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Uniform Approximation on Riemann Surfaces

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Abstract

This thesis consists of three contributions to the theory of complex approximation on Riemann surfaces.

It is known that if E is a closed subset of an open Riemann surface R and f is a holomorphic function on a neighbourhood of E, then it is "usually" not possible to approximate f uniformly by functions holomorphic on all of R. In Chapter 2, we show, however, that for every open Riemann surface R and every closed subset $E \subset R$, there is a closed subset $F \subset E$, which approximates E extremely well, and has the following property. Every function holomorphic on F can be approximated tangentially (much better than uniformly) by functions holomorphic on R.

In Chapter 3, given a function $f : E \to \overline{\mathbb{C}}$ from a closed subset of a Riemann surface R to the Riemann sphere $\overline{\mathbb{C}}$, we seek to approximate f in the spherical distance by functions meromorphic on R. As a consequence we generalize a recent extension of Mergelyan's theorem, due to Fragoulopoulou, Nestoridis and Papadoperakis [3.13]. The problem of approximating by meromorphic functions pole-free on E is equivalent to that of approximating by meromorphic functions zero-free on E, which in turn is related to Voronin's spectacular universality theorem for the Riemann zeta-function.

The reflection principles of Schwarz and Carathéodory give conditions under which holomorphic functions extend holomorphically to the boundary and the theorem of Osgood-Carathéodory states that a one-to-one conformal mapping from the unit disc to a Jordan domain extends to a homeomorphism of the closed disc onto the closed Jordan domain. In Chapter 4, we study similar questions on Riemann surfaces for holomorphic mappings.

Keywords: Riemann Surfaces, Holomorphic approximation, Meromorphic approximation, Zero-free approximation, Pole-free approximation, Chordal metric, Jordan region, Tangential approximation.

Co-Authorship

Fatemeh Sharifi was supervised initially by A. Boivin and later by P. M. Gauthier and G. Sinnamon over the course of this thesis work. This thesis contains revisions of previously submitted papers, each of which is co-authored by P. M. Gauthier and Fatemeh Sharifi. Both authors participated and contributed significantly in all new results presented in the thesis. The papers are accepted in the following journals: Canadian Mathematical Bulletin, [2.8]; Journal of Mathematical Analysis and Applications, [3.23]; Canadian Mathematical Bulletin[4.9].

Dedication

To our wonderful group; André, Paul, Gord, Myrto and my husband Shahrooz.

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Contents

Glossary of notation

$\mathbb{R}$	The real line.
$\mathbb{C}$	The complex plane.
$\overline{\mathbb{C}}$ and $\mathbb{C}^*$	The Riemann sphere.
	Note: "bar" and "star" may have other meanings, defined by chapter.
D	Unit disc.
R	An open Riemann surface.
$\mathrm{Hol}(R)$	Holomorphic functions on R.
E	A closed set.
K	A compact set.
$C(X)$	A class of complex-valued continuous functions on X.
$A(E)$	The space of functions in $C(E) \cap \mathrm{Hol}(E)$.
$H(K)$	Uniform Euclidean limits on K of holomorphic functions.
$R(K)$	Uniform Euclidean limits on K of rational functions, pole-free on K.
$M(K)$	Uniform Euclidean limits on K of meromorphic functions, pole-free on K.
R^*	One point compactification of R.
μ	A regular Borel measure.
X	A compact Hausdorff space.
∂E	The boundary of the set E.
$\|.,.\|$	Euclidean distance.
$\chi(.,.)$	Chordal distance.
$C(p,q)$	A Cauchy kernel.
Ω	A region in a Riemann surface R.
$\overline{\Omega}$	A compact Jordan region.
$\overline{D}$	A closed parametric disc.
α	Continuous analytic capacity.
$A_\chi(E)$	χ-continuous mapping on E, holomorphic on the interior E^0 of E.
$M_\chi(E)$	Spherical limits of meromorphic functions on E.
$\widetilde{A}$	$\{f \in A_\chi(E); f$ has no poles on $E^0\}$.
$\widetilde{M}$	Spherical uniform limits of meromorphic functions pole-free on E.
$A_0(K)$	Restrictions of function in $A(K)$ that are zero-free on K^0.
$\underline{d}(A)$	Lower density.

Chapter 1

Introduction and literature review

Complex Analysis is a rich and deep subject with a history going back to Cauchy in the 1820s. There are numerous interesting topics in complex analysis such as analytic number theory but we are interested in the very important subject "approximation theory". Surprisingly, some mathematicians get the impression that complex analysis, specifically approximation theory, is a dead subject!

In this introduction, we give a brief history of well-known results of approximation on the complex plane and Riemann surfaces. For that, firstly we study compact and closed subsets of the complex plane. The story begins with Runge's theorems, considering approximation by rational and polynomial functions. But it is quite interesting to mention that in the same year, 1885, Weierstrass also published his famous approximation theorem. Much of the work on compact sets has been extended to closed subsets of the complex plane. For example, in 1927, Carleman extended the result of Weierstrass.

Since the only holomorphic functions on compact Riemann surfaces are constant functions, we study non-compact Riemann surfaces, which are also called open Riemann surfaces. Many results on compact subsets of the complex plane have been extended to compact subsets of open Riemann surfaces. For instance, Behnke and Stein generalized Runge's theorem. But on closed sets, the situation is more complicated. Whether or not the genus of a Riemann surface is finite is an important factor. Paul M. Gauthier and Walter Hengartner in 1975, [1.16], gave a necessary condition for holomorphic approximation on closed subsets of an open Riemann surface which, surprisingly, was shown to be not sufficient in general.

It is known that if E is a closed subset of an open Riemann surface R and f is a holomorphic function on a neighbourhood of E, then it is "usually" not possible to approximate f uniformly by functions holomorphic on all of R. We show, however, that for every open Riemann surface R and every closed subset $E \subset R$, there is a closed subset $F \subset E$, which approximates E extremely well, such that every function holomorphic on F can be approximated much better than uniformly by functions holomorphic on R.

We study the approximation of a given function f, from a closed subset of a Riemann surface into the Riemann sphere. Consequently, we generalize a recent extension of Mergelyan's theorem, due to Fragoulopoulou, Nestoridis and Papadoperakis, [1.13]. We consider eight types of meromorphic approximation.

To this end, we study the extension to the boundary defined on subsets of Riemann surfaces of holomorphic function. We give a Carathéodory type reflection principle for bordered Riemann surfaces which are arbitrary. That is, we do not assume that they are compact, nor do we assume that they are of finite genus. From our Carathéodory type reflection principle we deduce a Schwarz type reflection principle as well as an Osgood-Carathéodory type theorem.

To simplify the notation for K an arbitrary compact subset of $\mathbb{C}$, we introduce the following function spaces: $C(K)$ denotes the space of complex-valued continuous functions on K with norm $||f|| = \max\{|f(z)| : z \in K\}$. By $\mathrm{Hol}(K)$ we mean the set of functions f holomorphic on a neighbourhood N_f of K and $H(K)$ denotes the set of uniform limits on K of functions in $\mathrm{Hol}(K)$.

Let $A(K) = C(K) \cap \mathrm{Hol}(K^0)$ (K^0 is the interior of K) denotes the set of functions continuous on K, which are holomorphic on K^0. Let $P(K)$ be functions f on K such that for each $\epsilon > 0$ there exists a polynomial p such that

$$|f(z) - p(z)| < \epsilon, \quad z \in K.$$

Let $R(K)$ be the closure in $C(K)$ of the rational functions which are holomorphic on K (without poles on K). Similarly, let $M(K)$ be the closure in $C(K)$ of all meromorphic functions on $\mathbb{C}$, which are holomorphic on K (without poles on K).

We naturally have the following inclusions:

$$P(K) \subseteq R(K) \subseteq M(K) \subseteq H(K) \subseteq A(K) \subseteq C(K). \tag{1.1}$$

Later, we will introduce corresponding spaces of functions for Riemann surfaces.

1.1 Approximation on compact subsets of the complex plane

Holomorphic approximation on compact sets K has been studied for polynomials and more generally, for rational functions. The story started with Runge's approximation theorems. Before stating Runge's theorems, we would like to recall some basic definitions. See [1.20].

Definition 1.1.1. A *topology* on set X is a collection τ of subsets of X having the following properties:

1) X and $\emptyset$ are in τ,

2) The union of the elements of any sub-collection of τ is in τ,

3) The intersection of the elements of any finite sub-collection of τ is in τ.

A set X for which a topology τ has been specified is called a *topological space.*

If X is a topological space with topology τ, we say that a subset U of X is an *open* set of X if U belongs to the collection τ.

It is routine to verify that for any subset Y of a topological space X with topology τ, the set $\{Y \cap U : U \in \tau\}$ is also a topology. It is called the *subspace topology* on Y and Y is called a *subspace* of X.

A subset E of a topological space X is said to be *closed* if the set $X \setminus E$ is open.

A space X is *connected* if and only if the only sets in X that are both open and closed are X and the empty set. A subset Y of a space X is a *connected subset* if Y is a connected subspace. Every topological space can be expressed as a disjoint union of connected subsets, called *components.*

Definition 1.1.2. Let $G \subset \mathbb{C}$ be open and connected. A function $f : G \to \mathbb{C}$ is *holomorphic* if it is continuously differentiable.

A collection $\mathcal{A}$ of subsets of a space X is said to *cover* X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition 1.1.3. A space X is said to be *compact* if every open covering $\mathcal{A}$ of X contains a finite sub-collection that also covers X.

A *metric space* is a pair (X, d) where X is a set and d is a function from $X \times X$ into $\mathbb{R}$, called a distance function or metric, that satisfies the following conditions for x, y and z in X:

1) $d(x, y) \geq 0$,

2) $d(x, y) = 0 \quad \text{if and only if} \quad x = y$,

3) $d(x, y) = d(y, x)$,

4) $d(x, z) \leq d(x, y) + d(y, z)$.

Let $a \in X$ and $r > 0$ be fixed. Then we denote $B(a; r) = \{y \in X : d(a, y) < r\}$.

Definition 1.1.4. A function f has an *isolated singularity* at $z = a$ if there is an $r > 0$ such that f is defined and holomorphic in $B(a; r) \setminus \{a\}$ but not in $B(a, r)$.

Definition 1.1.5. If $z = a$ is an isolated singularity of a function f then a is a *pole* of f if $\lim_{z \to a} |f(z)| = \infty$. That is, for any $M > 0$ there is a number $\epsilon > 0$ such that $|f(z)| \geq M$ whenever $0 < |z - a| < \epsilon$.

The components of the complement of K are called its *complementary components.* The following theorems may be found in [1.14] unless otherwise indicated.

Theorem 1.1.6 (Rational Runge Theorem)**.** *Let K be an arbitrary compact subset of $\mathbb{C}$. Let f be holomorphic on K and let $\epsilon > 0$. Then there exists a rational function r with poles only in $\mathbb{C} \setminus K$ such that*

$$|f(z) - r(z)| < \epsilon, \quad z \in K.$$

Theorem 1.1.7 (Polynomial Runge Theorem)**.** *Suppose K is compact in $\mathbb{C}$, $\mathbb{C} \setminus K$ is connected and f is holomorphic on K. Then, for each $\epsilon > 0$, there exists a polynomial p such that*

$$|f(z) - p(z)| < \epsilon, \quad z \in K.$$

Definition 1.1.8. The *hull* of K (denoted by $\hat{K}$) is defined as the union of K with the bounded complementary components of K, that is, $\hat{K} = K \cup$ (the bounded components of $\mathbb{C} \setminus K$).

In case $\mathbb{C} \setminus K$ is connected, then it has no bounded components, so $K = \hat{K}$ if and only if $\mathbb{C} \setminus K$ is connected; thus, Runge's polynomial theorem is often stated in the stronger form:

$$H(K) = P(K) \quad \text{if and only if} \quad K = \hat{K}.$$

If we weaken the hypothesis of Runge's theorem, namely, if we suppose that the function f is continuous on K and analytic at the interior points of K, we would get the very well-known interesting theorem proved by Mergelyan in 1951, which actually gives us a sufficient and necessary condition for polynomial approximation.

Theorem 1.1.9 (Mergelyan's Theorem)**.** *Suppose K is compact in the complex plane and $\mathbb{C} \setminus K$ is connected. Also suppose f is continuous on K and analytic in the interior K^0 of K. Then, for every $\epsilon > 0$, there exists a polynomial p such that*

$$|f(z) - p(z)| < \epsilon, \quad z \in K.$$

Mergelyan's theorem could be stated as

$$A(K) = P(K) \quad \textit{if and only if} \quad K = \hat{K}.$$

We note that Mergelyan's theorem is a generalization of the following well-known theorem:

Theorem 1.1.10 (Weierstrass's Theorem)**.** *If K is a (closed, bounded) segment of the real line, then $C(X) = P(X)$, that is, every continuous function on K can be uniformly approximated on K by polynomials.*

We now look at the rational approximation more closely having already stated Runge's theorem as one important theorem.

It is interesting to know that having many bounded complementary components for a compact set K makes it more difficult to approximate a function $f \in A(K)$ by rational functions. In 1938, the Swiss mathematician, Alice Roth, constructed a compact set K, called the Swiss Cheese, for which $\mathbb{C} \setminus K$ consists of infinitely many components and rational approximation for a certain function $f \in A(K)$ is not possible. To see this interesting example see [1.14].

So the question that comes to mind is whether or not rational approximation is possible when $\mathbb{C} \setminus K$ has only finitely many bounded components? To answer this question, we need the important Bishop localization theorem.

Theorem 1.1.11 (Bishop's Localization Theorem)**.** *Let K be a compact set in the complex plane and let f be continuous in $\mathbb{C}$. Suppose that for each $z \in K$ there exists a neighbourhood U_z such that $f|_{K \cap \overline{U}_z} \in R(K \cap \overline{U}_z)$. Then $f \in R(K)$.*

Theorem 1.1.12 (Mergelyan's Theorem 2 [1.7])**.** *Let K be a compact subset of $\mathbb{C}$, and let $\Omega_0 = \mathbb{C} \setminus \hat{K}$, $\Omega_1, \Omega_2, ...$ be the components of the complement of K; so that, $\mathbb{C} \setminus K = (\Omega_0) \cup (\cup_{s \geq 1} \Omega_s)$. If $d = \inf_s \{\operatorname{diam}(\Omega_s)\} > 0$ then $A(K) = R(K)$.*

Note that if $\Omega_0 = \mathbb{C}\setminus K$, i.e., $\mathbb{C}\setminus K$ is connected, then $d = \infty$. The preceding theorem has the following consequence.

Corollary 1.1.13. *A function $f \in A(K)$ can be uniformly approximated by rational functions if K has only finitely many bounded complementary components.*

Also to complete our task of describing approximation on compact subsets of the complex plane, we need to mention Vitushkin's theorem.

Theorem 1.1.14 (Vitushkin's Theorem)**.** *Let K be a compact subset of $\mathbb{C}$. Then the following are equivalent:*

1) $R(K) = A(K)$;

2) For every open disc D, $R(K \cap \overline{D}) = A(K \cap \overline{D})$, where $\overline{D}$ denotes the closure of D.

If $K^0 = \emptyset$ then the theorem can be stated for $C(K)$. Vitushkin's theorem can be stated in terms of *capacity*, which we will use later so it is important to mention this version. Denote the Riemann sphere by $\overline{\mathbb{C}}$. *Continuous analytic capacity* is defined as follows:

$$\alpha(K) = \sup\{|f'(\infty)| : f \in C(\overline{\mathbb{C}}), \quad f \in \mathrm{Hol}(\mathbb{C} \setminus K), \quad |f| \leq 1\}$$

where

$$f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$$

and

$$f(\infty) = \lim_{z\to\infty} f(z).$$

Theorem 1.1.15 (Equivalent statement of Vitushkin's Theorem)**.** *Let K be a compact subset of $\mathbb{C}$ then the following are equivalent:*

1) $R(K) = A(K)$;

2) For every open disc D, $\alpha(D \setminus K) = \alpha(D \setminus K^0)$.

The fusion lemma by Roth serves as a stepping stone to the study of approximation on closed sets, which will be treated in the next section.

Theorem 1.1.16 (Roth's Fusion Lemma)**.** *Let K_1, K_2 and K be compact subsets in the complex plane such that $K_1 \cap K_2 = \emptyset$. Then there exists a constant A, depending only on K_1, K_2, with the following property. If r_1, r_2 are rational functions, such that*

$$|r_1 - r_2| < \epsilon \quad on \quad K,$$

then there exists a rational function r such that

$$|r - r_j| \leq A \cdot \epsilon \quad on \quad K_j \cup K, \quad j = 1, 2.$$

1.2 Approximation on closed subsets of the complex plane

In this section we shall study the approximation of functions defined on a closed set E where the approximating functions are meromorphic or holomorphic in a neighbourhood N of E.

We use the same notation for function spaces on an arbitrary closed (but not necessarily compact) set that we use for compact sets. All the theorems above have been generalized to closed subsets of $\mathbb{C}$. But we would like to start with Roth's approximation theorem since it reduces the problem of approximating a function f on E by meromorphic functions to the problem of approximating functions on compact sets by rational functions.

Theorem 1.2.1 (Roth's theorem)**.** *Let E be a closed subset of the complex plane. A function f can be uniformly approximated on E by a function in $M(E)$ without poles in E if and only if*

$$f|_K \in R(K) \quad \text{for each compact subset} \quad K \subset E.$$

In 1938, she generalized the rational Runge theorem to closed sets. This was her doctoral thesis.

Theorem 1.2.2. *For every closed set E, $H(E) = M(E)$.*

In 1927, Carleman extended the result of Weierstrass.

Theorem 1.2.3 (Carleman's Theorem)**.** *For every function f continuous in $\mathbb{R}$ and for every positive continuous function ϵ on $\mathbb{R}$, there exists an entire function g such that*

$$|f(x) - g(x)| < \epsilon(x), \quad x \in \mathbb{R}.$$

Carleman generalized this theorem by replacing $\mathbb{R}$ by more general curves.

For a closed set E, if for every pair of functions $\{f(z), \epsilon(z)\}$, $f \in A(E)$ and $\epsilon(z)$ positive continuous function on E, there exists a function $g \in H(\mathbb{C})$ such that

$$|f(z) - g(z)| < \epsilon(z), \quad z \in E,$$

then E is called a *Carleman* set or a *set of holomorphic Carleman approximation*.

Carleman approximation is also called *tangential approximation*. Note that tangential and uniform approximation coincide for compact sets; simply replace the function ϵ by its minimum on the compact set.

The following necessary condition to characterize Carleman sets, was introduced in 1969 by Gauthier. A family $\mathcal{A}$ of sets in $\mathbb{C}$ satisfies the *long islands* condition if for each compact set K, there exists a compact set Q such that every element of $\mathcal{A}$ which meets K is contained in Q.

Later (1971) Nersesjan gave a complete characterization of holomorphic Carleman approximation on closed sets with respect to the long islands condition.

The long islands condition could apply to three categories: interior of a set, fine interior of a set and *Gleason part*. We will define it for a family of interior of a set later in Chapter 2.

Mergelyan's theorem was generalized by Arakelyan in 1964. The original paper is [1.2], but the theorem may also be found in [1.14]. Before stating the theorem some definitions need to be recalled.

A topological space S is called *locally connected* at $a \in S$ if, for each neighbourhood N of a, there exists a connected set $Z \subset N$ that contains a as an interior point.

We shall apply this definition to $\mathbb{C}^* \setminus E$ and $a = \infty$, where E is a closed set in $\mathbb{C}$. Since $\mathbb{C}^* \setminus E$ is open in $\mathbb{C}$, to show that it is locally connected in $\mathbb{C}^*$ it is enough to check the point at infinity.

We say that a continuous path $\gamma : [0, 1) \to \mathbb{C}$ starting at $z_0 \in \mathbb{C}$ connects z_0 with infinity (meaning $*$) if for every compact set $K \subset \mathbb{C}$ there is a point on γ after which γ does not meet K.

Lemma 1.2.4. *[1.14] The space $S = C^* \setminus E$ is locally connected at infinity, if and only if, the following holds: For every neighbourhood U of infinity, there exists a neighbourhood $V \subset U$ of infinity with the property that each point $z \in V \setminus E, z \neq \infty$, can be connected with infinity in $\mathbb{C}$ by a continuous path $\gamma \subset U \setminus E$.*

Theorem 1.2.5 (Arakelyan's Theorem)**.** *Let E be a closed set. For every $f \in A(E)$, for every $\epsilon > 0$, there exists a function $F \in \mathrm{Hol}(E)$ such that*

$$|f(z) - F(z)| < \epsilon, \quad z \in E,$$

if and only if the following conditions are satisfied:

1) $\mathbb{C}^* \setminus E$ is connected,

2) $\mathbb{C}^* \setminus E$ is locally connected at infinity.

1.3 Approximation on non-compact Riemann surfaces

Definition 1.3.1. A *complex chart* on a topological space S is a homeomorphism $\phi : U \to V$, where U is an open set in S and V is an open set in $\mathbb{C}$.

Charts are also called *local coordinates* or *uniformizing variables.* When two charts have overlapping domains, they need to be related.

Definition 1.3.2. Let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be two charts. We say that they are *compatible* if ether U_1 and U_2 are disjoint or $\phi_2 \circ \phi_1^{-1}$ is holomorphic on $\phi_1(U_1 \cap U_2)$.

Note that if $\phi_2 \circ \phi_1^{-1}$ is holomorphic on $\phi_1(U_1 \cap U_2)$ then $\phi_1 \circ \phi_2^{-1}$ is holomorphic on $\phi_2(U_1 \cap U_2)$. So the definition is symmetric in the two charts. We refer to the functions $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ as *transition functions.*

Definition 1.3.3. A *complex atlas* $\mathcal{U}$ on S is a collection $\mathcal{U} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ of compatible charts whose domains cover S, i.e., $S = \cup_\alpha U_\alpha$.

Two complex atlases $\mathcal{U}$ and $\mathcal{V}$ are *equivalent* if every chart in $\mathcal{U}$ is compatible with every chart in $\mathcal{V}$. When two atlases are compatible we can combine them to get another atlas that contains them both.

Definition 1.3.4. A *complex structure* on S is a maximal complex atlas on S. Equivalently it is an equivalence class of complex atlases on S.

Lemma 1.3.5. *(Zorn's Lemma) Suppose a partially ordered set has the property that every chain (i.e. totally ordered subset) has an upper bound. Then the set contains at least one maximal element.*

A Zorn's Lemma argument shows that every atlas is contained in a unique maximal atlas. Furthermore two atlases are compatible if and only if they are sub-collections of the same maximal atlas, i.e., two holomorphic atlases on S are compatible if and only if they generate the same maximal holomorphic atlas on S.

Definition 1.3.6. If S is a set, a *basis* for a topology on S is a collection $\mathcal{B}$ of subsets of S such that:

1) For $s \in S$ there is at least one basis element B containing s,

2) If s belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing s such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology τ generated by $\mathcal{B}$ as follows: A subset U of S is open in S, if for each element $s \in U$, there is a basis element $B \in \mathcal{B}$ such that $s \in B$ and $B \subset U$. Note that each basis element is itself an element of τ.

Definition 1.3.7. A space S is said to have a *countable basis* at x if there is a countable collection $\mathcal{B}$ of neighbourhoods of x such that each neighbourhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to be a *first-countable* space. If a space S has a countable basis for its topology, then S is said to satisfy the *second countability axiom*, or to be *second countable*. Obviously *second-countability* implies *first-countability*.

Recall that a topological space S is called a *Hausdorff* space if for each pair x_1, x_2 of distinct points of S, there exist neighbourhoods U_1, and U_2 of x_1 and x_2, respectively, that are disjoint.

Definition 1.3.8. A *Riemann surface* is a connected second countable Hausdorff topological space with a complex structure.

The complex plane is an example of a Riemann surface for the trivial chart $U = \mathbb{C}, V = \mathbb{C}, \phi(z) = z$. Also any connected open subset of the complex plane is a Riemann surface.

Since the charts of the complex structure are homeomorphisms, the topology on a Riemann surface is determined by its complex structure. But there is a finer topology, called the *fine topology*, which is sometimes of use. See [1.11] for a definition.

The next theorem gives the reason that questions of holomorphic approximation on Riemann surfaces are interesting only for open Riemann surfaces. It is known but rarely stated so we include a short proof.

Theorem 1.3.9. *Let R be a compact Riemann surface and R' be a Riemann surface. Let $f : R \to R'$ be a holomorphic mapping. Then f is either constant or surjective. In particular, the only holomorphic functions on a compact Riemann surface are the constant functions.*

Proof. Suppose f is non-constant. Since every non-constant holomorphic map is an open map, and R is open, $f(R)$ is an open set in R'. Also $f(R)$ is compact because it is the image of a compact set under a continuous map. Because R' is Hausdorff, $f(R)$ is closed. Then $f(R)$ is closed and open in the connected space R' which implies $f(R) = R'$, i.e., f is surjective. If f is a holomorphic function, R' is not compact, so f is not surjective. Then it must be constant. □

Now we can move on and talk about the generalization of the above stated theorems for both compact and closed subsets of non-compact Riemann surfaces.

We will use the same notation for function spaces over subsets of Riemann surfaces as we did for function spaces over subsets of $\mathbb{C}$, and recall that we denote by R^* the *one-point compactification* of a Riemann surface R.

The space $H_R(K)$ consists of the uniform limits on K of functions holomorphic on R; that is, $f \in H_R(K)$ if and only if there exists a sequence g_n of functions in $\mathrm{Hol}(R)$ such that $\lim_{n\to\infty} ||f - g_n||_K = 0$, i.e., uniform limits on K of "entire" functions.

Note that we required that the limit be uniform on all of K.

We have the following generalizations of the Runge and Mergelyan theorems.

Theorem 1.3.10. *[1.3] [Behnke-Stein Theorem] Let R be a non-compact Riemann surface and K be a compact subset of R. Then*

1) $H(K) = M(K)$, and

2) $H(K) = H_R(K)$ if and only if $R^* \setminus K$ is connected.

Theorem 1.3.11. *[1.4] [Bishop's Theorem]*

For each non-compact Riemann surface R and each compact subset K of R, we have

$$A(K) = H_R(K)$$

if and only if $R^* \setminus K$ is connected.

On closed sets the situation is more complicated than on compact sets. The genus of a Riemann surface plays a significant role. Recall that the *genus* of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. For instance, a sphere and a disc both have genus zero and a torus has genus one. (See [1.1] for the definition of "orientable surface"—all Riemann surfaces are orientable.)

Definition 1.3.12. A closed subset E of an open Riemann surface is said to be *essentially of finite genus* if there exists an open covering $\{U_i : i \in I\}$ of E such that $U_i \cap U_j = \emptyset$ if $i \neq j$ and each U_i is of finite genus.

If R is of finite genus, then every closed subset of R is automatically essentially of finite genus and if R is of infinite genus, then it is not essentially of finite genus, since Riemann surfaces are by definition connected.

Lemma 1.3.13. *[1.14] The space $S = R^* \setminus E$ is locally connected at infinity if and only if the following holds: For every compact $K \subset R$, there exists a larger compact $Q \subset R$, with the property that each point $z \in (R \setminus E) \setminus Q$, can be connected with infinity by a continuous path $\gamma \subset (R \setminus E) \setminus K$.*

Arakelyan, in [1.2] that also could be found in [1.17], showed that his theorem is also valid for every open subset of the complex plane. One could ask whether one could replace the complex plane by an arbitrary open Riemann surface in the Arakelyan theorem and have the theorem remain valid. The answer is no, but it is known that the Arakelyan theorem holds for an essentially of finite genus set, in particular for finite genus Riemann surface. More precisely, we have the following theorem.

Theorem 1.3.14 ([1.7])**.** *Suppose $E \subset R$ is closed and essentially of finite genus. Then the following are equivalent.*

1) $H(E) = H_R(E)$,

2) $A(E) = H_R(E)$,

3) $R^* \setminus E$ is connected and locally connected.

We may ask whether there is a Riemann surface of infinite genus, for which Arakelyan's theorem holds. We do not know the answer, but the following lemma suggests that we cannot use the previous theorem to find such a surface.

Lemma 1.3.15. *For every Riemann surface R of infinite genus, there is a proper closed subset E which is not essentially of finite genus and for which $R^* \setminus E$ is connected and locally connected.*

The lemma can be proved by the help of Stoilov-Kerjarto compactification, but since it has not been used to prove any investigation we omit the proof.

As we mentioned before, Nersesjan gave a characterization of holomorphic Carlemann approximation for a proper closed subset of the complex plane. A. Boivin, generalized his theorem for a proper closed subset of an open Riemann surface.

Theorem 1.3.16 (Boivin's Theorem)**.** *[1.6] Let E be a proper subset of an open Riemann surface R, then the following are equivalent:*

1) E is a set of holomorphic Carleman approximation,

2) E is a closed subset of R which satisfies the long islands condition such that $R^* \setminus E$ is connected and locally connected,

3) E is a set of uniform holomorphic approximation (i.e., $A(E) = H(E)$) that satisfies the long islands condition.

