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Determination of Lie superalgebras of supersymmetries of super differential equations

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Graduate Program in Applied Mathematics

A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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DETERMINATION OF LIE SUPERALGEBRAS OF SUPERSYMMETRIES OF
SUPER DIFFERENTIAL EQUATIONS

(Thesis Format: Monograph)

by

Xuan Liu

Graduate Program
in
Applied Mathematics

A thesis is submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

School of Graduate and Postdoctoral Studies
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London, Ontario, Canada

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Abstract

Superspaces are an extension of classical spaces that include certain (non-commutative) supervariables. Super differential equations are differential equations defined on superspaces, which arise in certain popular mathematical physics models. Supersymmetries of such models are superspace transformations which leave their sets of solutions invariant. They are important generalization of classical Lie symmetry groups of differential equations.

In this thesis, we consider finite-dimensional Lie supersymmetry groups of super differential equations. Such supergroups are locally uniquely determined by their associated Lie superalgebras, and in particular by the structure constants of those algebras. The main work of this thesis is providing an algorithmic method for finding the structure constants of such Lie superalgebras. The traditional method uses heuristic integrations to determine such structure constants. Two typical examples are used to demonstrate our algorithm for determining structure constants.

We also apply our method to a large class of super Lagrangians in $1 + 1$ dimensional space time. The supersymmetry classification of such a large class is impossible for hand calculation since it requires analysis of thousands of cases. We will show how to find hidden supersymmetry for such a class of super differential equations by our algorithms and the `Physics`, `DEtools`, `PDEtools` packages of Maple 17.

Dedicated to my beloved husband, Bo, the best event of my life.

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Chapter 1

Introduction

Symmetry methods provide powerful analytic tools for solving differential equations, especially nonlinear partial differential equations for which few analytic solution methods exist. Simply speaking, a symmetry is a transformation which maps solution manifolds to solution manifolds. Finite dimensional Lie symmetry groups are transformation groups that depend on finitely many parameters. They were initially developed in the works of Sophus Lie [1, 2, 3] in late nineteenth century. Lie also applied them to differential equations. For example by introducing invariants the number of independent variables in a PDE can be reduced. Locally a Lie group is characterized by its Lie algebra, which in fact is characterized by its structure constants. Indeed if the structure constants determine that an n -order ODE has n -dimensional solvable Lie algebra of symmetries, then it can be reduced to an $(n - r)$ -th order ODE.

Supersymmetry yields analogous results. For example, it transforms solution supermanifolds to solution supermanifolds. Over recent decades, researchers have extended many (Lie) symmetry properties to (Lie) supersymmetries. Supersymmetry originally arose from quantum field theory in 1960s and 1970s. In 1966, Miyazawa [4, 5] was the first physicist to use supersymmetry to relate mesons and baryons. In 1971, Gervais and Sakita [6] independently rediscovered supersymmetry with a consistent Lie algebraic graded structure arising in quantum field theory. Over a half century, supersymmetry has been prominent in physics. Various particle physics models have been developed

that predict new super particles under the action of supersymmetry. However, the Large Hadron Collider (LHC) has eliminated some popular supersymmetric models. Despite this, supersymmetry has been proved to be a powerful tool in simplifying the analytical solution of various well-established models, including classical models in quantum mechanics. For example, the non-commutative variables arising in such calculations are operators where components are complex functions.

In this thesis, supersymmetry is studied using symbolic computation. Our goal is to algorithmically determine the Lie superalgebra structure of the (maximal) group of supersymmetries of super differential equations. It is the generalization of Lie algebra structure determination methods invented by Reid [9, 10, 11], Lisle [12, 24], Boulton Wittkopf [12] in the 1990s. Moreover, using these techniques and symbolic algorithms in Maple we have determined new supersymmetries of a large class of physics models.

Next, we give a brief review of existing approaches and outline the contents of each chapter of this thesis.

1.1 Lie's infinitesimal symmetry method

Lie's profound discovery was that nonlinear analytic symmetry group transformations are uniquely locally determined by their linearized (infinitesimal) transformations. The infinitesimal transformations satisfy a linear homogeneous system of PDE called the defining or determining system for the symmetries.

The infinitesimal method for generating the defining system is introduced in this section for both classical (non-super) and super differential equations.

1.1.1 Lie's infinitesimal method for classical DEs

Olver has given a complete and detailed presentation of Lie's infinitesimal method and its applications to the differential equations in his book *Applications of Lie Groups to Differential Equations* [13]. Here we only concentrate on material for Lie's infinitesimal method for generating the symmetry defining system of differential equations whose solu-

tions determine the unknown Lie symmetries. Note that our notation is slightly changed from that in Olver [13] in order to later consistently generalize the same approach to super differential equations.

Consider a k -th order system of s differential equations

$$\Delta_\nu(X, A^{(k)}) = 0, \quad \nu = 1, \dots, s, \quad (1.1)$$

where $X = (x_1, \dots, x_m)$ are the independent variables and $A = (A^1, \dots, A^q)$ are the dependent variables. We will denote the system (1.1) by $\Delta = 0$.

In Lie theory, a one parameter local transformation has the form

$$\hat{X} = \Omega_\epsilon^1(X, A), \quad (1.2)$$

$$\hat{A} = \Omega_\epsilon^2(X, A). \quad (1.3)$$

Expanding each relations of (1.2) and (1.3) around the identity $\epsilon = 0$, one can generate the following infinitesimal (linearized) transformations.

$$\begin{aligned} \hat{x}_i &= x_i + \epsilon \Xi^i(X, A) + O(\epsilon^2), \quad i = 1, \dots, m, \\ \hat{A}^r &= A^r + \epsilon \Phi^r(X, A) + O(\epsilon^2), \quad r = 1, \dots, q, \end{aligned}$$

where the functions Ξ^i and Φ^r are the infinitesimals corresponding to the transformations for the independent variables x_i and dependent variables A^r .

A basis for corresponding symmetry algebra \mathcal{H} is denoted by the vector fields

$$V = \sum_{i=1}^m \Xi^i \frac{\partial}{\partial x_i} + \sum_{r=1}^q \Phi^r \frac{\partial}{\partial A^r}. \quad (1.4)$$

The action of a symmetry on (X, A) can be extended to the derivatives appearing in a differential equation $\Delta_\nu = 0$ by the process of prolongation (see Olver [13] for a detailed

description). The resulting prolonged system is

$$\text{pr}^{(k)}V\Delta_\nu = 0, \quad \nu = 1, \dots, s, \quad (1.5)$$

where the k -th prolongation of the vector field (1.4) is given by

$$\text{pr}^{(k)}V = V + \sum_{r=1}^q \sum_J \Phi_J^r \frac{\partial}{\partial A_J^r}. \quad (1.6)$$

Here $J = (j_1, \dots, j_\beta)$, $1 \leq j_\beta \leq m$ and $1 \leq \beta \leq m$ is multi-index notation for differentiations with respect to x_i 's. The coefficients Φ_J^r are given by

$$\Phi_J^r = D_J \left(\Phi^r - \sum_{i=1}^m \Xi^i A_i^r \right) + \sum_{i=1}^m \Xi^i A_{Ji}^r, \quad (1.7)$$

where $A_i^r = \partial A^r / \partial x_i$.

