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On the Dual Risk Models

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by

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Abstract

This thesis focuses on developing and computing ruin-related quantities that are potentially measurements for the dual risk models which was proposed to describe the annuity-type businesses from the perspective of the collective risk theory in 1950's. In recent years, the dual risk models are revisited by many researchers to quantify the risk of businesses similar to the annuity-type. The major extensions of this thesis consist of two aspects: the first is to search for new ruin-related quantities that are potentially indices of the risk for well-established dual models; the other aspect is to generalize the settings of the dual models instead of the ruin quantities. There are four separate articles in this thesis, in which the first (Chapter 2) and the last (Chapter 5) belong to the first type of extensions while the others (Chapter 3 and Chapter 4) belong to the generalizations of the dual models.

The first article (Chapter 2) studies the discounted moments of the surplus at the time of the last jump before ruin for the compound Poisson dual risk model. The idea comes from that the ruin of the compound Poisson dual models is caused by absence of positive jumps within a period with length being propotional to the surplus at the time of the last jump. As a quantity related to a non-stopping time, the explicit expression of the target quantity is obtained through solving integro-differential equations.

The second article (Chapter 3) investigate the Sparre-Andersen dual risk models in which the epochs are independently, identically distributed generalized Erlang-n random variables. An important difference between this model and some other models such as the Erlang-n dual risk models is that the roots to the generalized Lundberg's equation are not necessarily distinct. By taking the multiple roots into account, the explicit expressions of the Laplace transform of the time to ruin and expected discounted aggregate dividends under the threshold strategy and exponential distributed revenues are derived.

The third article (Chapter 4) revisits the the dual Lévy risk model. The target ruin quantity is the expected discounted aggregate dividends paid up to ruin under the threshold dividend strategy. The explicit expression is obtained in terms of the q-scale functions through constructing a new dividend strategy having the target ruin quantity converging to that under the threshold strategy. Also, the optimality of the threshold strategy among all the absolutely continuous stategies when evaluating the target quantity as a value function is discussed.

The fourth article (Chapter 5) initiate the study of the Parisian ruin problem for the general dual Lévy risk models. Unlike the regular ruin for the dual models, the deficit at Parisian ruin is not necessarily equal to zero. Hence we introduce the Gerber-Shiu expected discounted penalty function (EPDF) at the Parisian ruin and obtain an explicit expression for this function.

Keywords: Sparre-Andersen dual models, expected discounted aggregate dividends, dual Lévy risk models, Parisian ruin, Gerber-Shiu function

Co-Authorship Statement

The materials presented in this thesis are based on four joint-authored research articles, the first two of which are published on *Stochastic Models* and *Insurance: Mathematics and Economics* respectively. The third article is submitted to *Applied Stochastic Models for Business and Industry* and is now under revision. The fourth article will be submitted for publication in the near future. I am the lead author for all of these articles. I gratefully thank my supervisor Dr. Kristina P. Sendova for her important contributions to all of the aforementioned works, including suggesting research topics and solution approaches, crucial questions, extensive discussions, insightful comments and careful proofreading. Also, my co-author Dr. Zhong Li should be acknowledged for some useful discussions about the fourth article.

To my lord Jesus Christ.

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Chapter 1

Introduction

1.1 The risk model of interest

In ruin theory, Lundberg [1] was the first one to propose what is now known as the classical Cramér-Lundberg risk model that describes risks due to occasional losses called claims in the insurance businesses and studied the probability of ultimate ruin. This model is known as the theoretical foundation of ruin theory. Further investigation of the Cramér-Lundberg risk model was conducted by Cramér [2], after whom the model was named.

In general, we call the risk model with continuous income (premiums) but occasional losses (claims) classical risk model, whose surplus process $\{U(t) : t \geq 0\}$ is always given in the form

$$U(t) = u + X(t), \quad t \geq 0,$$

where $u > 0$ is the initial capital of the line of business and $\{X(t) : t \geq 0\}$, which is a càdlàg stochastic process issued from 0 (i.e. $X(0) \equiv 0$), is employed to characterize the model dynamics. For example, in the fundamental case of the Cramér-Lundberg risk model $\{X(t) : t \geq 0\}$ is assumed to have a linear drift minus a compound Poisson process. Namely,

$$X(t) = ct - \sum_{i=0}^{N(t)} Y_i, \quad Y_0 \equiv 0, \quad t \geq 0,$$

where $c > 0$ represents the premium rate, the claim counting process $\{N(t) : t \geq 0\}$ is a homogeneous Poisson process and the severities (sizes of each claim) $\{Y_i\}_{i=1}^{\infty}$ are a sequence of independent and identically distributed (i.i.d.) random variables with support on the positive half of the real line. Later in the 1950's, Andersen [3] generalized the classical Cramér-Lundberg model by extending $\{N(t) : t \geq 0\}$ to a general renewal process. Subsequently, the renewal risk model was named after him, i.e., Sparre-Andersen risk model. More recently, due to the rapid development of the fluctuation theory of Lévy processes, many important results including those for the reflected Lévy processes and refracted Lévy processes have already been introduced to the modern risk theory to calculate the probability of ultimate ruin and optimal dividends under various dividend strategies and dependence among the claim sizes, interclaim times and premium rate. In sum, the risk models with different model dynamics $\{X(t) : t \geq 0\}$ have been extensively studied in the past decade and many important ideas and results may be found in Asmussen and Albrecher [4] and Kyprianou [5].

This thesis focuses on the dual model of the ruin model, which was proposed by Cramér [6] to describe another type of insurance businesses which he called "pure annuity business" in his introduction of the fundamental assumptions of risk processes. An important special case of the pure annuity business is a life annuity business, under which the policy holder pays premiums as a lump sum at the date the policy is issued in exchange for the insurer's liability to paying annuities continuously. The death of the policyholder will cause instantaneous income by allowing the insurer to add the corresponding reserve free at the disposal of the insurer. Based on this idea, the surplus process of the life annuity business, denoted as $\{R(t) : t \geq 0\}$, can be described by the dual model of the classical risk models, namely,

$$R(t) = u - X(t), \quad t \geq 0, \tag{1.1.1}$$

where $u > 0$ is the initial capital (including the premium collected in advance) and $\{X(t) : t \geq 0\}$ defines the model dynamics for the classic risk model. Besides of the annuity business, the dual risk model may also be employed to describe the surplus process of businesses with income due to inventions and discoveries such as pharmaceutical or petroleum companies [see 7].

As a typical example of the dual risk models, it is necessary to discuss the dual model of the Cramér-Lundberg risk model in which $\{X(t) : t \geq 0\}$ is defined in the same way as that of the Cramér-Lundberg risk model but different meanings for the parameters, namely,

$$X(t) = ct - \sum_{i=0}^{N(t)} Y_i, \quad Y_0 \equiv 0, \quad t \geq 0,$$

where $c > 0$ is the annuity rate, compound Poisson process $\{N(t) : t \geq 0\}$ is the claim counting process and $\{Y_i\}_{i=1}^{\infty}$ are identically independently distributed income random variables (the reserves free of disposal). Then $\{R(t) : t \geq 0\}$ is called the surplus process of the dual model of the Cramér-Lundberg risk model. Figure 1.1 shows a sample path of $\{R(t) : t \geq 0\}$.

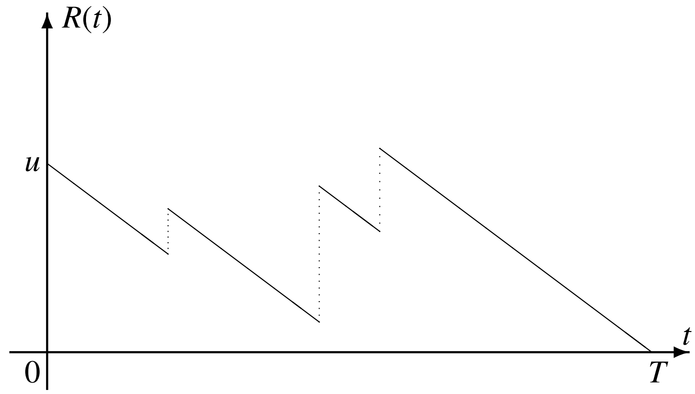

Figure 1.1: One sample path of the surplus process $R(t)$

Recent literatures related to the dual risk models focus on the optimality of certain strategies of dividend payments which was firstly proposed by de Finetti [8]. In particular, the optimality of the barrier strategy is proved in Bayraktar et al. [9] under the general dual Lévy risk models. However, in practice, the barrier strategy is not realistic due to that the probability or ultimate ruin is 1 [see 10]. Hence this thesis focuses on the threshold strategy which pays dividends at a fixed rate only when the surplus stays above some threshold.

1.2 General notation and assumptions

The research of this thesis is based on the surplus process (1.1.1) with various settings of $\{X(t) : t \geq 0\}$ defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathsf{P})$, which is a càdlàg skipfree upward stochastic process issued from 0. To avoid the trivial case that the surplus drifts to $-\infty$ with probability one, we employ the so-called *negative loading condition* or *net profit condition*, namely,

$$\mathbf{E}[X(1)] < 0, \tag{1.2.1}$$

as a basic assumption of this thesis. With the help of the condition (1.2.1) we could guarantee that the surplus process $\{R(t) : t \geq 0\}$ drifts to ∞ with probability one and thus the time to the ultimate ruin

$$T = \inf\{t \geq 0 : R(t) = 0\} \tag{1.2.2}$$

(with the convention that $\inf \varnothing = \infty$) may be equal to ∞ with a positive probability, i.e. the survival probability of $\{R(t) : t \geq 0\}$ is greater than 0. In this thesis, the most important ruin quantity is the Laplace-Stieltjes transform (L.S.T.) of the ruin time T as a function of the initial surplus $u > 0$, namely,

$$\phi_\delta(u) = \mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{T<\infty\}}\middle|R(0) = u\right], \tag{1.2.3}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and $\delta \geq 0$ is usually assumed to be a constant discount factor. In particular, the ruin probability as a function of the initial surplus $u > 0$ may be obtained straightforward as long as we have an explicit expression of $\phi_\delta(u)$.

One of other ruin-related quantities is the expected discounted aggregate dividends paid up to ruin. In this thesis, the dividends are paid according to the threshold strategy which pays dividends at a constant rate only when the surplus level is greater than or equal to a constant threshold $b > 0$. This quantity is denoted as

$$V_\delta(u; b) = \omega\mathbf{E}\left[\int_0^T e^{-\delta t}\mathbf{1}_{\{R(t)\geq b\}}\,\mathrm{d}t\middle|R(0) = u\right], \tag{1.2.4}$$

where $\omega > 0$ is the dividend rate.

Throughout this thesis we denote the Laplace-Stieltjes transform (L.S.T.) of a random variable Y as

$$\tilde{f}(s) = \int_{[0,\infty)} e^{-sy}\,\mathrm{d}F(y), \quad \mathrm{Re}(s) \geq 0,$$

whenever Y is nonnegative with the cumulative distribution function (C.D.F.) $F(y)$. In particular, if Y is a continuous nonnegative random variable with a probability density function (P.D.F.) $f(y)$, the L.S.T. of Y is denoted as

$$\tilde{f}(s) = \int_0^\infty e^{-sy} f(y)\,\mathrm{d}y, \quad \mathrm{Re}(s) \geq 0.$$

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Chapter 2

The discounted moments of the surplus after the last innovation before ruin under the dual risk model

2.1 Introduction

In this chapter, the surplus process $\{R(t) : t \geq 0\}$ is defined through letting $X(t) = ct - S(t)$ in (1.1.1), where $c > 0$ is called the expense rate and $\{S(t) : t \geq 0\}$ represents the aggregate revenue from time 0 up to time t. In insurance, this kind of models are used for the life-annuity business process where an insurance company is continuously paying out annuity instalments, while the death of a policyholder might bring the reserve of the annuity as an income [see 1]. Besides, the dual risk model might also be considered for modelling some other businesses with income due to inventions and discoveries such as pharmaceutical or petroleum companies [see 2, for more detail]. One particular case of this model in which $S(t)$ is defined as a compound Poisson process was considered in [2] in order to study the optimal-dividends problem under a barrier strategy. In that paper, the authors focus on models with jumps distributed as mixtures of exponential distributions. Also, later in [3], they work on the same topic for the dual compound Poisson risk model with diffusion. Inspired by their works, [4] study the time to ruin and the dividends' moments. Further, [5] generalizes the barrier strategy to a threshold strategy under which the expense rate level c_1 increases to c_2 once the surplus surpasses the threshold, instead of paying all dividends to the shareholders immediately after the surplus $R(t)$ reaches the barrier (the latter case may be seen as a special case with $c_2 = \infty$). Apart from the optimal-dividends problem, the dual risk model may also be utilized to solve other problems such as a tax-payment problem, which may also be interpreted as another generalization of the dual risk model with a barrier strategy [see 6]. A further generalization of this model is considered in [7] where the dual risk model with arbitrary jump distribution and innovation-time distribution is studied under a budget-restriction strategy. In the present paper, we do not generalize the model itself anymore. Instead, we are interested in a quantity of the classical dual risk model which is a generalization of the Laplace transform of the time to ruin.

The contents of this article is organized as follows: In Section 2, the compound Poisson dual risk model is formally defined. In Section 3, we provide the general solution to a certain homogeneous integro-differential equation and its relationship with the Laplace transform of the time to ruin. In Section 4, we derive the general solution to the non-homogeneous equation and then implement the results on the quantity of interest.

2.2 Notation and definitions

Let the number of innovations $\{N(t) : t \geq 0\}$ be a compound Poisson process with intensity parameter $\lambda > 0$ and the sequence of revenue random variables $\{Y_j\}_{j=1}^{\infty}$ be independent and identically distributed (i.i.d.) with C.D.F. $F(y) = 1 - \overline{F}(y)$, $y > 0$, while $F(0) = 0$, probability density function (p.d.f.) $f(y) = F'(y)$, $y \geq 0$, and Laplace transform

$$\tilde{f}(s) = \int_0^{\infty} e^{-sy} f(y)\,\mathrm{d}y, \quad s > 0.$$

Hence, the inter-innovation times $\{V_j\}_{j=1}^{\infty}$, independent of the revenue random variables $\{Y_j\}_{j=1}^{\infty}$, have an exponential distribution with mean $1/\lambda$. Also, the counting process $\{N(t) : t \geq 0\}$ and the jump sizes $\{Y_j\}_{j=1}^{\infty}$ are assumed independent. Thus, the surplus process may be written as

$$R(t) = u + ct - \sum_{j=1}^{N(t)} Y_j, \quad t \geq 0.$$

In most papers on renewal risk models, the time of ruin T is defined as $T = \inf\{t > 0 : R(t) < 0\}$ [see, for instance, 8]. However, under dual risk models, we usually define the time to ruin by (1.2.2) since ruin is caused by the continuous consumption at rate c, which means that ruin may only happen immediately after the surplus reaches 0 [5, 7]. For the same reason, it is more reasonable to consider the information of the last innovation before ruin rather than that at time T. Suppose the last innovation before ruin occurs at time τ, then the corresponding surplus is $R(\tau)$. If there is no innovation before ruin, then let $\tau = 0$. Due to the assumption that the expense rate is constant, we derive immediately a relationship between T and τ:

$$T = \tau + \frac{R(\tau)}{c}. \tag{2.2.1}$$

Our goal is to find a useful analog of the expected discounted penalty function that is introduced in [8]. We propose

$$\Phi_{w,\delta}(u) = \mathbf{E}\left[e^{-\delta\tau} w(R(\tau))\mathbf{1}_{\{T<\infty\}}\middle| R(0) = u\right], \quad u > 0, \tag{2.2.2}$$

where $w : [0, \infty) \to \mathbb{R}^+$ is the "penalty" function. The function $\Phi_{w,\delta}(u)$ is a generalization of $\phi_\delta(u)$ defined in (1.2.3), since if we let $w(x) = e^{-\frac{\delta}{c}x}$, $x \geq 0$ in (2.2.2), then by the relation (2.2.1) $\phi_\delta(u)$ coincides with $\Phi_{w,\delta}(u)$. The explicit form of the former is also obtained in previous works [see, for instance, 6, 5, 7].

Adapting to the *negative loading condition* (1.2.1), the expense rate c satisfies

$$\mathbf{E}[S(t)] - ct > 0 \tag{2.2.3}$$

[see 2], which is equivalent to $c < \lambda\mathbf{E}[Y_1]$.

In Section 3, we calculate some quantities by contour integrals in the complex plane. For this reason, we introduce the integral of $f(z)$ on a contour Γ in the counter-clockwise direction as

$$\oint_\Gamma f(z)\,\mathrm{d}z.$$

2.3 The Laplace transform of the time to ruin

The Laplace transform of the time to ruin is an important quantity in the literature related to ruin theory. From this quantity one may easily obtain the ruin probability of the surplus process by simply letting $\delta = 0$. Also, the Laplace transform of the time to ruin provides sufficient information about the defective distribution of the time to ruin. Apart from the above-mentioned reasons, another important role of this quantity is that it may be derived from a particular solution of a homogeneous integro-differential equation which we discuss later.

For initial capital $u > 0$, consider the Laplace transform of the time to ruin ϕ_δ defined by equation (1.2.2). We condition on the time (t) and the amount (y) of the first revenue and hence obtain two scenarios: either the surplus process starts all over with a new initial surplus $(u - ct + y)$, or the company is ruined ($u < ct$). In the latter case, the discount factor is $e^{-\delta u/c}$. Thus, the Total probability theorem yields

$$\phi_\delta(u) = \lambda \int_0^{u/c} \left[\int_0^\infty \phi_\delta(u - ct + y)\,\mathrm{d}F(y) \right] e^{-(\lambda+\delta)t}\,\mathrm{d}t + e^{-(\lambda+\delta)u/c}.$$

If we replace $u - ct$ by v, then

$$\phi_\delta(u) = \frac{\lambda}{c} e^{-(\lambda+\delta)u/c} \int_0^u \left[\int_0^\infty \phi_\delta(v + y)\,\mathrm{d}F(y) \right] e^{(\lambda+\delta)v/c}\,\mathrm{d}v + e^{-(\lambda+\delta)u/c}. \tag{2.3.1}$$

By differentiating both sides with respect to u, we obtain

$$\begin{aligned}\phi'_\delta(u) = &- \frac{\lambda+\delta}{c} \cdot \frac{\lambda}{c}\, e^{-(\lambda+\delta)u/c} \int_0^u \left[\int_0^\infty \phi_\delta(v + y)\,\mathrm{d}F(y) \right] e^{(\lambda+\delta)v/c}\,\mathrm{d}v \\ &+ \frac{\lambda}{c}\, e^{-(\lambda+\delta)u/c} \int_0^\infty \phi_\delta(u + y) e^{(\lambda+\delta)u/c}\,\mathrm{d}F(y) - \frac{\lambda+\delta}{c} e^{-(\lambda+\delta)u/c}.\end{aligned}$$

Since equation (2.3.1) may be presented also as

$$\frac{\lambda}{c} e^{-(\lambda+\delta)u/c} \int_0^u \left[\int_0^\infty \phi_\delta(v + y)\,\mathrm{d}F(y) \right] e^{(\lambda+\delta)v/c}\,\mathrm{d}v = \phi_\delta(u) - e^{-(\lambda+\delta)u/c},$$

we have as well

$$\begin{aligned}\frac{\lambda+\delta}{c}\, \phi_\delta(u) + \phi'_\delta(u) &= \frac{\lambda}{c} \int_0^\infty \phi_\delta(u + y)\,\mathrm{d}F(y) = \frac{\lambda}{c} \int_u^\infty \phi_\delta(y) f(y - u)\,\mathrm{d}y \\ &= \frac{\lambda}{c} \int_0^\infty \phi_\delta(y) k(u - y)\,\mathrm{d}y,\end{aligned} \tag{2.3.2}$$

where $k(z) = f(-z)\mathbf{1}_{\{z \le 0\}}$ is the convolution kernel of (2.3.2). To discuss the solution of (2.3.2), we first introduce the concept of the Hölder's condition.

Definition A function $h(x)$ satisfies the *Hölder's condition* if there exist constants $C > 0$ and $r \in (0, 1]$ such that for any real numbers x and y

$$\left|h(x) - h(y)\right| \leq C|x - y|^r$$

and for any $|x| > 1$ and $|y| > 1$

$$\left|h(x) - h(y)\right| \leq C\left|\frac{1}{x} - \frac{1}{y}\right|^r.$$

We shall denote H as the set of all the functions satisfying the Hölder's condition. Besides, by denoting the set of all functions $g(x)$ defined on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} |g(x)|^q \,\mathrm{d}x < \infty, \quad 1 \leq q \leq \infty$$

as $L_q(\mathbb{R})$ by the convention, we define another set of functions G as

$$G := \{g \in L_2(\mathbb{R}) : g_1 \in L_1(\mathbb{R})\},$$

where $g_1(x) = xg(x)$ for all $x \in \mathbb{R}$. Then it turns out that if $g(x) \in G$, then its Fourier transform, namely,

$$\hat{g}(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\imath sx} g(x)\,\mathrm{d}x, \quad s \in \mathbb{R}, \quad \imath^2 = -1$$

belongs to H. Since $f(x)$ is a density function defined on $(0, \infty)$ and

$$\int_{-\infty}^{\infty} k(z)\,\mathrm{d}z = 1, \qquad \int_{-\infty}^{\infty} |zk(z)|\,\mathrm{d}z = \mathbf{E}[Y_1],$$

$k \in G$ and hence $\hat{k} \in H$. Hence we may derive the general solution to (2.3.2) according to the guideline how to derive the general solution in the class

$$G_1 := \{g \in G : g' \in G\}$$

to an integro-differential equations of convolution type provided by [9, Section 7.2]. In particular, for the homogeneous case (2.3.2), the solution has the form $\phi_\delta(u) = \phi_\delta(0)e^{\alpha u}$, where α is a constant and $\phi_\delta(u) \leq 1$. The boundedness implies that $\alpha < 0$ and $\phi_\delta(0) \leq 1$. Moreover, replacing ϕ_δ by its solution in (2.3.2) and subsequently dividing by $\phi_\delta(0)e^{\alpha u}$, yields

$$\frac{\lambda + \delta}{c} + \alpha = \frac{\lambda}{c}\tilde{f}(-\alpha).$$

By letting $s = -\alpha$, we obtain the Lundberg's fundamental equation

$$\frac{\lambda + \delta}{c} - s = \frac{\lambda}{c}\tilde{f}(s). \tag{2.3.3}$$

Because of condition (2.2.3), this equation has a unique positive solution $\rho = \rho(\delta)$ and (for a sufficiently regular income-size distribution) a unique nonpositive solution $-r = -r(\delta)$. In particular, if $\delta = 0$, then $r(0) = 0$. Finally, it is established by the above reasoning that the solution of equation (2.3.2) has the form

$$\phi_\delta(u) = \phi_\delta(0)e^{-\rho u}, \quad u > 0.$$

If we set $u = 0$ in (2.3.1), we deduce that $\phi_\delta(0) = 1$. Therefore, the Laplace transform of the time of ruin T is

$$\phi_\delta(u) = \mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{T<\infty\}}\middle|R(0) = u\right] = e^{-\rho u}, \quad u > 0, \tag{2.3.4}$$

[see also Lemma 1 in 5]. This explicit expression is quite useful for determining the defective distribution of T.

Theorem 2.3.1 *The (defective) cumulative distribution function of the time of ruin T for the dual Poisson surplus process $\{R(t) : t \geq 0\}$ is*

$$F_T(t|u) = \begin{cases} e^{-\frac{\lambda}{c}u} + u\sum_{n=1}^{\infty}\left[\int_{(u/c,t]}\frac{\lambda^n}{n!}y^{n-1}e^{-\lambda y}f^{n*}(cy-u)\,\mathrm{d}y\right], & t \geq u/c \\ 0, & t < u/c \end{cases},$$

where f^{n*} is the n-fold convolution of the density p with itself. And the (defective) kth moment of T is

$$\mathbf{E}\left[T^k\mathbf{1}_{\{T<\infty\}}\middle|R(0) = u\right] = e^{-\frac{\lambda u}{c}}\left(\frac{u}{c}\right)^k + \sum_{n=1}^{\infty}\left(\frac{\lambda^n u}{n!}\int_{(u/c,\infty)}t^{n+k-1}e^{-\lambda t}f^{n*}(ct-u)\,\mathrm{d}t\right), \quad u > 0.$$

Proof Since for initial capital $u > 0$, ruin is impossible to happen before time u/c, then

$$\mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{T\leq t\}}\mathbf{1}_{\{T<\infty\}}|R(0) = u\right] = \mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{T\leq t\}}|R(0) = u\right] = 0 \tag{2.3.5}$$

for $t < u/c$. Also, the above reasoning indicates that

$$\mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{T\leq u/c, T<\infty\}}|R(0) = u\right] = \mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{T=u/c\}}|R(0) = u\right]\mathbf{P}(V_1 > u/c) = e^{-\frac{\lambda+\delta}{c}u}. \tag{2.3.6}$$

Hence by letting $\delta = 0$ in both (2.3.5) and (2.3.6), we obtain

$$F_T(t|u) = \begin{cases} 0 & t < u/c \\ e^{-\frac{\lambda}{c}u} & t = u/c \end{cases}. \tag{2.3.7}$$

Now, we only need to consider the case of $t > u/c$. As mentioned in [10], from the Lagrange's implicit-function theorem, for any analytic function $\eta(z)$, there exists an explicit form of $\eta(\rho)$ in terms of δ. Namely,

$$\eta(\rho) = \eta\left(\frac{\lambda+\delta}{c}\right) + \sum_{n=1}^{\infty}\left\{\left(\frac{\lambda}{c}\right)^n\frac{(-1)^n}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[\eta'(z)\tilde{f}^n(z)\right]\Big|_{z=\frac{\lambda+\delta}{c}}\right\}. \tag{2.3.8}$$

In particular, when $\eta(z) = e^{-zu}$, equation (2.3.8) reduces to

$$e^{-\rho u} = e^{-\frac{\lambda+\delta}{c}u} + \sum_{n=1}^{\infty}\left\{\left(\frac{\lambda}{c}\right)^n\frac{(-1)^n}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[-u\int_{(0,\infty)}e^{-z(u+y)}f^{n*}(y)\,\mathrm{d}y\right]\Big|_{z=\frac{\lambda+\delta}{c}}\right\}$$

$$= e^{-\frac{\lambda+\delta}{c}u} + \sum_{n=1}^{\infty}\left[\left(\frac{\lambda}{c}\right)^n \frac{u}{n!}\int_{(0,\infty)} (u+y)^{n-1} e^{-\frac{(\lambda+\delta)(u+y)}{c}} f^{n*}(y)\,\mathrm{d}y\right].$$

Now, let $t = \dfrac{u+y}{c}$. We then have

$$e^{-\rho u} = e^{-\frac{\lambda+\delta}{c}u} + \sum_{n=1}^{\infty}\left(\frac{\lambda^n u}{n!}\int_{(u/c,\infty)} t^{n-1} e^{-(\delta+\lambda)t} f^{n*}(ct-u)\,\mathrm{d}t\right). \tag{2.3.9}$$

Hence

$$\begin{aligned} e^{-\rho u} &= \mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{T<\infty\}}|R(0)=u\right] = \mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{T\le u/c\}}|R(0)=u\right] + \mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{u/c<T<\infty\}}|R(0)=u\right] \\ &= e^{-\frac{\lambda+\delta}{c}u} + \mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{u/c<T<\infty\}}|R(0)=u\right]. \end{aligned}$$

Comparing the above result with (2.3.9), we conclude

$$\mathbf{E}\left[e^{-\delta T}\mathbf{1}_{\{u/c<T<\infty\}}|R(0)=u\right] = \sum_{n=1}^{\infty}\left(\frac{\lambda^n u}{n!}\int_{(u/c,\infty)} t^{n-1} e^{-(\delta+\lambda)t} f^{n*}(ct-u)\,\mathrm{d}t\right). \tag{2.3.10}$$

If $f_T(t|u)$ is the defective conditional p.d.f. of T, then identity (2.3.10) may be presented as

$$\int_{(u/c,\infty)} e^{-\delta t} f_T(t|u)dt = \int_{(u/c,\infty)} \left\{\sum_{n=1}^{\infty}\left[\frac{\lambda^n u}{n!} t^{n-1} e^{-(\delta+\lambda)t} f^{n*}(ct-u)\right]\right\}\mathrm{d}t, \tag{2.3.11}$$

where we implemented Tonelli's theorem to exchange the order of summation and integration for integrands that are nonnegative. Equation (2.3.11) combined with (2.3.9) yields after inversion of the Laplace transforms with respect to δ

$$f_T(t|u)\mathbf{1}_{\{t>u/c\}} = \sum_{n=1}^{\infty}\left[\frac{\lambda^n u}{n!} t^{n-1} e^{-\lambda t} f^{n*}(ct-u)\right]\mathbf{1}_{\{t>u/c\}}$$

almost everywhere on $[0,\infty)$, which is equivalent to

$$f_T(t|u) = \sum_{n=1}^{\infty}\left[\frac{\lambda^n u}{n!} t^{n-1} e^{-\lambda t} f^{n*}(ct-u)\right]$$

almost everywhere on $(u/c,\infty)$. Hence for $t > u/c$, expression (2.3.7) implies

$$\begin{aligned} F_T(t|u) &= e^{-\frac{\lambda}{c}u} + \mathbf{P}\left[u/c < T \le t|R(0)=u\right] = e^{-\frac{\lambda}{c}u} + \lim_{a\downarrow u/c}\int_a^t f_T(y|u)\,\mathrm{d}y \\ &= e^{-\frac{\lambda}{c}u} + \lim_{a\downarrow u/c}\int_a^t \left\{\sum_{n=1}^{\infty}\left[\frac{\lambda^n u}{n!} y^{n-1} e^{-\lambda y} f^{n*}(cy-u)\right]\right\}\mathrm{d}y \\ &= e^{-\frac{\lambda}{c}u} + \int_{(u/c,t]} \left\{\sum_{n=1}^{\infty}\left[\frac{\lambda^n u}{n!} y^{n-1} e^{-\lambda y} f^{n*}(cy-u)\right]\right\}\mathrm{d}y. \end{aligned}$$

The last equality is due to the continuity of the infinite summation. Thus, combining with expression (2.3.7), the defective c.d.f. of T is

$$F_T(t|u) = \begin{cases} e^{-\frac{\lambda}{c}u} + u\sum_{n=1}^{\infty}\int_{(u/c,t]}\frac{\lambda^n}{n!}y^{n-1}e^{-\lambda y}f^{n*}(cy-u)\,\mathrm{d}y, & t \geq u/c \\ 0, & t < u/c \end{cases}.$$

Now, based on the defective distribution of T,

$$\begin{aligned}\mathbf{E}\left[T^k\mathbf{1}_{\{T<\infty\}}\middle|R(0)=u\right] &= \mathbf{E}\left[T^k\mathbf{1}_{\{T<\infty\}}\middle|T=u/c,R(0)=u\right]\mathbf{P}(T=u/c|R(0)=u) \\ &\quad + u\sum_{n=1}^{\infty}\left\{\lim_{a\downarrow u/c}\int_a^{\infty}\mathbf{E}\left[T^k\mathbf{1}_{\{T<\infty\}}\middle|T=t,R(0)=u\right]\frac{\lambda^n}{n!}t^{n-1}e^{-\lambda t}f^{n*}(ct-u)\,\mathrm{d}t\right\} \\ &= e^{-\frac{\lambda u}{c}}\left(\frac{u}{c}\right)^k + \sum_{n=1}^{\infty}\left(\frac{\lambda^n u}{n!}\int_{(u/c,\infty)}t^{n+k-1}e^{-\lambda t}f^{n*}(ct-u)\,\mathrm{d}t\right),\end{aligned}$$

which concludes the proof Theorem 2.3.1.

Although we have obtained a general solution for the defective kth moment of T in series form, there are still some practical difficulties to assign the exact value of the defective kth moment due to the n-fold convolution f^{n*}. Even in a very simple case, for example, when the jump size is an exponential random variable, a substantial amount of work is required for deducing the defective moments. Example 2.3 illustrates this idea. To proceed, we require an auxiliary result.

Lemma 2.3.2 *Suppose $x_1(s)$ and $x_2(s)$ ($x_1(s) > x_2(s)$) are the two (positive) roots to the quadratic equation*

$$x^2 - x + s = 0, \tag{2.3.12}$$

where $0 < s < \frac{1}{4}$. Then for $t > 0$, we have

$$\sum_{n=0}^{\infty}\left\{s^n\sum_{k=0}^{n}\left[\binom{2n-k-1}{n-k}\frac{t^k}{k!}\right]\right\} = \frac{x_1e^{x_2t}}{x_1-x_2}.$$

Proof Since $x_1(s)$ and $x_2(s)$ ($x_1(s) > x_2(s)$) are the two roots of the function $x^2 - x + s = 0$, then Vieta's formulae produce

$$\left| \begin{array}{l} x_1 + x_2 = 1 \\ x_1x_2 = s \end{array} \right. . \tag{2.3.13}$$

Since all summands are positive, by Tonelli's theorem, we may exchange the order of summation

$$\sum_{n=0}^{\infty}\left\{s^n\sum_{k=0}^{n}\left[\binom{2n-k-1}{n-k}\frac{t^k}{k!}\right]\right\} = \sum_{k=0}^{\infty}\left\{\frac{t^k}{k!}\sum_{n=k}^{\infty}\left[\binom{2n-k-1}{n-k}s^n\right]\right\} = \sum_{k=0}^{\infty}\left\{\frac{t^k}{k!}\sum_{m=0}^{\infty}\left[\binom{2m+k-1}{m}s^{m+k}\right]\right\}.$$

If $k = 0$, then Lemma 2.A.1 in Appendix 2.A implies that

$$\begin{aligned}\sum_{m=0}^{\infty}\left[\binom{2m-1}{m}s^m\right] &= 1 + \sum_{m=1}^{\infty}\left[\binom{2m-1}{m}s^m\right] = 1 + \sum_{m=1}^{\infty}\left[\frac{2^{2m-2}s^m}{\pi}\int_{-\pi}^{\pi}\cos^{2m}(\theta)\,\mathrm{d}\theta\right] \\ &= 1 + \sum_{m=1}^{\infty}\left[\frac{2^{2m-1}s^m}{\pi}\int_{0}^{\pi}\cos^{2m}(\theta)\,\mathrm{d}\theta\right] = \frac{1}{2} + \frac{1}{2\pi}\sum_{m=0}^{\infty}\left[4^m s^m\int_{0}^{\pi}\cos^{2m}(\theta)\,\mathrm{d}\theta\right] \\ &= \frac{1}{2} + \frac{1}{2\pi}\int_0^{\pi}\frac{\mathrm{d}\theta}{1-4s\cos^2(\theta)}\end{aligned}$$

by exchanging the order of summation and integration. By Vieta's formulae (2.3.13) and the trigonometric relationship $1 - 2\cos^2(\theta) = -\cos(2\theta)$,

$$1 - 4s\cos^2(\theta) = x_1^2 + x_2^2 - 2x_1x_2\cos(2\theta). \tag{2.3.14}$$

As a result, a change of variables from θ to $\theta/2$ implies

$$\frac{1}{2} + \frac{1}{2\pi}\int_0^{\pi}\frac{\mathrm{d}\theta}{1-4s\cos^2(\theta)} = \frac{1}{2} + \frac{1}{4\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta}{x_1^2 + x_2^2 - 2x_1x_2\cos(\theta)}. \tag{2.3.15}$$

Now, implementing Euler's formula, we replace the trigonometric function by the complex number $\cos(\theta) = (z + z^{-1})/2$ where $z = e^{\imath\theta} \in \mathbb{C}$, where $\imath = \sqrt{-1}$. Hence we have $\mathrm{d}z = \imath e^{\imath\theta}\,\mathrm{d}\theta = \imath z\,\mathrm{d}\theta$. As a result, the above integral becomes a contour integral in the complex plane. Then the entire expression reduces to

$$\begin{aligned}\frac{1}{2} + \frac{1}{4\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta}{x_1^2 + x_2^2 - 2x_1x_2\cos(\theta)} &= \frac{1}{2} + \frac{1}{4\pi}\oint_{|z|=1}\frac{1}{(x_1z^{-1} - x_2)(x_1z - x_2)}\cdot\frac{\mathrm{d}z}{\imath z} \\ &= \frac{1}{2} + \frac{1}{4\pi\imath}\oint_{|z|=1}\frac{\mathrm{d}z}{(x_1 - x_2z)(x_1z - x_2)}.\end{aligned}$$

Let $g(z) = (x_1 - x_2z)^{-1}$, which is a holomorphic function in $|z| \leq 1$. Then from the Cauchy's integral formula,

$$\frac{1}{2\pi\imath}\oint_{|z|=1}\frac{g(z)}{x_1z - x_2}\,\mathrm{d}z = \frac{1}{x_1}g\left(\frac{x_2}{x_1}\right) = \frac{1}{x_1}\cdot\frac{1}{x_1 - \dfrac{x_2^2}{x_1}} = \frac{1}{x_1^2 - x_2^2}. \tag{2.3.16}$$

Therefore, recalling once again Vieta's formulae (2.3.13), we deduce

$$\sum_{m=0}^{\infty}\left[\binom{2m-1}{m}s^m\right] = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{x_1^2 - x_2^2} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{x_1 - x_2} = \frac{x_1}{x_1 - x_2}.$$

Alternatively, if $k \geq 1$, following similar arguments as in the case $k = 0$, we deduce by implementing Lemma 2.A.1 and then exchanging the order of summation and integration that

$$\sum_{m=0}^{\infty}\left[\binom{2m+k-1}{m}s^{m+k}\right] = \sum_{m=0}^{\infty}\left[\frac{2^{2m+k-2}s^{m+k}}{\pi}\int_{-\pi}^{\pi}\cos((k-1)\theta)\cos^{2m+k-1}(\theta)\,\mathrm{d}\theta\right]$$

$$= \frac{(2s)^k}{2\pi} \int_0^{\pi} \frac{\cos((k-1)\theta)\cos^{k-1}(\theta)}{1 - 4s\cos^2(\theta)}\, \mathrm{d}\theta.$$

As before, we may implement identity (2.3.14) and then change variables from θ to $\theta/2$ to obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\left[\binom{2m+k-1}{m}s^{m+k}\right] &= \frac{(2s)^k}{4\pi}\int_0^{2\pi}\frac{\cos((k-1)\theta/2)\cos^{k-1}(\theta/2)}{x_1^2+x_2^2-2x_1x_2\cos(\theta)}\,\mathrm{d}\theta \\ &= \frac{s^k}{4\pi}\int_0^{2\pi}\frac{[2\cos((k-1)\theta/2)][2\cos(\theta/2)]^{k-1}}{x_1^2+x_2^2-2x_1x_2\cos(\theta)}\,\mathrm{d}\theta.\end{aligned}$$

We substitute $2\cos(\theta)$ by $z + z^{-1}$ with $z = e^{\imath\theta}$ according to Euler's formula and rewrite the denominator accordingly.

$$\begin{aligned}\sum_{m=0}^{\infty}\left[\binom{2m+k-1}{m}s^{m+k}\right] &= \frac{s^k}{4\pi\imath}\oint_{|z|=1}\frac{\left(z^{\frac{k-1}{2}}+z^{\frac{1-k}{2}}\right)\left(z^{\frac{1}{2}}+z^{-\frac{1}{2}}\right)^{k-1}}{(x_1-x_2z)(x_1z-x_2)}\,\mathrm{d}z \\ &= \frac{s^k}{4\pi\imath}\oint_{|z|=1}\frac{z^{\frac{k-1}{2}}\left(z^{\frac{1}{2}}+z^{-\frac{1}{2}}\right)^{k-1}+z^{\frac{1-k}{2}}\left(z^{\frac{1}{2}}+z^{-\frac{1}{2}}\right)^{k-1}}{(x_1-x_2z)(x_1z-x_2)}\,\mathrm{d}z \\ &= \frac{s^k}{4\pi i}\oint_{|z|=1}\frac{(z+1)^{k-1}}{(x_1-x_2z)(x_1z-x_2)}\,\mathrm{d}z + \frac{s^k}{4\pi i}\oint_{|z|=1}\frac{\left(z^{-1}+1\right)^{k-1}}{(x_1-x_2z)(x_1z-x_2)}\,\mathrm{d}z.\end{aligned}$$

Observe that whenever $|z| = 1$, then $z^{-1} = \bar{z}$. Therefore,

$$\begin{aligned}\oint_{|z|=1}\frac{\left(z^{-1}+1\right)^{k-1}}{(x_1-x_2z)(x_1z-x_2)}\,\mathrm{d}z &= \oint_{|z|=1}\frac{\left(z^{-1}+1\right)^{k-1}}{(x_1z^{-1}-x_2)(x_1-x_2z^{-1})}\,\mathrm{d}\left(-z^{-1}\right) \\ &= \oint_{|\bar{z}|=1}\frac{(\bar{z}+1)^{k-1}}{(x_1\bar{z}-x_2)(x_1-x_2\bar{z})}\,\mathrm{d}(\bar{z}).\end{aligned}$$

Hence, if we let $g(z;k) = \dfrac{(z+1)^{k-1}}{x_1 - x_2 z}$, which is a holomorphic function in $|z| \leq 1$, then the Cauchy's integral formula implies that

$$\sum_{m=0}^{\infty}\left[\binom{2m+k-1}{m}s^{m+k}\right] = \frac{s^k}{2\pi i}\oint_{|z|=1}\frac{g(z;k)}{x_1z-x_2}\,\mathrm{d}z = \frac{s^k}{x_1}g\left(\frac{x_2}{x_1};k\right) = \frac{s^k}{x_1^2-x_2^2}\left(\frac{x_2}{x_1}+1\right)^{k-1} = \frac{x_1x_2^k}{x_1-x_2}$$

after recalling Vieta's formulae (2.3.13) for x_1 and x_2. Therefore, the double summation becomes

$$\sum_{k=0}^{\infty}\left\{\frac{t^k}{k!}\sum_{m=0}^{\infty}\left[\binom{2m+k-1}{m}s^{m+k}\right]\right\} = \sum_{k=0}^{\infty}\frac{x_1(x_2t)^k}{k!(x_1-x_2)} = \frac{x_1e^{x_2t}}{x_1-x_2}$$

as needed.