It is not true that a set of uniform approximation is a set of holomorphic Carleman approximation if and only if all components of the interior are bounded. The following is an example in which all components of the interior are bounded but the set is not a set of holomorphic Carleman approximation.

$$E = \mathbb{R} \cup \bigcup_{n=1}^{\infty} \{z : |x| \leq n, 1/(2n+1) \leq y \leq 1/(2n)\}.$$

Definition 1.3.17. A collection $\{A_\alpha\}$ of subsets of S, is said to be *locally finite* if each point s of S has a neighbourhood that intersects A_α for only finitely many values of α.

Corollary 1.3.18. *Let E be a set of uniform holomorphic approximation in an open Riemann surface R. Then, E is a set of holomorphic Carleman approximation if the components of the interior of E are bounded and form a locally finite family.*

Proof. To prove the corollary we will show that the given condition implies the long islands condition, so that by Theorem 1.3.16, E is a set of holomorphic Carleman approximation.

Let $K \subset R$ be an arbitrary compact subset. Let $E^o = \cup_\alpha G_\alpha$ where the G_α are the components of E^o. Since the components form a locally finite family, K intersects G_α for only finitely many values of α. Also each component is bounded so we may take $Q = K \cup (\cup \overline{G}_\alpha)$ for each α that $G_\alpha \cap K \neq \emptyset$. □

It is known that an arbitrary unbounded closed subset E of an open Riemann surface is "usually" not a set of uniform holomorphic approximation. In the second chapter, we show, however, that there is always a subset of F, whose complement in E is as small as we please, on which we can approximate "extremely well".

To close this section we recall the notion of a regular exhaustion. This important technique is often used within arguments involving unbounded closed sets. We will apply it frequently in subsequent chapters.

Definition 1.3.19. Let R be a Riemann surface. We say a sequence $\{K_n\}$ of compact sets in R is a *regular exhaustion* for R if $R = \cup_n K_n$ and, for each n:

1) K_n is bounded by finitely many disjoint Jordan curves,

2) $K_n \subset K^o_{n+1}$,

3) $R^* \setminus K_n$ is connected.

A regular exhaustion always exists.[1.1]

For every open Riemann surface R, there exists a holomorphic mapping ρ of R into the complex plane which is a local homeomorphism, the mapping ρ induces a complex structure on R which is just the original complex structure R, since ρ is locally biholomorphic. We shall call ρ a *spreading* of R over $\mathbb{C}$. [1.15]

1.4 Characterization of topological conditions on closed subsets of a Riemann surface

In general, there does not exist a topological characterization of those sets for which the desired approximation can always be accomplished on Riemann surfaces of infinite genus.

If E is a closed subset of an open Riemann surface R, consider these statements about E as conditions which E may or may not satisfy:

a) Every function in $A(E)$ is the uniform limit on E of functions which are holomorphic on R,

b) $R^* \setminus E$ is connected,

c) $R^* \setminus E$ is locally connected at infinity,

d) E is essentially of finite genus.

By definition, (a) is a property of the pairs (R, E) and is a conformal invariant; that is, if one pair is related to another by analytic homeomorphism, then both pairs satisfy (a) or neither one does. It is natural to ask whether (a) is invariant for any other equivalence relations. By a topological invariant we mean an invariant for the equivalence defined by homeomorphism of pairs. The theorem of Bishop and Mergelyan implies that (a) is a topological invariant in the case E is compact, and the theorem of Arakelyan shows that (a) is a topological invariant when R is planar. Also it is known [1.22] that (a) is topological invariant when R is of finite genus, but it is not true for an infinite genus Riemann surface.

Recall that, a Riemann surface is said to be planar if it is homeomorphic to a subset of the complex plane.

In this part, we are going to give some examples to aid in understanding the characterization of Arakelyan's topological conditions. The first example may be found in [1.7]. It shows that Arakelyan's topological conditions are not enough to make E satisfy (a). The second example, from [1.21], shows that beside having conditions (b) and (c), adding condition (d) on the interior of E does not help to achieve approximation. The third example, also due to Scheinberg, in [1.22], shows that approximation is not invariant even under isotopy.

The second and third examples will be presented without proof, as they are quite technical. We will give a proof of the first example since it is a natural way to prove that a set is not a set of holomorphic approximation. This will require some preparation. Much of the preparatory material presented here will be repeated in Chapter 2, where it will be used to generate a new example, a set of approximation that cannot be "thickened" and remain a set of approximation.

Theorem 1.4.1 (Tietze's Extension Theorem [1.14, p. 99])**.** *Suppose X is a locally compact Hausdorff space, K is a compact subset of X and $f : K \to \mathbb{C}$ is continuous on K. Then there exists a function $F : X \to \mathbb{C}$, continuous on X and with compact support, such that $F(x) = f(x)$ for $x \in K$.*

Definition 1.4.2. A sequence of points $\{z_n\}$ inside the unit disc is said to satisfy the *Blaschke condition*, when $\Sigma_n(1 - |z_n|) < \infty$. The sequence $\{z_n\}$ is called a *Blaschke* sequence.

It is well known that if f is bounded holomorphic function on the unit disc and f is not identically zero, the zeros of f must satisfy the *Blaschke condition.* See, for example, [1.9].

We shall call an open Riemann surface R a *Myrberg surface* over the unit disc D (or more simply a *Myrberg surface*) if there is holomorphic function $\pi : R \to D$, realizing R as an n-sheeted, branched, fully covered surface of D. In other words, each point of D has exactly n pre-images, counting multiplicities . For further information the interested reader may consult [1.18] or [1.23].

We can construct a class of examples of Myrberg surfaces as follows:

Let $\{x_n\}$ be a strictly increasing sequence of positive real numbers converging to 1. Let R_1 and R_2 be two copies of the open unit disc each slit along the segments $[x_{2n-1}, x_{2n}]$, $n = 1, 2, \ldots$. Let R be the Riemann surface that results from joining R_1 and R_2 crossing along these segments in the usual way. There is a natural projection π from R onto the open unit disc D, which makes R into a Myrberg surface over D with a branch point of order two over each of the points $x_n, n = 1, 2, \ldots$.

Let $\pi : \widetilde{S} \to S$ be a (possibly branched) covering surface of a surface S. For simplicity, we sometimes say that $\widetilde{S}$ is a covering surface of S, where π is implicit. By abuse of notation, we write $*$ for both the ideal point of S and that of $\widetilde{S}$ and we extend the covering to $\pi : \widetilde{S}^* \to S^*$. For $p \in S^*$ and $A \subset S^*$, we denote $\widetilde{p} = \pi^{-1}(p)$ and $\widetilde{A} = \pi^{-1}(A)$. We say that $\widetilde{A}$ is the set "over A."

Let $S_\lambda = \{x + iy : \quad |y| \leq \lambda \quad \text{for} \quad \lambda \geq 0\}$, $E_\lambda = \pi^{-1}(S_\lambda)$. The function $\theta(z) = w$, where

$$\theta(z) = \frac{e^z - 1}{e^z + 1}, \quad \theta^{-1}(w) = \ln\left(\frac{1+w}{1-w}\right), \quad \ln 1 = 0,$$

maps the strip $E_{\pi/2}$ conformally onto the unit disc. In general, if the sequence $\{x_n\}$ is increasing to ∞, then

$$\sum_{n=1}^{\infty}(1 - |\theta(x_n)|) < +\infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{e^{x_n}} < +\infty. \tag{1.2}$$

In particular, if the sequence $\{x_n\}$ grows linearly, more precisely, if $x_n \geq a + nx_0$, for some $x_0 > 0$ and $a > 0$, then $u_n = \theta(x_n), n = 1, 2, \cdots$, is a Blaschke sequence. Indeed,

$$\sum_{n=1}^{\infty}(1 - |u_n|) = \sum_{n=1}^{\infty}\left(\frac{2}{e^{x_n} + 1}\right) \leq 2\sum_{n=1}^{\infty}\frac{1}{e^{na}} < \infty.$$

It follows that, if $\liminf(x_{n+1} - x_n) > 0$, then $\{\theta(x_n)\}$ is a Blaschke sequence. The converse does not hold. Indeed, let $u_n = 1 - 1/n^2$, then $\sum(1 - |u_n|) = \sum 1/n^2 < +\infty$, which is the Blaschke condition while,

$$x_{n+1} - x_n = \theta^{-1}(1 - \frac{1}{(n+1)^2}) - \theta^{-1}(1 - \frac{1}{n^2}) = \ln\left(\frac{2n^2 + 4n + 1}{2n^2 - 1}\right) \to 0.$$

To recapitulate, if $\liminf(x_{n+1} - x_n) > 0$, then $\{\theta(x_n)\}$ is a Blaschke sequence, but we may have $\lim(x_{n+1} - x_n) = 0$, for a Blaschke sequence $\{\theta(x_n)\}$. Equivalently, if $\{\theta(x_n)\}$ is not a Blaschke sequence (and hence $E_{\pi/2}$ is not a set of approximation), then $\lim(x_{n+1} - x_n) = 0$, but it is possible to also have $\lim(x_{n+1} - x_n) = 0$, with $\{\theta(x_n)\}$ a Blaschke sequence.

Consider $x_n = \delta \ln n$. Then

$$\sum \frac{1}{e^{x_n}} = \sum \frac{1}{n^\delta},$$

so $\{u_n\}$ is a Blaschke sequence if and only if $\delta > 1$. For $\delta \leq 1$ and $x_n = \delta \ln n$, $E_{\pi/2}$ is not a set of approximation. Notice that for every $\delta > 0$, the sequence $\{\delta \ln n\}$ is "tight" in both senses $x_{n+1} - x_n \to 0$ and $x_{n+1}/x_n \to 1$.

Lemma 1.4.3. *(Schwarz Lemma) Let $D = \{z : |z| < 1\}$ and suppose f is holomorphic on D with:*

1) $|f(z)| \leq 1$ for z in D,

2) $f(0) = 0$.

Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all z in the disc D. Moreover if $|f'(0)| = 1$ or if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then there is a constant c, $|c| = 1$, such that $f(z) = cz$ for all z in D.

Let $\theta : D \to D$ be a conformal mapping from the unit disc to itself, and such that $\theta(0) = 0$. By Schwarz's lemma $|\theta'(z)| \leq 1$.This reasoning applies to θ^{-1} as well, so that $|(\theta^{-1})'(0)| \leq 1$ and $|(\theta^{-1})'(0)| \geq 1$. We conclude that $|(\theta^{-1})'(0)| = 1$, so by the uniqueness part of Schwarz's lemma, θ must be a rotation. So there is a complex number w, with $|w| = 1$ such that

$$\theta(z) \equiv wz, \quad \forall z \in D.$$

It is often convenient to write a rotation as

$$\rho_\beta(z) \equiv e^{i\beta} z,$$

where we have set $w = e^{i\beta}$, with $0 \leq \beta \leq 2\pi$. In particular $e^{i\frac{3\pi}{2}} z$ is a self mapping sending $-1, 0, i$ to $i, 0, 1$ respectively. Let $\{x_n\}$ be a sequence of numbers between zero and one, which does not satisfy the Blaschke condition and let $u_n = e^{-i\frac{3\pi}{2}} x_n$. Since $|e^{-i\beta}| = 1$, the Blaschke condition is invariant under this conformal self mapping of the disc.

Now suppose $\theta(0) \neq 0$, then since θ is a bijection, $\theta(z_0) = 0$ for some $z_0 \neq 0$. Let $f(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$.

$$\left|\frac{z - z_0}{1 - \bar{z}_0 z}\right| < 1 \quad \text{if and only if} \quad |z - z_0|^2 < |1 - \bar{z}_0 z|^2$$

$$\text{if and only if} \quad |z|^2 - 2Re\bar{z}_0 z + |z_0|^2 < 1 - 2Re\bar{z}_0 z + |z_0||z|^2$$

$$\text{if and only if} \quad |z_0|^2(1 - |z|^2) < 1 - |z|^2 \quad \text{if and only if} \quad |z_0| < 1.$$

Then f is also bijective analytic map from D to D and $f(z_0) = 0$. Hence $\theta \circ f^{-1}(0) = 0$ so $\theta \circ f^{-1}$ is a rotation by the previous case then there exists β such that

$$\theta(z) = (\theta \circ f^{-1})(f(z)) = e^{i\beta} f(z) = e^{i\beta} \frac{z - z_0}{1 - \bar{z}_0 z}$$

Then every conformal self-map of the unit disc is a fractional linear transformation.

Let x_n be a sequence of numbers on the real line which does not satisfy the Blaschke condition, i.e., $\Sigma(1 - |x_n|) = \infty$ then $\Sigma(1 - |\theta(x_n)|) = \infty$ which means that the Blaschke condition is invariant under the conformal self-mapping from the unit disc.

Theorem 1.4.4 (Riemann Mapping Theorem)**.** *Let G be a simply connected region which is not all of $\mathbb{C}$ and let $w \in G$. Then there is a unique conformal mapping f of G onto the unit disc D such that $f(w) = 0$ and $f'(w) = 1$.*

Corollary 1.4.5. *Any two proper simply connected domains in $\mathbb{C}$ have a conformal map between them.*

For an arbitrary sequence of positive real numbers $X = \{x_j\}$, strictly increasing to $+\infty$, let R_X be the corresponding Riemann surface constructed by joining two copies of $\mathbb{C}$ having slits (x_{2j-1}, x_{2j}) and denote by π the projection $R_X \to \mathbb{C}$. For $S_\lambda = \{x + iy : |y| \leq \lambda(x)\}$, write $E_\lambda = \pi^{-1}(S_\lambda)$.

Remark. For each positive continuous function λ, on $\mathbb{R}$, there is a sequence X, such that S_λ is not a set of approximation in R_X. Indeed we can map S_λ onto the unit disc. Then if we take a sequence that does not satisfy the Blaschke condition and let X be the corresponding sequence in S_λ, then E_λ is not a set of holomorphic approximation.

Example 1.4.6. *([2.2]) There exist an open Riemann surface R and a closed subset E, such that $R^* \setminus E$ is connected and locally connected, but is not a set of holomorphic approximation.*

Proof. Let R be the surface obtained from two copies of the unit disc D, cutting each disc along the intervals $(\frac{2n-2}{2n-1}, \frac{2n-1}{2n}), n \geq 1$, removing the endpoints of the cuts, and joining the two copies along corresponding cuts in the usual way. Let $D^+ = D \cap \{z : Re(z) \geq 0\}$ and $E = \pi^{-1}(D^+)$, where π is the projection of R into the unit disc. Note that $R^* \setminus E$ is connected and locally connected. We proceed by contradiction. We will show that if approximation is possible, our constructed function behaves the same in each sheet of R. This will contradict its construction (having pole at one point).

For each function $g \in \mathrm{Hol}(E)$ we define the function g_Δ on D^+ as follows: We set $g_\Delta(z) = (g(z_1) - g(z_2))^2$, when $z \in \pi(E)$ and $\pi^{-1}(z) = \{z_1, z_2\}$. We set $g_\Delta(z) = 0$ when $z \in D \setminus \pi(E)$, i.e., when z is an endpoint of a cut. It is easy to see that g_Δ is holomorphic on D^+, and that g_Δ vanishes at the points $1 - \frac{1}{n}$; that is, at the endpoints of the cuts. It follows that if g_Δ is bounded on D^+, then g_Δ vanishes identically since the Blaschke condition on the zeros of g_Δ is not satisfied. (Strictly speaking, we must apply the Blaschke condition in the disk $\{z : |z - 1/2| < 1/2\}$ to obtain this conclusion. The details are omitted.) Now assume that f is a meromorphic function on R with a unique pole at p, where $\{p, q\} = \pi^{-1}(-\frac{1}{2})$. We now claim that if h is holomorphic on R then, $f - h$ must be unbounded on E and that consequently it is not possible to approximate f uniformly on E by entire functions. Indeed, assuming $f - h$ is bounded, we would have $(f - h)_\Delta$ is identically zero on D^+ and thus on $D \setminus \{-\frac{1}{2}\}$, but this is not possible since $f - h$ is holomorphic at the point q but has a pole at the point p.

□

Example 1.4.7. *[1.21] There is an open Riemann surface R with a closed set E which satisfies (b), (c), and E^0 satisfies (d) but E does not satisfy (a).*

Example 1.4.8. *[1.22] There exists an open Riemann surface R and a closed subset E such that E satisfies (a), $R^* \setminus E$ is connected, locally connected and E has infinite genus, and there exists a homotopy U of R onto itself such that while E is a set of holomorphic approximation, $U(E)$ is not.*

Spherical rational approximation on compact subsets of the complex plane and spherical meromorphic approximation on closed subsets is an interesting topic and has been studied, for example, in [1.13]. The analogue of this topic is to replace subsets of the complex plane by subsets of open Riemann surfaces. In Chapter 3 we consider pole-free meromorphic spherical approximation, giving results for both tangential and uniform approximation. In particular, we give an extension of the famous Mergelyan theorem for Jordan regions in open Riemann surfaces. The property of being a locally finite family is essential for this work.

The Osgood-Carathéodory theorem asserts that conformal mappings between Jordan domains extend to homeomorphisms between their closures. As we see in Chapter 4, similar results holds for multiply-connected domains on Riemann surfaces. One approach is to reduce them to the simply-connected case, but we find it simpler to deduce such results using a direct analogue of the Carathéodory reflection principle.

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Chapter 2

Luzin-type holomorphic approximation on closed subsets of open Riemann surfaces

Dedication: In memory of André Boivin.

2.1 Introduction

Undergraduate students are often first introduced to Riemann surfaces via so-called *concrete* Riemann surfaces, meaning surfaces constructed with paper, scissors and paste. *Abstract* Riemann surfaces, defined as manifolds, are usually encountered later in their studies.

A remarkable theorem of Gunning and Narasimhan [2.9] essentially asserts that every abstract non-compact Riemann surface can be represented as a concrete Riemann surface. Precisely, it says the following. For every open Riemann surface R, there exists a holomorphic mapping ρ of R into the complex plane which is a local homeomorphism, the mapping ρ induces a complex structure on R which is the initial complex structure on R, since ρ is locally biholomorphic. We shall call ρ a spreading of R over $\mathbb{C}$.

By λ we denote Lebesgue measure in $\mathbb{C} = \mathbb{R}^2$ and by μ the measure on R induced by ρ and λ in the sense that if $X = \cup X_n$ is the disjoint union of $X_n, n = 1, \cdots$ and each X_n is contained in a chart where ρ is injective, then $\mu(X) = \sum \lambda(\rho(X_n))$. One could also say that $\mu(X)$ is the Lebesgue measure of the projection $\rho(X)$ "counting multiplicities".

A subset E of a Riemann surface R is said to be *bounded* if the closure $\overline{E}$ in R is compact.

If a function can be approximated uniformly by holomorphic functions on a set E, then necessarily that function must be in the class $A(E)$ of continuous functions on E which are holomorphic on the interior E^0. Let us say that a closed set E in an open Riemann surface R is a set of *uniform approximation* if, for every $f \in A(E)$ and every *number* $\epsilon > 0$, there is a function g holomorphic on R, such that $|f(p) - g(p)| < \epsilon$, for all $p \in E$. Similarly, we say that E is a set of *tangential approximation* if, for every $f \in A(E)$ and every continuous *function* $\epsilon > 0$, there is a function g holomorphic on R, such that

$|f(p) - g(p)| < \epsilon(p)$, for all $p \in E$. A theorem of Carleman [2.5], which deserves to be better known, asserts that the real line is a set of tangential approximation in $\mathbb{C}$. For this reason, sets of tangential approximation are often called sets of Carleman approximation.

For the case that R is a planar domain G, Arakelian [2.2] gave a complete topological characterization of closed subsets $E \subset G$, for which E is a set of uniform approximation by functions holomorphic on G. Let us denote by $R^* = R \cup \{*\}$ the one-point compactification of an open Riemann surface R. Arakelian's theorem states that E is a set of uniform approximation in G if and only if $G^* \setminus E$ is connected and locally connected.

Gauthier and Hengartner [2.7] showed that the topological conditions of Arakelyan are still necessary in order for a closed set E in an open Riemann surface R to be a set of uniform approximation. That is, $R^* \setminus E$ must be connected and locally connected. In the same paper an example was given to show that these topological conditions of Arakelyan, although necessary, are not sufficient to guarantee that E be a set of uniform approximation in R. Thus, in passing from planar domains to open Riemann surfaces, Arakelyan's topological conditions no longer give a characterization of closed sets of uniform approximation. In fact, Scheinberg [2.11], showed that *no* topological conditions whatsoever could characterize sets of uniform approximation on Riemann surfaces.

If a closed set E in an open Riemann surface R is a set of tangential approximation, then of course it must *a fortiori* be a set of uniform approximation and so $R^* \setminus E$ must be connected and locally connected. A further condition, now called the long islands condition, was introduced by Gauthier in [2.6]. A closed subset $E \subset R$ is said to satisfy the *long islands* condition if for every compact set $K \subset R$, there exists a compact set $Q \subset R$ such that every component of the interior of E which meets K is contained in Q. If E is a closed set of uniform approximation in $\mathbb{C}$, then it was shown in [2.6] that the long islands condition is necessary in order for E to be a set of tangential approximation. Nersesjan [2.10] showed, that, in fact a closed set E of uniform approximation in a plane domain G is a set of tangential approximation in G if and only if the long islands condition is satisfied.

A closed strip in $\mathbb{C}$ of strictly positive width is a set of uniform approximation but not of tangential approximation. If the width becomes zero, then a straight line is a set of tangential approximation (and a fortiori of uniform approximation). At the end of the present chapter, I construct a Riemann Surface R where the real line is a set of tangential approximation (and a fortiori of uniform approximation) but a "strip" around it in R is even not a set of uniform approximation. See Example 2.3.3 , below.

Thus, for planar domains, we have a characterization of closed subsets of uniform approximation and also a characterization of closed subsets of tangential approximation. The problem of characterizing closed sets of uniform approximation on open Riemann surfaces is open, however, Boivin extended Nersesjan's result to open Riemann surfaces, thus giving a characterization of closed sets of tangential approximation in open Riemann surfaces. Here is Boivin's theorem.

Theorem 2.1.1 ([2.3])**.** *Let E be a proper closed subset of an open Riemann surface R, then the following are equivalent:*

1) E is a set of tangential approximation;

2) $R^* \setminus E$ is connected and locally connected and E satisfies the long islands condition;

3) E is a set of uniform approximation which satisfies the long islands condition.

Our principal result is the following Luzin-type theorem, which loosely speaking asserts that, for an arbitrary open Riemann surface R, an arbitrary (proper) closed set E in R, an arbitrary function $f \in A(E)$ and an arbitrary $\epsilon > 0$, although there is practically no chance that there exist a function g holomorphic on R, such that $|f - g| < \epsilon$, nevertheless we can always find a closed subset F of E which is most of E in the sense that $E \setminus F$ is small and becomes smaller at arbitrary speed as we approach the ideal boundary point $*$ and on F such approximations are possible, in fact with arbitrary speed. That is, $\epsilon(p)$ may decrease to zero with arbitrary speed as p tends to the ideal boundary. The precise statement is the following.

Theorem 2.1.2. *Let R be an arbitrary open Riemann surface and ρ be a spreading of R over $\mathbb{C}$. Let E be a closed subset of R. For every positive sequence δ_n, and every regular exhaustion $\{K_n\}$ of R, there exists a closed subset F of E such that F is a set of tangential approximation in R and :*

$$\mu((E \setminus F) \setminus K_n) < \delta_n, \quad n = 1, 2, \cdots .$$

Paraphrasing the 100% conjecture for the Riemann Hypothesis and denoting a proposition regarding a point p by $P(p)$, we shall say that the proposition $P(p)$ is true for 100% of the points in a set $E \subset R$ if, for some (equivalently every) exhaustion $\{K_n\}$ of R, we have

$$\lim_{n\to\infty} \frac{\mu\{p \in E \cap K_n : P(p)\}}{\mu(E \cap K_n)} = 1.$$

The following corollary is an easy consequence of Theorem 2.1.2.

Corollary 2.1.3. *If $\mu(E) \neq 0$, and E is unbounded, then for every $f \in A(E)$ and for every positive $\epsilon \in C(E)$, there exists $g \in H(R)$ such that, for 100% of the points p in E, we have $|f(p) - g(p)| < \epsilon(p)$.*

We cannot drop the hypothesis that E be unbounded. First of all, we note that, in case E is bounded, to say that $|f(p) - g(p)| < \epsilon(p)$ holds for 100% of the values of E is equivalent to saying that $|f(p) - g(p)| < \epsilon(p)$ a.e. on E. Now, since f, g and ϵ are continuous, this implies that $|f(p) - g(p)| \leq \epsilon(p)$ *everywhere* on E. Now let $R = \mathbb{C}$ and let E be the closed unit circle $\mathbb{T}$. Suppose, to obtain a contradiction, that for every $f \in A(\mathbb{T}) = C(\mathbb{T})$ and every $\epsilon > 0$, there is an entire function (equivalently a polynomial g) such that $|f(z) - g(z)| < \epsilon$, for 100% of the points of $\mathbb{T}$. We have seen that this implies that $|f(z) - g(z)| \leq \epsilon$, for every point of $\mathbb{T}$. We have shown that every continuous function on the unit circle is the uniform limit of polynomials. But this is well known to be false. This contradiction confirms our claim that the hypothesis that E be unbounded cannot be dropped.

A sequence g_n of almost everywhere finite measureable functions on a measureable set E is said to converge in measure to an almost everywhere finite measureable function f, if for each $\epsilon > 0$,

$$\mu\{p \in E : |g_n(p) - f(p)| > \epsilon\} \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

Corollary 2.1.4. *For every measureable subset $E \subset R$ and for every complex measureable function f on E, there exist a sequence $g_n \in H(R)$, such that $g_n \to f$ in measure.*

These results are new even for the case that R is the complex plane $\mathbb{C}$. We could state similar results for approximation by *meromorphic* functions on Riemann surfaces, but that is a topic for another time. The present chapter is concerned only with approximation by *holomorphic* functions.

In the following section we prove Theorem 2.1.2 and in the last section we briefly consider some so-called Myrberg surfaces, which are the most important source of examples where approximation fails.

2.2 Proof of Theorem 2.1.2

Proof. Let $E^c = R^* \setminus E$. Fix a regular exhaustion $\{K_n\}$ of R, with $K_0 = \emptyset$, and let $\{\delta_n\}$ be a sequence of positive numbers, which we may assume decreases strictly to zero.

Since $\{K_n\}$ is a regular exhaustion, for each $n = 1, 2, \cdots$, the open set $A_n = (K^0_{n+1} \setminus K_{n-1})$ has only finitely many components $U_{n,j}, j = 1, 2, \cdots, j_n$.

Claim 1. For each component $U_{n,j}$ of each A_n, we may assume that $E^c \cap U_{n,j}$ is nonempty. To see this, we form a closed subset E_1 of E as follows. For each $n = 1, 2, \cdots$, and each $j = 1, 2, \cdots, j_n$, we construct an open subset $V_{n,j}$ of $U_{n,j}$ so small that $\mu(V_{n,j}) < \delta_{n+1}/j_n 2^{n+2}$. Now, set

$$E_1 = E \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{j_n} V_{n,j}.$$

If we can show that E_1 satisfies the conclusion of the theorem, then it follows that E also satisfies the conclusion of the theorem. Thus, we assume that E^c meets each component $U_{n,j}$ of each A_n.

Claim 2. For each component $U_{n,j}$ of each A_n, we may assume that $E^c \cap U_{n,j}$ is connected. To see this, we note that the open set $E^c \cap U_{n,j}$ has at most countably many components and so there is a countable collection $J_{n,j}$ of compact smooth Jordan arcs in $U_{n,j}$ which connect all the components of $E^c \cap U_{n,j}$. For each such arc α, we have $\mu(\alpha) = 0$, since α is smooth. We may surround α by a Jordan domain G_α whose closure is contained in $U_{n,j}$, so small that, setting

$$H_{n,j} = \bigcup \{G_\alpha : \alpha \in J_{n,j}\}, \quad H_n = \bigcup_{j=1}^{j_n} H_{n,j},$$

we have

$$\mu(H_{n,j}) < \frac{\delta_{n+1}}{j_n 2^{n+2}}, \quad \mu(H_n) < \frac{\delta_{n+1}}{2^{n+2}}.$$

Now replace E by the closed subset

$$E_2 = E \setminus \bigcup_{n=1}^{\infty} H_n.$$

If E_2 satisfies the conclusion of the theorem, so does E and, for each component $U_{n,j}$ of each A_n, we have that $E_2^c \cap U_{n,j}$ is connected. We therefore assume that E itself has this property.