In addition we can further decompose the system of s equations by computing the coefficients of the monomials of A^r and its derivatives and equating these coefficients to zero. These expressions are the defining equations for symmetries. The defining system involves the infinitesimals Ξ^i, Φ^r and their partial derivatives with respect to x_i 's and A^r 's.

1.1.2 Lie's infinitesimal method for super DEs

Fortunately Lie's infinitesimal method is easily extended to super differential equations by methods that are very similar to those in Section 1.1.1. Ayari and Hussin used this method in their paper [20] in 1997. The main difference is to accommodate odd or non-commutative variables.

A super analytic system of Grassmann-valued differential equations or superequations of s equations of order $k = (k_1; k_2)$ is given by

$$\Delta_\nu(X, \Theta, A^{(k_1)}, Q^{(k_2)}) = 0, \quad \nu = 1, \dots, s, \quad (1.8)$$

with m independent even variables $X = (x_1, \dots, x_m)$, n independent odd variables $\Theta = (\theta_1, \dots, \theta_n)$, q even dependent variables $A = (A^1, \dots, A^q)$ and p odd dependent variables $Q = (Q^1, \dots, Q^p)$.

Lie's infinitesimal method for the defining system of super differential equations uses a similar procedure to the classical case. A brief verbal description is as follows.

1. Reduce to invariance under one-parameter Lie super transformation about the identity.
2. Apply the super prolongation formula to the super differential equations.
3. Simplify the results of Step 2.
4. Compute the coefficients of monomials of the dependent variables and their derivatives.
5. The determining equations for supersymmetries are the equations from Step 4.

More information on this procedure will be given in Chapter 3.

1.2 Existing supersymmetry related packages

Lie's infinitesimal method for generating the symmetry defining system of super differential equations has been implemented in Maple language by Ayari and Hussin [20]. They developed a Maple program GLie which can generate defining systems for Grassmann-valued partial differential equations. They also provided applications of GLie to a variety of models. The super KdV example used in this thesis is from Ayari's PhD thesis [19]. He found the Lie superalgebra structure by direct integration of the defining system. We will develop a new algorithm for determining the Lie superalgebra structure for supersymmetry without using integration.

Maple has its own built-in symmetry determining system generator `DeterminingPDE` as part of `PDEtools` package implemented by Chev-Terrab based on Chev-Terrab and

Bulow [15]. In 2011, Chev-Terrab extended `DeterminingPDE` to work with anticommutative variables as part of `Physics` package. The commands in these packages will be frequently used in my study of supersymmetric Lagrangian models.

There are also other symbolic computer languages which can handle anticommutative calculations. For example, Wolf [21, 22, 23, 25, 29] has made a powerful extension of his package `CRACK` in the computer algebra language `REDUCE`. `CRACK` can be used to find first order and higher order supersymmetry for polynomial super differential equations.

1.3 The `rifsimp` algorithm

In 1996, the `rifsimp` algorithm was introduced by Reid, Boulton and Wittkopf [16] and is part of distributed Maple since 2001. It is a powerful simplifier of systems of overdetermined DE. It can assist the determination of Lie point symmetry of ODE or PDE. In this thesis, we use `rifsimp` to help us simplify overdetermined super differential equations. In fact, `rifsimp` was designed only for commutative calculation. To able to apply `rifsimp`, we modify our super differential equations in order to apply `rifsimp` to the non-commutative case.

1.4 Existing algorithms for determining structure constants

Using the existence and uniqueness theorem [18], Lisle and Reid [24] developed algorithms for finding the structure constants for Lie symmetry of classical PDE. Briefly the existence and uniqueness theorem determines initial data. That uniquely determines the dimension d of the Lie algebra, and consequently that there exist d Lie supersymmetry operators L_1, \dots, L_d . The existence and uniqueness theorem then determines initial data that uniquely determines each L_j . Finally initial data of the commutator uniquely determines the commutator $[L_i, L_j]$ and specifically its structure constants. Most importantly,

this method does not depend on constructing solutions and so is algorithmic. Largely inspired by their methods, I am able to develop our algorithm for determine the structure constants for Lie superalgebra.

1.5 Outline of thesis

The fundamental mathematic definitions and computational rules are introduced in Chapter 2. We will give the definitions of superspace, superalgebra, Lie superalgebra, Grassmann algebra, super differential equations and differential rules for super differential equations. The main work in later chapters will be built on the concepts defined in Chapter 2.

The theory for the Lie infinitesimal method or the Lie supersymmetry method will be given in Chapter 3. Then we apply this method to two super differential equation examples to generate super defining system for their supersymmetry groups. Note that in Chapter 3, 4 and 5, for brevity we use the abbreviation defining system instead of super defining system. The first example is a simple super ordinary differential equation and the second example is well-known model, the super KdV equation, which is a super partial differential equation. After we obtain the defining system for these two examples, we will show how to find the Lie superalgebra structure or supercommutator table by integration. This is a heuristic process unlike the algorithmic method we will develop later.

In Chapter 4, we develop a new algorithm - the structure constant algorithm for finding Lie superalgebra structure. We introduce the concept of regular super differential equations, ones that can be solved for their highest derivatives. A technical difficulty for irregular super differential equations is addressed in this chapter. Moreover, we will show using existence and uniqueness theory, that the structure constants can be uniquely determined. We illustrate the new algorithm by applying it to our previous examples.

In Chapter 5, we apply supersymmetry analysis to a large class of super Lagrangians with general potential. The determination of hidden supersymmetry is executed for two extreme cases. One extreme case is with zero potential $F = 0$ and is easily solved. The

other extreme case is to incorporate maximum nonlinearity in F by letting the third order derivative of the coefficient of the leading term of F to be nonzero. By adding this constraint to F , we find a hidden non-trivial supersymmetry. Invariants corresponding to this supersymmetry are determined which reduce the Euler-Lagrange PDE system to an ODE system. During the demanding computations, we used Maple to help us to get the defining system of the Euler-Lagrange system of the input super Lagrangian. Then we sent the reduced defining system to `rifsimp` with the option `casesplit` to do the case analysis. Thousands of cases resulted from this step.

