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Gauss-Bonnet-Chern type theorem for the noncommutative four-sphere

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Abstract

We introduce a pseudo-Riemannian calculus of modules over noncommutative algebras in order to investigate to what extent the differential geometry of classical Riemannian manifolds can be extended to a noncommutative setting. In this framework, it is possible to prove an analogue of the Levi-Civita theorem. It states that there exists at most one connection, which satisfies torsion-free condition and metric compatibility condition, on a given smooth manifold with fixed metric. More significantly, the corresponding curvature operator has the same symmetry properties as the classical curvature tensors. We consider a pseudo-Riemannian calculus over the noncommutative 3-sphere and the noncommutative 4-sphere and explicitly determine the torsion-free and metric compatible connection, and we compute its scalar curvature. In the case of the noncommutative 4-sphere, we compute the scalar curvature of conformal perturbations of the round metric by localizing the algebra of noncommutative 4-sphere, which allows us to formulate and prove a Gauss-Bonnet-Chern type theorem. For the case of the noncommutative 4-torus, the Pfaffian of the curvature form for a conformal class of the flat metric is computed.

Keywords: Curvature, Gauss-Bonnet-Chern theorem, Levi-Civita connection, noncommutative 4-sphere, noncommutative geometry, noncommutative 3-sphere, noncommutative toric manifolds, pseudo-Riemannian calculus.

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To my family and friends.

Contents

Introduction

Over the last few years, there has been increasing interest in understanding the curvature of noncommutative manifolds. Starting from seminal work on the scalar curvature and Gauss-Bonnet type theorems for the noncommutative torus initiated by A. Connes, F. Fathizadeh, M. Khalkhali, H. Moscovici and P. Tretkoff [12, 17, 25], many interesting papers that discuss different aspects of curvature in the noncommutative setting have followed [1, 6, 19, 20, 23, 26, 27, 29, 30, 35]. Note that these are only examples of recent progress in the area; several authors have previously considered the curvature in this context [3, 7, 8, 13, 22, 31].

In a related approach, the formalism of curvature based on a Levi-Civita type connection on a finitely generated projective module was initiated by J. Rosenberg [40] where an analogue of the Levi-Civita theorem was proved. This aspect of noncommutative geometry was originally considered by A. Connes in [15]. In this paper Connes gives the definition of compatibility with a metric. However, the notion of torsion for a connection, and a Levi-Civita type theorem are due to J. Rosenberg [40].

Although connections on projective modules and their corresponding curvatures are natural objects in noncommutative geometry, classical objects that are built from the curvature tensor, like Ricci and scalar curvature, do not always have straightforward analogues. Therefore, it is interesting to study as to what extent such concepts are relevant for noncommutative geometry.

For Riemannian manifolds, the Gauss-Bonnet-Chern theorem provides an important link between geometry and topology. It states that the integral of the Pfaffian of the curvature form of a closed even dimensional oriented Riemannian manifold is proportional to its Euler characteristic, which is a topological invariant. For a two dimensional manifold, the Pfaffian is simply the Gaussian curvature, which reduces the Gauss-Bonnet-Chern theorem to the Gauss-Bonnet theorem. Therefore, to understand similar theorems for two dimensional noncommutative manifolds, one needs to find a suitable definition of the scalar curvature. For a Riemannian manifold, the asymptotic expansion of the heat kernel contains information about the scalar curvature in one of the coefficients. The expansion of the heat kernel makes sense even for a noncommutative manifold, and the very same coefficient serves as a definition of noncommutative scalar curvature. For the noncommutative torus, the scalar curvature corresponding to certain perturbations of the flat metric has been computed, and it is possible to show that a Gauss-Bonnet type theorem holds; that is, the trace of the scalar curvature is independent of the metric perturbation [17, 25]. However, for higher dimensional manifolds, it is not clear how to define an analogue of the Pfaffian of the curvature form in order to formulate the Gauss-Bonnet-Chern theorem.

In this dissertation we construct a differential calculus over the noncommutative 3-sphere, the noncommutative 4-sphere and the noncommutative 4-torus, in the framework of pseudo-Riemannian calculi [5, 6], and introduce projective modules that are in close analogy with the space of vector fields on the classical 3-sphere, 4-sphere and 4-torus. Pseudo-Riemannian calculus provides a sufficiently general framework to apply to essentially any noncommutative geometry. In particular, it allowed the computation of the scalar curvature in [6]. Moreover, via a suitable localization of the algebra, we find a local trivialization of the projective module and prove the existence of unique metric and torsion-free connections for a class of perturbations of the round metric for the noncommutative 4-sphere in [5]. Finally, we show that in this particular case, there exists a naive analogue of the Pfaffian of the curvature form, which allows us to prove a Gauss-Bonnet-Chern type theorem. Similarly, this construction applies to the noncommutative 4-torus and the naive analogue of the Pfaffian also exists.

This dissertation is organized as follows: Chapter 1 recalls the development of pseudo-Riemannian calculus and associated curvature operators in a general setting. Chapter 2 reviews explicit constructions of noncommutative spaces and geometries as examples. All noncommutative manifolds for our main results come from the noncommutative toric manifolds approach in Section 2.3, but the other two sections in Chapter 2 present also other mainstream constructions of noncommutative spaces and geometries. Chapter 3 is devoted to the study of the noncommutative 3-sphere, developing a suitable pseudo-Riemannian calculus that has enough resemblance with the classical vector fields for the ordinary 3-sphere. The scalar curvature à la Levi-Civita type connection is also computed in this chapter. Chapter 4 and Chapter 5 are our main results, in which a Gauss-Bonnet-Chern type theorem is proved through pseudo-Riemannian calculus for the noncommutative 4-sphere and the Pfaffian of the curvature form is computed for the noncommutative 4-torus, respectively.

Chapter 1

Pseudo-Riemannian calculus

In this chapter we construct a differential calculus over algebras in order to investigate as to what extent the calculus of classical Riemannian manifolds can be extended to a noncommutative setting. To achieve this, we introduce pseudo-Riemannian calculus on modules over noncommutative algebras, which is best suited for the study of geometric properties of the noncommutative 3-sphere and the noncommutative 4-sphere [5, 6]. In the framework of pseudo-Riemannian calculus, it is possible to prove an analogue of the Levi-Civita theorem, which states that there is at most one connection of Levi-Civita type for a given metric and to prove a Gauss-Bonnet-Chern theorem, which states that the analogue of the Euler characteristic of the noncommutative 4-sphere defined as the trace of a polynomial in the curvature tensor is a constant. More significantly, the corresponding curvature operator has the same symmetry properties as the classical Riemannian curvature. The metric torsion-free connection for the round metric and curvature tensors will be explicitly computed in the following chapters for the case of the noncommutative 3-sphere, the noncommutative 4-sphere and the noncommutative 4-torus. Moreover, an analogue of Gauss-Bonnet-Chern theorem will be proved for the noncommutative 4-sphere and the noncommutative 4-torus.

1.1 Pseudo-Riemannian calculus

In this section we fix notations that will be used throughout the rest of this dissertation.

Definition 1.1. Let A be an algebra over $\mathbb{C}$. A is called a $*$-algebra if it is endowed with an involution map $* : A \to A$, $a \mapsto a^*$ such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$

(i) $(a^*)^* = a$

(ii) $(a + b)^* = a^* + b^*$

(iii) $(ab)^* = b^* a^*$

(iv) $(\lambda a)^* = \bar{\lambda} a^*$.

If a $*$-algebra A is a complete normed space with respect to the metric $\|\cdot\|$ such that

(i) $\|ab\| \leq \|a\|\|b\|$

(ii) $\|a^*\| = \|a\|$

(iii) $\|aa^*\| = \|a\|^2$,

then A is called a C*-algebra. Over a $*$-algebra, $a \in A$ is called hermitian or self-adjoint if $a = a^*$, $z \in A$ is called normal if $zz^* = z^*z$. Furthermore, u is called unitary if A is a unital $*$-algebra with the unit 1 and $uu^* = u^*u = 1$.

From this point on, A denotes a unital $*$-algebra over $\mathbb{C}$ with the centre $Z(A)$ and, unless otherwise specified, all modules are right modules in this section. The set of all derivations of A is denoted by $\operatorname{Der}(A) := \{\delta \in \operatorname{End}(A) : \delta(ab) = \delta(a)b + a\delta(b) \ \ \forall\ a, b \in A\}$, and the adjoint δ^* of a derivation $\delta \in \operatorname{Der}(A)$ is defined by $\delta^*(a) = (\delta(a^*))^*$ for all $a \in A$. A derivation is called hermitian if $\delta^* = \delta$.

Note that unlike the commutative case, $\operatorname{Der}(A)$ is not an A-module in general. However, it is a module over the centre $Z(A)$. For instance if $ab \neq ba$ and $\delta(c) \neq 0$ for some $a, b, c \in A$ and $\delta \in \operatorname{Der}(A)$, then $a\delta(bc) = a\delta(b)c + ab\delta(c) \neq a\delta(b)c + ba\delta(c)$. Thus, $a\delta \notin \operatorname{Der}(A)$.

Although our definitions of metric and connection work for more general modules, we will be considering only finitely generated projective right A-modules in our examples. This is due to the Serre-Swan correspondence, which states that the category of vector bundles over a compact Hausdorff space X and the category of finitely generated projective A-modules, where $A = C(X)$, are in a one-to-one correspondence. In particular, a finitely generated free A-module $A^{\oplus n}$ admits a canonical generator given by

$$e_j = (0, \ldots, 0, \mathbb{1}, 0, \ldots, 0), \; j = 1, \ldots n$$

with the only nonzero element in the jth place. An element $U \in A^{\oplus n}$ can be written uniquely as $U = e_j U^j$, where the summation is assumed, for some elements $U^1, \ldots, U^n \in A$.

In the sequels, we make definitions attributed to [5, 6], which are reasonable generalizations of [40].

Definition 1.2. Let M be a finitely generated right A-module. A map $g : M \times M \to A$ is called a hermitian form on M if for all $U, V, W \in M$ and $a \in A$,

$$g(U, V + W) = g(U, V) + g(U, W)$$
$$g(U, Va) = g(U, V)a$$
$$g(U, V)^* = g(V, U).$$

A hermitian form is called *non-degenerate* if $g(U, V) = 0$ for all $V \in M$ implies $U = 0$. A metric on M is defined to be a non-degenerate hermitian form. The pair (M, g), where M is a right A-module and g is a hermitian form on M, is called a *right hermitian A-module.* If g is a metric, then (M, g) is called a *metric A-module*

Remark. In fact, if A is a C*-algebra Definition 1.2 is the definition given for pre-Hilbert C*-modules.

Now we introduce the definition of affine connections on a right A-module, which is suitable for the setting of this dissertation.

Definition 1.3. Let M be a right A-module and let $\mathfrak{g} \subseteq \mathrm{Der}(A)$ be a real Lie algebra of hermitian derivations. An affine connection on $(M, \mathfrak{g})$ is a map $\nabla : \mathfrak{g} \times M \to M$ such that

(i) $\nabla_\partial(U + V) = \nabla_\partial U + \nabla_\partial V$,

(ii) $\nabla_{\lambda\partial+\partial'} U = \lambda\nabla_\partial U + \nabla_{\partial'} U$ and

(iii) $\nabla_\partial(Ua) = (\nabla_\partial U)\, a + U\partial(a)$,

for all $U, V \in M$, $\partial, \partial' \in \mathfrak{g}$, $a \in A$ and $\lambda \in \mathbb{R}$.

Remark. Since we are considering affine connections with respect to a Lie subalgebra of $\mathrm{Der}(A)$, it does not make sense in general to impose $\nabla_{c\partial} U = c\nabla_\partial U$ for hermitian $c \in Z(A)$, since $\mathfrak{g}$ need not be closed under the left multiplication by $Z(A)$. However, the examples we consider will satisfy $\nabla_{c\partial} U = c\nabla_\partial U$ whenever $\partial, c\partial \in \mathfrak{g}$. In fact, this is a general statement that follows from Koszul's formula (1.5) as soon as φ in Definition 1.4 below is linear over $Z(A)$ in the above sense.

In differential geometry, every derivation on $C^\infty(M)$ corresponds to a unique vector field on the manifold M and vice versa. Hence, the elements $\partial \in \mathrm{Der}(C^\infty(M))$ and $U \in C^\infty(TM)$ in the definition of connection $\nabla_\partial U$ are interchangeable, which leads to a one-to-one correspondence $\nabla_\partial U \leftrightarrow \nabla_U \partial$ since there is a one-to-one correspondence between derivations and vector fields. More significantly, this makes the classical definition of torsion

$$T(U, V) = \nabla_U V - \nabla_V U - [U, V]$$

meaningful from an algebraic viewpoint. In a derivation based differential calculus over a noncommutative algebra (see for example, [21]), the arguments of a connection are fundamentally different. As an aforementioned example, the set of derivations does not form a module over a noncommutative algebra in general. Thus, there is no natural way to associate an element of the module to an arbitrary derivation.

This dissertation will investigate the consequences of introducing a correspondence, which assigns a unique element of a module to every derivation in a Lie algebra $\mathfrak{g} \subseteq \mathrm{Der}(A)$. This idea is formalized in the following definition.

Definition 1.4. Let (M, g) be a metric A-module, $\mathfrak{g} \subseteq \mathrm{Der}\,(A)$ a real Lie algebra of hermitian derivations and $\varphi : \mathfrak{g} \to M$ an $\mathbb{R}$-linear map. If the triple $(M, g, \mathfrak{g}_\varphi)$ where $\mathfrak{g}_\varphi$ denotes the pair $(\mathfrak{g}, \varphi)$ satisfies the following conditions:

(i) the image $M_\varphi = \varphi(\mathfrak{g})$ generates M as an A-module and

(ii) $g\,(E, E')^* = g\,(E, E')$ for all $E, E' \in M_\varphi$.

Then, the triple $(M, g, \mathfrak{g}_\varphi)$ is called a *real metric calculus*.

Classically, the condition that the elements in the image of φ have hermitian inner products corresponds to the fact that the metric is real, and that the inner product of two real vector fields, is again a real function. An important consequence of this assumption is that g is symmetric on the image of φ, that is, $g(E, E') = g(E', E)$ for all $E, E' \in M_\varphi$; this is a fact that will repeatedly be used in the sequel.

In our setting, a connection will be constructed on a real metric calculus, and we impose the following condition that the connection preserves the hermitian condition on M_φ.

Definition 1.5. Let $(M, g, \mathfrak{g}_\varphi)$ be a real metric calculus. If ∇ is an affine connection on $(M, \mathfrak{g})$ that satisfies

$$g(\nabla_\partial E, E') = g(\nabla_\partial E, E')^*$$

for all $E, E' \in M_\varphi$ and $\partial \in \mathfrak{g}$, then $(M, g, \mathfrak{g}_\varphi, \nabla)$ is called a *real connection calculus.*

For a real connection calculus it is straightforward to introduce the concept of a metric and torsion-free connection.

Definition 1.6. Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be a real connection calculus over M. The calculus is called *metric* if

$$\partial\left(g(U,V)\right) = g\left(\nabla_{\partial}U, V\right) + g\left(U, \nabla_{\partial}V\right)$$

for all $\partial \in \mathfrak{g}$, $U, V \in M$. It is called *torsion-free* if its torsion

$$T\left(\partial_1, \partial_2\right) := \nabla_{\partial_1}\varphi(\partial_2) - \nabla_{\partial_2}\varphi(\partial_1) - \varphi\left([\partial_1, \partial_2]\right) = 0$$

for all $\partial_1, \partial_2 \in \mathfrak{g}$. A torsion-free, metric and real connection calculus over M is called a *pseudo-Riemannian calculus over M.*

Pseudo-Riemannian calculi will be the main objects of interest to us because they provide a framework in which one may carry out computations in close analogy with classical Riemannian geometry.

The Levi-Civita theorem in Riemannian geometry states that there is a unique torsion-free and metric connection on the tangent bundle of a Riemannian manifold. In the current situation, the existence of such connection is not guaranteed but it can be proved that there exists at most one connection, which is both metric and torsion-free.

Theorem 1.7. *Let $(M, h, \mathfrak{g}_{\varphi})$ be a real metric calculus over M. Then there exists at most one affine connection ∇ on $(M, \mathfrak{g})$, such that $(M, h, \mathfrak{g}_{\varphi}, \nabla)$ is a pseudo-Riemannian calculus.*

Proof. Suppose ∇ and $\tilde{\nabla}$ are two connections such that $(M, g, \mathfrak{g}_{\varphi}, \nabla)$ and $(M, g, \mathfrak{g}_{\varphi}, \tilde{\nabla})$ are pseudo-Riemannian calculi, respectively. Let

$$\alpha(\partial, U) = \tilde{\nabla}_{\partial}U - \nabla_{\partial}U,$$

which implies

$$\alpha(\partial, Ua) = \left(\tilde{\nabla}_{\partial}U\right)a + U\partial a - \left(\nabla_{\partial}U\right)a - U\partial a = \alpha(\partial, U)a,$$

for $a \in A$ and

$$\alpha(\partial_1 + \lambda\partial_2, U) = \alpha(\partial_1, U) + \lambda\alpha(\partial_2, U)$$
$$\alpha(\partial, U + V) = \alpha(\partial, U) + \alpha(\partial, V),$$

for $\lambda \in \mathbb{R}$. By subtracting, the conditions that ∇ and $\tilde{\nabla}$ are metric implies

$$g\left(\alpha(\partial, U), V\right) = -g\left(U, \alpha(\partial, V)\right), \tag{1.1}$$

and the torsion-free condition implies

$$\alpha\left(\partial_1, \varphi(\partial_2)\right) = \alpha\left(\partial_2, \varphi(\partial_1)\right) \tag{1.2}$$

for $\partial_1, \partial_2 \in \mathfrak{g}$. The assumption that $g(\nabla_{\partial_1}\varphi(\partial_2), \varphi(\partial_3))$ and $g(\tilde{\nabla}_{\partial_1}\varphi(\partial_2), \varphi(\partial_3))$ are hermitian gives

$$g\left(\alpha(\partial_1, \varphi(\partial_2)), \varphi(\partial_3)\right)^* = g\left(\alpha(\partial_1, \varphi(\partial_2)), \varphi(\partial_3)\right). \tag{1.3}$$

By (1.1) and (1.2),

$$\begin{aligned} g\left(\alpha(\partial_1, E_2), E_3\right) &= g\left(\alpha(\partial_2, E_1), E_3\right) = -g\left(E_1, \alpha(\partial_2, E_3)\right) = -g\left(E_1, \alpha(\partial_3, E_2)\right) \\ &= g\left(\alpha(\partial_3, E_1), E_2\right) = g\left(\alpha(\partial_1, E_3), E_2\right) = -g\left(E_3, \alpha(\partial_1, E_2)\right), \end{aligned}$$

where $E_a = \varphi(\partial_a)$ for $a = 1, 2, 3$. This shows that

$$g\left(\alpha(\partial_1, \varphi(\partial_2)), \varphi(\partial_3)\right)^* = -g\left(\alpha(\partial_1, \varphi(\partial_2)), \varphi(\partial_3)\right). \tag{1.4}$$

Combining (1.3) and (1.4) yields

$$g\left(\alpha(\partial_1, \varphi(\partial_2)), \varphi(\partial_3)\right) = 0,$$

for all $\partial_1, \partial_2, \partial_3 \in \mathfrak{g}$. Since the image of φ generates M and g is non-degenerate,

$\alpha(\partial, U) = 0$ for all $U \in M$ and $\partial \in \mathfrak{g}$, which shows that

$$\tilde{\nabla}_\partial U = \nabla_\partial U$$

for all $\partial \in \mathfrak{g}$ and $U \in M$. □

In classical Riemannian geometry, the Levi-Civita connection can be constructed explicitly using Koszul's formula, which expresses the connection in terms of the metric tensor. For pseudo-Riemannian calculi, a corresponding formula also exists.

Proposition 1.8. *Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be a pseudo-Riemannian calculus and $\partial_1, \partial_2, \partial_3 \in \mathfrak{g}$. Then,*

$$\begin{aligned} 2g(\nabla_{\partial_1} E_2, E_3) &= \partial_1 g(E_2, E_3) + \partial_2 g(E_1, E_3) - \partial_3 g(E_1, E_2) \\ &\quad - g\left(E_1, \varphi([\partial_2, \partial_3])\right) + g\left(E_2, \varphi([\partial_3, \partial_1])\right) + g\left(E_3, \varphi([\partial_1, \partial_2])\right), \end{aligned} \tag{1.5}$$

where $E_a = \varphi(\partial_a)$ for $a = 1, 2, 3$.

Proof. Using the metric condition of ∇,

$$\partial_1 g(E_2, E_3) = g\left(\nabla_{\partial_1} E_2, E_3\right) + g\left(E_2, \nabla_{\partial_1} E_3\right) \tag{1.6}$$

$$\partial_2 g(E_3, E_1) = g\left(\nabla_{\partial_2} E_3, E_1\right) + g\left(E_3, \nabla_{\partial_2} E_1\right) \tag{1.7}$$

$$\partial_3 g(E_1, E_2) = g\left(\nabla_{\partial_3} E_1, E_2\right) + g\left(E_1, \nabla_{\partial_3} E_2\right), \tag{1.8}$$

and the torsion-free condition,

$$\begin{aligned} g(E_3, \nabla_{\partial_2} E_1) &= g(E_3, \nabla_{\partial_1} E_2) + g\left(E_3, \varphi([\partial_2, \partial_1])\right) \\ g(\nabla_{\partial_3} E_1, E_2) &= g(\nabla_{\partial_1} E_3, E_2) + g\left(\varphi([\partial_3, \partial_1]), E_2\right) \\ g(E_1, \nabla_{\partial_3} E_2) &= g(E_1, \nabla_{\partial_2} E_3) + g\left(E_1, \varphi([\partial_3, \partial_2])\right). \end{aligned}$$

Moreover, the fact that the connection is real enables us to rewrite the above equations in the following form

$$g(E_3, \nabla_{\partial_2} E_1) = g(\nabla_{\partial_1} E_2, E_3) + g\left(E_3, \varphi([\partial_2, \partial_1])\right) \tag{1.9}$$

$$g(\nabla_{\partial_3} E_1, E_2) = g(E_2, \nabla_{\partial_1} E_3) + g\left(\varphi([\partial_3, \partial_1]), E_2\right) \tag{1.10}$$

$$g(E_1, \nabla_{\partial_3} E_2) = g(\nabla_{\partial_2} E_3, E_1) + g\left(E_1, \varphi([\partial_3, \partial_2])\right). \tag{1.11}$$

Inserting (1.9) in (1.7), and (1.10) and (1.11) in (1.8) gives (together with (1.6))

$$\begin{aligned} g(\nabla_{\partial_1} E_2, E_3) &= \partial_1 g(E_2, E_3) - g(E_2, \nabla_{\partial_1} E_3) \\ g(\nabla_{\partial_1} E_2, E_3) &= \partial_2 g(E_3, E_1) - g(\nabla_{\partial_2} E_3, E_1) - g\left(E_3, \varphi([\partial_2, \partial_1])\right) \\ 0 &= -\partial_3 g(E_1, E_2) + g(E_2, \nabla_{\partial_1} E_3) + g\left(\varphi([\partial_3, \partial_1]), E_2\right) \\ &\quad + g(\nabla_{\partial_2} E_3, E_1) + g\left(E_1, \varphi([\partial_3, \partial_2])\right), \end{aligned}$$

and summing these three equations yields

$$\begin{aligned} 2g(\nabla_{\partial_1} E_2, E_3) &= \partial_1 g(E_2, E_3) + \partial_2 g(E_3, E_1) - \partial_3 g(E_1, E_2) \\ &\quad - g\left(E_3, \varphi([\partial_2, \partial_1])\right) + g\left(\varphi([\partial_3, \partial_1]), E_2\right) + g\left(E_1, \varphi([\partial_3, \partial_2])\right), \end{aligned}$$

which proves (1.5). □

Remark. Note that Proposition 1.8 gives an independent proof of the fact that the connection is unique, since the hermitian form g is assumed to be nondegenerate.

Now, let us show the converse of Proposition 1.8. That is, a connection satisfying (1.5) gives a pseudo-Riemannian calculus.

Proposition 1.9. *Let $(M, g, \mathfrak{g}_\varphi)$ be a real metric calculus, and ∇ be an affine connection on $(M, \mathfrak{g})$ such that Koszul's formula (1.5) holds. Then $(M, g, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus.*

Proof. From equation (1.5) it follows immediately that $g(\nabla_{\partial_1}\varphi(\partial_2), \varphi(\partial_3))$ is hermitian since every term in the right hand side is hermitian, due to the fact that

$(M, g, \mathfrak{g}_\varphi)$ is assumed to be a real metric calculus. This implies that $(M, g, \mathfrak{g}_\varphi, \nabla)$ is a real connection calculus. We now show that the connection is metric.

Let $\partial_1, \partial_2, \partial_3 \in \mathfrak{g}$ and $E_a = \varphi(\partial_a)$. Using equation (1.5) twice and the fact that $(M, g, \mathfrak{g}_\varphi)$ is a real metric calculus gives

$$g(\nabla_{\partial_1} E_2, E_3) + g(E_2, \nabla_{\partial_1} E_3) = \partial_1 g(E_2, E_3).$$

Since the image M_φ generates M, there exists a set $\{E_a = \varphi(\partial_a)\}_{a=1}^N$ that generates M and every $U \in M$ can be written as $U = E_a U^a$. It then follows that

$$\begin{aligned}
&g(\nabla_\partial U, V) + g(U, \nabla_\partial V) \\
&= g\left((\nabla_\partial E_a)U^a + E_a \partial U^a, E_b V^b\right) + g\left(E_a U^a, (\nabla_\partial E_b)V^b + E_b \partial V^b\right) \\
&= (U^a)^* \left(g(\nabla_\partial E_a, E_b) + g(E_a, \nabla_\partial E_b)\right) V^b + \partial (U^a)^* g(E_a, E_b) V^b + (U^a)^* g(E_a, E_b) \partial V^b \\
&= (U^a)^* \partial g(E_a, E_b) V^b + \partial (U^a)^* g(E_a, E_b) V^b + (U^a)^* g(E_a, E_b) \partial V^b \\
&= \partial \left((U^a)^* h(E_a, E_b) V^b\right) = \partial g(U, V),
\end{aligned}$$

which shows that the affine connection is metric. Finally, let us show that the connection is torsion-free. For $\partial_1, \partial_2, \partial_3 \in \mathfrak{g}$, with $E_a = \varphi(\partial_a)$, consider

$$T = g\left(\nabla_{\partial_1} E_2 - \nabla_{\partial_2} E_1 - \varphi([\partial_1, \partial_2]), E_3\right).$$

By using formula (1.5) for the first two terms, one obtains

$$T = g\left(E_3, \varphi([\partial_1, \partial_2])\right) - g\left(\varphi([\partial_1, \partial_2]), E_3\right) = 0.$$

Since the image M_φ of φ generates M one can conclude that

$$g\left(\nabla_{\partial_1} E_2 - \nabla_{\partial_2} E_1 - \varphi([\partial_1, \partial_2]), U\right) = 0$$

for all $U \in M$, which implies

$$\nabla_{\partial_1} E_2 - \nabla_{\partial_2} E_1 - \varphi([\partial_1, \partial_2]) = 0,$$

since g is nondegenerate. □

In particular examples, it is possible to use Koszul's formula to construct a metric and torsion-free connection. One of the cases, which is relevant to our examples, is when M is a free module. Free modules enable us to compute the values of components of connections in an unambiguous manner.