For each $n = 1, 2, \cdots$, and each $j = 1, 2, \cdots j_n$, choose a point $p_{n,j}$ in $E^c \cap U_{n,j}$. For each $j = 1, 2, \cdots, j_n$ and each $k = 1, 2, \cdots, k_{n+1}$, we shall say that $(n, j) < (n, k)$ if $p_{n,j}$ and $p_{n+1,k}$ are in the same component of $R \setminus K_{n-1}$. For each $(n, j) < (n, k)$, let $\beta_{j,k}$ be a smooth arc in $A_n \cup A_{n+1}$ from $p_{n,j}$ to $p_{n+1,k}$. Since there are finitely many such arcs, for fixed n, and each arc has μ-measure zero, we may surround each such arc by a Jordan domain $G_{n,j,k}$ in $A_n \cup A_{n+1}$, such that, for fixed n, setting

$$G_n = \bigcup\{G_{n,j,k} : (n, j) < (n, k)\},$$

we have

$$\mu(G_n) < \frac{\delta_{n+1}}{2^{n+2}}.$$

Now set $E_3 = E \setminus \cup_n G_n$. It is enough to show that E_3 satisfies the conclusion of the theorem.

We claim that $R^* \setminus E_3$ is locally connected at $*$. It is sufficient to show that, for each n, the set $(R^* \setminus E_3) \setminus K_{n-1}$ is connected. It is sufficient to show that for each $p \in (R \setminus E_3) \setminus K_{n-1}$, the component C_p of $(R \setminus E_3) \setminus K_{n-1}$ containing p is unbounded. The point p is contained in some $U_{n,j}$ and since $p \notin E_3$, it is in $E^c \cap U_{n,j}$ or in G_n. But G_n connects $E^c \cap U_{n,j}$ to some $E^c \cap U_{n+1,k}$. In any case C_p will contain some $E^c \cap U_{n+1,k}$. Since $E^c \cap U_{n+1,k}$ is connected to some $E^c \cap U_{n+2,\ell}$ by G_{n+1}, it follows by induction that the component C_p is unbounded. Thus $(R^* \setminus E_3) \setminus K_{n-1}$ is a connected neighbourhood of $*$. As n varies, these form a neighbourhood basis of $*$ in $R^* \setminus E_3$, so $R^* \setminus E_3$ is locally connected at $*$. Since $R^* \setminus E_3$ is clearly locally connected at each point of $R \setminus E_3$, it is locally connected. We have shown in passing that each point of $R \setminus E_3$ is connected to $*$ so $R^* \setminus E_3$ is not only locally connected but also connected.

There remains to perform one last modification to obtain the long islands condition. For each n, let $Q_{n,j}$ be a regular exhaustion of K_n^0 and choose j so large that, setting $W_n = K_n^0 \setminus Q_{n,j}$, we have

$$\mu(K_n^0 \setminus Q_{n,j}) < \frac{\delta_{n+1}}{2^{n+2}}.$$

Finally, set $F = E_3 \setminus \bigcup_{n=1}^{\infty} W_n$. Then, $R^* \setminus F$ continues to be connected and locally connected and moreover

$$\mu((E_3 \setminus F) \setminus K_n) < \delta_n, \quad n = 1, 2, \cdots.$$

By Theorem 2.1.1, F is a set of tangential approximation. □

Proof. (of the Corollary) For fixed $\delta > 0$, by Theorem 2.1.2, there exists a closed subset F of E such that F is a set of tangential approximation and

$$\mu((E \setminus F) \setminus K_m) < \delta,$$

for some integer $m > 0$. Now fix $f \in A(E)$ and ϵ a positive continuous function on E. Let $g \in H(R)$ such that $|f(p) - g(p)| < \epsilon(p)$, for $p \in F$ and denote by $P(p)$ the proposition that $|f(p) - g(p)| < \epsilon(p)$.

$$\lim_{n\to\infty} \frac{\mu(E \cap K_n : P(p))}{\mu(E \cap K_n)}$$

$$= \lim_{n\to\infty} \frac{\mu(E \cap K_m : P(p)) + \mu(E \cap (K_n \setminus K_m) : P(p))}{\mu(E \cap K_n)} \quad n > m$$

$$\geq \lim_{n\to\infty} \frac{\mu(E \cap (K_n \setminus K_m) : P(p))}{\mu(E \cap K_n)}$$

$$= \lim_{n\to\infty} \frac{\mu(E \cap (K_n \setminus K_m)) - \mu(E \cap (K_n \setminus K_m) :\sim P(p))}{\mu(E \cap K_n)}$$

$$\geq \frac{\mu(E \cap (K_n \setminus K_m)) - \mu(E \cap (K_n \setminus K_m) \setminus F)}{\mu(E \cap K_n)}$$

$$= \lim_{n\to\infty} \frac{\mu(E \cap (K_n \setminus K_m)) - \mu((E \setminus F) \cap (K_n \setminus K_m))}{\mu(E \cap K_n)}$$

$$\geq \lim_{n\to\infty} \frac{\mu(E \cap (K_n \setminus K_m)) - \mu((E \setminus F) \setminus K_m))}{\mu(E \cap K_n)}$$

$$\geq 1 - \lim_{n\to\infty} \frac{\delta}{\mu(E \cap K_n)} = 1.$$

□

Proof. (of Corollary 2) Let $G_k \nearrow R$ be a a regular exhaustion by smoothly bounded open sets and put $A_k = E \cap (G_k \setminus \overline{G}_{k-1})$.

For fixed $(k, n) \in \mathbb{N} \times \mathbb{N}$, by Luzin's Theorem, there exists a compact set $K_{k,n} \subset A_k$, with $\mu(A_k \setminus K_{k,n}) < 1/(2^k n)$, such that f restricted to $K_{k,n}$ is continuous.

We claim that we may assume $K^0_{k,n} = \emptyset$. First of all, there is a finite union

$$L_{k,n} = \bigcup \{L_{k,n,j} : j = 1, \dots, J(k, n)\}$$

of disjoint closed squares $L_{k,n,j} \subset K^0_{k,n}$, such that $\mu(L_{k,n})$ approximates $\mu(K^0_{k,n})$ as well as we please. Here, when we say that $L_{k,n,j}$ is a closed square on the Riemann surface R, we mean that ρ maps $L_{k,n,j}$ homeomorphically onto a square $I \times I \subset \mathbb{R}^2 = \mathbb{C}$, where I is a closed interval in $\mathbb{R}$. We may construct a Cantor-type set $B \subset I$, whose 1-dimensional measure approximates the length of I as well as we please. Then $B \times B$, is a "Cantor-square" whose measure approximates that of $\rho(L_{k,n,j})$ as well as we please. Thus $Q_{k,n,j} = \rho^{-1}(B \times B)$ is a compact nowhere dense subset of $L_{k,n,j}$ such that $\mu(Q_{k,n,j})$ approximates $\mu(L_{k,n,j})$ as well as we please. Consequently, the union

$$Q_{k,n} = \bigcup \{Q_{k,n,j} : j = 1, \dots, J(k, n)\}$$

is a compact nowhere dense subset of $K^0_{k,n}$ whose measure approximates the measure of $K^0_{k,n}$ as well as we please.

Now set $M_{k,n} = Q_{k,n} \cup \partial K_{k,n}$. Since $\mu(Q_{k,n})$ is a good approximation of $\mu(K^0_{k,n})$, $\mu(M_{k,n})$ is an equally good approximation of $\mu(K_{k,n})$. Since the compact set $M_{k,n}$ has empty interior, this proves our claim. Thus, we assume that $K_{k,n}$ has empty interior.

Let $E_n = \cup_k K_{k,n}$ and let f_n be the restriction of f to E_n. Since $f_n \in C(E_n)$, and $E^0_n = \emptyset$, we have $f_n \in A(E_n)$. By Theorem 2.1.2 , there is a closed subset $F_n \subset E_n$ with $\mu(E_n \setminus F_n) < 1/n$ and a function $g_n \in H(R)$, such that $|f_n - g_n| < 1/n$ on F_n. Since $f = f_n$ on E_n and $\mu(E \setminus E_n) < 1/n$, this completes the proof.

□

2.3 A Myrberg surface where approximation fails

We use the expression *Myrberg surface* loosely to refer to a Riemann surface obtained by taking two copies of the complex plane having identical slits and joining these two slit planes along the slits in the usual way.

Definition 2.3.1. A sequence of points $\{z_n\}$ inside the unit disc is said to satisfy the Blaschke condition, when $\Sigma_n(1 - |z_n|) < \infty$. The sequence $\{z_n\}$ is called a Blaschke sequence.

For $\lambda \geq 0$, we denote $S_\lambda = \{x + iy : |y| \leq \lambda\}$. The function $\theta(z) = w$, where

$$\theta(z) = \frac{e^z - 1}{e^z + 1}, \quad \theta^{-1}(w) = \ln\left(\frac{1+w}{1-w}\right), \quad \ln 1 = 0,$$

maps the strip $S_{\pi/2}$ conformally onto the unit disc. In general, if the sequence $\{x_n\}$ is increasing to ∞, then

$$\sum_{n=1}^{\infty}(1 - |\theta(x_n)|) < +\infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{e^{x_n}} < +\infty. \tag{2.1}$$

In particular, if the sequence X grows linearly, more precisely, if $x_n \geq a + nx_0$, for some $x_0 > 0$ and $a > 0$, then $u_n = \theta(x_n), n = 1, 2, \cdots$, is a Blaschke sequence. Indeed,

$$\sum_{n=1}^{\infty}(1 - |u_n|) = \sum_{n=1}^{\infty} \frac{2}{e^{x_n} + 1} \leq 2\sum_{n=1}^{\infty} \frac{1}{e^{na}} < \infty.$$

It follows that, if $\liminf(x_{n+1} - x_n) > 0$, then $\{\theta(x_n)\}$ is a Blaschke sequence. The converse does not hold. Indeed, let $u_n = 1 - 1/n^2$, then $\sum(1 - |u_n|) = \sum 1/n^2 < +\infty$, which is the Blaschke condition while,

$$x_{n+1} - x_n = \theta^{-1}\left(1 - \frac{1}{(n+1)^2}\right) - \theta^{-1}\left(1 - \frac{1}{n^2}\right) = \ln\left(\frac{2n^2 + 4n + 1}{2n^2 - 1}\right) \to 0.$$

To recapitulate, if $\liminf(x_{n+1} - x_n) > 0$, then $\{\theta(x_n)\}$ is a Blaschke sequence, but we may have $\lim(x_{n+1} - x_n) = 0$, for a Blaschke sequence $\{\theta(x_n)\}$. Equivalently, if

$\{\theta(x_n)\}$ is not a Blaschke sequence (and hence $S_{\pi/2}$ is not a set of approximation), then $\lim(x_{n+1} - x_n) = 0$, but it is possible to also have $\lim(x_{n+1} - x_n) = 0$, with $\{\theta(x_n)\}$ a Blaschke sequence.

Consider $x_n = \delta \ln n$. Then

$$\sum \frac{1}{e^{x_n}} = \sum \frac{1}{n^\delta},$$

so $\{u_n\}$ is a Blaschke sequence if and only if $\delta > 1$. For $\delta \leq 1$ and $x_n = \delta \ln n$, $E_{\pi/2}$ is not a set of approximation. Notice that for every $\delta > 0$, the sequence $\{\delta \ln n\}$ is "tight" in both senses $x_{n+1} - x_n \to 0$ and $x_{n+1}/x_n \to 1$.

The following lemma due to Scheinberg [2.12] gives us a Blaschke condition for a strip.

Lemma 2.3.2. *Let x_n be a sequence of distinct real numbers such that $|x_n| \to \infty$, $0 \leq \lambda < \infty$, θ be a conformal map which sends $S_\lambda = \{x + iy, |y| \leq \lambda\}$ to the unit disc, and $x_n = \theta^{-1}(u_n)$. Then*

$$\sum_{n=1}^{\infty} (1 - |u_n|) < \infty$$

if and only if

$$\sum_{n=1}^{\infty} a^{-|x_n|} < \infty, \quad \textit{where} \quad a = \exp\left(\frac{\pi}{2\lambda}\right).$$

This lemma allows us to construct the following example, where approximation fails.

Example 2.3.3. *There exists a Riemann surface R of infinite genus, formed by joining two copies of the complex plane with slits on the real axis, such that all closed strips "over" (see 1.4) the real axis are not sets of holomorphic approximation, while the set over the real axis itself is a set of tangential approximation.*

Proof. For $\lambda \geq 0$, we denote $S_\lambda = \{x + iy : |y| \leq \lambda\}$ and for $n = 1, 2, \cdots$, let θ_n be the conformal map of $S_{1/n}$ onto the unit disc, which maps $-\infty, 0, +\infty$ respectively to $-1, 0, +1$. Let $\{x_j\}$ be an increasing sequence of positive numbers tending to infinity, such that, for each n, the sequence $\{\theta_n(x_j)\}$ is not a Blaschke sequence. To obtain such a sequence, for each n let $\{x_{n,j}\}$ be a sequence of distinct real numbers tending to $+\infty$ with $x_{n,j} \geq n$, such that $\{\theta_n(x_{n,j})\}$ is not a Blaschke sequence. We may assume that these sequences are disjoint from each other. Now we may let $\{x_j\}$ be any sequence formed by combining all of the sequences $\{x_{n,j}\}$ into a single sequence.

We take two copies of the complex plane, remove the intervals (x_{2j-1}, x_{2j}) and join the slit planes together in the usual way to form a Riemann surface $R = R_X$, where X signifies the dependence on the sequence $X = \{x_j\}_j$. Let π be the projection map from R to $\mathbb{C}$ and put $E = E_\lambda := \pi^{-1}(S_\lambda)$. Let us show that $R^* \setminus E$ is connected. $R \setminus E$ has four connected components; each one is an open complex half-plane. None of these components is contained in any compact subset of R. Since every open neighbourhood of $*$ in R^* contains the complement of some compact subset of R, every neighbourhood of $*$ in R^* intersects each component of $R \setminus E$. Suppose $R^* \setminus E$ is disconnected. Then, it is the union of non-empty disjoint sets A and B, open in $R^* \setminus E$, whose union is $R^* \setminus E$. We may suppose that A contains $*$. Since A is an open neighbourhood of $*$ in $R^* \setminus E$, it is of the form $A = U \cap (R^* \setminus E)$, where U is an open neighbourhood of $*$ in R^*. We may assume that $U = R^* \setminus K$, where K is a compact subset of R. Thus, $A = R^* \setminus (K \cup E)$. Suppose some component H of $R \setminus E$ does not intersect A. Then $H \subset (K \cup E)$, which is precluded. Therefore A intersects each component of $R \setminus E$. But $B \cup (R \setminus E)$ is non-empty so some component H of $R \setminus E$ also intersects B. This shows that the half-plane H is the union of two non-empty disjoint open sets; $A \cap H$, and $B \cap H$. So H is disconnected, a contradiction. Therefore $R^* \setminus E$ is connected.

Now let us show that $R^* \setminus E$ is locally connected. Obviously, it is locally connected at points of $R \setminus E$. To show it is locally connected at infinity, let H_1, H_2, H_3, H_4 be the half-planes composing $R \setminus E$. For, $n = 1, 2, \cdots$, set $U_n = \{z : |z| > n\}, H_{j,n} = \pi^{-1}(U_n) \cap H_j$ and

$$V_n = \bigcup_{j=1}^{4} H_{j,n} \cup \{*\}.$$

Then, $\{V_n\}_n$ is a neighbourhood basis of $*$, all of whose members are connected, so $R^* \setminus E$ is locally connected at infinity. Thus $R^* \setminus E$ is locally connected.

The set E_0 has empty interior, so by Theorem 2.1.1 it is a set of holomorphic tangential approximation.

Fix $\lambda > 0$, let α be a point that is not in S_λ, and consider $\pi^{-1}(\alpha) = \{p_1, p_2\}$. Let f be a meromorphic function on R which has a pole at p_1 and only at p_1.

Suppose, to obtain a contradiction, that there exists a holomorphic function F on R such that $|F - f| \leq \varepsilon$ on E_λ. The function $g := F - f$ is meromorphic on R, has a pole at p_1, is holomorphic elsewhere, and is bounded on E_λ. Denote by X the set of values of the sequence $\{x_j\}$. Then $\widetilde{X} = \pi^{-1}(X)$ is the set of branch points of R. Let $\rho : R \setminus \widetilde{X} \to R \setminus \widetilde{X}$ be the involution, mapping each point of $R \setminus \widetilde{X}$ to the corresponding point on the other sheet having the same projection on $\mathbb{C}$. Set $g_1(p) = (g(p) - g(\rho(p)))^2$, for $p \in R \setminus \widetilde{X}$. Then, $G = g_1 \circ \pi^{-1}$ is a well-defined holomorphic function on $\mathbb{C} \setminus (X \cup \{\alpha\})$, which is bounded on $S_\lambda \setminus X$. Riemann's theorem on removable singularities implies that G extends holomorphically on $\mathbb{C} \setminus \{\alpha\}$ and vanishes at each point of the sequence $\{x_j\}$. Now by using the Blaschke condition, G is identically zero on S_λ, i.e., $g(p) = g(\rho(p))$, for $p \in E_\lambda \setminus \widetilde{X}$. By the uniqueness of meromorphic continuation, we obtain that $g(p) = g(\rho(p))$, for $p \in R \setminus \widetilde{X}$. In particular, g and hence f has a pole at p_1, which is a contradiction. Thus E_λ is not a set of approximation. $\square$

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Chapter 3

Uniform approximation in the spherical distance by functions meromorphic on Riemann surfaces

Given a function f defined on a closed subset E of a Riemann surface R, and given $\epsilon > 0$, a natural question is whether there exists a function f_ϵ meromorphic on R, such that $d(f(p), f_\epsilon(p)) < \epsilon$, for all $p \in E$, for a given distance function d. If ϵ is an arbitrary constant, we are speaking of uniform approximation, whereas, if $\epsilon = \epsilon(p)$ is an arbitrary positive continuous function, we call this tangential approximation. The two most natural distance functions here are the Euclidean distance $| \cdot - \cdot |$ and the chordal distance $\chi(\cdot, \cdot)$. If a function has no poles (respectively zeros) on E, we say that it is pole-free (respectively zero-free) on E. If the approximating functions f_ϵ are pole-free (respectively zero-free) on E, we call this pole-free (respectively zero-free) approximation. Unless explicitly stated otherwise, all approximations will be by meromorphic functions (equivalently, holomorphic mappings $R \to \overline{\mathbb{C}}$.) We shall consider eight types of meromorphic approximation:

- pole-free Euclidean uniform (respectively tangential) approximation,

- spherically uniform (respectively tangential) approximation,

- pole-free spherically uniform (respectively tangential) approximation,

- zero-free Euclidean uniform (respectively tangential) approximation.

Our main concern is with pole-free spherically tangential approximation, but we also discuss the other types of approximation, to situate our investigation in the general context of meromorphic approximation.

Spherical rational approximation on compact subsets of the plane was studied in [3.22] by Roth, Walsh and Gauthier. That paper concluded with a promise and two open problems. The promise was to consider in a later paper, spherical *meromorphic* approximation on *closed* sets. One open problem was to replace subsets of the plane by subsets of a Riemann surface. The other open problem was to ask whether we can spherically approximate a function f on a subset E by meromorphic functions pole-free on E, provided f is pole-free on E^0. The paper this chapter is based on finally addressed this promise and these open problems with a 39-year delay.

3.1 Pole-free Euclidean uniform and tangential approximation

Denote by $\overline{\mathbb{C}}$ the Riemann sphere $\mathbb{C} \cup \{\infty\}$. For a compact subset $K \subset \mathbb{C}$, the theory of rational approximation studies the class $R(K)$ of functions on K, which are uniform Euclidean limits on K of rational functions which are pole-free on K.

In this section we present extensions to Riemann surfaces of classical results on rational approximation. Some of these extensions are known, but since we shall need them in subsequent sections and since they are sometimes difficult to find in the literature and notations and terminology are different in various sources, we present them here using our notation and terminology, for the convenience of the reader.

If E is a closed subset of a Riemann surface R, we denote by $M(E)$ the space of functions $E \to \mathbb{C}$, which are Euclidean uniform limits of functions meromorphic on R, which are pole-free on E. The subject of uniform approximation by such pole-free functions has been studied extensively, although many problems remain open.

Let D be an open connected subset of a Riemann surface R, $\phi : D \to \mathbb{C}$ a one-to-one holomorphic function such that $\phi(D) = \{z : |z - z_0| < r\}$ in $\mathbb{C}$. If ϕ is one-to-one and holomorphic on a neighbourhood of $\overline{D}$, then $\overline{D}$ is a closed parametric disc.

We recall (see for example [3.20]) that every open Riemann surface admits a Cauchy kernel $C(p, q)$, that is, a meromorphic function on $R \times R$, whose only singularities are poles along the diagonal, such that, for some holomorphic function H on $R \times R$ and every point $p_0 \in R$, there is a parametric disc $\rho : D_0 \to \Delta$, where Δ is the unit disc in $\mathbb{C}$, such that, writing $(\rho \times \rho)(p, q) = (z, w)$,

$$C(p, q) = \frac{1}{z - w} + H\big((\rho \times \rho)^{-1}(z, w)\big), \quad (p, q) \in D_0 \times D_0.$$

A Bordered Riemann surface R is a connected Hausdorff space with an open covering $\{U_i\}$ and corresponding homeomorphisms h_i, $h_i : U_i \to V_i = h_i(U_i)$ where V_i is a relatively open subset of the closed upper half-plane. Also whenever $U_i \cap U_j \neq \emptyset$, the composition $h_i \circ h_j^{-1}$ is a conformal homeomorphism of the open set $h_j(U_i \cap U_j)$ on to $h_i(U_i \cap U_j)$.

Definition 3.1.1. A subset C of a Riemann surface R is called a 1-dimensional (real) sub-manifold if every $p \in C$ has an open neighbourhood Nwhich can be mapped homeomorphically onto $|z| < 1$ such that the intersection $N \cap C$ corresponds to the real interval $(-1, +1)$.

Definition 3.1.2. A region Ω in a Riemann surface R is *regularly embedded* if $\partial\Omega$ is a 1-dimensional sub-manifold and $\partial\Omega = \partial(R \setminus \overline{\Omega})$ and in this case we say that $\overline{\Omega}$ is a closed regularly embedded region. It is easy to see (cf. [3.4]) that $\overline{\Omega}$ is a bordered surface whose border $b\Omega$ coincides with its boundary $\partial\Omega$. By a *Jordan region* Ω in a Riemann surface R we mean a pre-compact regularly embedded region. By a *compact Jordan region*, we mean the closure $\overline{\Omega}$ of a Jordan region Ω.

Classical polynomial approximation on compact subsets K of $\mathbb{C}$ first considered the case where the complement $\mathbb{C}\setminus K$ is connected, that is, K has no bounded complementary components. The following theorem on a Riemann surface R may help to understand approximation on a subset, whose complement has only finitely many bounded components. We say that a subset of R is bounded if it is relatively compact.

Theorem 3.1.3. *Suppose a domain Ω in an open Riemann surface R has only finitely many bounded complementary components $Q_1, \ldots, Q_n$. Set $Q_\infty = R \setminus (\Omega \cup Q_1 \cup \cdots \cup Q_n)$ and $U_k = R \setminus Q_k$, for $k = \infty, 1, \ldots, n$. Let f be a function holomorphic in Ω. Then, f can be represented in the form*

$$f = f_\infty + f_1 + \cdots + f_n, \tag{3.1}$$

where f_k is holomorphic in U_k, for each k. Moreover, if $f \in A(\overline{\Omega})$, then $f_k \in A(\overline{U}_k)$.

Proof. We may exhaust Ω by smoothly bounded Jordan regions Ω_j, compatible with Ω, in the following sense. We may label the complementary components of Ω_j as $Q_{j,k}$, where $Q_{j,k} \supset Q_k$, $k = \infty, 1, \ldots, n$. We denote $\partial\Omega_{j,k} = \Gamma_{j,k}$; $k = \infty, 1, \ldots, n$. Let $C(p,q)dq$ be a Cauchy kernel on R. By the Cauchy formula,

$$f(p) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\Gamma_{j,k}} f(q)C(p,q)dq = \sum_{k=1}^{n} f_{j,k}(p), \quad \text{for} \quad p \in \Omega_j.$$

We note that $f_{j,k}$ is holomorphic on $R \setminus \Gamma_{j,k}$ and in particular on $R \setminus Q_{j,k}$. We may so choose the exhaustion that $\Gamma_{j,k}$ is homologous to $\Gamma_{i,k}$ in Ω, for each i and j, so that $f_{j,k} = f_{i,k}$ on $R \setminus Q_{j,k}$, for $i > j$. Hence, letting $j \to \infty$, we obtain well-defined functions f_k holomorphic on $R \setminus Q_k$ and we have (3.1).

Now suppose $f \in A(\overline{\Omega})$ and fix $k = 1, \ldots, n$. Since

$$f_k = f - \sum_{j \neq k} f_j,$$

where the f_j are respectively holomorphic on the open sets $U_j \supset \partial U_k$, it follows that f_k extends continuously to $\overline{U}_k$. □

Theorem 3.1.3 applies to the particularly interesting case where the domain Ω is a Jordan region.

The fusion lemma of Alice Roth is a fundamental tool in rational approximation and was extended to Riemann surfaces in [3.16] as follows.

Theorem 3.1.4 (Fusion)**.** *Let K_1, K_2 and K be compact subsets of a Riemann surface R such that $K_1 \cap K_2 = \emptyset$ and $K_1 \cup K_2 \neq R$. Then there exists a constant A, depending* *only on K_1, K_2 and K, with the following property. If m_1, m_2 are meromorphic functions on R, such that*

$$|m_1 - m_2| < \epsilon \quad on \quad K,$$

then there exists an m meromorphic on R such that

$$|m - m_j| \leq A \cdot \epsilon \quad on \quad K_j \cup K, \quad j = 1, 2.$$

For a closed subset $E \subset R$, recall that $M(E)$ is the family of functions f on E such that there exists a sequence of meromorphic functions f_n on R, pole-free on E, such that $|f - f_n| \to 0$ uniformly on E. In case E is compact, then by the theorem of Behnke-Stein, this is equivalent to such approximations by meromorphic functions having only finitely many poles. Functions in $M(E)$ necessarily lie in the set $A(E)$ of continuous functions $E \to \mathbb{C}$ which are holomorphic on E^0. Let us say that E is a *set of uniform Euclidean pole-free approximation* if $A(E) = M(E)$.

With the help of the Fusion Theorem 3.1.4, one can prove (see [3.20]) the following fundamental result, originally proved by Kodama [3.26].

Theorem 3.1.5 (Kodama)**.** *If K is a compact subset of a Riemann surface R and if for every $p \in K$ there is a closed neighbourhood W of p such that $f|W \in M(K \cap W)$, then $f \in M(K)$.*

Proof. Kodama proved this theorem for the case that R is an open Riemann surface.

Suppose R is compact. If $K = R$, the conclusion is trivial. This is because the property of being meromorphic is local. A function is meromorphic in an open set if and only if it is meromorphic in a neighbourhood of each point. Suppose $K \neq R$ and choose a point $q \in R \setminus K$. By hypothesis, for every $p \in K$ there is a neighbourhood W_p of p in R, such that $f|W_p \in M(K \cap W_p)$. Now we remove the point q to form the open Riemann surface R_q. It is easy to see that the set $U_p = W_p \setminus \{q\}$ is a closed neighbourhood of p in R_q, satisfying $f|U_p \in M(K \cap U_p)$. But Kodama proved the theorem for this case. Hence there is a sequence g_n of meromorphic functions in R_q which are pole-free on K such that $|f - g_n| < 1/n$ on K. By the Runge-type theorem for compact Riemann surfaces due to Köditz and Timmann [3.28], for each n, there is a meromorphic function h_n on R, which is pole-free on K, such that $|g_n - h_n| < 1/n$ on K. By the triangle inequality, the sequence of functions $\{h_n\}$, which are meromorphic on R and pole-free on K, converges uniformly on K to f. Thus, $f \in M(K)$. □

Corollary 3.1.6. *If $\overline{\Omega}$ is a compact Jordan region in a Riemann surface R, then $\overline{\Omega}$ is a set of Euclidean pole-free approximation.*

Proof. The complement $R \setminus \overline{\Omega}$ has only finitely many components, so if $\varphi : \overline{D} \to \mathbb{C}$ is a closed parametric disc, $\varphi(\overline{\Omega} \cap \overline{D})$ has only finitely many complementary components in $\mathbb{C}$. By a theorem of Mergelyan, $A(\varphi(\overline{\Omega} \cap \overline{D})) = R(\varphi(\overline{\Omega} \cap \overline{D}))$. By the Behnke-Stein Theorem, it follows that $A(\overline{\Omega} \cap \overline{D}) = M(\overline{\Omega} \cap \overline{D})$. From Theorem 3.1.5, then, $A(\overline{\Omega}) = M(\overline{\Omega})$. □

It follows from Theorem 3.1.5 that a compact set is a set of uniform Euclidean pole-free approximation if it is locally so. There is also, for closed sets, a theorem in the other direction due to Boivin and Jiang [3.9].