The last chapter is devoted to discussion and future work. In summary, three main contributions are made in this thesis. The first contribution is a method for getting the Lie superalgebra structure by integration. The second contribution shows how to get Lie superalgebra structure of the supersymmetry by an algorithms that avoids integration. The last contribution is an experimental search of hidden supersymmetry with the assistance of Maple.

At the end of this thesis, we list some Maple procedures in Appendix A.

Chapter 2

Background

Super spaces, variables, transformations etc, are fundamental objects in this thesis. In particle physics, there are two basic classes of elementary particles: bosons and fermions which are considered as even quantities and odd quantities.

Super objects possess a \mathbb{Z}_2 -grading, and consist of either even (0-graded) objects or odd (1-graded) objects. These quantities obey the following rules:

$$\text{even} \cdot \text{even} = \text{even},$$

$$\text{even} \cdot \text{odd} = \text{odd},$$

$$\text{odd} \cdot \text{odd} = \text{even}.$$

Every super concept, such as supervector, superspace, supersymmetry and superalgebra, admits its even partner as well as its odd partner. The even partners are just the usual vector space, and symmetry algebras etc, over \mathbb{R} or \mathbb{C} . Such super generalizations are often nontrivial, and certain crucial properties in the even case may be lost. They are of considerable interest to both physicists and mathematicians.

In this thesis we assume that the reader is familiar with manifolds, algebras, differential equations, symmetry groups and so on. We are working on superspace, superalgebra, super differential equations and supersymmetry groups. Simply speaking, these super concepts are the generalizations of those basic concepts in a ‘super’ sense by including

their super partners. The goal of this chapter is to introduce the mathematical definitions of superspace, superalgebra and super differential equations, as well as some necessary computation rules of super differential calculus. These ‘super’ concepts and rules are the important foundation of this thesis. More detailed explanation of odd variables, Grassmann algebra, Lie superalgebra and super differential equations can be found in Buchbinder and Kuzenko’s book [14] and Ayari’s PhD thesis [19].

2.1 Superspaces and superalgebras

2.1.1 Even and odd

This thesis is concerned with various super objects such as superspaces, superalgebras and supergroups. The essential feature of all of these is that they are graded. The simplest example of a graded structure is provided by the integers, each of which is either even or odd and:

$$\begin{aligned}
 \text{even integer} + \text{even integer} &= \text{even integer}, \\
 \text{even integer} + \text{odd integer} &= \text{odd integer}, \\
 \text{odd integer} + \text{odd integer} &= \text{even integer}.
 \end{aligned}
 \tag{2.1}$$

The operation of addition can be regarded as the group ‘product’ of the additive group of integers. Denoting this product by \cdot , the above addition rules (2.1) can be re-expressed as

$$\begin{aligned}
 \text{even integer} \cdot \text{even integer} &= \text{even integer}, \\
 \text{even integer} \cdot \text{odd integer} &= \text{odd integer}, \\
 \text{odd integer} \cdot \text{odd integer} &= \text{even integer}.
 \end{aligned}$$

Generally speaking, super objects obey the same rule as the integers

$$\begin{aligned} \text{even} \cdot \text{even} &= \text{even}, \\ \text{even} \cdot \text{odd} &= \text{odd}, \\ \text{odd} \cdot \text{odd} &= \text{even}. \end{aligned} \tag{2.2}$$

The first stage in applying the idea of grading to linear algebra is to define the concept of a graded vector space. To do this, suppose that V is a real or complex vector space of dimension $m + n$, where m and n are any two positive integers, and suppose that $\{a_1, a_2, \dots, a_{m+n}\}$ is a basis for V . Then any element a of V can be written in the form

$$a = \sum_{j=1}^{m+n} \mu_j a_j,$$

where the coefficients μ_j are real or complex numbers (as appropriate). A grading for this space is given by supposing that every element of the form

$$a = \sum_{j=1}^m \mu_j a_j,$$

is even, while every element of the form

$$a = \sum_{j=m+1}^{m+n} \mu_j a_j,$$

is said to be odd.

Definition 2.1.1 (Homogeneous). *Any element $a \in V$ that is either even or odd is said to be homogeneous.*

The degree (or parity) of such elements is defined by

$$\deg a = \begin{cases} 0, & \text{if } a \text{ is even,} \\ 1, & \text{if } a \text{ is odd.} \end{cases} \tag{2.3}$$

Definition 2.1.2 (Superspace). *The set of even elements of V form a subspace of V that is the even subspace and which will be denoted by V_0 . Similarly, the odd elements of V form the odd subspace V_1 . Clearly V is the direct sum of V_0 and V_1 , that is,*

$$V = V_0 + V_1,$$

which is called a \mathbb{Z}_2 -graded space or superspace.

2.1.2 Superalgebras and Lie superalgebras

A superspace is a \mathbb{Z}_2 -graded space $V = V_0 \oplus V_1$. A superalgebra is a \mathbb{Z}_2 -graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with a bilinear multiplication $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that

$$\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j},$$

where the integers i, j are taken module 2. A superalgebra is said to be supercommutative, if

$$ab = (-1)^{\deg(a)\deg(b)}ba$$

for all homogeneous a and b in the superalgebra; that is, if

$$ab = \begin{cases} -ba, & \text{if both } a \text{ and } b \text{ are odd,} \\ ba, & \text{otherwise.} \end{cases} \quad (2.4)$$

Definition 2.1.3 (Lie superalgebra). *Let \mathcal{L} be a real or complex graded vector space, with \mathcal{L}_0 and \mathcal{L}_1 being its even and odd subspaces, which are assumed to have dimension m and n respectively (where $m \geq 0, n \geq 0$ and $m + n \geq 1$). Suppose that for all $a, b \in \mathcal{L}$ there exists a generalized Lie bracket (Lie superbracket or supercommutator) $[a, b]$ with the following properties:*

- (i) $[a, b] \in \mathcal{L}$, for all $a, b \in \mathcal{L}$.

(ii) For all $a, b, c \in \mathcal{L}$ and any real or complex numbers α and β

$$[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]. \quad (2.5)$$

(iii) If a and b are homogeneous elements of \mathcal{L} then $[a, b]$ is also a homogeneous element of \mathcal{L} whose degree is $(\deg(a) + \deg(b)) \bmod 2$. So $[a, b]$ is odd if either a or b is odd. Also $[a, b]$ is even if a and b are both even or if a and b are both odd.

(iv) For any two homogeneous elements a and b of \mathcal{L}

$$[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a]. \quad (2.6)$$

(v) For any three homogeneous elements a, b and c of \mathcal{L}

$$[a, [b, c]](-1)^{\deg(a)\deg(c)} + [b, [c, a]](-1)^{\deg(b)\deg(a)} + [c, [a, b]](-1)^{\deg(c)\deg(b)} = 0. \quad (2.7)$$

Then \mathcal{L} is said to be a real or complex Lie superalgebra with even dimension m and odd dimension n .

