8-21-2015 12:00 AM

Completely monotone and Bernstein functions with convexity properties on their measures

Shen Shan, *The University of Western Ontario*

Supervisor: Hristo S. Sendov, *The University of Western Ontario*

A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Statistics and Actuarial Sciences

COMPLETELY MONOTONE AND BERNSTEIN FUNCTIONS WITH CONVEXITY PROPERTIES ON THEIR MEASURES

(Thesis format: Monograph)

by

Shen Shan

Graduate Program in Statistics and Actuarial Science

A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

The School of Graduate and Postdoctoral Studies
The University of Western Ontario
London, Ontario, Canada

Abstract

The concepts of completely monotone and Bernstein functions have been introduced near one hundred years ago. They find wide applications in areas ranging from stochastic Lévy processes and complex analysis to monotone operator theory. They have well-known Bernstein and Lévy-Khintchine integral representations through which there are one-to-one correspondences between them and Radon measures on $[0, \infty)$ or $(0, \infty)$, respectively. In this thesis, we investigate subclasses of completely monotone and Bernstein functions with various convexity properties on their measures. These subclasses have intriguing applications in probability theories and convex analysis.

The convexity properties we investigate include convexity, harmonic convexity and β-convexity of the cumulative distribution functions. We characterize measures with various convexity properties to obtain results analogous to the classical Pólya's Theorem. Then we apply these characterizations of the measures to derive integral representations for these classes of completely monotone and Bernstein functions that are variants of the classical Bernstein and Lévy-Khintchine integral representations.

To explore the connections among completely monotone and Bernstein functions with various convexity properties on their measures, we investigate the characterizations and obtain various necessary and sufficient conditions for a completely monotone or Bernstein function to belong to one of the subclasses. We also identify maps that transform completely monotone and Bernstein functions into one with certain convexity properties on their measures. Interesting parallels between completely monotone and Bernstein functions are observed. For example, the transformation that turn a Bernstein function into one having Lévy measure with harmonically concave tail is the same as the transformation that turns a completely monotone function into one having harmonically convex measure. To help understand these analogies, a criteria for completely monotone and Bernstein function to have measures with β-convexity property is obtained. That generalizes the conditions for both convexity and harmonic convexity.

Let $\mathcal{H}_{CM}$ be the set of all Bernstein functions h, such that $f \circ h$ is the Laplace transform of a harmonically convex measure for *any* completely monotone function f. Similarly, let $\mathcal{H}_{BF}$ be the set of all Bernstein functions h, such that $g \circ h$ has Lévy measure with harmonically concave tail for *any* Bernstein function g. Surprisingly, we show that $\mathcal{H}_{CM} = \mathcal{H}_{BF}$ and are non-empty. For example we prove that x^α is in $\mathcal{H}_{BF}$ for any $\alpha \in (0, 2/3]$. In other words, the Bernstein function $x \mapsto x^\alpha$ is a transformation that deforms the measure of any Bernstein (resp. completely monotone) function into one that not only has a continuous distribution function on $(0, \infty)$ but also a convenient concavity (reps. convexity) property. We give necessary and sufficient condition for a Bernstein function to be in $\mathcal{H}_{BF}$ in terms of its convolution semigroups of sub-probability measures. However, it is not well-understood what are the functions that "generate" this set. We hope to investigate such issues in the future.

Keywords: completely monotone function, Bernstein function, convexity, harmonic convexity, Laplace transform, convolution semigroups of sub-probability measures, Lévy processes, coupon collector's problem

Dedicated to my family
for their love, support and encouragement.

Acknowlegements

First and foremost, I would like to express my sincere gratitude to my supervisor, Dr. Hristo S. Sendov. This research would not have been possible without his talented insightful guidance and continuing motivation. His integrated knowledge system, ever lasting patience, and detailed academic principle have sustained the cumulative progress of my research. In addition, his humor and easy-going character make our meetings productive and joyful.

I also want convey my appreciation to Dr. John W. Braun, Dr. Hao Yu, and Dr. Ricard Zitikis. It was them who showed me the path into the kingdom of statistics when I began my PhD research. Their encouragement and assistance are highly appreciated and will always be remembered.

Besides, I take this chance to thank my family and my beloved wife. Their unconditional love, supportive encouragement and progressive motivations help me to achieve this academic accomplishment. It is them who always give me hope and make me feel loved. They are the reasons of everything.

Last but not least, I would like to thank all my friends and colleges. They provided great company and made my days enjoyable. The past four years are the best time I've ever had and will be remembered in all my life. Thank you all with my best wishes.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

For me everything started with the classical Coupon collector's problem and a related, still open, conjecture about its variance. Suppose there are n types of coupons, labeled $1, 2, \ldots, n$, and someone wants to collect at least one of each type. On each trial, a coupon is collected at random, and the probability that the i-th coupon is obtained is x_i for all $i = 1, \ldots, n$. The probabilities satisfy $x_i > 0$ and $x_1 + x_2 + \cdots + x_n = 1$. Define T to be the number of trials needed to get at least one coupon from each type. It is well-known that

$$E[T] = \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{1 \leq i < j \leq n} \frac{1}{x_i + x_j} + \cdots + (-1)^{n-1} \frac{1}{x_1 + \cdots + x_n}.$$

Baum and Billingsley discussed asymptotic distribution on the trials needed to have k different coupons collected for the first time in [5], $1 \leq k \leq n$. If $k = n$, then it is equivalent to the random variable T defined above. Moriarty and Neal in [57] considered the asymptotic distribution of T (appropriately normalized) as the number of coupons n approaches infinity under the assumptions that the probability for each coupon being draw is unequal. Holst obtained the asymptotic distributions of T from classical extreme value theory by embedding the sampling procedure in a Poisson point process and expressing its distribution using extremes of independent identically distributed random variables in [42]. Jocković and Mladenović in [44] studied the waiting time by certain stopping rules and investigated models for its asymptotic behavior. Doumas and Papanicolaou in [30] developed techniques for computing the asymptotic value of the first and second moments of the random variable T as n approaches infinity.

Some other aspects are investigated. Boneh and Hofri in [15] showed two computational paradigms that are well-suited for this type of problem. Adler and Ross generalized the coupon collector's problem by selecting a random subset of coupons each time, rather than collecting an individual coupon in [1]. They provided bounds for the number of trials needed to get at least one coupon of each kind as well.

The convexity of the expectation of this random variable T in terms of $(x_1, x_2, \ldots, x_n)$ is another popular topic. Caron, Hlynka, and McDonald in [23] minimized the expected number of trials in the coupon collector's problem, and they constructed conjectures on its convexity. Borwein, Affleck, and Girgensohn posed a problem in [17] regarding the shape of $E[T]$. It is shown in [18] that $E[T]$ is convex in $(x_1, \ldots, x_n)$ on $(0, \infty)^n$. Recently in [71], Sendov and

Zitikis considered a natural generalization

$$F[f](x_1, \ldots, x_n) := \sum_{i=1}^{n} f(x_i) - \sum_{1\le i<j\le n} f(x_i + x_j) + \cdots + (-1)^{n-1} f(x_1 + \cdots + x_n),$$

for a function f defined on a domain in $\mathbb{R}$ and investigated its convexity properties, when f is completely monotone and Bernstein function. Using this notation, we have that $E[T] = F[1/x]$. As for the variance of T, it can be showed that

$$\mathrm{Var}\,[T] = 2F[1/x^2] - F[1/x] - (F[1/x])^2,$$

see [30]. There is a conjecture saying that $\mathrm{Var}\,[T]$ is minimized when $x_1 = x_2 = \cdots = x_n = 1/n$. The difficulty of the conjecture lies in the fact that the variance is not a convex function but rather a difference of two convex functions. In fact, every C^2 function has this property, see [36]. In an attempt to attack the conjecture with tools from Convex Analysis, the following results were obtained in [71].

(a) Let $f(x)$ be a completely monotone function with measure μ. If μ is harmonically convex, then $F[f]$ is convex and non-negative on $\mathbb{R}^n_{++}$.

(b) Let $g(x)$ be a Bernstein function with measure ν. If ν has a harmonically concave tail, then the function $F[g]$ is concave and non-negative on $\mathbb{R}^n_{++}$.

A non-negative function $f(x) : (0, \infty) \to [0, \infty)$ is called *completely monotone*, if it is infinitely differentiable and

$$(-1)^n f^{(n)}(x) \ge 0 \text{ for all } x > 0 \text{ and } n \ge 1.$$

Closely related to completely monotone functions are the Bernstein functions. A non-negative function $g(x) : (0, \infty) \to [0, \infty)$ is a *Bernstein* function, if it is infinitely differentiable and

$$(-1)^{n-1} g^{(n)}(x) \ge 0 \text{ for all } x > 0 \text{ and } n \ge 1.$$

By definition, the first derivative of a Bernstein function is completely monotone.

A function $h(x) : (0, \infty) \to \mathbb{R}$ is called *harmonically convex* if and only if

$$h\left(\frac{2}{1/x + 1/y}\right) \le \frac{h(x) + h(y)}{2}$$

for every $x, y \in (0, \infty)$. This terminology, harmonic convexity, follows from the fact that $2/(1/x + 1/y)$ is the harmonic mean of x and y.

The main goal of this thesis is to provide a thorough investigation of the properties of completely monotone and Bernstein functions with various convexity properties on their measures. Monograph [62] is a classic reference on convexity.

Completely monotone function is a classical object finding numerous applications in analysis and probability theories. These functions are very common. In fact, every completely monotone function is the Laplace transform for a Radon measure. The monograph [80] is an excellent introduction to the subject. Reference [3] is a more recent review on Laplace transform.

Completely monotone function is closely related with Stieltjes functions, see [58] and [79]. They are also connect to operator monotone functions in the scope of matrix analysis, see [13] and [7]. Let $f(x)$ be a real function defined on interval I. Define $f(A)$ for a Hermitian matrix A whose eigenvalues are $\lambda_i \in I, i = 1, \ldots, n$ as follows.

$$f(A) = U f(D) U^*,$$

where $f(D) = \mathrm{Diag}\,(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n))$, and $A = UDU^*$, $D = \mathrm{Diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Note that U is unitary, that is $UU^* = I$ for its conjugate transpose U^*. A function $f(x)$ is called matrix monotone of order n if it is monotone with respect to $n \times n$ Hermitian matrices, that is $f(B) - f(A)$ is positive definite if $B - A$ is positive definite. If $f(x)$ is matrix monotone of order n for all n, then we call $f(x)$ operator monotone. For example, $f(x) = x^r, r \in [0, 1]$ is proven to be operator monotone on $[0, \infty)$ in [13]. Integral representation for operator monotone function is covered in [25]. In fact, operator monotone functions form a sub-class of Bernstein functions, which is called completely Bernstein functions.

Bernstein functions as well find numerous applications in different areas. The monograph [68] is a contemporary introduction to the subject. It is known that Bernstein functions correspond to the Laplace exponent of the subordinator Lévy processes, see [10]. Monographs [9], [67] and [63] are excellent references on Lévy processes.

Recent interest has focused on sub-classes of Bernstein functions and their intriguing properties. Examples of such sub-classes are the completely Bernstein functions, the special Bernstein functions, the Torin Bernstein functions. In [35], Fourati and Jedidi provide a unified view on Jurek Bernstein functions, self-decomposable Bernstein functions and completely Bernstein functions. In [73], Song and Vondraček investigated a class of subordinators, which are called special subordinators, and study their potential theory. Burridge, Kuznetsov, Kwaśnicki and Kyprianou use Kendalls classic identity for spectrally negative Lévy processes to construct new families of subordinators with explicit transition probability semigroups by considering stopping times in [22].

In [69], we proposed a class of completely monotone functions with harmonically convex measures and a related class of Bernstein functions with corresponding measures having harmonically concave tail. We used them to construct novel families of multivariable convex functions and to give an explicit representation of the higher-order moments of the Coupon collector's random variable T, as a difference of two convex functions.

Since composition with a Bernstein function preserves the completely monotone and the Bernstein functions, we asked the question if there is a Bernstein function that transforms every other completely monotoneor Bernstein function into one with corresponding convexity property on their measures. Surprisingly the answer was 'yes' and it turned out that one and the same transformations work for both the completely monotone and the Bernstein functions. We called this set of transformations $\mathcal{H}_{BF}$.

Our next goal was to understand the structure of the set $\mathcal{H}_{BF}$. This led us to conduct a systematic search for different ways to characterize the completely monotone functions with harmonically convex measures and the related class of Bernstein functions with corresponding measures having harmonically concave tail. A big part of the thesis is dedicated to this topic. Another attempt to clarify the structure of the set $\mathcal{H}_{BF}$ was done by generalizing the definition of harmonic convexity and harmonic concavity and this led us to the definition of β-convexity. Despite these attempts the structure of the set $\mathcal{H}_{BF}$ remains very much an open problem.

The thesis is organized as following. In Chapter 2, we introduce completely monotone and Bernstein functions in Definition 2.1.1 and Definition 2.2.1 with their classic characterization in Theorem 2.1.2 and Theorem 2.2.2. Their basic properties are also studied, together with their connections to convolution semigroups of sub-probability measures and Lévy processes. Some useful results, such as Laplace Inversion Formula, see Theorem 2.1.4 and identities regarding higher order derivatives (2.14), are introduced here.

Besides, we introduce harmonic convexity (concavity) in Definition 2.4.1 and define measures with various convexity properties in Definition 2.4.3. Examples are provided and some known results are presented. In addition, we introduce β-convexity and β-concavity in Definition 2.5.1 and 2.5.3, which is proven to be a generalization of convexity and harmonic convexity regarding completely monotone and Bernstein functions. The properties regarding their limits are investigated in Lemma 2.5.5 and 2.5.6.

In Chapter 3, we extensively investigate various convexity properties for functions defined on $(0, \infty)$. In the light of Pólya's theorem, we characterize measure μ on $[0, \infty)$ with different convexity properties, as well as measure ν on $(0, \infty)$ with convexity properties on its tail. The summary is shown in Table 3.1 and 3.2.

Applying these characterizations onto Bernstein measures and Lévy measures, we could represent completely monotone and Bernstein functions whose measure has certain convexity properties, see summary in Table 3.3 and 3.4.

In Chapter 4, we study the characterizations of completely monotone functions with various convexity properties on their measures. As summarized in Table 4.1 and 4.2, we present three different characterizations for them, that is the one involving derivatives, the derivative free one and the sequential one. Multiple proofs are provided in several scenarios, one of them implementing to the representation in Chapter 3. We also present the characterizations for completely monotone functions, whose measures are β-convex (β-concave). And different cases, such as harmonic convexity and convexity, are shown as corollaries for specific choice of $\beta = 1$ or $\beta = 0$.

What's more, we show that completely monotone functions with harmonically convex or convex measures are actually transformations of completely monotone functions with certain integrability conditions, see Proposition 4.5.1 and 4.5.2. In the end, we investigate a few transformations that could preserve certain convexity properties on the measures of completely monotone functions.

In Chapter 5, we study the characterizations of Bernstein functions with different convexity properties on the tail of their measures. The structure of this Chapter is surprisingly analogous to Chapter 4 as summarized in Table 5.1 and 5.2. One can sense some hints from Table 3.1 and 3.2. We present three characterizations for Bernstein functions with different convexity properties on the tail of their Lévy measures, which are the ones involving derivatives, derivative free ones, and sequential ones. Multiple proofs are also provided, one which connects to the representations in Chapter 3. We also characterize Bernstein functions with β-concavity (convexity) properties on the tail of their measures. Special cases for $\beta = 1$ and $\beta = 0$ are highlighted as well.

Additionally, it is shown that Bernstein functions whose Lévy measures have harmonically concave tail or convex tail are indeed transformations of certain Bernstein functions, see Proposition 5.5.1 and 5.5.2. At the end, we also investigate a few transformations that could preserve certain convexity on the tails of Lévy measures.

In Chapter 6, we start by introducing a family of completely monotone functions. Then we introduce sub-classes of Bernstein function, such that if composed with arbitrary completely monotone or Bernstein function, the composition is endowed with certain convexity properties on measures, see Definition 6.2.1. By the family covered in the first section, we are able to show that the sub-classes of Bernstein functions $\mathcal{H}_{CM}$ and $\mathcal{H}_{BF}$ are not trivial. In particular, if $\alpha \in [0, 2/3]$, then for any completely monotone function $f(x)$, the composition $f(x^\alpha)$ is completely monotone with harmonically convex measure, and for any Bernstein function $g(x)$, the composition $g(x^\alpha)$ is Bernstein whose Lévy measure has harmonically concave tail. As corollaries, we obtain representations of completely monotone and Bernstein functions with convexity on measures in Corollary 6.2.1 and 6.2.2.

Furthermore, we show sub-classes $\mathcal{H}_{CM}$ and $\mathcal{H}_{BF}$ are actually identical in Corollary 6.3.1, by connecting to convolution semigroups associated with Bernstein functions. Some basic properties for $\mathcal{H}_{BF}$ are discussed and examples are provided.

In Chapter 7, our work is applied to the coupon collector's problem and to spectral functions. Some straightforward results are obtained. In Chapter 8, remaining open questions are listed and future works are proposed.

Chapter 2

Preliminaries

In this Chapter, we introduce completely monotone and Bernstein functions with their classic characterizations. Their basic properties will also be studied, as well as the connections to Lévy processes. In addition, some useful results, such as Laplace Inversion Formula and higher order derivative identities, are introduced here. They will play an important role in the further developments.

2.1 Completely monotone functions

2.1.1 Definitions and basic properties

Definition 2.1.1 *A non-negative function $f(x) : (0,\infty) \to [0,\infty)$ is called completely monotone, if it is infinitely differentiable and*

$$(-1)^n f^{(n)}(x) \geq 0 \textit{ for all } x > 0 \textit{ and } n \geq 1.$$

The family of all completely monotone functions will be denoted by $\mathcal{CM}$. Completely monotone functions are classical objects finding numerous applications in analysis and probability. The monographs [80] and [68] are excellent introductions to the subject. Classic characterization exists for completely monotone functions, also known as Bernstein representation.

Theorem 2.1.2 (Bernstein) *If $f(x)$ is a completely monotone function on $(0,\infty)$, then it is the Laplace transform of a unique Radon measure μ on $[0,\infty)$, that is,*

$$f(x) = \int_{[0,\infty)} e^{-xt}\,\mu(dt) \tag{2.1}$$

for all $x > 0$. Conversely, if μ is a Radon measure on $[0,\infty)$ such that the above integral is convergent for $x > 0$, then it defines a completely monotone function.

The cumulative distribution function for measure μ on $[0,\infty)$ is denoted by

$$F_\mu(x) = \mu[0,x].$$

It is often written as $F(x)$ for simplicity. To facilitate the development in the following sections, it is worthwhile to mention that,

$$f^{(n)}(x) = (-1)^n \int_{(0,\infty)} e^{-xt} t^n \, \mu(dt) = (-1)^n \int_{(0,\infty)} e^{-xt} t^n \, dF(t). \tag{2.2}$$

for all $n \geq 1$. And therefore

$$(-1)^n f^{(n)}(x) = \int_{(0,\infty)} e^{-xt} t^n \, \mu(dt).$$

Now we investigate some limit properties for completely monotone functions. For any $M > 0$, integrating by parts, we have

$$f(x) \geq F(0) + \int_{(0,M)} e^{-xt} \, dF(t) = e^{-xM} F(M) + x \int_{(0,M)} e^{-xt} F(t) \, dt. \tag{2.3}$$

And therefore,

$$\int_{(0,M)} e^{-xt} F(t) \, dt \leq \frac{f(x)}{x}. \tag{2.4}$$

Letting M approach infinity in (2.4), using the monotone convergence theorem, we know

$$\int_{(0,\infty)} e^{-xt} F(t) \, dt$$

is convergent for any $x > 0$, which implies

$$\lim_{t \to \infty} e^{-xt} F(t) = 0. \tag{2.5}$$

In addition, letting M approach infinity in (2.3) and using (2.5), we obtain

$$f(x) = x \int_{(0,\infty)} e^{-xt} F(t) \, dt. \tag{2.6}$$

This variant of the Bernstein representation will be very helpful in our development, as well as the limit property (2.5). We can also represent $f(x)$ as following.

$$f(x) = F(0) + \int_{(x,\infty)} (-f'(t)) \, dt,$$

where $-f'(t)$ is completely monotone as well. As a non-increasing function that is integrable at infinity is $o(1/t)$ as t approaches infinity (see Lemma A.2.2), we obtain

$$\lim_{x \to \infty} x f'(x) = 0. \tag{2.7}$$

This limiting property can be generalized in the next lemma.

Lemma 2.1.3 *Suppose $f(x)$ is completely monotone with measure μ. Then*

$$\lim_{x \to \infty} x^n f^{(n)}(x) = 0, \tag{2.8}$$

for any $n \geq 1$.

Proof By Lemma A.1.8, we know $u^n e^{-u} \leq (n+1)^n e^{-n} e^{-u/(n+1)}$ for all $n \geq 1$ and $u > 0$. And by (2.1), we obtain

$$\begin{aligned}\left|x^n f^{(n)}(x)\right| &= x^n \int_{[0,\infty)} e^{-xt} t^n \,\mu(dt) = \int_{(0,\infty)} e^{-xt} (xt)^n \,\mu(dt)\\ &\leq \int_{(0,\infty)} (n+1)^n e^{-n} e^{-xt/(n+1)} \,\mu(dt) = (n+1)^n e^{-n}\left(f\left(\frac{x}{n+1}\right) - \mu(\{0\})\right).\end{aligned}$$

As $\lim_{x\to\infty} f(x) = \mu(\{0\})$, letting x approaches infinity, we get (2.8) for all $n \geq 1$. □

Analogously, we could have

$$\lim_{x\to\infty} x^{n+1}\left(\frac{f(x) - F(0)}{x}\right)^{(n)} = 0, \tag{2.9}$$

for completely monotone function $f(x)$ with measure μ, whose cumulative distribution function is denoted as $F(t)$. To show this, noticing (2.6) and following a similar argument, we could get

$$\left|x^{n+1}\left(\frac{f(x) - F(0)}{x}\right)^{(n)}\right| \leq (n+1)^{n+1} e^{-n}\left(f\left(\frac{x}{n+1}\right) - F(0)\right).$$

And (2.9) follows from $\lim_{x\to\infty} f(x) = F(0)$.

It is known that the set of completely monotone functions is a convex cone. In fact, a linear combination of completely monotone functions is still such. In other words,

$$t f_1(x) + s f_2(x) \in \mathcal{CM},$$

for any $s, t \geq 0$ and any $f_1(x), f_2(x) \in \mathcal{CM}$. Moreover, this set is also closed under multiplication and point-wise convergence. That is

$$f_1(x) f_2(x) \in \mathcal{CM}, \quad \text{and} \quad \lim_{n\to\infty} f_n(x) \in \mathcal{CM},$$

where $f_n(x) \in \mathcal{CM}$ for all $n \geq 1$ and their point-wise limit exists for any $x > 0$. See [68, Corollary 1.6] for details.

In particular, if $f_1(x)$ and $f_2(x)$ are completely monotone functions, with corresponding measures μ_1 and μ_2, then their product has corresponding measure being the convolution of μ_1 and μ_2. That is $f_1(x) f_2(x) \in \mathcal{CM}$ is associated with measure $\mu_1 * \mu_2$ given by

$$(\mu_1 * \mu_2)[0, x] := \int_{R_+^2} \mathbf{1}_{[0,x]}(s+t)\,\mu_1(ds)\,\mu_2(dt). \tag{2.10}$$

Here $\mathbb{R}_+^2$ stands for the non-negative orthant in $\mathbb{R}^2$ and $\mathbf{1}_{[0,x]}$ is the indicator function of the interval $[0, x]$.

We list some other facts on completely monotone functions that are used without further reference. They are trivially true or one can find proofs in [68].

1. If $f(x)$ is completely monotone, then $af(bx+c)+\lambda$ is also completely monotone for any $a, b > 0$ and $c, \lambda \geq 0$.

2. If $f(x)$ is completely monotone, then $f(x) - f(x+\lambda)$ is also completely monotone for any $\lambda \geq 0$.
3. If $f(x)$ is infinitely differentiable such that $f(x) \geq 0$, $f'(x) \leq 0$ and $(-1)^n f^{(n)}(x) \geq 0$ for infinitely many $n \in \mathbb{N}$, then $f(x)$ is completely monotone.

In addition, their higher order derivatives have the following convergence.

Proposition 2.1.1 (Corollary 1.7 in [68]) *Let $\{f_n(x)\}_{n\in\mathbb{N}}$ be a sequence of completely monotone functions such that their point-wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for x on $(0,\infty)$. Then $f(x)$ is completely monotone and*

$$f^{(k)}(x) = \lim_{n\to\infty} f_n^{(k)}(x)$$

locally uniformly on $(0,\infty)$ for all $k \in \mathbb{N}$.

2.1.2 Inverse Laplace transform

A completely monotone function $f(x)$ is the Laplace transform of certain σ-finite measure, and vice versa. As a result, measure μ is to a large extent determined by $f(x)$. Define the operator

$$L_n(f(x);t) := \frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} f^{(n)}\left(\frac{n}{t}\right) = \frac{(-1)^n}{n!} x^{n+1} f^{(n)}(x)\Big|_{x=n/t} \tag{2.11}$$

for any $n \in \mathbb{N}$ and $t > 0$. The following theory is the well-known real inversion formula for the Laplace-Lebesgue and the Laplace-Stieltjes integrals, see [80, Chapter VII, Theorems 6a & 7a].

Theorem 2.1.4 (Inversion formula) *Let $f(x)$ be completely monotone with measure μ. Then,*

$$\lim_{n\to\infty} \int_{(0,t]} L_n(f(x);u)\,du = \frac{\mu[0,t] + \mu[0,t)}{2} - \mu(\{0\}) \tag{2.12}$$

for every $t > 0$. Operator L_n is defined in (2.11). In particular, if μ has density $h(t)$, then

$$\lim_{n\to\infty} L_n(f(x);t) = h(t) \tag{2.13}$$

for every $t > 0$ in the Lebesgue set of $h(t)$.

We say that the set of values t, for which

$$\int_{(0,\lambda)} \left|h(t+s) - h(t)\right|\, ds = o(\lambda)$$

as λ approaches 0, is the Lebesgue set for the function $h(x)$. Note that points of continuity of a function are in its Lebesgue set. Also note that, if the cumulative distribution function $F(x) = \mu[0,x]$ is continuous at $x = t > 0$, then the right-hand side of equation (2.12) becomes $F(t) - F(0)$.

When applying the inversion formula, we need the next identity (see [3, Lemma 2.7.12] for a more general identity).

Lemma 2.1.5 *For any sufficiently differentiable function $r(x)$ and any integer $n \geq 0$, the following identity holds*

$$\left(x^{n+1}\left(\frac{r(x)}{x}\right)^{(n)}\right)' = x^n r^{(n+1)}(x). \tag{2.14}$$

Proof Identity (2.14) holds for $n = 0$ trivially. For all $n \geq 1$,

$$\left(x^{n+1}\left(\frac{r(x)}{x}\right)^{(n)}\right)' = (n+1)x^n\left(\frac{r(x)}{x}\right)^{(n)} + x^{n+1}\left(\frac{r(x)}{x}\right)^{(n+1)}.$$

On the other hand, noticing $(xf(x))^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$ by Lemma A.1.1, we obtain

$$\begin{aligned} r^{(n+1)}(x) &= \left(x\,\frac{r(x)}{x}\right)^{(n+1)} = \left(x\left(\frac{r(x)}{x}\right)' + \frac{r(x)}{x}\right)^{(n)} = \left(x\left(\frac{r(x)}{x}\right)^{(n+1)} + n\left(\frac{r(x)}{x}\right)^{(n)}\right) + \left(\frac{r(x)}{x}\right)^{(n)} \\ &= x\left(\frac{r(x)}{x}\right)^{(n+1)} + (n+1)\left(\frac{r(x)}{x}\right)^{(n)}. \end{aligned}$$

Therefore, (2.14) also holds for all $n \geq 1$. This completes the verification. □

A straightforward outcome from this Lemma is the following identity for $L_n(f(x);t)$ which is defined in (2.11). It will facilitate the development in our next result:

$$L_n'(f(x);t) = \frac{n+1}{n} L_{n+1}\left(xf(x); \frac{(n+1)t}{n}\right), \tag{2.15}$$

for all $n \geq 1$ and $t > 0$.

2.2 Bernstein functions

2.2.1 Definition and basic properties

Definition 2.2.1 *A non-negative function $g(x) : (0,\infty) \to [0,\infty)$ is a Bernstein function, if it is infinitely differentiable and*

$$(-1)^{n-1} g^{(n)}(x) \geq 0 \textit{ for all } x > 0 \textit{ and } n \geq 1.$$

The family of all Bernstein function will be denoted as $\mathcal{BF}$. The Lévy-Khintchine representation theorem characterizes Bernstein functions.

Theorem 2.2.2 (Lévy-Khintchine) *A function $g(x)$ is Bernstein if and only if it admits the representation*

$$g(x) = a + bx + \int_{(0,\infty)} (1 - e^{-tx})\, \nu(dt) \tag{2.16}$$

for some constants $a, b \geq 0$ and a Radon measure ν on $(0,\infty)$ which satisfies

$$\int_{(0,\infty)} (1 \wedge t)\, \nu(dt) < \infty. \tag{2.17}$$

The triplet (a, b, ν) uniquely determines the Bernstein function $g(x)$, and vice versa.

Here $1 \wedge t = \min(1, t)$. The measure ν in the Lévy-Khintchine representation theorem is usually called the *Lévy measure* of the Bernstein function $g(x)$ and the triplet (a, b, ν) is called *Lévy triplet*. The tail of the Lévy measure ν on $(0, \infty)$ is denoted by

$$\bar{\nu}(x) = \nu(x, \infty).$$

Condition (2.17) is equivalent to the convergence of the integral in (2.16). The coefficients a and b in formula (2.16) can be recognized in the following way.

$$a = \lim_{x\to 0^+} g(x) \quad \text{and} \quad b = \lim_{x\to\infty} g(x)/x = \lim_{x\to\infty} g'(x). \tag{2.18}$$

To facilitate the development in the following sections, it is worthwhile to mention that

$$g'(x) = b + \int_{(0,\infty)} e^{-xt} t\, \nu(dt) \quad \text{and} \quad g^{(n)}(x) = (-1)^{n-1} \int_{(0,\infty)} e^{-xt} t^n\, \nu(dt) \quad \text{for all } n > 1. \tag{2.19}$$

Clearly, the first derivative of a Bernstein function is completely monotone. Indeed, if $g(x)$ is a Bernstein function with Lévy triplet (a, b, ν), then the completely monotone function $g'(x)$ has measure

$$\mu(dt) = b\delta_0(dt) + t\nu(dt), \tag{2.20}$$

where δ_0 is the Dirac delta function. However, it is not true, in general, that every completely monotone function has a primitive that is Bernstein.

Proposition 2.2.1 (Proposition 3.4 in [68]) *Suppose $f(x)$ is completely monotone with measure μ. It has a primitive $g(x)$ which is Bernstein, if and only if the measure μ satisfies*

$$\int_{(0,\infty)} \frac{1}{1+t}\, \mu(dt) < \infty. \tag{2.21}$$

There are useful variants of the Lévy-Khintchine representation (2.16). Note that $\bar{\nu}(t)$ is non-increasing, right-continuous and the integrability condition (2.17) implies that $\bar{\nu}(t) < \infty$ for $t > 0$ and consequently

$$\lim_{t\to\infty} \bar{\nu}(t) = 0. \tag{2.22}$$

See Lemma A.1.10. An application of Fubini's theorem to (2.16) leads to

$$g(x) = a + bx + x \int_{(0,\infty)} e^{-xt} \bar{\nu}(t)\, dt. \tag{2.23}$$

Therefore for all $k \geq 1$, we have

$$\left(\frac{g(x) - a}{x}\right)^{(k)} = (-1)^k \int_{(0,\infty)} e^{-xt} \bar{\nu}(t) t^k\, dt.$$

Note $e^{-xt}\bar{\nu}(t)$ is non-negative and non-increasing for any $x > 0$. By Lemma A.2.1,

$$\lim_{t\to 0} t\bar{\nu}(t) = 0 \quad \text{for any} \quad x > 0. \tag{2.24}$$

Utilizing formula (2.23) one obtains

$$g(x) - xg'(x) = a + x^2 \int_{(0,\infty)} e^{-tx} t\bar{\nu}(t)\, dt. \tag{2.25}$$

This shows that for any $x > 0$, we have $0 \leq xg'(x) \leq g(x) - a$, and letting x approach 0, we obtain that for any Bernstein function $g(x)$

$$\lim_{x\to 0+} xg'(x) = 0. \tag{2.26}$$

It follows from here that if $g(x)$ is Bernstein, $xg'(x)$ can not be completely monotone unless $g(x)$ is constant. This limiting property can be used to show the next lemma.

Lemma 2.2.3 *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then*

$$\lim_{x\to 0} x^n g^{(n)}(x) = 0, \tag{2.27}$$

for any $n \geq 1$.

Proof The case for $n = 1$ is shown in (2.26). Consider $n \geq 2$. By Lemma A.1.8, we know $u^n e^{-u} \leq (n+1)^n e^{-n} e^{-u/(n+1)}$ for all $n \geq 2$ and $u > 0$. And by (2.19), we obtain

$$\left|x^n g^{(n)}(x)\right| = x \int_{(0,\infty)} e^{-xt}(xt)^{n-1} t\, \nu(dt) \leq x \int_{(0,\infty)} n^{n-1} e^{1-n} e^{-xt/n} t\, \nu(dt) = n^{n-1} e^{1-n} xg'(x/n).$$

Noticing (2.26), letting x approach zero, we get (2.27) for all $n \geq 2$. □

The following structural characterization is given by Bochner in [14], and was investigated in [68, Theorem 3.7].

Theorem 2.2.4 *Let $g(x) : (0,\infty) \to (0,\infty)$. The following statements are equivalent.*

(a) $g(x) \in \mathcal{BF}$*;*

(b) $f(g(x)) \in \mathcal{CM}$ *for all* $f(x) \in \mathcal{CM}$*;*

(c) $e^{-sg(x)} \in \mathcal{CM}$ *for all* $s > 0$*.*

It is known that the set of Bernstein functions is a convex cone. In fact, a linear combination of Bernstein functions with non-negative coefficients is still Bernstein, that is

$$tg_1(x) + sg_2(x) \in \mathcal{BF},$$

for all $s, t \geq 0$ and any $g_1(s), g_2(s) \in \mathcal{BF}$. This set is also closed under composition and point-wise convergence, that is

$$g_1(g_2(x)) \in \mathcal{BF} \quad \text{and} \quad \lim_{n\to\infty} g_n(x) \in \mathcal{BF},$$

where $g_n(x) \in \mathcal{BF}$ for all $n \geq 1$ and their point-wise limit exists for $x > 0$. See [68, Corollary 3.8] for details.

We list some facts on completely monotone and Bernstein functions here that are used without further reference. Refer to [68] for proofs.

1. If $g(x)$ is Bernstein then $ag(bx+c)+\lambda$ is Bernstein for any $a,b>0$ and $c,\lambda \geq 0$.
2. If $g(x)$ is Bernstein, then $g(x+\lambda)-g(x)$ is completely monotone for all $\lambda > 0$.
3. If $g(x)$ is Bernstein, then $g(x)+g(\lambda)-g(x+\lambda)$ is also Bernstein for all $\lambda > 0$.
4. If $g(x)$ is Bernstein then $g(x)/x$ is completely monotone.
5. If $f(x) \leq c$ is completely monotone, then $c - f(x)$ is Bernstein.
6. If $c - f(x)$ is Bernstein for some $c > 0$ and $f(x) > 0$, then $f(x)$ is completely monotone, bounded from above by c.
7. If $g(x)$ is Bernstein, then there exists some $c > 0$ such that $g(x) \leq cx$ for all $x > 1$.

In addition, if a sequence of Bernstein functions converges, their higher order derivatives also converge.

Proposition 2.2.2 (Corollary 3.9 in [68]) *Let $\{g_n(x)\}_{n\in\mathbb{N}}$ be a sequence of Bernstein functions such that their point-wise limit $g(x) = \lim_{n\to\infty} g_n(x)$ exists for x on $(0,\infty)$. Then $g(x)$ is Bernstein and*

$$g^{(n)}(x) = \lim_{n\to\infty} g_n^{(n)}(x)$$

locally uniformly in $x \in (0,\infty)$ for all $n \in \mathbb{N}$.

2.2.2 Convolution semigroups of sub-probability measures

Definition 2.2.5 (Definition 5.1 in [68]) *A vaguely continuous convolution semigroup of sub-probability measures on $[0,\infty)$ is a family of measure $\{\nu_t\}_{t\geq 0}$ satisfying the following properties:*

(a) $\nu_t[0,\infty) \leq 1$ *for all* $t \geq 0$*;*

(b) $\nu_{t+s} = \nu_t * \nu_s$ *for all* $t, s \geq 0$*;*

(c) vague- $\lim_{t\to 0} \nu_t = \delta_0$.

Convolution semigroups are closed connected with Bernstein functions, as shown in the next theorem.

Theorem 2.2.6 *Suppose $\{\nu_t\}_{t\geq 0}$ is a convolution semigroup of sub-probability measures on $[0,\infty)$. It is uniquely determined by a Bernstein function $g(x)$ in the following identity.*

$$\int_{[0,\infty)} e^{-xs}\, \nu_t(ds) = e^{-tg(x)}, \tag{2.28}$$

for all $t \geq 0$. Conversely, given Bernstein function $g(x)$, there is a unique convolution semigroup of sub-probability measures $\{\nu_t\}_{t\geq 0}$ on $[0,\infty)$ such that (2.28) holds.

This is the reason why probabilists often use the name *Laplace exponent* instead of Bernstein function. The following theorem describes the Lévy measure of the composition $f(g(x))$ of two Bernstein functions $f(x)$ and $g(x)$ in terms of their convolution semigroups, see [68, Theorem 5.27].

Theorem 2.2.7 *If $f(x)$ and $g(x)$ are Bernstein functions with Lévy triplets (a, b, μ) and (α, β, ν) respectively. Then*

$$f(g(x)) = f(\alpha) + \beta b x + \int_{(0,\infty)} (1 - e^{-xt})\, \eta(dt), \tag{2.29}$$

where the Lévy measure η is given by the vague integral

$$\eta(dt) = b\nu(dt) + \int_{(0,\infty)} \nu_s(dt)\, \mu(ds) \tag{2.30}$$

where $\{\nu_t\}_{t\geq 0}$ is the convolution semigroup corresponding to $g(x)$.

Trivial modifications lead to the representation of the composition of completely monotone and Bernstein functions. We include the short proof for completeness.

Theorem 2.2.8 *Suppose $f(x)$ is completely monotone and $g(x)$ is Bernstein. Then*

$$f(g(x)) = \int_{[0,\infty)} e^{-xt}\, \xi(dt), \tag{2.31}$$

where the measure ξ is given by the vague integral

$$\xi(dt) = \int_{[0,\infty)} \nu_s(dt)\, \mu(ds) \tag{2.32}$$

where μ is the measure of $f(x)$ and $\{\nu_t\}_{t\geq 0}$ is the convolution semigroup of $g(x)$.

Proof Using (2.28), we have

$$f(g(x)) = \int_{[0,\infty)} e^{-sg(x)}\, \mu(ds) = \int_{[0,\infty)} \int_{[0,\infty)} e^{-xt}\, \nu_s(dt)\, \mu(ds) = \int_{[0,\infty)} e^{-xt}\, \xi(dt).$$

The result follows from the uniqueness of the Bernstein representation theorem for completely monotone function $f(g(x))$. □

For any $g(x) \in \mathcal{BF}$, the completely monotone function $f(x) = e^{-tg(x)}$ satisfies

$$0 \leq f(x) - xf'(x) = e^{-tg(x)}(1 + txg'(x)) \leq 1, \tag{2.33}$$

for all $x, t \geq 0$. Non-negativity is trivial as $f(x) \in \mathcal{CM}$. We only need to verify the upper bound. Fix $x > 0$, consider the function

$$p(t) := e^{tg(x)} - 1 - txg'(x).$$

Note that $p(0) = 0$ and $p'(t) = g(x)e^{tg(x)} - xg'(x) \geq g(x) - xg'(x) \geq 0$ by identity (2.25). We conclude that $p'(t) \geq 0$ and $p(t)$ is non-decreasing on $(0, \infty)$. Thus $p(t) \geq 0$, implying $e^{-tg(x)}(1 + txg'(x)) \leq 1$.

2.3 Lévy Processes

In this section, we introduce the basics for Lévy processes, especially some sub-classes.

2.3.1 Definition and its characterization exponent

Definition 2.3.1 *A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}$ is a Lévy process if the following conditions are satisfied.*

(a) *$X_0 = 0$ almost surely.*

(b) *Independent increment: For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.*

(c) *Stationary increments: The distribution of $X_{t+s} - X_s$ does not depend on s.*

(d) *Continuity in probability: For any $\epsilon > 0$ and $t \geq 0$, it holds that*

$$\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| > \epsilon) = 0.$$

(e) *Function $t \to X_t$ is almost surely right continuous with left limits.*

Lévy processes can be recognized as generalization of random walks in continuous time, and the distribution μ_t of X_t is infinitely divisible for any $t > 0$, with characteristic function

$$\exp\{-s\Psi(\lambda)\} = \int_{\mathbb{R}^d} e^{i\lambda t}\,\mu_s(dt) = \mathbb{E}(e^{i\lambda X_s}).$$

Here $\Psi(\lambda)$ is also called characteristic exponent of the Lévy process X_t. Any infinitely divisible probability measure μ on $\mathbb{R}$ can be viewed as the distribution of a Lévy process evaluated at $t = 1$, and the converse is true as well. The well-known Lévy-Khintchine representation tells us that $\Psi(\lambda)$ can be expressed as

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda t} + i\lambda t \mathbf{1}_{|t|<1})\,\Pi(dt), \tag{2.34}$$

where $a \in \mathbb{R}, \sigma \geq 0$, $\mathbf{1}_{|t|<1}$ is the indicator function and Π is a sigma-finite measure concentrated on $\mathbb{R} \setminus \{0\}$, satisfying

$$\int_{\mathbb{R}} (1 \wedge t^2)\,\Pi(dt) < \infty.$$

By Lévy-Ito decomposition, the next proposition is shown as [47, Lemma 2.12].

Proposition 2.3.1 *A real valued Lévy process X_t with triplet (a, σ, Π) has path of bounded variation if and only if*

$$\sigma = 0 \quad \textit{and} \quad \int_{\mathbb{R}} (1 \wedge |t|)\,\Pi(dt) < \infty.$$

The finiteness of the integral $\int_{\mathbb{R}} (1 \wedge |t|)\,\Pi(dt)$ allows the Lévy exponent of any real-valued Lévy process of bounded variation to be rewritten as follows:

$$\Psi(\lambda) = -ib\lambda + \int_{\mathbb{R}} (1 - e^{i\lambda t})\,\Pi(dt) \tag{2.35}$$

where

$$b = -a - \int_{(-1,1)} t\,\Pi(dt). \tag{2.36}$$

The constant b is often referred to as *drift*. Particularly, a Lévy measure is a compound Poisson process with drift, if and only if its characteristic exponent $\Psi(\lambda)$ in (2.34) has $\sigma = 0$ and $\Pi(\mathbb{R}) < \infty$.

2.3.2 Sub-classes of Lévy processes

In this section, we introduce some popular sub-classes of Lévy processes. Within them, subordinators are closed related with Bernstein functions. Spectrally negative Lévy processes is connected to Wiener-Hopf satisfaction of Lévy processes. And stable process is linked to a special kind of Bernstein function x^α where $\alpha \in (0, 1)$.

Subordinators

A subordinator is a real-valued Lévy process which only takes nonnegative values. One can easily see the subordinator must have non-decreasing path almost surely. In fact, a Lévy process is a subordinator if and only if its characteristic exponent in (2.35) can be written as

$$\Psi(\lambda) = -ib\lambda + \int_{(0,\infty)} (1 - e^{i\lambda t})\,\Pi(dt).$$

where $b \geq 0$ and

$$\int_{(0,\infty)} (1 \wedge t)\,\Pi(dt) < \infty.$$

As $\Psi(\lambda)$ can be extended analytically on the complex upper half-plane. This yields its Laplace exponent

$$\phi(\lambda) = -\frac{1}{t}\log \mathbb{E}(e^{-\lambda X_t}) = \Psi(i\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\,\Pi(dt).$$

For every Bernstein function $g(x)$ it can be represented as

$$g(x) = a + bx + \int_{(0,\infty)} (1 - e^{-xt})\,\nu(dt).$$

It has been shown that Bernstein function is associated with convolution semigroups of sub probability measures $\{\nu_t\}_{t\geq 0}$. In particular, if ν_t is probability measure for all $t > 0$, then the constant $a = 0$. Its representation coincide with the Laplace exponent of subordinator. In other words, subordinator is associated with convolution semigroups of probability measures, and vice versa.

Spectrally negative Lévy processes

A real-valued Lévy process with no positive jumps is called spectrally negative. In other words, a Lévy process X_t is spectrally negative if $\Pi(0, \infty) = 0$ in (2.34). The degenerated cases include the negative of a subordinator and deterministic drift.

Although spectrally negative Lévy processes X_t may take values of both signs, its exponential moments are finite. That is

$$\mathbb{E}(e^{\lambda X_t}) < \infty, \quad \text{for all} \quad \lambda > 0.$$

See [9, Chapter VII] for detail. The characteristic function $\Psi(\lambda) = \mathbb{E}(e^{i\lambda X_1})$ can be extended analytically on the complex lower half-plane. This yield (2.34) to

$$\psi(\lambda) := -\Psi(-i\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda t} - 1 - \lambda t\mathbf{1}_{\{t>-1\}})\,\Pi(dt).$$

This $\psi(\lambda)$ is also called *Laplace exponent* of X_t as $\mathbb{E}(e^{\lambda X_t}) = e^{t\psi(\lambda)}$ holds for all λ whose real part is non-negative and $t > 0$.

If X_t has bounded variation, we may write

$$\psi(\lambda) = b\lambda - \int_{(-\infty,0)} (1 - e^{\lambda t}) \, \Pi(dt),$$

where b is the given in (2.36). In other words, a spectrally negative Lévy process of bounded variation is a drift minus a pure jump subordinator.

Stable processes

Stable processes are Lévy processes whose characteristic exponent correspond to stable distributions. We introduce the short-hand notation

$$X \overset{d}{=} Y$$

to indicates that the random variable X, Y have the same distribution. Let $X, X_1, X_2, \ldots$ be independent random variables with same distribution F.

Definition 2.3.2 (Definition 2.1 in [33]) *The distribution F is stable if for each n there exists constants $c_n > 0$, γ_n such that*

$$X_1 + X_2 + \cdots + X_n \overset{d}{=} c_n X + \gamma_n$$

and F is not concentrated at one point. F is stable in the strict sense if the above equation holds with $\gamma_n = 0$.

It is clear that stable distributions are infinitely divisible. And it is shown in [33] that $c_n = n^{1/\alpha}$ for $\alpha \in (0, 2]$. This constant α is called the characteristic exponent of F.

For example, Normal distribution is stable with $\alpha = 2$, and Cauchy distribution is stable with $\alpha = 1$. However, Poisson distribution is not stable. This can be shown by failing to match all the moments of $X_1 + X_2 + \cdots + X_n$ and $c_n X + \gamma_n$ for fixed n.

It is shown that the characteristic exponent $\Psi(\lambda)$ for stable processes with $\alpha \in (0, 1)$ or $\alpha \in (1, 2)$ is

$$\Psi(\lambda) = c|\lambda|^\alpha \, (1 - i\beta \, \mathrm{sign}(\lambda) \tan(\pi\alpha/2)), \quad \lambda \in \mathbb{R},$$

where $c > 0$ and $\beta \in [-1, 1]$. One can refer to [83], [9] or [47] for more details. In particular, Bernstein function $g(x) = x^\alpha$ for $\alpha \in (0, 1)$ is characteristic exponent for certain stable distributions.

2.4 Harmonic convexity and measures

In this section, we introduce harmonic convexity (concavity) and then apply these convexities on to measures. We also explore the basic properties of completely monotone and Bernstein functions with these convexities on their measures.

2.4.1 Definitions

Let I be a convex interval (open, closed, or half-close) of $\mathbb{R}$. A function $h(x) : I \to \mathbb{R}$ is *convex* on I if

$$h(\alpha x + (1-\alpha)y) \leq \alpha h(x) + (1-\alpha)h(y)$$

for any $x, y \in I$ and $\alpha \in [0,1]$. In particular, if $h(x)$ is convex, then

$$h\left(\frac{x+y}{2}\right) \leq \frac{h(x)+h(y)}{2}$$

for any $x, y \in I$. The function $h(x)$ is *concave* if $-h(x)$ is convex. It is well-known that every convex function is locally Lipschitz continuous on the interior of its domain. The left and the right directional derivatives

$$h'_+(x) := \lim_{t\to 0+} \frac{h(x+t)-h(x)}{t} \quad \text{and} \quad h'_-(x) = \lim_{t\to 0+} \frac{h(x)-h(x-t)}{t}$$

exist (in the extended sense) for every $x \in I$. (If x is a boundary point, chose the directional derivative that makes sense.) Both $h'_+(x)$ and $h'_-(x)$ are non-decreasing, finite on the interior of I and satisfy

$$h'_+(x) \leq h'_-(y) \leq h'_+(y) \leq h'_-(z) \tag{2.37}$$

for all $x < y < z$ in I, see [62, Theorem 24.1]. The function $h'_+(x)$ is right-continuous while $h'_-(x)$ is left continuous. Moreover, for any x, y in the interior of I we have

$$h(y) - h(x) = \int_{(x,y)} h'_+(t)\,dt = \int_{(x,y)} h'_-(t)\,dt, \tag{2.38}$$

see [62, Corollary 24.2.1]. Finally, if $h(x) \in C^2$, then h is convex on an open interval I if and only if $h''(x) \geq 0$ for all x in I.