Theorem 3.1.7 (Boivin-Jiang). *Let E be a closed subset of a Riemann surface R. If $A(E) = M(E)$, then, for each closed parametric disc $\overline{D}$, we have $A(E \cap \overline{D}) = M(E \cap \overline{D})$.*

The known examples [3.18] also show that the converse of Theorem 3.1.7 is false in general, without additional assumptions on E or R.

The following consequences of Theorem 3.1.7 are essentially obtained in [3.9].

Corollary 3.1.8. *For a compact subset K of a Riemann surface, the following are equivalent:*

1) $A(K) = M(K)$,

2) $A(K \cap \overline{D}) = M(K \cap \overline{D})$, for every closed parametric disc $\overline{D}$,

3) for each $p \in K$, and each sufficiently small closed parametric disc $\overline{D}_p$ centred at p, we have $A(K \cap \overline{D}_p) = M(K \cap \overline{D}_p)$,

4) $A(\varphi(K \cap \overline{D})) = R(\varphi(K \cap \overline{D}))$, for every closed parametric disc $(\varphi : \overline{D})$,

5) for each $p \in K$, and for each sufficiently small closed parametric disc $(\varphi_p : \overline{D}_p)$ centred at p, we have $A(\varphi_p(K \cap \overline{D}_p)) = R(\varphi_p(K \cap \overline{D}_p))$,

Proof. The preceding theorems give 1) $\leftrightarrow$ 2).

By the Runge-Behnke-Stein Theorem, 2) $\leftrightarrow$ 4) and 3) $\leftrightarrow$ 5). The implications 2) $\rightarrow$ 3) and 4) $\rightarrow$ 5) are trivial and the implication 3) $\rightarrow$ 1) follows from Theorem 3.1.5. $\square$

Corollary 3.1.9 ([3.16]). *For a closed subset E with empty interior in a Riemann surface, the following are equivalent:*

1) E is a set of Euclidean pole-free tangential approximation,

2) E is a set of Euclidean pole-free uniform approximation,

3) $C(E \cap \overline{D}) = M(E \cap \overline{D})$, for every closed parametric disc $\overline{D}$,

4) for each $p \in E$, and for each sufficiently small closed parametric disc $\overline{D}_p$ centred at p, we have $C(E \cap \overline{D}_p) = M(E \cap \overline{D}_p)$,

5) $C(\varphi(E \cap \overline{D})) = R(\varphi(E \cap \overline{D}))$, for every closed parametric disc $(\varphi : \overline{D})$,

6) for each $p \in E$, and for each sufficiently small closed parametric disc $(\varphi_p : \overline{D}_p)$ centred at p, we have $C(\varphi_p(E \cap \overline{D}_p)) = R(\varphi_p(E \cap \overline{D}_p))$,

Proof. The proof is similar to that of the previous corollary, with the help of [3.16]. $\square$

If we can approximate on each of two sets, it does not in general imply that we can approximate on the union. A counterexample is given in [3.14, p. 113], where one of the two sets is even a compact Jordan domain. However, in the following situations it is possible.

Theorem 3.1.10. *For $j = 1, 2$, let K_j be compact subsets of a Riemann surface such that $A(K_j) = M(K_j)$. If the compacta are disjoint, if one of them has empty interior or if one of them is a compact Jordan region with analytic boundary, then*

$$A(K_1 \cup K_2) = M(K_1 \cup K_2).$$

Proof. First Suppose $K_1 \cap K_2 = \emptyset$. Let $f \in A(K_1 \cup K_2)$ and $\epsilon > 0$. Then $f \in A(K_j)$. By hypothesis $A(K_j) = M(K_j)$. Then there exits a meromorphic function m_j such that $|f - m_j| < \epsilon/2 \quad \text{on} \quad K_j$. Since K_j is compact, there exits disjoint open neighbourhood U_j of K_j such that m_j is holomorphic on U_j. By the Behnke-Stein theorem, there exists a meromorphic function m such that $|m - m_j| < \epsilon/2$ on K_j, and so $|f - m| < \epsilon$ on $K_1 \cup K_2$.

Now suppose K_2 has analytic boundary. By Theorem 3.1.5, it is sufficient to show that, for each $p \in K_1 \cup K_2$, there is a closed parametric disc $\overline{D}_p$ centred at p, such that

$$A((K_1 \cup K_2) \cap \overline{D}_p) = M((K_1 \cup K_2) \cap \overline{D}_p). \tag{3.2}$$

Since $A(K_1) = M(K_1)$, by Theorem 3.1.7, for every closed parametric disc $\overline{D}_p$ centred at p, we have $A(K_1 \cap \overline{D}_p) = M(K_1 \cap \overline{D}_p)$. Since the boundary of K_2 is analytic, for sufficiently small $\overline{D}_p$, we have $K_2 \cap \overline{D}_p$ is a Jordan domain bounded by two analytic arcs, disjoint except for their common end points. So $K_2 \cap \overline{D}_p$ is a Lyapunov domain, therefore by Lemma 3 in [3.19],

$$A((K_1 \cap \overline{D}_p) \cup (K_2 \cap \overline{D}_p)) = M((K_1 \cap \overline{D}_p) \cup (K_2 \cap \overline{D}_p)),$$

Thus,

$$A((K_1 \cup K_2) \cap \overline{D}_p) = M((K_1 \cup K_2) \cap \overline{D}_p).$$

which is (3.2).

Now suppose $K_2^0 = \emptyset$. First we consider the case where K_1 and K_2 are compact sets in the complex plane. By the Vitushkin theorem, $A(K) = R(K)$ if and only if for each open disc D,

$$\alpha(D \setminus K) = \alpha(D \setminus K^0),$$

where, for an arbitrary Borel set E, $\alpha(E)$ denotes the continuous analytic capacity of E.

For $K = K_1 \cup K_2$,

$$\alpha(D \setminus (K_1 \cup K_2)^0) = \alpha(D \setminus K_1^0) = \alpha(D \setminus K_1),$$

The second equality is from the Vitushkin Theorem. The first equality is because $(K_1 \cup K_2)^0 = K_1^0$. To see this, because $K_2^0 = \emptyset$, it is sufficient to show that no point of $\partial K_1 \cap K_2$ can be in $(K_1 \cup K_2)^0 = K_1^0$. Suppose, to obtain a contradiction that z is such a point. Then z has an open neighbourhood $U \subset (K_1 \cup K_2)$. Since z is a boundary point of K_1, the open set $U \setminus K_1$ is nonempty and this nonempty open set is contained in $(K_1 \cup K_2) \setminus K_1$, which is contained in K_2. But this is a contradiction, since $K_2^0 = \emptyset$.

A second form of the Vitushkin Theorem states that $A(K) = R(K)$ if and only if, for every open set U, we have $\alpha(U \setminus K) = \alpha(U \setminus K^0)$. Applying this to the compact set K_2 and the open set $D \setminus K_1$, we have, since $K_2^0 = \emptyset$,

$$\alpha(D \setminus K_1) = \alpha(U) = \alpha(U \setminus K_2^0) = \alpha(U \setminus K_2) = \alpha(D \setminus (K_1 \cup K_2)).$$

Combining the previous two equations, we have $A(K_1 \cup K_2) = R(K_1 \cup K_2)$, by the first version of Vitushkin's theorem.

We now prove the theorem for the general case where K_1 and K_2 are compact subsets of a Riemann surface. By Corollary 3.1.8, it is sufficient to show that, for every closed parametric disc $\overline{D}$, $A(\varphi((K_1 \cup K_2) \cap \overline{D})) = R(\varphi((K_1 \cup K_2) \cap \overline{D}))$. By the Corollary, we have $A(\varphi(K_j \cap \overline{D})) = R(\varphi(K_j \cap \overline{D})), j = 1, 2$, so by the planar case,

$$A(\varphi((K_1 \cup K_2) \cap \overline{D})) = A(\varphi(K_1 \cap \overline{D}) \cup \varphi(K_2 \cap \overline{D})) =$$

$$R(\varphi(K_1 \cap \overline{D}) \cup \varphi(K_2 \cap \overline{D})) = R(\varphi((K_1 \cup K_2) \cap \overline{D})),$$

which concludes the proof. □

Definition 3.1.11. A closed subset E of an open Riemann surface is said to be *essentially of finite genus* if there exists an open covering $\{U_i : i \in I\}$ of E such that $U_i \cap U_j = \emptyset$ if $i \neq j$ and each U_i is of finite genus.

If R is of finite genus, then every closed subset of R is automatically essentially of finite genus and if R is of infinite genus, then it is not of essentially finite genus, since Riemann surfaces are by definition connected.

Theorem 3.1.12. *Let E be the union of a locally finite family of disjoint compact sets of uniform Euclidean pole-free approximation in a Riemann surface R. Then E is a set of Euclidean pole-free tangential approximation.*

Proof. By assumption $E = \cup E_n$, where $\{E_n\}$ is the locally finite family in the hypotheses. If the family is finite, then the conclusion follows from Theorem 3.1.10, so we suppose the family is infinite. It follows that R is open and E is essentially of finite genus. Theorem 2 in [3.16] states that, if E is a closed subset that is essentially of finite genus in an open Riemann surface R, and $R \setminus E$ is unbounded (this condition was omitted by error in [3.16]), then E is a set of Euclidean pole-free uniform approximation. This applies to our set E.

First of all, since $\{E_n\}$ is a locally finite family of disjoint compact sets, it follows that R is open. Suppose to obtain a contradiction that $R \setminus E$ is bounded. Then, at most finitely many of the E_n, say $E_1, \ldots, E_m$ can meet the compacts set $\overline{R \setminus E}$. Now

$$R = \left((\overline{R \setminus E}) \cup E_1 \cup \cdots \cup E_m\right) \cup \left(\bigcup_{n>m} E_n\right)$$

represents R as the union of two disjoint closed sets, contradicting the connectivity of R. Thus $R \setminus E$ is unbounded as claimed.

Indeed, we may cover E by a locally finite family $\{U_n\}$ of disjoint open sets of finite genus.

Fix an arbitrary positive continuous function ϵ on E. Set $\epsilon_n = \min\{\epsilon(p) : p \in E_n\}$. We define a holomorphic function on $U = \cup U_n$, by setting $h = 1 + 1/\epsilon_n$ on each U_n respectively. Let g_∞ be a meromorphic function on R such that $|g_\infty - h| < 1$ on E. The g_∞ exists by Runge- type theorem" Behnk-Stien" Then $|g_\infty| > 1/\epsilon_n$ on E_n, for each n.

Fix an arbitrary $f \in A(E)$. Let g_0 be a meromorphic function such that $|fg_\infty - g_0| < 1$ on E and set $g = g_0/g_\infty$. Then for $p \in E_n$,

$$|f(p) - g(p)| = |f(p)g_\infty(p) - g_o(p)| \cdot \frac{1}{|g_\infty(p)|} < \epsilon_n \leq \epsilon(p).$$

□

Corollary 3.1.13. *Let E be the union of a locally finite family of disjoint compact Jordan regions in a Riemann surface R. Then E is a set of Euclidean pole-free tangential approximation.*

Proof. This follows immediately from the theorem 3.1.12 and Corollary 3.1.6. □

In case the function to be approximated on a compact set K is holomorphic on a neighbourhood of K, the following result of Scheinberg [3.30] allows us to specify the location of poles of approximating meromorphic functions. It is a generalization of Runge's theorem to Riemann surfaces.

Theorem 3.1.14 (Scheinberg)**.** *Let K be a compact subset of an open Riemann surface R and let P be a set consisting of one point from each bounded complementary component of K. Then, each holomorphic function on K is the uniform limit of meromorphic functions all of whose poles lie in P.*

3.2 Spherically uniform and tangential approximation

The chordal distance χ in $\mathbb{C} \cup \{\infty\}$ is defined as follows:

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{(1 + |z_1|^2)^{\frac{1}{2}}(1 + |z_2|^2)^{\frac{1}{2}}}, \quad z_1, z_2 \in \mathbb{C},$$

$$\chi(z, \infty) = \frac{1}{(1 + |z|^2)^{\frac{1}{2}}}, \quad \chi(\infty, \infty) = 0.$$

If $f_n, f : E \to \overline{\mathbb{C}}$, $n = 1, 2, ...$ are mappings defined on a set E, then f_n is said to converge spherically uniformly (or χ-uniformly) to f on E, if ,

$$\sup_{z \in E} \chi(f_n(z), f(z)) \to 0, \quad as \quad n \to \infty.$$

For a closed set $E \subset R$, we denote by $M_\chi(E)$ the class of mappings $f : E \to \overline{\mathbb{C}}$, which are χ-uniform limits on E of holomorphic *mappings* $R \to \overline{\mathbb{C}}$, in other words of meromorphic *functions* $R \to \mathbb{C} \cup \{\infty\}$.

Clearly, $M_\chi(E) \subset A_\chi(E)$, where $A_\chi(E)$ denotes the family of all continuous mappings $f : E \to \overline{\mathbb{C}}$ which are holomorphic on the interior E^0. Equivalently, $A_\chi(E)$ is the family of spherically continuous functions $f : E \to \overline{\mathbb{C}} \cup \{\infty\}$, which on each component of E^0 are meromorphic or identically ∞.

An open problem is to characterize pairs (E, R), for which $M_\chi(E) = A_\chi(E)$, where E is a closed subset of a Riemann surface R. We call such a set E a set of uniform spherical approximation. We shall show that a compact set is a set of uniform spherical approximation if and only if it is a set of uniform Euclidean pole-free approximation.

For a point p in a Riemann surface R, we say that $\overline{D}_p$ is a *closed parametric disc* centred at p if there is an injective holomorphic function $h : U \to \mathbb{C}$ defined in a neighbourhood U of $\overline{D}_p$ such that $h(\overline{D}_p)$ is the closed unit disc and $h(p) = 0$. The following is a spherical version of the Bishop-Kodama Localization Theorem.

Theorem 3.2.1. *Let $f : K \to \overline{\mathbb{C}}$, where K is a compact subset of a Riemann surface R. Suppose that for each $p \in K$ there is a closed parametric disc $\overline{D}_p$ with center p such that*

$$f|_{(K\cap\overline{D}_p)} \in M_\chi(K \cap \overline{D}_p).$$

Then $f \in M_\chi(K)$.

Proof. Set

$$K_1 = \{p \in K : |f(p)| \le 1/2\}, \tag{3.3}$$

$$K_0 = \{p \in K : 1/2 \le |f(p)| \le 1\}, \quad \text{and} \tag{3.4}$$

$$K_2 = \{p \in K : |f(p)| \ge 1\}. \tag{3.5}$$

Let $a > 1$ be a positive constant associated with K_1 and K_2 by the Fusion Theorem 3.1.4. Since f is continuous and finite-valued in a neighbourhood of $K_1 \cup K_0$, for each $p \in K_1 \cup K_0$, there is a closed parametric disc $\overline{D}_p$ for which

$$f|_{(K\cap\overline{D}_p)} \in M(K \cap \overline{D}_p).$$

Thus, by Theorem 3.1.5 for each $\epsilon > 0$, there is a function r_1 holomorphic in a neighbourhood of $K_1 \cup K_0$, such that

$$|r_1 - f| < \epsilon/16a < \epsilon/2 \quad \text{on} \quad K_1 \cup K_0. \tag{3.6}$$

Similarly, there is a function r_2 holomorphic on $K_2 \cup K_0$, such that

$$|r_2 - 1/f| < \epsilon/16a < \epsilon/2 \quad \text{on} \quad K_2 \cup K_0. \tag{3.7}$$

Hence,

$$\chi(1/r_2, f) = \chi(r_2, 1/f) \le |r_2 - 1/f| < \epsilon/2 \quad \text{on} \quad K_2 \cup K_0. \tag{3.8}$$

Therefore,

$$|r_2| \ge 1/|f| - \epsilon/2 \ge 1/2 - \epsilon/2 > 1/4 \quad \text{on} \qquad K_0,$$

if $\epsilon < 1/2$. By (3.7),

$$|1/r_2 - f| < \frac{\epsilon}{16a}\frac{|f|}{|r_2|} < \frac{\epsilon}{4a} \quad \text{on} \quad K_0, \tag{3.9}$$

and by (3.6) and (3.9)

$$|r_1 - 1/r_2| < \frac{\epsilon}{2a} \quad \text{on} \quad K_0.$$

By the Fusion Theorem 3.1.4, there is an $r \in M(R)$ with

$$|r - r_1| < \epsilon/2 \quad \text{on} \quad K_1 \cup K_0; \tag{3.10}$$

$$|r - 1/r_2| < \epsilon/2 \quad \text{on} \quad K_0 \cup K_2. \tag{3.11}$$

Thus, on $K_1 \cup K_0$,

$$\chi(r, f) \leq |r - f| \leq |r - r_1| + |r_1 - f| < \epsilon,$$

and on $K_0 \cup K_2$, by (3.11) and (3.8),

$$\chi(r, f) \leq \chi(r, 1/r_2) + \chi(1/r_2, f) < \epsilon.$$

□

Corollary 3.2.2. *Let K be a compact subset of Riemann surface. If for each $p \in K$, there exists a closed parametric disc $\overline{D}_p$ centred at p such that $A_\chi(K \cap \overline{D}_p) = M_\chi(K \cap \overline{D}_p)$, then $A_\chi(K) = M_\chi(K)$.*

Proof. Let $f \in A_\chi(K)$. Then, for each $p \in K$ and each closed parametric disc $\overline{D}_p$, we have $f \in A_\chi(K \cap \overline{D}_p)$. By the preceding theorem $f \in M_\chi(K)$, so $A_\chi(K) \subset M_\chi(K)$. The opposite inclusion is obvious. □

The following was first proved in [3.22] for compact subsets of the Riemann sphere.

Theorem 3.2.3. *For a compact subset K of a Riemann surface, we have*

$$A_\chi(K) = M_\chi(K) \quad \textit{if and only if} \quad A(K) = M(K).$$

Proof. The implication from left to right is clear. Let $f \in A(E)$ and $\epsilon \in C(E)$ be positive. Since f is bounded on compact sets, there exists $\delta \in C(E)$ positive, such that, if $g : E \to \overline{\mathbb{C}}$ and $\chi(f, g) < \delta$, then g is finite-valued and $|f - g| < \epsilon$. If E is a set of spherical tangential approximation, there is a meromorphic g such that $\chi(f, g) < \delta$.

Suppose $A(K) = M(K)$ and let $f \in A_\chi(K)$. For each $p \in K$ we may choose a closed parametric disc $\overline{D}_p$ such that $f|_{(K \cap \overline{D}_p)} \in A(K \cap \overline{D}_p)$ or $1/f|_{(K \cap \overline{D}_p)} \in A(K \cap \overline{D}_p)$.

Since $M(K \cap \overline{D}_p) \subset M_\chi(K \cap \overline{D}_p)$, we have $f \in M_\chi(K \cap \overline{D}_p)$ or $1/f \in M_\chi(K \cap \overline{D}_p)$. Since $f \in M_\chi(K \cap \overline{D}_p)$ if and only if $1/f \in M_\chi(K \cap \overline{D}_p)$, it follows that $f|_{(K \cap \overline{D}_p)} \in M_\chi(K \cap \overline{D}_p)$. By Theorem 3.2.1, $f \in M_\chi(K)$. □

Combined with the previous section, this theorem yields the following corollaries.

Corollary 3.2.4. *For a compact subset K of a Riemann surface, the following are equivalent:*

1) $A_\chi(K) = M_\chi(K)$,

2) $A_\chi(K \cap \overline{D}) = M_\chi(K \cap \overline{D})$, for every closed parametric disc $\overline{D}$,

Proof. 1) → 2) Suppose $A_\chi(K) = M_\chi(K)$. Then, by Theorem 3.2.3, $A(K) = M(K)$ and so by Theorem 3.1.7, for every closed parametric disc $\overline{D}$, $A(K \cap \overline{D}) = M(K \cap \overline{D})$. The sets $K \cap \overline{D}$ are compact, so again by applying Theorem 3.2.3, $A_\chi(K \cap \overline{D}) = M_\chi(K \cap \overline{D})$.

2) → 1) Suppose $A_\chi(K \cap \overline{D}) = M_\chi(K \cap \overline{D})$, for every closed parametric disc $\overline{D}$ Then, by Theorem 3.2.3, $A(K \cap \overline{D}) = M(K \cap \overline{D})$, for every closed parametric disc $\overline{D}$. It follows easily from Theorem 3.1.5 that $A(K) = M(K)$, and so by Theorem 3.2.3, $A_\chi(K) = M_\chi(K)$. □

Corollary 3.2.5. *For $j = 1, 2$, let K_j be compact subsets of a Riemann surface such that $A_\chi(K_j) = M_\chi(K_j)$. If the compacta are disjoint, if one of them has empty interior or if one of them is a compact Jordan region with analytic boundary, then*

$$A_\chi(K_1 \cup K_2) = M_\chi(K_1 \cup K_2).$$

Proof. By Theorem 3.2.3, $A_\chi(K_j) = M_\chi(K_j)$ if and only if $A(K_j) = M(K_j)$. Then, by Theorem 3.1.10, $A(K_1 \cup K_2) = M(K_1 \cup K_2)$ in each case of the hypothesis, and so again by Theorem 3.2.3, $A_\chi(K_1 \cup K_2) = A_\chi(K_1 \cup K_2)$. □

The preceding theorem says that a compact set is a set of spherical uniform approximation if and only if it is a set of Euclidean uniform pole-free approximation. We now consider corresponding questions for closed sets.

Lemma 3.2.6. *Let E be a closed subset of a Riemann surface R and let $f \in C_\chi(E)$. Then, f can be extended to a continuous mapping $F \in C_\chi(U)$, where U is a neighbourhood of E.*

Proof. This follows directly from Theorem 2-35 in [3.25]. □

Theorem 3.2.7 (Whitney theorem)**.** *For every open subset U of $\mathbb{R}^n$, for every continuous $f : U \to \mathbb{R}^k$ and for every positive continuous function ϵ on U, there exists a function F, analytic on U, such that*

$$||F(x) - f(x)|| < \epsilon(x), \quad \textit{for all} \quad x \in U.$$

Lemma 3.2.8. *Let E be a closed subset of a Riemann surface R, let $f \in C_\chi(E)$ and ϵ be a positive continuous function on E. Then, there is an open neighbourhood N of E and a (real) analytic mapping $g : N \to \overline{\mathbb{C}}$, such that*

$$\sup_{p \in E} \chi(f(p), g(p)) < \epsilon(p).$$

Proof. We may consider that R is properly embedded in a Euclidean space $\mathbb{R}^n$, Rudy theorem and $\overline{\mathbb{C}}$ is the unit sphere S^2 centred at the origin in $\mathbb{R}^3$. By Lemma 3.2.6, we may extend f continuously as a mapping $f : U \to S^2$, where U is an open neighbourhood of E in $\mathbb{R}^n$. By the Tietze Extension Theorem, we may consider that ϵ is also defined on all of U. By choosing U smaller, we may assume that $\|f\| > 1/2$ on U. Thus $F = f/\|f\|$ is a continuous extension $U \to S^2$ of f. By the Whitney Approximation Theorem, for each positive continuous function η on U, there is an analytic mapping $h : U \to \mathbb{R}^3$ such that $\|F(p) - h(p)\| < \eta(p)$. We may assume that $\eta < 1/2$. Thus $|h| \geq |F| - |F - h| > 1/2$, so the mapping $g(p) = h(p)/\|h(p)\|$ is well-defined and analytic. Moreover, g takes its values on S^2. Since the projection $q \to q/\|q\|$ is uniformly continuous in the shell $1/2 \leq \|q\| \leq 3/2$, we may choose η to tend to zero so rapidly as $\|p\| \to \infty$, that $\|F(p) - g(p)\| < \epsilon(p)$ on U. Since $F(p)$ and $g(p)$ both lie on S^2, the Euclidean distance between them is the same as the chordal distance and so we have $\chi(F(p), g(p)) < \epsilon(p)$, for $p \in U$. Setting $N = U \cap R$, we have the desired conclusion. □

Theorem 3.2.9. *If a closed set E in a Riemann surface is a set of spherical tangential approximation, then it is also a set of Euclidean pole-free tangential approximation.*

Proof. Let $f \in A(E)$ and $\epsilon \in C(E)$ be positive. Since f is bounded on compact sets, there exists $\delta \in C(E)$ positive, such that, if $g : E \to \overline{\mathbb{C}}$ and $\chi(f,g) < \delta$, then g is finite-valued and $|f - g| < \epsilon$. If E is a set of spherical tangential approximation, there is a meromorphic g such that $\chi(f,g) < \delta$. Thus, E is a set of uniform Euclidean pole-free approximation. $\square$

Corollary 3.2.10. *If a closed set E in a Riemann surface is a set of spherical tangential approximation, then the family of components of the fine interior of E, and hence also the family of components of the interior, must satisfy the long islands condition.*

Proof. By Theorem 3.2.9, E is a set of Euclidean pole-free tangential approximation, so by Theorem 2 and a remark after that in [3.6] , the family of components of the fine interior satisfy the long islands condition. Since any subfamily of a family satisfying the long islands condition must also satisfy the long islands condition, it follows that the family of components of the interior of E must also satisfy the long islands condition. $\square$

We shall consider the case $E^0 = \emptyset$ in a more general context in the following section. The general situation, when $E^0 \neq \emptyset$ is not understood, but we have the following partial result.

Theorem 3.2.11. *Let E be the union of a locally finite family of disjoint compact sets of uniform spherical approximation in a Riemann surface R. Then E is a set of spherical tangential approximation.*

Proof. Let $f \in A_\chi(E)$ and $\epsilon > 0$ be an arbitrary continuous function on E. If $f \equiv \infty$, we may approximate f by the constant functions $f_n = n$, $n = 1, 2, \ldots$, so we shall assume that $f \not\equiv \infty$.

If the family of compacta is finite, then by Corollary 3.2.5, the conclusion holds, so we assume that the family is infinite.

By hypothesis $E = \cup E_j$ where E_j's are compact disjoint sets of uniform approximation which form a locally finite family. Let K_n be a regular exhaustion of R, where each K_n is bounded by finitely many disjoint analytic curves. Choose n_1 so large such that $K_{n_1} \cap E \neq \emptyset$. Let F_1 be the union of the E_j's which meet K_{n_1}. Now choose n_2 so large that $K_{n_2} \cap E \neq \emptyset$ and $F_1 \subset K^0_{n_2}$. Let F_2 be the union of the E_j's which meet K_{n_2}. We construct in this way sequences $\{K_{n_k}\}$ and $\{F_k\}$ and we put

$$H_k = K_{n_k} \cup F_{k+1} = \big(K_{n_k} \cup F_k\big) \cup (F_{k+1} \setminus F_k).$$

Note that F_k and $F_{k+1} \setminus F_k$ are sets of approximation, because they are finite unions of disjoint sets of approximation. Also, $K_{n_k} \cup F_k$ is a set of approximation, because it is the union of two sets of approximation, one of which is bounded by finitely many disjoint analytic curves. Therefore, H_k is a set of approximation, because it is the union of the two disjoint sets of approximation $K_{n_k} \cup F_k$ and $F_{k+1} \setminus F_k$.

Setting $F_0 = \emptyset$, we may choose a sequence ϵ_k of positive numbers decreasing so rapidly that for each $k = 0, 1, \dots,$

$$\epsilon_{k+1} + \epsilon_{k+2} + \cdots < \min\{\epsilon(p) : p \in F_{k+1} \setminus F_k\}/2.$$

Let g_1 be a meromorphic function on R such that $\chi(f, g_1) < \epsilon_1$ on F_1.

Define h_1 on H_1 by setting $h_1 = g_1$ on $K_{n_1} \cup F_1$ and $h_1 = f$ on $F_2 \setminus F_1$. Then $h_1 \in A_\chi(H_1)$. Let g_2 be a meromorphic function on R such that $\chi(h_1, g_2) < \epsilon_2$ on H_1. On $K_{n_1} \cup F_1$ we have $\chi(g_2, g_1) < \epsilon_2$. On F_1, we have $\chi(g_2, f) \leq \chi(g_2, g_1) + \chi(g_1, f) < \epsilon_2 + \epsilon_1$. On $F_2 \setminus F_1$, we have $\chi(g_2, f) < \epsilon_2$.

Define h_2 on H_2 by setting $h_2 = g_2$ on $K_{n_2} \cup F_2$ and $h_2 = f$ on $F_3 \setminus F_2$. Then $h_2 \in A_\chi(H_2)$. Let g_3 be a meromorphic function on R such that $\chi(h_2, g_3) < \epsilon_3$ on H_2. On $K_{n_2} \cup F_2$ we have $\chi(g_3, g_2) < \epsilon_3$. On F_1, we have $\chi(g_3, f) \leq \chi(g_3, g_2) + \chi(g_2, f) < \epsilon_3 + \epsilon_2 + \epsilon_1$. On $F_2 \setminus F_1$, we have $\chi(g_3, g_2) + \chi(g_2, f) < \epsilon_3 + \epsilon_2$. On $F_3 \setminus F_2$, we have $\chi(g_3, f) < \epsilon_3$.

