cap \tau| = k-1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

After these preparations, we claim that

$$\mathbf{T}(H) = \sum_{\dot\rho \in \mathbb{N}_{n,k-1}} L_{\dot\rho}(T_{\dot\rho} \circ H_{\dot\rho^c\dot\rho^c}). \tag{2.18}$$

To show (2.18), we are going to compare the diagonal and off-diagonal elements on both sides. Fix $\rho, \tau \in \mathbb{N}_{n,k}$.

If $|\rho \cap \tau| \leq k-2$, then, using (2.11), it can be seen that the elements in position (ρ, τ) on both sides are all zero.

If $|\rho \cap \tau| = k-1$, let $\dot{\rho} := \rho \cap \tau$. Then, ρ is an extension of $\dot{\rho}$ by ρ_i; and τ is an extension of $\dot{\rho}$ by τ_j for some i and j in $\mathbb{N}_k$. Then, using (2.11), we obtain

$$(\mathbf{T}(H))_{\rho,\tau} = (\mathbf{A} \circ \Delta_k(H))_{\rho,\tau} = (-1)^{i+j}\mathbf{A}_{\rho,\tau}H_{\rho_i\tau_j} = (L_{\dot{\rho}}(T_{\dot{\rho}} \circ H_{\dot{\rho}^c\dot{\rho}^c}))_{\rho,\tau}.$$

For any other $\dot{\zeta} \in \mathbb{N}_{n,k-1}$, with $\dot{\zeta} \neq \dot{\rho}$, we have that $\dot{\zeta}$ is not a subset of both ρ and τ. This implies $(L_{\dot{\zeta}}(T_{\dot{\zeta}} \circ H_{\dot{\zeta}^c\dot{\zeta}^c}))_{\rho,\tau} = 0$.

If $\rho = \tau$, then one obtains

$$(L_{\dot{\rho}}(T_{\dot{\rho}} \circ H_{\dot{\rho}^c\dot{\rho}^c}))_{\rho,\rho} = \begin{cases} J_{\rho,\rho_i}H_{\rho_i\rho_i} & \text{if } \dot{\rho} = \rho\backslash\{\rho_i\} \text{ for some } i \in \mathbb{N}_k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$(\mathbf{T}(H))_{\rho,\rho} = \left(\operatorname{Diag}(J(\operatorname{diag} H))\right)_{\rho,\rho} = (J(\operatorname{diag} H))_\rho = \sum_{i=1}^k J_{\rho,\rho_i}H_{\rho_i\rho_i} = \Bigg(\sum_{\dot{\rho}\in\mathbb{N}_{n,k-1}} L_{\dot{\rho}}(T_{\dot{\rho}} \circ H_{\dot{\rho}^c\dot{\rho}^c})\Bigg)_{\rho,\rho}.$$

The last equality holds, since if ρ is not an extension of $\dot{\rho}$ by ρ_i for some $i \in \mathbb{N}_k$, then the summand corresponding to $\dot{\rho}$ is zero.

Next, we claim that $T_{\dot{\rho}}$ is a principle minor of the matrix $\mathbf{T}(E^{\dot{\rho}})$. Indeed, if ρ is an extension of $\dot{\rho}$ by $\dot{\rho}^i$ and τ is an extension of $\dot{\rho}$ by $\dot{\rho}^j$, with $i \neq j$, then $\rho \neq \tau$ and we have

$$\left(\mathbf{T}(E^{\dot{\rho}})\right)_{\rho,\tau} = (\mathbf{A} \circ \Delta_k(E^{\dot{\rho}}))_{\rho,\tau} = (-1)^{\gamma(i)+\gamma(j)}\mathbf{A}_{\rho,\tau}(E^{\dot{\rho}})_{\dot{\rho}^i\dot{\rho}^j} = (-1)^{\gamma(i)+\gamma(j)}\mathbf{A}_{\rho,\tau} = (T_{\dot{\rho}})_{ij}.$$

For the second equality, we used (2.11). For the third equality, we used (2.15) and the fact that $\dot{\rho}^i, \dot{\rho}^j \in \dot{\rho}^c$. Finally, the last equality uses (2.17).

If ρ is an extension of $\dot{\rho}$ by $\dot{\rho}^i$, then one has

$$(\mathbf{T}(E^{\dot{\rho}}))_{\rho,\rho} = \left(\operatorname{Diag}(J(\operatorname{diag} E^{\dot{\rho}}))\right)_{\rho,\rho} = (J(\operatorname{diag} E^{\dot{\rho}}))_\rho = \sum_{s=1}^n J_{\rho,s}(E^{\dot{\rho}})_{ss} = J_{\rho,\dot{\rho}^i} = (T_{\dot{\rho}})_{ii},$$

where the fourth equality holds because $J_{\rho,s} = 0$, if $s \notin \rho$ and $(E^{\dot{\rho}})_{ss} = 0$, if $s \notin \dot{\rho}^c$. Thus, the only (possibly) non-zero term corresponds to $s = \dot{\rho}^i$.

We are now in a position to prove the theorem. One direction is easy: If $\mathbf{T}(H) \geq 0$ for all $H \geq 0$, then $\mathbf{T}(E^{\dot{\rho}}) \geq 0$ for all $\dot{\rho} \in \mathbb{N}_{n,k-1}$, since $E^{\dot{\rho}} \geq 0$.

For the other direction, suppose that $\mathbf{T}(E^{\dot{\rho}}) \geq 0$ for all $\dot{\rho} \in \mathbb{N}_{n,k-1}$. Then, since $T_{\dot{\rho}}$ is a principle minor of $\mathbf{T}(E^{\dot{\rho}})$, we obtain $T_{\dot{\rho}} \geq 0$ for any $\dot{\rho} \in \mathbb{N}_{n,k-1}$. Fix $H \geq 0$. Then, $H_{\dot{\rho}^c\dot{\rho}^c} \geq 0$, being a principle minor of H, for any $\dot{\rho} \in \mathbb{N}_{n,k-1}$. Theorem 5.2.1 in [8] shows that the Hadamard product of two positive semidefinite matrices is positive semidefinite, hence we have $T_{\dot{\rho}} \circ H_{\dot{\rho}^c\dot{\rho}^c} \geq 0$. The latter implies that $L_{\dot{\rho}}(T_{\dot{\rho}} \circ H_{\dot{\rho}^c\dot{\rho}^c}) \geq 0$, by the definition of $L_{\dot{\rho}}$, and $\mathbf{T}(H) \geq 0$ follows from (2.18).

The proof of Theorem 2.3.1 has several revealing features that are going to be important for us. We separate them in the next corollary.

Corollary 2.3.2 *Let* $\mathbf{A}$*,* J*, and* $\mathbf{T}$ *be as in Theorem 2.3.1. For any* $\dot{\rho} \in \mathbb{N}_{n,k-1}$*, define* $T_{\dot{\rho}} \in S^{n-k+1}$ *by*

$$\left(T_{\dot{\rho}}\right)_{ij} := \begin{cases} J_{\dot{\rho}+\dot{\rho}^i,\dot{\rho}^i} & \text{if } i = j, \\ (-1)^{\gamma(i)+\gamma(j)}\mathbf{A}_{\dot{\rho}+\dot{\rho}^i,\dot{\rho}+\dot{\rho}^j} & \text{if } i \neq j, \end{cases} \tag{2.19}$$

for all $i, j \in \mathbb{N}_{n-k+1}$*. Then,*

$$\mathbf{T}(H) = \sum_{\dot{\rho}\in\mathbb{N}_{n,k-1}} L_{\dot{\rho}}(T_{\dot{\rho}} \circ H_{\dot{\rho}^c\dot{\rho}^c})$$

and $\mathbf{T}(H) \geq 0$ *for all* $H \geq 0$*, if and only if* $T_{\dot{\rho}} \geq 0$ *for all* $\dot{\rho} \in \mathbb{N}_{n,k-1}$*.*

Comparing formulas (2.9) and (2.11), one arrives at the following representation of first differential of (generated) k-isotropic functions in the same pattern as (2.16).

Proposition 2.3.3 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a symmetric, C^1 function with corresponding (generated) k-isotropic function $F : S^n \to S^{\binom{n}{k}}$. Let $\mathcal{J}(x)$ be the $\binom{n}{k} \times n$ matrix defined by*

$$\mathcal{J}(x)_{\rho,s} := \begin{cases} \dfrac{\partial f}{\partial x_s}(x_\rho) & \text{if } s \in \rho, \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in \mathbb{R}^n$ has distinct coordinates, then the differential of F can be written as

$$\nabla F(\operatorname{Diag} x)[H] = \mathbf{A}(x) \circ \Delta_k(H) + \operatorname{Diag}(\mathcal{J}(x)(\operatorname{diag} H)), \tag{2.20}$$

where

$$\mathbf{A}(x)_{\rho,\tau} := \begin{cases} \dfrac{f(x_\rho) - f(x_\tau)}{x_{\rho_i} - x_{\tau_j}} & \text{if } |\rho \cap \tau| = k - 1 \text{ and } \rho_i \in \rho\backslash\tau, \tau_j \in \tau\backslash\rho, \\ 0 & \text{otherwise.} \end{cases}$$

Note that matrix $\mathcal{J}(x)$ is just the Jacobian of $\mathbf{f}(x)$, defined by (2.6).

For each $\dot{\rho} \in \mathbb{N}_{n,k-1}$ and each $x \in \mathbb{R}^n$ with distinct coordinates, define the divided difference matrix $T_{\dot{\rho}}(x) \in S^{n-k+1}$ by

$$T_{\dot{\rho}}(x)_{ij} := \begin{cases} \dfrac{\partial f}{\partial x_{\dot{\rho}^i}}(x_{\dot{\rho}+\dot{\rho}^i}) & \text{if } i = j, \\ (-1)^{\gamma(i)+\gamma(j)} \dfrac{f(x_{\dot{\rho}+\dot{\rho}^i}) - f(x_{\dot{\rho}+\dot{\rho}^j})}{x_{\dot{\rho}^i} - x_{\dot{\rho}^j}} & \text{if } i \neq j, \end{cases} \tag{2.21}$$

for all $i, j \in \mathbb{N}_{n-k+1}$. Note that $T_{\dot{\rho}}(x)$ is exactly (2.19) after replacing J by $\mathcal{J}(x)$ and $\mathbf{A}$ by $\mathbf{A}(x)$. Thus, Corollary 2.3.2 and (2.20) imply

$$\nabla F(\operatorname{Diag} x)[H] = \sum_{\dot{\rho} \in \mathbb{N}_{n,k-1}} L_{\dot{\rho}}(T_{\dot{\rho}}(x) \circ H_{\dot{\rho}^c \dot{\rho}^c}), \tag{2.22}$$

whenever $x \in \mathbb{R}^n$ has distinct coordinates.

The next lemma shows that any two divided difference matrices $T_{\dot{\rho}}(x)$ and $T_{\dot{\tau}}(x)$, where

$\dot\rho, \dot\tau \in \mathbb{N}_{n,k-1}$, are related to each other.

Lemma 2.3.4 *If $T_{\dot\tau}(x) \geq 0$ for some $\dot\tau \in \mathbb{N}_{n,k-1}$ and all $x \in \mathrm{dom}_n f$ with distinct coordinates, then $T_{\dot\rho}(x) \geq 0$ for all $\dot\rho \in \mathbb{N}_{n,k-1}$ and all $x \in \mathrm{dom}_n f$ with distinct coordinates.*

Proof Let $\dot\tau := \{n-k+2, \ldots, n\} \in \mathbb{N}_{n,k-1}$. Note that the function γ, defined by (2.14), corresponding to $\dot\tau$ is such that $\gamma(s) = 1$ for all $s \in \dot\tau^c$. Then, the divided difference matrix $T_{\dot\tau}(x)$ is given by

$$(T_{\dot\tau}(x))_{ij} = \begin{cases} \dfrac{\partial f}{\partial x_{\dot\tau^i}}(x_{\dot\tau+\dot\tau^i}) & \text{if } i = j, \\ \dfrac{f(x_{\dot\tau+\dot\tau^i}) - f(x_{\dot\tau+\dot\tau^j})}{x_{\dot\tau^i} - x_{\dot\tau^j}} & \text{if } i \neq j, \end{cases} \tag{2.23}$$

for $i, j \in \mathbb{N}_{n-k+1}$. Fix another index $\dot\rho \in \mathbb{N}_{n,k-1}$. We are going to relate $T_{\dot\rho}(x)$ to $T_{\dot\tau}(x)$ Let $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ be the permutation that sends $\dot\rho_i$ to $\dot\tau_i$ for $i \in \mathbb{N}_{k-1}$ and sends $\dot\rho^j$ to $\dot\tau^j$ for $j \in \mathbb{N}_{n-k+1}$. The corresponding permutation matrix P is

$$(Px)_{\dot\rho_i} = x_{\dot\tau_i} \quad \text{for } i \in \mathbb{N}_{k-1} \quad \text{and}$$
$$(Px)_{\dot\rho^i} = x_{\dot\tau^i} \quad \text{for } i \in \mathbb{N}_{n-k+1}.$$

Since f is a symmetric function, one can see

$$f((Px)_{\dot\rho+\dot\rho^i}) = f((Px)_{\dot\rho}, (Px)_{\dot\rho^i}) = f(x_{\dot\tau}, x_{\dot\tau^i}) = f(x_{\dot\tau+\dot\tau^i}) \quad \text{and}$$
$$\frac{\partial f}{\partial (Px)_{\dot\rho^i}}((Px)_{\dot\rho+\dot\rho^i}) = \frac{\partial f}{\partial x_{\dot\tau^i}}(x_{\dot\tau+\dot\tau^i}).$$

Using (2.21), for $i, j \in \mathbb{N}_{n-k+1}$ one obtains

$$(T_{\dot\rho}(Px))_{ij} = \begin{cases} \dfrac{\partial f}{\partial (Px)_{\dot\rho^i}}((Px)_{\dot\rho+\dot\rho^i}) & \text{if } i = j, \\ (-1)^{\gamma(i)+\gamma(j)} \dfrac{f((Px)_{\dot\rho+\dot\rho^i}) - f((Px)_{\dot\rho+\dot\rho^j})}{(Px)_{\dot\rho^i} - (Px)_{\dot\rho^j}} & \text{if } i \neq j, \end{cases} \tag{2.24}$$

$$= \begin{cases} \dfrac{\partial f}{\partial x_{\dot{\tau}^i}}(x_{\dot{\tau}+\dot{\tau}^i}) & \text{if } i = j, \\ (-1)^{\gamma(i)+\gamma(j)} \dfrac{f(x_{\dot{\tau}+\dot{\tau}^i}) - f(x_{\dot{\tau}+\dot{\tau}^j})}{x_{\dot{\tau}^i} - x_{\dot{\tau}^j}} & \text{if } i \neq j, \end{cases}$$

where the function γ is the one associated with $\dot{\rho}$. Define the matrix $A_{\dot{\rho}} \in S^{n-k+1}$ by

$$(A_{\dot{\rho}})_{ij} := (-1)^{\gamma(i)+\gamma(j)}.$$

Note that $A_{\dot{\rho}}$ is positive semidefinite, since

$$(A_{\dot{\rho}})_{ij} = (-1)^{\gamma(i)+\gamma(j)} = (-1)^{\gamma(1)+\gamma(i)}(-1)^{\gamma(1)+\gamma(j)} = (A_{\dot{\rho}})_{i1}(A_{\dot{\rho}})_{1j},$$

that is

$$A_{\dot{\rho}} = (A_{\dot{\rho}})_{*1}(A_{\dot{\rho}})_{*1}^{\top} \succeq 0,$$

where $(A_{\dot{\rho}})_{*1}$ denotes the first column of $A_{\dot{\rho}}$. Comparing (2.23) to (2.24), we see that

$$T_{\dot{\rho}}(Px) = T_{\dot{\tau}}(x) \circ A_{\dot{\rho}},$$

for all $x \in \mathrm{dom}_n f$ with distinct coordinates. Thus, if $T_{\dot{\tau}}(x) \succeq 0$, then by [8, Theorem 5.2.1], we conclude $T_{\dot{\rho}}(Px) \succeq 0$. The result follows from here.

Combining all of the results in this section, namely: Theorem 2.3.1, Proposition 2.3.3, Corollary 2.3.2, and Lemma 2.3.4, we obtain a characterization of the operator monotonicity of the (generated) k-isotropic function F in terms of its corresponding symmetric function f. Recall also Definition 2.1.11.

Corollary 2.3.5 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a symmetric, continuously differentiable function. Then, f is operator monotone, if and only if $T_{\dot{\rho}}(x) \succeq 0$, for some $\dot{\rho} \in \mathbb{N}_{n,k-1}$ and every $x \in \mathrm{dom}_n f$.*

Proof By Proposition 2.1.2, F is operator monotone, if and only if $\nabla F(A)[H] \succeq 0$ for all

$H \geq 0$ and all A in the domain of F. Using (2.10), one can see that this is equivalent to $\nabla F(\operatorname{Diag}\lambda(A))[H] \geq 0$ for all $H \geq 0$ and all A in the domain of F. (Since F is continuously differentiable, it suffices to consider matrices A in the domain of F with distinct eigenvalues.) Using (2.21), (2.22), and Corollary 2.3.2, the latter is equivalent to $T_{\dot\rho}(\lambda(A)) \geq 0$, for all $\dot\rho \in \mathbb{N}_{n,k-1}$ and every $A \in \operatorname{dom} F$ with distinct eigenvalues. Finally, by Lemma 2.3.4, the latter is equivalent to $T_{\dot\tau}(\lambda(A)) \geq 0$, for some $\dot\tau \in \mathbb{N}_{n,k-1}$ and every $A \in \operatorname{dom} F$ with distinct eigenvalues.

Note that Corollary 2.3.5 is a direct extension of Theorem 2.1.5. Indeed, when $k = n$, the (generated) k-isotropic function becomes a spectral function. The possible values of $\dot\rho$ are $\{1, \ldots, i-1, i+1, \ldots, n\}$ for $i = 1, \ldots, n$. If $\dot\rho = \{1, \ldots, i-1, i+1, \ldots, n\}$, then $T_{\dot\rho}(x)$ is a 1×1 matrix and (2.21) turns into

$$T_{\dot\rho}(x) = \frac{\partial f}{\partial x_i}(x).$$

Thus, the spectral function F is operator monotone, if and only if $\partial f(x)/\partial x_i \geq 0$ for all $i = 1, \ldots, n$.

Note as well that Corollary 2.3.5 is a direct extension of Theorem 2.1.8. Indeed, when $k = 1$, the (generated) k-isotropic function becomes a primary matrix function. We have $\dot\rho = \emptyset$ (the only choice for $\dot\rho$) and the divided difference matrix $T_{\dot\rho}(x)$ turns into (2.1).

Proof of Theorem 2.1.12 Fix $\dot x \in I^{k-1}$. Let $y := (y_1, \ldots, y_{n-k+1})$ be any vector from I^{n-k+1}. According to Theorem 2.1.8, the function $f(\cdot, \dot x)$ is operator monotone on I of order $n-k+1$, if and only if the divided difference matrix of $f(\cdot, \dot x)$:

$$D(y)_{ij} := \begin{cases} \dfrac{\partial f}{\partial y_i}(y_i, \dot x) & \text{if } i = j, \\ \dfrac{f(y_i, \dot x) - f(y_j, \dot x)}{y_i - y_j} & \text{if } i \neq j, \end{cases} \tag{2.25}$$

where $i, j \in \mathbb{N}_{n-k+1}$, is positive semidefinite for all $y \in I^{n-k+1}$. (By continuity, it suffices to consider only vectors $\dot x$ and y, such that together they have distinct coordinates.)

Let $\dot\tau := \{n-k+2, \ldots, n\}$. The divided difference matrix (2.21), with respect to $\dot\tau$, turns into

$$\left(T_{\dot\tau}(x)\right)_{ij} = \begin{cases} \dfrac{\partial f}{\partial x_{\dot\tau^i}}(x_{\dot\tau + \dot\tau^i}) & \text{if } i = j, \\ \dfrac{f(x_{\dot\tau+\dot\tau^i}) - f(x_{\dot\tau+\dot\tau^j})}{x_{\dot\tau^i} - x_{\dot\tau^j}} & \text{if } i \neq j, \end{cases} \tag{2.26}$$

for $i, j \in \mathbb{N}_{n-k+1}$. Note that $\dot\tau^i = i$ and $x_{\dot\tau+\dot\tau^i} = (x_i, x_{\dot\tau})$ for all $i \in \mathbb{N}_{n-k+1}$. Let $x := (y, \dot x) \in \mathbb{R}^n$ and observe that $x \in \operatorname{dom}_n f$. We also have that $x_{\dot\tau^i} = y_i$ and $x_{\dot\tau+\dot\tau^i} = (y_i, \dot x)$ for all $i \in \mathbb{N}_{n-k+1}$. Thus, comparing (2.25) and (2.26), we conclude that

$$D(y)_{ij} = \left(T_{\dot\tau}(x)\right)_{ij}$$

for all $i, j \in \mathbb{N}_{n-k+1}$.

If f is operator monotone, in the sense of Definition 2.1.11, then by Corollary 2.3.5 and Lemma 2.3.4, we obtain that $T_{\dot\tau}(x)$, and hence $D(y)$, is positive semidefinite. Thus, $f(\cdot, \dot x)$ is operator monotone on I of order $n - k + 1$.

If $f(\cdot, \dot x)$ is operator monotone on I of order $n - k + 1$, for all $\dot x \in \mathbb{R}^{k-1}$, then the divided difference matrix (2.25) is positive semidefinite for any $y \in I^{n-k+1}$ and the result follows by reversing the steps in the previous paragraph.

Since the above arguments hold for any $n \geq k$, the rest of the theorem follows.

Theorem 2.1.12 strengthens the connection between Theorem 2.1.5 and Theorem 2.1.8 and fully explains the apparent differences in the criteria for operator monotonicity. Taking $n = k$, the (generated) k-isotropic function becomes a spectral function. The fact that the symmetric function $f : \mathbb{R}^n \to \mathbb{R}$ is operator monotone of order n implies that it is operator monotone with respect to each argument of order one, that is, on 1×1 matrices. In that case, operator monotonicity is the same as being non-decreasing.

Taking $k = 1$, the (generated) k-isotropic function becomes a primary matrix function. In that case, the statement of Theorem 2.1.12 becomes trivial.

2.4 Applications and examples

2.4.1 Connections with operator convex functions

A function $F : S^n \to S^m$ is said to be *operator convex*, if the inequality

$$(1-t)F(A) + tF(B) \geq F((1-t)A + tB)$$

holds for all A, B in the domain of F and all $t \in [0,1]$. We say that F is *operator concave*, if $-F$ is operator convex.

Consider a function $f : \mathbb{R} \to \mathbb{R}$ and its corresponding primary matrix function $F : S^n \to S^n$. The function f is called *operator convex of order n*, if $F : S^n \to S^n$ is operator convex. The function $f : \mathbb{R} \to \mathbb{R}$ is called *operator convex*, if $F : S^n \to S^n$ is operator convex for all n.

Examples of operator convex functions are given in [2, Chapter V], among them are the functions x^r for $x > 0$ with $r \in [-1,0] \cup [1,2]$; and $x \log x$ for $x > 0$. A characterization of operator convex functions f of order n, in terms of the second order divided differences of f is given in [11]. There are many connections between the operator monotone functions and the operator convex functions. We now state the one that we are going to exploit, see [2, Theorem V.2.9].

Theorem 2.4.1 *Let $f : [0,\alpha) \to \mathbb{R}$ be continuous with $f(0) \leq 0$. The function f is operator convex, if and only if $x^{-1}f(x)$ is operator monotone on $(0,a)$.*

We should notice that unlike the characterization in [11], the characterization given in Theorem 2.4.1 does not rely on the second derivatives.

Extending the notion of operator convexity to symmetric functions $f : \mathbb{R}^k \to \mathbb{R}$ is now readily anticipated.

Definition 2.4.2 *Consider a symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ with its corresponding (generated) k-isotropic function $F : S^n \to S^{\binom{n}{k}}$. We say that f is operator convex of order n, if*

$F : S^n \to S^{\binom{n}{k}}$ *is operator convex. We say that f is operator convex, if* $F : S^n \to S^{\binom{n}{k}}$ *is operator convex for all* $n \geq k$.

A matrix $K \in \mathbb{R}^{n\times n}$ is a *contraction*, if $\lambda_1(KK^\top) \leq 1$. The proof of the next result and its corollary mimics parts of the proof of Theorem V.2.3 in [2].

Proposition 2.4.3 *Let* $f : [0, \alpha)^k \to \mathbb{R}$ *be symmetric and continuous with* $f(x) \leq 0$*, whenever* $x_i = 0$ *for some* $i \in \mathbb{N}_k$*. If a function f is operator convex, then the corresponding (generated) k-isotropic function F satisfies*

$$\left(K^{(k)}\right)^\top F(A)K^{(k)} \succeq F(K^\top AK) \tag{2.27}$$

for all A in the domain of F and all contraction matrices $K \in \mathbb{R}^{n\times n}$.

Proof Let $K \in \mathbb{R}^{n\times n}$ be a contraction and let $M_1 := (I - KK^\top)^{1/2}$ and $M_2 := (I - K^\top K)^{1/2}$. Define

$$U := \begin{pmatrix} K & M_1 \\ M_2 & -K^\top \end{pmatrix} \in O^{2n} \quad \text{and} \quad V := \begin{pmatrix} K & -M_1 \\ M_2 & K^\top \end{pmatrix} \in O^{2n}.$$

Fix any $A \in S^n$ and let $T := \operatorname{Diag}(A, 0) \in S^{2n}$. Then, matrices

$$U^\top TU = \begin{pmatrix} K^\top AK & K^\top AM_1 \\ M_1AK & M_1AM_1 \end{pmatrix} \quad \text{and} \quad V^\top TV = \begin{pmatrix} K^\top AK & -K^\top AM_1 \\ -M_1AK & M_1AM_1 \end{pmatrix},$$

satisfy

$$\frac{U^\top TU + V^\top TV}{2} = \begin{pmatrix} K^\top AK & 0 \\ 0 & M_1AM_1 \end{pmatrix}.$$

The following calculation shows inequality (2.27). In it we consider matrices of size $\binom{2n}{k} \times \binom{2n}{k}$. Note that every set in $\mathbb{N}_{n,k}$ can be viewed in a natural way as an element of $\mathbb{N}_{2n,k}$. We focus our attention on the $\binom{n}{k} \times \binom{n}{k}$ block of elements with indexes $\rho \in \mathbb{N}_{n,k}$. Without loss of generality, we may permute rows and columns so that this block is in the upper-left corner and write '$*$' to denote the remaining entries. Let $Q \in O^n$ be such that $A = Q(\operatorname{Diag}\lambda(A))Q^\top$. Since A is in the domain of F, we have $\lambda(A) \geq 0$ and this observation is used in the third equality below:

$$
\begin{aligned}
F(T) = F\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &= F\begin{pmatrix} Q(\operatorname{Diag}\lambda(A))Q^\top & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} Q^{(k)} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \operatorname{Diag}_\rho f(\lambda_\rho(A)) & 0 \\ 0 & * \end{pmatrix}\begin{pmatrix} Q^{(k)\top} & 0 \\ 0 & 0 \end{pmatrix} \\
&\preceq \begin{pmatrix} Q^{(k)} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \operatorname{Diag}_\rho f(\lambda_\rho(A)) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Q^{(k)\top} & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} F(A) & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned} \tag{2.28}
$$

where the third line is obtained using the fact that $f(x) \leq 0$, whenever $x_i = 0$ for some $i \in \mathbb{N}_k$. The notation $\operatorname{Diag}_\rho f(\lambda_\rho(A))$ stands for a diagonal matrix with the elements $\{f(\lambda_\rho(A)) : \rho \in \mathbb{N}_{n,k}\}$ on the diagonal, ordered lexicographically. We now continue the calculation. To justify the first equality, we use the definition of a (generated) k-isotropic function (together with the reordering convention used to bring the elements with indexes in $\mathbb{N}_{n,k}$ in the upper-left corner).

$$
\begin{aligned}
\begin{pmatrix} F(K^\top AK) & * \\ * & * \end{pmatrix} = F\begin{pmatrix} K^\top AK & 0 \\ 0 & M_1AM_1 \end{pmatrix} &= F\Big(\frac{U^\top TU + V^\top TV}{2}\Big) \\
&\preceq \frac{1}{2}F(U^\top TU) + \frac{1}{2}F(V^\top TV) \\
&= \frac{1}{2}(U^{(k)})^\top F\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}U^{(k)} + \frac{1}{2}(V^{(k)})^\top F\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}V^{(k)}
\end{aligned}
$$

$$\leq \frac{1}{2}\begin{pmatrix} (K^{(k)})^\top & * \\ * & * \end{pmatrix}\begin{pmatrix} F(A) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K^{(k)} & * \\ * & * \end{pmatrix}+$$
$$+\frac{1}{2}\begin{pmatrix} (K^{(k)})^\top & * \\ * & * \end{pmatrix}\begin{pmatrix} F(A) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K^{(k)} & * \\ * & * \end{pmatrix}$$
$$=\begin{pmatrix} (K^{(k)})^\top F(A)K^{(k)} & * \\ * & * \end{pmatrix}.$$

The second line is obtained using the operator convexity of F and the forth line is obtained using (2.28). Comparing the beginning with the end implies (2.27).