Definition 2.4.1 *A function $h : (0,\infty) \to \mathbb{R}$ is called harmonically convex (concave) if the function $h(1/x)$ is convex (concave) on $(0,\infty)$.*

Alternatively, we call $h(x) : (0,\infty) \to \mathbb{R}$ harmonically convex if and only if

$$h\left(\frac{2}{1/x+1/y}\right) \leq \frac{h(x)+h(y)}{2}$$

for every $x, y \in (0,\infty)$. This terminology comes from the fact that $2/(1/x+1/y)$ is the harmonic mean of x and y. And $h(x)$ is harmonically concave if $-h(x)$ is harmonically convex.

If a function $h(x)$ is harmonically convex, then its left and right directional derivatives also exit. Suppose the left (reps. right) directional derivative of $\varphi : (0,\infty) \to \mathbb{R}$ exists at x. Then, the right (reps. left) directional derivative of $h(x) := \varphi(1/x)$ exists at $1/x$ and

$$h'_+(x) = -\varphi'_-\left(\frac{1}{x}\right)\frac{1}{x^2}, \quad \text{resp. } h'_-(x) = -\varphi'_+\left(\frac{1}{x}\right)\frac{1}{x^2}. \tag{2.39}$$

Refer to A.1.2 for details of verifications.

Proposition 2.4.1 *Formula (2.38) is valid if the function h is harmonically convex on* $(0, \infty)$.

Proof For any $0 < x < y$, since the function h has bounded variation (on intervals away from 0), by Lemma 2.4.1, and Lemma A.2.5 we have

$$\begin{aligned} h(y) - h(x) &= \int_{(x,y)} dh(t) = \int_{(-1/x,-1/y)} dh(-1/s) = \int_{(-1/x,-1/y)} \big(h(-1/s)\big)'_- \, ds \\ &= \int_{(-1/x,-1/y)} h'_+(-1/s)\frac{1}{s^2}\, ds = \int_{(x,y)} h'_+(t)\, dt. \end{aligned}$$

Similarly, the equation holds for left derivative $h'_-(t)$. □

Harmonically convex (and harmonically concave) functions are not uncommon. Examples of these functions can be found in [55]. In fact, using the inequality $2/(1/x + 1/y) \le (x + y)/2$ which is valid for $x, y > 0$, it is easy to see that

(a) a non-decreasing convex function on $(0, \infty)$ is harmonically convex;

(b) a non-increasing concave function on $(0, \infty)$ is harmonically concave.

The first item is because

$$h\left(\frac{2}{1/x + 1/y}\right) \le h\left(\frac{x + y}{2}\right) \le \frac{h(x) + h(y)}{2}.$$

And the second item follows analogously. However, the converses are not true. Not every harmonically convex function is convex and increasing, while not every harmonically concave function is concave and decreasing.

In fact, by looking at (the graph of) a function, it is difficult to tell if it is harmonically convex (or harmonically concave). For example, $\log(x)$ is harmonically convex but not convex. More examples are listed below and depicted in Figure 2.1.

(a) $h(x) = \dfrac{1}{x^2}$.

(b) $h(x) = \dfrac{(x - 1)^2 + 1}{x}$.

(c) $h(x) = \begin{cases} \dfrac{1 - x}{x}, & 0 < x \le 1, \\ 0, & 1 < x \le 2, \\ \dfrac{x - 2}{x}, & x > 2. \end{cases}$

(d) $h(x) = x^{1/2}$.

Each of the four functions listed above is harmonically convex. However, the first two functions are not increasing, while the last two functions are not convex.

The following simple result (also see [55, Lemma 2.2]), helps to further clarify the relationship between convexity and harmonic convexity, as well as concavity and harmonic concavity.

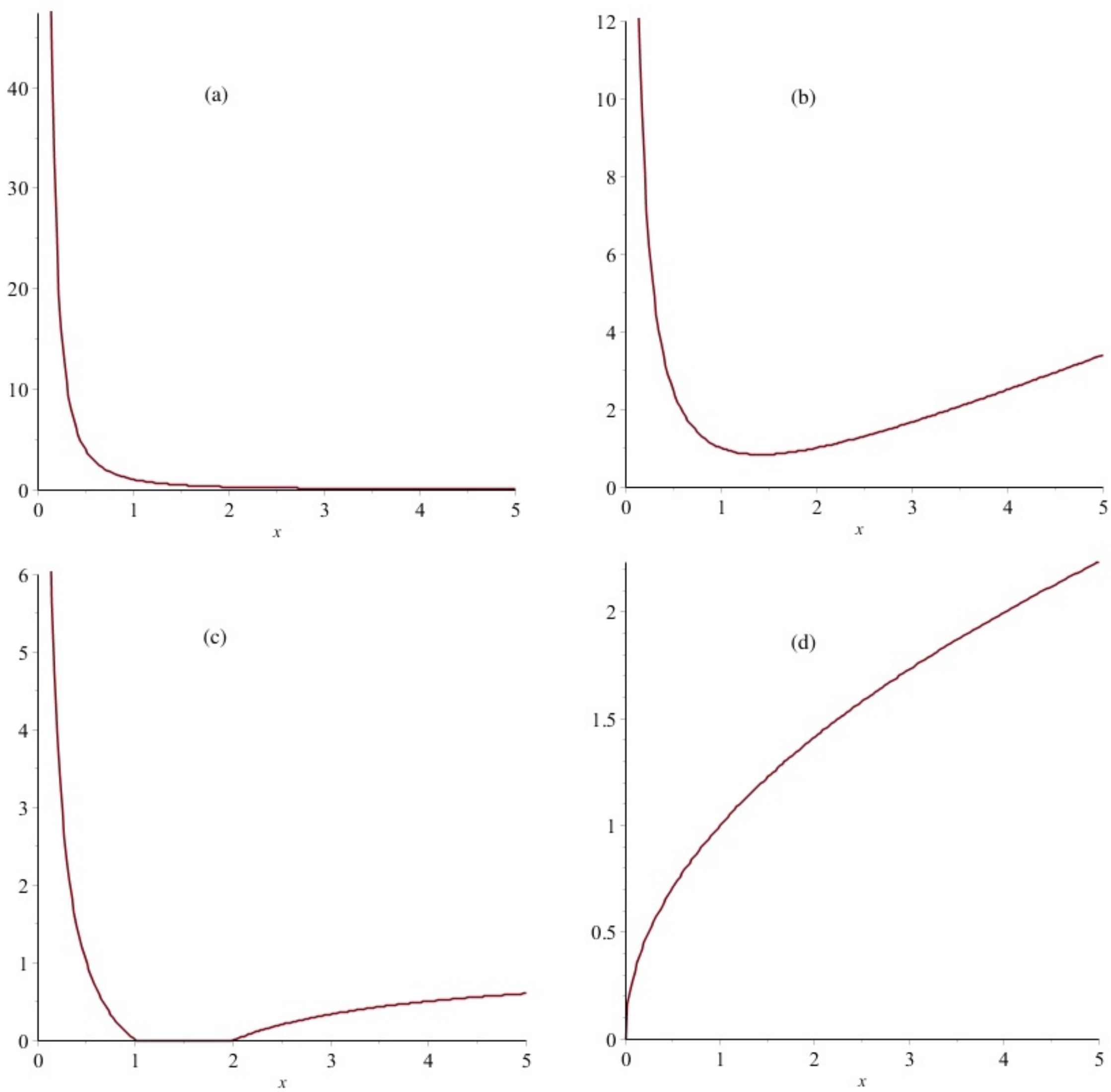

Figure 2.1: Examples of harmonically convex functions

Lemma 2.4.2 *A function $h : (0, \infty) \to R$ is convex (concave), if and only if $xh(1/x)$ is convex (concave).*

For function $h : (0, \infty) \to R$, this lemma implies it is harmonically convex if and only if $xh(x)$ is convex. This equivalence will be very useful in our further development. If $h(x) \in C^2$, then it is harmonically convex if and only if $2h'(x) + xh''(x) \geq 0$ for all $x > 0$.

2.4.2 Measures with harmonic convexity properties

Definition 2.4.3 *We call*

(a) a measure μ convex (concave, or harmonically convex), if the cumulative distribution function $F(x)$ is convex (concave, or harmonically convex);

(b) a measure ν having convex tail (harmonically convex tail, or harmonically concave tail), if the tail $\bar{\nu}(x) = \nu(x, \infty)$ is convex (harmonically convex, or harmonically concave).

In our thesis, we apply *convexities properties* on Bernstein measures and define completely monotone functions with convex (concave, or harmonically convex) measures. Besides, we apply *convexity of tails* on Lévy measures and define Bernstein functions whose Lévy measures have convex tail (harmonically convex tail, or harmonically concave tail). The reasoning is straightforward. Bernstein measures have well-defined cumulative distribution functions, while their tails may not be finite; and Lévy measures have well-defined tails, but they may not have finite cumulative distribution functions.

It is worth pointing out that a Bernstein measure μ on $[0, \infty)$ can not be harmonically concave unless μ is a point mass measure at zero, that is $\mu[0, \infty) = \mu(\{0\})$. Besides, a Lévy measure ν on $(0, \infty)$ can not have concave tail unless $\nu(0, \infty) = 0$.

Since the function $F(x) = \mu[0, x]$ is non-decreasing, if μ is a convex measure, then it is harmonically convex. By Lemma 2.4.2, a measure μ is harmonically convex if and only if the function $xF(x)$ is convex.

The following are standard example that we use on multiple occasions. See Lemma 3.2.3, Lemma 3.3.3 and Lemma 3.4.2, for more examples.

Example 2.4.4 *(a) Consider the completely monotone function $f(x) = x^{-\alpha}$ for $\alpha > 0$.*

1. *It has harmonically convex measure.*
2. *If $\alpha \geq 1$, then $f(x)$ has convex measure.*
3. *If $\alpha \leq 1$, then $f(x)$ has concave measure.*

(b) Consider the Bernstein function $g(x) = x^\alpha$ for $\alpha \in (0, 1)$. It has Lévy measure with harmonically concave tail, as well as convex tail.

Proof (a) For $f(x) = x^{-\alpha}$ where $\alpha > 0$, its integral representation is

$$x^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_{[0,\infty)} e^{-xt} t^{\alpha-1}\, dt = \int_{[0,\infty)} e^{-xt}\, \mu(dt).$$

This shows that its Bernstein measure μ is given by $\mu(dt) = (t^{\alpha-1}/\Gamma(\alpha))\,dt$. In particular, it has with no mass at $\{0\}$, and has cumulative distribution function

$$F(x) = \mu[0,x] = \frac{1}{\Gamma(\alpha)} \int_{[0,x]} t^{\alpha-1}\,dt = \frac{x^\alpha}{\alpha\Gamma(\alpha)}.$$

Since $\mu[0,1/x]$ is in proportion to $x^{-\alpha}$, which is convex, $f(x)$ has harmonically convex measure for all $\alpha > 0$. And it is trivial that if $\alpha \geq 1$, then $f(x)$ has convex measure, and if $0 < \alpha \leq 1$, $f(x)$ has concave measure.

(b) For $g(x) = x^\alpha$ where $0 < \alpha < 1$, its integral representation is

$$x^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_{(0,\infty)} (1-e^{-xt})t^{-\alpha-1}\,dt = \int_{(0,\infty)} (1-e^{-xt})\,\nu(dt).$$

This shows that the Lévy measure of $g(x)$ is $\nu(dt) = (\alpha/\Gamma(1-\alpha))t^{-\alpha-1}\,dt$ where the tail of the measure ν is given by

$$\nu(x,\infty) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{(x,\infty)} t^{-\alpha-1}\,dt = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}.$$

It is easy to see that $g(x)$ has convex tail measure for all $\alpha \in (0,1)$. Note that $\bar{\nu}(x) = \nu(1/x,\infty)$ is in proportion to x^α, which is concave, hence $g(x)$ has harmonically concave tail measure as well for $\alpha \in (0,1)$. □

Example 2.4.5 *Function* $\ln(x) - \psi(x)$ *is completely monotone with convex measure having no mass at zero, where* $\psi(x)$ *is the digamma function, which is defined as the logarithmic derivative of the gamma function, that is*

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Particularly, we have the following representation.

$$\ln(x) - \psi(x) = \int_{(0,\infty)} e^{-xt}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)dt.$$

Proof By the relationship between digamma function $\psi(x)$ and polygamma function $\psi'(x)$,

$$(\ln(x) - \psi(x))' = \frac{1}{x} - \psi'(x) = \int_{(0,\infty)} e^{-xt}\,dt - \int_{(0,\infty)} e^{-xt}\frac{t}{1-e^{-t}}\,dt = \int_{(0,\infty)} e^{-xt}\left(1 - \frac{t}{1-e^{-t}}\right)dt.$$

Notice $t \geq 1 - e^{-t}$ for all $t > 0$ by Lemma A.1.4 (a), function $-(\ln(x) - \psi(x))'$ is completely monotone and $\ln(x) - \psi(x)$ is non-increasing. Using harmonic number H_n and Euler-Mascheroni constant γ, we obtain

$$\begin{aligned}\lim_{x\to\infty} \ln(x) - \psi(x) &= \lim_{n\to\infty} \ln(n) - \psi(n) = \lim_{n\to\infty} \ln(n) - H_{n-1} + \gamma \\ &= \lim_{n\to\infty} \ln(n) - \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n} + \gamma = \lim_{n\to\infty} \ln(n) - \sum_{k=1}^{n} \frac{1}{k} + \gamma = -\gamma + \gamma = 0.\end{aligned}$$

Therefore, by Fubini's theorem,

$$\begin{aligned}\ln(x)-\psi(x) &= \int_{(x,\infty)} -(\ln(s)-\psi(s))'\,ds = \int_{(x,\infty)}\int_{(0,\infty)} e^{-st}\Big(\frac{t}{1-e^{-t}}-1\Big)\,dt\,ds \\ &= \int_{(0,\infty)}\Big(\frac{t}{1-e^{-t}}-1\Big)\int_{(x,\infty)} e^{-st}\,ds\,dt = \int_{(0,\infty)} e^{-xt}\Big(\frac{1}{1-e^{-t}}-\frac{1}{t}\Big)\,dt.\end{aligned}$$

It is trivial that this completely monotone function has measure with no mass at zero. To show the measure is convex, it suffices to show its density is non-decreasing. Notice the derivative for its density is

$$\Big(\frac{1}{1-e^{-t}}-\frac{1}{t}\Big)' = \frac{-e^{-t}t^2+e^{-2t}-2e^{-t}+1}{t^2(1-e^{-t})^2}.$$

By lemma A.1.4 (c), this function is non-negative for all $t>0$, which implies the density is non-decreasing. We conclude that $\ln(x)-\psi(x)$ has convex measure. □

Example 2.4.6 *Function $f_\lambda(x) := \ln(x)-\psi(x+\lambda+1)+(\lambda+1)/x$ is completely monotone function with convex measure having no mass at zero for all $\lambda>0$. Here $\psi(x)$ is the digamma function. Particularly, it has the following representation.*

$$f_\lambda(x) = \int_{(0,\infty)} e^{-xt}\Big(\frac{e^{-\lambda t}}{1-e^{-t}}-\frac{1}{t}+1+\lambda-e^{-\lambda t}\Big)\,dt. \tag{2.40}$$

Proof Using the property of digamma function that $\psi(x+1)=\psi(x)+1/x$, we rewrite

$$f_\lambda(x) = \ln(x)-\psi(x+\lambda)+\frac{\lambda+1}{x}-\frac{1}{x+\lambda}.$$

It is easy to see

$$\frac{\lambda+1}{x}-\frac{1}{x+\lambda} = \int_{(0,\infty)} e^{-xt}(1+\lambda-e^{-\lambda t})\,dt. \tag{2.41}$$

On the other hand, notice that

$$\lim_{x\to\infty}\ln(x)-\psi(x+\lambda) = \lim_{x\to\infty}\ln\Big(\frac{x}{x+\lambda}\Big)+\ln(x+\lambda)-\psi(x+\lambda) = \lim_{x\to\infty}\ln(x+\lambda)-\psi(x+\lambda).$$

And we know from Example (2.4.5) that $\lim_{x\to\infty}\ln(x)-\psi(x)=0$. This imply

$$\lim_{x\to\infty}\ln(x)-\psi(x+\lambda)=0$$

for any $\lambda>0$. Also notice

$$(\ln(x)-\psi(x+\lambda))' = \frac{1}{x}-\psi'(x+\lambda) = \frac{1}{x}-\int_{(0,\infty)} e^{-(x+\lambda)t}\frac{t}{1-e^{-t}}\,dt = \int_{(0,\infty)} e^{-xt}\Big(1-\frac{te^{-\lambda t}}{1-e^{-t}}\Big)\,dt.$$

Therefore, we obtain

$$\ln(x)-\psi(x+\lambda) = -\int_{(x,\infty)}(\ln(s)-\psi(s+\lambda))'\,ds = \int_{(x,\infty)}\int_{(0,\infty)} e^{-st}\Big(\frac{te^{-\lambda t}}{1-e^{-t}}-1\Big)\,dt\,ds.$$

The last double integral can interchange by lemma A.2.4. Therefore,

$$\ln(x) - \psi(x+\lambda) = \int_{(0,\infty)} e^{-xt}\Big(\frac{e^{-\lambda t}}{1-e^{-t}} - \frac{1}{t}\Big)\,dt. \tag{2.42}$$

Equation (2.40) follows from (2.41) and (2.42). To show $f_\lambda(x)$ is completely monotone with convex measure having no mass at zero, denote

$$D_\lambda(t) := \frac{e^{-\lambda t}}{1-e^{-t}} - \frac{1}{t} + 1 + \lambda - e^{-\lambda t}.$$

It suffices to show $D_\lambda(t)$ is non-negative, and non-decreasing on $(0,\infty)$. By L'Hopital rule,

$$\begin{aligned}\lim_{t\to 0} D_\lambda(t) &= \lim_{t\to 0} \frac{te^{-\lambda t} - 1 + e^{-t}}{t(1-e^{-t})} + \lambda = \lim_{t\to 0}\frac{e^{-\lambda t} - t\lambda e^{-\lambda t} - e^{-t}}{1 - e^{-t} + te^{-t}} + \lambda \\ &= \lim_{t\to 0}\frac{-2\lambda e^{-\lambda t} + t\lambda^2 e^{-\lambda t} + e^{-t}}{2e^{-t} - te^{-t}} + \lambda = \frac{-2\lambda+1}{2} + \lambda = \frac{1}{2} > 0,\end{aligned}$$

for all $\lambda > 0$. Thus it suffices to show $D_\lambda(t)$ is non-decreasing. Consider

$$D'_\lambda(t) = \frac{e^{-\lambda t-2t}t^2\lambda - \lambda e^{-\lambda t - t}t^2 - e^{-\lambda t - t}t^2 + e^{-2t} - 2e^{-t} + 1}{t^2(1-e^{-t})^2}.$$

To show $D'_\lambda(t) \geq 0$, we need to show its numerator

$$\begin{aligned} N_\lambda(t) &:= e^{-\lambda t-2t}t^2\lambda - \lambda e^{-\lambda t - t}t^2 - e^{-\lambda t - t}t^2 + e^{-2t} - 2e^{-t} + 1 \\ &= -t^2 e^{-t} e^{-\lambda t}(\lambda(1-e^{-t}) + 1) + (1-e^{-t})^2 \geq 0 \end{aligned} \tag{2.43}$$

on $(0,\infty)$ for $\lambda > 0$. By Lemma A.1.5, we know $e^{-\lambda t}(\lambda(1-e^{-t}) + 1)$ is decreasing in terms of λ for all $t > 0$. So $N_\lambda(t)$ is increasing with respect to λ for all $t > 0$. Its limit is

$$\lim_{\lambda\to 0} N_\lambda(t) = -t^2e^{-t} + (1-e^{-t})^2 = -t^2e^{-t} + 1 - 2e^{-t} + e^{-2t} > 0.$$

By Lemma A.1.4 (c). So we can conclude (2.43) holds and close the proof. □

The next lemma from [71, Corollary 4.2 (b)], plays important role in our development. We include the proof not only for completeness, but also for our different context.

Lemma 2.4.7 *Let $f_1(x)$ and $f_2(x)$ be completely monotone functions. If one of them has a convex measure with no mass at $\{0\}$, then their completely monotone product $f_1(x)f_2(x)$ has convex measure.*

Proof Let μ_1 and μ_2 be the measures corresponding to $f_1(x)$ and $f_2(x)$ respectively. Without loss of generality, suppose μ_1 is convex with no mass at $\{0\}$. Using equation (2.10), in order to show that the function $x \mapsto (\mu_1 * \mu_2)[0,x]$ is convex, we calculate

$$(\mu_1 * \mu_2)[0,x] = \int_{R_+^2} \mathbf{1}_{[0,x]}(s+t)\,\mu_1(ds)\,\mu_2(dt) = \int_{R_+^2} \mathbf{1}_{[0,x-t]}(s)\mathbf{1}_{[0,x]}(t)\,\mu_1(ds)\,\mu_2(dt)$$

$$= \int_{[0,\infty)} \mu_1[0, x-t]\mathbf{1}_{[0,x]}(t)\,\mu_2(dt) = \int_{[0,\infty)} \mu_1[0, \phi_t(x)]\,\mu_2(dt), \tag{2.44}$$

where $\phi_t(x) = \max\{x-t, 0\}$. The fact that μ_1 has no mass at $\{0\}$ is used to produce equality (2.44). The functions ϕ_t and $x \mapsto \mu_1[0, x]$ are convex and non-decreasing, and so is their composition $x \mapsto \mu_1[0, \phi_t(x)]$ for every fixed $t \geq 0$. Hence, the integral in (2.44) is a convex function of x. □

Since every convex measure on $[0, \infty)$ is harmonically convex, under the conditions of Lemma 2.4.7, the product $f_1(x)f_2(x)$ has harmonically convex measure. Lemma 2.4.7 fails if the convex measure has mass at $\{0\}$, as the following example shows.

Example 2.4.8 *Let $f_1(x) = e^{-x}$ and $f_2(x) = 1$. It can be shown that f_1 and f_2 are completely monotone functions with Bernstein representations:*

$$f_1(x) = \int_{[0,\infty)} e^{-tx}\,\delta_1(dt), \quad \text{and} \quad f_2(x) = \int_{[0,\infty)} e^{-tx}\,\delta_0(dt),$$

where δ_0 and δ_1 are the Dirac delta function. Note that measure δ_0 is convex on $[0, \infty)$ and has mass at $\{0\}$. Their product is $f_1(x)$ and it has measure δ_1, which is not convex on $[0, \infty)$.

Next two propositions are some known connections between completely monotone (Bernstein) function and its derivatives, regarding the convexity properties on their measures. One can find their proofs in [71].

Proposition 2.4.2 (Theorem 3.1 in [71]) *Let $f(x)$ be a completely monotone function with measure μ. If μ is (harmonically) convex, then for all integers $n \geq 1$, the measure corresponding to the function $(-1)^n f^{(n)}(x)$ is also (harmonically) convex.*

Proposition 2.4.3 (Lemma 4.1 in [71]) *If the measure ν on $(0, \infty)$ has a harmonically concave tail, then the measure μ defined by the equation $\mu(dt) = t\nu(dt)$ is harmonically convex on $(0, \infty)$.*

2.5 β-convexity and β-concavity

In this section, we introduce β-convexity. This convexity is a generalization for convexity and harmonic convexity. We also include some basic discussions on measures with β-convexities, as well as related completely monotone and Bernstein functions. We consider $\beta \in [0, 1]$ in the following content without further notice.

2.5.1 Definitions and basic properties

Definition 2.5.1 *Let $\beta \in [0, 1]$. A function $h : (0, \infty) \to \mathbb{R}$ is called β-convex (β-concave) if $x^\beta h(x)$ is convex (concave) on $(0, \infty)$.*

It is clear that function $h(x)$ on $(0,\infty)$ is 0-convex if it is convex; and $h(x)$ is 1-convex if it is harmonically convex by Lemma 2.4.2. The following equivalence is also an immediate result from Lemma 2.4.2.

Corollary 2.5.1 *A function $h(x)$ is β-convex (β-concave), if and only if $h(1/x)$ is $(1-\beta)$-convex ($(1-\beta)$-concave, respectively).*

If $h : (0,\infty) \to \mathbb{R}$ is convex (concave), then the directional derivatives of $x^p h(x)$ exist for all $p \in \mathbb{R}$. More precisely, it can be shown that

$$(x^p h(x))'_+ = px^{p-1}h(x) + x^p h'_+(x) \quad \text{and} \quad (x^p h(x))'_- = px^{p-1}h(x) + x^p h'_-(x). \tag{2.45}$$

See A.1.3 for verifications. Next lemma helps to understand β-convexity when β changes.

Lemma 2.5.2 *Suppose $f : (0,\infty) \to (0,\infty)$ is non-decreasing and $f(0+) = 0$. If $f(x)$ is convex, then it is β-convex for any $\beta \in [0,1]$.*

Proof To show $x^\beta f(x)$ is convex, it suffices to show its right derivative is non-decreasing. By (2.45), we have

$$(x^\beta f(x))'_+ = \beta x^{\beta-1} f(x) + x^\beta f'_+(x) = \beta x^\beta \frac{f(x)}{x} + x^\beta f'_+(x).$$

The second term $x^\beta f'(x)$ is non-decreasing because both $f'_+(x)$ and x^β are non-negative and non-decreasing for $\beta > 0$. It suffices to show $f(x)/x$ is non-decreasing. Indeed, for any $x_1 > x_2 > 0$, as $f(x)$ is convex, increasing and $f(0+) = 0$, so

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \geq \frac{f(x_2) - 0}{x_2 - 0}.$$

It implies $f(x_1)/x_1 \geq f(x_2)/x_2$. So $f(x)/x$ is non-decreasing and proof is complete. □

This lemma fails if we remove the condition non-decreasing or $f(0+) = 0$. Consider

$$f_1(x) = (x-1)^2 - 1,$$

which is convex on $(0,\infty)$ and $f_1(0+) = 0$. However, it is not 1/2-convex. See Figure 2.2 for the plot of $x^{1/2} f_1(x)$. On the other hand, consider

$$f_2(x) = \mathbf{1}_{\{x<1\}} + ((x-1)^2 + 1)\mathbf{1}_{\{x\geq 1\}},$$

which is non-decreasing and convex on $(0,\infty)$. However, it is not 1/2-convex. also see Figure 2.2 for the plot of $x^{1/2} f_2(x)$.

Analogous to Definition 2.4.3, we could introduce β-convexity onto measures.

Definition 2.5.3 *We call*

(a) a measure μ to be β-convex (β-concave), if its cumulative distribution function $F(x)$ is β-convex (β-concave);

(b) a measure ν to have β-convex (β-concave) tail, if its tail $\bar{\nu}(x)$ is β-convex (β-concave).

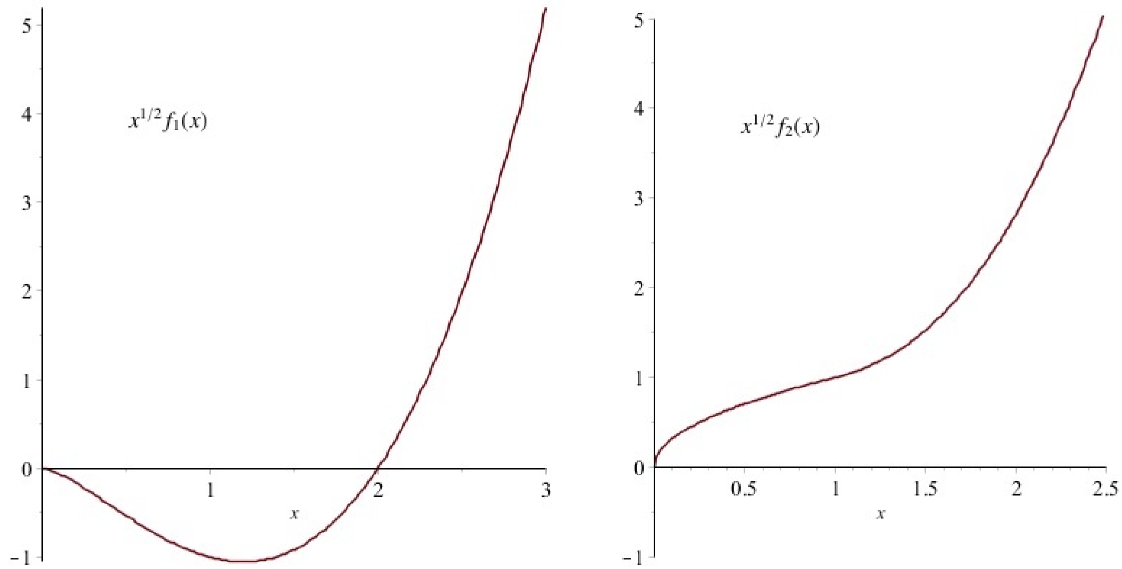

Figure 2.2: Plot of $x^{1/2}f_1(x)$ and $x^{1/2}f_2(x)$

Similarly, we mainly consider completely monotone functions with β-convex (β-concave) measures, and Bernstein functions whose Lévy measures have β-convex tail (β-concave tail) because they are well-defined.

The following are some standard examples of completely monotone and Bernstein functions with β-convexity properties on their measures.

Example 2.5.4 *(a) Consider completely monotone function $f(x) = x^{-\alpha}$ for $\alpha > 0$.*

1. *If $\alpha \geq 1$, then $f(x)$ has β-convex measure for any $0 \leq \beta \leq 1$.*
2. *If $\alpha \leq 1$, then $f(x)$ has β-convex measure for $\beta \geq 1 - \alpha$, and it has β-concave measure for $\beta \leq 1 - \alpha$.*

(b) Consider the Bernstein function $g(x) = x^{\alpha}$ for $0 < \alpha < 1$.

1. *It has Lévy measure with β-convex tail if $\beta \leq \alpha$.*
2. *It has Lévy measure with β-concave tail if $\alpha \leq \beta$.*

By the proof of Example 2.4.4, we can see that the cumulative distribution function of the measure for $f(x) = x^{-\alpha}$ is $F(x) = x^{\alpha}/(\alpha\Gamma(\alpha))$, and the tail of the Lévy measure ν for $g(x) = x^{\alpha}$ is given by $\bar{\nu}(x) = x^{-\alpha}/\Gamma(1-\alpha)$. Their β-convexity properties follow immediately.

From Lemma 2.5.2, if completely monotone function $f(x)$ has β_1-convex measure for some $\beta_1 > 0$, then it is β_2-convex for any $1 \geq \beta_2 > \beta_1$. In addition, suppose $f(x)$ has measure with no mass at $\{0\}$, if $f(x)$ has convex measure, then it has β-convex measure for all $1 \geq \beta \geq 0$.

2.5.2 Limiting properties for measures with β-convexity or β-concavity

In this section, we study some limiting properties of measures associated with completely monotone and Bernstein functions. These measures are assumed to have β-convexity type properties.

Lemma 2.5.5 *Suppose $f(x)$ is completely monotone with measure μ. If measure μ is β-convex (or β-concave), then the cumulative distribution function $F(t)$ of μ satisfies*

$$\lim_{t\to 0} e^{-xt}t^{2-\beta}r(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} e^{-xt}t^{2-\beta}r(t) = 0, \tag{2.46}$$

for any $x > 0$, where $r(t) = (t^\beta F(t))'$ is the right derivative.

Proof Since μ is β-convex (or β-concave), $t^\beta F(t)$ is convex (or concave). Also note it is non-negative and non-decreasing. Thus $r(x)$ is non-negative and non-decreasing (or non-increasing, respectively). By Lemma A.2.9 and Remark A.2.8,

$$\int_{(0,\infty)} e^{-xt}t^{1-\beta}r(t)\,dt = \int_{(0,\infty)} e^{-xt}t^{1-\beta}\,d(t^\beta F(t)) = \int_{(0,\infty)} e^{-xt}t\,dF(t) + \beta\int_{(0,\infty)} e^{-xt}F(t)\,dt.$$

Observing (2.6) and (2.2), we obtain

$$\int_{(0,\infty)} e^{-xt}t^{1-\beta}r(t)\,dt = \beta\frac{f(x)}{x} - f'(x) < \infty. \tag{2.47}$$

This verifies the integrability of the integral above, which implies $\lim_{t\to\infty} e^{-xt}t^{1-\beta}r(t) = 0$ for all $x > 0$. It further implies the second limit in (2.46) as following,

$$\lim_{t\to\infty} e^{-xt}t^{2-\beta}r(t) = \lim_{t\to\infty}(e^{-xt/2}t)(e^{-xt/2}t^{1-\beta}r(t)) = 0.$$

The first limit requires more insights. Note that (2.47) also indicates

$$\frac{1}{2-\beta}\int_{(0,1)} r(t)\,d(t^{2-\beta}) = \int_{(0,1)} t^{1-\beta}r(t)\,dt < \infty. \tag{2.48}$$

If μ is β-convex, then $t^\beta F(t)$ is convex and non-decreasing. Thus, $r(t)$ is non-negative and non-decreasing, so is $t^{2-\beta}r(t)$. The limit exists as t approaches 0. Suppose

$$\lim_{t\to 0} t^{2-\beta}r(t) = c \geq 0.$$

We have

$$\int_{(0,1)} t^{1-\beta}r(t)\,dt = \int_{(0,1)} \frac{t^{2-\beta}r(t)}{t}\,dt \geq \int_{(0,1)} \frac{c}{t}\,dt.$$

Integrability (2.48) indicates $c = 0$ and the first limit in (2.46) follows.

If μ is β-concave, then $t^\beta F_\mu(t)$ is concave and non-decreasing. Thus, $r(t)$ non-negative and non-increasing. Applying Lemma A.2.1 on the first integral in (2.48), we conclude $r(t)$ is $o(1/t^{2-\beta})$ as t approaches 0. In other words,

$$\lim_{t\to 0} t^{2-\beta}r(t) = 0.$$

Here follows the first limit in (2.46). This closes the proof. □

Lemma 2.5.6 *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). If measure ν has β-convex (or β-concave) tail, then the tail of the measure ν satisfies*

$$\lim_{t\to 0} e^{-x/t} t^{\beta-1} h(t) = 0 \quad \textit{and} \qquad \lim_{t\to\infty} e^{-x/t} t^{\beta-1} h(t) = 0, \tag{2.49}$$

for any $x > 0$, where $h(t) = (t^{1-\beta}\bar{\nu}(1/t))'$ is the right derivative.

Proof Without loss of generality, we can assume $a = b = 0$. Since ν has β-convex tail (or β-concave tail), we also know $t^{1-\beta}\bar{\nu}(1/t)$ is convex (or concave) by Corollary 2.5.1, which implies the existence of $h(t)$. Because $\bar{\nu}(t)$ is non-increasing, $t^{1-\beta}\bar{\nu}(1/t)$ is non-decreasing and thus $h(t) \geq 0$. To show (2.49), it is equivalent to show

$$\lim_{t\to\infty} e^{-xt} t^{1-\beta} h(1/t) = 0 \quad \text{and} \qquad \lim_{t\to 0} e^{-xt} t^{1-\beta} h(1/t) = 0. \tag{2.50}$$

By Lemma A.2.9 and Remark A.2.8,

$$\begin{aligned}
\int_{(0,\infty)} e^{-x/t} t^{\beta-2} h(t)\, dt &= \int_{(0,\infty)} e^{-x/t} t^{\beta-2} (t^{1-\beta}\bar{\nu}(1/t))'\, dt = \int_{(0,\infty)} e^{-x/t} t^{\beta-2}\, d(t^{1-\beta}\bar{\nu}(1/t)) \\
&= \int_{(0,\infty)} e^{-x/t} t^{-1}\, d\bar{\nu}(1/t) + (1-\beta) \int_{(0,\infty)} e^{-x/t} t^{-2} \bar{\nu}(1/t)\, dt \\
&= -\int_{(0,\infty)} e^{-xs} s\, d\bar{\nu}(s) + (1-\beta) \int_{(0,\infty)} e^{-xs} \bar{\nu}(s)\, ds.
\end{aligned}$$

Variable is changed by letting $s = 1/t$ in the last equation by Lemma A.2.5. Observing (2.23) and (2.19), we obtain

$$\int_{(0,\infty)} e^{-x/t} t^{\beta-2} h(t)\, dt = g'(x) + (1-\beta) \frac{g(x)}{x}.$$

This verifies the integrability of the integral above. Change variable again, we have

$$\int_{(0,\infty)} e^{-xt} t^{-\beta} h(1/t)\, dt < \infty, \tag{2.51}$$

for all $x > 0$. From the integrability follows that $\lim_{t\to\infty} e^{-xt} t^{-\beta} h(1/t) = 0$, which implies

$$\lim_{t\to\infty} e^{-xt} t^{1-\beta} h(1/t) = \lim_{t\to\infty} (e^{-xt/2} t)(e^{-xt/2} t^{-\beta} h(1/t)) = 0.$$

Thus the first limit in (2.50) holds . The second limit needs more detailed discussion. Note that integrability (2.51) also indicates

$$\int_{(0,1)} t^{-\beta} h(1/t)\, dt < \infty. \tag{2.52}$$

If ν has β-convex tail, it is shown that $t^{1-\beta}\bar{\nu}(1/t)$ is also convex and non-decreasing. So $h(t)$ is non-negative and non-decreasing. And $h(1/t)$ is non-increasing, so is $t^{-\beta}h(1/t)$. Observing its integrability at zero, we conclude that $t^{-\beta}h(1/t)$ is $o(1/t)$ as t approaches zero by Lemma A.2.1. Therefore,

$$\lim_{t\to 0} t^{1-\beta}h(1/t) = 0,$$

and the second limit holds in (2.50).

If ν has β-concave tail, $t^{1-\beta}\bar{\nu}(1/t)$ is concave and non-decreasing. Thus $h(t)$ is non-negative and non-increasing, implying $h(1/t)$ is non-decreasing and so is $t^{1-\beta}h(1/t)$. The limit exists when t approaches 0. Suppose

$$\lim_{t\to 0} t^{1-\beta}h(1/t) = c \geq 0.$$

We have

$$\int_{(0,1)} t^{-\beta}h(1/t)\,dt = \int_{(0,1)} \frac{t^{1-\beta}h(1/t)}{t}\,dt \geq \int_{(0,1)} \frac{c}{t}\,dt.$$

Integrability (2.52) indicates $c = 0$. Here follows the second limit in (2.50). $\square$

Chapter 3

Measures and Convexity

In this Chapter, we characterize measure with various convexity properties introduced in Section 2.4. We also apply these characterizations on completely monotone and Bernstein functions whose measures have related convexity properties. The measures are all Radon measures if without further specification. In particular, they are σ-finite, and finite on compact sets, see [82, Definition 19.15]

3.1 Pólya's criterion

Theorem 3.1.1 (Polya's criterion) *Suppose a function $\varphi : [0, \infty) \mapsto [0, \infty)$ is non-increasing, $\varphi(0) = 1$, convex on $(0, \infty)$, with*

$$\lim_{x \to 0^+} \varphi(x) = 1, \quad \text{and} \quad \lim_{x \to \infty} \varphi(x) = 0.$$

Then, it admits the representation

$$\varphi(x) = \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds), \tag{3.1}$$

for some probability measure τ on $(0, \infty)$. Here $x^+ := \max\{x, 0\}$. Furthermore, $\varphi(|x|)$ is the characteristic function of a symmetric distribution.

In fact, this criterion can be extended on $f : \mathbb{R} \to [0, \infty)$ if it is even, continuous, convex on $(0, \infty)$, with $f(0) = 1$ and $\lim_{x \to \infty} f(x) = 0$. See [52, Theorem 4.3.1] for details.

The theorem fails without the assumption $\lim_{x \to 0+} \varphi(x) = 1$. For example, the indicator function $x \mapsto \mathbf{1}_{\{0\}}(x)$ satisfies all other conditions in the theorem, but does not admit such integral representation.

Notice that the cumulative distribution function of a probability measure is non-decreasing and bounded from above. It can not be convex unless $\mu(\{0\}) = 1$ and $\mu(0, \infty) = 0$. Pólya's criterion is not applicable in this scenario. However, it can be used to characterize harmonically convex measures. Applying Theorem 3.1.1, we could have

Corollary 3.1.1 *A probability measure μ on $(0,\infty)$ is harmonically convex, if and only if there is a probability measure τ on $(0,\infty)$ such that*

$$\mu(0,x] = \int_{(0,\infty)} \left(1 - \frac{1}{xs}\right)^+ \tau(ds).$$

Proof Define function $\varphi(x)$ on $[0,\infty)$ by

$$\varphi(x) := \begin{cases} \mu(0,1/x], & \text{if } x > 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Apply Theorem 3.1.1 and the proof is completed after replacing x by $1/x$ in (3.1). □

More generally, not only probability measures, but general Radon measures with various convexity properties could be characterized by the same techniques as well. We will elaborate the identifications in detail in the following sections, and will apply the characterizations to completely monotone and Bernstein functions with convexity properties on their measures to construct representations. Our main results are summarized in Table 3.1, 3.2, 3.3, and 3.4.

Table 3.1: Characterization of measure μ on $[0,\infty)$ with convexity properties

Property on μ	Characterization	Reference
harmonically convex	$\mu[0,x] = a + \int_{(0,\infty)} \left(1 - \frac{1}{xs}\right)^+ \tau(ds)$	Thm 3.2.1
concave	$\mu[0,x] = a + bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds)$	Thm 3.3.1
convex	$\mu[0,x] = a + bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^+ \tau(ds)$	Thm 3.4.1
harmonically concave	$\mu[0,x] = a$	N/A

3.2 Harmonically convex measures and harmonically concave tail measures

Following the techniques used in the proof of Polya's criterion, we characterize harmonically convex measures and harmonically concave tail measures in Theorem 3.2.1 and Theorem 3.2.2.

Table 3.2: Characterization of measure ν on $(0,\infty)$ with convexity properties on its tail

Property on ν	Characterization	Reference
harmonically concave tail	$\nu(x,\infty) = \frac{b}{x} + \int_{(0,\infty)} \left(1 \wedge \frac{1}{xs}\right) \tau(ds)$	Thm 3.2.2
convex tail	$\nu(x,\infty) = \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds)$	Thm 3.3.2
concave tail	$\nu(x,\infty) = 0$	N/A
harmonically convex tail	$\nu(x,\infty) = \frac{b}{x} + \int_{(0,\infty)} \left(\frac{1}{xs} - 1\right)^+ \tau(ds)$	Thm 3.5.1

Table 3.3: Representation for $f(x) \in \mathcal{CM}$ with convexity properties on its measure μ

Property on μ	Representation	Reference
harmonically convex	$f(x) = a + \int_{(0,\infty)} \frac{x}{s} k\left(\frac{x}{s}\right) \tau(ds)$	Prop 3.2.1
concave	$f(x) = a + \frac{b}{x} + \int_{(0,\infty)} r(xs)\, \tau(ds)$	Prop 3.3.1
convex	$f(x) = a + \frac{b}{x} + \int_{(0,\infty)} l(xs)\, \tau(ds)$	Prop 3.4.1
harmonically concave	$f(x) = a$	N/A

3.2.1 Characterizations

Theorem 3.2.1 *A measure μ on $[0,\infty)$ is harmonically convex, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\mu[0,x] = a + \int_{(0,\infty)} \left(1 - \frac{1}{xs}\right)^+ \tau(ds), \tag{3.2}$$

where $a \geq 0$ and τ satisfies $\tau(1,\infty) < \infty$.

Proof First, we show the sufficiency. If (3.2) holds for any $x > 0$, then

$$\mu[0,1/x] = a + \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds).$$

The integral is well-defined because $\mu[0,1/x] \leq a+\tau(x,\infty) < \infty$, for any $x > 0$. As the function $x \mapsto (1 - x/s)^+$ is convex for any $s > 0$, the integral is also convex. Hence, μ is harmonically convex.

Table 3.4: Representation for $g(x) \in \mathcal{BF}$ with convexity properties on its Lévy measure ν

Property on ν	Representation	Reference
harmonically concave tail	$g(x) = a + bx + \int_{(0,\infty)} \left(1 - \frac{x}{s}k\left(\frac{x}{s}\right)\right)\tau(ds)$	Prop 3.2.2
convex tail	$g(x) = a + bx + \int_{(0,\infty)} (1 - r(xs))\,\tau(ds)$	Prop 3.3.2
concave tail	$g(x) = a + bx$	N/A
harmonically convex tail	$g(x) = a + bx$	Cor 3.5.1 b)

Now, we show the necessity. If μ is harmonically convex, then $\varphi(x) := \mu[0, 1/x]$ is convex, non-increasing and non-negative on $(0, \infty)$. The following limits exist

$$a := \lim_{x\to\infty} \varphi(x) = \mu(\{0\}) \geq 0, \quad \lim_{x\to\infty} \varphi'_+(x) = 0, \quad \lim_{x\to 0+} \varphi'_+(x) \geq -\infty.$$

Define a Radon measure on $(0, \infty)$ by

$$\mu^*(p, q] = \begin{cases} \varphi'_+(q) - \varphi'_+(p), & \text{if } 0 < p < q < \infty, \\ \varphi'_+(q) - \lim_{x\to 0+} \varphi'_+(x), & \text{if } 0 = p < q < \infty. \end{cases}$$

For all $t > 0$, we have

$$\mu^*(t, \infty) = 0 - \varphi'_+(t) = -\varphi'_+(t).$$

Consider the measure τ defined on $(0, \infty)$ by $\tau(dt) = t\mu^*(dt)$. By (2.38), for $M > 0$,

$$\varphi(x) = \varphi(x) - \varphi(M) + \varphi(M) = -\int_{(x,M)} \varphi'_+(t)\,dt + \varphi(M).$$

Letting M approach infinity, and using Fubini's theorem, we obtain

$$\begin{aligned} \varphi(x) &= a - \int_{(x,\infty)} \varphi'_+(t)\,dt = a + \int_{(x,\infty)} \int_{(t,\infty)} \mu^*(ds)\,dt = a + \int_{(x,\infty)} \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt \\ &= a + \int_{(x,\infty)} \frac{1}{s} \int_{(x,s)} dt\,\tau(ds) = a + \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds). \end{aligned}$$

A change of variable leads to (3.2). The integrability condition on τ follows from

$$\tau(1, \infty) = \int_{(1,\infty)} \tau(ds) \leq \int_{(1,\infty)} \left(2 - \frac{1}{s}\right)\tau(ds) \leq 2\varphi(1/2) < \infty.$$

The proof is complete. $\square$

Theorem 3.2.2 *A measure ν on $(0,\infty)$ has harmonically concave tail, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\nu(x,\infty) = \frac{b}{x} + \int_{(0,\infty)} \left(1 \wedge \frac{1}{xs}\right) \tau(ds), \tag{3.3}$$

where $b \geq 0$ and τ satisfies

$$\int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right) \tau(ds) < \infty. \tag{3.4}$$

Proof First, we show the sufficiency. If (3.3) holds for any $x > 0$, then

$$\nu(1/x,\infty) = bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds).$$

The integral is well-defined for any $x > 0$, since τ satisfies condition (3.4). As the function $x \mapsto \min(1, x/s)$ is concave, the integral is concave, which implies measure ν has harmonically concave tail.

Now, we show the necessity. If measure ν has harmonically concave tail, then function $\varphi(x) := \nu(1/x,\infty)$ is non-negative, non-decreasing and concave on $(0,\infty)$. The following limits exist.

$$\lim_{x\to 0+} \varphi(x) = 0, \quad \lim_{x\to 0^+} \varphi'_+(x) \leq \infty, \quad b := \lim_{x\to\infty} \varphi'_+(x) < \infty.$$

The first limit follows from Lemma A.1.10. Define a Radon measure μ^* on $(0,\infty)$ by

$$\mu^*(p,q] = \begin{cases} \varphi'_+(p) - \varphi'_+(q), & \text{if } 0 < p < q < \infty, \\ \lim\limits_{x\to 0+} \varphi'_+(x) - \varphi'_+(q), & \text{if } 0 = p < q < \infty. \end{cases}$$

For all $t > 0$, we have

$$\mu^*(t,\infty) = \varphi'_+(t) - b.$$

Consider the measure τ on $(0,\infty)$ defined by $\tau(dt) = t\mu^*(dt)$. By (2.38), for $\epsilon > 0$,

$$\varphi(x) = \varphi(x) - \varphi(\epsilon) + \varphi(\epsilon) = \int_{(\epsilon,x)} \varphi'_+(t)\, dt + \varphi(\epsilon).$$

Letting ϵ approach 0, and using Fubini's Theorem, we obtain

$$\begin{aligned}
\varphi(x) &= \int_{(0,x)} \varphi'_+(t)\, dt = bx + \int_{(0,x)} (\varphi'_+(t) - b)\, dt \\
&= bx + \int_{(0,x)} \int_{(t,\infty)} \mu^*(ds)\, dt = bx + \int_{(0,x)} \int_{(t,\infty)} \frac{1}{s}\, \tau(ds)\, dt \\
&= bx + \int_{(0,x]} \int_{(0,s)} \frac{1}{s}\, dt\, \tau(ds) + \int_{(x,\infty)} \int_{(0,x)} \frac{1}{s}\, dt\, \tau(ds) = bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds).
\end{aligned}$$

Representation (3.3) follows, while the integrability condition (3.4) follows the fact that $\varphi(1) = \nu(1,\infty) < \infty$. This concludes the proof. □

Denote Dirac delta measure on $[0,\infty)$ by $\delta_0(dx)$, which has mass 1 at $\{0\}$ and mass 0 everywhere else. Next lemma extends to completely monotone and Bernstein functions. Recall that the completely monotone functions are associated with Bernstein measures, and Bernstein functions are associated with Lévy measures.