Example Suppose the distribution of the revenue random variables Y_j, $j = 1, 2, \ldots,$ is exponential with mean μ. Then $f(y) = e^{-y/\mu}/\mu$ and thus $f^{n*}(y) = y^{n-1}e^{-y/\mu}/[\mu^n(n-1)!]$ is the Erlang(n) p.d.f. Hence, we may calculate the defective mean of the time to ruin. We begin by implementing Theorem 2.3.1 and then changing the variable of integration form t to $y = ct - u$

$$\begin{aligned}
\mathbf{E}[T\mathbf{1}_{\{T<\infty\}}|R(0)=u] &= \frac{u}{c}\,e^{-\frac{\lambda u}{c}} + \sum_{n=1}^{\infty}\left[\frac{\lambda^n u}{n!}\int_{(u/c,\infty)} t^n e^{-\lambda t} f^{n*}(ct-u)\,\mathrm{d}t\right] \\
&= \frac{u}{c}\,e^{-\frac{\lambda u}{c}}\left\{1+\sum_{n=1}^{\infty}\left[\frac{1}{n!}\left(\frac{\lambda}{c}\right)^n \int_{(0,\infty)} (y+u)^n e^{-\frac{\lambda}{c}y} f^{n*}(y)\,\mathrm{d}y\right]\right\} \\
&= \frac{u}{c}\,e^{-\frac{\lambda u}{c}}\left\{1+\sum_{n=1}^{\infty}\left[\left(\frac{\lambda}{c}\right)^n\left(\frac{1}{\mu}\right)^n \frac{1}{n!(n-1)!}\int_0^{\infty}(y+u)^n y^{n-1} e^{-\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)y}\,\mathrm{d}y\right]\right\} \\
&= \frac{u}{c}\,e^{-\frac{\lambda u}{c}}\left\{1+\sum_{n=1}^{\infty}\left\{\left(\frac{\lambda}{c}\right)^n\left(\frac{1}{\mu}\right)^n \frac{1}{n!(n-1)!}\left[\sum_{k=0}^{n}\binom{n}{k}u^k\int_0^{\infty} y^{2n-k-1} e^{-\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)y}\,\mathrm{d}y\right]\right\}\right\} \\
&= \frac{u}{c}\,e^{-\frac{\lambda u}{c}}\sum_{n=0}^{\infty}\left\{\left(\frac{\lambda}{c}\right)^n\left(\frac{1}{\mu}\right)^n\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-2n}\left[\sum_{k=0}^{n}\binom{2n-k-1}{n-k}\frac{u^k}{k!}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^k\right]\right\}.
\end{aligned}$$

Condition (2.2.3) is equivalent to $\lambda/c > 1/\mu$. Thus, if we let $x_1 = \lambda/\left[c\,(\lambda/c + 1/\mu)\right]$ and $x_2 = 1/\left[\mu\,(\lambda/c + 1/\mu)\right]$ in Lemma 2.3.2, then

$$\begin{aligned}
\mathbf{E}[T\mathbf{1}_{\{T<\infty\}}|R(0)=u] &= \frac{u}{c}\,e^{-\frac{\lambda u}{c}}\sum_{n=0}^{\infty}\left\{x_1^n x_2^n \sum_{k=0}^{n}\left[\binom{2n-k-1}{n-k}\frac{u^k}{k!}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^k\right]\right\} \\
&= \frac{u}{c}\,e^{-\frac{\lambda u}{c}}\cdot\frac{x_1}{x_1-x_2}\,e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u} = \frac{u}{c}\,e^{-\frac{\lambda u}{c}}\cdot\frac{(\lambda/c)e^{\frac{u}{\mu}}}{\lambda/c-1/\mu} = \frac{\lambda\mu u}{c(\lambda\mu-c)}\,e^{-\left(\frac{\lambda}{c}-\frac{1}{\mu}\right)u}.
\end{aligned}$$

Alternatively, one may obtain the defective mean of the time to ruin by differentiating the Lundberg's equation

$$\lambda + \delta - c\rho = \lambda\tilde{f}(\rho) = \frac{\lambda}{1+\mu\rho}. \tag{2.3.17}$$

Thus, when $\delta = 0$, the positive root $\rho(0) = \lambda/c - 1/\mu > 0$ by inequality (2.2.3). Now, by differentiating both sides of (2.3.17) with respect to δ, we obtain

$$1 - c\rho'(\delta) = -\frac{\lambda\mu\rho'(\delta)}{[1+\mu\rho(\delta)]^2}.$$

Hence, $\rho'(0) = \dfrac{\lambda\mu}{c(\lambda\mu - c)}$. Therefore, the defective mean of T is found by formula (2.3.4) to be

$$\mathbf{E}[T\mathbf{1}_{\{T<\infty\}}|R(0)=u] = -\frac{\partial}{\partial\delta}\phi_\delta(u)\Big|_{\delta=0} = -\frac{\partial}{\partial\delta}\,e^{-\rho(\delta)u}\Big|_{\delta=0} = u\rho'(0)e^{-\rho(0)u} = \frac{\lambda\mu u}{c(\lambda\mu-c)}\,e^{-\left(\frac{\lambda}{c}-\frac{1}{\mu}\right)u}.$$

Since the ruin probability is equal to $\phi_\delta(u)\big|_{\delta=0} = e^{-\rho(0)u} = e^{-\left(\frac{\lambda}{c}-\frac{1}{\mu}\right)u}$, the proper mean of the time to ruin T, given that $T < \infty$, is

$$\mathbf{E}[T|T < \infty, R(0) = u] = \frac{\lambda\mu u}{c(\lambda\mu - c)}.$$

Similarly, the second (defective) moment of T may be expressed by Theorem 2.3.1 with a subsequent change of variables from t to $y = ct - u$

$$\begin{aligned}
&\mathbf{E}\left[T^2\mathbf{1}_{\{T<\infty\}}|R(0) = u\right] \\
&= \left(\frac{u}{c}\right)^2 e^{-\frac{\lambda u}{c}} + \frac{u}{c^2}\, e^{-\frac{\lambda u}{c}} \sum_{n=1}^{\infty}\left[\frac{1}{n!}\left(\frac{\lambda}{c}\right)^n \int_{(0,\infty)} (y+u)^{n+1} e^{-\frac{\lambda}{c}y} f^{n*}(y)\,\mathrm{d}y\right] \\
&= \left(\frac{u}{c}\right)^2 e^{-\frac{\lambda u}{c}} + \frac{u}{c^2}\, e^{-\frac{\lambda u}{c}} \sum_{n=1}^{\infty}\left[\left(\frac{\lambda}{c}\right)^n\left(\frac{1}{\mu}\right)^n \frac{1}{n!(n-1)!}\int_0^\infty (y+u)^{n+1} y^{n-1} e^{-\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)y}\,\mathrm{d}y\right] \\
&= \left(\frac{u}{c}\right)^2 e^{-\frac{\lambda u}{c}} + \frac{u}{c^2}\, e^{-\frac{\lambda u}{c}} \sum_{n=1}^{\infty}\left\{\left(\frac{\lambda}{c}\right)^n\left(\frac{1}{\mu}\right)^n \frac{1}{n!(n-1)!}\sum_{k=0}^{n+1}\left[\binom{n+1}{k} u^k \int_0^\infty y^{2n-k} e^{-\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)y}\,\mathrm{d}y\right]\right\} \\
&= \left(\frac{u}{c}\right)^2 e^{-\frac{\lambda u}{c}} + \frac{u}{c^2}\, e^{-\frac{\lambda u}{c}} \sum_{n=1}^{\infty}\left\{(n+1)x_1^n x_2^n \sum_{k=0}^{n+1}\left[\binom{2n-k}{n+1-k}\frac{u^k}{k!}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{k-1}\right]\right\} \\
&= \frac{u}{c}\, e^{-\frac{\lambda u}{c}}\left\{\frac{u}{c} + \frac{1}{c}\sum_{n=1}^{\infty}\left\{(n+1)x_1^n x_2^n\left[\binom{2n}{n+1}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} + \sum_{k=1}^{n+1}\binom{2n-k}{n+1-k}\frac{u^k}{k!}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{k-1}\right]\right\}\right\},
\end{aligned}$$

where $x_1 = \lambda/\left[c\left(\lambda/c + 1/\mu\right)\right]$ and $x_2 = 1/\left[\mu\left(\lambda/c + 1/\mu\right)\right]$ as before. Then for $s = x_1x_2$, it is easy to verify that $x_1(s)$ and $x_2(s)(x_1(s) > x_2(s))$ are the roots to the quadratic equation (2.3.12). If we let

$$S_n(u) = \sum_{k=0}^{n}\left[\binom{2n-k-1}{n-k}\frac{u^k}{k!}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^k\right],$$

then

$$\mathbf{E}\left[T^2\mathbf{1}_{\{T<\infty\}}|R(0) = u\right] = \frac{u}{c}\, e^{-\frac{\lambda u}{c}}\left\{\frac{u}{c} + \frac{1}{c}\frac{\partial}{\partial s}\left\{\sum_{n=1}^{\infty}\left[\binom{2n}{n+1}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} + \int_0^u S_n(t)\,\mathrm{d}t\right]s^{n+1}\right\}\Bigg|_{s=x_1x_2}\right\}.$$

Observe that by convention, $\binom{0}{j} = 0$ for all $j > 0$ and that $\cos(2\theta)\cos^{2n}(\theta)$ is an even function. Hence, by Lemma 2.A.1 and after changing the order of summation and integration we deduce

$$\begin{aligned}
\sum_{n=1}^{\infty}\left[\binom{2n}{n+1}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} s^{n+1}\right] &= \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1}\sum_{n=0}^{\infty}\left[\frac{2^{2n-1}s}{\pi}\int_{-\pi}^{\pi}\cos(2\theta)\cos^{2n}(\theta)s^n\right]\mathrm{d}\theta \\
&= \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1}\sum_{n=0}^{\infty}\left[\frac{2^{2n}s}{\pi}\int_0^{\pi}\cos(2\theta)\cos^{2n}(\theta)s^n\right]\mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} \frac{s}{\pi} \int_0^{\pi} \cos(2\theta) \left\{ \sum_{n=0}^{\infty} \left[4s\cos^2(\theta)\right]^n \right\} \mathrm{d}\theta \\
&= \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} \frac{s}{\pi} \int_0^{\pi} \frac{\cos(2\theta)\,\mathrm{d}\theta}{1-4s\cos^2(\theta)} \\
&= \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} \frac{1}{2\pi} \int_0^{2\pi} \frac{x_1x_2\cos(\theta)\,\mathrm{d}\theta}{x_1^2+x_2^2-2x_1x_2\cos(\theta)},
\end{aligned}
$$

where identity (2.3.14) was implemented to deduce the last equality. Slight rearrangements, followed by application of (2.3.15) and (2.3.16) indicate that

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \frac{x_1x_2\cos(\theta)\,\mathrm{d}\theta}{x_1^2+x_2^2-2x_1x_2\cos(\theta)} &= \frac{1}{4\pi} \int_0^{2\pi} \left[\frac{x_1^2+x_2^2}{x_1^2+x_2^2-2x_1x_2\cos(\theta)} - 1\right] \mathrm{d}\theta \\
&= \frac{1}{2}\left[\frac{x_1^2+x_2^2}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta}{x_1^2+x_2^2-2x_1x_2\cos(\theta)} - 1\right] \\
&= \frac{1}{2}\left(\frac{x_1^2+x_2^2}{x_1^2-x_2^2} - 1\right) = \frac{x_2^2}{x_1^2-x_2^2} = \frac{x_2^2}{x_1-x_2}.
\end{aligned}
$$

Therefore, the infinite sum reduces to

$$
\sum_{n=1}^{\infty} \left[\binom{2n}{n+1} \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} s^{n+1}\right] = \frac{x_2^2}{x_1-x_2} \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1}.
$$

Also, by Lemma 2.3.2, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} \left\{ \left[\int_0^u S_n(t)\,\mathrm{d}t\right] s^{n+1} \right\} &= s\int_0^u \left\{ \sum_{n=1}^{\infty} [S_n(t)s^n]\,\mathrm{d}t \right\} = s\int_0^u \left[\frac{x_1}{x_1-x_2} e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)t} - 1\right] \mathrm{d}t \\
&= \frac{x_1^2}{x_1-x_2} \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u} - \frac{x_1^2}{x_1-x_2} \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} - su.
\end{aligned}
$$

Hence, the total is

$$
\sum_{n=1}^{\infty} \left[\binom{2n}{n+1} \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} + \int_0^u S_n(t)\,\mathrm{d}t\right] s^{n+1} = \frac{x_1^2}{x_1-x_2} \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u} - \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} - su.
$$

We now need to determine the derivative with respect to s of the right-hand side of this equality. Namely,

$$
\frac{\partial}{\partial s}\left[\frac{x_1^2}{x_1-x_2} e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u}\right] = \frac{\partial}{\partial x_1}\left[\frac{x_1^2}{x_1-x_2} e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u}\right]\frac{\mathrm{d}x_1}{\mathrm{d}s} + \frac{\partial}{\partial x_2}\left[\frac{x_1^2}{x_1-x_2} e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u}\right]\frac{\mathrm{d}x_2}{\mathrm{d}s}.
$$

As a root to the quadratic equation (2.3.12), x_1 satisfies $x_1^2 - x_1 + s = 0$, which implies the relation

$$
(2x_1-1)\frac{\mathrm{d}x_1}{\mathrm{d}s} + 1 = 0,
$$

or equivalently,

$$\frac{\mathrm{d}x_1}{\mathrm{d}s} = \frac{1}{x_2 - x_1}.$$

Similarly,

$$\frac{\mathrm{d}x_2}{\mathrm{d}s} = \frac{1}{x_1 - x_2}.$$

As a result,

$$\begin{aligned}
\frac{\partial}{\partial s}\left[\frac{x_1^2}{x_1 - x_2}\, e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u}\right] =& \left[-\frac{x_1^2}{(x_1-x_2)^2} + \frac{2x_1}{x_1-x_2}\right]\frac{1}{x_2-x_1}\, e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u} \\
&+ \left[\frac{x_1^2}{(x_1-x_2)^2} + \frac{x_1^2 u}{x_1-x_2}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)\right]\frac{1}{x_1-x_2}\, e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u} \\
=& \left[\frac{2x_1^2}{(x_1-x_2)^3} - \frac{2x_1}{(x_1-x_2)^2} + \frac{x_1^2 u}{(x_1-x_2)^2}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)\right] e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u} \\
=& \left\{\frac{2x_1[x_1-(x_1-x_2)]}{(x_1-x_2)^3} + \frac{x_1^2 u}{(x_1-x_2)^2}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)\right\} e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u} \\
=& \left[\frac{2x_1x_2}{(x_1-x_2)^3} + \frac{x_1^2 u}{(x_1-x_2)^2}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)\right] e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u} \\
=& \left(\frac{\lambda}{c}+\frac{1}{\mu}\right)\left[\frac{2\lambda(c\mu)^2}{(\lambda\mu-c)^2} + \frac{(\lambda\mu)^2}{(\lambda\mu-c)^2}\right] e^{\frac{u}{\mu}}
\end{aligned}$$

by the definitions of x_1 and x_2 in terms of the model parameters. Finally,

$$\begin{aligned}
\mathbf{E}\left[T^2\mathbf{1}_{\{T<\infty\}}|R(0)=u\right] &= \frac{u}{c^2}\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)^{-1} e^{-\frac{\lambda u}{c}}\frac{\partial}{\partial s}\left[\frac{x_1^2}{x_1-x_2}e^{x_2\left(\frac{\lambda}{c}+\frac{1}{\mu}\right)u}\right]\Bigg|_{s=x_1x_2} \\
&= \left[\frac{2\lambda\mu^2 u}{(\lambda\mu-c)^3} + \frac{(\lambda\mu u)^2}{c^2(\lambda\mu-c)^2}\right] e^{-\left(\frac{\lambda}{c}-\frac{1}{\mu}\right)u}.
\end{aligned}$$

Again, the alternative way of calculating the second (defective) moment is by differentiating twice with respect to δ the Laplace transform of the time to ruin (2.3.4)

$$\begin{aligned}
\mathbf{E}\left[T^2\mathbf{1}_{\{T<\infty\}}|R(0)=u\right] =& \frac{\partial^2}{\partial\delta^2}\phi_\delta(u)\Big|_{\delta=0} = \frac{\partial^2}{\partial\delta^2}\, e^{-\rho(\delta)u}\Big|_{\delta=0} \\
=& u e^{-\rho(0)u}\{u[\rho'(0)]^2 - \rho''(0)\}.
\end{aligned}$$

Differentiating Lundberg's equation (2.3.17) twice with respect to δ and then letting $\delta = 0$ yields

$$-c\rho''(0) = -\frac{\partial}{\partial\delta}\frac{\lambda\mu\rho'(\delta)}{[1+\mu\rho(\delta)]^2}\Big|_{\delta=0} = \frac{2\lambda[\mu\rho'(0)]^2}{[1+\mu\rho(0)]^3} - \frac{\lambda\mu\rho''(0)}{[1+\mu\rho(0)]^2} = \frac{2\mu c}{(\lambda\mu-c)^2} - \frac{c^2\rho''(0)}{\lambda\mu},$$

which indicates that $\rho''(0) = -\dfrac{2\lambda\mu^2}{(\lambda\mu-c)^3}$. Thus, the second moment is

$$\mathbf{E}\left[T^2\mathbf{1}_{\{T<\infty\}}|R(0)=u\right] = \left[\frac{(\lambda\mu u)^2}{c^2(\lambda\mu-c)^2} + \frac{2\lambda\mu^2 u}{(\lambda\mu-c)^3}\right] e^{-\left(\frac{\lambda}{c}-\frac{1}{\mu}\right)u}.$$

Hence, the proper second moment of the time to ruin T, given $T < \infty$, is

$$\mathbf{E}[T^2|T < \infty, R(0) = u] = \frac{(\lambda\mu u)^2}{c^2(\lambda\mu - c)^2} + \frac{2\lambda\mu^2 u}{(\lambda\mu - c)^3}.$$

Therefore, the variance of T, given $T < \infty$, is

$$\mathrm{Var}[T|T < \infty, R(0) = u] = \frac{2\lambda\mu^2 u}{(\lambda\mu - c)^3}.$$

2.4 The discounted moments of the surplus at the time of the last jump

Suppose that $k \in \mathbb{N}$ and define the defective discounted kth moment of $R(\tau)$ as

$$\Phi_\delta(u;k) = \mathbf{E}\left[e^{-\delta\tau}R(\tau)^k \mathbf{1}_{\{T<\infty\}}\middle|R(0) = u\right], \quad u > 0, \quad \delta > 0, \tag{2.4.1}$$

where τ represents the time at which the last innovation before ruin happens. In fact, equation (2.2.1) implies that $\Phi_\delta(u;k)$ is a particular case of $\Phi_{w,\delta}(u)$ with $w(x) = e^{\frac{\delta}{c}x}x^k$, $x \geq 0$. Notice that for $u = 0$, ruin happens without any positive jump almost surely. Hence $\tau = 0$ with probability 1. This leads to

$$\Phi_\delta(0;k) = \mathbf{E}\left[e^{-\delta\tau}R(\tau)^k \mathbf{1}_{\{T<\infty\}}\middle|R(0) = 0\right] = \mathbf{E}\left[e^{-\delta\tau}R(\tau)^k \mathbf{1}_{\{T<\infty\}}\middle|\tau = 0, R(0) = 0\right] = \mathbf{1}_{\{k=0\}}. \tag{2.4.2}$$

Now, we derive the explicit form of $\Phi_\delta(u;k)$.

Theorem 2.4.1 *$\Phi_\delta(u;k)$ defined by (2.4.1) satisfies the following nonhomogeneous integro-differential equation (IDE):*

$$(\lambda + \delta)\Phi_\delta(u;k) + c\frac{\mathrm{d}}{\mathrm{d}u}\Phi_\delta(u;k) = \lambda \int_0^\infty \Phi_\delta(u + y;k)\,\mathrm{d}F(y) + (\delta u + ck)u^{k-1}e^{-\frac{\lambda}{c}u} \tag{2.4.3}$$

along with boundary condition (2.4.2).

Proof Conditioning on the time (t) and the amount (y) of the first gain, there are two possible scenarios: either the surplus process starts all over with a new initial surplus ($u - ct + y$), or the company is ruined ($u < ct$). In the latter case, $\tau = 0$ and $R(\tau)^k = R(0)^k = u^k$. Thus, the Total probability theorem yields

$$\begin{aligned}
\Phi_\delta(u;k) &= \int_0^{u/c} e^{-\delta t}\left[\int_0^\infty \Phi_\delta(u - ct + y;k)\,\mathrm{d}F(y)\right]\lambda e^{-\lambda t}\,\mathrm{d}t + \int_{u/c}^\infty u^k \cdot \lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \lambda \int_0^{u/c}\left[\int_0^\infty \Phi_\delta(u - ct + y;k)\,\mathrm{d}F(y)\right]e^{-(\lambda+\delta)t}\,\mathrm{d}t + u^k e^{-\frac{\lambda}{c}u} \\
&= \frac{\lambda}{c}e^{-\frac{\lambda+\delta}{c}u}\int_0^\infty \int_0^u e^{\frac{\lambda+\delta}{c}x}\Phi_\delta(x + y;k)\,\mathrm{d}x\,\mathrm{d}F(y) + u^k e^{-\frac{\lambda}{c}u}
\end{aligned}$$

by a change of variables form t to $x = u - ct$. Applying the operator $\left(\lambda + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)$ on both sides of the above equation produces the required result.

In order to solve (2.4.3), we introduce a $(k+1)$-dimensional function space W_k^{ξ}, which is defined as

$$W_k^{\xi} := \mathrm{Span}\{e^{\xi u}, ue^{\xi u}, \ldots, u^k e^{\xi u}\}.$$

Since the differentiation operator and the integration operator with respect to the c.d.f. $P(y)$ are linear transformations on W_k^{ξ}, we may find a particular solution of (2.4.3) in the space $W_k^{-\frac{\lambda}{c}}$ by determining the inverse transformation if it exists.

Theorem 2.4.2 *If* $\delta \neq \lambda \tilde{f}\left(\frac{\lambda}{c}\right)$ *and*

$$\int_0^{\infty} y^k e^{-\frac{\lambda}{c}y}\,\mathrm{d}F(y) < \infty,$$

then equation (2.4.3) *has a particular solution* $\Phi_\delta^*(u;k) = \sum_{i=0}^{k} a_i u^{k-i} e^{-\frac{\lambda}{c}u}$, *where the constants* $\{a_i\}_{i=1}^k$ *may be obtained by the following recursive formula:*

1. $a_0 = \dfrac{\delta}{\delta - \lambda \tilde{f}\left(\frac{\lambda}{c}\right)}$;
2. $a_1 = \dfrac{k\left[c - \lambda a_0 \tilde{f}'\left(\frac{\lambda}{c}\right) - ca_0\right]}{\delta - \lambda \tilde{f}\left(\frac{\lambda}{c}\right)}$;
3. $a_i = \dfrac{1}{\delta - \lambda \tilde{f}\left(\frac{\lambda}{c}\right)}\left[\lambda \sum_{j=0}^{i-1}\binom{k-j}{i-j}(-1)^{i-j} a_j \tilde{f}^{(i-j)}\left(\frac{\lambda}{c}\right) - c(k-i+1)a_{i-1}\right]$ *for all* $2 \leq i \leq k$.

Proof We only need to verify that $\Phi^*(u;k)$ solves equation (2.4.3). On one hand, its left-hand side is

$$\begin{aligned}(\lambda+\delta)\Phi_\delta^*(u;k) + c\frac{\mathrm{d}}{\mathrm{d}u}\Phi_\delta^*(u;k) &= \left[(\lambda+\delta)\sum_{i=0}^{k} a_i u^{k-i} - \lambda \sum_{i=0}^{k} a_i u^{k-i} + c\sum_{i=1}^{k}(k-i+1)a_{i-1}u^{k-i}\right] e^{-\frac{\lambda}{c}u} \\ &= \delta \sum_{i=0}^{k} a_i u^{k-i} e^{-\frac{\lambda}{c}u} + c\sum_{i=1}^{k}(k-i+1)a_{i-1}u^{k-i}e^{-\frac{\lambda}{c}u} \\ &= \delta a_0 u^k e^{-\frac{\lambda}{c}u} + \sum_{i=1}^{k}\left[\delta a_i + c(k-i+1)a_{i-1}\right] u^{k-i} e^{-\frac{\lambda}{c}u}.\end{aligned}$$

On the other hand, the right-hand side is

$$\lambda \int_0^{\infty}\left[\sum_{i=0}^{k} a_i (u+y)^{k-i}\right] e^{-\frac{\lambda}{c}(u+y)}\,\mathrm{d}F(y) + (\delta u + ck)u^{k-1}e^{-\frac{\lambda}{c}u}$$

$$
\begin{aligned}
&=\lambda e^{-\frac{\lambda}{c}u}\int_0^\infty \sum_{i=0}^{k}\left[a_i\sum_{l=0}^{k-i}\binom{k-i}{l}u^l y^{k-i-l}\right]e^{-\frac{\lambda}{c}y}\,\mathrm{d}F(y)+(\delta u+ck)u^{k-1}e^{-\frac{\lambda}{c}u}\\
&=\lambda e^{-\frac{\lambda}{c}u}\sum_{l=0}^{k}u^l\sum_{i=0}^{k-l}\left[a_i\binom{k-i}{l}\int_0^\infty y^{k-l-i}e^{-\frac{\lambda}{c}y}\,\mathrm{d}F(y)\right]+(\delta u+ck)u^{k-1}e^{-\frac{\lambda}{c}u}\\
&=\lambda e^{-\frac{\lambda}{c}u}\sum_{l=0}^{k}u^l\sum_{i=0}^{k-l}\left[(-1)^{k-l-i}\binom{k-i}{l}a_i\tilde{f}^{(k-l-i)}\left(\frac{\lambda}{c}\right)\right]+(\delta u+ck)u^{k-1}e^{-\frac{\lambda}{c}u}\\
&=\lambda e^{-\frac{\lambda}{c}u}\sum_{j=0}^{k}u^{k-j}\left[\sum_{i=0}^{j}\binom{k-i}{k-j}(-1)^{j-i}a_i\tilde{f}^{(j-i)}\left(\frac{\lambda}{c}\right)\right]+(\delta u+ck)u^{k-1}e^{-\frac{\lambda}{c}u}\\
&=\lambda e^{-\frac{\lambda}{c}u}\sum_{j=0}^{k}u^{k-j}\left[\sum_{i=0}^{j}\binom{k-i}{j-i}(-1)^{j-i}a_i\tilde{f}^{(j-i)}\left(\frac{\lambda}{c}\right)\right]+(\delta u+ck)u^{k-1}e^{-\frac{\lambda}{c}u} \qquad (2.4.4)
\end{aligned}
$$

by exchanging the order of summation and by some subsequent changes of variables. Notice that from the recursive formula in the case $2 \leq i \leq k$, we obtain

$$
\lambda\sum_{j=0}^{i}\binom{k-j}{i-j}(-1)^{i-j}a_j\tilde{f}^{(i-j)}\left(\frac{\lambda}{c}\right)=\delta a_i+c(k-i+1)a_{i-1}.
$$

Hence, expression (2.4.4) may be rewritten as

$$
\begin{aligned}
&\lambda a_0\tilde{f}\left(\frac{\lambda}{c}\right)u^k e^{-\frac{\lambda}{c}u}+\lambda\left[a_1\tilde{f}\left(\frac{\lambda}{c}\right)-ka_0\tilde{f}'\left(\frac{\lambda}{c}\right)\right]u^{k-1}e^{-\frac{\lambda}{c}u}+\sum_{j=2}^{k}\left[\delta a_j+c(k-j+1)a_{j-1}\right]u^{k-j}e^{-\frac{\lambda}{c}u}\\
&\quad+(\delta u+ck)u^{k-1}e^{-\frac{\lambda}{c}u}\\
&=\left[\lambda a_0\tilde{f}\left(\frac{\lambda}{c}\right)+\delta\right]u^k e^{-\frac{\lambda}{c}u}+\left[\lambda a_1\tilde{f}\left(\frac{\lambda}{c}\right)-\lambda ka_0\tilde{f}'\left(\frac{\lambda}{c}\right)+ck\right]u^{k-1}e^{-\frac{\lambda}{c}u}\\
&\quad+\sum_{j=2}^{k}\left[\delta a_j+c(k-j+1)a_{j-1}\right]u^{k-j}e^{-\frac{\lambda}{c}u}. \qquad (2.4.5)
\end{aligned}
$$

By the definitions of a_0 and a_1 we have

$$
\begin{aligned}
\lambda a_0\tilde{f}\left(\frac{\lambda}{c}\right)+\delta &=\delta a_0\\
\lambda a_1\tilde{f}\left(\frac{\lambda}{c}\right)-\lambda ka_0\tilde{f}\left(\frac{\lambda}{c}\right)+ck &=\delta a_1+cka_0.
\end{aligned}
$$

Thus, expression (2.4.5) reduces to

$$
\begin{aligned}
&\delta a_0u^ke^{-\frac{\lambda}{c}u}+(\delta a_1+cka_0)\,u^{k-1}e^{-\frac{\lambda}{c}u}+\sum_{j=2}^{k}\left[\delta a_j+c(k-j+1)a_{j-1}\right]u^{k-j}e^{-\frac{\lambda}{c}u}\\
&=\delta a_0u^ke^{-\frac{\lambda}{c}u}+\sum_{j=1}^{k}\left[\delta a_j+c(k-j+1)a_{j-1}\right]u^{k-j}e^{-\frac{\lambda}{c}u},
\end{aligned}
$$

which is exactly the left-hand side of (2.4.3) as demonstrated above.

In the particular case $k = 0$, we recover the Laplace transform of the time of the last revenue before ruin.

Corollary 2.4.3 *Under the compound Poisson dual risk model, if* $\delta \neq \lambda \tilde{f}\left(\frac{\lambda}{c}\right)$, *then*

$$\mathbf{E}\left[e^{-\delta\tau}\mathbf{1}_{\{T<\infty\}}|R(0)=u\right] = \frac{\delta e^{-\frac{\lambda}{c}u} - \lambda \tilde{f}\left(\frac{\lambda}{c}\right) e^{-\rho u}}{\delta - \lambda \tilde{f}\left(\frac{\lambda}{c}\right)}, \quad u > 0. \tag{2.4.6}$$

Proof Since

$$\mathbf{E}\left[e^{-\delta\tau}\mathbf{1}_{\{T<\infty\}}|R(0)=u\right] = \Phi_\delta(u;0),$$

Theorem 2.4.1 implies that $\mathbf{E}\left[e^{-\delta\tau}\mathbf{1}_{\{T<\infty\}}|R(0)=u\right]$ satisfies (2.4.3) with $k = 0$. Thus, Theorem 2.4.2 provides a particular solution of (2.4.3) with $k = 0$, which is

$$\Phi^*_\delta(u;0) = \frac{\delta e^{-\frac{\lambda}{c}u}}{\delta - \lambda \tilde{f}\left(\frac{\lambda}{c}\right)}.$$

Hence,

$$\Phi^*_\delta(0;0) = \frac{\delta}{\delta - \lambda \tilde{f}\left(\frac{\lambda}{c}\right)} \neq 1.$$

Thus, we need to adjust $\Phi^*(u;0)$ by involving the general solution of the corresponding homogeneous integro-differential equation. Notice that $0 \leq \mathbf{E}\left[e^{-\delta\tau}\mathbf{1}_{\{T<\infty\}}|R(0)=u\right] \leq 1$ for all $u > 0$, we need to exclude one of the particular solutions of the corresponding homogeneous integro-differential equation, namely, e^{ru}. As a result,

$$\mathbf{E}\left[e^{-\delta\tau}\mathbf{1}_{\{T<\infty\}}|R(0)=u\right] = \Phi^*_\delta(u;0) + ae^{-\rho u} = \frac{\delta e^{-\frac{\lambda}{c}u}}{\delta - \lambda \tilde{f}\left(\frac{\lambda}{c}\right)} + ae^{-\rho u}.$$

where a is a constant that needs to be specified. To do so, we recall the initial condition (2.4.2), which implies that $\Phi_\delta(0;0) = 1$. Thus,

$$a = -\frac{\lambda \tilde{f}\left(\frac{\lambda}{c}\right)}{\delta - \lambda \tilde{f}\left(\frac{\lambda}{c}\right)}.$$

and the desired equation (2.4.6) is recovered.