We proceed by induction passing from step m to step m+1 in the same way as we went from step 1 to step 2 and from step 2 to step 3. In this way we obtain a sequence of meromorphic functions $\{g_k\}$ having the following properties. For $k = 1, 2, 3, \dots,$

$$\chi(g_{k+1}, g_k) < \epsilon_{k+1} \quad \text{on} \quad K_{n_k} \cup F_k;$$

$$\chi(g_{k+1}, f) < \epsilon_{k+1} + \ldots + \epsilon_1, \quad \text{on} \quad F_1;$$

$$\chi(g_{k+1}, f) < \epsilon_{k+1} + \ldots + \epsilon_2, \quad \text{on} \quad F_2 \setminus F_1;$$

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.

.

$$\chi(g_{k+1}, f) < \epsilon_{k+1} \quad \text{on} \quad F_{k+1} \setminus F_k.$$

The sequence $\{g_k\}$ is now inductively defined for all $k = 1, 2, \dots$. The sequence $\{g_k\}$ is spherically uniformly Cauchy on compact subsets and hence converges to a function g identically infinite or meromorphic on R.

Now, let us estimate $\chi(f, g)$. Fix $p \in E$. Then $p \in F_{k+1} \setminus F_k$, for some k, and

$$\chi(f(p), g(p)) \leq \chi(f(p), g_{k+1}(p)) + \sum_{j=k+1}^{m} \chi(g_j(p), g_{j+1}(p)) + \chi(g_{m+1}(p), g(p)) <$$

$$\epsilon_{k+1} + \sum_{j=k+2}^{m+2} \epsilon_j + \chi(g_{m+1}(p), g(p)).$$

Letting $m \to +\infty$, we have

$$\chi(f(p), g(p)) \leq \sum_{j=k+1}^{\infty} \epsilon_j < \epsilon(p).$$

□

The following case is of particular interest for us in view of the main result of the next section. Notice here that there is no restriction on the genus of the respective Jordan regions, whereas in the next section, the corresponding result will be for Jordan regions of genus zero.

Corollary 3.2.12. *Let E be the union of a locally finite family of disjoint compact Jordan regions in a Riemann surface R. Then E is a set of tangential spherical approximation.*

Proof. We have proved that compact Jordan regions are sets of uniform Euclidean pole-free approximation. Later we proved that for compact sets, it is equivalent to be a set of uniform spherical approximation. Thus it satisfies the condition of previous theorem. □

In case E has empty interior, the following theorem states that the possibilities of (tangential, uniform, local) Euclidean pole-free and spherical approximations are equivalent.

Theorem 3.2.13. *If E is a closed subset of a Riemann surface R such that $E^0 = \emptyset$, then the following are equivalent:*

1a) E is a set of spherical tangential approximation,

1b) E is a set of spherical uniform approximation,

1c) $E \cap K$ is a set of spherical uniform approximation, for every compact K,

2a) E is a set of Euclidean pole-free tangential approximation,

2b) E is a set of Euclidean pole-free uniform approximation,

2c) $E \cap K$ is a set of Euclidean pole-free uniform approximation, for every compact K.

Proof. $1a) \to 1b)$ is obvious and

$1c) \leftrightarrow 2c)$ follows from Theorem 3.2.3.

$2a) \to 2b) \to 2c) \to 2a)$. First of all $2a \to 2b)$ is trivial. $2b) \to 2c)$ follows from the fact that every continuous function on $E \cap K$ can be extended to E. $2c) \to 2a)$ follows from [3.16].

$1a) \to 2a)$ is Theorem 3.2.9.

$1b) \to 2a)$ Suppose E is a set of spherically uniform approximation. Then, in particular, every $f \in C(E)$ can be spherically uniformly approximated by meromorphic functions. In particular, every bounded $f \in C(E)$ can be spherically uniformly approximated by meromorphic functions. But, for bounded functions, spherically uniform approximation is equivalent to Euclidean uniform approximation. If $E^0 = \emptyset$, we claim that $C(E \cap K) = M(E \cap K)$ for every compact K. Indeed, let K be compact and $f \in C(E \cap K)$. We may extend, by Teitz extension theorem, f to a bounded function $F \in C(E)$. Given $\epsilon > 0$, there is a meromorphic function G such that $|F - G| < \epsilon$. In particular, $|f - G| < \epsilon$. This proves the claim. Since $C(E \cap K) = M(E \cap K)$ for every compact K, it follows from [3.16] that E is a set of Euclidean pole-free tangential approximation.

$1c) \to 1a)$ Let $f \in C_\chi(E)$ and ϵ be a positive continuous function on E. Let G_n be a regular exhaustion of R and set $\epsilon_n(z) = \inf_{z \in E \cap G_n} \epsilon(z)$. By the hypothesis there exists a meromorphic function g_1 such that $\chi(f, g_1) < \epsilon_2/2^2$ on $E \cap \overline{G}_1$. We proceed the proof by induction. Set $g_0 = g_1$ and $G_0 = \emptyset$. Suppose we have meromorphic functions $g_j; j = 1, \cdots, n-1$, such that

$$\chi(g_j, f) < \frac{\epsilon_{j+1}}{2^{j+1}} \quad \text{on} \quad E \cap \partial G_j, \tag{3.12}$$

$$\chi(g_j, g_{j-1}) < \frac{\epsilon_{j+1}}{2^{j+1}} \quad \text{on} \quad \overline{G}_{j-1}, \tag{3.13}$$

$$\chi(g_j, f) < \frac{\epsilon_j}{2^j} + \frac{\epsilon_{j+1}}{2^{j+1}} \quad \text{on} \quad E \cap (G_j \setminus G_{j-1}). \tag{3.14}$$

We shall construct g_n. If we identify the sphere $\overline{\mathbb{C}}$ with the unit sphere S in $\mathbb{R}^3$, then we may consider $g_{n-1}(p)$ and $f(p)$ as vectors in $\mathbb{R}^3$ lying on S. Define f_n to be g_{n-1} on G_{n-1} and to be f on ∂G_n. Consider the vector function $\lambda(p)$ which on ∂G_n is 0 and on ∂G_{n-1} is $g_{n-1}(p) - f(p)$, considered as a vector in the ball of radius $\epsilon_{n-1}/2^{n-1}$ in $\mathbb{R}^3$. Let $\Lambda(p)$ be a continuous extension of $\lambda(p)$ to R, whose values remain in the ball of radius $\epsilon_n/2^n$ in $\mathbb{R}^3$. On $E \cap (G_n \setminus G_{n-1})$, define

$$f_n(p) = \frac{f(p) + \Lambda(p)}{|f(p) + \Lambda(p)|}.$$

Since $f_n(p)$ is of norm 1, on $E \cap (G_n \setminus G_{n-1})$, we may consider it as a continuous map to the Riemann sphere. Combining the three definitions of f_n, we have that $f_n \in A_\chi(\overline{G}_{n-1} \cup (E \cap \overline{G}_n))$ and

$$\chi(f_n, f) < \epsilon_n/2^n$$

on $E \cap (G_n \setminus G_{n-1})$.

By Corollary 3.1.6, Theorem 3.2.3 and Corollary 3.2.5,

$$A_\chi(\overline{G}_{n-1} \cup (E \cap \overline{G}_n)) = M_\chi(\overline{G}_{n-1} \cup (E \cap \overline{G}_n)).$$

Then there exists a meromorphic function g_n such that

$$\chi(g_n, f_n) < \epsilon_{n+1}/2^{n+1}.$$

on $\overline{G}_{n-1} \cup (E \cap \overline{G}_n)$. We have

$$\chi(g_n, f) \le \chi(g_n, f_n) + \chi(f_n, f) < \epsilon_{n+1}/2^{n+1} + \epsilon_n/2^n,$$

on $E \cap (G_n \setminus G_{n-1})$;

$$\chi(g_n, g_{n-1}) < \epsilon_{n+1}/2^{n+1} \quad \text{on} \quad \overline{G}_{n-1};$$

and

$$\chi(g_n, f) < \epsilon_{n+1}/2^{n+1} \quad \text{on} \quad E \cap \partial G_n.$$

We have established by induction the existence of a sequence $\{g_j\}$ of meromorphic functions satisfying (3.12), (3.13) and (3.14).

The sequence g_n is spherically uniform Cauchy on compact subsets and so converges to a function g identically infinite or meromorphic on R. For $p \in E \cap (G_n \setminus G_{n-1})$ and for all $m > n$,

$$\chi(f(p), g_m(p)) \le \chi(f(p), g_n(p)) + \Sigma_{j=n+1}^{m-1} \chi(g_j(p), g_{j-1}(p)) <$$

$$\epsilon_n/2^n + \epsilon_{n+1}/2^{n+1} + \Sigma_{j=n+1}^{m-1} \epsilon_{j+1}/2^{j+1} < \epsilon_n.$$

Letting m goes to infinity, $\chi(f, g) \le \epsilon_n \le \epsilon(p)$. Hence, 1c)$\to$ 1a). $\square$

3.3 Pole-free spherically uniform and tangential approximation

For a mapping $f : E \to \overline{\mathbb{C}}$ on a subset E of a Riemann surface R, we say that f has a pole at a point p if $f(p) = \infty$ and we say that f is pole-free on E if f omits the value ∞ on E. In this section, we study the space $\widetilde{M}(E)$ of functions $f : E \to \overline{\mathbb{C}}$, for which there is a sequence of functions f_n meromorphic on R and pole-free on E, such that $f_n \to f$ χ-uniformly on E. Of course, such a function f must be in $A_\chi(E)$ and by the Hurwitz theorem, in each component V of E^0, the mapping f is either pole-free or identically ∞. We denote by $\widetilde{A}(E)$ the space of such functions. Let us say that E is a set of pole-free (uniform) spherical approximation if $\widetilde{M}(E) = \widetilde{A}(E)$. Thus, E is a set of pole-free uniform spherical approximation if and only if, for each $f \in \widetilde{A}(E)$ and for each *number* $\epsilon > 0$, there is a function g, meromorphic on R and pole-free on E, such that $\chi(f(p), g(p)) < \epsilon$, for all $p \in E$. Moreover, we shall say that E is a set of *tangential* pole-free spherical approximation, if we may take ϵ to be an arbitrary positive continuous *function*. Thus, $\chi(f(p), g(p)) < \epsilon(p)$, for all $p \in E$.

It is in general impossible to approximate continuous functions to the Riemann sphere by continuous finite-valued functions, but the following lemma has a sufficiently strong hypothesis.

Lemma 3.3.1. *Let E be a closed subset of a Riemann surface R with $E^o = \emptyset$ and U an open neighbourhood of E. Let f be a function meromorphic on U and let ϵ be a positive continuous function on U. Then there is a function $g \in C_\chi(U)$ pole-free on E (thus $g|_E \in C(E)$), such that $\chi(f(p), g(p)) < \epsilon(p)$, for all $p \in E$.*

Proof. Let $\{p_n\}$ be an enumeration of the poles of f which lie on E and, for each n, let $\overline{D}_n$ be a closed parametric disc at p_n, such that p_n is the only pole of f in $\overline{D}_n$ and the family $\{\overline{D}_n\}$ is locally finite in U. We may assume that the D_n are so small that

$$\max\{\chi(f(p), f(q)) : p, q \in \overline{D}_n\} < \min\{\epsilon(x) : x \in \overline{D}_n\}.$$

Choose a point $q_n \in D_n \setminus E$. Let $\eta_n : \overline{D}_n \to \overline{D}_n$ be a continuous map which fixes ∂D_n such that $\eta(q_n) = p_n$. Define $g(p) = f(p)$, for $p \notin \cup_n D_n$ and $g = f \circ \eta_n$ on D_n, for each n. Then g has the required property. □

The chordal distance is not linear, however the following lemma is a helpful substitute for linearity.

Lemma 3.3.2. *Given $M > 0$ and $\epsilon > 0$, there exists $\delta > 0$, such that, if $a, b \in \overline{\mathbb{C}}$, with $\chi(a, \infty), \chi(b, \infty) < \delta$ and $c, d \in \mathbb{C}$, with $|c|, |d| < M$, then $\chi(a + c, b + d) < \epsilon$.*

Proof. By the triangle inequality,

$$\chi(a + c, b + d) \leq \chi(a + c, \infty) + \chi(\infty, b + d),$$

so it is sufficient to show there is a δ such that both $\chi(a+c, \infty) < \epsilon/2$ and $\chi(b+d, \infty) < \epsilon/2$. Both cases are the same, so we show the first.

$$\chi(a + c, \infty) \leq \chi(a + c, a) + \chi(a, \infty) < o(1) + \delta, \quad \text{as} \quad \delta \to 0.$$

Thus, for all sufficiently small δ, we have $\chi(a+c,a)<\epsilon/4$. If we choose δ with the further property that $\delta<\epsilon/4$, the proof is complete. □

The following lemma enables us to "push" poles.

Lemma 3.3.3. *Suppose E is a closed subset of an open Riemann surface R and f is meromorphic in an open neighbourhood U of E and has a single pole at a point $p_0\in E$. Then, for each compact $K\subset E$ and for each $\epsilon>0$, there is a neighbourhood N of p_0 in U, such that, for each $q\in N$, there is a function f_q meromorphic on U, whose only pole on $K\cup N$ is at q and*

$$\max_{p\in K}\chi(f(p),f_q(p))<\epsilon.$$

Proof. Let $\rho(p)=z$ be a local coordinate near p_0, such that $\rho(p_0)=0$. Let $C(p,q)$ be a Cauchy kernel on R. For p and q near p_0, we may write

$$C(p,q)=\frac{1}{z-w}+H(z,w),$$

where $z=\rho(p), w=\rho(q)$ and H is holomorphic. In particular,

$$C(p,p_0)=\frac{1}{z}+H(z,0).$$

In this local coordinate,

$$f(z)=P\left(\frac{1}{z}\right)+h(z),$$

where P is the principal part of $f(z)$ at 0 and h is holomorphic. Then, for p near p_0, we may write

$$f(p)=P(C(p,p_0))+h(p),$$

where, by abuse of notation, we write $f(p)=f(\rho(p))=f(z)$ and $h(p)=h(\rho(p))=h(z)$.

We claim that, for each compact set K and each $\epsilon>0$, there is a neighbourhood N_ϵ of p_0, such that

$$\max_{p\in K}\chi(C(p,q),C(p,p_0))<\epsilon,\quad\text{for all}\quad p\in K,q\in N_\epsilon. \tag{3.15}$$

We may choose a parametric disc $z:D_\delta\to(|z|<\delta)$ for p_0, such that $C(p,q)$ is holomorphic on $(R\times R)\setminus(D_\delta\times D_\delta)$ and in local coordinates in $D_\delta\times D_\delta$, we have $C(p,q)=(z-w)^{-1}+H(z,w)$. In particular, we have, by Lemma 3.3.2, and for sufficiently small δ,

$$\chi(C(p,q),C(p,p_0))=\chi\left(\frac{1}{z-w}+H(z,w),\frac{1}{z}+H(z,0)\right)<\epsilon\quad\text{for}\quad p,q\in D_\delta. \tag{3.16}$$

Now, we consider the situation, when $p\notin D_\delta$. If we fix an arbitrary strictly smaller parametric disc

$z:D_\eta\to(|z|<\eta)$ at p_0, then, since $C(p,q)$ is holomorphic on $(R\setminus D_\delta)\times D_\delta$, it is uniformly continuous on $(K\setminus D_\delta)\times\overline{D}_\eta$. Hence, if η is small enough,

$$\chi(C(p,q),C(p,p_0))\le|C(p,q)-C(p,p_0)|<\epsilon,\quad\text{for}\quad p\in K\setminus D_\delta,q\in D_\eta. \tag{3.17}$$

Combining the estimates (3.16) and (3.17), we have (3.15).

Since polynomials are (uniformly) continuous mappings $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, it follows from (3.15) that

$$\max_{p\in K} \chi(P(C(p,p_0)), P(C(p,q))) \to 0, \quad \text{as} \quad q \to p_0.$$

Set $a = P(C(p,p_0)), b = P(C(p,q))), c = d = f - a$ and $f_q(p) = f(p) - P(C(p,p_0)) + P(C(p,q))$. Then, by Lemma 3.3.2, since $c = c(p)$ remains bounded for $p \in K \cup D_\eta$,

$$\chi(f(p), f_q(p)) = \chi(a+c, b+c) \to 0, \quad \text{as} \quad q \to p_0.$$

To obtain the conclusion of the Lemma, it is sufficient to set $N = D_\eta$, for sufficiently small η. $\square$

Lemma 3.3.4. *Suppose E is a closed subset of an open Riemann surface R and f is meromorphic in an open neighbourhood U of E and pole-free on E^0. Then, $f|_K \in \widetilde{M}(K)$, for each compact $K \subset E$.*

Proof. We claim that for each $\epsilon > 0$, there is a function f_K meromorphic on U and pole-free on K, such that

$$\max_{p\in K} \chi(f(p), f_K(p)) < \epsilon.$$

The function f has at most finitely many poles $p_1, \cdots, p_n$ on K. Setting $p_0 = p_1$ in the previous lemma, we obtain a function f_1 meromorphic on U such that $\chi(f, f_1) < \epsilon/n$ on K and f_1 has the same poles on K, except for the pole p_1 which has been shifted to q_1. We may choose as q_1 any point sufficiently close to p_1. Since $p_1 \in \partial E$, we may choose $q_1 \notin E$. The function f_1 thus has one less pole on K. We apply the same procedure to f_1 to remove the pole p_2. After finitely many steps, we arrive at the function f_n, which we define as f_K. By the Behnke-Stein theorem, there is a function g meromorphic in R, such that $|f_K - g| < \epsilon$ on K. Of course g is pole-free on K and $\chi(f_K, g) < \epsilon$ on K. By the triangle inequality, $\chi(f, g) < 2\epsilon$ on K, which concludes the proof. $\square$

For a compact set with empty interior, the following theorem states that the possibilities of Euclidean pole-free, spherical, and spherical pole-free approximations are equivalent.

Theorem 3.3.5. *Let K be a compact subset of an open Riemann surface and $K^0 = \emptyset$. Then $M_\chi(K) = \widetilde{M}(K)$, and*

$$A(K) = M(K) \quad \Leftrightarrow \quad A_\chi(K) = M_\chi(K) \quad \Leftrightarrow \quad \widetilde{A}(K) = \widetilde{M}(K). \tag{3.18}$$

Proof. By definition, $\widetilde{M}(K) \subset M_\chi(K)$. Now, suppose $K^o = \emptyset$ and $f \in M_\chi(K)$. Then, for each $n = 1, 2, \cdots$, there is a function f_n, meromorphic on R, such that $\chi(f, f_n) < 1/n$ on K. By Lemma 3.3.4, for each n, there exists a function g_n meromorphic on R and pole-free on K, such that

$$\max_{p\in K} \chi(f_n(p), g_n(p)) < 1/n.$$

Thus, $g_n \to f$ χ-uniformly on K and so $f|_K \in \widetilde{M}(K)$.

By Theorem 3.2.3, the first equivalence in (3.18) is true even if $K^o \neq \emptyset$. Now, suppose $K^o = \emptyset$ and $A_\chi(K) = M_\chi(K)$. If $f \in \widetilde{A}(K)$, then $f \in A_\chi(K)$, so $f \in M_\chi(K) = \widetilde{M}(K)$ (since $K^0 = \emptyset$) and hence $\widetilde{A}(K) \subset \widetilde{M}(K)$. Since we always have the opposite inclusion, we have the third equality in (3.18). We have shown that the second equality implies the third.

Now, suppose we have the third equality for a compact set K having empty interior, and let $f \in A(K)$. Then, there is a sequence g_n of meromorphic functions on R, pole-free on K, which converges χ-uniformly on K to f. Since f is bounded, the convergence is also Euclidean uniform, so $f \in M(K)$ and we have shown that $A(K) \subset M(K)$. Since the opposite inclusion is always true, we have shown that $A(K) = M(K)$. We have shown that the third equality in (3.18) implies the first. This completes the proof of (3.18) and of the theorem. □

Lemma 3.3.6. *Suppose a closed subset E of a Riemann surface is a set of spherically pole-free uniform approximation and $f \in A(E)$. Then $f|_K \in M(K)$, for each compact set $K \subset E$.*

Proof. Since a function $f \in A(E)$ can be uniformly spherically approximated by meromorphic function pole-free on E, it follows that the restriction of f to an arbitrary compact subset of E can be spherically uniformly approximated by meromorphic functions pole-free on E. Since such a restriction is bounded, it follows that it can be uniformly approximated in the Euclidean distance by meromorphic functions pole-free on E. □

Our next theorem will be phrased in terms of Gleason parts.

Definition 3.3.7. For a compact subset K of a Riemann surface R, the space $Spec(M(K))$ of non-zero (continuous) multiplicative linear functionals on the algebra $M(K)$ can be identified with K itself [3.29], where the action of a point of K on $M(K)$ is that of evaluation at p. We define an equivalence relation on K as follows. For any two points $p, q \in K$, we write $p \sim q$ if and only if

$$\|p - q\| \equiv \sup\{|f(p) - f(q)| : f \in M(K), |f| \leq 1\} < 2.$$

The equivalence classes are called Gleason parts for the algebra $M(K)$. See [3.10].

The following lemma can be found in [3.10, p. 130].

Lemma 3.3.8. *If W is a component of K^0, then W is contained in a single Gleason part of $M(K)$.*

Wilken showed that, if K is a compact subset of $\overline{\mathbb{C}}$, the Gleason parts of the algebra $R(K)$ are connected. Boivin [3.7] extended this result to Riemann surfaces as follows.

Lemma 3.3.9. *If P is a Gleason part for $M(K)$, then $\overline{P}$ is connected.*

The previous definition defines Gleason parts for $M(K)$, when K is compact. For a closed subset E of a Riemann surface R, Boivin [3.6] introduced the following notion of Gleason parts for the algebra $M(E)$.

Definition 3.3.10. Let E be a closed subset of R and let K_n be an exhaustion of R by compacta. We define the parts of E to be the limit as n tends to ∞ of the Gleason parts of $M(E \cap K_n)$; that is, two points of E are in the same part of E if eventually they are in the same Gleason part of $M(E \cap K_n)$. A part is called trivial if it consists of only one point.

It is shown in [3.6] that this definition makes sense and is independent of the choice of the exhaustion K_n.

Definition 3.3.11. Let $\mathcal{P} \subset K \subset R$. If K is compact and $\mathcal{P}$ is an (at most countable) union of non-trivial Gleason parts of $M(K)$, we say $(\mathcal{P}, K)$ is a Nersesjan pair.

Lemma 3.3.12. *[3.6, Prop. 2] If the Gleason parts of a closed subset E of R satisfy the long islands condition, then there exists an exhaustion D_n of R, such that for all n : $\overline{D}_n$ is compact and connected; $\overline{D}_n \subset D^0_{n+1}$; and D_n is the union of (at most countably many) non-trivial Gleason parts of $M(\overline{D}_n \cup (E \cap \overline{D}_{n+1}))$.*

Boivin does not explicitly say that the $\overline{D}_n$ are connected, but it can be deduced from his construction using Lemma 3.3.9.

We shall need the following easily verified fact, which is also used in [3.6] .

Lemma 3.3.13. *Let $\{D_n\}$ be as in the previous lemma. There exists an exhaustion by compact sets $\widehat{D}_n$ with $(\widehat{D}_n)^0$ connected, such that $\overline{D}_n \subset (\widehat{D}_n)^0, \widehat{D}_n \subset (D_{n+1})^0$.*

Boivin does not say explicitly that the $\widehat{D}_n$ can be chosen to be compact with $(\widehat{D}_n)^0$ connected, but this follows easily from the compactness and connectivity of the D_n.

The following two technical lemmas will assist us in pushing poles.

Lemma 3.3.14. *Let V be a bounded complementary component of $E \cap (\widehat{D}_n \setminus (\widehat{D}_{n-2})^0)$. Suppose v is a meromorphic function pole-free on E which has a pole p in the intersection of V with $\overline{D}_{n-2} \cup (E \cap (\widehat{D}_{n-2})^0)$ Then, $V \not\subset \overline{D}_{n-2} \cup (E \cap (\widehat{D}_{n-2})^0)$.*

Proof. First of all, $V \not\subset E$, since v is pole-free on E. Nor can we have $V \subset \overline{D}_{n-2}$ since $\overline{D}_{n-2}$ is connected and strictly contained in $(\widehat{D}_{n-2})^0$ which is a subset of the complement of $E \cap (\widehat{D}_n \setminus (\widehat{D}_{n-2})^0)$. There would exist a connected open set W, with

$$\overline{D}_{n-2} \subsetneq W \subset (\widehat{D}_{n-2})^0.$$

Thus W would be a connected open set in the complement of $E \cap (\widehat{D}_n \setminus (\widehat{D}_{n-2})^0)$, which strictly contains V. This would contradict the fact that V is a component. Therefore $V \setminus \overline{D}_{n-2} \neq \emptyset$.

Suppose to obtain a contradiction that $V \cap (\widehat{D}_{n-2} \setminus \overline{D}_{n-2}) \subset E$ and $V \cap D_{n-2} \neq \emptyset$. Then, there is a connected open set W, such that

$W \subset \overline{D}_{n-2} \cup (E \cap \widehat{D}_{n-1})$, $W \cap D_{n-2} \neq \emptyset$ and $W \not\subset \overline{D}_{n-2}$. By Lemma 3.3.8, W is contained in a non-trivial Gleason part of $M(\overline{D}_{n-2} \cup (E \cap \widehat{D}_{n-1}))$. Since $W \cap D_{n-2} \neq \emptyset$, this contradicts the fact that D_{n-2} is a union of non-trivial Gleason parts of $M(\overline{D}_{n-2} \cup (E \cap \widehat{D}_{n-1}))$. Thus, either $V \cap (\widehat{D}_{n-2} \setminus \overline{D}_{n-2}) \not\subset E$ or $V \cap D_{n-2} = \emptyset$. There are two cases.

Case 1. $V \cap (\widehat{D}_{n-2} \setminus \overline{D}_{n-2}) \not\subset E$. Then

$$V \cap \widehat{D}_{n-2} = (V \cap \overline{D}_{n-2}) \cup (V \cap (\widehat{D}_{n-2} \setminus \overline{D}_{n-2})) \not\subset \overline{D}_{n-2} \cup (E \cap \widehat{D}_{n-2}).$$

Thus, $V \not\subset \overline{D}_{n-2} \cup (E \cap \widehat{D}_{n-2})$ and we are through.

Case 2. $V \cap (\widehat{D}_{n-2} \setminus \overline{D}_{n-2}) \subset E$ and $V \cap D_{n-2} = \emptyset$. Then the pole p is in E which is a contradiction. □

Lemma 3.3.15. *Suppose ν is meromorphic on R and pole-free on E. Then, ν is the Euclidean uniform limit on $E \cap (\widehat{D}_n \setminus \widehat{D}^0_{n-2})$ of meromorphic functions having no poles on $\overline{D}_{n-2} \cup (E \cap \widehat{D}^0_{n-2})$.*

Proof. This follows directly from Lemma 3.3.14 and Theorem 3.1.14. □

By definition, $D_n = \cup P_j$, where each P_j is a non-trivial Gleason part of $M(\overline{D}_n \cup (E \cap \overline{D}_{n+1}))$. Set

$$K_n = (\overline{D}_n \cup (E \cap \overline{D}_{n+1})) \cup J_n = \overline{D}_n \cup (E \cap \widehat{D}_{n+1}),$$

where $J_n = E \cap (\widehat{D}_{n+1} \setminus D^o_{n+1})$. By [3.6, Lemma 2] , each P_j is a non-trivial Gleason part of $M(K_n)$. Hence (D_n, K_n) is a Nersesjan pair.

Lemma 3.3.16. *Let $\{\delta_n\}$ be a sequence of positive numbers. For each n, there exists a meromorphic function η_n such that*

$$|\eta_n| < 1 \quad on \quad \overline{D}_{n-1} \cup (E \cap \widehat{D}_n), \tag{3.19}$$

$$|1 - \eta_n| < \delta_n \quad on \quad (\widehat{D}_n \setminus \widehat{D}_{n-1}) \cap E, \tag{3.20}$$

$$|\eta_n| < \delta_n \quad on \quad \widehat{D}_{n-2}. \tag{3.21}$$

Proof. This follows from [3.6, Cor. 2]. The last inequality is true since $\widehat{D}_{n-2}$ is a compact subset of D_{n-1} and D_{n-1} is the union of some Gleason parts of $M(K)$, where $K := K_n = \overline{D}_{n-1} \cup (E \cap \widehat{D}_n)$, so III of [3.6, Cor. 2] yields (3.20) □

Theorem 3.3.17. *If a closed subset E of a Riemann surface is a set of pole-free spherically uniform approximation and the Gleason parts of E satisfy the long islands condition, then E is a set of Euclidean pole-free tangential approximation.*

Proof. Our proof is inspired by [3.6], but we must be careful at each step to specify the location of the poles. For this, we employ the previous lemmas.