Corollary 2.4.4 *Let $f : [0,\alpha)^k \to \mathbb{R}$ be symmetric and continuous with $f(x) \leq 0$, whenever $x_i = 0$ for some $i \in \mathbb{N}_k$. If f is operator convex, then $(x_1 \cdots x_k)^{-1} f(x)$ is operator monotone on $(0,\alpha)^k$.*

Proof Let $A, B \in S_+^n$ be invertible with $A \geq B$. Apply Proposition 2.4.3 to the contraction $K := A^{-1/2}B^{1/2}$. The left-hand side of (2.27) turns into

$$(K^{(k)})^\top F(A)K^{(k)} = (B^{1/2}A^{-1/2})^{(k)}F(A)(A^{-1/2}B^{1/2})^{(k)} = (B^{1/2})^{(k)}(A^{-1/2})^{(k)}F(A)(A^{-1/2})^{(k)}(B^{1/2})^{(k)}.$$

The right-hand side of (2.27) turns into

$$F(K^\top AK) = F(B^{1/2}A^{-1/2}AA^{-1/2}B^{1/2}) = F(B).$$

Thus, (2.27) implies that

$$(B^{1/2})^{(k)}(A^{-1/2})^{(k)}F(A)(A^{-1/2})^{(k)}(B^{1/2})^{(k)} \geq F(B),$$

that is,

$$(A^{-1/2})^{(k)}F(A)(A^{-1/2})^{(k)} \geq (B^{-1/2})^{(k)}F(B)(B^{-1/2})^{(k)}.$$

Since $(A^{-1/2})^{(k)}$ and $F(A)$ are co-axial (simultaneously diagonalized), they commute. Thus,

$$(A^{-1})^{(k)} F(A) \geq (B^{-1})^{(k)} F(B),$$

whenever $A \geq B$. The proof follows since the k-isotropic function $(A^{-1})^{(k)} F(A)$ is is generated by the symmetric function $(x_1 \cdots x_k)^{-1} f(x)$.

2.4.2 Constructing operator monotone functions on $(0, \alpha)^k$

To construct operator monotone functions, we use Theorem 2.1.12.

Example 2.4.5 *Let I be an interval. If $g : I \to \mathbb{R}$ is operator monotone, then*

$$f(x_1, \ldots, x_k) := g(x_1) + \cdots + g(x_k)$$

is operator monotone on I^k.

If $g : I \to [0, \infty)$ is operator monotone, then

$$f(x_1, \ldots, x_k) := g(x_1) \cdots g(x_k)$$

is operator monotone on I^k. More generally, for any $1 \leq m \leq k$, we have that

$$f(x_1, \ldots, x_k) := \sum_{\rho \in \mathbb{N}_{k,m}} g(x_{\rho_1}) \cdots g(x_{\rho_m})$$

is operator monotone on I^k. ∎

Our extended notion of operator monotonicity for symmetric functions on k variables allows us to see Theorem V.3.10 in [2] in an entirely new light.

Theorem 2.4.6 (Theorem V.3.10 in [2]) *Let I be an interval. If $f \in C^2(I)$ and f is operator* *convex, then for each $y \in I$, the function*

$$g(x) := \begin{cases} f'(x) & \text{if } x = y, \\ \dfrac{f(x) - f(y)}{x - y} & \text{if } x \neq y, \end{cases} \tag{2.29}$$

is operator monotone on I.

The function on the right-hand side of (2.29) is symmetric and is operator monotone with respect to each argument. Thus, using Theorem 2.1.12 we obtain the following interpretation.

Corollary 2.4.7 *Let I be an interval. If $f \in C^2(I)$ and f is operator convex, then the function*

$$g(x, y) := \begin{cases} f'(x) & \text{if } x = y, \\ \dfrac{f(x) - f(y)}{x - y} & \text{if } x \neq y, \end{cases}$$

is operator monotone on I^2.

Conversely, g is operator monotone on $(0, \alpha)^2$, then f is operator convex on $[0, \alpha)$.

Proof The first part follows by Theorem 2.1.12. For the converse, if $g(x, y)$ is operator monotone on $(0, \alpha)^2$, $g(x, 0)$ is operator monotone on $(0, \alpha)$ by Theorem 2.1.12. Since $f(x) - f(0) = xg(x, 0)$ is operator convex on $[0, \alpha)$, by Theorem 2.4.1, and so is $f(x)$.

Finally, we need another connection between operator monotone and operator convex functions, see Theorem V.2.5 in [2].

Lemma 2.4.8 *A function $g : [0, +\infty) \to [0, +\infty)$ is operator monotone, if and only if g is operator concave.*

In the next example, we use Theorem 2.1.9 and combine Corollary 2.4.7 with Lemma 2.4.8 inductively to obtain a family of symmetric operator monotone functions on k variables.

Example 2.4.9 *Let $g : (0,\alpha) \to (0,\infty)$ be an operator monotone function, with integral representation (2.2) for some $a \in \mathbb{R}, b \geq 0$, and a positive Borel measure μ on $\mathbb{R}$ with zero mass on $(0,\alpha)$, satisfying (2.3). By Lemma 2.4.8, g is operator concave, hence $-g$ is operator convex. Then, by Corollary 2.4.7, the symmetric function*

$$g_1(x_1,x_2) := \begin{cases} -g'(x_1) & \text{if } x_1 = x_2, \\ -\dfrac{g(x_1)-g(x_2)}{x_1-x_2} & \text{if } x_1 \neq x_2, \end{cases}$$

is operator monotone on $(0,\alpha)^2$. Using (2.2), it is easy to see that

$$g_1(x_1,x_2) = -b - \int_{-\infty}^{+\infty} (\lambda - x_1)^{-1}(\lambda - x_2)^{-1}\, d\mu(\lambda).$$

By Theorem 2.4.1, the fact that $g_1(x_1,x_2)$ is operator monotone implies that $f_1(x_2) := x_2 g_1(x_1,x_2)$ is operator convex on $[0,\alpha)$ with integral representation

$$f_1(x_2) = -bx_2 - x_2\int_{-\infty}^{+\infty} (\lambda - x_1)^{-1}(\lambda - x_2)^{-1}\, d\mu(\lambda).$$

Let

$$g_2(x_1,x_2,x_3) := \begin{cases} f_1'(x_2) & \text{if } x_2 = x_3, \\ \dfrac{f_1(x_2)-f_1(x_3)}{x_2-x_3} & \text{if } x_2 \neq x_3. \end{cases}$$

The function $g_2(x_1,x_2,x_3)$ is symmetric, since it is continuous and for any distinct x_1,x_2,x_3, we have

$$g_2(x_1,x_2,x_3) = -\frac{x_1}{(x_1-x_2)(x_1-x_3)}g(x_1) - \frac{x_2}{(x_2-x_1)(x_2-x_3)}g(x_2) - \frac{x_3}{(x_3-x_2)(x_3-x_1)}g(x_3).$$

By Corollary 2.4.7, it is operator monotone on $(0,\alpha)^3$ with integral representation

$$g_2(x_1,x_2,x_3) = -b - \int_{-\infty}^{+\infty} \lambda(\lambda - x_1)^{-1}(\lambda - x_2)^{-1}(\lambda - x_3)^{-1}\, d\mu(\lambda).$$

Define functions $f_k : [0,\alpha) \to \mathbb{R}$ *and* $g_{k+1} : (0,\alpha)^{k+2} \to \mathbb{R}$, *for* $k \geq 2$, *iteratively by*

$$f_k(x_{k+1}) := x_{k+1} g_k(x_1, \ldots, x_{k+1}),$$

$$g_{k+1}(x_1, \ldots, x_{k+2}) := \begin{cases} f_k'(x_{k+1}) & \text{if } x_{k+1} = x_{k+2}, \\ \dfrac{f_k(x_{k+1}) - f_k(x_{k+2})}{x_{k+1} - x_{k+2}} & \text{if } x_{k+1} \neq x_{k+2}. \end{cases}$$

By induction, for any natural number k, we have for any distinct $x_1, \ldots, x_{k+2}$

$$g_{k+1}(x_1, \ldots, x_{k+2}) = -\sum_{i=1}^{k+2} x_i^k g(x_i) \prod_{j \neq i} (x_i - x_j)^{-1}$$

with integral representation

$$g_{k+1}(x_1, \ldots, x_{k+2}) = -b - \int_{-\infty}^{+\infty} \lambda^n \prod_{i=1}^{k+2} (\lambda - x_i)^{-1} \, d\mu(\lambda).$$

This shows that $g_{k+1}(x_1, \ldots, x_{k+2})$ *is a symmetric function.*

If g_k *is operator monotone on* $(0,\alpha)^{k+1}$, *then by Theorem 2.4.1,* f_k *is operator convex on* $[0,\alpha)$. *Then, by Corollary 2.4.7 the divided difference* g_{k+1} *is symmetric and operator monotone on* $(0,\alpha)^{k+2}$. *This completes the inductive step.* ∎

2.4.3 A connection with D-type linear maps

We conclude this work with a curious connection to the mathematics used in the field of quantum entanglement.

A linear map $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is called *a D-type linear map*, if it has the representation

$$T(H) = -H + \mathrm{Diag}\,(D(\mathrm{diag}\, H)) \quad \text{for all } H \in \mathbb{R}^{n\times n} \tag{2.30}$$

for some $D \in \mathbb{R}^{n\times n}$ with non-negative elements. Such maps are studied in [10] in connection with k-positive maps. A linear map $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is called *diagonal D-type linear map*, if in addition $D \in \mathbb{R}^{n\times n}$ is diagonal.

The linear map (2.16) extends diagonal D-type maps, as we now explain.

Corollary 2.4.10 *Linear map* (2.16) *turns into a diagonal D-type linear map, when* $k = 1$, $J_{ii} \geq -1$ *for* $i = 1, \ldots, n$, *and*

$$\mathbf{A}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ -1 & \text{otherwise.} \end{cases}$$

Proof Let $k = 1$. Then, matrix J in (2.16) becomes diagonal matrix and we have that $\Delta_k(H) = H$ and $\mathbf{A} \circ H = -H + \operatorname{Diag}(\operatorname{diag} H)$. Then, (2.16) turns into

$$\begin{aligned} \mathbf{T}(H) &= -H + \operatorname{Diag}(\operatorname{diag} H) + \operatorname{Diag}(J(\operatorname{diag} H)) \\ &= -H + \operatorname{Diag}((J + I)(\operatorname{diag} H)) = -H + \operatorname{Diag}(D(\operatorname{diag} H)), \end{aligned}$$

where $D := J + I$ is diagonal with non-negative elements.

In [9] and [19], a necessary and sufficient condition on the matrix D is given, so that the D-type linear map is positive. In the case when D is diagonal the condition turns into

$$\sum_{i=1}^{n} \frac{1}{D_{ii}} \leq 1 \quad \text{and} \quad D_{ii} > 0 \text{ for } i = 1, \ldots, n. \tag{2.31}$$

This is equivalent with the condition obtained in Theorem 2.3.1, that is, $T(E_n) \succeq 0$, where E_n is the $n \times n$ all-one matrix. We conclude by showing the equivalence.

Proposition 2.4.11 *Consider the map (2.30) for a diagonal* $D \in S^n_+$. *Then,* $T(E_n) \succeq 0$, *if and only if (2.31) holds.*

Proof Suppose that $T(E_n) = D - E_n$ is positive semidefinite. Then, $D_{ii} \geq 1$ for all i. Consider vector $d := (1/D_{11}, \ldots, 1/D_{nn})$ to obtain

$$0 \leq d^\top (D - E_n) d = \sum_{i=1}^{n} \frac{1}{D_{ii}} - \Big(\sum_{i=1}^{n} \frac{1}{D_{ii}}\Big)^2,$$

and (2.31) follows.

Suppose now that (2.31) holds. It can be shown that

$$\det(D - E_n) = D_{11} \cdots D_{nn} - \sum_{i=1}^{n} \prod_{j \neq i} D_{jj}.$$

Dividing by $D_{11} \cdots D_{nn}$, one concludes that $\det(D-E_n) \geq 0$. One can show analogously, that the determinant of every principle minor is non-negative. Hence, $D - E_n$ is positive semidefinite.

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Chapter 3

On the analyticity of k-isotropic functions

3.1 Introduction

Denote by $\mathbb{R}^{n\times n}$ the space of all $n \times n$ real matrices and denote by S^n the subspace of all $n \times n$ symmetric matrices with inner product $\langle A, B\rangle := \mathrm{Tr}(AB)$ and Frobenius norm $\|A\| := \sqrt{\mathrm{Tr}(AA)}$. Let O^n be the group of $n \times n$ orthogonal matrices: $A \in O^n$ if and only if $A^\top A = I$. Denote the convex cone in $\mathbb{R}^n$ of all vectors with non-increasing coordinates by $\mathbb{R}^n_{\geq}$. Then for any $A \in S^n$, let $\lambda(A) \in \mathbb{R}^n_{\geq}$ be the vector of ordered eigenvalues of A:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

For any vector $x \in \mathbb{R}^n$, let $\mathrm{Diag}\, x$ be the $n \times n$ matrix with diagonal elements x and zeros elsewhere. Denote by P^n the collection of all $n \times n$ permutation matrices.

The focus of this paper is to show the analyticity of a class of orthogonally invariant matrix-valued functions that captures as a special case three previously investigated classes of orthogonally invariant matrix-valued functions. We begin by familiarizing the reader with the three special cases.

Definition 3.1.1 *A real-valued function $F : S^n \to \mathbb{R}$ is called a spectral function if*

$$F(UAU^\top) = F(A)$$

holds for all $U \in O^n$ and all $A \in S^n$ in the domain of F.

The spectral functions have been extensively studied and applied in various areas ranging from optimization and variational analysis, see [15], to engineering [25] and material science, see [21], where spectral functions are also called *scalar-valued isotropic functions*. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *symmetric* if for any $x \in \mathbb{R}^n$, $f(Px) = f(x)$ for any permutation matrix $P \in P^n$. The following representation theorem is well-known, and can be found in [7] and [22].

Theorem 3.1.2 *A real-valued function $F : S^n \to \mathbb{R}$ is a spectral function, if and only if there exists a unique symmetric function $f : \mathbb{R}^n \to \mathbb{R}$ such that $F(A) = (f \circ \lambda)(A)$ for all $A \in S^n$.*

Properties of the symmetric function f and its corresponding spectral function are tightly connected and many investigations have focused on particular properties. For example, even though the eigenvalue map $\lambda : S^n \to \mathbb{R}^n$ is not in general differentiable, F is differentiable at A if and only if f is such at $\lambda(A)$. One may refer to [14], [16], [19], [20], [23], and [24] for an example of the evolution of such studies. The analyticity of spectral functions is shown in [24] following the same pattern: F is is analytic at A if and only if f is analytic at $\lambda(A)$.

We now define the second class of orthogonally invariant functions that concern us.

Definition 3.1.3 *A function $F : S^n \to S^n$ is called a primary matrix function if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$F(A) = U(\operatorname{Diag}(f(\lambda_1(A)), \ldots, f(\lambda_n(A))))U^\top, \tag{3.1}$$

where $U \in O^n$ is such that $A = U(\operatorname{Diag}\lambda(A))U^\top$.

It can be shown that primary matrix functions are well-defined. That is, the right-hand side of (3.1) does not depend on the choice of the orthogonal matrix U diagonalizing A.

Derivatives, operator monotonicity, and operator convexity of primary matrix functions are studied and characterized in terms of the underlying function f, see for example [5, Chapter V] and [10, Chapter 6]. If f is analytic, then (3.1) becomes

$$F(A) = \oint_{\Gamma} f(z)(zI - A)^{-1}\, dz,$$

where Γ is a Jordan curve in the complex plane enclosing the eigenvalues of A. Thus, the primary matrix function F is analytic if and only if f is analytic, see [12, Chapter 7].

Primary matrix functions are also known as *Löwner's operator* functions. They are a special case of the following class of maps.

Definition 3.1.4 *A function $F : S^n \to S^n$ is called a tensor-valued isotropic function if*

$$F(UAU^\top) = UF(A)U^\top,$$

for all $U \in O^n$ and all $A \in S^n$ in the domain of F.

An example of tensor-valued isotropic function found in [21] is the Piola-Kirchhoff stress function in an isotropic solid.

We say a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *symmetric* if for any $x \in \mathbb{R}^n$, we have $f(Px) = Pf(x)$ for any permutation matrix $P \in P^n$. The following representation theorem for tensor-valued isotropic functions can be found in [21] and [22].

Theorem 3.1.5 *A function $F : S^n \to S^n$ is a tensor-valued isotropic function if and only if there exists a symmetric function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$F(A) = U(\mathrm{Diag}\, f(\lambda(A)))U^\top,$$

where $U \in O^n$ is such that $A = U(\operatorname{Diag}\lambda(A))U^\top$.

Note that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that $f(x) = (g(x_1), \ldots, g(x_n))$ for some $g : \mathbb{R} \to \mathbb{R}$, then the tensor-valued isotropic function becomes primary matrix function (3.1).

The construction in works [2], [3], [8], [9], and [26] generalizes primary matrix functions to several operator arguments. We introduce the setting as follows. Let $\mathbb{N}$ be the set of natural numbers and let

$$\mathbb{N}_n := \{1, 2, \ldots, n\}$$

be the set of the first n natural numbers. For any fixed $n_1, \ldots, n_k$, consider the set of k-tuples in $\mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$ endowed with the lexicographical order.

Any function $f : \mathbb{R}^k \to \mathbb{R}$ defines an operator map $F : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$ by

$$F(A_1, \ldots, A_k) := (\otimes_{i=1}^k U_i)(\operatorname{Diag}_l f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k)))(\otimes_{i=1}^k U_i)^\top, \tag{3.2}$$

where $l = (l_1, \ldots, l_k)$ is in $\mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$ and $U_i \in O^{n_i}$ are such that $A_i = U_i(\operatorname{Diag}\lambda(A_i))U_i^\top$ for $i = 1, \ldots, k$. Here, Diag_l denotes a diagonal matrix, where on the main diagonal we have the values $f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$ ordered lexicographically. The right-hand side of (3.2) does not depend on the choice of the diagonalizing matrices U_i, $i = 1, \ldots, k$. When $k = 1$, the map (3.2) becomes a primary matrix function (3.1).

Note that the map (3.2) has the following invariance property

$$F(U_1 A_1 U_1^\top, \ldots, U_k A_k U_k^\top) = (\otimes_{i=1}^k U_i) F(A_1, \ldots, A_k)(\otimes_{i=1}^k U_i)^\top,$$

for any $U_i \in O^{n_i}, i = 1, \ldots, k$. While construction (3.2) extends primary matrix functions to the multi-variable setting, it cannot capture the class of spectral functions nor tensor-valued isotropic functions. The focus of this paper is the class of orthogonally invariant maps introduced in [1] and [17]. It captures all three classes of functions and the special case of (3.2), when $f : \mathbb{R}^k \to \mathbb{R}$ is a symmetric function. Next two sections introduce the back ground, notation, and the construction.

3.2 Background

3.2.1 Tensor and anti-symmetric tensor power of $\mathbb{R}^n$

Denote by $\otimes^k\mathbb{R}^n$ the k-th tensor product of $\mathbb{R}^n$. This is the linear space of dimension n^k of all formal finite linear combinations of products $\{x_1 \otimes \cdots \otimes x_k \,:\, x_1, \ldots, x_k \in \mathbb{R}^n\}$, where all the necessary identifications are made so that the product is linear in each argument separately. The inner product in $\otimes^k\mathbb{R}^n$ of any vectors $u_1 \otimes \cdots \otimes u_k$ and $v_1 \otimes \cdots \otimes v_k$ is given by

$$\langle u_1 \otimes \cdots \otimes u_k, v_1 \otimes \cdots \otimes v_k\rangle = \langle u_1, v_1\rangle \cdots \langle u_k, v_k\rangle.$$

Given k linear operators $A_1, \ldots, A_k$ on $\mathbb{R}^n$, their tensor product $A_1 \otimes \ldots \otimes A_k$ is a linear operator on $\otimes^k\mathbb{R}^n$ defined by

$$(A_1 \otimes \cdots \otimes A_k)(x_1 \otimes \cdots \otimes x_k) := A_1x_1 \otimes \cdots \otimes A_kx_k$$

and then extended by linearity. It is well-known that there are no inconsistencies in the extension process. Important properties of the tensor product include

$$(A_1 \otimes \cdots \otimes A_k)(B_1 \otimes \cdots \otimes B_k) = A_1B_1 \otimes \cdots \otimes A_kB_k, \quad (A_1 \otimes \cdots \otimes A_k)^* = A_1^* \otimes \cdots \otimes A_k^*.$$

When $A_1, \ldots, A_k$ are invertible, so is their tensor product and

$$(A_1 \otimes \cdots \otimes A_k)^{-1} = A_1^{-1} \otimes \cdots \otimes A_k^{-1}.$$

When all operators are the same, we use the short-hand notation $\otimes^k A := A \otimes \cdots \otimes A$.

For any vectors $x_1, \dots, x_k \in \mathbb{R}^n$, the k-th anti-symmetric tensor product is defined by

$$x_1 \wedge \cdots \wedge x_k := \frac{1}{\sqrt{k!}} \sum_{\sigma:\mathbb{N}_k \to \mathbb{N}_k} \epsilon_\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}, \tag{3.3}$$

where ϵ_σ is equal to 1 if the permutation σ is even and is equal to -1 if the permutation σ is odd. It is easy to see that this product is anti-commutative:

$$x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_n = -x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_i \wedge \cdots \wedge x_n.$$

Denote by $\wedge^k \mathbb{R}^n$ the subspace of $\otimes^k \mathbb{R}^n$ spanned by all k-th anti-symmetric tensor products. The dimension of $\wedge^k \mathbb{R}^n$ is $\binom{n}{k}$. The inner product in $\wedge^k \mathbb{R}^n$ is the restriction of the inner product on $\otimes^k \mathbb{R}^n$. Explicitly, the inner product between $u_1 \wedge \cdots \wedge u_k$ and $v_1 \wedge \cdots \wedge v_k$ is given by

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det\left(\langle u_i, v_j \rangle_{i,j=1}^k\right).$$

The subspace $\wedge^k \mathbb{R}^n$ is invariant under the operator $\otimes^k A$, allowing one to denote by $\wedge^k A$ the restriction of $\otimes^k A$ to the space $\wedge^k \mathbb{R}^n$. This is known as the *k-th anti-symmetric tensor power of A*. It can be shown that

$$(\wedge^k A)(x_1 \wedge \cdots \wedge x_k) = Ax_1 \wedge \cdots \wedge Ax_k.$$

Note that anti-symmetric tensor product between operators is not defined when the operators are not all equal. The antisymmetric tensor power of an operator shares similar properties to the tensor power:

$$(\wedge^k A)(\wedge^k B) = \wedge^k (AB), \quad (\wedge^k A)^* = \wedge^k A^*. \tag{3.4}$$

When A is invertible, so is its antisymmetric tensor power and

$$(\wedge^k A)^{-1} = \wedge^k A^{-1}. \tag{3.5}$$

3.2.2 Indexing in the spaces $\mathbb{R}^{n^k}$ and $\mathbb{R}^{\binom{n}{k}}$

Let

$$\mathbb{N}_n^k := \mathbb{N}_n \times \cdots \times \mathbb{N}_n \text{ (k times)},$$

and assume that the elements in $\mathbb{N}_n^k$ are ordered lexicographically. We use $\mathbb{N}_n^k$ to index the coordinates of vectors in $\mathbb{R}^{n^k}$ and the entries of matrices in $\mathbb{R}^{n^k \times n^k}$. If $\mathbf{x} \in \mathbb{R}^{n^k}$, the $\mathbf{x}_l$ denotes the l-th element of $\mathbf{x}$, for $l \in \mathbb{N}_n^k$; and if $\mathbf{A} \in \mathbb{R}^{n^k \times n^k}$, then $\mathbf{A}_{l,m}$ denotes the element in the l-th row and m-th column, for $l, m \in \mathbb{N}_n^k$.

Given $n \times n$ matrices A and B, their tensor product is defined by

$$A \otimes B := \begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{pmatrix}$$

Extend this inductively to products $A_1 \otimes \cdots \otimes A_k$ of more than two matrices. We denote the k-tensor power of a matrix A by

$$A^{\otimes k} := A \otimes \cdots \otimes A.$$

(This notation distinguishes between the k-tensor power $\otimes^k A$ of an operator A.) It is easy to see that

$$(A^{\otimes k})_{l,m} = A_{l_1 m_1} \cdots A_{l_k m_k} \quad \text{for any } l, m \in \mathbb{N}_n^k.$$

The tensor product between matrices has analogous properties as those between operators on $\mathbb{R}^n$.

Let $\mathbb{N}_{n,k}$ be the set of all subsets of $\mathbb{N}_n$ of size k with elements ordered non-decreasingly, here $1 \le k \le n$. That is, if $\rho := \{\rho_1, \dots, \rho_k\} \in \mathbb{N}_{n,k}$, assume that

$$\rho_1 < \rho_2 < \cdots < \rho_k.$$

This assumption allows us to view $\mathbb{N}_{n,k}$ as a subset of $\mathbb{N}_n^k$.

Order the elements in $\mathbb{N}_{n,k}$ lexicographically. In this way they are used to index the coordinates of vectors in $\mathbb{R}^{\binom{n}{k}}$ and the entries of matrices in $\mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$. For example, if $\mathbf{x} \in \mathbb{R}^{\binom{n}{k}}$, then for any $\rho \in \mathbb{N}_{n,k}$, denote by $\mathbf{x}_\rho$ the ρ-th element in vector $\mathbf{x}$; and if $\mathbf{A} \in \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$, for any $\rho, \tau \in \mathbb{N}_{n,k}$, $\mathbf{A}_{\rho,\tau}$ denotes the element in the ρ-th row and τ-th column of $\mathbf{A}$.

If $A \in \mathbb{R}^{n\times n}$, then for any $\rho, \tau \in \mathbb{N}_{n,k}$, denote by $A_{\rho\tau}$, the $k \times k$ minor of A obtained at the intersection of rows with indexes $\rho_1, \dots, \rho_k$ and columns with indexes $\tau_1, \dots, \tau_k$.