For $k > 0$, let $\Phi^*(u;k)$ be the particular solution for (2.4.3) obtained by Theorem 2.4.2. Then $\Phi^*(0;k) = a_k \neq 0$, which means that we should adjust $\Phi^*(u;k)$ by involving the general solution of the corresponding homogeneous integro-differential equation. By condition (2.2.1), we have

$$R(\tau) \leq cT.$$

Equality holds if and only if $V_1 > u/c$. Thus,

$$\Phi_\delta(u;k) = \mathbf{E}\left[e^{-\delta\tau}R(\tau)^k \mathbf{1}_{\{T<\infty\}}|R(0) = u\right] \le c^k\,\mathbf{E}\left[T^k \mathbf{1}_{\{T<\infty\}}|R(0) = u\right].$$

Notice that identity (2.3.4) implies that

$$\mathbf{E}\left[T^k \mathbf{1}_{\{T<\infty\}}|R(0) = u\right] = \frac{\partial^k}{\partial\delta^k}\, e^{-\rho(\delta)u}\bigg|_{\delta=0} = O\left(u^k e^{-\rho(0)u}\right),$$

we have

$$\lim_{u\to\infty} \mathbf{E}\left[e^{-\delta\tau}R(\tau)^k \mathbf{1}_{\{T<\infty\}}|R(0) = u\right] = 0.$$

Thus, we need to exclude the particular solution e^{ru} to the homogeneous integro-differential equation in the expression of $\Phi_\delta(u;k)$. Therefore, the boundary condition (2.4.2) leads to

$$\Phi_\delta(u;k) = \Phi^*_\delta(u;k) - a_k e^{-\rho u}, \quad u > 0. \tag{2.4.7}$$

Example Suppose that the revenue random variable has exponential distribution with mean μ so that $\tilde{f}(s) = 1/(1+\mu s)$. Then,

$$a_0 = \frac{\delta}{\delta - \frac{\lambda}{1+\frac{\lambda\mu}{c}}} = \frac{\delta(1+\frac{\lambda\mu}{c})}{\delta(1+\frac{\lambda\mu}{c}) - \lambda} = \frac{\delta c + \delta\lambda\mu}{\delta c + \delta\lambda\mu - c\lambda}.$$

To calculate the discounted mean of the surplus at the last jump before ruin, we also require the constant

$$a_1 = \frac{c + \frac{a_0\lambda\mu}{(1+\frac{\lambda\mu}{c})^2} - ca_0}{\delta - \frac{\lambda}{1+\mu\frac{\lambda}{c}}} = \frac{c^2\lambda\mu\delta - c^2\lambda(c+\lambda\mu)}{(\delta c + \delta\lambda\mu - c\lambda)^2}.$$

Thus,

$$\mathbf{E}\left[e^{-\delta\tau}R(\tau)\mathbf{1}_{\{T<\infty\}}|R(0) = u\right] = \frac{\delta c + \delta\lambda\mu}{\delta c + \delta\lambda\mu - c\lambda} u e^{-\frac{\lambda}{c}u} + \frac{c^2\delta\lambda\mu - c^2\lambda(c+\lambda\mu)}{(\delta c + \delta\lambda\mu - c\lambda)^2}\left(e^{-\frac{\lambda}{c}u} - e^{-\rho u}\right).$$

Appendix

2.A The trigonometric integral representation of binomial coefficients

Lemma 2.A.1 *Consider any $l, j \in \mathbb{N}$, then the binomial coefficient $\binom{j}{l}$ is equal to a trigonometric integral*

$$\binom{j}{l} = \frac{2^{j-1}}{\pi}\int_{-\pi}^{\pi} \cos\left((2l-j)\theta\right)(\cos(\theta))^j \,\mathrm{d}\theta. \tag{2.A.1}$$

Proof Since by the Euler's formula and the binomial expansion we have

$$(\cos(\theta))^j = \left[\frac{e^{\imath\theta}+e^{-\imath\theta}}{2}\right]^j = 2^{-j}\sum_{l=0}^{j}\binom{j}{l}e^{\imath\theta l}\cdot e^{-\imath\theta(j-l)} = 2^{-j}\sum_{l=0}^{j}\binom{j}{l}e^{\imath\theta(2l-j)},$$

we have

$$(\cos(\theta))^j = 2^{1-j}\sum_{l=0}^{j}\binom{j}{l}\cos\left((2l-j)\theta\right). \tag{2.A.2}$$

Notice that for the Fourier expansion of $(\cos(\theta))^j$ we have

$$(\cos(\theta))^j = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n\cos(n\theta) + \sum_{n=1}^{\infty}b_n\sin(n\theta), \tag{2.A.3}$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos(\theta))^j\,\mathrm{d}\theta \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos(\theta))^j\cos(n\theta)\,\mathrm{d}\theta, \quad n=1,2,\ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos(\theta))^j\sin(n\theta)\,\mathrm{d}\theta, \quad n=1,2,\ldots \end{aligned}$$

Therefore, comparing the coefficient of the term $\cos\left((2l-j)\theta\right)$ in (2.A.2) and (2.A.3) yields (2.A.1).

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Chapter 3

The ruin time under the Sparre-Andersen dual model

3.1 Introduction

The dual ruin model is defined through letting

$$X(t) = ct - S(t)$$

in (1.1.1), where $c > 0$ is the constant expense rate and $\{S(t) : t \geq 0\}$ is the aggregate revenue from time 0 up to time t. This kind of models is widely used in modelling the surplus processes of companies with continuous expense but occasional income due to contingent events [see 1, 2, 3]. One particular case of this model is the compound Poisson dual risk model, which is studied thoroughly in many other papers, including the dividend payment problem with barrier [see 1] or threshold strategy [see 2] and the tax payment problem [see 4]. Besides, Landriault and Sendova [3] generalize the Sparre Andersen dual risk model with Erlang-n inter-innovation times by adding a budget-restriction strategy. Recently, in Rodríguez et al. [5], an explicit form of the Laplace transform of the ruin time under the Erlang-n dual risk model is provided. In this paper, we are mainly interested in the explicit form of the Laplace transform of the time to ruin under the Sparre-Andersen dual model with generalized Erlang-n inter-innovation times. As shown in Ji and Zhang [6], under the Erlang-n dual risk model, the roots to the Lundberg's equation are distinct. However, under the generalized Erlang-n dual risk model, this is not the case any longer (see Example 3.5). Instead, the multiplicity of the roots should be considered when we derive an explicit form of the Laplace transform of the ruin time.

The contents of this article are organized as follows: Section 2 introduces the notation and model settings. In Section 3, we derive a homogeneous integro-differential equation for an auxiliary quantity related to the Laplace transform of the ruin time. In Section 4, we discuss the number of roots of Lundberg's equation with positive real part in order to find the general solution of the integro-differential equation deduced in Section 3. Section 5 provides the explicit expression of the Laplace transform of the time to ruin. In Section 6, we apply similar arguments for analyzing the threshold-dividend-strategy problem and obtain the explicit form of the expected discounted dividends under the dual risk model with exponential jumps.

3.2 Notation and model settings

Let the independently and identically distributed (i.i.d.) positive random variables $\{Y_1, Y_2, \ldots\}$ represent the amounts of the occasional revenue and denote their common cumulative distribution function (c.d.f.) by $P(y)$, $y \geq 0$, with $P(0) = 0$, their probability density function (p.d.f.) by $p(y) = P'(y)$, $y \geq 0$, and their Laplace transform by $\tilde{p}(s) = \int_0^\infty e^{-sy} dP(y)$, $s \geq 0$. The renewal reward process $\{S(t) : t \geq 0\}$ with i.i.d. inter-event times $\{V_i\}_{i=1}^\infty$ is constructed as

$$S(t) = \sum_{i=1}^{N(t)} Y_i,$$

where $N(t) = \max\{k \in \mathbb{N} : V_1 + V_2 + \ldots + V_k \leq t\}$ is the number of gains from time 0 up to time t. By convention, $S(t) = 0$ whenever $N(t) = 0$. In this paper, we assume that the inter-event times V_j, $j = 1, 2, \ldots$, (we may also call them inter-innovation times) have a generalized Erlang-n distribution with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$, i.e. V_1, in particular, may be expressed as

$$V_1 \stackrel{d}{=} \sum_{j=1}^{n} W_j,$$

where W_j is an exponential random variable with mean $1/\lambda_j$, $j = 1, 2, \ldots, n$. Hence, if we denote the probability distribution function of V_1 by $f(t)$, $t \geq 0$, then the corresponding Laplace transform of $f(t)$ has the form

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt = \prod_{j=1}^{n} \frac{\lambda_j}{\lambda_j + s}, \quad \mathrm{Re}(s) \geq 0. \tag{3.2.1}$$

Futhermore, if we define by $f_c(t)$ the p.d.f of the random variable cV_1, then $f_c(t) = \frac{1}{c} f(t/c)$ and hence by the change of scale property of the Laplace transform, we have

$$\tilde{f}_c(s) = \tilde{f}(cs) = \prod_{j=1}^{n} \frac{\lambda_j}{\lambda_j + cs}, \quad \mathrm{Re}(s) \geq 0. \tag{3.2.2}$$

Now define auxiliary function

$$g_c(t) = e^{-\delta t/c} f_c(t) \tag{3.2.3}$$

then by the first translation property of the Laplace transform

$$\tilde{g}_c(s) = \tilde{f}_c\left(s + \frac{\delta}{c}\right) = \prod_{j=1}^{n} \frac{\lambda_j}{\lambda_j + cs + \delta}, \quad \mathrm{Re}(s) \geq 0. \tag{3.2.4}$$

Since the dual model describes the surplus process of some kind of business which we do not want to bancrupt with probability 1, adapting to the *net-profit condition* (1.2.1), the expense rate c satisfies,

$$c\mathbf{E}[V_1] < \mathbf{E}[Y_1]$$

as one of the basic assumptions for our model. The *net-profit condition* is one of the basic assumptions in many articles related to the dual model such as Avanzi et al. [1] and Landriault and Sendova [3]. Furthermore, in Section 4, the *net-profit condition* plays an important role in determining the number of roots with positive real part to the generalized Lundberg's equation when there is a simple root on the boundary.

Since the expectation of V_1 is

$$\mathbf{E}[V_1] = \sum_{j=1}^{n} \mathbf{E}[W_j] = \sum_{j=1}^{n} \frac{1}{\lambda_j},$$

if we denote by $\mu = \mathbf{E}[Y_1]$, the *net-profit condition* becomes

$$\sum_{j=1}^{n} \frac{1}{\lambda_j} < \frac{\mu}{c}. \tag{3.2.5}$$

Now define the ruin time $T := \inf\{t \geq 0 : R(t) = 0\}$ and the ruin probability with given initial capital u

$$\varphi_0(u) = \mathbf{E}[\mathbf{1}_{\{T<\infty\}} | R(0) = u], \quad u > 0,$$

where $\mathbf{1}_{\{E\}}$ is the indicator function of an event E. Then

$$\varphi_0(u) < 1$$

for all $u > 0$ only if the *net-profit condition* (3.2.5) holds. More generally, the Laplace transform of the ruin time, given initial capital u, is defined as

$$\varphi_\delta(u) = \mathbf{E}\left[e^{-\delta T} \mathbf{1}_{\{T<\infty\}} | R(0) = u\right], \quad u > 0.$$

Our goal is to find an explicit form of $\varphi_\delta(u)$ by solving an integro-differential equation.

In addition, we introduce the Fourier transform of $\varphi_\delta(u)$

$$\hat{\varphi}_\delta(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_\delta(u) e^{-\imath \xi u} \, \mathrm{d}u, \quad \xi \in \mathbb{R},$$

where $\imath = \sqrt{-1}$.

3.3 An integro-differential equation

In most literature related to ruin theory, the Laplace transform of the ruin time satisfies some integro-differential equation derived by conditioning on the amount and the time of the first innovation. We apply this approach here too. Namely,

$$\varphi_\delta(u) = \int_0^{u/c} e^{-\delta t} \left[\int_0^{\infty} \varphi_\delta(u - ct + y) \, \mathrm{d}P(y) \right] f(t) \, \mathrm{d}t + \int_{u/c}^{\infty} e^{-\delta \frac{u}{c}} f(t) \, \mathrm{d}t.$$

With $v = u - ct$ the above equation becomes

$$\begin{aligned}\varphi_\delta(u) =& \frac{1}{c}\int_0^u e^{-\delta\frac{u-v}{c}}\left[\int_0^\infty \varphi_\delta(v+y)\,\mathrm{d}P(y)\right] f\left(\frac{u-v}{c}\right)\mathrm{d}v + e^{-\delta\frac{u}{c}}\overline{F}\left(\frac{u}{c}\right)\\ =& \int_0^u\left[\int_0^\infty \varphi_\delta(v+y)\,\mathrm{d}P(y)\right] g_c(u-v)\,\mathrm{d}v + e^{-\delta\frac{u}{c}}\overline{F}\left(\frac{u}{c}\right),\end{aligned}$$

where $\overline{F}(t)$ is the tail distribution of the density function $f(t)$. Hence, $\varphi_\delta(u)$ satisfies a convolution-type integro-differential equation of the form

$$\zeta(u) = \int_0^u \mathcal{I}\left[\zeta\right](v) g_c(u-v)\,\mathrm{d}v + G(u). \tag{3.3.1}$$

with $G(t) = e^{-\delta t/c}\overline{F}(t/c)$ and operator $\mathcal{I} : C_{(0,\infty)} \mapsto C_{(0,\infty)}$ defined as

$$\mathcal{I}\left[\zeta\right](u) = \int_0^\infty \zeta(u+y)\,\mathrm{d}P(y).$$

For integro-differential equation (3.3.1), we have the following theorem for a particular class of functions $G(u)$.

Theorem 3.3.1 *For a function*

$$G(u) \in \left\{ h \in C^n_{(0,\infty)} : \left[\prod_{j=1}^n \left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right] h(u) = 0\right\}, \tag{3.3.2}$$

if $\zeta(u)$ satisfies (3.3.1), then $\zeta(u)$ also satisfies the homogeneous integro-differential equation

$$\left[\prod_{j=1}^n \left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\zeta(u) = \left(\prod_{j=1}^n \lambda_j\right)\mathcal{I}\left[\zeta\right](u), \quad u > 0, \tag{3.3.3}$$

with boundary conditions $\zeta^{(i)}(0) = G^{(i)}(0)$ for $i = 0, 1, \ldots, n-1$.

Proof Since equation (3.3.1) is of the convolution type, if we denote the Laplace transform of $\mathcal{I}\left[\zeta\right](u)$ by $\widetilde{\mathcal{I}}[\zeta](s)$, then we have

$$\tilde{\zeta}(s) = \widetilde{\mathcal{I}}[\zeta](s)\tilde{g}_c(s) + \widetilde{G}(s),$$

which may be rewritten as

$$\left[\prod_{i=1}^n (\lambda_i + \delta + cs)\right]\left[\tilde{\zeta}(s) - \widetilde{G}(s)\right] = \left(\prod_{i=1}^n \lambda_i\right)\widetilde{\mathcal{I}}[\zeta](s)$$

implementing identity (3.2.4). For convenience, let $\psi(u) = \zeta(u) - G(u)$. Then the above equation reduces to

$$\left[\prod_{i=1}^n (\lambda_i + \delta + cs)\right]\tilde{\psi}(s) = \left(\prod_{i=1}^n \lambda_i\right)\widetilde{\mathcal{I}}[\zeta](s). \tag{3.3.4}$$

Now, define the elementary symmetric functions for $\lambda_1+\delta, \lambda_2+\delta, \ldots, \lambda_n+\delta$, namely,

$$\begin{cases}\sigma_0 & \equiv 1\\ \sigma_1 & = (\lambda_1+\delta)+(\lambda_2+\delta)+\ldots+(\lambda_n+\delta)\\ \sigma_2 & = (\lambda_1+\delta)(\lambda_2+\delta)+(\lambda_1+\delta)(\lambda_3+\delta)+\ldots\\ & \quad +(\lambda_{n-1}+\delta)(\lambda_n+\delta)\\ \vdots & \\ \sigma_n & = \prod_{i=1}^{n}(\lambda_i+\delta)\end{cases} \tag{3.3.5}$$

[also mentioned at the end of Section 6 in 7]. Then (3.3.4) may be expanded as

$$\sum_{i=0}^{n}\sigma_i(cs)^i\tilde{\psi}(s)=\left(\prod_{i=1}^{n}\lambda_i\right)\widetilde{\mathcal{I}}[\zeta](s), \tag{3.3.6}$$

and thus the Laplace transform of $\left[\prod_{j=1}^{n}\left(\lambda_j+\delta+c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\psi(u)$ is

$$\begin{aligned}\int_0^\infty e^{-su}\left[\prod_{j=1}^{n}\left(\lambda_j+\delta+c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\psi(u)\,\mathrm{d}u &= \int_0^\infty e^{-su}\left[\sum_{i=0}^{n}\sigma_i c^i\psi^{(i)}(u)\right]\mathrm{d}u\\ =\sum_{i=0}^{n}\sigma_i c^i\int_0^\infty e^{-su}\psi^{(i)}(u)\,\mathrm{d}u &\\ =\sum_{i=0}^{n}\sigma_i(cs)^i\tilde{\psi}(s)-\sum_{i=2}^{n}\left[\sigma_i c^i\left(\sum_{j=1}^{i-1}s^j\psi^{(i-1-j)}(0)\right)\right]-\sum_{i=1}^{n}\sigma_i c^i\psi^{(i-1)}(0) &\\ =\left(\prod_{i=1}^{n}\lambda_i\right)\widetilde{\mathcal{I}}[\zeta](s)-\sum_{j=1}^{n-1}\left[s^j\sum_{i=j+1}^{n}\sigma_i c^i\psi^{(i-1-j)}(0)\right]-\sum_{i=1}^{n}\sigma_i c^i\psi^{(i-1)}(0) &\end{aligned} \tag{3.3.7}$$

due to the relationship

$$\int_0^\infty e^{-sy}\psi^{(i)}(u)\,\mathrm{d}u = s^i\tilde{\psi}(s)-s^{i-1}\psi(0)-\ldots-s\psi^{(i-2)}(0)-\psi^{(i-1)}(0)=s^i\tilde{\psi}(s)-\sum_{j=0}^{i-1}s^j\tilde{\psi}^{(i-1-j)}(s)$$

for $i=1,2,\ldots,n-1$. Therefore, by letting $s\to\infty$ we deduce

$$0=-\lim_{s\to\infty}\sum_{j=1}^{n-1}\left[s^j\sum_{i=j+1}^{n}\sigma_i c^i\psi^{(i-1-j)}(0)\right]-\sum_{i=1}^{n}\sigma_i c^i\psi^{(i-1)}(0), \tag{3.3.8}$$

since the left-hand side of (3.3.7) is a Laplace transform. Now, we obtain that the limit of a polynomial with respect to s is 0 when $s\to\infty$, which implies that all the coefficients of this polynomial are 0. More precisely, if we denote

$$a_j = -\sum_{i=j+1}^{n} \sigma_i c^i \psi^{i-1-j}(0), \quad j = 0, 1, \ldots, n-1,$$

then equation (3.3.8) may be written in terms of $\{a_j\}_{j=0}^{n-1}$ as

$$\lim_{s\to\infty} \left(a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0\right) = 0.$$

Thus,

$$a_{n-1} = -\lim_{s\to\infty} \frac{a_{n-2}s^{n-2} + a_{n-3}s^{n-3} + \ldots + a_1 s + a_0}{s^{n-1}} = 0$$

and for the same reason,

$$a_{n-2} = a_{n-3} = \ldots = a_1 = a_0 = 0.$$

Consequently, we obtain a homogeneous linear-equation system satisfied by $\psi(0), \psi'(0), \ldots, \psi^{(n-1)}(0)$

$$\mathbf{\Sigma}_c\vec{\psi} = \begin{bmatrix} \sigma_n c^n & 0 & 0 & \ldots & 0 \\ \sigma_{n-1}c^{n-1} & \sigma_n c^n & 0 & \ldots & 0 \\ \sigma_{n-2}c^{n-2} & \sigma_{n-1}c^{n-1} & \sigma_n c^n & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_1 c & \sigma_2 c^2 & \sigma_3 c^3 & \ldots & \sigma_n c^n \end{bmatrix} \begin{bmatrix} \psi(0) \\ \psi'(0) \\ \psi''(0) \\ \vdots \\ \psi^{(n-1)}(0) \end{bmatrix} = \vec{0}. \tag{3.3.9}$$

Notice that in (3.3.9) the matrix $\mathbf{\Sigma}_c$ is such that $\det(\mathbf{\Sigma}_c) = (\sigma_n c^n)^n \neq 0$. Hence, the homogeneous linear equation system (3.3.9) has unique solution, which is the trivial solution

$$\vec{\psi} = \vec{0}. \tag{3.3.10}$$

Therefore, by (3.3.7) we deduce

$$\int_0^\infty e^{-su}\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\psi(u)\,\mathrm{d}u = \left(\prod_{i=1}^{n}\lambda_i\right)\widetilde{\mathcal{I}}[\zeta](s).$$

By (3.3.2) we may simplify the latter equation to

$$\int_0^\infty e^{-su}\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\zeta(u)\,\mathrm{d}u = \left(\prod_{i=1}^{n}\lambda_i\right)\widetilde{\mathcal{I}}[\zeta](s).$$

Therefore, inversion of the Laplace transforms yields the required result (3.3.3). Also, from (3.3.10) we may conclude that $\zeta^{(i)}(0) = G^{(i)}(0)$ for all $i = 0, 1, \ldots, n-1$.

In particular, we have the following corollary for $\varphi_\delta(u)$.

Corollary 3.3.2 *The Laplace transform of the time to ruin $\varphi_\delta(u)$ satisfies*

$$\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\varphi_\delta(u) = \left(\prod_{i=1}^{n}\lambda_i\right)\int_0^\infty \varphi_\delta(u+y)\,\mathrm{d}P(y) \tag{3.3.11}$$

with boundary conditions $\varphi_\delta^{(i)}(0) = \left(-\frac{\delta}{c}\right)^i$ *for* $i = 0, 1, \ldots, n-1$.

Proof Since $\varphi_\delta(u)$ satisfies (3.3.1) with $G(u) = e^{-\delta u/c}\overline{F}\left(\frac{u}{c}\right)$ and

$$\mathcal{I}\left[\varphi_\delta\right](u) = \int_0^\infty \varphi_\delta(u+y)\,\mathrm{d}P(y),$$

we only need to verify that

$$\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\left[e^{-\delta u/c}\overline{F}\left(\frac{u}{c}\right)\right] = 0 \tag{3.3.12}$$

and that

$$\frac{\partial^i}{\partial u^i}\left[e^{-\delta u/c}\overline{F}\left(\frac{u}{c}\right)\right]\bigg|_{u=0} = \left(-\frac{\delta}{c}\right)^i.$$

Since by definition

$$e^{-\delta u/c}\overline{F}\left(\frac{u}{c}\right) = e^{-\delta u/c}\int_{u/c}^\infty f(s)\,\mathrm{d}s = e^{-\delta u/c}\int_u^\infty f_c(s)\,\mathrm{d}s = \int_0^\infty e^{\delta s/c} g_c(u+s)\,\mathrm{d}s,$$

by Lemma 3.A.2

$$\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\left[e^{-\delta u/c}\overline{F}\left(\frac{u}{c}\right)\right] = \int_0^\infty \left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c\frac{\partial}{\partial u}\right)\right] g(u+s)e^{\delta s}\,\mathrm{d}s = 0,$$

which confirms that (3.3.12) holds. Thus, $\varphi_\delta(u)$ satisfies (3.3.3) due to Theorem 3.3.1. Finally, by the Leibniz's rule,

$$\frac{\partial^i}{\partial u^i}\left[e^{-\delta u/c}\overline{F}\left(\frac{u}{c}\right)\right]\bigg|_{u=0} = \frac{1}{c^i}\left[(-\delta)^i\overline{F}\left(\frac{u}{c}\right) - \sum_{k=0}^{i-1}\binom{i}{k}(-\delta)^k e^{-\delta u/c} f^{(i-k-1)}\left(\frac{u}{c}\right)\right]\bigg|_{u=0} = \left(-\frac{\delta}{c}\right)^i \tag{3.3.13}$$

as $\overline{F}(0) = 1$ and Lemma 3.A.1 implies that $f^{(j)}(0) = g_1^{(j)}(0)|_{\delta=0} = 0,\ j = 0, 1, \ldots, n-2$, when $n = 2, 3, \ldots$ When $n = 1$,

$$\varphi_\delta(0) = e^{-\delta u/c}\overline{F}\left(\frac{u}{c}\right)\bigg|_{u=0} = 1 = \left(-\frac{\delta}{c}\right)^0$$

as needed.

Lastly, the implied boundary conditions are $\varphi_\delta^{(i)}(0) = \left(-\frac{\delta}{c}\right)^i$ for $i = 0, 1, \ldots, n-1$ and $n = 1, 2, \ldots$

Now, by taking Fourier transforms on both sides of equation (3.3.3), we obtain

$$\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c\imath\xi\right)\right]\hat{\varphi}_\delta(\xi) = \left(\prod_{j=1}^{n}\lambda_j\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty\left[\int_0^\infty \varphi_\delta(u+y)\,\mathrm{d}P(y)\right]e^{-\imath\xi u}\,\mathrm{d}u$$

$$=\left(\prod_{j=1}^{n}\lambda_j\right)\int_0^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi_\delta(u+y)e^{-\iota\xi u}\,\mathrm{d}u\right]\mathrm{d}P(y)$$
$$=\left(\prod_{j=1}^{n}\lambda_j\right)\int_0^{\infty}e^{\iota\xi y}\hat{\varphi}_\delta(\xi)\,\mathrm{d}P(y)=\left(\prod_{j=1}^{n}\lambda_j\right)\hat{\varphi}_\delta(\xi)\tilde{p}(-\iota\xi).$$

Thus, if ξ is not a root of

$$\prod_{j=1}^{n}\left(\lambda_j+\delta+c\iota\xi\right)=\left(\prod_{j=1}^{n}\lambda_j\right)\tilde{p}(-\iota\xi),$$

then $\hat{\varphi}_\delta(\xi)=0$. With $s=-\iota\xi$ the latter equation reduces to

$$\prod_{j=1}^{n}\left(1+\frac{\delta-cs}{\lambda_j}\right)=\tilde{p}(s), \tag{3.3.14}$$

which is the generalized Lundberg's equation under the generalized Erlang-n dual risk model. Hence, the roots of (3.3.14) are a key element in the solution of (3.3.3).

3.4 Roots to the generalized Lundberg's equation

The roots with positive real parts of the generalized Lundberg's equation (3.3.14) were first studied by Gerber and Shiu [7]. We include their conclusion in this section as our first theorem.

Theorem 3.4.1 (Gerber and Shiu [7]) *For $\delta>0$, the generalized Lundberg's equation (3.3.14) has exactly n roots with positive real part (counting possible multiplicity of the roots).*

Proof Since $\delta>0$, consider a domain D that is a half disk centered at 0, lying in the right half of the complex plane, with a sufficiently large radius R. For convenience, we choose some $R>\max_{1\le i\le n}\left\{\frac{2\lambda_i+\delta}{c}\right\}$. Denote

$$\gamma(s)=\prod_{i=1}^{n}\left[\left(1+\frac{\delta}{\lambda_i}\right)-\frac{cs}{\lambda_i}\right]$$

and

$$\eta(s)=\prod_{i=1}^{n}\left[\left(1+\frac{\delta}{\lambda_i}\right)-\frac{cs}{\lambda_i}\right]-\tilde{p}(s).$$

Then for s located on the half circle $|s|=R$, $\mathrm{Re}(s)\ge 0$, we have

$$\left|\left(1+\frac{\delta}{\lambda_i}\right)-\frac{cs}{\lambda_i}\right|\ge\left|\left|1+\frac{\delta}{\lambda_i}\right|-\left|\frac{cs}{\lambda_i}\right|\right|=\left|1+\frac{\delta}{\lambda_i}-\frac{cR}{\lambda_i}\right|=\frac{cR}{\lambda_i}-\left(1+\frac{\delta}{\lambda_i}\right)$$
$$>\frac{c}{\lambda_i}\cdot\frac{2\lambda_i+\delta}{c}-\left(1+\frac{\delta}{\lambda}\right)=1,$$

for any $i = 1, 2, \ldots, n$. Thus, $|\gamma(s)| > 1$ on the the semicircle $|s| = R$, $\mathrm{Re}(s) \geq 0$. On the other hand, for $\mathrm{Re}(s) = 0$, i.e., s locates on the imaginary axis,

$$|\gamma(s)| = \prod_{i=1}^{n} \left| \left(1 + \frac{\delta}{\lambda_i}\right) - \frac{cs}{\lambda_i} \right| \geq \prod_{i=1}^{n} \left(1 + \frac{\delta}{\lambda_i}\right) > 1.$$

Therefore, $|\gamma(s)| > 1$ on the boundary of the half disk ∂D. Since for $\mathrm{Re}(s) \geq 0$,

$$|\eta(s) - \gamma(s)| = |\tilde{p}(s)| \leq 1 < |\gamma(s)|$$

on the boundary of the half disk, by Rouché's theorem, $\eta(s)$ has the same number of roots in the half disk as $\gamma(s)$. Therefore, $\eta(s)$ has exactly n roots in the half disk with radius R and thus, by letting $R \to \infty$ we conclude that $\eta(s)$ has exactly n roots in the right half of the complex plane.

Remark This simple result does not require any additional assumption, which makes it valid for both the generalized Erlang-n Sparre-Andersen risk model and its dual model.

The idea for proving Theorem 3.4.1 relies on Rouché's theorem, which requires that there are no roots located on the boundary of some domain. However, we are not able to construct a suitable domain to prove the same result in the case $\delta = 0$ because $s = 0$ is a root of the generalized Lundberg's equation (3.3.14). As a result, we choose the modified Rouché's theorem developed by Klimenok [8] to prove the following theorem.

Theorem 3.4.2 *For $\delta = 0$, the generalized Lundberg's equation* (3.3.14) *has exactly n roots with positive real part (counting possible multiplicity of the roots) and one simple root $s = 0$ under the net-profit condition* (3.2.5).

Proof Since $\delta = 0$, let

$$\gamma(s) = \prod_{j=1}^{n} \left(1 - \frac{cs}{\lambda_j}\right),$$

then $\gamma(0) = \tilde{p}(0) = 1$. Thus, the function $\gamma(s) - \tilde{p}(s)$ has a root $s = 0$. Now, consider two functions

$$\begin{cases} \phi_1(z) & = \gamma(R(1-z)) \\ \phi_2(z) & = -\tilde{p}(R(1-z)) \end{cases}$$

on the open disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and its boundary $\partial D = \{z \in \mathbb{C} : |z| = 1\}$, where R is a sufficiently large positive number (for example, let $R > \max_{1 \leq j \leq n} \left\{\lambda_j\right\} / c$). Then we immediately obtain

$$\phi_1(1) = -\phi_2(1) = 1 \neq 0. \tag{3.4.1}$$

Besides, ϕ_1 and ϕ_2 are analytic functions on D and continuous on the boundary ∂D. On one hand, for those arguments $z \in \partial D$ such that $z \neq 1$, we deduce

$$\left| 1 - \frac{cR(1-z)}{\lambda_j} \right| > \left| \left| \frac{cRz}{\lambda_j} \right| - \left| \frac{cR}{\lambda_j} - 1 \right| \right| = \left| \frac{cR}{\lambda_j} - \left(\frac{cR}{\lambda_j} - 1\right) \right| = 1$$

for $i = 1, 2, \ldots, n$. Thus, for $z \in \partial D$ and $z \neq 1$,

$$|\phi_1(z)| = \left|\prod_{j=1}^{n}\left(1 - \frac{cR(1-z)}{\lambda_j}\right)\right| = \prod_{j=1}^{n}\left|1 - \frac{cR(1-z)}{\lambda_j}\right| > 1.$$

On the other hand, for $z \in \partial D$ and $z \neq 1$,

$$|\phi_2(z)| = |-\tilde{p}(R(1-z))| = \left|-\int_0^\infty e^{-R(1-z)y}\,\mathrm{d}P(y)\right| \leq \int_0^\infty |e^{-R(1-z)y}|\,\mathrm{d}P(y) \leq 1.$$

Therefore, we have

$$|\phi_1(z)| > |\phi_2(z)|, \quad z \in \partial D, z \neq 1. \tag{3.4.2}$$

Moreover,

$$\phi_1'(1) = -R\gamma'(0) = \sum_{j=1}^{n}\frac{cR}{\lambda_j} = cR\mathbf{E}[V_1]$$

and $\phi_2'(1) = R\tilde{p}'(0) = -R\mu$. Hence, by the *net-profit condition* (3.2.5),

$$\frac{\phi_1'(1) + \phi_2'(1)}{\phi_1(1)} = R(c\mathbf{E}[V_1] - \mu) < 0. \tag{3.4.3}$$

Therefore, by conditions (3.4.1), (3.4.2) and (3.4.3), we conclude that $\phi_1(z) + \phi_2(z)$ has the same number of zeros as $\phi_1(z)$ on the open disk D [see 8, Corollary 2]. Since the zeros of $\phi_1(z)$ are

$$1 - \frac{\lambda_j}{cR} \in (0,1) \subset D, \quad j = 1, 2, \ldots, n,$$

$\phi_1(z)$ has n zeros in D. Thus, the function $\phi_1(z) + \phi_2(z)$ also has n zeros in D, denoted by $z_1, \ldots, z_n$. Now, consider the conformal mapping $h(z) := R(1-z)$. Let $s_j = h(z_j) \in h(D)$, $j = 1, 2, \ldots, n$. Then $s_1, \ldots, s_n$, which are independent of R, are all the roots of (3.3.14) such that $s \in h(D)$. Since $h(D) = \{z \in \mathbb{C} : |z - R| < R\}$, then $h(D) \to \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ as $R \to \infty$. Therefore, $s_1, \ldots, s_n$ are the only roots of (3.3.14) such that $\mathrm{Re}(s) > 0$. In addition, (3.4.3) also guarantees that $s = 0$ is a simple root of (3.3.14) on the boundary of the right half of the complex plane.

If we combine Theorems 3.4.1 and 3.4.2, we may conclude that for any $\delta \geq 0$, the generalized Lundberg'sequation (3.3.14) has exactly n roots (including their multiplicity) with positive real parts under the *net-profit condition* (3.2.5).

3.5 The Laplace transform of the ruin time

As we established in the previous section, there are n roots of Lundberg's equation (3.3.14) in the right half of the complex plane whenever $\delta \geq 0$ under the net-profit condition (3.2.5). Assume that among them m are distinct ($m \leq n$) and are denoted by $\rho_1, \ldots, \rho_m$ with respective multiplicities $\nu_1, \ldots, \nu_m$. Then we shall derive an explicit expression for the Laplace transform of the ruin time by implementing the boundary conditions appearing in Theorem 3.3.1. Before doing so, we proceed with two auxiliary results.

Lemma 3.5.1 *Suppose that $h(x)$ is an arbitrary polynomial of degree $n \in \mathbb{N}$, namely,*

$$h(x) = h_0 + h_1 x + \cdots + h_n x^n, \quad h_0, h_1, \ldots, h_n \in \mathbb{C}, \quad h_n \neq 0.$$

Then by defining

$$h\left(\frac{\mathrm{d}}{\mathrm{d}u}\right) f(u) = h_0 f(u) + h_1 f'(u) + \ldots + h_n f^{(n)}(u), \quad f \in C^n(-\infty, \infty),$$

we have

$$h\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(u^i e^{\xi u}\right) = e^{\xi u} \sum_{j=0}^{n} \frac{h^{(j)}(\xi)}{j!} \frac{\mathrm{d}^j}{\mathrm{d}u^j}(u^i), \tag{3.5.1}$$

for any $i \in \mathbb{N}$ and $\xi \in \mathbb{R}$.

Proof By the linearity of the differentiation operator, we obtain

$$\begin{aligned} h\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(u^i e^{\xi u}\right) &= \sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} \frac{\mathrm{d}^k}{\mathrm{d}u^k}\left(u^i e^{\xi u}\right) \\ &= \sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^i\right) \frac{\mathrm{d}^{k-j}}{\mathrm{d}u^{k-j}}\left(e^{\xi u}\right) \\ &= e^{\xi u} \sum_{k=0}^{n} h^{(k)}(0) \sum_{j=0}^{k} \frac{\xi^{k-j} \frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^i\right)}{j!(k-j)!} \\ &= e^{\xi u} \sum_{j=0}^{n} \frac{\frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^i\right)}{j!} \sum_{k=j}^{n} \frac{h^{(k)}(0)\xi^{k-j}}{(k-j)!}. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{k=j}^{n} \frac{h^{(k)}(0)\xi^{k-j}}{(k-j)!} &= \sum_{k=j}^{n} \left[\frac{h^{(k)}(0)}{k!} \frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^k\right)\right]\Bigg|_{u=\xi} \\ &= \sum_{k=0}^{n} \left[\frac{h^{(k)}(0)}{k!} \frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^k\right)\right]\Bigg|_{u=\xi} \\ &= \frac{\mathrm{d}^j}{\mathrm{d}u^j}\left[\sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} u^k\right]\Bigg|_{u=\xi} = h^{(j)}(\xi). \end{aligned}$$

Therefore, identity (3.5.1) is proved.

Lemma 3.5.1 is useful for determining the solution of the integro-differential equation (3.3.11) satisfied by the Laplace transform of the time of ruin.