Therefore a Lie algebra is a Lie superalgebra with trivial odd part. The most obvious example of a Lie superalgebra is that of linear maps on a \mathbb{Z}_2 -graded vector space.

Example 2.1.4. Let $V = V_0 \oplus V_1$ be a \mathbb{Z}_2 -graded vector space. Consider the associative algebra $gl(V)$ of endomorphism of V . It has a natural \mathbb{Z}_2 -grading:

$$gl(V)_0 = \{f \in gl(V) : f(V_0) \subseteq V_0 \text{ and } f(V_1) \subseteq V_1\}, \quad (2.8)$$

$$gl(V)_1 = \{f \in gl(V) : f(V_0) \subseteq V_1 \text{ and } f(V_1) \subseteq V_0\}. \quad (2.9)$$

The Lie superbracket is defined as follows:

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)}ba, \quad (2.10)$$

or equivalently,

$$[a, b] = \begin{cases} ab - ba, & \text{if } a \text{ or } b \in gl(V)_0, \\ ab + ba, & \text{if } a, b \in gl(V)_1. \end{cases}$$

2.2 Grassmann algebras

Definition 2.2.1 (Associative superalgebra). *Suppose that V is a graded vector space.*

(i) *For every pair of elements a and b in V , there exists a product ab that is also in V , and this product satisfies the grading multiplication rule.*

(ii) *For all $a, b, a', b' \in V$ and $\mu, \lambda, \mu', \lambda'$ of the field of V (\mathbb{R} or \mathbb{C}),*

$$(\mu a + \mu' a')(\lambda b + \lambda' b') = \mu\lambda(ab) + \mu\lambda'(ab') + \mu'\lambda(a'b) + \mu'\lambda'(a'b').$$

(iii) *For all $a, b, c \in V$,*

$$(ab)c = a(bc).$$

Then V is called an associative superalgebra.

Grassmann algebras are particular examples of associative algebras that will play a very important part in the developments of this thesis.

Definition 2.2.2 (Grassmann algebra). *Consider a set of N generators $\theta_1, \theta_2, \dots, \theta_N$, which are assumed to have products $\theta_i\theta_j$ such that*

(i) *For all $i, j, k = 1, \dots, N$,*

$$(\theta_i\theta_j)\theta_k = \theta_i(\theta_j\theta_k). \tag{2.11}$$

(ii) *For all $i, j = 1, \dots, N$,*

$$\theta_i\theta_j = -\theta_j\theta_i. \tag{2.12}$$

(iii) *Each non-zero product*

$$\theta_{j_1}\theta_{j_2} \cdots \theta_{j_r}$$

involving r generators is linearly independent of products involving less than r generators.

It should be noted that (2.12) implies that

$$\theta_i \theta_i = \theta_i^2 = 0 \quad (2.13)$$

for all $i = 1, \dots, N$.

This set of generators and products may be supplemented by introducing an identity, which is denoted by 1, and which is assumed to be such that

$$11 = 1 \quad (2.14)$$

and

$$1\theta_j = \theta_j 1 = \theta_j \quad (2.15)$$

for all $i = 1, \dots, N$. It follows that

$$1(\theta_{j_1} \theta_{j_2} \cdots \theta_{j_r}) = (\theta_{j_1} \theta_{j_2} \cdots \theta_{j_r})1 = \theta_{j_1} \theta_{j_2} \cdots \theta_{j_r} \quad (2.16)$$

for any product of generators.

The product $\theta_i \theta_j$ is sometimes written in the literature as the *wedge product* $\theta_i \wedge \theta_j$. The resulting algebras are sometimes called *exterior algebras*.

Example 2.2.3. For $N = 3$ there are three generators θ_1, θ_2 and θ_3 . By (2.13), $(\theta_1)^2 = 0, (\theta_2)^2 = 0$ and $(\theta_3)^2 = 0$. With the identity included, the independent products of generators are

$$1, \theta_1, \theta_2, \theta_3, \theta_1\theta_2, \theta_1\theta_3, \theta_2\theta_3, \theta_1\theta_2\theta_3. \quad (2.17)$$

It leads to a 8-dimensional Grassmann algebra. Note that N generators lead to a finite-dimensional Grassmann algebra of dimension 2^N .

Remark 2.2.4. There also exist infinite dimensional Grassmann algebras which are generated by infinitely many generators. In this thesis, we only consider the finite dimen-

sional Grassmann algebras (i.e. Grassmann algebras generated by finitely many generators).

For a fixed value of N , let σ be an index set that contains $N(\sigma)$ different integers with value between 1 and N inclusive. Thus

$$\sigma = \{j_1, j_2, \dots, j_{N(\sigma)}\},$$

where the integers $j_1, j_2, \dots, j_{N(\sigma)}$ are assumed to be ordered in such a way that

$$1 \leq j_1 < j_2 < j_3 < \dots < j_{N(\sigma)} \leq N.$$

Define θ_σ by

$$\theta_\sigma = \theta_{j_1} \theta_{j_2} \dots \theta_{j_{N(\sigma)}}. \quad (2.18)$$

Hence any element in the Grassmann algebra which is generated by θ_σ and 1 can be written as

$$B = \sum_{\sigma} B_{\sigma} \theta_{\sigma}, \quad (2.19)$$

where the coefficients B_{σ} are either real or complex numbers. In this thesis the vector space is real. This structure is a real associative superalgebra which is known as a real Grassmann algebra. It is denoted by $\mathbb{R}B_N$ and has dimension 2^N . The subset of even elements of $\mathbb{R}B_N$ and the subset of odd elements of $\mathbb{R}B_N$ both form real vector spaces of dimension 2^{N-1} . They will be denoted by $\mathbb{R}B_{N_0}$ and $\mathbb{R}B_{N_1}$ respectively. Hence $\mathbb{R}B_N = \mathbb{R}B_{N_0} \oplus \mathbb{R}B_{N_1}$.