Corollary 1.10. *Let $(M, h, \mathfrak{g}_\varphi)$ be a real metric calculus and let $\{\partial_1, \ldots, \partial_n\}$ be a basis of $\mathfrak{g}$ such that $\{E_a = \varphi(\partial_a)\}_{a=1}^n$ generates M as a free module. If there exist $U_{ab} \in M$, $1 \leq a, b \leq n$ such that*

$$\begin{aligned} 2g(U_{ab}, E_c) =& \partial_a g(E_b, E_c) + \partial_b g(E_a, E_c) - \partial_c g(E_a, E_b) \\ & - g\left(E_a, \varphi([\partial_b, \partial_c])\right) + g\left(E_b, \varphi([\partial_c, \partial_a])\right) + g\left(E_c, \varphi([\partial_a, \partial_b])\right) \end{aligned} \tag{1.12}$$

for $a, b, c = 1, \ldots, n$, then there exists a connection ∇, given by $\nabla_{\partial_a} E_b = U_{ab}$, such that $(M, g, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus.

Proof. Assuming that such elements $U_{ab} \in M$ exist, define

$$\nabla_{\partial_a} E_b = U_{ab}$$

and extend ∇ to $\mathfrak{g}$ by linearity. Since $\{E_a\}_{a=1}^n$ generates M as an A-module, every element $U \in M$ has a unique expression $U = E_a U^a$, and we extend ∇ to M by linearity and Leibniz's rule

$$\nabla_\partial U = \left(\nabla_\partial E_a\right) U^a + E_a \partial(U^a),$$

which then defines an affine connection on $(M, \mathfrak{g})$. From Proposition 1.9 it follows that $(M, g, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus. □

1.2 Curvature of pseudo-Riemannian calculi

This section is devoted to a study of symmetries of the curvature tensor of a pseudo-Riemannian calculus and its associated scalar curvature. The curvature tensors are defined for the metric and torsion-free connection, as in classical geometry. It turns out that in order to recover the full symmetry of the curvature tensor as expected from the classical geometry, a hermitian condition need be assumed. Namely, although a real connection calculus satisfies the requirement that $g(\nabla_{\partial_1}E_1, E_2)$ is hermitian, in general, $g(\nabla_{\partial_1}\nabla_{\partial_2}E_1, E_2)$ need not be hermitian. However, this mild assumption proves to be powerful in the sense that this assumption enables us to prove all the familiar symmetries of the curvature tensor hold (cf. Proposition 1.15). Pseudo-Riemannian calculi fulfilling this extra condition will appear often in the sequel, and therefore, we make the following definition.

Definition 1.11. A pseudo-Riemannian calculus $(M, g, \mathfrak{g}_\varphi, \nabla)$ is said to be *real* if $g(\nabla_{\partial_1}\nabla_{\partial_2}E_1, E_2)$ is hermitian for all $\partial_1, \partial_2 \in \mathfrak{g}$ and $E_1, E_2 \in M_\varphi$.

For later convenience, let us provide a slight reformulation of the condition in the definition above.

Lemma 1.12. *Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be a pseudo-Riemannian calculus. Then the following statements are equivalent:*

(i) $g(\nabla_{\partial_1}\nabla_{\partial_2}E_1, E_2)$ *is hermitian for all* $\partial_1, \partial_2 \in \mathfrak{g}$ *and* $E_1, E_2 \in M_\varphi$,

(ii) $g(\nabla_{\partial_1}E_1, \nabla_{\partial_2}E_2)$ *is hermitian for all* $\partial_1, \partial_2 \in \mathfrak{g}$ *and* $E_1, E_2 \in M_\varphi$.

Proof. Since the connection is metric, one may write

$$\partial_2 g(\nabla_{\partial_1}E_1, E_2) = g(\nabla_{\partial_2}\nabla_{\partial_1}E_1, E_2) + g(\nabla_{\partial_1}E_1, \nabla_{\partial_2}E_2).$$

Now, $\partial_2 g(\nabla_{\partial_1}E_1, E_2)$ is hermitian (since ∇ is real and ∂_2 is hermitian), and it follows that if one of $g(\nabla_{\partial_2}\nabla_{\partial_1}E_1, E_2)$ and $g(\nabla_{\partial_1}E_1, \nabla_{\partial_2}E_2)$ is hermitian, then the other one is also hermitian (since it is then a sum of two hermitian elements). □

For a pseudo-Riemannian calculus $(M, g, \mathfrak{g}_\varphi, \nabla)$, there is a definition of the curvature operator in a natural way as follows.

$$R(\partial_1, \partial_2)U = \nabla_{\partial_1}\nabla_{\partial_2}U - \nabla_{\partial_2}\nabla_{\partial_1}U - \nabla_{[\partial_1,\partial_2]}U$$

for $\partial_1, \partial_2 \in \mathfrak{g}$ and $U \in M$. The operator $R(\partial_1, \partial_2)$ has a trivial antisymmetry when exchanging its arguments ∂_1, ∂_2 and, furthermore, due to the torsion-free condition, the first Bianchi identity holds.

Proposition 1.13. *Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be a pseudo-Riemannian calculus with curvature operator R. Then*

(i) $g(U, R(\partial_1, \partial_2)V) = -g(U, R(\partial_2, \partial_1)V)$

(ii) $R(\partial_1, \partial_2)\varphi(\partial_3) + R(\partial_2, \partial_3)\varphi(\partial_1) + R(\partial_3, \partial_1)\varphi(\partial_2) = 0,$

for $U, V \in M$ and $\partial_1, \partial_2, \partial_3 \in \mathfrak{g}$.

Proof. Property (i) follows immediately from the definition of the curvature operator. To prove (ii), one uses the torsion-free condition twice (set $E_a = \varphi(\partial_a)$):

$$\begin{aligned}
&R(\partial_1, \partial_2)E_3 + R(\partial_2, d_3)E_1 + R(\partial_3, \partial_1)E_2 \\
&\quad= \nabla_{\partial_1}\left(\nabla_{\partial_2}E_3 - \nabla_{\partial_3}E_2\right) + \nabla_{\partial_2}\left(\nabla_{\partial_3}E_1 - \nabla_{\partial_1}E_3\right) + \nabla_{\partial_3}\left(\nabla_{\partial_1}E_2 - \nabla_{\partial_2}E_1\right) \\
&\qquad - \nabla_{[\partial_1,\partial_2]}E_3 - \nabla_{[\partial_2,\partial_3]}E_1 - \nabla_{[\partial_3,\partial_1]}E_2 \\
&\quad= \nabla_{\partial_1}\varphi\left([\partial_2, \partial_3]\right) + \nabla_{\partial_2}\varphi\left([\partial_3, \partial_1]\right) + \nabla_{\partial_3}\varphi\left([\partial_1, \partial_2]\right) \\
&\qquad - \nabla_{[\partial_1,\partial_2]}E_3 - \nabla_{[\partial_2,\partial_3]}E_1 - \nabla_{[\partial_3,\partial_1]}E_2 \\
&\quad= \varphi\left([\partial_1, [\partial_2, \partial_3]]\right) + \varphi\left([\partial_2, [\partial_3, \partial_1]]\right) + \varphi\left([\partial_3, [\partial_1, \partial_2]]\right) = 0,
\end{aligned}$$

where the last equality follows from the Jacobi identity, and the fact that φ is an additive map. □

As already mentioned, the full symmetry of the curvature operator is recovered in the case of *real* pseudo-Riemannian calculi. This is stated in Proposition 1.15, and in the proof we shall need the following short lemma.

Lemma 1.14. *If* $(M, g, \mathfrak{g}_\varphi, \nabla)$ *is a pseudo-Riemannian calculus, then*

$$\partial\left(g(E,E)\right) = 2g(E, \nabla_\partial E)$$

for all $\partial \in \mathfrak{g}$ *and* $E \in M_\varphi$.

Proof. Since ∇ is a metric connection

$$\partial\left(g(E,E)\right) = g\left(\nabla_\partial E, E\right) + g\left(E, \nabla_\partial E\right),$$

and, as ∇ is real, it follows that $g(E, \nabla_\partial E) = g(\nabla_\partial E, E)$, which implies that

$$\partial\left(g(E,E)\right) = 2g(E, \nabla_\partial E)$$

for all $E \in M_\varphi$ and $\partial \in \mathfrak{g}$. □

Note that, for the sake of completeness, the results of Proposition 1.13 are repeated in the formulation below.

Proposition 1.15. *Let* $(M, g, \mathfrak{g}_\varphi, \nabla)$ *be a real pseudo-Riemannian calculus, with curvature operator* R. *Then*

(a) $g(U, R(\partial_1, \partial_2)V) = -g(U, R(\partial_2, \partial_1)V)$,

(b) $g(E_1, R(\partial_1, \partial_2)E_2) = -g(E_2, R(\partial_1, \partial_2)E_1)$,

(c) $R(\partial_1, \partial_2)\varphi(\partial_3) + R(\partial_2, \partial_3)\varphi(\partial_1) + R(\partial_3, \partial_1)\varphi(\partial_2) = 0$,

(d) $g\left(\varphi(\partial_1), R(\partial_3, \partial_4)\varphi(\partial_2)\right) = g\left(\varphi(\partial_3), R(\partial_1, \partial_2)\varphi(\partial_4)\right)$,

for all $U, V \in M$, $E_1, E_2 \in M_\varphi$ *and* $\partial_1, \partial_2, \partial_3, \partial_4 \in \mathfrak{g}$.

Proof. Properties (a) and (c) are contained in the statement of Proposition 1.13, which is valid for an arbitrary pseudo-Riemannian calculus. We now show that (b) holds, by proving that $h(E, R(d_1, d_2)E) = 0$ for all $E \in M_\varphi$. By using the fact that

∇ is metric, one computes

$$
\begin{aligned}
g\left(E, R(\partial_1, \partial_2)E\right) &= g\big(E, \nabla_{\partial_1}\nabla_{\partial_2}E - \nabla_{\partial_2}\nabla_{\partial_1}E - \nabla_{[\partial_1,\partial_2]}E\big) \\
&= \partial_1 g(E, \nabla_{\partial_2}E) - \partial_2 g(E, \nabla_{\partial_1}E) - g(E, \nabla_{[\partial_1,\partial_2]}E),
\end{aligned}
$$

using the result in Lemma 1.12 (and the fact that the pseudo-Riemannian calculus is assumed to be real). Next, it follows from Lemma 1.14 that

$$
g\left(E, R(\partial_1, \partial_2)E\right) = \frac{1}{2}\partial_1\partial_2 g(E,E) - \frac{1}{2}\partial_2\partial_1 g(E,E) - \frac{1}{2}\left[\partial_1, \partial_2\right] g(E,E) = 0.
$$

Finally, we prove (d) by using (c) to write (again, $E_a = \varphi(d_a)$)

$$
\begin{aligned}
0 =& g\left(E_1, R(\partial_2, \partial_3)E_4 + R(\partial_3, \partial_4)E_2 + R(\partial_4, \partial_2)E_3\right) \\
&+ g\left(E_2, R(\partial_3, \partial_4)E_1 + R(\partial_4, \partial_1)E_3 + R(\partial_1, \partial_3)E_4\right) \\
&+ g\left(E_3, R(\partial_4, \partial_1)E_2 + R(\partial_1, \partial_2)E_4 + R(\partial_2, \partial_4)E_1\right) \\
&+ g\left(E_4, R(\partial_1, \partial_2)E_3 + R(\partial_2, \partial_3)E_1 + R(\partial_3, \partial_1)E_2\right) \\
&= 2g\left(E_1, R(\partial_4, \partial_2)E_3\right) + 2g\left(E_2, R(\partial_1, \partial_3)E_4\right),
\end{aligned}
$$

by using (b) and (a). Consequently, by using (a) once more, relation (d) follows. □

1.3 Scalar curvature

Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be a real pseudo-Riemannian calculus, and let $\{\partial_1, \ldots, \partial_n\}$ be a basis of $\mathfrak{g}$. Setting $E_a = \varphi(\partial_a)$ one introduces the components of the metric and the curvature tensor relative to this basis via

$$
\begin{aligned}
g_{ab} &= g(E_a, E_b) \\
R_{abpq} &= g\left(E_a, R(\partial_p, \partial_q)E_b\right),
\end{aligned}
$$

and we note that $(R_{abpq})^* = R_{abpq}$ (using the fact that the pseudo-Riemannian calculus is real). Proposition 1.15 implies

$$R_{abpq} = -R_{abqp}, \tag{1.13}$$

$$R_{abpq} = -R_{bapq}, \tag{1.14}$$

$$R_{abpq} = R_{pqab}, \tag{1.15}$$

$$R_{apqr} + R_{aqrp} + R_{arpq} = 0. \tag{1.16}$$

In the traditional definition of scalar curvature $S = g^{ab}g^{pq}R_{apbq}$ one makes use of the inverse of the metric to contract indices of the curvature tensor. For an arbitrary algebra, the metric g_{ab} may fail to be invertible; i.e., there does not exist g^{ab} such that $g^{ab}g_{bc} = \delta^a_c\mathbb{1}$. However, one might be in the situation where there exist $G \in A$ and $\hat{g}^{ab}$ such that

$$\hat{g}^{ab}g_{bc} = g_{cb}\hat{g}^{ba} = \delta^a_c G.$$

If G is hermitian and regular (that is, G is not a zero divisor), then g_{ab} is said to have a pseudo-inverse $(\hat{g}^{ab}, G)$.

Lemma 1.16. *If $(\hat{g}^{ab}, G)$ and $(\hat{h}^{ab}, H)$ are pseudo-inverses for (g_{ab}) then the following holds:*

(i) if $G = H$ then $\hat{g}^{ab} = \hat{h}^{ab}$,

(ii) $[g_{ab}, G] = [\hat{g}^{ab}, G] = 0$,

(iii) $\left(\hat{g}^{ab}\right)^* = \hat{g}^{ba}$,

(iv) $\hat{g}^{ab}H = G\hat{h}^{ab}$ and $H\hat{g}^{ab} = \hat{g}^{ab}G$,

(v) if $[H, \hat{g}^{ab}] = 0$ then $[G, \hat{h}^{ab}] = [H, G] = 0$.

Proof. To prove (i), suppose $(\hat{g}^{ab}, G)$ and $(\hat{h}^{ab}, G)$ are two pseudo-inverses of g. Then by assumption, $\left(\hat{g}^{ab} - \hat{h}^{ab}\right) g_{bc} = 0$, which yields

$$\left(\hat{g}^{ap} - \hat{h}^{ap}\right) G = 0$$

when multiplied $\hat{g}^{cp}$ on the right. Since G is a regular element, $\hat{g}^{ap} = \hat{g}^{ap}$.

By using the definition of pseudo-inverse, the two expression $\hat{h}^{ab} g_{bc} \hat{g}^{cp}$ can be rewritten in two ways. Namely,

$$\left(\hat{h}^{ab} g_{bc}\right) \hat{g}^{cp} = H \delta^a_c \hat{g}^{cp} = H \hat{g}^{ap}$$
$$\hat{h}^{ab} \left(g_{bc} \hat{g}^{cp}\right) = \hat{h}^{ab} G \delta^p_b = \hat{h}^{ap} G$$

proving (iv). Consider $\hat{g}^{ab} g_{bc} \hat{h}^{cp}$ for the second part of the statement. Setting $(\hat{g}, H) = (\hat{g}, G)$ in the above result immediately gives $[G, \hat{g}^{ab}] = 0$. Together with

$$g_{ab} G = g_{ap} \delta^p_b G = g_{ap} \hat{g}^{pc} g_{cb} = G \delta^c_a g_{cb} = G g_{ab}.$$

This proves (ii).

Let us consider property (v). If $[G, \hat{h}^{ab}] = 0$ then (iv) implies

$$H \hat{g}^{ab} = \hat{h}^{ab} G = G \hat{h}^{ab} = \hat{g}^{ab} H.$$

Moreover,

$$\hat{h}^{ab} G - G \hat{h}^{ab} = 0 \implies g_{ca} \hat{h}^{ab} G - g_{ca} G \hat{h}^{ab} = 0 \implies \text{(using (ii))}$$
$$g_{ca} \hat{h}^{ab} G - G g_{ca} \hat{h}^{ab} = 0 \implies (HG - GH) \delta^b_c = 0 \implies [H, G] = 0,$$

which concludes the proof of (v).

Finally, to prove (iii) one considers the hermitian conjugates of $g_{ab} \hat{g}^{bc} = G \delta^c_a$ and $\hat{g}^{ab} g_{bc} = G \delta^a_c$, which give

$$\left(\hat{g}^{bc}\right)^* g_{ba} = G \delta^c_a$$

$$g_{cb}\left(\hat{g}^{ab}\right)^* = G\delta^a_c,$$

by using $g^*_{ab} = g_{ba}$. The above equations show that if $k^{ab} = (\hat{g}^{ba})^*$ then (k^{ab}, G) is a pseudo-inverse for g_{ab}. Since $(\hat{g}^{ab}, G)$ and (k^{ab}, G) are pseudo-inverses for g_{ab}, it follows from (i) that $\hat{g}^{ab} = k^{ab} = (\hat{g}^{ba})^*$. □

Definition 1.17. Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be a real pseudo-Riemannian calculus such that g_{ab} has a pseudo-inverse $(\hat{g}^{ab}, G)$ with respect to a basis of $\mathfrak{g}$. A *scalar curvature of* $(M, g, \mathfrak{g}_\varphi, \nabla)$ *with respect to* $(\hat{g}^{ab}, G)$ is an element $S \in A$ such that

$$\hat{g}^{ab}R_{apbq}\hat{g}^{pq} = GSG.$$

Remark. Note that it is easy to show that $\hat{g}^{ab}R_{apbq}\hat{g}^{pq}$ and, hence, the scalar curvature with respect to $(\hat{g}^{ab}, G)$, is independent of the choice of basis in $\mathfrak{g}$.

Proposition 1.18. *Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be a real pseudo-Riemannian calculus, and let $(\hat{g}^{ab}, G)$ be a pseudo-inverse of g_{ab} with respect to a basis of $\mathfrak{g}$. Then there exists at most one scalar curvature of $(M, g, \mathfrak{g}_\varphi, \nabla)$ with respect to $(\hat{g}^{ab}, G)$ and, furthermore, the scalar curvature is hermitian.*

Proof. Uniqueness of the scalar curvature follows immediately from the fact that G is regular; namely,

$$GSG = GS'G \quad \Leftrightarrow \quad G(S - S')G = 0,$$

which then implies that $S = S'$ by the regularity of G.

As noted in the beginning of this section, R_{abcd} is hermitian. Furthermore, Lemma 1.16 states that $\left(\hat{g}^{ab}\right)^* = \hat{g}^{ba}$, which implies that

$$\left(\hat{g}^{ab}R_{apbq}\hat{g}^{pq}\right)^* = \hat{g}^{qp}R_{apbq}\hat{g}^{ba} = \hat{g}^{qp}R_{qbpa}\hat{g}^{ba} = \hat{g}^{ab}R_{apbq}\hat{g}^{pq},$$

by using (1.13)–(1.15). From the definition of scalar curvature, this implies that

$$GSG = (GSG)^* = GS^*G \quad \Leftrightarrow \quad G(S - S^*)G = 0.$$

Since G is assumed to be regular, it follows that $S = S^*$. □

If S is the scalar curvature with respect to a pseudo-inverse $(\hat{g}^{ab}, G)$, in which G is central, then any scalar curvature (with respect to an arbitrary pseudo-inverse) coincides with S, giving a unique hermitian scalar curvature of a real pseudo-Riemannian calculus.

Proposition 1.19. *Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be a real pseudo-Riemannian calculus with scalar curvature S with respect to $(\hat{g}^{ab}, G)$. If $G \in Z(A)$ then the scalar curvature is unique; i.e. if S' is the scalar curvature with respect to $(\hat{g'}^{ab}, G')$, then $S' = S$.*

Proof. If $G \in Z(A)$ then property (iv) of Lemma 1.16 implies that

$$G'(GSG)G' = G'\hat{g}^{ab}R_{apbq}\hat{g}^{pq}G' = G\hat{g'}^{ab}R_{apbq}\hat{g'}^{pq}G = G(G'S'G')G,$$

and since $[G, G'] = 0$ one obtains

$$GG'(S - S')G'G = 0 \implies S = S',$$

since G and G' are assumed to be regular. □

Remark. In particular, if the metric g_{ab} is invertible, then it admits a pseudo-inverse $(\hat{g}^{ab}, \mathbb{1})$ and by Proposition 1.19, there exists a unique scalar curvature of the corresponding real pseudo-Riemannian calculus.

Although our main examples are the noncommutative 3-sphere and the noncommutative 4-sphere, the constructions in [5, 6] can be applied to the noncommutative torus [40]. For instance, Section 5 in [6] discusses the construction of pseudo-Riemannian calculus on the noncommutative 2-torus, which mimics the construction of Levi-Civita type connection in [40]. Thus, our construction is more widely applicable to other noncommutative settings than the noncommutative tori.

Chapter 2

Constructions of noncommutative spaces and geometries

This chapter will discuss various explicit constructions of geometries. For instance, by a well-known theorem of Kontsevich, if a manifold M is equipped with a Poisson structure, then its algebra of smooth functions $C^\infty(M)$ admits an algebraic deformation of the pointwise multiplication into a noncommutative product [28]. This point of view in the theory of noncommutative geometry is called the deformation quantization program. There is a corresponding strict deformation quantization program in the context of operator algebras as advanced by Marc Rieffel [37]. For example, a vast generalization of noncommutative tori A_θ comes from a type of strict deformation quantization of manifolds that admit an isospectral action by $\mathbb{R}^n$, $n \geq 2$. In those approaches, the algebra of smooth functions on the smooth manifold is deformed into a noncommutative algebra.

2.1 Deformation quantization

This section is devoted to the study of deformation quantization of Poisson manifolds. The deformation quantization is a way in which the algebra of smooth functions $A = C^\infty(M)$ on a Poisson manifold $(M, \{\cdot, \cdot\})$ is quantized into a noncommutative algebra $A[\![\hbar]\!]$ of formal power series in $\hbar$ with a formal product $\star_\hbar$. More generally, the following definition can be used.

Definition 2.1. A Poisson algebra A is a commutative associative algebra with a multiplication$(a, b) \mapsto a \cdot b$, and a Lie algebra structure $(a, b) \mapsto \{a, b\}$ such that for all $a, b, c \in A$

$$\{ab, c\} = a\{b, c\} + \{a, c\}b. \tag{2.1}$$

Remark. For noncommutative algebras, Definition 2.1 still makes sense but the condition (2.1) is known to be unduly restrictive. Nevertheless, definitions of Poisson algebra for noncommutative algebras exist. See, for example, [9].

Example 2.2. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a Lie bracket $[\ ,\ ]$ and $\mathfrak{g}^* =$Hom$(\mathfrak{g}, \mathbb{R})$ its dual. Then, $A = C^\infty(\mathfrak{g}^*)$ admits a canonical Poisson structure. In fact, since $\mathfrak{g} \cong (\mathfrak{g}^*)^* \cong T^*_\lambda \mathfrak{g}^*$, $d_\lambda f$ can be viewed as an element of $\mathfrak{g}$ for all $f \in A$ and $\lambda \in \mathfrak{g}^*$. Let $[\cdot, \cdot]$ be the Lie bracket on $\mathfrak{g}$. Then one defines the Poisson structure on A by the formula*

$$\{f, g\}(\lambda) = \langle \lambda, [d_\lambda f, d_\lambda g] \rangle, \tag{2.2}$$

where $\langle \lambda, X \rangle$ is the pairing of a linear functional $\lambda \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

Definition 2.3. A formal deformation or a star product of a Poisson algebra A is defined to be a map

$$\star_\hbar : A \otimes A \longrightarrow A[\![\hbar]\!] \tag{2.3}$$

$$a \otimes b \mapsto \sum_{n=0}^{\infty} B_n(a, b)\,\hbar^n \tag{2.4}$$

that satisfies the following conditions:

(i) Each B_n is a bilinear map $A \otimes A \to A$,

(ii) associativity in the sense that $(a \star_\hbar b) \star_\hbar c = a \star_\hbar (b \star_\hbar c)$ or equivalently,

$$\sum_{m+n=l} \left[B_m\left(B_n\left(a,b\right),c\right) - B_m\left(a, B_n\left(b,c\right)\right)\right] = 0, \forall l \geq 0,$$

(iii) $B_0(a,b) = ab$ and

(iv) $\frac{1}{2}\left(B_1(a,b) - B_1(b,a)\right) = \{a,b\}$.

Note that $A[\![\hbar]\!]$ is the formal power series in $\hbar$ with coefficients in A.

Example 2.4. *Let $M = \mathbb{R}^n$ and*

$$\pi = \frac{1}{2}\pi^{jk}\frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k}, \quad \pi^{jk} = -\pi^{kj} \in \mathbb{R}$$

its Poisson structure. That is,

$$\{f,g\} = \pi^{jk}\frac{\partial f}{\partial x^j}\frac{\partial g}{\partial x^k}$$

The Moyal $\star_\hbar$-product is given by

$$f \star_\hbar g = \mu\left(\exp\left(i\frac{\hbar}{2}\pi^{jk}\frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k}\right) f \otimes g\right) \tag{2.5}$$

where $\mu(f \otimes g) = fg$.

In fact, the product $\star_\hbar$ can be extended to a product on $A[\![\hbar]\!]$ at the formal level, although, the formula for the product is formidable.

In [28], Kontsevich proved the existence of a vast class of noncommutative spaces which are deformation quantizations of Poisson manifolds. That is, every Poisson structure lifts to a deformation quantization.

Theorem 2.5 (Kontsevich)**.** *Let M be a smooth Poisson manifold and $A = C^\infty(M)$. Then, there is a star-product $\star_\hbar : A \otimes A \to A[\![\hbar]\!]$ on A.*

The proof which Kontsevich gave is constructive for the case of $M = \mathbb{R}^n$. In particular, as in Example 2.2, the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ always admits a formal deformation quantization.