Lemma 3.5.2 *Suppose that $\rho \in \mathbb{C}$ with $\mathrm{Re}(\rho) > 0$ is a root with multiplicity ν of Lundberg's equation (3.3.14). Then for any polynomial $\pi_\nu(u)$ of degree $\nu - 1$, $\varphi^*(u) = \pi_\nu(u)e^{-\rho u}$ is a solution of (3.3.11).*

Proof Denote $\pi_\nu(u) = r_0 + r_1 u + \ldots + r_{\nu-1}u^{\nu-1}$ and $h(x) = \prod_{j=1}^{n}\left(\lambda_j + \delta + cx\right)$, where $r_{\nu-1} \neq 0$. Then

$$\begin{aligned}\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right]\varphi^*(u) &= h\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(\sum_{i=0}^{\nu-1} r_i u^i e^{-\rho u}\right)\\ &= \sum_{i=0}^{\nu-1} r_i h\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(u^i e^{-\rho u}\right)\\ &= \sum_{i=0}^{\nu-1} r_i e^{-\rho u}\sum_{j=0}^{n}\frac{h^{(j)}(-\rho)}{j!}\frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^i\right)\end{aligned}$$

by Lemma 3.5.1 since $h(x)$ is a polynomial of degree n. Furthermore, equation (3.3.14) is equivalent to

$$h(-s) - \left(\prod_{j=1}^{n}\lambda_j\right)\tilde{p}(s) = 0, \tag{3.5.2}$$

and ρ is a root of (3.5.2) with multiplicity ν. Thus, we may write

$$h(-s) - \left(\prod_{j=1}^{n}\lambda_j\right)\tilde{p}(s) = (s-\rho)^\nu \eta(s), \tag{3.5.3}$$

where $\eta(s)$ is an analytic function such that $\eta(\rho) \neq 0$. Hence, for $j = 1, 2, \ldots, \nu - 1$, differentiation j times of (3.5.3) yields

$$(-1)^j h(-\rho) - \left(\prod_{j=1}^{n}\lambda_j\right)\tilde{p}^{(j)}(\rho) = \sum_{k=0}^{j}\left[\binom{j}{k}\frac{\nu!}{(\nu-k)!}(s-\rho)^{\nu-k}\eta^{(j-k)}(s)\right]\Bigg|_{s=\rho} = 0,$$

i.e.,

$$h^{(j)}(-\rho) = (-1)^j\left(\prod_{j=1}^{n}\lambda_j\right)\tilde{p}^{(j)}(\rho), \quad j = 0, 1, \ldots, \nu - 1. \tag{3.5.4}$$

By identity (3.5.1), we conclude that

$$\sum_{i=0}^{\nu-1} r_i h\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(u^i e^{-\rho u}\right) = \sum_{i=0}^{\nu-1} r_i e^{-\rho u}\sum_{j=0}^{n}\frac{h^{(j)}(-\rho)}{j!}\frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^i\right). \tag{3.5.5}$$

Notice that in the right-hand side of the latter equation, $i \leq \nu - 1 < n$, which implies $\frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^i\right) = 0$ for all $i < j \leq n$. Hence, implementation of equation (3.5.4) produces

$$\sum_{j=0}^{n}\frac{h^{(j)}(-\rho)}{j!}\frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^i\right) = \left(\prod_{j=1}^{n}\lambda_j\right)\sum_{j=0}^{i}\frac{(-1)^j\tilde{p}^{(j)}(\rho)}{j!}\frac{\mathrm{d}^j}{\mathrm{d}u^j}\left(u^i\right)$$

$$= \left(\prod_{j=1}^{n} \lambda_j\right) \sum_{j=0}^{i} \binom{i}{j} (-1)^j \tilde{p}^{(j)}(\rho) u^{i-j}.$$

Thus,

$$\sum_{i=0}^{\nu-1} r_i h\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(u^i e^{-\rho u}\right) = \left(\prod_{j=1}^{n} \lambda_j\right) \sum_{i=0}^{\nu-1} r_i e^{-\rho u} \sum_{j=0}^{i} \binom{i}{j} (-1)^j \tilde{p}^{(j)}(\rho) u^{i-j}. \tag{3.5.6}$$

Furthermore, the relationship

$$(-1)^j \tilde{p}^{(j)}(\rho) = \int_0^\infty y^j e^{-\rho y} \,\mathrm{d}P(y)$$

combined with equation (3.5.6) indicates that

$$\begin{aligned}\sum_{i=0}^{\nu-1} r_i h\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\left(u^i e^{-\rho u}\right) &= \left(\prod_{j=1}^{n} \lambda_j\right) \sum_{i=0}^{\nu-1} r_i e^{-\rho u} \int_0^\infty (u+y)^i e^{-\rho y} \,\mathrm{d}P(y) \\ &= \left(\prod_{j=1}^{n} \lambda_j\right) \int_0^\infty \varphi^*(u+y) \,\mathrm{d}P(y).\end{aligned}$$

Therefore,

$$\left[\prod_{j=1}^{n} \left(\lambda_j + \delta + c\frac{\mathrm{d}}{\mathrm{d}u}\right)\right] \varphi^*(u) = \left(\prod_{j=1}^{n} \lambda_j\right) \int_0^\infty \varphi^*(u+y) \,\mathrm{d}P(y),$$

i.e., $\varphi^*(u)$ is a solution of equation (3.3.11). Since $\pi_\nu(u)$ is an arbitrary polynomial of degree $\nu - 1$, Lemma 3.5.2 is proved.

Lemma 3.5.2 indicates that the solution of the integro-differential equation (3.3.11) has the form

$$\varphi_\delta(u) = \sum_{j=1}^{m} \left(\sum_{k=0}^{\nu_j - 1} r_{j,k} u^k\right) e^{-\rho_j u} + r\mathbf{1}_{\{\delta=0\}}$$

with coefficients satisfying the boundary conditions $\varphi_\delta^{(i)}(0) = \left(-\frac{\delta}{c}\right)^i$ for $i = 0, 1, \ldots, n-1$.

Theorem 3.5.3 *The Laplace transform $\varphi_\delta(u)$ of the time of ruin under the Sparre-Andersen dual model with generalized Erlang-n inter-innovation times, given initial surplus $u > 0$, has the explicit form*

$$\varphi_\delta(u) = \sum_{j=1}^{m} \left(\sum_{k=0}^{\nu_j - 1} r_{j,k} u^k\right) e^{-\rho_j(\delta) u},$$

where $\rho_1(\delta), \rho_2(\delta), \ldots, \rho_m(\delta)$ are the roots of Lundberg's equation (3.3.14) lying on the right half of the complex plane with multiplicity $\nu_1, \nu_2, \ldots, \nu_m$, respectively. All coefficients $r_{j,k}$ may be obtained by solving the linear-equation system

$$\sum_{k=0}^{i} \sum_{\{j:\nu_j>k\}} \frac{i!(-1)^k}{(i-k)!} \rho_j^{i-k}(\delta) r_{j,k} = \left(\frac{\delta}{c}\right)^i, \tag{3.5.7}$$

where $i = 0, 1, \ldots, n-1$.

Proof According to Lemma 3.5.2,

$$\varphi_\delta(u) = \sum_{j=1}^{m} \left[\sum_{k=0}^{\nu_j-1} r_{j,k} u^k e^{-\rho_j(\delta)u} \right] + r\mathbf{1}_{\{\delta=0\}}$$

with boundary conditions $\varphi_\delta^{(i)}(0) = \left(-\frac{c}{\delta}\right)^i$ for $i = 0, 1, \ldots, n-1$. Hence, because $\varphi_\delta(u) \to 0$ as $u \to \infty$ [see 4, Lemma 2.1], we have $r = 0$, i.e., the Laplace transform of the ruin time T has the form

$$\varphi_\delta(u) = \sum_{j=1}^{m} \left(\sum_{k=0}^{\nu_j-1} r_{j,k} u^k \right) e^{-\rho_j(\delta)u} \tag{3.5.8}$$

with boundary conditions

$$\varphi_\delta^{(i)}(0) = \left(-\frac{\delta}{c}\right)^i, \quad i = 0, 1, \ldots, n-1. \tag{3.5.9}$$

For convenience, denote $\rho_j = \rho_j(\delta)$ for $j = 1, 2, \ldots, m$. Then for $i = 0, 1, \ldots, n-1$,

$$\varphi_\delta^{(i)}(0) = \sum_{j=1}^{m} \frac{\mathrm{d}^i}{\mathrm{d}u^i} \left[\left(\sum_{k=0}^{\nu_j-1} r_{j,k} u^k \right) e^{-\rho_j u} \right] \Bigg|_{u=0} = \sum_{j=1}^{m} \sum_{k=0}^{\nu_j-1} \left[r_{j,k} \frac{\mathrm{d}^i}{\mathrm{d}u^i} \left(u^k e^{-\rho_j u} \right) \Big|_{u=0} \right] = \left(-\frac{\delta}{c}\right)^i. \tag{3.5.10}$$

Since

$$\frac{\mathrm{d}^i}{\mathrm{d}u^i} \left(u^k e^{\xi u} \right) \Big|_{u=0} = \frac{i!\xi^{i-k}}{(i-k)!} \mathbf{1}_{\{i \geq k\}},$$

equation (3.5.10) may be simplified to

$$\sum_{j=1}^{m} \sum_{k=0}^{\min\{\nu_j-1, i\}} \frac{i!(-1)^k}{(i-k)!} \rho_j^{i-k} r_{j,k} = \left(\frac{\delta}{c}\right)^i. \tag{3.5.11}$$

Exchanging the order of summation in (3.5.11) yields the required linear system (3.5.7).

The following examples provide the Laplace transform of the ruin time for some special cases.

Example Consider the special case $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. Then the model is just an Erlang-n dual risk model. Hence, as mentioned in Rodríguez et al. [5], all roots of the generalized Lundberg's fundamental equation (3.3.14) are distinct, i.e., $m = n$ and $\nu_1 = \nu_2 = \ldots = \nu_n = 1$. Thus, the Laplace transform of the ruin time T is

$$\varphi_\delta(u) = \sum_{j=1}^{n} \frac{(-1)^{j-1} \prod_{i=1, i\neq j}^{n} \left(\rho_j - \frac{\delta}{c}\right)}{\left[\prod_{i=1}^{j-1} \left(\rho_j - \rho_i\right)\right]\left[\prod_{i=j+1}^{n} \left(\rho_i - \rho_j\right)\right]} e^{-\rho_j u}$$

for $j = 1, 2, \ldots, n$, which coincides with Rodríguez et al. [5, Theorem 4.2].

Example Consider a dual Sparre Andersen ruin model with exponentially distributed claims with mean $\mu = 1$, while the inter-innovation times are i.i.d. generalized Erlang-3 random variables with parameters $\lambda_1 = 9$, $\lambda_2 = 25$ and $\lambda_3 = 32$. Now, let $\delta = 0$ and the expense rate $c = 1$. Then the generalized Lundberg's equation is

$$\left(1 - \frac{s}{9}\right)\left(1 - \frac{s}{25}\right)\left(1 - \frac{s}{32}\right) = \frac{1}{1+s}.$$

After some algebra, it is rewritten as

$$s^4 - 65s^3 + 1247s^2 - 5887s = s(s-7)(s-29)^2 = 0.$$

Hence, the generalized Lundberg's equation has a zero root $s_0 = 0$, two positive roots $s_1 = 7$ and $s_2 = 29$. In particular, the multiplicity of the root $s_2 = 29$ is 2. Since $\delta = 0$, $\varphi_0(u)$ is actually the ruin probability for initial capital $u > 0$. Thus,

$$\varphi_0(u) = r_1 e^{-7u} + (r_{2,0} + r_{2,1}u)e^{-29u}$$

where coefficients r_1, $r_{2,0}$, and $r_{2,1}$ satisfy

$$\left|\begin{array}{ll} r_1 + r_{2,0} & = 1 \\ 7r_1 + 29r_{2,0} - r_{2,1} & = 0 \\ (7)^2 r_1 + (29)^2 r_{2,0} - 2(29) r_{2,1} & = 0 \end{array}\right. .$$

Therefore, $r_1 = \frac{841}{484}$, $r_{2,0} = -\frac{357}{484}$, and $r_{2,1} = -\frac{203}{22}$, which leads to

$$\varphi(u) = \frac{841}{484} e^{-7u} - \frac{357 + 4466u}{484} e^{-29u}, \quad u > 0.$$

3.6 Expected discounted dividends under a model with a threshold strategy

3.6.1 A set of integro-differential equations

In this subsection, we generalize some existing results concerning the expected discounted dividends. More precisely, we consider the Sparre-Andersen dual risk model with generalized Erlang-n inter-event times' distribution.

Consider a dividend strategy with threshold $b > 0$ and different expense rates c_1 and c_2, where $c_2 > c_1$, depending on whether the current surplus is below or above the threshold [for further detail on this model, see 2]. Moreover, as an expense rate without dividend payments, usually c_1 is assumed to satisfy condition (3.2.5) with c replaced by c_1. Then the total discounted dividends until ruin are defined as

$$D_\delta(b) = (c_2 - c_1) \int_0^T e^{-\delta t} \mathbf{1}_{\{R(t) \geq b\}} \,\mathrm{d}t$$

and the expected total dividends paid until ruin are

$$V_\delta(u, b) = \mathbf{E}[D_\delta(b) | R(0) = u].$$

Depending on the value of the initial surplus, define

$$V_\delta(u, b) = \begin{cases} V_{1,\delta}(u, b), & 0 < u < b \\ V_{2,\delta}(u, b), & u > b \end{cases}.$$

Then we have the following main results for this section.

Theorem 3.6.1 *The function $V_{1,\delta}(u, b)$ satisfies the integro-differential equation*

$$\begin{aligned} &\left[\prod_{j=1}^{n} \left(\lambda_j + \delta + c_1 \frac{\partial}{\partial u}\right)\right] V_{1,\delta}(u, b) \\ &= \left(\prod_{j=1}^{n} \lambda_j\right) \left[\int_0^{b-u} V_{1,\delta}(u + y, b) \,\mathrm{d}P(y) + \int_{b-u}^{\infty} V_{2,\delta}(u + y, b) \,\mathrm{d}P(y)\right] \end{aligned} \tag{3.6.1}$$

with boundary conditions $\left.\frac{\partial^i}{\partial u^i} V_{1,\delta}(u, b)\right|_{u=0} = 0,\ i = 0, 1, \ldots, n-1$, *while the function* $V_{2,\delta}(u, b)$ *satisfies the integro-differential equation*

$$\begin{aligned} &\left[\prod_{j=1}^{n} \left(\lambda_j + \delta + c_2 \frac{\partial}{\partial u}\right)\right] \left[V_{2,\delta}(u, b) - \frac{c_2 - c_1}{\delta}\right] \\ &= \left(\prod_{j=1}^{n} \lambda_j\right) \int_0^{\infty} \left[V_{2,\delta}(u + y, b) - \frac{c_2 - c_1}{\delta}\right] \mathrm{d}P(y) \end{aligned} \tag{3.6.2}$$

with boundary conditions

$$\left.\frac{\partial^i}{\partial u^i} V_{2,\delta}(u, b)\right|_{u=b} = \left(\frac{c_1}{c_2}\right)^i \left.\frac{\partial^i}{\partial u^i} V_{1,\delta}(u, b)\right|_{u=b} - \frac{c_2 - c_1}{\delta} \left(-\frac{\delta}{c_2}\right)^i \tag{3.6.3}$$

for $i = 0, 1, \ldots, n-1$.

Proof Firstly, we consider the function $V_{1,\delta}(u,b)$. Conditioning on the time and the amount of the first innovation, we have

$$V_{1,\delta}(u,b) = \int_0^{\frac{u}{c_1}} e^{-\delta t}\left[\int_0^{b-u+c_1 t} V_{1,\delta}(u-c_1 t+y,b)\,\mathrm{d}P(y) + \int_{b-u+c_1 t}^{\infty} V_{2,\delta}(u-c_1 t+y,b)\,\mathrm{d}P(y)\right] f(t)\,\mathrm{d}t.$$

The change of variables $v = u - c_1 t$ then yields the equivalent representation

$$V_{1,\delta}(u,b) = \int_0^u \left[\int_0^{b-v} V_{1,\delta}(v+y,b)\,\mathrm{d}P(y) + \int_{b-v}^{\infty} V_{2,\delta}(v+y,b)\,\mathrm{d}P(y)\right] g_{c_1}(u-v)\,\mathrm{d}v \qquad (3.6.4)$$

for all $0 < u \le b$, which is of the form (3.3.1) with

$$\mathcal{I}\left[V_\delta\right](v,b) = \int_0^{\infty} V_\delta(v+y,b)\,\mathrm{d}P(y).$$

and $G(u) \equiv 0$ which is from the set (3.3.2).

Therefore, Theorem 3.3.1 implies equation (3.6.1) with the required boundary conditions $\left.\frac{\partial^i}{\partial u^i} V_{1,\delta}(u,b)\right|_{u=0} = 0$ for $i = 0, 1, \ldots, n-1$.

Secondly, consider the case $u > b$. Again conditioning on the first jump, we have three scenarios:

1. If the jump happens within time $[0,(u-b)/c_2]$, then the expected discounted dividend, given the time of first jump t, is
$$e^{-\delta t}\int_0^{\infty} V_{2,\delta}(u-c_2 t+y,b)\,\mathrm{d}P(y) + (c_2-c_1)\bar{a}_{\overline{t}|\delta}.$$
2. If the jump happens within time $((u-b)/c_2,(u-b)/c_2+b/c_1]$, then the expected discounted dividend, given the time of first jump t, is
$$\begin{aligned}&e^{-\delta t}\int_0^{c_1\left(t-\frac{u-b}{c_2}\right)} V_{1,\delta}\left(b-c_1\left(t-\frac{u-b}{c_2}\right)+y,b\right)\mathrm{d}P(y)\\ &+e^{-\delta t}\int_{c_1\left(t-\frac{u-b}{c_2}\right)}^{\infty} V_{2,\delta}\left(b-c_1\left(t-\frac{u-b}{c_2}\right)+y,b\right)\mathrm{d}P(y)\\ &+(c_2-c_1)\bar{a}_{\overline{\frac{u-b}{c_2}}|\delta}.\end{aligned}$$
3. If there is no jump before ruin, i.e. the time of the first jump is greater than $(u-b)/c_2 + b/c_1$, then the expected discounted dividend is $(c_2-c_1)\bar{a}_{\overline{\frac{u-b}{c_2}}|\delta}$.

Hence, by letting $v = u - b > 0$ for convenience, the total probability theorem yields

$$V_{2,\delta}(v+b,b) = \int_0^{v/c_2} e^{-\delta t}\left[\int_0^{\infty} V_{2,\delta}(v+b-c_2 t+y,b)\,\mathrm{d}P(y)\right] f(t)\,\mathrm{d}t$$

$$
\begin{aligned}
&+\int_0^{v/c_2} e^{-\delta t}(c_2-c_1)\bar{a}_{\overline{t}|\delta} f(t)\,\mathrm{d}t\\
&+\int_{v/c_2}^{v/c_2+b/c_1} e^{-\delta t}\left[\int_0^{c_1\left(t-\frac{v}{c_2}\right)} V_{1,\delta}\left(b-c_1\left(t-\frac{v}{c_2}\right)+y,b\right)\mathrm{d}P(y)\right] f(t)\,\mathrm{d}t\\
&+\int_{v/c_2}^{v/c_2+b/c_1} e^{-\delta t}\left[\int_{c_1\left(t-\frac{v}{c_2}\right)}^{\infty} V_{2,\delta}\left(b-c_1\left(t-\frac{v}{c_2}\right)+y,b\right)\mathrm{d}P(y)\right] f(t)\,\mathrm{d}t\\
&+\int_{v/c_2}^{v/c_2+b/c_1} e^{-\delta t}(c_2-c_1)\bar{a}_{\overline{\frac{v}{c_2}}|\delta} f(t)\,\mathrm{d}t\\
&+(c_2-c_1)\bar{a}_{\overline{\frac{v}{c_2}}|\delta}\overline{F}\left(\frac{v}{c_2}+\frac{b}{c_1}\right)\\
=&\int_0^{v/c_2}\left[\int_0^{\infty} V_{2,\delta}(v+b-c_2t+y,b)\,\mathrm{d}P(y)-\frac{c_2-c_1}{\delta}\right] e^{-\delta t} f(t)\,\mathrm{d}t\\
&+\frac{c_2-c_1}{\delta}\int_0^{v/c_2} f(t)\,\mathrm{d}t\\
&+\int_{v/c_2}^{v/c_2+b/c_1}\left[\int_0^{c_1(t-v/c_2)} V_{1,\delta}\left(b-c_1(t-v/c_2)+y,b\right)\mathrm{d}P(y)\right.\\
&+\left.\int_{c_1[t-v/c_2]}^{\infty} V_{2,\delta}\left(b-c_1(t-v/c_2)+y,b\right)\mathrm{d}P(y)\right] e^{-\delta t} f(t)\,\mathrm{d}t\\
&+(c_2-c_1)\bar{a}_{\overline{\frac{v}{c_2}}|\delta}\overline{F}\left(\frac{v}{c_2}\right).
\end{aligned}
$$

Since

$$
\frac{c_2-c_1}{\delta}\int_0^{v/c_2} f(t)\,\mathrm{d}t+\frac{c_2-c_1}{\delta}\overline{F}\left(\frac{v}{c_2}\right)=\frac{c_2-c_1}{\delta},
$$

by the substitutions $\tau_1 = v - c_2 t$ in the first integral and $\tau_2 = b - c_1(t - v/c_2)$ in the third and fourth integrals, we deduce

$$
\begin{aligned}
V_{2,\delta}(v+b,b)=&\int_0^{v}\left[\int_0^{\infty} V_{2,\delta}(b+\tau_1+y,b)\,\mathrm{d}P(y)-\frac{c_2-c_1}{\delta}\right] g_{c_2}(v-\tau_1)\,\mathrm{d}\tau_1\\
&+\frac{c_2}{c_1}\int_0^{b}\mathcal{I}\left[V_\delta\right](\tau_2,b)\,g_{c_2}\left(\frac{c_2(b-\tau_2)}{c_1}+v\right)\mathrm{d}\tau_2\\
&-\frac{c_2-c_1}{\delta}e^{-\delta v/c_2}\overline{F}\left(\frac{v}{c_2}\right)+\frac{c_2-c_1}{\delta}.
\end{aligned}
\tag{3.6.5}
$$

Notice that

$$
\frac{c_2-c_1}{\delta}=(c_2-c_1)\,\bar{a}_{\overline{\infty}|\delta}\geq V_{2,\delta}(v+b,b),\quad v\geq 0,
$$

we rearrange (3.6.5) as follows to make sure that both sides of the equation are nonnegative

$$
\frac{c_2-c_1}{\delta}-V_{2,\delta}(v+b,b)=\int_0^{v}\left[\int_0^{\infty}\left(\frac{c_2-c_1}{\delta}-V_{2,\delta}(b+\tau_1+y,b)\,\mathrm{d}P(y)\right)\right] g_{c_2}(v-\tau_1)\,\mathrm{d}\tau_1
$$

$$
\begin{aligned}
&- \frac{c_2}{c_1} \int_0^b \mathcal{I}\left[V_\delta\right](\tau_2, b) g_{c_2}\left(\frac{c_2(b-\tau_2)}{c_1} + v\right) \mathrm{d}\tau_2 \\
&+ \frac{c_2 - c_1}{\delta} e^{-\delta v / c_2} \overline{F}\left(\frac{v}{c_2}\right) + \frac{c_2 - c_1}{\delta}.
\end{aligned}
$$

Since Lemma 3.A.2 implies that

$$
\begin{aligned}
&\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c_2 \frac{\partial}{\partial v}\right)\right] \int_0^b \mathcal{I}\left[V_\delta\right](\tau_2, b) g_{c_2}\left(\frac{c_2(b-\tau_2)}{c_1} + v\right) \mathrm{d}\tau_2 \\
=& \int_0^b \mathcal{I}\left[V_\delta\right](\tau_2, b) \left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c_2 \frac{\partial}{\partial v}\right)\right] g_{c_2}\left(\frac{c_2(b-\tau_2)}{c_1} + v\right) \mathrm{d}\tau_2 = 0,
\end{aligned}
$$

and (3.3.12) indicates that

$$
\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c_2 \frac{\partial}{\partial v}\right)\right]\left[\frac{c_2 - c_1}{\delta} e^{-\delta v / c_2} \overline{F}\left(\frac{v}{c_2}\right)\right] = 0,
$$

then if we let

$$
G(v) = -\frac{c_2}{c_1} \int_0^b \mathcal{I}\left[V_\delta\right](\tau_2, b) g_{c_2}\left(\frac{c_2(b-\tau_2)}{c_1} + v\right) \mathrm{d}\tau_2 + \frac{c_2 - c_1}{\delta} e^{-\delta v / c_2} \overline{F}\left(\frac{v}{c_2}\right),
$$

we have

$$
\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c_2 \frac{\partial}{\partial v}\right)\right] G(v) = 0.
$$

Therefore, Theorem 3.3.1 implies

$$
\begin{aligned}
&\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c_2 \frac{\partial}{\partial v}\right)\right]\left[\frac{c_2 - c_1}{\delta} - V_{2,\delta}(v+b, b)\right] \\
&= \left(\prod_{j=1}^{n} \lambda_j\right) \int_0^\infty \left[\frac{c_2 - c_1}{\delta} - V_{2,\delta}(b+v+y, b)\right] \mathrm{d}P(y)
\end{aligned}
\tag{3.6.6}
$$

with boundary conditions

$$
\frac{\partial^i}{\partial v^i}\left[\frac{c_2 - c_1}{\delta} - V_{2,\delta}(v+b, b)\right]\Bigg|_{v=0} = -\frac{c_2}{c_1} \int_0^b \mathcal{I}\left[V_\delta\right](\tau_2, b) g_{c_2}^{(i)}\left(\frac{c_2(b-\tau_2)}{c_1}\right) \mathrm{d}\tau_2 + \frac{c_2 - c_1}{\delta}\left(-\frac{\delta}{c_2}\right)^i
$$

for $i = 0, 1, \ldots, n-1$, due to (3.3.13). Since

$$
\begin{aligned}
\frac{\partial}{\partial v} V_{2,\delta}(v+b, b) =& \frac{\partial}{\partial(v+b)} V_{2,\delta}(v+b, b) \cdot \frac{\mathrm{d}(v+b)}{\mathrm{d}v} \\
=& \frac{\partial}{\partial(v+b)} V_{2,\delta}(v+b, b) = \frac{\partial}{\partial u} V_{2,\delta}(u, b)\Big|_{u=v+b},
\end{aligned}
$$

we conclude that

$$\frac{\partial^i}{\partial v^i} V_{2,\delta}(v+b,b) = \frac{\partial^i}{\partial u^i} V_{2,\delta}(u,b)\Big|_{u=v+b}$$

for $i = 1, 2, \ldots$ by induction. Thus, equation (3.6.6) is equivalent to (3.6.2) with corresponding version of boundary conditions

$$\frac{\partial^i}{\partial u^i}\left[\frac{c_2-c_1}{\delta} - V_{2,\delta}(u,b)\right]\Big|_{u=b} = -\frac{c_2}{c_1}\int_0^b \mathcal{I}\left[V_\delta\right](\tau_2,b) g_{c_2}^{(i)}\left(\frac{c_2(b-\tau_2)}{c_1}\right) \mathrm{d}\tau_2 + \frac{c_2-c_1}{\delta}\left(-\frac{\delta}{c_2}\right)^i$$

for $i = 0, 1, \ldots, n-1$. Furthermore, we observe that

$$f(t) = c_2 e^{\delta t} g_{c_2}(c_2 t),$$

which yields

$$g_{c_1}(t) = \frac{1}{c_1} e^{-\delta t/c_1} f\left(\frac{t}{c_1}\right) = \frac{c_2}{c_1} g_{c_2}\left(\frac{c_2 t}{c_1}\right).$$

Thus, (3.6.4) becomes with v replaced by τ_2

$$\begin{aligned} V_{1,\delta}(u,b) &= \int_0^u \mathcal{I}\left[V_\delta\right](\tau_2,b) g_{c_1}(u-\tau_2)\,\mathrm{d}\tau_2 \\ &= \frac{c_2}{c_1}\int_0^u \mathcal{I}\left[V_\delta\right](\tau_2,b) g_{c_2}\left(\frac{c_2(u-\tau_2)}{c_1}\right) \mathrm{d}\tau_2, \end{aligned} \tag{3.6.7}$$

Differentiating both sides of (3.6.7) i times with respect to u yields

$$\frac{\partial^i}{\partial u^i} V_{1,\delta}(u,b) = \left(\frac{c_2}{c_1}\right)^{i+1} \int_0^u \mathcal{I}\left[V_\delta\right](\tau_2,b) g_{c_2}^{(i)}\left(\frac{c_2(u-\tau_2)}{c_1}\right) \mathrm{d}\tau_2$$

for $i = 0, 1, \ldots, n-1$. Hence, by letting $u = b$ we have

$$\frac{c_2}{c_1}\int_0^b \mathcal{I}\left[V_\delta\right](\tau_2,b) g_{c_2}^{(i)}\left(\frac{c_2(b-\tau_2)}{c_1}\right) \mathrm{d}\tau_2 = \left(\frac{c_1}{c_2}\right)^i \frac{\partial^i}{\partial u^i} V_{1,\delta}(u,b)\Big|_{u=b}.$$

Therefore, the required initial conditions (3.6.3) are obtained.

Since (3.6.2) has the same form as (3.3.11), Lemma 3.5.2 produces

$$\frac{c_2-c_1}{\delta} - V_{2,\delta}(u,b) = \sum_{j=1}^{m}\sum_{k=0}^{\nu_j-1} r_{j,k}(u-b)^k e^{-R_j(\delta)(u-b)}, \quad u > b,$$

for $\delta > 0$, where $R_1(\delta), R_2(\delta), \ldots, R_m(\delta)$ are roots of the equation

$$\prod_{j=1}^{n}\left(1 + \frac{\delta - c_2 s}{\lambda_j}\right) = \tilde{p}(s)$$

lying on the right half of the complex plane with multiplicity $\nu_1, \nu_2, \ldots, \nu_m$, respectively. All coefficients $r_{j,k}$ are determined by the boundary conditions involving $V_{1,\delta}(u,b)$ and $V_{2,\delta}(u,b)$. Namely, the boundary conditions are

$$\sum_{k=0}^{i} \sum_{j:\nu_j>k} \frac{i!(-1)^k}{(i-k)!} R_j^{i-k}(\delta) r_{j,k} = -\left(-\frac{c_1}{c_2}\right)^i \frac{\partial^i}{\partial u^i} V_{1,\delta}(u,b)\Big|_{u=b} + \frac{c_2-c_1}{\delta}\left(\frac{\delta}{c_2}\right)^i \tag{3.6.8}$$

for $i = 0, 1, \ldots, n-1$.

Remark Under the compound Poisson dual risk model with a threshold strategy, there is only one boundary condition

$$V_{2,\delta}(b,b) = \frac{c_2 - c_1}{\delta} - r = V_{1,\delta}(b,b),$$

which is then called "the continuity condition" [see 2].

3.6.2 Exponential jump distributions

Suppose that the amounts of revenue have an exponential distribution with density function $p(y) = \beta e^{-\beta y}$. Then

$$\begin{aligned}\mathcal{I}[V_\delta](u,b) &= \int_0^{b-u} V_{1,\delta}(u+y,b)\beta e^{-\beta y}\,\mathrm{d}y + \int_{b-u}^{\infty} V_{2,\delta}(u+y,b)\beta e^{-\beta y}\,\mathrm{d}y \\ &= \beta e^{\beta u}\int_u^b V_{1,\delta}(y,b)e^{-\beta y}\,\mathrm{d}y + \beta e^{\beta u}\int_b^\infty V_{2,\delta}(y,b)e^{-\beta y}\,\mathrm{d}y, \quad 0 < u \le b.\end{aligned}$$

Hence, applying the operator $\left(\frac{\partial}{\partial u} - \beta\right)$ on $\mathcal{I}[V_\delta](u,b)$ yields

$$\left(\frac{\partial}{\partial u} - \beta\right)\mathcal{I}[V_\delta](u,b) = -\beta V_{1,\delta}(u,b)$$

[see also Subsection 2.1 in 2]. Thus, for $0 < u \le b$ we have

$$\left(\frac{\partial}{\partial u} - \beta\right)\left[\prod_{j=1}^{n}\left(\lambda_j + \delta + c_1\frac{\partial}{\partial u}\right)\right] V_{1,\delta}(u,b) + \left(\prod_{j=1}^{n}\lambda_j\right)\beta V_{1,\delta}(u,b) = 0. \tag{3.6.9}$$

Hence, the related characteristic equation reduces to

$$\prod_{j=1}^{n}\left(1 + \frac{\delta + c_1 x}{\lambda_j}\right) = \frac{\beta}{\beta - x} = \tilde{p}(-x).$$

If we let $\bar{x} = -x$, we actually obtain another Lundberg's fundamental equation for the generalized Erlang-n dual risk model

$$\prod_{j=1}^{n}\left(1 + \frac{\delta - c_1\bar{x}}{\lambda_j}\right) = \frac{\beta}{\beta + \bar{x}}.$$

By the fundamental theorem of algebra, the above equation has $n+1$ roots in $\mathbb{C}$. Besides, our previous discussion at the end of Section 3.4 demonstrates that only one of the $n+1$ roots is located in the left half of the complex plane, which means it is a negative real number denoted as s_0. Suppose that the roots with positive real part are $s_1, \dots, s_l$ with multiplicities $\kappa_1, \dots, \kappa_l$. Thus, the theory of ordinary differential equations implies that

$$V_{1,\delta}(u,b) = q_0 e^{-s_0 u} + \sum_{j=1}^{l} \left(\sum_{k=0}^{\kappa_j - 1} q_{j,k} u^k \right) e^{-s_j u}, \quad 0 < u \le b.$$

In order to determine the coefficients q_0 and $q_{j,k}$'s, we have n boundary conditions

$$\frac{\partial^i}{\partial u^i} V_{1,\delta}(u,b) \bigg|_{u=0} = 0, \quad i = 0, \dots, n-1.$$

Hence, $q_{j,k}$'s may be evaluated in terms of q_0 through the linear-equation system

$$\sum_{k=0}^{i} \sum_{j:\kappa_j > k} \frac{i!(-1)^k}{(i-k)!} s_j^{i-k} q_{j,k} = -q_0 s_0^i \tag{3.6.10}$$

for $i = 0, 1, \dots, n-1$. Besides, by Lemma 3.5.1 we may calculate i-th derivatives of $V_{1,\delta}(u,b)$

$$\begin{aligned} \frac{\partial^i}{\partial u^i} V_{1,\delta}(u,b) \bigg|_{u=b} &= (-s_0)^i q_0 e^{-s_0 b} + \sum_{j=1}^{l} \sum_{k=0}^{\kappa_j - 1} q_{j,k} \frac{\mathrm{d}^i}{\mathrm{d}u^i} \left(u^k e^{-s_j u} \right) \bigg|_{u=b} \\ &= (-s_0)^i q_0 e^{-s_0 b} + \sum_{j=1}^{l} \sum_{k=0}^{\kappa_j - 1} q_{j,k} e^{-s_i b} \sum_{z=0}^{i \wedge k} \binom{i}{z} \frac{k!}{(k-z)!} b^{k-z}, \end{aligned}$$

Hence the boundary conditions of (3.6.2), namely, equations (3.6.8) are

$$\begin{aligned} &\frac{c_2 - c_1}{\delta} \left(\frac{\delta}{c_2} \right)^i - \sum_{k=0}^{i} \sum_{j:\nu_j > k} \frac{i!(-1)^k}{(i-k)!} R_j^{i-k}(\delta) r_{j,k} \\ &= \left(-\frac{c_1}{c_2} \right)^i \left[(-s_0)^i q_0 e^{-s_0 b} + \sum_{j=1}^{l} \sum_{k=0}^{\kappa_j - 1} q_{j,k} e^{-s_i b} \sum_{z=0}^{i \wedge k} \binom{i}{z} \frac{k!}{(k-z)!} (-s_j)^{i-z} b^{k-z} \right] \end{aligned} \tag{3.6.11}$$

for $i = 0, 1, \dots, n-1$. Now, we only have $2n$ linear equations to solve for $2n+1$ unknown parameters q_0, $q_{j,k}$'s and $r_{j,k}$'s. For this reason we need one more linear equation by reviewing

$$\left[\prod_{j=1}^{n} \left(\lambda_j + \delta + c_1 \frac{\partial}{\partial u} \right) \right] V_{1,\delta}(u,b) = \left(\prod_{j=1}^{n} \lambda_j \right) \beta e^{\beta u} \left[\int_u^b V_{1,\delta}(y,b) e^{-\beta y} \, \mathrm{d}y + \int_b^\infty V_{2,\delta}(y,b) e^{-\beta y} \, \mathrm{d}y \right].$$

Since Lemma 3.5.2 implies that

$$\left[\prod_{j=1}^{n} \left(\lambda_j + \delta + c_1 \frac{\partial}{\partial u} \right) \right] \left[V_{1,\delta}(u) - q_0 e^{-s_0 u} \right] = \left[\prod_{j=1}^{n} \left(\lambda_j + \delta + c_1 \frac{\partial}{\partial u} \right) \right] \left(\sum_{j=1}^{l} \sum_{k=0}^{\kappa_j - 1} q_{j,k} u^k e^{-s_j u} \right)$$

$$=\left(\prod_{i=1}^{n}\lambda_i\right)\beta e^{\beta u}\int_u^{\infty}\left(\sum_{j=1}^{l}\sum_{k=0}^{\kappa_j-1}q_{j,k}y^k e^{-s_j y}\right)e^{-\beta y}\,\mathrm{d}y,$$

we obtain

$$\prod_{j=1}^{n}\left(1+\frac{\delta-c_1 s_0}{\lambda_j}\right)q_0 e^{-s_0 u}+\beta e^{\beta u}\int_u^{\infty}\left(\sum_{j=1}^{l}\sum_{k=0}^{\kappa_j-1}q_{j,k}y^k e^{-s_j y}\right)e^{-\beta y}\,\mathrm{d}y$$

$$=q_0\beta e^{\beta u}\int_u^{b}e^{(-s_0-\beta)y}\,\mathrm{d}y+\beta e^{\beta u}\int_u^{b}\left(\sum_{j=1}^{l}\sum_{k=0}^{\kappa_j-1}q_{j,k}y^k e^{-s_j y}\right)e^{-\beta y}\,\mathrm{d}y+\beta e^{\beta u}\int_b^{\infty}V_{2,\delta}(y,b)e^{-\beta y}\,\mathrm{d}y.$$

Thus,

$$\prod_{j=1}^{n}\left(1+\frac{\delta-c_1 s_0}{\lambda_j}\right)q_0 e^{-s_0 u}=\frac{q_0\beta\left(e^{(-s_0-\beta)b+\beta u}-e^{-s_0 u}\right)}{-s_0-\beta}$$
$$+\beta e^{\beta u}\int_b^{\infty}\left[V_{2,\delta}(y)-\sum_{j=1}^{l}\sum_{k=0}^{\kappa_j-1}q_{j,k}y^k e^{-s_j y}\right]e^{-\beta y}\,\mathrm{d}y.$$

Since $e^{-s_0 u}$ is one of the general solutions of (3.6.9), we have

$$\prod_{j=1}^{n}\left(1+\frac{\delta-c_1 s_0}{\lambda_j}\right)q_0 e^{-s_0 u}=\frac{q_0\beta e^{-s_0 u}}{s_0+\beta}.$$

Thus the coefficient of term $e^{\beta u}$ must be zero, i.e.,

$$\frac{q_0 e^{(-s_0-\beta)b}}{s_0+\beta}=\int_b^{\infty}\left[V_{2,\delta}(y,b)-\sum_{j=1}^{l}\sum_{k=0}^{\kappa_j-1}q_{j,k}y^k e^{-s_j y}\right]e^{-\beta y}\,\mathrm{d}y$$
$$=\frac{c_2-c_1}{\beta\delta}e^{-\beta b}-\sum_{j=1}^{m}\sum_{k=0}^{\nu_j-1}\frac{k!\,r_{j,k}e^{-\beta b}}{(\beta+R_j(\delta))^{k+1}}-\sum_{j=1}^{l}\sum_{k=0}^{\kappa_j-1}\frac{\Gamma\left(k+1,b(s_j+\beta)\right)q_{j,k}}{(\beta+s_j)^{k+1}},\tag{3.6.12}$$

where $\Gamma(n,x)=\int_x^{\infty}t^{n-1}e^{-t}\,\mathrm{d}t$ is the upper incomplete Gamma function. Therefore, we conclude that

$$V_\delta(u,b)=\begin{cases}q_0 e^{-s_0 u}+\sum_{j=1}^{l}\left(\sum_{k=0}^{\kappa_j-1}q_{j,k}u^k\right)e^{-s_j u}, & 0<u\le b\\ \frac{c_2-c_1}{\delta}-\sum_{j=1}^{m}\left[\sum_{k=0}^{\nu_j-1}r_{j,k}(u-b)^k\right]e^{-R_j(\delta)(u-b)}, & u\ge b\end{cases},$$

where the $2n+1$ unknown parameters $q_{j,k}$, q_0 and $r_{j,k}$ are obtained by solving the linear-equation system that consists of $2n+1$ equations and includes systems (3.6.10) and (3.6.11) together with equations (3.6.12).