Corollary 3.2.1 *(a) A measure μ is harmonically convex on $[0,\infty)$, if and only if*

$$\mu(dx) = a\delta_0(dx) + \frac{1}{x^2}\Big(\int_{(1/x,\infty)} \frac{1}{s}\,\tau(ds)\Big)dx, \tag{3.5}$$

for some constant $a \geq 0$ and measure τ on $(0,\infty)$ satisfying $\tau(1,\infty) < \infty$.

(b) A Bernstein measure μ is harmonically convex, if and only if (3.5) holds and the measure τ on $(0,\infty)$ satisfies

$$\tau(1,\infty) < \infty, \quad \textit{and} \quad \int_{(0,1]}\int_{(0,1)} e^{-x/ts}\,dt\,\tau(ds) < \infty \quad \textit{for all} \quad x > 0. \tag{3.6}$$

(c) A measure ν has harmonically concave tail on $(0,\infty)$, if and only if

$$\nu(dx) = \frac{1}{x^2}\Big(b + \int_{(1/x,\infty)} \frac{1}{s}\,\tau(ds)\Big)dx, \tag{3.7}$$

for some constant $b \geq 0$ and measure τ on $(0,\infty)$ satisfying (3.4).

(d) A Lévy measure ν has harmonically concave tail, if and only if (3.7) holds with $b = 0$ and the measure τ on $(0,\infty)$ satisfies

$$\tau(0,1] < \infty \quad \textit{and} \quad \int_{(1,\infty)} \frac{\ln(s)}{s}\,\tau(ds) < \infty. \tag{3.8}$$

Proof (a) Formula (3.5) follows from the representation

$$\mu[0,x] = a + \int_{(1/x,\infty)}\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt,$$

which is inferred from the proof of Theorem 3.2.1. Here, measure τ satisfies the integrability condition $\tau(1,\infty) < \infty$.

(b) Suppose now μ is a Bernstein measure and it is harmonically convex. By part (a), it suffices to verify the second integrability conditions in (3.6). By (3.5) and Fubini's theorem, we have

$$\begin{aligned}\int_{[0,\infty)} e^{-xu}\,\mu(du) &= a + \int_{(0,\infty)} e^{-xu}\,\mu(du) = a + \int_{(0,\infty)} e^{-xu}\frac{1}{u^2}\int_{(1/u,\infty)}\frac{1}{s}\,\tau(ds)\,du\\ &= a + \int_{(0,\infty)}\frac{1}{s}\int_{(1/s,\infty)} e^{-xu}\frac{1}{u^2}\,du\,\tau(ds) = a + \int_{(0,\infty)}\int_{(0,1)} e^{-x/ts}\,dt\,\tau(ds),\end{aligned}$$

where in the last equality, we changed the variable $u = 1/ts$. Since this Laplace transform is well-defined for all $x > 0$, the second integrability condition in (3.6) follows.

Conversely, suppose a measure μ on $[0,\infty)$ have the representation (3.5) with condition (3.6). By part a), measure μ is harmonically convex. Reversing the steps above and noticing

$$\int_{(1,\infty)}\int_{(0,1)} e^{-x/ts}\,dt\,\tau(ds) \le \tau(1,\infty) < \infty,$$

we can see that the Laplace transform is well-defined for all $x > 0$. Therefore, μ is a Bernstein measure.

(c) Formula (3.7) follows immediately from the following representation

$$\nu(x,\infty) = \frac{b}{x} + \int_{(0,1/x)}\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt,$$

inferred from the proof of Theorem 3.2.2. Here, measure τ satisfies the integrability condition (3.4).

(d) Suppose now ν is a Lévy measure with harmonically concave tail. Then, from (3.7) and Fubini's Theorem, we obtain

$$\begin{aligned}\int_{(0,1]} t\,\nu(dt) &= \int_{(0,1]} \frac{b}{t}\,dt + \int_{(0,1]}\int_{(1/t,\infty)} \frac{1}{ts}\,\tau(ds)\,dt = \int_{(0,1]} \frac{b}{t}\,dt + \int_{(1,\infty)}\int_{(1/s,1]} \frac{1}{ts}\,dt\,\tau(ds)\\ &= \int_{(0,1]} \frac{b}{t}\,dt + \int_{(1,\infty)} \frac{\ln(s)}{s}\,\tau(ds).\end{aligned}$$

By (2.17), we can see $b = 0$ and

$$\int_{(1,\infty)} \frac{\ln(s)}{s}\,\tau(ds) < \infty.$$

On the other hand,

$$\begin{aligned}\int_{(1,\infty)} \nu(dt) &= \int_{(1,\infty)}\int_{(1/t,\infty)} \frac{1}{t^2 s}\,\tau(ds)\,dt = \int_{(0,1)}\int_{(1/s,\infty)} \frac{1}{t^2 s}\,dt\,\tau(ds) + \int_{[1,\infty)}\int_{(1,\infty)} \frac{1}{t^2 s}\,dt\,\tau(ds)\\ &= \int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right)\tau(ds) \ge \tau(0,1].\end{aligned}$$

By (2.17) again, we conclude $\tau(0,1] < \infty$.

Conversely, suppose a measure ν on $(0,\infty)$ have the representation (3.7) with $b = 0$ and condition (3.8). By part c), measure ν has harmonically concave tail. We also have

$$\int_{(0,1]} t\,\nu(dt) = \int_{(1,\infty)} \frac{\ln(s)}{s}\,\tau(ds) \quad \text{and} \quad \int_{(1,\infty)} \nu(dt) = \int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right)\tau(ds).$$

They are integrable by (3.8). Hence ν is a Lévy measure and the proof is complete. □

As an application of the above representations, we characterize completely monotone functions with harmonically convex measure and Bernstein functions having Lévy measure with harmonically concave tail. We need the well-known *exponential integral function* on $(0,\infty)$, given by

$$\mathrm{E}_1(x) := \int_{(x,\infty)} \frac{e^{-t}}{t}\,dt,$$

and use it to define the kernel

$$k(x) := -\mathrm{E}_1'(x) - \mathrm{E}_1(x) = \frac{e^{-x}}{x} - \int_{(x,\infty)} \frac{e^{-t}}{t}\, dt. \tag{3.9}$$

Function $k(x)$ plays a role in the next developments, hence we briefly summarize its properties.

Lemma 3.2.3 *Function $k(x)$ defined by (3.9) has the following properties.*

(a) It satisfies the inequality $0 \le k(x) \le e^{-x}/x$.

(b) It is completely monotone with convex measure. More precisely,

$$k(x) = \int_{(0,\infty)} e^{-xt} \frac{(t-1)^+}{t}\, dt. \tag{3.10}$$

(c) The product $xk(x)$ is completely monotone with harmonically convex measure. Moreover, $xk(x)$ is the derivative of a Bernstein function. Explicitly,

$$xk(x) = \int_{(1,\infty)} e^{-xt} \frac{1}{t^2}\, dt = \left(\int_{(1,\infty)} (1 - e^{-xt}) \frac{1}{t^3}\, dt \right)'. \tag{3.11}$$

(d) The function $1 - xk(x)$ is Bernstein, whose Lévy measure has harmonically concave tail. More precisely

$$1 - xk(x) = \int_{(1,\infty)} (1 - e^{-xt}) \frac{1}{t^2}\, dt. \tag{3.12}$$

Proof (a) The first inequality follows from

$$\mathrm{E}_1(x) = \int_{(x,\infty)} \frac{e^{-t}}{t}\, dt \le \frac{1}{x} \int_{(x,\infty)} e^{-t}\, dt = \frac{e^{-x}}{x} = -\mathrm{E}_1'(x),$$

while the second inequality is straightforward.

(b) Representation (3.10) is readily verified. The Bernstein measure for $k(x)$ is convex, as its cumulative distribution function is $(x - \ln(x) - 1) \cdot \mathbf{1}_{(x>1)}$ for $x \in [0, \infty)$, which is convex , as its derivative is non-decreasing.

(c) The Bernstein measure μ of $xk(x)$ is harmonically convex, since $x\mu[0, x] = (x-1)\cdot\mathbf{1}_{(x>1)}$. It is convex on $(0, \infty)$. One can readily verify (3.11) and see that $xk(x)$ is the derivative of a Bernstein function.

(d) The Lévy measure ν of $1 - xk(x)$ has harmonically concave tail, since $x\nu(x, \infty) = x \cdot \mathbf{1}_{(x\le1)} + \mathbf{1}_{(x>1)}$, and it is concave. $\square$

Note that, in general, part c) of Lemma 3.2.3 implies part b). Since $k(x) \in \mathcal{CM}$ with $\lim_{x\to\infty} k(x) = 0$ and $xk(x) \in \mathcal{CM}$, we conclude $k(x)$ has convex measure by Theorem 4.1.5.

3.2.2 Applications on completely monotone and Bernstein functions

Proposition 3.2.1 *Suppose $f(x)$ is completely monotone with measure μ. Then, measure μ is harmonically convex, if and only if there exists a unique measure τ on $(0,\infty)$ such that*

$$f(x) = a + \int_{(0,\infty)} \frac{x}{s} k\Big(\frac{x}{s}\Big) \tau(ds), \tag{3.13}$$

where $k(x)$ is defined in (3.9), $a = \mu(\{0\}) \geq 0$ is constant, and τ satisfies (3.6).

Proof For the necessity, let $a := \mu(\{0\}) \geq 0$. By Corollary 3.2.1 part b), for all $x > 0$, we have

$$\begin{aligned} f(x) &= \int_{[0,\infty)} e^{-xu} \mu(du) = a + \int_{(0,\infty)} e^{-xu} \mu(du) \\ &= a + \int_{(0,\infty)} e^{-xu} \frac{1}{u^2} \int_{(1/u,\infty)} \frac{1}{s} \tau(ds)\, du = a + \int_{(0,\infty)} \frac{1}{s} \int_{(1/s,\infty)} e^{-xu} \frac{1}{u^2}\, du\, \tau(ds), \end{aligned}$$

where τ satisfies conditions (3.6). The change of variable $u = t/s$ gives

$$f(x) = a + \int_{(0,\infty)} \int_{(1,\infty)} e^{-xt/s} \frac{1}{t^2}\, dt\, \tau(ds).$$

Finally, by (3.11), we obtain representation (3.13). The uniqueness of measure τ follows from the uniquness of μ.

For the sufficiency, suppose (3.13) holds. By Lemma 3.2.3 part c), we know $xk(x)$ is completely monotone with harmonically convex measure. So is $(x/s)k(x/s)$ for all $s > 0$. Indeed, if $f(x)$ is completely monotone function with harmonically convex measure μ, then so is the measure of the completely monotone $f(xs)$ for any $s > 0$, as its cumulative distribution function is $F_\mu(x/s)$. The proof is complete. □

We point out that the measures μ and τ in Proposition 3.2.1 are related through Equation (3.5).

Combining Corollary 3.2.1 part b), Proposition 3.2.1, and Proposition 2.2.1, we arrive at the following corollary.

Corollary 3.2.2 *Suppose $f(x)$ is completely monotone with harmonically convex measure μ. It has a Bernstein primitive, if and only if the measure τ in (3.13) satisfies*

$$\int_{(0,\infty)} \Big(1 - \frac{\ln(s+1)}{s}\Big) \tau(ds) < \infty. \tag{3.14}$$

Proof We only have to verify the integrability condition (3.14) on τ is equivalent to (2.21) on μ. Using (3.5), we can observe

$$\begin{aligned} \int_{(0,\infty)} \frac{1}{1+t} \mu(dt) &= \int_{(0,\infty)} \frac{1}{(1+t)t^2} \Big(\int_{(1/t,\infty)} \frac{1}{s} \tau(ds) \Big) dt \\ &= \int_{(0,\infty)} \Big(\int_{(1/s,\infty)} \frac{1}{t^2(1+t)}\, dt \Big) \frac{1}{s} \tau(ds) = \int_{(0,\infty)} \Big(1 - \frac{\ln(s+1)}{s}\Big) \tau(ds), \end{aligned}$$

where in the second equality, we changed the order of integration using Fubini's theorem. The equivalence follows. □

The next proposition characterizes Bernstein functions, whose Lévy measure has harmonically concave tail.

Proposition 3.2.2 *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, measure ν has harmonically concave tail, if and only if there exists a unique triplet (a, b, τ) such that*

$$g(x) = a + bx + \int_{(0,\infty)} \left(1 - \frac{x}{s} k\left(\frac{x}{s}\right)\right) \tau(ds), \tag{3.15}$$

where $k(x)$ is defined in (3.9), $a, b \geq 0$ are constants, and τ is a measure on $(0, \infty)$, satisfying (3.8).

Proof We show necessity first. If ν has harmonically concave tail, by Corollary 3.2.1 part d), for all $x > 0$, we have

$$\begin{aligned} g(x) &= a + bx + \int_{(0,\infty)} (1 - e^{-xu})\, \nu(du) = a + bx + \int_{(0,\infty)} (1 - e^{-xu}) \frac{1}{u^2} \int_{(1/u,\infty)} \frac{1}{s}\, \tau(ds)\, du \\ &= a + bx + \int_{(0,\infty)} \frac{1}{s} \int_{(1/s,\infty)} (1 - e^{-xu}) \frac{1}{u^2}\, du\, \tau(ds), \end{aligned}$$

where τ satisfies (3.8). The change of variable $u = t/s$ gives

$$g(x) = a + bx + \int_{(0,\infty)} \int_{(1,\infty)} (1 - e^{-xt/s}) \frac{1}{t^2}\, dt\, \tau(ds).$$

Apply now (3.12) to get representation (3.15). The uniqueness of the triplet (a, b, τ) follows from the uniqueness of the Lévy triplet (a, b, ν).

For the sufficiency, suppose (3.15) holds. By part d) of Lemma 3.2.3 we know $1 - xk(x)$ is a Bernstein function with Lévy measure having harmonically concave tail. So is $1 - (x/s)k(x/s)$ for all $s > 0$. The proof follows from here. $\square$

3.3 Concave measures and convex tail measures

In this section we characterize concave measures and measures with convex tail. They are accomplished in Theorems 3.3.1 and 3.3.2. The development parallels that of Section 3.2.

3.3.1 Characterizations

Theorem 3.3.1 *A measure μ on $[0, \infty)$ is concave, if and only if there is a measure τ on $(0, \infty)$ such that*

$$\mu[0, x] = a + bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds), \tag{3.16}$$

where $a, b \geq 0$ and τ satisfies

$$\int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right) \tau(ds) < \infty. \tag{3.17}$$

Proof First, we show the sufficiency. If (3.16) and (3.17) hold, then the integral in (3.16) is convergent for all $x > 0$. Since the function $x \mapsto \min(1, x/s)$ is concave for all $s > 0$, the integral is concave, which implies that μ is concave measure.

Now, we show necessity. Since $F(x) := \mu[0, x]$ is non-decreasing and concave, the following limits of right derivatives exist:

$$b := \lim_{x\to\infty} F'_+(x) < \infty, \quad \lim_{x\to 0+} F'_+(x) \le \infty.$$

Denote $a := F(0) = \mu(\{0\}) \ge 0$ and define the Radon measure μ^* on $(0, \infty)$ by

$$\mu^*(p, q] = \begin{cases} F'_+(p) - F'_+(q), & \text{if } 0 < p < q < \infty, \\ \lim_{x\to 0+} F'_+(x) - F'_+(q), & \text{if } 0 = p < q < \infty. \end{cases}$$

Note that for all $t > 0$, we have

$$\mu^*(t, \infty) = F'_+(t) - \lim_{x\to\infty} F'_+(x) = F'_+(t) - b.$$

Consider the measure τ defined by $\tau(dt) = t\mu^*(dt)$ on $(0, \infty)$. By (2.38), for $\epsilon > 0$,

$$F(x) = F(x) - F(\epsilon) + F(\epsilon) = \int_{(\epsilon,x)} F'_+(t)\, dt + F(\epsilon).$$

Letting ϵ approaching 0, and using Fubini's theorem, we obtain

$$\begin{aligned} F(x) &= \int_{(0,x)} F'_+(t)\, dt + F(0) = a + bx + \int_{(0,x)} (F'_+(t) - b)\, dt \\ &= a + bx + \int_{(0,x)} \int_{(t,\infty)} \mu^*(ds)\, dt = a + bx + \int_{(0,x]} \int_{(0,s)} \frac{1}{s}\, dt\, \tau(ds) + \int_{(x,\infty)} \int_{(0,x)} \frac{1}{s}\, dt\, \tau(ds) \\ &= a + bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds). \end{aligned}$$

Representation (3.16) follows, while $F(1) < \infty$ implies the integrability condition (3.17). This completes the proof. □

Theorem 3.3.2 *A measure ν on $(0, \infty)$ has convex tail, if and only if there is a measure τ on $(0, \infty)$ such that*

$$\nu(x, \infty) = \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds), \tag{3.18}$$

where τ satisfies $\tau(1, \infty) < \infty$.

Proof We show sufficiency first. If (3.18) holds for measure ν with $\tau(1, \infty) < \infty$, then $\nu(x, \infty)$ is well-defined and since the function $x \mapsto (1 - x/s)^+$ is convex for all $s > 0$, the integral is convex, implying that ν has convex tail.

Now, we show the necessity. If measure ν has convex tail, then $\bar{\nu}(x) = \nu(x,\infty)$ is non-increasing, convex, and non-negative. The following limits exist:

$$\lim_{x\to\infty} \bar{\nu}(x) = 0, \quad \lim_{x\to\infty} \bar{\nu}'_+(x) = 0, \quad \lim_{x\to 0+} \bar{\nu}'_+(x) \geq -\infty.$$

The first limit follows from Lemma A.1.10. Define the Radon measure μ^* on $(0,\infty)$

$$\mu^*(p,q] = \begin{cases} \bar{\nu}'_+(q) - \bar{\nu}'_+(p), & \text{if } 0 < p < q < \infty, \\ \bar{\nu}'_+(q) - \lim\limits_{x\to 0+} \bar{\nu}'_+(x), & \text{if } 0 = p < q < \infty. \end{cases}$$

Note that for all $t > 0$, we have

$$\mu^*(t,\infty) = \lim_{x\to\infty} \bar{\nu}'_+(x) - \bar{\nu}'_+(t) = -\bar{\nu}'_+(t).$$

Consider the measure τ defined by $\tau(dt) = t\mu^*(dt)$ on $(0,\infty)$. By (2.38), for $M > 0$,

$$\nu(x,\infty) = \bar{\nu}(x) - \bar{\nu}(M) + \bar{\nu}(M) = -\int_{(x,M)} \bar{\nu}'_+(t)\,dt + \bar{\nu}(M).$$

Letting M approaching infinity, and using Fubini's theorem, we obtain

$$\begin{aligned} \nu(x,\infty) &= -\int_{(x,\infty)} \bar{\nu}'_+(t)\,dt = \int_{(x,\infty)} \int_{(t,\infty)} \mu^*(ds)\,dt = \int_{(x,\infty)} \int_{(x,s)} dt\,\mu^*(ds) \\ &= \int_{(x,\infty)} \frac{s-x}{s}\,\tau(ds) = \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds). \end{aligned}$$

The condition $\tau(1,\infty) < \infty$ follows from the existence of $\bar{\nu}(x)$. Indeed,

$$\tau(1,\infty) = \int_{(1,\infty)} \tau(ds) \leq \int_{(1,\infty)} \left(2 - \frac{1}{s}\right) \tau(ds) \leq 2\bar{\nu}(1/2) < \infty.$$

The proof is completed. $\square$

Corollary 3.3.1 *(a) A measure μ is concave on $[0,\infty)$, if and only if*

$$\mu(dx) = a\delta_0(dx) + \left(b + \int_{(x,\infty)} \frac{1}{s}\,\tau(ds)\right) dx, \tag{3.19}$$

for some constant $a \geq 0$ and measure τ on $(0,\infty)$ satisfying (3.17). In particular, every concave measure on $[0,\infty)$ is a Bernstein measure.

(b) A measure ν has convex tail on $(0,\infty)$, if and only if

$$\nu(dx) = \left(\int_{(x,\infty)} \frac{1}{s}\,\tau(ds)\right) dx, \tag{3.20}$$

for some measure τ on $(0,\infty)$ satisfying $\tau(1,\infty) < \infty$.

(c) A Lévy measure ν has convex tail, if and only if (3.20) holds and the measure τ on $(0,\infty)$ is another Lévy measure, satisfying (2.17).

Proof (a) Formula (3.19) follows immediately from the following representation

$$\mu[0,x] = a + bx + \int_{(0,x)} \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt,$$

which is inferred from the proof of Theorem 3.3.1. Here, measure τ satisfies (3.17).

Consider the Laplace transform of such measure μ. Using Fubini's theorem,

$$\begin{aligned}\int_{[0,\infty)} e^{-tx}\,\mu(dt) &= a + \int_{(0,\infty)} e^{-xt}\Big(b + \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\Big)dt = a + \frac{b}{x} + \int_{(0,\infty)}\int_{(t,\infty)} e^{-xt}\frac{1}{s}\,\tau(ds)\,dt\\ &= a + \frac{b}{x} + \int_{(0,\infty)}\int_{(0,s)} e^{-xt}\frac{1}{s}\,dt\,\tau(ds) = a + \frac{b}{x} + \frac{1}{x}\int_{(0,\infty)} \frac{1-e^{-xs}}{s}\,\tau(ds).\end{aligned}$$

If (3.17) holds, the last integral is convergent, because $(1-e^{-xs})/s \le 1/s$ on $[1,\infty)$, and $(1-e^{-xs})/s \le x$ on $(0,1)$ for any $x > 0$. Thus, a Bernstein measure is concave, if and only if it admits representation (3.19) with condition (3.17).

(b) Formula (3.20) follows immediately from the following representation

$$\nu(x,\infty) = \int_{(x,\infty)}\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt,$$

inferred from the proof of Theorem 3.3.2. Here, measure τ satisfies $\tau(1,\infty) < \infty$.

(c) Suppose now ν is Lévy measure with convex tail. By part b) we know that ν admits representation (3.20) with $\tau(1,\infty) < \infty$. On the other hand, using Fubini's theorem, we obtain

$$\begin{aligned}\int_{(0,1]} t\,\nu(dt) &= \int_{(0,1]} t\Big(\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\Big)dt = \int_{(0,1]}\int_{(0,s)} \frac{t}{s}\,dt\,\tau(ds) + \int_{(1,\infty)}\int_{(0,1)} \frac{t}{s}\,dt\,\tau(ds)\\ &= \frac{1}{2}\int_{(0,1]} s\,\tau(ds) + \frac{1}{2}\int_{(1,\infty)} \frac{1}{s}\,\tau(ds).\end{aligned}$$

The integrability (2.17) for ν implies that

$$\int_{(0,1]} s\,\tau(ds) < \infty.$$

Therefore, measure τ is a Lévy measure as well.

Conversely, suppose ν has representation (3.20) while τ is a Lévy measure. By part b), ν has convex tail. Reversing the steps above shows that

$$\int_{(0,1]} t\,\nu(dt) = \frac{1}{2}\int_{(0,\infty)} \Big(s \wedge \frac{1}{s}\Big)\tau(ds) \le \frac{1}{2}\int_{(0,\infty)} (s\wedge 1)\,\tau(ds) < \infty.$$

Also note that

$$\int_{(1,\infty)} \nu(dt) = \int_{(1,\infty)}\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt = \int_{(1,\infty)} \frac{1}{s}\int_{(1,s)} dt\,\tau(ds) \le \int_{(1,\infty)} \tau(ds) < \infty.$$

Therefore, measure ν is a Lévy measure by (2.17). $\square$

As an application of the above representations, we characterize completely monotone functions with concave measures and Bernstein functions whose Lévy measures have convex tail. We first define the following kernel on $(0,\infty)$.

$$r(x) := \frac{1-e^{-x}}{x}. \tag{3.21}$$

Below we list several properties of $r(x)$ that will be needed.

Lemma 3.3.3 *Function $r(x)$ defined by (3.21) has the following properties.*

(a) It satisfies the inequalities $0 \le r(x) \le \min(1, 1/x)$.

(b) It is completely monotone with concave measure. More precisely,

$$r(x) = \int_{(0,1)} e^{-xt}\, dt. \tag{3.22}$$

(c) The function $1 - r(x)$ *is Bernstein, whose Lévy measure has convex tail. More precisely,*

$$1 - r(x) = \int_{(0,1)} (1 - e^{-xt})\, dt. \tag{3.23}$$

Proof (a) We only need to show that $r(x) \le 1/x$ and $r(x) \le 1$. The first inequality is trivial. The second inequality is equivalent to $1 - e^{-x} \le x$ for $x > 0$.

(b) The Bernstein measure μ for $r(x)$ is concave as $\mu[0,x] = \min(x,1)$.

(c) The Lévy measure ν for $1 - r(x)$ is convex, as $\bar{\nu}(x) = (1-x)^+$. □

3.3.2 Applications on completely monotone and Bernstein functions

Proposition 3.3.1 *Suppose $f(x)$ is completely monotone with measure μ. Then, measure μ is concave, if and only if there exists a unique measure τ on $(0,\infty)$ such that*

$$f(x) = a + \frac{b}{x} + \int_{(0,\infty)} r(xs)\,\tau(ds), \tag{3.24}$$

where $r(x)$ is defined in (3.21), $a, b \ge 0$ are constants and measure τ satisfies (3.17).

Proof For the necessity, let $a := \mu(\{0\}) \ge 0$. By Corollary 3.3.1 part a), for all $x > 0$, we have

$$\begin{aligned}\int_{[0,\infty)} e^{-xt}\,\mu(dt) &= a + \int_{(0,\infty)} e^{-xt}\Big(b + \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\Big)\,dt = a + \frac{b}{x} + \int_{(0,\infty)}\int_{(0,s)} e^{-xt}\frac{1}{s}\,dt\,\tau(ds)\\ &= a + \frac{b}{x} + \int_{(0,\infty)} \frac{1-e^{-xs}}{xs}\,\tau(ds) = a + \frac{b}{x} + \int_{(0,\infty)} r(xs)\,\tau(ds),\end{aligned}$$

where τ satisfies conditions (3.17). The uniqueness of measure τ follows the uniqueness of measure μ.

For the sufficiency, suppose (3.24) holds. By Lemma 3.3.3 part b), completely monotone function $r(x)$ has concave measure. So is the completely monotone function $r(xs)$ for all $s > 0$. Since $1/x$ is also completely monotone with concave measure, the proof follows. □

Combining Corollary 3.3.1 part a), Proposition 3.3.1 and Proposition 2.2.1, we have the following corollary.

Corollary 3.3.2 *Suppose $f(x)$ is completely monotone with concave measure. It has a Bernstein primitive, if and only if (3.24) holds with $b = 0$ and τ satisfies*

$$\int_{(0,\infty)} \frac{\ln(s+1)}{s}\,\tau(ds) < \infty. \tag{3.25}$$

Proof Observe that condition (3.25) implies condition (3.17), since

$$\frac{1}{2}\Big(1 \wedge \frac{1}{s}\Big) \le \frac{\log(s+1)}{s} \quad \text{for } s > 0.$$

Let μ be the Bernstein measure of $f(x)$. Using Fubini's theorem, we have

$$\begin{aligned}\int_{(0,\infty)} \frac{1}{1+t}\,\mu(dt) &= \int_{(0,\infty)} \frac{1}{1+t}\Big(b + \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\Big)\,dt\\ &= \int_{(0,\infty)} \frac{b}{1+t}\,dt + \int_{(0,\infty)}\int_{(0,s)} \frac{1}{(1+t)s}\,dt\,\tau(ds)\\ &= \int_{(0,\infty)} \frac{b}{1+t}\,dt + \int_{(0,\infty)} \frac{\ln(s+1)}{s}\,\tau(ds).\end{aligned}$$

This shows that, condition (2.21) is equivalent to (3.25) with $b = 0$ in (3.24) □

Next proposition characterizes Bernstein functions whose Lévy measures have convex tails.

Proposition 3.3.2 *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, measure ν has convex tail, if and only if there exists a unique triplet (a, b, τ) such that*

$$g(x) = a + bx + \int_{(0,\infty)} (1 - r(xs))\,\tau(ds), \tag{3.26}$$

where $r(x)$ is defined in (3.21), $a, b \ge 0$ are constants, and τ is also a Lévy measure.

Proof We show necessity first. If ν has convex tail, by Corollary 3.3.1 part c), for all $x > 0$, we have

$$\begin{aligned}g(x) &= a + bx + \int_{(0,\infty)} (1 - e^{-xt})\,\nu(dt) = a + bx + \int_{(0,\infty)} (1 - e^{-xt}) \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt\\ &= a + bx + \int_{(0,\infty)}\int_{(0,s)} \frac{1 - e^{-xt}}{s}\,dt\,\tau(ds) = a + bx + \int_{(0,\infty)} \Big(1 - \frac{1 - e^{-xs}}{xs}\Big)\tau(ds)\\ &= a + bx + \int_{(0,\infty)} (1 - r(xs))\,\tau(ds),\end{aligned}$$

where τ is a Lévy measure. The uniqueness of the triplet (a, b, τ) follows from the uniqueness of the triplet (a, b, ν).

For the sufficiency, suppose (3.26) holds. By Lemma 3.3.3 part c) we have that $1 - r(x)$ is a Bernstein function having Lévy measure with convex tail. So is Bernstein function $1 - r(xs)$ for all $s > 0$. The proof follows. □

3.4 Convex measures

In this section we characterize convex measure on $[0,\infty)$ in Theorem 3.4.1 and investigate the properties of the related completely monotone functions.

We do not consider measures with concave tail as a counterpart, because Radon measures on $(0,\infty)$ can not have concave tail unless $\nu(0,\infty) = 0$. This is because the tail function $\bar{\nu}(x) = \nu(x,\infty)$ is non-negative and non-increasing. It is concave if and only if $\bar{\nu}(x)$ is constant. This trivial case is not of interest.

3.4.1 Characterizations

Theorem 3.4.1 *A measure μ on $[0,\infty)$ is convex, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\mu[0,x] = a + bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^+ \tau(ds), \tag{3.27}$$

where $a, b \geq 0$ and τ satisfies

$$\int_{(0,1]} \frac{1}{s}\,\tau(ds) < \infty. \tag{3.28}$$

Proof First we show the sufficiency. Condition (3.28) implies the convergence of the integral in (3.27), for any $x > 0$. Since function $x \mapsto (x/s - 1)^+$ is convex for all $s > 0$, the integral is convex, which implies that μ is a convex measure.

Now, we show necessity. If μ is a convex measure, it cumulative distribution function $F(x) := \mu[0,x]$ is non-decreasing and convex, hence the following limit exists:

$$b := \lim_{x\to 0+} F'_+(x) < \infty.$$

Denote $a := \lim_{x\to 0+} F(x) = \mu(\{0\})$ and define the Radon measure μ^* on $(0,\infty)$ by

$$\mu^*(p,q] = \begin{cases} F'_+(q) - F'_+(p), & \text{if } 0 < p < q < \infty, \\ F'_+(q) - \lim\limits_{x\to 0+} F'_+(x), & \text{if } 0 = p < q < \infty. \end{cases}$$

Note that for any $t > 0$, we have

$$\mu^*(0,t] = F'_+(t) - \lim_{x\to 0+} F'_+(x) = F'_+(t) - b.$$

Consider the measure τ defined by $\tau(dt) = t\mu^*(dt)$ on $(0,\infty)$. By (2.38) for $\epsilon > 0$,

$$F(x) = F(x) - F(\epsilon) + F(\epsilon) = \int_{(\epsilon,x)} F'_+(t)\,dt + F(\epsilon).$$

Letting ϵ approaching zero, and using Fubini's theorem, we obtain

$$F(x) = a + \int_{(0,x)} F'_+(t)\,dt = a + bx + \int_{(0,x)} (F'_+(t) - b)\,dt = a + bx + \int_{(0,x)}\int_{(0,t]} \mu^*(ds)\,dt$$

$$= a + bx + \int_{(0,x)} \int_{[s,x)} \frac{1}{s}\, dt\, \tau(ds) = a + bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^{+} \tau(ds).$$

The integrability condition (3.28) follows from

$$\int_{(0,1]} \frac{1}{s}\, \tau(ds) \le \int_{(0,1]} \left(\frac{2}{s} - 1\right) \tau(ds) \le \int_{(0,\infty)} \left(\frac{2}{s} - 1\right)^{+} \tau(ds) \le F(2) < \infty.$$

This closes the proof. □

Corollary 3.4.1 *(a) A measure μ is convex on $[0, \infty)$, if and only if*

$$\mu(dx) = a\delta_0(dx) + \left(b + \int_{(0,x]} \frac{1}{s}\, \tau(ds)\right) dx, \tag{3.29}$$

for some constant $a, b \ge 0$ and measure τ on $(0, \infty)$ satisfying (3.28).

(b) A Bernstein measure μ on $[0, \infty)$ is convex, if and only if (3.29) holds and the measure τ on $(0, \infty)$ satisfies

$$\int_{(0,\infty)} \frac{e^{-xs}}{s}\, \tau(ds) < \infty, \quad \textit{for all} \quad x > 0. \tag{3.30}$$

Proof (a) Formula (3.29) follows immediately from the following representation

$$\mu[0, x] = a + bx + \int_{(0,x)} \int_{(0,t]} \frac{1}{s}\, \tau(ds)\, dt,$$

inferred from the proof of Theorem 3.4.1. Here, measure τ satisfies condition (3.28).

(b) Suppose now Bernstein measure μ is convex. It admits representation (3.29). On the other hand,

$$\begin{aligned}\int_{[0,\infty)} e^{-xt}\, \mu(dt) &= a + \int_{(0,\infty)} e^{-xt}\left(b + \int_{(0,t]} \frac{1}{s}\, \tau(ds)\right) dt \\ &= a + \frac{b}{x} + \int_{(0,\infty)} \frac{1}{s} \int_{[s,\infty)} e^{-xt}\, dt\, \tau(ds) = a + \frac{b}{x} + \frac{1}{x} \int_{(0,\infty)} \frac{e^{-xs}}{s}\, \tau(ds).\end{aligned}$$

The convergence of the Laplace transform of μ indicates τ satisfies (3.30).

Conversely, if a measure μ on $[0, \infty)$ has representation (3.29) with condition (3.30), then it is convex by part a), because (3.30) implies (3.28). Its Laplace transform is well-defined for all $x > 0$, by reversing the steps above. Thus μ is a Bernstein measure. □

As an application of the above representation, we characterize completely monotone functions with convex measures. Define the kernel

$$l(x) := \frac{e^{-x}}{x}, \quad x > 0. \tag{3.31}$$

The necessary properties of function $l(x)$ are listed below.

Lemma 3.4.2 *Function $l(x)$ defined in (3.31) has the following properties.*

(a) It satisfies the inequalities $0 \le l(x) \le 1/x$.

(b) It is completely monotone with convex measure. More precisely,

$$l(x) = \int_{(1,\infty)} e^{-xt}\, dt. \tag{3.32}$$

Proof (a) The inequalities are trivial.

(b) Note that the cumulative distribution function of the measure of $l(x)$ is $x \mapsto (x-1)^+$, which is convex. □

3.4.2 Applications on completely monotone and Bernstein functions

Proposition 3.4.1 *Suppose $f(x)$ is completely monotone with measure μ. Then, measure μ is convex, if and only if there exists a unique measure τ on $(0,\infty)$ such that*

$$f(x) = a + \frac{b}{x} + \int_{(0,\infty)} l(xs)\, \tau(ds), \tag{3.33}$$

where $l(x)$ is defined in (3.31), $a, b \ge 0$ are constants, and τ satisfies (3.30).

Proof For the necessity, let $a := \mu(\{0\}) \ge 0$. By Corollary 3.4.1 part b), for all $x > 0$, we have

$$\begin{aligned}
f(x) &= \int_{[0,\infty)} e^{-xt}\, \mu(dt) = a + \int_{(0,\infty)} e^{-xt}\Big(b + \int_{(0,t]} \frac{1}{s}\, \tau(ds)\Big)\, dt \\
&= a + \frac{b}{x} + \int_{(0,\infty)} \int_{[s,\infty)} e^{-xt}\frac{1}{s}\, dt\, \tau(ds) = a + \frac{b}{x} + \int_{(0,\infty)} \frac{e^{-xs}}{xs}\, \tau(ds) \\
&= a + \frac{b}{x} + \int_{(0,\infty)} l(xs)\, \tau(ds),
\end{aligned}$$

where τ satisfies conditions (3.30). The uniqueness of τ follows from the uniqueness of μ.

For the sufficiency, suppose (3.33) holds. By Lemma 3.4.2 part b) we know $l(x)$ is completely monotone with convex measure. So is completely monotone function $l(sx)$ for all $s > 0$. Since $1/x$ is also completely monotone with convex measure, the proof follows. □

Combining Corollary 3.4.1 part b) with Proposition 2.2.1, we have the following corollary.

Corollary 3.4.2 *Suppose $f(x)$ is completely monotone with convex measure μ. It can not have a Bernstein primitive, unless $f(x)$ is a constant.*

Proof We only have to verify the integrability condition (2.21) can not be satisfies unless μ vanishes on $(0,\infty)$. By (3.29), we have

$$\begin{aligned}
\int_{(0,\infty)} \frac{1}{1+t}\, \mu(dt) &= \int_{(0,\infty)} \frac{1}{1+t}\Big(b + \int_{(0,t]} \frac{1}{s}\, \tau(ds)\Big)\, dt \\
&= \int_{(0,\infty)} \frac{b}{1+t}\, dt + \int_{(0,\infty)} \frac{1}{s} \int_{[s,\infty)} \frac{1}{1+t}\, dt\, \tau(ds).
\end{aligned}$$

However, $1/(1+t)$ is not integrable on $[s,\infty)$ for any $s > 0$. So $f(x)$ can not have a primitive which is Bernstein, unless $b = 0$ and τ vanishes on $(0,\infty)$. That implies that μ vanishes on $(0,\infty)$ and $f(x)$ is a constant. □

In [71, Theorem 3.1, Lemma 4.1] the following hereditary properties are shown.

(a) If a completely monotone function $f(x)$ has (harmonically) convex measure, then $-f'(x)$ also has (harmonically) convex measure; and

(b) If a Bernstein function $g(x)$ has Lévy measure with harmonically concave tail, then $g'(x)$ has harmonically convex measure.

Alternatively, Corollary 3.4.2 says that the derivative of a Bernstein function $g(x)$, cannot have convex measure, unless $g(x)$ is affine. Curiously, we have a partial converse of (a).

Proposition 3.4.2 *Suppose $f(x)$ is completely monotone. If $-f'(x)$ has convex measure, then $f(x)$ has harmonically convex measure.*

Proof Note that the measure for completely monotone function $-f'(x)$ has no mass at $\{0\}$. Then, by Theorem 4.1.5, functions $-xf'(x)$ is completely monotone. So is $f(x) - xf'(x)$. Theorem 4.1.1 now shows that $f(x)$ has harmonically convex measure. □

However, some other converses fail. For completely monotone function $f(x)$, even if $-f'(x)$ has convex measure, $f(x)$ may not have convex measure. Consider completely monotone function $f(x) = 2x^{-1/2}$ with $-f'(x) = x^{-3/2}$. It is clear that $-f'(x)$ has convex measure. But $f(x)$ does not have convex measure. See Example 2.4.4.

Meanwhile, if $g(x)$ is Bernstein, then $g'(x)$ having harmonically convex measure does not necessarily implies $g(x)$ have Lévy measure with harmonically concave tail. Consider Bernstein function and its completely monotone derivative

$$g(x) = \int_{(0,\infty)} (1 - e^{-xt}) \frac{1}{t(t+1)^2}\, dt \quad \text{and} \quad g'(x) = \int_{(0,\infty)} e^{-xt} \frac{1}{(t+1)^2}\, dt.$$

It can be shown that $g(x)$ is well-defined, as

$$\int_{(0,1)} \frac{1}{(t+1)^2}\, dt < 1 \quad \text{and} \quad \int_{(1,\infty)} \frac{1}{t(t+1)^2}\, dt = \ln(2) - \frac{1}{2}.$$

Note that $g'(x)$ has harmonically convex measure, because its cumulative distribution function is $F(t) = t/(t+1)$ and $(tF(t))'' = 2/(t+1)^3 \geq 0$. However, the Lévy measure ν for $g(x)$ does not have harmonically concave tail. Because the tail is $\bar{\nu}(t) = \ln(t+1) - \ln(t) - 1/(t+1)$, and $\bar{\nu}(1/t)$ is not concave. Indeed,

$$\frac{d}{dt^2}\bar{\nu}(1/t) = \frac{1-t}{(t+1)^3}.$$

It changes sign on $(0, \infty)$.

3.5 Harmonically convex tail measures

This section is devoted to prove Theorem 3.5.1, which characterize measure with harmonically convex tail. Then, Corollary 3.5.1 shows that there are no Bernstein function whose measure has harmonically convex tail, unless it degenerates to affine function.

We do not consider harmonically concave measure as a counterpart, as Radon measures on $[0,\infty)$ can not be harmonically concave unless $\mu[0,\infty)$ is constant. This is because the cumulative function $F(x)$ is non-negative and increasing. $F(1/x)$ is concave if and only if $F(1/x)$ is constant. This trivial case is not interesting.

Theorem 3.5.1 *A measure ν on $(0,\infty)$ has harmonically convex tail, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\nu(x,\infty) = \frac{b}{x} + \int_{(0,\infty)} \left(\frac{1}{xs} - 1\right)^+ \tau(ds), \tag{3.34}$$

where $b \geq 0$ and τ satisfies

$$\int_{(0,1]} \frac{1}{s}\,\tau(ds) < \infty. \tag{3.35}$$

Proof First, we show the sufficiency. If (3.34) and (3.35) hold, then

$$\nu(1/x,\infty) = bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^+ \tau(ds).$$

And the integral is convergent for all $x > 0$. Since the function $x \mapsto (x/s - 1)^+$ is convex, the integral on the right hand side is convex, which implies that ν has harmonically convex tail.

Now, we show necessity. If ν has harmonically convex tail, then function $\varphi(x) := \nu(1/x,\infty)$ is non-negative, non-decreasing and convex. The following limits exist and are non-negative:

$$\lim_{x\to 0+} \varphi(x) = 0, \quad b := \lim_{x\to 0+} \varphi'_+(x) < \infty.$$

Define the Radon measure μ^* on $(0,\infty)$ by

$$\mu^*(p,q] = \begin{cases} \varphi'_+(q) - \varphi'_+(p), & \text{if } 0 < p < q < \infty, \\ \varphi'_+(q) - \lim\limits_{x\to 0+} \varphi'_+(x), & \text{if } 0 = p < q < \infty. \end{cases}$$

Note that for any $t > 0$, we have

$$\mu^*(0,t] = \varphi'_+(t) - b.$$

Consider the measure τ defined by $\tau(dt) = t\mu^*(dt)$ on $(0,\infty)$. By (2.38), for $\epsilon > 0$,

$$\varphi(x) = \varphi(x) - \varphi(\epsilon) + \varphi(\epsilon) = \int_{(\epsilon,x)} \varphi'_+(t)\,dt + \varphi(\epsilon).$$

Letting ϵ approaching 0, and using Fubini's theorem, we obtain

$$\begin{aligned} \varphi(x) &= \int_{(0,x)} \varphi'_+(t)\,dt = bx + \int_{(0,x)} (\varphi'_+(t) - b)\,dt = bx + \int_{(0,x)} \int_{(0,t]} \mu^*(ds)\,dt \\ &= bx + \int_{(0,x)} \int_{[s,x)} \frac{1}{s}\,dt\,\tau(ds) = bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^+ \tau(ds). \end{aligned}$$

Representation (3.34) follows. The integrability condition (3.35) holds, because

$$\int_{(0,1]} \frac{1}{s}\,\tau(ds) \leq \int_{(0,1]} \left(\frac{2}{s} - 1\right) \tau(ds) \leq \int_{(0,\infty)} \left(\frac{2}{s} - 1\right)^+ \tau(ds) \leq \varphi(2) < \infty.$$

The proof is completed. □

Corollary 3.5.1 *(a) A measure ν on $(0,\infty)$ is convex, if and only if*

$$\nu(dx) = \frac{1}{x^2}\Big(b + \int_{(0,1/x]} \frac{1}{s}\,\tau(ds)\Big)\,dx, \tag{3.36}$$

for some constant $b \geq 0$ and measure τ satisfying (3.35).

(b) A Lévy measure ν can not have harmonically convex tail, unless $\nu(0,\infty) = 0$.

Proof (a) Formula (3.36) follows immediately from the following representation

$$\nu(x,\infty) = \frac{b}{x} + \int_{(0,1/x)} \int_{(0,t]} \frac{1}{s}\,\tau(ds)\,dt,$$

inferred from the proof of Theorem 3.5.1, and τ satisfies (3.35).

(b) If Lévy measure ν has harmonically convex tail, representation (3.36) holds. As ν is a Lévy meausre, by (2.17), the following is finite:

$$\begin{aligned}\int_{(0,1]} t\,\nu(dt) &= \int_{(0,1]} \frac{1}{t}\Big(b + \int_{(0,1/t]} \frac{1}{s}\,\tau(ds)\Big)dt = \int_{(0,1]} \frac{b}{t}\,dt + \int_{(0,1]} \frac{1}{t} \int_{(0,1/t]} \frac{1}{s}\,\tau(ds)\,dt \\ &= \int_{(0,1]} \frac{b}{t}\,dt + \int_{(1,\infty)} \frac{1}{s} \int_{(0,1/s]} \frac{1}{t}\,dt\,\tau(dt) + \int_{(0,1]} \frac{1}{s} \int_{(0,1]} \frac{1}{t}\,dt\,\tau(dt).\end{aligned}$$

The above is convergent if and only if $b = 0$ and τ vanishes on $(0,\infty)$, which is equivalent to $\nu(0,\infty) = 0$. □

Chapter 4

Completely monotone functions with convexity on their measures

In this chapter, we consider completely monotone functions with various convexity properties on their measures. Different characterizations are given and the connections are stressed.

Note that completely monotone function could not have harmonically concave measure unless it is degenerated to constant. In fact, as mentioned in the paragraph below Definition 2.4.3 and in the introduction of Section 3.5, Radon measures on $[0, \infty)$ have harmonically concave measure if and only if it vanishes on $(0, \infty)$.

Suppose $f(x)$ is completely monotone with measure μ. Define a non-negative sequence $A_n(x)$ as

$$A_n(x) := \frac{(-1)^n}{n!} f^{(n)}(x)$$

for $x > 0$ and $n \geq 1$. Also define function on $(0, \infty)$ by

$$M_\beta(x) := \beta(\beta-1)\frac{f(x)}{x} - 2(\beta-1)f'(x) + xf''(x) - \beta(\beta-1)\frac{\mu(\{0\})}{x}.$$

As a summary, our main results are listed in Table 4.1 and Table 4.2.

4.1 Characterizations

In this section, we characterize completely monotone functions with various convexity properties. The results correspond to the ones numbered (a) in Table 3.3.

4.1.1 Harmonically convex measure

Theorem 4.1.1 (Harmonically convex measure) *Suppose $f(x)$ is completely monotone with measure μ. Then, measure μ is harmonically convex if and only if $f(x) - xf'(x) \in \mathcal{CM}$.*

Proof Let $F(t) = \mu[0, t]$ be the cumulative distribution function of μ. Without loss of generality, assume $a := F(0) = \mu(\{0\}) = 0$. Otherwise, consider the shifted function $f(x) - a \in \mathcal{CM}$.