2.2 Strict deformation quantization

As discussed in the previous section, the algebra $A^\infty = C^\infty(M)$ of smooth functions on a Poisson manifold with pointwise multiplication admits a deformation into a noncommutative algebra $(C^\infty(M)[\![\hbar]\!], \star_\hbar)$, which consists of formal power series in $\hbar$ over A^∞. The analytical aspect of the manifold was completely neglected in this type of deformation, which raises the question of convergence. Strict deformation quantization is a refinement of deformation quantization by considering the parameter $\hbar$ not as a formal parameter but as a real number. The $\star_\hbar$-product on A extends to elements of $A[\![\hbar]\!]$ as follows. If $f = \sum_{n=0}^\infty f_n \hbar^n, g = \sum_{m=0}^\infty g_n \hbar^m \in A[\![\hbar]\!]$, $f_n, g_n \in A^\infty$, then

$$f \star_\hbar g = \sum_{n=0}^{\infty} \left(\sum_{n=m+k+r} B_r\left(f_k, g_m\right) \right) \hbar^n \tag{2.6}$$

Note that the associativity of the product in this way is manifested by the associativity of $\star_\hbar$ on A as in Definition 2.3 (ii).

To rectify the analytical aspect, let α denote a strongly continuous action of the Abelian Lie group $V = \mathbb{R}^n$ on a locally compact manifold M, which induces an action on the algebra $A^\infty = C_0^\infty(M)$ of smooth functions vanishing at infinity ($A = C_0(M)$) and $J : V \to V$ such that $J^T = -J$. In fact, J determines a Poisson structure on M [37]. Let $\mathfrak{v}$ be the Lie algebra with a basis $\{e_j\}$ of V. Then,

$$\{f, g\} = \sum J_{jk} \alpha_{e_j}(f)\, \alpha_{e_k}(g)$$

where $\alpha_X : A^\infty \to A^\infty$ is the induced action of $\mathfrak{v}$.

Definition 2.6. $f \in A$ is defined to be smooth if the map $V \to \mathbb{R}$

$$v \mapsto \alpha_v(f)$$

is smooth with respect to $v \in V$.

Now, by Theorem 2.5, A admits a deformation quantization. Alternatively, the action by the Abelian Lie group V can be used to deform the product on A^∞ using the formula

$$f \star_\hbar g = \int\int_{V\times V} \alpha_{\hbar Jx}(f)\,\alpha_y(g)\,e^{2\pi i x^T y} dxdy.$$

This integral is known to converge for $\hbar \in I \subset \mathbb{R}$ [37] where I is an interval and $\hbar \in \mathbb{R}$ is regarded as a parameter. In particular, the product $\star_\hbar$ gives a pre C^*-algebra structure on $A^\infty[\![\hbar]\!]$ as a dense subspace of C*-algebra $A[\![\hbar]\!] = A_\hbar$ on this interval. That is,

Theorem 2.7 (Rieffel, [37])**.** *Let α, V, A and J as above. Then, $\{A_\hbar\}$ is a continuous field of C^*-algebras.*

Example 2.8. *Let $M = \mathbb{R}^2$, $V = \mathbb{R}^2$, $A = C_0(\mathbb{R}^2)$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and α be given by $\alpha_x(f) = f(v+x)$, $v \in \mathbb{R}^2$ and $x \in \mathbb{R}^2$ thereby giving the product*

$$f \star_\hbar g(v) = \int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(v+\hbar Jx)g(v+x)e^{2\pi i x^T y} dxdy$$

It is known that the linear span of $x^m y^n e^{-\frac{x^2+y^2}{2}}$ where $(x,y) \in \mathbb{R}^2$ is dense in $L^2(\mathbb{R}^2)$. Then for instance, the $\star_\hbar$-product of $f(u,v) = ue^{-\frac{u^2+v^2}{2}}$ and $g(u,v) = ve^{-\frac{u^2+v^2}{2}}$ is given by

$$\begin{aligned} ue^{-\frac{u^2+v^2}{2}} \star_\hbar ve^{-\frac{u^2+v^2}{2}} &= \int\int_{\mathbb{R}^2\times\mathbb{R}^2} (u-\hbar q)e^{-\frac{(u-\hbar q)^2+(v+\hbar p)^2}{2}}(v+x)e^{-\frac{(u-y)^2+(v+x)^2}{2}} e^{2\pi i(xp+yq)} dxdydpdq \\ &= e^{-\frac{\hbar^2 u^2+\hbar^2 v^2}{1+\hbar^2}} \frac{\hbar}{(1+\hbar^2)^3}\left(-i\hbar^3 uv - \hbar^2 v^2 - \hbar^2 u^2 + i\hbar uv + 1\right), \end{aligned}$$

which can be regarded as a convergent power series in $\hbar \in (-1,1)$.

2.3 Noncommutative toric manifolds

Broadly speaking, deformation along an isometric action by a n-torus is a special case of Rieffel's strict deformation quantization of manifolds. In this section, a slightly more general version of the theorem than [11] is discussed. [11] considers spin manifolds endowed with isometric actions of a torus. This admits a slight generalization to the case of a compact Abelian Lie group which acts isometrically on a compact Riemannian manifold M. An action $\alpha : G \times M \to M$ by a Lie group G is called smooth if it is smooth as a map of manifolds.

Theorem 2.9. *Let M be a compact Riemannian manifold endowed with a smooth action $\alpha : G \times M \longrightarrow M$ by a compact Abelian Lie group G. Then, the algebra $C^\infty(M)$ of smooth functions admits a deformation $C^\infty(M)_\theta$ endowed with an involution. Moreover, $C^\infty(M)_0 = C^\infty(M)$.*

Proof. Any $f \in C^\infty(M)$ admits an isotypic decomposition with respect to the induced action $g \cdot f(x) = f(g^{-1} \cdot x)$. That is,

$$f = \sum_{\chi \in \hat{G}} f_\chi, \quad f_\chi = \int_G f(g^{-1} \cdot x)\chi(g)dg,$$

where $\hat{G}$ is the Pontryagin dual of G and the integration is with respect to the normalized Haar measure. Let $\theta(\cdot,\cdot) : \hat{G} \wedge \hat{G} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$ be an alternating bicharacter. That is, θ is a character in each variable and $\theta(\lambda, \rho) = -\theta(\rho, \lambda)$. Then,

$$f \times_\theta g = \sum_{\chi,\psi} e^{i\theta(\chi,\psi)} f_\chi g_\psi \tag{2.7}$$

is a deformed product.

To see that the new product is associative,

$$\begin{aligned}
(f \times_\theta g) \times_\theta h &= \left(\sum_{\chi,\psi} \exp(i\theta(\chi,\psi)) f_\chi g_\psi \right) \times_\theta \left(\sum_\omega h_\omega \right) \\
&= \sum_{\omega,\rho} \exp(i\theta(\rho,\omega)) \left(\sum_{\chi+\psi=\rho} \exp(i\theta(\chi,\psi)) f_\chi g_\psi \right) h_\omega \\
&= \sum_{\omega,\chi,\psi} \exp(i\theta(\chi+\psi,\omega)) \exp(i\theta(\chi,\psi)) f_\chi g_\psi h_\omega \\
&= \sum_{\omega,\chi,\psi} \exp(i\theta(\chi,\omega)) \exp(i\theta(\psi,\omega)) \exp(i\theta(\chi,\psi)) f_\chi g_\psi h_\omega
\end{aligned}$$

while a similar calculation yields

$$f \times_\theta (g \times_\theta h) = \sum_{\omega,\chi,\psi} \exp(i\theta(\chi,\psi)) \exp(i\theta(\chi,\omega)) \exp(i\theta(\psi,\omega)) f_\chi g_\psi h_\omega \,.$$

This shows that the deformed product is associative. Moreover,

$$f^* = \sum_{\chi \in \hat{G}} \bar{f}_{\chi^{-1}} \tag{2.8}$$

defines an involution. Finally,

$$\begin{aligned}
(f \times_\theta g)^* &= \left(\sum_{\chi,\psi \in \hat{G}} \exp\left(i\theta\left(\chi,\psi\right)\right) f_\chi g_\psi \right)^* \\
&= \sum_{\chi,\psi \in \hat{G}} \exp\left(i\theta\left(\chi,\psi\right)\right)^* \left(f_\chi\right)^* \left(g_\psi\right)^* \\
&= \sum_{\chi,\psi \in \hat{G}} \exp\left(i\theta\left(\psi,\chi\right)\right) f^*_{\chi^{-1}} g^*_{\psi^{-1}} \\
&= \sum_{\chi,\psi \in \hat{G}} \exp\left(i\theta\left(\psi^{-1},\chi^{-1}\right)\right) f^*_{\chi^{-1}} g^*_{\psi^{-1}} \\
&= \sum_{\chi,\psi \in \hat{G}} \exp\left(i\theta\left(\psi,\chi\right)\right) g^*_\psi f^*_\chi = g^* \times_\theta f^*
\end{aligned}$$

□

Remark. In the literature, this algebra is often denoted by $C^\infty(M_\theta) = C^\infty(M)_\theta$. Although M_θ is a serious abuse of notation, as there is no underlying space to the algebra, thereby no such object, and the deformation takes place on the algebra, it is used consistently in literature as though such a space M_θ exists.

Example 2.10. *Let $M = \mathbb{T}^2$ and $G = \mathbb{R}^2/\mathbb{Z}^2$. Then, $\hat{G} = \mathbb{Z}^2$ and $C^\infty(M)$ is generated as an algebra by the unitaries $u_{(n,m)} := u_{(n,m)}(x,y) = e^{2\pi i(nx+my)}$. The isotypic component of the action defined by $\alpha_{(\xi_1,\xi_2)}(u_{(n,m)}) = e^{i(\xi_1 n+\xi_2 m)} u_{(n,m)}$ are precisely $u_{(n,m)}$ and the Fourier theory implies that the subspace spanned by functions $\{u_{(n,m)}\}$*

forms a dense subspace. Then, with $\theta(m,n,m'n') = mn' - m'n$,

$$u_{(n,m)} \times_\theta u_{(n',m')} = e^{i\theta(mn'-m'n)} u_{(n,m)} u_{(n',m')}, \tag{2.9}$$

which constructs the noncommutative 2-tori $\mathbb{T}^2_\theta$.

In fact, the noncommutative n*-tori* $\mathbb{T}^n_\theta$ *can be constructed, analogously.*

Example 2.11. *Let* $M = S^n$, $n \geq 3$. *Then, the isometry group of* M *is* $O(n)$, *which contains* $\mathbb{T}^{\lfloor \frac{n}{2} \rfloor}$ *as a subgroup where* $\lfloor x \rfloor$ *is the greatest integer not exceeding* $x \in \mathbb{R}$. *For instance,* $G = \mathbb{T}^2$ *acts isometrically on* S^3 *as follows. Regarded as* $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$, $\mathbb{T}^2$ *acts by* $(e^{i\zeta_1}, e^{i\zeta_2}) \cdot (z,w) = (e^{i\zeta_1} z, e^{i\zeta_2} w)$ *for all* $(z,w) \in \mathbb{C}^2$ *and* $(e^{i\zeta_1}, e^{i\zeta_2}) \in \mathbb{T}^2$. *The isotypic components of this action are constant multiples of*

$$\begin{aligned} z &:= z(\xi,\eta,\varphi) = e^{2\pi i\xi} \sin\varphi \\ w &:= w(\xi,\eta,\varphi) = e^{2\pi i\eta} \cos\varphi, \end{aligned} \tag{2.10}$$

and the products and powers of z *and* w.

For instance with $\theta(m,n,m'n') = mn' - m'n$,

$$z \times_\theta w = e^{2\pi i\theta} wz.$$

Chapter 3

Scalar curvature of the noncommutative 3-sphere

In this chapter, a version of noncommutative 3-spheres is studied as a prominent example of noncommutative spaces. We focus on the deformation of the sphere as a noncommutative toric manifold and some geometric invariants such as connections and curvatures will be computed. This is achieved via use of pseudo-Riemannian calculus, which was developed in Chapter 1.

3.1 Noncommutative 3-sphere

As discussed in Example 2.11, the noncommutative 3-sphere presents a class of the most prominent examples in noncommutative geometry besides the noncommutative tori $\mathbb{T}^2_\theta$. In fact, S^3_θ and S^4_θ were defined in [32, 33] prior to [11].

For our purposes, the noncommutative 3-sphere S^3_θ [32, 33] is a unital $*$-algebra generated by z, z^*, w, w^* subject to some relations. More precisely,

Definition 3.1. The noncommutative 3-sphere S^3_θ is defined to be the $*$-algebra generated by the normal elements z and w subject to the relation

$$\begin{aligned} zz^* &= z^*z & ww^* &= w^*w \\ wz &= e^{2\pi i\theta} zw & zz^* + ww^* &= \mathbb{1} \end{aligned} \tag{3.1}$$

In the above definition, the commutation relations between z^*w and wz^*, and the relations between zw^* and w^*z is not clear. However, we will see that these commutation relations can be obtained by the Putnam's theorem in [36].

Theorem 3.2 (Putnam 1951)**.** *Let A and B be normal operators and X be an operator on a Hilbert space such that $AX = XB$. Then, $A^*X = XB^*$.*

In fact, S^3_θ admits a representation on a Hilbert space and completion into a C*-algebra. For instance, the proof of [11] shows that the L^2-sections of the spinor bundle on S^3 would be a Hilbert space on which S^3_θ acts as bounded operators. By taking $A = z$, $B = e^{2\pi i\theta}z$ and $X = w$, it is easy to see that

$$z^*w = w\left(e^{2\pi i\theta}z\right)^* = e^{-2\pi i\theta}wz^*.$$

Similarly, taking $A = w$, $B = e^{2\pi i\theta}w$ and $X = z$ gives

$$w^*z = z\left(e^{2\pi i\theta}w\right)^* = e^{-2\pi i\theta}zw^*.$$

Of course, $z^*w^* = (wz)^* = \left(e^{2\pi i\theta}zw\right)^* = e^{-2\pi i\theta}w^*z^*$. These are a fact which will be used repeatedly in the sequel.

3.2 Hopf coordinates for the 3-sphere

As we shall work in close analogy with differential geometry, we give a brief review of the geometry of the ordinary 3-sphere. The 3-sphere can be described as an embedded manifold in $\mathbb{C}^2$ with the two complex coordinates $z = x^1 + ix^2$ and $w = x^3 + ix^4$, satisfying $|z|^2 + |w|^2 = \mathbb{1}$, which can be realized by

$$\begin{aligned} z &= e^{i\xi_1}\sin\eta \\ w &= e^{i\xi_2}\cos\eta, \end{aligned} \tag{3.2}$$

giving

$$\begin{aligned} x^1 &= \cos\xi_1 \sin\eta & x^2 &= \sin\xi_1 \sin\eta \\ x^3 &= \cos\xi_2 \cos\eta & x^4 &= \sin\xi_2 \cos\eta \end{aligned} \tag{3.3}$$

where $0 \leq \xi_1, \xi_2 < 2\pi$ and $0 \leq \eta < \pi/2$. The elements ξ_1, ξ_2, η in the open set given by the strict inequalities can be taken as coordinates. Moreover, the tangent space is spanned by the three vectors

$$\begin{aligned} E_1 &= \partial_1 \left(x^1, x^2, x^3, x^4\right) = \left(-x^2, x^1, 0, 0\right) \\ E_2 &= \partial_2 \left(x^1, x^2, x^3, x^4\right) = \left(0, 0, -x^4, x^3\right) \\ E_\eta &= \partial_\eta \left(x^1, x^2, x^3, x^4\right) = \left(\cos\xi_1 \cos\eta, \sin\xi_1 \cos\eta, -\cos\xi_2 \sin\eta, -\sin\xi_2 \sin\eta\right) \end{aligned} \tag{3.4}$$

where $\partial_1 = \partial_{\xi_1}$, $\partial_2 = \partial_{\xi_2}$. Instead of ∂_η, we introduce the derivation $\partial_3 = |z||w|\partial_\eta$, which gives

$$E_3 = \partial_3 \left(x^1, x^2, x^3, x^4\right) = \left(x^1 |w|^2, x^2 |w|^2, -x^3 |z|^2, -x^4 |z|^2\right). \tag{3.5}$$

E_1, E_2 and E_3 forms an orthonormal frame for the tangent bundle. The action of ∂_1, ∂_2 and ∂_3 on z and w is given by

$$\partial_1(z) = iz \qquad \partial_1(w) = 0 \tag{3.6}$$

$$\partial_2(z) = 0 \qquad \partial_2(w) = iw \tag{3.7}$$

$$\partial_3(z) = z|w|^2 \qquad \partial_3(w) = -w|z|^2. \tag{3.8}$$

The induced metric with respect to the basis $\{E_1, E_2, E_3\}$ of T_pS^3 is

$$(g_{ab}) = \begin{pmatrix} |z|^2 & 0 & 0 \\ 0 & |w|^2 & 0 \\ 0 & 0 & |z|^2 |w|^2 \end{pmatrix} \tag{3.9}$$

3.3 A pseudo-Riemannian calculus for S^3_θ

In this section we compute the scalar curvature of the noncommutative 3-sphere S^3_θ using pseudo-Riemannian calculus. Elements of S^3_θ are finite linear combinations of products of z, z^*, w, w^*. That is, each $a \in S^3_\theta$ can be written as a finite sum

$$a = \sum c_{i,j,k,l} z^i (z^*)^j w^k (w^*)^l, \quad c_{i,j,k,l} \in \mathbb{C}.$$

In this sum, products such as $w^* z z^*$ does not appear because z (or z^*) and w (or w^*) commute up to a phase factor $e^{2\pi i\theta}$. In fact, it follows from the defining relations in Definition 3.1 that an element in S^3_θ is given by the linear combination of monomials

$$z^i (z^*)^j w^{(k)}$$

for $i, j \geq 0$ and $k \in \mathbb{Z}$, where

$$w^{(k)} = \begin{cases} w^k \text{ if } k \geq 0 \\ (w^*)^{-k} \text{ if } k < 0 \end{cases}$$

because we can use the relation $ww^* = \mathbb{1} - zz^*$ to rewrite $zz^*ww^* = zz^* - (zz^*)^2$. Some properties of S^3_θ that will be useful for this dissertation are as follows.

Proposition 3.3. *If $a \in S^3_\theta$ then*

1. $zz^* a = 0 \implies a = 0,$

2. $ww^* a = 0 \implies a = 0.$

Moreover, zz^* and ww^* are central elements of S^3_θ.

Proof. Let us prove that zz^* commutes with every element of S^3_θ (the proof for ww^* is analogous). From the defining relations of the algebra, it is clear that zz^* commutes with z and z^*. Let us check that zz^* commutes with w and w^*. Using $zz^* = \mathbb{1} - ww^*$,

$$wzz^* = w(\mathbb{1} - ww^*) = (\mathbb{1} - ww^*)w = zz^*w$$
$$w^*zz^* = w^*(\mathbb{1} - ww^*) = (\mathbb{1} - ww^*)w^* = zz^*w^*$$

because w and w^* both commute with $\mathbb{1}$ and ww^*. Next, let us show that neither zz^* nor ww^* is a zero divisor. An arbitrary element $a \in S^3_\theta$ may be written as

$$a = \sum_{i,j\geq 0, k\in\mathbb{Z}} a_{ijk} z^i (z^*)^j w^{(k)}$$

for $a_{ijk} \in \mathbb{C}$, and it follows that

$$zz^*a = \sum_{i,j\geq 0, k\in\mathbb{Z}} a_{ijk} z^{i+1} (z^*)^{j+1} w^{(k)}$$

since $[z, z^*] = 0$. As $z^i(z^*)^j w^{(k)}$ is a basis for S^3_θ, setting $zz^*a = 0$ implies that $a_{ijk} = 0$ for all $i, j \geq 0$ and $k \in \mathbb{Z}$, which implies that $a = 0$. Similarly,

$$\begin{aligned} ww^*a = (\mathbb{1} - zz^*)a &= \sum_{i,j\geq 0, k\in\mathbb{Z}} \left(a_{ijk} z^i (z^*)^j w^{(k)} - a_{ijk} z^{i+1} (z^*)^{j+1} w^{(k)}\right) \\ &= \sum_{j\geq 0, k\in\mathbb{Z}} a_{0jk} (z^*)^j w^{(k)} + \sum_{i\geq 1, k\in\mathbb{Z}} a_{i0k} z^i w^{(k)} \\ &\quad + \sum_{i,j\geq 1, k\in\mathbb{Z}} \left(a_{ijk} - a_{i-1,j-1,k}\right) z^i (z^*)^j w^{(k)}, \end{aligned}$$

which can easily be seen to give $a_{ijk} = 0$ upon setting $ww^*a = 0$. □

Let us introduce the notation

$$\begin{aligned} X^1 &= \frac{1}{2}(z + z^*) & X^2 &= \frac{1}{2i}(z - z^*) \\ X^3 &= \frac{1}{2}(w + w^*) & X^4 &= \frac{1}{2i}(w - w^*) \\ |z|^2 &= zz^* & |w|^2 &= ww^*, \end{aligned}$$

and note that $|z|^2 = (X^1)^2 + (X^2)^2$ and $|w|^2 = (X^3)^2 + (X^4)^2$, as well as

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = |z|^2 + |w|^2 = \mathbb{1}.$$

In the following, we shall construct a real pseudo-Riemannian calculus for S^3_θ. Let us start by introducing a metric module (M, g) in close analogy with the Hopf parametrization in Section 3.2. Therefore, we let E_1, E_2, E_3 be the following elements of the right free module $(S^3_\theta)^4$:

$$\begin{aligned} E_1 &= (-X^2, X^1, 0, 0) \\ E_2 &= (0, 0, -X^4, X^3) \\ E_3 &= (X^1|w|^2, X^2|w|^2, -X^3|z|^2, -X^4|z|^2), \end{aligned} \tag{3.10}$$

and let M be the module generated by $\{E_1, E_2, E_3\}$.

Proposition 3.4. *The module $M = \{E_1a + E_2b + E_3c : a, b, c \in S^3_\theta\}$ is a free right S^3_θ-module with a free generating set $\{E_1, E_2, E_3\}$.*

Proof. By construction $\{E_1, E_2, E_3\}$ are generators of M. To prove that M is a free module, we assume that

$$E = E_1a + E_2b + E_3c = 0$$

for some $a, b, c \in S^3_\theta$, then $a = b = c = 0$. The requirement that $E = 0$ is equivalent to

$$\begin{aligned} -X^2a + X^1|w|^2c = 0 \qquad & X^1a + X^2|w|^2c = 0 \\ -X^4b - X^3|z|^2c = 0 \qquad & X^3b - X^4|z|^2c = 0, \end{aligned}$$

and multiplying the first two equations by X^1 and X^2, respectively, and summing them yields (using that $[X^1, X^2] = 0$)

$$\big((X^1)^2 + (X^2)^2\big)\,|w|^2c = 0 \quad \Leftrightarrow \quad |z|^2|w|^2c = 0.$$

It follows from Proposition 3.3 that $c = 0$, and the system of equations becomes

$$X^2 a = 0 \qquad X^1 a = 0$$
$$X^4 b = 0 \qquad X^3 b = 0,$$

from which it follows that $((X^1)^2 + (X^2)^2)\, a = 0$ and $((X^3)^2 + (X^4)^2)\, b = 0$, which is equivalent to

$$|z|^2 a = 0 \qquad |w|^2 b = 0.$$

Again, it follows from Proposition 3.3 that $a = b = 0$. This shows that $\{E_1, E_2, E_3\}$ is a free generating set for M. □

In the differential geometric setting, the three tangent vectors E_1, E_2, E_3 are associated to the three derivations $\partial_1, \partial_2, \partial_3$, as given in (3.6)–(3.8). These derivations have noncommutative analogues.

Proposition 3.5. *There exist hermitian derivations* $\partial_1, \partial_2, \partial_3 \in \mathrm{Der}(S^3_\theta)$ *such that*

$$\partial_1(z) = iz \qquad \partial_1(w) = 0$$
$$\partial_2(z) = 0 \qquad \partial_2(w) = iw$$
$$\partial_3(z) = z|w|^2 \qquad \partial_3(w) = -w|z|^2,$$

and $[\partial_a, \partial_b] = 0$ *for* $a, b = 1, 2, 3$.

Proof. Let us show that ∂_3 exists; the proof that ∂_1, ∂_2 exist is analogous. If ∂_3 exists, the fact that it is hermitian, together with $\partial_3(z) = z|w|^2$ and $\partial_3(w) = -w|z|^2$ completely determines ∂_3 via

$$\partial_3(z) = z|w|^2 \qquad \partial_3(w) = -w|z|^2$$
$$\partial_3(z^*) = z^*|w|^2 \qquad \partial_3(w^*) = -w^*|z|^2,$$

since the action on an arbitrary element of S^3_θ is given by applying Leibniz' rule repeatedly. Conversely, one may try to define ∂_3 via the above relations and extend it to S^3_θ through Leibniz' rule. However, to show that ∂_3 is a derivation on S^3_θ, one needs to check that it respects all the relations between z and w (given in (3.1)). For instance, applying Leibniz' rule to $\partial_3(wz - e^{2\pi i\theta}zw)$ gives

$$\begin{aligned}
\partial_3(wz - e^{2\pi i\theta}zw) &= (\partial_3 w)z + w(\partial_3 z) - e^{2\pi i\theta}(\partial_3 z)w - e^{2\pi i\theta}z(\partial_3 w)\\
&= -w|z|^2 z + wz|w|^2 - e^{2\pi i\theta}z|w|^2 w + e^{2\pi i\theta}zw|z|^2\\
&= -(wz - e^{2\pi i\theta}zw)|z|^2 + (wz - e^{2\pi i\theta}zw)|w|^2 = 0,
\end{aligned}$$

as required (using that $|z|^2$ and $|w|^2$ are central). In the same way, one may check that ∂_3 is compatible with all the relations in S^3_θ (given in (3.1)), which shows that ∂_3 is indeed a derivation on S^3_θ. To prove that $[\partial_a, \partial_b] = 0$ one simply shows that

$$[\partial_a, \partial_b](z) = [\partial_a, \partial_b](z^*) = [\partial_a, \partial_b](w) = [\partial_a, \partial_b](w^*) = 0,$$

which, by Leibniz' rule, implies $[\partial_a, \partial_b](a) = 0$ for all $a \in S^3_\theta$. For instance

$$\begin{aligned}
[\partial_1, \partial_3](z) &= \partial_1(\partial_3(z)) - \partial_3(\partial_1(z)) = \partial_1\left(z|w|^2\right) - \partial_3(iz)\\
&= \partial_1(z)|w|^2 + z\partial_1(|w|^2) - iz|w|^2 = z\partial_1(ww^*) = 0.
\end{aligned}$$

The remaining computations are carried out in the same manner, all giving 0. □

Next, let us construct a real metric calculus over S^3_θ. As the metric module we choose the free module M defined in Proposition 3.4, together with the hermitian form

$$g(U, V) = \sum_{a,b=1}^{3} (U^a)^* g_{ab} V^b$$

where $U = E_a U^a$, $V = E_a V^a$ and

$$(g_{ab}) = \begin{pmatrix} |z|^2 & 0 & 0 \\ 0 & |w|^2 & 0 \\ 0 & 0 & |z|^2|w|^2 \end{pmatrix}. \qquad (3.11)$$

Note that g is induced from the canonical metric on the free module $(S^3_\theta)^4$; that is, $g_{ab} = \sum_{i=1}^4 (E^i_a)^* E^i_b$, where $E_a = e_i E^i_a$. Furthermore, let $\mathfrak{g}$ be the Abelian Lie algebra generated by the derivations $\partial_1, \partial_2, \partial_3$ (in Proposition 3.5) and set $\varphi(\partial_a) = E_a$ (and extend it as a linear map over $\mathbb{R}$).