To avoid confusion, vectors in (resp. operators on) $\mathbb{R}^{\binom{n}{k}}$ are denoted in bold font, such as $\mathbf{x}$ and $\mathbf{A}$, while vectors in (resp. operators on) $\mathbb{R}^n$ are in plain italics: x and A.

For $A \in \mathbb{R}^{n\times n}$, the *k-th multiplicative compound matrix of* A, denoted by $A^{(k)} \in \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$, is defined by

$$(A^{(k)})_{\rho,\tau} := \det(A_{\rho\tau}) \quad \text{for any } \rho, \tau \in \mathbb{N}_{n,k}.$$

The following properties are well-known:

$$A^{(k)}B^{(k)} = (AB)^{(k)}, (A^{(k)})^\top = (A^\top)^{(k)}, \text{ and } (A^{(k)})^{-1} = (A^{-1})^{(k)},$$

corresponding to properties (3.4) and (3.5).

3.2.3 Identification of $\mathbb{R}^{n^k}$ with $\otimes^k\mathbb{R}^n$ and of $\mathbb{R}^{\binom{n}{k}}$ with $\wedge^k\mathbb{R}^n$

Denote the standard orthonormal basis in $\mathbb{R}^n$ by $\{e^1, \dots, e^n\}$. Denote by $\{\mathbf{e}^l : l \in \mathbb{N}_n^k\}$ the standard orthonormal basis in $\mathbb{R}^{n^k}$ and by $\{\mathbf{e}^\rho : \rho \in \mathbb{N}_{n,k}\}$ the one in $\mathbb{R}^{\binom{n}{k}}$. Fixing a basis, allows one to view any linear operator on $\mathbb{R}^n$ as a matrix.

The standard isometry between $\mathbb{R}^{n^k}$ and $\otimes^k \mathbb{R}^n$ is denoted by $\mathcal{T}$ and defined by

$$\mathcal{T}(\mathbf{e}^l) := e^{l_1} \otimes \cdots \otimes e^{l_k}, \text{ for all } l \in \mathbb{N}_n^k,$$

extended by linearity. For an $n \times n$ matrix A (viewed also as an operator on $\mathbb{R}^n$), we have the relationship:

$$\mathcal{T}((A^{\otimes k})\mathbf{x}) = (\otimes^k A)(\mathcal{T}\mathbf{x}), \text{ for any } \mathbf{x} \in \mathbb{R}^{n^k}.$$

The standard isometry, call it $\mathcal{W}$, between $\mathbb{R}^{\binom{n}{k}}$ and $\wedge^k \mathbb{R}^n$ is given by

$$\mathcal{W}(\mathbf{e}^\rho) := e^{\rho_1} \wedge \cdots \wedge e^{\rho_k}, \text{ for all } \rho \in \mathbb{N}_{n,k}$$

and then extended by linearity. For an $n \times n$ matrix A (viewed also as an operator on $\mathbb{R}^n$), we have the relationship:

$$\mathcal{W}(A^{(k)}\mathbf{x}) = (\wedge^k A)(\mathcal{W}\mathbf{x}), \text{ for any } \mathbf{x} \in \mathbb{R}^{\binom{n}{k}}.$$

Equivalently, we have the following commuting relationships:

$$\mathcal{W}A^{(k)} = (\wedge^k A)\mathcal{W} \quad \text{or} \quad A^{(k)}\mathcal{W}^{-1} = \mathcal{W}^{-1}(\wedge^k A).$$

3.2.4 Permutation matrices

For any permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, the corresponding permutation matrix P is such that $Px = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})^\top$ for any vectors $x \in \mathbb{R}^n$ or in terms of basis vectors $Pe^{\sigma(i)} = e^i$ for all $i = 1, \ldots, n$.

Every permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ defines a permutation on $\mathbb{N}_{n,k}$ denoted by $\sigma^{(k)}$ in the following way

$$\sigma^{(k)}(\rho) := \text{the increasing rearrangement of } \{\sigma(\rho_1), \ldots, \sigma(\rho_k)\},$$

for any $\rho \in \mathbb{N}_{n,k}$. The corresponding $\binom{n}{k} \times \binom{n}{k}$ permutation matrix $\mathbf{P}$ of $\sigma^{(k)}$ satisfies

$$\mathbf{P}\mathbf{e}^{\sigma^{(k)}(\rho)} = \mathbf{e}^{\rho}, \tag{3.6}$$

for any $\rho \in \mathbb{N}_{n,k}$. Let $\epsilon_{\sigma,\rho}$ be $+1$ if an even number of transpositions are required to order vector $(\sigma(\rho_1), \ldots, \sigma(\rho_k))$ increasingly; and let it be equal to -1 otherwise. The ralation between matrix $\mathbf{P}$ and $P^{(k)}$ is:

$$\begin{aligned} P^{(k)}\mathbf{e}^{\sigma^{(k)}(\rho)} &= P^{(k)}\mathcal{W}^{-1}(\epsilon_{\sigma,\rho} e^{\sigma(\rho_1)} \wedge \cdots \wedge e^{\sigma(\rho_k)}) = \epsilon_{\sigma,\rho}\mathcal{W}^{-1}(\wedge^k P)(e^{\sigma(\rho_1)} \wedge \cdots \wedge e^{\sigma(\rho_k)}) \\ &= \epsilon_{\sigma,\rho}\mathcal{W}^{-1}(e^{\rho_1} \wedge \cdots \wedge e^{\rho_k}) = \epsilon_{\sigma,\rho}\mathbf{e}^{\rho} = \epsilon_{\sigma,\rho}\mathbf{P}\mathbf{e}^{\sigma^{(k)}(\rho)}, \end{aligned}$$

for any $\rho \in \mathbb{N}_{n,k}$.

3.2.5 Final remark

Let E be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$. An isomorphism $T : E \to E^*$ between E and its dual is defined by $T(x)(a) := \langle x, a \rangle$. Then, one can view that $x \otimes y \in E \otimes E$ has a linear map $E \to E$ defined by

$$(x \otimes y)(a) := T(x)(a) \otimes y = \langle x, a \rangle y.$$

Similarly, one can view that $x \otimes y \in E \otimes E$ has a bilinear map $T(x) \otimes T(y) \in E^* \otimes E^*$ defined by

$$(x \otimes y)(a, b) := T(x) \otimes T(y)(a, b) = T(x)(a)T(y)(b) = \langle x, a \rangle \langle y, b \rangle.$$

3.3 The main definition

Definition 3.3.1 *A function $G : S^n \to S^{\binom{n}{k}}$ is called k-isotropic if*

$$G(UAU^\top) = U^{(k)}G(A)(U^{(k)})^\top$$

for all $U \in O^n$ and $A \in S^n$ in the domain of G.

Definition 3.3.2 *A function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is called symmetric if the equation*

$$g(Px) = \mathbf{P}g(x),$$

holds for all $x \in \mathbb{R}^n$ and all permutation matrices $P \in P^n$, with corresponding $\mathbf{P}$ defined by (3.6).

The representation theorem of k-isotropic functions, stated in [17], is as follows.

Theorem 3.3.3 *The function $G : S^n \to S^{\binom{n}{k}}$ is k-isotropic, if and only if there is a unique symmetric function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ such that*

$$G(A) = U^{(k)}(\mathrm{Diag}\, g(\lambda(A)))(U^{(k)})^\top, \tag{3.7}$$

for all $A \in S^n$ and $U \in O^n$ such that $A = U(\mathrm{Diag}\,\lambda(A))U^\top$.

The matrix $G(A)$ corresponds to a self-adjoint operator on $\wedge^k\mathbb{R}^n$ given by

$$\mathcal{W} \circ G(A) \circ \mathcal{W}^{-1} = \sum_{\rho \in \mathbb{N}_{n,k}} g_\rho(\lambda(A))(u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}) \otimes (u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}), \tag{3.8}$$

where $\{u_1, \ldots, u_n\}$ are the columns of U such that $A = U(\mathrm{Diag}\,\lambda(A))U^\top$.

Example 3.3.4 *When $k = 1$, map (3.7) reduces to tensor-valued isotropic function $G : S^n \to S^n$, since $U^{(1)} = U$ and $G(A) = U(\mathrm{Diag}\, g(\lambda(A)))U^\top$ for a unique symmetric $g : \mathbb{R}^n \to \mathbb{R}^n$.* ∎

For any $x \in \mathbb{R}^n$ and any $\rho \in \mathbb{N}_{n,k}$, denote

$$x_\rho := (x_{\rho_1}, \dots, x_{\rho_k}) \in \mathbb{R}^k.$$

Example 3.3.5 *Let $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ be symmetric and define $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ by*

$$g_\rho(x) := \tilde{g}(x_\rho), \quad \textit{for all } x \in \mathbb{R}^n \textit{ and all } \rho \in \mathbb{N}_{n,k}.$$

It was shown in [1], that g is symmetric in the sense of Definition 3.3.2. In this situation, we say that g is generated by $\tilde{g}$. Using Theorem 3.3.3, consider the following subclass of k-isotropic maps:

$$G(A) := U^{(k)}(\operatorname{Diag} g(\lambda(A)))(U^{(k)})^\top = U^{(k)}(\operatorname{Diag}_\rho \tilde{g}(\lambda_\rho(A)))(U^{(k)})^\top, \tag{3.9}$$

where $U \in O^n$ is such that $A = U(\operatorname{Diag} \lambda(A))U^\top$.

The map G defined in (3.9) *becomes a spectral function if we take $k = n$. In that case $U^{(k)} =$* $\det(U) = \pm 1$, *since U is orthogonal. The set $\mathbb{N}_{n,n}$ contains only one element $\rho = \{1, 2, \dots, n\}$, and thus*

$$G(A) = \tilde{g}(\lambda_1(A), \dots, \lambda_n(A)).$$

The map G defined in (3.9) *becomes primary matrix function if we take $k = 1$. In that case, $U^{(1)} = U$. The set $\mathbb{N}_{n,1}$ contains n elements $\{1\}, \dots, \{n\}$ and thus*

$$G(A) = U(\operatorname{Diag}(\tilde{g}(\lambda_1(A)), \dots, \tilde{g}(\lambda_n(A))))U^T.$$

One can specialize even further. Taking

$$\tilde{g}(x_1, \dots, x_k) = x_1 + \cdots + x_k,$$

(3.9) *becomes the well-known k-th additive compound matrix, see [11, page 19]. While taking*

$$\tilde{g}(x_1, \ldots, x_k) = x_1 \cdots x_k,$$

(3.9) *becomes the well-known k-th multiplicative compound matrix, see [18].* ∎

The main result in [17] states the following.

Theorem 3.3.6 *Suppose $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is symmetric and $G : S^n \to S^{\binom{n}{k}}$ is its corresponding k-isotropic function. Then G is C^r if and only if g is C^r for any $r = 1, \ldots, \infty$.*

Following the construction in Example 3.3.5, Theorem 3.3.6 extends the main result in [4], [20], and [23] (see also [21]), when one puts $k = n$, and it extends one of the main results in [6] (see also [19]), when one puts $k = 1$. When $k = n$, a spectral function is C^r if and only if the corresponding symmetric function is C^r for $r = 1, \ldots, \infty$.

In the more general case when g is symmetric but not generated by a symmetric $\tilde{g}$, as in Example 3.3.5, then Theorem 3.3.6, applied with $k = 1$, extends Theorem 8.1.9 in [21] about the C^r differentiability of tensor-valued isotropic functions, where $r = 1, \ldots, \infty$.

The technique used in [17] is not suitable for showing the analyticity of k-isotropic functions. That is precisely the focus of this note. The main result, see Theorem 3.7.4 below, states necessary and sufficient conditions for the analyticity of k-isotropic functions that are generated by a function $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$, as explained in Example 3.3.5.

3.4 Additional notation and lemmas

A partition of $\mathbb{N}_n$ is a collection of non-empty, pairwise disjoint subsets of $\mathbb{N}_n$ with union $\mathbb{N}_n$. A set in a partition is called a *block*. In this note, partitions of $\mathbb{N}_n$ are generally denoted by the letter I.

Every $x \in \mathbb{R}^n$ defines a partition I^x on $\mathbb{N}_n$ by having i and j in the same block if and only if $x_i = x_j$. The blocks of a partition determined by x are denoted by $I^x = \{I_1^x, \ldots, I_r^x\}$, that is, r

denotes the number of blocks.

For example, if $x \in \mathbb{R}^n_{\geq}$ and

$$x_1 = \cdots = x_{k_1} > x_{k_1+1} = \cdots = x_{k_2} > \cdots > x_{k_{r-1}+1} = \cdots = x_{k_r},$$

then

$$I^x_1 = \{1, \dots, k_1\}, I^x_2 = \{k_1 + 1, \dots, k_2\}, \dots, I^x_r = \{k_{r-1} + 1, \dots, k_r\}.$$

We now explain how any $x \in \mathbb{R}^n$ generates a partition $\mathbf{I}^x$ on $\mathbb{N}^k_n$. The blocks of this partition are labelled by the elements of $\mathbb{N}^k_r$, as follows. For any $s = (s_1, \dots, s_k)$ in $\mathbb{N}^k_r$, define a block in the partition of $\mathbb{N}^k_n$ by

$$\mathbf{I}^x_s := \{l \in \mathbb{N}^k_n : l_i \in I^x_{s_i}, \text{ for any } i \in \mathbb{N}_k\},$$

where $I^x_{s_i}$ are the blocks of the partition I^x. Equivalently, $l, m \in \mathbb{N}^k_n$ are in the same block of the partition $\mathbf{I}^x$ if and only if $x_l = x_m$, where for $x \in \mathbb{R}^n$ and $l \in \mathbb{N}^k_n$ we denoted

$$x_l := (x_{l_1}, \dots, x_{l_k}) \in \mathbb{R}^k.$$

For example the cardinality of $\mathbf{I}^x_s$ is

$$|\mathbf{I}^x_s| = |I^x_{s_1}| \cdots |I^x_{s_k}|.$$

For any fixed $x \in \mathbb{R}^n_{\geq}$, we say that a matrix A is *I^x-block-diagonal* if A can be written as $A = \mathrm{Diag}\,(A_1, A_2, \dots, A_r)$ with A_i of size $|I^x_i|$ for all $i = 1, \dots, r$.

Lemma 3.4.1 *If a matrix A is I^x-block-diagonal, then $A^{\otimes k}$ is $\mathbf{I}^x$-block-diagonal with blocks of size $|\mathbf{I}^x_s|$ for $s \in \mathbb{N}^k_n$.*

Proof For any $l, m \in \mathbb{N}_n^k$, the entry of $A^{\otimes k}$ in l-th column and m-th row is calculated by $(A^{\otimes k})_{l,m} = \prod_{i=1}^k A_{l_i m_i}$. For any $a, b \in \mathbb{N}_r^k$ with $a \neq b$, let $l \in \mathbf{I}_a^x$ and $m \in \mathbf{I}_b^x$. Then, there exists at least one $i \in \mathbb{N}_k$ such that a_i and b_i are in different blocks of the partition I^x. Thus, $A_{l_i m_i} = 0$ and we have $(A^{\otimes k})_{l,m} = 0$.

For any $x \in \mathbb{R}^n$, we say that $\mathbf{x} \in \mathbb{R}^{n^k}$ is $\mathbf{I}^x$-*block-constant*, if $\mathbf{x}_l = \mathbf{x}_m$ for any l, m in the same block of $\mathbf{I}^x$. Equivalently, for any $l, m \in \mathbb{N}_n^k$ we have $\mathbf{x}_l = \mathbf{x}_m$, whenever $x_l = x_m$.

We say that a function $f \colon \mathbb{R}^n \to \mathbb{R}^{n^k}$ is *block-constant*, if vector $f(x)$ is $\mathbf{I}^x$-block-constant, for all $x \in \mathbb{R}^n$. In other words, a function $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ is block-constant if and only if

$$f_l(x) = f_m(x), \text{ whenever } x_l = x_m$$

for all $x \in \mathbb{R}^n$ and all $l, m \in \mathbb{N}_n^k$.

It is easy to see that, for any fixed $x \in \mathbb{R}^n$, we have $(\operatorname{Diag} x)U = U(\operatorname{Diag} x)$, for all $U \in O^n$ if and only if x has equal coordinates. We need a result analogous to this one about matrices commuting with tensor powers of U.

For any $x \in \mathbb{R}^n_{\geq}$, define the following subgroup of O^n:

$$O^n_x := \{W \in O^n : W \text{ is } I^x\text{-block-diagonal}\}.$$

The following lemma gives a necessary sufficient condition for diagonal matrices to commute with tensor powers of matrices from O^n_x.

Lemma 3.4.2 *For any fixed $x \in \mathbb{R}^n_{\geq}$ and $\mathbf{x} \in \mathbb{R}^{n^k}$, we have*

$$(\operatorname{Diag} \mathbf{x})U^{\otimes k} = U^{\otimes k}(\operatorname{Diag} \mathbf{x}), \tag{3.10}$$

for all $U \in O^n_x$, if and only if $\mathbf{x} \in \mathbb{R}^{n^k}$ is $\mathbf{I}^x$-block-constant.

Proof Suppose $x \in \mathbb{R}^n_{\geq}$ is fixed. Equation (3.10) holds, if and only if for any fixed $l, m \in \mathbb{N}^k_n$,

$$((\mathrm{Diag}\,\mathbf{x})U^{\otimes k})_{l,m} = (U^{\otimes k}(\mathrm{Diag}\,\mathbf{x}))_{l,m},$$

holds, that is,

$$(\mathrm{Diag}\,\mathbf{x})_{l,*}(U^{\otimes k})_{*,m} = (U^{\otimes k})_{l,*}(\mathrm{Diag}\,\mathbf{x})_{*,m}, \tag{3.11}$$

holds where $\mathbf{A}_{l,*}$ is the l-th row of $\mathbf{A}$ and $\mathbf{A}_{*,m}$ is the m-th column of $\mathbf{A}$ for all $\mathbf{A} \in \mathbb{R}^{n^k \times n^k}$. Simplify (3.11) to obtain equivalently

$$\mathbf{x}_l(U^{\otimes k})_{l,m} = \mathbf{x}_m(U^{\otimes k})_{l,m}.$$

The last equality holds, if $\mathbf{x}$ is $\mathbf{I}^x$-block-constant. In the other direction, choose $U \in O^n_x$, such that $(U^{\otimes k})_{l,m} \neq 0$, then one can conclude $\mathbf{x}_l = \mathbf{x}_m$.

Lemma 3.4.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ and let $F : S^n \to S^{n^k}$ be given by*

$$F(A) := U^{\otimes k}(\mathrm{Diag}\, f(\lambda(A)))(U^{\otimes k})^\top, \tag{3.12}$$

where $U \in O^n$ is such that $A = U(\mathrm{Diag}\,\lambda(A))U^\top$. The map F is well-defined, whenever f is block-constant function.

Proof For this map (3.12) to be well-defined its value must not depend on the choice of U in the spectral decomposition of A. Note that if $U \in O^n$ is one matrix such that $A = U(\mathrm{Diag}\,\lambda(A))U^\top$ then

$$O^n_{\lambda(A)} \cdot U := \{UW : W \in O^n_{\lambda(A)}\}$$

are all orthogonal matrices that give the ordered spectral decomposition of A. Thus, map (3.12) is well-defined, whenever

$$U^{\otimes k}(\operatorname{Diag} f(\lambda(A)))(U^{\otimes k})^\top = V^{\otimes k}(\operatorname{Diag} f(\lambda(A)))(V^{\otimes k})^\top$$

for any U, V giving ordered spectral decomposition of A. The properties of tensor powers give equivalently

$$(V^\top U)^{\otimes k}(\operatorname{Diag} f(\lambda(A))) = (\operatorname{Diag} f(\lambda(A)))(V^\top U)^{\otimes k}.$$

Since $V^\top U$ can be any element of $O^n_{\lambda(A)}$, by Lemma 3.4.2, map (3.12) is well-defined, whenever f is block-constant.

Matrix $F(A)$ corresponds to an operator on $\otimes^k\mathbb{R}^n$ given by

$$\mathcal{T} \circ F(A) \circ \mathcal{T}^{-1} = \sum_{l\in\mathbb{N}_n^k} f_l(\lambda(A))(u_{l_1} \otimes \cdots \otimes u_{l_k}) \otimes (u_{l_1} \otimes \cdots \otimes u_{l_k}), \tag{3.13}$$

where $\{u_1, \dots, u_n\}$ are the columns of U such that $A = U(\operatorname{Diag}\lambda(A))U^\top$.

3.5 Lifting k-isotropic functions

Theorem 3.5.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ be a block-constant function and let $F : S^n \to S^{n^k}$ be defined by* (3.12)*. Suppose that the following condition holds*

$$f_l(x) = f_\rho(x), \text{ for all } l \in \mathbb{N}_n^k, \rho \in \mathbb{N}_{n,k} \text{ with } l^\uparrow = \rho \text{ and all } x \in \mathbb{R}^n, \tag{3.14}$$

where $l^\uparrow$ is the non-decreasing rearrangements of l. Then, operator $\mathcal{T} \circ F(A) \circ \mathcal{T}^{-1}$ on $\otimes^k\mathbb{R}^n$ preserves subspace $\wedge^k\mathbb{R}^n$.

Proof We only have to show that for any $x_1 \wedge \cdots \wedge x_k \in \wedge^k\mathbb{R}^n$ and any $A \in S^n$, vector

$\mathcal{T} \circ F(A) \circ \mathcal{T}^{-1}(x_1 \wedge \cdots \wedge x_k)$ is in $\wedge^k \mathbb{R}^n$. By (3.13), we have

$$\begin{aligned} \mathcal{T} \circ F(A) \circ \mathcal{T}^{-1}(x_1 \wedge \cdots \wedge x_k) &= \sum_{l \in \mathbb{N}_n^k} f_l(\lambda(A))(u_{l_1} \otimes \cdots \otimes u_{l_k}) \otimes (u_{l_1} \otimes \cdots \otimes u_{l_k})(x_1 \wedge \cdots \wedge x_k) \\ &= \sum_{l \in \mathbb{N}_n^k} f_l(\lambda(A)) \langle u_{l_1} \otimes \cdots \otimes u_{l_k}, x_1 \wedge \cdots \wedge x_k \rangle (u_{l_1} \otimes \cdots \otimes u_{l_k}). \end{aligned} \tag{3.15}$$

Using definition (3.3) of the wedge product, one obtains

$$\begin{aligned} \langle u_{l_1} \otimes \cdots \otimes u_{l_k}, x_1 \wedge \cdots \wedge x_k \rangle &= \frac{1}{\sqrt{k!}} \langle u_{l_1} \otimes \cdots \otimes u_{l_k}, \sum_{\sigma : \mathbb{N}_k \to \mathbb{N}_k} \epsilon_\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)} \rangle \\ &= \frac{1}{\sqrt{k!}} \sum_{\sigma : \mathbb{N}_k \to \mathbb{N}_k} \epsilon_\sigma \langle u_{l_1} \otimes \cdots \otimes u_{l_k}, x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)} \rangle \\ &= \frac{1}{\sqrt{k!}} \sum_{\sigma : \mathbb{N}_k \to \mathbb{N}_k} \epsilon_\sigma \prod_{i=1}^{k} \langle u_{l_i}, x_{\sigma(i)} \rangle, \end{aligned}$$

where ϵ_σ is $+1$ if the permutation σ is even and is -1 if the permutation σ is odd. Notice that

$$\sum_{\sigma : \mathbb{N}_k \to \mathbb{N}_k} \epsilon_\sigma \prod_{i=1}^{k} \langle u_{l_i}, x_{\sigma(i)} \rangle = \det\left(\langle u_{l_i}, x_j \rangle_{i,j=1}^k \right)$$

by definition of determinant and notice that

$$\det\left(\langle u_{l_i}, x_j \rangle_{i,j=1}^k \right) = 0, \text{ whenever } l_i = l_j \text{ for some } i \neq j.$$

Thus, in the sum (3.15), the possibly non-zero terms are those corresponding to indexes $l \in \mathbb{N}_n^k$ with distinct elements. For every $l \in \mathbb{N}_n^k$ with distinct elements, there exists a $\rho \in \mathbb{N}_{n,k}$ such that $l^{\uparrow} = \rho$. Equivalently, there exist a $\rho \in \mathbb{N}_{n,k}$ and a permutation $\sigma : \mathbb{N}_k \to \mathbb{N}_k$ (depending on l) such that $\rho_{\sigma(i)} = l_i$. Using (3.14), we continue

$$\mathcal{T} \circ F(A) \circ \mathcal{T}^{-1}(x_1 \wedge \cdots \wedge x_k) = \sum_{l \in \mathbb{N}_n^k} f_l(\lambda(A)) \det\left(\langle u_{l_i}, x_j \rangle_{i,j=1}^k \right) \frac{1}{\sqrt{k!}} (u_{l_1} \otimes \cdots \otimes u_{l_k})$$

$$= \sum_{\rho \in \mathbb{N}_{n,k}} f_\rho(\lambda(A)) \det\left(\langle u_{\rho_i}, x_j \rangle_{i,j=1}^k\right) \sum_{l : l^\uparrow = \rho} \epsilon_\sigma \frac{1}{\sqrt{k!}} (u_{l_1} \otimes \cdots \otimes u_{l_k}), \tag{3.16}$$

since

$$\det(\langle u_{l_i}, x_j \rangle_{i,j=1}^k) = \det(\langle u_{\rho_{\sigma(i)}}, x_j \rangle_{i,j=1}^k) = \epsilon_\sigma \det(\langle u_{\rho_i}, x_j \rangle_{i,j=1}^k),$$

holds, where ϵ_σ is the sign of the permutation σ that orders l non-increasingly. Notice that for any fixed $\rho \in \mathbb{N}_{n,k}$, we have

$$\sum_{l : l^\uparrow = \rho} \epsilon_\sigma \frac{1}{\sqrt{k!}} (u_{l_1} \otimes \cdots \otimes u_{l_k}) = \sum_{\sigma : \mathbb{N}_k \to \mathbb{N}_k} \epsilon_\sigma \frac{1}{\sqrt{k!}} (u_{\rho_{\sigma(1)}} \otimes \cdots \otimes u_{\rho_{\sigma(k)}}) = u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}.$$

Finally, substituting into (3.16), gives

$$\mathcal{T} \circ F(A) \circ \mathcal{T}^{-1}(x_1 \wedge \cdots \wedge x_k) = \sum_{\rho \in \mathbb{N}_{n,k}} f_\rho(\lambda(A)) \det\left(\langle u_{\rho_i}, x_j \rangle_{i,j=1}^k\right)(u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}), \tag{3.17}$$

which is in the subspace $\wedge^k \mathbb{R}^n$.

Corollary 3.5.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ be a block-constant function and let $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ be a symmetric function. Suppose that f satisfies condition* (3.14) *and*

$$f_\rho(x) = g_\rho(x) \text{ for all } x \in \mathbb{R}^n \text{ and all } \rho \in \mathbb{N}_{n,k}.$$

Let $F : S^n \to S^{n^k}$ be defined by (3.12) *and let $G : S^n \to S^{\binom{n}{k}}$ be the k-isotropic function corresponding to g, see (3.7). Then,*

$$\mathcal{T} \circ F(A) \circ \mathcal{T}^{-1}\Big|_{\wedge^k \mathbb{R}^n} = \mathcal{W} \circ G(A) \circ \mathcal{W}^{-1}, \tag{3.18}$$

for all $A \in S^n$.