Table 4.1: Characterization for $f(x) \in \mathcal{CM}$ with convexity properties on its measure μ

Property on μ	No.	Characterization	Reference
harmonically convex	(a)	$f(x) - xf'(x) \in \mathcal{CM}$	Thm 4.1.1
	(b)	$\lambda f(x) - x(f(x+\lambda) - f(x)) \in \mathcal{CM}, \quad \forall \lambda > 0$	Thm 4.2.1
	(c)	$(n-1)A_n(x) \leq (n+1)xA_{n+1}(x), \quad \forall n \geq 1$	Thm 4.3.1
concave	(a)	$f(x) + xf'(x) \in \mathcal{CM}$	Thm 4.1.3
	(b)	$\lambda f(x) + x(f(x+\lambda) - f(x)) \in \mathcal{CM}, \quad \forall \lambda > 0$	Thm 4.2.2
	(c)	$A_n(x) \geq xA_{n+1}(x), \quad \forall n \geq 1$	Thm 4.3.2
convex	(a)	$x(f(x) - \mu(\{0\})) \in \mathcal{CM}$	Thm 4.1.5
	(b)	N/A	N/A
	(c)	$A_n(x) \leq xA_{n+1}(x), \quad \forall n \geq 1$	Thm 4.3.3
harmonically concave	N/A	$f(x) = a$	N/A

For sufficiency, suppose $f(x) - xf'(x)$ is completely monotone. We show $F(t)$ is harmonically convex, which is equivalent to show $tF(t)$ is convex. Define the functions

$$G_n(t) := \int_{(0,t]} L_n(f(x); u)\, du,$$

where the operator $L_n(f(x); u)$ is defined by (2.11). By Theorem 2.1.4, we know $\lim_{n\to\infty} G_n(t) = F(t)$ at every point of continuity of $F(t)$. By Lemma A.1.12, it suffices to show $tG_n(t)$ is convex on $(0, \infty)$ for $n \geq 1$. Indeed, observing for all $n \geq 1$,

$$G_n'(t) := \frac{(-1)^n}{n!} x^{n+1} f^{(n)}(x)\Big|_{x=n/t},$$

and

$$\begin{aligned} G_n''(t) &= \frac{(-1)^n}{n!}\Big((n+1)x^n f^{(n)}(x) + x^{n+1} f^{(n+1)}(x)\Big)\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) \\ &= \frac{(-1)^{n+1}}{n!\cdot n}\Big((n+1)x^{n+2} f^{(n)}(x) + x^{n+3} f^{(n+1)}(x)\Big)\Big|_{x=n/t}. \end{aligned}$$

Thus by $(tG_n(t))'' = 2G_n'(t) + tG_n''(t)$, we obtain

$$(tG_n(t))'' = \frac{(-1)^n}{n!} 2x^{n+1} f^{(n)}(x) + \frac{(-1)^{n+1}}{n!\cdot x}\Big((n+1)x^{n+2} f^{(n)}(x) + x^{n+3} f^{(n+1)}(x)\Big)\Big|_{x=n/t}$$

Table 4.2: Characterization for $f(x) \in \mathcal{CM}$ with β-convexity on its measure μ

Property on μ	Characterization	Reference
β-convex	$M_\beta(x) \in \mathcal{CM}$	Thm 4.4.1 part a)
β-concave	$-M_\beta(x) \in \mathcal{CM}$	Thm 4.4.1 part b)

$$= \frac{(-1)^n}{n!} x^{n+1}\Big((1-n)f^{(n)}(x) - xf^{(n+1)}(x)\Big)\Big|_{x=n/t}$$
$$= \frac{(-1)^n}{n!} x^{n+1}\Big(f(x) - xf'(x)\Big)^{(n)}\Big|_{x=n/t}.$$

The last equation follows from Corollary A.1.1. As $f(x) - xf'(x) \in \mathcal{CM}$, we get $(tG_n(t))'' \geq 0$ for all $n \geq 1$. Therefore μ is harmonically convex.

Conversely, suppose $F(t)$ is harmonically convex on $(0, \infty)$. So $(tF(t))'$ is non-decreasing. Note that by Lemma A.2.7 and Lemma A.2.9,

$$\begin{aligned}
f(x) - xf'(x) &= x\int_{(0,\infty)} e^{-xt}F(t)\,dt + x\int_{(0,\infty)} e^{-xt}t\,dF(t) = x\int_{(0,\infty)} e^{-xt}\,d(tF(t)) \\
&= x\int_{(0,\infty)} e^{-xt}(tF(t))'\,dt = -e^{-xt}(tF(t))'\Big|_{t=0}^{\infty} + \int_{(0,\infty)} e^{-xt}\,d(tF(t))' \\
&= \lim_{t\to 0+}(tF(t))' + \int_{[0,\infty)} e^{-xt}\,d(tF(t))'.
\end{aligned}$$

The first equation follows from (2.6) and the last equation follows from Lemma 2.5.5. As $tF(t)$ is non-decreasing and convex, we know $(tF(t))'$ is non-decreasing and $\lim_{t\to 0+}(tF(t))' \geq 0$ is finite. Thus $f(x) - xf'(x)$ is completely monotone. The proof is complete. □

Notice that we have already represented completely monotone functions with harmonically convex measures in Proposition 3.2.1. There could be an alternative proof.

Alternative proof for Theorem 4.1.1 Suppose measure μ has mass a at $\{0\}$ in the proof of both directions.

If $f(x)$ has harmonically convex measure, by Proposition 3.2.1, we know $f(x)$ has representation (3.13). Thus, using [31, Theorem A.5.2] for differentiating under the integral, we have

$$f(x) - xf'(x) = a - \int_{(0,\infty)} \frac{x^2}{s^2}k'\Big(\frac{x}{s}\Big)\tau(ds) = a + \int_{(0,\infty)} e^{-x/s}\,\tau(ds),$$

where $a \geq 0$ and τ satisfies integrability condition (3.6), which ensures the convergence of the last integral. Thus $f(x) - xf'(x)$ defines a completely monotone function.

Conversely, if $f(x) - xf'(x)$ is completely monotone, we have

$$f(x) - xf'(x) = a' + \int_{(0,\infty)} e^{-xt}\,\eta(dt),$$

for $a' \geq 0$ and Bernstein measure η on $(0,\infty)$. Therefore, we obtain

$$f''(x) = \frac{(f(x) - xf'(x))'}{-x} = \frac{1}{x}\int_{(0,\infty)} e^{-xt} t\,\eta(dt).$$

Integrating both sides twice using Fubini's theorem,

$$\begin{aligned} f(x) - a &= \int_{(x,\infty)}\int_{(v,\infty)} f''(u)\,du\,dv = \int_{(x,\infty)}\int_{(v,\infty)}\int_{(0,\infty)} \frac{e^{-ut}t}{u}\,\eta(dt)\,du\,dv \\ &= \int_{(0,\infty)}\Big(\int_{(x,\infty)}\int_{(x,u)} \frac{e^{-ut}}{u}\,dv\,du\Big)t\,\eta(dt) = \int_{(0,\infty)}\Big(\int_{(x,\infty)} e^{-ut}\,du - x\int_{(x,\infty)}\frac{e^{-ut}}{u}\,du\Big)t\,\eta(dt) \\ &= \int_{(0,\infty)}\Big(\frac{e^{-xt}}{t} - x\mathrm{E}_1(xt)\Big)t\,\eta(dt) = \int_{(0,\infty)} xtk(xt)\,\eta(dt). \end{aligned}$$

Here $k(x)$ is defined in (3.9). Let $F_\eta(x) = \eta(0,x]$ and note that $-F_\eta(1/s)$ is a non-decreasing function. Define measure τ on $(0,\infty)$ by $\tau(ds) := d(-F_\eta(1/s))$. Thus, change variable by $t = 1/s$, we have

$$f(x) = a + \int_{(0,\infty)} \frac{x}{s}k\Big(\frac{x}{s}\Big)\tau(ds).$$

By Proposition 3.2.1, to show $f(x)$ has harmonically convex measure, it suffices to show τ satisfies (3.6). Indeed,

$$\tau(1,\infty) = \int_{(1,\infty)} d(-F_\eta(1/s)) = F_\eta(1) - \lim_{s\to\infty} F_\eta(1/s) = F_\eta(1) < \infty.$$

And

$$\int_{(0,1]}\int_{(0,1)} e^{-x/ts}\,dt\,\tau(ds) = \int_{(0,1]}\int_{(0,1)} e^{-x/ts}\,dt\,d(-F_\eta(1/s)) = \int_{(1,\infty)}\int_{[1,\infty)} \frac{e^{-xuv}}{v^2}\,dF_\eta(u)\,dv,$$

where in the last equality, we changed the variables by $v = 1/t$ and $u = 1/s$, and also interchange the order of integration using Fubini's theorem. Therefore,

$$\int_{(1,\infty)}\int_{[1,\infty)} \frac{e^{-xuv}}{v^2}\,dF_\eta(u)\,dv \leq \Big(\int_{(1,\infty)} \frac{1}{v^2}\,dv\Big)\Big(\int_{(0,\infty)} e^{-xu}\,\eta(du)\Big) = f(x) - xf'(x) - a' < \infty.$$

Measure τ satisfies (3.6). The proof is completed. □

Example 4.1.2 *Function*

$$k_t(x) = x\int_{(x,\infty)} \frac{e^{-st}}{s^2}\,ds$$

is completely monotone with harmonically convex measure for all $t > 0$.

Proof This function is non-negative. Its first derivative is

$$k_t'(x) = \int_{(x,\infty)} \frac{e^{-st}}{s^2}\,ds - \frac{e^{-xt}}{x} \leq e^{-xt}\int_{(x,\infty)} \frac{1}{s^2}\,ds - \frac{e^{-xt}}{x} = 0,$$

and its second derivative is

$$k_t''(x) = -\frac{e^{-xt}}{x^2} - \frac{-xte^{-xt} - e^{-xt}}{x^2} = \frac{te^{-xt}}{x} \in \mathcal{CM}.$$

Thus, $k_t(x)$ is completely monotone. Notice that $k_t(x) - xk_t'(x) = e^{-xt} \in \mathcal{CM}$. By Theorem 4.1.1, $k_t(x)$ has harmonically convex measure for all $t > 0$. □

It is known that the product of two completely monotone functions is still such, from where follows the next corollary.

Corollary 4.1.1 *Suppose $f(x)$ is completely monotone with measure μ. For $\alpha \in \mathbb{R}$, consider*

$$x^\alpha(f(x) - xf'(x)). \tag{4.1}$$

(a) If for some $\alpha \geq 0$, (4.1) is completely monotone, then, measure μ is harmonically convex.

(b) If measure μ is harmonically convex, then, (4.1) is completely monotone for any $\alpha \leq 0$.

It can be shown that the set of completely monotone functions with harmonically convex measures is a convex cone, and it is closed under non-negative scalar multiplication and positive scaling. That is, if $f_1(x)$ and $f_2(x)$ are completely monotone with harmonically convex measures μ_1 and μ_2, then so are

$$\lambda f_1(x) + (1-\lambda) f_2(x) \quad \text{and} \quad cf_1(\rho x),$$

for $\lambda \in [0, 1]$, $c \geq 0$, and $\rho > 0$. They follow trivially from the fact that the cumulative distribution functions for the measure of above completely monotone functions are

$$\lambda F_1(t) + (1-\lambda)F_2(t) \quad \text{and} \quad cF_1(t/\rho),$$

where $F_1(t), F_2(t)$ are the cumulative distribution function of μ_1 and μ_2 respectively. One can also use Theorem 4.1.1 to verify. In addition, Theorem 4.1.1 implies that the aforementioned set is closed under point-wise limit.

Corollary 4.1.2 *Let $\{f_n(x)\}_{n\in\mathbb{N}}$ be a sequence of completely monotone functions such that their point-wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for x on $(0, \infty)$. If $f_n(x)$ has harmonically convex measure for all $n \geq 1$, then $f(x)$ is also completely monotone with harmonically convex measure.*

Proof It is shown in Proposition 2.1.1 that $f(x) \in \mathcal{CM}$ and $f'(x) = \lim_{n\to\infty} f_n'(x)$. Notice that $f_n(x) - xf_n'(x) \in \mathcal{CM}$ for all $n \geq 1$ by Theorem 4.1.1. Therefore, for all $x \in (0, \infty)$ we obtain

$$f(x) - xf'(x) = \lim_{n\to\infty} f_n(x) - xf_n'(x).$$

As the $\mathcal{CM}$ is closed under point-wise limit, we conclude $f(x) - xf'(x) \in \mathcal{CM}$. This implies $f(x)$ has harmonically convex measure by Theorem 4.1.1. □

4.1.2 Concave measures

Theorem 4.1.3 (concave measure) *Suppose $f(x)$ is completely monotone with measure μ. Then, measure μ is concave if and only if $f(x) + xf'(x) \in \mathcal{CM}$.*

Proof Let $F(t) = \mu[0, t]$ be the cumulative distribution function of μ. Without loss of generality, assume $a := F(0) = \mu(\{0\}) = 0$. Otherwise, consider the shifted function $f(x) - a \in \mathcal{CM}$.

For sufficiency, suppose $f(x) + xf'(x)$ is completely monotone. We show $F(t)$ is concave. Define the functions

$$G_n(t) := \int_{(0,t]} L_n(f(x); u)\, du,$$

where the operator $L_n(f(x); u)$ is defined by (2.11). By Theorem 2.1.4, we know $\lim_{n\to\infty} G_n(t) = F(t)$ for every point of continuity of $F(t)$. By Lemma A.1.12, it suffice to show $G_n(t)$ is concave on $(0, \infty)$ for all $n \geq 1$. Indeed, observing for all $n \geq 1$,

$$G_n'(t) := \frac{(-1)^n}{n!} x^{n+1} f^{(n)}(x)\Big|_{x=n/t},$$

and

$$\begin{aligned} G_n''(t) &= \frac{(-1)^n}{n!}\Big((n+1)x^n f^{(n)}(x) + x^{n+1} f^{(n+1)}(x)\Big)\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) \\ &= \frac{(-1)^{n+1}}{n! \cdot n}\Big((n+1)x^{n+2} f^{(n)}(x) + x^{n+3} f^{(n+1)}(x)\Big)\Big|_{x=n/t} = \frac{(-1)^{n+1}}{n! \cdot n} x^{n+2}\Big(f(x) + xf'(x)\Big)^{(n)}\Big|_{x=n/t}. \end{aligned}$$

The last equation utilizes Lemma A.1.1. As $f(x) + xf'(x) \in \mathcal{CM}$, we get $G_n''(t) \leq 0$ for all $n \geq 1$. This shows that $G_n(t)$ is concave on $(0, \infty)$. And thus μ is concave.

Conversely, suppose now that $F(t)$ is concave on $(0, \infty)$. To show $f(x) + xf'(x) \in \mathcal{CM}$, it suffices to show $xf(x) \in \mathcal{BF}$ because $f(x) + xf'(x) = (xf(x))'$. By (2.6) and Lemma A.2.9,

$$\begin{aligned} xf(x) &= x \int_{(0,\infty)} e^{-xt}\, dF(t) = x \int_{(0,\infty)} e^{-xt} F'(t)\, dt \\ &= (1 - e^{-xt})F'(t)\Big|_{t=0+}^{\infty} + \int_{(0,\infty)} (1 - e^{-xt})\, d(-F'(t)). \end{aligned}$$

As $F(t)$ is concave, $F'(t)$ is non-negative and non-increasing, thus $\lim_{t\to\infty} F'(t) = c \geq 0$ is finite. The second equation implies $F'(t)$ is integrable at zero. By Lemma A.2.1, we obtain $\lim_{t\to 0} tF'(t) = 0$. As a result, $\lim_{t\to 0+}(1 - e^{-xt})F'(t) = 0$, and

$$xf(x) = c + \int_{(0,\infty)} (1 - e^{-xt})\, d(-F'(t)).$$

The finiteness of left side ensures the integrability of the last integral. One can also verify this without much difficulty. Therefore, $xf(x)$ is Bernstein and the proof is complete. $\square$

Notice that we have already represented completely monotone functions with concave measures in Proposition 3.24. There could be an alternative proof. In fact, we could prove the next corollary

Corollary 4.1.3 *Suppose $f(x)$ is completely monotone with measure μ. The following conditions are equivalent:*

(a) μ is concave;

(b) $xf(x) \in \mathcal{BF}$;

(c) $f(x) + xf'(x) \in \mathcal{CM}$.

Proof The equivalence between (b) and (c) is trivial given $f(x) \in \mathcal{CM}$. We only need to show (a) and (b) are equivalent. If $f(x)$ has concave measure, by Proposition 3.3.1, we have

$$xf(x) = ax + b + \int_{(0,\infty)} (1 - e^{-xs})\frac{1}{s}\,\tau(ds),$$

where $a, b \geq 0$ and τ satisfies (3.17). This shows that $xf(x)$ is Bernstein with Lévy measure $\tau(ds)/s$. Conversely, if $xf(x)$ is Bernstein with triplet (a', b', η), we know

$$f(x) = \frac{a'}{x} + b' + \int_{(0,\infty)} \frac{1 - e^{-xs}}{x}\,\eta(ds) = \frac{a'}{x} + b' + \int_{(0,\infty)} r(xs)\, s\,\eta(ds),$$

where $a', b' \geq 0$, $r(x)$ is defined in (3.21) and η satisfies (2.17). Define measure τ on $(0, \infty)$ by $\tau(ds) = s\eta(ds)$. The integrability condition on η implies that τ satisfies (3.17). By Proposition 3.3.1, completely monotone function $f(x)$ has concave measure. This shows the equivalence between (a) and (b). □

Example 4.1.4 *Function*

$$r_t(x) = \frac{(1 - e^{-xt})}{xt}$$

is completely monotone with concave measure for all $t > 0$.

Proof It is trivial that $r_t(x)$ is completely monotone because $1 - e^{-xt} \in \mathcal{BF}$. Notice

$$k_t(x) + xk_t'(x) = \frac{(1 - e^{-xt})}{xt} + \frac{e^{-xt}xt - (1 - e^{-xt})}{xt} = e^{-xt} \in \mathcal{CM}.$$

By Theorem 4.1.3, we know $k_t(x)$ has concave measure for $t > 0$. □

The next corollary follows from the fact that the set of completely monotone functions is closed under product. This is analogous to Corollary 4.1.1

Corollary 4.1.4 *Suppose $f(x)$ is completely monotone with measure μ. For $\alpha \in \mathbb{R}$, consider*

$$x^{\alpha}(f(x) + xf'(x)). \tag{4.2}$$

(a) If for some $\alpha \geq 0$, (4.2) is completely monotone, then the measure μ is concave.

(b) If the measure μ is concave, then (4.2) is completely monotone for any $\alpha \leq 0$.

The set of completely monotone functions with concave measures is a convex cone, and it is closed under non-negative scalar multiplication and positive scaling. That is, if $f_1(x)$ and $f_2(x)$ are completely monotone with concave measures μ_1 and μ_2, then so are

$$\lambda f_1(x) + (1-\lambda) f_2(x) \quad \text{and} \quad c f_1(\rho x),$$

for $\lambda \in [0,1]$, $c \geq 0$, and $\rho > 0$. They follow trivially from the fact that the cumulative distribution functions for the measure of above completely monotone functions are

$$\lambda F_1(t) + (1-\lambda) F_2(t) \quad \text{and} \quad c F_1(t/\rho),$$

where $F_1(t), F_2(t)$ are the cumulative distribution function of μ_1 and μ_2 respectively. One can also use Theorem 4.1.3 to verify. In addition, Theorem 4.1.3 implies that the aforementioned set is closed under point-wise limit.

Corollary 4.1.5 *Let $\{f_n(x)\}_{n\in\mathbb{N}}$ be a sequence of completely monotone functions such that their point-wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for x on $(0,\infty)$. If $f_n(x)$ has concave measure for all $n \geq 1$, then $f(x)$ is also completely monotone with concave measure.*

Proof It is shown in Proposition 2.1.1 that $f(x) \in \mathcal{CM}$ and $f'(x) = \lim_{n\to\infty} f_n'(x)$. Notice that $f_n(x) + x f_n'(x) \in \mathcal{CM}$ for all $n \geq 1$ by Theorem 4.1.3. Therefore, for all $x \in (0,\infty)$ we obtain

$$f(x) + x f'(x) = \lim_{n\to\infty} f_n(x) + x f_n'(x).$$

As the $\mathcal{CM}$ is closed under point-wise limit, we conclude $f(x) + x f'(x) \in \mathcal{CM}$. This implies $f(x)$ has concave measure by Theorem 4.1.3. □

4.1.3 Convex measures

Next Theorem extends Proposition 2.7.13 from [3] for completely monotone functions with convex measure.

Theorem 4.1.5 (Convex measure) *Suppose $f(x)$ is completely monotone with measure μ having mass a at $\{0\}$. Then, measure μ is convex if and only if $x(f(x) - a) \in \mathcal{CM}$.*

Proof Let $F(t) = \mu[0,t]$ be the cumulative distribution function of μ. Without loss of generality, assume $a := F(0) = \mu(\{0\}) = 0$. Otherwise, consider the shifted function $f(x) - a \in \mathcal{CM}$.

For sufficiency, suppose $x f(x)$ is completely monotone. We want to show $F(t)$ is convex. Define the functions

$$G_n(t) := \int_{(0,t]} L_n(f(x); u)\, du,$$

where the operator $L_n(f(x); u)$ is defined by (2.11). By Theorem 2.1.4, we know $\lim_{n\to\infty} G_n(t) = F(t)$ at every point of continuity of $F(t)$. By Lemma A.1.12, it suffices to show $G_n(t)$ is convex on $(0,\infty)$ for $n \geq 1$. Indeed, observing

$$G_n'(t) := \frac{(-1)^n}{n!} x^{n+1} f^{(n)}(x)\Big|_{x=n/t},$$

and

$$
\begin{aligned}
G_n''(t) &= \frac{(-1)^n}{n!}\Big((n+1)x^n f^{(n)}(x) + x^{n+1} f^{(n+1)}(x)\Big)\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) \\
&= \frac{(-1)^{n+1}}{n!\cdot n}\Big((n+1)x^{n+2} f^{(n)}(x) + x^{n+3} f^{(n+1)}(x)\Big)\Big|_{x=n/t} = \frac{(-1)^{n+1}}{n!\cdot n} x^{n+2}\Big(xf(x)\Big)^{(n+1)}\Big|_{x=n/t}.
\end{aligned}
$$

The last equation implements Lemma A.1.1. Thus, using $xf(x) - ax \in \mathcal{CM}$, we get $G_n''(t) \geq 0$ for all $n \geq 1$. This shows that $G_n(t)$ is convex on $(0,\infty)$ and thus μ is convex.

Conversely, suppose now that $F(t)$ is convex on $(0,\infty)$. By (2.6) and Lemma A.2.9, we have

$$
\begin{aligned}
x(f(x)-a) &= x\int_{(0,\infty)} e^{-xt}\,\mu(dt) = x\int_{(0,\infty)} e^{-xt}\,dF(t) = x\int_{(0,\infty)} e^{-xt}F'(t)\,dt \\
&= -\int_{(0,\infty)} F'_+(t)\,d(e^{-xt}) = -e^{-xt}F'(t)\Big|_{t=0+}^{\infty} + \int_{(0,\infty)} e^{-xt}\,dF'(t).
\end{aligned}
$$

The integrability implies $\lim_{t\to\infty} e^{-xt}F'(t) = 0$. As $F(t)$ is non-decreasing and convex, $F'(t)$ is non-negative and non-decreasing, which indicates $\lim_{t\to 0} F'(t) = c \geq 0$ is finite. Therefore,

$$
x(f(x)-a) = c + \int_{(0,\infty)} e^{-xt}\,dF'(t).
$$

The function $F'(t)$ is non-decreasing and right-continuous, hence it defines a measure on $(0,\infty)$. We conclude that $x(f(x)-a)$ is completely monotone. The proof is complete. □

Notice that we have already represented completely monotone functions with convex measures in Proposition 3.4.1. There could be an alternative proof.

Alternative proof for Theorem 4.1.5 If $f(x)$ has convex measure, by Proposition 3.4.1, we have

$$
f(x) = a + \frac{b}{x} + \int_{(0,\infty)} \frac{e^{-xs}}{xs}\,\tau(ds),
$$

where $a, b \geq 0$ and τ satisfies (3.30). It is clear that $\lim_{x\to\infty} f(x) = a$, which is the mass at $\{0\}$ for the Bernstein measure of $f(x)$. Therefore,

$$
x(f(x)-a) = b + \int_{(0,\infty)} e^{-xs}\frac{1}{s}\,\tau(ds),
$$

which defines a completely monotone function.

Conversely, if $x(f(x)-a)$ is completely monotone with measure η, then

$$
f(x) = a + \frac{\eta\{0\}}{x} + \int_{(0,\infty)} \frac{e^{-xs}}{xs} s\,\eta(ds),
$$

where η is a Bernstein measure. Define measure τ on $(0,\infty)$ by $\tau(ds) = s\eta(ds)$. The integrability condition on η implies that τ satisfies (3.30). By Proposition 3.4.1, completely monotone function $f(x)$ has convex measure. □

In fact, by Lemma 2.4.7, we can easily see that if $f(x) \in \mathcal{CM}$ and $xf(x) \in \mathcal{CM}$, then $f(x)$, the product of $xf(x)$ and $1/x$, has convex measure. Because $1/x$ is completely monotone and its measure has no mass at $\{0\}$.

Theorem 4.1.5 provides us an easier way to prove Theorem 4.1.1 as below.

Another proof for Theorem 4.1.1 Let $F(x) = \mu[0, x]$ be cumulative distribution function of the measure μ. Extend this function by defining $F(x) = 0$ for $x < 0$. This extension is right continuous and with bounded variation on compact sets. By definition, the harmonic convexity of μ means that $F(1/t)$ is convex on $(0, \infty)$ and as pointed out, that is equivalent to the convexity of $tF(t)$. Integrate by parts by Theorem A.2.6, we have

$$\int_{[0,\infty)} e^{-xt}\, d(tF(t)) = \lim_{b\to\infty}\left(e^{-xt}tF(t)\Big|_{t=0^-}^{b} - \int_{[0,b]} e^{-xt}(-x)tF(t)\, dt\right),$$
$$\int_{[0,\infty)} e^{-xt}t\, dF(t) = \lim_{b\to\infty}\left(e^{-xt}tF(t)\Big|_{t=0^-}^{b} - \int_{[0,b]} e^{-xt}F(t)(1-xt)\, dt\right).$$

Subtracting the second equation, both sides of which are finite, from the first gives

$$\begin{aligned} x\int_{[0,\infty)} e^{-xt}\, d(tF(t)) &= x\int_{[0,\infty)} e^{-xt}F(t)\, dt + x\int_{[0,\infty)} e^{-xt}t\, dF(t) \\ &= \left(-e^{-xt}F(t)\Big|_{t=0^-}^{\infty} + \int_{[0,\infty)} e^{-xt}\, dF(t)\right) - x\int_{[0,\infty)} e^{-xt}(-t)\, dF(t) \\ &= f(x) - xf'(x). \end{aligned}$$

For the second equality, we used the integration by parts Theorem A.2.6 again. Noticing the measure defined by $d(tF(t))$ has no mass at $\{0\}$, applying Theorem 4.1.5, we know μ is harmonically convex if and only if $f(x) - xf'(x) \in \mathcal{CM}$. □

Next corollary follows from Theorem 4.1.5 and the fact that the set of completely monotone functions is closed under product.

Corollary 4.1.6 *Suppose $f(x)$ is completely monotone with measure μ having mass a at $\{0\}$.*

(a) If for some $\alpha \geq 1$, the function $x^\alpha(f(x) - a) \in \mathcal{CM}$, then the measure μ is convex.

(b) If the measure μ is convex, then $x^\alpha(f(x) - a) \in \mathcal{CM}$, for any $\alpha \leq 1$.

It can be shown that the set of completely monotone functions with convex measures is a convex cone, and it is closed under non-negative scalar multiplication and positive scaling. If $f_1(x)$ and $f_2(x)$ are completely monotone with convex measures μ_1 and μ_2, then so are

$$\lambda f_1(x) + (1-\lambda)f_2(x) \quad \text{and} \quad cf_1(\rho x),$$

for $\lambda \in [0, 1]$, $c \geq 0$, and $\rho > 0$. They follow trivially from the fact that the cumulative distribution functions for the measure of above completely monotone functions are

$$\lambda F_1(t) + (1-\lambda)F_2(t) \quad \text{and} \quad cF_1(t/\rho),$$

where $F_1(t), F_2(t)$ are the cumulative distribution function of μ_1 and μ_2 respectively. One can also use Theorem 4.1.5 to verify. In addition, Theorem 4.1.5 implies this set is closed under point-wise limit.

Corollary 4.1.7 *Let $\{f_n(x)\}_{n\in\mathbb{N}}$ be a sequence of completely monotone functions such that their point-wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for x on $(0,\infty)$. If $f_n(x)$ has convex measure for all $n \geq 1$, then $f(x)$ is also completely monotone with convex measure.*

Proof Denote $f_n(x)$ is associated with measure μ_n having mass a_n at $\{0\}$. It is shown in Proposition 2.1.1 that $f(x) \in \mathcal{CM}$. Denote the measure for $f(x)$ has mass a at $\{0\}$. by Theorem 4.1.5, to see $f(x)$ has convex measure, it suffice to show

$$x(f(x) - a) \in \mathcal{CM}.$$

Observe $x(f(x) - a) \geq 0$ and its first derivative is $f(x) - a + xf'(x)$. By Lemma 2.1.3, we have

$$\lim_{x\to\infty} f(x) - a + xf'(x) = 0.$$

Therefore, it suffice to show its second derivative $2f'(x) + xf''(x)$ is completely monotone. Notice $x(f_n(x) - a_n) \in \mathcal{CM}$ for all $n \geq 1$ by Theorem 4.1.5. So is $2f_n'(x) + xf_n''(x)$. By Proposition 2.1.1, we observe

$$2f'(x) + xf''(x) = \lim_{n\to\infty} 2f_n'(x) + xf_n''(x).$$

As the $\mathcal{CM}$ is closed under point-wise limit, we conclude $2f'(x)+xf''(x)$ is completely monotone. The proof is complete. □

4.2 Derivative free characterizations

In this section, we try to remove the derivatives from the characterizations for completely monotone functions with harmonically convex measure and concave measure. The results correspond to the ones listed under (b) in Table 4.1.

4.2.1 Harmonically convex measures

Theorem 4.2.1 *Suppose $f(x)$ is completely monotone with measure μ. Then μ is harmonically convex, if and only if $\lambda f(x) - x(f(x+\lambda) - f(x)) \in \mathcal{CM}$ for all $\lambda > 0$.*

Proof We show sufficiency first. If $\lambda f(x) - x(f(x+\lambda) - f(x))$ is completely monotone for all $\lambda > 0$, so is

$$f(x) - \frac{x(f(x+\lambda) - f(x))}{\lambda}.$$

Letting λ approaching 0, we know that $f(x) - xf'(x)$ is completely monotone, which implies μ is harmonically convex by Theorem 4.1.1.

For necessity, assume $f(x)$ has harmonically convex measure. We want to show $\lambda f(x) - x(f(x+\lambda) - f(x))$ is completely monotone for all $\lambda > 0$. Denote

$$\Lambda(x) := \lambda f(x) - x(f(x+\lambda) - f(x)) = (x+\lambda)f(x) - xf(x+\lambda).$$

As $f(x)$ is decreasing, we know $K(x) \geq 0$. Thus, it suffices to show $-\Lambda'(x) \in \mathcal{CM}$.

$$\begin{aligned}
-\Lambda'(x) &= -f(x) - (x+\lambda)f'(x) + f(x+\lambda) + xf'(x+\lambda) \\
&= (f(x) - xf'(x)) - (f(x+\lambda) - (x+\lambda)f'(x+\lambda)) \\
&\quad - 2f(x) - \lambda f'(x) + 2f(x+\lambda) - \lambda f'(x+\lambda).
\end{aligned}$$

Because $f(x)$ has harmonically convex measure, $f(x) - xf'(x)$ is completely monotone, indicating $(f(x) - xf'(x)) - (f(x+\lambda) - (x+\lambda)f'(x+\lambda))$ is also completely monotone. It suffices to show

$$-2f(x) - \lambda f'(x) + 2f(x+\lambda) - \lambda f'(x+\lambda) \in \mathcal{CM}.$$

By Bernstein representation (2.1), we obtain

$$-2f(x) - \lambda f'(x) + 2f(x+\lambda) - \lambda f'(x+\lambda) = \int_{[0,\infty)} e^{-xt}(-2 + \lambda t + 2e^{-\lambda t} + \lambda t e^{-\lambda t})\,\mu(dt).$$

By Lemma A.1.4 d), we know $-2 + \lambda t + 2e^{-\lambda t} + \lambda t e^{-\lambda t} \geq 0$ for all $\lambda \geq 0$. Thus, above integral defines a completely monotone function, and the proof is complete. □

4.2.2 Concave measures

Lemma 4.2.2 *Suppose $f(x)$ is completely monotone with measure μ. Then, measure μ has concave measure if and only if $\lambda f(x) + x(f(x+\lambda) - f(x)) \in \mathcal{CM}$ for all $\lambda > 0$.*

Proof We show sufficiency first. If $\lambda f(x) + x(f(x+\lambda) - f(x))$ is completely monotone for all $\lambda > 0$, so is

$$f(x) + \frac{x(f(x+\lambda) - f(x))}{\lambda}.$$

Letting λ approach 0, we know that $f(x) + xf'(x)$ is completely monotone. By Theorem 4.1.3, $f(x)$ has concave measure.

For necessity, assume $f(x)$ has concave measure. Denote

$$\Lambda(x) := xf(x+\lambda) - (x-\lambda)f(x) = xf(x+\lambda) + (\lambda - x)f(x).$$

We show $\Lambda(x)$ is completely monotone for all $\lambda > 0$. As $f(x)$ is convex,

$$\begin{aligned}
\Lambda(x) &= xf(x+\lambda) - (x-\lambda)f(x) \geq x(f(x) + \lambda f'(x)) - (x-\lambda)f(x) \\
&= \lambda(xf'(x) + f(x)) \geq 0,
\end{aligned}$$

where $f(x)+xf'(x)$ is completely monotone, by Theorem 4.1.3. It suffices to show $\Lambda'(x) \in \mathcal{CM}$.

$$\begin{aligned}
-\Lambda'(x) &= -f(x+\lambda) - xf'(x+\lambda) + f(x) + (x-\lambda)f'(x) \\
&= (f(x) + xf'(x) - f(x+\lambda) - (x+\lambda)f'(x+\lambda)) + \lambda(f'(x+\lambda) - f'(x)).
\end{aligned}$$

As $f(x) + xf'(x)$ and $-f'(x)$ are completely monotone, we know

$$f(x) + xf'(x) - f(x+\lambda) - (x+\lambda)f'(x+\lambda) \in \mathcal{CM}, \quad \text{and} \quad f'(x+\lambda) - f'(x) \in \mathcal{CM}.$$

Thus, we obtain $-\Lambda'(x) \in \mathcal{CM}$ for all $\lambda > 0$, and so is $\Lambda(x)$. This closes the proof. □

4.3 Sequential characterizations

In this section, we characterize same objectives in terms of sequences. This idea finds its root in the definition of Hirsch Class. Suppose $f(x)$ is completely monotone. Define a non-negative sequence $A_n(x)$ as

$$A_n(x) = \frac{(-1)^n}{n!} f^{(n)}(x)$$

for every $x > 0$ and $n \geq 0$. If $\{A_n(x)\}_{n\geq 0}$ is log-convex, then $f(x)$ is in Hirsch class, see [68, Definition 11.22]. Analogously, The shape of the measures can also be characterized by $\{A_n(x)\}_{n\geq 1}$. The results corresponds to the items listed by (c) in Table 4.1.

4.3.1 Harmonically convex measures

Theorem 4.3.1 *Suppose $f(x)$ is completely monotone with measure μ. Then, measure μ is harmonically convex, if and only if the sequence $\{A_n(x)\}_{n\geq 1}$ satisfies*

$$\frac{n-1}{n+1} A_n(x) \leq x A_{n+1}(x) \tag{4.3}$$

for all $x > 0$ and $n \geq 1$.

Proof Proof of sufficiency. We need to show $F(t)$ is harmonically convex given (4.3). This proof utilizes Inverse Laplace transformation formula in Theorem 2.1.4. Define

$$G_n(t) := \int_{(0,t]} L_n(f(x); u)\, du.$$

where the operator L_n is defined in (2.11). Therefore, we know $\lim_{n\to\infty} G_n(t) = F(t) - F(0)$ at every point of continuity of $F(t)$. To show $F(t)$ is harmonically convex, it suffices to show $tF(t)$ is convex. By lemma A.1.12, it suffices to show $tG_n(t)$ is convex for all $n \geq 1$. Notice that $G_n(t)$ is infinitely differentiable. Observe

$$G_n'(t) = \frac{(-1)^n}{n!} x^{n+1} f^{(n)}(x)\Big|_{x=n/t} = x^{n+1} A_n(x)\Big|_{x=n/t},$$

and

$$\begin{aligned}
G_n''(t) &= \frac{(-1)^n}{n!}\Big((n+1)x^n f^{(n)}(x) + x^{n+1} f^{(n+1)}(x)\Big)\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) \\
&= \frac{(-1)^{n+1}}{n!\cdot n}\Big((n+1)x^{n+2} f^{(n)}(x) + x^{n+3} f^{(n+1)}(x)\Big)\Big|_{x=n/t} \\
&= -\frac{n+1}{n} x^{n+2} A_n(x) + \frac{(n+1)}{n} x^{n+3} A_{n+1}(x)\Big|_{x=n/t}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
(tG_n(t))'' &= 2G_n'(t) + tG_n''(t) \\
&= 2x^{n+1} A_n(x) - \frac{n}{x}\frac{n+1}{n} x^{n+2} A_n(x) + \frac{n}{x}\frac{(n+1)}{n} x^{n+3} A_{n+1}(x)\Big|_{x=n/t}
\end{aligned}$$

$$\geq (1-n)x^{n+1}A_n(x) + (n-1)x^{n+1}A_n(x) = 0.$$

We used (4.3) in the last inequality. So $tG_n(t)$ is convex for all $n \geq 1$.

Proof of necessity. Suppose measure μ is harmonically convex. We need to show inequality (4.3) holds. The case $n = 1$ is trivial as $A_n(x) \geq 0$. Consider $n \geq 2$. As μ is harmonically convex, function $F(1/t)$ is convex, which implies its left derivative $(F(1/t))'_- = -F'(1/t)/t^2$ is non-decreasing, indicating $F'(t)t^2$ is non-decreasing. Therefore, by Lemma A.2.9

$$\begin{aligned}
A_n(x) &= \frac{(-1)^n}{n!} f^{(n)}(x) = \frac{1}{n!}\int_{(0,\infty)} e^{-xt}t^n\,dF(t) = \int_{(0,\infty)} \frac{t^n}{n!}e^{-xt}F'(t)\,dt \\
&= \frac{t^{n+1}}{(n+1)!}e^{-xt}F'(t)\Big|_{t=0+}^{\infty} - \int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}\,d(e^{-xt}F'(t)).
\end{aligned}$$

Lemma 2.5.5 implies for any $n \geq 1$,

$$\lim_{t\to\infty} t^{n+1}e^{-xt}F'(t) = 0,$$

and

$$\lim_{t\to 0} t^{n+1}e^{-xt}F'(t) = \lim_{t\to 0} t^{n-1}e^{-xt}F'(t)t^2 = 0.$$

Therefore by lemma A.2.7 and noticing the finiteness of $A_n(x)$ for any $n \geq 1$,

$$\begin{aligned}
A_n(x) &= -\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}\,d(e^{-xt}t^{-2}t^2F'(t)) \\
&= x\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}e^{-xt}F'(t)\,dt + 2\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}e^{-xt}t^{-3}t^2F'(t)\,dt \\
&\quad - \int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}e^{-xt}t^{-2}\,d(t^2F'(t)) \\
&\leq x\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}e^{-xt}\,dF(t) + \frac{2}{n+1}\int_{(0,\infty)} \frac{t^n}{n!}e^{-xt}\,dF(t) \\
&= xA_{n+1}(x) + \frac{2}{n+1}A_n(x),
\end{aligned}$$

which simplifies into (4.3). This closes the proof. □

In fact, the condition (4.3) is equivalent to $f(x)-xf'(x) \in \mathcal{CM}$. By Lemma A.1.1, we know $f(x)-xf'(x) \in \mathcal{CM}$ is equivalent to

$$(-1)^n\Big(-xf^{(n+1)}(x) - (n-1)f^{(n)}(x)\Big) \geq 0,$$

for all $n \geq 1$. Rewriting the above inequality in terms of $A_n(x)$ reveals (4.3).

4.3.2 Concave measures

Theorem 4.3.2 *Suppose $f(x)$ is completely monotone with measure μ. Then, measure μ is concave, if and only if the sequence $\{A_n(x)\}_{n\geq 1}$ satisfies*

$$A_n(x) \geq xA_{n+1}(x) \tag{4.4}$$

for all $x > 0$ and $n \geq 1$.

Proof Proof of sufficiency. Given (4.4) for all $n \geq 1$, we need to show measure μ is concave. This proof utilizes the Inverse Laplace transformation formula. Define

$$G_n(t) := \int_{(0,t]} L_n(f(x); u)\, du$$

where the operator L_n is defined in (2.11). By Theorem 2.1.4, we know $\lim_{n\to\infty} G_n(t) = F(t) - F(0)$ at every point of continuity of $F(t)$. By lemma A.1.12, to show $F(t)$ is concave, it suffices to show $G_n(t)$ is concave for all $n \geq 1$. Notice that $G_n(t)$ is infinitely differentiable, and

$$G_n'(t) = L_n(f(x); t) = \frac{(-1)^n}{n!} x^{n+1} f^{(n)}(x)\Big|_{x=n/t}.$$

Therefore,

$$\begin{aligned}
G_n''(t) &= \frac{(-1)^n}{n!}\Big((n+1)x^n f^{(n)}(x) + x^{n+1} f^{(n+1)}(x)\Big)\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) \\
&= \frac{(-1)^{n+1}}{n! \cdot n}\Big((n+1)x^{n+2} f^{(n)}(x) + x^{n+3} f^{(n+1)}(x)\Big)\Big|_{x=n/t} \\
&= -\frac{n+1}{n} x^{n+2} A_n(x) + \frac{(n+1)}{n} x^{n+3} A_{n+1}(x)\Big|_{x=n/t} \\
&\leq -\frac{n+1}{n} x^{n+2} A_n(x) + \frac{(n+1)}{n} x^{n+2} A_n(x)\Big|_{x=n/t} = 0.
\end{aligned}$$

The last inequality utilizes (4.4). This indicates $G_n(t)$ is concave for all $n \geq 1$.

Proof of necessity. Suppose measure μ is convex. We want to show (4.4). As $F(t)$ is concave, its right derivative $F'(t)$ exists and is non-increasing. Therefore, for any $n \geq 1$,

$$\begin{aligned}
A_n(x) &= \frac{(-1)^n}{n!} f^{(n)}(x) = \frac{1}{n!}\int_{(0,\infty)} e^{-xt} t^n\, dF(t) = \int_{(0,\infty)} \frac{t^n}{n!} e^{-xt} F'(t)\, dt \\
&= \frac{t^{n+1}}{(n+1)!} e^{-xt} F'(t)\Big|_{t=0}^{\infty} - \int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}\, d(e^{-xt} F'(t)).
\end{aligned}$$

Lemma 2.5.5 implies

$$\lim_{t\to 0} t^{n+1} e^{-xt} F'(t) = \lim_{t\to\infty} t^{n+1} e^{-xt} F'(t) = 0,$$

for all $x > 0$ and $n \geq 1$. Therefore by Lemma A.2.7,

$$\begin{aligned}
A_n(x) &= x\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt} F'(t)\, dt - \int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt}\, dF'(t) \\
&\geq x\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt}\, dF(t) = xA_{n+1}(x).
\end{aligned}$$

The proof is complete. □

In fact, the condition (4.4) is equivalent to $f(x) + xf'(x) \in \mathcal{CM}$. By Lemma A.1.1, we know $f(x) + xf'(x) \in \mathcal{CM}$ is equivalent to

$$(-1)^n\Big((n+1)f^{(n)}(x) + xf^{(n+1)}(x)\Big) \geq 0,$$

for all $n \geq 1$. Rewriting the above inequality in terms of $A_n(x)$ reveals (4.4).

4.3.3 Convex measures

Theorem 4.3.3 *Suppose $f(x)$ is completely monotone Then, measure μ is convex, if and only if the sequence $\{A_n(x)\}_{n\geq 1}$ satisfies*

$$A_n(x) \leq xA_{n+1}(x) \tag{4.5}$$

for all $x > 0$ and $n \geq 1$.

Proof Proof of sufficiency. Given (4.5) for all $n \geq 1$, we need to show measure μ is convex. This proof utilizes the Inverse Laplace transformation formula. Define

$$G_n(t) := \int_{(0,t]} L_n(f(x); u)\,du.$$

where the operator L_n is defined in (2.11). By Theorem 2.1.4 b), we know $\lim_{n\to\infty} G_n(t) = F(t) - F(0)$ at every point of continuity of $F(t)$. By lemma A.1.12, to show $F(t)$ is convex, it suffices to show $G_n(t)$ is convex for all $n \geq 1$. Notice that $G_n(t)$ is infinitely differentiable, and

$$G_n'(t) = L_n(f(x); t) = \frac{(-1)^n}{n!} x^{n+1} f^{(n)}(x)\Big|_{x=n/t}.$$

Therefore,

$$\begin{aligned}
G_n''(t) &= \frac{(-1)^n}{n!}\Big((n+1)x^n f^{(n)}(x) + x^{n+1} f^{(n+1)}(x)\Big)\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) \\
&= \frac{(-1)^{n+1}}{n!\cdot n}\Big((n+1)x^{n+2} f^{(n)}(x) + x^{n+3} f^{(n+1)}(x)\Big)\Big|_{x=n/t} \\
&= -\frac{n+1}{n} x^{n+2} A_n(x) + \frac{n+1}{n} x^{n+3} A_{n+1}(x)\Big|_{x=n/t} \\
&\geq -\frac{n+1}{n} x^{n+2} A_n(x) + \frac{n+1}{n} x^{n+2} A_n(x)\Big|_{x=n/t} = 0.
\end{aligned}$$

The last inequality utilizes (4.5). This indicates $G_n(t)$ is convex for all $n \geq 1$.

Proof of necessity. Supposing measure μ is convex, we show (4.5) holds. As $F(t)$ is convex, its right derivative $F'(t)$ exists, and is non-decreasing. Therefore, for any $n \geq 1$,

$$\begin{aligned}
A_n(x) &= \frac{(-1)^n}{n!} f^{(n)}(x) = \frac{1}{n!}\int_{(0,\infty)} e^{-xt} t^n\,dF(t) = \int_{(0,\infty)} \frac{t^n}{n!} e^{-xt} F'(t)\,dt \\
&= \frac{t^{n+1}}{(n+1)!} e^{-xt} F'(t)\Big|_{t=0+}^{\infty} - \int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}\,d(e^{-xt}F'(t)).
\end{aligned}$$

Lemma 2.5.5 implies for all $n \geq 1$,

$$\lim_{t\to\infty} t^{n+1} e^{-xt} F'(t) = 0, \quad \text{and} \quad \lim_{t\to 0} t^{n+1} e^{-xt} F'(t) = 0.$$

Therefore by Lemma A.2.7,

$$\begin{aligned}
A_n(x) &= x\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt} F'(t)\,dt - \int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt}\,dF'(t) \\
&\leq x\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt}\,dF(t) = xA_{n+1}(x).
\end{aligned}$$

The proof is complete. $\square$

The condition (4.5) is equivalent to $x(f(x)-a) \in \mathcal{CM}$ where $a = \mu\{0\} = \lim_{x\to\infty} f(x) \geq 0$. By Lemma A.1.1, we know $x(f(x)-a) \in \mathcal{CM}$ is equivalent to

$$(-1)^n\Big((n)f^{(n-1)}(x) + xf^{(n)}(x)\Big) + a1_{\{n=1\}} \geq 0,$$

for all $n \geq 1$. Rewriting the above inequality in terms of $A_n(x)$, we have

$$xA_n(x) + a1_{\{n=1\}} \geq A_{n-1}(x).$$

This implies (4.4). Conversely, we only need to verify $-(f(x)+xf'(x)) + a \geq 0$. It is trivial, as

$$-(f(x)+xf'(x)) = \int_{(x,\infty)} (2f'(t) + tf''(t))\,dt = 2\int_{(x,\infty)} (tA_2(t) - A_1(t))\,dt \geq 0.$$

In addition, notice that (4.5) implies (4.3), which agrees with the fact that convex measure is harmonically convex. Also notice condition (4.4) and (4.5) hold simultaneously if $A_n(x) = xA_{n+1}(x)$, which corresponds to completely monotone function $f(x) = a + b/x$.

However, there is a non-trivial sub-class of completely monotone functions such that both (4.4) and (4.3) hold, in which the measures are both concave and harmonically convex. For example, completely monotone function $f(x) = x^{-\alpha}$ has measure which is both concave and harmonically convex if $\alpha \in (0,1)$.

4.4 Completely monotone functions with β-convexity or β-concavity

In this section, we study completely monotone functions with β-convexity type properties. See Section (2.5) for definitions. These results generalize the characterizations in previous sections in this chapter.

4.4.1 Characterizations

Theorem 4.4.1 *Suppose $f(x)$ is completely monotone with measure μ. Consider the function*

$$M_\beta(x) = \beta(\beta-1)\frac{f(x)}{x} - 2(\beta-1)f'(x) + xf''(x) - \beta(\beta-1)\frac{\mu(\{0\})}{x}. \tag{4.6}$$

(a) Measure μ is β-convex, if and only if $M_\beta(x)$ is completely monotone.

(b) Measure μ is β-concave, if and only if $-M_\beta(x)$ is completely monotone.

Proof Notice $M_\beta(x)$ can be rewritten as

$$M_\beta(x) = \beta(\beta-1)\frac{f(x)-\mu(\{0\})}{x} - 2(\beta-1)f'(x) + xf''(x).$$

Without loss of generality, we can assume $\mu(\{0\}) = 0$. Otherwise consider $f(x) - \mu(\{0\})$.