3.6.3 A numerical example

We consider a Sparre-Andersen dual model with a generalized Erlang-3 distributed inter-innovation times with parameters $\lambda_1 = 0.12$, $\lambda_2 = 0.6$, and $\lambda_3 = 0.81$. If we assume that the income distribution is an exponential with $\beta = 0.1$ and the discount factor is $\delta = 0.08$, then under the threshold strategy with $c_1 = 0.5$ and $c_2 = 1$, the generalized Lundberg's equation for initial capital larger than the threshold is

$$\left(1 + \frac{0.08 - s}{0.12}\right)\left(1 + \frac{0.08 - s}{0.6}\right)\left(1 + \frac{0.08 - s}{0.81}\right) = \frac{1}{10s + 1}. \tag{3.6.13}$$

By solving (3.6.13), we obtain that apart from a negative root $R_- = -0.07$, there are two positive roots $\rho_1 = 0.14$ and $\rho_2 = 0.8$ with multiplicities $\nu_1 = 1$ and $\nu_2 = 2$, respectively. Thus, for all $u > b$ the expected discounted total dividend paid before ruin $V_{0.08}(u, b)$ has the form

$$V_{0.08}(u, b) = \frac{1 - 0.5}{0.08} - r_1 e^{-0.14(u-b)} + \left[r_{2,0} + r_{2,1}(u - b)\right] e^{-0.8(u-b)}, \quad u > b, \tag{3.6.14}$$

with the unknown parameters r_1, $r_{2,0}$ and $r_{2,1}$ to be determined. Similarly, by solving equation

$$\left(1 + \frac{0.08 - 0.5s}{0.12}\right)\left(1 + \frac{0.08 - 0.5s}{0.6}\right)\left(1 + \frac{0.08 - 0.5s}{0.81}\right) = \frac{1}{10s + 1}$$

we obtain the values of s_j, $j = 0, 1, 2, 3$ as listed in Table 3.1 below.

s_0	s_1	s_2	s_3
-0.0614	0.3272	1.4464	1.7277

Table 3.1: The values of s_j for $j = 0, 1, 2, 3$

Therefore, when $0 < u < b$, the expected discounted dividend paid before ruin has the form

$$V_{0.08}(u, b) = q_0 e^{0.0614u} + q_1 e^{-0.3272u} + q_2 e^{-1.4464u} + q_3 e^{-1.7277u}, \quad 0 < u < b, \tag{3.6.15}$$

where q_1, q_2, and q_3 satisfy the following Vandermonde linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.3272 & 1.4464 & 1.7277 \\ 0.3272^2 & 1.4464^2 & 1.7277^2 \end{bmatrix} \begin{bmatrix} -q_1/q_0 \\ -q_2/q_0 \\ -q_3/q_0 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.0614 \\ (-0.0614)^2 \end{bmatrix}. \tag{3.6.16}$$

The solution to (3.6.16) suggests that for $0 < u < b$,

$$V_{0.08}(u, b) = q_0\left(e^{0.0614u} - 1.7210e^{-0.3272u} + 2.2083e^{-1.4464u} - 1.4873e^{-1.7277u}\right). \tag{3.6.17}$$

Now, there are only four unknown parameters, r_1, $r_{2,0}$, $r_{2,1}$, and q_0 to be determined by specifying the threshold b and then solving the linear equation system comprised by (3.6.11) and (3.6.12), namely,

$$\begin{bmatrix} 1 & 1 & 0 & \chi(b) \\ 0.14 & 0.8 & -1 & \chi'(b) \\ 0.14^2 & 0.8^2 & -0.16 & \chi''(b) \\ 0.24^{-1} & 0.9^{-1} & 0.9^{-2} & \zeta(b) \end{bmatrix} \begin{bmatrix} r_1 \\ r_{2,0} \\ r_{2,1} \\ q_0 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 0.5 \\ 0.04 \\ 62.5 \end{bmatrix},$$

where

$$\chi(x) = e^{0.0614x} - 1.7210e^{-0.3272x} + 2.2083e^{-1.4464x} - 1.4873e^{-1.7277x}$$

and

$$\zeta(x) = \frac{e^{0.0614x}}{0.1 - 0.0614} - \frac{1.7210e^{-0.3272x}}{0.1 + 0.3272} + \frac{2.2083e^{-1.4464x}}{0.1 + 1.4464} - \frac{1.4873e^{-1.7277x}}{0.1 + 1.7277}.$$

We calculate the values of r_1, $r_{2,0}$, $r_{2,1}$, and q_0 for $b = 4$, 8, 12, 16, and 20. The results are given in Table 3.2.

b	r_1	$r_{2,0}$	$r_{2,1}$	q_0
4	5.4401	-0.2170	-0.0521	1.2545
8	4.9567	-0.2210	-0.0538	1.0040
12	4.8516	-0.2233	-0.0544	0.7894
16	4.8293	-0.2238	-0.0545	0.6182
20	4.8246	-0.2239	-0.0545	0.4838

Table 3.2: The values of r_1, $r_{2,0}$, $r_{2,1}$, and q_0 for $b = 4$, 8, 12, 16, and 20

Figure 3.1 displays the graph of $V_{0.08}(u, b)$ for $b = 4$, 8, 12, 16, and 20 as well as $V_{0.08}(u, u)$ as a function of u. The function $V_{0.08}(u, u)$ is of some interest not only because it comprises the non-differentiable point of $V_{0.08}(u, b)$ for different $b > 0$ but also because it is important in calculating the optimal threshold [see 2, Section 5.]. Finally, all the five curves share the same properties as those in [2, Fig. 5]:

1. If u is fixed, then $V_{0.08}(u, b)$ decreases as b increases.

2. If b is fixed, then $V_{0.08}(u, b)$ increases up to the upper bound $\dfrac{c_2 - c_1}{\delta} = 6.25$ as u increases.

3. All curves are non-differentiable at $u = b$ due to (3.6.3), which shows that the left and the right derivatives of V do not coinside.

4. $V_{0.08}(u, u)$ is a monotone increasing function of u.

Appendix

3.A Some auxiliary results

Lemma 3.A.1 *Suppose that $f(t)$ is the density function of the generalized Erlang-n distribution with parameters $\lambda_1, \ldots, \lambda_n > 0$ and $g_c(t)$ is defined by (3.2.3). Let $g_c^{(i)}(t)$ represent the i-th derivative of $g_c(t)$, then $g_c^{(i)}(0) = 0,\ i = 0, 1, \ldots, n-2$, and $g_c^{(n-1)}(0) = \frac{1}{c^n}\prod_{i=1}^{n} \lambda_i$.*

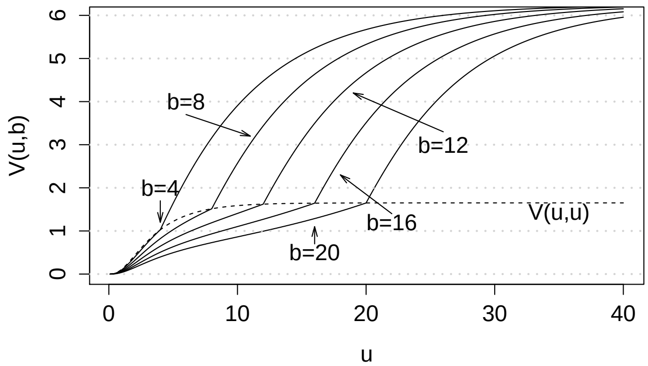

Figure 3.1: $V_{0.08}(u, b)$ for $b = 4$, 8, 12, 16 and 20.

Proof The Laplace transform of $g_c(t)$ is provided by formula (3.2.4). And the Laplace transform of $g_c^{(n)}(t)$ is presented as

$$\int_0^\infty e^{-st} g_c^{(n)}(t)\,\mathrm{d}t = s^n \tilde{g}_c(s) - \sum_{i=0}^{n-1} s^{n-1-i} g_c^{(i)}(0). \tag{3.A.1}$$

By letting $s \to \infty$, the left hand side of (3.A.1) has to go to 0. Thus we obtain

$$\begin{aligned} 0 &= \lim_{s\to\infty}\left[s^n \tilde{g}_c(s) - \sum_{i=0}^{n-2} s^{n-1-i} g_c^{(i)}(0)\right] - g_c^{(n-1)}(0) \\ &= \lim_{s\to\infty}\prod_{i=1}^{n} \frac{\lambda_i}{\frac{\lambda_i+\delta}{s} + c} - \lim_{s\to\infty}\sum_{i=0}^{n-2} s^{n-1-i} g_c^{(i)}(0) - g_c^{(n-1)}(0) \\ &= \frac{1}{c^n}\prod_{i=1}^{n}\lambda_i - \lim_{s\to\infty}\sum_{i=0}^{n-2} s^{n-1-i} g_c^{(i)}(0) - g_c^{(n-1)}(0), \end{aligned}$$

which yields $g_c(0) = g_c'(0) = \ldots = g_c^{(n-2)}(0) = 0$ and $g_c^{(n-1)}(0) = \frac{1}{c^n}\prod_{i=1}^{n}\lambda_i$ as needed.

Lemma 3.A.2 *Suppose that $f(t)$ is the density function of the generalized Erlang-n distribution with parameters $\lambda_1, \ldots, \lambda_n > 0$ and $g_c(t)$ is defined by (3.2.3). Then for all $\alpha \in \mathbb{R}$,*

$$\left[\prod_{i=1}^{n}\left(\lambda_i + \delta + c\frac{\mathrm{d}}{\mathrm{d}t}\right)\right] g_c(t+\alpha) = 0. \tag{3.A.2}$$

Proof We first consider the case $\alpha = 0$, i.e.

$$\left[\prod_{i=1}^{n}\left(\lambda_i + \delta + c\frac{\mathrm{d}}{\mathrm{d}t}\right)\right] g_c(t) = 0. \tag{3.A.3}$$

Consider the elementary symmetric functions of $\lambda_1 + \delta, \lambda_2 + \delta, \ldots, \lambda_n + \delta$ defined by (3.3.5). Then

$$\left[\prod_{i=1}^{n}\left(\lambda_i + \delta + c\frac{\mathrm{d}}{\mathrm{d}t}\right)\right] g_c(t) = \sum_{i=0}^{n} \sigma_i c^i g_c^{(i)}(t).$$

Hence, taking Laplace transforms on both sides yields

$$\int_0^\infty e^{-st}\left[\prod_{i=1}^{n}\left(\lambda_i + \delta + c\frac{\mathrm{d}}{\mathrm{d}t}\right)\right] g_c(t)\,\mathrm{d}t = \sum_{i=0}^{n} \sigma_i c^i \int_0^\infty e^{-st} g_c^{(i)}(t)\,\mathrm{d}t.$$

Lemma 3.A.1 indicates that

$$\int_0^\infty e^{-st} g_c^{(i)}(t)\,\mathrm{d}t = \begin{cases} s^i \tilde{g}_c(s), & i = 0, 1, \ldots, n-1 \\ s^n \tilde{g}_c(s) - \dfrac{1}{c^n}\displaystyle\prod_{i=1}^{n} \lambda_i, & i = n \end{cases},$$

and therefore,

$$\begin{aligned} \int_0^\infty e^{-st}\left[\prod_{i=1}^{n}\left(\lambda_i + \delta + c\frac{\mathrm{d}}{\mathrm{d}t}\right)\right] g_c(t)\,\mathrm{d}t &= \tilde{g}_c(s) \sum_{i=0}^{n} \sigma_i c^i s^i - \prod_{i=1}^{n} \lambda_i \\ &= \left(\prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + cs + \delta}\right) \prod_{i=1}^{n} (\lambda_i + cs + \delta) - \prod_{i=1}^{n} \lambda_i = 0. \end{aligned}$$

Finally, (3.A.3) follows by the inversion of the Laplace transforms.

For the general case $\alpha \neq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g_c(t+\alpha) = \frac{\mathrm{d}}{\mathrm{d}(t+\alpha)} g_c(t+\alpha) \cdot \frac{\mathrm{d}(t+\alpha)}{\mathrm{d}t} = g_c'(x)\big|_{x=t+\alpha}$$

by the chain rule. We deduce then by induction that

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i} g_c(t+\alpha) = \frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}}\left(g_c'(x)\big|_{x=t+\alpha}\right) = \ldots = g^{(i)}(x)\big|_{x=t+\alpha}.$$

Therefore,

$$\begin{aligned} \left[\prod_{i=1}^{n}\left(\lambda_i + \delta + c\frac{\mathrm{d}}{\mathrm{d}t}\right)\right] g_c(t+\alpha) &= \sum_{i=0}^{n} \sigma_i c^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} g_c(t+\alpha) = \sum_{i=0}^{n} \sigma_i c^i \left(g^{(i)}(x)\big|_{x=t+\alpha}\right) \\ &= \left(\sum_{i=0}^{n} \sigma_i c^i g^{(i)}(x)\right)\Bigg|_{x=t+\alpha} = 0. \end{aligned}$$

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Chapter 4

On the threshold strategy for paying dividends under the dual Lévy risk model

4.1 Introduction

There is a substantial body of literature focusing on the dividend problem in recent years. Of main interest are the expected discounted total dividends paid to shareholders and the optimal strategy that maximizes these dividends. In particular, the dividend problem under the classical compound Poisson model is studied thoroughly. The extensive results are mainly due to the fact that the surplus process is essentially a spectrally negative Lévy process with a positive drift. Namely,

$$U(t) = u + X(t), \quad t \geq 0,$$

where $u > 0$ is the initial capital of the insurance company and the spectrally negative Lévy process $\{X(t) : t \geq 0\}$ describing the evolution in surplus does not have monotone sample paths. Various dividend strategies including the barrier strategy and the threshold strategy are discussed for this type of models. Under the barrier strategy, an elegant solution is found by Avram et al. [1] in identity (5.1), while Frostig [2] develops a more general result for Markovian arrival risk processes. In addition, Wang and Yin [3] study the moment generating function of the discounted aggregate dividend payments until absolute ruin by allowing borrowing money with a debit interest rate when the surplus is negative. It is worth mentioning that the optimal dividend strategy might be a non-barrier strategy (see Azcue and Muler [4, Section 10.1]). The threshold dividend strategy is proved to be optimal when the dividend rate is restricted by a constant (see Kyprianou et al. [5]). Also, the expression satisfied by the expected discounted aggregate dividends is in terms of defective renewal equations under a Sparre-Andersen model perturbed by diffusion (see Meng et al. [6]).

The dividend problem under the dual model, which is used to describe the life-annuity insurance business (see Cramér [7, Section 5.13]), is first considered in Avanzi et al. [8]. Under this type of models, the surplus process is

$$R(t) = u - ct + S(t), \quad t \geq 0, \tag{4.1.1}$$

where $u > 0$ is the total single premium of some life annuities, $c > 0$ is assumed to be the rate at which annuitants are paid continuously, and $\{S(t) : t \geq 0\}$ is the gross reserve that is freed due to the death of annuitants by time t. Model (4.1.1) further assumes that $S(t)$ is a compound Poisson process. Apart from the life-annuity interpretation, the dual model may also be utilized to describe the surplus of companies whose income is due to inventions or discoveries, for instance, pharmaceutical or petroleum companies (see Avanzi et al. [8], Landriault and Sendova [9], Song et al. [10]). In Avanzi and Gerber [11], the authors study the expected discounted dividends until ruin for a diffusion-perturbed version of model (4.1.1) under a barrier strategy by integro-differential equations. Consequently, they obtain an explicit solution when the occasional income follows an exponential distribution and a mixture of exponential distributions, respectively.

The optimal dividend strategy under the dual model is the barrier strategy (see Bayraktar et al. [12, Theorem 2.1]) as long as the surplus process belongs to the class of spectrally positive Lévy process without monotonic sample paths. In practice, however, it is not advisable to implement this strategy as ruin occurs with probability one. For this reason, Ng [13] proposes the threshold strategy that pays dividends at a rate $\omega > 0$ instead of paying to the shareholders the entire excess ($\omega = \infty$) of the current surplus over the threshold. This modification makes the threshold strategy more flexible. Based on the analysis of a threshold dividend strategy, one may easily extend the results to a more general case, namely, the multi-threshold dividend strategy.

A particular difficulty that was not resolved in the literature is to deduce an explicit expression of the expected discounted dividends when the initial capital is below the barrier or threshold. More precisely, identity (2.4) in Avanzi and Gerber [11] is a second order integro-differential equation satisfied by the expected discounted dividends up to ruin but the authors provide an explicit solution only when the gains' distribution is a finite mixture of exponential distributions. Similar is the situation in the simpler case when $\sigma = 0$ in (4.1.1), which is considered by Ng [13]. The author provides an explicit solution to the first order integro-differential equation satisfied by the expected discounted dividends (see Ng [13, equation (5)]) when the initial surplus is greater than the threshold and deduces an explicit solution for an initial surplus lower than the threshold in the case of gains that have a distribution that is a finite mixture of exponential distributions.

In this paper, we consider the threshold dividend strategy under the dual risk model with general Lévy assumptions, which is a natural generalization of the dual model with surplus process (4.1.1). We attempt to resolve the above difficulty by approaching the problem from a different angle. Namely, the presence of the threshold allows us to build connections between the dual model and a certain spectrally negative Lévy process and to obtain subsequently explicit results through fluctuation identities. Consequently, we are able to fill in the gap in the existing literature by providing explicit solutions for the quantities of interest when the initial capital is below the threshold.

This paper is organized as follows: Section 4.2 provides further details regarding the model. Then, for a given dividend threshold $b > 0$, the expected discounted dividends under the dual Lévy risk model with initial surplus $u > 0$ are discussed in Sections 4.3. In Section 4.4, we derive the optimal threshold when the threshold strategy is applied and verify the optimality of the threshold strategy assuming a ceiling dividend rate. Finally, an example of the dual Lévy risk model with gain-size probability density function (p.d.f.) that has a rational Laplace transform is discussed in Section 4.5.

4.2 Model settings

4.2.1 The threshold dividend strategy

Suppose the surplus process of a business may be described by the dual model $\{R(t) : t \geq 0\}$ which is defined as

$$R(t) = u - X(t), \quad t \geq 0, \tag{4.2.1}$$

where $\{X(t) : t \geq 0\}$ is a spectrally negative Lévy process defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbf{P})$ with $\{\mathscr{F}_t\}_{t\geq 0}$ being the natural filtration. The Laplace exponent of $\{X(t) : t \geq 0\}$ is defined by

$$\psi(\theta) = \log\left(\mathbf{E}\left[e^{\theta X(1)}\right]\right) = \frac{\sigma^2\theta^2}{2} + \gamma\theta - \int_{(0,\infty)} \left(1 - e^{-\theta x} - \theta x \mathbf{1}_{\{x<1\}}\right) \nu(\mathrm{d}x) \tag{4.2.2}$$

where $\theta \geq 0$, $\gamma \in \mathbb{R}$, $\sigma \geq 0$, $\mathbf{1}_{\{\cdot\}}$ is the indicator function, and the Lévy measure ν satisfies $\nu(-\infty, 0) = 0$ and

$$\int_{(0,\infty)} \left(1 \wedge x^2\right) \nu(\mathrm{d}x) < \infty.$$

We denote the law and the expectation with respect to $\{X(t) : t \geq 0\}$ issued at $x \in \mathbb{R}$ (i.e., $X(0) = x$ almost surely) by $\mathbf{P}_x$ and $\mathbf{E}_x$, respectively. If $\{X(t) : t \geq 0\}$ has paths of bounded variation, then $\{X(t) : t \geq 0\}$ can be written as $X(t) = ct - S(t)$ uniquely where

$$c = \gamma + \int_{(0,1)} x\nu(\mathrm{d}x) \in \left(0, \mathbf{E}[S(1)]\right), \quad \mathbf{E}[S(1)] = \int_0^\infty x\nu(\mathrm{d}x). \tag{4.2.3}$$

Condition (4.2.3) is called *negative loading condition*, which prevents the sample paths of $\{R(t) : t \geq 0\}$ from drifting to $-\infty$ with probability 1.

Now, suppose a threshold dividend strategy with constant threshold $b > 0$ and dividend rate $\omega > 0$ is applied to $\{R(t) : t \geq 0\}$, which results in the modified surplus process $\{R_b(t) : t \geq 0\}$ after the dividends are paid. This modified surplus process is driven by the stochastic differential equation

$$\mathrm{d}R_b(t) = -\omega \mathbf{1}_{\{R_b(t)>b\}}\,\mathrm{d}t - \mathrm{d}X(t), \quad R_b(0) = u. \tag{4.2.4}$$

If we define $\{X_\omega(t) : t \geq 0\}$ by $X_\omega(t) = X(t) + \omega t$, then equation (4.2.4) may also be written as

$$\mathrm{d}R_b(t) = \omega \mathbf{1}_{\{R_b(t)\leq b\}}\,\mathrm{d}t - \mathrm{d}X_\omega(t), \quad R_b(0) = u. \tag{4.2.5}$$

where $\{X_\omega(t) : t \geq 0\}$ is also a spectrally negative Lévy process. The corresponding Laplace exponent $\psi_\omega(\theta)$ of $\{X_\omega(t) : t \geq 0\}$ is defined by

$$\psi_\omega(\theta) = \psi(\theta) + \omega\theta, \quad \theta \geq 0 \tag{4.2.6}$$

and in particular, $\psi_0(\theta) = \psi(\theta)$ for all $\theta \geq 0$.

Remark The modified process $\{R_b(t) : t \geq 0\}$ is called *refracted Lévy processes* and was first discussed by Kyprianou and Loeffen [14], who consider the spectrally negative case.

For the surplus process $\{R_b(t) : t \geq 0\}$, the first passage times of some level $x \in \mathbb{R}$ are defined as

$$T_x^+ = \inf\{t > 0 : R_b(t) \geq x\} \quad \text{and} \quad T_x^- = \inf\{t > 0 : R_b(t) \leq x\}, \quad x \in \mathbb{R},$$

(with the convention that $\inf \varnothing = \infty$) which allows us to formally define the expected discounted aggregate dividends before ruin as

$$V^{(\delta)}(u,b) = \mathbf{E}\left[\omega \int_0^{T_0^-} e^{-\delta t} \mathbf{1}_{\{R_b(t)>b\}} \, \mathrm{d}t \middle| R_b(0) = u\right].$$

If we define the running maximum and running minimum of some stochastic process $\{Y(t) : t \geq 0\}$ as

$$\overline{Y}(t) = \sup_{s \in [0,t]} Y(s) \quad \text{and} \quad \underline{Y}(t) = \inf_{s \in [0,t]} Y(s),$$

then the function $V^{(\delta)}(u,b)$ may also be expressed as

$$V^{(\delta)}(u,b) = \omega \int_0^{\infty} e^{-\delta t} \, \mathbf{P}\left[R_b(t) > b, \underline{R}_b(t) \geq 0 | R_b(0) = u\right] \mathrm{d}t. \tag{4.2.7}$$

In general, $V^{(\delta)}(u,b)$ has different representations depending on whether $0 < u < b$ or $u > b$. Hence, for mathematical clarity, we rewrite $V^{(\delta)}(u,b)$ as

$$V^{(\delta)}(u,b) = \begin{cases} V_1^{(\delta)}(u,b), & 0 < u < b \\ V_2^{(\delta)}(u,b), & u > b \end{cases}$$

(see Ng [13, p. 316]). Observe that for the special case $u = b$, $V^{(\delta)}(b,b)$ is a constant that needs to be determined additionally.

4.2.2 Scale functions

Most quantities of interest in ruin theory under Lévy risk models can be expressed explicitly in terms of scale functions. The most basic scale function is $W^{(q)}(x) : \mathbb{R} \mapsto [0,\infty)$ which is defined by its Laplace transform

$$\int_0^{\infty} e^{-\theta x} W^{(q)}(x) \, \mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q), \tag{4.2.8}$$

with $\Phi(q) = \Phi_0(q)$ where

$$\Phi_\omega(q) = \sup\{\xi \geq 0 : \psi_\omega(\xi) = q\} \tag{4.2.9}$$

more generally. Moreover, $W^{(q)}(x) = 0$ for all $x \in (-\infty, 0)$. In ruin theory, the function $W^{(q)}(x)$ in the bounded variation case plays the same role as the function $v(u)$ in equation (3.3) in Lin et al. [15]. In addition to $W^{(q)}(x)$, another important function is

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) \, \mathrm{d}y, \quad x \in \mathbb{R}, \tag{4.2.10}$$

which is also widely used in fluctuation theory of Lévy process (see Avram et al. [1, p. 160]). The functions $W^{(q)}(x)$ and $Z^{(q)}(x)$ are called *q-scale functions* and their versions under the exponential change of measure with respect to any $a \in \mathbb{R}$

$$\left.\frac{\mathrm{d}\mathbf{P}_x^a}{\mathrm{d}\mathbf{P}_x}\right|_{\mathscr{F}_t} = e^{a[X(t)-x]-\psi(a)t}$$

are denoted as $W_a^{(q)}(x)$ and $Z_a^{(q)}(x)$ throughout this article. Then Kyprianou [16, Lemma 8.4] provides the following relation between $W^{(q)}(x)$ and $W_a^{(q)}(x)$:

$$W_a^{(q)}(x) = e^{-ax} W^{(q+\psi(a))}(x), \quad x \in \mathbb{R}, \tag{4.2.11}$$

implying that

$$Z_a^{(q)}(x) = 1 + q\int_0^x e^{-ay} W^{(q+\psi(a))}(y)dy.$$

Remark Observe that notation with a subscript ω or a such as $X_\omega(t), \psi_\omega(\theta), \Phi_\omega(q), W_a^{(q)}(x)$, and $Z_a^{(q)}(x)$ reduce to their versions without a subscript when that subscript equals 0. (See, for instance, comments regarding identities (4.2.6) and (4.2.9).)

We note the following result which is required for further derivations.

Lemma 4.2.1 *Suppose that* $\{X(t) : t \geq 0\}$ *is a spectrally negative Lévy process without monotonic sample paths that is issued at* $x \in (0, a)$ *for some* $a > 0$*. If* $\theta \geq 0$ *and* $q > 0$ *are such that* $q + \psi(\theta) > 0$ *and* $\tau = \inf\{t > 0 : X(t) \notin (0, a)\}$*, then for a fixed* $t > 0$,

$$M_\theta^{(q)}(\tau \wedge t) = e^{-[q+\psi(\theta)](\tau\wedge t)+\theta X(\tau\wedge t)} Z_\theta^{(q)}\left(X(\tau \wedge t)\right), \tag{4.2.12}$$

is a $\mathbf{P}_x -$ *martingale.*

Proof Since $t > 0$, there exists s such that $0 \leq s < t$. Then

$$\mathbf{E}_x\left[M_\theta^{(q)}(\tau \wedge t)\middle|\mathscr{F}_s\right] = M_\theta^{(q)}(\tau \wedge t)\mathbf{1}_{\{s\geq\tau\wedge t\}} + \mathbf{E}_x\left[M_\theta^{(q)}(\tau \wedge t)\middle|\mathscr{F}_s\right]\mathbf{1}_{\{s<\tau\wedge t\}}.$$

For $s < \tau \wedge t$, we have by the the strong Markov property that

$$\mathbf{E}_x\left[M_\theta^{(q)}(\tau \wedge t)\middle|\mathscr{F}_s\right] = e^{-[q+\psi(\theta)]s}\mathbf{E}_{X(s)}\left[M_\theta^{(q)}\left(\tau \wedge (t-s)\right)\right]. \tag{4.2.13}$$

Then by the exponential change of measure

$$\mathbf{E}_{X(s)}\left[M_\theta^{(q)}\left(\tau \wedge (t-s)\right)\right] = e^{\theta X(s)}\mathbf{E}_{X(s)}^\theta\left[e^{-q[\tau\wedge(t-s)]}Z_\theta^{(q)}\left(X(\tau \wedge (t-s))\right)\right]. \tag{4.2.14}$$

Since $e^{-q(\tau\wedge t)}Z_\theta^{(q)}\left(X(\tau \wedge t)\right)$ is a $\mathbf{P}_x^\theta$-martingale (see Avram et al. [17, p. 220]), we have

$$\mathbf{E}_{X(s)}^\theta\left[e^{-q[\tau\wedge(t-s)]}Z_\theta^{(q)}\left(X(\tau \wedge (t-s))\right)\right] = Z_\theta^{(q)}\left(X(s)\right), \quad s < t. \tag{4.2.15}$$

Thus, combining (4.2.13), (4.2.14) and (4.2.15), we obtain

$$\mathbf{E}_x\left[M_\theta^{(q)}(\tau \wedge t)\middle|\mathscr{F}_s\right] = M_\theta^{(q)}(s), \quad s < \tau \wedge t,$$

and therefore

$$\mathbf{E}_x\left[M_\theta^{(q)}(\tau \wedge t)\middle|\mathscr{F}_s\right] = M_\theta^{(q)}(\tau \wedge t)\mathbf{1}_{\{s\geq\tau\wedge t\}} + M_\theta^{(q)}(s)\mathbf{1}_{\{s<\tau\wedge t\}} = M_\theta^{(q)}(\tau \wedge t \wedge s) = M_\theta^{(q)}(\tau \wedge s)$$

as required.

4.3 The expected discounted aggregate dividends

In this section, we focus on deriving an explicit formula for $V^{(\delta)}(u,b)$ for all $u>0$. Our main result is stated in the following theorem.

Theorem 4.3.1 *Under the threshold dividend strategy with threshold $b>0$, dividend rate $\omega>0$, and discount factor $\delta>0$, the expected discounted aggregate dividends paid by the line of business $\{R(t):t\geq 0\}$ until ruin with initial surplus $u>0$ satisfy*

$$V^{(\delta)}(u,b)=\frac{\omega}{\delta}\left[Z^{(\delta)}(b-u)-\frac{Z^{(\delta)}(b)}{Q_{\omega}^{(\delta)}(b)}Q_{\omega}^{(\delta)}(b-u)\right],\quad u>0, \tag{4.3.1}$$

where

$$Q_{\omega}^{(\delta)}(x)=e^{\Phi_{\omega}(\delta)x}Z_{\Phi_{\omega}(\delta)}^{(\omega\Phi_{\omega}(\delta))}(x)=e^{\Phi_{\omega}(\delta)x}+\omega\Phi_{\omega}(\delta)\int_0^x e^{\Phi_{\omega}(\delta)(x-y)}W^{(\delta)}(y)\,\mathrm{d}y. \tag{4.3.2}$$

Proof Consider a dividend payment strategy π^{ε} with a parameter $\varepsilon\geq 0$ constructed as follows. The strategy π^{ε} pays dividends at rate $\omega>0$ whenever the corresponding modified surplus process $\left\{R_b^{\varepsilon}(t):t\geq 0\right\}$ stays above level $b>0$. Otherwise, no dividend is paid if the surplus level stays below $b-\varepsilon>0$. If the surplus at time t falls into the interval $[b-\varepsilon,b)$, then dividends are paid at rate $\omega>0$ only when the surplus enters the interval from above, i.e. neither $R_b^{\varepsilon}(0)=u\in[b-\varepsilon,b)$ nor the surplus entering $[b-\varepsilon,b)$ from below will lead to dividend payments. A graphical comparison of the modified surplus processes between dividend strategy π^{ε} (black solid line) and the threshold dividend strategy (purple solid line if it is different from the surplus process of π^{ε}) is given in Figure 4.1. In Figure 4.1, when for the first time $\left\{R_b^{\varepsilon}(t):t\geq 0\right\}$ enters

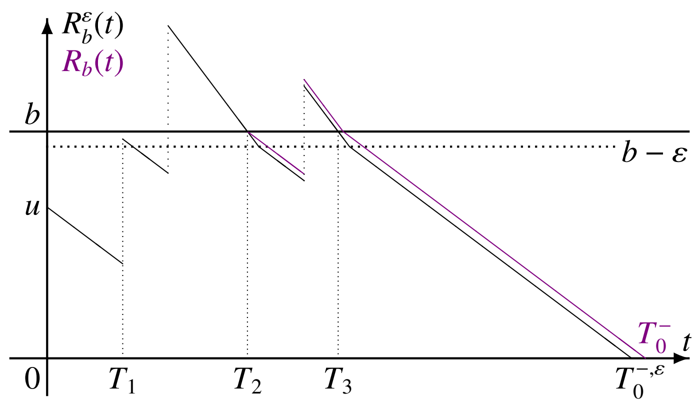

Figure 4.1: Comparison of sample paths of the modified surplus processes

$[b-\varepsilon,b)$ from below at T_1, no dividends are paid. However, when the surplus process revisits the interval $[b-\varepsilon,b)$ from above at T_2 and T_3, dividends are paid at rate $\omega>0$ as long as the surplus remains in the interval $[b-\varepsilon,b)$. Based on our construction of dividend strategy π^{ε}, we have

$$R_{b-\varepsilon}(t)\leq R_b^{\varepsilon}(t)\leq R_b(t)$$

almost surely for all $t \geq 0$. Moreover, the difference between $\{R_b(t) : t \geq 0\}$ and $\{R_{b-\varepsilon}(t) : t \geq 0\}$ is

$$R_b(t) - R_{b-\varepsilon}(t) = \omega \int_0^t \mathbf{1}_{\{R_{b-\varepsilon}(s) \in [b-\varepsilon, b)\}} \, \mathrm{d}s,$$

which is an increasing function of t and thus

$$\sup_{s \in [0,t]} \left| R_{b-\varepsilon}(s) - R_b(s) \right| = R_b(t) - R_{b-\varepsilon}(t) = \omega \int_0^t \mathbf{1}_{\{R_{b-\varepsilon}(s) \in [b-\varepsilon, b)\}} \, \mathrm{d}s \to 0$$

as $\varepsilon \downarrow 0$ in almost sure sense, which results in

$$\sup_{s \in [0,t]} \left| R_b^\varepsilon(s) - R_b(s) \right| \leq \sup_{s \in [0,t]} \left| R_{b-\varepsilon}(s) - R_b(s) \right| \to 0$$

as $\varepsilon \downarrow 0$ in almost sure sense. Therefore, by the arguments in Kyprianou and Loeffen [14, p. 38] we may conclude

$$\lim_{\varepsilon \downarrow 0} \left(R_b^\varepsilon(t), \underline{R}_b^\varepsilon(t) \right) = \left(R_b(t), \underline{R}_b(t) \right)$$

almost surely for all $t \geq 0$.