Proof Straightforward calculation using (3.8), gives

$$\begin{aligned}
&\mathcal{W} \circ G(A) \circ \mathcal{W}^{-1}(x_1 \wedge \cdots \wedge x_k) \\
&\qquad = \sum_{\rho \in \mathbb{N}_{n,k}} g_\rho(\lambda(A))(u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}) \otimes (u_{\rho_1} \wedge \cdots \wedge u_{\rho_k})(x_1 \wedge \cdots \wedge x_k) \\
&\qquad = \sum_{\rho \in \mathbb{N}_{n,k}} g_\rho(\lambda(A))\langle u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}, x_1 \wedge \cdots \wedge x_k\rangle(u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}) \\
&\qquad = \sum_{\rho \in \mathbb{N}_{n,k}} g_\rho(\lambda(A)) \det\left(\langle u_{\rho_i}, x_j\rangle_{i,j=1}^k\right)(u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}) \\
&\qquad = \sum_{\rho \in \mathbb{N}_{n,k}} f_\rho(\lambda(A)) \det\left(\langle u_{\rho_i}, x_j\rangle_{i,j=1}^k\right)(u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}) \\
&\qquad = \mathcal{T} \circ F(A) \circ \mathcal{T}^{-1}(x_1 \wedge \cdots \wedge x_k),
\end{aligned}$$

where (3.17) was used for the last equality.

3.6 The structure of symmetric functions $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$

In this section, we examine the structure of a symmetric function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$. For any $\rho \in \mathbb{N}_{n,k}$, define $\rho^c := \mathbb{N}_n \backslash \rho$ ordered increasingly as a member of $\mathbb{N}_{n,n-k}$. For $x \in \mathbb{R}^n$, let

$$x_{\rho^c} := (x_{\rho_1^c}, \ldots, x_{\rho_{n-k}^c}).$$

For any $\rho, \tau \in \mathbb{N}_{n,k}$, and any permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, such that $\rho = \sigma^{(k)}(\tau)$, we have that σ sends the elements of τ^c to the elements of ρ^c as well.

Theorem 3.6.1 *A function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is symmetric if and only if there exists a function $h : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}$ symmetric in $\mathbb{R}^k$ and in $\mathbb{R}^{n-k}$ respectively such that*

$$g_\rho(x) = h(x_\rho; x_{\rho^c}),$$

holds for all $x \in \mathbb{R}^n$ and all $\rho \in \mathbb{N}_{n,k}$.

Proof Suppose function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is symmetric. Take a permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, with corresponding matrix P, which sends $\{1,\dots,k\}$ to $\{1,\dots,k\}$ and sends $\{k+1,\dots,n\}$ to $\{k+1,\dots,n\}$. By Definition 3.3.2, for all such permutations σ, using (3.6), we have

$$g_{\{1,\dots,k\}}(Px) = (\mathbf{P}g(x))_{\{1,\dots,k\}} = g_{\sigma^{(k)}(\{1,\dots,k\})}(x) = g_{\{1,\dots,k\}}(x),$$

holds for all $x \in \mathbb{R}^n$. Thus, the coordinate functions $g_{\{1,\dots,k\}}(x)$ is symmetric in $(x_1,\dots,x_k)$ as well as in $(x_{k+1},\dots,x_n)$.

Define $h : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}$ by

$$h(x_{\{1,\dots,k\}}; x_{\{k+1,\dots,n\}}) := g_{\{1,\dots,k\}}(x),$$

for all $x \in \mathbb{R}^n$. For any $\rho \in \mathbb{N}_{n,k}$, there exists a permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, with corresponding matrix P, such that $\sigma^{(k)}(\{1,\dots,k\}) = \rho$ and sending $\{k+1,\dots,n\}$ to ρ^c. Calculate $g_\rho(x)$ by using (3.6) again:

$$\begin{aligned} g_\rho(x) &= g_{\sigma^{(k)}(\{1,\dots,k\})}(x) = (\mathbf{P}g(x))_{\{1,\dots,k\}} = g_{\{1,\dots,k\}}(Px) = h((Px)_{\{1,\dots,k\}}; (Px)_{\{k+1,\dots,n\}}) \\ &= h(x_{\sigma(1)},\dots,x_{\sigma(k)}; x_{\sigma(k+1)},\dots,x_{\sigma(n)}) = h(x_\rho; x_{\rho^c}), \end{aligned}$$

where in the last equality, we used that h is symmetric with respect to its first k, as well as its last $n-k$, arguments. This is what we had to show.

For the other direction, let $h : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}$ be symmetric in $\mathbb{R}^k$ and in $\mathbb{R}^{n-k}$ separately. Define function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ by

$$g_\rho(x) := h(x_\rho; x_{\rho^c}), \text{ for all } x \in \mathbb{R}^n \text{ and all } \rho \in \mathbb{N}_{n,k}.$$

To show that g is symmetric in the sense of Definition 3.3.2, take any permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, with corresponding matrix P. Then, for any $\tau \in \mathbb{N}_{n,k}$, let $\rho := \sigma^{(k)}(\tau)$, and consider

$$\begin{aligned} g_\tau(Px) &= h((Px)_\tau; (Px)_{\tau^c}) = h((x_{\sigma(1)}, \ldots, x_{\sigma(n)})_\tau; (x_{\sigma(1)}, \ldots, x_{\sigma(n)})_{\tau^c}) \\ &= h(x_{\sigma(\tau_1)}, \ldots, x_{\sigma(\tau_k)}; x_{\sigma(\tau_1^c)}, \ldots, x_{\sigma(\tau_{n-k}^c)}) = h(x_\rho; x_{\rho^c}) = g_\rho(x), \end{aligned}$$

where the penultimate equality holds because h is symmetric in $\mathbb{R}^k$ and in $\mathbb{R}^{n-k}$ separately. Since this holds for all $x \in \mathbb{R}^n$, the proof is complete.

Corollary 3.6.2 *If a function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is symmetric, then*

$$g_\rho(x) = g_\tau(x), \text{ whenever } x_\rho = x_\tau$$

for all $\rho, \tau \in \mathbb{N}_{n,k}$ and all $x \in \mathbb{R}^n$.

Example 3.6.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be symmetric and continuously differentiable. It is easy to show that the gradient $\nabla f(x) : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric. By the Theorem 3.6.1, function $h : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ defined by*

$$h(x_1; x_2, \ldots, x_n) := \frac{\partial f}{\partial x_1}(x) \text{ for all } x \in \mathbb{R}^n$$

is symmetric with respect to its last $n-1$ arguments. Every other partial derivative of f can be expressed as

$$\frac{\partial f}{\partial x_i}(x) = h(x_i; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

for $i = 1, \ldots, n$. ∎

Theorem 3.6.1 allows us to characterize the symmetric functions $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ that are generated in the sense of Example 3.3.5.

Corollary 3.6.4 *Let $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ be symmetric and continuously differentiable. Then, g is generated by a symmetric function $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ if and only if for any $\rho \in \mathbb{N}_{n,k}$*

$$\frac{\partial g_\rho}{\partial x_i}(x) = 0, \text{ whenever } i \in \rho^c \text{ for all } x \in \mathbb{R}^n. \tag{3.19}$$

Proof Suppose that g is generated by symmetric function $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$. For any $\rho \in \mathbb{N}_{n,k}$, we have

$$\frac{\partial g_\rho}{\partial x_i}(x) = \frac{\partial \tilde{g}}{\partial x_i}(x_\rho) = 0, \text{ whenever } i \in \rho^c \text{ for all } x \in \mathbb{R}^n.$$

For the other direction, since g is symmetric, by Theorem 3.6.1, we have $g_\rho(x) = h(x_\rho; x_{\rho^c})$ for any $\rho \in \mathbb{N}_{n,k}$ and any $x \in \mathbb{R}^n$. By (3.19) we have

$$\frac{\partial h}{\partial x_i}(x_\rho; x_{\rho^c}) = \frac{\partial g_\rho}{\partial x_i}(x) = 0, \text{ holds whenever } i \in \rho^c \text{ for all } x \in \mathbb{R}^n.$$

This implies that $h(x_\rho; x_{\rho^c}) = h(x_\rho)$, so g is generated by $\tilde{g} := h$.

3.7 Analyticity of generated k-isotropic functions

Notice that antisymmetric of operators $A_1, \ldots, A_k$ is only defined when $A_1 = \cdots = A_k$. One can see that a (generated) k-isotropic function cannot be represented by the Dunford-Taylor integral directly, when the corresponding symmetric function $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ has a Cauchy integral representation. Thus, we lift a (generated) k-isotropic function to a function $F : S^n \to S^{n^k}$ defined by (3.12) and find the Dunford-Taylor representation of F, see Theorem 3.7.3. By applying the identity (3.18), one can prove the analyticity of (generated) k-isotropic functions, see Theorem 3.7.4.

3.7.1 Lifting a symmetric function

We begin this section with a general extension problem, that is of some interest to us. Given an analytic, symmetric function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$, is there an analytic function $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ that is

1. block-constant;
2. satisfies condition (3.14); and
3. $f_\rho(x) = g_\rho(x)$ for all $x \in \mathbb{R}^n$ and all $\rho \in \mathbb{N}_{n,k}$.

The last condition, says that the following diagram commutes

$$\begin{tikzcd} & \mathbb{R}^{n^k} \arrow[d, "\Pi"] \\ \mathbb{R}^n \arrow[ur, "f"] \arrow[r, "g"'] & \mathbb{R}^{\binom{n}{k}} \end{tikzcd} \tag{3.20}$$

where $\Pi : \mathbb{R}^{n^k} \to \mathbb{R}^{\binom{n}{k}}$ is the projection defined by $\Pi(\mathbf{x}) := (\mathbf{x}_\rho)_{\rho \in \mathbb{N}_{n,k}}$.

As the next example shows, constructing such an analytic extension f is easy when g is generated by a symmetric function $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$.

Example 3.7.1 *Let $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ be symmetric and analytic, then the generated symmetric function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$, given by $g_\rho(x) := \tilde{g}(x_\rho)$ for all $\rho \in \mathbb{N}_{n,k}$, is analytic.*

Define $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ by $f_l(x) := \tilde{g}(x_l)$ for all $l \in \mathbb{N}^k_n$. The three conditions are trivially verified. Indeed, if $l, m \in \mathbb{N}^k_n$ are in the same partition of $\mathbf{I}^x$ then $x_l = x_m$, so $f_l(x) = \tilde{g}(x_l) = \tilde{g}(x_m) = f_m(x)$. The fact that $\tilde{g}$ is symmetric guarantees that condition (3.14) *holds. Finally, f is analytic since it is the composition of the analytic function $\tilde{g}$ and the projection map $x \in \mathbb{R}^n \mapsto x_l \in \mathbb{R}^k$.* ∎

The next example shows that not only generated, symmetric functions g can be extended analytically in the required way.

Example 3.7.2 *Let $q : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric and analytic and let $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ be symmetric and analytic. Define $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ by*

$$g_\rho(x) := \tilde{g}(q_\rho(x)), \text{ for all } x \in \mathbb{R}^n \text{ and all } \rho \in \mathbb{N}_{n,k}.$$

Function g is analytic since it is the composition of $\tilde{g}$, the projection map $x \in \mathbb{R}^n \mapsto x_\rho \in \mathbb{R}^k$, and q and they are all analytic.

Function g is not generated since for any $\rho \in \mathbb{N}_{n,k}$ and $i \in \rho^c$, the derivative

$$\frac{\partial g_\rho}{\partial x_i}(x) = \sum_{j=1}^{k} \tilde{g}'_j(q_\rho(x)) \frac{\partial q_{\rho_j}}{\partial x_i}(x)$$

is not necessarily zero, see Corollary 3.6.4.

Function g is symmetric since for any $\rho \in \mathbb{N}_{n,k}$ and permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, with corresponding permutation matrix P, we have

$$g_\rho(Px) = \tilde{g}(q_\rho(Px)) = \tilde{g}(q_{\rho_1}(Px), \ldots, q_{\rho_k}(Px)).$$

Next, since q is symmetric we have $q(Px) = Pq(x)$. That is, for any $i = 1, \ldots, k$, we have $q_{\rho_i}(Px) = (Pq(x))_{\rho_i} = q_{\sigma(\rho_i)}(x)$ and we continue

$$g_\rho(Px) = \tilde{g}(q_{\sigma(\rho_1)}(x), \ldots, q_{\sigma(\rho_k)}(x)) = \tilde{g}(q_{\sigma^{(k)}(\rho)}(x)) = g_{\sigma^{(k)}(\rho)}(x) = (\mathbf{P}g(x))_\rho.$$

Define the function $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ by

$$f_l(x) := \tilde{g}(q_l(x)), \text{ for } l \in \mathbb{N}_n^k.$$

It is clearly analytic since it is the composition of analytic functions.

To show that f is block-constant, fix $x \in \mathbb{R}^n$ and choose $l, m \in \mathbb{N}_n^k$ such that $x_l = x_m$. We are going to show that $f_l(x) = f_m(x)$. Consider the transposition $\sigma^i : \mathbb{N}_n \to \mathbb{N}_n$ that maps l_i to m_i

and vice versa, keeping the rest of the elements of $\mathbb{N}_n$ *fixed, for all* $i = 1, \ldots, k$. *Let* P^i *be its corresponding transposition matrix. Since* $x_l = x_m$, *we have that* $P^i x = x$ *for all* i. *The fact that* q *is symmetric implies*

$$q_{l_i}(x) = q_{l_i}(P^i x) = (P^i q(x))_{l_i} = q_{\sigma(l_i)}(x) = q_{m_i}(x),$$

for all $i = 1, \ldots, k$. *Hence,*

$$f_l(x) = \tilde{g}(q_{l_1}(x), \ldots, q_{l_k}(x)) = \tilde{g}(q_{m_1}(x), \ldots, q_{m_k}(x)) = f_m(x).$$

Function f *satisfies* (3.14) *because* $\tilde{g}$ *is symmetric. Finally, diagram (3.20) trivially commutes.* ∎

Note that Example 3.7.2 reduces to Example 3.7.1, when we let $q(x) := x$ for all $x \in \mathbb{R}^n$.

3.7.2 Analyticity of generated k-isotropic functions

The following lemma shows the analyticity of a sub-class of map $F : S^n \to S^{n^k}$ defined by (3.12).

Theorem 3.7.3 *Let* $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ *be symmetric and let* $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ *be defined by*

$$f_l(x) := \tilde{g}(x_l) \text{ for all } x \in \mathbb{R}^n \text{ and } l \in \mathbb{N}_n^k. \tag{3.21}$$

Then, function $F : S^n \to S^{n^k}$ *defined by* (3.12) *is analytic if and only if* $\tilde{g}$ *is.*

Proof Suppose $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ is analytic. Then, the Cauchy integral representation of $\tilde{g}$ holds:

$$\tilde{g}(x_1, \ldots, x_k) = \frac{1}{(2\pi i)^k} \oint_\Gamma \cdots \oint_\Gamma \frac{\tilde{g}(z_1, \ldots, z_k)}{\prod_{i=1}^k (z_i - x_i)}\, dz_1 \cdots dz_k,$$

where Γ is a positively oriented circle in the complex plane enclosing the points $\{x_1, \ldots, x_k\}$. Function f, defined by (3.21) is clearly analytic, since $\tilde{g}$ is and the map $x \mapsto x_l$ is analytic for all $l \in \mathbb{N}_n^k$.

For any $\mathbf{x} \in \mathbb{R}^{n^k}$, denote by $\mathrm{Diag}_l\, \mathbf{x}_l$ the $n^k \times n^k$ diagonal matrix with l-th diagonal element $\mathbf{x}_l$. The Dunford-Taylor integral representation of $F(A)$ for any $A \in S^n$ is

$$\begin{aligned}
F(A) &= U^{\otimes k}(\mathrm{Diag}\, f(\lambda(A)))(U^{\otimes k})^\top = U^{\otimes k}(\mathrm{Diag}_l\, \tilde{g}(\lambda_l(A)))(U^{\otimes k})^\top \\
&= U^{\otimes k}\Big(\mathrm{Diag}_l \frac{1}{(2\pi i)^k} \oint_\Gamma \cdots \oint_\Gamma \frac{\tilde{g}(z_1, \ldots, z_k)}{\prod_{i=1}^k (z_i - \lambda_{l_i}(A))}\, dz_1 \cdots dz_k\Big)(U^{\otimes k})^\top \\
&= \frac{1}{(2\pi i)^k} \oint_\Gamma \cdots \oint_\Gamma \tilde{g}(z_1, \ldots, z_k) U^{\otimes k}\Big(\mathrm{Diag}_l \prod_{i=1}^k (z_i - \lambda_{l_i}(A))^{-1}\Big)(U^{\otimes k})^\top\, dz_1 \cdots dz_k,
\end{aligned}$$

where $U \in O^n$ is such that $A = U(\mathrm{Diag}\, \lambda(A))U^\top$ and Γ is a positively oriented circle in the complex plane enclosing all eigenvalues $\{\lambda_1(A), \ldots, \lambda_n(A)\}$. Notice that

$$U^{\otimes k}(\mathrm{Diag}_l \prod_{i=1}^k (z_i - \lambda_{l_i}(A))^{-1})(U^{\otimes k})^\top = (z_1 I - A)^{-1} \otimes \cdots \otimes (z_k I - A)^{-1},$$

holds. Thus, one obtains the integral representation

$$F(A) = \frac{1}{(2\pi i)^k} \oint_\Gamma \cdots \oint_\Gamma \tilde{g}(z_1, \ldots, z_k)((z_1 I - A)^{-1} \otimes \cdots \otimes (z_k I - A)^{-1})\, dz_1 \cdots dz_k,$$

where Γ is a positively oriented circle in the complex plane enclosing all eigenvalues of A. Since the eigenvalue map $A \mapsto \lambda(A)$ is a continuous function, the circle Γ also encloses eigenvalues of all matrices B in a small open neighbourhood of A. Thus,

$$F(B) = \frac{1}{(2\pi i)^k} \oint_\Gamma \cdots \oint_\Gamma \tilde{g}(z_1, \ldots, z_k)((z_1 I - B)^{-1} \otimes \cdots \otimes (z_k I - B)^{-1})\, dz_1 \cdots dz_k$$

holds for every B in an open neighbourhood of A. Since for any fixed $z_1, \ldots, z_k \in \Gamma$, the map $B \mapsto (z_1 I - B)^{-1} \otimes \cdots \otimes (z_k I - B)^{-1}$ is also analytic in that open neighbourhood, the result follows.

For the opposite direction, one just needs to restrict $F : S^n \to S^{n^k}$ to the subspace of S^n of diagonal matrices.

As a corollary, we obtain the analyticity of the k-isotropic functions defined by (3.7) in the case when the function g is generated by a symmetric function $\tilde{g}$.

Theorem 3.7.4 *Let $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ be symmetric and let $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ be defined by*

$$g_\rho(x) := \tilde{g}(x_\rho) \textit{ for all } x \in \mathbb{R}^n \textit{ and } \rho \in \mathbb{N}_{n,k}.$$

Then, the k-isotropic function $G : S^n \to S^{\binom{n}{k}}$ defined by (3.7) is analytic if and only if $\tilde{g}$ is.

Proof Suppose that $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ is analytic. Define $f : \mathbb{R}^n \to \mathbb{R}^{n^k}$ as in (3.21) and $F : S^n \to S^{n^k}$ by (3.12). By Example 3.7.1, function f satisfies the conditions in Corollary 3.5.2. By Theorem 3.7.3, F is analytic, hence identity (3.18) implies the analyticity of G.

For the opposite direction, one just needs to restrict $G : S^n \to S^{\binom{n}{k}}$ to the subspace of S^n of diagonal matrices.

Theorem 3.7.4 contains two well-known special cases. Taking $k = 1$ in it, one obtains the the analyticity of primary matrix functions, see for example [12, Chapter 7]. While taking $k = n$ in it, one obtains the analyticity of spectral functions, see [24].

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Chapter 4

Canonical representation of k-tensor isotropic functions

4.1 Introduction

Denote by $\mathbb{R}^{n\times n}$ the space of all $n\times n$ real matrices and by S^n the subspace of all $n\times n$ symmetric matrices with inner product $\langle A, B\rangle := \mathrm{Tr}(AB)$ and Frobenius norm $\|A\| := \sqrt{\mathrm{Tr}(AA)}$. Let O^n be the group of $n \times n$ orthogonal matrices: $A \in O^n$, if and only if $A^\top A = I$. Denote by $\mathbb{R}^n_\geq$ the convex cone in $\mathbb{R}^n$ of all vectors with non-increasing coordinates. For any $A \in S^n$, let $\lambda(A) \in \mathbb{R}^n_\geq$ be the ordered vector of eigenvalues of A:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

For any vector $x \in \mathbb{R}^n$, let $\mathrm{Diag}\, x$ be the $n \times n$ matrix with diagonal elements x and zeros elsewhere. Denote by P^n the collection of all $n \times n$ permutation matrices.

There are several well-studied classes of matrix-valued functions that have orthogonally invariant properties. We give a short summary.

Definition 4.1.1 *A function $F : S^n \to S^n$ is called a primary matrix function, if there exists a* *function* $f : \mathbb{R} \to \mathbb{R}$*, such that*

$$F(A) = U(\mathrm{Diag}\,(f(\lambda_1(A)), \ldots, f(\lambda_n(A))))U^\top, \tag{4.1}$$

where $U \in O^n$ *is such that* $A = U(\mathrm{Diag}\,\lambda(A))U^\top$.

It is easy to see that the primary matrix functions are well-defined, see [6, Chapter V]. That is, the right-hand side of (4.1) does not depend on the choice of the orthogonal matrix U diagonalizing A. For example, the map $F(A) = \exp(A)$ is a primary matrix function. Primary matrix functions are also known as *Löwner's operator* functions. Works [12] and [19] characterize operator monotone primary matrix functions in terms of Pick functions. Operator convex primary matrix functions are characterized in [17]. Other properties are studied in [2], [3], [10], [11], and [13]. For more examples, see the monographs [6, Chapter V] and [14, Chapter 6].

Primary matrix functions are a special case of the following class of maps.

Definition 4.1.2 *A function* $F : S^n \to S^n$ *is called a tensor-valued isotropic function, if*

$$F(UAU^\top) = UF(A)U^\top$$

for all $U \in O^n$ *and all* $A \in S^n$ *in the domain of* F.

The monographs [15] and [22] list numerous applications of the tensor-valued isotropic functions in mechanics and engineering. For example, the Piola-Kirchhoff stress function in an isotropic solid, given in [15, page 128] and [22, page 173] is an example of tensor-valued isotropic function.

We say that a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *symmetric*, if $f(Px) = Pf(x)$ for any $x \in \mathbb{R}^n$ and any permutation matrix $P \in P^n$. The following representation theorem for tensor-valued isotropic functions can be found in [22] and [23].

Theorem 4.1.3 *A function $F : S^n \to S^n$ is a tensor-valued isotropic function, if and only if there is a unique symmetric function $f : \mathbb{R}^n \to \mathbb{R}^n$, such that*

$$F(A) = U(\operatorname{Diag} f(\lambda(A)))U^\top, \tag{4.2}$$

for all $A \in S^n$ and $U \in O^n$, such that $A = U(\operatorname{Diag}\lambda(A))U^\top$.

Note that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that $f(x) = (g(x_1), \ldots, g(x_n))$ for some $g : \mathbb{R} \to \mathbb{R}$, then the tensor-valued isotropic function, given by (4.2), becomes primary matrix function, given by (4.1).

In [1] and [20] an generalization of the tensor-valued isotropic functions was proposed. For an $n\times n$ matrix A, let $A^{(k)}$ denote the *k-th multiplicative compound matrix of* A, where $1 \le k \le n$, see [21].

Definition 4.1.4 *A function $G : S^n \to S^{\binom{n}{k}}$ is called k-isotropic, if*

$$G(UAU^\top) = U^{(k)}G(A)(U^{(k)})^\top$$

for all $U \in O^n$ and $A \in S^n$ in the domain of G.

Since $U^{(1)} = U$, the k-isotropic functions turn into the tensor-valued isotropic functions, when $k = 1$. Moreover, when $k = n$ the k-isotropic functions become spectral functions, see [18] for an overview of their properties and applications. The k-isotropic functions have a representation in the spirit of (4.2). In order to describe it, we introduce the following notation.

Let $\mathbb{N}_n := \{1, 2, \ldots, n\}$ and let $\mathbb{N}_{n,k}$ be the set of all subsets of $\mathbb{N}_n$ of size k, ordered non-decreasingly, where $k \in \mathbb{N}_n$. Order the elements in $\mathbb{N}_{n,k}$ lexicographically. In this way, they are used to index the coordinates of vectors in $\mathbb{R}^{\binom{n}{k}}$.

Every permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ defines a permutation on $\sigma^{(k)} : \mathbb{N}_{n,k} \to \mathbb{N}_{n,k}$ in the following way:

$$\sigma^{(k)}(\rho) := \text{the non-decreasing rearrangement of } \{\sigma(\rho_1), \ldots, \sigma(\rho_k)\}.$$

Definition 4.1.5 *A function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is called symmetric, if*

$$g_\rho(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = g_{\sigma^{(k)}(\rho)}(x_1, \ldots, x_n)$$

holds for all $x \in \mathbb{R}^n$ and all $\rho \in \mathbb{N}_{n,k}$.

The representation theorem of k-isotropic functions, stated in [20], is as follows.

Theorem 4.1.6 *A function $G : S^n \to S^{\binom{n}{k}}$ is k-isotropic, if and only if there is a unique symmetric function $g : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$, such that*

$$G(A) = U^{(k)}(\operatorname{Diag} g(\lambda(A)))(U^{(k)})^\top$$

for all $A \in S^n$ and $U \in O^n$, such that $A = U(\operatorname{Diag} \lambda(A))U^\top$.

Several authors have pursued another direction for generalizing the primary matrix functions. For example, in [4], [5], [8], [9], and [24] the primary matrix functions have been generalized to function on several matrix arguments (not necessarily of the same size). We now proceed to describe that extension.

Any function $f : \mathbb{R}^k \to \mathbb{R}$ defines a map $F : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$ by

$$F(A_1, \ldots, A_k) := (\otimes_{i=1}^k U_i)(\operatorname{Diag}_l f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k)))(\otimes_{i=1}^k U_i)^\top, \tag{4.3}$$

where $l = (l_1, \ldots, l_k)$ is in $\mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$ and $U_i \in O^{n_i}$ is such that $A_i = U_i(\operatorname{Diag} \lambda(A_i))U_i^\top$ for all $i = 1, \ldots, k$. In addition, Diag_l denotes a diagonal matrix with the indicated values on the main diagonal ordered lexicographically with respect to the k-tuples l.

When $k = 1$, the function given by (4.3) turns into the primary matrix function given by (4.1). While (4.3) extends primary matrix functions to the multi-operator setting, it does not capture the class of tensor-valued isotropic functions.

The map defined by (4.3) has the following invariance property:

$$F(U_1 A_1 U_1^\top, \dots, U_k A_k U_k^\top) = (\otimes_{i=1}^k U_i) F(A_1, \dots, A_k) (\otimes_{i=1}^k U_i)^\top \tag{4.4}$$

for any $U_i \in O^{n_i}$, $i = 1, \dots, k$. The opposite is not true. That is, if map $F : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$ satisfies (4.4), then one cannot conclude the existence of $f : \mathbb{R}^k \to \mathbb{R}$, such that (4.3) holds.

A natural goal is to formulate a representation theorem, in the spirit of Theorems 4.1.3 and 4.1.6, for functions $F : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$ that satisfy (4.4). At present time, this problem appears to be too challenging. A particular case is to let $n_1 = \cdots = n_k =: n$, and assume that the invariance (4.4) holds whenever $A_1 = \cdots = A_k =: A \in S^n$. Further specialization is to formulate a representation theorem for maps $F : S^n \to S^{n^k}$ satisfying

$$F(UAU^\top) = (\otimes_{i=1}^k U) F(A) (\otimes_{i=1}^k U)^\top$$

for all $U \in O^n$ and all A in the domain of F. This work addresses the latter problem, and it answers it fully in the case $k = 2$ and any n. That result is formulated in Corollary 4.4.4.