Let $f \in A(E)$ and let ϵ be a positive continuous function on E. Set

$$\epsilon_k = \min_{p \in (\widehat{D}_k \cap E)} \epsilon(p).$$

It follows from Lemma 3.3.6 that there is a function u_1 meromorphic on R and pole-free on E, such that

$$|f - u_1| < \frac{\epsilon_1}{4} \quad E \cap \widehat{D}_1.$$

Also, from Lemma 3.3.6, there exists a function ν_2 meromorphic on $\mathbb{R}$ and pole-free on E such that

$$|(f-u_1)-\nu_2| < \frac{\epsilon_2}{4} \quad \text{on} \quad (\widehat{D}_2 \setminus \widehat{D}_0^0) \cap E.$$

By Lemma 3.3.15 we may assume that ν_2 is also pole-free on $\overline{D}_0$.

By Lemma 3.3.16, there exists, for each $\delta > 0$, a meromorphic function η_2 (depending on δ) such that

$$|\eta_2| < \delta \quad \text{on} \quad \widehat{D}_0,$$

$$|1-\eta_2| < \delta \quad \text{on} \quad (\widehat{D}_2 \setminus \widehat{D}_1) \cap E,$$

$$|\eta_2| \leq 1 \quad \text{on} \quad D_1 \cup (E \cap \widehat{D}_2).$$

Choose δ so small that

$$|(f-u_1)-\eta_2\nu_2| < \frac{\epsilon_2}{4} \quad \text{on} \quad (\widehat{D}_2 \setminus \widehat{D}_1) \cap E,$$

$$|(f-u_1)-\eta_2\nu_2| < |f-u_1| + |\eta_2||\nu_2| < \frac{\epsilon_1}{4} + 1\left(\frac{\epsilon_1}{4} + \frac{\epsilon_2}{4}\right)$$

$$< \epsilon_1 \quad \text{on} \quad (\widehat{D}_1 \setminus \widehat{D}_0) \cap E,$$

$$|(f-u_1)-\eta_2\nu_2| < \frac{\epsilon_1}{4} + \delta|\nu_2| < \epsilon_0 \quad \text{on} \quad \widehat{D}_0 \cap E$$

and

$$|\eta_2\nu_2| < \delta|\nu_2| < \frac{1}{2^2} \quad \text{on} \quad D_0.$$

Set $u_2 = \eta_2\nu_2$. By Lemma 3.3.6 there exists a function ν_3 meromorphic on R and pole-free on E such that

$$|(f-u_1-u_2)-\nu_3| < \frac{\epsilon_3}{4} \quad \text{on} \quad (\widehat{D}_3 \setminus \widehat{D}_1) \cap E.$$

By Lemma 3.3.15 we may assume that ν_3 is also pole-free on $\overline{D}_1$.

Again, by Lemma 3.3.16. for each $\delta > 0$, there is a function η_3 meromorphic on R such that

$$|\eta_3| < \delta \quad \text{on} \quad \widehat{D}_1,$$

$$|1-\eta_3| < \delta \quad \text{on} \quad (\widehat{D}_3 \setminus \widehat{D}_2) \cap E$$

$$|\eta_3| \leq 1 \quad \text{on} \quad \overline{D}_2 \cup (E \cap \widehat{D}_3).$$

Choose δ so small that

$$|f-u_1-u_2-\eta_3\nu_3| < \frac{\epsilon_3}{4} \quad \text{on} \quad (\widehat{D}_3 \setminus \widehat{D}_2) \cap E,$$

$$|f-u_1-u_2-\eta_3\nu_3| < \epsilon_2 \quad \text{on} \quad (\widehat{D}_2 \setminus \widehat{D}_1) \cap E,$$

$$|f-u_1-u_2-\eta_3\nu_3| < \epsilon_1 \quad \text{on} \quad (\widehat{D}_1 \setminus \widehat{D}_0) \cap E,$$

$$|f-u_1-u_2-\eta_3\nu_3| < \epsilon_0 \quad \text{on} \quad \widehat{D}_0 \cap E,$$

and

$$|\eta_3\nu_3| < \frac{1}{2^3} \quad \text{on} \quad D_1.$$

Set $u_3 = \eta_3\nu_3$.

Thus, by induction, we can find u_k meromorphic on R and pole-free on E such that

$$|u_k| < \frac{1}{2^k} \quad \text{on} \quad D_{k-2}, \quad k \geq 2,$$

$$|f - u_1 - u_2 - \cdots - u_k| < \epsilon_j \quad \text{on} \quad E \cap (\widehat{D}_j \setminus D_{j-1}), \quad j = 1, 2, \ldots, k-1.$$

$$|f - u_1 - u_2 - \cdots - u_k| < \frac{\epsilon_k}{4} \quad \text{on} \quad E \cap (\widehat{D}_k \setminus D_{k-1}).$$

Then, the function $g = \sum_{k=1}^{\infty} u_k$ is meromorphic on R and satisfies

$$|f(p) - g(p)| < \epsilon(p), \quad p \in E.$$

□

Suppose Ω is a compact Jordan region in an open Riemann surface R. Denote by $K_1, \ldots, K_n$ the bounded complementary components and by K_∞ the union of the unbounded complementary components. Let $f \in \widetilde{A}(\overline{\Omega})$. Then f can be represented in the form

$$f = f_\infty + \cdots + f_n,$$

such that $f_k \in \widetilde{A}(R \setminus K_k)$. Indeed, since the interior of $\overline{\Omega}$ has only the single component Ω, the function f is either identically ∞ or pole-free on Ω. The case where $f \equiv \infty$ is trivial. If $f \not\equiv \infty$, then by Theorem 3.1.3,

$$f = f_\infty + f_1 + \cdots + f_n,$$

where f_k is holomorphic in $R \setminus K_k$. For fixed $k \neq j$, the functions f_j are holomorphic on ∂K_k, so $f_k = f - \Sigma_{j \neq k} f_j$ is spherically continuous on ∂K_k. Therefore $f_k \in \widetilde{A}(R \setminus K_k)$.

It was shown in [3.13] that if K is a finite union of disjoint compact Jordan regions in $\mathbb{C}$, then K is a set of pole-free spherical meromorphic approximation. The following result generalizes this to certain closed subsets of Riemann surfaces (and in particular, to a class of closed subsets of $\mathbb{C}$).

Theorem 3.3.18. *Let E be the union of a locally finite family of disjoint compact sets of uniform spherical pole-free approximation in a Riemann surface R. Then E is a set of tangential spherical pole-free approximation.*

Proof. Let $f \in \widetilde{A}(E)$ and ϵ be a positive continuous function on $E = \cup_n E_n$. Set $f_n = f|_{E_n}$. Then, $f_n \in \widetilde{A}(E_n)$. Take $\epsilon_n = \min_{p \in E_n} \epsilon(p)$. By hypothesis, for each n, the set E_n is a set of uniform spherical pole-free approximation. That means there exists a function m_n meromorphic on R and pole-free on E_n, such that $\chi(m_n, f_n) < \epsilon_n/2$. Since the E_n are disjoint and compact, we may construct a locally finite family U_n of open neighbourhoods of the E_n with disjoint closures, such that, for each n, the function m_n is holomorphic on U_n. Then, we obtain a holomorphic function m on E by setting $m = m_n$ on U_n for each n. By Behnke-Stein and Theorem 3.1.12, then E is a set of tangential Euclidean pole-free approximation, i.e., there is a meromorphic function M on R, pole-free on E, such that $|M - m| < \epsilon_n/2$ on E_n for each n. For $p \in E$, there is a unique n such that $p \in E_n$. We have

$$\chi(M(p), f(p)) \leq \chi(M(p), m_n(p)) + \chi(m_n(p), f_n(p)) < \epsilon_n < \epsilon(p).$$

$\square$

If $\psi : U \to V$ is a biholomorphic mapping of an open neighbourhood U of a compact Jordan region $\overline{G}$ of the complex plane onto an open set V in a Riemann surface R, we say that $\Omega = \psi(G)$ is a parametric Jordan region in R and $\overline{\Omega}$ a closed parametric Jordan region in R. We note that a closed parametric Jordan region is compact.

Lemma 3.3.19. *Let $\overline{\Omega}$ be a parametric compact Jordan region in a Riemann surface R. Then $\overline{\Omega}$ is a set of pole-free uniform spherical approximation.*

Proof. Suppose $\overline{\Omega}$ is a parametric compact Jordan region . Let $f \in \widetilde{A}(\overline{\Omega})$ and $\epsilon > 0$ be given. From the definition of a parametric Jordan region, there exists a compact Jordan region $\overline{G} \subset \mathbb{C}$ and a biholomorphic mapping ψ from a neighbourhood of $\overline{G}$ onto a neighbourhood of $\overline{\Omega}$ such that $\psi : \overline{G} \to \overline{\Omega}$, We have $f \circ \psi \in \widetilde{A}(\overline{G})$. It follows from a theorem in [3.13] that there exists a rational function r pole-free on $\overline{G}$, such that

$$\chi(r, f \circ \psi) < \epsilon/2 \quad \text{on} \quad \overline{G}.$$

Thus, $g = r \circ \psi^{-1} \in A(\overline{\Omega})$ and

$$\chi(g, f) < \epsilon/2 \quad \text{on} \quad \overline{\Omega}.$$

By Corollary 3.1.6, there is a meromorphic function h on R, pole-free on E, such that $|h - g| < \epsilon/2$ and hence $\chi(h, g) < \epsilon/2$. By the triangle inequality, $\chi(f, h) < \epsilon$. $\square$

Theorem 3.3.20. *Let E be the union of a locally finite family of disjoint parametric compact Jordan regions in a Riemann surface R. Then E is a set of pole-free tangential spherical approximation.*

Proof. This follows immediately from Theorem 3.3.18 and the preceding lemma. $\square$

Having considered parametric Jordan regions, we now turn to general Jordan regions.

Lemma 3.3.21. *If Ω is a Jordan region in a Riemann surface R, then $\overline{\Omega}$ and every complementary component of Ω is a bordered surface.*

Proof. Ω is a regularly embedded region and every complementary component of a regularly embedded region is also a regularly embedded region and hence a bordered surface. $\square$

Let S be a topological space and B is a subset of S. Following Brown [3.11], we say that B is *collared* in S if there exists a homeomorphism h from $B \times [0, 1)$ onto a neighbourhood of B such that $h(b, 0) = b$ for all $b \in B$. If B can be covered by a collection of subsets relatively open in B each of which is collared in S, then B is said to be locally collared in S.

A bordered n-manifold is a connected metrizable topological space such that each point has a closed neighbourhood homeomorphic to the closed n-ball.

Theorem 3.3.22 (Brown [3.11]). *The border of a bordered n-manifold M is collared in M.*

Lemma 3.3.23. *If $\overline{\Omega}$ is a compact Jordan region whose interior Ω is of genus zero in a Riemann surface R, then $\overline{\Omega}$ has a planar neighbourhood.*

Proof. By the previous theorem, both $\overline{\Omega}$ and $R \setminus \Omega$ are collared. Hence, there is an open neighbourhood W of $\partial\Omega$ and a homeomorphism

$$h : \partial\Omega \times (-1, +1) \longrightarrow W,$$

with

$$h\left(\partial\Omega \times (-1, 0]\right) = \overline{\Omega} \cap W, \quad h(p, 0) = p, \quad h(\partial\Omega \times [0, +1)) = W \setminus \Omega.$$

The function $\phi(t) = -1/2 + 3(t + 1/2)$ defines a homeomorphism $\phi : [-1/2, 0) \to [-1/2, +1)$, which induces a homeomorphism

$$\Phi : \partial\Omega \times [-1/2, 0) \longrightarrow \partial\Omega \times [-1/2, +1),$$

given by $\Phi(p, t) = (p, \phi(t))$. Set

$$C = h(\partial\Omega \times \{-1/2\}), \quad V = h(\partial\Omega \times [-1/2, 0]), \quad U = h(\partial\Omega \times [-1/2, +1)).$$

The function $G = h \circ \Phi \circ h^{-1}$ defines a homeomorphism of V onto U, which fixes points of C. Denoting $N = \Omega \cup U$, we have a homeomorphism $H : \Omega \to N$, defined by setting $H(p) = p$, for $p \in \Omega \setminus V$ and $H(p) = G(p)$, for $p \in V$. Since Ω is of genus zero it is planar. The stolow kerekyardo compactification represents Ω as an open subset of the sphere with g handle added, where g is a genus. Ω is of genus zero, so there is no handle. then it is homemorphic to a proper subset of the plane. and since N is homeomorphic to Ω, the neighbourhood N is also planar, which completes the proof. $\square$

Lemma 3.3.24. *If $\overline{\Omega}$ is a compact Jordan region whose interior is of genus zero in a Riemann surface R, then $\overline{\Omega}$ is a set of pole-free uniform spherical approximation.*

Proof. By the preceding lemma, $\overline{\Omega}$ has a planar neighbourhood N. By the general uniformization theorem of Koebe, every planar Riemann surface is biholomorphic to a plane domain, so there is a biholomorphic mapping $h : N \to W$ mapping the open set N in R onto an open set Win the complex plane. Thus $\overline{\Omega}$ is a parametric compact Jordan region and so the lemma follows from Lemma 3.3.19. $\square$

Theorem 3.3.25. *Let E be the union of a locally finite family of disjoint compact Jordan regions of genus zero in a Riemann surface R. Then E is a set of pole-free tangential spherical approximation.*

Proof. We have $E = \cup_n E_n$, where $\{E_n\}$ is a locally finite family of disjoint compact Jordan regions with genus zero. Let $f \in \widetilde{A}(E)$. Set $f_n = f|_{E_n}$. By the lemma, for each n, E_n is a set of pole-free meromorphic uniform approximation. Then, for every $\epsilon_n > 0$ there exists a function g_n meromorphic on R and pole-free on E_n, such that $\chi(f_n, g_n) < \epsilon_n/2$. Since E_n compact and disjoint, we can construct a locally finite family U_n of disjoint neighbourhood of the E_n such that g_n is holomorphic on U_n for each n.

Set $g = g_n$ on U_n. Then g is a holomorphic function on E.

And so by [3.16], there exists a meromorphic function G on R which is pole-free on E, such that, $|g - G| < \epsilon_n/2$ on E_n for each n

So $\chi(g, G) < |g - G| < \epsilon_n/2$ on E_n. For $p \in E$, there exists n such that $p \in E_n$ and so,

$$\chi(G(p), f(p)) \leq \chi(G(p), g_n(p)) + \chi(g_n(p), f_n(p)) < \epsilon_n.$$

□

3.4 Zero-free uniform and tangential approximation

The problem of pole-free spherical approximation was raised in [3.22]. Since the chordal distance is invariant with respect to the inversion $z \mapsto 1/z$, this problem is in an obvious sense equivalent to that of approximating spherically continuous functions on E which are zero-free in E^0, by meromorphic functions on R which are zero-free on E. For example, the following theorem is essentially a reformulation of Theorem 3.3.20.

Theorem 3.4.1. *Let E be the union of a locally finite family of disjoint parametric compact Jordan regions in a Riemann surface R. Then E is a set of zero-free tangential spherical approximation.*

This allows us to obtain a similar result on zero-free tangential Euclidean meromorphic approximation.

Theorem 3.4.2. *Let E be the union of a locally finite family of disjoint parametric compact Jordan regions in a Riemann surface R. Then E is a set of zero-free tangential Euclidean meromorphic approximation.*

Proof. We may write $E = \cup E_n$, where the E_n are pairwise disjoint compact Jordan regions. Fix $f \in A(E)$ zero-free on E^0 and a sequence $\{\epsilon_n\}, \epsilon_n > 0$. We must show the existence of a function g meromorphic on R and zero-free on E such that $|f - g| < \epsilon_n$ on E_n, for each n.

By Theorem 3.4.1, for each sequence $\{\delta_n\}, \delta_n > 0$, there is a meromorphic function g, zero-free on E, such that $\chi(f, g) < \delta_n$ on E_n, for each n.

Since, for each n the function f is bounded on E_n, we may choose δ_n so small that on E_n the function g is also bounded and $|f - g| < \epsilon_n$. □

We have briefly considered the problem of Euclidean uniform approximation on a closed set E by *meromorphic* functions zero-free on E. The analogous problem of Euclidean uniform approximation on a compact subset $K \subset \mathbb{C}$ by *polynomials* zero-free on K has been investigated by several authors recently: [3.1], [3.2], [3.3], [3.12], [3.17], [3.21], and, not so recently, [3.31].

For a compact set $K \subset \mathbb{C}$, denote by $R_0(K)$ the uniform Euclidean limits on K of rational functions zero-free on K. By Hurwitz's theorem, $R_0(K)$ is contained in the family $A_0(K)$ of functions in $A(K)$ having no isolated zeros in K^0. The problem of zero-free Euclidean approximation is that of describing those compacta for which $A_0(K) = R_0(K)$. If we denote by $P(K)$ the uniform Euclidean limits on K of polynomials and by $P_0(K)$ the uniform Euclidean limits on K of polynomials zero-free on K, then the problem of zero-free Euclidean polynomial approximation is that of determining those K for which $P_0(K) = A_0(K)$. The following was recently conjectured by Andersson [3.2].

Zero-free approximation conjecture.

$$A_0(K) = P_0(K) \quad \Longleftrightarrow \quad A(K) = P(K).$$

Denote as usual the Riemann zeta-function by $\zeta(z)$. Let A be a subset of $[0, +\infty)$. The lower density $\underline{d}(A)$ of A is defined as follows.

$$\underline{d}(A) = \liminf_{T \to +\infty} \frac{m(A \cap [0, T])}{T},$$

where m denotes Lebesgue measure.

Theorem 3.4.3. *The following assertions are equivalent and true. For each compact set K in the strip $1/2 < \Re z < 1$ with $\mathbb{C} \setminus K$ connected, for each zero-free $f \in A(K)$ and for each $\epsilon > 0$:*

$$\{t : \max_{z \in K} |\zeta(z + it) - f(z)| < \epsilon\} \quad \textit{has positive lower density,} \tag{3.22}$$

$$\{t : \max_{z \in K} \left| \frac{1}{\zeta(z + it)} - f(z) \right| < \epsilon\} \quad \textit{has positive lower density,} \tag{3.23}$$

$$\{t : \max_{z \in K} \chi(\zeta(z + it), f(z)) < \epsilon\} \quad \textit{has positive lower density,} \tag{3.24}$$

$$\{t : \max_{z \in K} \chi\left(\frac{1}{\zeta(z + it)}, f(z) \right) < \epsilon\} \quad \textit{has positive lower density.} \tag{3.25}$$

The assertion (3.22) is the spectacular theorem of Voronin on the universality of the Riemann zeta-function [3.31]. We have stated the other (clearly equivalent) forms to suggest that this is perhaps related to our present investigation.

Indeed, surprisingly, Andersson [3.2] has shown that the zero-free polynomial approximation conjecture is equivalent to a strengthening of the Voronin universality theorem, namely it is equivalent to replacing the hypothesis that $f \in A(K)$ is zero-free by the weaker hypothesis that it is merely zero-free on K^0. Such f are in $A_0(K)$. We could also say that f is in $A_\chi(K)$ and pole-free on K^0 or equivalently f is in $\widetilde{A}(K)$ and pole-free on K^0.

Theorem 3.4.4. *For $K \subset \mathbb{C}$ compact,*

$$\widetilde{A}(K) = \widetilde{R}(K) \quad \Longrightarrow \quad A_0(K) = R_0(K).$$

Proof. Suppose the left side and let $f \in A_0(K)$. Then $1/f \in \widetilde{A}(K)$ and there exists a rational function $1/r$ pole-free on K such that $\chi(1/r, 1/f) < \epsilon$ on K. Thus, $\chi(r, f) < \epsilon$ and r is zero-free on K. Since ϵ is arbitrary and f is bounded on K, there exists a rational r such that $|r - f| < \epsilon$ on K. □

Corollary 3.4.5. *If $\mathbb{C} \setminus K$ is connected,*

$$\widetilde{A}(K) = \widetilde{R}(K) \quad \Longrightarrow \quad A_0(K) = P_0(K).$$

Proof. For $f \in A_0(K)$ and $\epsilon > 0$, there is rational function r zero-free on K such that $|f - r| < \epsilon/2$. Let $m = \min |r|$ on K. By Runge's theorem, there is a polynomial p such that

$$|r - p| < \min\{m/2, \epsilon/2\}.$$

Then, by the triangle inequality, $|f - p| < \epsilon$ and since $|r - p| < m/2$, the polynomial p is zero-free on K. □

From the corollary, we have that a complete solution to the spherical pole-free approximation problem would yield a complete solution of the zero-free Euclidean polynomial approximation problem and hence the strengthening of the Voronin Universality Theorem.

The Voronin Universality Theorem states that translates of the Riemann zeta-function approximate in a very strong sense all zero-free holomorphic functions in the strip $1/2 < \Re z < 1$. Bagchi [3.5] has shown that the assertion that the zeta-function can also approximate itself in this manner is equivalent to the Riemann Hypothesis. This chain of implications suggests that a complete solution of the spherical pole-free approximation problem may be difficult.

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Chapter 4

The Carathéodory reflection principle and Osgood-Carathéodory theorem on Riemann surfaces

4.1 Introduction

The reflection principles of Schwarz and Carathéodory give conditions under which holomorphic functions extend holomorphically to the boundary and the theorem of Osgood-Carathéodory states that a one-to-one conformal mapping from the unit disc to a Jordan domain extends to a homeomorphism of the closed disc onto the closed Jordan domain. In this chapter, we study similar questions on Riemann surfaces for holomorphic mappings. We give a Carathéodory type reflection principle for bordered Riemann surfaces which are arbitrary. That is, we do not assume that they are compact; nor do we assume that they are of finite genus. From this follows a Schwarz type reflection principle as well as an Osgood-Carathéodory type theorem.

When we speak of a conformal mapping f from a domain Ω_1 of one Riemann surface R_1 to a domain Ω_2 in another Riemann surface R_2, we always mean an orientation preserving conformal mapping which is one-to-one, but not necessarily onto. The expressions "one-to-one conformal mapping onto" and "biholomorphic mapping" will be used interchangeably. For an overview of conformal mappings in the plane, see [4.5], [4.8], [4.13], [4.15] and [4.16].

A Riemann surface is said to be planar if it is homeomorphic to a subset of the complex plane $\mathbb{C}$. In extending results from the complex plane to Riemann surfaces, the following General Uniformization Theorem of Koebe is extremely helpful.

Theorem 4.1.1. *Every planar Riemann surface is conformally equivalent to a plane domain.*

It will also be helpful to recall that meromorphic functions on Riemann surfaces are the same as holomorphic mappings to the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

For a domain $G \subset \mathbb{C}$, a function $f : G \to \overline{\mathbb{C}}$ and a boundary point $\zeta \in \partial G$, Carathéodory defines a point $\alpha \in \overline{\mathbb{C}}$ to be a boundary value of f at ζ, if there is a sequence z_ν in G for which the equations

$$\lim_{\nu\to\infty} z_\nu = \zeta \quad \text{and} \quad \lim_{\nu\to\infty} f(z_\nu) = \alpha$$

hold. The set of boundary values of f at ζ is precisely the cluster set $C(f,\zeta)$. Also, for a subset $E \subset \partial G$, we denote

$$C(f,E) \equiv \bigcup_{\zeta\in E} C(f,\zeta).$$

For a set $E \subset \overline{\mathbb{C}}$, let us set $E^* = \{\overline{z} : z \in E\}$, where $\infty^* = \infty$.

The first part of the following theorem is the Carathéodory reflection principle [4.4].

Theorem 4.1.2. *Let V be a domain in the open upper half-plane $\{\Im z > 0\}$ and suppose I be the interior of $\{z \in \partial V : \Im z = 0\}$ in the topology of $\mathbb{R}$. Set $\widehat{V} = V \cup I \cup V^*$.Let f be meromorphic in V and suppose all boundary values of f on I are real or ∞. Then f extends to a surjective meromorphic function $\widehat{f} : \widehat{V} \to f(V) \cup C(f,I) \cup f(V)^*$. Suppose, moreover that $f(V)$ is contained in the open upper half-plane $H^+ = \{w \in \mathbb{C} : \Im w > 0\}$. If f is respectively locally conformal, or conformal, then so is $\widehat{f}$.*

Proof. The final two sentences are not in Carathéodory's formulation of the theorem, but, as we shall see, this final portion follows from the first part.

Suppose, then, that $f(V)$ is contained in the open upper half-plane. Then combining the first part of the theorem with the Schwarz reflection principle, we conclude that, if f is locally conformal (respectively conformal) in V, then $\widehat{f}$ is locally conformal (respectively conformal) in V^*. Suppose, for some value $p \in I$, that $\widehat{f}(p)$ were assumed with multiplicity greater than 1. Then, at p all angles would be multiplied by p which contradicts the assumption that the image by $\widehat{f}$ of any upper half-disc "centred" at p is contained in the open upper half-plane. Thus, $\widehat{f}$ is locally conformal at each point of I. Now, suppose f is conformal. We have already verified that $\widehat{f}$ is injective on $V \cup V^*$. Since $\widehat{f}(V)$, $\widehat{f}(V^*)$ and $\widehat{f}(I)$ are disjoint, it is sufficient to show that $\widehat{f}$ is injective on I. Then, $\widehat{f}$ will be injective and hence conformal. Suppose, to obtain a contradiction, that $\widehat{f}(p) = \widehat{f}(q)$, for $p \neq q$ in I. Let U_p and U_q be disjoint neighbourhoods of p and q respectively, sufficiently small that $\widehat{f}$ is conformal, hence injective, in U_p and U_q. Since $\widehat{f}$ is an open mapping and $\widehat{f}(I)$ is of measure zero, it follows that there are points a and b in $U_p \setminus I$ and $U_q \setminus I$ respectively such that $\widehat{f}(a) = \widehat{f}(b)$. This contradicts the fact that $\widehat{f}$ is injective on $V \cup V^*$. Thus, $\widehat{f}$ is injective on I.

□

The example $f(z) = z^2$ shows that, if we omit the assumption that $f(V)$ is contained in the open upper half-plane, it does not always follow that $\widehat{f}$ is locally conformal, when f is locally conformal, in fact, not even when if f is conformal.

4.2 A few facts on conformal mappings in the plane

An open (respectively compact) *Jordan arc* is defined as the homeomorphic image of the interval $(0, 1)$ (respectively the interval [0,1]). A *Jordan curve* is the homeomorphic image of a circle by Ahlfors, I.7A, this definition is for general topological space S not only for Riemann sphere.

and a *Jordan domain* in $\overline{\mathbb{C}}$ is a domain whose boundary is a Jordan curve. By the Jordan curve theorem, if J is a Jordan curve in $\overline{\mathbb{C}}$, then its complement $\overline{\mathbb{C}} \setminus J$ consists of two disjoint Jordan domains, both having J as boundary. A closed Jordan domain is the closure of a Jordan domain. By the Schoenflies theorem, a closed Jordan domain is the homeomorphic image of the closed unit disc. The Schoenflies theorem could be phrased as follows. A homeomorphism from the boundary of the disc to the boundary of a Jordan domain extends to a homeomorphism of the interiors. The Osgood-Carathéodory theorem goes in the opposite direction, and has as a consequence that a conformal mapping of the unit disc onto a Jordan domain (which of course is a homeomorphism) extends to a homeomorphism of the boundaries.

More precisely, the Osgood-Carathéodory theorem states that a conformal mapping from the open unit disc onto a Jordan domain in the Riemann sphere $\overline{\mathbb{C}}$ extends to a homeomorphism of the closed disc onto the closed Jordan domain. If we think of a Jordan domain U as the complement of a closed Jordan domain $\overline{V}$, then a natural generalization would be to replace $\overline{V}$ by a compact Jordan arc J (thinking of a Jordan arc as a "compressed" Jordan domain). In this spirit, we shall consider to what extent we can obtain an analogue of the Osgood-Carathéodory theorem, if we map the unit disc to the complement of a compact Jordan arc. The following discussion describes the situation.

A topological space is said to be locally connected if every point has a fundamental system of connected neighbourhoods. The continuous image of a locally connected space need not be locally connected. For example, the closure of the curve

$$\gamma(t) = \left|\sin\left(\frac{2\pi}{t}\right)\right| e^{it}, \quad 0 < t \leq 1$$

is not locally connected.