Now, define the dividend-rate process $\left\{ J_b^\varepsilon(t) : t \geq 0 \right\} \in \{0, \omega\}$ corresponding to the dividend strategy π^ε. The corresponding expected discounted aggregate dividends are defined by

$$V^{(\delta)}(u, b, \varepsilon) = \mathbf{E} \left[\int_0^\infty e^{-\delta t} J_b^\varepsilon(t) \mathbf{1}_{\{\underline{R}_b^\varepsilon(t) > 0\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right].$$

Then, based on our construction of π^ε, we have

$$\omega \mathbf{1}_{\{R_b^\varepsilon(t) \geq b\}} \leq J_b^\varepsilon(t) \leq \omega \mathbf{1}_{\{R_b^\varepsilon(t) \geq b-\varepsilon\}}$$

almost surely. Hence, by (4.2.7), Fubini's theorem, and the dominated convergence theorem we have

$$V^{(\delta)}(u, b, \varepsilon) \geq \mathbf{E} \left[\omega \int_0^\infty e^{-\delta t} \mathbf{1}_{\{R_b^\varepsilon(t) \geq b, \underline{R}_b^\varepsilon(t) > 0\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right] \to V^{(\delta)}(u, b)$$

as $\varepsilon \downarrow 0$. On the other hand, we also have

$$\begin{aligned}
V^{(\delta)}(u, b, \varepsilon) \leq & \mathbf{E} \left[\omega \int_0^\infty e^{-\delta t} \mathbf{1}_{\{R_b^\varepsilon(t) \geq b-\varepsilon, \underline{R}_b^\varepsilon(t) > 0\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right] \\
= & \mathbf{E} \left[\omega \int_0^\infty e^{-\delta t} \mathbf{1}_{\{R_b^\varepsilon(t) \in [b-\varepsilon, b), \underline{R}_b^\varepsilon(t) > 0\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right] + \mathbf{E} \left[\omega \int_0^\infty e^{-\delta t} \mathbf{1}_{\{R_b^\varepsilon(t) \geq b, \underline{R}_b^\varepsilon(t) > 0\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right] \\
\leq & \mathbf{E} \left[\omega \int_0^\infty e^{-\delta t} \mathbf{1}_{\{R_b^\varepsilon(t) \in [b-\varepsilon, b)\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right] + \mathbf{E} \left[\omega \int_0^\infty e^{-\delta t} \mathbf{1}_{\{R_b^\varepsilon(t) \geq b, \underline{R}_b^\varepsilon(t) > 0\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right] \\
= & \omega \int_0^\infty e^{-\delta t} \mathbf{P} \left[R_b^\varepsilon(t) \in [b-\varepsilon, b) | R_b^\varepsilon(0) = u \right] \mathrm{d}t + \mathbf{E} \left[\omega \int_0^\infty e^{-\delta t} \mathbf{1}_{\{R_b^\varepsilon(t) \geq b, \underline{R}_b^\varepsilon(t) > 0\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right] \\
= & \frac{\omega}{\delta} \mathbf{P} \left[R_b^\varepsilon(\boldsymbol{e}_\delta) \in [b-\varepsilon, b) | R_b^\varepsilon(0) = u \right] + \mathbf{E} \left[\omega \int_0^\infty e^{-\delta t} \mathbf{1}_{\{R_b^\varepsilon(t) \geq b, \underline{R}_b^\varepsilon(t) > 0\}} \, \mathrm{d}t \middle| R_b^\varepsilon(0) = u \right] \\
\to & V^{(\delta)}(u, b)
\end{aligned}$$

as $\varepsilon \downarrow 0$, where $\boldsymbol{e}_\delta$ is an exponential random variable with mean $1/\delta$, which is independent of all other random variables. In the above derivation, the second-to-the-last equality is obtained by exchanging the order of expectation and integration. Subsequently, we employ the newly defined r.v. $\boldsymbol{e}_\delta$.

Finally, the two limiting results yield

$$\lim_{\varepsilon \downarrow 0} V^{(\delta)}(u, b, \varepsilon) = V^{(\delta)}(u, b), \quad u > 0. \tag{4.3.3}$$

We now discuss the explicit expression of $V^{(\delta)}(u, b, \varepsilon)$ in the cases $u > b$ and $0 < u < b$ separately.

Case 1: If $u > b > 0$, we denote $V^{(\delta)}(u, b, \varepsilon) = V_2^{(\delta)}(u, b, \varepsilon)$ and the first passage time of level x from above as

$$T_x^{-,\varepsilon} = \inf\{t > 0 : R_b^\varepsilon(t) \le x\}, \quad x \in \mathbb{R}.$$

Then the expected discounted aggregate dividends $V^{(\delta)}(u, b, \varepsilon)$ may be rewritten as

$$V^{(\delta)}(u, b, \varepsilon) = \mathbf{E}\left[\int_0^{T_0^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon(t)\, \mathrm{d}t \middle| R_b^\varepsilon(0) = u\right].$$

Notice that the sample paths of $\left\{R_b^\varepsilon(t) : t \ge 0\right\}$ are skip-free downward, which results in $T_0^{-,\varepsilon} > T_{b-\varepsilon}^{-,\varepsilon}$ almost surely. Hence, given $R_b^\varepsilon(0) = u > b > 0$, we have

$$\int_0^{T_0^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon(t)\, \mathrm{d}t = \int_0^{T_{b-\varepsilon}^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon(t)\, \mathrm{d}t + \int_{T_{b-\varepsilon}^{-,\varepsilon}}^{T_0^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon(t)\, \mathrm{d}t. \tag{4.3.4}$$

Since for all $t \in \left[0, T_{b-\varepsilon}^{-,\varepsilon}\right]$, we have $J_b^\varepsilon(t) = \omega$ as $u > b$, and then (4.3.4) reduces to

$$\int_0^{T_0^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon(t)\, \mathrm{d}t = \frac{\omega}{\delta}\left(1 - e^{-\delta T_{b-\varepsilon}^{-,\varepsilon}}\right) + e^{-\delta T_{b-\varepsilon}^{-,\varepsilon}} \int_0^{T_0^{-,\varepsilon} - T_{b-\varepsilon}^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon\left(T_{b-\varepsilon}^{-,\varepsilon} + t\right) \mathrm{d}t.$$

By taking conditional expectation on $\mathscr{F}_{T_{b-\varepsilon}^{-,\varepsilon}}$, the strong Markov property of the Lévy process $\{X_\omega(t) : t \ge 0\}$ implies

$$\mathbf{E}\left[\int_0^{T_0^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon(t)\, \mathrm{d}t \middle| \mathscr{F}_{T_{b-\varepsilon}^{-,\varepsilon}}\right] = \frac{\omega}{\delta}\left(1 - e^{-\delta T_{b-\varepsilon}^{-,\varepsilon}}\right) + e^{-\delta T_{b-\varepsilon}^{-,\varepsilon}} V^{(\delta)}(b - \varepsilon, b, \varepsilon). \tag{4.3.5}$$

Thus, taking expectations on both sides of (4.3.5) yields

$$V_2^{(\delta)}(u, b, \varepsilon) = \left(V^{(\delta)}(b - \varepsilon, b, \varepsilon) - \frac{\omega}{\delta}\right) \mathbf{E}\left[e^{-\delta T_{b-\varepsilon}^{-,\varepsilon}} \middle| R_b^\varepsilon(0) = u\right] + \frac{\omega}{\delta}. \tag{4.3.6}$$

Given $R_b^\varepsilon(0) = u > b > 0$, we have $R_b^\varepsilon(t) = u - X_\omega(t)$ for all $0 \le t \le T_{b-\varepsilon}^{-,\varepsilon}$ and thus

$$T_{b-\varepsilon}^{-,\varepsilon} = \inf\{t > 0 : X_\omega(t) \ge u - b + \varepsilon\} =: \tau_{u-b+\varepsilon}^{+,\omega},$$

which is the exiting time of $(-\infty, u-b+\varepsilon)$ for $\{X_\omega(t) : t \geq 0\}$ issued at 0. Therefore,

$$\mathbf{E}\left[e^{-\delta T_{b-\varepsilon}^{-,\varepsilon}} \Big| R_b^\varepsilon(0) = u\right] = \mathbf{E}\left[e^{-\delta \tau_{u-b+\varepsilon}^{+,\omega}}\right] = e^{-\Phi_\omega(\delta)(u-b+\varepsilon)} \tag{4.3.7}$$

[see 16, Section 8.1]. In sum, we obtain the explicit expression of $V_2^{(\delta)}(u, b, \varepsilon)$ by incorporating (4.3.7) into (4.3.6). Namely,

$$V_2^{(\delta)}(u, b, \varepsilon) = \left[V^{(\delta)}(b-\varepsilon, b, \varepsilon) - \frac{\omega}{\delta}\right] e^{-\Phi_\omega(\delta)(u-b+\varepsilon)} + \frac{\omega}{\delta}. \tag{4.3.8}$$

Also, since $Q_\omega^{(\delta)}(b-u) = e^{-\Phi_\omega(\delta)(u-b)}$ and $Z^{(\delta)}(b-u) = 1$ for $u > b$, identity (4.3.8) may be rewritten as

$$V_2^{(\delta)}(u, b, \varepsilon) = H^{(\delta)}(b-u, b, \varepsilon), \quad u > b, \tag{4.3.9}$$

where

$$H^{(\delta)}(x, b, \varepsilon) = e^{-\Phi_\omega(\delta)\varepsilon}\left[V^{(\delta)}(b-\varepsilon, b, \varepsilon) - \frac{\omega}{\delta}\right] Q_\omega^{(\delta)}(x) + \frac{\omega}{\delta} Z^{(\delta)}(x), \quad x \in \mathbb{R}. \tag{4.3.10}$$

Then it is clear that

$$V^{(\delta)}(b, b) = V^{(\delta)}(b, b, 0) = H^{(\delta)}(0, b, 0). \tag{4.3.11}$$

Identity (4.3.11) is used to obtain the unknown constant $V^{(\delta)}(b, b)$ in the following derivations.

Case 2: Now, we discuss the case $0 < u < b$ as well as the constant $V^{(\delta)}(b-\varepsilon, b, \varepsilon)$. Since there is no dividend payment from the business line $\{R(t) : t \geq 0\}$ if $T_0^{-,\varepsilon} < T_b^{+,\varepsilon}$ where

$$T_x^{+,\varepsilon} = \inf\{t > 0 : R_b^\varepsilon(t) \geq x\},$$

for $R_b^\varepsilon(0) = u \in (0, b)$, we have

$$\int_0^{T_0^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon(t)\, \mathrm{d}t = e^{-\delta T_b^{+,\varepsilon}} \mathbf{1}_{\{T_0^{-,\varepsilon} > T_b^{+,\varepsilon}\}} \int_{T_b^{+,\varepsilon}}^{T_0^{-,\varepsilon}} e^{-\delta(t - T_b^{+,\varepsilon})} J_b^\varepsilon(t)\, \mathrm{d}t.$$

By taking conditional expectation on $\mathscr{F}_{T_b^{+,\varepsilon}}$, we obtain

$$\mathbf{E}\left[\int_0^{T_0^{-,\varepsilon}} e^{-\delta t} J_b^\varepsilon(t)\, \mathrm{d}t \Big| \mathscr{F}_{T_b^{+,\varepsilon}}\right] = e^{-\delta T_b^{+,\varepsilon}} \mathbf{1}_{\{T_0^{-,\varepsilon} > T_b^{+,\varepsilon}\}} V_2^{(\delta)}\left(R_b^\varepsilon(T_b^{+,\varepsilon}), b, \varepsilon\right). \tag{4.3.12}$$

Thus, by recalling (4.3.9) we have,

$$V_1^{(\delta)}(u, b, \varepsilon) = \mathbf{E}\left[e^{-\delta T_b^{+,\varepsilon}} \mathbf{1}_{\{T_0^{-,\varepsilon} > T_b^{+,\varepsilon}\}} H^{(\delta)}\left(b - R_b^\varepsilon(T_b^{+,\varepsilon}), b, \varepsilon\right) \Big| R_b^\varepsilon(0) = u\right].$$

As $\left\{R_b^\varepsilon(t) : t \geq 0\right\}$ is equivalent to $\{R(t) : t \geq 0\}$ for all $t \in [0, T_b^{+,\varepsilon}]$, identity (4.2.1) produces

$$V_1^{(\delta)}(u, b, \varepsilon) = \mathbf{E}\left[e^{-\delta T_b^{+,\varepsilon}} H^{(\delta)}\left(b - u + X(T_b^{+,\varepsilon}), b, \varepsilon\right) \mathbf{1}_{\{T_0^{-,\varepsilon} > T_b^{+,\varepsilon}\}}\right]. \tag{4.3.13}$$

Notice that the event $T_0^{-,\varepsilon} < T_b^{+,\varepsilon} | R_b^\varepsilon(0) = u$ is equivalent to

$$\inf\{t > 0 : u - X(t) \leq 0\} < \inf\{t > 0 : u - X(t) \geq b\},$$

hence the its complement $T_0^{-,\varepsilon} \geq T_b^{+,\varepsilon} | R_b^\varepsilon(0) = u$ is equivalent to

$$\inf\{t > 0 : b - u + X(t) \geq b\} \geq \inf\{t > 0 : b - u + X(t) \leq 0\}.$$

Since

$$T_b^{+,\varepsilon} = \inf\{t > 0 : R_b^\varepsilon(t) \geq b\} = \inf\{t > 0 : b - u + X(t) \leq 0\}$$

then it follows (4.3.13) by letting

$$\tau_x^- = \inf\{t > 0 : X(t) \leq x\} \quad \text{and} \quad \tau_x^+ = \inf\{t > 0 : X(t) \geq x\}$$

for any $x \in \mathbb{R}$ that

$$V_1^{(\delta)}(u, b, \varepsilon) = \mathbf{E}_{b-u}\left[e^{-\delta\tau_0^-} H^{(\delta)}\left(X(\tau_0^-), b, \varepsilon\right) \mathbf{1}_{\{\tau_0^- < \tau_b^+\}}\right].$$

Hence, we obtain by equation (9) in Avram et al. [17]

$$\begin{aligned} \mathbf{E}_{b-u}\left[e^{-\delta(\tau_0^- \wedge \tau_b^+)} H^{(\delta)}\left(X(\tau_0^- \wedge \tau_b^+), b, \varepsilon\right)\right] &= V_1^{(\delta)}(u, b, \varepsilon) + H^{(\delta)}(b, b, \varepsilon)\mathbf{E}_{b-u}\left[e^{-\delta\tau_b^+}\mathbf{1}_{\{\tau_b^+ < \tau_0^-\}}\right] \\ &= V_1^{(\delta)}(u, b, \varepsilon) + H^{(\delta)}(b, b, \varepsilon)\frac{W^{(\delta)}(b-u)}{W^{(\delta)}(b)}. \end{aligned} \tag{4.3.14}$$

The left-hand side of (4.3.14) may be obtained via the fact that for all $t \geq 0$

$$\mathbf{E}_{b-u}\left[e^{-\delta(\tau_0^- \wedge \tau_b^+ \wedge t)} H^{(\delta)}\left(X(\tau_0^- \wedge \tau_b^+ \wedge t), b, \varepsilon\right)\right] \tag{4.3.15}$$

is a $\mathbf{P}$-martingale. To see this, since

$$Q_\omega^{(\delta)}(\tau_0^- \wedge \tau_b^+ \wedge t) = M_{\Phi_\omega(\delta)}^{(\omega\Phi_\omega(\delta))}(\tau_0^- \wedge \tau_b^+ \wedge t) \quad \text{and} \quad Z^{(\delta)}(\tau_0^- \wedge \tau_b^+ \wedge t) = M_0^{(\delta)}(\tau_0^- \wedge \tau_b^+ \wedge t),$$

we may conclude that $Q_\omega^{(\delta)}(\tau_0^- \wedge \tau_b^+ \wedge t)$ and $Z^{(\delta)}(\tau_0^- \wedge \tau_b^+ \wedge t)$ are $\mathbf{P}$-martingales by Lemma 4.2.1. Thus, (4.3.15) is a $\mathbf{P}$-martingale by the linearity. Therefore,

$$\mathbf{E}_{b-u}\left[e^{-\delta(\tau_0^- \wedge \tau_b^+)} H^{(\delta)}\left(X(\tau_0^- \wedge \tau_b^+), b, \varepsilon\right)\right] = H^{(\delta)}(b-u, b, \varepsilon), \quad 0 < u < b.$$

As a result, we have

$$V_1^{(\delta)}(u, b, \varepsilon) = H^{(\delta)}(b-u, b, \varepsilon) - \frac{H^{(\delta)}(b, b, \varepsilon)}{W^{(\delta)}(b)} W^{(\delta)}(b-u), \quad 0 < u < b. \tag{4.3.16}$$

Hence, by letting $u = b - \varepsilon$, we have

$$V^{(\delta)}(b-\varepsilon, b, \varepsilon) = H^{(\delta)}(\varepsilon, b, \varepsilon) - \frac{H^{(\delta)}(b, b, \varepsilon)}{W^{(\delta)}(b)} W^{(\delta)}(\varepsilon), \quad \forall \varepsilon \geq 0. \tag{4.3.17}$$

If $\{X(t) : t \geq 0\}$ has sample paths of bounded variation, then $\{X(t) : t \geq 0\}$ may be written in the form $X(t) = ct - S(t)$ uniquely, where $c > 0$ and $\{S(t) : t \geq 0\}$ is a pure jump subordinator. Thus by letting $\varepsilon = 0$ in (4.3.17) we obtain

$$V^{(\delta)}(b, b, 0) = V^{(\delta)}(b, b) = H^{(\delta)}(0, b, 0) - \frac{H^{(\delta)}(b, b, 0)}{cW^{(\delta)}(b)}$$

which leads to $H^{(\delta)}(b, b, 0) = 0$ by (4.3.11). Therefore,

$$V^{(\delta)}(b, b) = \frac{\omega}{\delta}\left[1 - \frac{Z^{(\delta)}(b)}{Q_\omega^{(\delta)}(b)}\right]. \tag{4.3.18}$$

If $\{X(t) : t \geq 0\}$ has sample paths of unbounded variation, then we calculate $V^{(\delta)}(b - \varepsilon, b, \varepsilon)$ from (4.3.17), namely,

$$V^{(\delta)}(b-\varepsilon, b, \varepsilon) = \frac{\omega}{\delta}\frac{\left\{\left[Z^{(\delta)}(\varepsilon) - e^{-\Phi_\omega(\delta)\varepsilon}Q_\omega^{(\delta)}(\varepsilon)\right] - \left[Z^{(\delta)}(b) - e^{-\Phi_\omega(\delta)\varepsilon}Q_\omega^{(\delta)}(b)\right]\frac{W^{(\delta)}(\varepsilon)}{W^{(\delta)}(b)}\right\}}{1 - e^{-\Phi_\omega(\delta)\varepsilon}\left[Q_\omega^{(\delta)}(\varepsilon) - \frac{Q_\omega^{(\delta)}(b)}{W^{(\delta)}(b)}W^{(\delta)}(\varepsilon)\right]}.$$

Hence by letting $\varepsilon \downarrow 0$ and the L'Hoptial rule, the continuity condition [13, see] requires

$$\begin{aligned}
V^{(\delta)}(b, b) &= V^{(\delta)}(b, b, 0) = \lim_{\varepsilon\downarrow 0} V^{(\delta)}(b-\varepsilon, b, \varepsilon)\\
&= \frac{\omega}{\delta}\lim_{\varepsilon\downarrow 0}\frac{\left[\delta e^{\Phi_\omega(\delta)\varepsilon} - \omega\Phi_\omega(\delta) - \frac{Q_\omega^{(\delta)}(b)}{W^{(\delta)}(b)}\Phi_\omega(\delta)\right]W^{(\delta)}(\varepsilon) + \frac{Q_\omega^{(\delta)}(b) - Z^{(\delta)}(b)e^{\Phi_\omega(\delta)\varepsilon}}{W^{(\delta)}(b)}W^{(\delta)\prime}(\varepsilon)}{\frac{Q_\omega^{(\delta)}(b)}{W^{(\delta)}(b)}W^{(\delta)\prime}(\varepsilon) - \left[\omega\Phi_\omega(\delta) + \frac{Q_\omega^{(\delta)}(b)}{W^{(\delta)}(b)}\Phi_\omega(\delta)\right]W^{(\delta)}(\varepsilon)}\\
&= \frac{\omega}{\delta}\lim_{\varepsilon\downarrow 0}\frac{\left[\delta e^{\Phi_\omega(\delta)\varepsilon} - \omega\Phi_\omega(\delta) - \frac{Q_\omega^{(\delta)}(b)}{W^{(\delta)}(b)}\Phi_\omega(\delta)\right]\frac{W^{(\delta)}(\varepsilon)}{W^{(\delta)\prime}(\varepsilon)} + \frac{Q_\omega^{(\delta)}(b) - Z^{(\delta)}(b)e^{\Phi_\omega(\delta)\varepsilon}}{W^{(\delta)}(b)}}{\frac{Q_\omega^{(\delta)}(b)}{W^{(\delta)}(b)} - \left[\omega\Phi_\omega(\delta) + \frac{Q_\omega^{(\delta)}(b)}{W^{(\delta)}(b)}\Phi_\omega(\delta)\right]\frac{W^{(\delta)}(\varepsilon)}{W^{(\delta)\prime}(\varepsilon)}}\\
&= \frac{\omega}{\delta}\left(1 - \frac{Z^{(\delta)}(b)}{Q_\omega^{(\delta)}(b)}\right)
\end{aligned}$$

[see 18, Lemma 3.1 and Lemma 3.2], which corresponds to (4.3.18).

Therefore, (4.3.18) is correct for the surplus process driven by all spectrally negative Lévy processes without monotonic sample paths. Based on (4.3.18), by letting $\varepsilon = 0$ in (4.3.16) and (4.3.9) we obtain (4.3.1) finally.

4.4 The optimality

4.4.1 The optimal threshold b^*

In this section, we obtain the optimal threshold b^* so that we could maximize the expected net present value of the dividends under a threshold strategy. We have the following result.

Theorem 4.4.1 *The optimal dividend threshold b^* for any initial surplus $u > 0$ is*

$$b^* = \inf\left\{b \geq 0 : \delta Q_\omega^{(\delta)}(b) > \omega\Phi_\omega(\delta)Z^{(\delta)}(b)\right\}.$$

Proof Let $G^{(\delta)}(x) = \delta Q_\omega^{(\delta)}(x) - \omega\Phi_\omega(\delta)Z^{(\delta)}(x)$, then

$$G^{(\delta)\prime}(x) = \delta\Phi_\omega(\delta)Q_\omega^{(\delta)}(x) + \delta\omega\Phi_\omega(\delta)W^{(\delta)}(x) - \omega\Phi_\omega(\delta)\delta W^{(\delta)}(x) = \delta\Phi_\omega(\delta)Q_\omega^{(\delta)}(x) > 0,$$

which means $G^{(\delta)}(x)$ is a strictly monotone increasing function. Hence $G^{(\delta)}(b) > G^{(\delta)}(b^*)$ for all $b > b^* \geq 0$. Notice that definitions of $Z^{(\delta)}(x)$ and $Q_\omega^{(\delta)}(x)$ produce

$$\frac{Z^{(\delta)}(b)}{Q_\omega^{(\delta)}(b)e^{-\Phi_\omega(\delta)b}} = \frac{\delta\displaystyle\int_0^b W^{(\delta)}(x)\,\mathrm{d}x + 1}{\omega\Phi_\omega(\delta)\displaystyle\int_0^b e^{-\Phi_\omega(\delta)x}W^{(\delta)}(x)\,\mathrm{d}x + 1},$$

which allows us to find the derivative

$$\frac{\mathrm{d}}{\mathrm{d}b}\left[\frac{Z^{(\delta)}(b)}{Q_\omega^{(\delta)}(b)e^{-\Phi_\omega(\delta)b}}\right] = \frac{W^{(\delta)}(b)}{\left[Q_\omega^{(\delta)}(b)\right]^2 e^{-\Phi_\omega(\delta)b}}G^{(\delta)}(b).$$

Thus,

$$\begin{aligned}
\frac{\partial}{\partial b}V^{(\delta)}(u,b) &= \frac{\omega}{\delta}\left\{Z^{(\delta)\prime}(b-u) - \frac{\partial}{\partial b}\left[\frac{Z^{(\delta)}(b)}{Q_\omega^{(\delta)}(b)e^{-\Phi_\omega(\delta)b}} \cdot e^{-\Phi_\omega(\delta)b}Q_\omega^{(\delta)}(b-u)\right]\right\}\\
&= \frac{\omega}{\delta}\left\{\delta W^{(\delta)}(b-u) - \frac{W^{(\delta)}(b)}{\left[Q_\omega^{(\delta)}(b)\right]^2}G^{(\delta)}(b)Q_\delta(b-u) - \frac{d\Phi_\omega(\delta)Z^{(\delta)}(b)W^{(\delta)}(b-u)}{Q_\omega^{(\delta)}(b)}\right\}\\
&= \frac{\omega}{\delta Q_\omega^{(\delta)}(b)}\left[W^{(\delta)}(b-u) - \frac{W^{(\delta)}(b)}{Q_\omega^{(\delta)}(b)}Q_\omega^{(\delta)}(b-u)\right]G^{(\delta)}(b)\\
&= -\frac{\omega W^{(\delta)}(b)}{\delta\left[Q_\omega^{(\delta)}(b)\right]^2}\left[Q_\omega^{(\delta)}(b-u) - \frac{Q_\omega^{(\delta)}(b)}{W^{(\delta)}(b)}W^{(\delta)}(b-u)\right]G^{(\delta)}(b)\\
&= -\frac{\omega W^{(\delta)}(b)}{\delta\left[Q_\omega^{(\delta)}(b)\right]^2}e^{\Phi_\omega(\delta)(b-u)}\left[Z_{\Phi_\omega(\delta)}^{(d\Phi_\omega(\delta))}(b-u) - \frac{Z_{\Phi_\omega(\delta)}^{(d\Phi_\omega(\delta))}(b)}{W_{\Phi_\omega(\delta)}^{(d\Phi_\omega(\delta))}(b)}W_{\Phi_\omega(\delta)}^{(d\Phi_\omega(\delta))}(b-u)\right]G^{(\delta)}(b)\\
&= -\frac{\omega W^{(\delta)}(b)}{\delta\left[Q_\omega^{(\delta)}(b)\right]^2}e^{(b-u)\Phi_\omega(\delta)}\mathbf{E}_{b-u}^{\Phi_\omega(\delta)}\left[e^{-d\Phi_\omega(\delta)\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]G^{(\delta)}(b).
\end{aligned}$$

Therefore, for $b \geq b^*$, we have $G^{(\delta)}(b) > G^{(\delta)}(b^*) \geq 0$ and thus $V^{(\delta)}(u, b)$ is monotone decreasing with respect to b on $b \in (b^*, \infty)$. Hence if $b^* = 0$, then the optimality is guaranteed. Otherwise, $G^{(\delta)}(b^*)$ must be 0 and for those $b \in (0, b^*)$ we have $G^{(\delta)}(b) < 0$, which indicates $V^{(\delta)}(u, b)$ is monotone increasing with respect to b on $b \in [0, b^*)$. As a result, b^* is the optimal threshold.

4.4.2 The variational inequality

In this section, we will discuss the optimality of the threshold dividend strategy among the set of strategies $\Pi_\omega \subseteq \Pi$, namely,

$$\Pi_\omega = \left\{ \pi \in \Pi : D^\pi(t) = \int_0^t \omega^\pi(s)\,\mathrm{d}s, 0 \leq \omega^\pi(s) \leq \omega. \right\},$$

where Π represents the set of all admissible dividend strategies $\pi := \{D^\pi(t) : t \geq 0\}$. The value function is the expected discounted aggregate dividends until ruin, namely,

$$v_\pi(u) = \mathbf{E}\left[\int_0^{\sigma^\pi} e^{-\delta t}\,\mathrm{d}D^\pi(t) \middle| R(0) = u \right], \quad u > 0,$$

where $\sigma^\pi = \inf\{t \geq 0 : R(t) < D^\pi(t)\}$. The corresponding optimal value function is

$$v_*(u) := \sup_{\pi \in \Pi_\omega} v_\pi(u).$$

Heuristically, by forcing the surplus process paying dividends by an arbitrary strategy $\pi \in \Pi_\omega$ during time interval $[0, h)$ for small h the optimal value function $v_*(u)$ has to satisfy

$$\begin{aligned} v_*(u) &\geq \mathbf{E}\left[e^{-\delta h} v_*\left(R^\pi(h)\right) + \int_0^h e^{-\delta s}\,\mathrm{d}D^\pi(s) \middle| R(0) = u \right] \\ &= \mathbf{E}_{-u}\left[e^{-\delta h} v_*\left(-X(h) - D^\pi(h)\right) + \int_0^h e^{-\delta s}\,\mathrm{d}D^\pi(s) \right]. \end{aligned}$$

Since $e^{-\delta h} \approx 1 - \delta h$, $D^\pi(h) \approx \omega^\pi(h)h$ and $\int_0^h e^{-\delta s}\,\mathrm{d}D^\pi(s) \approx \omega^\pi(h)h$, we have

$$\begin{aligned} v_*(u) &\geq \mathbf{E}_{-u}\Big[(1 - \delta h)\left[v_*\left(-X(h)\right) - v_*'\left(-X(h)\right)\omega^\pi(h)h \right] + \omega^\pi(h)h \Big] + o(h) \\ &= (1 - \delta h)\,\mathbf{E}_{-u}\left[v_*\left(-X(h)\right) \right] - \mathbf{E}_{-u}\left[\left(v_*'\left(-X(h)\right) - 1 \right)\omega^\pi(h) \right] h + o(h) \end{aligned}$$

for all $\pi \in \Pi_\omega$. By the definition of infinitesimal generator of $\{X(t) : t \geq 0\}$ we have for any suitable function $f(x)$ that

$$\mathbf{E}_x\left[f\left(X(t)\right) \right] = t \cdot \Gamma f(x) + f(x) + o(t), \quad t \to 0,$$

where for any sufficiently smooth function $f(x)$, the infinitesimal generator of $\{X(t) : t \geq 0\}$ is defined through

$$\Gamma f(x) = \frac{\sigma^2}{2} f''(x) + \gamma f'(x) + \int_0^\infty \left[f(x - z) - f(x) + f'(x) z \mathbf{1}_{\{z<1\}} \right] \nu(\mathrm{d}z).$$

Thus,

$$\begin{aligned} v_*(u) &\geq (1-\delta h)\left[h\Gamma v_*(-x) + v_*(-x)\right]\Big|_{x=-u} + \left(1 - v_*'(-x)\right)\Big|_{x=-u}\omega^\pi(h)h + o(h) \\ &= h\hat{\Gamma}v_*(u) + v_*(u) - \delta v_*(u)h + \left[1 - v_*'(u)\right]\omega^\pi(h)h + o(h), \end{aligned}$$

where $\hat{\Gamma}$ is the infinitesimal generator of the spectrally positive Lévy process $\{-X(t) : t \geq 0\}$, namely,

$$\hat{\Gamma}f(x) = \frac{\sigma^2}{2}f''(x) - \gamma f'(x) + \int_0^\infty \left[f(x+z) - f(x) - f'(x)z\mathbf{1}_{\{z<1\}}\right]\nu(\mathrm{d}z)$$

for any sufficiently smooth function $f(x)$. Therefore, for any dividend strategy $\pi \in \Pi_\omega$,

$$\hat{\Gamma}v_*(u) - \delta v_*(u) + \left(1 - v_*'(u)\right)\omega^\pi(h) + o(1) \leq 0.$$

Thus, the variational inequality of Hamilton-Jacobi-Bellman (HJB) type

$$\sup_{0\leq l\leq d}\left\{(\hat{\Gamma}-\delta)v_*(u) + l\left(1 - v_*'(u)\right)\right\} \leq 0, \tag{4.4.1}$$

provides a sufficient condition for optimality.

Remark For $\Pi_\infty = \bigcup_{\omega>0}\Pi_\omega$, the variational equation (4.4.1) is

$$\max\{(\hat{\Gamma}-\delta)v_*(u), 1 - v_*'(u)\} \leq 0$$

as discussed in Bayraktar et al. [12].

4.4.3 Verification

In this section we show that the threshold strategy with optimal threshold b^* is an optimal strategy in Π through verifying the value function $V^{(\delta)}(u, b^*)$ satisfies the HJB equation (4.4.1). As we discussed in the previous section, for all $u \geq b^*$,

$$V^{(\delta)}(u, b^*) = \frac{\omega}{\delta}\left[1 - \frac{Z^{(\delta)}(b^*)}{Q_\omega^{(\delta)}(b^*)}e^{-\Phi_\omega(\delta)(u-b^*)}\right].$$

Notice that $e^{-\Phi_\omega(\delta)u}$ is an eigenfunction of the operator $\hat{\Gamma} - \omega\frac{\partial}{\partial u}$ with respect to the eigenvalue $\delta > 0$. On the other hand, the operator $(\hat{\Gamma} - \omega\frac{\partial}{\partial u})f(u) \equiv 0$ as long as $f(u)$ is a constant. Thus,

$$\left(\Gamma - \omega\frac{\partial}{\partial u}\right)V^{(\delta)}(u, b^*) + \omega - \delta V^{(\delta)}(u, b^*) = \omega - \delta\cdot\frac{\omega}{\delta} = 0. \tag{4.4.2}$$

Now let us assume $b^* > 0$. In this case, we have $G^{(\delta)}(b^*) = 0$ as we discussed in subsection 4.4.1. Thus we have

$$V^{(\delta)}(u, b^*) = \frac{\omega}{\delta}\left[Z^{(\delta)}(b^*-u) - \frac{\delta}{\omega\Phi_\omega(\delta)}Q_\omega^{(\delta)}(b^*-u)\right], \quad u > 0.$$

The optimality of $V^{(\delta)}(u, b^*)$ under the situation $u > b^*$ is verified by (4.4.2). Hence we only need to consider the case $0 < u < b^*$. First, we introduce the following result which is also mentioned in Avram et al. [17, p. 224].

Proposition 4.4.2 *Suppose $\theta \geq 0$ satisfies $\psi(\theta) < \infty$, then for $q > 0$ such that $q > -\psi(\theta)$, the function $f(x) = e^{\theta x} Z_\theta^{(q)}(x)$ is an eigenfunction of Γ with respect to the eigenvalue $\lambda = q + \psi(\theta)$.*

Based on Proposition 4.4.2, by letting $v = 0$ and $v = \Phi_\omega(\delta)$ respectively, we may conclude that both $Z^{(\delta)}(x)$ and $Q_\omega^{(\delta)}(x)$ are eigenfunctions of Γ with respect to eigenvalue $\delta > 0$. Thus, $(\Gamma - \delta)Z^{(\delta)}(x) = (\Gamma - \delta)Q_\omega^{(\delta)}(x) = 0$ for all $x \geq 0$, which implies

$$(\Gamma - \delta)H^{(\delta)}(x; b^*) = \frac{\omega}{\delta}\left[(\Gamma - \delta)Z^{(\delta)}(x) - \frac{\delta}{\omega\Phi_\omega(\delta)}(\Gamma - \delta)Q_\omega^{(\delta)}(x)\right] = 0,$$

for all $x \geq 0$. Therefore, $(\hat{\Gamma} - \delta)V^{(\delta)}(u, b^*) = (\Gamma - \delta)H^{(\delta)}(x, b^*)\big|_{x=b^*-u} = 0$. Besides, notice that the first derivative at any point $u \in (0, b^*)$ is

$$\frac{\partial}{\partial u}V^{(\delta)}(u, b^*) = Q_\omega^{(\delta)}(b^* - u) > 1, \quad u \in (0, b^*).$$

Therefore,

$$\left(\hat{\Gamma} - \delta\right) V^{(\delta)}(u, b^*) + \omega\left[1 - \frac{\partial}{\partial u}V^{(\delta)}(u, b^*)\right] < 0, \quad u \in (0, b^*). \tag{4.4.3}$$

Combining (4.4.2) and (4.4.3) completes the verification of the optimality of the threshold strategy.