2.3 The superspace $\mathbb{R}B_N^{m,n}$

As a Grassmann generalization of \mathbb{R}^m , consider the space $\mathbb{R}B_N^{m,n}$, which is defined to consist of m copies of the even space $\mathbb{R}B_{N_0}$ of the real Grassmann algebra $\mathbb{R}B_N$ and n copies of the odd space $\mathbb{R}B_{N_1}$ of $\mathbb{R}B_N$. The m copies of $\mathbb{R}B_{N_0}$ will be denoted by x_1, x_2, \dots, x_m and the n copies of $\mathbb{R}B_{N_1}$ will be indicated by $\theta_1, \theta_2, \dots, \theta_n$. It is convenient

to make the notation more concise by regarding x_1, x_2, \dots, x_m as elements of m -component quantity X , and $\theta_1, \theta_2, \dots, \theta_n$ as elements of a n -component quantity Θ . Then, $(X; \Theta)$, a typical element of $\mathbb{R}B_N^{m,n}$, is defined by

$$(X; \Theta) = (x_1, x_2, \dots, x_m, \theta_1, \theta_2, \dots, \theta_n).$$

As $\mathbb{R}B_{N_0}$ and $\mathbb{R}B_{N_1}$ are both 2^{N-1} dimensional real vector spaces, $\mathbb{R}B_N^{m,n}$ is a real vector space of dimension $(m+n)2^{N-1}$.

To allow analysis to be performed on the space $\mathbb{R}B_N^{m,n}$, it has to be provided with a metric. Let B be any element of $\mathbb{R}B_N$ of the form

$$B = \sum_{\mu} B_{\mu} \theta_{\mu}.$$

We define the norm as

$$\|B\| = \sum_{\mu} |B_{\mu}|. \quad (2.20)$$

For $\mathbb{R}B_N^{m,n}$, the norm corresponding to (2.20) may be defined by

$$\|(X; \Theta)\| = \sum_{j=1}^m \sum_{\mu} |x_{\mu}^j| + \sum_{k=1}^n \sum_{\mu} |\theta_{\mu}^k| \quad (2.21)$$

The metric d associated with the norm (2.21) is

$$d((X; \Theta), (X'; \Theta')) = \|(X; \Theta) - (X'; \Theta')\| \quad (2.22)$$

for any $(X; \Theta)$ and $(X'; \Theta')$ in $\mathbb{R}B_N^{m,n}$. It immediately follows that d satisfies

(i) For all $(X; \Theta)$ and $(X'; \Theta')$ in $\mathbb{R}B_N^{m,n}$,

$$d((X; \Theta), (X'; \Theta')) = d((X'; \Theta'), (X; \Theta)).$$

(ii) For all $(X; \Theta)$ in $\mathbb{R}B_N^{m,n}$,

$$d((X; \Theta), (X; \Theta)) = 0.$$

(iii) If $(X; \Theta) \neq (X'; \Theta')$,

$$d((X; \Theta), (X'; \Theta')) > 0.$$

(iv) If $(X; \Theta)$, $(X'; \Theta')$ and $(X''; \Theta'')$ are any three points in $\mathbb{R}B_N^{m,n}$ then

$$d((X; \Theta), (X''; \Theta'')) \leq d((X; \Theta), (X'; \Theta')) + d((X'; \Theta'), (X''; \Theta'')).$$

For metric d an open sphere of radius r centered at the point $(X'; \Theta')$ is defined to be the set of points $(X; \Theta)$ of $\mathbb{R}B_N^{m,n}$ such that

$$d((X; \Theta), (X'; \Theta')) < r.$$

A set of points U of $\mathbb{R}B_N^{m,n}$ is said to form an open set of $\mathbb{R}B_N^{m,n}$ if for every point $(X'; \Theta')$ of U there exists an open sphere centered on $(X'; \Theta')$ of some radius r (which may depend on $(X'; \Theta')$) that is completely contained in U .

2.4 Differential functions on $\mathbb{R}B_N^{m,n}$

Two types of Grassmann-valued functions will now be discussed. One is defined on an open set of \mathbb{R}^m , the other on an open set of $\mathbb{R}B_N^{m,n}$. Although the latter is more important in applications to Lie supergroups, the former will be considered first as it is more straightforward.

A Grassmann-valued function \mathfrak{F} can be defined on an open set V of \mathbb{R}^m by assigning to each element $X = (x_1, \dots, x_m)$ of V an element $\mathfrak{F}(X)$ of the Grassmann algebra $\mathbb{R}B_N$.

Such a function can be expanded in the form

$$\mathfrak{F}(X) = \sum_{\mu} \mathfrak{F}_{\mu}(X)\theta_{\mu}, \quad (2.23)$$

where the $\mathfrak{F}_{\mu}(X)$ are all real-valued functions of X in V and the sum is over all index sets μ . $\mathfrak{F}_{\mu}(X)$ is said to be even if only the even basis elements θ_{μ} of $\mathbb{R}B_N$ appear in this expansion, and to be odd if only odd θ_{μ} appear. So every Grassmann-valued function $\mathfrak{F}(X)$ can be written as the sum of an even and an odd function.

This idea can be generalized immediately to a Grassmann-valued function defined on an open set of $\mathbb{R}B_N^{m,n}$ rather than \mathbb{R}^m . Such a function may be defined by assigning to each point $(X; \Theta) = (x_1, x_2, \dots, x_m, \theta_1, \theta_2, \dots, \theta_n)$ in an open set U of $\mathbb{R}B_N^{m,n}$ an element $F(X; \Theta)$ of the Grassmann algebra $\mathbb{R}B_N$. The analogue of (2.23) is

$$\mathfrak{F}(X; \Theta) = \sum_{\mu} F_{\mu}(X; \Theta)\theta_{\mu}, \quad (2.24)$$

where each of the $F_{\mu}(X; \Theta)$ is a real-valued function of $(X; \Theta)$ in U and the sum is over all index sets μ . Again $F(X; \Theta)$ is said to be even if only the even basis θ_{μ} of $\mathbb{R}B_N$ appear in this expansion, and to be odd if only odd θ_{μ} appear. Thus every Grassmann-valued function $F(X; \Theta)$ on U can again be written as the sum of an even and an odd function.

Definition 2.4.1 (Continuous super function). *The function $F(X; \Theta)$ defined on the open set U of $\mathbb{R}B_N^{m,n}$ is said to be continuous at a point $(X'; \Theta')$ of U if $F(X; \Theta) \rightarrow F(X'; \Theta')$ as $(X; \Theta) \rightarrow (X'; \Theta')$.*

This can be expressed more precisely in terms of the metric for $\mathbb{R}B_N^{m,n}$ that was introduced above. In particular, F is continuous at $(X'; \Theta')$ for any real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$d(F(X; \Theta), F(X'; \Theta')) < \varepsilon$$

for all $(X; \Theta) \in U$ for which $d((X; \Theta), (X'; \Theta')) < \delta$.

The concept of differentiability for a function defined on $\mathbb{R}B_N^{m,n}$ is more subtle, and requires careful definition and discussion. The difficulty is that it involves dividing the Grassmann-valued quantity $F(X') - F(X)$ by the real number $x_j - x'_j$. Division of the corresponding quantity $F(X'; \Theta') - F(X; \Theta)$ by an element $(X'; \Theta') - (X; \Theta)$ of $\mathbb{R}B_N^{m,n}$ is not defined. The following definition was first given by Rogers (1980).