Proposition 3.6. *$(M, g, \mathfrak{g}_\varphi)$ is a real metric calculus over S^3_θ.*

Proof. Let us first prove that (M, g) is a metric module. By definition, g is a hermitian form, thereby it remains to show that it is non-degenerate. Assume that $g(U, V) = 0$ for all $V \in M$. In particular, one may choose $V = E_a$, which gives

$$0 = g(U, E_1) = (U^1)^* h_{11} = (U^1)^* |z|^2$$
$$0 = g(U, E_2) = (U^2)^* h_{22} = (U^2)^* |w|^2$$
$$0 = g(U, E_3) = (U^3)^* h_{33} = (U^3)^* |z|^2 |w|^2,$$

and from Proposition 3.3 it follows that $U^1 = U^2 = U^3 = 0$. Hence, g is non-degenerate, which shows that (M, g) is a metric module.

Moreover, it is clear that $\varphi(\mathfrak{g})$ generates M since $E_a = \varphi(\partial_a)$, for $a = 1, 2, 3$, is in the image of φ. Finally, for $E, E' \in M_\varphi$, it is easy to see that $g(E, E')$ is hermitian since g_{ab} is central and hermitian from relations in (3.11). □

Since M is a free module, and $E_a = \varphi(\partial_a)$ is a generator for M, one may use Corollary 1.10 to construct a metric and torsion-free connection on $(M, g, \mathfrak{g}_\varphi)$.

Proposition 3.7. *There exists a (unique) connection ∇ on $(M, g, \mathfrak{g}_\varphi)$ such that*

$(M, g, \mathfrak{g}_\varphi, \nabla)$ *is a real pseudo-Riemannian calculus. The connection is given by*

$$\begin{aligned}
&\nabla_1 E_1 = -E_3 && \nabla_1 E_2 = 0 && \nabla_1 E_3 = E_1 |w|^2 \\
&\nabla_2 E_1 = 0 && \nabla_2 E_2 = E_3 && \nabla_2 E_3 = -E_2 |z|^2 \\
&\nabla_3 E_1 = E_1 |w|^2 && \nabla_3 E_2 = -E_2 |z|^2 && \nabla_3 E_3 = E_3(|w|^2 - |z|^2),
\end{aligned}$$

where $\nabla_a \equiv \nabla_{\partial_a}$.

Proof. It is clear that $(M, g, \mathfrak{g}_\varphi)$ satisfies the prerequisites of Corollary 1.10. Furthermore, it is a straightforward exercise to check that $U_{ab} = \nabla_a E_b$ satisfy equation (4.11), which then implies that there exists a connection ∇ on $(M, \mathfrak{g})$, given by $\nabla_a E_b$ above, such that $(M, g, \mathfrak{g}_\varphi)$ is a pseudo-Riemannian calculus.

Let us now show that the pseudo-Riemannian calculus is real; i.e, that the elements $g(\nabla_a \nabla_b E_p, E_q)$ are hermitian for all $a, b, p, q \in \{1, 2, 3\}$. We introduce the connection coefficients $\Gamma^c_{ab} \in S^3_\theta$ through

$$\nabla_a E_b = E_c \Gamma^c_{ab},$$

and note that Γ^c_{ab} is central and hermitian for all $a, b, c \in \{1, 2, 3\}$. It follows that

$$\begin{aligned}
&\nabla_a \nabla_b E_p = \nabla_a \left(E_r \Gamma^r_{bp} \right) = \left(\nabla_a E_r \right) \Gamma^r_{bp} + E_r \partial_a \Gamma^r_{bp} \implies \\
&g(\nabla_a \nabla_b E_p, E_q) = g(\nabla_a E_r, E_q) \Gamma^r_{bp} + g(E_r, E_q) \left(\partial_a \Gamma^r_{bp} \right).
\end{aligned}$$

Since $(M, g, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus and Γ^r_{bp} is central and hermitian, it follows that the first term is hermitian. Furthermore, since ∂_a is a hermitian derivation, and the derivative of a central element is again central, also the second term is hermitian. This shows that $g(\nabla_a \nabla_b E_p, E_q)$ is hermitian and, hence, that $(M, g, \mathfrak{g}_\varphi, \nabla)$ is a real pseudo-Riemannian calculus. □

Let us proceed to compute the curvature of $(M, g, \mathfrak{g}_\varphi, \nabla)$. Recall that since the pseudo-Riemannian calculus is real, Proposition 1.15 implies that the curvature operator has all the classical symmetries.

Proposition 3.8. *The curvature of the pseudo-Riemannian calculus* $(M, g, \mathfrak{g}_\varphi, \nabla)$ *over* S^3_θ *is given by*

$$
\begin{aligned}
&R(\partial_1,\partial_2)E_1 = -E_2|z|^2, && R(\partial_1,\partial_2)E_2 = E_1|w|^2, && R(\partial_1,\partial_2)E_3 = 0\\
&R(\partial_1,\partial_3)E_1 = -E_3|z|^2, && R(\partial_1,\partial_3)E_2 = 0, && R(\partial_1,\partial_3)E_3 = E_1|z|^2|w|^2\\
&R(\partial_2,\partial_3)E_1 = 0, && R(\partial_2,\partial_3)E_2 = -E_3|w|^2, && R(\partial_2,\partial_3)E_3 = E_2|z|^2|w|^2,
\end{aligned}
$$

from which it follows that the nonzero curvature components can be obtained from

$$R_{1212} = |z|^2|w|^2 \qquad R_{1313} = \left(|z|^2\right)^2|w|^2 \qquad R_{2323} = |z|^2\left(|w|^2\right)^2.$$

Moreover, the (unique) scalar curvature is given by $S = 6\cdot\mathbb{1}$*.*

Proof. First, it is straightforward to compute $R(\partial_a,\partial_b)E_c$ by using the results in Proposition 3.7. For instance (recall that $[\partial_a,\partial_b]=0$)

$$
\begin{aligned}
R(\partial_1,\partial_3)E_3 &= \nabla_1\nabla_3E_3 - \nabla_3\nabla_1E_3 = \nabla_1\left(E_3(|w|^2-|z|^2)\right) - \nabla_3\left(E_1|w|^2\right)\\
&= (\nabla_1E_3)\left(|w|^2-|z|^2\right) - (\nabla_3E_1)\,|w|^2 - E_1\partial_3|w|^2\\
&= E_1|w|^2\left(|w|^2-|z|^2\right) - E_1(|w|^2)^2 - E_1\left(-2|w|^2|z|^2\right)\\
&= E_1|z|^2|w|^2.
\end{aligned}
$$

The components are easily computed as well; for example,

$$R_{1212} = g\left(E_1, R(d_1,d_2)E_2\right) = g(E_1,E_1|w|^2) = g(E_1,E_1)|w|^2 = |z|^2|w|^2$$

and

$$R_{1223} = g\left(E_1, R(\partial_2,\partial_3)E_2\right) = g(E_1,-E_3|w|^2) = -g(E_1,E_3)|w|^2 = 0.$$

Computing R_{abcd} for $a,b,c,d\in\{1,2,3\}$ (using the symmetries in Proposition 1.15 to reduce the number of computations that need to be performed) gives R_{1212}, R_{1313} and R_{2323} (together with the ones obtained by symmetry from these) as the only nonzero components.

Finally, let us show that there is a unique scalar curvature. The metric g_{ab} has a pseudo-inverse $(\hat{g}^{ab}, G)$, given by

$$(\hat{g}^{ab}) = \begin{pmatrix} |w|^2 & 0 & 0 \\ 0 & |z|^2 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad G = |z|^2|w|^2.$$

From the computation

$$\begin{aligned} &\hat{g}^{ab}R_{apbq}\hat{g}^{pq} = \hat{g}^{11}R_{1p1q}\hat{g}^{pq} + \hat{g}^{22}R_{2p2q}\hat{g}^{pq} + \hat{g}^{33}R_{3p3q}\hat{g}^{pq} = \\ &\hat{g}^{11}\left(R_{1212}\hat{g}^{22} + R_{1313}\hat{g}^{33}\right) + \hat{g}^{22}\left(R_{2121}\hat{g}^{11} + R_{2323}\hat{g}^{33}\right) + \hat{g}^{33}\left(R_{3131}\hat{g}^{11} + R_{3232}\hat{g}^{22}\right) \\ &= 2|w|^2|z|^2|w|^2|z|^2 + 2|w|^2(|z|^2)^2|w|^2 + 2|z|^2|z|^2(|w|^2)^2 = H(6\cdot\mathbb{1})H, \end{aligned}$$

one concludes that the scalar curvature with respect to $(\hat{g}^{ab}, G)$ is given by $6\cdot\mathbb{1}$. Since G is central, it follows from Proposition 1.19 that this is indeed the unique scalar curvature of $(M, g, \mathfrak{g}_\varphi, \nabla)$. □

3.4 Aspects of localization on S^3_θ

Classically, the 3-sphere S^3 is regarded as an embedded submanifold of $\mathbb{R}^4$ and the projection of $\mathbb{R}^4 = T_x\mathbb{R}^4$ onto the normal space of $T_xS^3 \subset T_x\mathbb{R}^4$ is given by the map

$$\Pi(U)^i = \Pi^{ij}U^j = x^ix^jU^j,$$

where $x^1, \ldots, x^4$ are the embedding coordinates of S^3 into $\mathbb{R}^4$, which satisfy

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1.$$

Hence, sections of the tangent bundle can be identified with the projective module

$$\Gamma\left(TS^3\right) = \mathcal{P}\left(\left(C^\infty\left(S^3\right)\right)^4\right)$$

where $\mathcal{P} = \mathbb{1} - \Pi$, giving TS^3 as a subspace of $T\mathbb{R}^4 = \mathbb{R}^8$. It is well known that S^3 is parallelizable, which is equivalent to stating that $\Gamma(TS^3)$ is a free module, and one may explicitly give a global basis for the vector fields as follows:

$$v_1 = (-x^4, x^3, -x^2, x^1) \quad v_2 = (-x^3, -x^4, x^1, x^2) \quad v_3 = (-x^2, x^1, x^4, -x^3).$$

The global vector fields E_1, E_2, E_3

$$\begin{aligned} &E_1 = (-x^2, x^1, 0, 0) \qquad E_2 = (0, 0, -x^4, x^3) \\ &E_3 = \left(x^1|w|^2, x^2|w|^2, -x^3|z|^2, -x^4|z|^2\right) \end{aligned}$$

as defined in Section 3.2 are linearly independent at every point where $|z|^2 = (x^1)^2 + (x^2)^2 \neq 0$ and $|w|^2 = (x^3)^2 + (x^4)^2 \neq 0$, which can easily be seen by computing the determinant

$$\begin{vmatrix} -x^2 & x^1 & 0 & 0 \\ 0 & 0 & -x^4 & x^3 \\ x^1|w|^2 & x^2|w|^2 & -x^3|z|^2 & -x^4|z|^2 \\ x^1 & x^2 & x^3 & x^4 \end{vmatrix} = -|z|^2|w|^2,$$

giving a condition for $E_1, E_2, E_3, \vec{n} = (x^1, x^2, x^3, x^4)$ to be linearly independent. Thus, the vector fields E_1, E_2, E_3 provide a globalization of the corresponding vector fields in the local chart defined by the Hopf coordinates, and one may use them for computations, keeping in mind that they do not span the tangent space at points $(x^1, x^2, x^3, x^4) \in S^3$ where $x^1 = x^2 = 0$ or $x^3 = x^4 = 0$. However, in this case, the set of points on S^3 which are not covered by this chart has measure zero, which implies that certain results, for example, results involving integration over the manifold, are not sensitive to the difference between $\{E_1, E_2, E_3\}$ and $\{v_1, v_2, v_3\}$.

It is easy to check that since

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = zz^* + ww^* = \mathbb{1}$$

the above situation admits a straightforward generalization. Namely, for $U = e_i U^i \in (S^3_\theta)^4$, the linear map $\mathcal{P} : (S^3_\theta)^4 \to (S^3_\theta)^4$ defined as

$$\mathcal{P}(U) = \sum_{i,j=1}^{4} e_i \mathcal{P}^{ij} U^j$$

where $\mathcal{P}^{ij} = \delta^{ij}\mathbb{1} - X^i X^j$ is a S^3_θ-module, which satisfies $\mathcal{P}^2(U) = \mathcal{P}(U)$. Hence, $TS^3_\theta = \mathcal{P}\left((S^3_\theta)^4\right)$ is a projective module in close analogy with the module of vector fields on S^3. Let us now further study the structure of TS^3_θ. We start by proving the following lemma.

Lemma 3.9. *Let $q = e^{2\pi i\theta}$. In S^3_θ*

$$X^2X^4 + X^1X^3 = q\left(X^4X^2 + X^3X^1\right) \tag{3.12}$$

$$X^2X^4 - X^1X^3 = \bar{q}\left(X^4X^2 - X^3X^1\right) \tag{3.13}$$

$$X^2X^3 + X^1X^4 = \bar{q}\left(X^3X^2 + X^4X^1\right) \tag{3.14}$$

$$X^2X^3 - X^1X^4 = q\left(X^3X^2 - X^4X^1\right). \tag{3.15}$$

Proof. The proof is a straightforward computation; for instance,

$$\begin{aligned} X^2X^3 + X^1X^4 &= \frac{1}{2i}(z - z^*)\frac{1}{2}(w + w^*) + \frac{1}{2}(z + z^*)\frac{1}{2i}(w - w^*) \\ &= \frac{1}{2i}\left(zw - z^*w^*\right) = \frac{\bar{q}}{2i}\left(wz - w^*z^*\right) \\ &= \frac{1}{2}(w + w^*)\frac{1}{2i}(z - z^*) + \frac{1}{2i}(w - w^*)\frac{1}{2}(z + z^*) \\ &= \bar{q}\left(X^3X^2 + X^4X^1\right), \end{aligned}$$

and the remaining computations are completely analogous. □

The next statement corresponds to the fact that S^3 is a parallelizable manifold.

Proposition 3.10. *The right S^3_θ-module TS^3_θ is a free module with a free generating* *set*

$$
\begin{aligned}
F_1 &= (-X^4, X^3, -qX^2, qX^1)\\
F_2 &= (-X^3, -X^4, qX^1, qX^2)\\
F_3 &= (-X^2, X^1, X^4, -X^3).
\end{aligned}
$$

Proof. Let us start by showing that $\Pi(F_a) = 0$, which implies that $F_a \in TS^3_\theta$. Since $\Pi^{ij} = X^i X^j$, it is enough to show that $X^i F^i_a = 0$ for $a = 1, 2, 3$:

$$
\begin{aligned}
X^i F^i_1 &= -X^1X^4 + X^2X^3 - qX^3X^2 + qX^4X^1 = 0\\
X^i F^i_2 &= -X^1X^3 - X^2X^4 + qX^3X^1 + qX^4X^2 = 0\\
X^i F^i_3 &= -X^1X^2 + X^2X^1 + X^3X^4 - X^4X^3 = 0,
\end{aligned}
$$

by using (3.15), (3.12) in Lemma 3.9, and the fact that $[X^1, X^2] = [X^3, X^4] = 0$.

Next, we show that F_1, F_2, F_3 generate TS^3_θ. It is sufficient to show that $\mathcal{P}(e_i)$ where $\{e_i\}^4_{i=1}$ denotes the canonical generator of $(S^3_\theta)^4$ can be written as linear combination of F_1, F_2, F_3, for $i = 1, 2, 3, 4$. In fact, one can show that

$$
\begin{aligned}
\mathcal{P}(e_1) &= \big(\mathbb{1} - (X^1)^2, -X^2X^1, -X^3X^1, -X^4X^1\big) = -F_1X^4 - F_2X^3 - F_3X^2\\
\mathcal{P}(e_2) &= \big(-X^1X^2, \mathbb{1} - (X^2)^2, -X^3X^2, -X^4X^2\big) = F_1X^3 - F_2X^4 + F_3X^1\\
\mathcal{P}(e_3) &= \big(-X^1X^3, -X^2X^3, \mathbb{1} - (X^3)^2, X^4X^3\big) = -\bar{q}F_1X^2 + \bar{q}F_2X^1 + F_3X^4\\
\mathcal{P}(e_4) &= \big(-X^1X^4, -X^2X^4, -X^3X^4, \mathbb{1} - (X^4)^2\big) = \bar{q}F_1X^1 + \bar{q}F_2X^2 - F_3X^3.
\end{aligned}
$$

For instance,

$$
\begin{aligned}
-F_1X^4 - F_2X^3 - F_3X^2 &= \big((X^2)^2 + (X^3)^2 + (X^4)^2, -X^3X^4 + X^4X^3 - X^1X^2,\\
&\qquad qX^2X^4 - qX^1X^3 - X^4X^2, -qX^1X^4 - qX^2X^3 + X^3X^2\big)\\
&= \big(\mathbb{1} - (X^1)^2, -X^2X^1, -X^3X^1, -X^4X^1\big) = \mathcal{P}(e_1),
\end{aligned}
$$

by using (3.13), (3.14) (in the third and fourth component, respectively) and the fact that $[X^1, X^2] = [X^3, X^4] = 0$. Finally, let us show that F_1, F_2, F_3 are free generators. For $a, b, c \in S^3_\theta$, we assume that

$$F_1 a + F_2 b + F_3 c = 0,$$

which is equivalent to

$$\begin{cases} -X^4 a - X^3 b - X^2 c &= 0 \\ X^3 a - X^4 b + X^1 c &= 0 \\ -qX^2 a + qX^1 b + X^4 c &= 0 \\ qX^1 a + qX^2 b - X^3 c &= 0. \end{cases}$$

Multiplying these equations (from the left) by $-X^2$, X^1, X^4 and $-X^3$, respectively, and summing them yields $c = 0$, by using (3.12) and (3.15). Setting $c = 0$ in the above equations gives

$$\begin{aligned} X^4 a + X^3 b = 0 \qquad & X^3 a - X^4 b = 0 \\ -X^2 a + X^1 b = 0 \qquad & X^1 a + X^2 b = 0, \end{aligned}$$

which implies that

$$\begin{aligned} (X^4)^2 a = -X^4 X^3 b \qquad & (X^3)^2 a = X^3 X^4 b \\ (X^2)^2 a = X^2 X^1 b \qquad & (X^1)^2 a = -X^1 X^2 b. \end{aligned}$$

Summing these equations gives $a = 0$, which then (via a similar argument) implies that $b = 0$. This shows that F_1, F_2, F_3 are linearly independent. □

It is easy to check that the elements E_1, E_2, E_3, as defined in (3.10), fulfill $\mathcal{P}(E_a) = E_a$ for $a = 1, 2, 3$, implying that they are elements of TS^3_θ. Hence, the module M, of the pseudo-Riemannian calculus for S^3_θ, is a submodule of TS^3_θ, providing a noncommutative analogue of the globalization of the local vector fields in the Hopf coordinates as described in the beginning of the section.

As is well known, every projective module is endowed with a canonical affine connection; namely, the module $(S^3_\theta)^4$ has an affine connection given by

$$\bar{\nabla}_\partial V = e_i \partial(V^i)$$

where $V = e_i V^i \in (S^3_\theta)^4$ and $\partial \in \mathrm{Der}(S^3_\theta)$, and it follows that

$$\hat{\nabla}_\partial V = \mathcal{P}(\bar{\nabla}_\partial V)$$

is an affine connection on TS^3_θ. Since we have argued in analogy with differential geometry, where M is a sub-module of TS^3 and the connection on M is merely the restriction of the connection on TS^3, it is natural to ask if the connection $\hat{\nabla}$ (restricted to M) coincides with ∇ (as given by the pseudo-Riemannian calculus over M).

Proposition 3.11. *Let $(M, g, \mathfrak{g}_\varphi, \nabla)$ be the pseudo-Riemannian calculus over S^3_θ introduced in this chapter. The affine connection $\hat{\nabla}_\partial U = \mathcal{P}(\bar{\nabla}_\partial U)$, restricted to $M \subseteq TS^3_\theta$, coincides with ∇; that is, $\hat{\nabla}_d\, U = \nabla_\partial U$ for $\partial \in \mathfrak{g}$ and $U \in M$.*

Proof. The proof is easily done by a straightforward computation, where one computes $\hat{\nabla}_a E_b$ for $a, b = 1, 2, 3$, and compares it with the result in Proposition 3.7. For instance,

$$\begin{aligned}
\hat{\nabla}_1 E_1 &= \mathcal{P}\left((-\partial_1 X^2, \partial_1 X^1, 0, 0)\right) = \mathcal{P}\left((-X^1, -X^2, 0, 0)\right) \\
&= (-X^1, -X^2, 0, 0) - (X^1, X^2, X^3, X^4)\left(-(X^1)^2 - (X^2)^2\right) \\
&= \left(X^1(|z|^2 - \mathbb{1}), X^2(|z|^2 - \mathbb{1}), X^3|z|^2, X^4|z|^2\right) \\
&= \left(-X^1|w|^2, -X^2|w|^2, X^3|z|^2, X^4|z|^2\right) = -E_3,
\end{aligned}$$

which coincides with $\nabla_1 E_1$. □

In order to take the analogy with localization one step further, let us introduce a localized algebra $S^3_{\theta,\mathrm{loc}}$ constructed by formally adjoining the inverses of $|z|^2$ and $|w|^2$ to the algebra S^3_θ. This can be achieved by a localization process due to Ore [34].

Definition 3.12. Let R be a not necessarily commutative ring and S be a subset of A. The subset S is called a right (respectively left) multiplicative set if it satisfies the following three conditions for every $a, b \in R$, and $s, t \in S$:

(i) $st \in S$,

(ii) $aS \cap sR$ is not empty and

(iii) If $sa = 0$, then there is some $u \in S$ with $au = 0$.

It is a result of Ore [34] that if S is a right (resp. left) multiplicative set, then one can construct the ring of right (resp. left) fractions RS^{-1} similarly to the commutative case.

In particular, the multiplicative set $S \subset S^3_\theta$ generated by $|z|^2, |w|^2, \mathbb{1}$ trivially satisfies the (right and left) Ore condition (since it consists of central elements) and the fact that $|z|^2, |w|^2$ are regular elements (cf. Proposition 3.3) implies that the Ore localization at S exists (see for instance [14]). If we consider TS^3_θ and M as (right) $S^3_{\theta,\mathrm{loc}}$-modules, they coincide, which we show by explicitly finding a relation between the two sets of generators.

Proposition 3.13. *Consider the following elements of* $(S^3_{\theta,\mathrm{loc}})^4$*:*

$$
\begin{aligned}
F_1 &= (-X^4, X^3, -qX^2, qX^1) \qquad & E_1 &= (-X^2, X^1, 0, 0)\\
F_2 &= (-X^3, -X^4, qX^1, qX^2) \qquad & E_2 &= (0, 0, -X^4, X^3)\\
F_3 &= (-X^2, X^1, X^4, -X^3) \qquad & E_3 &= (X^1|w|^2, X^2|w|^2, -X^3|z|^2, -X^4|z|^2).
\end{aligned}
$$

Thens

$$
\begin{aligned}
F_1 &= E_1|z|^{-2}\left(X^1X^3 + X^2X^4\right) + E_2|w|^{-2}\left(X^1X^3 + X^2X^4\right)\\
&\quad + E_3|z|^{-2}|w|^{-2}\left(X^2X^3 - X^1X^4\right)\\
F_2 &= E_1|z|^{-2}\left(X^2X^3 - X^1X^4\right) + E_2|w|^{-2}\left(X^2X^3 - X^1X^4\right)\\
&\quad - E_3|z|^{-2}|w|^{-2}\left(X^1X^3 + X^2X^4\right)\\
F_3 &= E_1 - E_2.
\end{aligned}
$$

Proof. Let us show that F_1 can be written as a linear combination of E_1, E_2, E_3, as given in the statement. Namely, introducing W^i through

$$e_i w^i = E_1 |z|^{-2} \left(X^1 X^3 + X^2 X^4\right) + E_2 |w|^{-2} \left(X^1 X^3 + X^2 X^4\right) + E_3 |z|^{-2} |w|^{-2} \left(X^2 X^3 - X^1 X^4\right),$$

gives

$$W^1 = -X^2 |z|^{-2} (X^1 X^3 + X^2 X^4) + X^1 |z|^{-2} (X^2 X^3 - X^1 X^4),$$
$$W^2 = X^1 |z|^{-2} (X^1 X^3 + X^2 X^4) + X^2 |z|^{-2} (X^2 X^3 - X^1 X^4),$$
$$W^3 = -X^4 |w|^{-2} (X^1 X^3 + X^2 X^4) - X^3 |w|^{-2} (X^2 X^3 - X^1 X^4) \text{ and}$$
$$W^4 = X^3 |w|^{-2} (X^1 X^3 + X^2 X^4) - X^4 |w|^{-2} (X^2 X^3 - X^1 X^4).$$

Using the fact that $[X^1, X^2] = 0$ (in W^1, W^2), together with (3.12) and (3.15) (in W^3, W^4), yields

$$W^1 = -|z|^{-2} \left((X^2)^2 + (X^1)^2\right) X^4 = -|z|^{-2} |z|^2 X^4 = -X^4$$
$$W^2 = |z|^{-2} \left((X^1)^2 + (X^2)^2\right) X^3 = |z|^{-2} |z|^2 X^3 = X^3$$
$$W^3 = -q |w|^{-2} \left((X^4)^2 + (X^3)^2\right) X^2 = -q |w|^{-2} |w|^2 X^2 = -q X^2$$
$$W^4 = q |w|^{-2} \left((X^4)^2 + (X^3)^2\right) X^1 = q |w|^{-2} |w|^2 X^1 = q X^1,$$

which shows that $e_i W^i = F_1$. □

Chapter 4

Gauss-Bonnet-Chern type theorem for the noncommutative 4-sphere

In this chapter a pseudo-Riemannian calculus over the noncommutative 4-sphere is constructed and a projective module, which is in close analogy with the space of vector fields on the classical 4-sphere is introduced. Moreover, via a suitable localization of the algebra, we find a local trivialization of the projective module and prove that a conformal class of perturbations of the round metric admits a unique metric and torsion-free connection. Finally, we show that in this particular case, a naive analogue of the Pfaffian of the curvature form exists. The existence of an analogue of the Pfaffian allows us to prove a Gauss-Bonnet-Chern type theorem for the noncommutative 4-sphere.

4.1 The 4-sphere S^4 as an embedded manifold in $\mathbb{R}^5$

The geometric constructions for the noncommutative 4-sphere will closely follow that of classical geometry. Therefore, let us review an explicit parametrization of S^4, giving a chart that covers almost all of the manifold. Furthermore, we present a particular basis for vector fields over that chart.