(a) For sufficiency, suppose $M_\beta(x)$ is completely monotone. Anticipating the use of Inversion formula in Theorem 2.1.4, define

$$G_n(t) := \int_{(0,t]} L_n(f(x); u)\, du, \tag{4.7}$$

where the operator L_n is defined in (2.11). By Theorem 2.1.4, we have $\lim_{n\to\infty} G_n(t) = F(t)$ at every point of continuity of $F(t)$. By Lemma A.1.12, it suffices to show $G_n(t)$ is β-convex on $(0, \infty)$ for every $n \geq 2$, that is to show

$$(t^\beta G_n(t))'' = t^{\beta-2}(\beta(\beta-1)G_n(t) + 2\beta t G_n(t)' + t^2 G_n''(t)) \geq 0.$$

Indeed, one can see that

$$\begin{aligned} G_n'(t) &= L_n(f(x); t) = \frac{(-1)^n}{n!} x^{n+1} f^{(n)}(x)\Big|_{x=n/t}, \\ G_n''(t) &= \frac{(-1)^n}{n!}\Big(x^{n+1} f^{(n)}(x)\Big)'\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) = \frac{(-1)^{n+1}}{n \cdot n!}\, x^{n+2}(xf(x))^{(n+1)}\Big|_{x=n/t}. \end{aligned}$$

The second equation for $G_n''(t)$ utilizes Lemma 2.14. Using identity (2.15) reversely and by Lemma 2.7, we observe

$$\begin{aligned} G_n(t) &= \int_{(0,t]} L_n(f(x); u)\, du = \frac{n-1}{n} \int_{(0,t]} L_{n-1}'\Big(\frac{f(x)}{x}; \frac{(n-1)u}{n}\Big) du \\ &= L_{n-1}\Big(\frac{f(x)}{x}; \frac{(n-1)u}{n}\Big)\Big|_{u=0+}^{t} = \frac{(-1)^{(n-1)}}{(n-1)!} x^n \Big(\frac{f(x)}{x}\Big)^{(n-1)}\Big|_{x=n/t}. \end{aligned}$$

Therefore, we show

$$(t^\beta G_n(t))'' = \frac{(-1)^{(n-1)} t^{\beta-2}}{(n-1)!} x^n M_\beta^{(n-1)}(x)\Big|_{x=n/t}.$$

As $M_\beta(x)$ is completely monotone, we know that $(-1)^{n-1} M_\beta^{(n-1)}(x) \geq 0$ for all $x > 0$ and $n \geq 2$, which implies $(t^\beta G_n(t))'' \geq 0$.

Now we show necessity. Suppose measure μ is β-convex, we prove $M_\beta(x)$ is completely monotone. First, by (2.6) and using [31, Theorem A.5.2] to differentiate under the integral, we have

$$\begin{aligned} f'(x) &= \int_{(0,\infty)} e^{-xt} F(t)\, dt - x \int_{(0,\infty)} e^{-xt} t F(t)\, dt, \\ f''(x) &= -2 \int_{(0,\infty)} e^{-xt} t F(t)\, dt + x \int_{(0,\infty)} e^{-xt} t^2 F(t)\, dt. \end{aligned}$$

To simplify the notation, denote

$$a_n(x) = \int_{(0,\infty)} e^{-xt} t^n F(t)\, dt, \quad b_m(x) = \int_{(0,\infty)} e^{-xt} t^m\, d(t^\beta F(t)).$$

Note that $a_n(x)$ is finite for all $n \geq 0$ and $b_m(x)$ is also convergent for all $m \geq 1-\beta$, which can be verified by Lemma A.2.7. With these notations, we can rewrite

$$\frac{f(x)}{x} = a_0(x), \quad f'(x) = a_0(x) - xa_1(x), \quad xf''(x) = -2xa_1(x) + x^2 a_2(x).$$

Therefore,

$$\begin{aligned} M_\beta(x) &= \beta(\beta-1)a_0(x) - 2(\beta-1)(a_0(x) - xa_1(x)) - 2xa_1(x) + x^2 a_2(x) \\ &= (\beta-2)(\beta-1)a_0(x) + 2(\beta-2)xa_1(x) + x^2 a_2(x). \end{aligned}$$

By Lemma A.2.7 and (2.6) , we have

$$\begin{aligned} xa_1(x) &= x\int_{(0,\infty)} e^{-xt} tF(t)\,dt = -\int_{(0,\infty)} tF(t)\,d(e^{-xt}) = tF(t)e^{-xt}\Big|_{t=0+}^{\infty} + \int_{(0,\infty)} e^{-xt}\,d(t^\beta F(t)t^{1-\beta}) \\ &= \int_{(0,\infty)} e^{-xt}t^{1-\beta}\,d(t^\beta F(t)) + (1-\beta)\int_{(0,\infty)} e^{-xt}F(t)\,dt = b_{1-\beta}(x) + (1-\beta)a_0(x). \end{aligned}$$

In addition, by Lemma 2.5.5 and Lemma A.2.7, we obtain

$$\begin{aligned} x^2 a_2(x) &= x^2\int_{(0,\infty)} e^{-xt}t^2F(t)\,dt = -xt^2F(t)e^{-xt}\Big|_{t=0}^{\infty} + x\int_{(0,\infty)} e^{-xt}\,d(t^\beta F(t)t^{2-\beta}) \\ &= (2-\beta)x\int_{(0,\infty)} e^{-xt}tF(t)\,dt + x\int_{(0,\infty)} e^{-xt}t^{2-\beta}\,d(t^\beta F(t)) \\ &= (2-\beta)xa_1(x) + x\int_{(0,\infty)} e^{-xt}t^{2-\beta}(t^\beta F(t))'\,dt \\ &= (2-\beta)xa_1(x) - t^{2-\beta}(t^\beta F(t))'e^{-xt}\Big|_{t=0+}^{\infty} + \int_{(0,\infty)} e^{-xt}\,d(t^{2-\beta}(t^\beta F(t))') \\ &= (2-\beta)xa_1(x) + \int_{(0,\infty)} e^{-xt}t^{2-\beta}\,d(t^\beta F(t))' + (2-\beta)\int_{(0,\infty)} e^{-xt}(t^\beta F(t))'t^{1-\beta}\,dt \\ &= (2-\beta)xa_1(x) + \int_{(0,\infty)} e^{-xt}t^{2-\beta}\,d(t^\beta F(t))' + (2-\beta)b_{1-\beta}. \end{aligned}$$

Note the above equations also hold if μ is β-concave. Therefore, it can be shown that

$$M_\beta(x) = \int_{(0,\infty)} e^{-xt}t^{2-\beta}\,d(t^\beta F(t))'.$$

As $t^\beta F(t)$ is convex, $(t^\beta F(t))'$ is non-decreasing. Above is the Laplace transform for Radon measure $t^{2-\beta}\,d(t^\beta F(t))'$ on $(0,\infty)$. By Bernstein representation, $M_\beta(x)$ is completely monotone.

(b) The proof is very much analogous to the proof for part a), so we only provide a sketch. One can easily recover the full details by comparing with the proof for part a).

For sufficiency, suppose $-M_\beta(x)$ is completely monotone. Define $G_n(t)$ as (4.7). Without any further assumption, we could have for all $n \geq 2$,

$$(t^\beta G_n(t))'' = \frac{(-1)^{(n-1)}t^{\beta-2}}{(n-1)!}x^n M_\beta^{(n-1)}(x)\Big|_{x=n/t}.$$

As $-M_\beta(x)$ is completely monotone, we know that $t^\beta G_n(t)$ is concave. By Lemma A.1.12, we conclude that measure μ is β-concave.

For necessity, suppose μ is β-concave, we prove $-M_\beta$ is completely monotone. Applying the notation $a_n(x)$ and $b_m(x)$ in part a), we also have

$$M_\beta(x) = \beta(\beta-1)a_0(x) - 2(\beta-1)(a_0(x) - xa_1(x)) - 2xa_1(x) + x^2a_2(x).$$

And the following identities hold as well given μ is β-concave.

$$\begin{aligned} xa_1(x) &= b_{1-\beta}(x) + (1-\beta)a_0(x), \\ x^2a_2(x) &= (2-\beta)xa_1(x) + \int_{(0,\infty)} e^{-xt}t^{2-\beta}\, d(t^\beta F(t))' + (2-\beta)b_{1-\beta}. \end{aligned}$$

Thus, we obtain

$$-M_\beta(x) = \int_{(0,\infty)} e^{-xt}t^{2-\beta}\, d(-t^\beta F(t))'.$$

As $t^\beta F(t)$ is concave, $-(t^\beta F(t))'$ is non-decreasing. Above is the Laplace transform for Radon measure $t^{2-\beta}\, d(-t^\beta F(t))'$ on $(0,\infty)$. Hence, function $-M_\beta(x)$ is completely monotone. □

4.4.2 Connections with (harmonic) convexities

In this section, we prove several theorem from Section 4.1 as corollaries. They reveals β-convexity is indeed a generalization for both harmonic convexity and convexity.

Corollary 4.4.1 (Theorem 4.1.1) *Suppose $f(x)$ is completely monotone with measure μ. Then measure μ is harmonically convex, if and only if $f(x) - xf'(x) \in \mathcal{CM}$.*

Proof Without loss of generality, we can assume μ has no mass at $\{0\}$. By Theorem 4.4.1 part (a), measure μ is harmonically convex, if and only if $xf''(x)$ is completely monotone. We show this condition is equivalent to $f(x) - xf'(x)$ being completely monotone.

If $f(x) - xf'(x)$ is completely monotone, then

$$xf''(x) = -(f(x) - xf'(x))' \in \mathcal{CM}.$$

Conversely, if $xf''(x)$ is completely monotone, to see $f(x) - xf'(x)$ is completely monotone, it suffices to show its non-negativity. This is trivial, because $f(x) \geq 0$ and $f'(x) \leq 0$. □

Corollary 4.4.2 (Theorem 4.1.3) *Suppose $f(x)$ is completely monotone with measure μ. Then measure μ is concave, if and only if $f(x) + xf'(x) \in \mathcal{CM}$.*

Proof Without loss of generality, we can assume μ has no mass at $\{0\}$. By Theorem 4.4.1 part (b), measure μ is concave, if and only if $-2xf'(x) - xf''(x)$ is completely monotone. We show this condition is equivalent to $f(x) + xf'(x)$ being completely monotone.

If $f(x) + xf'(x)$ is completely monotone, then

$$-2xf'(x) - xf''(x) = -(f(x) + xf'(x))' \in \mathcal{CM}.$$

Conversely, if $-2xf'(x) - xf''(x)$ is completely monotone, to see $f(x) + xf'(x)$ is completely monotone, it suffices to show it is non-negative. As its derivative is non-positive, $f(x) + xf'(x)$ is non-increasing. By (2.7), we obtain

$$\lim_{x\to\infty} f(x) + xf'(x) = 0.$$

So $f(x) + xf'(x) \geq 0$. This closes the proof. □

Corollary 4.4.3 (Theorem 4.1.5) *Suppose $f(x)$ is completely monotone with measure μ having mass a at $\{0\}$. Then, measure μ is convex, if and only if $x(f(x) - a) \in \mathcal{CM}$.*

Proof Consider the shifted function $f(x) - a$. By Theorem 4.4.1 part (a), measure μ is convex, if and only if $2f'(x) + xf''(x)$ is completely monotone. We show this condition is equivalent to $x(f(x) - a)$ being completely monotone.

If $x(f(x) - a)$ is completely monotone, then

$$2f'(x) + xf''(x) = (x(f(x) - a))'' \in \mathcal{CM}.$$

Conversely, suppose $2f'(x) + xf''(x)$ is completely monotone. To see $x(f(x) - a)$ is completely monotone, we only have to show

$$x(f(x) - a) \geq 0 \quad \text{and} \quad xf'(x) + f(x) - a \leq 0.$$

The first inequality is trivial, because $f(x) - a \geq 0$. For the second inequality, as $2f'(x) + xf''(x) \geq 0$, we know $xf'(x) + f(x) - a$ is non-decreasing. By (2.7), we obtain

$$\lim_{x\to\infty} xf'(x) + f(x) - a = 0.$$

The second inequality follows. □

Corollary 4.4.4 *Suppose $f(x)$ is completely monotone with measure μ having mass a at $\{0\}$. Then, measure μ is harmonically concave, if and only if $f(x) = a$.*

Proof Sufficiency is trivial. We show necessity. By Theorem 4.4.1 part b), if measure μ is harmonically concave, then $-xf''(x)$ is completely monotone. Notice that $f''(x) \geq 0$. So $f''(x) = 0$, which implies $-f'(x) \in \mathcal{CM}$ is linear. Therefore $f(x) = a$. □

4.5 Convex shape preserving transformations

In this section, we revisit [69, Corollary 5.4], and investigate transformations that maps a completely monotone function into another with certain convexity properties on its measure.

In addition, we investigate some transformations that can preserve certain convexity properties on measures. Let $f : (0, \infty) \mapsto \mathbb{R}$. For any $x > 0$, define operator

$$H[f](x) := x \int_{(x,\infty)} \frac{f(s)}{s^2}\, ds. \tag{4.8}$$

Obviously, the integrability needs more insight on $f(x)$. Fortunately, $H[f](x)$ is well-defined for all $f(x) \in \mathcal{CM}$.

Proposition 4.5.1 *The set of $\mathcal{CM}$ with harmonically convex measure is*

$$\Big\{H[f](x) : f(x) \in \mathcal{CM}\Big\}.$$

Proof It is trivial that the above set is well-defined for any completely monotone function $f(x)$. To show $H[f](x)$ is completely monotone with harmonically convex measure, recall (2.1) to rewrite $H[f](x)$ as

$$H[f](x) = x \int_{(x,\infty)} \int_{[0,\infty)} \frac{e^{-st}}{s^2} \mu(dt)\, ds = \int_{[0,\infty)} x \int_{(x,\infty)} \frac{e^{-st}}{s^2}\, ds\, \mu(dt).$$

It suffices to verify

$$k_t(x) := x \int_{(x,\infty)} \frac{e^{-st}}{s^2}\, ds$$

is completely monotone with harmonically convex measure for all $t > 0$. This is true by Example 4.1.2.

Conversely, suppose $h(x) \in \mathcal{CM}$ has harmonically convex measure with mass a at $\{0\}$. By Theorem 4.1.1, we know $f(x) := h(x) - xh'(x) \in \mathcal{CM}$ and $\lim_{x\to\infty} f(x) = a$ by (2.7). Note that $1/x$ is an integrating factor, solving the differential equation

$$h'(x) - (1/x)h(x) + (1/x)f(x) = 0$$

reveals $h(x) = H[f](x) + cx$ for some constant $c \in \mathbb{R}$. L'Hopital rule indicates

$$\lim_{x\to\infty} H[f](x) = \lim_{x\to\infty} \frac{-f(x)/x^2}{-1/x^2} = a.$$

And applying the limit condition $\lim_{x\to\infty} h(x) = a$, the constant can be determined to be $c = 0$. Therefore, there is a completely monotone function $f(x)$ such that $h(x) = H[f](x)$. □

To describe the set of completely monotone functions with concave measure, we need the next operator. Let $f : (0, \infty) \mapsto \mathbb{R}$. For any $x > 0$, define operator

$$K[f](x) := \frac{1}{x} \int_{(0,x)} f(s)\, ds. \tag{4.9}$$

Again, the integrability need more insight on function $f(x)$.

Proposition 4.5.2 *The set of $\mathcal{CM}$ with concave measure is*

$$\Big\{K[f](x) : f(x) \in \mathcal{CM}, \textit{ such that the integral is convergent}\Big\}.$$

Proof Suppose $f(x)$ is completely monotone with measure μ, such that $K[f](x)$ is well-defined. By Fubini's theorem, it can be rewritten as

$$K[f](x) = \frac{1}{x} \int_{(0,x)} \int_{[0,\infty)} e^{-st}\, \mu(dt)\, ds = \int_{[0,\infty)} \frac{1 - e^{-xt}}{xt}\, \mu(dt).$$

It suffices to show $k_t(x) := (1 - e^{-xt})/xt$ is completely monotone with concave measure for all $t > 0$. This is true due to Example 4.1.4.

Conversely, if $h(x) \in \mathcal{CM}$ has concave measure, then $f(x) := h(x) + xh'(x)$ is completely monotone. Solving the differential equation

$$h'(x) + (1/x)h(x) - (1/x)f(x) = 0$$

reveals $h(x) = K[f](x) + a/x + c/x$ for $a = \lim_{x\to 0} xh(x)$ and some constant $c \in \mathbb{R}$. The existence of a follows from $xh(x) \in \mathcal{BF}$, see Corollary 4.1.3. The integrability of $K[f](x)$ follows form the finiteness of $h(x)$. Applying the limit condition again, we obtain

$$a = \lim_{x\to 0} xh(x) = \lim_{x\to 0} xK[f](x) + a + c = a + c.$$

This implies $c = 0$. Therefore, we know $h(x) = K[f](x)$ for some $f(x) \in \mathcal{CM}$. □

Tauberian Theorem also provides us with an sufficient condition such that $K[f](x)$ is well-defined. If there is some constant $\gamma \in (0, 1)$, such that the cumulative function $F(t)$ for the measure of $f(x)$ satisfies $F(t) = O(t^\gamma)$, as $t \to \infty$, then $f(x) = O(1/x^\gamma)$ as x approaches zero, by [80, Corollary 1a, Chapter V]. As $\gamma \in (0, 1)$, we obtain the integrability of $f(x)$ at zero, which is equivalent to $K[f](x)$ being well-defined.

For completeness, we state the next proposition, which follows from Theorem 4.1.5.

Proposition 4.5.3 *The set of $\mathcal{CM}$ with convex measure is*

$$\left\{\frac{f(x)}{x} + a : f(x) \in \mathcal{CM} \quad \text{and} \quad a \geq 0\right\}.$$

Next we consider transformations that preserve certain convexity properties on measures.

Proposition 4.5.4 *Suppose $f(x)$ is completely monotone.*

(a) $f(x)$ has harmonically convex measure, if and only if $f(x) - f(x + \lambda)$ has harmonically convex measure for all $\lambda > 0$.

(b) $f(x)$ has concave measure, if and only if $f(x + \lambda)$ has concave measure for all $\lambda > 0$.

(c) $f(x)$ has convex measure, if and only if $f(x) - f(x + \lambda)$ has convex measure for all $\lambda > 0$.

Proof (a) Sufficiency is trivial by taking λ approach infinity. To see necessity, assume $f(x)$ has harmonically convex measure. Thus, $f(x) - xf'(x) \in \mathcal{CM}$ by Theorem 4.1.1. Consider

$$\begin{aligned}&(f(x) - f(x + \lambda)) - x(f(x) - f(x + \lambda))' \\ &= (f(x) - xf'(x) - (f(x + \lambda) - (x + \lambda)f'(x + \lambda))) - \lambda f'(x + \lambda).\end{aligned}$$

It is completely monotone, as both terms are. Hence, completely monotone function $f(x) - f(x + \lambda)$ has harmonically convex measure by Theorem 4.1.1.

(b) Sufficiency is trivial by taking λ approach zero. To see necessity, assume $f(x)$ has concave measure. Thus, $f(x) + xf'(x) \in \mathcal{CM}$ by Theorem 4.1.3. Consider

$$f(x + \lambda) + xf'(x + \lambda) = (f(x + \lambda) + (x + \lambda)f'(x + \lambda)) - \lambda f'(x + \lambda).$$

It is completely monotone, as both terms are. Hence $f(x + \lambda)$ has concave measure by Theorem 4.1.3.

(c) Sufficiency is trivial by taking λ approach infinity. To see necessity, assume $f(x)$ has convex measure with mass a at $\{0\}$. Thus, $x(f(x) - a) \in \mathcal{CM}$ by Theorem 4.1.5. Clearly, completely monotone function $f(x) - f(x + \lambda)$ has measure with no mass at $\{0\}$. Consider

$$x(f(x) - f(x + \lambda)) = (x(f(x) - a) - (x + \lambda)(f(x + \lambda) - a)) + \lambda(f(x + \lambda) - a).$$

It is completely monotone, as both terms are such. Therefore, completely monotone function $f(x) - f(x + \lambda)$ has convex measure by Theorem 4.1.5. □

From this proposition, we can see that $f(x) - f(x + \lambda)$ preserve harmonic convexity and convexity property on the measure of $f(x)$. Hence, if completely monotone function $f(x)$ has convex measure, $f(x) - f(x + \lambda)$ have harmonically convex measure for any $\lambda > 0$. However, if completely monotone function $f(x)$ has harmonically convex measure, $f(x) - f(x + \lambda)$ may not have convex measure for any $\lambda > 0$.

Example 4.5.1 *Completely monotone function*

$$f(x) = \int_{(0,\infty)} e^{-xt} \frac{1}{t+1}\, dt$$

has harmonically convex measure. However $f(x) - f(x + \lambda)$ does not have convex measure.

In fact, the measure for $f(x)$ has cumulative function $F(t) = \ln(t + 1)$, which is harmonically convex, as $(tF(t))'' = (t + 2)/(t + 1)^2 \geq 0$. Meanwhile,

$$f(x) - f(x + \lambda) = \int_{[0,\infty)} e^{-xt}(1 - e^{-\lambda t}) \frac{1}{t+1}\, dt.$$

This completely monotone function has convex measure, if and only if $(1 - e^{-\lambda t})/(t + 1)$ is non-decreasing for any $\lambda > 0$. However, it can be easily shown negatively. See Figure 4.1.

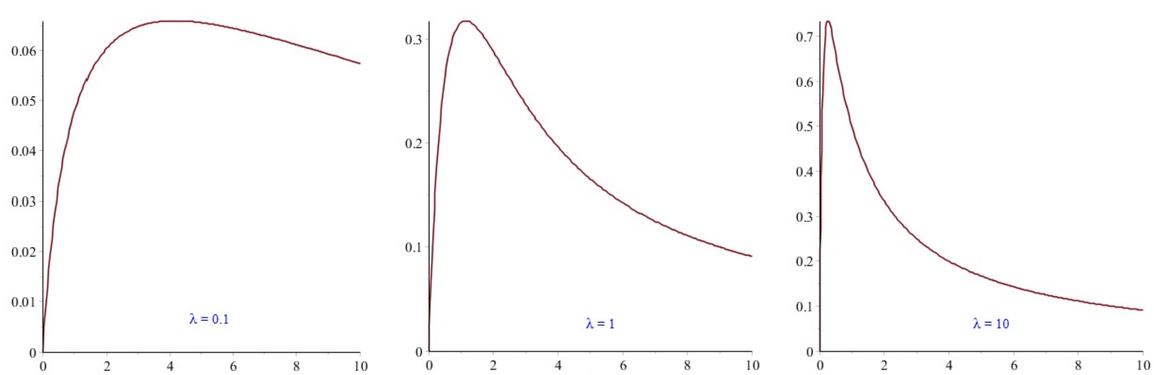

Figure 4.1: Function $f(x) = (1 - e^{-\lambda t})/(t + 1)$ for $\lambda = 0.1,\ 1,\ 10$.

Chapter 5

Bernstein functions with convexity on their measures

In this chapter, we consider Bernstein functions with various convexity properties on the tail of their measures. Different characterizations are given and the connections are studied.

Note that Bernstein function could not have Lévy measure with concave tail, nor harmonically convex tail, unless it degenerates to affine function. In fact, as mentioned in the paragraph below Definition 2.4.3 in Section 3.4, as well as Corollary 3.5.1 part b), a Lévy measures on $(0, \infty)$ have concave tail or harmonically convex tail if and only if it vanishes on $(0, \infty)$.

Throughout this chapter, readers will find how surprisingly resemble it is to Chapter 4. As shown in Table 5.1 and Table 4.1, the characterization for Bernstein function whose Lévy measure has harmonically concave tail is closely related with the one for completely monotone functions with harmonically convex measures, while Bernstein function whose Lévy measure has convex tail connects to completely monotone functions with concave measures. These analogies trace back to Chapter 3, where Table 3.1 and Table 3.2 could give us a clue, as well as Table 3.3 and Table 3.4.

Suppose $g(x)$ is Bernstein Lévy triplet (a, b, ν). Define non-negative sequence $\{B_n(x)\}_{n\geq 1}$ as

$$B_n(x) = \frac{(-1)^{n+1}}{n!} g^{(n)}(x)$$

for all $x > 0$ and $n \geq 1$. Also define function on $(0, \infty)$ by

$$N_\beta(x) := \beta(\beta - 1)\frac{g(x)}{x} - 2(\beta - 1)g'(x) + xg''(x) - \beta(\beta - 1)\frac{a}{x} - (\beta - 1)(\beta - 2)b.$$

A summary of our main results are given in Table 5.1 and Table 5.2.

5.1 Characterizations

In this section, we characterize Bernstein functions with various convexity properties on their tails. The results correspond to the ones numbered (a) in Table 5.1.

Table 5.1: Characterization for $g(x) \in \mathcal{BF}$ with convexity properties on its Lévy measure ν

Property on ν	No.	Characterization	Reference
harmonically concave tail	(a)	$g(x) - xg'(x) \in \mathcal{BF}$	Thm 5.1.1
	(b)	$\lambda g(x) - x(g(x+\lambda) - g(x)) \in \mathcal{BF}, \quad \forall \lambda > 0$	Thm 5.2.1
	(c)	$(n-1)B_n(x) \leq (n+1)xB_{n+1}(x), \quad \forall n \geq 1$	Thm 5.3.1
convex tail	(a)	$g(x) + xg'(x) \in \mathcal{BF}$	Thm 5.1.3
	(b)	$\lambda g(x) + x(g(x+\lambda) - g(x)) \in \mathcal{BF}, \quad \forall \lambda > 0$	Thm 5.2.2
	(c)	$B_n(x) \geq xB_{n+1}(x), \quad \forall n \geq 1$	Thm 5.3.2
concave tail		$g(x) = a + bx$	N/A
harmonically convex tail		$g(x) = a + bx$	Cor 3.5.1 b)

Table 5.2: Characterization for $g(x) \in \mathcal{BF}$ with β-convexity convexity on its Lévy measure ν

Property on ν	Characterization	Reference
β-convex tail	$N_\beta(x) \in \mathcal{CM}$	Thm 5.4.1 a)
β-concave tail	$-N_\beta(x) \in \mathcal{CM}$	Thm 5.4.1 b)

5.1.1 Measures with harmonically concave tail

Theorem 5.1.1 (Harmonically concave tail) *Suppose $g(x)$ is Bernstein with triplet (a, b, ν). Then, measure ν has harmonically concave tail if and only if $g(x) - xg'(x) \in \mathcal{BF}$.*

Proof By Lemma 2.4.2, measure ν has harmonically concave tail if and only if $t\bar{\nu}(t)$ is concave. Recall formula (2.25).

$$g(x) - xg'(x) = a + x^2 \int_{(0,\infty)} e^{-tx} t\bar{\nu}(t)\, dt.$$

First suppose $g(x) - xg'(x) \in \mathcal{BF}$. So is $r(x) := g(x) - xg'(x) - a$. We prove that $t\bar{\nu}(t)$ is concave. One can see that $r(x)/x^2$ is the Laplace transform of $t\bar{\nu}(t)$. Define the functions

$$G_n(t) := L_n\left(\frac{r(x)}{x^2}; t\right) = \frac{(-1)^n}{n!} x^{n+1} \left(\frac{r(x)}{x^2}\right)^{(n)} \Big|_{x=n/t},$$

where the operator $L_n(\cdot;t)$ is defined by (2.11). By Theorem 2.1.4, we know $\lim_{n\to\infty} G_n(t) = t\bar{\nu}(t)$ at every point of continuity $\bar{\nu}(t)$. By Lemma A.1.12, to show $t\bar{\nu}(t)$ is concave, it suffices to show $G_n(t)$ is concave for all $n \geq 1$. Indeed, by (2.14), we obtain

$$G_n'(x) = \frac{(-1)^n}{n!}\left(x^{n+1}\left(\frac{r(x)}{x^2}\right)^{(n)}\right)'\bigg|_{x=n/t}\left(-\frac{n}{t^2}\right) = \frac{(-1)^{n+1}}{n\cdot n!}x^{n+2}\left(\frac{r(x)}{x}\right)^{(n+1)}\bigg|_{x=n/t}.$$

As well as

$$G_n''(x) = \frac{(-1)^{n+1}}{n\cdot n!}\left(x^{n+2}\left(\frac{r(x)}{x}\right)^{(n+1)}\right)'\bigg|_{x=n/t}\left(-\frac{n}{t^2}\right) = \frac{(-1)^{n+2}}{n^2\cdot n!}x^{n+3}r^{(n+2)}(x)\bigg|_{x=n/t}.$$

As $r(x) \in \mathcal{BF}$, we conclude that $G_n''(x) \leq 0$, implying $G_n(x)$ is convex for all $n \geq 1$. Therefore, ν has harmonically concave tail.

Conversely, suppose now ν has harmonically concave tail, we prove that $g(x) - xg'(x)$ is Bernstein. Since $t\bar{\nu}(t)$ is concave and non-negative on $(0,\infty)$, it has to be non-decreasing. Recall (2.25), and taking limit after applying Theorem A.2.6:

$$\begin{aligned} g(x) - xg'(x) - a &= x^2\int_{(0,\infty)} e^{-tx}t\bar{\nu}(t)\,dt = -x\int_{(0,\infty)} t\bar{\nu}(t)\,d(e^{-tx}) \\ &= -xt\bar{\nu}(t)e^{-tx}\Big|_{t=0+}^{\infty} + x\int_{(0,\infty)} e^{-tx}\,d(t\bar{\nu}(t)). \end{aligned}$$

By (2.22), (2.24) and Lemma A.2.9, we have

$$\begin{aligned} g(x) - xg'(x) - a &= x\int_{(0,\infty)} e^{-tx}\,d(t\bar{\nu}(t)) = x\int_{(0,\infty)} e^{-tx}(t\bar{\nu}(t))'\,dt \\ &= (1-e^{-tx})(t\bar{\nu}(t))'\Big|_{t=0+}^{\infty} + \int_{(0,\infty)}(1-e^{-tx})\,d(-(t\bar{\nu}(t))'). \end{aligned}$$

As noted above, $t\bar{\nu}(t)$ is non-negative and non-decreasing. Therefore, $\lim_{t\to\infty}(t\bar{\nu}(t))' = c \geq 0$. Also note the above calculation implies $(t\bar{\nu}(t))'$ is integrable on $(0,1)$, and by Lemma A.2.1, $\lim_{t\to0+}(1-e^{-tx})(t\bar{\nu}(t))' = \lim_{t\to0+} t(t\bar{\nu}(t))' = 0$. Therefore,

$$g(x) - xg'(x) = (a+c) + \int_{(0,\infty)}(1-e^{-xt})\,d(-(t\bar{\nu}(t))').$$

The left-hand side defines a Bernstein function. The closes the proof. □

Notice that we have already represent Bernstein functions whose measure has harmonically concave tail in Proposition 3.2.2. There could be an alternative proof.

Alternative proof for Theorem 5.1.1 Suppose $g(x)$ has Lévy triplet (a,b,ν). If ν has harmonically concave tail, by Proposition 3.2.2, we know $g(x)$ has representation (3.15). Thus, using [31, Theorem A.5.2] for differentiating under the integral, we have

$$g(x) - xg'(x) = a + \int_{(0,\infty)}\left(1 + \frac{x^2}{s^2}k'\left(\frac{x}{s}\right)\right)\tau(ds) = a + \int_{(0,\infty)}(1-e^{-x/s})\,\tau(ds),$$

where $a \geq 0$ and τ satisfies integrability condition (3.8). The integral is convergent and $g(x) - xg'(x)$ is Bernstein.

Conversely, suppose $g(x) - xg'(x)$ is Bernstein with Lévy-Khintchine representation:

$$g(x) - xg'(x) = a' + b'x + \int_{(0,\infty)} (1 - e^{-xt})\,\eta(dt),$$

where $a', b' \geq 0$, and η satisfies (2.17). Notice that

$$b' = \lim_{x\to\infty} \frac{g(x) - xg'(x)}{x} = \lim_{x\to\infty} \frac{g(x)}{x} - \lim_{x\to\infty} g'(x) = b - b = 0.$$

Therefore, we obtain

$$-g''(x) = \frac{(g(x) - xg'(x))'}{x} = \frac{1}{x}\int_{(0,\infty)} e^{-xt} t\,\eta(dt).$$

Integrating twice both sides of this equality and using Fubini's theorem, we obtain

$$\begin{aligned}
g(x) - a - bx &= \int_{(0,x)}\int_{(v,\infty)} -g''(u)\,du\,dv = \int_{(0,x)}\int_{(v,\infty)}\int_{(0,\infty)} \frac{e^{-ut}t}{u}\,\eta(dt)\,du\,dv \\
&= \int_{(0,\infty)} \Big(\int_{(0,x]}\int_{(0,u)} \frac{e^{-ut}}{u}\,dv\,du + \int_{(x,\infty)}\int_{(0,x)} \frac{e^{-ut}}{u}\,dv\,du \Big) t\,\eta(dt) \\
&= \int_{(0,\infty)} \Big(\frac{1 - e^{-xt}}{t} + x\mathrm{E}_1(xt) \Big) t\,\eta(dt) = \int_{(0,\infty)} (1 - xtk(xt))\,\eta(dt).
\end{aligned}$$

Here $k(x)$ is defined in (3.9). The penultimate equality gives the following identity, which is needed later.

$$g(z) - a - bz = \int_{(0,\infty)} (1 - e^{-zt})\,\eta(dt) + z\int_{(0,\infty)} t\mathrm{E}_1(zt)\,\eta(dt). \tag{5.1}$$

Denote $\bar{\eta}(s) = \eta(s,\infty)$. Utilizing the fact that $\bar{\eta}(1/s)$ is non-decreasing, define measure τ on $(0,\infty)$ by $\tau(ds) = d\bar{\eta}(1/s)$. Thus, changing variable $t = 1/s$, we have

$$g(x) = a + bx + \int_{(0,\infty)} \Big(1 - \frac{x}{s}k\Big(\frac{x}{s}\Big)\Big)\,\tau(ds).$$

We only need to show that τ satisfies (3.8). Since η is a Lévy measure,

$$\tau(0,1] = \int_{(0,1]} d\bar{\eta}(1/s) = \bar{\eta}(1) - \lim_{s\to 0^+} \bar{\eta}(1/s) = \bar{\eta}(1) < \infty.$$

Besides, by Lemma A.1.6, we have $-\ln(t) \leq e^x\mathrm{E}_1(xt)$ for every $t \in (0,1)$ and $x > 0$. Therefore,

$$\begin{aligned}
\int_{(1,\infty)} \frac{\ln(s)}{s}\,\tau(ds) &= \int_{(1,\infty)} \frac{\ln(s)}{s}\,d\bar{\eta}(1/s) = \int_{(0,1)} -t\ln(t)\,\eta(dt) \\
&\leq e^x \int_{(0,1)} t\mathrm{E}_1(xt)\,\eta(dt) \leq e^x \int_{(0,\infty)} t\mathrm{E}_1(xt)\,\eta(dt).
\end{aligned}$$

The finiteness of the last integral follows from (5.1). We conclude τ satisfies (3.8) and close the proof. $\square$

Example 5.1.2 *Function*

$$l_t(x) = x\int_{(x,\infty)} \frac{(1-e^{-st})}{s^2}\,ds$$

is Bernstein whose Lévy measure has harmonically convex tail for all $t > 0$.

Proof First notice $l_t(x) \geq 0$. And the first derivative is

$$l_t'(x) = \int_{(x,\infty)} \frac{(1-e^{-st})}{s^2}\,ds - \frac{(1-e^{-xt})}{x} \geq (1-e^{-xt})\int_{(x,\infty)} \frac{1}{s^2}\,ds - \frac{(1-e^{-xt})}{x} = 0.$$

The second derivative is

$$-l_t''(x) = \frac{(1-e^{-xt})}{x^2} + \frac{e^{-xt}xt - (1-e^{-xt})}{x^2} = \frac{e^{-xt}t}{x} \in \mathcal{CM}.$$

By definition, $l_t(x)$ is Bernstein. Notice

$$l_t(x) - xl_t'(x) = 1 - e^{-xt} \in \mathcal{BF}.$$

By Theorem 5.1.1, we know $l_t(x)$ has Lévy measure with harmonically convex tail. □

As the composition of two Bernstein functions is still such, it follows the next corollary.

Corollary 5.1.1 *Suppose* $g(x)$ *is Bernstein with Lévy measure* ν*. For any real* α*, consider*

$$(g(x) - xg'(x))^\alpha. \tag{5.2}$$

(a) If (5.2) is Bernstein for some $\alpha \geq 1$*, then the measure* ν *has harmonically concave tail.*

(b) If the measure ν *has harmonically concave tail, then (5.2) is Bernstein for any* $0 \leq \alpha \leq 1$.

It can be shown that the set of Bernstein functions whose Lévy measures have harmonically concave tails is a convex cone, and it is closed under non-negative scalar multiplication and scaling. That is, if $g_1(x)$ and $g_2(x)$ are Bernstein functions whose Lévy measures ν_1 and ν_2 have harmonically concave tails, then so are

$$\lambda g_1(x) + (1-\lambda)g_2(x) \quad \text{and} \quad cg_1(\rho x),$$

for $\lambda \in [0,1]$ and $c, \rho \geq 0$. They follow trivially from the fact that the tail for the measure of above Bernstein functions are

$$\lambda\bar{\nu}_1(t) + (1-\lambda)\bar{\nu}_2(t) \quad \text{and} \quad c\bar{\nu}_1(t/\rho).$$

One can also use Theorem 5.1.1 to verify. In addition, Theorem 5.1.1 implies that the aforementioned set is closed under point-wise limit.

Corollary 5.1.2 *Let* $\{g_n(x)\}_{n\in\mathbb{N}}$ *be a sequence of Bernstein functions such that their point-wise limit* $g(x) = \lim_{n\to\infty} g_n(x)$ *exists for* x *on* $(0,\infty)$*. If* $g_n(x)$ *has Lévy measure with harmonically concave tail for all* $n \geq 1$*, then* $g(x)$ *is also Bernstein whose Lévy measure has harmonically concave tail.*

Proof It is shown in Proposition 2.2.2 that $g(x) \in \mathcal{BF}$ and $g'(x) = \lim_{n\to\infty} g_n'(x)$. Notice that $g_n(x) - xg_n'(x) \in \mathcal{BF}$ for all $n \geq 1$ by Theorem 5.1.1. Therefore, for all $x \in (0, \infty)$ we obtain

$$g(x) - xg'(x) = \lim_{n\to\infty} g_n(x) - xg_n'(x).$$

As $\mathcal{BF}$ is closed under point-wise limit, we conclude $g(x) - xg'(x) \in \mathcal{BF}$. This implies $g(x)$ has Lévy measure with harmonically concave tail by Theorem 5.1.1. □

We conclude this subsection with another curious fact. On one hand, if $g(x) - xg'(x)$ is Bernstein, then, so is $g(x) - xg'(x) - a$ and hence $(g(x) - xg'(x) - a)/x$ is completely monotone, where a is from the Lévy triplet of $g(x)$; on the other hand, if $t\bar{\nu}(t)$ is concave and positive on $(0, \infty)$ then it is non-decreasing, and hence

$$x \mapsto \int_{(0,x]} t\bar{\nu}(t)\, dt \tag{5.3}$$

is a convex function. Theorem 5.1.1 shows that the premises in both 'if-then' clauses are equivalent. The next proposition shows that the conclusions are also necessary and sufficient for each other.

Proposition 5.1.1 *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). Then, function (5.3) is convex if and only if $(g(x) - a)/x - g'(x)$ is completely monotone.*

Proof Consider the function

$$V(x) := \int_{(0,x]} t\bar{\nu}(t)\, dt$$

which is finite for all $x > 0$ by (2.24). Dividing both sides of (2.25) by x gives

$$(g(x) - a)/x - g'(x) = x \int_{(0,\infty)} e^{-tx} t\bar{\nu}(t)\, dt = x \int_{(0,\infty)} e^{-tx}\, dV(x).$$

By Theorem 4.1.5, function $V(x)$ is convex if and only if $(g(x) - a)/x - g'(x)$ is completely monotone. □

5.1.2 Measures with convex tail

Theorem 5.1.3 (Convex tail) *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, ν has convex tail, if and only if $g(x) + xg'(x) \in \mathcal{BF}$.*

Proof First suppose $g(x) + xg'(x) \in \mathcal{BF}$. We want to show $\bar{\nu}(t)$ is convex. This proof utilizes Inverse Laplace transformation formula in Theorem 2.1.4. By (2.23), we have

$$\frac{g(x) - a}{x} - b = \int_{(0,\infty)} e^{-xt}\bar{\nu}(t)\, dt.$$

Define for every $n \geq 1$,

$$G_n(t) := L_n\Big(\frac{g(x) - a}{x} - b; t\Big).$$

Here operator $L_n(\cdot, t)$ is defined by (2.11). Notice $L_n(a/x; t) = a$, $L_n(b; t) = 0$ for all $n \geq 1$. The above formula simplifies into

$$G_n(t) = L_n\Big(\frac{g(x)}{x}; t\Big) - a = \frac{(-1)^n}{n!} x^{n+1}\Big(\frac{g(x)}{x}\Big)^{(n)}\Big|_{x=n/t} - a.$$

By Theorem 2.1.4, we know that $\lim_{n\to\infty} G_n(t) = \bar{\nu}(t)$ at every point of continuity of $\bar{\nu}(t)$. Notice $\bar{\nu}(t)$ is non-increasing and right continuous. By Lemma A.1.12, it suffices to show $G_n(t)$ is convex for all $n \geq 1$. Observe

$$G_n'(t) = \frac{(-1)^n}{n!}\Big(x^{n+1}\Big(\frac{g(x)}{x}\Big)^{(n)}\Big)'\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) = \frac{(-1)^{n+1}}{n \cdot n!}\Big(x^{n+2} g^{(n+1)}(x)\Big)\Big|_{x=n/t}.$$

The second equation is due to (2.14). And also

$$G_n''(t) = \frac{(-1)^{n+1}}{n \cdot n!}\Big(x^{n+2} g^{(n+1)}(x)\Big)'\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) = \frac{(-1)^{n+2}}{n^2 \cdot n!} x^{n+3}\Big(xg(x)\Big)^{(n+2)}\Big|_{x=n/t}.$$

As $(xg(x))' = g(x) + xg'(x) \in \mathcal{BF}$, we know $(-1)^{n+2}(xg(x))^{(n+2)} \geq 0$ for all $n \geq 1$. So $G_n(t)$ is convex for all $n \geq 1$.

Conversely, suppose now $g(x)$ is Bernstein with Lévy triplet (a, b, ν) where ν has convex tail. We prove that $g(x) + xg'(x)$ is Bernstein. Using (2.23), (2.25) and Lemma A.2.7,

$$\begin{aligned}
g(x) + xg'(x) &= 2g(x) - (g(x) - xg'(x)) \\
&= a + 2bx + \int_{(0,\infty)} (1 - e^{-xt})\, \nu(dt) + x \int_{(0,\infty)} e^{-xt}\bar{\nu}(t)\, dt - x^2 \int_{(0,\infty)} e^{-tx} t\bar{\nu}(t)\, dt \\
&= a + 2bx + \int_{(0,\infty)} (1 - e^{-xt})\, \nu(dt) + x \int_{(0,\infty)} \bar{\nu}(t)\, d(te^{-xt}) \\
&= a + 2bx + \int_{(0,\infty)} (1 - e^{-xt})\, \nu(dt) + x\bar{\nu}(t)te^{-xt}\Big|_{t=0}^{\infty} - x \int_{(0,\infty)} te^{-xt}\, d\bar{\nu}(t).
\end{aligned}$$

The limits (2.22) and (2.24) implies $\lim_{t\to\infty} \bar{\nu}(t)te^{-xt} = \lim_{t\to 0} \bar{\nu}(t)te^{-xt} = 0$. As $\bar{\nu}(t)$ is convex,

$$\begin{aligned}
g(x) + xg'(x) &= a + 2bx + \int_{(0,\infty)} (1 - e^{-xt})\, \nu(dt) - x \int_{(0,\infty)} te^{-xt}\, d\bar{\nu}(t) \\
&= a + 2bx - \int_{(0,\infty)} (1 - e^{-xt})\bar{\nu}'(t)\, dt - x \int_{(0,\infty)} te^{-xt}\bar{\nu}'(t)\, dt \\
&= a + 2bx - \int_{(0,\infty)} \bar{\nu}'(t)\, d(t(1 - e^{-xt})) \\
&= a + 2bx - \bar{\nu}'(t)t(1 - e^{-xt})\Big|_{t=0}^{\infty} + \int_{(0,\infty)} t(1 - e^{-xt})\, d\bar{\nu}'(t).
\end{aligned}$$

Integrability condition (2.17) and the convexity of $\bar{\nu}(t)$ implies

$$\frac{1}{2}\int_{(0,1)} (-\bar{\nu}'(t))\, dt^2 = \int_{(0,1)} t(-\bar{\nu}'(t))\, dt < \infty \quad \text{and} \quad \int_{(1,\infty)} (-\bar{\nu}'(t))\, dt < \infty.$$

Notice $(-\bar{\nu}'(t)) \geq 0$ is non-increasing. The first inequality, by Lemma A.2.1, implies

$$\lim_{t\to 0} \bar{\nu}'(t)t(1-e^{-xt}) = \lim_{t\to 0} \bar{\nu}'(t)t^2 = 0,$$

while the second inequality implies $\lim_{t\to\infty} t\bar{\nu}'(t) = 0$ by Lemma A.2.2. Therefore, we obtain

$$g(x) + xg'(x) = a + 2bx + \int_{(0,\infty)} (1-e^{-xt})t\, d\bar{\nu}'(t).$$

Notice that $\bar{\nu}'(t)$ is non-decreasing. So $g(x) + xg'(x)$ is Bernstein with Lévy measure defined by $t\, d\bar{\nu}'(t)$. Its integrability follows from the existence of $g(x) + xg'(x)$. Indeed, integrating by parts again, one could verify the measure $td\bar{\nu}'(t)$ satisfies (2.17). The proof is complete. □

Notice that we have already represent Bernstein functions whose Lévy measure has harmonically concave tail in Proposition 3.3.2. There could be an alternative proof.

Alternative proof for Theorem 5.1.3 Suppose $g(x)$ has Lévy triplet (a, b, ν). If ν has convex tail, using representation (3.26), we have that

$$g(x) + xg'(x) = a + 2bx + \int_{(0,\infty)} (1 - r(xs) - xsr'(xs))\, \tau(ds) = a + 2bx + \int_{(0,\infty)} (1-e^{-xs})\, \tau(ds).$$

Here $r(x)$ is defined in (3.21). Since τ is a Lévy measure by Proposition 3.3.2, we obtain $g(x) + xg'(x)$ is Bernstein.

Conversely, if $g(x) + xg'(x)$ is Bernstein with Lévy triplet (a', b', η), we know

$$g(x) + xg'(x) = a' + b'x + \int_{(0,\infty)} (1-e^{-xt})\, \eta(dt).$$

Note that $g(x) + xg'(x) = (xg(x))'$ and $\lim_{x\to 0} xg(x) = 0$ by (2.18). Hence, we have

$$\begin{aligned} xg(x) &= \int_{(0,x)} \Big(a' + b'u + \int_{(0,\infty)} (1-e^{-ut})\, \eta(dt)\Big)\, du = a'x + \frac{b'}{2}x^2 + \int_{(0,\infty)} \int_{(0,x)} (1-e^{-ut})\, du\, \eta(dt) \\ &= a'x + \frac{b'}{2}x^2 + \int_{(0,\infty)} \Big(x - \frac{1-e^{-xt}}{t}\Big)\eta(dt). \end{aligned}$$

Dividing both sides by x gives

$$g(x) = a' + \frac{b'}{2}x + \int_{(0,\infty)} (1 - r(xt))\, \eta(dt),$$

where $a', b' \geq 0$, $r(x)$ is defined in (3.21), and η is a Lévy measure. Proposition 3.3.2 implies that ν has convex tail. □

Example 5.1.4 *Function*

$$l_t(x) = 1 - \frac{1-e^{-xt}}{xt}$$

is Bernstein function whose Lévy measure has convex tail for all $t > 0$.

Proof Notice $(1 - e^{-xt})/xt \in \mathcal{CM}$ and is bounded from above by 1, as $1 - e^{-xt} \leq xt$ for $x > 0$ and $t > 0$. So $l_t(x)$ is Bernstein. Note that

$$l_t(x) + xl_t'(x) = 1 - e^{-xt} \in \mathcal{BF}.$$

By Theorem 5.1.3, we know the Lévy measure for $l_t(x)$ has convex tail. □

As composition of Bernstein functions are Bernstein, next corollary follows trivially.

Corollary 5.1.3 *Suppose $g(x)$ is Bernstein with Lévy measure ν.*

(a) if $(xg(x))^\alpha$ is Bernstein for some $\alpha \geq 1$, then ν has convex tail;

(b) if ν has convex tail, then $(xg(x))^\alpha$ is Bernstein for all $0 \leq \alpha \leq 1$.

It can be shown that the set of Bernstein functions whose Lévy measures have convex tails is a convex cone, and it is closed under non-negative scalar multiplication and scaling. That is, if $g_1(x)$ and $g_2(x)$ are Bernstein functions whose Lévy measures ν_1 and ν_2 have convex tails, then so are

$$\lambda g_1(x) + (1 - \lambda)g_2(x) \quad \text{and} \quad cg_1(\rho x),$$

for $\lambda \in [0, 1]$ and $c, \rho \geq 0$. They follow trivially from the fact that the tail for the measure of above Bernstein functions are

$$\lambda\bar{\nu}_1(t) + (1 - \lambda)\bar{\nu}_2(t) \quad \text{and} \quad c\bar{\nu}_1(t/\rho).$$

One can also use Theorem 5.1.3 to verify. In addition, Theorem 5.1.3 implies that the aforementioned set is closed under point-wise limit.