Definition 2.4.2 (Differential super function). *Let $F(X; \Theta)$ be a continuous function that takes values in $\mathbb{R}B_N$ and is defined on an open set U of $\mathbb{R}B_N^{m,n}$. Let $j = 1, 2, \dots, m$ and $k = 1, 2, \dots, n$. Suppose that there exist m functions $\partial F(X; \Theta)/\partial x_j$ and n functions $\partial F(X; \Theta)/\partial \theta_k$ that all have values in $\mathbb{R}B_N$, and are defined for all $(X; \Theta)$ in U and are such that*

$$\begin{aligned} F(X + Y; \Theta + \Psi) &= F(X; \Theta) + \sum_{j=1}^m Y^j \frac{\partial F(X; \Theta)}{\partial x_j} \\ &\quad + \sum_{k=1}^n \Psi^k \frac{\partial F(X; \Theta)}{\partial \theta_k} + \|(Y; \Psi)\| \eta(Y; \Psi). \end{aligned} \quad (2.25)$$

In (2.25), $(X; \Theta)$ and $(X + Y; \Theta + \Psi)$ are points in U , and $\eta(Y; \Psi)$ is a function defined on $\mathbb{R}B_N^{m,n}$ with values in $\mathbb{R}B_N$:

$$\|\eta(Y; \Psi)\| \rightarrow 0 \quad \text{as} \quad \|(Y; \Psi)\| \rightarrow 0. \quad (2.26)$$

Then the function $F(X; \Theta)$ is said to be differentiable in U and the quantities $\partial F(X; \Theta)/\partial x_j$ and $\partial F(X; \Theta)/\partial \theta_k$ are called its partial derivatives.

One immediate consequence of Definition 2.4.2 is that if $F(X; \Theta)$ is an even function then its derivatives $\partial F(X; \Theta)/\partial x_j$ are all even and its derivatives $\partial F(X; \Theta)/\partial \theta_k$ are all odd. In contrast, if $F(X; \Theta)$ is an odd function then the $\partial F(X; \Theta)/\partial x_j$ are all odd and the $\partial F(X; \Theta)/\partial \theta_k$ are all even. Note that

$$\frac{\partial \theta_j}{\partial \theta_i} = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

2.5 Basic differential rules for superdifferentiable functions

Given two superfunctions $F(X; \Theta), G(X; \Theta) \in \mathbb{R}B_N^{m,n}$, for $j = 1, 2, \dots, m$ and $k = 1, 2, \dots, n$, we have:

Rule 1

$$\begin{aligned}\frac{\partial(F(X; \Theta) + G(X; \Theta))}{\partial x_j} &= \frac{\partial F(X; \Theta)}{\partial x_j} + \frac{\partial G(X; \Theta)}{\partial x_j}; \\ \frac{\partial(F(X; \Theta) + G(X; \Theta))}{\partial \theta_k} &= \frac{\partial F(X; \Theta)}{\partial \theta_k} + \frac{\partial G(X; \Theta)}{\partial \theta_k}.\end{aligned}$$

Rule 2

$$\begin{aligned}\frac{\partial}{\partial x_i}(F(X; \Theta)G(X; \Theta)) &= \frac{\partial F(X; \Theta)}{\partial x_i}G(X; \Theta) + F(X; \Theta)\frac{\partial G(X; \Theta)}{\partial x_i}; \\ \frac{\partial}{\partial \theta_k}(F(X; \Theta)G(X; \Theta)) &= \frac{\partial F(X; \Theta)}{\partial \theta_k}G(X; \Theta) + (-1)^{\deg(F(X; \Theta))}F(X; \Theta)\frac{\partial G(X; \Theta)}{\partial \theta_k},\end{aligned}$$

if $F(X; \Theta)$ is homogenous.

Rule 3 For any real number λ ,

$$\begin{aligned}\frac{\partial(\lambda F(X; \Theta))}{\partial x_j} &= \lambda \frac{\partial F(X; \Theta)}{\partial x_j}; \\ \frac{\partial(\lambda F(X; \Theta))}{\partial \theta_k} &= \lambda \frac{\partial F(X; \Theta)}{\partial \theta_k}.\end{aligned}$$

Next we deal with higher derivatives of superdifferentiable function. For $j, j' \in \{1, 2, \dots, m\}$ and $k, k' \in \{1, 2, \dots, n\}$ with $k \neq k'$, we have:

Rule 1

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_{j'}} F(X; \Theta) = \frac{\partial}{\partial x_{j'}} \frac{\partial}{\partial x_j} F(X; \Theta).$$

Rule 2

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial \theta_k} F(X; \Theta) = \frac{\partial}{\partial \theta_k} \frac{\partial}{\partial x_j} F(X; \Theta).$$

Rule 3

$$\frac{\partial}{\partial \theta_k} \frac{\partial}{\partial \theta_k} F(X; \Theta) = 0,$$

and

$$\frac{\partial}{\partial \theta_k} \frac{\partial}{\partial \theta_{k'}} F(X; \Theta) = -\frac{\partial}{\partial \theta_{k'}} \frac{\partial}{\partial \theta_k} F(X; \Theta).$$

Rule 4 Note that every superfunction $F(X; \Theta)$ can be written in the form

$$F(X; \Theta) = \sum_{\Lambda} F^{\Lambda}(X) \Theta^{\Lambda}, \quad (2.27)$$

where $\Lambda = \{k_1, k_2, \dots, k_{N(\Lambda)}\}$ with $1 \leq k_1 < k_2 < \dots < k_{N(\Lambda)} \leq n$ and Θ^{Λ} is a product of θ_k factors of the form

$$\Theta^{\Lambda} = \theta_{k_1} \theta_{k_2} \dots \theta_{k_{N(\Lambda)}}.$$

With the expansion (2.27) for $F(X; \Theta)$

$$\frac{\partial}{\partial \theta_n} \frac{\partial}{\partial \theta_{n-1}} \dots \frac{\partial}{\partial \theta_2} \frac{\partial}{\partial \theta_1} F(X; \Theta) = (-1)^{n \deg(F^{(1,2,\dots,n-1,n)})} F^{(1,2,\dots,n-1,n)}(X).$$

The above calculation rules happen on every step of the calculation which involves odd quantities. The degree of both even and odd quantities may not be shown in the future calculation, but the parity of them is taken into account carefully in every step.