Theorem 4.2.1 (Continuity theorem [4.15])**.** *Let f be a conformal mapping of the open unit disc Δ onto a domain $G \subset \overline{C}$. The function f has a continuous extension to $\overline{\Delta}$ if and only if ∂G is locally connected.*

Lemma 4.2.2. *Let $f : \Delta \to G$ be conformal, with ∂G locally connected. Then, the continuous extension to $\overline{\Delta}$ maps the circle $\mathbb{T}$ onto ∂G.*

Proof. If $w \in \partial G$, there is a sequence $z_n \in \Delta$, such that $f(z_n) \to w$. By choosing a subsequence, we may assume that z_n converges to a point ζ of the unit circle. Then $f(\zeta) = w$, so $f(\mathbb{T}) \supset \partial G$. Conversely, if $\zeta \in \mathbb{T}$ and $z_n \in D$ converges to ζ, then $f(z_n)$ is eventually outside of every compact subset of $f(\Delta) = G$, so $f(\zeta) \in \partial G$. Thus, $f(\mathbb{T}) \subset \partial G$. □

Lemma 4.2.3. *Let ϕ be a conformal mapping of the open unit disc Δ onto the complement J^c of a compact Jordan arc J in $\overline{\mathbb{C}}$. Then ϕ extends to a continuous mapping of $\overline{\Delta}$ onto $\overline{\mathbb{C}}$, which maps the unit circle $\mathbb{T}$ onto J.*

Proof. Since J is locally connected, the lemma follows from the previous theorem and lemma. □

Let E be a locally connected continuum. We say that $a \in E$ is a cut point of E if $E \setminus \{a\}$ is no longer connected. For a Jordan arc, all points except end points are cut-points.

Lemma 4.2.4. *Let ϕ be as in the previous lemma. Then, for $a \in J$, the set $\phi^{-1}(a)$ is a singleton if and only if a is an end point of J.*

Proof. By [4.15, Proposition 2.5], if ϕ is a conformal mapping of Δ onto a bounded domain G, where ∂G is locally connected, then, for each $a \in \partial\Omega$, the set $f^{-1}(a)$ is a singleton if and only if a is not a cut-point of ∂G. In our situation, J^c is not a bounded domain in C, but the proof can be easily modified to apply to our case. Since $a \in J$ is not a cut-point of J if and only if a is an end point, the lemma follows. □

Lemma 4.2.5. *Let ϕ be as in the previous lemma. Let p and q be the ends of J and J^0 be the inner points of J. There are points a and b on the unit circle, such that $\phi(a) = p$, $\phi(b) = q$ and ϕ maps each of the two arcs comprising $\mathbb{T} \setminus \{a, b\}$ onto J^0. The map ϕ is one to one on each arc with end points a and b.*

Proof. From the previous lemma, $\phi^{-1}(p)$ is a singleton $\{a\}$ and $\phi^{-1}(q)$ is a singleton $\{b\}$. Let A be one of the two arcs comprising $\mathbb{T} \setminus \{a, b\}$. Since $\phi(A)$ is a connected subset of the (open) Jordan arc J^0, it is a point or an arc. It cannot be a point, for then, ϕ would be constant on the arc A and hence constant by uniqueness theorems. Hence, $\phi(A)$ is a sub-arc of J^0. Since a and b are in the closure of $\phi(A)$, the arc $\phi(A)$ must be all of J^0. □

A cross-cut C of an open set G is an open Jordan arc in G such that $\overline{C} = C \cup \{a, b\}$ with $a, b \in \partial G$. We allow that $a = b$ (see [4.1]).

Lemma 4.2.6. *Let J be a compact Jordan arc in $\overline{\mathbb{C}}$. Then, for every neighbourhood G of J, there is a Jordan domain $W \subset \overline{\mathbb{C}}$, such that $J^0 \subset W \subset \overline{W} \subset G$ and J is a cross-cut of W. That is, J is contained in W, except for the end points, which (of course) lie on ∂W.*

Proof. It follows from the Jordan arc separation theorem that $J^c = \overline{\mathbb{C}} \setminus J$ is connected. For a proof, see for example [4.2, Lemma 4]. Let $\phi : \Delta \to J^c$ be a conformal map. By lemma 4.2.3 , ϕ extends to a continuous mapping (which we continue to denote by ϕ) of $\overline{\Delta}$ onto $\overline{\mathbb{C}}$ which maps $\mathbb{T}$ onto J. There are two points $a, b \in \mathbb{T}$ which are mapped to the end points of J and the two arcs of $\mathbb{T} \setminus \{a, b\}$ are mapped onto J^0. We may assume that $\{a, b\} = \{-1, +1\}$. Let G be a neighbourhood of J. The neighbourhood $\phi^{-1}(G)$ of $\mathbb{T}$ contains an annulus $A_r = (r \le |z| \le 1)$, for some $r > 0$. Let L be a "lens domain" in Δ such that $\overline{L} \cap \mathbb{T} = \{-1, +1\}$ and the disc $\overline{D}_r = (|z| \le r)$ is contained in L. Then, $\Gamma = \phi(\partial L)$ is a Jordan curve in $\overline{\mathbb{C}}$, which, separates $\overline{\mathbb{C}}$ into two Jordan domains with boundary Γ. One of these domains $\phi(L)$ contains $\phi(\overline{D}_r)$, so the other Jordan domain, call it W, is contained in $\phi(A_r) \subset G$. Since $\partial L \subset A_r$, we also have $\phi(\partial L) = \Gamma \subset G$. Hence $\overline{W} = W \cup \Gamma \subset G$. Since ϕ maps the two semicircles $\mathbb{T} \setminus \{-1, +1\}$ onto J^0, and these semicircles are disjoint from ∂L, it follows that $J^0 \subset W$. Since $\phi(\pm 1)$ are the end points of J and they lie on $\Gamma = \partial W$, it follows that J is a cross-cut of W. □

A domain $W \subset \overline{\mathbb{C}}$ is called a *circular domain* if ∂W consists of finitely many disjoint spherical circles. A domain is non-degenerate if no component of its complement is a single point. The following theorem of Koebe states that circular domains are conformally canonical for the class of non-degenerate n-connected domains.

Theorem 4.2.7. *Every non-degenerate n-connected domain in $\overline{\mathbb{C}}$ is conformally equivalent to a circular domain.*

We define a (finitely connected) Jordan region Ω in $\overline{\mathbb{C}}$ to be a domain bounded by finitely many disjoint Jordan curves and, if Ω is a Jordan region, we say that $\overline{\Omega}$ is a closed Jordan region. If there is only one boundary curve, then we call the Jordan region a Jordan domain.

Occasionally, the Osgood-Carathéodory theorem is invoked not only for Jordan domains, but also (implicitly) for Jordan regions (for example in [4.11]). The following extension of the Osgood-Carathéodory theorem for Jordan regions in $\overline{\mathbb{C}}$ was proved in [4.14] (see also [4.5, Ch. 15]) and can be deduced from the simply-connected case.

Theorem 4.2.8. *If G and Ω are two Jordan regions and $f : \Omega \to G$ is a conformal equivalence, then f extends to a homeomorphism of $\overline{\Omega}$ onto $\overline{G}$.*

If two plane domains are conformally equivalent, then their automorphism groups are isomorphic. Thus, by the Riemann mapping theorem, for simply connected plane domains, we only need to understand the automorphism groups of the disc and the plane, which are well known.

If a plane domain is not simply connected, the group $\mathrm{Aut}(\Omega)$ of conformal self-maps is "in general small".

However, for a given domain, there may be many conformally equivalent domains which are presented in very different ways. For example, let Ω be a Jordan region in $\mathbb{C}$ and let D be a disc containing $\overline{\Omega}$. Now let f be an arbitrary conformal mapping of D onto a simply connected domain. Then, $f(\Omega)$ is conformally equivalent to Ω, but may appear quite different as a subset of $\mathbb{C}$.

For $n > 2$, an example of an n-connected Jordan region $\overline{\Omega} \subset \mathbb{C}$ for which $\mathrm{Aut}(\Omega)$ is not trivial, is obtained by choosing $0 < r < 1$, and taking as Ω the unit disc Δ, from which we have removed $n - 1$ disjoint closed discs of the same small radius, whose centres are equidistributed on the circle $|z| = r$. Clearly, rotations of angle $j2\pi/(n-1), j = 0, 1, \ldots, n-1$, are distinct elements of $\mathrm{Aut}(\Omega)$.

4.3 Bordered Riemann surfaces

Let us denote a bordered Riemann surface with interior Ω and border $b\Omega$ by $\widetilde{\Omega} = \Omega \cup b\Omega$. Every bordered Riemann surface is a bordered surface, so there is an open cover $\{U_\alpha\}$

of $\widetilde{\Omega}$ and corresponding homeomorphismas $h_\alpha : \overline{U}_\alpha \to \overline{\Delta}_\alpha$, which we call closed charts, where each Δ_α is either a disc, whose closure is contained in the open upper half-plane of $\mathbb{C}$ or an upper half-disc $\{w : |w - t| < r, \Im w \geq 0\}$, for some real "center" t and positive radius r. Points of $\widetilde{\Omega}$ that correspond to points on the real line form the border $b\Omega$ and the remaining points, which correspond to points of the open upper half-plane, form the "interior" Ω of $\widetilde{\Omega}$. The changes of charts $h_\beta^{-1} \circ h_\alpha$, when defined, preserve interior points and border points, and are clearly homeomorphisms. If $\widetilde{\Omega}$ is, not only a bordered surface, but also a bordered Riemann surface, then we require in addition that these changes of charts be conformal. At interior points the meaning of conformal is obvious and at border points we ask that $h_\beta^{-1} \circ h_\alpha$ be the restriction of a conformal mapping in an open subset of the complex plane.

Lemma 4.3.1. *If $\widetilde{\Omega} = \Omega \cup b\Omega$ is a bordered Riemann surface, then each border point $p \in b\Omega$ has a neighbourhood system, given by closed border charts $h_r : \overline{U}_r \to \overline{\Delta}_r^+, 0 < r < 1$, where Δ_r^+ is the open upper half-disc $\{z : |z| < r, \Im z > 0\}$. Set $U_r = h_r^{-1}(\Delta_r^+)$. Each closed neighbourhood $\overline{U}_r$ is thus a closed Jordan domain, where the Jordan curve $\overline{U}_r \setminus U_r$ consists of an open border arc $\beta_r \subset b\Omega$ and a cross-cut C_r of $\widetilde{\Omega}$ having the same end points as β_r.*

Proof. Fix $p \in b\Omega$. Let $h_p : \overline{U}_p \to \overline{\Delta}_p^+$ be a closed chart at p where $\Delta_p^+ = \{z : |z - t| < r, \Im z > 0\}$ for some real centre t and positive radius r. Without loss of generality, we may suppose $t = 0, r = 1$, $\Delta_p^+ = \{z : |z| < 1, \Im z > 0\}$ and h_p sends p to zero. Denote by Δ_r^+ the open upper half-disc $\{z : |z| < r, \Im z > 0\}$ and U_r the inverse image $h_p^{-1}(\Delta_r^+)$. Since $\overline{\Delta}_r^+, 0 < r < 1$, is a neighbourhood system of 0 in the closed upper half-plane and h_p is a homeomorphism, it follows that the $\overline{U}_r, 0 < r < 1$, are closed Jordan domains and form a neighbourhood system of p. The Jordan curve ∂U_r consists of the open border arc $\beta_r = h_p^{-1}\{(-r, r)\}$ and the cross-cut $h_p^{-1}(c_r)$, where c_r is the closed semi-circle $\{z : |z| = r, \Im z \geq 0\}$. If we denote by h_r the restriction of h_p to $\overline{U}_r$, then $h_r : \overline{U}_r \to \overline{\Delta}_r^+, 0 < r < 1$, are closed border charts at p. □

Given a bordered Riemann surface $\widetilde{\Omega} = \Omega \cup b\Omega$, we construct a bordered Riemann surface $\widetilde{\Omega}^*$, called the conjugate of $\widetilde{\Omega}$ (see [4.1]). The conjugate $\widetilde{\Omega}^*$ of $\widetilde{\Omega}$ is a topological copy of $\widetilde{\Omega}$. For each α, denote by U_α^* the corresponding topological copy of the U_α and for each $p \in \widetilde{\Omega}$ by p^* the corresponding point in $\widetilde{\Omega}^*$. The space $\widetilde{\Omega}^*$ is endowed with the complex structure obtained by replacing the closed charts $h_\alpha : \overline{U}_\alpha \to \overline{\Delta}_\alpha$ of $\widetilde{\Omega}$ by the charts $h_\alpha^* : \overline{U}_\alpha^* \to \overline{\Delta}_\alpha^*$, where $h_\alpha^*(p^*) = -\overline{h}_\alpha(p)$.

We now form the *double* $\widehat{\Omega}$ of the bordered Riemann surface $\widetilde{\Omega}$ by welding $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ together by the identity mapping on $b\Omega$. The double of a bordered Riemann surface is a Riemann surface (not a bordered Riemann surface). The complex structure of the double $\widehat{\Omega}$ is given by charts $\widehat{h}_\alpha : \widehat{U}_\alpha \to \widehat{\Delta}_\alpha$, which we now describe. If U_α is contained in the interior Ω, then we set $\widehat{U}_\alpha = U_\alpha$ and $\widehat{h}_\alpha = h_\alpha$. Similarly, if U_α^* is contained in the interior of $\widetilde{\Omega}^*$, we set $\widehat{h}_\alpha = h_\alpha^*$ and $\overline{\Delta}_\alpha^* = h_\alpha^*(U_\alpha^*)$. There remains to define charts at points of $b\widetilde{\Omega} = b\widetilde{\Omega}^*$. If U_α corresponds to a half-disc, then we denote by $\widehat{U}_\alpha$ the set obtained by the welding together of U_α and U_α^*. We define the function $\widehat{h}_\alpha$ on the closure of $\widehat{U}_\alpha$ by setting $\widehat{h}_\alpha = h_\alpha$ on $\overline{U}_\alpha$ and $\widehat{h}_\alpha = -h_\alpha^* = -(-\overline{h}) = \overline{h}$ on $\overline{U}_\alpha^*$.

A manifold need not be second countable (consider the long line), but it is a profound property (Rado's theorem) of Riemann surfaces that they are second countable. They are therefore σ-compact, that is, they can be represented as a countable union of compacta. Similar properties hold for bordered Riemann surfaces but, since non-compact bordered Riemann surfaces are less familiar, we state the following result, which makes it easier to see these properties (and many others) for bordered Riemann surfaces.

Theorem 4.3.2. *Every bordered Riemann surface is homeomorphic to a closed subset of* $\mathbb{R}^3$.

Proof. Let $\widetilde{\Omega}$ be a bordered Riemann surface. The remarkable result of Rüedy [4.17] states that every Riemann surface admits a smooth proper conformal embedding into $\mathbb{R}^3$. Let $h : \widehat{\Omega} \to \mathbb{R}^3$ be such an embedding. Since $\widetilde{\Omega}$ is closed in $\widehat{\Omega}$, it follows that $h(\widetilde{\Omega})$ is closed in $h(\widehat{\Omega})$ and, since $h(\widehat{\Omega})$ is closed in $\mathbb{R}^3$, it follows that $h(\widetilde{\Omega})$ is also closed in $\mathbb{R}^3$. $\square$

A subset of a Riemann surface or bordered Riemann surface is said to be *bounded* if its closure is compact.

Corollary 4.3.3. *In a bordered Riemann surface $\widetilde{\Omega}$, a subset is compact if and only if it is closed and bounded. Hence, a closed subset is non-compact if and only if it contains a sequence which tends to infinity (the Alexandroff point of $\widetilde{\Omega}$).*

4.4 A reflection principle for bordered Riemann surfaces

Theorem 4.4.1. *Let $\widetilde{\Omega} = \Omega \cup b\Omega$ be a bordered Riemann surface. Bordered Riemann surface could be compact or non-compact. We care of the latest one. Let f be meromorphic in Ω and suppose all boundary values of f on $b\Omega$ are real or ∞. Then f extends to a meromorphic function $\widehat{f}$ on $\widehat{\Omega}$. Suppose $f(\Omega)$ is contained in the open upper half-plane. Then, if f is locally conformal, so is $\widehat{f}$ and, if f is conformal, so is $\widehat{f}$.*

Proof. First, we shall extend f to a point p of the border $b\Omega$. At p considered as a point of $\widehat{\Omega}$, there is a chart $\widehat{h} : \widehat{U} \to \Delta$, where Δ is the open unit disc. Set $\Delta^+ = \{w : |w| < 1, \Im w \geq 0\}$ and $\Delta^- = \{w : |w| < 1, \Im w \leq 0\}$. Setting $U^+ = \widehat{h}^{-1}(\Delta^+)$ and $U^- = \widehat{h}^{-1}(\Delta^-)$, we have $\widehat{U} = U^+ \cup U^-$. Moreover, $\widehat{h}|_{U^+} = h : U^+ \to \Delta^+$ and $\widehat{h}|_{U^-} = -h^* : U^- \to -\Delta^-$ are border charts of p in $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ respectively.

Denote $\Delta_0^+ = \{w : |w| < 1, \Im w > 0\}$. The meromorphic function

$$f \circ \widehat{h}^{-1} : \Delta_0^+ \longrightarrow \mathbb{C}$$

satisfies the hypotheses of Theorem 4.1.2 and so extends meromorphically to the open disc Δ. Consequently f extends meromorphically to the neighbourhood $\widehat{U}$ of p.

If p and q are two border points and $\widehat{U}_p$ and $\widehat{U}_q$ are corresponding neighbourhoods as above which intersect, then the corresponding meromorphic extensions agree, since they agree on $\widehat{U}_p^+ \cap \widehat{U}_q^+$. Setting

$$\widehat{U}_b = \bigcup\{\widehat{U}_p : p \in b\widetilde{\Omega}\},$$

we obtain a meromorphic extension $\widehat{f}$, defined on the neighbourhood $\widehat{U}_b$ of $b\widetilde{\Omega}$.

Since this extension to the neighbourhood $\widehat{U}_b$ of the common border $b\widetilde{\Omega}$ is defined explicitly on $\widetilde{\Omega}^* \cap \widehat{U}_b$ by the formula $\widehat{f}(p^*) = \overline{f}(p)$, we may extend $\widehat{f}$ to all of $\widetilde{\Omega}^*$ by the same formula. Namely, we set $\widehat{f}(p^*) = \overline{f}(p)$, for all $p^* \in \widetilde{\Omega}^*$.

From this formula, we see that if f is locally conformal on Ω, then $\widehat{f}$ is locally conformal on $\Omega \cup \Omega^*$.

Now, suppose $f(\Omega)$ is contained in the open upper half-plane. The proof that $\widehat{f}$ is locally conformal or conformal, if f is respectively locally conformal or conformal, is the same as the proof of the corresponding portion of Theorem 4.1.2. □

Let $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ be two bordered Riemann surfaces and consider a holomorphic mapping $f : \Omega_1 \to \widehat{\Omega}_2$. For $p \in b\Omega_1$, denote the set of boundary values (the cluster set) of f at p by $C(f, p)$. For $B \subset b\Omega_1$ we define the cluster set at B as

$$C(f, B) = \{q \in \widetilde{\Omega}_2 : \exists p \in B, \exists p_n \to p, p_n \in \Omega_1, f(p_n) \to q\} = \bigcup_{p \in B} C(f, p).$$

If $\widetilde{\Omega}_2$ is compact, then $C(f, p)$ is not empty f is holomorphic, so continuous, but if $\widetilde{\Omega}_2$ is not compact, $C(f, p)$ may be empty. For example, this is the case for $C(f, 0)$, when $\widetilde{\Omega}_1 = \widetilde{\Omega}_2$ is the closed upper half-plane and $f(z) = 1/z$.

We shall say that the mapping f sends the border $b\Omega_1$ to the border $b\Omega_2$, if for every sequence $p_j \in \Omega_1$ converging to a point of $b\Omega_1$, the sequence $f(p_j)$ has a limit point in $b\Omega_2$. If f sends the border to the border, then $C(f, p)$ is a non-empty subset of $b\Omega_2$ and since $C(f, p)$ is connected it lies in a single component of $b\Omega_2$. Similarly, we shall say that f sends a border component B_1 of $b\Omega_1$ to the border $b\Omega_2$, if for every sequence $p_j \in \Omega_1$ converging to a point of B_1, the sequence $f(p_j)$ has a limit point in $b\Omega_2$. Also, we shall say that f sends a border component B_1 of $b\Omega_1$ to a border component B_2 of $b\Omega_2$, if for every sequence $p_j \in \Omega_1$ converging to a point of B_1, the sequence $f(p_j)$ has a limit point in B_2.

For B a closed subset of $b\Omega_1$, the cluster set $C(f, B)$ may not be closed, even if f is continuous and sends the border to the border. For example, let $\widetilde{\Omega}_2$ be the closed upper half-plane, $\widetilde{\Omega}_1$ the closed upper half-plane less the point 0 and $f(z) = z$. Then, for $B = b\Omega_1$, the set B is closed in $b\Omega_1$, but $C(f, B)$ is not closed in $b\Omega_2$.

Lemma 4.4.2. *Let $\widetilde{\Omega}_j = \Omega_j \cup b\Omega_j$, $j = 1, 2$, be bordered Riemann surfaces. Let $f : \Omega_1 \to \widehat{\Omega}_2$ be a continuous mapping which sends the border to the border and let $B \subset b\Omega_1$. If B is compact or connected, then $C(f, B)$ is respectively compact or connected.*

Proof. Suppose B is compact. Since each component of $b\Omega_1$ is clopen (closed and open) in $b\Omega_1$, it follows that B is contained in the union $b_1 \cup \cdots \cup b_n$ of finitely many components of $b\Omega_1$ and that each $B_j = B \cap b_j$ is compact. Since

$$C(f, B) = \bigcup_{k=1}^{n} C(f, B_j),$$

we may assume that B is contained in a single component b of $b\Omega_1$.

Let $h : b \times [0, 1) \to H_b$ be a collar of b in $\widetilde{\Omega}_1$. By [4.3] it exists and it is a neighbourhood of b such that $h(b_b, 0) = b_b$, $\forall b_b \in b$. Let I_n be a nested sequence of open subsets of b such that each $\overline{I}_n$ is compact and

$$B \subset I_n \subset \overline{I}_n \subset b, \quad B = \bigcap_{n=1}^{\infty} I_n$$

and put $U_n = h(I_n \times (0, 1/n])$. Then,

$$C(f, B) = \bigcap_{n \geq 1} \overline{f(U_n)} \tag{4.1}$$

and we see that $C(f, B)$ is closed.

To see that $C(f, B)$ is compact, it is sufficient to show that $\overline{f(U_1)}$ is compact. Suppose $\overline{f(U_1)}$ is not compact. Then, since $\overline{f(U_1)}$ is closed, $\widehat{\Omega}_2$ is surely not compact because then $\overline{f(U_1)}$ is compact which we assume to be not and there is a sequence $q_n \in \overline{f(U_1)}$, such that $q_n \to *_2$, where $*_2$ is the ideal (Alexandroff) point of $\widehat{\Omega}_2$. By a diagonal process, we can construct a sequence $p_n \in U_1$, such that $f(p_n) \to *_2$. By choosing a subsequence, if necessary, we may assume that p_n converges to a point $p \in \overline{I}_n$. This contradicts the assumption that f sends the border to the border. Thus, $\overline{f(U_1)}$ is compact. Since $C(f, B)$ is a closed subset of the compact set $\overline{f(U_1)}$, it follows that $C(f, B)$ is also compact.

Suppose that B is not only compact but also connected. Then, we may take the I_n to be connected. Recall that, since f sends the border to the border, $C(f, B) \neq \emptyset$. The sets $\overline{f(U_n)}$ are connected subsets of the compact Hausdorff space $\overline{f(U_1)}$ and

$$\liminf \overline{f(U_n)} = \limsup \overline{f(U_n)} = C(f, B) \neq \emptyset.$$

It follows [4.10, Th. 2-101] that $C(f, B)$ is connected.

We have shown that if B is compact, and if B is moreover connected, then $C(f, B)$ is also connected.

Now, we show that if B is connected, then $C(f, B)$ is connected, even if B is not compact. Since B is connected, it is contained in a single border component b. The only connected subsets of b are Jordan arcs and Jordan curves. Jordan curves are compact, so we may assume that B is a Jordan arc, possibly containing one or both end points. In any case, we may write B as the union of an increasing sequence of compact Jordan arcs $\overline{I}_n$. Since the $\overline{I}_n$ are compact and connected, we have shown that the sets $C(f, \overline{I}_n)$ are connected. Now,

$$C(f, B) = \bigcup_{n} C(f, \overline{I}_n)$$

and the $C(f, \overline{I}_n)$ are increasing, so $C(f, B)$ is connected. □

Lemma 4.4.3. *Let $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ be two bordered Riemann surfaces and $f : \Omega_1 \to \widehat{\Omega}_2$ a holomorphic map. Then, f sends a border component B_1 to the border $b\Omega_2$ if and only if it sends B_1 to some border component B_2.*

Proof. By the definition, direction "if" is obvious. Now suppose f sends a border component B_1 of $b\Omega_1$ to the border $b\Omega_2$. It suffices to show that $C(f, B_1)$ is connected, but since B_1 is a border component, it is connected, and so by the previous lemma, $C(f, B)$ is connected □

The following result extends the Carathéodory reflection principle to bordered Riemann surfaces.

Theorem 4.4.4. *For $j = 1, 2$, let $\widetilde{\Omega}_j = \Omega_j \cup b\Omega_j$ be bordered Riemann surfaces with respective interiors Ω_j, respective borders $b\Omega_j$ and respective doubles $\widehat{\Omega}_j$. Let $f : \Omega_1 \to \widehat{\Omega}_2$ be a holomorphic mapping which sends the border $b\Omega_1$ to the border $b\Omega_2$. Then, there is a holomorphic surjective extension*

$$\widehat{f} : \widehat{\Omega}_1 \longrightarrow f(\Omega_1) \cup C(f, b\Omega_1) \cup f(\Omega_1)^* \subset \widehat{\Omega}_2,$$

such that $\widehat{f}(b\Omega_1) = C(f, b\Omega_1)$.

Proof. Fix $p \in b\Omega_1$. By Lemma 4.4.3, $C(f, p)$ is contained in a single component B_2 of the border $b\Omega_2$. We consider two cases, depending on whether B_2 is an open Jordan arc or a Jordan curve. B_2 can not be closed at one end and open at the other end because from the definition of border, every point of B_2 must have a neighbourhood homeomorphic to $(-1, +1)$, so B_2 has no end points.

Suppose first that B_2 is a Jordan arc. By the proof of lemma 4.4.2, there is some compact Jordan arc $[\alpha, \beta] \subset B_2$ such that $C(f, p)$ is contained in the open Jordan arc (α, β). We may choose a closed arc $[a, b]$ about p in $b\Omega_1$, such that $C(f, q) \subset (\alpha, \beta)$, for each $q \in [a, b]$.

Construct a closed Jordan domain $\overline{G}_2$ in $\widetilde{\Omega}_2$, such that the Jordan curve $\overline{G}_2 \setminus G_2$, consists of the closed arc $[\alpha, \beta]$ and a cross-cut γ_2 of $\widetilde{\Omega}_2$. To see that this is possible, use a collar of B_2. Similarly, (see also Lemma 4.3.1) we may construct a closed border chart $\overline{G}_1$ for p, which is a closed Jordan domain in $\widetilde{\Omega}_1$, such that the Jordan curve $\overline{G}_1 \setminus G_1$ consists of a closed arc $[a, b]$ in $b\Omega_1$ and a cross-cut γ_1 of $\widetilde{\Omega}_1$. Let ϕ be the restriction of f to G_1. Denote by $\widetilde{G}_2$ the bordered Riemann surface whose interior is G_2 and whose border is (α, β). By Lemma 4.3.1, we may further assume that G_1 is so small that $\phi(G_1) \subset \widehat{G}_2$, and all boundary values of ϕ on (a, b) lie in (α, β).

Let h be a conformal mapping of G_2 onto the upper half-plane H^+. By Theorem 4.4.1, h extends to a conformal mapping $\widehat{h} : \widehat{G}_2 \to \widehat{h}(G_2) \subset \overline{\mathbb{C}}$. The function $h \circ \phi$ also satisfies the hypotheses of Theorem 4.4.1, so $h \circ \phi$ extends to a meromorphic function $\widehat{h \circ \phi} : \widehat{G}_1 \to \overline{\mathbb{C}}$. Since meromorphic functions on Riemann surfaces are the same as holomorphic maps to the Riemann sphere, this extension can be considered as a holomorphic mapping $\widehat{G}_1 \to \overline{\mathbb{C}}$. On G_1 we have

$$\phi = h^{-1} \circ h \circ \phi = (\widehat{h})^{-1} \circ \widehat{h} \circ \phi = (\widehat{h})^{-1} \circ (\widehat{h \circ \phi}).$$

Hence, ϕ extends to a holomorphic mapping $\widehat{\phi}:\widehat{G}_1\to\widehat{G}_2$. Since ϕ is the restriction of f to G_1, this gives a holomorphic extension of f which we denote by f_p and $f_p:\widehat{G}_1\to\widehat{G}_2$. Moreover, the value $f_p(p)$ lies on B_2, since $C(f,p)\subset B_2$.