4.2 Background and notation

Denote by $\otimes^k \mathbb{R}^n$ the k-th tensor power of $\mathbb{R}^n$. That is the linear space of all formal finite linear combinations of products $\{x_1 \otimes \cdots \otimes x_k : x_1, \dots, x_k \in \mathbb{R}^n\}$, with all necessary identifications made so that the product is linear in each argument separately. The dimension of $\otimes^k \mathbb{R}^n$ is n^k.

The tensor product of k linear operators $A_1, \dots, A_k$ on $\mathbb{R}^n$, $A_1 \otimes \cdots \otimes A_k$, is a linear operator on $\otimes^k \mathbb{R}^n$ defined by

$$(A_1 \otimes \cdots \otimes A_k)(x_1 \otimes \cdots \otimes x_k) := A_1 x_1 \otimes \cdots \otimes A_k x_k$$

and extended by linearity. Important properties of the tensor product are as follows:

$$(A_1 \otimes A_2)(B_1 \otimes B_2) = A_1 B_1 \otimes A_2 B_2, \text{ and } (A_1 \otimes A_2)^* = A_1^* \otimes A_2^*. \tag{4.5}$$

When A_1 and A_2 are invertible, so is their tensor product and

$$(A_1 \otimes A_2)^{-1} = A_1^{-1} \otimes A_2^{-1}. \tag{4.6}$$

The notation $\otimes^k A := A \otimes \cdots \otimes A$ is used, when $A_1, \dots, A_k$ are all equal to A.

The tensor product of two $n \times n$ matrices A, B is defined by

$$A \otimes B := \begin{pmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \dots & A_{nn}B \end{pmatrix}$$

and can be extended to a tensor product of $A_1, \dots, A_k$ by $A_1 \otimes \cdots \otimes A_k$.

Analogous properties to (4.5) and (4.6) hold for tensor product of matrices as well. Taking $A_1 = \cdots = A_k =: A$, the tensor product $A_1 \otimes \cdots \otimes A_k$ turns into k-th tensor power of A, denoted by $A^{\otimes k}$.

We now give the main definition of this work.

Definition 4.2.1 *A function $F\colon S^n \to S^{n^k}$ is called k-tensor isotropic, if*

$$F(UAU^\top) = (U^{\otimes k})F(A)(U^{\otimes k})^\top$$

holds for all $U \in O^n$ and $A \in S^n$ in the domain of F.

A collection of non-empty, pairwise disjoint subsets of $\mathbb{N}_n$ with union $\mathbb{N}_n$ is called a partition of $\mathbb{N}_n$. A set in a partition is called a *block*. We generally use the letter I to denote a partition of $\mathbb{N}_n$. Every $x \in \mathbb{R}^n$, defines a partition I^x on $\mathbb{N}_n$ by taking i and j in the same block, if and only if $x_i = x_j$. Denote by $I^x = \{I^x_1, \ldots, I^x_r\}$ the partition, and its blocks, determined by x, where r is the number of blocks.

Let

$$\mathbb{N}_n^k := \mathbb{N}_n \times \cdots \times \mathbb{N}_n \text{ (k times)}$$

and assume that the elements in $\mathbb{N}_n^k$ are ordered lexicographically. The set $\mathbb{N}_n^k$ is used to index the coordinates of vectors in $\mathbb{R}^{n^k}$ and the entries of matrices in $\mathbb{R}^{n^k \times n^k}$. If $\mathbf{x} \in \mathbb{R}^{n^k}$, then $\mathbf{x}_l$ denotes the l-th element of $\mathbf{x}$, for $l \in \mathbb{N}_n^k$; and if $\mathbf{A} \in \mathbb{R}^{n^k \times n^k}$, then $\mathbf{A}_{l,m}$ denotes the element in the l-th row and m-th column, for $l, m \in \mathbb{N}_n^k$. Then, the entry in the l-th row and m-th column of $A^{\otimes k}$ is

$$(A^{\otimes k})_{l,m} = \prod_{i=1}^{k} A_{l_i m_i}.$$

On the other hand, if $x \in \mathbb{R}^n$ and $l \in \mathbb{N}_n^k$, then define the subvector

$$x_l := (x_{l_1}, \ldots, x_{l_k}).$$

Any $x \in \mathbb{R}^n_{\geq}$ generates a partition $\mathbf{I}^x$ on $\mathbb{N}_n^k$. The elements of $\mathbb{N}_r^k$ label the blocks of this partition as follows. For any $s = (s_1, \ldots, s_k) \in \mathbb{N}_r^k$, define a block in the partition of $\mathbb{N}_n^k$ by

$$\mathbf{I}^x_s := \{l \in \mathbb{N}_n^k : l_i \in I^x_{s_i}, \text{ for any } i \in \mathbb{N}_k\},$$

where $I^x_{s_i}$ is the s_i-th block of the partition I^x for $i = 1, \ldots, k$. Equivalently, $l, m \in \mathbb{N}_n^k$ are in the same block of the partition $\mathbf{I}^x$, if and only if $x_l = x_m$. The cardinality of $\mathbf{I}^x_s$ is

$$|\mathbf{I}^x_s| = |I^x_{s_1}| \cdots |I^x_{s_k}|.$$

Denote the standard orthonormal basis in $\mathbb{R}^n$ by $\{e^1, \ldots, e^n\}$ and denote the standard orthonormal basis in $\mathbb{R}^{n^k}$ by $\{\mathbf{e}^l : l \in \mathbb{N}_n^k\}$. A natural isometry $\mathcal{T} : \mathbb{R}^{n^k} \to \otimes^k \mathbb{R}^n$ is defined by

$$\mathcal{T}(\mathbf{e}^l) := e^{l_1} \otimes \cdots \otimes e^{l_k} \text{ for all } l \in \mathbb{N}_n^k$$

and extended by linearity.

For any fixed $x \in \mathbb{R}^n_\geq$, matrix $A \in \mathbb{R}^{n\times n}$ is called *I^x-block-diagonal*, if A can be written as $A = \operatorname{Diag}(A_1, \ldots, A_r)$ with A_i of size $|I_i^x|$ for all $i = 1, \ldots, r$. It is not difficult to see that, if matrix A is I^x-block-diagonal, then $A^{\otimes k}$ is $\mathbf{I}^x$-block-diagonal with blocks of size $|\mathbf{I}_s^x|$ for $s \in \mathbb{N}_n^k$.

For any $x \in \mathbb{R}^n_\geq$, a vector $\mathbf{x} \in \mathbb{R}^{n^k}$ is called *$\mathbf{I}^x$-block-constant*, if $\mathbf{x}_l = \mathbf{x}_m$ for any l, m in the same block of $\mathbf{I}^x$. Alternatively, $\mathbf{x} \in \mathbb{R}^{n^k}$ is $\mathbf{I}^x$-block-constant, if $\mathbf{x}_l = \mathbf{x}_m$, whenever $x_l = x_m$ for all $l, m \in \mathbb{N}_n^k$. We say that a function $f : \mathbb{R}^n_\geq \to \mathbb{R}^{n^k}$ is *block-constant*, if vector $f(x)$ is $\mathbf{I}^x$-block-constant for all $x \in \mathbb{R}^n_\geq$. In other words, a function $f : \mathbb{R}^n_\geq \to \mathbb{R}^{n^k}$ is block-constant, if and only if $f_l(x) = f_m(x)$, whenever $x_l = x_m$ for all $x \in \mathbb{R}^n_\geq$ and all $l, m \in \mathbb{N}_n^k$.

We use the following notation throughout. For any $x \in \mathbb{R}^n_\geq$, let

$$\begin{aligned}
O^{n,k} &:= \{V^{\otimes k} : V \in O^n\};\\
O^n_x &:= \{V \in O^n : \ V \text{ is } I^x\text{-block-diagonal}\};\\
O^{n,k}_x &:= \{V^{\otimes k} : V \in O^n_x\}.
\end{aligned}$$

Observe that $O^n_x = O^n$ holds, if x contains equal coordinates. A k-tensor isotropic function $F : S^n \to S^{n^k}$ is well-defined, if

$$(V^\top U)^{\otimes k} F(\operatorname{Diag}\lambda(A)) = F(\operatorname{Diag}\lambda(A))(V^\top U)^{\otimes k}$$

for any $A \in S^n$ and any $U, V \in O^n$, such that

$$A = U(\operatorname{Diag}\lambda(A))U^\top = V(\operatorname{Diag}\lambda(A))V^\top. \tag{4.7}$$

It is easy to see that, if $U, V \in O^n$ satisfy (4.7), then $V^\top U \in O^n_{\lambda(A)}$. Moreover, every matrix in $O^n_{\lambda(A)}$ can be obtained in this way, that is, if $W \in O^n_{\lambda(A)}$ and $U \in O^n$ satisfies (4.7), then $V := UW^\top$ also satisfies (4.7) and $W = V^\top U$. Thus, $F(\operatorname{Diag} \lambda(A))$ is in the centralizer of $O^{n,k}_{\lambda(A)}$. Finding a representation theorem for k-tensor isotropic functions reduces to finding the centralizer of $O^{n,k}_{\lambda(A)}$. With that in mind, denote the *centralizer* and the *symmetric centralizer* of a collection $\mathcal{A}$ of $n \times n$ matrices by

$$C(\mathcal{A}) := \{B \in \mathbb{R}^{n\times n} : AB = BA \text{ for all } A \in \mathcal{A}\},$$
$$C_S(\mathcal{A}) := \{B \in S^n : AB = BA \text{ for all } A \in \mathcal{A}\}.$$

The orthogonal group O^n has two connected components. One, consisting of orthogonal matrices with determinant $+1$, forms a group called *special orthogonal group* and denoted by SO^n. The other connected component consists of orthogonal matrices with determinant -1. We need to specialize our notation further. For any fixed $x \in \mathbb{R}^n_{\geq}$, let

$$SO^n_x := \{V \in SO^n : \ V \text{ is } I^x\text{-block-diagonal}\},$$
$$SO^{n,k}_x := \{V^{\otimes k} : V \in SO^n_x\}.$$

A matrix is called *sign-identity*, if it is a diagonal matrix with $+1$ or -1 on the main diagonal. Denote by $I^n_\pm$ the set of all $n \times n$ sign-identity matrices, and let

$$I^{n,k}_\pm := \{V^{\otimes k} : V \in I^n_\pm\}.$$

If $x \in \mathbb{R}^n_{\geq}$ has distinct coordinates, then $I^n_\pm = O^n_x$.

4.3 The centralizer of $O_x^{n,k}$

The goal of this section is to obtain a system of linear equations that characterize the matrices in the centralizer $C(O_x^{n,k})$, for $x \in \mathbb{R}^n_{\geq}$. That is the statement of Theorem 4.3.11.

It is well-known that, for $x \in \mathbb{R}^n$, Diag x is in the centralizer of O^n, if and only if x has equal coordinates. Similarly, we have

$$C(O_x^n) = \{\mathrm{Diag}\, y : y \in \mathbb{R}^n \text{ is } I^x\text{-block-constant}\}.$$

Next lemma gives an analogous statement for the group $O_x^{n,k}$, see Lemma 4.2 in [16].

Lemma 4.3.1 *For $\mathbf{x} \in \mathbb{R}^{n^k}$, we have*

$$\mathrm{Diag}\, \mathbf{x} \in C(O_x^{n,k}), \tag{4.8}$$

if and only if $\mathbf{x}$ is $\mathbf{I}^x$-block-constant.

4.3.1 A representation of $C(O_x^{n,k})$

Lemma 4.3.2 *For any $A \in \mathbb{R}^{n\times n}$, we have $A \in C(O^n)$, if and only if $A \in C(SO^n)$ and A commutes with one orthogonal matrix with determinant -1.*

Proof Let I_n^- be an $n \times n$ orthogonal matrix with determinant -1. Suppose that $AI_n^- = I_n^- A$ and $AV = VA$ holds for all $V \in SO^n$. Notice that for any $V \in O^n$ with determinant -1, we have $VI_n^- \in SO^n$. Hence, $A(VI_n^-) = VAI_n^- = (VI_n^-)A$ holds for all $V \in O^n$ with determinant -1. Thus, $AU = UA$ holds for all $U \in O^n$.

The other direction is obvious.

Analogous proof shows more generally, that for any $x \in \mathbb{R}^n_{\geq}$, we have

$$C(O_x^n) = C(I_\pm^n) \cap C(SO_x^n).$$

Lemma 4.3.3 *For any $x \in \mathbb{R}^n_{\geq}$, we have $C(O_x^{n,k}) = C(I_{\pm}^{n,k}) \cap C(SO_x^{n,k})$.*

Proof Fix $x \in \mathbb{R}^n_{\geq}$. It is clear that the centralizer on the left-hand side is contained in the intersection on the right. In the other direction, for any $B \in O_x^n$, there exists a $B_1 \in SO_x^n$ and a $B_2 \in I_{\pm}^n$, such that $B = B_1 B_2$. Then, for any matrix $\mathbf{A} \in C(SO_x^{n,k}) \cap C(I_{\pm}^{n,k})$, we have

$$\mathbf{A}B^{\otimes k} = \mathbf{A}B_1^{\otimes k}B_2^{\otimes k} = B_1^{\otimes k}\mathbf{A}B_2^{\otimes k} = B_1^{\otimes k}B_2^{\otimes k}\mathbf{A} = B^{\otimes k}\mathbf{A},$$

completing the argument.

The next goal is to compute $C(I_{\pm}^{n,k})$ and $C(SO_x^{n,k})$ separately.

4.3.2 The centralizer of $I_{\pm}^{n,k}$

Lemma 4.3.4 *We have that $A \in C(I_{\pm}^n)$, if and only if A is diagonal.*

Proof Suppose $A \in C(I_{\pm}^n)$. Then, for any $B \in I_{\pm}^n$ and $i, j \in \mathbb{N}_n$, we have

$$0 = (AB - BA)_{ij} = A_{ij}B_{jj} - B_{ii}A_{ij} = A_{ij}(B_{jj} - B_{ii}).$$

Choosing B, such that $B_{jj} \neq B_{ii}$, whenever $i \neq j$, shows that $A_{ij} = 0$. The opposite direction is trivial.

To characterize $C(I_{\pm}^{n,k})$, we need the following notation.

1. For any $l := (l_1, \ldots, l_k) \in \mathbb{N}_n^k$ and a permutation $\sigma \colon \mathbb{N}_k \to \mathbb{N}_k$, let

$$\sigma(l) := (l_{\sigma(1)}, \ldots, l_{\sigma(k)}).$$

2. Define the map $\alpha \colon \mathbb{N}_n^k \to \{0,1\}^n$ by

$$\alpha_j(l) := \begin{cases} 0 & \text{if } j \text{ appears even number of times in } l, \\ 1 & \text{if } j \text{ appears odd number of times in } l, \end{cases} \tag{4.9}$$

for $j = 1, \ldots, n$.

In other words, if $l = (l_1, \ldots, l_k)$, then we have

$$\alpha(l) = \sum_{i=1}^{k} e^{l_i} \,(\mathrm{mod}\, 2).$$

The map α is transitive, in the sense that, if $\alpha(l^1) = \alpha(l^2)$ and $\alpha(l^2) = \alpha(l^3)$, then $\alpha(l^1) = \alpha(l^3)$ holds, for any $l^1, l^2, l^3 \in \mathbb{N}_n^k$. We are now ready to describe $C(I_{\pm}^{n,k})$.

Proposition 4.3.5 *We have that* $\mathbf{A} \in C(I_{\pm}^{n,k})$*, if and only if*

$$\mathbf{A}_{l,m} = 0, \ \textit{whenever}\ \alpha(l) \neq \alpha(m)$$

holds for all $l, m \in \mathbb{N}_n^k$.

Proof Matrix $\mathbf{A}$ is in $C(I_{\pm}^{n,k})$, if and only if for any $B \in I_{\pm}^{n}$,

$$0 = (\mathbf{A}B^{\otimes k} - B^{\otimes k}\mathbf{A})_{l,m} = \mathbf{A}_{l,m}((B^{\otimes k})_{m,m} - (B^{\otimes k})_{l,l}).$$

Thus, $\mathbf{A} \in C(I_{\pm}^{n,k})$, if and only if $\mathbf{A}_{l,m} = 0$, whenever $(B^{\otimes k})_{m,m} \neq (B^{\otimes k})_{l,l}$ for some $B \in I_{\pm}^{n}$. Since

$$(B^{\otimes k})_{l,l} = B_{l_1 l_1} B_{l_2 l_2} \cdots B_{l_k l_k} \quad \text{and} \quad (B^{\otimes k})_{m,m} = B_{m_1 m_1} B_{m_2 m_2} \cdots B_{m_k m_k},$$

one sees that $(B^{\otimes k})_{m,m} = (B^{\otimes k})_{l,l}$ for all $B \in I_{\pm}^{n}$, if and only if $\alpha(l) = \alpha(m)$. The result follows.

For the rest of the subsection, we focus on some special matrices from $C(I_{\pm}^{n,k})$. For any $l, m \in \mathbb{N}_n^k$ and a permutation $\sigma : \mathbb{N}_k \to \mathbb{N}_k$, define the $n^k \times n^k$ matrix $\mathbf{P}_\sigma$ by

$$(\mathbf{P}_\sigma)_{l,m} := \begin{cases} 1 & \text{if } \sigma(l) = m, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Notice that for any $l \in \mathbb{N}_n^k$ and a permutation σ on $\mathbb{N}_k$, we have $\alpha(\sigma(l)) = \alpha(l)$. Thus, using Proposition 4.3.5, one can see that $\mathbf{P}_\sigma \in C(I_\pm^{n,k})$. For example, when $k = 1$, $\mathbf{P}_\sigma$ is identity, since the only permutation on $\mathbb{N}_1$ is the identity. It should be clear that $\mathbf{P}_\sigma$ is a permutation matrix, and its action on the standard basis of $\mathbb{R}^{n^k}$ is

$$\mathbf{P}_\sigma \mathbf{e}^m = \mathbf{e}^{\sigma^{-1}(m)}$$

for all $m \in \mathbb{N}_n^k$. Thus, $\mathbf{P}_\sigma$ is the permutation matrix corresponding to the permutation on $\mathbb{N}_n^k$ induced by σ. For later reference we record the fact:

$$\mathbf{P}_\sigma{}^\top = \mathbf{P}_{\sigma^{-1}}.$$

Of course, not every $n^k \times n^k$ permutation matrix is of the form (4.10). What makes (4.10) special is that it (or rather its corresponding operator) permutes vectors within the tensor products in $\otimes^k \mathbb{R}^n$. Indeed, for any $l \in \mathbb{N}_n^k$, we have

$$\mathcal{T} \circ \mathbf{P}_\sigma \circ \mathcal{T}^{-1}(e^{l_{\sigma(1)}} \otimes \cdots \otimes e^{l_{\sigma(k)}}) = \mathcal{T} \circ \mathbf{P}_\sigma \mathbf{e}^{\sigma(l)} = \mathcal{T}(\mathbf{e}^l) = e^{l_1} \otimes \cdots \otimes e^{l_k}.$$

Hence, by linearity

$$\mathcal{T} \circ \mathbf{P}_\sigma \circ \mathcal{T}^{-1}(x^{\sigma(1)} \otimes \cdots \otimes x^{\sigma(k)}) = x^1 \otimes \cdots \otimes x^k$$

for any $x^i \in \mathbb{R}^n, i = 1, \ldots, k$.

4.3.3 Linearizing the problem

Fix any $x \in \mathbb{R}^n_\geq$. By Lemma 4.3.3, in order to characterize the matrices in $C(O_x^{n,k})$, we need to find those in $C(I_\pm^{n,k})$ that also commute with the matrices in $SO_x^{n,k}$. A direct approach appears to be futile. Instead, we carry the calculations in the tangent space to the manifold $SO_x^{n,k}$ at the identity matrix I. The following proposition is standard, we include the proof for completeness.

Proposition 4.3.6 *Let $t \in (-\epsilon, \epsilon) \mapsto V(t) \in SO^n_x$ be a differentiable path with $V(0) = I$. Then, $V'(0)$ is a skew-symmetric, I^x-block-diagonal matrix. In fact, any skew-symmetric, I^x-block-diagonal matrix can be obtained in this way.*

Proof Take the derivative on both sides of $V(t)V(t)^\top = I$ with respect to t at $t = 0$ to obtain $V(0)V'(0)^\top + V'(0)V(0)^\top = 0$, implying that $V'(0)^\top = -V'(0)$. Since $V(t)$ is I^x-block-diagonal matrix, so is $V'(0)$. For any $n \times n$ skew-symmetric, I^x-block-diagonal matrix A, the smooth curve $V(t) := \exp(tA) \in SO^n_x$ is such that $V(0) = I$ and $V'(0) = A$.

Let

$$
\begin{aligned}
H^n &:= \{A \in \mathbb{R}^{n\times n} : A \text{ is skew-symmetric}\},\\
H^n_x &:= \{A \in \mathbb{R}^{n\times n} : A \text{ is skew-symmetric and } I^x\text{-block-diagonal}\}.
\end{aligned}
$$

Define a linear map $\mathcal{S}_k : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^k\times n^k}$ by

$$\mathcal{S}_k(A) := A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A.$$

The next step is to show that the centralizer of $SO^{n,k}_x$ is equal to the centralizer of

$$\mathcal{S}_k(H^n_x) := \{\mathcal{S}_k(A) : A \in H^n_x\}.$$

For the next lemma, recall that $\exp(A + B) = \exp(A)\exp(B)$ whenever A, B are in $\mathbb{R}^{n\times n}$ with $AB = BA$, see [7, page 32].

Lemma 4.3.7 *For any $A \in \mathbb{R}^{n\times n}$, the following equality holds,*

$$\exp(\mathcal{S}_k(A)) = \big(\exp(A)\big)^{\otimes k}. \tag{4.11}$$

Proof Since $A \otimes I \otimes \cdots \otimes I, \ldots, I \otimes \cdots \otimes I \otimes A$ commute, the left-hand side of (4.11) becomes

$$\begin{aligned}
\exp(\mathcal{S}_k(A)) &= \exp(A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A) \\
&= \exp(A \otimes I \otimes \cdots \otimes I) \cdots \exp(I \otimes \cdots \otimes I \otimes A) \\
&= \Big(\sum_{i=0}^{\infty} \frac{(A \otimes I \otimes \cdots \otimes I)^i}{i!} \Big) \cdots \Big(\sum_{i=0}^{\infty} \frac{(I \otimes \cdots \otimes I \otimes A)^i}{i!} \Big) \\
&= \sum_{i_1, \ldots, i_k = 0}^{\infty} \frac{(A \otimes I \otimes \cdots \otimes I)^{i_1} \cdots (I \otimes \cdots \otimes I \otimes A)^{i_k}}{i_1! \cdots i_k!} \\
&= \sum_{i_1, \ldots, i_k = 0}^{\infty} \frac{(A^{i_1} \otimes I \otimes \cdots \otimes I) \cdots (I \otimes \cdots \otimes I \otimes A^{i_k})}{i_1! \cdots i_k!} \\
&= \sum_{i_1, \ldots, i_k = 0}^{\infty} \frac{A^{i_1} \otimes A^{i_2} \otimes \cdots \otimes A^{i_k}}{i_1! \cdots i_k!} = \Big(\sum_{i=0}^{\infty} \frac{A^i}{i!} \Big)^{\otimes k} = (\exp(A))^{\otimes k},
\end{aligned}$$

which is what we had to show.

It is well-known that the map

$$A \in H_x^n \mapsto \exp(A) \in SO_x^n$$

is onto, see [7, page 58]. For the next theorem, recall that $A \exp(B) = \exp(B) A$, whenever A, B are in $\mathbb{R}^{n \times n}$ with $AB = BA$, see [7, page 31].

Corollary 4.3.8 *We have $C(SO_x^{n,k}) = C(\mathcal{S}_k(H_x^n))$.*

Proof Let $\mathbf{A} \in C(SO_x^{n,k})$. Take a smooth curve $V(t) \in SO_x^n$ for $t \in \mathbb{R}$ with $V(0) = I$. Differentiate both sides of

$$\mathbf{A} V(t)^{\otimes k} = V(t)^{\otimes k} \mathbf{A}$$

with respect to t at $t = 0$ to obtain

$$\mathbf{A} \mathcal{S}_k(V'(0)) = \mathcal{S}_k(V'(0)) \mathbf{A}.$$

Since any matrix in H^n_x is equal to $V'(0)$ for some smooth curve, we conclude that $\mathbf{A} \in C(\mathcal{S}_k(H^n_x))$.

For the opposite inclusion, let $\mathbf{A} \in C(\mathcal{S}_k(H^n_x))$. For any $A \in H^n_x$, by Lemma 4.3.7, we have

$$\mathbf{A}(\exp(A))^{\otimes k} = \mathbf{A}\exp(\mathcal{S}_k(A)) = \exp(\mathcal{S}_k(A))\mathbf{A} = (\exp(A))^{\otimes k}\mathbf{A}.$$

Since every matrix in $SO^{n,k}_x$ is of the form $(\exp(A))^{\otimes k}$ for some $A \in H^n_x$, we are done.

Since $SO^{n,k}_x \subseteq SO^{n,k} \subseteq O^{n,k}$, we have

$$C(O^{n,k}) \subseteq C(SO^{n,k}) \subseteq C(SO^{n,k}_x).$$

The next proposition shows that every matrix $\mathbf{P}_\sigma$ is in $C(O^{n,k})$.

Proposition 4.3.9 *We have $\mathbf{P}_\sigma \in C(O^{n,k})$, for every permutation $\sigma : \mathbb{N}_k \to \mathbb{N}_k$.*

Proof Since $\mathbf{P}_\sigma$ is in $C(I^{n,k}_\pm)$, we only need to show that $\mathbf{P}_\sigma$ is in $C(SO^{n,k})$. That is, we have to show that $\mathbf{P}_\sigma\mathcal{S}_k(A) = \mathcal{S}_k(A)\mathbf{P}_\sigma$ for all $A \in H^n$ and all permutations $\sigma : \mathbb{N}_k \to \mathbb{N}_k$. For any $l, m \in \mathbb{N}^k_n$, we have

$$(\mathbf{P}_\sigma\mathcal{S}_k(A))_{l,m} - (\mathcal{S}_k(A)\mathbf{P}_\sigma)_{l,m} = (\mathbf{P}_\sigma)_{l,\sigma(l)}(\mathcal{S}_k(A))_{\sigma(l),m} - (\mathcal{S}_k(A))_{l,\sigma^{-1}(m)}(\mathbf{P}_\sigma)_{\sigma^{-1}(m),m}.$$

Now,

$$(\mathbf{P}_\sigma)_{l,\sigma(l)} = 1 \text{ and } (\mathbf{P}_\sigma)_{\sigma^{-1}(m),m} = 1$$

and

$$(\mathcal{S}_k(A))_{\sigma(l),m} = \sum_{i=1}^{k} A_{l_{\sigma(i)}m_i} \prod_{j\neq i} I_{l_{\sigma(j)}m_j} = \sum_{i=1}^{k} A_{l_i m_{\sigma^{-1}(i)}} \prod_{j\neq i} I_{l_j m_{\sigma^{-1}(j)}} = (\mathcal{S}_k(A))_{l,\sigma^{-1}(m)}.$$

The proof follows from here.

4.3.4 The centralizer of $O_x^{n,k}$

The following function allows us to compare two given indexes. Define $\theta : \mathbb{N}_n^k \times \mathbb{N}_n^k \to \{0, 1\}$ by

$$\theta(l, m) := \begin{cases} 1 & \text{if } l_{i^*} \neq m_{i^*} \text{ for some } i^* \in \mathbb{N}_k \text{ and } l_i = m_i \text{ for all } i \neq i^*, \\ 0 & \text{otherwise,} \end{cases} \tag{4.12}$$

for any $l, m \in \mathbb{N}_n^k$. It is obvious that $\theta(l, m) = \theta(m, l)$, for any $l, m \in \mathbb{N}_n^k$.