As a subset of $\mathbb{R}^5$, the 4-dimensional sphere is defined as

$$S^4 = \left\{(x^1, x^2, x^3, x^4, x^5) \in \mathbb{R}^5 : (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 = 1\right\},$$

and we let $U_0 \subseteq S^4$ denote the chart of S^4 given by

$$\begin{aligned} x^1 &= \cos(\xi_1)\cos(\varphi)\cos(\psi) & x^2 &= \sin(\xi_1)\cos(\varphi)\cos(\psi) \\ x^3 &= \cos(\xi_2)\sin(\varphi)\cos(\psi) & x^4 &= \sin(\xi_2)\sin(\varphi)\cos(\psi) \\ x^5 &= \sin(\psi), \end{aligned}$$

where $0 < \xi_1, \xi_2 < 2\pi$, $0 < \varphi < \pi/2$ and $-\pi/2 < \psi < \pi/2$. Equivalently, one may consider $z = x^1 + ix^2$, $w = x^3 + ix^4$ and $t = x^5$ with

$$\begin{aligned} z &= e^{i\xi_1}\cos(\varphi)\cos(\psi) \\ w &= e^{i\xi_2}\sin(\varphi)\cos(\psi) \\ t &= \sin(\psi). \end{aligned}$$

At each point $p \in U_0$, the tangent space T_pS^4 is spanned by the vectors

$$\begin{aligned} \partial_{\xi_1}\vec{x} &= (-\sin\xi_1\cos\varphi\cos\psi, \cos\xi_1\cos\varphi, \cos\psi, 0, 0) = (-x^2, x^1, 0, 0, 0) \\ \partial_{\xi_2}\vec{x} &= (0, 0, -\sin\xi_2\sin\varphi\cos\psi, \cos\xi_2\sin\varphi\cos\psi, 0) = (0, 0, -x^4, x^3, 0) \\ \partial_{\varphi}\vec{x} &= (-\cos\xi_1\sin\varphi\cos\psi, -\sin\xi_1\sin\varphi\cos\psi, \cos\xi_2\cos\varphi\cos\psi, \sin\xi_2\cos\varphi\cos\psi, 0) \\ \partial_{\psi}\vec{x} &= (-\cos\xi_1\cos\varphi\sin\psi, -\sin\xi_1\cos\varphi\sin\psi, \\ &\qquad -\cos\xi_2\sin\varphi\sin\psi, -\sin\xi_2\sin\varphi\sin\psi, \cos\psi). \end{aligned}$$

These vector fields are defined in the local chart U_0 and we would like to extend them to global vector fields on S^4 (however, *not* providing a basis at each point of S^4). As written above, $\partial_{\xi_1}\vec{x}$ and $\partial_{\xi_2}\vec{x}$ may be extended to all of S^4, since all components can be expressed in terms of $x^1, \dots, x^5$. By rescaling $\partial_\varphi \vec{x}$ and $\partial_\psi \vec{x}$ one obtains

$$
\begin{aligned}
-|z||w|\partial_\varphi \vec{x} &= \left(x^1|w|^2, x^2|w|^2, -x^3|z|^2, -x^4|z|^2, 0\right)\\
-\cos\psi\, \partial_\psi \vec{x} &= \left(x^1 t, x^2 t, x^3 t, x^4 t, -|z|^2 - |w|^2\right),
\end{aligned}
$$

which are well defined as vector fields on S^4. Thus, the globally defined vector fields given by

$$
\begin{aligned}
e_1 &= \left(-x^2, x^1, 0, 0, 0\right) & e_2 &= \left(0, 0, -x^4, x^3, 0\right)\\
e_3 &= \left(x^1|w|^2, x^2|w|^2, -x^3|z|^2, -x^4|z|^2, 0\right) & e_4 &= \left(x^1 t, x^2 t, x^3 t, x^4 t, -|z|^2 - |w|^2\right),
\end{aligned}
$$

span the space of vector fields over U_0. For later comparison, let us write down the action of the derivations that corresponds to the above vector fields are:

$$
\begin{aligned}
&\partial_1 z = iz && \partial_1 w = 0 && \partial_1 t = 0\\
&\partial_2 z = 0 && \partial_w = iw && \partial_2 t = 0\\
&\partial_3 z = z|w|^2 && \partial_3 w = -w|z|^2 && \partial_3 t = 0\\
&\partial_4 z = zt && \partial_4 w = wt && \partial_4 t = t^2 - 1.
\end{aligned}
\tag{4.1}
$$

4.2 The noncommutative 4-sphere

4.2.1 Basic properties of S^4_θ

For $\theta \in [0, 1)$, we let S^4_θ denote the unital $*$-algebra (over $\mathbb{C}$) generated by Z, W and T, satisfying the relations [11, 18]

$$
\begin{aligned}
&WZ = qZW \qquad W^*Z = \bar{q}ZW^*\\
&ZZ^* + WW^* + T^2 = \mathbb{1}\\
&T^* = T \qquad [T, Z] = [T, W] = [W, W^*] = [Z, Z^*] = 0,
\end{aligned}
\tag{4.2}
$$

where $q = e^{i2\pi\theta}$. Furthermore, $ZZ^* \in Z(S^4_\theta)$ and $WW^* \in Z(S^4_\theta)$ where $Z(S^4_\theta)$ denotes the center of S^4_θ. It follows from (4.2) that a linear basis for S^4_θ is given by the elements

$$Z^j(Z^*)^kW^l(W^*)^mT^\epsilon$$

for $j, k, l, m \in \{0, 1, 2, \ldots\}$ and $\epsilon \in \{0, 1\}$ (where, e.g., higher powers of T are eliminated by using the relation $T^2 = \mathbb{1} - ZZ^* - WW^*$). For convenience, let us introduce the multi-index notation $I = (j, k, l, m, \epsilon)$ and

$$e^I = Z^j(Z^*)^kW^l(W^*)^mT^\epsilon, \tag{4.3}$$

by which, in this notation, every element $a \in S^4_\theta$ admits the unique expression

$$a = \sum_I a_I e^I$$

with $a_I \in \mathbb{C}$. It is useful to develop the multi-index notation a bit further. Namely, for $I = (j, k, l, m, \epsilon)$ we write $I = (\hat{I}, \epsilon)$ with $\hat{I} = (j, k, l, m)$. Furthermore, we introduce

$$1_Z = (1, 1, 0, 0, 0) = (\hat{1}_Z, 0) \quad \text{and} \quad 1_W = (0, 0, 1, 1, 0) = (\hat{1}_W, 0),$$

and we write $I + J$ for component-wise addition of multi-indices. Let us now state the result of multiplying two elements of the form (4.3) in the following lemma:

Lemma 4.1. *If $I_1 = (j_1, k_1, l_1, m_1, \epsilon_1)$ and $I_2 = (j_2, k_2, l_2, m_2, \epsilon_2)$ then*

$$e^{I_1}e^{I_2} = \begin{cases} q^{(l_1-m_1)(j_2-k_2)}e^{I_1+I_2} & \text{if } \epsilon_1 + \epsilon_2 \leq 1 \\ q^{(l_1-m_1)(j_2-k_2)}\left(e^{(\hat{I}_1+\hat{I}_2,0)} - e^{(\hat{I}_1+\hat{I}_2+\hat{1}_Z,0)} - e^{(\hat{I}_1+\hat{I}_2+\hat{1}_W,0)}\right) & \text{if } \epsilon_1 + \epsilon_2 = 2. \end{cases}$$

Proof. Using (4.2) one obtains

$$\begin{aligned} e^Ie^J &= Z^{j_1}(Z^*)^{k_1}W^{l_1}(W^*)^{m_1}T^{\epsilon_1}Z^{j_2}(Z^*)^{k_2}W^{l_2}(W^*)^{m_2}T^{\epsilon_2} \\ &= q^{j_2(l_1-m_1)}Z^{j_1+j_2}(Z^*)^{k_1}W^{l_1}(W^*)^{m_1}(Z^*)^{k_2}W^{l_2}(W^*)^{m_2}T^{\epsilon_1+\epsilon_2} \end{aligned}$$

$$= q^{j_2(l_1-m_1)} q^{k_2(m_1-l_1)} Z^{j_1+j_2}(Z^*)^{k_1+k_2} W^{l_1}(W^*)^{m_1} W^{l_2}(W^*)^{m_2} T^{\epsilon_1+\epsilon_2}$$
$$= q^{(l_1-m_1)(j_2-k_2)} Z^{j_1+j_2}(Z^*)^{k_1+k_2} W^{l_1+l_2}(W^*)^{m_1+m_2} T^{\epsilon_1+\epsilon_2}.$$

Now, if $\epsilon_1 + \epsilon_2 \leq 1$ then the statement in the lemma is proved. If $\epsilon_1 + \epsilon_2 = 2$, then the statement follows after using that $T^2 = \mathbb{1} - ZZ^* - WW^*$, and the fact that both ZZ^* and WW^* are central. □

Let us now proceed to state a few properties of S^4_θ that we shall need in the following.

Proposition 4.2. *The elements ZZ^*, WW^* and $\mathbb{1} - T^2$ are regular (i.e. none of them is a zero divisor).*

Proof. Let us first prove that ZZ^* is not a zero divisor. Thus, let a be an element of S^4_θ, given as

$$a = \sum_I a_I e^I$$

and compute (by using Lemma 4.1)

$$ZZ^* a = \sum_I a_I e^{1_Z} e^I = \sum_I q^{(0-0)(j-k)} a_I e^{I+1_Z} = \sum_I a_I e^{I+1_Z}.$$

Clearly, setting $ZZ^*a = 0$ gives $a_I = 0$ for all I since $\{e^I\}$ is a basis for S^4_θ. Similarly, we consider

$$WW^* a = \sum_I a_I e^{1_W} e^I = \sum_I q^{(1-1)(j-k)} e^{I+1_W} = \sum_I a_I e^{I+1_W}$$

and conclude that $WW^*a = 0$ gives $a = 0$. Finally, we compute

$$(\mathbb{1} - T^2)a = (|Z|^2 + |W|^2)a = \sum_I a_I \left(e^{1_Z} + e^{1_W}\right) e^I = \sum_I a_I e^{I+1_Z} + \sum_I a_I e^{I+1_W}$$
$$= \sum_{j=0,\, l,m\geq 1} a_{I-1_W} e^I + \sum_{k=0,\, j,l,m\geq 1} a_{I-1_W} e^I + \sum_{l=0,\, j,k\geq 1} a_{I-1_Z} e^I$$
$$+ \sum_{m=0,\, j,k,l\geq 1} a_{I-1_Z} e^I + \sum_{j,k,l,m\geq 1} \left(a_{I-1_Z} + a_{I-1_W}\right) e^I.$$

Note that in the above expression, every basis element appears at most once. Therefore, setting $(\mathbb{1} - T^2)a = 0$ immediately gives $a_{j,k,l,m,\epsilon} = 0$ if at least one of j, k, l, m is zero. If $j, k, l, m \geq 1$ one gets

$$a_{I-(0,0,1,1,0)} = -a_{I-(1,1,0,0,0)} \implies a_I = -a_{I+(1,1,-1,-1)},$$

which, by iteration, gives

$$a_I = (-1)^n a_{I+(n,n,-n,-n)} \quad \text{for } 0 \leq n \leq \min(l, m).$$

Hence, since $a_{j,k,l,m,\epsilon} = 0$ if at least one of j, k, l, m is zero, one concludes that

$$a_{(j,k,l,m,\epsilon)} = \begin{cases} (-1)^l a_{j+l,k+l,0,m-l,\epsilon} = 0 \text{ if } l \leq m \\ (-1)^m a_{j+m,k+m,l-m,0,\epsilon} = 0 \text{ if } l \geq m \end{cases}$$

which, together with the previous observation, shows that $a = 0$. □

It was already noted that ZZ^*, WW^* and T are central elements. The next results shows that if θ is an irrational number, then these elements generate the center of S^4_θ.

Proposition 4.3. *If θ is irrational then $Z(S^4_\theta)$ is generated by ZZ^*, WW^* and T. That is, every $a \in Z(S^4_\theta)$ can be uniquely written as*

$$a = \sum_{j,k,\epsilon} a_{jk\epsilon} (ZZ^*)^j (WW^*)^k T^\epsilon$$

where $a_{jk\epsilon} \in \mathbb{C}$, $j, k \in \{0, 1, 2, \ldots\}$ and $\epsilon \in \{0, 1\}$.

Proof. Let a be a nonzero central element of S^4_θ and write

$$a = \sum_I a_I e^I.$$

In particular, a has to commute with Z, and one computes

$$[a,Z]=\sum_I a_I\left(e^I e^{(1,0,0,0,0)}-e^{(1,0,0,0,0)}e^I\right)=\sum_I a_I(q^{l-m}-1)e^{I+(1,0,0,0,0)}.$$

Demanding that $[a,Z]=0$ gives $(q^{l-m}-1)a_I=0$. If $a\neq 0$, there exists an I such that $a_I\neq 0$, which implies that $q^{l-m}=1$. Since θ is assumed to be irrational it follows that $l=m$. Similarly, if a commutes with W then

$$0=[a,W]=\sum_I a_I\left(e^I e^{(0,0,1,0,0)}-e^{(0,0,1,0,0)}e^I\right)=\sum_I a_I\left(1-q^{j-k}\right)e^{I+(0,0,1,0,0)}$$

giving $j=k$ in analogy with the previous case. Thus, an element $a\in Z(S^4_\theta)$ must be of the following form

$$a=\sum_{j,k,\epsilon} a_{j,k,\epsilon}\,(ZZ^*)^j\,(WW^*)^k\,T^\epsilon,$$

and it is clear that any element of the above form is in $Z(S^4_\theta)$ since ZZ^*, WW^* and T are central. □

Remark. Note that Proposition 4.3 does not hold if θ is rational. For instance, if $q^N=1$ then both Z^N and W^N are central elements.

Let us introduce

$$\begin{aligned}
X^1&=\frac{1}{2}(Z+Z^*) && X^2=\frac{1}{2i}(Z-Z^*)\\
X^3&=\frac{1}{2}(W+W^*) && X^4=\frac{1}{2i}(W-W^*)\\
|Z|^2&=ZZ^* \quad |W|^2=WW^* && X^5=T,
\end{aligned}$$

and note that $|Z|^2=(X^1)^2+(X^2)^2$ and $|W|^2=(X^3)^2+(X^4)^2$, as well as

$$(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2+(X^5)^2=|Z|^2+|W|^2+T^2=\mathbb{1}.$$

Moreover, the normality of Z and W is equivalent to $[X^1,X^2]=[X^3,X^4]=0$.

Next, let us show that there exist noncommutative analogues of the four derivations appearing in (4.1).

Proposition 4.4. *There exist hermitian derivations* $\tilde{\partial}_1$, $\tilde{\partial}_2$, $\tilde{\partial}_3$, $\tilde{\partial}_4$ *such that*

$$
\begin{array}{lll}
\tilde{\partial}_1 Z = iZ & \tilde{\partial}_1 W = 0 & \tilde{\partial}_1 T = 0 \\
\tilde{\partial}_2 Z = 0 & \tilde{\partial}_2 W = iW & \tilde{\partial}_2 T = 0 \\
\tilde{\partial}_3 Z = Z|W|^2 & \tilde{\partial}_3 W = -W|Z|^2 & \tilde{\partial}_3 T = 0 \\
\tilde{\partial}_4 Z = ZT & \tilde{\partial}_4 W = WT & \tilde{\partial}_4 T = T^2 - \mathbb{1},
\end{array}
$$

and it follows that

$$
\begin{aligned}
&[\tilde{\partial}_1, \tilde{\partial}_2] = [\tilde{\partial}_1, \tilde{\partial}_3] = [\tilde{\partial}_1, \tilde{\partial}_4] = 0 \\
&[\tilde{\partial}_2, \tilde{\partial}_3] = [\tilde{\partial}_2, \tilde{\partial}_4] = 0 \\
&[\tilde{\partial}_3, \tilde{\partial}_4] = -2T\tilde{\partial}_3.
\end{aligned}
$$

Proof. If the derivations exist, the relations given above (together with the fact that they are hermitian derivations), completely determine their actions via Leibniz' rule. However, for these derivations to be well-defined, one has to check that they respect the defining relations (4.2) of S^4_θ. For instance

$$
\begin{aligned}
\tilde{\partial}_1(WZ - qZW) &= (\tilde{\partial}_1 W)Z + W(\tilde{\partial}_1 Z) - q(\tilde{\partial}_1 Z)W - qZ(\tilde{\partial}_1 W) \\
&= iWZ - iqZW = i(WZ - qZW) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\partial}_3(WZ - qZW) &= (\tilde{\partial}_3 W)Z + W(\tilde{\partial}_3 Z) - q(\tilde{\partial}_3 Z)W - qZ(\tilde{\partial}_3 W) \\
&= -W|Z|^2 Z + WZ|W|^2 - qZ|W|^2 W + qZW|Z|^2 \\
&= (WZ - qZW)|W|^2 - (WZ - qZW)|Z|^2 = 0
\end{aligned}
$$

(using that $|Z|^2$ and $|W|^2$ are central). In this way, relations (4.2) can be checked for the derivations $\tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3, \tilde{\partial}_4$. □

4.2.2 A real metric calculus over S^4_θ

This section will introduce a differential calculus over S^4_θ in close analogy with the classical parametrization in Section 4.1. The calculus will be constructed in the framework of pseudo-Riemannian calculi, as developed in [6], and briefly reviewed in Section 4.3.

To this end, we introduce four elements of the free right module $(S^4_\theta)^5$ that correspond to the classical vector fields e_1, e_2, e_3, e_4 in Section 4.1. However, in order to properly define a connection, one needs to slightly rescale e_1 and e_2. Thus, we consider the following elements of $(S^4_\theta)^5$:

$$
\begin{aligned}
E_1 &= (-X^2(\mathbb{1} - T^2), X^1(\mathbb{1} - T^2), 0, 0, 0) \\
E_2 &= (0, 0, -X^4(\mathbb{1} - T^2), X^3(\mathbb{1} - T^2), 0) \\
E_3 &= (X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2, 0) \\
E_4 &= (X^1T, X^2T, X^3T, X^4T, T^2 - \mathbb{1}),
\end{aligned}
$$

and let M be the submodule of $(S^4_\theta)^5$ generated by $\{E_1, E_2, E_3, E_4\}$. Note that there are no ordering ambiguities when defining these elements, since $|Z|^2$, $|W|^2$ and T are central. This module is the analogue of the local vector fields over the chart U_0, and the corresponding local triviality is reflected in the following result.

Proposition 4.5. *The module $M = \{E_1a + E_2b + E_3c + E_4d : a, b, c, d \in S^4_\theta\}$ is a free right S^4_θ-module of rank 4, and $\{E_1, E_2, E_3, E_4\}$ is a free generating set for M.*

Proof. By definition, $\{E_1, E_2, E_3, E_4\}$ generates M. To prove that $\{E_1, E_2, E_3, E_4\}$ is a free generating set, we assume that

$$E_1a + E_2b + E_3c + E_4d = 0 \tag{4.4}$$

and show that this implies that $a = b = c = d = 0$. Relation (4.4) is equivalent to the equations

$$
\begin{aligned}
&-X^2(\mathbb{1}-T^2)a + X^1|W|^2c + X^1Td = 0\\
&X^1(\mathbb{1}-T^2)a + X^2|W|^2c + X^2Td = 0\\
&-X^4(\mathbb{1}-T^2)b - X^3|Z|^2c + X^3Td = 0\\
&X^3(\mathbb{1}-T^2)b - X^4|Z|^2c + X^4Td = 0\\
&(\mathbb{1}-T^2)d = 0,
\end{aligned}
$$

which immediately implies that $d = 0$ (since $\mathbb{1} - T^2$ is not a zero divisor by Proposition 4.2), and the remaining equations may be written as

$$-X^2(\mathbb{1}-T^2)a + X^1|W|^2c = 0 \tag{4.5}$$

$$X^1(\mathbb{1}-T^2)a + X^2|W|^2c = 0 \tag{4.6}$$

$$-X^4(\mathbb{1}-T^2)b - X^3|Z|^2c = 0 \tag{4.7}$$

$$X^3(\mathbb{1}-T^2)b - X^4|Z|^2c = 0. \tag{4.8}$$

The sum of (4.5), multiplied from the left with X^1, and (4.6), multiplied from the left by X^2 gives

$$\big((X^1)^2 + (X^2)^2\big)\,|W|^2c = |Z|^2|W|^2c = 0$$

(using that $[X^1, X^2] = 0$), which implies that $c = 0$ since neither $|Z|^2$ nor $|W|^2$ is a zero divisor (by Proposition 4.2). Hence, one is left with the equations

$$
\begin{aligned}
X^2(\mathbb{1}-T^2)a &= 0 \qquad & X^1(\mathbb{1}-T^2)a &= 0\\
X^4(\mathbb{1}-T^2)b &= 0 \qquad & X^3(\mathbb{1}-T^2)b &= 0,
\end{aligned}
$$

and since $\mathbb{1} - T^2$ is not a zero divisor one obtains

$$X^2a = 0 \qquad X^1a = 0$$

$$X^4 b = 0 \qquad X^3 b = 0,$$

giving

$$\big((X^1)^2 + (X^2)^2\big)\, a = |Z|^2 a = 0$$
$$\big((X^3)^2 + (X^4)^2\big)\, b = |W|^2 b = 0,$$

which implies that $a = b = 0$. Thus, we have shown that $E_1 a + E_2 b + E_3 c + E_4 d = 0$ necessarily gives $a = b = c = d = 0$, which proves that $\{E_1, E_2, E_3, E_4\}$ is indeed a free generating set for M. □

4.3 Pseudo-Riemannian calculus for S^4_θ

In the module M, we introduce the restriction of the canonical metric on $(S^4_\theta)^5$:

$$g(U, V) = \sum_{a,b=1}^{4} (U^a)^* h_{ab} V^b$$

for $U = E_a U^a$ and $V = E_b V^b$, where

$$g_{ab} = \sum_{i=1}^{5} (E^i_a)^* (E^i_b),$$

which gives

$$(g_{ab}) = \begin{pmatrix} |Z|^2(\mathbb{1} - T^2)^2 & 0 & 0 & 0 \\ 0 & |W|^2(\mathbb{1} - T^2)^2 & 0 & 0 \\ 0 & 0 & |Z|^2|W|^2(\mathbb{1} - T^2) & 0 \\ 0 & 0 & 0 & \mathbb{1} - T^2 \end{pmatrix}.$$

As we shall be interested in perturbations of the standard metric, we introduce

$$g^\delta = \delta g$$

where $\delta \in S^4_\theta$ is assumed to be a hermitian, central and regular element. Since g^δ is diagonal, and each diagonal element is regular, it follows immediately that g^δ is non-degenerate on M; i.e.

$$g(U,V) = 0 \text{ for all } V \in M \implies U = 0.$$

Thus, the pair (M, g^δ) is a metric module (cf. Definition 1.2). To construct a real metric calculus over (M, g^δ) (cf. Definition 1.4), we need to associate derivations to E_1, E_2, E_3, E_4. In analogy with the classical situation, we consider the following derivations

$$\begin{aligned} \partial_1 &= (\mathbb{1} - T^2)\tilde{\partial}_1 & \partial_2 &= (\mathbb{1} - T^2)\tilde{\partial}_2 \\ \partial_3 &= \tilde{\partial}_3 & \partial_4 &= \tilde{\partial}_4, \end{aligned}$$

with $\tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3, \tilde{\partial}_4$ given as in Proposition 4.4. (Note that ∂_1 and ∂_2 are derivations since $\mathbb{1} - T^2$ is central.) These derivations generate an infinite-dimensional Lie algebra.

Proposition 4.6. *For any integer $n \geq 0$, the hermitian derivations*

$$\partial_1^{(n)} = T^n(\mathbb{1} - T^2)\tilde{\partial}_1, \quad \partial_2^{(n)} = T^n(\mathbb{1} - T^2)\tilde{\partial}_2, \quad \partial_3^{(n)} = T^n\tilde{\partial}_3, \quad \partial_4 = \tilde{\partial}_4$$

span an infinite-dimensional Lie algebra, where

$$\begin{aligned} &[\partial_1^{(n)}, \partial_2^{(n)}] = [\partial_1^{(n)}, \partial_3^{(n)}] = [\partial_2^{(n)}, \partial_3^{(n)}] = 0 \\ &[\partial_4, \partial_i^{(n)}] = (n+2)\partial_i^{(n+1)} - n\partial_i^{(n-1)}, \end{aligned}$$

for $i = 1, 2, 3$ (with the convention that $n\partial_i^{(n-1)} = 0$ if $n = 0$). Moreover, it follows that

$$\begin{aligned} &\partial_1|Z|^2 = 0 & &\partial_1|W|^2 = 0 & &\partial_1(\mathbb{1} - T^2) = 0 \\ &\partial_2|Z|^2 = 0 & &\partial_2|W|^2 = 0 & &\partial_2(\mathbb{1} - T^2) = 0 \\ &\partial_3|Z|^2 = 2|Z|^2|W|^2 & &\partial_3|W|^2 = -2|Z|^2|W|^2 & &\partial_3(\mathbb{1} - T^2) = 0 \end{aligned}$$

$$\partial_4|Z|^2 = 2|Z|^2T \qquad \partial_4|W|^2 = 2|W|^2T \qquad \partial_4(\mathbb{1} - T^2) = 2T(\mathbb{1} - T^2),$$

where $\partial_i \equiv \partial_i^{(0)}$ *for* $i = 1, 2, 3$.

Proof. The proof consists of straight-forward computations using the definition of $\tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3, \tilde{\partial}_4$ in Proposition 4.4. □

We let $\mathfrak{g}$ denote the (real) Lie algebra spanned by $\partial_1^{(n)}, \partial_2^{(n)}, \partial_3^{(n)}, \partial_4$, and let $\varphi : \mathfrak{g} \to M$ be the $\mathbb{R}$-linear map defined by

$$\varphi(\partial_i^{(n)}) = E_i T^n \quad \text{for } i = 1, 2, 3,$$
$$\varphi(\partial_4) = E_4.$$

The pair $(\mathfrak{g}, \varphi)$ is denoted by $\mathfrak{g}_\varphi$.

Proposition 4.7. *The triple* $(M, g^\delta, \mathfrak{g}_\varphi)$ *is a real metric calculus over* S^4_θ*.*

Proof. As already noted, the metric g^δ is non-degenerate on M and, by definition, $\{E_1, E_2, E_3, E_4\}$ generates M, which implies that the image of φ generates M. Finally, since every component of g^δ is hermitian, it follows that $g^\delta(E, E')$ is hermitian for all E, E' in the image of φ. This shows that the triple $(M, g^\delta, \mathfrak{g}_\varphi)$ satisfies all the requirements of a real metric calculus. □

Given a real metric calculus $(M, g^\delta, \mathfrak{g}_\varphi)$, there exists at most one metric and torsion-free connection on the module M (cf. Theorem 1.7). In Section 4.4.1 we proceed to show that such a connection exists, but let us first discuss certain aspects of localization on S^4_θ.