4.5 Applications: Dual Cramér-Lundberg model perturbed by diffusion

In this section, we are interested in the numerical example in Avanzi and Gerber [11] equipped with an optimal threshold dividend strategy instead of the optimal barrier strategy. As a special case of spectrally positive Lévy processes with jumps of rational transform, the Laplace transform of the scale function $W^{(\delta)}(x)$ of the pertubed dual Cramér-Lundberg model may be expressed as

$$\int_0^\infty e^{-\theta x} W^{(\delta)}(x)\, \mathrm{d}x = \frac{1}{\psi(\theta) - \delta} = \frac{1}{\psi'(\Phi(\delta))[\theta - \Phi(\delta)]} + \sum_{k=1}^n \frac{C_k}{\theta - r_k}, \quad \Re(\theta) > \Phi(\delta), \tag{4.5.1}$$

where $n + 1 \in \mathbb{N}^+$ is the degree of the polynomial in the numerator of $\psi(\theta) - \delta$, $\{r_k\}_{k=1}^n$ are distinct roots with negative real part (for simplicity) to the equation $\psi(\theta) - \delta = 0$ and $\{C_k\}_{k=1}^n$ can be obtained by the partial fraction decomposition, namely,

$$C_k = \frac{1}{\psi'(r_k)}, \quad k = 1, \ldots, n,$$

(see Kuznetsov et al. [18, p. 170]). Hence if we let $\theta = 0$ without considering the domain $\{\theta \in \mathbb{C} : \Re(\theta) > \Phi(\delta)\}$ in (4.5.1) we obtain by denoting $r_0 = \Phi(\delta)$ and $C_0 = \left(\psi'(r_0)\right)^{-1}$

$$\sum_{k=0}^n \frac{C_k}{r_k} = \frac{1}{\delta}.$$

Hence, definition (4.2.10) yields

$$Z^{(\delta)}(x) = 1 + \delta \sum_{k=0}^{n} C_k \frac{e^{r_k x} - 1}{r_k} \mathbf{1}_{\{x>0\}} = \delta \sum_{k=0}^{n} \frac{C_k}{r_k} e^{r_k (x \vee 0)}, \quad x \in \mathbb{R}$$

Also, by (4.3.2) we obtain an explicit expression for $Q_\omega^{(\delta)}(x)$

$$Q_\omega^{(\delta)}(x) = \omega \Phi_\omega(\delta) \sum_{k=0}^{n} C_k \frac{e^{r_k x} - e^{\Phi_\omega(\delta) x}}{r_k - \Phi_\omega(\delta)} \mathbf{1}_{\{x>0\}} + e^{\Phi_\omega(\delta) x}.$$

Since by letting $\theta = \Phi_\omega(\delta)$ in (4.5.1) we obtain

$$\sum_{k=0}^{n} \frac{C_k}{\Phi_\omega(\delta) - r_k} = \frac{1}{\psi(\Phi_\omega(\delta)) - \delta} = \frac{1}{\psi_\omega(\Phi_\omega(\delta)) - \omega \Phi_\omega(\delta) - \delta} = -\frac{1}{\omega \Phi_\omega(\delta)}.$$

Thus

$$Q_\omega^{(\delta)}(x) = \omega \Phi_\omega(\delta) \sum_{k=0}^{n} \frac{C_k}{r_k - \Phi_\omega(\delta)} e^{r_k (x \vee 0) + \Phi_\omega(\delta)(x \wedge 0)}, \quad x \in \mathbb{R}.$$

Therefore, our main result (4.3.1) indicates

$$V^{(\delta)}(u,b) = \omega \sum_{k=0}^{n} C_k e^{r_k (b-u)_+} \left[\frac{1}{r_k} - \frac{\sum_{l=0}^{n} \frac{C_l}{r_l} e^{r_l b}}{\sum_{l=0}^{n} \frac{C_l}{r_l - \Phi_\omega(\delta)} e^{r_l b}} \cdot \frac{e^{-\Phi_\omega(\delta)(u-b)_+}}{r_k - \Phi_\omega(\delta)} \right], \quad u > 0,$$

where $(x)_+ = x \vee 0$. To obtain the optimal threshold b^*, if $\delta \geq d\Phi_\omega(\delta)$ then $b^* = 0$. Otherwise, b^* is the unique positive root to

$$\sum_{k=0}^{n} C_k \left[\frac{1}{r_k - \Phi_\omega(\delta)} - \frac{1}{r_k} \right] e^{r_k x} = 0.$$

Example Consider a threshold strategy under the dual model perturbed by diffusion with fixed parameters $b = 10$, $\nu(\mathrm{d}x) = \lambda p(x)\,\mathrm{d}x$ where $\lambda = 1$, $p(x) = e^{-x}$, $x > 0$, $c = 0.75$, $\sigma = 0.5$ and $\delta = 0.005$. These are the same parameter values as in Avanzi and Gerber [11, Illustration 3.1] where the barrier strategy is studied. Then we derive Table 4.1 and Table 4.2 by choosing d such that $c + \omega = 1, 10, 100, 1000, 10000$. Notice the last line with $\omega = \infty$ coincides with the barrier strategy with $b = 10$ discussed by Avanzi and Gerber [11, Illustration 3.1].

Usually, we are interested in the optimal threshold for different values of d under the above model. Hence, we derive Table 4.3 to show how the optimal dividend threshold changes as d increases. The last line recovers Avanzi and Gerber [11, Illustration 6.1].

$c+\omega$	$V^{(\delta)}(10,10)$	$V^{(\delta)}(u,10)$
1	34.144	$-29.64807e^{-0.2979284391u} + 29.64807e^{0.01844367193u} + 3\cdot10^{-34}e^{7.279484767u}$
10	33.482	$-29.08062e^{-0.2979284391u} + 29.08062e^{0.01844367193u} + 1\cdot10^{-33}e^{7.279484767u}$
100	33.222	$-28.85514e^{-0.2979284391u} + 28.85515e^{0.01844367193u} + 1\cdot10^{-33}e^{7.279484767u}$
1,000	33.198	$-28.83437e^{-0.2979284391u} + 28.83423e^{0.01844367193u} + 1\cdot10^{-33}e^{7.279484767u}$
10,000	33.196	$-28.83323e^{-0.2979284391u} + 28.83211e^{0.01844367193u} + 1\cdot10^{-33}e^{7.279484767u}$
∞	33.196	$-28.83199e^{-0.2979284391u} + 28.83199e^{0.01844367193u} - 3\cdot10^{-34}e^{7.279484767u}$

Table 4.1: The expected net present value of dividends with threshold $b = 10$ for $0 < u < b$

$c+\omega$	$V^{(\delta)}(10,10)$	$V^{(\delta)}(u,10)$
1	34.144	$50 - 31.505e^{-0.687u}$
10	33.482	$1,850 - 1,826.6e^{-5.555\cdot10^{-3}u}$
100	33.222	$19,850 - 19,827e^{-5.05\cdot10^{-4}u}$
1,000	33.198	$1.9985\cdot10^{5} - 1.99827\cdot10^{5}e^{-5.005\cdot10^{-5}u}$
10,000	33.196	$1.99985\cdot10^{6} - 1.99983\cdot10^{6}e^{-5\cdot10^{-6}u}$
∞	33.196	$u + 23.196$

Table 4.2: The expected net present value of dividend under threshold barrier with $b = 10$ for $u > b$

$c+\omega$	b^*
1	11.10482123
10	16.73973727
100	16.83025628
1,000	16.83852844
10,000	16.83916094
∞	16.84

Table 4.3: The optimal threshold ($\lambda = 1$, $p(x) = e^{-x}$, $c = 0.75$, $\sigma = 0.5$ and $\delta = 0.005$)

Appendix

4.A Proof of Proposition 4.4.2

Proof The proposition can be proved by verifying the Laplace transform. For convenience, we define the identity operator by $\mathcal{I}$. Also, we denote the Laplace transform of any function g such that $g(x) = 0$ for all $x \le 0$ and the convoluation of g with Lévy measure ν by

$$\mathcal{L}_s g = \int_0^\infty e^{-sx} g(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{C}_\nu g(x) = \int_0^\infty g(x-z)\nu(\mathrm{d}z).$$

Denote

$$f(x) = e^{\theta x} Z_\theta^{(q)}(x) = e^{\theta x} + q e^{\theta x} \int_0^x W_\theta^{(q)}(y)\,\mathrm{d}y = e^{\theta x} + q e^{\theta x} \int_0^x e^{-\theta y} W^{(q+\psi(\theta))}(y)\,\mathrm{d}y.$$

Then $f(x) - e^{\theta x} = 0$ for all $x \le 0$ and the Laplace transform of $f(x) - e^{\theta x}$ is

$$\mathcal{L}_s\left[f(x) - e^{\theta x}\right] = \frac{q}{(s-\theta)(\psi(s) - \psi(\theta) - q)}, \qquad s > \Re(\theta).$$

Hence, when $\nu(0,\infty) < \infty$, we have by the identities $f(0) - 1 = 0$ and $q\sigma^2 W^{(q+\psi(\theta))}(0) = 0$ (see Kuznetsov et al. [18, Lemma 3.1]) that

$$\begin{aligned}
\mathcal{L}_s\Gamma\left(f(x) - e^{\theta x}\right) &= \left[\frac{\sigma^2}{2}\mathcal{L}_s\frac{\partial^2}{\partial x^2} + \gamma\mathcal{L}_s\frac{\partial}{\partial x} + \mathcal{L}_s\mathcal{C}_\nu - \nu(0,\infty)\mathcal{L}_s + \int_0^1 z\nu(\mathrm{d}z)\mathcal{L}_s\mathcal{D}_x\right]\left[f(x) - e^{\theta x}\right] \\
&= \left[\frac{\sigma^2}{2}s^2 + cs + \int_0^\infty e^{-sz}\nu(\mathrm{d}z) - \nu(0,\infty)\right]\mathcal{L}_s\left[f(x) - e^{\theta x}\right] \\
&= \psi(s)\mathcal{L}_s\left(f(x) - e^{\theta x}\right) = \frac{q\psi(s)}{(s-\theta)[\psi(s) - \psi(\theta) - q]}, \qquad s > \Re(\theta).
\end{aligned}$$

On the other hand, $\Gamma\left(e^{\theta x}\right) = \psi(\theta)e^{\theta x}$ which implies

$$\mathcal{L}_s\Gamma\left(e^{\theta x}\right) = \frac{\psi(\theta)}{s-\theta}, \qquad s > \Re(\theta).$$

Therefore,

$$\mathcal{L}_s\Gamma f = \frac{q\psi(s)}{(s-\theta)[\psi(s)-\psi(\theta)-q]} + \frac{\psi(\theta)}{s-\theta} = [\psi(\theta)+q]\left\{\frac{1}{s-\theta} + \frac{q}{(s-\theta)[\psi(s)-\psi(\theta)-q]}\right\},$$

which implies by the uniqueness of the inverse Laplace transform that $\Gamma f(x) = [\psi(\theta)+q]\,f(x)$. If $\nu(0,\infty) = \infty$, then define an approximating sequence $\{\nu_n\}_{n=1}^\infty$ of the Lévy measure ν by $\nu_n(\mathrm{d}x) = \mathbf{1}_{\{x>n^{-1}\}}\nu(\mathrm{d}x)$. Obviously, $\nu_n(0,\infty) < \infty$ for all $n \in \mathbb{N}^+$. Thus by defining the corresponding $\psi_n(\theta)$ and Γ_n we have

$$\Gamma_n f(x) = [\psi_n(\theta) + q]\,f(x).$$

Notice that

$$\Gamma_n f(x) = \frac{\sigma^2}{2} f''(x) + \gamma f'(x) + \int_1^{\infty} [f(x-z) - f(x)]\, \nu(\mathrm{d}z) + \int_{(0,1)} g_n(z;x)\nu(\mathrm{d}z), \qquad (4.A.1)$$

where $g_n(z;x) = [f(x-z) - f(x) + f'(x)z]\,\mathbf{1}_{\{z>n^{-1}\}} = \frac{f''(\xi_z)}{2}\mathbf{1}_{\{z>n^{-1}\}}$ for some $\xi_z \in (x-z, x)$. Since $\int_0^1 z\nu(\mathrm{d}z) = \infty$, we have by Kuznetsov et al. [18, Lemma 2.3 and Lemma 2.4] that

$$f''(x) = \theta^2 f(x) + qW^{(q+\psi(\theta))}(x)\left[\theta + \frac{W^{(q+\psi(\theta))\prime}(x)}{W^{(q+\psi(\theta))}(x)}\right] \geq 0, \quad \forall x \in \mathbb{R},$$

Thus we have $0 \leq g_n(z;x) \leq g_{n+1}(z;x)$ for all $z \in (0,1)$. Therefore, by Lebesgue's monotone convergence theorem,

$$\lim_{n\to\infty} \int_{(0,1)} g_n(z;x)\nu(\mathrm{d}z) = \int_{(0,1)} \left(f(x-z) - f(x) + f'(x)z\right) \nu(\mathrm{d}z),$$

which leads to through (4.A.1)

$$\Gamma f(x) = \lim_{n\to\infty} \Gamma_n f(x) = \lim_{n\to\infty} [\psi_n(\theta) + q]\, f(x) = (\psi(\theta) + q)\, f(x).$$

In sum, $f(x) = e^{\theta x} Z_\theta^{(q)}(x)$ is an eigenfunction of Γ with respect to eigenvalue $\psi(\theta) + q$.

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Chapter 5

On the Parisian ruin of the dual Lévy risk models

5.1 Introduction

In the literature on collective risk theory, dual models are collective risk models with the surplus process defined by (1.1.1), where $\{X(t) : t \geq 0\}$ is an upward skip-free stochastic process such that $X(0) = 0$ and sample paths that are non-monotonic. One fundamental case of the setup (1.1.1) is the dual model of the Cramér-Lundberg type, which was first proposed by Cramér [1] to describe the behaviour of businesses selling annuities. In recent years, the dual model (1.1.1) was revisited by many authors with the intention of modelling a broader class of businesses with continuous expenses and occasional gains, such as brokerage, pharmaceutical and petroleum companies [see 2]. The majority of the discussions about the dual model focus on the optimal dividend strategy and the Laplace transform of the time to ruin. Compared to the ruin model for insurance companies, there are fewer quantities under the dual model that may be used as meaningful measures of vulnerability to the solvency of the analyzed company. As a result, the analysis under the dual model focuses on the Laplace transform of the time to ruin and its special case, the probability of (ultimate) ruin [see 3, 4, 5, for instance].

In this article, we introduce a more recent variant of the notion of ruin, called *Parisian ruin*, and related quantities to be studied under the dual model. All previous works on Parisian ruin are either in financial or in insurance context. Here we make an attempt to broaden the set of measures of vulnerability to the solvency considered under the dual model. We assume a fairly general setting where $\{X(t) : t \geq 0\}$ is a spectrally negative Lévy process.

The concept of Parisian ruin comes from the Parisian barrier option proposed by Chesney et al. [6], for which the option does not loose value if the price of the underlying asset reaches and remains constantly below a fixed barrier within a predetermined period (we refer to this period as the *Parisian delay*). Later, an analog of this concept is introduced in insurance context by Dassios and Wu [7] who allow the surplus level to be negative within a prefixed period. By extending the concept of regular ruin to Parisian ruin, Dassios and Wu [8] obtain the Laplace transform of the time of Parisian ruin under the diffusion-perturbed Cramér-Lundberg model with exponentially distributed jumps. More recently, an elegant expression of the Parisian ruin probability under a general Lévy insurance risk model is derived by Loeffen et al. [9].

To formally define the quantities of interest related to Parisian ruin, we introduce a class of stopping times

$$\kappa_p^{\varepsilon} = \inf\{t \geq p : t - \eta_t^{\varepsilon} \geq p\} \tag{5.1.1}$$

with the convention $\inf \varnothing = \infty$. Here $\eta_t^{\varepsilon} = \sup\{s \in [0, t] : R(s) \geq -\varepsilon\}$, $\varepsilon \geq 0$, with the convention $\sup \varnothing = 0$. We denote the Parisian delay by $p > 0$. Then κ_p^{ε} represents the smallest time such that the surplus drops and remains below $-\varepsilon$ for a period with length larger than or equal to p. In particular, we denote the Parisian ruin time $\kappa_p^0 = \kappa_p$ for short. Then the occurence of Parisian ruin is equivalent to the event $\kappa_p < \infty$. Furthermore, for the dual models, the deficit at Parisian ruin is not necessarily 0. Hence, we may define an analog of the well-known Gerber-Shiu expected discounted penalty function (EDPF) via

$$\phi_{\delta,w,p}(u) = \mathbf{E}\left[e^{-\delta\kappa_p} w\left(|R(\kappa_p)|\right) \mathbf{1}_{\{\kappa_p<\infty\}} \middle| R(0) = u\right], \quad u \in \mathbb{R}, \tag{5.1.2}$$

where $w : \mathbb{R} \mapsto [0, \infty)$ is assumed to be bounded and continuous on $(0, \infty)$ with $w(x) = 0$ for all $x < 0$ [see 10]. Our goal is to derive an explicit expression of the EDPF defined via (5.1.2).

The rest of this article is organized as follows: In Section 5.2, we introduce additional assumptions and notation related to our study, which are crucial to the derivations in Section 5.3. Also, the main idea and the full derivation will be given in Section 5.3. In Section 5.4, we obtain the expected discounted moments of the deficit at Parisian ruin, including the Parisian ruin probability for the compound Poisson dual risk model with exponentially distributed jumps both analytically and numerically. The last section is dedicated to conclusions.

5.2 Preliminaries

We consider a spectrally negative Lévy processes $\{X(t) : t \geq 0\}$ defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbf{P})$ with Lévy measure ν that is absolutely continuous with respect to the Lebesgue measure, satisfying $\nu(-\infty, 0) = 0$ and

$$\int_{(0,\infty)} (1 \wedge x^2)\nu(\mathrm{d}x) < \infty.$$

Let the associated Laplace exponent of $\{X(t) : t \geq 0\}$ be $\psi(s) = \log(\mathbf{E}[e^{sX(1)}])$ for all $s \geq 0$. Further assumptions include that the sample paths of $\{X(t) : t \geq 0\}$ are not monotonically decreasing but drift to $-\infty$ eventually with probability 1. The latter assumption is equivalent to

$$\psi'(0+) = \mathbf{E}[X(1)] < 0, \tag{5.2.1}$$

which is called the *negative loading condition 5*. With the aforementioned assumptions about $\{X(t) : t \geq 0\}$, the absolute continuity of the Lévy measure ν allows us to define the probability density function $f : (-\infty, 0) \times [0, \infty) \mapsto [0, \infty)$ such that for all $x < 0$,

$$\mathbf{P}(X(t) \in (x, x + \mathrm{d}x]) = \mathbf{P}(X(t) \in \mathrm{d}x) = f(x, t)\,\mathrm{d}x, \quad t \geq 0. \tag{5.2.2}$$

In general, if $\{X(t) : t \geq 0\}$ has sample paths of unbounded variation, then the assumption (5.2.2) could be generalized to $x \in \mathbb{R}$ for a given $t \geq 0$. However, the assumption (5.2.2) is not always true for those Lévy processes having sample paths of bounded variation on $(0,\infty)$. To see this, consider the following spectrally negative Lévy process

$$X(t) = ct - S(t), \quad t \geq 0, \tag{5.2.3}$$

where $c > 0$ and $\{S(t) : t \geq 0\}$ is a compound Poisson process with rate $\lambda > 0$. Then it is obvious that

$$\mathbf{P}(X(t) = ct) = e^{-\lambda t} > 0, \quad t \geq 0,$$

which implies a point mass of the distribution of $\{X(t) : t \geq 0\}$ at $ct > 0$ for a given $t \geq 0$.

Now, suppose $X(0) = x \in \mathbb{R}$, which is not necesarily equal to 0. Then the law of $\{X(t) : t \geq 0\}$ given $X(0) = x$ is defined through $\mathbf{P}_x(\cdot) = \mathbf{P}(\cdot|X(0) = x)$. The expectation with respect to $\mathbf{P}_x$ is denoted as $\mathbf{E}_x$. With these notations, we may define the exponential change of measure or the Esscher transform through the Radon-Nikodym derivative, namely,

$$\left.\frac{\mathrm{d}\mathbf{P}_x^a}{\mathrm{d}\mathbf{P}_x}\right|_{\mathscr{F}_t} = e^{a\left(X(t)-x\right)-\psi(a)t}, \quad t \geq 0, \tag{5.2.4}$$

for all $x, a \in \mathbb{R}$. Moreover, the duality (1.1.1) between $\{R(t) : t \geq 0\}$ and $\{X(t) : t \geq 0\}$ indicates an equivalent definition of (5.1.2), namely,

$$\phi_{\delta,w,p}(u) = \mathbf{E}_{-u}\left[e^{-\delta\kappa_p} w\left(X(\kappa_p)\right)\mathbf{1}_{\{\kappa_p<\infty\}}\right], \quad u \in \mathbb{R}, \tag{5.2.5}$$

where κ_p here is redefined via

$$\kappa_p = \inf\{t \geq p : t - \eta_t \geq p\}, \quad p > 0$$

correspondingly, where η_t is redefined by $\eta_t = \sup\{s \in [0,t] : X(s) \leq 0\}$.

To calculate the EDPF defined in (5.1.2) or (5.2.5), we shall introduce the q-scale function $W^{(q)}(x)$ through its Laplace transform, namely,

$$\int_0^\infty e^{-sx} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(s) - q}, \quad s > \Phi(q),$$

where $\Phi(q) = \sup\{s \geq 0 : \psi(s) = q\}$. On $(-\infty, 0)$ the q-scale function $W^{(q)}(x)$ is always equal to 0. With the help of the q-scale function $W^{(q)}(x)$ we may obtain an explicit formula for the q-potential density of $\{X(t) : t \geq 0\}$ denoted as $\theta^{(q)}(x)$, namely

$$\theta^{(q)}(x) = \Phi'(q)e^{\Phi(q)x} - W^{(q)}(x), \quad x \in \mathbb{R}, \tag{5.2.6}$$

where $\theta^{(q)}(x)$ is defined through

$$\int_0^\infty e^{-qt}\mathbf{P}\left(X(t) \in -\mathrm{d}x\right)\mathrm{d}t = \theta^{(q)}(x)\,\mathrm{d}x, \quad x \in \mathbb{R} \tag{5.2.7}$$

[see 11, Corollary 8.9.]. In particular, we have

$$\theta^{(q)}(x) = \int_0^\infty e^{-qt} f(x,t)\,\mathrm{d}t, \quad x < 0.$$

Last, we denote the first passage times of $\{X(t) : t \geq 0\}$

$$\tau_y^+ = \inf\{t > 0 : X(t) > y\} \quad \text{and} \quad \tau_y^- = \inf\{t > 0 : X(t) < y\}$$

for some level $y \in \mathbb{R}$ with the convention $\inf\varnothing = \infty$. Then the Kendall's identity [see 12, Corollary VII.3] may be written as

$$s\mathbf{P}\left(\tau_x^+ \in \mathrm{d}s\right)\mathrm{d}x = x\mathbf{P}\left(X(s) \in \mathrm{d}x\right)\mathrm{d}s, \quad s \geq 0, \quad x > 0. \tag{5.2.8}$$

5.3 Main result

In this section, we derive a representation of the EDPF defined in (5.1.2) or (5.2.5) in terms of some known functions and measures. A similar limiting technique used in Loeffen et al. [9] will be considered in our proof, namely, we will derive the EDPF with respected to the stopping time

$$\tilde{\kappa}_p^{\varepsilon} = \inf\left\{t \geq p : t - \inf\{s > \eta_t : R(s) < -\varepsilon \mathbf{1}_{\{\eta_t > 0\}}\} > p\right\},$$

instead of κ_p, where $\varepsilon \geq 0$ and $\eta_t = \sup\{s \in [0, t] : R(s) \geq 0\}$. Then it is clear that $\tilde{\kappa}_p^0 = \kappa_p$ defined in (5.1.1). Now we shall explain how the event $\tilde{\kappa}_p^{\varepsilon} < \infty$ occurs. First we consider the scenario that $R(0) = u \leq 0$ in which it is possible for $\tilde{\kappa}_p^{\varepsilon}$ to be equal to p due to that the surplus process $\{R(t) : t \geq 0\}$ never attains level 0 during the period $[0, p]$. One sample path under the situation $\tilde{\kappa}_p^{\varepsilon} = p$ is shown in Figure 5.1. If the surplus process $\{R(t) : t \geq 0\}$ with

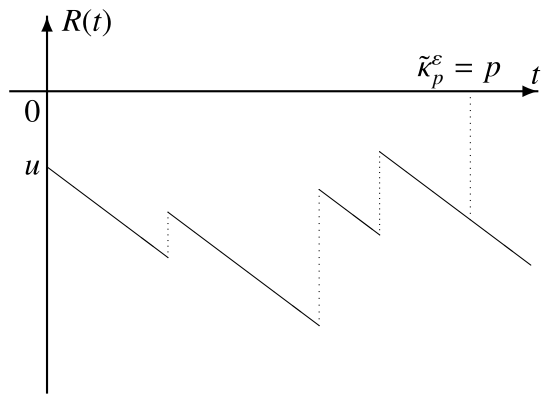

Figure 5.1: Sample path of $R(t)$ corresponding to the case $\tilde{\kappa}_p^{\varepsilon} = p$

negative initial value enters $(0, \infty)$ during $[0, p]$, whenever the surplus level becomes positive, the starting point of the Parisian delay is chosen to be the next time that the surplus level enters $(-\infty, -\varepsilon)$ instead of the next passage time of $\{R(t) : t \geq 0\}$ at level 0. One sample path of this situation is shown in Figure 5.2. If the initial surplus $u > 0$, then $\tilde{\kappa}_p^{\varepsilon} > p$ for sure and the event $\tilde{\kappa}_p^{\varepsilon} < \infty$ occurs in the same way as in the situation $u \leq 0$ with $\tilde{\kappa}_p^{\varepsilon} > p$. Now we define the EDPF corresponding to $\tilde{\kappa}_p^{\varepsilon}$ as

$$\phi_{\delta,w,p}(u, \varepsilon) = \mathbf{E}\left[e^{-\delta \tilde{\kappa}_p^{\varepsilon}} w\left(|R(\tilde{\kappa}_p^{\varepsilon})|\right) \mathbf{1}_{\{\tilde{\kappa}_p^{\varepsilon} < \infty\}} \middle| R(0) = u\right], \quad u \in \mathbb{R}. \tag{5.3.1}$$

Recall that the EDPF defined by (5.1.2) has an equivalent definition (5.2.5), likewise, the EDPF defined by (5.3.1) has the following equivalent definition

$$\phi_{\delta,w,p}(u, \varepsilon) = \mathbf{E}_{-u}\left[e^{-\delta \tilde{\kappa}_p^{\varepsilon}} w\left(X(\tilde{\kappa}_p^{\varepsilon})\right) \mathbf{1}_{\{\tilde{\kappa}_p^{\varepsilon} < \infty\}}\right], \quad u \in \mathbb{R} \tag{5.3.2}$$

with $\tilde{\kappa}_p^{\varepsilon}$ redefined by

$$\tilde{\kappa}_p^{\varepsilon} = \inf\left\{t \geq p : t - \inf\{s > \eta_t : X(s) > \varepsilon \mathbf{1}_{\{\eta_t > 0\}}\} > p\right\},$$

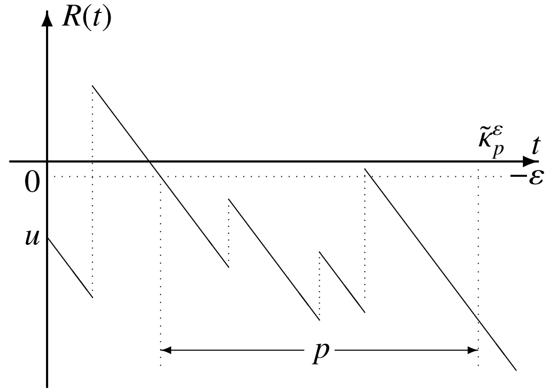

Figure 5.2: Sample path of $R(t)$ corresponding to the case $\tilde{\kappa}_p^{\varepsilon} > p$

where $\eta_t = \sup\{s \in [0,t] : X(s) \le 0\}$. Hence we have the following result of the EDPF defined by (5.3.2).

Theorem 5.3.1 *For given initial surplus $u \in \mathbb{R}$, discount factor $\delta \ge 0$, Parisian delay $p > 0$ and suitable penalty function $w(x)$, the EDPF $\phi_{\delta,w,p}(u)$ has the following representation for $\varepsilon > 0$*

$$\phi_{\delta,w,p}(u,\varepsilon) = \frac{A_{w,p}(\varepsilon)}{B_{\delta,p}(\varepsilon)}\left(e^{-\Phi(\delta)u} - e^{-\delta p}B_{\delta,p}(-u)\right) + e^{-\delta p}A_{w,p}(-u), \quad u \in \mathbb{R}, \tag{5.3.3}$$

where

$$A_{w,p}(y) = \int_0^{\infty} w(x)\Lambda_{y,p}(\mathrm{d}x) \quad \text{and} \quad B_{\delta,p}(y) = \int_0^{\infty} e^{\Phi(\delta)x}\Lambda_{y,p}(\mathrm{d}x), \quad y \in \mathbb{R}$$

and

$$\Lambda_{y,p}(\mathrm{d}x) = \left[\mathbf{P}_y\left(X(p) \in \mathrm{d}x\right) - x\int_0^p f(-y, p-t)\mathbf{P}\left(X(t) \in \mathrm{d}x\right)\frac{\mathrm{d}t}{t}\right]\mathbf{1}_{\{y \wedge x > 0\}}.$$

Proof If the initial surplus $u > 0$, then $\eta_t > 0$ for all $t \ge 0$. To obtain the EDPF given by (5.3.1), by conditioning on the first passage time τ_ε^+, we have by the strong Markov property and Kyprianou [11, Section 8.1]

$$\phi_{\delta,w,p}(u,\varepsilon) = \phi_{\delta,w,p}(-\varepsilon,\varepsilon)\mathbf{E}_{-u}\left[e^{-\delta\tau_\varepsilon^+}\mathbf{1}_{\{\tau_\varepsilon^+ < \infty\}}\right] = \phi_{\delta,w,p}(-\varepsilon,\varepsilon)e^{-\Phi(\delta)(u+\varepsilon)}. \tag{5.3.4}$$

The unknown function $\phi_{\delta,w,p}(-\varepsilon,\varepsilon)$ will be determined later. Now we consider the case $u \le 0$. For convenience we denote $v = -u \ge 0$. Under this situation, either $\tilde{\kappa}_p^{\varepsilon} > p$ or $\tilde{\kappa}_p^{\varepsilon} = p$, corresponding to the situations $\tau_0^- \le p$ and $\tau_0^- > p$, respectively. Then the EDPF defined in (5.3.2) can be splited into two parts, namely,

$$\phi_{\delta,w,p}(-v,\varepsilon) = \mathbf{E}_v\left[e^{-\delta\tau_0^-}\phi_{\delta,w,p}(-X(\tau_0^-),\varepsilon)\mathbf{1}_{\{\tau_0^- \le p\}}\right] + e^{-\delta p}\mathbf{E}_v\left[w(X(p))\mathbf{1}_{\{\tau_0^- > p\}}\right]. \tag{5.3.5}$$

Since we have (5.3.4) for the cases that $u > 0$, by the Stong Markov property we obtain

$$\mathbf{E}_v\left[e^{-\delta\tau_0^-}\phi_{\delta,w,p}(-X(\tau_0^-),\varepsilon)\mathbf{1}_{\{\tau_0^- \le p\}}\right] = \phi_{\delta,w,p}(-\varepsilon,\varepsilon)\mathbf{E}_v\left[e^{-\delta\tau_0^- + \Phi(\delta)\left(X(\tau_0^-)-\varepsilon\right)}\mathbf{1}_{\{\tau_0^- \le p\}}\right]. \tag{5.3.6}$$

And the applying the exponential exchange of measure on the right handside of (5.3.6) yields

$$\mathbf{E}_v\left[e^{-\delta\tau_0^-}\phi_{\delta,w,p}(-X(\tau_0^-),\varepsilon)\mathbf{1}_{\{\tau_0^-\le p\}}\right] = \phi_{\delta,w,p}(-\varepsilon,\varepsilon)e^{\Phi(\delta)(v-\varepsilon)}\mathbf{P}_v^{\Phi(\delta)}\left(\tau_0^-\le p\right). \tag{5.3.7}$$

While the other term in the right handside of (5.3.5) can be written as

$$e^{-\delta p}\mathbf{E}_v\left[w(X(p))\mathbf{1}_{\{\tau_0^->p\}}\right] = e^{-\delta p}\int_0^\infty w(x)\mathbf{P}_v\left(X(p)\in \mathrm{d}x, \underline{X}(p)>0\right), \tag{5.3.8}$$

where $\underline{X}(t) = \inf_{s\in[0,t]} X(s)$. Thus, by denoting

$$\Lambda_{v,p}(\mathrm{d}x) = \mathbf{P}_v\left(X(p)\in \mathrm{d}x, \underline{X}(p)>0\right),$$

equations (5.3.7), (5.3.8) and (5.3.5) yields

$$\phi_{\delta,w,p}(-v,\varepsilon) = \phi_{\delta,w,p}(-\varepsilon,\varepsilon)e^{\Phi(\delta)v}\mathbf{P}_v^{\Phi(\delta)}\left(\tau_0^-\le p\right) + e^{-\delta p}\int_0^\infty w(x)\Lambda_{v,p}(\mathrm{d}x). \tag{5.3.9}$$

We shall start with taking Laplace transform with respect to p for the measure $\Lambda_{v,p}(\mathrm{d}x)$. Consider an independent exponential random variable with intensity $q>0$ denoted by $\boldsymbol{e}_q$. Then for $v, x>0$, Bertoin [13, Lemma 1.] provides

$$\mathbf{P}_v\left(X(\boldsymbol{e}_q)\in \mathrm{d}x, \underline{X}(\boldsymbol{e}_q)>0\right) = q\left(e^{-\Phi(q)x}W^{(q)}(v) - W^{(q)}(v-x)\right)\mathrm{d}x. \tag{5.3.10}$$

Notice that with the help of (5.2.6), (5.3.10) reduces to

$$\mathbf{P}_v\left(X(\boldsymbol{e}_q)\in \mathrm{d}x, \underline{X}(\boldsymbol{e}_q)>0\right) = q\left(\theta^{(q)}(v-x) - e^{-\Phi(q)x}\theta^{(q)}(v)\right)\mathrm{d}x, \quad x>0. \tag{5.3.11}$$

Therefore, by the inverse Laplace transform on both sides of (5.3.11) we have for $v>0$,

$$\Lambda_{v,p}(\mathrm{d}x) = \mathbf{P}_v\left(X(p)\in \mathrm{d}x\right) - \int_0^p f(-v,p-t)\mathbf{P}\left(\tau_x^+\in \mathrm{d}t\right)\mathrm{d}x, \quad x>0. \tag{5.3.12}$$

By applying the Kendall's indentity (5.2.8) on (5.3.12), we obtain for $v>0$

$$\Lambda_{v,p}(\mathrm{d}x) = \mathbf{P}_v\left(X(p)\in \mathrm{d}x\right) - x\int_0^p f(-v,p-t)\mathbf{P}\left(X(t)\in \mathrm{d}x\right)\frac{\mathrm{d}t}{t}, \quad x>0. \tag{5.3.13}$$

Therefore, (5.3.9) may reduce to

$$\phi_{\delta,w,p}(-v,\varepsilon) = \phi_{\delta,w,p}(-\varepsilon,\varepsilon)e^{\Phi(\delta)v}\mathbf{P}_v^{\Phi(\delta)}\left(\tau_0^-\le p\right) + e^{-\delta p}A_{w,p}(v), \quad v>0. \tag{5.3.14}$$

Notice that under the exponential change of measure,

$$\mathbf{P}_v^{\Phi(\delta)}\left(X(p)\in \mathrm{d}x, \underline{X}(p)>0\right) = e^{-\delta p+\Phi(\delta)(x-v)}\mathbf{P}_v\left(X(p)\in \mathrm{d}x, \underline{X}(p)>0\right).$$

Hence,

$$e^{\Phi(\delta)(v-\varepsilon)}\mathbf{P}_v^{\Phi(\delta)}\left(\tau_0^-\le p\right) = e^{\Phi(\delta)(v-\varepsilon)}\mathbf{P}_v^{\Phi(\delta)}\left(\underline{X}(p)\le 0\right) = e^{\Phi(\delta)(v-\varepsilon)} - e^{-\delta p-\Phi(\delta)\varepsilon}B_{\delta,p}(v). \tag{5.3.15}$$

Thus, combining (5.3.14) and (5.3.15) yields

$$\phi_{\delta,w,p}(-v,\varepsilon) = \phi_{\delta,w,p}(-\varepsilon,\varepsilon)\left(e^{\Phi(\delta)(v-\varepsilon)} - e^{-\delta p-\Phi(\delta)\varepsilon}B_{\delta,p}(v)\right) + e^{-\delta p}A_{w,p}(v), \quad v > 0. \tag{5.3.16}$$

By letting $v = \varepsilon > 0$ in (5.3.16) we obtain

$$\phi_{\delta,w,p}(-\varepsilon,\varepsilon) = e^{\Phi(\delta)\varepsilon}\frac{A_{w,p}(\varepsilon)}{B_{\delta,p}(\varepsilon)}. \tag{5.3.17}$$

Therefore, if we replace $\phi_{\delta,w,p}(-\varepsilon,\varepsilon)$ in (5.3.16) with (5.3.17) and recall $v = -u$, we obtain (5.3.3) for $u < 0$. For $u = v = 0$, $\Lambda_{v,p}(\mathrm{d}x) = 0$ and thus $A_{w,p}(0) = B_{\delta,p}(0) = 0$, which completes the proof.

Theorem 5.3.1 provides a representation of the EDPF with respect to $\tilde{\kappa}_p^\varepsilon$ for $\varepsilon > 0$. Since $A_{w,p}(0) = B_{\delta,p}(0) = 0$, the case $\varepsilon = 0$ should be defined in a limiting sense. The next theorem provides the limit of $\phi_{\delta,w,p}(0,\varepsilon)$ as $\varepsilon \downarrow 0$.