Lemma 4.3.10 *For any $A \in H^n$, $\mathcal{S}_k(A)$ is skew-symmetric and its elements are*

$$(\mathcal{S}_k(A))_{l,m} = \begin{cases} A_{l_{i^*} m_{i^*}} & \text{if } \theta(l, m) = 1 \text{ with } l_{i^*} \neq m_{i^*}, \\ 0 & \text{if } \theta(l, m) = 0, \end{cases}$$

for any $l, m \in \mathbb{N}_n^k$.

Proof The fact that $\mathcal{S}_k(A)$ is skew-symmetric follows from the properties of tensor product. Without loss of generality, it suffices to consider the following three cases.

Case 1. If $l = m$, then $\theta(l, m) = 0$, and since

$$(A \otimes I \otimes \cdots \otimes I)_{l,m} = \cdots = (I \otimes I \otimes \cdots \otimes A)_{l,m} = 0,$$

we have $(\mathcal{S}_k(A))_{l,m} = 0$.

Case 2. If $l_1 \neq m_1$ and $l_i = m_i$ for all $i \neq 1$, then $\theta(l, m) = 1$. Using that

$$\begin{cases} (A \otimes I \otimes \cdots \otimes I)_{l,m} & = A_{l_1 m_1} I_{l_2 m_2} \cdots I_{l_k m_k} = A_{l_1 m_1}, \\ (I \otimes A \otimes \cdots \otimes I)_{l,m} & = I_{l_1 m_1} A_{l_2 m_2} \cdots I_{l_k m_k} = 0, \\ & \vdots \\ (I \otimes I \otimes \cdots \otimes A)_{l,m} & = I_{l_1 m_1} I_{l_2 m_2} \cdots A_{l_k m_k} = 0, \end{cases}$$

we obtain that $(\mathcal{S}_k(A))_{l,m} = A_{l_1 m_1}$.

Case 3. If $l_1 \neq m_1, l_2 \neq m_2$ and $l_i = m_i$ for all $i = 3, \ldots, k$, then $\theta(l, m) = 0$. Using that

$$(A \otimes I \otimes \cdots \otimes I)_{l,m} = \cdots = (I \otimes I \otimes \cdots \otimes A)_{l,m} = 0,$$

we obtain $(\mathcal{S}_k(A))_{l,m} = 0$.

Consider the following skew-symmetric matrices in H^n_x. For any $s, t \in \mathbb{N}_n$, with $s \neq t$, such that s and t are in the same block of I^x, let

$$(A^{st}_x)_{ij} := \begin{cases} 1 & \text{if } i = s \text{ and } j = t, \\ -1 & \text{if } i = t \text{ and } j = s, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, any $A \in H^n_x$ can be written as

$$A = \sum_{s>t} A_{st} A^{st}_x,$$

where $s, t \in \mathbb{N}_n$ are in the same block of I^x. If the coordinates of vector x are all equal, then $H^n_x = H^n$, so we drop the index x and write simply A^{st}, for all $s, t \in \mathbb{N}_n$ with $s \neq t$.

Define the set

$$\beta(l, m) := \{l^1 \in \mathbb{N}^k_n : \alpha(l) = \alpha(l^1) \text{ and } \theta(l^1, m) = 1\},$$

where α is given by (4.9) and θ is given by (4.12). Note that, in general, the sets $\beta(l, m)$ and $\beta(m, l)$ are not equal.

Theorem 4.3.11 *For any $x \in \mathbb{R}^n_{\geq}$, we have $\mathbf{A} \in C(O^{n,k}_x)$, if and only if $\mathbf{A} \in C(I^{n,k}_{\pm})$ and satisfies*

$$\sum_{l^1 \in \beta(l,m)} \mathbf{A}_{l,l^1} (\mathcal{S}_k(A^{st}_x))_{l^1,m} = \sum_{l^1 \in \beta(m,l)} (\mathcal{S}_k(A^{st}_x))_{l,l^1} \mathbf{A}_{l^1,m} \tag{4.13}$$

for all $l, m \in \mathbb{N}^k_n$ and all s, t in a same block of I^x, with $s \neq t$.

Proof By Lemma 4.3.3, $\mathbf{A} \in C(O_x^{n,k})$, if and only if $\mathbf{A} \in C(I_\pm^{n,k}) \cap C(SO_x^{n,k})$. By Corollary 4.3.8, $C(SO_x^{n,k}) = C(S_k(H_x^n))$ and by the linearity of the map $\mathcal{S}_k$, we have that $\mathbf{A} \in C(S_k(H_x^n))$, if and only if

$$\sum_{l^1 \in \mathbb{N}_n^k} \mathbf{A}_{l,l^1} (\mathcal{S}_k(A_x^{st}))_{l^1,m} = \sum_{l^1 \in \mathbb{N}_n^k} (\mathcal{S}_k(A_x^{st}))_{l,l^1} \mathbf{A}_{l^1,m} \tag{4.14}$$

for all $l, m \in \mathbb{N}_n^k$ and all s, t in a same block of I^x, with $s \neq t$. If $l^1 \notin \beta(l, m)$ then either $\alpha(l) \neq \alpha(l^1)$ or $\theta(l^1, m) = 0$. In the first case, by Proposition 4.3.5, we get $\mathbf{A}_{l,l^1} = 0$. In the latter case, $(\mathcal{S}_k(A_x^{st}))_{l^1,m} = 0$. Thus, the left-hand side of (4.14) is equal to the left-hand side of (4.13).

Analogously, the terms on the right-hand side of (4.14) corresponding to $l^1 \notin \beta(m, l)$ are zero.

Lemma 4.3.12 *Fix any $l, m \in \mathbb{N}_n^k$ and $s, t \in \mathbb{N}_n$, such that s and t are in a same block of the partition I^x and $s \neq t$. Then, the left-hand side of* (4.13) *is not identically zero, if and only if at least one coordinate of m is s or t and*

$$\alpha(l) = \alpha(m) + e^s + e^t \pmod 2. \tag{4.15}$$

Proof By Lemma 4.3.10, one can see

$$(\mathcal{S}_k(A_x^{st}))_{l^1,m} = \begin{cases} (A_x^{st})_{l^1_{i^*} m_{i^*}} & \text{if } \theta(l^1, m) = 1 \text{ with } l^1_{i^*} \neq m_{i^*}, \\ 0 & \text{if } \theta(l^1, m) = 0, \end{cases}$$

$$= \begin{cases} 1 & \text{if } \theta(l^1, m) = 1 \text{ with } l^1_{i^*} = s, m_{i^*} = t, \\ -1 & \text{if } \theta(l^1, m) = 1 \text{ with } l^1_{i^*} = t, m_{i^*} = s, \\ 0 & \text{otherwise,} \end{cases} \tag{4.16}$$

for any $l^1, m \in \mathbb{N}_n^k$, where the second equality is obtained after observing that

$$(A_x^{st})_{ij} = \begin{cases} 1 & \text{if } i = s, j = t, \\ -1 & \text{if } i = t, j = s. \end{cases}$$

Suppose that the left-hand side of (4.13) is not identically zero. This implies that $\beta(l, m) \neq \emptyset$ and there is an $l^1 \in \beta(l, m)$, such that $(\mathcal{S}_k(A_x^{st}))_{l^1,m} \neq 0$. By (4.16), at least one coordinate of $m \in \mathbb{N}_n^k$ is s or t. Let $m := (m_1, m_2, \ldots, m_k)$ and $l^1 := (l_1^1, l_2^1, \ldots, l_k^1)$. The k-tuples l^1 and m are almost the same, except that they differ in one position, namely i^*. In fact, $(l_{i^*}^1, m_{i^*})$ is equal to (s, t) or (t, s). Thus,

$$\begin{aligned} \alpha(l^1) &= e^{l_{i^*}^1} + \sum_{\substack{i=1 \\ i \neq i^*}}^{k} e^{l_i^1} \pmod 2 = e^{l_{i^*}^1} + \sum_{\substack{i=1 \\ i \neq i^*}}^{k} e^{m_i} \pmod 2 \\ &= e^{m_{i^*}} + e^s + e^t + \sum_{\substack{i=1 \\ i \neq i^*}}^{k} e^{m_i} \pmod 2 = \alpha(m) + e^s + e^t \pmod 2. \end{aligned}$$

Since $l^1 \in \beta(l, m)$, we also have $\alpha(l) = \alpha(l^1)$.

For the other direction, suppose that m contains s or t and l and m satisfy (4.15). We only need to find an $l^1 \in \beta(l, m)$ satisfying $(\mathcal{S}_k(A_x^{st}))_{l^1,m} \neq 0$.

Since m contains s or t, without loss of generality, assume $m = (s, m_2, \ldots, m_k)$. Taking $l^1 = (t, m_2, \ldots, m_k)$, one can see by (4.16), that $(\mathcal{S}_k(A_x^{st}))_{l^1,m} \neq 0$. The latter implies also that $\theta(l^1, m) = 1$. By (4.15), one can see

$$\alpha(l^1) = e^t + \sum_{i=2}^{k} e^{m_i} = e^s + \sum_{i=2}^{k} e^{m_i} + e^t + e^s \pmod 2 = \alpha(m) + e^t + e^s \pmod 2 = \alpha(l).$$

Thus $l^1 \in \beta(l, m)$, completing the proof.

4.4 The main results

In this section, we specialize to the case $k = 2$ and to considering only symmetric matrices in the centralizer of $O_x^{n,2}$. It is easy to see $C_S(O_x^{n,k}) = C_S(I_\pm^{n,k}) \cap C_S(SO_x^{n,k})$. Since $C(SO_x^{n,k}) = C(\mathcal{S}_k(H_x^n))$, it is easy to see that $C_S(SO_x^{n,k}) = C_S(\mathcal{S}_k(H_x^n))$. We apply Theorem 4.3.11 to obtain explicit representation of the matrices in $C_S(O_x^{n,2})$. To simplify the presentation, we first obtain explicit representation of the matrices in $C_S(O^{n,2})$. This is achieved in the next theorem, whose proof is given in the Appendix.

Theorem 4.4.1 *Any matrix* $\mathbf{A} \in C_S(O^{n,2})$ *can be expressed as*

$$\mathbf{A} = a_1\mathbf{Q} + a_2\mathbf{P}_{(1,2)} + a_3\mathbf{P}_{(1)(2)}$$

for some $a_1, a_2, a_3 \in \mathbb{R}$, *where*

$$\mathbf{Q}_{l,m} := \begin{cases} 1 & \textit{if } l_1 = l_2 \textit{ and } m_1 = m_2, \\ 0 & \textit{otherwise.} \end{cases}$$

The permutation matrices $\mathbf{P}_{(1,2)}$ *and* $\mathbf{P}_{(1)(2)}$ *in* $\mathbb{R}^{n^2\times n^2}$ *are defined by* (4.10) *and correspond to the transposition and the identity permutations on* $\mathbb{N}_2$.

For easy reference, we state $\mathbf{P}_{(1,2)}$ and $\mathbf{P}_{(1)(2)}$ explicitly:

$$(\mathbf{P}_{(1,2)})_{l,m} := \begin{cases} 1 & \text{if } l_1 = m_2 \text{ and } l_2 = m_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\mathbf{P}_{(1)(2)})_{l,m} := \begin{cases} 1 & \text{if } l_1 = m_1 \text{ and } l_2 = m_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathbf{Q}$ and $\mathbf{P}_{(1,2)}$ are symmetric and $\mathbf{P}_{(1)(2)}$ is the identity matrix.

We now state the representation theorem for the matrices in $C_S(O_x^{n,2})$. The proof is in the Appendix.

Theorem 4.4.2 *Fix any $x \in \mathbb{R}^n_{\geq}$. A matrix $\mathbf{A} \in C_S(O^{n,2}_x)$ has the following representation*

$$\mathbf{A} = \mathbf{Q}(\mathbf{x}_1) + (\operatorname{Diag} \mathbf{x}_2)\mathbf{P}_{(1,2)} + (\operatorname{Diag} \mathbf{x}_3)\mathbf{P}_{(1)(2)}, \tag{4.17}$$

where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^{n^2}$ are $\mathbf{I}^x$-block-constant, such that

$$(\mathbf{x}_1)_{(l_1,l_2)} = (\mathbf{x}_1)_{(l_2,l_1)} \quad \textit{and} \quad (\mathbf{x}_2)_{(l_1,l_2)} = (\mathbf{x}_2)_{(l_2,l_1)}$$

for all $l \in \mathbb{N}^2_n$. The map $\mathbf{Q} : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2 \times n^2}$ is defined by

$$(\mathbf{Q}(\mathbf{x}))_{l,m} := \begin{cases} \mathbf{x}_{(l_1,m_1)} & \text{if } l_1 = l_2 \text{ and } m_1 = m_2, \\ 0 & \text{otherwise,} \end{cases} \tag{4.18}$$

for all $l, m \in \mathbb{N}^2_n$.

One can verify that all three matrices on the right-hand side of (4.17) are in S^{n^2}.

Remark 4.4.3 *Representation* (4.17) *may not be unique, if the partition I^x of $\mathbb{N}_n$ contains a block with one element. Indeed, from the proof of Theorem 4.4.2, one can see that, if $\{s\} \in I^x$ is a block of size one, then one can choose the values of $(\mathbf{x}_1)_{(s,s)}$, $(\mathbf{x}_2)_{(s,s)}$, and $(\mathbf{x}_3)_{(s,s)}$ freely, as long as they satisfy*

$$\mathbf{A}_{(s,s),(s,s)} = (\mathbf{x}_1)_{(s,s)} + (\mathbf{x}_2)_{(s,s)} + (\mathbf{x}_3)_{(s,s)}.$$

To resolve this issue and make representation (4.17) *unique, for every block $\{s\} \in I^x$ of size one, we set*

$$(\mathbf{x}_1)_{(s,s)} := 0, \quad (\mathbf{x}_2)_{(s,s)} := 0, \quad \text{and } (\mathbf{x}_3)_{(s,s)} := \mathbf{A}_{(s,s),(s,s)}.$$

Vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ remain $\mathbf{I}^x$-block-constant.

The representation of 2-tensor isotropic functions is as follows.

Corollary 4.4.4 *A function $F : S^n \to S^{n^2}$ is 2-tensor isotropic, if and only if there exist block-constant functions $f_1, f_2, f_3 : \mathbb{R}^n_{\geq} \to \mathbb{R}^{n^2}$, such that*

$$F(A) = U^{\otimes 2}(\mathbf{Q}(f_1(\lambda(A))) + (\mathrm{Diag}\, f_2(\lambda(A)))\mathbf{P}_{(1,2)} + (\mathrm{Diag}\, f_3(\lambda(A)))\mathbf{P}_{(1)(2)})(U^{\otimes 2})^\top \tag{4.19}$$

for any $U \in O^n$, such that $A = U(\mathrm{Diag}\,\lambda(A))U^\top$. In addition, the functions f_1 and f_2 satisfy

$$(f_1(x))_{(l_1,l_2)} = (f_1(x))_{(l_2,l_1)} \text{ and } (f_2(x))_{(l_1,l_2)} = (f_2(x))_{(l_2,l_1)}$$

for all $l \in \mathbb{N}^2_n$. The map $\mathbf{Q} : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2 \times n^2}$ is defined as in (4.18).

Proof The right-hand side of (4.19) does not depend on the choice of the diagonalizing matrix $U \in O^n$, since the expression

$$\mathbf{Q}(f_1(\lambda(A))) + (\mathrm{Diag}\, f_2(\lambda(A)))\mathbf{P}_{(1,2)} + (\mathrm{Diag}\, f_3(\lambda(A)))\mathbf{P}_{(1)(2)}$$

is in the centralizer $C_S(O^{n,2}_{\lambda(A)})$ by Theorem 4.4.2. Thus, the value of the right-hand side of (4.19) is well-defined.

Suppose that the value of $F(A)$ is given by (4.19). For any $V \in O^n$, we have that

$$\begin{aligned} F(VAV^\top) &= (VU)^{\otimes 2}(\mathbf{Q}(f_1(\lambda(A))) + (\mathrm{Diag}\, f_2(\lambda(A)))\mathbf{P}_{(1,2)} + \mathrm{Diag}\, f_3(\lambda(A)))((VU)^{\otimes 2})^\top \\ &= V^{\otimes 2}U^{\otimes 2}(\mathbf{Q}(f_1(\lambda(A))) + (\mathrm{Diag}\, f_2(\lambda(A)))\mathbf{P}_{(1,2)} + \mathrm{Diag}\, f_3(\lambda(A)))(U^{\otimes 2})^\top (V^{\otimes 2})^\top \\ &= V^{\otimes 2}F(A)(V^{\otimes 2})^\top, \end{aligned}$$

implying that F is 2-tensor isotropic.

For the other direction, suppose that the function $F : S^n \to S^{n^2}$ is 2-tensor isotropic. For each $A \in S^n$, it is easy to see that $F(\mathrm{Diag}\,\lambda(A))$ is in the centralizer $C_S(O^{n,2}_{\lambda(A)})$. Hence, by Theorem 4.4.2 and Remark 4.4.3, there are unique $\mathbf{I}^{\lambda(A)}$-block-constant vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, depending only on $\lambda(A)$, such that

$$F(\operatorname{Diag}\lambda(A)) = \mathbf{Q}(\mathbf{x}_1) + (\operatorname{Diag}\mathbf{x}_2)\mathbf{P}_{(1,2)} + (\operatorname{Diag}\mathbf{x}_3)\mathbf{P}_{(1)(2)}.$$

Defining $f_i(\lambda(A)) := \mathbf{x}_i$ for all $i = 1, 2, 3$, completes the proof.

4.5 Appendix

Proof of Theorem 4.4.1 We use (4.16) repeatedly without further mention. We should notice that $(\mathcal{S}_2(A^{st}))_{l,m} = 0$, whenever no entry of m is s or t. We characterize the symmetric matrices in $C_S(O^{n,2})$, using Theorem 4.3.11. That is, we solve the following system of linear equations in the entries of $\mathbf{A} \in S^{n^2}$:

$$\sum_{l^1 \in \beta(l,m)} \mathbf{A}_{l,l^1}(\mathcal{S}_2(A^{st}))_{l^1,m} = \sum_{l^1 \in \beta(m,l)} (\mathcal{S}_2(A^{st}))_{l,l^1}\mathbf{A}_{l^1,m} \tag{4.20}$$

for all $l, m \in \mathbb{N}_n^2$ and all $s, t \in \mathbb{N}_n$ with $s \neq t$. By the definition of the set $\beta(l, m)$, system (4.20) does not involve entries $\mathbf{A}_{l,m}$ from $\mathbf{A}$ for which $\alpha(l) \neq \alpha(m)$. Such entries are necessarily zero by Proposition 4.3.5.

Also, if $\beta(l, m) = \beta(m, l) = \emptyset$, then both sides of (4.20) are zero. Thus, we only need to consider $l, m \in \mathbb{N}_n^2$, such that $\beta(l, m) \neq \emptyset$, or $\beta(m, l) \neq \emptyset$, or both to find all constrains on $\mathbf{A}$. Note that, if we transpose l and m in equation (4.20), then its both sides switch places and change sign. Thus, we only need to consider cases when the left-hand side of (4.20) is non-zero. Fix $s, t \in \mathbb{N}_n$ with $s \neq t$. Since at least one entry of m is s or t, the possible values of $m \in \mathbb{N}_n^2$ are

$$(s, s),\ (t, t),\ (s, t),\ (t, s),\ (m_1, s),\ (m_1, t),\ (s, m_2),\ \text{and } (t, m_2),$$

where $m_1, m_2 \notin \{s, t\}$.

Case 1. Suppose $m = (s, s)$. By Lemma 4.3.12, the possible values for l are (s, t) and (t, s).

Case 1.a Suppose $m = (s, s)$ and $l = (s, t)$. Then, we have

$$\beta(l, m) = \{(t, s), (s, t)\} \text{ and } \beta(m, l) = \{(s, s), (t, t)\}.$$

Equation (4.20) turns into

$$\begin{aligned}\mathbf{A}_{(s,t),(t,s)}(\mathcal{S}_2(A^{st}))_{(t,s),(s,s)} + \mathbf{A}_{(s,t),(s,t)}(\mathcal{S}_2(A^{st}))_{(s,t),(s,s)}\\ = (\mathcal{S}_2(A^{st}))_{(s,t),(s,s)}\mathbf{A}_{(s,s),(s,s)} + (\mathcal{S}_2(A^{st}))_{(s,t),(t,t)}\mathbf{A}_{(t,t),(s,s)},\end{aligned}$$

which simplifies to

$$\mathbf{A}_{(s,t),(t,s)}(-1) + \mathbf{A}_{(s,t),(s,t)}(-1) = (-1)\mathbf{A}_{(s,s),(s,s)} + (+1)\mathbf{A}_{(t,t),(s,s)},$$

that is

$$\mathbf{A}_{(s,s),(s,s)} = \mathbf{A}_{(s,t),(s,t)} + \mathbf{A}_{(s,t),(t,s)} + \mathbf{A}_{(t,t),(s,s)}. \tag{4.21}$$

Case 1.b Suppose $m = (s, s)$ and $l = (t, s)$. Then, we have

$$\beta(l, m) = \{(t, s), (s, t)\} \text{ and } \beta(m, l) = \{(s, s), (t, t)\}.$$

Equation (4.20) turns into

$$\begin{aligned}\mathbf{A}_{(t,s),(t,s)}(\mathcal{S}_2(A^{st}))_{(t,s),(s,s)} + \mathbf{A}_{(t,s),(s,t)}(\mathcal{S}_2(A^{st}))_{(s,t),(s,s)}\\ = (\mathcal{S}_2(A^{st}))_{(t,s),(s,s)}\mathbf{A}_{(s,s),(s,s)} + (\mathcal{S}_2(A^{st}))_{(t,s),(t,t)}\mathbf{A}_{(t,t),(s,s)},\end{aligned}$$

which implies that

$$\mathbf{A}_{(t,s),(t,s)}(-1) + \mathbf{A}_{(t,s),(s,t)}(-1) = (-1)\mathbf{A}_{(s,s),(s,s)} + (+1)\mathbf{A}_{(t,t),(s,s)},$$

that is

$$\mathbf{A}_{(s,s),(s,s)} = \mathbf{A}_{(t,s),(t,s)} + \mathbf{A}_{(t,s),(s,t)} + \mathbf{A}_{(t,t),(s,s)}. \tag{4.22}$$

Comparing (4.21) and (4.22), and using that $\mathbf{A}$ is symmetric matrix, one obtains

$$\mathbf{A}_{(s,t),(s,t)} = \mathbf{A}_{(t,s),(t,s)}. \tag{4.23}$$

Case 2. Suppose $m = (t,t)$. By Lemma 4.3.12, the possible values for l are (s,t) and (t,s).

Case 2.a Suppose $m = (t,t)$ and $l = (s,t)$. Then, we have

$$\beta(l,m) = \{(t,s),(s,t)\} \text{ and } \beta(m,l) = \{(s,s),(t,t)\}.$$

Equation (4.20) turns into

$$\begin{aligned}\mathbf{A}_{(s,t),(t,s)}&(\mathcal{S}_2(A^{st}))_{(t,s),(t,t)} + \mathbf{A}_{(s,t),(s,t)}(\mathcal{S}_2(A^{st}))_{(s,t),(t,t)}\\ &= (\mathcal{S}_2(A^{st}))_{(s,t),(s,s)}\mathbf{A}_{(s,s),(t,t)} + (\mathcal{S}_2(A^{st}))_{(s,t),(t,t)}\mathbf{A}_{(t,t),(t,t)},\end{aligned}$$

which implies that

$$\mathbf{A}_{(s,t),(t,s)}(+1) + \mathbf{A}_{(s,t),(s,t)}(+1) = (-1)\mathbf{A}_{(s,s),(t,t)} + (+1)\mathbf{A}_{(t,t),(t,t)},$$

that is

$$\mathbf{A}_{(t,t),(t,t)} = \mathbf{A}_{(s,t),(t,s)} + \mathbf{A}_{(s,t),(s,t)} + \mathbf{A}_{(s,s),(t,t)}. \tag{4.24}$$

Case 2.b Suppose $m = (t,t)$ and $l = (t,s)$. Then, we have

$$\beta(l,m) = \{(t,s),(s,t)\} \text{ and } \beta(m,l) = \{(s,s),(t,t)\}.$$

Equation (4.20) turns into

$$\begin{aligned}&\mathbf{A}_{(t,s),(t,s)}(\mathcal{S}_2(A^{st}))_{(t,s),(t,t)} + \mathbf{A}_{(t,s),(s,t)}(\mathcal{S}_2(A^{st}))_{(s,t),(t,t)}\\&\quad = (\mathcal{S}_2(A^{st}))_{(t,s),(s,s)}\mathbf{A}_{(s,s),(t,t)} + (\mathcal{S}_2(A^{st}))_{(t,s),(t,t)}\mathbf{A}_{(t,t),(t,t)},\end{aligned}$$

which implies that

$$\mathbf{A}_{(t,s),(t,s)}(+1) + \mathbf{A}_{(t,s),(s,t)}(+1) = (-1)\mathbf{A}_{(s,s),(t,t)} + (+1)\mathbf{A}_{(t,t),(t,t)},$$

that is

$$\mathbf{A}_{(t,t),(t,t)} = \mathbf{A}_{(t,s),(t,s)} + \mathbf{A}_{(t,s),(s,t)} + \mathbf{A}_{(s,s),(t,t)}.$$

This equation is a consequence of (4.23), (4.24), and the fact that $\mathbf{A}$ is symmetric, so we discard it.

Case 3. Suppose $m = (s,t)$. By Lemma 4.3.12, the possible values for l are (l_1, l_1) for $l_1 \in \mathbb{N}_n$.

Case 3.a Suppose $m = (s,t)$ and $l = (s,s)$. This case is the same as Case 1.a after transposing l and m.

Case 3.b Suppose $m = (s,t)$ and $l = (t,t)$. This case is the same as Case 2.a after transposing l and m.

Case 3.c Suppose $m = (s,t)$ and $l = (l_1, l_1)$ for $l_1 \notin \{s,t\}$. Then, we have

$$\beta(l,m) = \{(s,s),(t,t)\} \text{ and } \beta(m,l) = \emptyset.$$

Equation (4.20) turns into

$$\mathbf{A}_{(l_1,l_1),(s,s)}(\mathcal{S}_2(A^{st}))_{(s,s),(s,t)} + \mathbf{A}_{(l_1,l_1),(t,t)}(\mathcal{S}_2(A^{st}))_{(t,t),(s,t)} = 0,$$

which implies that

$$\mathbf{A}_{(l_1,l_1),(s,s)}(+1) + \mathbf{A}_{(l_1,l_1),(t,t)}(-1) = 0,$$

that is

$$\mathbf{A}_{(l_1,l_1),(s,s)} = \mathbf{A}_{(l_1,l_1),(t,t)} \text{ for all } l_1 \notin \{s,t\}. \tag{4.25}$$

Case 4. Suppose $m = (t,s)$. By Lemma 4.3.12, the possible values for l are (l_1,l_1) for $l_1 \in \mathbb{N}_n$.

Case 4.a Suppose $m = (t,s)$ and $l = (s,s)$. This case is the same as Case 1.b after transposing l and m.

Case 4.b Suppose $m = (t,s)$ and $l = (t,t)$. This case is the same as Case 2.b after transposing l and m.