4.4 The local algebra $S^4_{\theta,\mathrm{loc}}$

For the classical 4-sphere, the vector fields corresponding to E_1, E_2, E_3, E_4 are linearly independent in the chart given in Section 4.1. Thus, as already mentioned, the module M does not correspond to the module of vector fields of S^4, but rather to a local trivialization in the chart U_0. In this chart, the functions $|w|^2$, $|z|^2$ and $1-t^2$ are invertible, and in analogy with this situation we shall introduce a localization of the algebra S^4_θ in order to be able to perform computations in a "noncommutative chart". Moreover, let us also consider the inverse of $\mathbb{1}+T^2$ (which is globally invertible as in the classical setting) as it is an algebraic prototype of the kind of perturbations of the metric that we will consider. To this end, we let S be the multiplicative subset of S^4_θ generated by $\mathbb{1}$, $|Z|^2$, $|W|^2$, $\mathbb{1}-T^2$ and $\mathbb{1}+T^2$. Since every element of S is central, S trivially fulfills the left (and right) Ore condition [34]. Hence, the localization of S^4_θ at S exists, and we denote it by $S^4_{\theta,\text{loc}}$. In other words, $S^4_{\theta,\text{loc}}$ is constructed from S^4_θ by adding the formal inverses of $|Z|^2$, $|W|^2$, $\mathbb{1}-T^2$ and $\mathbb{1}+T^2$. Clearly, $(M, g^\delta, \mathfrak{g}_\varphi)$, as constructed above, is also a real metric calculus over $S^4_{\theta,\text{loc}}$. In what follows, we shall discuss the two algebras in parallel.

Let us take a closer look at the structure of the noncommutative localization we have introduced. The algebra S^4_θ has been localized to include elements, which are classically not globally defined, and the corresponding free module M has been defined, which we claim to be the local trivialization of the module of vector fields. Now, the existence of a global module of vector fields, for which M is a localization is a natural question. For the noncommutative 4-sphere, a particular projective module presents itself as a natural candidate. Defining $\mathcal{P} : (S^4_\theta)^5 \to (S^4_\theta)^5$ as

$$\mathcal{P}(U) = \sum_{j=1}^{5} \left(\delta^{ij}\mathbb{1} - X^i X^j\right) U^j \tag{4.9}$$

where $U = e_i U^i$, it is easy to check that $\mathcal{P}^2 = \mathcal{P}$ since

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 = \mathbb{1}.$$

Let us denote by TS^4_θ the image of $\mathcal{P}$, which is, by definition, a finitely generated projective module. In classical geometry, $\mathcal{P}$ is the projector that defines the module of vector fields on S^4. Let us now show that, over the local algebra $S^4_{\theta,\text{loc}}$, this module is isomorphic to the module of the real metric calculus we have previously constructed.

Proposition 4.8. *The localization of TS^4_θ and M obtained by inverting $|Z|^2$, $|W|^2$ and $1 \pm T^2$ are isomorphic as right $S^4_{\theta,\mathrm{loc}}$-modules.*

Proof. First of all, it is easy to check that $E_1, E_2, E_3, E_4 \in TS^4_\theta$; for instance,

$$\sum_{i=1}^{5} X^i E^i_1 = X^1(-X^2) + X^2 X^1 = 0,$$

since $[X^1, X^2] = 0$, which implies that $\mathcal{P}(E_1) = E_1$ and $E_1 \in TS^4_\theta$. Thus, it follows that $M \subseteq TS^4_\theta$. Next, we will show that $TS^4_\theta \subseteq M$, by explicitly writing $\mathcal{P}(e_i)$ (for $i = 1, 2, 3, 4, 5$) as linear combinations of E_1, E_2, E_3, E_4. Since $\{\mathcal{P}(e_i)\}_{i=1}^5$ generates TS^4_θ, this shows that every element of TS^4_θ can be written in terms of E_1, E_2, E_3, E_4. We claim that

$$\begin{aligned}
\mathcal{P}(e_1) &= -E_1 X^2 |Z|^{-2} (\mathbb{1} - T^2)^{-1} + E_3 X^1 |Z|^{-2} (\mathbb{1} - T^2)^{-1} + E_4 X^1 T (\mathbb{1} - T^2)^{-1} \\
\mathcal{P}(e_2) &= E_1 X^1 |Z|^{-2} (\mathbb{1} - T^2)^{-1} + E_3 X^2 |Z|^{-2} (\mathbb{1} - T^2)^{-1} + E_4 X^2 T (\mathbb{1} - T^2)^{-1} \\
\mathcal{P}(e_3) &= -E_2 X^4 |W|^{-2} (\mathbb{1} - T^2)^{-1} - E_3 X^3 |W|^{-2} (\mathbb{1} - T^2)^{-1} + E_4 X^3 T (\mathbb{1} - T^2)^{-1} \\
\mathcal{P}(e_4) &= E_2 X^3 |W|^{-2} (\mathbb{1} - T^2)^{-1} - E_3 X^4 |W|^{-2} (\mathbb{1} - T^2)^{-1} + E_4 X^4 T (\mathbb{1} - T^2)^{-1} \\
\mathcal{P}(e_5) &= -E_4.
\end{aligned}$$

Let us show that $\mathcal{P}(e_1)$ can be written as the linear combination given above. The proof of the other four identities is analogous. First, one checks that

$$\mathcal{P}(e_1) = \left(1 - (X^1)^2, -X^2 X^1, -X^3 X^1, -X^4 X^1, -X^5 X^1\right).$$

Next, write

$$\begin{aligned}
U &= -E_1 X^2 |Z|^{-2} (\mathbb{1} - T^2)^{-1} + E_3 X^1 |Z|^{-2} (\mathbb{1} - T^2)^{-1} + E_4 X^1 T (\mathbb{1} - T^2)^{-1} \\
&= (U^1, U^2, U^3, U^4, U^5),
\end{aligned}$$

and compute the components one by one

$$\begin{aligned}
U^1 &= (X^2)^2|Z|^{-2} + (X^1)^2|W|^2|Z|^{-2}(\mathbb{1}-T^2)^{-1} + (X^1)^2T^2(\mathbb{1}-T^2)^{-1} \\
&= (X^2)^2|Z|^{-2} + (X^1)^2|W|^2|Z|^{-2}(\mathbb{1}-T^2)^{-1} \\
&\qquad - (X^1)^2(\mathbb{1}-T^2)(\mathbb{1}-T^2)^{-1} + (X^1)^2(\mathbb{1}-T^2)^{-1} \\
&= -(X^1)^2 + |Z|^{-2}(\mathbb{1}-T^2)^{-1}\left((X^2)^2(\mathbb{1}-T^2) + (X^1)^2(|Z|^2+|W|^2)\right) \\
&\qquad \text{(using } |Z|^2+|W|^2+T^2=\mathbb{1}\text{)} \\
&= -(X^1)^2 + |Z|^{-2}(\mathbb{1}-T^2)^{-1}\left((X^2)^2(\mathbb{1}-T^2) + (X^1)^2(\mathbb{1}-T^2)\right) \\
&= -(X^1)^2 + |Z|^{-2}\left((X^1)^2+(X^2)^2\right) = \mathbb{1}-(X^1)^2, \\
U^2 &= -X^1X^2|Z|^{-2} + X^2X^1|W|^2|Z|^{-2}(\mathbb{1}-T^2)^{-1} + X^2X^1T^2(\mathbb{1}-T^2)^{-1} \\
&\qquad \text{(using } [X^1,X^2]=0\text{)} \\
&= -X^2X^1|Z|^{-2}(\mathbb{1}-T^2)^{-1}\left(\mathbb{1}-T^2-|W|^2\right) + X^2X^1T^2(\mathbb{1}-T^2)^{-1} \\
&= -X^2X^1|Z|^{-2}(\mathbb{1}-T^2)^{-1}|Z|^2 + X^2X^1T^2(\mathbb{1}-T^2)^{-1} \\
&= -X^2X^1(\mathbb{1}-T^2)^{-1}\left(\mathbb{1}-T^2\right) = -X^2X^1, \\
U^3 &= -X^3X^1(\mathbb{1}-T^2)^{-1} + X^3X^1T^2(\mathbb{1}-T^2)^{-1} \\
&= -X^3X^1(\mathbb{1}-T^2)^{-1}(\mathbb{1}-T^2) = -X^3X^1, \\
U^4 &= -X^4X^1(\mathbb{1}-T^2)^{-1} + X^4X^1T^2(\mathbb{1}-T^2)^{-1} \\
&= -X^4X^1(\mathbb{1}-T^2)^{-1}(\mathbb{1}-T^2) = -X^4X^1, \\
U^5 &= (T^2-\mathbb{1})X^1T(\mathbb{1}-T^2)^{-1} = -X^1T = -X^1X^5.
\end{aligned}$$

Thus, we have shown that

$$\mathcal{P}(e_1) = -E_1X^2|Z|^{-2}(\mathbb{1}-T^2)^{-1} + E_3X^1|Z|^{-2}(\mathbb{1}-T^2)^{-1} + E_4X^1T(\mathbb{1}-T^2)^{-1},$$

which, together with the other four analogous computations, shows that TS^4_θ is contained in M. Combined with the fact that $M \subseteq TS^4_\theta$ one can conclude that $TS^4_\theta = M$ as right $S^4_{\theta,\mathrm{loc}}$-modules. □

4.4.1 Pseudo-Riemannian calculus

To construct a connection ∇ on M, such that $(M, g^\delta, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus, we consider the following class of perturbations. Let us assume that

$$\partial_a \delta = 2\alpha_a \delta,$$

where $\alpha_a \in S^4_{\theta,\mathrm{loc}}$ is hermitian, for $a = 1, 2, 3, 4$. The connection will be constructed over $S^4_{\theta,\mathrm{loc}}$, but we shall see that perturbations in certain directions give connections over S^4_θ.

Proposition 4.9. *Let $\delta \in S^4_{\theta,\mathrm{loc}}$ be a hermitian, regular and central element, such that $\partial_a \delta = 2\alpha_a \delta$, for $a = 1, 2, 3, 4$, where $\alpha_a \in S^4_{\theta,\mathrm{loc}}$ and $\alpha_a^* = \alpha_a$. Then there exists a unique connection ∇, such that $(M, g^\delta, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus over $S^4_{\theta,\mathrm{loc}}$, and ∇ is given by*

$$\begin{aligned}
\nabla_1 E_1 &= E_1\alpha_1 - E_2\alpha_2|Z|^2|W|^{-2} - E_3\left(\alpha_3|W|^{-2} + \mathbb{1}\right)(\mathbb{1} - T^2) \\
&\qquad - E_4(\alpha_4 + T)|Z|^2(\mathbb{1} - T^2) \\
\nabla_1 E_2 &= \nabla_2 E_1 = E_1\alpha_2 + E_2\alpha_1 \\
\nabla_1 E_3 &= \nabla_3 E_1 = E_1(\alpha_3 + |W|^2) + E_3\alpha_1 \\
\nabla_1 E_4 &= E_1(\alpha_4 + T) + E_4\alpha_1 \\
\nabla_4 E_1 &= E_1(\alpha_4 + 3T) + E_4\alpha_1 \\
\nabla_2 E_2 &= -E_1\alpha_1|W|^2|Z|^{-2} + E_2\alpha_2 - E_3\left(\alpha_3|Z|^{-2} - \mathbb{1}\right)(\mathbb{1} - T^2) \\
&\qquad - E_4(\alpha_4 + T)|W|^2(\mathbb{1} - T^2) \\
\nabla_2 E_3 &= \nabla_3 E_2 = E_2(\alpha_3 - |Z|^2) + E_3\alpha_2 \\
\nabla_2 E_4 &= E_2(\alpha_4 + T) + E_4\alpha_2 \\
\nabla_4 E_2 &= E_2(\alpha_4 + 3T) + E_4\alpha_2 \\
\nabla_3 E_3 &= -E_1\alpha_1|W|^2(\mathbb{1} - T^2)^{-1} - E_2\alpha_2|Z|^2(\mathbb{1} - T^2)^{-1} \\
&\qquad + E_3(\alpha_3 + |W|^2 - |Z|^2) - E_4(\alpha_4 + T)|Z|^2|W|^2 \\
\nabla_3 E_4 &= E_3(\alpha_4 + T) + E_4\alpha_3
\end{aligned}$$

$$\nabla_4 E_3 = E_3(\alpha_4 + 3T) + E_4\alpha_3$$

$$\nabla_4 E_4 = -E_1\alpha_1|Z|^{-2}(\mathbb{1} - T^2)^{-1} - E_2\alpha_2|W|^{-2}(\mathbb{1} - T^2)^{-1} - E_3\alpha_3|Z|^{-2}|W|^{-2} + E_4(\alpha_4 + T),$$

and

$$\nabla_{\partial_i^{(n)}} E_a = (\nabla_i E_a)\, T^n$$

for $i = 1, 2, 3,$ $a = 1, 2, 3, 4,$ *where* $\nabla_a \equiv \nabla_{\partial_a}$.

Proof. Let us recall (cf. [6]) that Koszul's formula

$$\begin{aligned} 2g(\nabla_{\partial_1} E_2, E_3) &= \partial_1 g(E_2, E_3) + \partial_2 g(E_3, E_1) - \partial_3 g(E_1, E_2) \\ &\quad - g\left(E_1, \varphi([\partial_2, \partial_3])\right) + g\left(E_2, \varphi([\partial_3, \partial_1])\right) + g\left(E_3, \varphi([\partial_1, \partial_2])\right), \end{aligned} \tag{4.10}$$

where $E_1, E_2, E_3 \in M_\varphi$ and $\partial_1, \partial_2, \partial_3 \in \mathfrak{g}$, gives a straight-forward way of finding a connection on M such that $(M, g^\delta, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus. Namely, if one finds $U_{ab} \in M$ such that

$$\begin{aligned} 2g(U_{ab}, E_c) &= \partial_a g(E_b, E_c) + \partial_b g(E_a, E_c) - \partial_c g(E_a, E_b) \\ &\quad - g\left(E_a, \varphi([\partial_b, \partial_c])\right) + g\left(E_b, \varphi([\partial_c, \partial_a])\right) + g\left(E_c, \varphi([\partial_a, \partial_b])\right) \end{aligned} \tag{4.11}$$

for all $a, b, c \in \{1, 2, 3, 4\}$ then (since the module M is free) one may set $\nabla_a E_b = U_{ab}$, and it follows that $(M, g^\delta, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus (see Corollary 3.8 in [6]). It is straight-forward to check that the expressions given in Proposition 4.9 fulfill (4.11). For instance, to check Koszul's formula for $\nabla_1 E_1$ one sets

$$K_a = g^\delta(\nabla_1 E_1, E_a) - \partial_1 g^\delta(E_1, E_a) + \frac{1}{2}\partial_a g^\delta(E_1, E_1) + g^\delta\left(E_1, \varphi([\partial_1, \partial_a])\right),$$

which gives

$$K_1 = g^\delta(\nabla_1 E_1, E_1) - \alpha_1\delta|Z|^2(\mathbb{1} - T^2)^2$$

$$
\begin{aligned}
&= \alpha_1 g^\delta(E_1, E_1) - \alpha_1\delta|Z|^2(\mathbb{1} - T^2)^2 \\
&= \alpha_1\delta|Z|^2(\mathbb{1} - T^2)^2 - \alpha_1\delta|Z|^2(\mathbb{1} - T^2)^2 = 0, \\
K_2 &= g^\delta(\nabla_1 E_1, E_2) + \alpha_2\delta|Z|^2(\mathbb{1} - T^2)^2 \\
&= -\alpha_2|Z|^2|W|^{-2}g^\delta(E_2, E_2) + \alpha_2\delta|Z|^2(\mathbb{1} - T^2)^2 \\
&= -\alpha_2|Z|^2|W|^{-2}|W|^2(\mathbb{1} - T^2)^2 + \alpha_2\delta|Z|^2(\mathbb{1} - T^2)^2 = 0, \\
K_3 &= g^\delta(\nabla_1 E_1, E_3) + \tfrac{1}{2}\partial_3\left(\delta|Z|^2(\mathbb{1} - T^2)^2\right) \\
&= -(\mathbb{1} + \alpha_3|W|^{-2})(\mathbb{1} - T^2)g^\delta(E_3, E_3) + \left(\alpha_3\delta|Z|^2 + \delta|Z|^2|W|^2\right)(\mathbb{1} - T^2)^2 \\
&= -(\mathbb{1} + \alpha_3|W|^{-2})\delta|Z|^2|W|^2(\mathbb{1} - T^2)^2 + \left(\alpha_3\delta|Z|^2 + \delta|Z|^2|W|^2\right)(\mathbb{1} - T^2)^2 = 0, \\
K_4 &= g^\delta(\nabla_1 E_1, E_4) + \tfrac{1}{2}\partial_4\left(\delta|Z|^2(\mathbb{1} - T^2)^2\right) - g^\delta(E_1, E_1)2T \\
&= -(\alpha_4 + T)\delta|Z|^2(\mathbb{1} - T^2)^2 + (\alpha_4 + 3T)\,\delta|Z|^2(\mathbb{1} - T^2)^2 - 2\delta|Z|^2T(\mathbb{1} - T^2)^2 \\
&= 0.
\end{aligned}
$$

This shows that $\nabla_1 E_1$ satisfies Koszul's formula (4.11). The other connection components can be checked in an analogous way.

Let us now consider the claim that

$$
\nabla_{\partial_i^{(n)}} E_a = (\nabla_i E_a)\, T^n.
$$

This fact is easily derived from Koszul's formula. Namely, note that

$$
\varphi\left([\partial_a, \partial_i^{(n)}]\right) = \varphi\left([\partial_a, \partial_i]\right) T^n + E_i(\partial_a T^n)
$$

and compute with Koszul's formula:

$$
\begin{aligned}
2g^\delta\left(\nabla_{\partial_i^{(n)}} E_b, E_c\right) &= \left(\partial_i g^\delta(E_b, E_c)\right) T^n + \partial_b\left(g^\delta(E_c, E_i T^n)\right) - \partial_c\left(g^\delta(E_i T^n, E_b)\right) \\
&\quad - g^\delta\left(E_i, \varphi([\partial_b, \partial_c])\right) T^n + g^\delta\left(E_b, \varphi([\partial_c, \partial_i^{(n)}])\right) + g^\delta\left(E_c, \varphi([\partial_i^{(n)}, \partial_b])\right) \\
&= \left(\partial_i g^\delta_{bc} + \partial_b g^\delta_{ci} - \partial_c g^\delta_{ib}\right) T^n + g^\delta_{ci}(\partial_b T^n) - g^\delta_{ib}(\partial_c T^n) - g^\delta\left(E_i, \varphi([\partial_b, \partial_c])\right) T^n \\
&\quad + g^\delta\left(E_b, \varphi([\partial_c, \partial_i])\right) T^n + g^\delta_{bi}(\partial_c T^n) + g^\delta\left(E_c, \varphi([\partial_i, \partial_b])\right) T^n - g^\delta_{ci}(\partial_b T^n)
\end{aligned}
$$

$$= 2g^{\delta}\left(\nabla_{\partial_i}E_b, E_c\right)T^n = 2g^{\delta}\left((\nabla_{\partial_i}E_b)T^n, E_c\right),$$

using that $g^{\delta}_{ab} = g^{\delta}_{ba}$ and the fact that T is hermitian and central. Since the metric is non-degenerate, it follows that

$$\nabla_{\partial_i^{(n)}}E_b = (\nabla_{\partial_i}E_b)T^n. \qquad \square$$

Note that if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, the connection in Proposition 4.9 only involves elements of S^4_θ and is therefore a valid connection for $(M, g^{\delta}, \mathfrak{g}_{\varphi}, \nabla)$ over S^4_θ. In particular, this is true for the unperturbed metric; i.e. for $\delta = \mathbb{1}$.

In Section 4.4 we constructed the projective module TS^4_θ and showed that it is isomorphic to M (as a right $S^4_{\theta,\mathrm{loc}}$-module) in Proposition 4.8. It is well known that a projective module defined by a projector $\mathcal{P}$ admits a connection of the form

$$\bar{\nabla}_{\partial}U = \mathcal{P}\left(e_i\partial(U^i)\right),$$

which is compatible with the canonical metric on the free module. Thus, having argued that one may regard the module M as a localization of the (global) module TS^4_θ, it is natural to ask if the connection on TS^4_θ, defined in the above manner, coincides with the connection found in Proposition 4.9 for the unperturbed metric.

Proposition 4.10. *Let $U = e_iU^i$ be an element of $TS^4_\theta = \mathcal{P}((S^4_\theta)^5)$ (as defined in* (4.9)*) and set*

$$\bar{\nabla}_a U = \mathcal{P}\left(e_i\partial_a(U^i)\right),$$

for $a = 1, 2, 3, 4$. Then $\bar{\nabla}_a E_b = \nabla_a E_b$ for $a, b = 1, 2, 3, 4$ and $\delta = \mathbb{1}$.

Proof. Let us prove the statement by computing $\bar{\nabla}_a E_b$ for $a, b = 1, 2, 3, 4$ (i.e. 16 components in total) and compare it with Proposition 4.9 for $\delta = \mathbb{1}$. Since the calculations are straight-forward we shall only present one of them here to illustrate how they are performed. Thus,

$$\begin{aligned}
\bar{\nabla}_1 E_1 &= \mathcal{P}\left(\partial_1(-X^2(\mathbb{1}-T^2), X^1(\mathbb{1}-T^2), 0, 0, 0)\right) \\
&= \mathcal{P}\left((-X^1, -X^2, 0, 0, 0)\right)(\mathbb{1}-T^2)^2 \\
&= (-X^1, -X^2, 0, 0, 0)(\mathbb{1}-T^2)^2 - e_i X^i\left(-(X^1)^2 - (X^2)^2\right)(\mathbb{1}-T^2)^2 \\
&= (-X^1, -X^2, 0, 0, 0)(\mathbb{1}-T^2)^2 + (X^1, X^2, X^3, X^4, T)|Z|^2(\mathbb{1}-T^2)^2 \\
&= \left(X^1(|Z|^2-\mathbb{1}), X^2(|Z|^2-\mathbb{1}), X^3|Z|^2, X^4|Z|^2, T\right)(\mathbb{1}-T^2)^2.
\end{aligned}$$

Now, for comparison, we find $\nabla_1 E_1$ from Proposition 4.9 when $\delta = \mathbb{1}$:

$$\begin{aligned}
\nabla_1 E_1 &= -E_3(\mathbb{1}-T^2) - E_4 T|Z|^2(\mathbb{1}-T^2) \\
&= -(X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2, 0)(\mathbb{1}-T^2) \\
&\qquad - (X^1T, X^2T, X^3T, X^4T, T^2-\mathbb{1})T|Z|^2(\mathbb{1}-T^2) \\
&= |W|^2 + T^2|Z|^2 = \mathbb{1} - |Z|^2 - T^2 + T^2|Z|^2 = (\mathbb{1}-T^2)(\mathbb{1}-|Z|^2) \\
&= -\left(X^1(\mathbb{1}-|Z|^2), X^2(\mathbb{1}-|Z|^2), -X^3|Z|^2, -X^4|Z|^2, -T\right)(\mathbb{1}-T^2)^2,
\end{aligned}$$

which equals $\bar{\nabla}_1 E_1$. The remaining computations are done in an analogous way. □

4.5 The Gauss-Bonnet-Chern theorem

4.5.1 The trace

A trace corresponds to integration on a classical manifold. A trace can be assigned to S^4_θ in an analogous way as integration is defined for S^4. Namely, for a given basis element e^I with $I = (j, k, l, m, \epsilon)$ (in the notation of Section 4.2.1) one defines a linear map $\phi : S^4_\theta \to C^\infty(S^4)$ via

$$\phi(e^I) = e^{i(j-k)\xi_1}\left(\cos\varphi\cos\psi\right)^{j+k} e^{i(l-m)\xi_2}\left(\sin\varphi\cos\psi\right)^{l+m}\left(\sin\psi\right)^{\epsilon}$$

and

$$\tau(e^I) = \int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 \int_{-\pi/2}^{\pi/2} d\psi \int_0^{\pi/2} d\varphi\, \phi(e^I) \sin\varphi \cos\varphi \cos^3\psi,$$

which are extended to S^4_θ as linear maps (cf. [41] for a similar approach in the unperturbed case). The volume element of the round metric g_0 on S^4 is given by $\sin\varphi\cos\varphi\cos^3\psi\, d\xi_1 d\xi_2 d\psi d\varphi$ and for the perturbed metric δg_0 one obtains

$$dV = \delta^2 \sin\varphi\cos\varphi\cos^3\psi\, d\xi_1 d\xi_2 d\psi d\varphi.$$

In order to reflect the fact that one would like to integrate with respect to the perturbed metric, we introduce

$$\tau_\delta(a) = \tau\left(\delta a \delta\right).$$

Let us note a few properties of the linear functional τ_δ. We start with the following lemma:

Lemma 4.11. *Assume that $\theta \notin \mathbb{Q}$ and $\delta \in Z(S^4_\theta)$. If $e^I \notin Z(S^4_\theta)$ then $\tau_\delta(e^I) = 0$.*

Proof. Let us start by considering $\tau_\delta(e^I)$ when $I = (j,k,l,m,0)$. Assuming that $\delta \in Z(S^4_\theta)$ and $\theta \notin \mathbb{Q}$, one may write

$$\delta^2 = \sum_{i_1 i_2 \epsilon} a_{i_1 i_2 \epsilon} (|Z|^2)^{i_1} |W|^2)^{i_2} T^\epsilon$$

by Proposition 4.3, and

$$\begin{aligned}\tau_\delta(e^I) &= \sum_{i_1 i_2 \epsilon} a_{i_1 i_2 \epsilon} \tau\left(e^{(j,k,l,m,0)} (|Z|^2)^{i_1} (|W|^2)^{i_2} T^\epsilon\right) \\ &= \sum_{i_1 i_2 \epsilon} a_{i_1 i_2 \epsilon} \tau\left(e^{(j+i_1, k+i_1, l+i_2, m+i_2, \epsilon)}\right).\end{aligned}$$

Since

$$\int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 e^{ik_1\xi_1} e^{ik_2\xi_2} = \begin{cases} 4\pi^2 \text{ if } k_1 = k_2 = 0, \\ 0 \text{ otherwise.} \end{cases}$$

We conclude that $\tau_\delta(e^{(j,k,l,m,0)}) = 0$ if $j \neq k$ or $l \neq m$, which is equivalent to $e^{(j,k,l,m,0)} \notin Z(S^4_\theta)$. Similarly, for $I = (j,k,l,m,1)$, terms proportional to $a_{i_1 i_2 1}$ are of the form

$$a_{i_1 i_2 1} \tau \big(e^{(j+i_1, k+i_1, l+i_2, m+i_2, 0)} - e^{(j+i_1+1, k+i_1+1, l+i_2, m+i_2, 0)} \\ - e^{(j+i_1, k+i_1, l+i_2+1, m+i_2+1, 0)} \big).$$

The same argument as above implies that $\tau_\delta(e^{(j,k,l,m,1)}) = 0$ if $j \neq k$ or $l \neq m$. □

Proposition 4.12. *If $\delta \in Z(S^4_\theta)$ and $\theta \notin \mathbb{Q}$, then τ_δ satisfies*

(i) $\tau_\delta([a,b]) = 0$,

(ii) $\tau_\delta(a^*) = \overline{\tau_\delta(a)}$,

for all $a, b \in S^4_\theta$.