Theorem 5.3.2 *For known functions $A_{w,p}(v)$ and $B_{\delta,p}(v)$, we have*

$$\lim_{\varepsilon\downarrow 0}\phi_{\delta,w,p}(0,\varepsilon) = \lim_{\varepsilon\downarrow 0}\frac{A_{w,p}(\varepsilon)}{B_{\delta,p}(\varepsilon)} = \frac{\displaystyle\int_0^\infty xw(x)\mathbf{P}\left(X(p)\in \mathrm{d}x\right)}{\displaystyle\int_0^\infty xe^{\Phi(\delta)x}\mathbf{P}\left(X(p)\in \mathrm{d}x\right)}. \tag{5.3.18}$$

Proof We only need to study the limiting behavior of $A_{w,\delta}(\varepsilon)$ as $\varepsilon \downarrow 0$ because $B_{\delta,p}(\varepsilon)$ is a particular case of $A_{w,p}(\varepsilon)$ that $w(x) = e^{\Phi(\delta)x}$. Let us first discuss the case that $\{X(t) : t \geq 0\}$ has sample paths of bounded variation, under which $\{X(t) : t \geq 0\}$ has a unique representation as (5.2.3) where $c > 0$, $\{S(t) : t \geq 0\}$ is a pure-jump subordinator. Then by the standard Tauberian theorem of Laplace transform, we have

$$\lim_{\varepsilon\downarrow 0}A_{w,p}(\varepsilon) = \lim_{\zeta\uparrow\infty}\Phi(\zeta)\int_0^\infty e^{-\Phi(\zeta)y}A_{w,p}(y)\,\mathrm{d}y. \tag{5.3.19}$$

Now consider the double Laplace transform of $\Lambda_{y,p}(\mathrm{d}x)$ with respect to z and p, namely,

$$\begin{aligned}\int_0^\infty e^{-\Phi(\zeta)y}\left(\int_0^\infty e^{-qp}\Lambda_{y,p}(\mathrm{d}x)\,\mathrm{d}p\right)\mathrm{d}x &= \int_0^\infty e^{-\Phi(\zeta)y}\left(e^{-\Phi(q)x}W^{(q)}(y) - W^{(q)}(y-x)\right)\mathrm{d}y\,\mathrm{d}x\\ &= \frac{e^{-\Phi(q)x}-e^{-\Phi(\zeta)x}}{\zeta - q}, \quad \Phi(\zeta) > q.\end{aligned}$$

Besides, Kyprianou [11, Section 8.1] provides

$$e^{-\Phi(\zeta)x} = \mathbf{E}\left[e^{-\zeta\tau_x^+}\mathbf{1}_{\{\tau_x^+<\infty\}}\right] = \int_0^\infty e^{-\zeta t}\mathbf{P}\left(\tau_x^+\in \mathrm{d}t\right).$$

Thus, by inverting the double Laplace transform with respect to p we obtain

$$\int_0^\infty e^{-\Phi(\zeta)y}\Lambda_{y,p}(\mathrm{d}x) = \int_p^\infty e^{-\zeta(t-p)}\mathbf{P}\left(\tau_x^+\in \mathrm{d}t\right)\mathrm{d}x. \tag{5.3.20}$$

Hence by (5.3.20), (5.2.8) and Tonelli's theorem we have

$$\begin{aligned}\int_0^\infty e^{-\Phi(\zeta)y}A_{w,p}(y)\,\mathrm{d}y &= \int_0^\infty e^{-\Phi(\zeta)y}\left(\int_0^\infty w(x)\Lambda_{y,p}(\mathrm{d}x)\right)\mathrm{d}y\\ &= \int_0^\infty w(x)\left(\int_0^\infty e^{-\Phi(\zeta)y}\Lambda_{y,p}(\mathrm{d}x)\,\mathrm{d}y\right)\\ &= \int_0^\infty xw(x)\left(\int_p^\infty e^{-\zeta(t-p)}\mathbf{P}\left(\tau_x^+\in\mathrm{d}t\right)\right)\mathrm{d}x\\ &= \int_0^\infty e^{-\zeta t}\left(\int_0^\infty xw(x)\mathbf{P}\left(X(t+p)\in\mathrm{d}x\right)\right)\frac{\mathrm{d}t}{t+p}. \end{aligned} \tag{5.3.21}$$

Thus with the help of Kuznetsov et al. [14, Eq.(50)], by (5.3.19) and (5.3.21) we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}A_{w,p}(\varepsilon) &= \lim_{\zeta\uparrow\infty}\frac{\Phi(\zeta)}{\zeta}\cdot\lim_{\zeta\uparrow\infty}\zeta\int_0^\infty e^{-\zeta t}\left(\int_0^\infty w(x)\mathbf{P}\left(X(t+p)\in\mathrm{d}x\right)\right)\frac{\mathrm{d}t}{t+p} \qquad (5.3.22)\\ &= \frac{1}{cp}\int_0^\infty xw(x)\mathbf{P}\left(X(p)\in\mathrm{d}x\right)\end{aligned}$$

and in particular

$$\lim_{\varepsilon\downarrow 0}B_{\delta,p}(\varepsilon) = \frac{1}{cp}\int_0^\infty xe^{\Phi(\delta)x}\mathbf{P}\left(X(p)\in\mathrm{d}x\right).$$

Therefore, we obtain (5.3.18) for the bounded variation case. Next, we shall discuss the case that $\{X(t):t\ge 0\}$ contains a Gaussian component with coefficient $\sigma>0$ in which the sample paths of $\{X(t):t\ge 0\}$ have unbounded variation. Hence from (5.3.22) we know

$$\lim_{\varepsilon\downarrow 0}A_{w,p}(\varepsilon)=0$$

and so is the limit of $B_{\delta,p}(\varepsilon)$ as $\varepsilon\downarrow 0$. This reminds us of the L'Hopital rule. Consider the limit of $A'_{w,p}(\varepsilon)$ as $\varepsilon\downarrow 0$, namely, by the Tauberian theorem and integration by parts,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}A'_{w,p}(\varepsilon) &= \lim_{\zeta\uparrow\infty}\Phi(\zeta)\int_0^\infty e^{-\Phi(\zeta)y}A'_{w,p}(y)\,\mathrm{d}y\\ &= \lim_{\zeta\uparrow\infty}\Phi(\zeta)^2\int_0^\infty e^{-\Phi(\zeta)y}A_{w,p}(y)\,\mathrm{d}y\\ &= \lim_{\zeta\uparrow\infty}\frac{\Phi(\zeta)^2}{\zeta}\cdot\lim_{\zeta\uparrow\infty}\zeta\int_0^\infty e^{-\zeta t}\left(\int_0^\infty xw(x)\mathbf{P}\left(X(t+p)\in\mathrm{d}x\right)\right)\frac{\mathrm{d}t}{t+p}\\ &= \frac{2}{\sigma^2}\int_0^\infty xw(x)\mathbf{P}\left(X(p)\in\mathrm{d}x\right).\end{aligned}$$

The last equality comes from the proof of Lemma 3.2 in Kuznetsov et al. [14] and identity (5.3.22). In particular, we also have

$$\lim_{\varepsilon\downarrow 0}B'_{\delta,p}(\varepsilon) = \frac{2}{\sigma^2}\int_0^\infty xe^{\Phi(\delta)x}\mathbf{P}\left(X(t)\in\mathrm{d}x\right).$$

Therefore, using L'Hôpital's rule, we obtain (5.3.18) when $\sigma > 0$. Now we have the last case to discuss, namely, $\{X(t) : t \geq 0\}$ has sample paths of unbounded variation but no Gaussian component. Then (5.3.18) may be proved through constructing a strong approximating sequence $\{X_n(t) : t \geq 0\}_{n=1}^{\infty}$ defined by

$$X_n(t) = X(t) + n^{-1}B(t), \quad t \geq 0$$

for all $n \in \mathbb{N}$, where $\{B(t) : t \geq 0\}$ is a standard Brownian motion. Since Whitt [15, Lemma 13.4.1] implies

$$\left|\underline{X}_n(t) - \underline{X}(t)\right| \leq \sup_{s\in[0,t]} \left|X_n(s) - X(s)\right| \to 0$$

almost surely for all $t \geq 0$, we have in almost sure sense,

$$\lim_{n\to\infty} \left(X_n(t), \underline{X}_n(t)\right) = \left(X(t), \underline{X}(t)\right), \quad t \geq 0.$$

Thus, if we denote $\Lambda_{z,p}^{(n)}(\mathrm{d}x) = \mathbf{P}_z\left(X_n(p) \in \mathrm{d}x, \underline{X}_n(p) > 0\right)$, then by construction $\Lambda_{z,p}^{(n)}$ is absolutely continuous with respect to the Lebesgue measure for all $n \in \mathbb{N}$ and $\Lambda_{z,p}^{(n)} \to \Lambda$ as $n \to \infty$ in setwise sense. Hence for all $E \in \mathcal{B}(\mathbb{R})$ with Lebesgue measure 0, $\Lambda_{z,p}^{(n)}(E) = 0$ for all $n \in \mathbb{N}$ while $\Xi_{z,p}(E) := \mathbf{P}_z(X(p) \in E) = 0$ due to the unbounded variation sample paths. Therefore, $\Lambda_{z,p}^{(n)}(E) = \Xi_{z,p}(E)$ for all $n \in \mathbb{N}$ and $E \in \mathcal{B}(\mathbb{R})$ with Lebesgue measure 0. Now consider $E \in \mathcal{B}(\mathbb{R})$ with nonzero Lebesgue measure, then $\Lambda_{z,p}^{(n)}(E) > 0$, $\Lambda_{z,p}(E) > 0$ and $\Xi_{z,p}(E) > 0$. Let $\Delta_{z,p}(E) = \Xi_{z,p}(E) - \Lambda_{z,p}(E)$, then there exists $N_{z,p}(E) \in \mathbb{N}$ such that $\Lambda_{z,p}^{(n)}(E) < \Lambda_{z,p}(E) + \Delta_{z,p}(E) = \Xi_{z,p}(E)$ for all $n > N_{z,p}(E)$. Notice that for $z > 0$

$$\int_0^{\infty} w(x)\Xi_{z,p}(\mathrm{d}x) = \mathbf{E}_z\left[w\left(X(p)\right)\mathbf{1}_{\{X(p)>0\}}\right] < \infty.$$

Therefore, by the Dominated Convergence Theorem for measures [see 16, Theorem 2.1(b)] we have for $z > 0$

$$\lim_{n\to\infty} A_{w,p}^{(n)}(z) := \lim_{n\to\infty} \int_0^{\infty} w(x)\Lambda_{z,p}^{(n)}(\mathrm{d}x) = A_{z,p}(z).$$

and in particular,

$$\lim_{n\to\infty} B_{\delta,p}^{(n)}(z) := \lim_{n\to\infty} \int_0^{\infty} e^{\Phi(\delta)x}\Lambda_{z,p}^{(n)}(\mathrm{d}x) = B_{\delta,p}(z).$$

Thus by Lemma 5.A.1 we have

$$\lim_{\varepsilon\downarrow 0} \frac{A_{w,p}(\varepsilon)}{B_{\delta,p}(\varepsilon)} = \lim_{n\to\infty} \lim_{\varepsilon\downarrow 0} \frac{A_{w,p}^{(n)}(\varepsilon)}{B_{\delta,p}^{(n)}(\varepsilon)} = \frac{\displaystyle\int_0^{\infty} xw(x)\mathbf{P}\left(X(p) \in \mathrm{d}x\right)}{\displaystyle\int_0^{\infty} xe^{\Phi(\delta)x}\mathbf{P}\left(X(p) \in \mathrm{d}x\right)},$$

which completes the proof of that (5.3.18) is true for all spectrally negative Lévy processes without monotone sample paths.

To obtain the representation of the EDPF defined by (5.2.5), we only have to show that

$$\lim_{\varepsilon\downarrow 0} \phi_{\delta,w,p}(u, \varepsilon) = \phi_{\delta,w,p}(u), \tag{5.3.23}$$

which is equivalent to the following theorem due to (5.3.4) and (5.3.9).

Theorem 5.3.3 *For fixed $\delta, p \geq 0$ and $w(x)$ bounded and continuous on $\mathbb{R}$ satisfying $w(x) = 0$ for all $x < 0$, we have*

$$\lim_{\varepsilon \downarrow 0} e^{-\Phi(\delta)\varepsilon} \phi_{\delta,w,p}(-\varepsilon, \varepsilon) = \lim_{\varepsilon \downarrow 0} \phi_{\delta,w,p}(0, \varepsilon) = \phi_{\delta,w,p}(0). \tag{5.3.24}$$

Proof Recall the stopping time κ_p^ε defined in (5.1.1). By redefining $\eta_t^\varepsilon = \inf\left\{t \geq p : t - \eta_t^\varepsilon > p\right\}$, we have $\eta_t^\varepsilon > \eta_t$, which leads to

$$\inf\{s > \eta_t : X(s) > \varepsilon \mathbf{1}_{\{\eta_t > 0\}}\} \leq \inf\{s > \eta_t : X(s) > \varepsilon\} \leq \inf\{s > \eta_t^\varepsilon : X(s) > \varepsilon\} = \eta_t^\varepsilon.$$

Therefore, $\kappa_p \leq \tilde{\kappa}_p^\varepsilon \leq \kappa_p^\varepsilon$ and thus by the spatial homogeneity

$$\mathbf{E}\left[e^{-\delta\kappa_p}\mathbf{1}_{\{\kappa_p<\infty\}}\right] \geq \mathbf{E}\left[e^{-\delta\tilde{\kappa}_p^\varepsilon}\mathbf{1}_{\{\tilde{\kappa}_p^\varepsilon<\infty\}}\right] \geq \mathbf{E}\left[e^{-\delta\kappa_p^\varepsilon}\mathbf{1}_{\{\kappa_p^\varepsilon<\infty\}}\right] = e^{-\Phi(\delta)\varepsilon}\mathbf{E}\left[e^{-\delta\kappa_p}\mathbf{1}_{\{\kappa_p<\infty\}}\right].$$

Consequently, we obtain for all $\delta \geq 0$

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}\left[e^{-\delta\tilde{\kappa}_p^\varepsilon}\mathbf{1}_{\{\tilde{\kappa}_p^\varepsilon<\infty\}}\right] = \mathbf{E}\left[e^{-\delta\kappa_p}\mathbf{1}_{\{\tilde{\kappa}_p<\infty\}}\right].$$

Hence, we have

$$\lim_{\varepsilon \downarrow 0} \tilde{\kappa}_p^\varepsilon \to \kappa_p$$

in the sense of weak convergence [see 17, Theorem 1]. Then due to the definition of weak convergence, if we denote

$$\Phi_w(t) = \mathbf{E}\left[e^{-\delta t} w\left(X(t)\right)\right],$$

which is obviously bounded and continuous, then (5.3.23) is proved for all $w(x)$ that is bounded and continuous on $\mathbb{R}$.

Finally, by combining Theorem 5.3.1, Theorem 5.3.2 and Theorem 5.3.3 we may conclude

$$\phi_{\delta,w,p}(u) = \frac{\displaystyle\int_0^\infty x w(x) \mathbf{P}\left(X(p) \in \mathrm{d}x\right)}{\displaystyle\int_0^\infty x e^{\Phi(\delta)x} \mathbf{P}\left(X(p) \in \mathrm{d}x\right)} \left(e^{-\Phi(\delta)u} - e^{-\delta p} B_{\delta,p}(-u)\right) + e^{-\delta p} A_{w,p}(-u), \quad u \in \mathbb{R}, \tag{5.3.25}$$

for all bounded and almost everywhere continuous $w(x)$. In particular, if we let $w(x) = e^{-\rho x}$ for some $\rho > 0$, we may obtain the double Laplace transform of κ_p and $X(\kappa_p)$, namely,

$$\begin{aligned}\phi_{\delta,\rho,p}(u) &= \mathbf{E}_{-u}\left[e^{-\delta\kappa_p - \rho X(\kappa_p)}\mathbf{1}_{\{\kappa_p<\infty\}}\right] \\ &= \frac{\displaystyle\int_0^\infty x e^{-\rho x} \mathbf{P}\left(X(p) \in \mathrm{d}x\right)}{\displaystyle\int_0^\infty x e^{\Phi(\delta)x} \mathbf{P}\left(X(p) \in \mathrm{d}x\right)} \left(e^{-\Phi(\delta)u} - e^{-\delta p} B_{\delta,p}(-u)\right) \\ &\quad + e^{-\delta p} \int_0^\infty e^{-\rho x} \Lambda_{-u,p}(\mathrm{d}x), \quad u \in \mathbb{R}.\end{aligned} \tag{5.3.26}$$

5.4 Parisian Ruin Quantities

In this section, we focus on the particular case $w(x) = x^k$ where $k = 0, 1, \ldots,$ under the negative loading condition $\psi'(0+) \le 0$. The corresponding EDPF, which is the discounted kth moment of the deficit at Parisian ruin, is then defined as

$$\phi_{\delta,k,p}(u) = \mathbf{E}\left[e^{-\delta\kappa_p}\left|R(\kappa_p)\right|^k \mathbf{1}_{\{\kappa_p<\infty\}}\middle|R(0) = u\right].$$

Consequently, by taking kth derivative with respect to ρ on both sides of (5.3.26), mutiplying both sides by $(-1)^k$ and letting $\rho = 0$, we deduce

$$\phi_{\delta,k,p}(u) = \frac{\int_0^\infty x^{k+1}\mathbf{P}\left(X(p) \in \mathrm{d}x\right)}{\int_0^\infty xe^{\Phi(\delta)x}\mathbf{P}\left(X(p) \in \mathrm{d}x\right)}\left(e^{-\Phi(0)u} - e^{-\delta p}B_{\delta,p}(-u)\right) + e^{-\delta p}A_{k,p}(-u), \tag{5.4.1}$$

for all $u \in \mathbb{R}$, where

$$A_{k,p}(-u) = \int_0^\infty x^k\Lambda_{-u,p}(\mathrm{d}x).$$

Then the following proposition is helpful for obtaining the quantity $\phi_{\delta,p,k}(u)$.

Proposition 5.4.1 *For fixed $p > 0$ and $k \in \mathbb{N}$, we have the following Laplace transforms*

$$\int_0^\infty e^{-\theta p}\int_0^\infty \frac{x^{k+1}}{p}\mathbf{P}\left(X(p) \in \mathrm{d}x\right)\mathrm{d}p = \frac{k!}{\Phi(\theta)^{k+1}} \tag{5.4.2}$$

and

$$\int_0^\infty e^{-\theta p}\int_0^\infty \frac{x}{p}e^{\Phi(\delta)x}\mathbf{P}\left(X(p) \in \mathrm{d}x\right)\mathrm{d}p = \frac{1}{\Phi(\theta) - \Phi(\delta)}. \tag{5.4.3}$$

Proof The proof is straightforward using Kendall's identity (5.2.8).

Proposition 5.4.1 shows that we may evaluate the quantity $\phi_{\delta,p,k}(0)$ through numerical inversion of Laplace transforms.

5.4.1 Compound Poisson dual risk processes with exponential jumps

Assume that $\{S(t) : t \ge 0\}$ is a compound Poisson process with intensity $\lambda > 0$ and continuous secondary distribution with probability density function $\beta e^{-\beta y}$, $y \ge 0$. Then the Laplace exponent

$$\psi(\theta) = c\theta - \lambda + \frac{\lambda\beta}{\theta + \beta}$$

and hence

$$\Phi(\delta) = \frac{1}{2}\left[\left(\frac{\lambda}{c} - \beta\right) + \frac{\delta}{c} + \sqrt{\left(\frac{\lambda}{c} - \beta + \frac{\delta}{c}\right)^2 + \frac{4\delta\beta}{c}}\right], \quad \frac{\lambda}{c} > \beta.$$

We also have for $t \geq 0$ and $x \geq 0$

$$c\delta_{ct}(\mathrm{d}x)\,\mathrm{d}t = \delta_{x/c}(\mathrm{d}t)\,\mathrm{d}x.$$

To see this, consider the Laplace transform with respect to t on the left handside of the above equation, namely,

$$\int_0^\infty e^{-\theta t}\delta_{ct}(\mathrm{d}x)\,\mathrm{d}t = \int_0^\infty e^{-\theta t}\mathbf{1}_{\{x<ct\leq x+\mathrm{d}x\}}\,\mathrm{d}t = \frac{e^{-x\theta/c}}{\theta}\left(1-e^{-\theta\,\mathrm{d}x/c}\right) = c^{-1}e^{-x\theta/c}\,\mathrm{d}x.$$

On the other hand,

$$\int_0^\infty e^{-\theta t}\delta_x(c\,\mathrm{d}t)\,\mathrm{d}x = \int_0^\infty e^{-\theta t}\delta_{x/c}(\mathrm{d}t)\,\mathrm{d}x = e^{-x\theta/c}\,\mathrm{d}x.$$

Now we denote for some $\rho > 0$

$$h(x,t,\rho) = e^{-\rho x}\sqrt{\frac{\lambda\beta t}{x}}I_1\left(2\sqrt{\lambda\beta t x}\right)$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind, we have

$$\begin{aligned} f(u,p-t) &= e^{-\lambda(p-t)}\sum_{k=1}^\infty \beta^k\frac{(c(p-t)-u)^{k-1}\,e^{-\beta\left(c(p-t)-u\right)}}{(k-1)!}\frac{(\lambda(p-t))^k}{k!} \\ &= e^{-\lambda(p-t)}h\left(c(p-t)-u,p-t,\beta\right), \quad u<0 \end{aligned} \tag{5.4.4}$$

and

$$\begin{aligned} \mathbf{P}(X(t)\in \mathrm{d}x) &= e^{-\lambda t}\left(\delta_0(ct-\mathrm{d}x)+\sum_{k=1}^\infty \beta^k\frac{(ct-x)^{k-1}e^{-\beta(ct-x)}}{(k-1)!}\frac{(\lambda t)^k}{k!}\,\mathrm{d}x\right) \\ &= e^{-\lambda t}\left(\delta_{ct}(\mathrm{d}x)+h(ct-x,t,\beta)\,\mathrm{d}x\right), \quad x\in[0,ct] \end{aligned} \tag{5.4.5}$$

Hence, by setting $w(x) = x^k$, we obtain

$$\int_0^\infty xw(x)\mathbf{P}\left(X(p)\in \mathrm{d}x\right) = e^{-\lambda p}\left((cp)^{k+1}+\int_0^{cp}x^{k+1}h(cp-x,p,\beta)\,\mathrm{d}x\right). \tag{5.4.6}$$

Define

$$Q_k(z) := e^{-\lambda p}\left(z^{k+1}+\int_0^z x^{k+1}h(z-x,p,\beta)\,\mathrm{d}x\right), \quad z\geq 0,$$

then its Laplace transform is

$$\tilde{q}_k(s) = (k+1)!s^{-(k+2)}e^{-\lambda p+\frac{\lambda\beta p}{s+\beta}}.$$

Hence, if we denote the Laplace transform operator as $\mathcal{L}$, then (5.4.6) may be expressed as

$$\int_0^\infty xw(x)\mathbf{P}\left(X(p)\in \mathrm{d}x\right) = (k+1)!e^{-\lambda p}\mathcal{L}^{-1}\left[s^{-(k+2)}e^{\frac{\lambda\beta p}{s+\beta}}\right](cp).$$

Likewise,

$$\int_0^\infty x e^{\Phi(\delta)x}\mathbf{P}\left(X(p)\in \mathrm{d}x\right) = e^{-\left(\lambda - c\Phi(\delta)\right)p}\mathcal{L}^{-1}[s^{-2}e^{\frac{\lambda\beta p}{s+\beta+\Phi(\delta)}}](cp).$$

Thus,

$$\frac{\displaystyle\int_0^\infty x^{k+1}\mathbf{P}\left(X(p)\in \mathrm{d}x\right)}{\displaystyle\int_0^\infty x e^{\Phi(\delta)x}\mathbf{P}\left(X(p)\in \mathrm{d}x\right)} = (k+1)! e^{-c\Phi(\delta)p}\frac{\mathcal{L}^{-1}[s^{-(k+2)}e^{\frac{\lambda\beta p}{s+\beta}}](cp)}{\mathcal{L}^{-1}[s^{-2}e^{\frac{\lambda\beta p}{s+\beta+\Phi(\delta)}}](cp)}.$$

Moreover,

$$\begin{aligned}\Lambda_{-u,p}(\mathrm{d}x) &= \mathbf{P}_{-u}\left(X(p)\in \mathrm{d}x\right) - x\int_{x/c}^{p} f(u,p-t)\mathbf{P}\left(X(t)\in \mathrm{d}x\right)\frac{\mathrm{d}t}{t}\\ &= e^{-\lambda p}\delta_{cp-u}(\mathrm{d}x) + e^{-\lambda p}h(cp-u-x,p,\beta)\,\mathrm{d}x\\ &- e^{-\lambda p}\left(h(cp-x-u,p-x/c,\beta) + x\int_{x/c}^{p}\frac{h\left(c(p-t)-u,p-t,\beta\right)h(ct-x,t,\beta)}{t}\,\mathrm{d}t\right)\mathrm{d}x.\end{aligned}$$

Thus, we may obtain specific expressions for the functions $A_{k,p}(-u)$ and $B_{\delta,p}(-u)$. Namely,

$$\begin{aligned}A_{k,p}(-u) = e^{-\lambda p}\Bigg((cp-u)^k &+ \int_0^{cp-u} x^k h(cp-u-x,p,\beta)\,\mathrm{d}x\\ &- \int_0^{cp}(cp-x)^k h(x-u,x/c,\beta)\,\mathrm{d}x\\ &- \int_0^{cp} x^{k+1}\int_{x/c}^{p} h(c(p-t)-u,p-t,\beta)h(ct-x,t,\beta)\frac{\mathrm{d}t}{t}\,\mathrm{d}x\Bigg)\end{aligned} \tag{5.4.7}$$

and

$$\begin{aligned}B_{\delta,p}(-u) = e^{-\left(\lambda-c\Phi(\delta)\right)p-\Phi(\delta)u}\Bigg(1 &+ \int_0^{cp-u} h\left(cp-u-x,p,\beta+\Phi(\delta)\right)\mathrm{d}x\\ &- \int_0^{cp} h\left(x-u,x/c,\beta+\Phi(\delta)\right)\mathrm{d}x\\ &+ \int_0^{cp} x\int_{x/c}^{p} h\left(c(p-t)-u,p-t,\beta+\Phi(\delta)\right)h\left(ct-x,t,\beta+\Phi(\delta)\right)\frac{\mathrm{d}t}{t}\,\mathrm{d}x\Bigg).\end{aligned} \tag{5.4.8}$$

Example We specify the model parameters as $c=\beta=1$ and $\lambda=2$, which are chosen to satisfy the negative loading condition. Then for Parisian delays with different periods ($p=0.1$, 1 and 5, respectively), we obtain the corresponding Parisian ruin probability (by assuming $k=0$ and $\delta=0$) as a function of the initial surplus $u\in\mathbb{R}$. The result is shown below in Figure 5.3.

In Figure 5.3, we see that on one hand, as the Parisian delay period p is approaching 0, the Parisian ruin probability seems to be approaching the classical ruin probability of the dual risk model. Namely,

$$\phi(u) = e^{-\Phi(0)(u)_+}\quad \text{with } (u)_+ = \max\{u,0\}.$$

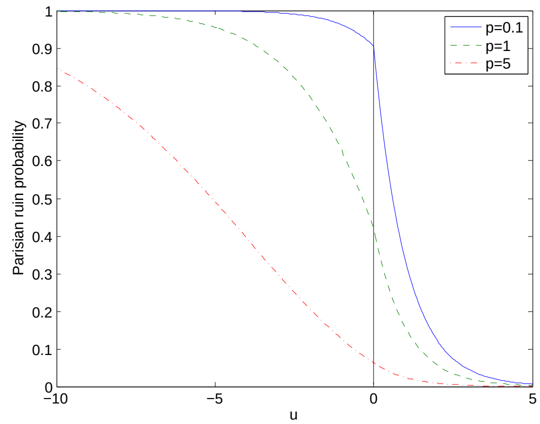

Figure 5.3: Parisian ruin probability of the compound Poisson dual model

On the other hand, as the Parisian delay period p is approaching ∞, the Parisian ruin seems not to occur due to the negative loading condition which results in

$$\lim_{t\to\infty} R(t) = \infty$$

almost surely [see 11, Theorem 7.2(i)]. Therefore, the Parisian ruin probability for this model is actually not a good measure of vulnerability to the solvency because the ruin probability may be reduced by increasing p.

Similarly, we may obtain the expected discounted deficit at Parisian ruin (assuming discount factor $\delta = 0.05$ and $k = 1$), which is shown in Figure 5.4. Figure 5.4 also emphasises

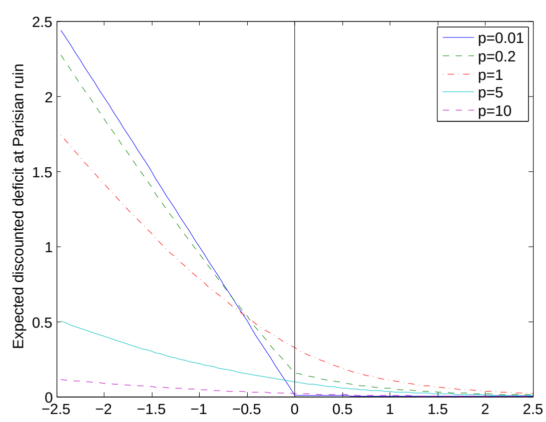

Figure 5.4: Expected discounted deficit at Parisian ruin under the compound Poisson dual model

the consistency between Parisian ruin and regular ruin. That is, when p is approaching 0, the expected discounted deficit at Parisian ruin as a function of u approaches $(u)_- = -\min\{u, 0\}$, which may be evaluated as the deficit at ruin (under dual models, the deficit at ruin is always

0 when the initial surplus is positive; otherwise, ruin occurs immediately and the nonpositive initial surplus is interpreted as deficit). Another interesting observation is that the expected discounted deficit at Parisian ruin for a nonnegative initial surplus becomes a decreasing function of p as long as p is large enough, which indicates that the expected discounted deficit at Parisian ruin might not be a proper measure of vulnerability to the solvency because it may also be reduced by increasing p as the lines for $p = 5$ and $p = 10$ show.

5.4.2 Diffusion-perturbed Compound Poisson dual risk processes with exponential jumps

Assume that $\{S(t) : t \geq 0\}$ is a diffusion-perturbed compound Poisson process with intensity $\lambda > 0$, $\sigma > 0$ and continuous secondary distribution with probability density function $\beta e^{-\beta y}$, $y \geq 0$. Then the Laplace exponent

$$\psi(\theta) = c\theta + \frac{\sigma^2}{2}\theta^2 - \frac{\lambda\theta}{\theta + \beta}$$

and hence $\Phi(\delta)$ is the largest positive root to

$$\frac{\sigma^2}{2}\theta^3 + \left(\frac{\sigma^2}{2}\beta + c\right)\theta^2 + (c\beta - \lambda - \delta)\,\theta - \delta\beta = 0.$$

The derivation of the explicit formula is omitted due to its complexity. Instead, we give an example to show how the diffusion coefficient affects the Parisian ruin probability.

Example Assume the same model parameters as in Example 5.4.1 except that $p = 1$ is fixed. Then we compute the Parisian ruin probability for nonnegative initial surplus employing Proposition 5.4.1 and applying the Gaver-Stehfest algorithm for the inversion of the Laplace transforms with respect to different choices of the value of σ. The results are shown in Figure 5.5. In Figure 5.5 we see that for a small initial surplus the Parisian ruin probability is smaller for larger σ. This might be because small initial surplus tends to bring the surplus level into the interval $(-\infty, 0)$, while large σ seems to help bringing the surplus level back to $(0, \infty)$. However, due to the negative loading condition, the surplus level for a large initial surplus may not drop below 0 easily without a larger value of σ.

5.5 Conclusions

In this article, we obtain an explicit expression of EDPF as defined in the (5.1.2) and assuming Parisian ruin.

Based on the numerical illustrations shown in the previous section, we may conclude that the Parisian ruin quantities, probability of ruin and expected discounted deficit, may not be employed as proper measures of vulnerability to the solvency under dual models unless the Parisian delay is predetermined due to other considerations. Otherwise, the EDPF may be reduced by increasing the length of the Parisian delay p. Such a strategy may not be implemented in practice where external regulations restrict the waiting period for reporting bankruptcy. A

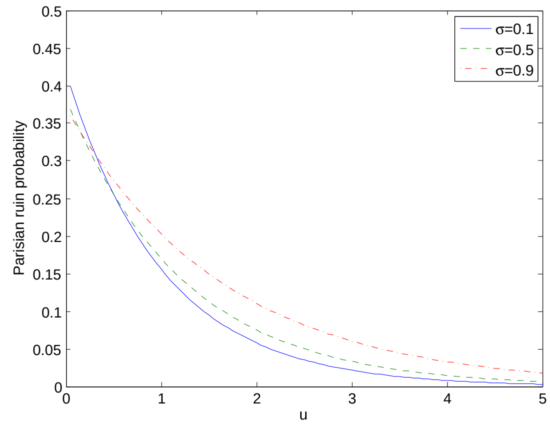

Figure 5.5: Parisian ruin probability under the compound Poisson dual model perturbed by diffusion

potential way to render Parisian ruin quantities useful for evaluating the risk of default is to implement certain types of penalties for a large Parisian delay p. One natural choice is to incorporate a debit interest rate for negative surplus. As a result, the strong Markov property will no longer apply to the surplus process, while the surplus is negative and hence, the problem will become more challenging.

Appendix

5.A Interchange limit operators

Lemma 5.A.1 *Assume sequences of functions $\{f_n(x)\}_{n=1}^{\infty}$ and $\{g_n(x)\}_{n=1}^{\infty}$ converge to $f(x)$ and $g(x)$ in pointwise sense respectively and*

$$\lim_{n\uparrow\infty}\lim_{x\to x_0} f_n(x) = \lim_{x\to x_0}\lim_{n\uparrow\infty} f_n(x) = \lim_{x\to x_0} f(x) = 0$$

and

$$\lim_{n\uparrow\infty}\lim_{x\to x_0} g_n(x) = \lim_{x\to x_0}\lim_{n\uparrow\infty} g_n(x) = \lim_{x\to x_0} g(x) = 0,$$

besides, for all $n \in \mathbb{N}$,

$$\lim_{x\to x_0}\frac{f_n(x)}{g_n(x)} = y_n \in (0,\infty), \quad \lim_{n\uparrow\infty} y_n = y \in (0,\infty),$$

then we have

$$\lim_{x\to x_0}\lim_{n\uparrow\infty}\frac{f_n(x)}{g_n(x)} = \lim_{n\uparrow\infty}\lim_{x\to x_0}\frac{f_n(x)}{g_n(x)} = y. \tag{5.A.1}$$

Proof Notice that the conclusion (5.A.1) is obviously true if $\lim_{x\to x_0} f(x) \neq 0$ and $\lim_{x\to x_0} g(x) \neq 0$. Hence we consider sequences of functions $\{f_n(x,a)\}_{n=1}^{\infty}$ and $\{g_n(x,a)\}_{n=1}^{\infty}$ such that

$$f_n(x,a) = f_n(x) + a y_n \quad \text{and} \quad g_n(x,a) = g_n(x) + a$$

for some $a \in \mathbb{R}\backslash\{0\}$. Then immediately we have

$$f(x,a) = f(x) + ay \quad \text{and} \quad g(x,a) = g(x) + a$$

and

$$\lim_{x\to x_0}\lim_{n\uparrow\infty}\frac{f_n(x,a)}{g_n(x,a)} = \lim_{n\uparrow\infty}\lim_{x\to x_0}\frac{f_n(x,a)}{g_n(x,a)}. \tag{5.A.2}$$

Then (5.A.1) may be obtained by letting $a \to 0$ on both sides of (5.A.2) and the fact that $f_n(x,a)$ and $g_n(x,a)$ converge to $f_n(x)$ and $g_n(x)$ uniformly for all $n \in \mathbb{N}$ and $f(x,a)$ and $g(x,a)$ converge to $f(x)$ and $g(x)$ uniformly.

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Chapter 6

Future work

So far as we obtain in Chapter 5, a Gerber-Shiu type function with respect to the Parisian ruin of the dual Lévy risk model has been discovered. It is natural to consider the de Finetti's optimal dividend problem based on the Parisian ruin. In Czarna and Palmowski [1], the authors provided a very nice expression for the expected discounted aggregate dividends paid up to the Parisian ruin under the barrier strategy and proved the optimality of the barrier strategy under certain conditions for the general classical Lévy risk models. Encouraged by their results, we strongly believe the optimality of the barrier/threshold strategy for the dual Lévy risk model holds for the Parisian ruin time without any further specific conditions.

Besides, we will keep an eye on any potential ruin-related quantities of interest. Recently, Liu and Cheung [2] develop a new type of Gerber-Shiu function for a Semi-Markovian risk model involving the quantities $R(\tau)$ defined in Chapter 2, $R(\tau-)$, the running minimum at the non-stopping time τ (probably should be denoted as $\underline{R}(\tau)$), and the deficit right before the first jump (probably could be denoted as $|R(\tau_{N(T)+1}-)|$ where $\tau_n := \sum_{i=1}^{n} V_i$) from a negative surplus if we allow the business to continue even when the surplus is negative. This piece of work reminds me of a potential generalization of the Gerber-Shiu type function defined via (5.1.2), namely,

$$\phi_{\delta,w,p}(u) = \mathbf{E}\left[e^{-\delta\kappa_p} w\left(|R(\kappa_p)|, \left|\underline{R}(\kappa_p)\right|\right) \mathbf{1}_{\{\kappa_p<\infty\}} \middle| R(0) = u\right], \quad u \in \mathbb{R} \tag{6.0.1}$$

for the general dual Lévy risk models.

Moreover, we could consider the generalization of the dual Lévy risk models to the dual Markov additive risk models, for which the underlying risk process is a spectrally positive Markov addtive process (MAP). Since a serie of fluctuation identities related to the spectrally negative MAP are given in Kyprianou and Palmowski [3, Theorem 1, Theorem 3], it is possible to consider the de Finetti's optimal dividend problem for the dual Markov additive risk models.

Last but not least, to approximate the Parisian ruin related quantities, the numerical inversion of Laplace transforms is required, which might be computational consuming. Hence we could consider to apply Erlangian approximation technique discussed in Asmussen et al. [4] to approximate the fixed Parisian delay with a random variable having an Erlang(n) distribution. Certainly, further study of major interest relies on the convergence rate of this type of approxmation as well.

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