Case 4.c Suppose $m = (t,s)$ and $l = (l_1,l_1)$ for $l_1 \notin \{s,t\}$. Then, we have

$$\beta(l,m) = \{(s,s),(t,t)\} \text{ and } \beta(m,l) = \emptyset,$$

Equation (4.20) turns into

$$\mathbf{A}_{(l_1,l_1),(s,s)}(\mathcal{S}_2(A^{st}))_{(s,s),(t,s)} + \mathbf{A}_{(l_1,l_1),(t,t)}(\mathcal{S}_2(A^{st}))_{(t,t),(t,s)} = 0,$$

which implies that

$$\mathbf{A}_{(l_1,l_1),(s,s)}(+1) + \mathbf{A}_{(l_1,l_1),(t,t)}(-1) = 0,$$

that is

$$\mathbf{A}_{(l_1,l_1),(s,s)} = \mathbf{A}_{(l_1,l_1),(t,t)} \text{ for all } l_1 \notin \{s,t\},$$

which is the same as (4.25).

Case 5. Suppose $m = (m_1, s)$ for some $m_1 \notin \{s, t\}$. By Lemma 4.3.12, the possible values for l are (m_1, t) and (t, m_1).

Case 5.a Suppose $m = (m_1, s)$ and $l = (m_1, t)$. Then, we have

$$\beta(l, m) = \{(m_1, t)\} \text{ and } \beta(m, l) = \{(m_1, s)\}.$$

Equation (4.20) turns into

$$\mathbf{A}_{(m_1,t),(m_1,t)}(\mathcal{S}_2(A^{st}))_{(m_1,t),(m_1,s)} = (\mathcal{S}_2(A^{st}))_{(m_1,t),(m_1,s)}\mathbf{A}_{(m_1,s),(m_1,s)},$$

which implies that

$$\mathbf{A}_{(m_1,t),(m_1,t)}(-1) = (-1)\mathbf{A}_{(m_1,s),(m_1,s)},$$

that is

$$\mathbf{A}_{(m_1,t),(m_1,t)} = \mathbf{A}_{(m_1,s),(m_1,s)} \text{ for all } m_1 \notin \{s, t\}. \tag{4.26}$$

Case 5.b Suppose $m = (m_1, s)$ and $l = (t, m_1)$. Then, we have

$$\beta(l, m) = \{(m_1, t)\} \text{ and } \beta(m, l) = \{(s, m_1)\}.$$

Equation (4.20) turns into

$$\mathbf{A}_{(t,m_1),(m_1,t)}(\mathcal{S}_2(A^{st}))_{(m_1,t),(m_1,s)} = (\mathcal{S}_2(A^{st}))_{(t,m_1),(s,m_1)}\mathbf{A}_{(s,m_1),(m_1,s)},$$

which implies that

$$\mathbf{A}_{(t,m_1),(m_1,t)}(-1) = (-1)\mathbf{A}_{(s,m_1),(m_1,s)},$$

that is

$$\mathbf{A}_{(t,m_1),(m_1,t)} = \mathbf{A}_{(s,m_1),(m_1,s)} \text{ for all } m_1 \notin \{s,t\}. \tag{4.27}$$

Case 6. Suppose $m = (m_1, t)$ for $m_1 \notin \{s, t\}$. By Lemma 4.3.12, the possible values for l are (m_1, s) and (s, m_1).

Case 6.a Suppose $m = (m_1, t)$ and $l = (m_1, s)$. This case is the same as Case 5.a after transposing l and m.

Case 6.b Suppose $m = (m_1, t)$ and $l = (s, m_1)$. Then, we have

$$\beta(l,m) = \{(m_1, s)\} \text{ and } \beta(m,l) = \{(t, m_1)\}.$$

Equation (4.20) turns into

$$\mathbf{A}_{(s,m_1),(m_1,s)}(\mathcal{S}_2(A^{st}))_{(m_1,s),(m_1,t)} = (\mathcal{S}_2(A^{st}))_{(s,m_1),(t,m_1)}\mathbf{A}_{(t,m_1),(m_1,t)},$$

which implies that

$$\mathbf{A}_{(s,m_1),(m_1,s)}(+1) = (+1)\mathbf{A}_{(t,m_1),(m_1,t)},$$

that is

$$\mathbf{A}_{(s,m_1),(m_1,s)} = \mathbf{A}_{(t,m_1),(m_1,t)} \text{ for all } m_1 \notin \{s,t\},$$

which was obtained in (4.27).

Case 7. Suppose $m = (s, m_2)$ for $m_2 \notin \{s, t\}$. By Lemma 4.3.12, the possible values for l are (t, m_2) and (m_2, t).

Case 7.a Suppose $m = (s, m_2)$ and $l = (t, m_2)$. Then, we have

$$\beta(l, m) = \{(t, m_2)\} \text{ and } \beta(m, l) = \{(s, m_2)\}.$$

Equation (4.20) turns into

$$\mathbf{A}_{(t,m_2),(t,m_2)}(\mathcal{S}_2(A^{st}))_{(t,m_2),(s,m_2)} = (\mathcal{S}_2(A^{st}))_{(t,m_2),(s,m_2)}\mathbf{A}_{(s,m_2),(s,m_2)},$$

which implies that

$$\mathbf{A}_{(t,m_2),(t,m_2)}(-1) = (-1)\mathbf{A}_{(s,m_2),(s,m_2)},$$

that is

$$\mathbf{A}_{(t,m_2),(t,m_2)} = \mathbf{A}_{(s,m_2),(s,m_2)} \text{ for all } m_1 \notin \{s, t\}. \tag{4.28}$$

Case 7.b Suppose $m = (s, m_2)$ and $l = (m_2, t)$. This case is the same as Case 6.b after transposing l and m.

Case 8. Suppose $m = (t, m_2)$ for $m_2 \notin \{s, t\}$. By Lemma 4.3.12, the possible values for l are (s, m_2) and (m_2, s).

Case 8.a Suppose $m = (t, m_2)$ and $l = (s, m_2)$. This case is the same as Case 7.a after transposing l and m.

Case 8.b Suppose $m = (t, m_2)$ and $l = (m_2, s)$. This case is the same as Case 5.b after transposing l and m.

Now, we summarize the constrains (4.22), (4.23), (4.24), (4.25), (4.26), (4.27), and (4.28) that we found and neglect the constrains which can be derived from the listed ones. Thus, matrix $\mathbf{A} \in S^{n^2}$ satisfies the following conditions:

$$\begin{aligned}
\mathbf{A}_{(s,s),(s,s)} &= \mathbf{A}_{(t,s),(t,s)} + \mathbf{A}_{(t,s),(s,t)} + \mathbf{A}_{(t,t),(s,s)}, \\
\mathbf{A}_{(s,t),(s,t)} &= \mathbf{A}_{(t,s),(t,s)}, \\
\mathbf{A}_{(t,t),(t,t)} &= \mathbf{A}_{(t,s),(t,s)} + \mathbf{A}_{(t,s),(s,t)} + \mathbf{A}_{(s,s),(t,t)}, \\
\mathbf{A}_{(q,q),(s,s)} &= \mathbf{A}_{(q,q),(t,t)}, \\
\mathbf{A}_{(q,t),(q,t)} &= \mathbf{A}_{(q,s),(q,s)}, \\
\mathbf{A}_{(t,q),(q,t)} &= \mathbf{A}_{(s,q),(q,s)}, \\
\mathbf{A}_{(t,q),(t,q)} &= \mathbf{A}_{(s,q),(s,q)},
\end{aligned} \tag{4.29}$$

for all distinct $q, s, t \in \mathbb{N}_n$. As a consequence of these relationships, using the fact that $\mathbf{A}$ is symmetric, one can obtain that there are constants a_1, a_2, a_3, such that

$$\mathbf{A}_{(t,t),(s,s)} = a_1, \quad \mathbf{A}_{(s,t),(t,s)} = a_2, \quad \mathbf{A}_{(t,s),(t,s)} = a_3, \tag{4.30}$$

for any distinct $s, t \in \mathbb{N}_n$, and

$$\mathbf{A}_{(s,s),(s,s)} = a_3 + a_2 + a_1 \quad \text{for all } s \in \mathbb{N}_n.$$

Note that, if $\mathbf{A}_{l,m}$ is such that $\alpha(l) = \alpha(m)$, then $\mathbf{A}_{l,m}$ is of the form $\mathbf{A}_{(t,t),(s,s)}$, $\mathbf{A}_{(s,t),(t,s)}$, $\mathbf{A}_{(t,s),(t,s)}$, or $\mathbf{A}_{(s,s),(s,s)}$ for some t and s. As mentioned above $\mathbf{A}_{l,m} = 0$, whenever $\alpha(l) \neq \alpha(m)$. Thus, all entries of $\mathbf{A}$ are accounted for together with all relationships between them. The result follows from here.

Proof of Theorem 4.4.2 Fix $x \in \mathbb{R}^n_{\geq}$. By Theorem 4.3.11, in order to characterize the matrices in $C_S(O^{n,2}_x)$, we need to solve the following system of linear equations in the entries of $\mathbf{A} \in$

$C_S(I_{\pm}^{n,2})$:

$$\sum_{l^1 \in \beta(l,m)} \mathbf{A}_{l,l^1} (\mathcal{S}_2(A_x^{st}))_{l^1,m} = \sum_{l^1 \in \beta(m,l)} (\mathcal{S}_2(A_x^{st}))_{l,l^1} \mathbf{A}_{l^1,m} \tag{4.31}$$

for all $l, m \in \mathbb{N}_{n,k}$ and all $s, t \in \mathbb{N}_n$, with $s \neq t$, in the same block of I^x.

Notice that, for any fixed s, t in the same block of I^x, equation (4.31) is the same as equation (4.20), since $A_x^{st} = A^{st}$. Hence, relationships (4.30) obtained in the proof of Theorem 4.4.1 still hold, but now the constants a_1, a_2, a_3 depend on the block. Suppose that partition I^x contains r blocks and let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^{r^2}$ be three vectors (one may also view them as $r \times r$ matrices) whose entries are free variables. Thus, if s, t are in the i-th block of I^x, then one can obtain

$$\mathbf{A}_{(t,t),(s,s)} = (\mathbf{a}_1)_{(i,i)}, \quad \mathbf{A}_{(s,t),(t,s)} = (\mathbf{a}_2)_{(i,i)}, \quad \mathbf{A}_{(t,s),(t,s)} = (\mathbf{a}_3)_{(i,i)},$$

and

$$\mathbf{A}_{(s,s),(s,s)} = (\mathbf{a}_1)_{(i,i)} + (\mathbf{a}_2)_{(i,i)} + (\mathbf{a}_3)_{(i,i)}. \tag{4.32}$$

If $s \in \mathbb{N}_n$ is in the i-th block of I^x and q is in the j-th block of I^x, then the last four relations in (4.29) imply that

$$\begin{aligned} \mathbf{A}_{(q,q),(s,s)} &= (\mathbf{a}_1)_{(j,i)}, & \mathbf{A}_{(s,s),(q,q)} &= (\mathbf{a}_1)_{(i,j)}, \\ \mathbf{A}_{(s,q),(q,s)} &= (\mathbf{a}_2)_{(i,j)}, & \mathbf{A}_{(q,s),(s,q)} &= (\mathbf{a}_2)_{(j,i)}, \\ \mathbf{A}_{(q,s),(q,s)} &= (\mathbf{a}_3)_{(j,i)}, & \mathbf{A}_{(s,q),(s,q)} &= (\mathbf{a}_3)_{(i,j)}. \end{aligned} \tag{4.33}$$

Since $\mathbf{A}$ is symmetric, we conclude that

$$(\mathbf{a}_1)_{(j,i)} = (\mathbf{a}_1)_{(i,j)} \quad \text{and} \quad (\mathbf{a}_2)_{(i,j)} = (\mathbf{a}_2)_{(j,i)}.$$

A clarification is in order. The relations in (4.29) were derived under the assumption that we can choose distinct s and t in a same block of I^x. If s is the only element in the i-th block and q is the only element in the j-th block, then there are no restrictions on the elements $\mathbf{A}_{(q,q),(s,s)}$, $\mathbf{A}_{(s,q),(q,s)}$, $\mathbf{A}_{(q,s),(q,s)}$, $\mathbf{A}_{(s,q),(s,q)}$, and $\mathbf{A}_{(s,s),(s,s)}$, so we may *assume* that (4.33) and (4.32) hold as well.

Finally, define the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^{n^2}$ as follows

$$
\begin{aligned}
&(\mathbf{x}_1)_{(s,t)} := (\mathbf{a}_1)_{(i,i)}, \quad (\mathbf{x}_1)_{(s,q)} := (\mathbf{a}_1)_{(i,j)},\\
&(\mathbf{x}_2)_{(s,t)} := (\mathbf{a}_2)_{(i,i)}, \quad (\mathbf{x}_2)_{(s,q)} := (\mathbf{a}_2)_{(i,j)},\\
&(\mathbf{x}_3)_{(s,t)} := (\mathbf{a}_3)_{(i,i)}, \quad (\mathbf{x}_3)_{(s,q)} := (\mathbf{a}_3)_{(i,j)}, \quad (\mathbf{x}_3)_{(q,s)} := (\mathbf{a}_3)_{(j,i)}
\end{aligned}
$$

for all s, t in the i-th block and q in the j-th block, where $i, j \in \mathbb{N}_r$ and $i \neq j$. The verification of (4.17) is routine.

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Chapter 5

On differentiability of a class of orthogonally invariant functions on several operator variables

5.1 Introduction

Let $\mathbb{N}_n := \{1, \ldots, n\}$. Denote by S^n the space of all $n \times n$ symmetric matrices with inner product $\langle A, B \rangle := \mathrm{Tr}\,(AB)$. Let O^n be the group of $n \times n$ orthogonal matrices. Denote by $\mathbb{R}^n_{\geq}$ the convex cone in $\mathbb{R}^n$ of all vectors with non-increasingly ordered coordinates. For any $A \in S^n$, let $\lambda(A) \in \mathbb{R}^n_{\geq}$ be the ordered vector of eigenvalues of A. Let $\mathrm{Diag}\, x$ be the $n \times n$ matrix with $x \in \mathbb{R}^n$ on the main diagonal.

Fix natural numbers $n_1, \ldots, n_k$ and assume that the k-tuples in $\mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$ are ordered lexicographically. For any function $f : \mathbb{R}^k \to \mathbb{R}$ define

$$F^H : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k} \text{ by}$$

$$F^H(A_1, \ldots, A_k) := (\otimes_{i=1}^k U_i)(\mathrm{Diag}_l\, f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k)))(\otimes_{i=1}^k U_i)^\top,$$

where $U_i \in O^{n_i}$ is such that $A_i = U_i(\operatorname{Diag} \lambda(A_i))U_i^\top$ for $i \in \mathbb{N}_k$. Here, $\operatorname{Diag}_l \mathbf{x}_l$ denotes the diagonal matrix with vector $\mathbf{x} \in \mathbb{R}^{n_1 \cdots n_k}$ on the main diagonal and $l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$.

Several properties of these functions have been studied. For example operator monotonicity and operator convexity are extensively studied in [2], [3], [6], [7], [9], [11], and [12]. In [6], the author shows that, for values $m = 1, 2$, function F^H is C^m at $(A_1, \ldots, A_k)$, if the underlying f is C^p, where $p > m + k/2$, at $(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$ for all $l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$.

To introduce the second class of functions, denote by $\mathbb{N}_{n,k}$ the set of all subsets of $\mathbb{N}_n$ of size k with elements ordered increasingly, where $1 \le k \le n$. The elements of the set $\mathbb{N}_{n,k}$ are ordered lexicographically and used to index the coordinates of vectors in $\mathbb{R}^{\binom{n}{k}}$. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a *symmetric* function, that is invariant under permutations of its arguments. Define $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ by

$$\mathbf{f}_\rho(x) := f(x_{\rho_1}, \ldots, x_{\rho_k})$$

for all $x \in \mathbb{R}^n$ and all $\rho \in \mathbb{N}_{n,k}$. Finally, let $U^{(k)}$ be the *k-th multiplicative compound matrix* of an $n \times n$ matrix U. It is known that $U^{(k)}$ is orthogonal, whenever U is, see Section 5.2 for more details. For any symmetric $f : \mathbb{R}^k \to \mathbb{R}$, define a function $F : S^n \to S^{\binom{n}{k}}$, called *(generated) k-isotropic*, by

$$F(A) := U^{(k)}(\operatorname{Diag} \mathbf{f}(\lambda(A)))(U^{(k)})^\top,$$

where $U \in O^n$ is such that $A = U(\operatorname{Diag} \lambda(A))U^\top$.

Function F is well-defined and satisfies $F(UAU^\top) = U^{(k)}F(A)(U^{(k)})^\top$ for any $U \in O^n$ and any A in the domain of F, as shown in [10].

A characterization of C^1 (generated) k-isotropic functions was obtained in [1] and that was extended in in [10] to C^m for a larger class, called *k-isotropic functions*. The (generated) k-isotropic function F is C^m at A, if and only if the underlying symmetric function f is C^m at $\lambda_\rho(A)$ for all $\rho \in \mathbb{N}_{n,k}$. That result holds for $m = 0, 1, \ldots$ Later on, [8] showed that, F is analytic at A, if and only if the underlying symmetric function f is analytic at $\lambda_\rho(A)$ for all $\rho \in \mathbb{N}_{n,k}$.

The main goal in this work is to connect F^H and F, when the underlying function f is symmetric. This allows us to characterize differentiability of F^H in terms of symmetric f, using the corresponding known properties of F. In addition, we characterize the analyticity of F^H in terms of f, not necessarily symmetric.

5.2 Main definition

5.2.1 Tensor products

Denote by $\otimes_{i=1}^k \mathbb{R}^{n_i}$ the tensor product of $\mathbb{R}^{n_i}$, $i \in \mathbb{N}_k$. This is a linear space of dimension $n_1 \cdots n_k$ consisting of formal finite linear combinations of $\{x_1 \otimes \cdots \otimes x_k : x_i \in \mathbb{R}^{n_i}, i \in \mathbb{N}_k\}$, with all necessary identifications made so that the product is multi-linear. The inner product between $u_1 \otimes \cdots \otimes u_k$ and $v_1 \otimes \cdots \otimes v_k$ in $\otimes_{i=1}^k \mathbb{R}^{n_i}$ is $\langle u_1, v_1\rangle \cdots \langle u_k, v_k\rangle$. The tensor product $A_1 \otimes \cdots \otimes A_k$, between operators A_i on $\mathbb{R}^{n_i}$, $i \in \mathbb{N}_k$, is a linear operator on $\otimes_{i=1}^k \mathbb{R}^{n_i}$ defined by

$$(A_1 \otimes \cdots \otimes A_k)(x_1 \otimes \cdots \otimes x_k) := (A_1 x_1) \otimes \cdots \otimes (A_k x_k)$$

and extended by linearity. For short introduction to tensor product and its properties, see [4, Chapter I].

Denote by $\{e_i^1, \dots, e_i^{n_i}\}$ the standard orthonormal basis in $\mathbb{R}^{n_i}$ for $i \in \mathbb{N}_k$. Let $\{\mathbf{e}^l : l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}\}$ denote the standard orthonormal basis in $\mathbb{R}^{n_1 \cdots n_k}$. An isometry $\mathcal{T} : \mathbb{R}^{n_1 \cdots n_k} \to \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_k}$ is defined by

$$\mathcal{T}(\mathbf{e}^l) := e_1^{l_1} \otimes \cdots \otimes e_k^{l_k} \quad \text{for all } l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}.$$

Given any $n_i \times n_i$ matrix A_i for all $i \in \mathbb{N}_k$ and any $\mathbf{x} \in \mathbb{R}^{n_1 \cdots n_k}$, we have

$$\mathcal{T}((\otimes_{i=1}^k A_i)\mathbf{x}) = (\otimes_{i=1}^k A_i)(\mathcal{T}\mathbf{x}),$$

where on the right-hand side, A_i is viewed as an operator on $\mathbb{R}^{n_i}$ with respect to the standard basis for all $i \in \mathbb{N}_k$.

For any $A_i \in S^{n_i}$ for all $i \in \mathbb{N}_k$, the self-adjoint operator corresponding to the symmetric matrix $F^H(A_1, \ldots, A_k)$ is

$$\begin{aligned}\mathcal{F}^H(A_1, \ldots, A_k) :=& \mathcal{T} \circ F^H(A_1, \ldots, A_k) \circ \mathcal{T}^{-1} \\ =& \sum_{l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))(\otimes_{i=1}^k u_i^{l_i}) \otimes (\otimes_{i=1}^k u_i^{l_i}),\end{aligned}$$

where $U_i \in O^{n_i}$ is such that $A_i = U_i(\operatorname{Diag} \lambda(A_i))U_i^\top$ and $u_i^{l_i}$ denotes the l_i-th column of U_i for all $i \in \mathbb{N}_k$.

5.2.2 Anti-symmetric tensor products

The k-tuples in $\mathbb{N}_{n,k}$ are ordered lexicographically and used to index the coordinates of vectors in $\mathbb{R}^{\binom{n}{k}}$ and matrices of dimension $\binom{n}{k} \times \binom{n}{k}$. For example, $\mathbf{x}_\rho$ is the ρ-th coordinate of a vector $\mathbf{x}$ in $\mathbb{R}^{\binom{n}{k}}$ and $\mathbf{A}_{\rho,\tau}$ is the (ρ, τ)-th element of an $\binom{n}{k} \times \binom{n}{k}$ matrix $\mathbf{A}$. But if $x \in \mathbb{R}^n$, then let $x_\rho := (x_{\rho_1}, \ldots, x_{\rho_k}) \in \mathbb{R}^k$ for any $\rho \in \mathbb{N}_{n,k}$ and if A is an $n \times n$ matrix, let $A_{\rho\tau}$ (without a comma) be the $k \times k$ minor of an A with elements at the intersections of rows $\rho_1, \ldots, \rho_k$ and columns $\tau_1, \ldots, \tau_k$ for any $\rho, \tau \in \mathbb{N}_{n,k}$.

The *k-th multiplicative compound matrix* of $n \times n$ matrix A is an $\binom{n}{k} \times \binom{n}{k}$ matrix, denoted by $A^{(k)}$, such that $A^{(k)}_{\rho,\tau} := \det(A_{\rho\tau})$ for any $\rho, \tau \in \mathbb{N}_{n,k}$. For properties of k-th multiplicative compound matrix, see for example [5].

For any vectors $x_1, \ldots, x_k \in \mathbb{R}^n$, their *k-th anti-symmetric tensor product* (*wedge product*) is defined by

$$x_1 \wedge \cdots \wedge x_k := \frac{1}{\sqrt{k!}} \sum_{\sigma: \mathbb{N}_k \to \mathbb{N}_k} \epsilon_\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

where the summation is over all permutations σ on $\mathbb{N}_k$ and ϵ_σ is defined to be $+1$, if σ is even and to be -1, if σ is odd. The wedge product is multi-linear and anti-commutative. Denote by $\wedge^k\mathbb{R}^n$ the $\binom{n}{k}$-dimensional subspace of $\otimes^k\mathbb{R}^n$ spanned by all k-th anti-symmetric tensor products with inherited inner product

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k\rangle = \det\left(\langle u_i, v_j\rangle_{i,j=1}^k\right).$$

If A is an operator on $\mathbb{R}^n$, then $\otimes^k A$ keeps the subspace $\wedge^k\mathbb{R}^n$ invariant. Denote by $\wedge^k A$ the restriction of $\otimes^k A$ onto $\wedge^k\mathbb{R}^n$. It is called the *k-th anti-symmetric tensor power (wedge power) of A* and satisfies

$$(\wedge^k A)(x_1 \wedge \cdots \wedge x_k) = (Ax_1) \wedge \cdots \wedge (Ax_k).$$

For properties of k-th wedge power of A, see [5].

Denote by $\{\mathbf{e}^\rho : \rho \in \mathbb{N}_{n,k}\}$ the standard orthonormal basis in $\mathbb{R}^{\binom{n}{k}}$. An isometry $\mathcal{W} : \mathbb{R}^{\binom{n}{k}} \to \wedge^k\mathbb{R}^n$ is defined by

$$\mathcal{W}(\mathbf{e}^\rho) := e^{\rho_1} \wedge \cdots \wedge e^{\rho_k} \text{ for all } \rho \in \mathbb{N}_{n,k}$$

and extended by linearity. The relationship between $A^{(k)}$ and $\wedge^k A$ is:

$$\mathcal{W}(A^{(k)}\mathbf{x}) = (\wedge^k A)(\mathcal{W}\mathbf{x}) \quad \text{for any } \mathbf{x} \in \mathbb{R}^{\binom{n}{k}},$$

where A is viewed as an operator and a matrix with respect to the standard basis.

For future reference, the self-adjoint operator on $\wedge^k\mathbb{R}^n$, corresponding to the symmetric matrix $F(A)$, is

$$\mathcal{F}(A) := \mathcal{W} \circ F(A) \circ \mathcal{W}^{-1} = \sum_{\rho\in\mathbb{N}_{n,k}} f(\lambda_\rho(A))(u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}) \otimes (u_{\rho_1} \wedge \cdots \wedge u_{\rho_k}),$$

where $\{u_1, \ldots, u_n\}$ are the columns of $U \in O^n$, such that $A = U(\operatorname{Diag}\lambda(A))U^\top$.

5.2.3 Operator functions on k variables

We are ready to introduce a class of operator functions on several variables by restricting (generated) k-isotropic functions to block-diagonal matrices. Henceforth, we assume that $n = n_1 + \cdots + n_k$.

Let $F^* : S^{n_1} \times \cdots \times S^{n_k} \to S^{\binom{n}{k}}$ be defined by

$$F^*(A_1, \dots, A_k) := F(A_1 \oplus \cdots \oplus A_k),$$

where $A_1 \oplus \cdots \oplus A_k$ denotes the block-diagonal matrix with blocks A_i, $i \in \mathbb{N}_k$. The corresponding self-adjoint operator is

$$\mathcal{F}^*(A_1, \dots, A_k) := \mathcal{W} \circ F(A_1 \oplus \cdots \oplus A_k) \circ \mathcal{W}^{-1}.$$

5.2.4 Note about domains

We assume that the domain of the symmetric function $f : \mathbb{R}^k \to \mathbb{R}$, denoted by $\operatorname{dom} f \subseteq \mathbb{R}^k$, is a symmetric and open set. Then, it is easy to see that the set $\operatorname{dom}_n f := \{x \in \mathbb{R}^n : x_\rho \in \operatorname{dom} f \text{ for all } \rho \in \mathbb{N}_{n,k}\}$ is also symmetric and open. Then, the domain of a (generated) k-isotropic function $F : S^n \to S^{\binom{n}{k}}$ corresponding to $f : \mathbb{R}^k \to \mathbb{R}$ is

$$\operatorname{dom} F := \{A \in S^n : \lambda(A) \in \operatorname{dom}_n f\}.$$

It is not too difficult to see that for any $l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$, there is a $\rho \in \mathbb{N}_{n,k}$, such that $\lambda_\rho(A_1 \oplus \cdots \oplus A_k)$ is a permutation of $(\lambda_{l_1}(A_1), \dots, \lambda_{l_k}(A_k))$. Since the set $\operatorname{dom} f$ is symmetric, we see that $A_1 \oplus \cdots \oplus A_k \in \operatorname{dom} F$ implies that $(A_1, \dots, A_k) \in \operatorname{dom} F^H$. Hence, the domain of F^* is the set of all k-tuples $(A_1, \dots, A_k)$ from $S^{n_1} \times \cdots \times S^{n_k}$ that satisfy $A_1 \oplus \cdots \oplus A_k \in \operatorname{dom} F$, and this set is sufficient for our needs.

5.3 Connecting F^H to F^*

5.3.1 Introducing the linear map Π

We introduce a linear map Π that links the operator functions $\mathcal{F}^H$ and $\mathcal{F}^*$, whenever both of them are defined in terms of the same symmetric function $f : \mathbb{R}^k \to \mathbb{R}$.