Proof. To prove (1), we show that $\tau_\delta([e^{I_1}, e^{I_2}]) = 0$. By using Lemma 4.1 one obtains

$$\tau_\delta\left([e^{I_1}, e^{I_2}]\right) = \left(q^{(l_1-m_1)(j_2-k_2)} - q^{(l_2-m_2)(j_1-k_1)}\right) \tau_\delta\left(e^{I_1+I_2}\right)$$

if $\epsilon_1 + \epsilon_2 \leq 1$, and

$$\tau_\delta([e^{I_1}, e^{I_2}]) = \left(q^{(l_1-m_1)(j_2-k_2)} - q^{(l_2-m_2)(j_1-k_1)}\right) \times \\ \left(e^{(\hat{I}_1+\hat{I}_2,0)} - e^{(\hat{I}_1+\hat{I}_2+\hat{1}_Z,0)} - e^{(\hat{I}_1+\hat{I}_2+\hat{1}_W,0)}\right) \tag{4.12}$$

if $\epsilon_1 + \epsilon_2 = 2$. From Lemma 4.11 it follows that if $j_1 + j_2 \neq k_1 + k_2$ or $l_1 + l_2 \neq m_1 + m_2$ then $\tau_\delta([e^{I_1}, e^{I_2}]) = 0$. On the other hand, if $j_1 + j_2 = k_1 + k_2$ and $l_1 + l_2 = m_1 + m_2$ then

$$(l_1 - m_1)(j_2 - k_2) = (l_2 - m_2)(j_1 - k_1)$$

which gives $\tau_\delta([e^{I_1}, e^{I_2}]) = 0$ from (4.12).

For (2), we again consider $a = \sum_I a_I e^I$ and find

$$\tau_\delta(a^*) = \sum_I \overline{a_I} \tau_\delta\left((e^I)^*\right) = \sum_I q^{(j-k)(l-m)} \overline{a_I} \tau_\delta(e^I).$$

Since $\tau_\delta(e^I) = 0$ if $j \neq k$ or $l \neq m$ (by Lemma 4.11), the above sum equals

$$\tau_\delta(a^*) = \sum_I \overline{a_I} \tau_\delta(e^I) = \overline{\tau_\delta(a)}$$

using that $\tau_\delta(e^I) \in \mathbb{R}$ when $j = k$ and $l = m$. □

For the forthcoming discussion of the Gauss-Bonnet-Chern theorem, we extend τ_δ to the commutative subalgebra $Z_{\text{loc}} \subseteq S^4_{\theta,\text{loc}}$ given by

$$Z_{\text{loc}} = \mathbb{C}\left\langle \mathbb{1}, |Z|^2, |Z|^{-2}, |W|^2, |W|^{-2}, T, (\mathbb{1} - T^2)^{-1}, (\mathbb{1} + T^2)^{-1} \right\rangle,$$

by defining a homomorphism (of commutative $*$-algebras) $\phi_0 : Z_{\text{loc}} \to C^\infty(U_0)$ as

$$\phi_0(|Z|^2) = \cos^2(\varphi)\cos^2(\psi) \qquad \phi_0(|W|^2) = \sin^2(\varphi)\cos^2(\psi)$$
$$\phi_0(\mathbb{1}) = 1 \qquad \phi_0(T) = \sin(\psi)$$

as well as

$$\begin{aligned}
\phi_0\left((\mathbb{1} - T^2)^{-1}\right) &= \frac{1}{\cos^2(\psi)} = \frac{1}{\phi_0(\mathbb{1} - T^2)} \\
\phi_0\left((\mathbb{1} + T^2)^{-1}\right) &= \frac{1}{1 + \sin^2(\psi)} = \frac{1}{\phi_0(\mathbb{1} + T^2)} \\
\phi_0\left(|Z|^{-2}\right) &= \frac{1}{\cos^2(\varphi)\cos^2(\psi)} = \frac{1}{\phi_0(|Z|^2)} \\
\phi_0\left(|W|^{-2}\right) &= \frac{1}{\sin^2(\varphi)\cos^2(\psi)} = \frac{1}{\phi_0(|W|^2)}.
\end{aligned}$$

For ϕ_0 to be well-defined, one needs to check that the above definition is compatible with the relations in Z_{loc}. The only nontrivial relation to check is

$$\begin{aligned}\phi_0(|Z|^2+|W|^2+T^2-\mathbb{1}) &= \cos^2(\varphi)\cos^2(\psi)+\sin^2(\varphi)\cos^2(\psi)+\sin^2(\psi)-1\\ &= \cos^2(\psi)+\sin^2(\psi)-1=0,\end{aligned}$$

which shows that ϕ_0 is indeed well-defined. Note that ϕ_0 coincides with ϕ on $Z(S^4_\theta)$. Finally, for $\delta \in Z_{\text{loc}}$, we define

$$\tau_{\delta,\text{loc}}(a) = \int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 \int_{-\pi/2}^{\pi/2} d\psi \int_0^{\pi/2} d\varphi\, \phi_0(a)\phi_0(\delta^2)\cos^3\psi \sin\varphi\cos\varphi,$$

for $a \in Z_{\text{loc}}$, whenever the above integral is convergent. (For instance, the integral does not exists when $a = (\mathbb{1}-T^2)^{-2}$.)

4.5.2 The Gauss-Bonnet-Chern theorem

For a closed surface Σ, the Gauss-Bonnet theorem states that the integral of the Gaussian curvature over Σ is proportional to the Euler characteristic of Σ. This provides an important link between topology and Riemannian geometry. In particular, since the Euler characteristic is independent of any metric tensor, the integral gives the same value if we perturb the metric. This theorem has been generalized to closed even dimensional Riemannian manifolds, where the scalar curvature is replaced by the Pfaffian of the curvature form. In case of a closed four dimensional manifold M, the Gauss-Bonnet-Chern theorem states that

$$\chi(M) = \frac{1}{32\pi^2}\int_M \left(R^{abcd}R_{abcd} - 4Ric_{ab}Ric^{ab} + S^2\right) d\mu \tag{4.13}$$

where R_{abcd} is the Riemann curvature tensor, Ric_{ab} is the Ricci curvature, S denotes the scalar curvature and $\chi(M)$ is the Euler characteristic of M. (Recall that $\chi(S^4) = 2$.) In this section, we will show that there exists an analogue of the Gauss-Bonnet-Chern theorem for the pseudo-Riemannian calculus of S^4_θ we have developed. Our approach is based on the fact that all coefficients of the curvature tensor lie in the commutative subalgebra Z_{loc}, which allows us to compute directly the Pfaffian of the curvature form.

Let us consider a metric perturbation $\delta \in Z_{\mathrm{loc}}$ that is a polynomial in T, and such that δ is invertible in Z_{loc}. It follows that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (in the notation of Section 4.4.1), since $\partial_1 T = \partial_2 T = \partial_3 T = 0$. Moreover,

$$\partial_4 \delta = \delta'(T)(\partial_4 T)\delta^{-1}\delta = -\delta'(T)(\mathbb{1} - T^2)\delta^{-1}\delta$$

where $\delta'(T)$ denotes the (formal) derivative of the polynomial $\delta(T)$ with respect to T, which implies that

$$\alpha \equiv \alpha_4 = -\frac{1}{2}(\mathbb{1} - T^2)\delta'\delta^{-1}.$$

An example of such a perturbation is given by $\delta = (\mathbb{1} + T^2)^N$ which gives

$$\alpha = -NT(\mathbb{1} - T^2)(\mathbb{1} + T^2)^{-1}.$$

Moreover, by α' we shall denote the (formal) derivative of $\alpha(T)$ with respect to T. Recall the formulas from Proposition 4.9 in the situation where $\alpha_1 = \alpha_2 = \alpha_3 = 0$ are:

$$\begin{aligned}
\nabla_1 E_1 &= -E_3(\mathbb{1} - T^2) - E_4(\alpha + T)|Z|^2(\mathbb{1} - T^2)\\
\nabla_2 E_2 &= E_3(\mathbb{1} - T^2) - E_4(\alpha + T)|W|^2(\mathbb{1} - T^2)\\
\nabla_3 E_3 &= E_3(|W|^2 - |Z|^2) - E_4(\alpha + T)|Z|^2|W|^2\\
\nabla_4 E_4 &= E_4(\alpha + T)
\end{aligned}$$

$$\begin{aligned}
&\nabla_1 E_2 = \nabla_2 E_1 = 0 && \nabla_1 E_3 = \nabla_3 E_1 = E_1|W|^2\\
&\nabla_1 E_4 = E_1(\alpha + T) && \nabla_4 E_1 = E_1(\alpha + 3T)\\
&\nabla_2 E_3 = -E_2|Z|^2 && \nabla_3 E_2 = -E_2|Z|^2\\
&\nabla_2 E_4 = E_2(\alpha + T) && \nabla_4 E_2 = E_2(\alpha + 3T)
\end{aligned}$$

$$\nabla_3 E_4 = E_3(\alpha + T) \qquad \nabla_4 E_3 = E_3(\alpha + 3T).$$

It is now straight-forward to compute the curvature:

$$R(\partial_1, \partial_2)E_1 = -E_2\left(\mathbb{1} - (\alpha + T)^2\right)|Z|^2(\mathbb{1} - T^2)$$
$$R(\partial_1, \partial_2)E_2 = E_1\left(\mathbb{1} - (\alpha + T)^2\right)|W|^2(\mathbb{1} - T^2)$$
$$R(\partial_1, \partial_2)E_3 = 0 \qquad R(\partial_1, \partial_2)E_4 = 0$$
$$R(\partial_1, \partial_3)E_1 = -E_3\left(\mathbb{1} - (\alpha + T)^2\right)|Z|^2(\mathbb{1} - T^2)$$
$$R(\partial_1, \partial_3)E_3 = E_1\left(\mathbb{1} - (\alpha + T)^2\right)|Z|^2|W|^2$$
$$R(\partial_1, \partial_3)E_2 = 0 \qquad R(\partial_1, \partial_3)E_4 = 0$$
$$R(\partial_1, \partial_4)E_1 = -E_4\left(\mathbb{1} + \alpha'\right)|Z|^2(\mathbb{1} - T^2)^2$$
$$R(\partial_1, \partial_4)E_4 = E_1(\mathbb{1} + \alpha')(\mathbb{1} - T^2)$$
$$R(\partial_1, \partial_4)E_2 = 0 \qquad R(\partial_1, \partial_4)E_3 = 0$$
$$R(\partial_2, \partial_3)E_2 = -E_3\left(\mathbb{1} - (\alpha + T)^2\right)|W|^2(\mathbb{1} - T^2)$$
$$R(\partial_2, \partial_3)E_3 = E_2\left(\mathbb{1} - (\alpha + T)^2\right)|Z|^2|W|^2$$
$$R(\partial_2, \partial_3)E_1 = 0 \qquad R(\partial_2, \partial_3)E_4 = 0$$
$$R(\partial_2, \partial_4)E_2 = -E_4(\mathbb{1} + \alpha')|W|^2(\mathbb{1} - T^2)^2$$
$$R(\partial_2, \partial_4)E_4 = E_2(\mathbb{1} + \alpha')(\mathbb{1} - T^2)$$
$$R(\partial_2, \partial_4)E_1 = 0 \qquad R(\partial_2, \partial_4)E_3 = 0$$
$$R(\partial_3, \partial_4)E_3 = -E_4(\mathbb{1} + \alpha')|Z|^2|W|^2(\mathbb{1} - T^2)$$
$$R(\partial_3, \partial_4)E_4 = E_3(\mathbb{1} + \alpha')(\mathbb{1} - T^2)$$
$$R(\partial_3, \partial_4)E_1 = 0 \qquad R(\partial_3, \partial_4)E_2 = 0$$

and the only non-zero curvature components $R_{abpq} = g^\delta(E_a, R(\partial_p, \partial_q)E_b)$ turn out to be

$$R_{1212} = \delta\left(\mathbb{1} - (\alpha + T)^2\right)|Z|^2|W|^2(\mathbb{1} - T^2)^3$$
$$R_{1313} = \delta\left(\mathbb{1} - (\alpha + T)^2\right)|Z|^4|W|^2(\mathbb{1} - T^2)^2$$

$$\begin{aligned}
R_{1414} &= \delta(\mathbb{1}+\alpha')|Z|^2(\mathbb{1}-T^2)^3 \\
R_{2323} &= \delta\left(\mathbb{1}-(\alpha+T)^2\right)|Z|^2|W|^4(\mathbb{1}-T^2)^2 \\
R_{2424} &= \delta(\mathbb{1}+\alpha')|W|^2(\mathbb{1}-T^2)^3 \\
R_{3434} &= \delta(\mathbb{1}+\alpha')|Z|^2|W|^2(\mathbb{1}-T^2)^2.
\end{aligned}$$

In the local algebra Z_{loc}, the metric g^δ is invertible since δ is invertible. Moreover, every component of the metric, as well as of the curvature, is central, which implies that there exists a naive analogue of the integrand in (4.13). Setting

$$\begin{aligned}
R^{abcd} &= (g^\delta)^{ap}(g^\delta)^{bq}(g^\delta)^{cr}(g^\delta)^{ds}R_{pqrs} \\
Ric_{ab} &= (g^\delta)^{pq}R_{apbq} \\
Ric^{ab} &= (g^\delta)^{ap}(g^\delta)^{bq}Ric_{pq} \\
S &= (g^\delta)^{ab}Ric_{ab}
\end{aligned}$$

one finds that

$$R^{abcd}R_{abcd} - 4Ric_{ab}Ric^{ab} + S^2 = 24\left(\mathbb{1}-(\alpha+T)^2\right)(\mathbb{1}+\alpha')(\mathbb{1}-T^2)^{-1}\delta^{-2}. \quad (4.14)$$

The following is one of the simplest cases. Suppose δ is a polynomial in T and $\partial_4\delta = \alpha\delta$. Since $\partial_4 T = T^2 - \mathbb{1}$, α is again a polynomial in T. Indeed, $\phi_0(\alpha)\big|_{\psi=\frac{\pi}{2}} = \phi_0(\alpha)\big|_{\psi=-\frac{\pi}{2}} = 0$ is a sufficient condition.

Theorem 4.13. *Let $\delta(T)$ be an invertible polynomial in Z_{loc} and define α via the relation $\partial_4\delta = 2\alpha\delta$ (and we assume $\partial_j(\delta) = 0$, $j = 1, 2, 3$). If*

$$\phi_0(\alpha)\big|_{\psi=\frac{\pi}{2}} = \phi_0(\alpha)\big|_{\psi=-\frac{\pi}{2}} = 0,$$

then

$$\chi(S^4_\theta) = \frac{1}{32\pi^2}\tau_{\delta,\text{loc}}\left(R^{abcd}R_{abcd} - 4Ric_{ab}Ric^{ab} + S^2\right) = 2.$$

Proof. Since δ is a polynomial in T and $\partial_4 T = T^2 - \mathbb{1}$, one can express α in terms of T and, by a slight abuse of notation, we let $\alpha(t)$ be such that $\phi_0(\alpha) = \alpha(\sin\psi)$. In this notation, the assumption on $\phi_0(\alpha)$ may be stated as $\alpha(1) = \alpha(-1) = 0$.

From the definition of $\tau_{\delta,\text{loc}}$ it follows that

$$\begin{aligned}\chi &= \frac{1}{32\pi^2}\tau_{\delta,\text{loc}}\left(R^{abcd}R_{abcd} - 4Ric_{ab}Ric^{ab} + S^2\right)\\ &= I_\psi \int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 \int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi,\end{aligned}$$

where

$$I_\psi = \frac{24}{32\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - (\alpha(\sin\psi) + \sin\psi)^2\right)(1 + \alpha'(\sin\psi))\cos\psi d\psi.$$

Substituting $t = \sin\psi$ gives

$$I_\psi = \frac{24}{32\pi^2}\int_{-1}^{1}\left(1 - (\alpha(t) + t)^2\right)(1 + \alpha'(t))dt,$$

which can easily be integrated to

$$I_\psi = \frac{24}{32\pi^2}\left[\alpha(t) + t - \tfrac{1}{3}\left(\alpha(t) + t\right)^3\right]_{-1}^{1} = \frac{24}{32\pi^2}\left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right) = \frac{1}{\pi^2},$$

since $\alpha(1) = \alpha(-1) = 0$. Finally, one obtains

$$\begin{aligned}\chi = I_\psi &\int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 \int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi\\ &= \frac{1}{\pi^2}\int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 \int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi = \frac{1}{\pi^2}\cdot 4\pi^2 \cdot \frac{1}{2} = 2,\end{aligned}$$

which proves the statement. □

In this chapter, we have preferred to stay in the purely algebraic regime, and have thus not considered any smooth completion of S^4_θ, in order to stress the point that our results do not depend on the analytic structure. However, we expect that Theorem 4.13 holds true even for more general perturbations in a potentially larger algebra. For instance, if $\delta = e^{\lambda T}$ exists for all $\lambda \in \mathbb{R}$, one obtains $\alpha = \frac{\lambda}{2}(T^2 - \mathbb{1})$ which clearly fulfills the conditions of Theorem 4.13. Moreover, one may consider perturbations given, not only as functions of T, but as more general elements of Z_{loc}. Although our approach to the Gauss-Bonnet-Theorem may be too naive to have any impact on the general problem, we hope that our investigations will contribute to the growing understanding of Riemannian curvature in noncommutative geometry.

Chapter 5

On the Gauss-Bonnet-Chern type theorem for the noncommutative 4-torus

5.1 The noncommutative 4-torus

After having developed a general framework for Riemannian curvature of a real metric calculus for the noncommutative 3-sphere and the noncommutative 4-sphere, we consider another example the noncommutative 4-torus to probe the competence of a pseudo-Riemannian calculus (cf. [1] for a related approach for $\mathbb{T}^2_\theta$ that uses the concrete embedding of the torus into $\mathbb{R}^4$). For the noncommutative torus, our construction of a Levi-Civita connection and its corresponding curvature is similar to the approach taken in [40].

As we shall work in close analogy with differential geometry, let us briefly review the geometry of the 4-torus. We consider the 4-torus as embedded in $\mathbb{R}^8$ with the induced flat metric. Concretely, let us consider the following parametrization

$$\vec{x} = (x^1, x^2, x \ldots, x^8) = (\cos u_1, \sin u_1, \ldots, \cos u_4, \sin u_4),$$

which implies that the tangent space at each point is spanned by

$$\partial_{u_1}\vec{x} = (-\sin u_1, \cos u_1, 0,0,0,0,0,0) = (-x^2, x^1, 0,0,0,0,0,0),$$
$$\partial_{u_2}\vec{x} = (0,0,-\sin u_2, \cos u_2, 0,0,0,0) = (0,0,-x^4, x^3, 0,0,0,0),$$
$$\partial_{u_3}\vec{x} = (0,0,0,0,-\sin u_3, \cos u_3, 0,0) = (0,0,0,0,-x^6, x^5, 0,0),$$
$$\partial_{u_4}\vec{x} = (0,0,0,0,0,0,-\sin u_4, \cos u_4) = (0,0,0,0,0,0,-x^8, x^7),$$

from which the induced metric is obtained as

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Setting $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$, $z^3 = x^5 + ix^6$ and $z^4 = x^7 + ix^8$, and $\partial_j = \partial_{u_j}$ yield

$$\begin{array}{llll} \partial_1 z^1 = iz^1 & \partial_1 z^2 = 0 & \partial_1 z_3 = 0 & \partial_1 z_4 = 0 \\ \partial_2 z^1 = 0 & \partial_2 z^2 = iz^2 & \partial_2 z_3 = 0 & \partial_2 z_4 = 0 \\ \partial_3 z^1 = 0 & \partial_1 z^2 = 0 & \partial_1 z_3 = iz^3 & \partial_1 z_4 = 0 \\ \partial_1 z^1 = 0 & \partial_1 z^2 = 0 & \partial_1 z_3 = 0 & \partial_1 z_4 = iz^4. \end{array}$$

As the noncommutative torus $\mathbb{T}^4_\theta$, we consider the unital $*$-algebra generated by four unitary operators $U_1, \ldots, U_4$ satisfying $U_j U_k = q_{jk} U_k U_j$ with $q_{jk} = e^{2\pi i \theta_{jk}}$. Note that necessarily $\theta_{jk} = -\theta_{kj}$. We introduce

$$\begin{array}{ll} X^1 = \frac{1}{2}(U_1 + U_1^*) & X^2 = \frac{1}{2i}(U_1 - U_1^*) \\ X^3 = \frac{1}{2}(U_2 + U_2^*) & X^4 = \frac{1}{2i}(U_2 - U_2^*) \\ X^5 = \frac{1}{2}(U_3 + U_3^*) & X^6 = \frac{1}{2i}(U_3 - U_3^*) \\ X^7 = \frac{1}{2}(U_4 + U_4^*) & X^8 = \frac{1}{2i}(U_4 - U_4^*). \end{array}$$

In analogy with the geometrical setting, let M be the right submodule of $(\mathbb{T}^4_\theta)^4$ generated by

$$E_1 = (-x^2, x^1, 0, 0, 0, 0, 0, 0),$$
$$E_2 = (0, 0, -x^4, x^3, 0, 0, 0, 0),$$
$$E_3 = (0, 0, 0, 0, -x^6, x^5, 0, 0),$$
$$E_4 = (0, 0, 0, 0, 0, 0, -x^8, x^7),$$

and for $V, W \in M$, with $V = E_a V^a$ and $W = E_a W^a$ we set

$$g(V, W) = \sum_{a=1}^{4} (V^a)^* W^a.$$

Proposition 5.1. *Let E_j, $j = 1, 2, 3, 4$ be as above. The set $\{E_1, E_2, E_3, E_4\} \subset M$ provides a free generating set for M as a free module and g is a nondegenerate hermitian form on M. Therefore, (M, g) is a free metric $\mathbb{T}^4_\theta$-module.*

Proof. First, let us show that E_1, E_2, E_3 and E_4 are free generators.

$$\begin{aligned}
&E_1 a + E_2 b + E_3 c + E_4 d = 0\\
&\implies (-x^2 a, x^1 a, -x^4 b, x^3 b, -x^6 c, x^5 c, -x^8 d, x^7 d) = (0, 0, 0, 0)\\
&\implies \begin{cases} ((X^1)^2 + (X^2)^2)\, a = 0\\ ((X^3)^2 + (X^4)^2)\, b = 0\\ ((X^5)^2 + (X^6)^2)\, c = 0\\ ((X^7)^2 + (X^8)^2)\, d = 0 \end{cases} \Leftrightarrow \begin{cases} U_1 U_1^* a = 0\\ U_2 U_2^* b = 0\\ U_3 U_3^* c = 0\\ U_4 U_4^* d = 0 \end{cases} \Leftrightarrow \quad a = b = c = d = 0.
\end{aligned}$$

Next, we prove that g is nondegenerate on M. Let $V, W \in M$ and write $V = E_a V^a$. Assume that $g(V, W) = 0$ for all $W \in M$, which may be equivalently stated as $g(V, E_a) = 0$ for $a = 1, 2, 3, 4$. The last equation immediately gives $V^1 = V^2 = V^3 = V^4 = 0$. □

Next, we let $\mathfrak{g}$ be the real Abelian Lie algebra generated by the four hermitian derivations $\partial_1, \partial_2, \partial_3$ and ∂_4, given by

$$\partial_k U_k = iU_k \quad \text{and} \quad \partial_k U_\ell = 0 \text{ if } k \neq \ell.$$

Note that $[\partial_k, \partial_\ell] = 0$ for all $k, \ell = 1, 2, 3, 4$. Together with the map $\varphi : \mathfrak{g} \to M$ defined by $\varphi(\partial_a) = E_a$ extended linearly, it is easy to check that $(M, g, \mathfrak{g}_\varphi)$ is a real metric calculus over $\mathbb{T}^4_\theta$. Furthermore, we note that with respect to the basis $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ of $\mathfrak{g}$ the metric can be written as

$$(g_{ab}) = (g(E_a, E_b)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and is invertible.

One is now in a position to use Corollary 1.10 to find a unique connection ∇ on M such that $(M, g, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus. However, since $g(E_a, E_b) = \delta_{ab}\mathbb{1}$ and $[\partial_a, \partial_b] = 0$, the only solution to

$$\begin{aligned} 2g(V_{ab}, E_c) &= \partial_a g(E_b, E_c) + \partial_b g(E_a, E_c) - \partial_c g(E_a, E_b) \\ &\quad - g\left(E_a, \varphi([\partial_b, \partial_c])\right) + g\left(E_b, \varphi([\partial_c, \partial_a])\right) + g\left(E_c, \varphi([\partial_a, \partial_b])\right) \end{aligned}$$

is $V_{ab} = 0$, which gives $\nabla_\partial U = 0$ for all $\partial \in \mathfrak{g}$ and $U \in M$. Hence, the curvature of the corresponding pseudo-Riemannian calculus vanishes identically, and its scalar curvature is 0.

As done in [40] one can obtain more intricate results by conformally perturbing the flat metric on the noncommutative 4-torus

$$g_\alpha(V, W) = \sum_{a=1}^{4} (V^a)^* \, e^\alpha W^a$$

for some hermitian element $\alpha = \alpha^* \in \mathbb{T}^4_\theta$. Of course, to define

$$e^\alpha = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!},$$

one needs to consider the smooth part of the C^*-algebra generated by U_j. One can easily check that $(M, g_\alpha, \mathfrak{g}_\varphi)$ is a real metric calculus, and one may find a connection ∇ (using Corollary 1.10) such that $(M, g_\alpha, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus. However, unless α is central, it will not be a real pseudo-Riemannian calculus in general.

On the other hand, it is well known that $\mathbb{T}^4_\theta$ has no nontrivial central elements for $\theta \in \mathbb{R} \setminus \mathbb{Q}$. That is, $Z(\mathbb{T}^4_\theta) = \mathbb{C}$. To probe our setting, we proceed the construction for $\theta \in \mathbb{Q}$ in which case the centre of $\mathbb{T}^4_\theta$ is not trivial.