Corollary 5.1.4 *Let $\{g_n(x)\}_{n\in\mathbb{N}}$ be a sequence of Bernstein functions such that their point-wise limit $g(x) = \lim_{n\to\infty} g_n(x)$ exists for x on $(0, \infty)$. If $g_n(x)$ has Lévy measure with convex tail for all $n \geq 1$, then $g(x)$ is also Bernstein whose Lévy measure has convex tail.*

Proof It is shown in Proposition 2.2.2 that $g(x) \in \mathcal{BF}$ and $g'(x) = \lim_{n\to\infty} g_n'(x)$. Notice that $g_n(x) + xg_n'(x) \in \mathcal{BF}$ for all $n \geq 1$ by Theorem 5.1.3. Therefore, for all $x \in (0, \infty)$ we obtain

$$g(x) + xg'(x) = \lim_{n\to\infty} g_n(x) + xg_n'(x).$$

As $\mathcal{BF}$ is closed under point-wise limit, we conclude $g(x) + xg'(x) \in \mathcal{BF}$. This implies $g(x)$ has Lévy measure with convex tail by Theorem 5.1.3. □

5.2 Derivative free characterizations

In this section, we try to remove the derivatives in the characterizations in previous Section 4.1. The results correspond to the ones numbered (b) in Table 3.4.

5.2.1 Measures with harmonically concave tail

Lemma 5.2.1 *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). Then, Lévy measure ν has harmonically concave tail if and only if $\lambda g(x) - x(g(x+\lambda) - g(x))$ is Bernstein for all $\lambda > 0$.*

Proof We show sufficiency first. If $\lambda g(x) - x(g(x+\lambda) - g(x))$ is Bernstein for all $\lambda > 0$, so is

$$g(x) - \frac{x(g(x+\lambda) - g(x))}{\lambda}.$$

Letting λ approach 0, we know $g(x) - xg'(x)$ is Bernstein. By Theorem 5.1.1, measure ν has harmonically concave tail.

For necessity, assume measure ν has harmonically concave tail. Denote

$$\Lambda(x) := \lambda g(x) - x(g(x+\lambda) - g(x)) = (\lambda + x)g(x) - xg(x+\lambda).$$

We show $\Lambda(x)$ is Bernstein for all $\lambda > 0$. As $g(x)$ is concave, and $g(x) - xg'(x) \in \mathcal{BF}$ by Theorem 5.1.1,

$$\Lambda(x) = (\lambda + x)g(x) - xg(x+\lambda) \geq (\lambda + x)g(x) - x[g(x) + \lambda g'(x)] = \lambda[g(x) - xg'(x)] \geq 0.$$

In addition,

$$\begin{aligned}\Lambda'(x) &= (\lambda + x)g'(x) + g(x) - g(x+\lambda) - xg'(x+\lambda) \\ &= \Big[\Big(g(x+\lambda) - (x+\lambda)g'(x+\lambda)\Big) - \Big(g(x) - xg'(x)\Big)\Big] \\ &\quad - 2g(x+\lambda) + \lambda g'(x+\lambda) + 2g(x) + \lambda g'(x).\end{aligned}$$

As $g(x) - xg'(x) \in \mathcal{BF}$, we know $(g(x+\lambda) - (x+\lambda)g'(x+\lambda)) - (g(x) - xg'(x))$ is completely monotone by fact 6 in section 2.2.1. It suffices to show

$$-2g(x+\lambda) + \lambda g'(x+\lambda) + 2g(x) + \lambda g'(x) \in \mathcal{CM}.$$

By Lévy-Khintchine representation (2.16), we obtain

$$-2g(x+\lambda) + \lambda g'(x+\lambda) + 2g(x) + \lambda g'(x) = \int_{(0,\infty)} e^{-xt}(-2 + 2e^{-\lambda t} + \lambda t + \lambda t e^{-\lambda t})\, \nu(dt).$$

By Lemma A.1.4, we know $-2 + 2e^{-\lambda t} + \lambda t + \lambda t e^{-\lambda t} \geq 0$ for all $t > 0$ and $\lambda > 0$. Therefore, the above function is completely monotone and the proof is complete. □

5.2.2 Measures with convex tail

Lemma 5.2.2 *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). Then measure ν has convex tail, if and only if $\lambda g(x) + x(g(x+\lambda) - g(x))$ is Bernstein for all $\lambda > 0$.*

Proof We show sufficiency first. If $\lambda g(x) + x(g(x+\lambda) - g(x))$ is Bernstein for all $\lambda > 0$, so is

$$g(x) + \frac{x(g(x+\lambda) - g(x))}{\lambda}.$$

Letting λ approach 0, we know $g(x) + xg'(x)$ is Bernstein. By Theorem 5.1.3, measure ν has convex tail.

For necessity, assume measure ν has convex tail. Denote

$$\Lambda(x) := \lambda g(x) + x(g(x+\lambda) - g(x)) = xg(x+\lambda) - (x-\lambda)g(x).$$

We show that $\Lambda(x)$ is Bernstein for all $\lambda > 0$. It is trivial that $\Lambda(x) \geq 0$ for $x \leq \lambda$. For $x > \lambda$, as $g(x)$ is concave, we obtain

$$\begin{aligned}\Lambda(x) &= xg(x+\lambda) - (x-\lambda)g(x)\\ &\geq xg(x+\lambda) - (x-\lambda)[g(x+\lambda) - \lambda g'(x+\lambda)] = \lambda g(x+\lambda) + (x-\lambda)\lambda g'(x+\lambda) \geq 0.\end{aligned}$$

Thus, it suffices to show $\Lambda'(x)$ is completely monotone for all $\lambda > 0$.

$$\begin{aligned}\Lambda'(x) &= g(x+\lambda) + xg'(x+\lambda) - (x-\lambda)g'(x) - g(x)\\ &= g(x+\lambda) + (x+\lambda)g'(x+\lambda) - g(x) - xg'(x) + \lambda(g'(x) - g'(x+\lambda)).\end{aligned}$$

As $g(x)+xg'(x)$ is Bernstein, by Theorem 5.1.3 we know $g(x+\lambda)+(x+\lambda)g'(x+\lambda)-g(x)-xg'(x)$ is completely monotone. In addition, as $g'(x)$ is completely monotone, we know $g'(x)-g'(x+\lambda) \in \mathcal{CM}$, so is $\Lambda'(x)$. The proof is complete. □

5.3 Sequential characterizations

In this section, we characterize same subjects in term of sequences. Suppose $g(x)$ is Bernstein. Define a non-negative sequence $\{B_n(x)\}$ as

$$B_n(x) = \frac{(-1)^{n+1}}{n!} g^{(n)}(x)$$

for all $x > 0$ and $n \geq 1$. The shape of its Lévy measure can be characterized by $\{B_n(x)\}_n$. The results correspond to the ones numbered (c) in Table 5.1.

5.3.1 Measures with harmonically concave tail

Theorem 5.3.1 *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, measure ν has harmonically concave tail, if and only if the sequence $\{B_n(x)\}_{n\geq 1}$ satisfies*

$$\frac{n-1}{n+1} B_n(x) \leq xB_{n+1}(x) \tag{5.4}$$

for all $x > 0$ and $n \geq 1$.

Proof Proof of sufficiency. We need to show $\bar{\nu}(t)$ is harmonically concave, given (5.4). It suffices to show $t\bar{\nu}(t)$ is concave. This proof utilizes Inverse Laplace transformation formula in Theorem 2.1.4. Noticing (2.23), we have

$$-\Big(\frac{g(x)-a}{x}\Big)' = \int_{(0,\infty)} e^{-xt} t\bar{\nu}(t)\, dt.$$

Define for every $n \geq 1$,

$$G_n(t) := L_n\Big(-\Big(\frac{g(x)-a}{x}\Big)';t\Big).$$

Notice that

$$L_n\Big(\Big(\frac{a}{x}\Big)';t\Big) = \frac{(-1)^n}{n!}x^{n+1}\Big(\frac{a}{x}\Big)^{(n+1)}\Big|_{x=n/t} = -\frac{(n+1)t}{n}.$$

So $G_n(x)$ simplifies into

$$G_n(t) := \frac{(-1)^{n+1}}{n!}x^{n+1}\Big(\frac{g(x)}{x}\Big)^{(n+1)}\Big|_{x=n/t} - \frac{(n+1)t}{n}.$$

By Theorem 2.1.4, $\lim_{n\to\infty} G_n(t) = t\bar{\nu}(t)$ at every point of continuity of $\bar{\nu}(t)$. By lemma (A.1.12), it suffices to show $G_n(t)$ is concave for all $n \geq 1$. Notice $G_n(t)$ is infinitely differentiable. Observe

$$G_n'(t) = \frac{(-1)^{n+2}}{n\cdot n!}\Big((n+1)x^{(n+2)}\Big(\frac{g(x)}{x}\Big)^{(n+1)} + x^{n+3}\Big(\frac{g(x)}{x}\Big)^{(n+2)}\Big)\Big|_{x=n/t} - \frac{n+1}{n}.$$

By (2.14), we obtain

$$\begin{aligned}
G_n''(t) &= \frac{(-1)^{n+2}}{n\cdot n!}\Big((n+1)x^{n+1}g^{(n+2)}(x) + x^{n+2}g^{(n+3)}(x)\Big)\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big)\\
&= \frac{(-1)^{n+3}}{n^2\cdot n!}\Big((n+1)x^{n+3}g^{(n+2)}(x) + x^{n+4}g^{(n+3)}(x)\Big)\Big|_{x=n/t}\\
&= \frac{(n+1)^2(n+2)}{n^2}x^{n+3}B_{n+2}(x) - \frac{(n+1)(n+2)(n+3)}{n^2}x^{n+4}B_{n+3}(x)\Big|_{x=n/t}\\
&\leq \frac{(n+1)(n+2)(n+3)}{n^2}\Big(x^{n+4}B_{n+3}(x) - x^{n+4}B_{n+3}(x)\Big)\Big|_{x=n/t} = 0.
\end{aligned}$$

The last inequality implements (5.4). So $G_n(t)$ is concave for all $n \geq 1$.

Proof of necessity. Suppose measure ν has harmonically concave tail. We want to show inequality (5.4) holds. The case $n = 1$ is trivial as $B_n(x) \geq 0$. Consider $n \geq 2$.

As ν has harmonically concave tail, function $\bar{\nu}(1/t)$ is concave, which implies its left derivative $(\bar{\nu}(1/t))'_- = -\bar{\nu}'(1/t)/t^2$ is non-increasing, indicating $\bar{\nu}'(t)t^2$ is non-increasing. Noticing $\lim_{t\to\infty} \bar{\nu}(t) = 0$ and taking limits in (2.38), we obtain

$$-\bar{\nu}(t) = \int_{(t,\infty)} \bar{\nu}'(s)\,ds.$$

Therefore, for any $n \geq 2$,

$$\begin{aligned}
B_n(x) = \frac{(-1)^{n+1}}{n!}g^{(n)}(x) &= \frac{1}{n!}\int_{(0,\infty)} e^{-xt}t^n\,\nu(dt) = -\int_{(0,\infty)}\frac{t^n}{n!}e^{-xt}\bar{\nu}'(t)\,dt\\
&= -\frac{t^{n+1}}{(n+1)!}e^{-xt}\bar{\nu}'(t)\Big|_{t=0}^{t=\infty} + \int_{(0,\infty)}\frac{t^{n+1}}{(n+1)!}\,d(e^{-xt}\bar{\nu}'(t)).
\end{aligned}$$

By Lemma A.1.11, we have

$$\lim_{t\to\infty} t^{n+1}e^{-xt}\bar{\nu}'(t) = 0, \quad \text{and} \quad \lim_{t\to 0} t^{n+1}e^{-xt}\bar{\nu}'(t) = \lim_{t\to 0} t^{n-1}e^{-xt}\bar{\nu}'(t)t^2 = 0.$$

for $n \ge 2$ and $x > 0$. Therefore by Lemma A.2.7 and the finiteness of $B_n(x)$,

$$\begin{aligned}
B_n(x) &= \int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!}\, d(e^{-xt}t^{-2}t^2\bar{\nu}'(t)) \\
&= -x\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt}\bar{\nu}'(t)\,dt - 2\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt}t^{-3}t^2\bar{\nu}'(t)\,dt \\
&\quad + \int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt}t^{-2}\, d(t^2\bar{\nu}'(t)) \\
&\le x\int_{(0,\infty)} \frac{t^{n+1}}{(n+1)!} e^{-xt}\,\nu(dt) + \frac{2}{n+1}\int_{(0,\infty)} \frac{t^n}{n!} e^{-xt}\,\nu(dt) \\
&= xB_{n+1}(x) + \frac{2}{n+1}B_n(x).
\end{aligned}$$

It simplifies into (5.4). This closes the proof. □

In fact, the condition (5.4) is equivalent to $g(x) - xg'(x) \in \mathcal{BF}$. By Lemma A.1.1, we know $g(x) - xg'(x) \in \mathcal{BF}$ is equivalent to

$$(-1)^n(-xg^{(n+1)}(x) - (n-1)g^{(n)}(x)) \le 0,$$

for all $n \ge 1$. Rewriting the above inequality in terms of $B_n(x)$ reveals (5.4).

5.3.2 Measures with convex tails

Theorem 5.3.2 *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, measure ν has convex tail, if and only if the sequence $\{B_n(x)\}_{n\ge1}$ satisfies*

$$B_n(x) \ge xB_{n+1}(x) \tag{5.5}$$

for all $x > 0$ and $n \ge 1$.

Proof Proof of sufficiency. We need to show $\bar{\nu}(t)$ is convex given (5.5). This proof utilizes Inverse Laplace transformation formula in Theorem 2.1.4. By (2.23), we have

$$\frac{g(x) - a}{x} - b = \int_{(0,\infty)} e^{-xt}\bar{\nu}(t)\,dt.$$

Define for every $n \ge 1$,

$$G_n(t) := L_n\Big(\frac{g(x) - a}{x} - b; t\Big).$$

Notice that $L_n(a/x; t) = a$ and $L_n(b; t) = 0$ for all $n \ge 1$. The above formula simplifies into

$$G_n(t) = \frac{(-1)^n}{n!} x^{n+1}\Big(\frac{g(x)}{x}\Big)^{(n)}\Big|_{x=n/t} - a.$$

By Theorem 2.1.4, we know $\lim_{n\to\infty} G_n(t) = \bar{\nu}(t)$ at every point of continuity for $\bar{\nu}(t)$. To show ν has convex tail, by lemma A.1.12, it suffices to show $G_n(t)$ is convex for all $n \ge 1$. Notice $G_n(x)$ is infinitely differentiable. Observe

$$G_n'(t) = \frac{(-1)^{n+1}}{n \cdot n!} x^{n+2} g^{(n+1)}(x)\Big|_{x=n/t}.$$

And also

$$\begin{aligned}
G_n''(t) &= \frac{(-1)^{n+1}}{n\cdot n!}\Big((n+2)x^{n+1}g^{(n+1)}(x) + x^{n+2}g^{(n+2)}(x)\Big)\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big)\\
&= \frac{(-1)^{n+2}}{n^2\cdot n!}\Big((n+2)x^{n+3}g^{(n+1)}(x) + x^{n+4}g^{(n+2)}(x)\Big)\Big|_{x=n/t}\\
&= \frac{(n+2)!}{n^2\cdot n!}x^{n+3}B_{n+1}(x) - \frac{(n+2)!}{n^2\cdot n!}x^{n+4}B_{n+2}(x)\Big|_{x=n/t}\\
&\geq \frac{(n+2)!}{n^2\cdot n!}\Big(x^{n+4}B_{n+2}(x) - x^{n+4}B_{n+2}(x)\Big)\Big|_{x=n/t} = 0.
\end{aligned}$$

We use (5.4) in the last inequality. So $K_n(t)$ is convex for all $n \geq 1$.

Proof of necessity. Suppose measure ν has convex tail. We show inequality (5.5) holds. It suffices to show (5.5) for $a = b = 0$ in (2.16). As $\bar{\nu}(t)$ is convex, its right derivative $\bar{\nu}'(t)$ is non-decreasing. Noticing $\lim_{t\to\infty}\bar{\nu}(t) = 0$ and taking limits in (2.38), we get

$$-\bar{\nu}(t) = \int_{(t,\infty)} \bar{\nu}'(s)\,ds.$$

Therefore, for any $n \geq 1$,

$$\begin{aligned}
B_n(x) &= \frac{(-1)^{n+1}}{n!}g^{(n)}(x) = \frac{1}{n!}\int_{(0,\infty)} e^{-xt}t^n\,\nu(dt) = -\int_{(0,\infty)}\frac{t^n}{n!}e^{-xt}\bar{\nu}'(t)\,dt\\
&= -\frac{t^{n+1}}{(n+1)!}e^{-xt}\bar{\nu}'(t)\Big|_{t=0}^{t=\infty} + \int_{(0,\infty)}\frac{t^{n+1}}{(n+1)!}\,d(e^{-xt}\bar{\nu}'(t)).
\end{aligned}$$

As $t^n e^{-xt}\bar{\nu}'(t)$ is integrable at infinity, its limit is zero when t approaches infinity for all $x > 0$, which implies

$$\lim_{t\to\infty} t^{n+1}e^{-xt}\bar{\nu}'(t) = 0,$$

for all $x > 0$. On the other hand, from (2.17), we obtain

$$2\int_{(0,1)} t\,\nu(dt) = -\int_{(0,1)} 2t\bar{\nu}'(t)\,dt = \int_{(0,1)} -\bar{\nu}'(t)\,dt^2 < \infty.$$

Noticing $-\bar{\nu}'(t) \geq 0$ is non-increasing, by Lemma A.2.1, we obtain $\bar{\nu}'(t)$ is $o(1/t^2)$ when t approaches zero. That is $\lim_{t\to 0}\bar{\nu}'(t)t^2 = 0$. Thus,

$$\lim_{t\to 0} t^{n+1}e^{-xt}\bar{\nu}'(t) = \lim_{t\to 0} t^{n-1}e^{-xt}\bar{\nu}'(t)t^2 = 0,$$

for all $n \geq 1$ and $x > 0$. Therefore, by Lemma A.2.7 and noticing the finiteness of $B_{n+1}(x)$, we obtain

$$\begin{aligned}
B_n(x) &= -x\int_{(0,\infty)}\frac{t^{n+1}}{(n+1)!}e^{-xt}\bar{\nu}'(t)\,dt + \int_{(0,\infty)}\frac{t^{n+1}}{(n+1)!}e^{-xt}\,d\bar{\nu}'(t)\\
&\geq x\int_{(0,\infty)}\frac{t^{n+1}}{(n+1)!}e^{-xt}\,\nu(dt) = xB_{n+1}(x).
\end{aligned}$$

It simplifies into (5.5). This closes the proof. $\square$

In fact, the condition (5.5) is equivalent to $g(x) + xg'(x) \in \mathcal{BF}$. Note that $g(x) + xg'(x) \geq 0$ holds trivially. By Lemma A.1.1, we know $g(x) + xg'(x) \in \mathcal{BF}$ is equivalent to

$$(-1)^{n+1}(xg^{(n+1)}(x) + (n+1)g^{(n)}(x)) \geq 0,$$

for all $n \geq 1$. Rewriting the above inequality in terms of $B_n(x)$ connects to (5.5).

In addition, notice that there is a sub-class of Bernstein functions such that both (5.4) and (5.5) hold, in which the Lévy measures have both harmonically concave tail and convex tail. For example, Bernstein function $g(x) = x^\alpha$ has Lévy measure which has both harmonically concave tail and convex tail if $\alpha \in (0, 1)$.

5.4 Bernstein functions with β-convexity or β-concavity

In this section, we characterize Bernstein functions with β-convexity properties on the tails of Lévy measures. See Section (2.5) for definitions. These results generalize the characterizations in previous sections this chapter.

5.4.1 Characterizations

In Theorem 5.4.1, Bernstein functions whose measures have β-concave tail and β-convex tail are considered.

Theorem 5.4.1 *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). Consider the function*

$$N_\beta(x) = \beta(\beta-1)\frac{g(x)}{x} - 2(\beta-1)g'(x) + xg''(x) - \beta(\beta-1)\frac{a}{x} - (\beta-1)(\beta-2)b. \tag{5.6}$$

(a) Measure ν has β-convex tail, if and only if $N_\beta(x)$ is completely monotone.

(b) Measure ν has β-concave tail, if and only if $-N_\beta(x)$ is completely monotone.

Proof Notice $N_\beta(x)$ can be rewritten as

$$N_\beta(x) := \beta(\beta-1)\frac{g(x) - a - bx}{x} - 2(\beta-1)(g'(x) - b) + xg''(x).$$

Without loss of generality, we can assume $a = b = 0$. Otherwise consider $g(x) - a - bx$. By (2.23) we have

$$g(x) = x\int_{(0,\infty)} e^{-xt}\bar{\nu}(t)\,dt. \tag{5.7}$$

(a) We show sufficiency first. Suppose $N_\beta(x)$ is completely monotone. Anticipating the use of Inversion formula in Theorem 2.1.4, define

$$G_n(t) := L_n\left(\frac{g(x)}{x}; t\right) = (-1)^n x^{n+1}\left(\frac{g(x)}{x}\right)^{(n)}\Big|_{x=n/t}, \tag{5.8}$$

where the operator L_n is defined in (2.11). Theorem 2.1.4 part b) shows that $\lim_{n\to\infty} G_n(t) = \bar{\nu}(t)$ for every point of continuity of $\bar{\nu}(t)$. By Lemma A.1.12, it suffices to show $t^\beta G_n(t)$ is convex on $(0, \infty)$ for every $n \geq 1$. Notice that by (2.14),

$$G_n'(t) = (-1)^n \Big(x^{n+1}\Big(\frac{g(x)}{x}\Big)^{(n)}\Big)'\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) = (-1)^{n+1}\frac{1}{n}\,x^{n+2}g(x)^{(n+1)}\Big|_{x=n/t},$$

and

$$G_n''(t) = (-1)^{n+1}\frac{1}{n}\Big(x^{n+2}g(x)^{(n+1)}\Big)'\Big|_{x=n/t}\Big(-\frac{n}{t^2}\Big) = (-1)^{n+2}\frac{1}{n^2}\,x^{n+3}(xg(x))^{(n+2)}\Big|_{x=n/t}.$$

So we have

$$(t^\beta G_n(t))'' = t^{\beta-2}\Big(\beta(\beta-1)G_n(t) + 2\beta t G_n(t)' + t^2 G_n(t)''\Big) = t^{\beta-2}x^{n+1}(-1)^n N_\beta^{(n)}(x)\Big|_{x=n/t}.$$

As $N_\beta(x)$ is completely monotone, we know $(-1)^n N_\beta^{(n)}(x) \geq 0$ for all $x > 0$ and $n \geq 1$, which implies $t^\beta G_n(t)$ is convex.

Now we show necessity. Suppose measure ν has β-convex tail. As a result, function $t^\beta\bar{\nu}(t)$ is convex and $s^{1-\beta}\bar{\nu}(1/s)$ is also convex by Corollary 2.5.1. We prove that $N_\beta(x)$ is completely monotone. By (5.7) and change of variable $s = 1/t$,

$$\frac{g(x)}{x} = \int_{(0,\infty)} e^{-xt}\bar{\nu}(t)\,dt = \int_{(0,\infty)} e^{-x/s}s^{-2}\bar{\nu}(1/s)\,ds.$$

Therefore, differentiating under integral by [31, Theorem A.5.2], we have

$$g'(x) = \int_{(0,\infty)} e^{-x/s}s^{-2}\bar{\nu}(1/s)\,ds - x\int_{(0,\infty)} e^{-x/s}s^{-3}\bar{\nu}(1/s)\,ds,$$
$$g''(x) = -2\int_{(0,\infty)} e^{-x/s}s^{-3}\bar{\nu}(1/s)\,ds + x\int_{(0,\infty)} e^{-x/s}s^{-4}\bar{\nu}(1/s)\,ds,$$

To simplify the notation, denote

$$c_n(x) = \int_{(0,\infty)} e^{-x/s}s^{-2-n}\bar{\nu}(1/s)\,ds, \quad d_m(x) = \int_{(0,\infty)} e^{-x/s}s^{-2+m}\,d(s^{1-\beta}\bar{\nu}(1/s)).$$

Note that $c_n(x)$ is finite for all $n \geq 0$ and $d_m(x)$ is also convergent for all $m \leq \beta$, which could be verified by Lemma A.2.7. With these notations, we can rewrite

$$\frac{g(x)}{x} = c_0(x), \quad g'(x) = c_0(x) - xc_1(x), \quad xg''(x) = -2xc_1(x) + x^2c_2(x).$$

Hence

$$N_\beta(x) = \beta(\beta-1)c_0(x) - 2(\beta-1)(c_0(x) - xc_1(x)) - 2xc_1(x) + x^2c_2(x).$$

By (2.22), Lemma 2.5.6 and Lemma A.2.7, we have

$$xc_1(x) = x\int_{(0,\infty)} e^{-x/s}s^{-3}\bar{\nu}(1/s)\,ds = \int_{(0,\infty)} s^{-1}\bar{\nu}(1/s)\,d(e^{-x/s})$$

$$
\begin{aligned}
&= s^{-1}\bar{\nu}(1/s)e^{-x/s}\Big|_{s=0+}^{\infty} - \int_{(0,\infty)} e^{-x/s}\, d(s^{1-\beta}\bar{\nu}(1/s)s^{\beta-2}) \\
&= -\int_{(0,\infty)} e^{-x/s}s^{\beta-2}\, d(s^{1-\beta}\bar{\nu}(1/s)) + (2-\beta)\int_{(0,\infty)} e^{-x/s}s^{-2}\bar{\nu}(1/s)\, ds \\
&= -d_\beta(x) + (2-\beta)c_0(x).
\end{aligned}
$$

In addition, by Lemma 2.5.6, Lemma A.2.7 and Lemma A.2.9,

$$
\begin{aligned}
x^2c_2(x) &= x^2\int_{(0,\infty)} e^{-x/s}s^{-4}\bar{\nu}(1/s)\, ds = xs^{-2}\bar{\nu}(1/s)e^{-x/s}\Big|_{s=0+}^{\infty} - x\int_{(0,\infty)} e^{-x/s}\, d(s^{1-\beta}\bar{\nu}(1/s)s^{\beta-3}) \\
&= (3-\beta)x\int_{(0,\infty)} e^{-x/s}s^{-3}\bar{\nu}(1/s)\, ds - x\int_{(0,\infty)} e^{-x/s}s^{\beta-3}\, d(s^{1-\beta}\bar{\nu}(1/s)) \\
&= (3-\beta)xc_1(x) - x\int_{(0,\infty)} e^{-x/s}s^{\beta-3}(s^{1-\beta}\bar{\nu}(1/s))'\, ds \\
&= (3-\beta)xc_1(x) - s^{\beta-1}(s^{1-\beta}\bar{\nu}(1/s))'e^{-x/s}\Big|_{s=0+}^{\infty} + \int_{(0,\infty)} e^{-x/s}\, d(s^{\beta-1}(s^{1-\beta}\bar{\nu}(1/s))') \\
&= (3-\beta)xc_1(x) + \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d(s^{1-\beta}\bar{\nu}(1/s))' + (\beta-1)\int_{(0,\infty)} e^{-x/s}s^{\beta-2}\, d(s^{1-\beta}\bar{\nu}(1/s)) \\
&= (3-\beta)xc_1(x) + \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d(s^{1-\beta}\bar{\nu}(1/s))' + (\beta-1)d_\beta.
\end{aligned}
$$

Note that the above equations also hold if ν has β-concave tail. Therefore, it can be shown that

$$
N_\beta(x) = \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d(s^{1-\beta}\bar{\nu}(1/s))'.
$$

As $s^{1-\beta}\bar{\nu}(1/s)$ is convex, $(s^{1-\beta}\bar{\nu}(1/s))'$ is non-decreasing. It defines a Radon measure. One can see $N_\beta(x)$ is completely monotone by definition.

(b) The proof is very much analogous to the proof for part a), so we only provide the sketch of the proof. One can recover the full details without much difficulty. For sufficiency, suppose $-N_\beta(x)$ is completely monotone. Define $G_n(t)$ as (5.8). Without any further assumption,

$$
(t^\beta G_n(t))'' = t^{\beta-2}x^{n+1}(-1)^nN_\beta^{(n)}(x)\Big|_{x=n/t}.
$$

As $-N_\beta(x)$ is completely monotone, we know that $t^\beta G_n(t)$ is concave. By Lemma A.1.12, we conclude that ν has β-concave tail.

For necessity, suppose ν has β-concave tail, we prove $-N_\beta(x)$ is completely monotone. Applying the notation $c_n(x)$ and $d_m(x)$ in part a), we also have

$$
\begin{aligned}
xc_1(x) &= -d_\beta(x) + (2-\beta)c_0(x), \\
x^2c_2(x) &= (3-\beta)xc_1(x) + \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d((s^{1-\beta}\bar{\nu}(1/s))') + (\beta-1)d_\beta.
\end{aligned}
$$

Thus, we obtain

$$
-N_\beta(x) = \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d(-(s^{1-\beta}\bar{\nu}(1/s))').
$$

As function $s^{\beta}\bar{\nu}(s)$ is concave, we know $s^{1-\beta}\bar{\nu}(1/s)$ is concave by Corollary 2.5.1, which implies $-(s^{1-\beta}\bar{\nu}(1/s))'$ is non-decreasing. Hence it defines a Radon measure and we conclude $-N_{\beta}(x)$ is completely monotone. □

5.4.2 Connections with (harmonic) convexities on tails

In this section, we prove several theorems in Section 5.1 as corollaries. They reveals β-convexity on tail is indeed a generalization for both harmonically concave tail and convex tail.

Corollary 5.4.1 (Theorem 5.1.1) *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). Then, measure ν has harmonically concave tail, if and only if $g(x) - xg'(x) \in \mathcal{BF}$.*

Proof Consider the shifted Bernstein function $g(x)-a-bx$. By Theorem 5.4.1 part b), measure ν has harmonically concave tail, if and only if $-xg''(x)$ is completely monotone. We show this condition is equivalent to $g(x) - xg'(x)$ being Bernstein. If $g(x) - xg'(x)$ is Bernstein, then

$$-xg''(x) = (g(x) - xg'(x))' \in \mathcal{CM}.$$

Conversely, If $-xg''(x)$ is completely monotone, then, to show $g(x) - xg'(x)$ is Bernstein, it suffices to show it is non-negative. As its derivative is non-negative, $g(x) - xg'(x)$ is non-decreasing. Noticing $\lim_{x\to 0+} xg'(x) = 0$ by (2.26), we obtain

$$\lim_{x\to 0} g(x) - xg'(x) = a \geq 0.$$

So $g(x) - xg'(x) \geq 0$, and this closes the proof. □

Corollary 5.4.2 (Theorem 5.1.3) *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). Then, measure ν has convex tail, if and only if $g(x) + xg'(x) \in \mathcal{BF}$.*

Proof Consider the shifted Bernstein function $g(x)-a-bx$. By Theorem 5.4.1 part a), measure ν has convex tail, if and only if $2g'(x) - 2b + xg''(x)$ is completely monotone. We show this condition is equivalent to $g(x) + xg'(x)$ being Bernstein.

If $g(x) + xg'(x)$ is Bernstein, then

$$2g'(x) + xg''(x) = (g(x) + xg'(x))' \in \mathcal{CM}.$$

It suffice to show $2g'(x) + xg''(x) \geq 2b$. This is true, as $2g'(x) + xg''(x)$ is non-increasing and by (2.18)

$$\lim_{x\to\infty} 2g'(x) + xg''(x) = \lim_{x\to\infty}(g(x) + xg'(x))' = \lim_{x\to\infty} \frac{g(x) + xg'(x)}{x} = 2b.$$

The second equation follows from $g(x) + xg'(x) \in \mathcal{BF}$.

Conversely, If $2(g'(x) - b) + xg''(x)$ is completely monotone, then it suffices to show $g(x) + xg'(x) \geq 0$ to see $g(x) + xg'(x)$ is Bernstein. This is trivial, because $g(x) \geq 0$ and $g'(x) \geq 0$. □

It could also be used to show Lévy measure could not have concave tail or harmonically convex tail unless it vanishes on $(0, \infty)$.

Corollary 5.4.3 *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). Then, measure ν has harmonically convex tail, if and only if $g(x) = a + bx$.*

Proof Sufficiency is trivial. For necessity, by Theorem 5.4.1 part a), measure ν has harmonically convex tail, then $xg''(x)$ is completely monotone. Because $g(x)$ is Bernstein, $g''(x) \leq 0$. Thus, we obtain $g''(x) = 0$, which implies $g(x) = a + bx$. □

Corollary 5.4.4 *Suppose $g(x)$ is Bernstein with Lévy triplet (a, b, ν). Then, measure ν has concave tail, if and only if $g(x) = a + bx$.*

Proof Sufficiency is trivial. For necessity, by Theorem 5.4.1 part b), measure ν has concave tail, then $-2(g'(x) - b) - xg''(x)$ is completely monotone. Thus, its anti-derivative $g(x) - a - bx + x(g'(x) - b)$ is non-increasing. Notice

$$\lim_{x \to 0} g(x) - a - bx + x(g'(x) - b) = 0.$$

We know $g(x) - a - bx + x(g'(x) - b) \leq 0$. Note that $g(x) - a - bx \geq 0$ and $g'(x) - b \geq 0$ by definition. Thus, both terms are identical to 0, indicating $g(x) = a + bx$. □

5.5 Convex shape preserving transformations

In this section, we revisit [69, Corollary 5.4 and 6.2], and investigate transformations that maps a Bernstein function into another with certain convexity properties on the tail of its measure.

In addition, we investigate some transformations that can preserve certain convexity properties. For any function $g : (0, \infty) \mapsto \mathbb{R}$, recall operator $H[g](x)$ is defined in (4.8) as following.

$$H[g](x) := x \int_{(x,\infty)} \frac{g(s)}{s^2} \, ds.$$

Proposition 5.5.1 *The set of $\mathcal{BF}$ whose Lévy measure has harmonically concave tail is*

$$\Big\{ H[g](x) : g(x) \in \mathcal{BF}, \textit{such that the integral is convergent} \Big\}.$$

Proof Suppose $g(x) \in \mathcal{BF}$ has Lévy triplet (a, b, ν) such that $H[g](x)$ is convergent. It can be rewritten as

$$H[g](x) = x \int_{(x,\infty)} \frac{a}{s^2} \, ds + x \int_{(x,\infty)} \frac{b}{s} \, ds + x \int_{(x,\infty)} \int_{(0,\infty)} \frac{(1 - e^{-st})}{s^2} \, \nu(dt) \, ds.$$

Integrability implies $b = 0$ and the last integral is convergent. By Fubini's Theorem,

$$H[g](x) = a + x \int_{(0,\infty)} \int_{(x,\infty)} \frac{(1 - e^{-st})}{s^2} \, ds \, \nu(dt) = a + \int_{(0,\infty)} x \int_{(x,\infty)} \frac{(1 - e^{-st})}{s^2} \, ds \, \nu(dt).$$

It suffices to show

$$l_t(x) := x \int_{(x,\infty)} \frac{(1 - e^{-st})}{s^2}\, ds$$

is Bernstein with measure having harmonically concave tail for all $t > 0$. This is true by Example 5.1.2.

Conversely, suppose $h(x) \in \mathcal{BF}$ with Lévy triplet (a', b', η). If measure η has harmonically concave tail, then, by Theorem 5.1.1, we know $g(x) := h(x) - xh'(x) \in \mathcal{BF}$. It can be shown that $h(x) = H[g](x) + cx$ is the solution to the above differential equation, where $c \in \mathbb{R}$ is constant. Using the limit condition $\lim_{x\to\infty} h(x)/x = b'$, we could get $c = 0$. Therefore we obtain $h(x) = H[g](x)$, implying $h(x)$ belongs to the above set. This closes the proof. □

Tauberian Theorem provides us with an sufficient condition such that $H[g](x)$ is well-defined. Suppose $g(x) \in \mathcal{BF}$ with Lévy triplet (a, b, ν). If $b = 0$ and there is some constant $\gamma > 0$, such that measure ν satisfies

$$\int_{(0,u)} t\,\nu(dt) = O(u^\gamma), \quad \text{as} \quad u \to 0,$$

then $g'(x) = O(1/x^\gamma)$ as x approaches infinity, by [80, Corollary 1a, Chapter V]. Hence, integrating by parts, we obtain

$$\begin{aligned}\int_{(x,\infty)} \frac{g(s)}{s^2}\, ds &= -\int_{(x,\infty)} g(s)\, d(1/s) = -\frac{g(s)}{s}\Big|_{s=x}^{\infty} + \int_{(x,\infty)} \frac{g'(s)}{s}\, ds \\ &= -b + \frac{g(x)}{x} + \int_{(x,\infty)} \frac{g'(s)}{s}\, ds < \infty.\end{aligned}$$

This means $H[g](x)$ being well-defined.

Bernstein functions with convex tail measure also entitled with a similar description. For function $g : (0, \infty) \mapsto \mathbb{R}$ recall operator $K[g](x)$ is defined in (4.9) as following.

$$K[g](x) := \frac{1}{x} \int_{(0,x)} g(s)\, ds.$$

Proposition 5.5.2 *The subset of $\mathcal{BF}$ with convex tail measure is*

$$\Big\{K[g](x) : g(x) \in \mathcal{BF}\Big\}.$$

Proof It is trivial that the set above is well-defined for any Bernstein function. Suppose $g(x)$ has Lévy triplet (a, b, ν), then by Fubini's theorem

$$K[g](x) = a + \frac{1}{2}bx + \int_{(0,\infty)} \frac{1}{x} \int_{(0,x)} (1 - e^{-st})\, ds\, \nu(dt) = a + \frac{1}{2}bx + \int_{(0,\infty)} \Big(1 - \frac{1 - e^{-xt}}{xt}\Big) \nu(dt).$$

It suffices to show

$$l_t(x) := 1 - \frac{1 - e^{-xt}}{xt}$$

is Bernstein whose Lévy measure has convex tail for all $t > 0$. This is true by Example 5.1.4.

Conversely, if $h(x) \in \mathcal{BF}$ has convex tail measure, then by Theorem 5.1.3, we know $g(x) := h(x) + xh'(x)$ is Bernstein. It can be shown that $h(x) = K[g](x) + c/x$, where $c \in \mathbb{R}$ is constant. Using the condition $\lim_{x\to 0} xh(x) = 0$, we conclude $c = 0$. Therefore $h(x) = K[g](x)$, lying in the above set. This closes the proof. □

Next we consider transformations that preserve certain convexity properties on measures. Part a) in the following proposition generalizes Lemma 2.4.3.

Proposition 5.5.3 *Suppose $g(x) \in \mathcal{BF}$. If it has harmonically concave tail Lévy measure, then*

(a) $g(x+\lambda) - g(x) \in \mathcal{CM}$ has harmonically convex measure for all $\lambda > 0$.

(b) $g(x) + g(\lambda) - g(x+\lambda) \in \mathcal{BF}$ also has Lévy measure with harmonically concave tail for all $\lambda > 0$.

Proof (a) By fact 2 listed in Section 2.2.1, we know $g(x+\lambda) - g(x)$ is completely monotone. To show it has harmonically convex measure, consider

$$\begin{aligned} & g(x+\lambda) - g(x) - x(g(x+\lambda) - g(x))' \\ = {} & g(x+\lambda) - (x+\lambda)g'(x+\lambda) - g(x) + xg'(x) + \lambda g'(x+\lambda). \end{aligned}$$

As $g(x) - xg'(x) \in \mathcal{BF}$ by Theorem 5.1.1, we know

$$g(x+\lambda) - (x+\lambda)g'(x+\lambda) - g(x) + xg'(x) \in \mathcal{CM}.$$

Note that $g'(x+\lambda) \in \mathcal{CM}$. Thus, the above function is completely monotone and $g(x+\lambda)-g(x)$ has harmonically convex measure by Theorem 4.1.1.

(b) By Theorem 5.1.1, we know $g(x) - xg'(x)$ is Bernstein and hence $-xg''(x) \in \mathcal{CM}$. Also by Theorem 5.1.1, it suffices to show $g(x) + g(\lambda) - g(x+\lambda) - x(g'(x) - g'(x+\lambda))$ is Bernstein. Notice

$$\lim_{x\to 0} g(x) + g(\lambda) - g(x+\lambda) - x(g'(x) - g'(x+\lambda)) = \lim_{x\to 0} g(x) - xg'(x) = g(0+) \geq 0.$$

It suffices to show $-x(g''(x) - g''(x+\lambda)) \in \mathcal{CM}$. We obtain

$$-x(g''(x) - g''(x+\lambda)) = -xg''(x) + (x+\lambda)g''(x+\lambda) - \lambda g''(x+\lambda).$$

Since $-xg''(x) + (x+\lambda)g''(x+\lambda) \in \mathcal{CM}$ and $-g''(x+\lambda) \in \mathcal{CM}$, the above function is completely monotone. This closes the proof. □

In general, if Bernstein function $g(x)$ has Lévy measure with convex tail, then the above transformations do not have such convexity properties.

Example 5.5.1 *Consider Bernstein function*

$$g(x) = 1 - \frac{1 - e^{-x}}{x}$$

Its Lévy measure has convex tail, and

(a) the measure for $g(x+1) - g(x) \in \mathcal{CM}$ is not concave measure, nor convex measure;

(b) the Lévy measure for $g(x) + g(1) - g(x+1) \in \mathcal{BF}$ does not have harmonically concave tail, nor convex tail.

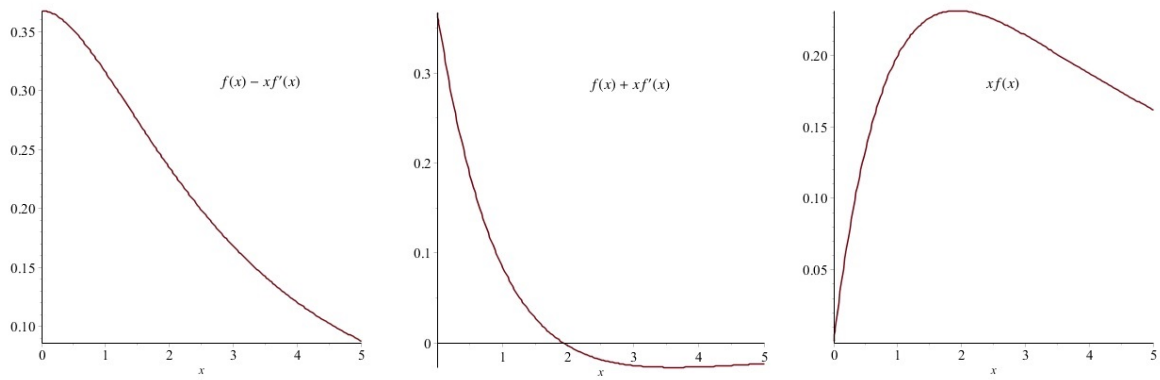

Figure 5.1: Function $f(x) - xf'(x)$, $f(x) + xf'(x)$ and $xf(x)$ where $f(x) = g(x+1) - g(x)$ and $g(x) = 1 - (1 - e^{-x})/x$

By Theorem 5.1.3, noticing $g(x) + xg'(x) = 1 - e^{-x} \in \mathcal{BF}$, we know $g(x)$ has Lévy measure with convex tail. Consider completely monotone function $f(x) := g(x+1) - g(x)$. Note that $\lim_{x\to\infty} f(x) = 0$. In addition, we know

$$f(x) - xf'(x) \notin \mathcal{CM}, \quad f(x) + xf'(x) \notin \mathcal{CM}, \quad \text{and} \quad xf(x) \notin \mathcal{CM}.$$

See Figure 5.1. By Theorem 4.1.1, Theorem 4.1.3 and Theorem 4.1.5, $f(x)$ does not have harmonically convex measure, concave measure, or convex measure.

Consider Bernstein function $h(x) := g(x) + g(1) - g(x+1)$. It can be shown that

$$h(x) - xh'(x) \notin \mathcal{BF} \quad \text{and} \quad h(x) + xh'(x) \notin \mathcal{BF}.$$

See Figure 5.2. By Theorem 5.1.1 and Theorem 5.1.3, $h(x)$ does not have Lévy measure with harmonically concave tail, or convex tail.

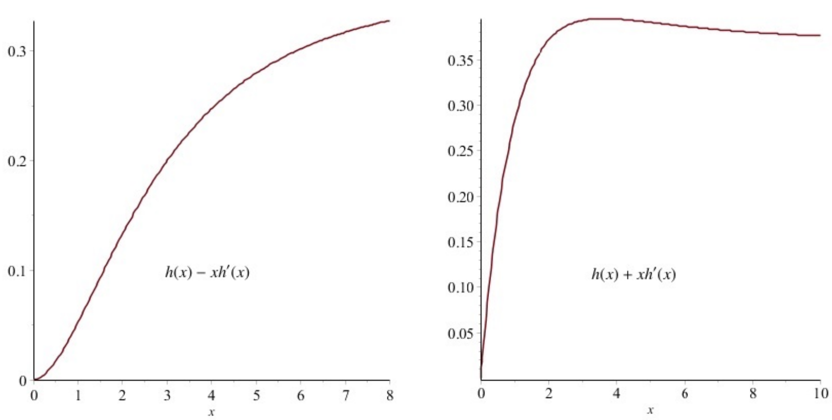

Figure 5.2: Function $h(x) - xh'(x)$ and $h(x) + xh'(x)$ where $h(x) = g(x) + g(1) - g(x+1)$ and $g(x) = 1 - (1 - e^{-x})/x$

Chapter 6

Sub-classes of Bernstein functions

In this section, we introduce a family of completely monotone functions. And then we identify a sub-class of Bernstein functions which enable us to construct completely monotone and Bernstein functions with convexity properties on their measures by compositions. It is shown that these sub-classes can be characterized by their convolution semigroups.

6.1 A family of completely monotone functions

In this section, we identify a class of completely monotone functions which play important roles

Lemma 6.1.1 *For any* $\alpha \in [0, 2/3]$ *and* $s \geq 0$*, the function*

$$f(x) = e^{-sx^\alpha}(1 + s\alpha x^\alpha)$$

is completely monotone.

Proof By the definition of completely monotone function, it suffices to verify the n-th derivatives satisfy $(-1)^n f^{(n)}(x) \geq 0$ for all $n \geq 0$. This is trivially true for $n = 0$. For $n \geq 1$, we will use mathematical induction to show that

$$f^{(n)}(x) = s\alpha e^{-sx^\alpha} x^{\alpha-n} P_n(x^\alpha), \tag{6.1}$$

where $P_n(x) = a_0^{(n)} + a_1^{(n)}x + \cdots + a_n^{(n)}x^n$, whose coefficients satisfy

$$(-1)^n a_k^{(n)} \geq 0 \text{ for all } k = 0, 1, 2, \ldots, n. \tag{6.2}$$

For $n = 1$, it can be shown that

$$\begin{aligned} f'(x) &= e^{-sx^\alpha}(-s\alpha x^{\alpha-1})(1 + s\alpha x^\alpha) + e^{-sx^\alpha} s\alpha^2 x^{\alpha-1} \\ &= s\alpha e^{-sx^\alpha} x^{\alpha-1}(-(1-\alpha) - s\alpha x^\alpha) = s\alpha e^{-sx^\alpha} x^{\alpha-1} P_1(x^\alpha), \end{aligned}$$

where we defined $P_1(x) = a_0^{(1)} + a_1^{(1)}x := -(1-\alpha) - s\alpha x$. Since both coefficients $a_0^{(1)}$ and $a_1^{(1)}$ are negative, we are done in this case.

For $n = 2$, we have

$$\begin{aligned}
f''(x) &= \frac{d}{dx}\Big(s\alpha e^{-sx^\alpha}[-(1-\alpha)x^{\alpha-1} - s\alpha x^{2\alpha-1}]\Big) \\
&= s\alpha e^{-sx^\alpha}(-s\alpha x^{\alpha-1})[-(1-\alpha)x^{\alpha-1} - s\alpha x^{2\alpha-1}] \\
&\quad + s\alpha e^{-sx^\alpha}[-(1-\alpha)(\alpha-1)x^{\alpha-2} - s\alpha(2\alpha-1)x^{2\alpha-2}] \\
&= s\alpha e^{-sx^\alpha}x^{\alpha-2}[(1-\alpha)^2 + s\alpha(2-3\alpha)x^\alpha + s^2\alpha^2x^{2\alpha}] \\
&= s\alpha e^{-sx^\alpha}x^{\alpha-2}P_2(x^\alpha),
\end{aligned}$$

where we defined $P_2(x) = a_0^{(2)} + a_1^{(2)}x + a_2^{(2)}x^2 := (1-\alpha)^2 + s\alpha(2-3\alpha)x + s^2\alpha^2x^2$. Using the condition $\alpha \in [0, 2/3]$ one readily sees that the three coefficients $a_0^{(2)}$, $a_1^{(2)}$ and $a_2^{(2)}$ are non-negative, concluding the case.