Chapter 3

Symmetrygroups of Grassmann-valued differential equations

The symmetry group of a system of differential equations is the local group of transformations acting on the independent and dependent variables of the system with property that it transforms solutions of the system to other solutions. See the book by Olver [13] for background on symmetry groups for differential equations. In particular [13] is our main source on Lie theory and infinitesimal techniques to get the determining system of the symmetry group of a system of differential equations. Other good references for this material are the books by Bluman and Cole [7] and Bluman, Cheviakov and Anco [26]. In this thesis, we say that this is how one gets the determining system in the *usual case*.

For us, the good news is that the procedure of getting the determining system of the supersymmetry group of a system of super differential equations (*super case*) is very similar to the usual case. Both cases follow the same steps to get the determining system.

1. Reduce to one-parameter Lie (super) transformations about the identity.
2. Apply the (super) prolongation formula to the (super) differential equations.
3. Replace the highest derivatives in the system from Step 2.

4. Compute the coefficients of independent monomials of the dependent variables and their derivatives.
5. The coefficients from Step 4 are determining equations for supersymmetries.

In this chapter, we will introduce Lie's infinitesimal method for generating the defining system of super differential equations. We apply this method to two typical example super differential equations. For each example, the determination of structure constants of the Lie superalgebra of supersymmetries will be done by the traditional method which uses heuristic integration. Note that, later in Chapter 4, we will develop an alternative method to find the Lie supercommutator table that is algorithmic and avoids heuristic integration.

3.1 Supersymmetry group of super differential equations

The symmetry group of a system of Grassmann-valued differential equations is the local group of transformations acting on the independent and dependent variables of the system with property that it transforms solutions of the system to other solutions.

Let us consider the general case of a nonlinear system of Grassmann-valued differential equations or superequations of s equations of order $k = (k_1; k_2)$ denoted by

$$\Delta_\nu(X, \Theta, A^{(k_1)}, Q^{(k_2)}) = 0, \quad \nu = 1, \dots, s, \quad (3.1)$$

with m independent even variables $X = (x_1, \dots, x_m)$, n independent odd variables $\Theta = \{\theta_1, \dots, \theta_n\}$, q dependent even variables $A = (A^1, \dots, A^q)$ and p dependent odd variables $Q = (Q^1, \dots, Q^p)$.

In the spirit of Lie theory, a one parameter ε local transformation has the form

$$\begin{aligned}\bar{X} &= \Omega_\varepsilon^1(X, \Theta, A, Q), \\ \bar{\Theta} &= \Omega_\varepsilon^2(X, \Theta, A, Q), \\ \bar{A} &= \Omega_\varepsilon^3(X, \Theta, A, Q), \\ \bar{Q} &= \Omega_\varepsilon^4(X, \Theta, A, Q),\end{aligned}\tag{3.2}$$

where ε is an homogeneous Grassmann variable and denotes the supergroup parameter. The supervector valued functions $\Omega_\varepsilon^i, i = 1, \dots, 4$ depend only on the variables X, Θ, A and Q (and not on the derivatives of A and Q). The supersymmetry group of a system of Grassmann-valued differential equations is the maximal supergroup of transformations leaving (3.1) invariant.

Expanding each relations in (3.3) around the identity $\varepsilon = 0$, one can generate the following infinitesimal transformations

$$\begin{aligned}\bar{x}_i &= x_i + \varepsilon \Xi^i(X, \Theta, A, Q) + O(\varepsilon^2), \quad i = 1, \dots, m, \\ \bar{\theta}_j &= \theta_j + \varepsilon \Gamma^j(X, \Theta, A, Q) + O(\varepsilon^2), \quad j = 1, \dots, n, \\ \bar{A}^r &= A^r + \varepsilon \Phi^r(X, \Theta, A, Q) + O(\varepsilon^2), \quad r = 1, \dots, q, \\ \bar{Q}^l &= Q^l + \varepsilon \Lambda^l(X, \Theta, A, Q) + O(\varepsilon^2), \quad l = 1, \dots, p,\end{aligned}$$

where the functions Ξ^i, Γ^j, Φ^l and Λ^l are the infinitesimals of the transformations for the independent and dependent (even and odd) variables.

A basis for the corresponding symmetry superalgebra \mathcal{H} is given in terms of supervector fields

$$V = \sum_{i=1}^m \Xi^i \frac{\partial}{\partial x_i} + \sum_{j=1}^n \Gamma^j \frac{\partial}{\partial \theta_j} + \sum_{r=1}^q \Phi^r \frac{\partial}{\partial A^r} + \sum_{l=1}^p \Lambda^l \frac{\partial}{\partial Q^l}.\tag{3.3}$$

Thus, the infinitesimal criterion for the invariance of (3.1) under the superalgebra \mathcal{H} may be expressed as

$$\text{pr}^{(k)} V \Delta_\nu|_{\Delta=0} = 0,\tag{3.4}$$

where the $k = (k_1; k_2)$ -th superprolongation of the vector field (3.3) is given by

$$\text{pr}^{(k)}V = V + \sum_{r=1}^q \sum_J \Phi_J^r \frac{\partial}{\partial A_J^r} + \sum_{l=1}^p \sum_K \Lambda_K^l \frac{\partial}{\partial Q_K^l}. \quad (3.5)$$

In (3.5) $J = (J_1; J_2) = (j_1^1, \dots, j_\alpha^1; j_1^2, \dots, j_\mu^2)$, $K = (K_1; K_2) = (k_1^1, \dots, k_\beta^1; k_1^2, \dots, k_\nu^2)$ and $1 \leq j_\alpha^1, j_\mu^2, k_\beta^1, k_\nu^2 \leq m + n$ are the multi-indices notations for differentiations with respect to the x_i and θ_j variables. Additional explanation of multi-indices notation is given in Section 3.1.1. The coefficients Φ_J^r and Λ_K^l are given by

$$\Phi_J^r = D_J \left(\Phi^r - \sum_{i=1}^m \Xi^i A_i^r - \sum_{j=1}^n \Gamma^j A_j^r \right) + \sum_{i=1}^m \Xi^i A_{J,i}^r + \sum_{j=1}^n \Gamma^j A_{J,j}^r \quad (3.6)$$

and

$$\Lambda_K^l = D_K \left(\Lambda^l - \sum_{i=1}^m \Xi^i Q_i^l - \sum_{j=1}^n \Gamma^j Q_j^l \right) + \sum_{i=1}^m \Xi^i Q_{K,i}^l + \sum_{j=1}^n \Gamma^j Q_{K,j}^l, \quad (3.7)$$

where $A_i^k = \partial A^r / \partial x_i$, $A_j^r = \partial A^r / \partial \theta_j$, $Q_i^l = \partial Q^l / \partial x_i$ and $Q_j^l = \partial Q^l / \partial \theta_j$.