Proposition 5.2. *Let $\theta = p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$, $\alpha = \alpha^* \in Z(\mathbb{T}^4_\theta) \setminus \mathbb{C}$ and $(M, g_\alpha, \mathfrak{g}_\varphi, \nabla)$ as above. Then, $(M, g_\alpha, \mathfrak{g}_\varphi, \nabla)$ is a real metric calculus.*

Proof. By definition, it is clear that g_α is a hermitian form. To show that it is non-degenerate, assume $g(V, W) = 0$ for all $W \in M$. In particular, one may choose $W = E_a$, which gives

$$\begin{aligned}
0 = g(V, E_1) &= \left(V^1\right)^* (g_\alpha)_{11} = \left(V^1\right)^* e^\alpha \\
0 = g(V, E_2) &= \left(V^2\right)^* (g_\alpha)_{22} = \left(V^2\right)^* e^\alpha \\
0 = g(V, E_3) &= \left(V^3\right)^* (g_\alpha)_{33} = \left(V^3\right)^* e^\alpha \\
0 = g(V, E_4) &= \left(V^4\right)^* (g_\alpha)_{44} = \left(V^4\right)^* e^\alpha,
\end{aligned}$$

while e^α is invertible with the inverse $e^{-\alpha}$. Hence, $V^a = 0$ for all $a = 1, 2, 3, 4$ and g is non-degenerate, which shows that (M, g) is a metric module. Moreover, it is clear that $\varphi(\mathfrak{g})$ generates M as a free module since $E_a = \varphi(\partial_a)$, $a = 1, 2, 3, 4$, is in the image of φ. Lastly, if $E, E' \in M$, then $g(E, E') = g(E', E)$ because the elements $g_{ab} = \delta_{ab} e^\alpha$ are central and hermitian. $\square$

Since M is a free module, and $E_a = \varphi(\partial_a)$ generate M, a metric and torsion-free connection on $(M, g, \mathfrak{g}_\varphi)$ can be constructed. Note that $\partial_a(e^\alpha) = \partial_a(\alpha)e^\alpha$ since α is central. Furthermore, the derivations of α are e^α are central.

Proposition 5.3. *Let $(M, g, \mathfrak{g}_\varphi)$ be as above. Then, there exists a unique connection ∇ on $(M, g, \mathfrak{g}_\varphi)$ such that $(M, g, \mathfrak{g}_\varphi, \nabla)$ is a real pseudo-Riemannian calculus. The connection is given by*

$$\begin{aligned}
\nabla_1 E_1 &= \frac{1}{2}E_1\partial_1(\alpha) - \frac{1}{2}E_2\partial_2(\alpha) - \frac{1}{2}E_3\partial_3(\alpha) - \frac{1}{2}E_4\partial_4(\alpha)\\
\nabla_1 E_2 &= \nabla_2 E_1 = \frac{1}{2}E_1\partial_2(\alpha) + \frac{1}{2}E_2\partial_1(\alpha)\\
\nabla_1 E_3 &= \nabla_3 E_1 = \frac{1}{2}E_1\partial_3(\alpha) + \frac{1}{2}E_3\partial_1(\alpha)\\
\nabla_1 E_4 &= \nabla_4 E_1 = \frac{1}{2}E_1\partial_4(\alpha) + \frac{1}{2}E_4\partial_1(\alpha)\\
\nabla_2 E_2 &= -\frac{1}{2}E_1\partial_1(\alpha) + \frac{1}{2}E_2\partial_2(\alpha) - \frac{1}{2}E_3\partial_3(\alpha) - \frac{1}{2}E_4\partial_4(\alpha)\\
\nabla_2 E_3 &= \nabla_3 E_2 = \frac{1}{2}E_2\partial_3(\alpha) + \frac{1}{2}E_3\partial_2(\alpha)\\
\nabla_2 E_4 &= \nabla_4 E_2 = \frac{1}{2}E_2\partial_4(\alpha) + \frac{1}{2}E_4\partial_2(\alpha)\\
\nabla_3 E_3 &= -\frac{1}{2}E_1\partial_1(\alpha) - \frac{1}{2}E_2\partial_2(\alpha) + \frac{1}{2}E_3\partial_3(\alpha) - \frac{1}{2}E_4\partial_4(\alpha)\\
\nabla_3 E_4 &= \nabla_4 E_3 = \frac{1}{2}E_3\partial_4(\alpha) + \frac{1}{2}E_4\partial_3(\alpha)\\
\nabla_4 E_4 &= -\frac{1}{2}E_1\partial_1(\alpha) - \frac{1}{2}E_2\partial_2(\alpha) - \frac{1}{2}E_3\partial_3(\alpha) + \frac{1}{2}E_4\partial_4(\alpha).
\end{aligned}$$

The computation of the connection for $\mathbb{T}^4_\theta$ goes exactly the same as Proposition 3.7.

We now compute the curvature of $(M, g, \mathfrak{g}_\varphi, \nabla)$ for $\mathbb{T}^4_\theta$. Recall that since the pseudo-Riemannian calculus is real, Proposition 1.15 implies that the curvature operator has all the same symmetries expected from the classical geometry. More importantly, since the metric $g_{ab} = e^\alpha\delta_{ab}$ is invertible, the pseudo-inverse $(\hat{g}^{ab}, G)$ of g is given by $(e^{-\alpha}\delta_{ab}, \mathbb{1})$.

Proposition 5.4. *The curvature of the pseudo-Riemannian calculus $(M, g_\alpha, \mathfrak{g}_\varphi, \nabla)$*

over $\mathbb{T}^4_\theta$, $\theta \in \mathbb{Q}$ *and* $\alpha = \alpha^* \in Z(\mathbb{T}^4_\theta)$, *is given by*

$$
\begin{aligned}
R(\partial_1,\partial_2)E_1 &= E_2\left((\alpha_3)^2+(\alpha_4)^2\right) - E_3\alpha_2\alpha_3 - E_4\alpha_2\alpha_4\\
&\quad E_2\left(\partial_1(\alpha_1)+\partial_2(\alpha_2)\right) + E_3\partial_2(\alpha_3) + E_4\partial_2(\alpha_4)\\
R(\partial_1,\partial_2)E_2 &= -E_1\left((\alpha_3)^2+(\alpha_4)^2\right) + E_3\alpha_1\alpha_3 + E_4\alpha_1\alpha_4\\
&\quad - E_1\left(\partial_1(\alpha_1)+\partial_2(\alpha_2)\right) - E_3\partial_1(\alpha_3) - E_4\partial_1(\alpha_4)\\
R(\partial_1,\partial_2)E_3 &= E_1\left(\alpha_2\alpha_3 - \partial_2(\alpha_3)\right) + E_2\left(-\alpha_1\alpha_3 + \partial_1(\alpha_3)\right)\\
R(\partial_1,\partial_2)E_4 &= E_1\left(\alpha_2\alpha_4 - \partial_2(\alpha_4)\right) + E_2\left(-\alpha_1\alpha_4 + \partial_1(\alpha_4)\right)\\
R(\partial_1,\partial_3)E_1 &= -E_2\alpha_2\alpha_3 + E_3\left((\alpha_2)^2+(\alpha_4)^2\right) - E_4\alpha_4\alpha_3\\
&\quad + E_2\partial_3(\alpha_2) + E_3\left(\partial_1(\alpha_1)+\partial_3(\alpha_3)\right) + E_4\partial_4(\alpha_3)\\
R(\partial_1,\partial_3)E_2 &= E_1\left(\alpha_2\alpha_3 - \partial_3(\alpha_2)\right) + E_3\left(-\alpha_2\alpha_1 + \partial_1(\alpha_2)\right)\\
R(\partial_1,\partial_3)E_3 &= -E_1\left((\alpha_2)^2+(\alpha_4)^2\right) + E_2\alpha_1\alpha_2 + E_4\alpha_1\alpha_4\\
&\quad - E_1\left(\partial_1(\alpha_1)+\partial_3(\alpha_3)\right) - E_2\partial_1(\alpha_2) - E_4\partial_1(\alpha_4)\\
R(\partial_1,\partial_3)E_4 &= E_1\left(\alpha_3\alpha_4 - \partial_3(\alpha_4)\right) + E_3\left(-\alpha_1\alpha_4 + \partial_1(\alpha_4)\right)\\
R(\partial_1,\partial_4)E_1 &= -E_2\alpha_2\alpha_4 - E_3\alpha_3\alpha_4 + E_4\left((\alpha_2)^2+(\alpha_3)^2\right)\\
&\quad + E_2\partial_4(\alpha_2) + E_3\partial_4(\alpha_3) + E_4\left(\partial_1(\alpha_1)+\partial_4(\alpha_4)\right)\\
R(\partial_1,\partial_4)E_2 &= E_1\left(\alpha_2\alpha_4 - \partial_2(\alpha_4)\right) + E_4\left(-\alpha_1\alpha_2 + \partial_1(\alpha_2)\right)\\
R(\partial_1,\partial_4)E_3 &= E_1\left(\alpha_3\alpha_4 - \partial_4(\alpha_3)\right) + E_4\left(-\alpha_3\alpha_1 + \partial_1(\alpha_3)\right)\\
R(\partial_1,\partial_4)E_4 &= E_1\left((\alpha_2)^2+(\alpha_3)^2\right) + E_2\alpha_2\alpha_1 + E_3\alpha_3\alpha_1\\
&\quad - E_1\left(\partial_1(\alpha_1)+\partial_4(\alpha_4)\right) - E_2\partial_1(\alpha_2) - E_3\partial_1(\alpha_3)\\
R(\partial_2,\partial_3)E_1 &= E_2\left(\alpha_1\alpha_3 - \partial_1(\alpha_3)\right) + E_3\left(-\alpha_1\alpha_2 + \partial_1(\alpha_2)\right)\\
R(\partial_2,\partial_3)E_2 &= -E_1\alpha_1\alpha_3 + E_3\left((\alpha_1)^2+(\alpha_4)^2\right) - E_4\alpha_3\alpha_4\\
&\quad + E_1\partial_3(\alpha_1) + E_3\left(\partial_2(\alpha_2)+\partial_3(\alpha_3)\right) + E_4\partial_3(\alpha_4)\\
R(\partial_2,\partial_3)E_3 &= -E_1\alpha_2\alpha_1 - E_2\left((\alpha_1)^2+(\alpha_4)^2\right) + E_4\alpha_2\alpha_4\\
&\quad - E_1\partial_2(\alpha_1) - E_2\left(\partial_2(\alpha_2)+\partial_3(\alpha_3)\right) - E_4\partial_2(\alpha_4)\\
R(\partial_2,\partial_3)E_4 &= E_2\left(\alpha_4\alpha_3 - \partial_3(\alpha_4)\right) + E_3\left(-\alpha_4\alpha_2 + \partial_2(\alpha_4)\right)\\
R(\partial_2,\partial_4)E_1 &= E_2\left(\alpha_1\alpha_4 - \partial_1(\alpha_4)\right) + E_4\left(-\alpha_1\alpha_2 + \partial_1(\alpha_2)\right)
\end{aligned}
$$

$$
\begin{aligned}
R(\partial_2,\partial_4)E_2 &= -E_1\alpha_1\alpha_4 - E_3\alpha_3\alpha_4 + E_4\left((\alpha_1)^2+(\alpha_3)^2\right)\\
&\quad + E_1\partial_4(\alpha_1) + E_3\partial_4(\alpha_3) + E_4\left(\partial_2(\alpha_2)+\partial_4(\alpha_4)\right)\\
R(\partial_2,\partial_4)E_3 &= E_2\left(\alpha_3\alpha_4 - \partial_3(\alpha_4)\right) + E_4\left(-\alpha_3\alpha_2 + \partial_3(\alpha_2)\right)\\
R(\partial_2,\partial_4)E_4 &= E_1\alpha_1\alpha_2 - E_2\left((\alpha_1)^2+(\alpha_3)^2\right) + E_3\alpha_3\alpha_2\\
&\quad - E_1\partial_2(\alpha_1) - E_2\left(\partial_2(\alpha_2)+\partial_4(\alpha_4)\right) - E_3\partial_2(\alpha_3)\\
R(\partial_3,\partial_4)E_1 &= E_3\left(\alpha_1\alpha_4 - \partial_1(\alpha_4)\right) + E_4\left(-\alpha_1\alpha_3 + \partial_1(\alpha_3)\right)\\
R(\partial_3,\partial_4)E_2 &= E_3\left(\alpha_2\alpha_4 - \partial_2(\alpha_4)\right) + E_4\left(-\alpha_2\alpha_3 + \partial_2(\alpha_3)\right)\\
R(\partial_3,\partial_4)E_3 &= -E_1\alpha_1\alpha_4 - E_2\alpha_2\alpha_4 + E_4\left((\alpha_1)^2+(\alpha_2)^2\right)\\
&\quad + E_1\partial_4(\alpha_1) + E_2\partial_4(\alpha_2) + E_4\left(\partial_3(\alpha_3)+\partial_4(\alpha_4)\right)\\
R(\partial_3,\partial_4)E_4 &= E_1\alpha_1\alpha_3 + E_2\alpha_2\alpha_3 - E_3\left((\alpha_1)^2+(\alpha_2)^2\right)\\
&\quad - E_1\partial_3(\alpha_1) - E_2\partial_3(\alpha_2) - E_3\left(\partial_3(\alpha_3)+\partial_4(\alpha_4)\right)
\end{aligned}
$$

where $\alpha_j = \frac{1}{2}\partial_j(\alpha)$, *from which we have the nonzero Riemann curvature components*

$$
\begin{aligned}
R_{1212} &= \left((\alpha_3)^2+(\alpha_4)^2+\partial_1(\alpha_1)+\partial_2(\alpha_2)\right)e^\alpha\\
R_{1313} &= \left((\alpha_2)^2+(\alpha_4)^2+\partial_1(\alpha_1)+\partial_3(\alpha_3)\right)e^\alpha\\
R_{1414} &= \left((\alpha_2)^2+(\alpha_3)^2+\partial_1(\alpha_1)+\partial_4(\alpha_4)\right)e^\alpha\\
R_{2323} &= \left((\alpha_1)^2+(\alpha_4)^2+\partial_2(\alpha_2)+\partial_3(\alpha_3)\right)e^\alpha\\
R_{2424} &= \left((\alpha_1)^2+(\alpha_3)^2+\partial_2(\alpha_2)+\partial_4(\alpha_4)\right)e^\alpha\\
R_{3434} &= \left((\alpha_1)^2+(\alpha_2)^2+\partial_3(\alpha_3)+\partial_4(\alpha_4)\right)e^\alpha
\end{aligned}
$$

$$
\begin{aligned}
R_{1213} &= (-\alpha_2\alpha_3+\partial_2(\alpha_3)))e^\alpha & R_{1214} &= (-\alpha_2\alpha_4+\partial_2(\alpha_4))e^\alpha\\
R_{1223} &= (\alpha_1\alpha_3-\partial_1(\alpha_3))e^\alpha & R_{1224} &= (\alpha_1\alpha_4-\partial_1(\alpha_4))e^\alpha\\
R_{1314} &= (-\alpha_3\alpha_4+\partial_4(\alpha_3))e^\alpha & R_{1323} &= (-\alpha_1\alpha_2+\partial_1(\alpha_2))e^\alpha\\
R_{1334} &= (\alpha_1\alpha_4-\partial_1(\alpha_4))e^\alpha & R_{1424} &= (-\alpha_1\alpha_2+\partial_1(\alpha_2))e^\alpha\\
R_{1434} &= (-\alpha_1\alpha_3+\partial_1(\alpha_3))e^\alpha & R_{2324} &= (-\alpha_3\alpha_4+\partial_3(\alpha_4))e^\alpha\\
R_{2334} &= (\alpha_2\alpha_4-\partial_2(\alpha_4))e^\alpha & R_{2434} &= (-\alpha_2\alpha_3+\partial_2(\alpha_3))e^\alpha.
\end{aligned}
$$

Moreover, the Ricci curvautre components are computed as follows.

$$R_{11} = 2\left((\alpha_2)^2 + (\alpha_3)^2 + (\alpha_4)^2\right) + 3\partial_1(\alpha_1) + \partial_2(\alpha_2) + \partial_3(\alpha_3) + \partial_4(\alpha_4)$$
$$R_{22} = 2\left((\alpha_1)^2 + (\alpha_3)^2 + (\alpha_4)^2\right) + \partial_1(\alpha_1) + 3\partial_2(\alpha_2) + \partial_3(\alpha_3) + \partial_4(\alpha_4)$$
$$R_{33} = 2\left((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_4)^2\right) + \partial_1(\alpha_1) + \partial_2(\alpha_2) + 3\partial_3(\alpha_3) + \partial_4(\alpha_4)$$
$$R_{44} = 2\left((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2\right) + \partial_1(\alpha_1) + \partial_2(\alpha_2) + \partial_3(\alpha_3) + 3\partial_4(\alpha_4)$$

and

$$R_{12} = 2(\alpha_1\alpha_2 - \partial_1(\alpha_2)) \quad R_{13} = 2(\alpha_1\alpha_3 - \partial_1(\alpha_3)) \quad R_{14} = 2(\alpha_1\alpha_4 - \partial_1(\alpha_4))$$
$$R_{23} = 2(\alpha_2\alpha_3 - \partial_2(\alpha_3)) \quad R_{24} = 2(\alpha_2\alpha_4 - \partial_2(\alpha_4)) \quad R_{34} = 2(\alpha_3\alpha_4 - \partial_3(\alpha_4)).$$

Finally, the scalar curvature is given by

$$S = 6\left(((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 + (\alpha_4)^2 + \partial_1(\alpha_1) + \partial_2(\alpha_2) + \partial_3(\alpha_3) + \partial_4(\alpha_4)\right) e^{-\alpha}.$$

5.2 The Gauss-Bonnet-Chern theorem for the non-commutative 4-torus

The formula (4.13) in Section 4.5 can be used again to state an analogue of Gauss-Bonnet-Chern theorem for the noncommutative 4-torus. Recall that the Euler characteristics for a smooth orientable 4-dimensional Riemannian manifold is given by the formula

$$\chi(M) = \frac{1}{32\pi^2}\int_M \left(R^{abcd}R_{abcd} - 4\,\mathrm{Ric}_{ab}\,\mathrm{Ric}^{ab} + S^2\right) d\mu,$$

where R_{abcd} is the Riemann curvature tensor, Ric_{ab} the Ricci curvature, S the scalar curvature and $\chi(M)$ is the Euler characteristic of M. (Recall that $\chi(\mathbb{T}^4) = 0$.) In this section, we state an analogue of the Gauss-Bonnet-Chern theorem for the pseudo-Riemannian calculus of $\mathbb{T}^4_\theta$. Unlike the case of $\mathbb{T}^4_\theta$, we do not need to localize the algebra $\mathbb{T}^4_\theta$ because the entries of the metric are all invertible. Pfaffian of the curvature form can be computed without localization.

The trace of $\mathbb{T}^4_\theta$ is defined as

$$\tau(a) = a_{0000}, \quad a = \sum_{n\in\mathbb{Z}^4} c_n U_1^{n_1} U_2^{n_2} U_3^{n_3} U_4^{n_4}. \tag{5.1}$$

It is well known that this map is a trace on the 4-noncommutative torus and it enjoys the following property.

Proposition 5.5. *Let* $\tau : \mathbb{T}^4_\theta \to \mathbb{C}$ *be defined as in* (5.1)*. Then the following property holds.*

$$\tau(a\partial_j(b)) = -\tau(\partial_j(a)b) \qquad \text{for all} \quad a, b \in \mathbb{T}^4_\theta, \quad j = 1, 2, 3, 4. \tag{5.2}$$

Intuitively speaking, the equation (5.2) is an analogue of integration by parts. The proposition 5.5 can be used to compute the trace of the Pfaffian in our setting.

Recall that

$$\begin{aligned} R^{abcd} &= g^{ap} g^{bq} g^{cr} g^{ds} R_{pqrs} \\ \mathrm{Ric}_{ab} &= g^{pq} R_{apbq} \\ \mathrm{Ric}^{ab} &= g^{ap} g^{bq} \,\mathrm{Ric}_{pq} \\ S &= g^{ab} \,\mathrm{Ric}_{ab} \end{aligned}$$

where $g^{ab} = \delta_{ab} e^{-h}$ in the current setting. In this case,

$$R^{abcd} = e^{-4\alpha} R_{abcd} \quad \text{and} \quad R^{ab} = e^{-2\alpha} R_{ab}.$$

Using Proposition 5.4, we compute

$$R^{abcd} R_{abcd} - 4 \,\mathrm{Ric}_{ab} \,\mathrm{Ric}^{ab} + S^2 = 4\Big(\big((\alpha_3)^2 + (\alpha_4)^2 + \partial_1(\alpha_1) + \partial_2(\alpha_2)\big)^2$$
$$+ \big((\alpha_2)^2 + (\alpha_4)^2 + \partial_1(\alpha_1) + \partial_3(\alpha_3)\big)^2$$

$$
\begin{aligned}
&+ \big((\alpha_2)^2 + (\alpha_3)^2 + \partial_1(\alpha_1) + \partial_4(\alpha_4)\big)^2 \\
&+ \big((\alpha_1)^2 + (\alpha_4)^2 + \partial_2(\alpha_2) + \partial_3(\alpha_3)\big)^2 + \big((\alpha_1)^2 + (\alpha_3)^2 + \partial_2(\alpha_2) + \partial_4(\alpha_4)\big)^2 \\
&+ \big((\alpha_1)^2 + (\alpha_2)^2 + \partial_3(\alpha_3) + \partial_4(\alpha_4)\big)^2 + 2\,(\alpha_2\alpha_3)^2 + 2\,(\alpha_2\alpha_4)^2 + 2\,(\alpha_1\alpha_3)^2 \\
&+ 2\,(\alpha_1\alpha_4)^2 + 2\,(\alpha_3\alpha_4)^2 + 2\,(\alpha_1\alpha_2)^2 + 2\,(\alpha_1\alpha_2)^2 \\
&+ 2\,(\alpha_1\alpha_3)^2 + 2\,(\alpha_3\alpha_4)^2 + 2\,(\alpha_2\alpha_3)^2 \Big)e^{-2\alpha} \\
&- 4\Big(\big(2\left((\alpha_2)^2 + (\alpha_3)^2 + (\alpha_4)^2\right) + 3\partial_1(\alpha_1) + \partial_2(\alpha_2) + \partial_3(\alpha_3) + \partial_4(\alpha_4)\big)^2 \\
&+ \big(2\left((\alpha_1)^2 + (\alpha_3)^2 + (\alpha_4)^2\right) + \partial_1(\alpha_1) + 3\partial_2(\alpha_2) + \partial_3(\alpha_3) + \partial_4(\alpha_4)\big)^2 \\
&+ \big(2\left((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_4)^2\right) + \partial_1(\alpha_1) + \partial_2(\alpha_2) + 3\partial_3(\alpha_3) + \partial_4(\alpha_4)\big)^2 \\
&+ \big(2\left((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2\right) + \partial_1(\alpha_1) + \partial_2(\alpha_2) + \partial_3(\alpha_3) + 3\partial_4(\alpha_4)\big)^2 \\
&+ 4(\alpha_1\alpha_2 - \partial_1(\alpha_2))^2 + 4(\alpha_1\alpha_3 - \partial_1(\alpha_3))^2 + 4(\alpha_1\alpha_4 - \partial_1(\alpha_4))^2 \\
&+ 4(\alpha_2\alpha_3 - \partial_2(\alpha_3))^2 + 4(\alpha_2\alpha_4 - \partial_2(\alpha_4))^2 + 4(\alpha_3\alpha_4 - \partial_3(\alpha_4))^2\Big)e^{-2\alpha} \\
&+ 36\Big(\big((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 + (\alpha_4)^2 \\
&\quad + \partial_1(\alpha_1) + \partial_2(\alpha_2) + \partial_3(\alpha_3) + \partial_4(\alpha_4)\big)e^{-\alpha}\Big)^2 \\
&= \cdots \\
&= -8(\alpha_1)^2(\alpha_2)^2 - 8(\alpha_1)^2(\alpha_3)^2 - 8(\alpha_1)^2(\alpha_4)^2 \\
&- 8(\alpha_2)^2(\alpha_3)^2 - 8(\alpha_2)^2(\alpha_4)^2 - 8(\alpha_3)^2(\alpha_4)^2 \\
&+ 24(\alpha_1)^2\partial_1(\alpha_1) + 8(\alpha_1)^2\partial_2(\alpha_2) + 8(\alpha_1)^2\partial_3(\alpha_3) + 8(\alpha_1)^2\partial_4(\alpha_4) \\
&+ 8(\alpha_2)^2\partial_1(\alpha_1) + 24(\alpha_2)^2\partial_2(\alpha_2) + 8(\alpha_2)^2\partial_3(\alpha_3) + 8(\alpha_2)^2\partial_4(\alpha_4) \\
&+ 8(\alpha_3)^2\partial_1(\alpha_1) + 8(\alpha_3)^2\partial_2(\alpha_2) + 24(\alpha_3)^2\partial_3(\alpha_3) + 8(\alpha_3)^2\partial_4(\alpha_4) \\
&+ 8(\alpha_4)^2\partial_1(\alpha_1) + 8(\alpha_4)^2\partial_2(\alpha_2) + 8(\alpha_4)^2\partial_3(\alpha_3) + 24(\alpha_4)^2\partial_4(\alpha_4) \\
&+ 48\alpha_1\alpha_2\partial_1(\alpha_2) + 48\alpha_1\alpha_3\partial_1(\alpha_3) + 48\alpha_1\alpha_4\partial_1(\alpha_4) \\
&+ 48\alpha_2\alpha_3\partial_2(\alpha_3) + 48\alpha_2\alpha_4\partial_2(\alpha_4) + 48\alpha_3\alpha_4\partial_3(\alpha_4) \\
&- 24(\partial_1(\alpha_2))^2 - 24(\partial_1(\alpha_3))^2 - 24(\partial_1(\alpha_4))^2 \\
&- 24(\partial_2(\alpha_3))^2 - 24(\partial_2(\alpha_4))^2 - 24(\partial_3(\alpha_4))^2 \\
&+ 16\partial_1(\alpha_1)\partial_2(\alpha_2) + 16\partial_1(\alpha_1)\partial_3(\alpha_3) + 16\partial_1(\alpha_1)\partial_4(\alpha_4) \\
&+ 16\partial_2(\alpha_2)\partial_3(\alpha_3) + 16\partial_2(\alpha_2)\partial_4(\alpha_4) + 16\partial_3(\alpha_3)\partial_4(\alpha_4).
\end{aligned}
$$

Hence, an appropriate version of the Gauss-Bonnet-Chern theorem for the noncommutative 4-torus would state that the trace of the above expression multiplied by the volume form is identically 0. That is, one should expect the following equation to hold:

$$\chi(\mathbb{T}^4_\theta) = \frac{1}{32\pi^2}\tau\left(\left(R^{abcd}R_{abcd} - 4\operatorname{Ric}_{ab}\operatorname{Ric}^{ab} + S^2\right)e^{2\alpha}\right) = 0 \tag{5.3}$$

for a central self-adjoint element $\alpha = \alpha^* \in \mathbb{T}^4_\theta$ and $(M, g, \mathfrak{g}_\varphi, \nabla)$ the pseudo-Riemannian calculus for the noncommutative 4-torus $\mathbb{T}^4_\theta$ as defined above.

In our computation, since α is central, there is no ambiguity in the order of multiplication. For instance, $\alpha_1\alpha_2 = \alpha_2\alpha_1$. In principle, the above computation should carry on for a general self-adjoint element $\alpha \in \mathbb{T}^4_\theta$, although the noncommutativity will be an immense obstruction to the final formula for the Pfaffian. If α is not central, then it is not clear whether the trace of the Pfaffian (against the volume form) will vanish or not.

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