Now we need to consider the case that B_2 is a Jordan curve. Let $\widetilde{C}_2=C_2\cup B_2$ be a collar about B_2 in $\widetilde{\Omega}_2$. The interior C_2 of $\widetilde{C}_2$ is planar and so, by the General Uniformization Theorem 4.1.1 and the Koebe theorem on circular domains 4.2.7 , there is a conformal mapping h of C_2 onto a circular domain $A=H^+\setminus K$, where K is a closed disc in H^+. By Lemma 4.4.3, we may assume that h sends B_2 to $\mathbb{R}\cup\{\infty\}$. By Theorem 4.4.1, we may extend h to a meromorphic function $\widehat{h}:\widehat{C}_2\to\overline{\mathbb{C}}$. Let B_1 be the border component containing p. Then, by Lemma 4.4.3, $C(f,q)\subset B_2$, for each $q\in B_1$. Hence, if we fix a sufficiently small open arc α in B_1 which contains p and which is pre-compact in B_1, then, we may construct a collar $\widetilde{C}_1=C_1\cup\alpha$ of α in $\widetilde{\Omega}_1$, such that $f(C_1)\subset\widehat{C}_2$. Let ϕ be the restriction of f to C_1. As for the case that B_2 was not compact, the function h extends meromorphically to $\widehat{C}_2$ and $h\circ\phi$ extends meromorphically to $\widehat{C}_1$. Consequently f extends to a holomorphic mapping $f_p:\widehat{C}_1\to\widehat{C}_2$ and $f_p(p)\in B_2$. By the construction, $\widehat{C}_2\subset\widehat{\Omega}_2$.

From the preceding, it follows that, for every $p\in b\Omega_1$, There is a closed Jordan domain $\overline{U}_p\subset\widetilde{\Omega}_1$, such that the Jordan curve $\overline{U}_p\setminus U_p$ consists of an open border arc α_p containing p and a cross-cut σ_p of Ω_1 and there is a closed Jordan domain $\overline{V}_p\subset\widetilde{\Omega}_2$, such that the Jordan curve $\overline{V}_p\setminus V_p$ consists of an open border arc β_p and a cross-cut τ_p of Ω_2, such that, denoting $\widetilde{U}_p=U_p\cup\alpha_p$ and $\widetilde{V}_p=V_p\cup\beta_p$, f restricted to U_p extends to a holomorphic mapping $f_p:\widehat{U}_p\to\widehat{V}_p$, where $\widehat{U}_p$ is the double of $\widetilde{U}_p$ and $\widehat{V}_p$ is the double of $\widetilde{V}_p$. Moreover, $f_p(\alpha_p)\subset\beta_p$. We may assume that we have a closed border chart $h_p:\widetilde{V}_p\to\overline{\Delta}^+$.

These various holomorphic extensions $f_p, p\in b\Omega_1$ are compatible. That is, suppose p ad q are two arbitrary points in the border $b\Omega_1$ of Ω_1, with corresponding holomorphic extensions $f_p:\widehat{U}_p\to\widehat{V}_p$ and $f_q:\widehat{U}_q\to\widehat{V}_q$. Suppose $\alpha_p\cap\alpha_q\neq\emptyset$. Then, $f_p=f_q$ on $\widehat{U}_p\cap\widehat{U}_q$, by the uniqueness of holomorphic continuation.

It follows that there is an open neighbourhood of $b\Omega_1$ in $\widetilde{\Omega}_1$, which is a bordered surface of the form $\widetilde{U}=U\cup bU$, with interior $U\subset\Omega_1$ and border $bU=b\Omega_1$ and there is a holomorphic extension $\widehat{f}:\Omega_1\cup\widehat{U}\to\widehat{\Omega}_2$, such that $\widehat{f}(b\Omega_1)\subset b\Omega_2$. Since, for $p^*\in U^*$, this extension is given by $\widehat{f}(p^*)=f(p)^*$, we may define the extension on all of Ω_1^* by the same formula. We now have a holomorphic extension $\widehat{f}:\widehat{\Omega}_1\to\widehat{\Omega}_2$. □

As we already mentioned, the previous theorem, for maps $f:\Omega_1\to\widehat{\Omega}_2$, can be considered as an extension of the Carathéodory reflection principle to Riemann surfaces. In the following, we consider the particular case that $f(\Omega_1)\subset\Omega_2$ and obtain a generalization of the Schwarz reflection principle.

Theorem 4.4.5. *For $j=1,2$, let $\widetilde{\Omega}_j=\Omega_j\cup b\Omega_j$ be bordered Riemann surfaces; let $f:\Omega_1\to\Omega_2$ be a holomorphic map which sends the border $b\Omega_1$ to the border $b\Omega_2$ and let $\widehat{f}:\widehat{\Omega}_1\to\widehat{\Omega}_2$ be the holomorphic extension given by Theorem 4.4.4. If f is locally conformal (respectively conformal), then so is $\widehat{f}$.*

If f is conformal and onto and $C(f,b\Omega_1)=b\Omega_2$, then $\widehat{f}$ is a biholomorphic mapping of $\widehat{\Omega}_1$ onto $\widehat{\Omega}_2$.

For an example which is not onto but satisfies the other conditions, take both bordered surfaces to be the upper half-plane with the positive real axis. Take f to be the branch of the square root which sends the point 1 to itself.

For an example which satisfies the first two conditions but not the last, take Ω_2 to be the closed upper half-plane and Ω_1 to be the upper half-plane with the positive real axis. Take $f(z) = z$.

Proof. Let $\widehat{U}, \widehat{U}_p, h_p$ and f_p be the same as in the proof of the previous theorem.

From the formula $\widehat{f}(p^*) = f(p)^*$, it clear that if f is locally conformal on Ω_1, then $\widehat{f}$ is locally conformal on $\Omega_1 \cup \Omega_1^*$ and we claim that it is also locally conformal on $\widehat{U}$. It is sufficient to show that it is locally conformal on each $\widehat{U}_p$. Clearly, $(h_p \circ f_p)(\widehat{U_p})$ is contained in the open upper half-plane. Hence, so is $h_p \circ f_p$ restricted to U_p and so, by Theorem 4.4.1, $\widehat{h_p \circ f_p}$ is locally conformal on $\widehat{U}_p$. Consequently also f_p is locally conformal on $\widehat{U}_p$. It follows that $\widehat{f}$ is locally conformal.

The proof that, if f is conformal, then $\widehat{f}$ is also conformal, is similar to that of the analogous statement in Theorems 4.1.2 and 4.4.1.

If f is conformal and onto and $C(f, b\Omega_1) = b\Omega_2$, then $\widehat{f}$ is conformal and onto and hence is a biholomorphic mapping of $\widehat{\Omega}_1$ onto $\widehat{\Omega}_2$. □

Let Ω be the interior of a compact bordered Riemann surface $\widetilde{\Omega}$. Let $p \in \Omega$ and H_p be the family of holomorphic functions from Ω to the unit disc which take p to zero, and which have, in a fixed coordinate chart, a non-negative derivative at p. The Ahlfors function for Ω and p is the unique function A in H_p such that

$$A'(p) = \max_{f \in H_p} Ref'(p)$$

It is a non-trivial fact that every Ahlfors function is a proper mapping of Ω onto the unit disc Δ.

The Ahlfors function for a Jordan region in $\mathbb{C}$ is presented in [4.8, Ch. VI]. For a monumental treatment of Ahlfors functions, see [4.7].

Corollary 4.4.6. *Let $\widetilde{\Omega}$ be a compact bordered Riemann surface and let $f : \Omega \to \Delta$ be an Ahlfors function of Ω onto the open unit disc Δ. Then f extends to a meromorphic function $\widehat{f} : \widehat{\Omega} \to \mathbb{C} \cup \{\infty\}$.*

For Riemann surfaces Ω, Ω_1 and Ω_2, let us denote by $\mathrm{Iso}(\Omega_1, \Omega_2)$ the space of biholomorphic mappings $\Omega_1 \to \Omega_2$ and by $\mathrm{Aut}(\Omega)$ the automorphism group $\mathrm{Iso}(\Omega, \Omega)$. Similarly, for bordered Riemann surfaces $\widetilde{\Omega}, \widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$, let us denote by $\mathrm{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2)$ the space of homeomorphisms $\widetilde{\Omega}_1 \to \widetilde{\Omega}_2$ whose restrictions to Ω_1 are in $\mathrm{Iso}(\Omega_1, \Omega_2)$ and by $\mathrm{Aut}(\widetilde{\Omega})$ the space $\mathrm{Iso}(\widetilde{\Omega}, \widetilde{\Omega})$.

Theorem 4.4.7 (Schwarz 1879)**.** *The automorphism group of every compact Riemann surface of genus $g \geq 2$ is finite.*

A compact bordered Riemann surface is said to be of type (g, n), if it is of genus g and the number of border components is n.

Corollary 4.4.8. *If $\widetilde{\Omega}$ is a compact bordered Riemann surface of type (g, n) and $2g+n \geq 3$, then $\mathrm{Aut}(\widetilde{\Omega})$ is finite.*

Proof. It follows from the theorem 4.4.5 that every $\phi \in \mathrm{Aut}(\widetilde{\Omega})$ extends to $\widehat{\phi} \in \mathrm{Aut}(\widehat{\Omega})$. Since ψ is an automorphism it is biholomorphic so both ψ and its inverse are continuous. But a continuous function takes compact sets to compact sets, so ψ is proper. But here $\tilde{\Omega}$ is compact so ψ takes the border to the border. The genus of the double $\widehat{\Omega}$ is $2g+n-1$, which is greater than or equal to 2. By the Schwarz Theorem, $\mathrm{Aut}(\widehat{\Omega})$ is finite. Consequently, since the mapping $\phi \mapsto \widehat{\phi}$ is injective, $\mathrm{Aut}(\widetilde{\Omega})$ is also finite. $\square$

The hypothesis on the type is satisfied if the genus g is not zero or if the genus is zero and the number n of border components is at least 3.

The restriction mapping gives a natural embedding $\mathrm{Aut}(\widetilde{\Omega}) \hookrightarrow \mathrm{Aut}(\Omega)$, but this need not be surjective. For example, if $\widetilde{\Delta}$ is the bordered Riemann surface, whose interior is the open unit disc Δ and whose border is an arc $e^{i\theta}, 0 < \theta < \beta$, for some $\beta \in (0, 2\pi)$, then $\mathrm{Aut}(\widetilde{\Delta})$ is the proper subgroup of $\mathrm{Aut}(\Delta)$ described as follows. Fix $\alpha \in (0, \beta)$. The group $\mathrm{Aut}(\widetilde{\Delta})$ consists of the elements $\phi_\gamma \in \mathrm{Aut}(\Delta)$ which send the points $1, e^{i\alpha}, e^{i\beta}$ to the points $1, e^{i\gamma}, e^{i\beta}$ respectively, for $0 < \gamma < \beta$. They are thus parametrized by the values $\gamma, 0 < \gamma < \beta$.

If $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are bordered Riemann surfaces and $\mathrm{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2) \neq \emptyset$, then every element f of $\mathrm{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2)$ induces bijections

$$\mathrm{Aut}(\widetilde{\Omega}_1) \longrightarrow \mathrm{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2), \quad \phi \mapsto f \circ \phi$$

and

$$\mathrm{Aut}(\widetilde{\Omega}_2) \longrightarrow \mathrm{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2), \quad \psi \mapsto \psi \circ f.$$

In this situation, the groups $\mathrm{Aut}(\widetilde{\Omega}_1)$ and $\mathrm{Aut}(\widetilde{\Omega}_2)$ are isomorphic and have the same cardinality as the family $\mathrm{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2)$. Of course, for "most" Riemann surfaces Ω, the group $\mathrm{Aut}(\Omega)$ is trivial. Similarly, for most bordered Riemann surfaces $\widetilde{\Omega}$, the group $\mathrm{Aut}(\Omega)$ is trivial. For such $\widetilde{\Omega}$, the subgroup $\mathrm{Aut}(\widetilde{\Omega})$ is of course also trivial.

4.5 Bordered regions in Riemann surfaces

We wish to show the equivalence between bordered Riemann surfaces and certain domains in Riemann surfaces together with a portion of their boundary. We shall call these bordered domains and they include Jordan domains as the prime example.

Let Ω be a domain in a Riemann surface R. An open Jordan arc $A \subset \partial\Omega$ is called a *free boundary arc* of the domain Ω if, for each point $p \in A$, there is an open set $U \subset \Omega$ and a homeomorphism $h_p : \overline{U} \to \overline{\Delta}^+$, where Δ^+ is the open upper half-disc $\{|z| < 1, \Im z > 0\}$, $h(\overline{U} \cap A) = (-1, +1)$ and $h_p(p) = 0$. The maps h_p are similar to border charts in a bordered Riemann surface, where, the h_p were additionally required to have a certain analyticity property.

An open arc $A \subset \partial\Omega$ is called a *doubly free boundary arc* of the domain Ω if, for each point $p \in A$, there is an open set $U \subset R$ and a homeomorphism $h : \overline{U} \to \overline{\Delta}$, where Δ is the open unit disc, $h(U \cap A) = (-1, +1)$, $h(U \cap \Omega) = \Delta^+$, $h(U \setminus \overline{\Omega}) = \Delta^-$ and $h(p) = 0$.

As an example, if Ω is a Jordan domain in $\overline{\mathbb{C}}$, then it follows from the Schoenflies theorem that $\partial\Omega$ is doubly free.

Lemma 4.5.1. *If A is a doubly free boundary arc of a domain Ω in a Riemann surface, then A is a free boundary arc of Ω.*

Proof. Fix $p \in A$. By the definition there exists an open set $N \subset R$, and a homeomorphism $g : \overline{N} \to \overline{\Delta}$ such that $g(N \cap A) = (-1, 1)$, $g(N \cap \Omega) = \Delta^+$ and $g(p) = 0$. Take $U := N \cap \Omega$ and $h := g|_{\overline{U}}$ in the definition of free boundary arc. $\square$

Let us say that a subset $E \subset \partial\Omega$ is a *doubly free boundary set* of Ω, if each point of E is contained in a doubly free boundary arc of E.

If Ω is a bounded domain (open connected set) in a Riemann surface R and B is a (non empty) doubly-free boundary set of set Ω, then we shall say that $\widetilde{\Omega} = \Omega \cup B$ is a bordered region in R. We note that a bordered region $\widetilde{\Omega} = \Omega \cup B$ is compact if and only if $\widetilde{\Omega} = \overline{\Omega}$. In this case, $B = \partial\Omega$ and B consists of finitely many disjoint Jordan curves. For this reason, we call a compact bordered region a closed Jordan region. A closed Jordan region of genus zero, whose boundary is a single Jordan curve is a closed Jordan domain. The following theorem asserts that every bordered region can be considered to be a bordered Riemann surface, thus giving us a multitude of bordered Riemann surfaces. It is similar to a result in [4.1], where there is the further hypothesis that the border B is locally analytic arc.

Theorem 4.5.2. *Suppose $\widetilde{\Omega} = \Omega \cup B$ is a bordered region in a Riemann surface R. Then, $\widetilde{\Omega}$ admits the structure of a bordered Riemann surface with interior Ω and border B. The complex structures on Ω as interior of the bordered Riemann surface and as domain in R are the same. In the other direction, if $\widetilde{\Omega}$ is a bordered Riemann surface, then $\widetilde{\Omega}$ may be considered as a bordered region in the double $\widehat{\Omega}$.*

Proof. Fix a point $p \in B$. Since B is a doubly free boundary set of Ω, there is an open set $U \subset R$ and a homeomorphism $h : \overline{U} \to \overline{\Delta}$, where Δ is the open unit disc, $h(U \cap B) = (-1, +1)$, $h(U \cap \Omega) = \Delta^+$, $h(U \setminus \overline{\Omega}) = \Delta^-$ and $h(p) = 0$. Since U is planar, it follows from the General Uniformization Theorem 4.1.1, that there is a biholomorphic mapping ϕ_p of U onto a plane domain G_p. We may assume that $\phi_p(p) = 0$. Let $A \subset U \cap B$ be a compact Jordan arc containing p not as an end point. Then, $J = \phi_p(A)$ is a compact Jordan arc in G_p, containing 0, not as an end point. By Lemma 4.2.6, there is a closed Jordan domain $\overline{W}_p$ in G_p, such that $\overline{W}_p \cap \phi_p(U \cap B) = J$ and J is a cross-cut of W_p. That is, J is contained in W_p except for its end points, which lie on the Jordan curve ∂W_p. By the Jordan curve theorem, J separates $\overline{W}_p$ into two closed Jordan domains, whose intersection is J. By construction, ϕ_p^{-1} maps one of these, call it $\overline{G}_p^+$, homeomorphically to a closed Jordan domain $\overline{V}_p \subset \widetilde{\Omega}$. We note that $\overline{V}_p$ is a closed neighbourhood of p in $\widetilde{\Omega}$; $\phi_p : \overline{V}_p \to \overline{G}_p^+$ is a homeomorphism; $\phi_p : V_p \to G_p^+$ is biholomorphic; ϕ_p maps $\overline{V}_p \cap B$ onto J; and $\phi_p(p) = 0$. By the Riemann mapping theorem and the Osgood-Carathéodory theorem, there is a conformal mapping $\sigma_p : G_p^+ \to \Delta^+$, which extends to a homeomorphism $\overline{G}_p^+ \to \overline{\Delta}^+$, such that $\sigma_p(J) = [-1, +1]$ and $\sigma_p(0) = 0$.

Set $\eta_p = \sigma_p \circ \phi_p$. We may consider the family of maps: $\eta_p : \overline{V}_p \to \overline{\Delta}^+, p \in B$, as closed border charts and if, for every $p \in \Omega$, we add to this family a chart $\eta_p : V_p \to \Delta^+$ at p, for the Riemann surface Ω, then these combined charts give $\widetilde{\Omega} = \Omega \cup B$ the desired structure of a bordered Riemann surface. Although, the subset B of $\partial\Omega$ is locally an arc, these arcs may be non-analytic. Nevertheless, the change of border charts $\eta_q \circ \eta_p^{-1}$ is analytic on $\eta_p(B \cap (\overline{V}_p \cap \overline{V}_q))$, by the Schwarz reflection principle,. This completes the proof of the first part of the theorem.

The other direction is almost immediate. □

A particular consequence of the preceding theorem is that every bordered region $\widetilde{\Omega} = \Omega \cup B$ in $\mathbb{C}$ can be endowed with the structure of a bordered Riemann surface and the restriction of this structure to Ω is compatible with the given holomorphic structure on Ω. This is striking, considering that the curves which comprise the border of Ω need not be analytic. Nevertheless, the change of border charts

$$\phi_q \circ \phi_p^{-1}, \quad p, q \in B \subset \partial\Omega,$$

which maps the real interval $\phi_p(\partial\Omega \cap (V_p \cap V_q))$ to the real interval $\phi_q(\partial\Omega \cap (V_p \cap V_q))$ is analytic. Of course, an illustration of this is the Riemann mapping theorem (with the Osgood-Carathéodory theorem), which sends an arbitrary closed Jordan domain $\overline{\Omega}$ to the closed unit disc. If the Jordan curve $\partial\Omega$ is not analytic, the structure of a bordered Riemann surface we give to $\overline{\Omega}$ is definitely *not* the restriction to $\overline{\Omega}$ of the complex structure of $\mathbb{C}$, although the restriction to Ω of both structures are the same.

Theorem 4.5.3. *If $\widetilde{\Omega} = \Omega \cup B$ is a bordered region, whose interior is of genus zero in a Riemann surface R, Ω is planar Riemann surface then $\widetilde{\Omega}$ has a planar neighbourhood.*

Proof. If A is a component of the exterior $R \setminus \overline{\Omega}$ whose boundary meets B, denote by B_A the intersection $B \cap \partial A$. Then $\widetilde{A} = A \cup B_A$ is also a bordered region.

Denote by $X\Omega$ the union of all components A of the exterior of Ω, whose boundaries meet B. Since the border of every bordered manifold is collared [3.11], each set B_A is collard in both $\Omega \cup B_A$ and $A \cup B_A$.

Hence, there is an open neighbourhood W of B and a homeomorphism

$$h : B \times (-1, +1) \longrightarrow W,$$

with

$$h\,(B \times (-1, 0]) = \widetilde{\Omega} \cap W, \quad h(p, 0) = p, \quad h(B \times [0, +1)) = W \setminus \Omega.$$

The function $\phi(t) = -1/2 + 3(t + 1/2)$ defines a homeomorphism $\phi : [-1/2, 0) \to [-1/2, +1)$, which induces a homeomorphism

$$\Phi : B \times [-1/2, 0) \longrightarrow B \times [-1/2, +1),$$

given by $\Phi(p, t) = (p, \phi(t))$. Set

$$C = h(B \times \{-1/2\}), \quad V = h(B \times [-1/2, 0]), \quad U = h(B \times [-1/2, +1)).$$

The function $G = h \circ \Phi \circ h^{-1}$ defines a homeomorphism of V onto U, which fixes points of C. Denoting $N = \Omega \cup U$, we have a homeomorphism $H : \Omega \to N$, defined by setting $H(p) = p$, for $p \in \Omega \setminus V$ and $H(p) = G(p)$, for $p \in V$. Since Ω is of genus zero it is planar and since N is homeomorphic to Ω, the neighbourhood N is also planar, which completes the proof. $\square$

The next theorem may be considered as a generalization of the Osgood-Carathéodory theorem to bordered regions in Riemann surfaces.

Theorem 4.5.4. *For $j = 1, 2$, let $\widetilde{\Omega}_j = \Omega_j \cup B_j$ be bordered regions in Riemann surfaces R_j with respective interiors Ω_j and respective borders B_j. Let $f : \Omega_1 \to \Omega_2$ be a holomorphic mapping, which sends B_1 to B_2. Then, f extends to a (unique) continuous surjective mapping*

$$\widetilde{f} : \widetilde{\Omega}_1 \longrightarrow f(\Omega_1) \cup C(f, B_1) \subset \widetilde{\Omega}_2.$$

If f is locally conformal or conformal, then $\widetilde{f}$ is respectively locally injective or injective. If f is conformal and onto and $C(f, B_1) = B_2$, then $\widetilde{f}$ is a homeomorphism of $\widetilde{\Omega}_1$ onto $\widetilde{\Omega}_2$.

Let Ω_1 be the upper half plane and positive real axis, Ω_2 be the closed upper half plane. $f(z) = z$. Then f is conformal and onto but $C(f, B_1)$ is not B_2.

Proof. By Theorem 4.5.2, each $\widetilde{\Omega}_j$ can be endowed with the structure of a bordered Riemann surface, with interior Ω_j and border B_j, and such that on Ω_j this structure is compatible with the given holomorphic structure. This implies that f is also holomorphic, when considered as a mapping between the interiors of the bordered Riemann surfaces. By the Carathéodory reflection principle for bordered Riemann surfaces (Theorem 4.4.4), the mapping f extends to a holomorphic mapping $\widehat{f} : \widehat{\Omega}_1 \to \widehat{\Omega}_2$.

We claim that the restriction $\widetilde{f}$ of $\widehat{f}$ to $\widetilde{\Omega}_1$ has the desired properties. First of all, since $\widehat{f}$ is continuous, the restriction $\widetilde{f}$ is certainly a *continuous* extension of f. Since Ω_1 is dense in $\widetilde{\Omega}_1$, the continuous extension of f is unique. By Theorem 4.4.4, $\widehat{f}(B_1) = C(f, B_1)$, so $\widetilde{f}$ is surjective onto $f(\Omega_1) \cup C(f, B_1)$. Since f sends B_1 to B_2, this image is certainly contained in $\widetilde{\Omega}_2$.

If f is locally conformal or conformal, then by Theorem 4.4.5, the mapping $\widehat{f}$ is locally conformal or conformal respectively and hence $\widetilde{f}$ is locally injective or injective respectively.

It follows that, if f is conformal onto and $C(f, B_1) = B_2$, then $\widetilde{f}$ is a continuous bijection $\widetilde{\Omega}_1 \to \widetilde{\Omega}_2$. From Theorem 4.3.2, $\widetilde{\Omega}_1$ has an exhaustion by compact sets $K_j, j = 1, 2, \ldots$, such that, in the relative topology of Ω_1,

$$K_j^0 \subset K_j \subset K_{j+1}^0, \quad j = 1, 2, \ldots.$$

Let g_j be the restriction of $\widetilde{f}$ to K_j. Since

$$g_j : K_j \longrightarrow g_j(K_j) = \widetilde{f}(K_j)$$

is a continuous injective mapping of a compact space onto a Hausdorff space, it is a homeomorphism [3.25]. Thus g_j^{-1} is a continuous mapping on $g_j(K_j^0) = \widetilde{f}(K_j)^0$. Since $\widetilde{f}$

is a bijection, the inverse mapping $\widetilde{f}^{-1}$ is well-defined. For each j, the restriction of $\widetilde{f}^{-1}$ to the set $\widetilde{f}(K_j)^0$ is the continuous function g_j^{-1}. Since the family $\widetilde{f}(K_j)^0, j = 1, 2, \ldots,$ is an open cover of $\widetilde{\Omega}_2$, it follows that $\widetilde{f}^{-1}$ is continuous. Hence $\widetilde{f}$ is a homeomorphism. $\square$

Since the interior of every bordered region in a Riemann surface can be viewed as a Riemann surface, it follows that if $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are two such bordered regions, the family $\mathrm{Iso}(\Omega_1, \Omega_2)$ is usually empty and, if not, then it has the same cardinality as $\mathrm{Aut}(\widetilde{\Omega}_1)$ and $\mathrm{Aut}(\widetilde{\Omega}_2)$. For a general Riemann surface, and in particular for a general domain Ω in a Riemann surface, the group $\mathrm{Aut}(\Omega)$ is usually trivial.

There are interesting exceptional bordered regions $\widetilde{\Omega}$ of infinite genus, for which $\mathrm{Aut}(\widetilde{\Omega})$ is infinite. For example, consider the bordered region in $\mathbb{C}$:

$$\widetilde{\Omega} = \mathbb{C} \setminus \bigcup_{-\infty}^{+\infty} \Delta_j,$$

where Δ_j is the open disc of center j and radius $1/3$. Then, the interior Ω is of infinite connectivity and $\mathrm{Aut}(\Omega)$ is clearly infinite.

We can easily modify this example to obtain an example of infinite genus. Take two copies of $\mathbb{C}$, from which we have removed the slits $z = x + i : j < x < j + 1/2, j = 0, \pm 1, \pm 2, \ldots$ and let R be the Riemann surface obtained by gluing these two slit domains along the slits in the usual way. Let $\widetilde{W}$ be the bordered region in R, obtained by removing the open discs Δ_j from each sheet of R. Then $\widetilde{W}$ is of infinite genus, has infinitely many border components and $\mathrm{Aut}(W)$ is again clearly infinite.

For more information regarding domains with infinite automorphism groups, see for example [4.12].

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Chapter 5

Conclusion

In this thesis we have discussed approximating holomorphic functions in compact and closed subsets of the complex plane and open Riemann surfaces. Mainly holomorphic and meromorphic functions were considered as approximating functions. At the beginning, compact subsets and closed subsets of the complex plane were studied and later we moved on to work on Riemann surfaces.

We have discussed topological conditions and the Blaschke condition for holomorphic approximation on a closed subset of an open Riemann surface of infinite genus. We also mentioned some known examples constructed based on Myrberg surfaces to ease understanding the topological conditions of Arakelyan's approximation.

Gauthier and Hengartner showed that Arakelyan's topological conditions are not enough for uniform holomorphic approximation on an arbitrary Riemann surface. We have shown that for an arbitrary open Riemann surface R, and an arbitrary (proper) closed set E in R, although uniform approximation is not to be expected there always exists a closed subset F of E which is most of E in the sense that $E \setminus F$ is small and becomes smaller at arbitrary speed as we approach the ideal boundary point $*$ and F is also "better" than E because it is a set of tangential approximation.

Euclidean and spherical distances are two natural distances for the complex plane and Riemann sphere, respectively. In this work, pole-free approximation with respect to both of these distances has been studied for compact and closed subsets of an open Riemann surface as well as for (compact) Jordan regions with interior of genus zero and for parametric Jordan regions. Consequently, a recent extension of Mergelyan's theorem, due to Fragoulopoulou, Nestoridis and Papadoperakis (see [3.13] in Chapter 3) has been generalized. Also, a discussion of zero-free approximation was included, since approximating by pole-free meromorphic functions is equivalent to approximating by zero-free meromorphic functions.

The reflection principles of Schwarz and Carathéodory give conditions under which holomorphic functions extend holomorphically to the boundary and the theorem of Osgood-Carathéodory states that a one-to-one conformal mapping from the unit disc to a Jordan domain extends to a homeomorphism of the closed disc onto the closed Jordan domain. In the last chapter, we studied similar questions on Riemann surfaces for holomorphic mappings. We gave a Carathéodory type reflection principle for bordered Riemann surfaces which are arbitrary. That is, we did not assume that they are compact; nor did we assume that they are of finite genus. From this follows a Schwarz type reflection principle as well as an Osgood-Carathéodory type theorem.

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