Let the linear map $\Pi_i : \mathbb{R}^{n_i} \to \oplus_{j=1}^k \mathbb{R}^{n_j}$ for $i \in \mathbb{N}_k$, be the embedding of $\mathbb{R}^{n_i}$ in $\oplus_{j=1}^k \mathbb{R}^{n_j}$:

$$\Pi_i(u) := 0 \oplus \cdots \oplus u \oplus \cdots \oplus 0 \quad \text{for any } u \in \mathbb{R}^{n_i},$$

where u appears in the i-th place of the direct sum.

Let $\Pi : \otimes_{j=1}^k \mathbb{R}^{n_j} \to \wedge^k(\oplus_{j=1}^k \mathbb{R}^{n_j})$ be a linear map defined by

$$\Pi(e_1^{l_1} \otimes \cdots \otimes e_k^{l_k}) := \Pi_1(e_1^{l_1}) \wedge \cdots \wedge \Pi_k(e_k^{l_k}) = (e_1^{l_1} \oplus \cdots \oplus 0) \wedge \cdots \wedge (0 \oplus \cdots \oplus e_k^{l_k}).$$

It can be extended by linearity to any vector in $u_1 \otimes \cdots \otimes u_k \in \otimes_{i=1}^k \mathbb{R}^{n_i}$:

$$\Pi(u_1 \otimes \cdots \otimes u_k) = \Pi_1(u_1) \wedge \cdots \wedge \Pi_k(u_k). \tag{5.1}$$

Next, we show that Π preserves the inner product.

Lemma 5.3.1 *For any $s_1, \ldots, s_k \in \mathbb{N}_k$ with $s_1 \le \cdots \le s_k$, let $u_j \in \mathbb{R}^{n_j}$ and $v_j \in \mathbb{R}^{n_{s_j}}$ for $j \in \mathbb{N}_k$. Define $\mathbf{u} := \Pi_1(u_1) \wedge \cdots \wedge \Pi_k(u_k)$ and $\mathbf{v} := \Pi_{s_1}(v_1) \wedge \cdots \wedge \Pi_{s_k}(v_k)$. The inner product between $\mathbf{u}$ and $\mathbf{v}$ is given by*

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{cases} \prod_{j=1}^k \langle u_j, v_j \rangle & \text{if } s_1, \ldots, s_k \text{ are distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, Π preserves the inner product.

Proof **Case I.** If all $s_1, \ldots, s_k$ are distinct, then $s_i = i$ for $i \in \mathbb{N}_k$ and we have

$$\mathbf{v} = \Pi_1(v_1) \wedge \cdots \wedge \Pi_k(v_k)$$

with $v_i \in \mathbb{R}^{n_i}$ for $i \in \mathbb{N}_k$. Calculate $\langle \mathbf{u}, \mathbf{v} \rangle$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \det\left(\langle \Pi_i(u_i), \Pi_j(v_j) \rangle_{i,j=1}^k\right) = \prod_{j=1}^k \langle u_j, v_j \rangle,$$

since $\langle \Pi_i(u_i), \Pi_j(v_j) \rangle = 0$, whenever $i \neq j$.

Case II. Suppose now that $s_1, \ldots, s_k$ are not distinct. There exists a $j \in \mathbb{N}_k$ such that $s_i \neq j$ for all $i \in \mathbb{N}_k$. Then, we calculate the inner product between $\mathbf{u}$ and $\mathbf{v}$ by

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= \langle \Pi_1(u_1) \wedge \cdots \wedge \Pi_{k-1}(u_{k-1}) \wedge \Pi_k(u_k), \Pi_{s_1}(v_1) \wedge \cdots \wedge \Pi_{s_{k-1}}(v_{k-1}) \wedge \Pi_{s_k}(v_k) \rangle \\
&= \langle (u_1 \oplus \cdots \oplus 0) \wedge \cdots \wedge (0 \oplus \cdots \oplus u_{k-1} \oplus 0) \wedge (0 \oplus \cdots \oplus u_k), \\
&\qquad \Pi_{s_1}(v_1) \wedge \cdots \wedge \Pi_{s_{k-1}}(v_{k-1}) \wedge \Pi_{s_k}(v_k) \rangle \\
&= \det(\langle \Pi_i(u_i), \Pi_{s_j}(v_j) \rangle_{i,j=1}^k) = 0,
\end{aligned}$$

since the j-th row of the determinant is zero.

For any $u := u_1 \otimes \cdots \otimes u_k, v := v_1 \otimes \cdots \otimes v_k \in \otimes_{i=1}^k \mathbb{R}^{n_i}$, we have

$$\langle u, v \rangle = \langle u_1 \otimes \cdots \otimes u_k, v_1 \otimes \cdots \otimes v_k \rangle = \prod_{i=1}^k \langle u_i, v_i \rangle = \langle \Pi(u), \Pi(v) \rangle,$$

hence, the linear map Π preserves the inner product.

The linear map $\Pi : \otimes_{i=1}^k \mathbb{R}^{n_i} \to \wedge^k(\oplus_{i=1}^k \mathbb{R}^{n_i})$ is an injection, so we can consider the inverse map Π^{-1}, defined on the range of Π. That is, $\Pi^{-1} \circ \Pi(u) = u$ for any $u \in \otimes_{i=1}^k \mathbb{R}^{n_i}$.

5.3.2 Connecting F^H to F^*

Let $n := \sum_{i=1}^{k} n_i$ and let $A := A_1 \oplus \cdots \oplus A_k \in S^n$, where $A_i \in S^{n_i}$ for all $i \in \mathbb{N}_k$. Let $U_i \in O^{n_i}$ be such that $A_i = U_i(\mathrm{Diag}\,\lambda(A_i))U_i^\top$ for all $i \in \mathbb{N}_k$. Denote by $u_i^{l_i}$ the l_i-th column of U_i and $u_i^{l_i}$ is the eigenvector corresponding to $\lambda_{l_i}(A_i)$ for all $i \in \mathbb{N}_k$. Matrix A is diagonalized by

$$A = (U_1 \oplus \cdots \oplus U_k)(\mathrm{Diag}\,\tilde{\lambda}(A))(U_1 \oplus \cdots \oplus U_k)^\top, \tag{5.2}$$

where $\tilde{\lambda}(A) := (\lambda(A_1), \ldots, \lambda(A_k))$ is not necessarily ordered. For any $t \in \mathbb{N}_n$, the t-th column of $U_1 \oplus \cdots \oplus U_k$ is denoted by $\tilde{u}_t$. For any such t, there exist unique $i \in \mathbb{N}_k$ and $l_i \in \mathbb{N}_{n_i}$, such that $t = \sum_{j=1}^{i-1} n_j + l_i$ and

$$\tilde{u}_t = \Pi_i(u_i^{l_i}). \tag{5.3}$$

This notation allows us to obtain the following representation of $\mathcal{F}^*$.

Recall that any symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ defines a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ by $\mathbf{f}_\rho(x) := f(x_\rho)$ for all x in the domain of f and all $\rho \in \mathbb{N}_{n,k}$. Such function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ is symmetric in the sense of

$$\mathrm{Diag}\,\mathbf{f}(Px) = P^{(k)}(\mathrm{Diag}\,\mathbf{f}(x))P^{(k)\top} \tag{5.4}$$

for all $x \in \mathbb{R}^n$ and all $n \times n$ permutation matrix P, see [1].

Proposition 5.3.2 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric. For any $A := A_1 \oplus \cdots \oplus A_k$ with $A_i \in S^{n_i}$ for $i \in \mathbb{N}_k$, we have*

$$\mathcal{F}^*(A_1, \ldots, A_k) = \sum_{\rho \in \mathbb{N}_{n,k}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k}) \otimes (\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k}).$$

Proof Let P be an $n \times n$ permutation matrix, such that $\lambda(A) = P\tilde{\lambda}(A)$. Using (5.2), we obtain

$$\begin{aligned} A &= (U_1 \oplus \cdots \oplus U_k)(\operatorname{Diag} \tilde{\lambda}(A))(U_1 \oplus \cdots \oplus U_k)^\top \\ &= (U_1 \oplus \cdots \oplus U_k)(\operatorname{Diag} P^\top \lambda(A))(U_1 \oplus \cdots \oplus U_k)^\top \\ &= (U_1 \oplus \cdots \oplus U_k)P^\top(\operatorname{Diag} \lambda(A))P(U_1 \oplus \cdots \oplus U_k)^\top. \end{aligned}$$

Let $U := (U_1 \oplus \cdots \oplus U_k)P^\top$ and let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}}$ be defined by $\mathbf{f}_\rho(x) := f(x_\rho)$ for all $\rho \in \mathbb{N}_{n,k}$ and all $x \in \mathbb{R}^n$. Such $\mathbf{f}$ is symmetric in the sense of (5.4). Then, we have

$$\begin{aligned} F^*(A_1, \ldots, A_k) = F(A) = U^{(k)}(\operatorname{Diag} \mathbf{f}(\lambda(A)))U^{(k)\top} &= U^{(k)}(\operatorname{Diag} \mathbf{f}(P\tilde{\lambda}(A)))U^{(k)\top} \\ = U^{(k)}P^{(k)}(\operatorname{Diag} \mathbf{f}(\tilde{\lambda}(A)))P^{(k)\top}U^{(k)\top} &= (UP)^{(k)}(\operatorname{Diag} \mathbf{f}(\tilde{\lambda}(A)))(UP)^{(k)\top} \\ = (U_1 \oplus \cdots \oplus U_k)^{(k)}(\operatorname{Diag} \mathbf{f}(\tilde{\lambda}(A)))&(U_1 \oplus \cdots \oplus U_k)^{(k)\top}, \end{aligned}$$

where we used (5.4). The rest follows.

Denote by $\mathcal{M}$ the collection of all $\rho \in \mathbb{N}_{n,k}$ satisfying

$$\rho_i \in \{(n_1 + \cdots + n_{i-1}) + 1, \ldots, (n_1 + \cdots + n_{i-1}) + n_i\} \quad \text{for all } i \in \mathbb{N}_k$$

and let $\mathcal{M}^c := \mathbb{N}_{n,k} \setminus \mathcal{M}$. Define the operator $\mathcal{F}^*_{\mathcal{M}}$ by

$$\mathcal{F}^*_{\mathcal{M}}(A_1, \ldots, A_k) := \sum_{\rho \in \mathcal{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k}) \otimes (\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k})$$

and let $\mathcal{F}^*_{\mathcal{M}^c}(A_1, \ldots, A_k) := \mathcal{F}^*(A_1, \ldots, A_k) - \mathcal{F}^*_{\mathcal{M}}(A_1, \ldots, A_k)$.

The relationship between $\mathcal{F}^H$ and $\mathcal{F}^*$ is given in the next theorem.

Theorem 5.3.3 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric and consider the corresponding operators $\mathcal{F}^H$,*

$\mathcal{F}^*$, and $\mathcal{F}^*_{\mathcal{M}}$. Then, for any $A_i \in S^{n_i}$, $i \in \mathbb{N}_k$, the following diagram commutes

$$\begin{tikzcd} \otimes_{i=1}^k \mathbb{R}^{n_i} \arrow[r, "\mathcal{F}^H(A_1,\ldots,A_k)"] \arrow[d, "\Pi"] & \otimes_{i=1}^k \mathbb{R}^{n_i} \arrow[d, "\Pi"] \\ \wedge^k(\oplus_{i=1}^k \mathbb{R}^{n_i}) \arrow[r, "\mathcal{F}^*(A_1,\ldots,A_k)"] & \wedge^k(\oplus_{i=1}^k \mathbb{R}^{n_i}) \end{tikzcd}$$

Moreover, the operators $\mathcal{F}^*(A_1,\ldots,A_k)$ and $\mathcal{F}^*_{\mathcal{M}}(A_1,\ldots,A_k)$ coincide on subspace $\Pi(\otimes_{i=1}^k \mathbb{R}^{n_i})$.

Proof We need to show that for any $v := v_1 \otimes \cdots \otimes v_k \in \otimes_{i=1}^k \mathbb{R}^{n_i}$, we have

$$\Pi \circ \mathcal{F}^H(A_1,\ldots,A_k)(v) = \mathcal{F}^*(A_1,\ldots,A_k) \circ \Pi(v). \tag{5.5}$$

For the right-hand side, we have

$$\mathcal{F}^*(A_1,\ldots,A_k) \circ \Pi(v) = \mathcal{F}^*_{\mathcal{M}}(A_1,\ldots,A_k) \circ \Pi(v) + \mathcal{F}^*_{\mathcal{M}^c}(A_1,\ldots,A_k) \circ \Pi(v). \tag{5.6}$$

Note that the elements ρ of $\mathcal{M}$ have the property that for any $\rho_i \in \rho$, there exists a unique $l_i \in \mathbb{N}_{n_i}$, such that $\rho_i = \sum_{j=1}^{i-1} n_j + l_i$. Thus, using (5.3) and (5.1), we obtain

$$\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k} = \Pi_1(u_1^{l_1}) \wedge \cdots \wedge \Pi_k(u_k^{l_k}) = \Pi(u_1^{l_1} \otimes \cdots \otimes u_k^{l_k}). \tag{5.7}$$

With that we express the first term on the right-hand side of (5.6) as

$$\begin{aligned}
&\mathcal{F}^*_{\mathcal{M}}(A_1,\ldots,A_k) \circ \Pi(v) \\
&= \sum_{\rho\in\mathcal{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k}) \otimes (\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k})\big(\Pi_1(v_1) \wedge \cdots \wedge \Pi_k(v_k)\big) \\
&= \sum_{\rho\in\mathcal{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k})\langle \tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k}, \Pi_1(v_1) \wedge \cdots \wedge \Pi_k(v_k)\rangle \\
&= \sum_{\rho\in\mathcal{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k})\langle \Pi_1(u_1^{l_1}) \wedge \cdots \wedge \Pi_k(u_k^{l_k}), \Pi_1(v_1) \wedge \cdots \wedge \Pi_k(v_k)\rangle \\
&= \sum_{\rho\in\mathcal{M}} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k}) \prod_{j=1}^k \langle u_j^{l_j}, v_j\rangle
\end{aligned}$$

$$= \sum_{l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k)) \prod_{j=1}^{k} \langle u_j^{l_j}, v_j \rangle \Pi(u_1^{l_1} \otimes \cdots \otimes u_k^{l_k}),$$

where in the last three equalities we used (5.7) and Lemma 5.3.1.

We now turn our attention to the second term on the right-hand side of (5.6). Note that the elements ρ of $\mathcal{M}^c$ have the property that for any $\rho_i \in \rho$, there exists unique $s_i \in \mathbb{N}_k$ and $l_i \in \mathbb{N}_{n_{s_i}}$, such that $\rho_i = \sum_{j=1}^{s_i-1} n_j + l_i$. The important observation is that the indexes $s_1, \ldots, s_k$ are not distinct and

$$\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k} = \Pi_{s_1}(u_{s_1}^{l_1}) \wedge \cdots \wedge \Pi_{s_k}(u_{s_k}^{l_k}).$$

With that, we calculate $\mathcal{F}^*_{\mathcal{M}^c}(A_1, \ldots, A_k) \circ \Pi(v)$ by

$$\begin{aligned}
&\mathcal{F}^*_{\mathcal{M}^c}(A_1, \ldots, A_k) \circ \Pi(v) \\
&= \sum_{\rho \in \mathcal{M}^c} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k}) \otimes (\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k})(\Pi_1(v_1) \wedge \cdots \wedge \Pi_k(v_k)) \\
&= \sum_{\rho \in \mathcal{M}^c} f(\tilde{\lambda}_\rho(A))(\tilde{u}_{\rho_1} \wedge \cdots \wedge \tilde{u}_{\rho_k}) \langle \Pi_{s_1}(u_{s_1}^{l_1}) \wedge \cdots \wedge \Pi_{s_k}(u_{s_k}^{l_k}), \Pi_1(v_1) \wedge \cdots \wedge \Pi_k(v_k) \rangle \\
&= 0,
\end{aligned}$$

where the last equality is obtained using Lemma 5.3.1.

Combining the results for the two terms on the right-hand side of (5.6), gives

$$\mathcal{F}^*(A_1, \ldots, A_k) \circ \Pi(v) = \sum_{l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k)) \prod_{j=1}^{k} \langle u_j^{l_j}, v_j \rangle \Pi(u_1^{l_1} \otimes \cdots \otimes u_k^{l_k}).$$

This also shows that $\mathcal{F}^*(A_1, \ldots, A_k)$ and $\mathcal{F}^*_{\mathcal{M}}(A_1, \ldots, A_k)$ are the same map when restricted to the subspace $\Pi(\otimes_{i=1}^k \mathbb{R}^{n_i})$.

For the left-hand side of (5.5), we have

$$\Pi \circ \mathcal{F}^H(A_1, \ldots, A_k)(v)$$

$$
\begin{aligned}
&= \Pi \circ \sum_{l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))(\otimes_{i=1}^k u_i^{l_i}) \otimes (\otimes_{i=1}^k u_i^{l_i})(v_1 \otimes \cdots \otimes v_k) \\
&= \Pi \circ \sum_{l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))\langle \otimes_{i=1}^k u_i^{l_i}, v_1 \otimes \cdots \otimes v_k \rangle (\otimes_{i=1}^k u_i^{l_i}) \\
&= \Pi \circ \sum_{l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k)) \prod_{i=1}^k \langle u_i^{l_i}, v_j \rangle (u_1^{l_1} \otimes \cdots \otimes u_k^{l_k}) \\
&= \sum_{l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}} f(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k)) \prod_{i=1}^k \langle u_i^{l_i}, v_j \rangle \Pi(u_1^{l_1} \otimes \cdots \otimes u_k^{l_k}).
\end{aligned}
$$

This shows that the diagram commutes.

Theorem 5.3.3 shows that

$$
\begin{aligned}
\mathcal{F}^H(A_1, \ldots, A_k) &= \Pi^{-1} \circ \mathcal{F}^*(A_1, \ldots, A_k) \circ \Pi \\
&= \Pi^{-1} \circ \mathcal{F}(A_1 \oplus \cdots \oplus A_k) \circ \Pi \quad \text{and} \\
\mathcal{F}(A_1 \oplus \cdots \oplus A_k) &= \Pi \circ \mathcal{F}^H(A_1, \ldots, A_k) \circ \Pi^{-1},
\end{aligned}
\tag{5.8}
$$

where both sides of the last equality are assumed to be restricted to $\Pi(\otimes_{i=1}^k \mathbb{R}^{n_i})$.

Thus, one can use (5.8) to infer properties of $\mathcal{F}^H$ from those of $\mathcal{F}$.

5.4 Differentiability properties of F^H

In this section, we study the differentiability properties of F^H. We start with those associated with a symmetric function $f : \mathbb{R}^k \to \mathbb{R}$. The following is a special case of Theorem 5.1 in [10], which was proven for the more general k-isotropic functions.

Theorem 5.4.1 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric with corresponding (generated) k-isotropic function $F : S^n \to S^{\binom{n}{k}}$. Then, F is C^m at A, if and only if f is C^m at $\lambda_\rho(A)$ for any $\rho \in \mathbb{N}_{n,k}$. Here, $m = 0, 1, \ldots$*

Theorem 5.4.1, together with (5.8), allows us to see the following corollary.

Corollary 5.4.2 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric with corresponding $F^H : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$. The function F^H is C^m at $(A_1, \ldots, A_k)$, whenever f is C^m at $\lambda_\rho(A_1 \oplus \cdots \oplus A_k)$ for any $\rho \in \mathbb{N}_{n,k}$. Here, $m = 0, 1, \ldots$*

Proof Suppose that f is C^m at $\lambda_\rho(A_1 \oplus \cdots \oplus A_k)$ for any $\rho \in \mathbb{N}_{n,k}$. Using Theorem 5.4.1, one can obtain that the corresponding (generated) k-isotropic function is C^m at $A_1 \oplus \cdots \oplus A_k$. Using (5.8), the corresponding F^H is C^m at $(A_1, \ldots, A_k)$.

Restricting F^H to diagonal matrices, we get the following converse.

Corollary 5.4.3 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be symmetric with corresponding $F^H : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$. The function f is C^m at $(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$ for any $l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$, whenever F^H is C^m at $(A_1, \ldots, A_k)$. Here, $m = 0, 1, \ldots$*

An inductive formula for the first and higher-order derivatives of k-isotropic functions was the focus of [10]. A formula for just the first derivative of (generated) k-isotropic functions was obtained in [1]. Thus, at least in theory, one can obtain the formula for the derivatives of F^H using (5.8).

Now, address the analyticity of F^H. Here, the symmetricity of $f : \mathbb{R}^k \to \mathbb{R}$ is not necessary.

Theorem 5.4.4 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a function with corresponding $F^H : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_1 \cdots n_k}$. The function F^H is analytic at $(A_1, \ldots, A_k)$, if and only if f is analytic at $(\lambda_{l_1}(A_1), \ldots, \lambda_{l_k}(A_k))$ for all $l \in \mathbb{N}_{n_1} \times \cdots \times \mathbb{N}_{n_k}$.*

Proof Suppose $f : \mathbb{R}^k \to \mathbb{R}$ is analytic. Then, the Cauchy integral representation of f holds as follows

$$f(x_1, \ldots, x_k) = \frac{1}{(2\pi i)^k} \oint_{\Gamma_k} \cdots \oint_{\Gamma_1} \frac{f(z_1, \ldots, z_k)}{\prod_{j=1}^k (z_j - x_j)} \, dz_1 \cdots dz_k,$$

where Γ_j is a positively oriented circle in the complex plane enclosing the points x_j for all $j \in \mathbb{N}_k$. The Dunford-Taylor integral representation of $F^H(A_1, \ldots, A_k)$ for any $A_j \in S^{n_j}$, $j \in \mathbb{N}_k$

is

$$
\begin{aligned}
F^H(A_1,\dots,A_k) &= (\otimes_{i=1}^k U_i)(\mathrm{Diag}_l\, f(\lambda_{l_1}(A_1),\dots,\lambda_{l_k}(A_k)))(\otimes_{i=1}^k U_i)^\top \\
&= (\otimes_{j=1}^k U_j)\Big(\mathrm{Diag}_l\, \frac{1}{(2\pi i)^k}\oint_{\Gamma_k}\cdots\oint_{\Gamma_1} \frac{f(z_1,\dots,z_k)}{\prod_{j=1}^k (z_j-\lambda_{l_j}(A_j))}\, dz_1\cdots dz_k\Big)(\otimes_{j=1}^k U_j)^\top \\
&= \frac{1}{(2\pi i)^k}\oint_{\Gamma_k}\cdots\oint_{\Gamma_1} f(z_1,\dots,z_k)(\otimes_{j=1}^k U_j)\Big(\mathrm{Diag}_l \prod_{j=1}^k (z_j-\lambda_{l_j}(A_j))^{-1}\Big)(\otimes_{j=1}^k U_j)^\top\, dz_1\cdots dz_k,
\end{aligned}
$$

where $U_j \in O^{n_j}$ is such that $A_j = U_j(\mathrm{Diag}\,\lambda(A_j))U_j^\top$ and Γ_j is a positively oriented circle in the complex plane enclosing all eigenvalues $\{\lambda_{l_j}(A_j) : l_j \in \mathbb{N}_{n_j}\}$ for all $j \in \mathbb{N}_k$. Notice that

$$
(\otimes_{j=1}^k U_j)\Big(\mathrm{Diag}_l \prod_{j=1}^k (z_j-\lambda_{l_j}(A_j))^{-1}\Big)(\otimes_{j=1}^k U_j)^\top = (z_1 I - A_1)^{-1}\otimes\cdots\otimes(z_k I - A_k)^{-1},
$$

holds. Thus, we have the integral representation

$$
F^H(A_1,\dots,A_k) = \frac{1}{(2\pi i)^k}\oint_{\Gamma_k}\cdots\oint_{\Gamma_1} f(z_1,\dots,z_k)\big((z_1 I - A_1)^{-1}\otimes\cdots\otimes(z_k I - A_k)^{-1}\big)\, dz_1\cdots dz_k.
$$

Since the eigenvalue map $A_j \mapsto \lambda(A_j)$ is a continuous function, the circle Γ_j encloses the eigenvalues of all matrices in a small neighbourhood of A_j for all $j \in \mathbb{N}_k$. It is easy to see then, that for each fixed $(z_1,\dots,z_k)$, the integrand is analytic in $(A_1,\dots,A_k)$, and so is F^H.

For the other direction, restrict the function F^H to diagonal matrices.

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Chapter 6

Conclusion and further extensions of research

6.1 Contributions

In this thesis, we study the orthogonally invariant functions, called k-isotropic functions. This class of functions captures previously well-studied classes of functions, for example, spectral, primary, and tensor-valued isotropic functions. We extend the notion of operator monotonicity and analyticity.

We emphasize that the apparent differences in operator monotonicity and analyticity are explained in this thesis by characterizing such properties of (generated) k-isotropic functions. The first differential is used to characterize the operator monotonicity with a decomposition represented by a series of divided difference matrices. This allows us to also decompose the first differentials of spectral and primary matrix functions to connect the their results. To characterized the analyticity of (generated) k-isotropic functions, we lift the underlying simpler function to a higher dimension so that we can use Dunford-Taylor integral to represent the k-isotropic function.

Then, we consider another orthogonally invariant class of functions, k-tensor isotropic functions that captures primary and tensor-valued isotropic functions. The k-isotropic functions can be obtained when we restrict sub-class of k-tensor isotropic functions to the space spanned by all k-th anti-symmetric tensor products. The representation is found when $k = 2$ in the spirit of the result in [8].

Finally, we connect (generated) k-isotropic functions to orthogonally invariant functions F^H studied in [1], [2], [4], [5], [6], [9], and [10], when the underlying function f is symmetric. We characterize differentiability of F^H in terms of symmetric f, using the corresponding known properties of (generated) k-isotropic functions. Characterization of the analyticity of F^H in terms of f is obtained, where f is not necessarily symmetric.

6.2 Further extensions of research

There are limitations and imperfections in the results we obtained, which shows potential research directions. We list them as follows.

1. Characterization of operator monotonicity of all k-isotropic functions is unknown. Notice that the results of operator monotonicity are obtained when we consider a sub-class of k-isotropic functions. We may study characterization of operator monotonicity of tensor-valued isotropic functions first and then extend the result to k-isotropic functions.

2. Operator monotonicity can be extended to operator monotonicity on several variables naturally. That is, we say that $F : S^{n_1} \times \cdots \times S^{n_k} \to S^m$ is operator monotone, if $A_i \geq B_i, i \in \mathbb{N}_k$ implies $F(A_1, \ldots, A_k) \geq F(B_1, \ldots, B_k)$ for any $A_i, B_i \in S^{n_i}, i \in \mathbb{N}_k$. Korányi proposed a different notion of operator monotonicity on two variables, see [6] and Hansen proposed another notion of operator monotonicity on several variables that extends Theorem V.2.3 in [3], see [5]. The connections among different notions of operator monotonicity are worth studying.

3. The characterization of operator convexity of k-isotropic functions is unknown. One

should notice that operator monotonicity and operator convexity are closely related, see [3, Chapter V]. There are first-order condition and second-order condition to characterize operator convexity. For k-isotropic functions, we may prefer to calculate the second differential to study the characterization. Then, we would be able to generalize the Lieb's inequality

$$\wedge^k (A+B)^{1/k} \geq \wedge^k A^{1/k} + \wedge^k B^{1/k}$$

for all $A, B \in S^n_+$, see [7].

4. Characterization of analyticity of all k-isotropic functions is unknown. Also, notice that the result for analyticity is obtained when we consider (generated) k-isotropic functions. We may study the characterization of analyticity of tensor-valued isotropic functions first and then extend the result to k-isotropic functions.

5. For k-tensor isotropic functions, the representation when $k > 2$ is unknown. Properties of differentiability and analyticity of k-tensor isotropic functions are also worth studying.

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