Suppose representation (6.1) and conditions (6.2) hold for $n \geq 2$. We need to show they also hold for $n + 1$.

$$\begin{aligned}
f^{(n+1)}(x) &= \frac{d}{dx}f^{(n)}(x) = \frac{d}{dx}\Big(s\alpha e^{-sx^\alpha}x^{\alpha-n}(a_0^{(n)} + a_1^{(n)}x^\alpha + \cdots + a_n^{(n)}x^{n\alpha})\Big) \\
&= \frac{d}{dx}\Big(s\alpha e^{-sx^\alpha}(a_0^{(n)}x^{\alpha-n} + a_1^{(n)}x^{2\alpha-n} + \cdots + a_n^{(n)}x^{(n+1)\alpha-n})\Big) \\
&= s\alpha e^{-sx^\alpha}\Big(-s\alpha x^{\alpha-1}\Big)\Big(a_0^{(n)}x^{\alpha-n} + a_1^{(n)}x^{2\alpha-n} + \cdots + a_n^{(n)}x^{(n+1)\alpha-n}\Big) \\
&\quad + s\alpha e^{-sx^\alpha}\Big(a_0^{(n)}(\alpha-n)x^{\alpha-(n+1)} + a_1^{(n)}(2\alpha-n)x^{2\alpha-(n+1)} \\
&\qquad\qquad + \cdots + a_n^{(n)}((n+1)\alpha-n)x^{(n+1)\alpha-(n+1)}\Big) \\
&= s\alpha e^{-sx^\alpha}x^{\alpha-(n+1)}\Big[-a_0^{(n)}s\alpha x^\alpha - a_1^{(n)}s\alpha x^{2\alpha} - \cdots - a_n^{(n)}s\alpha x^{(n+1)\alpha} \\
&\quad + a_0^{(n)}(\alpha-n) + a_1^{(n)}(2\alpha-n)x^\alpha + \cdots + a_n^{(n)}((n+1)\alpha-n)x^{n\alpha}\Big] \\
&= s\alpha e^{-sx^\alpha}x^{\alpha-(n+1)}\Big[a_0^{(n)}(\alpha-n) + \Big(a_1^{(n)}(2\alpha-n) - a_0^{(n)}s\alpha\Big)x^\alpha \\
&\quad + \cdots + \Big(a_n^{(n)}((n+1)\alpha-n) - a_{n-1}^{(n)}s\alpha\Big)x^{n\alpha} + a_n^{(n)}(-s\alpha)x^{(n+1)\alpha}\Big] \\
&= s\alpha e^{-sx^\alpha}x^{\alpha-(n+1)}P_{n+1}(x^\alpha),
\end{aligned}$$

where the coefficients $a_0^{(n+1)}, a_1^{(n+1)}, \dots, a_{n+1}^{(n+1)}$ of the polynomial $P_{n+1}(x)$ are defined inductively by

$$\begin{aligned}
a_0^{(n+1)} &= (\alpha-n)a_0^{(n)}, \\
a_k^{(n+1)} &= ((k+1)\alpha-n)a_k^{(n)} - s\alpha a_{k-1}^{(n)} \quad \text{for} \quad k = 1, 2, \dots, n, \\
a_{n+1}^{(n+1)} &= -s\alpha a_n^{(n)}.
\end{aligned}$$

Utilizing conditions (6.2), we obtain

$$\begin{aligned}
(-1)^{(n+1)}a_0^{(n+1)} &= (-1)^{(n+1)}(\alpha-n)a_0^{(n)} = (n-\alpha)(-1)^{(n)}a_0^{(n)} \geq 0, \\
(-1)^{(n+1)}a_{n+1}^{(n+1)} &= (-1)^{(n+1)}(-s\alpha a_n^{(n)}) = s\alpha(-1)^{(n)}a_n^{(n)} \geq 0.
\end{aligned}$$

Next, fixing an index $k \in \{1, 2, \dots, n\}$, we have

$$(-1)^{(n+1)}a_k^{(n+1)} = (n-(k+1)\alpha)(-1)^{(n)}a_k^{(n)} + s\alpha(-1)^{(n)}a_{k-1}^{(n)}.$$

Since $n \geq 2$, we have

$$\frac{n}{k+1} \geq \frac{n}{n+1} \geq \frac{2}{3} \geq \alpha.$$

Together with (6.2) this implies that $(-1)^{(n+1)} a_k^{(n+1)} \geq 0$ and concludes the induction. □

Next proposition shows that the set of all numbers α in $[0, 1]$, for which (8.1) is a completely monotone function for all $s \geq 0$, is a closed interval. Because completely monotone functions are closed under point-wise convergence.

Proposition 6.1.1 *If for some $r \in (0, 1]$ the function*

$$f_r(x) := e^{-sx^r}(1 + srx^r)$$

is completely monotone for all $s \geq 0$, then $f(x) = e^{-sx^\alpha}(1 + s\alpha x^\alpha)$ is completely monotone for all $\alpha \in [0, r]$ and all $s \geq 0$.

Proof Since $0 \leq \alpha/r \leq 1$, the function

$$f_r(x^{\alpha/r}) = e^{-sx^\alpha}(1 + srx^\alpha)$$

is the composition of the completely monotone function $f_r(x)$ and the Bernstein function $x^{\alpha/r}$, hence, it is completely monotone for all $s \geq 0$. The function that we are interested in is

$$f(x) = e^{-sx^\alpha}(1 + s\alpha x^\alpha) = e^{-s(1-\frac{\alpha}{r})x^\alpha} e^{-\frac{s\alpha}{r}x^\alpha}\left(1 + \frac{s\alpha}{r} r x^\alpha\right). \tag{6.3}$$

Note that $e^{-s(1-\alpha/r)x^\alpha}$ is the composition of the completely monotone function $e^{-s(1-\alpha/r)x}$ and the Bernstein function x^α, hence it is completely monotone. Thus, (6.3) expresses the left-hand side as a product of two completely monotone functions, hence it is completely monotone for all $s \geq 0$. □

Our numerical experiments show that the largest number r such that $f_r(x) \in \mathcal{CM}$ for all $s \geq 0$ is around 0.7424. In addition, Example 6.1.2 implies that it is less than 1.

Example 6.1.2 *Function $f(x) = e^{-x^{0.9}}(1 + 0.9x^{0.9}) \notin \mathcal{CM}$.*

This can be verified easily, see Figure 6.1.

6.2 Sub-classes of Bernstein functions

By Theorem 2.2.4, we know that for any $f(x) \in \mathcal{CM}$ and $g(x) \in \mathcal{BF}$, their composition $f(g(x)) \in \mathcal{BF}$. And it is also know that $\mathcal{BF}$ is closed under composition. In this section, we identify sub-classes of $\mathcal{BF}$, such that their compositions with completely monotone or Bernstein functions have particular convexity properties on their measures. Notations are defined below.

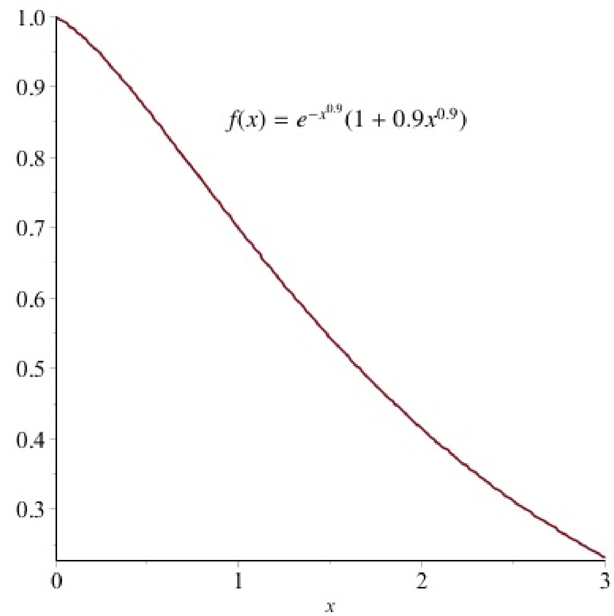

Figure 6.1: Function $f(x) = e^{-x^{0.9}}(1 + 0.9x^{0.9})$

Definition 6.2.1 *Sub-classes $\mathcal{H}_{CM}, \mathcal{H}_{BF} \subset \mathcal{BF}$ are defined as following.*

$$\mathcal{H}_{CM} := \{g(x) \in \mathcal{BF} : f(g(x)) \textit{ has harmonically convex measure}, \forall f(x) \in \mathcal{CM}\};$$
$$\mathcal{H}_{BF} := \{g(x) \in \mathcal{BF} : h(g(x)) \textit{ has harmonically concave tail Lévy measure}, \forall h(x) \in \mathcal{BF}\}.$$

From the definition follows some trivial properties of these sub-classes.

(a) They are not empty. Non-negative constant functions lie in every sub-classes.

(b) They are closed under composition. For example if $g(x), h(x) \in \mathcal{H}_{CM}$, then $g \circ h \in \mathcal{H}_{CM}$.

(c) They are closed under left composition with $\mathcal{BF}$. For example if $h(x) \in \mathcal{BF}$ and $g(x) \in \mathcal{H}_{CM}$, then $h \circ g \in \mathcal{H}_{CM}$.

(d) For any $g(x) \in \mathcal{H}_{BF}$, it has Lévy measure with harmonically concave tail.

Next theorems imply that these sets are not trivial. Moreover, we will show that $\mathcal{H}_{CM} = \mathcal{H}_{BF}$ in Thereom 6.3.1.

Theorem 6.2.2 *Let $g(x) = x^\alpha$, where $\alpha \in [0, 2/3]$. Then for any $f(x) \in \mathcal{CM}$, the composition $f \circ g(x) = f(x^\alpha)$ is completely monotone with harmonically convex measure.*

Proof The composition $f(g(x))$ is completely monotone, we need to show its measure is harmonically convex. Consider

$$h(x) = f(g(x)) - x(f(g(x)))' = f(x^\alpha) - \alpha x^\alpha f'(x^\alpha).$$

By Theorem 4.1.1, it suffices to verify that $h(x)$ is completely monotone. Notice that

$$h(x) = \int_{[0,\infty)} e^{-tx^\alpha}\,\mu(dt) + \alpha x^\alpha \int_{[0,\infty)} e^{-tx^\alpha}\,t\,\mu(dt) = \int_{[0,\infty)} e^{-tx^\alpha}(1 + t\alpha x^\alpha)\,\mu(dt).$$

By Lemma 6.1.1, the integrand $e^{-tx^\alpha}(1 + t\alpha x^\alpha)$ is completely monotone with for all $t \geq 0$. Therefore $h(x)$ is also completely monotone. The proof is complete. □

A quick observation from Theorem 6.2.2 is that the set $\mathcal{H}_{CM}$ is not trivial, and

$$\{x^\alpha, \alpha \in [0, 2/3]\} \subset \mathcal{H}_{CM}.$$

As corollary, we can represent completely monotone functions in terms of harmonically convex measure.

Corollary 6.2.1 *For any completely monotone function $f(x)$ and a number $\alpha \in (0, 2/3]$, there exists a unique harmonically convex measure μ_α on $[0,\infty)$, such that*

$$f(x) = \int_{[0,\infty)} e^{-tx^{1/\alpha}}\,\mu_\alpha(dt).$$

This corollary follows immediately by a simple change of variables in the Bernstein representation for the completely monotone function $f(x^\alpha)$.

It is worth to point out that Theorem 6.2.2 fails when α is close to 1. Consider completely monotone function $f(x) = e^{-x}$ and Bernstein function $g(x) = x^{0.9}$. It can be show by Theorem 4.1.1 and Example 6.1.2 that $f(g(x)) = e^{-x^{0.9}}$ does not have harmonically convex measure.

Theorem 6.2.3 *Let $g(x) = x^\alpha$, where $\alpha \in [0, 2/3]$. Then, for any $h(x) \in \mathcal{BF}$, the composition $h \circ g(x) = h(x^\alpha)$ is Bernstein whose Lévy measure has harmonically concave tail.*

Proof Let $h(x)$ be any Bernstein function determined by a Lévy triplet (a, b, η). Suppose $\alpha \in [0, 2/3]$. We want to show that the Lévy measure corresponding to the composition $h \circ g$ has harmonically concave tail. By Theorem 5.1.1, consider

$$\begin{aligned} h(g(x)) - x(h(g(x)))' &= h(x^\alpha) - \alpha x^\alpha h'(x^\alpha) \\ &= a + bx^\alpha + \int_{(0,\infty)} (1 - e^{-tx^\alpha})\,\eta(dt) - \alpha x^\alpha \Big(b + \int_{(0,\infty)} e^{-tx^\alpha}\,t\,\eta(dt)\Big) \\ &= a + b(1-\alpha)x^\alpha + \int_{(0,\infty)} \Big(1 - e^{-tx^\alpha}(1 + t\alpha x^\alpha)\Big)\,\eta(dt). \end{aligned}$$

It suffices to show this function is Bernstein. By Lemma 6.1.1, $e^{-tx^\alpha}(1 + t\alpha x^\alpha)$ is completely monotone for every t. By (2.33), we know $1 - e^{-tx^\alpha}(1 + t\alpha x^\alpha)$ is Bernstein for every $t \geq 0$. The proof is completed. □

A quick conclusion from Theorem 6.2.3 is that the set $\mathcal{H}_{BF}$ is not trivial as well, and

$$\{x^\alpha, \alpha \in [0, 2/3]\} \subset \mathcal{H}_{BF}.$$

It also implies the next corollary, where Bernstein functions are represented in terms of measure with harmonically concave tail.

Corollary 6.2.2 *For any Bernstein function $g(x)$ and a number $\alpha \in (0, 2/3]$, there exists a unique triplet (a, b, ν_α), such that*

$$g(x) = a + bx^{1/\alpha} + \int_{(0,\infty)} (1 - e^{-tx^{1/\alpha}})\, \nu_\alpha(dt),$$

where $a, b \geq 0$ are constants, and ν_α is a measure on $(0, \infty)$ with harmonically concave tail. The measure ν_α satisfies

$$\int_{(0,\infty)} (1 \wedge t)\, \nu_\alpha(dt) < \infty.$$

Theorem 6.2.2 follows immediately by a simple change of variables in the Lévy-Khintchine representation for the Bernstein function $g(x^\alpha)$.

Analogously, Theorem 6.2.3 fails when α is close to 1. Consider Bernstein functions $h(x) = \log(1 + x)$ and $g(x) = x^{0.9}$. It can be shown that the Lévy measure for $h(g(x))$ does not have harmonically concave tail by Theorem 5.1.1 and Example 6.1.2.

6.3 Connections with convolution semigroups

In this section, we characterize the sets $\mathcal{H}_{CM}$ and $\mathcal{H}_{BF}$ in terms of convolution semigroups. First recall convolution semigroups ν_t is uniquely associated with Bernstein functions $g(x)$ by (2.28). See Section 2.2.2 for details.

Lemma 6.3.1 *A Bernstein function $g(x)$ is in $\mathcal{H}_{CM}$ if and only if the measure ν_t in its convolution semigroup is harmonically convex for all $t \geq 0$.*

Proof If $g(x) \in \mathcal{H}_{CM}$, then $e^{-tg(x)}$ is completely monotone with harmonically convex measure for all $t \geq 0$. Harmonic convexity of measure ν_t follows from definitions.

Conversely, for any completely monotone function $f(x)$, the composition $f(g(x))$ has measure ξ given by Theorem 2.2.8, implying

$$\xi[0, x] = \int_{[0,\infty)} \nu_s[0, x]\, \mu(ds), \tag{6.4}$$

where μ is the measure of $f(x)$ and $\{\nu_t\}_{t\geq 0}$ is the convolution semigroup of $g(x)$. Hence, harmonic convexity of ξ follows from the harmonic convexity of ν_t for all $t \geq 0$. □

Lemma 6.3.2 *A Bernstein function $g(x)$ is in $\mathcal{H}_{BF}$ if and only if the measure ν_t in its convolution semigroup is harmonically convex for all $t \geq 0$.*

Proof If $g(x)$ is in $\mathcal{H}_{BF}$, then $1 - e^{-tg(x)}$ is Bernstein whose Lévy measure has harmonically concave tail for all $t \geq 0$. By Theorem 5.1.1,

$$1 - e^{-tg(x)} - xtg'(x)e^{-tg(x)} = 1 - e^{-tg(x)}(1 + txg'(x)) \in \mathcal{BF}. \tag{6.5}$$

This implies $e^{-tg(x)}(1 + txg'(x))$ is completely monotone by inequality (2.33) and fact 6 on Bernstein functions in Section 2.2.1. Therefore, by Theorem 4.1.1, we concludes that ν_t is harmonically convex for all $t \geq 0$.

Conversely, suppose ν_t is harmonically convex for all $t \geq 0$. For any Bernstein function $h(x)$ with Lévy triplet (a, b, η), their composition $h(g(x))$ has Lévy measure γ given by Theorem 2.2.7, implying

$$\gamma(x,\infty) = b\nu(x,\infty) + \int_{(0,\infty)} \nu_s(x,\infty)\,\eta(ds), \tag{6.6}$$

where ν is the Lévy measure for $g(x)$.

Since ν_t is sub-probability measure, it is finite on $[0,\infty)$. Therefore ν_t is harmonically convex if and only if it has harmonically concave tail, because $\nu_t[0,x] = \nu_t[0,\infty) - \nu_t(x,\infty)$. Thus ν_t has harmonically concave tail for $t \geq 0$ and the integral in (6.6) is harmonically concave. It suffices to show that ν has harmonically concave tail as well. Theorem 4.1.1 implies that

$$e^{-tg(x)}(1 + txg'(x)) \in \mathcal{CM}.$$

By (2.33), we know $1 - e^{-tg(x)}(1 + txg'(x))$ is Bernstein. Therefore, for any $t > 0$,

$$\frac{1 - e^{-tg(x)}(1 + txg'(x))}{t} \in \mathcal{BF}.$$

As Bernstein functions are closed under point-wise convergence, we obtain

$$\lim_{t\to 0+} \frac{1 - e^{-tg(x)}(1 + txg'(x))}{t} = -\frac{d}{dt}\Big(e^{-tg(x)}(1 + txg'(x))\Big)\Big|_{t=0+} = g(x) - xg'(x) \in \mathcal{BF}.$$

Theorem 5.1.1 shows that ν has harmonically concave tail. The proof is complete. □

Combining Lemma 6.3.1 and Lemma 6.3.2, we arrive at the following equivalence.

Corollary 6.3.1 *We have $\mathcal{H}_{CM} = \mathcal{H}_{BF}$.*

Applying derivative free characterizations for completely monotone function with harmonically convex measure, we have the following corollary.

Corollary 6.3.2 *Suppose $g(x)$ is Bernstein. The following statements are equivalent:*

(a) $g(x) \in \mathcal{H}_{BF}$;

(b) for all $t \geq 0$, function $e^{-tg(x)}(1 + txg'(x)) \in \mathcal{CM}$;

(c) for all $t \geq 0$ and $\lambda > 0$, function $(x+\lambda)e^{-tg(x)} - xe^{-tg(x+\lambda)} \in \mathcal{CM}$.

We've shown $\{x^\alpha : \alpha \in [0, 2/3]\} \subset \mathcal{H}_{BF}$. Following examples show some more.

Example 6.3.3 *Let $\alpha_k \in [0, 2/3]$ for all $k = 1, \ldots, n$, then $\sum_{k=1}^n x^{\alpha_k} \in \mathcal{H}_{BF}$.*

Indeed, let $\alpha := \max\{\alpha_1, \ldots, \alpha_n\} \leq 2/3$. Consider the Bernstein functions $f(x) := \sum_{k=1}^n x^{\alpha_k/\alpha}$ and $g(x) = x^\alpha$. Since $g(x) \in \mathcal{H}_{BF}$ and $\mathcal{H}_{BF}$ is closed under left composition with Bernstein functions, we conclude

$$f(g(x)) = \sum_{k=1}^n x^{\alpha_k} \in \mathcal{H}_{BF}.$$

Assigning coefficients and taking limit, we have

Example 6.3.4 *Suppose $c(\alpha) \geq 0$ is defined on $\alpha \in [0, 2/3]$. Then*

$$\int_{[0,2/3]} c(\alpha) x^{\alpha}\, d\alpha \in \mathcal{H}_{BF},$$

if the above integral is convergent.

Next two examples need detailed verifications.

Example 6.3.5 *Consider the Lévy measure ν given by density*

$$\pi(s) = \frac{1}{\Gamma(1+s)} s^{s-1} e^{-s}, \quad s > 0.$$

The Bernstein function $g(x)$ defined by Lévy triplet $(0, 0, \nu)$ is a member of $\mathcal{H}_{BF}$.

Proof In [22], the authors provide explicit densities for the measure ν_t from the convolution semigroups for $g(x)$ as following:

$$p_t(s) = \frac{t}{\Gamma(1+t+s)} s^{t+s-1} e^{-s}, \quad s > 0.$$

By Lemma 6.3.2 , to show $g(x) \in \mathcal{H}_{BF}$, it suffices to show the measures ν_t is harmonically convex for all $t \geq 0$. In terms of its density $p_t(s)$, it is equivalent to verify

$$2p_t(s) + s p_t'(s) \geq 0. \tag{6.7}$$

Note

$$2p_t(s) + s p_t'(s) = \frac{t s^{s+t} e^{-s}}{\Gamma(1+t+s)} \left(\ln(s) - \psi(s+t+1) + \frac{(t+1)}{s} \right).$$

Here $\psi(s)$ is the digamma function. Denote

$$f_t(s) := \ln(s) - \psi(s+t+1) + (t+1)/s.$$

By Example (2.4.6), we know $f_t(s)$ is completely monotone with convex measure having no mass at zero, which indicates that (6.7) holds on $(0, \infty)$ for all $t \geq 0$. This closes the proof. □

Example 6.3.6 *Consider the Lévy measure ν given by density*

$$\pi(s) = \frac{c\theta^{-1}}{\Gamma(1+cs)} \left(\frac{s}{\theta}\right)^{cs-1} e^{-s/\theta}, \quad s > 0,$$

where $c, \theta > 0$ are constants . The Bernstein function $g(x)$ defined by Lévy triplet $(0, 0, \nu)$ is a member of $\mathcal{H}_{BF}$ if and only if $c\theta = 1$.

Proof In [22], the authors provide explicit densities for measures ν_t in its convolution semigroups for $g(x)$ as following.

$$p_t(s) = \frac{c\theta^{-1} t}{\Gamma(1+c(t+s))} \left(\frac{s}{\theta}\right)^{c(t+s)-1} e^{-s/\theta + at}, \quad s, t > 0.$$

Here $a = 0$ if $c\theta \leq 1$ and $a = -1/\theta - cW_{-1}(-e^{-1/c\theta}/(c\theta))$ if $c\theta > 1$, and W_{-1} is one of the two real branches of the Lambert W-function. See [22, Section 3.2] for details.

By Lemma 6.3.2 , Bernstein function $g(x) \in \mathcal{H}_{BF}$ if and only if the measure ν_t is h-convex for all $t \geq 0$, which is equivalent to verify

$$2p_t(s) + sp_t'(s) \geq 0. \tag{6.8}$$

Notice that

$$2p_t(s) + yp_t'(s) = \frac{c^2t(s/\theta)^{c(s+t)}e^{(a\theta t-s)/\theta}}{\Gamma(c(t+s)+1)}\left(\ln\left(\frac{s}{\theta}\right) - \Psi(c(s+t)+1) + \frac{ct+1}{cs} - \frac{1}{c\theta} + 1\right).$$

Change variable by letting $u = cs, T = ct$, and $\xi = c\theta$, we can rewrite

$$\ln\left(\frac{s}{\theta}\right) - \Psi(c(s+t)+1) + \frac{ct+1}{cs} - \frac{1}{c\theta} + 1 = \ln(u) - \Psi(u+T+1) + \frac{(T+1)}{u} + 1 - \frac{1}{\xi} - \ln(\xi).$$

By Example 2.4.6, we observe that $\ln(u) - \Psi(u+t+1) + (t+1)/u$ is completely monotone in u with convex measure having no mass at zero. By A.1.7, the constant $1 - 1/\xi - \ln(\xi) \leq 0$ for $\xi > 0$, while it takes value zero if and only if $\xi = 1$. So (6.8) is non-negative on $(0, \infty)$ if and only if $\xi = 1$, which implies $g(x)$ is a member of $\mathcal{H}_{BF}$ if and only if $c\theta = 1$. $\square$

Remark 6.3.7 *Regarding different convexity properties, we could also define other sub-classes $C_{CM}, C_{BF}, \mathcal{V}_{CM} \subset \mathcal{BF}$ as following.*

$$\begin{aligned}
C_{CM} &:= \{g(x) \in \mathcal{BF} : f(g(x)) \textit{ has concave measure}, \forall f(x) \in \mathcal{CM}\};\\
C_{BF} &:= \{g(x) \in \mathcal{BF} : h(g(x)) \textit{ has convex tail Lévy measure}, \forall h(x) \in \mathcal{BF}\};\\
\mathcal{V}_{CM} &:= \{g(x) \in \mathcal{BF} : f(g(x)) \textit{ has convex measure}, \forall f(x) \in \mathcal{CM}\}.
\end{aligned}$$

However, these sub-classes are trivial. That is,

$$C_{CM} = C_{BF} = \mathcal{V}_{CM} = \{g(x) = c, c > 0\}.$$

Suppose $g(x) \in \mathcal{BF}$ and its convolution semigroups is $\{\nu_t\}_{t\geq 0}$. Lemma 6.3.1 and 6.3.2 can be modified to shown that

(a) $g(x) \in C_{CM}$ if and only if the measure ν_t is concave for all $t \geq 0$.

(b) $g(x) \in C_{BF}$ if and only if the measure ν_t is concave for all $t \geq 0$.

(c) $g(x) \in \mathcal{V}_{CM}$ if and only if the measure ν_t is convex for all $t \geq 0$.

Therefore, $C_{CM} = C_{BF}$. However, if measure ν_t is concave for $t \geq 0$, by Theorem 4.1.3,

$$e^{-tg(x)}(1 - txg'(x)) \in \mathcal{CM}.$$

This is true if and only if $g'(x) = 0$. Otherwise, if there is some x_0 such that $g'(x_0) > 0$, the above function would be negative for t large enough. This implies $g(x)$ is constant.

On the other hand, if measure ν_t is convex for $t \geq 0$, by Theorem 4.1.5,

$$x(e^{-tg(x)} - \lim_{x\to\infty} e^{-tg(x)}) \in \mathcal{CM}.$$

Notice the limit for above function is zero when x approaches zero. Hence it is identical to zero on $(0, \infty)$. This implies $g(x)$ is a constant.

Chapter 7

Applications

7.1 Coupon collector's problem and BAG functions

In the Coupon Collector's problem, an agent is interested in obtaining at least one copy of n different coupons. Every time the agent purchases a coupon, they get coupon i with probability x_i. Let T be the minimum number of purchases required to obtain at least one coupon of each type. It is well-known that

$$E[T] = \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{1\le i<j\le n} \frac{1}{x_i + x_j} + \cdots + (-1)^{n-1} \frac{1}{x_1 + \cdots + x_n}.$$

It is shown in [18] that this function is convex in $(x_1, \ldots, x_n)$ on $(0, \infty)^n$. Of interest to us is the minimum variance conjecture for the Coupon Collector's problem, see [30]. It says that $\mathrm{Var}\,[T]$ is mininized when $x_1 = \cdots = x_n = 1/n$. In order to express the variance in a succinct way we introduce the following notation. For a function f defined on a domain in $\mathbb{R}$, let

$$F[f](x_1, \ldots, x_n) := \sum_{i=1}^{n} f(x_i) - \sum_{1\le i<j\le n} f(x_i + x_j) + \cdots + (-1)^{n-1} f(x_1 + \cdots + x_n).$$

Thus, $E[T] = F[1/x]$ and for the variance of T, we have, see [30], that

$$\mathrm{Var}\,[T] = 2F[1/x^2] - F[1/x] - \Big(F[1/x]\Big)^2.$$

We quote the following result from [71].

Theorem 7.1.1 *Let $f(x)$ be a completely monotone function with measure μ and $g(x)$ be a Bernstein function with Lévy measure ν.*

(a) If μ is harmonically convex, then $F[f]$ is convex and non-negative on $\mathbb{R}^n_{++}$.

(b) If ν has a harmonically concave tail, then the function $F[g]$ is concave and non-negative on $\mathbb{R}^n_{++}$.

Before we continue with the Coupon Collector's problem, we note the following curious corollary, obtained by combining Theorems 6.2.2, 6.2.3, and 7.1.1.

Corollary 7.1.1 *Let α be any number in* $[0, 2/3]$.

(a) For any completely monotone function f, the function $F[f(x^\alpha)]$ is convex and non-negative on $\mathbb{R}^n_{++}$.

(b) For any Bernstein function g, the function $F[g(x^\alpha)]$ is concave and non-negative on $\mathbb{R}^n_{++}$.

Theorem 7.1.1 together with Example 2.4.4 show that $\mathrm{Var}\,[T]$ is the difference of two convex functions

$$\mathrm{Var}\,[T] = 2F[1/x^2] - \Big(F[1/x] + \big(F[1/x]\big)^2\Big).$$

The fact that $\mathrm{Var}\,[T]$ can be represented as difference of convex functions is not surprising since every C^2 function has this property, see [36]. The explicit representation is the key. It is conceivable that the conjecture may benefit from duality techniques in optimization.

Our next goal is to give an explicit representation of any (central) moment of T as a difference of two convex functions. In [30], it is given that

$$G(z) := E[z^{-T}] = 1 + (z^{-1} - 1)\sum_{k=1}^{\infty} z^{-(k-1)} P\{T \geq k\} = 1 + (z-1)\sum_{\substack{J \subset \{1,\ldots,n\} \\ J \neq \emptyset}} \frac{(-1)^{|J|}}{z - 1 + \sum_{j \in J} x_j}.$$

Denote

$$g(z) := \sum_{\substack{J \subset \{1,\ldots,n\} \\ J \neq \emptyset}} \frac{(-1)^{|J|}}{z - 1 + \sum_{j \in J} x_j}.$$

Therefore,

$$G^{(k)}(z) = E[(-1)^k T(T+1)\cdots(T+k-1) z^{-T-k}], \text{ and}$$
$$g^{(k)}(z) = (-1)^k \sum_{\substack{J \subset \{1,\ldots,n\} \\ J \neq \emptyset}} \frac{(-1)^{|J|}}{(z - 1 + \sum_{j \in J} x_j)^{k+1}}.$$

Observing that $G^{(k)}(z) = kg^{(k-1)}(z) + (z-1)g^{(k)}(z)$, we obtain

$$G^{(k)}(1) = kg^{(k-1)}(1) = k(-1)^{k-1} \sum_{\substack{J \subset \{1,\ldots,n\} \\ J \neq \emptyset}} \frac{(-1)^{|J|}}{(\sum_{j \in J} x_j)^k}.$$

Next, letting

$$N_{m,k} := E[(T+k)\cdots(T+1)T^m],$$

we have, for all $m > 1$ and $k \geq 0$, the recursive relationship

$$N_{m,k} = N_{m-1,k+1} - (k+1)N_{m-1,k}, \tag{7.1}$$

with the initial condition, for all $k \geq 0$,

$$N_{1,k} = E[(T+k)\cdots(T+1)T] = (-1)^{k+1} G^{(k+1)}(1)$$

$$= (k+1) \sum_{\substack{J \subset \{1,\dots,n\} \\ J \neq \emptyset}} \frac{(-1)^{|J|-1}}{(\sum_{j \in J} x_j)^{k+1}} = (k+1) F\left[\frac{1}{x^{k+1}}\right].$$

Theorem 7.1.1 together with Example 2.4.4 show that $N_{1,k}$ is a non-negative, convex function. Noting that $N_{m,0} = E[T^m]$ is the m-th moment of T, one sees that $E[T^m]$ is linear combination of $N_{1,0}, N_{1,1}, \dots, N_{1,m-1}$, thus, one can group the terms with positive and negative coefficients together to express $E[T^m]$ as a difference of two convex functions. For example:

$$\begin{aligned} E[T] &= N_{1,0}, \\ E[T^2] &= N_{2,0} = N_{1,1} - N_{1,0}, \\ E[T^3] &= N_{3,0} = N_{1,2} - 3N_{1,1} + N_{1,0}. \end{aligned}$$

To get a rather general expression for the m-th moment of T, define the infinite dimensional vectors and the infinite matrix A, having diagonal $(-1, -2, \dots)$, a superdiagonal $(1, 1, \dots)$, and zeros everywhere else. Then, using the recursive relationship (7.1), it can be shown that, for any $m \geq 1$, one has

$$N_m = A^{m-1} N_1.$$

More work is required to express the central moments of T as a difference of two convex functions. Note that

$$E[(T - E[T])^m] = \sum_{i=0}^{m} (-1)^i \binom{m}{i} E[T^{m-i}](E[T])^i = \sum_{i=0}^{m} (-1)^i \binom{m}{i} N_{m-i,0} N_{1,0}^i.$$

As discussed above, $N_{m-i,0}$ is a linear combination of the functions $N_{1,0}, N_{1,1}, \dots, N_{1,m-i-1}$. So, it suffices to show that $N_{1,k} N_{1,0}^s$ is a convex function for every $k \geq 0$ and $s \geq 0$. We show a bit more with the use of the following particular case of [71, Theorem 5.1].

Theorem 7.1.2 *Let f be a completely monotone function with measure μ. If the function $x \mapsto$* $\log(\mu[0, x])$ *is harmonically convex, then the function* $\log(F[f])$ *is non-negative and convex on* $\mathbb{R}^n_{++}$.

Proposition 7.1.1 *For any integer $n, m \geq 0$ and real $r, s \geq 0$, the function $N_{1,n}^r N_{1,m}^s$ is convex.*

Proof Since logarithmic convexity of a function implies its convexity, it is enough to show that $\log(N_{1,n})$ is convex for any integer $n \geq 0$. By Theorem 7.1.2, we need to show that $x \mapsto \log(\mu[0, x])$ is harmonically convex, where μ is the measure corresponding to $f(x) = 1/x^{n+1}$. By Example 2.4.4, we have

$$\log(\mu[0, x]) = (n+1)\log(x) - \log((n+1)\Gamma(n+1)),$$

which is clearly harmonically convex.

7.2 Spectral functions

Let S^n denote the Euclidean space of $n \times n$ symmetric matrices with inner product $\langle X, Y \rangle := \operatorname{tr}(XY)$. For a vector $x \in \mathbb{R}^n$, denote by $\operatorname{Diag}(x)$ the diagonal matrix with x on the main diagonal. It is well-known that every $X \in S^n$ has n real eigenvalues, counting multiplicities, denoted by

$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X).$$

Define the map $\lambda : X \to \mathbb{R}^n$ by $\lambda(X) := (\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X))$. If all eigenvalues of X are positive (reps. non-negative) then X is called positive definite (reps. semidefinite) matrix. The set of all such matrices is a convex cone denoted by S^n_{++} (reps. S^n_{+}).

The spectral decomposition theorem says that for every $X \in S^n$, there is an orthogonal U, such that

A function $g : \mathbb{R}^n \to \mathbb{R}$ is called symmetric if it is invariant under arbitrary permutations of its arguments. The following result is attributed to [28] see also [48, Corollary 2.7].

Theorem 7.2.1 *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a symmetric and convex function defined on a (convex) domain $D \subset \mathbb{R}^n$. Then the function $G : S^n \to \mathbb{R}$ defined by*

$$G(X) := g(\lambda(X))$$

is convex on the (convex) domain $\{X \in S^n : \lambda(X) \in D\}$.

Combining Theorem 7.2.1 with Theorem 7.1.1 we obtain the next results.

Theorem 7.2.2 *Let $f(x)$ be a completely monotone function with measure μ and $g(x)$ be a Bernstein function with Lévy measure ν.*

(a) If μ is harmonically convex, then $F[f] \circ \lambda$ is convex and non-negative on S^n_{++}.

(b) If ν has a harmonically concave tail, then the function $F[g] \circ \lambda$ is concave and non-negative on S^n_{++}.

The following corollary is obtained by combining Theorem 7.2.2 and Corollary 7.1.1.

Corollary 7.2.1 *Let α be any number in* $[0, 2/3]$.

(a) For any completely monotone function f, the function $F[f(x^\alpha)] \circ \lambda$ is convex and non-negative on S^n_{++}.

(b) For any Bernstein function g, the function $F[g(x^\alpha)] \circ \lambda$ is concave and non-negative on S^n_{++}.

The matrix theoretic interpretation of the function $F[f] \circ \lambda$ is interesting in its own right and this is what we proceed to describe next. Let $f : R \to R$ be a function defined on an interval I. It defines a *primary matrix* function on the domain

$$\{X \in S^n : \lambda_i(X) \in I \text{ for all } i = 1, \ldots, n\} \tag{7.2}$$

by It can be shown, see [13, Chapter V], that this definition does not depend on the choice of U. Hence, the primary matrix function is defined on the domain (7.2) and takes its values in S^n.

There are deep connections between the primary matrix functions and the Bernstein functions. For example, the *operator monotone* functions $f : (0,\infty) \to [0,\infty)$ (that is, having the property that $f(A)-f(B) \in S^n_+$ whenever $A-B \in S^n_+$) are precisely the so-called *complete* Bernstein functions (defined by the property that their Lévy measure has a completely monotone density). See for example [68, Theorem 11.17].

We now proceed to describe the additive compound matrices, see [54, Chapter 19]. Consider the lexicographical ordering between all ordered k-tuples $(i_1,\dots,i_k)$, where $1 \le i_1 < \cdots < i_k \le n$. Given an $n \times n$ matrix A, the k-th *multiplicative compound* matrix, denoted $A^{(k)}$, is the $\binom{n}{k} \times \binom{n}{k}$ matrix whose rows and columns are labeled by ordered k-tuples and the entry in roll $(i_1,\dots,i_k)$ and column $(j_1,\dots,j_k)$ is the $k\times k$ determinant of A formed from the elements at the intersection of rolls $i_1,\dots,i_k$ with columns $j_1,\dots,j_k$. If $\lambda_1(A),\dots,\lambda_n(A)$ are the eigenvalues of A, then the eigenvalues of $A^{(k)}$ are $\{\lambda_{i_1}(A)\cdots\lambda_{i_k}(A) : 1 \le i_1 < \cdots < i_k \le n\}$, hence the word 'multiplicative' in the name of the matrix $A^{(k)}$. Note that $A^{(1)} = A$ and $A^{(n)} = \det A$.

The k-th *additive compound* matrix, denoted $A^{[k]}$, is the $\binom{n}{k} \times \binom{n}{k}$ matrix appearing as the coefficient of t in the expansion

$$(I + tA)^{(k)} = I + tA^{[k]} + \cdots .$$

The eigenvalues of $A^{[k]}$ are $\{\lambda_{i_1}(A) + \cdots + \lambda_{i_k}(A) : 1 \le i_1 < \cdots < i_k \le n\}$, hence the word 'additive' in the name of the matrix $A^{(k)}$. Note that $A^{[1]} = A$ and $A^{[n]} = \operatorname{tr} A$.

With this notation, Theorem 7.2.2 and Corollary 7.2.1 may be rewritten as follows.

Theorem 7.2.3 *Let $f(x)$ be a completely monotone function with measure μ and $g(x)$ be a Bernstein function with Lévy measure ν.*

(a) If μ is harmonically convex, then

$$A \mapsto \sum_{k=1}^{n} (-1)^{k-1} \operatorname{tr} f(A^{[k]}) \tag{7.3}$$

is convex and non-negative on S^n_{++}.

(b) Let $g(x)$ be a Bernstein function with measure ν. If ν has a harmonically concave tail, then the function

$$A \mapsto \sum_{k=1}^{n} (-1)^{k-1} \operatorname{tr} g(A^{[k]}) \tag{7.4}$$

is concave and non-negative on S^n_{++}.

Corollary 7.2.2 *Let α be any number in* $[0, 2/3]$.

(a) For any completely monotone function $\tilde f$, let $f(x) := \tilde f(x^\alpha)$, then function (7.3) is convex and non-negative on S^n_{++}.

(b) For any Bernstein function $\tilde g$, let $g(x) := \tilde g(x^\alpha)$, then function (7.4) is concave and non-negative on S^n_{++}.

Chapter 8

Open questions and future works

8.1 Open questions

We would like to list some open questions which are encountered in our researches.

Problem 8.1.1 *Find the largest number $r \in [0, 1]$ such that the function*

$$f(x) = e^{-sx^{\alpha}}(1 + s\alpha x^{\alpha}) \tag{8.1}$$

is completely monotone for every $\alpha \in [0, r]$ and $s \geq 0$.

This problem roots in Section 6.1. Proposition 6.1.1 indicates this open question is well-defined, as the set of all $\alpha \in [0, 1]$, for which (8.1) is completely monotone for all $s \geq 0$, is a continuous closed interval. We shown that $r > 2/3$ in Lemma 6.1.1 and $r < 0.9$ in Example 6.1.2. However, the exact number of r is unknown, though our numerical experiments show that r is around 0.7424.

Problem 8.1.2 *Suppose $f(x)$ is completely monotone. Is the inverse function of $f(x)/x$ completely monotone? Suppose $g(x)$ is Bernstein, Is the inverse of $xg(x)$ Bernstein?*

This problem rises when we want to find more ways to construct completely monotone and Bernstein functions. It also connects to the Lambert W-function, see [26] and [22]. Lambert W-function is defined as the inverse to the function $w \to we^{w}$. When $z \neq 0$, the equation $we^{w} = z$ has infinitely many solutions, including two real branches. The increasing branch is denoted as W_0 and the decreasing branch is denoted as W_{-1}. See [26] for details. We wonder whether the increasing real branch of the Lambert W-function is Bernstein on $(0, \infty)$.

Problem 8.1.3 *Suppose measures μ and ν on $[0, \infty)$ are both β-convex measures for $\beta \in [0, 1]$. Does their convolution $\mu * \nu$ have γ-convexity for some $\gamma \in [0, 1]$?*

It is shown in Lemma 2.4.7 that if Bernstein measure μ is convex and $\mu(\{0\}) = 0$, then $\mu * \nu$ is convex for any Bernstein measure ν. This connects to the case $\beta = \gamma = 0$ in the above question. However, the other cases are unknown. Investigation of this problem contributes to understand the product of completely monotone functions and the shape of their measures.

8.2 Future works

Besides these specific open questions, we have some other general topics for future researches.

8.2.1 The structure of $\mathcal{H}_{BF}$

It is shown in Example 6.3.3 and Example 6.3.4 that the linear combination

$$c_1x^{\alpha_1} + c_2x^{\alpha_2} + \cdots + c_nx^{\alpha_n} \in \mathcal{H}_{BF},$$

for $c_i \geq 0, \alpha_i \in [0, 2/3], i = 1, 2, \ldots, n$. It remains unknown if this property can be extended onto the entire set of $\mathcal{H}_{BF}$. In other words, it is interesting to show if the following function is from $\mathcal{H}_{BF}$:

$$c_1g_1(x) + c_2g_2(x) + \cdots + c_ng_n(x),$$

where $c_i \geq 0$ and $g_i(x) \in \mathcal{H}_{BF}$ for all $i = 1, 2, \ldots, n$.

A special case of this linear combination is the convex combination. In this scenario, the objective is to investigate whether the set $\mathcal{H}_{BF}$ is a convex cone, that is to prove or disprove

$$\lambda g_1(x) + (1 - \lambda)g_2(x) \in \mathcal{H}_{BF},$$

for $g_1(x), g_2(x) \in \mathcal{H}_{BF}$ and $\lambda \in [0, 1]$.

Another aspect regarding the structure of $\mathcal{H}_{BF}$ is to identify its generator. Function $h(x) \in \mathcal{H}_{BF}$ is called a generator, if it can not be represented as $h(x) = g(h^*(x))$ for some non-affine Bernstein function $g(x) \in \mathcal{BF}$ and some $h^* \in \mathcal{H}_{BF}$. In other words, if $h(x)$ is a generator of $\mathcal{H}_{BF}$, then for any identity such that

$$h(x) = g(h^*(x)),$$

where $g(x) \in \mathcal{BF}$ and $h^*(x) \in \mathcal{H}_{BF}$, the function $g(x)$ must have Lévy $\nu(0, \infty) = 0$. For example, consider $\{x^\alpha : \alpha \in [0, 2/3]\} \subset \mathcal{H}_{BF}$. Its generator is $h(x) = x^{2/3}$.

It is unclear what are the generators of $\mathcal{H}_{BF}$. However, if we could identify the set of generators, denoted as $\mathcal{GH}_{BF}$, then we have identified all functions in $\mathcal{H}_{BF}$ as the following:

$$\mathcal{H}_{BF} = \{g \circ h(x) : g(x) \in \mathcal{BF} \text{ and } h(x) \in \mathcal{GH}_{BF}\}.$$

8.2.2 Connections to complete Bernstein functions

Definition 8.2.1 *Suppose $g(x) \in \mathcal{BF}$. It is complete Bernstein function if its Lévy measure ν has a completely monotone density $m(t)$ with respect to Lebesgue measure.*

In other words, a function $g(x)$ is complete Bernstein function if it admits

$$g(x) = a + bx + \int_{(0,\infty)} (1 - e^{-xt})m(t)\, dt,$$

for $a, b \geq 0$ and $m(t)$ is completely monotone, satisfying

$$\int_{(0,\infty)} (t \wedge 1)m(t)\, dt < \infty.$$

The set of complete Bernstein function is denoted as $\mathcal{CBF}$. It is easy to verify that for any $g(x) \in \mathcal{CBF}$, it has Lévy measure with convex tail. However, the converse is not true. Hence the Bernstein functions whose Lévy measures have convex tail generalize $\mathcal{CBF}$.

There are a lot of interesting properties for complete Bernstein functions. For example, the set $\mathcal{CBF}$ is closes under composition. That is, if $g_1(x), g_2(x) \in \mathcal{CBF}$, then $g_1(g_2(x)) \in \mathcal{CBF}$. See [68, Corollary 7.6]. Another example can be the characterization of the set

$$\mathcal{CBF}^{\alpha} := \{g^{\alpha}(x) : \; g(x) \in \mathcal{CBF}\} = \{g(x) \in \mathcal{CBF} \, : \, x^{1-\alpha} g(x) \in \mathcal{CBF}\}.$$

It is unknown how these properties could be generalized onto the set of Bernstein functions whose Lévy measures have convex tails.

8.2.3 Connections to subordinators

It is shown in Section 2.3.2 that Bernstein function is closely related with subordinator processes. They differ from the Laplace exponent of subordinators by a constant $a \geq 0$.

We would like to try to characterize Bernstein functions whose Lévy measures have various convexity properties in terms of associated subordinators. Besides, it is interesting to investigate the emerging properties of the subordinators with these convexity properties on their Lévy measures.

Bibliography

[1] Adler, I. and Ross, S. M. (2001). The coupon subset collection problem. *J. Appl. Probab.* **38,** 737–746.

[2] Affleck, I. (2000). Convex! solution of part (b) of problem 99-002. *Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania*. Online http://www.siam.org/journals/problems/downloadfiles/99-002s.pdf.

[3] Arendt, W. (1987). *Vector-valued Laplace transforms and Cauchy problems* vol. 59.

[4] Bauer, H. (2001). *Measure and integration theory* vol. 26 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin. Translated from the German by Robert B. Burckel.

[5] Baum, L. E. and Billingsley, P. (1965). Asymptotic distributions for the coupon collector's problem. *Ann. Math. Statist.* **36,** 1835–1839.

[6] Berg, C. (1979). The Stieltjes cone is logarithmically convex. In *Complex analysis Joensuu 1978 (Proc. Colloq., Univ. Joensuu, Joensuu, 1978).* vol. 747 of *Lecture Notes in Math.* Springer, Berlin pp. 46–54.

[7] Berg, C., Boyadzhiev, K. and Delaubenfels, R. (1993). Generation of generators of holomorphic semigroups. *Journal of the Australian Mathematical Society (Series A)* **55,** 246–269.

[8] Bertoin, J. (1992). Factorizing Laplace exponents in a spectrally positive Lévy process. *Stochastic Process. Appl.* **42,** 307–313.

[9] Bertoin, J. (1996). *Lévy processes* vol. 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge.

[10] Bertoin, J. (1999). *Subordinators: examples and applications* vol. 1717 of *Lecture Notes in Math.* Springer, Berlin.

[11] Bertoin, J., Roynette, B. and Yor, M. (2004). Some connections between (sub)critical branching mechanisms and bernstein functions. *ArXiv Mathematics e-prints*. Online http://arxiv.org/abs/math/0412322.

[12] Bertoin, J. and Yor, M. (2005). Exponential functionals of Lévy processes. *Probab. Surv.* **2,** 191–212.

[13] Bhatia, R. (1997). *Matrix analysis* vol. 169 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[14] Bochner, S. (1955). *Harmonic analysis and the theory of probability*. University of California Press, Berkeley and Los Angeles.

[15] Boneh, A. and Hofri, M. (1997). The coupon-collector problem revisited—a survey of engineering problems and computational methods. *Comm. Statist. Stochastic Models* **13,** 39–66.

[16] Borak, S., Härdle, W. and Weron, R. (2005). Stable distributions. In *Statistical tools for finance and insurance*. Springer, Berlin pp. 21–44.

[17] Borwein, J., Affleck, I. and Girgensohn, R. (2000). Convex? problem 99-002. *Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania*. Online `http://www.siam.org/journals/problems/downloadfiles/99-002.pdf`.

[18] Borwein, J. and Hijab, O. (2000). Convex! II solution of problem 99-002. *Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania*. Online `http://www.siam.org/journals/problems/downloadfiles/99-5sii.pdf`.

[19] Borwein, J. M. and Lewis, A. S. (2010). *Convex analysis and nonlinear optimization: theory and examples*. Springer Science & Business Media.

[20] Bosch, P. and Simon, T. (2015). A proof of bondessons conjecture on stable densities. *Arkiv fr Matematik* 1–8.

[21] Bryan, K. M. (1999). Elementary inversion of the laplace transform. *Mathematical Sciences Technical Reports (MSTR)*. Paper 114. `http://scholar.rose-hulman.edu/math_mstr/114`.

[22] Burridge, J., Kuznetsov, A., Kwaśnicki, M. and Kyprianou, A. E. (2014). New families of subordinators with explicit transition probability semigroup. *Stochastic Process. Appl.* **124,** 3480–3495.

[23] Caron, R., Hlynka, M. and McDonald, J. (1988). Minimizing the expected number of trials in the coupon collector's problem. WMR 88-03, Department of Mathematics and Statistics, University of Windsor, Windsor, ON Canada.

[24] Carter, M. and van Brunt, B. (2000). *The Lebesgue-Stieltjes integral*. Undergraduate Texts in Mathematics. Springer-Verlag, New York. A practical introduction.

[25] Chansangiam, P. (2013). Operator Monotone Functions: Characterizations and Integral Representations. *ArXiv e-prints*.

[26] Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J. and Knuth, D. E. (1996). On the Lambert *W* function. *Adv. Comput. Math.* **5,** 329–359.

[27] Daleckiĭ, J. L. (1960). Integration and differentiation of functions of hermitian operators depending on a parameter. *Amer. Math. Soc. Transl. (2)* **16,** 396–400.

[28] Davis, C. (1957). All convex invariant functions of hermitian matrices. *Arch. Math.* **8,** 276–278.

[29] Doob, J. L. (2001). *Classical potential theory and its probabilistic counterpart*. Classics in Mathematics. Springer-Verlag, Berlin. Reprint of the 1984 edition.

[30] Doumas, A. V. and Papanicolaou, V. G. (2012). The coupon collector's problem revisited: asymptotics of the variance. *Adv. in Appl. Probab.* **44,** 166–195.

[31] Durrett, R. (2010). *Probability: theory and examples* fourth ed. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge.

[32] Feller, W. (1957). *An introduction to probability theory and its applications. Vol. I.* John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London. 2nd ed.

[33] Feller, W. (1971). *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney.

[34] Flanders, H. (1973). Differentiation under the integral sign. *Amer. Math. Monthly* **80,** 615–627; correction, ibid. 81 (1974), 145.

[35] Fourati, S. and Jedidi, W. (2011). Some remarks on the class of bernstein functions and some sub-classes.

[36] Ginchev, I. and Martínez-Legaz, J.-E. (2011). Characterization of d.c. functions in terms of quasidifferentials. *Nonlinear Anal.* **74,** 6781–6787.

[37] Gneiting, T. (2000). Kuttner's problem and a Pólya type criterion for characteristic functions. *Proc. Amer. Math. Soc.* **128,** 1721–1728.

[38] Grimmett, G. R. and Stirzaker, D. R. (2001). *Probability and random processes* third ed. Oxford University Press, New York.

[39] Halmos, P. (1976). *Measure Theory*. Graduate Texts in Mathematics. Springer New York.

[40] Henstock, R. (1973). Integration by parts. *Aequationes Math.* **9,** 1–18.

[41] Hiriart-Urruty, J.-B. and Ye, D. (1995). Sensitivity analysis of all eigenvalues of a symmetric matrix. *Numer. Math.* **70,** 45–72.

[42] Holst, L. (2001). Extreme value distributions for random coupon collector and birthday problems. *Extremes* **4,** 129–145 (2002).

[43] Hubalek, F. and Kyprianou, E. (2011). Old and new examples of scale functions for spectrally negative Lévy processes. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*. vol. 63 of *Progr. Probab.* Birkhäuser/Springer Basel AG, Basel pp. 119–145.

[44] Jocković, J. and Mladenović, P. (2011). Coupon collector's problem and generalized Pareto distributions. *J. Statist. Plann. Inference* **141,** 2348–2352.

[45] Jones, F. (1993). *Lebesgue integration on Euclidean space*. Jones and Bartlett Publishers, Boston, MA.

[46] Kuznetsov, A. and Pardo, J. C. (2013). Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. *Acta Appl. Math.* **123,** 113–139.

[47] Kyprianou, A. E. (2014). *Fluctuations of Lévy processes with applications* second ed. Universitext. Springer, Heidelberg. Introductory lectures.

[48] Lewis, A. S. (1996). Convex analysis on the Hermitian matrices. *SIAM J. Optim.* **6,** 164–177.

[49] Lewis, A. S. (1996). Nonsmooth analysis of eigenvalues: a summary. *Rend. Sem. Mat. Fis. Milano* **66,** 33–41 (1998).

[50] Lewis, A. S. (1999). Nonsmooth analysis of eigenvalues. *Math. Program.* **84,** 1–24.

[51] Lewis, A. S. and Sendov, H. S. (2001). Twice differentiable spectral functions. *SIAM J. Matrix Anal. Appl.* **23,** 368–386 (electronic).

[52] Lukacs, E. (1970). *Characteristic functions*. Hafner Publishing Co., New York. Second edition, revised and enlarged.

[53] Magnus, J. R. and Neudecker, H. (1995). Matrix differential calculus with applications in statistics and econometrics.

[54] Marshall, A. W. and Olkin, I. (1979). *Inequalities: theory of majorization and its applications* vol. 143 of *Mathematics in Science and Engineering*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London.

[55] Merkle, M. (2004). Reciprocally convex functions. *J. Math. Anal. Appl.* **293,** 210–218.

[56] Mimica, A. (2010). Laplace transforms and exponential behavior of representing measures. Online `http://web.math.pmf.unizg.hr/ amimica/pub/lapl.pdf`.

[57] Moriarty, J. and Neal, P. The generalized coupon collector problem 2008.

[58] Nakamura, Y. (1989). Classes of operator monotone functions and Stieltjes functions. In *The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988)*. vol. 41 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel pp. 395–404.

[59] Niezgoda, M. and Pečarić, J. (2012). Hardy-Littlewood-Pólya-type theorems for invex functions. *Comput. Math. Appl.* **64,** 518–526.

[60] Penson, K. A. and Górska, K. (2010). Exact and explicit probability densities for one-sided Lévy stable distributions. *Phys. Rev. Lett.* **105,** 210604, 4.

[61] Rice, J. (2006). *Mathematical statistics and data analysis*. Cengage Learning.

[62] Rockafellar, R. T. (1970). *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J.

[63] Rogers, L. C. G. (1983). Wiener-Hopf factorization of diffusions and Lévy processes. *Proc. London Math. Soc. (3)* **47,** 177–191.

[64] Ross, S. (2010). *A First Course in Probability*. Pearson Prentice Hall.

[65] Rudin, W. (1976). *Principles of mathematical analysis* third ed. McGraw-Hill Book Co., New York-Auckland-Düsseldorf. International Series in Pure and Applied Mathematics.

[66] Rudin, W. (1987). *Real and complex analysis* third ed. McGraw-Hill Book Co., New York.

[67] Sato, K.-i. (1999). *Lévy processes and infinitely divisible distributions* vol. 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge. Translated from the 1990 Japanese original, Revised by the author.

[68] Schilling, R. L., Song, R. and Vondraček, Z. (2012). *Bernstein functions* second ed. vol. 37 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin. Theory and applications.

[69] Sendov, H. and Shan, S. (2014). New representation theorems for completely monotone and Bernstein functions with convexity properties on their measures. *Journal of Theoretical Probability* 1–37.

[70] Sendov, H. S. (2007). The higher-order derivatives of spectral functions. *Linear Algebra Appl.* **424,** 240–281.

[71] Sendov, H. S. and Zitikis, R. (2014). The shape of the Borwein-Affleck-Girgensohn function generated by completely monotone and Bernstein functions. *J. Optim. Theory Appl.* **160,** 67–89.

[72] Song, R. Lectures on the potential theory of subordinate brownian motions. Department of Mathematics, University of Illinois, Urbana.

[73] Song, R. and Vondraček, Z. (2006). Potential theory of special subordinators and subordinate killed stable processes. *J. Theoret. Probab.* **19,** 817–847.

[74] Song, R. and Vondraček, Z. (2009). Potential theory of subordinate brownian motion. In *Potential Analysis of Stable Processes and its Extensions*. Springer pp. 87–176.

[75] Stroock, D. W. (2011). *Essentials of integration theory for analysis* vol. 262 of *Graduate Texts in Mathematics*. Springer, New York.

[76] Sun, D. and Sun, J. (2008). Löwner's operator and spectral functions in Euclidean Jordan algebras. *Math. Oper. Res.* **33,** 421–445.

[77] Talvila, E. (2001). Necessary and sufficient conditions for differentiating under the integral sign. *Amer. Math. Monthly* **108,** 544–548.

[78] Tao, P. D. and An, L. T. H. (1997). Convex analysis approach to d.c. programming: theory, algorithms and applications. *Acta Math. Vietnam.* **22,** 289–355.

[79] Widder, D. V. (1938). The Stieltjes transform. *Trans. Amer. Math. Soc.* **43,** 7–60.

[80] Widder, D. V. (1941). *The Laplace Transform*. Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J.

[81] Yan, M. (2012). Extension of convex function. *arXiv preprint arXiv:1207.0944*.

[82] Yeh, J. (2006). *Real Analysis: Theory of Measure and Integration*. World Scientific.

[83] Zolotarev, V. M. (1986). *One-dimensional stable distributions* vol. 65 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.

Appendix A

Auxiliary facts and their proofs

In this appendix, we list some facts that are used in our contents. Some of the statements are easy to comprehend but requires prolonged close arguments to prove, while some others may be of more interests with wide applications.

A.1 On functions

Lemma A.1.1 *For infinite differentiable function $f(x)$, we have*

$$(xf(x))^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x).$$

It could be easily verified by mathematical induction. The next corollaries follow trivially.

Corollary A.1.1 *For infinite differentiable function*

$$(f(x) - xf'(x))^{(n)} = -xf^{(n+1)}(x) - (n-1)f^{(n)}(x).$$

Corollary A.1.2 *For infinite differentiable function*

$$(f(x) + xf'(x))^{(n)} = xf^{(n+1)}(x) + (n+1)f^{(n)}(x).$$

Lemma A.1.2 *Suppose the left (right) directional derivative of $\varphi : (0,\infty) \to \mathbb{R}$ exists at x. Then, the right (left) directional derivative of $h(x) := \varphi(1/x)$ exists at $1/x$ and*

$$h'_+(x) = -\varphi'_-\Big(\frac{1}{x}\Big)\frac{1}{x^2}, \quad \textit{resp. } h'_-(x) = -\varphi'_+\Big(\frac{1}{x}\Big)\frac{1}{x^2}.$$

Proof We only need to verify the case for $h'_+(x)$. The other follows similarly. By definition,

$$\begin{aligned} h'_+(x) &= \lim_{\lambda\to 0+} \frac{\varphi\Big(\frac{1}{x+\lambda}\Big) - \varphi\Big(\frac{1}{x}\Big)}{\lambda} = \lim_{\lambda\to 0+} \frac{\varphi\Big(\frac{1}{x} - \Big(\frac{1}{x} - \frac{1}{x+\lambda}\Big)\Big) - \varphi\Big(\frac{1}{x}\Big)}{\frac{1}{x} - \frac{1}{x+\lambda}} \cdot \frac{\frac{1}{x} - \frac{1}{x+\lambda}}{\lambda} \\ &= -\frac{1}{x^2}\lim_{\delta\to 0+} \frac{\varphi\Big(\frac{1}{x} - \delta\Big) - \varphi\Big(\frac{1}{x}\Big)}{\delta} = -\frac{1}{x^2}\varphi'_-\Big(\frac{1}{x}\Big). \end{aligned}$$

We change variable by letting $\delta = 1/x - 1/(x+\lambda)$ in second line. The verification is closed. □

Lemma A.1.3 *If function $h : (0, \infty) \to \mathbb{R}$ is convex (concave), then the directional derivatives of $x^p h(x)$ exist for all $p \in \mathbb{R}$. More precisely, it can be shown that*

$$(x^p h(x))'_+ = px^{p-1}h(x) + x^p h'_+(x) \quad \textit{and} \quad (x^p h(x))'_- = px^{p-1}h(x) + x^p h'_-(x).$$

Proof We only need to verify the case for $(x^p h(x))'_+$. The other follows similarly. By definition,

$$\begin{aligned}
(x^p h(x))'_+ &= \lim_{\lambda\to 0+} \frac{(x+\lambda)^p h(x+\lambda) - x^p h(x)}{\lambda} \\
&= \lim_{\lambda\to 0+} \frac{(x+\lambda)^p h(x+\lambda) - x^p h(x+\lambda)}{\lambda} + \lim_{\lambda\to 0+} \frac{x^p h(x+\lambda) - x^p h(x)}{\lambda} \\
&= px^{p-1}h(x) + x^p h'_+(x).
\end{aligned}$$

The last equation utilizes the continuity of $h(x)$. The verification is closed. □

Lemma A.1.4 *The following functions are positive on* $(0, \infty)$.

(a) $f_0(s) = s - 1 + e^{-s}$;

(b) $f_1(s) = s^2 - 2s - 2e^{-s} + 2$;

(c) $f_2(s) = -s^2 e^{-s} + e^{-2s} - 2e^{-s} + 1$;

(d) $f_3(s) = -se^{-s} - e^{-s} + 1$;

(e) $f_4(s) = se^{-s} + 2e^{-s} + s - 2$;

(f) $f_5(s) = (1 - e^{-s})^2 - 2e^{-s}(e^{-s} + s - 1)$.

Proof (a) This is trivial because

$$f_0(s) = s - (1 - e^{-s}) = \int_{(0,s)} (1 - e^{-u})\, du.$$

(b) Notice $f_1'(s) = 2f_0(s) > 0$ for $s > 0$ by (a). Thus $f_1(s)$ is increasing on $(0, \infty)$. Noticing $\lim_{s\to 0} f_1(s) = 0$, we can conclude $f_1(s) > 0$ on $(0, \infty)$.

(c) Notice $f_2'(s) = e^{-s} f_1(s) > 0$ for $s > 0$ by (b). Thus $f_2(s)$ is increasing on $(0, \infty)$. Noticing $\lim_{s\to 0} f_2(s) = 0$, we can conclude $f_2(s) > 0$ on $(0, \infty)$.

(d) Notice $f_3'(s) = se^{-s} > 0$ for $s > 0$. Thus $f_3(s)$ is increasing on $(0, \infty)$. Noticing $\lim_{s\to 0} f_3(s) = 0$, we can conclude $f_3(s) > 0$ on $(0, \infty)$.

(e) Notice $f_4'(s) = f_3(s) > 0$ for $s > 0$ by (d). Thus $f_4(s)$ is increasing on $(0, \infty)$. Noticing $\lim_{s\to 0} f_4(s) = 0$, we can conclude $f_4(s) > 0$ on $(0, \infty)$.

(f) Notice $f_5'(s) = 2e^{-s}(s - (1 - e^{-s})) > 0$ for $s > 0$. Thus $f_5(s)$ is increasing on $(0, \infty)$. Noticing $\lim_{s\to 0}(1 - e^{-s})^2 - 2e^{-s}(e^{-s} + s - 1) = 0$, we conclude $f_5(s) > 0$ on $(0, \infty)$. □

Lemma A.1.5 *Function $f_s(t) = e^{-st}(t - te^{-s} + 1)$ is decreasing on* $(0, \infty)$ *for any* $s > 0$.

Proof Consider its derivative, which is $f_s'(t) = e^{-st}(-st + ste^{-s} - s + 1 - e^{-s})$. It suffices to show $f_s'(t) < 0$ for $s, t > 0$. Notice

$$-st + ste^{-s} - s + 1 - e^{-s} = -(1 - e^{-s})st + 1 - e^{-s} - s.$$

It is linear in t with negative slope and negative intercept for any $s > 0$. Hence $f_s'(t) < 0$ and $f_s(t)$ is decreasing on $(0, \infty)$. □

Lemma A.1.6 *Consider the exponential integral function* $\mathrm{E}_1(x)$ *on* $(0, \infty)$*, defined by*

$$\mathrm{E}_1(x) := \int_{(x,\infty)} \frac{e^{-t}}{t}\, dt.$$

We have $0 < -\ln(t) < \mathrm{E}_1(st)e^s$ *for every* $t \in (0, 1)$ *and* $s > 0$.

Proof To prove this, define

$$q_s(t) := e^{-s}\ln(t) + \mathrm{E}_1(st).$$

Note that $q_s'(t) = (e^{-s} - e^{-st})/t < 0$ for $t \in (0, 1)$ and $s > 0$. Hence $q_s(t)$ is decreasing on $(0, 1)$. Also note that $q_s(1) = \mathrm{E}_1(s) > 0$. Hereby we conclude $q_s(t) > 0$, which implies $0 < -\ln(t) < \mathrm{E}_1(st)e^s$ for every $t \in (0, 1)$ and $s > 0$. □

Lemma A.1.7 *Function* $f(x) = 1 - 1/x - \ln(x)$ *is non-positive on* $(0, \infty)$*. In particular,* $f(x) = 0$ *if and only if* $x = 1$.

Proof Notice that

$$f'(x) = \frac{1 - x}{x^2}.$$

It is positive on $(0, 1)$ and negative on $(1, \infty)$. Thus $f(x)$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. The global maximum takes place at $x = 1$ with $f(1) = 0$. □

Lemma A.1.8 *Function* $f_k(x) = x^k e^{-xk/(k+1)}$ *is bounded from above by* $(k + 1)^k e^{-k}$ *on* $(0, \infty)$ *for all* $k \geq 1$*. In particular,* $f_k(x) = (k + 1)^k e^{-k}$ *if and only if* $x = k + 1$.

Proof Notice that

$$f_k'(x) = kx^{k-1}e^{-xk/(k+1)} - \frac{k}{k + 1}x^k e^{-xk/(k+1)} = kx^{k-1}e^{-xk/(k+1)}\left(1 - \frac{x}{k + 1}\right).$$

It is positive if $x < k + 1$ and negative if $x > k + 1$. Thus $f_k(x)$ is increasing on $(0, k + 1)$ and decreasing on $(k + 1, \infty)$. The global maximum takes place at $x = k + 1$ with $f_k(k + 1) = (k + 1)^k e^{-k}$. □

Lemma A.1.9 *Consider the function*

$$f_t(s) = 1 - \frac{se^{-ts}}{1 - e^{-s}}$$

on $(0, \infty)$*, where* $t > 0$*. Then we have the following properties.*

(a) If $t \geq 1/2$, then $f_t(s) \geq 0$;

(b) If $0 < t < 1/2$, then there is only one solution for s on $(0, \infty)$ such that $f_t(s) = 0$.

Proof (a) Suppose $t \geq 1/2$. L'Hopital rule implies $\lim_{s\to 0} f_t(s) = 0$. To verify $f_t(s) \geq 0$, it suffices to show its derivative is non-negative. Notice

$$f_t'(s) = \frac{e^{-ts}((1-e^{-s})st + e^{-s} + se^{-s} - 1)}{(1-e^{-s})^2}.$$

It is equivalent to show $(1-e^{-s})st + e^{-s} + se^{-s} - 1 \geq 0$. Notice that this function is linear in t with positive slope. We just need to show it is non-negative when $t = 1/2$, which is equivalent to show

$$\frac{s}{2} + \frac{se^{-s}}{1-e^{-s}} - 1 \geq 0.$$

By L'Hopital rule, we obtain

$$\lim_{s\to 0}\left(\frac{s}{2} + \frac{se^{-s}}{1-e^{-s}} - 1\right) = 0.$$

Thus it suffices to show this function in non-decreasing. Indeed, notice

$$\left(\frac{s}{2} + \frac{se^{-s}}{1-e^{-s}} - 1\right)' = \frac{1}{2} - \frac{e^{-s}(e^{-s}+s-1)}{(1-e^{-s})^2} = \frac{(1-e^{-s})^2 - 2e^{-s}(e^{-s}+s-1)}{2(1-e^{-s})^2}.$$

The numerator is non-negative by Lemma A.1.4 (f).

(b) Suppose $0 < t < 1/2$, we want to show there is an unique solution to $f_t(s) = 0$. Rewrite $f_t(s) = 0$ in the following way,

$$t = \frac{1}{s}\ln\left(\frac{s}{1-e^{-s}}\right) =: h(s).$$

By L'Hopital rule,

$$\lim_{s\to\infty} h(s) = \lim_{s\to\infty}\frac{1-e^{-s}-se^{-s}}{s(1-e^{-s})} = 0,$$
$$\lim_{s\to 0} h(s) = \lim_{s\to 0}\frac{1-e^{-s}-se^{-s}}{s(1-e^{-s})} = \lim_{s\to 0}\frac{se^{-s}}{1-e^{-s}+se^{-s}} = \lim_{s\to 0}\frac{-e^{-s}+se^{-s}}{-2e^{-s}+se^{-s}} = \frac{1}{2}.$$

To show $f_t(s) = 0$ has one unique solution on $(0, \infty)$ for any $0 < t < 1/2$, if suffices to show $h(s)$ is strictly decreasing on $(0, \infty)$. Consider its derivative

$$h'(s) = \frac{1}{s^2}\left(\frac{1-(s+1)e^{-s}}{1-e^{-s}} - \ln\left(\frac{s}{1-e^{-s}}\right)\right) =: \frac{1}{s^2}g(s).$$

It suffices to show $g(s) > 0$ for all $s > 0$. Notice

$$\lim_{s\to 0}\frac{1-(s+1)e^{-s}}{1-e^{-s}} = \lim_{s\to 0}\frac{1-e^{-s}-se^{-s}}{e^{-s}} = 0, \quad \text{and} \quad \lim_{s\to 0}\ln\left(\frac{s}{1-e^{-s}}\right) = \ln\left(\lim_{s\to 0}\frac{s}{1-e^{-s}}\right) = 0.$$

Therefore, $\lim_{s\to 0} g(s) = 0$ and it suffices to prove $g(s)$ is strictly increasing. Consider

$$g'(s) = \frac{1-e^{-s}-se^{-s}}{s(1-e^{-s})} - \frac{e^{-s}(s-1+e^{-s})}{(1-e^{-s})^2} = \frac{e^{-2s} - s^2e^{-s} - 2e^{-s} + 1}{s(1-e^{-s})^2}.$$

The numerator is positive on $(0, \infty)$ by A.1.4 (c). Therefore, $g'(s) > 0$ on $(0, \infty)$ and there exists one unique solution to $f_t(s) = 0$ on $(0, \infty)$ for $0 < t < 1/2$. □

Lemma A.1.10 *If ν is a Radon measure on $(0,\infty)$ and $\bar{\nu}(x_0) < \infty$ for some $x_0 > 0$. Then* $\lim_{x\to\infty} \bar{\nu}(x) = 0$.

Proof First notice that $\bar{\nu}(x)$ is non-negative and non-increasing. Hence the limit exists as x approaches infinity. As ν is inner regular, we obtain $\nu[x_0+1,\infty) = \lim_{x\to\infty} \nu[x_0+1, x]$.

Note $\nu[x_0+1,\infty) = \nu[x_0+1,x] + \bar{\nu}(x)$ for x large. Letting x approaching infinity, using $\nu[x_0+1,\infty) < \bar{\nu}(x_0)$ being finite, we get $\lim_{x\to\infty} \bar{\nu}(x) = 0$. □

Lemma A.1.11 *Suppose ν is Lévy measure with harmonically concave tail. Then*

(a) $\lim\limits_{t\to\infty} t\bar{\nu}'_+(t) = 0$;

(b) $\lim\limits_{t\to 0+} t^2\bar{\nu}'_+(t) = 0$.

Proof (a) Because ν has harmonically concave tail, the function $t\bar{\nu}(t)$ is concave on $(0,\infty)$. Its right derivative $\bar{\nu}(t) + t\bar{\nu}'_+(t)$ is decreasing. Thus, $\lim\limits_{t\to\infty}[\bar{\nu}(t) + t\bar{\nu}'_+(t)]$ exists in $[-\infty,\infty)$ and using formula (2.22) we obtain that $\lim_{t\to\infty} t\bar{\nu}'_+(t)$ exists in $[-\infty, 0]$ because $\bar{\nu}'_+(t)$ is non-positive. Noticing formula (2.38) holds for harmonically convex (concave) functions on $(0,\infty)$, integrability condition (2.17) gives

$$\infty > \int_{(1,\infty)} \nu(dt) = -\int_{(1,\infty)} d\bar{\nu}(t)$$

We extend $\bar{\nu}(x)$ onto $(-\infty, 0)$ by defining $\bar{\nu}(-x) = \bar{\nu}(x)$. Then,

$$\begin{aligned}\int_{(1,\infty)} d\bar{\nu}(t) &= \int_{(-1,-\infty)} d\bar{\nu}(t) = \int_{(1,0)} d\bar{\nu}\Big(-\frac{1}{s}\Big) = \int_{(1,0)} \Big(\bar{\nu}\Big(-\frac{1}{s}\Big)\Big)'_+ ds \\ &= -\int_{(1,0)} \bar{\nu}'_-\Big(-\frac{1}{s}\Big)\frac{1}{s^2}\, ds = \int_{(1,\infty)} \bar{\nu}'_-(-t)\, dt = \int_{(1,\infty)} \bar{\nu}'_+(t)\, dt\end{aligned}$$

The first equation in second line is the property of even extension. The second equation follows [80, Theorem 11a] by change of variable $t = -1/s$, and Lemma 2.39 implies the first equation in second line. Change of variable is applied again by $s = 1/t$ in the next equation and the last one is the property of even extension. We obtain

$$\infty > -\int_{(1,\infty)} d\bar{\nu}(t) = -\int_{(1,\infty)} \bar{\nu}'_+(t)\, dt = \int_{(1,\infty)} \frac{1}{t}(-t\bar{\nu}'_+(t))\, dt \geq 0$$

The convergence of this integral implies that $\lim\limits_{t\to\infty} t\bar{\nu}'_+(t) = 0$.

(b) As ν has harmonically concave tail, function $\bar{\nu}(1/t)$ is concave and non-decreasing. By Lemma 2.39, and the basic properties of concave functions, we have that

$$(\bar{\nu}(1/t))'_- = -\bar{\nu}'_+(1/t)/t^2$$

is non-increasing and non-negative.

Hence, $t^2\bar{\nu}'_+(t)$ is non-increasing and non-positive. $\lim_{t\to 0+}(-t^2\bar{\nu}'_+(t))$ exists in $[0,\infty]$. Integration by parts shows, for $\epsilon \in (0,1)$, that

$$\int_{(\epsilon,1]} \bar{\nu}(t)\, dt = \int_{(\epsilon,1]} t\,\nu(dt) + \bar{\nu}(1) - \epsilon\bar{\nu}(\epsilon).$$

As $t\bar{\nu}(t)$ is concave, its right derivative

$$(t\bar{\nu}(t))'_+ = \bar{\nu}(t) + t\bar{\nu}'_+(t)$$

satisfies

$$\int_{(\epsilon,1]} t\bar{\nu}'_+(t) + \bar{\nu}(t)\,dt = \bar{\nu}(1) - \epsilon\bar{\nu}(\epsilon).$$

Therefore,

$$0 \le -\int_{(\epsilon,1]} t\bar{\nu}'_+(t)\,dt = \int_{(\epsilon,1]} \bar{\nu}(t)\,dt + \epsilon\bar{\nu}(\epsilon) - \bar{\nu}(1) = \int_{(\epsilon,1]} t\,\nu(dt) \le \int_{(0,1]} t\,\nu(dt) < \infty,$$

where the last inequality follows from (2.17). Letting ϵ approach 0, we obtain

$$0 \le -\int_{(0,1]} t\bar{\nu}'_+(t)\,dt = \int_{(0,1]} \frac{1}{t}(-t^2\bar{\nu}'_+(t))\,dt < \infty.$$

The convergence of the last integral, reveals that $\lim_{t\to 0+} t^2\bar{\nu}'_+(t) = 0$. □

Lemma A.1.12 *Suppose $F(x) : (0,\infty) \to (0,\infty)$ is right continuous, non-decreasing (or non-increasing), and there exists a sequence of functions $\{G_n(x)\}$ such that $\lim_{n\to\infty} G_n(x) = x^\alpha F(x)$ at every point of continuity of $F(x)$ for some $\alpha \ge 0$.*

(a) If $G_n(x)$ is convex for all $n \ge 1$, then $x^\alpha F(x)$ is also convex;

(b) If $G_n(x)$ is concave for all $n \ge 1$, then $x^\alpha F(x)$ is also concave.

Proof First note that the point of continuity of $x^\alpha F(x)$ is the same as $F(x)$ for all $\alpha \ge 0$.
(a) Suppose $F(t)$ to be non-decreasing. Let $x, y \in (0,\infty)$ be points of continuity of $F(t)$. Choose $\lambda \in [0,1]$, such that $(1-\lambda)x + \lambda y$ is also a point of continuity of $F(t)$. So we have

$$\lim_{n\to\infty} G_n(t) = t^\alpha F(t),$$

for every $t \in \{x, y, (1-\lambda)x + \lambda y\}$ and since G_n is convex for every n, we obtain

$$((1-\lambda)x + \lambda y)^\alpha F((1-\lambda)x + \lambda y) \le (1-\lambda)x^\alpha F(x) + \lambda y^\alpha F(y). \tag{A.1}$$

It suffices to show that $F(t)$ is continuous function on $(0,\infty)$. Indeed, suppose that $u \in (0,\infty)$ is a jump point for the right-continuous, non-decreasing functions $F(t)$, that is, $F(u-) < F(u)$. Let $\{y_n\}$ be a decreasing sequence of points of continuity of $F(t)$, converging to u. Such sequence exists, since the number of jumps of $F(t)$ is at most countable. Using the density of the points of continuity, for every *n large enough*, we can choose an $\lambda_n \in [1/4, 3/4]$ and a point of continuity $x_n < u$ such that the following three conditions are satisfied:

1) the sequence $\{x_n\}$ converges to u from the left;
2) $(1-\lambda_n)x_n + \lambda_n y_n$ is a point of continuity for every n; and
3) $u \le (1-\lambda_n)x_n + \lambda_n y_n$ for every n.

So we get for every *n large enough*,

$$((1-\lambda_n)x_n+\lambda_n y_n)^\alpha F((1-\lambda_n)x_n+\lambda_n y_n) \leq (1-\lambda_n)x_n^\alpha F(x_n)+\lambda_n y_n^\alpha F(y_n). \tag{A.2}$$

Note that $\{\lambda_n\}$ is bounded away from 1 and 0. Without loss of generality, assume the sequence $\{\lambda_n\}$ converges to some $\lambda \in [1/4, 3/4]$. Otherwise select a subsequence. Taking the limit as n approaches infinity in (A.2), using that $F(t)$ is right-continuous, we obtain

$$u^\alpha F(u) \leq (1-\lambda)u^\alpha F(u-)+\lambda u^\alpha F(u).$$

Since $\lambda \neq 1$ we reach the contradiction $F(u) \leq F(u-)$.

If $F(t)$ is non-increasing, inequality (A.1) still holds for points of continuity. It suffices to show $F(t)$ is continuous on $(0,\infty)$. If there is a jump point u, such that $F(u-) > F(u)$, then we can also construct sequences $\{x_n\}, \{y_n\}$ and $\{\lambda_n\}$, satisfying the conditions above with modified condition 3) as $(1-\lambda_n)x_n + \lambda_n y_n \leq u$ for every n. Taking limit in (A.2), using that $F(t)$ is right-continuous, we obtain

$$u^\alpha F(u-) \leq (1-\lambda)u^\alpha F(u-)+\lambda u^\alpha F(u).$$

Since $\lambda \neq 0$ we reach the contradiction $F(u-) \leq F(u)$.

(b) The proof is analogous. Suppose $F(t)$ to be non-decreasing. We could have

$$((1-\lambda)x+\lambda y)^\alpha F((1-\lambda)x+\lambda y) \geq (1-\lambda)x^\alpha F(x)+\lambda y^\alpha F(y). \tag{A.3}$$

Here $x, y, (1-\lambda)x+\lambda y$ are points of continuity of $F(t)$. It suffices to show $F(t)$ is continuous on $(0,\infty)$. If there is a jump point u, such that $F(u-) < F(u)$, we could also find sequences $\{x_n\}, \{y_n\}$ and $\{\lambda_n\}$ satisfying the conditions above with modified condition 3) as $(1-\lambda_n)x_n + \lambda_n y_n \leq u$ for every n. Taking limit in

$$((1-\lambda_n)x_n+\lambda_n y_n)^\alpha F((1-\lambda_n)x_n+\lambda_n y_n) \geq (1-\lambda_n)x_n^\alpha F(x_n)+\lambda_n y_n^\alpha F(y_n), \tag{A.4}$$

and using that $F(t)$ is right-continuous, we obtain

$$u^\alpha F(u-) \geq (1-\lambda)u^\alpha F(u-)+\lambda u^\alpha F(u).$$

Since $\lambda \neq 0$ we reach the contradiction $F(u-) \geq F(u)$.

If $F(t)$ to be non-increasing. We could have (A.3) for points of continuity as well. It suffices to show $F(t)$ is continuous on $(0,\infty)$. If there is a jump point u, such that $F(u-) > F(u)$, we can also construct sequences $\{x_n\}, \{y_n\}$ and $\{\lambda_n\}$, satisfying the conditions as listed. Taking limit in (A.2), using that $F(t)$ is right-continuous, we obtain

$$u^\alpha F(u) \geq (1-\lambda)u^\alpha F(u-)+\lambda u^\alpha F(u).$$

Since $\lambda \neq 1$ we reach the contradiction $F(u) \geq F(u-)$. □

A.2 On integrals

In this sections, we show some properties of Lebesgue-Stieltjes integrals. Several of them are not easy to find, but are quite useful in our developments.

Lemma A.2.1 *Suppose $f(t) \geq 0$ is non-increasing. If for some $T > 0$*

$$\int_{(0,T)} f(t)\,dt < \infty,$$

then $f(t)$ is $o(1/t)$ as t approaches zero.

Proof Suppose $f(t)$ is not $o(1/t)$ as t approaches zero. Then for some $\epsilon > 0$, we have a decreasing sequence $\{x_n\}$ approaching zero, such that $x_1 < T$ and $x_n f(x_n) > \epsilon$ for all $n \geq 1$. Without loss of generality, we can assume $x_{n+1} < x_n/2$, otherwise choose a subsequence.

As $f(x)$ is non-increasing, we know that

$$\int_{(0,T)} f(t)\,dt \geq \sum_{n=1}^{\infty}(x_n - x_{n+1})f(x_n) \geq \epsilon \sum_{n=1}^{\infty} \frac{x_n - x_{n+1}}{x_n} \geq \epsilon \sum_{n=1}^{\infty}\left(1 - \frac{x_{n+1}}{x_n}\right).$$

As it is assumed that $x_{n+1} < x_n/2$, we know $1 - x_{n+1}/x_n > 1/2$ and the above series does not converge, which contradicts the integrability condition. So we know $f(t)$ is $o(1/t)$ as t approaches 0. □

Lemma A.2.2 *Suppose $f(t) \geq 0$ is non-increasing. If for some $T > 0$,*

$$\int_{(T,\infty)} f(t)\,dt < \infty,$$

then $f(t)$ is $o(1/t)$ as t approaches infinity.

Proof Suppose $f(t)$ is not $o(1/t)$ as t approaches infinity. Then for some $\epsilon > 0$, we have an increasing sequence of $\{x_1\}$ such that $x_n > T$ and $x_n f(x_n) > \epsilon$ for all $n \geq 1$.

Denote $y_0 = x_1$ and $y_n = x_{n+1} - x_n$. Without loss of generality, we can assume y_n is increasing and approaches infinity, otherwise we choose a subsequence of x_n. As $f(x)$ is non-increasing, we know that

$$\int_{(T,\infty)} f(t)\,dt \geq \sum_{n=1}^{\infty}(x_n - x_{n+1})f(x_n) \geq \epsilon \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{x_{n+1}} \geq \epsilon \sum_{n=1}^{\infty} \frac{y_n}{y_n + y_{n-1} + \cdots + y_1 + y_0}$$
$$= \epsilon \sum_{n=1}^{\infty} \frac{1}{n+1} \cdot \frac{y_n}{\dfrac{y_n + y_{n-1} + \cdots + y_1 + y_0}{n+1}} \geq \epsilon \sum_{n=1}^{\infty} \frac{1}{n+1}.$$

The last series does not converge. Thus it contradicts our integrability assumption. So we know $f(t)$ is $o(1/t)$ as t approaches infinity. □

Lemma A.2.3 *Suppose $f(t) \geq 0$ is non-increasing and $g(t)$ is strictly increasing with $g(0) = 0$. If for some $T > 0$,*

$$\int_{(0,T)} f(t)\, d(g(t)) < \infty,$$

then $f(t)$ is $o(1/g(t))$ as t approaches zero.

Proof As $g(t)$ is strictly increasing with $g(0) = 0$, its inverse function $g^{-1}(t)$ is also strictly increasing with $g^{-1}(0) = 0$. Change variable by setting $t = g^{-1}(s)$. Note $f(g^{-1}(s))$ is non-increasing and

$$\int_{(0,g(T))} f(g^{-1}(s))\, ds < \infty.$$

So $f(g^{-1}(s))$ is $o(1/s)$ as s approaches zero, which implies $\lim_{t\to 0} f(t)g(t) = 0$. □

By taking $g(t) = t^p$ in the above lemma, we could have the next handy corollary.

Corollary A.2.1 *Suppose $f(t) \geq 0$ is non-increasing. If there exists some $T > 0$*

$$\int_{(0,T)} f(t)\, d(t^p) < \infty,$$

for $p > 0$. Then $f(t)$ is $o(1/t^p)$ as t approaches zero.

Lemma A.2.4 *For any $t > 0$, we have the following equation:*

$$\int_{(y,\infty)} \int_{(0,\infty)} e^{-us}\left(1 - \frac{se^{-ts}}{1 - e^{-s}}\right) ds\, du = \int_{(0,\infty)} \int_{(y,\infty)} e^{-us}\left(1 - \frac{se^{-ts}}{1 - e^{-s}}\right) du\, ds.$$

Proof If $t \geq 1/2$, by Lemma A.1.9, we know $1 - se^{-ts}/(1 - e^{-s}) \geq 0$ on $(0, \infty)$. By Fubini theorem, the above equation holds.

If $0 < t < 1/2$, to interchange the integral by Fubini-Tonelli theorem, it suffices to verify

$$\int_{(0,\infty)} \int_{(y,\infty)} \left| e^{-us}\left(1 - \frac{se^{-ts}}{1 - e^{-s}}\right) \right| du\, ds = \int_{(0,\infty)} \left| \frac{1}{s} - \frac{e^{-ts}}{1 - e^{-s}} \right| e^{-ys}\, ds < \infty.$$

Denote

$$h_t(s) := \frac{1}{s} - \frac{e^{-ts}}{1 - e^{-s}}.$$

Consider the shape of $h_t(s)$. By L'Hopital rule, we observe that,

$$\lim_{s\to\infty} sh_t(s) = 1 - \lim_{s\to\infty} \frac{se^{-ts}}{1 - e^{-s}} = 1,$$

$$\lim_{s\to 0} h_t(s) = \lim_{s\to 0} \frac{1 - e^{-s} - se^{-ts}}{s(1 - e^{-s})} = \lim_{s\to 0} \frac{-e^{-ts} + ste^{-ts} + e^{-s}}{1 - e^{-s} + se^{-s}} = \lim_{s\to 0} \frac{2te^{-ts} - st^2e^{-ts} - e^{-s}}{2e^{-s} - se^{-s}} = t - \frac{1}{2}.$$

The first limit implies $sh_t(s) > 0$ for s large. By Lemma A.1.9, we know that there is an unique solution for s on $(0, \infty)$ such that $sh_t(s) = 0$. Since $h_t(s)$ approaches $t - 1/2 < 0$ as s approaches zero, we know $h_t(s)$ is negative on $(0, s^*)$ and positive on (s^*, ∞), where s^* is the unique solution of $sh_t(s) = 0$. Therefore, we obtain

$$\int_{(0,\infty)} \left| \frac{1}{s} - \frac{e^{-ts}}{1-e^{-s}} \right| e^{-ys}\, ds = \int_{(0,s^*]} -h_t(s)e^{-ys}\, ds + \int_{(s^*,\infty)} h_t(s)e^{-ys}\, ds.$$

It is clear that the first term is finite, because $h_t(s^*) = 0$ and $\lim_{s\to 0} -h_t(s)e^{-ys} = 1/2 - t$. The second term is also finite, because $\lim_{s\to\infty} h_t(s) = 0$ and therefore $h_t(s)$ is bounded on (s^*, ∞). Its Laplace transformation is well-defined. Hence for any $y > 0$

$$\int_{(0,\infty)} \left|h_t(s)\right| e^{-ys}\, ds < \infty.$$

Fubini-Tonelli Theorem is applicable to interchange the integral. $\square$

The following lemma is a particular case of the change of variable formula for Lebesgue-Stieltjes integrals, see [80, Theorem 11a].

Lemma A.2.5 *Suppose $f(x)$ is continuous on $(0, \infty)$ and $g(x)$ has bounded variation on $(0, \infty)$. Then*

$$\int_{(0,\infty)} f(x)\, dg(x) = -\int_{(0,\infty)} f(1/t)\, dg(1/t).$$

The following theorem describes integration by parts for Lebesgue-Stieltjes integrals on finite intervals, see [24, Theorem 6.2.2]. They can be extend onto $(0, \infty)$ by taking limits.

Theorem A.2.6 *Let $f, g : I \to \mathbb{R}$ be right-continuous functions of bounded variation. If f is also continuous, for any $[a, b] \subset I$, we have*

$$\begin{aligned}
\int_{[a,b]} f\, dg + \int_{[a,b]} g\, df &= f(b)g(b) - f(a-)g(a-),\\
\int_{(a,b]} f\, dg + \int_{(a,b]} g\, df &= f(b)g(b) - f(a)g(a),\\
\int_{(a,b)} f\, dg + \int_{(a,b)} g\, df &= f(b-)g(b-) - f(a)g(a).
\end{aligned}$$

Lemma A.2.7 *Suppose $f(x)$ is continuous on $(0, \infty)$ and $g(x)$ is right-continuous with bounded variation on $(0, \infty)$. For right-continuous and non-negative function $m(x)$ on $(0, \infty)$,*

$$\int_{(0,\infty)} m(x)\, d(f(x)g(x)) = \int_{(0,\infty)} m(x)f(x)\, dg(x) + \int_{(0,\infty)} m(x)g(x)\, df(x), \tag{A.5}$$

given one of the integrals on the right hand side is convergent,

Proof Step 1: Show that A.5 holds for $g(x)$ increasing on close interval $[a, b] \subset (0, \infty)$. For partition $a = x_0 < x_1 < \cdots < x_n = b$, we have

$$\int_{[a,b]} m(x)\, d(f(x)g(x))$$

$$
\begin{aligned}
&= \lim_{\Delta\to 0}\sum_{i=0}^{n-1} m(x_i)(f(x_{i+1})g(x_{i+1}) - f(x_i)g(x_i)) \\
&= \lim_{\Delta\to 0}\sum_{i=0}^{n-1} m(x_i)f(x_i)(g(x_{i+1}) - g(x_i)) + \lim_{\Delta\to 0}\sum_{i=0}^{n-1} m(x_i)g(x_i)(f(x_{i+1}) - f(x_i)) \\
&\qquad + \lim_{\Delta\to 0}\sum_{i=0}^{n-1} m(x_i)(f(x_{i+1}) - f(x_i))(g(x_{i+1}) - g(x_i)) \\
&= \int_{[a,b]} m(x)f(x)\,dg(x) + \int_{[a,b]} m(x)g(x)\,df(x) \\
&\qquad + \lim_{\Delta\to 0}\sum_{i=0}^{n-1} m(x_i)(f(x_{i+1}) - f(x_i))(g(x_{i+1}) - g(x_i)).
\end{aligned}
$$

Here $\Delta = \max_{i=0,\dots,n-1}|x_{i+1} - x_i|$. Notice that $f(x)$ continuous, thus uniformly continuous on $[a,b]$. For any $\epsilon > 0$, there exists $\delta > 0$, such that for any $t, s \in [a,b]$ and $|t-s| < \delta$, we have $\left|f(t) - f(s)\right| < \epsilon$. For any partition such that $\Delta < \delta$, we obtain

$$
\begin{aligned}
&\lim_{\Delta\to 0}\left|\sum_{i=0}^{n-1} m(x_i)[f(x_{i+1}) - f(x_i)][g(x_{i+1}) - g(x_i)]\right| \\
&\leq \epsilon \lim_{\Delta\to 0}\sum_{i=0}^{n-1}\left|m(x_i)[g(x_{i+1}) - g(x_i)]\right| = \epsilon \int_{[a,b]} m(x)\,dg(x).
\end{aligned}
$$

So this limit can be arbitrary small, which indicates

$$
\int_{[a,b]} m(x)\,d(f(x)g(x)) = \int_{[a,b]} m(x)f(x)\,dg(x) + \int_{[a,b]} m(x)g(x)\,df(x).
$$

Step 2: For any $g(x)$ with bounded variation on $[a,b]$, A.5 holds, as such $g(x)$ can be represented as the difference of two increasing functions.

Step 3: Taking a approaches zero and b approaching infinity, A.5 holds on $(0,\infty)$, given one of the integrals on right hand side is convergent. □

Remark A.2.8 *From step 3 above, we can see that if $f(x)$ and $g(x)$ are both non-negative and non-decreasing, then A.5 holds by the monotone convergence theorem, in which case, we can remove the condition that one of the integral on the right is convergent.*

Lemma A.2.9 *Suppose real valued function $g(x)$ is non-decreasing and convex (or concave) on $(0,\infty)$. Then,*

$$
\int_{(0,\infty)} m(x)\,dg(x) = \int_{(0,\infty)} m(x)g'(x)\,dx, \tag{A.6}
$$

where $m(x)$ is continuous and non-negative on $(0,\infty)$.

Proof First consider interval $[a,b] \subset (0,\infty)$. For partition $a = x_0 < x_1 < \cdots < x_n = b$, by (2.38), we have

$$\begin{aligned}
&\int_{[a,b]} m(x)\,dg(x) - \int_{[a,b]} m(x)g'(x)\,dx \\
&= \lim_{\Delta\to 0} \sum_{i=0}^{n-1} m(x_i)(g(x_{i+1}) - g(x_i)) - \sum_{i=0}^{n-1} \int_{(x_i,x_{i+1})} m(x)g'(x)\,dx \\
&= \lim_{\Delta\to 0} \sum_{i=0}^{n-1} \int_{(x_i,x_{i+1})} (m(x_i) - m(x))g'(x)\,dx.
\end{aligned}$$

Here $\Delta = \max_{i=0,\dots,n-1}|x_{i+1} - x_i|$. As $g'(x)$ is right-continuous on $[a,b]$, it is bounded, that is $\left|g'(x)\right| \le M$ for some $M > 0$.

On the other hand, as $m(x)$ is continuous on $(0,\infty)$, it is uniformly continuous on $[a,b]$. For any $\epsilon > 0$, there exists $\delta > 0$, such that for any $t,s \in [a,b]$ and $|t-s| < \delta$, we have $\left|m(t) - m(s)\right| < \epsilon$. For any partition such that $\Delta < \delta$, we obtain

$$\int_{(x_i,x_{i+1})} \left|(m(x_i) - m(x))g'(x)\right|\,dx \le M \int_{(x_i,x_{i+1})} \left|(m(x_i) - m(x))\right|\,dx \le \epsilon M(x_{i+1} - x_i).$$

Therefore, we obtain

$$\left|\int_{[a,b]} m(x)\,dg(x) - \int_{[a,b]} m(x)g'(x)\,dx\right| \le \epsilon M(b-a).$$

As ϵ could be arbitrarily small, we know

$$\int_{[a,b]} m(x)\,dg(x) = \int_{[a,b]} m(x)g'(x)\,dx.$$

Taking limit and letting a approach zero and b approaches infinity, we could have (A.6) by monotone convergence theorem. $\square$

Curriculum Vitae

Name:	Shen Shan
Post-Secondary Education and Degrees:	Ph.D. in Statistics (Financial Modelling), 2011 - 2015 University of Western Ontario, London, ON
	M.S. in Statistics (Financial Mathematic), 2008 - 2011 Shanghai Jiao Tong University, Shanghai, China
	B.S. in Math and Applied Math, 2004 - 2008 Shanghai Jiao Tong University, Shanghai, China
Honours and Awards:	Western Graduate Research Scholarship, 2011-2015
	Liu Yuan Scholarship, 2010
	Excellent Graduate of Shanghai Universities, 2008
Related Work Experience:	Lecturer, Jan 2015 - Apr 2015 The University of Western Ontario
	Teaching Assistant, 2011 - 2015 The University of Western Ontario

Publications:

[1] Hristo S. Sendov and Shen Shan. New representation theorems for completely monotone and Bernstein functions with convexity properties on their measures. *Journal of Theoretical Probability*, 2014. DOI 10.1007/s10959-014-0557-9
[2] Hristo S. Sendov and Shen Shan. Properties of completely monotone and Bernstein functions related to the shape of their measures. *submitted*, 2015.
[3] Hristo S. Sendov and Shen Shan. Properties of completely monotone and Bernstein functions related to the shape of their measures, part II. *working paper*, 2015.