Since the input equation (3.1) must be satisfied everywhere, the infinitesimal criterion equation (3.4) is simplified with respect to (3.1). Then the dependencies on derivatives of A^r and Q^l are eliminated by decomposition into coefficients of the monomials of the derivatives of A^r and Q^l . Equating these coefficients to zero forms the symmetry defining system. In Section 3.2, instead of trying to exactly solve the defining system, we will algorithmically find the structure constants without using integrations.

3.1.1 Multi-index notation

In the superprolongation formula (3.5) and the coefficient formulae (3.6) and (3.7), the even dependent variables Φ^r and odd dependent variables Λ^l can be differentiated with respect to any of the independent variables (x, θ) . Hence we need multi-index notation to denote the differentiations. The multi-index notation may look complicated, however, the following small example will help the reader to easily understand formulae expressed in this notation.

Example 3.1.1. Consider a super differential equation with 1 even independent variable x , 1 odd independent variable θ , 1 even dependent variable A and 1 odd dependent variable. The corresponding infinitesimals are Ξ, Γ, Φ and Λ , respectively.

Then the supervector field is

$$V = \Xi \frac{\partial}{\partial x} + \Gamma \frac{\partial}{\partial \theta} + \Phi \frac{\partial}{\partial A} + \Lambda \frac{\partial}{\partial Q}.$$

If this is a first order super differential equation, then we need the first order super-prolongation of V

$$\text{pr}^{(1)}V = V + \Phi^x \frac{\partial}{\partial A_x} + \Phi^\theta \frac{\partial}{\partial A_\theta} + \Lambda^x \frac{\partial}{\partial Q_x} + \Lambda^\theta \frac{\partial}{\partial Q_\theta},$$

where

$$\Phi^x = D_x(\Phi - \Xi A_x - \Gamma A_\theta) + \Xi A_{xx} + \Gamma A_{x\theta},$$

$$\Phi^\theta = D_\theta(\Phi - \Xi A_x - \Gamma A_\theta) + \Xi A_{\theta x} + \Gamma A_{\theta\theta},$$

$$\Lambda^x = D_x(\Lambda - \Xi Q_x - \Gamma Q_\theta) + \Xi Q_{xx} + \Gamma Q_{x\theta},$$

$$\Lambda^\theta = D_\theta(\Lambda - \Xi Q_x - \Gamma Q_\theta) + \Xi Q_{\theta x} + \Gamma Q_{\theta\theta}.$$

Obviously, $\Gamma A_{\theta\theta}$ and $\Gamma Q_{\theta\theta}$ vanish.

Remark 3.1.2. Here we follow Olver's notation [13] to avoid numeric indices, replacing these with superscripts or subscripts (x, θ , etc). Throughout this thesis, the lower subscript with variables such as A_{xx} means the usual derivative. We use superscript to denote the coefficient notation to distinguish from our derivative subscript notation.

If we are considering a second order super differential equation, then we need the

second order superprolongation of V which is given by

$$\begin{aligned} \text{pr}^{(2)}V &= V + \Phi^x \frac{\partial}{\partial A_x} + \Phi^\theta \frac{\partial}{\partial A_\theta} + \Lambda^x \frac{\partial}{\partial Q_x} + \Lambda^\theta \frac{\partial}{\partial \Lambda_\theta} \\ &\quad + \Phi^{xx} \frac{\partial}{\partial A_{xx}} + \Phi^{x\theta} \frac{\partial}{\partial A_{x\theta}} + \Phi^{\theta\theta} \frac{\partial}{\partial A_{\theta\theta}} \\ &\quad + \Lambda^{xx} \frac{\partial}{\partial Q_{xx}} + \Lambda^{x\theta} \frac{\partial}{\partial Q_{x\theta}} + \Lambda^{\theta\theta} \frac{\partial}{\partial Q_{\theta\theta}}, \end{aligned}$$

where $\Phi^{\theta\theta} = 0$ and $\Lambda^{\theta\theta} = 0$. The coefficients Φ^x , Φ^θ , Λ^x and Λ^θ are already known. The remaining coefficients are

$$\begin{aligned} \Phi^{xx} &= (D_x)^2(\Phi - \Xi A_x - \Gamma A_\theta) + \Xi A_{xxx} + \Gamma A_{xx\theta}, \\ \phi^{x\theta} &= D_x D_\theta(\Phi - \Xi A_x - \Gamma A_\theta) + \Xi A_{x\theta x} + \Gamma A_{x\theta\theta}, \\ \Lambda^{xx} &= (D_x)^2(\Lambda - \Xi Q_x - \Gamma Q_\theta) + \Xi Q_{xxx} + \Gamma Q_{xx\theta}, \\ \Lambda^{x\theta} &= D_x D_\theta(\Lambda - \Xi Q_x - \Gamma Q_\theta) + \Xi Q_{x\theta x} + \Gamma Q_{x\theta\theta}, \end{aligned}$$

where $\Gamma A_{x\theta\theta}$ and $\Gamma Q_{x\theta\theta}$ vanish.

3.2 Finding structure constants using integration

We will use two examples to illustrate how to get the symmetry determining system and how to find the structure constants in the traditional way - using heuristic integrations.

The first example is the simple and easily solved second order super differential equation

$$Q_{xx} = 0, \tag{3.8}$$

where x is the even dependent variable and Q is the odd dependent variable. We will use this as our simplest illustrative example.

Our second and more complicated example is a well-known model, the super KdV equation,

$$Q_t = Q_{xxx} - a\theta Q Q_{xx} + aQ Q_{\theta x} + (6 - 3a)Q_\theta Q_x, \tag{3.9}$$

which is integrable when $a = 3$. It has two even independent variables, x and t , one odd independent variable θ , and one odd dependent variable Q . The reader is especially directed to the thesis of Ayari [19], where its supersymmetries are determined. In Chapter 4, we will use it to illustrate and address a difficulty that can occur in the algorithmic determination of supersymmetries.

3.2.1 Supersymmetries of $Q_{xx} = 0$

We determine the supersymmetry group of the first example (3.8) using the approach of Section 3.1.

Let

$$V = \Xi(x, Q) \frac{\partial}{\partial x} + \Lambda(x, Q) \frac{\partial}{\partial Q}$$

be the supervector field on $X \times Q$. We wish to determine all possible coefficient functions $\Xi(x, Q)$ and $\Lambda(x, Q)$ so that the corresponding one-parameter group $\exp(\varepsilon V)$ is a (super) symmetry group of the second order super differential equation. Hence we need to know the second superprolongation. Recall the superprolongation formula (3.5) which is

$$\text{pr}^{(k)}V = V + \sum_{r=1}^q \sum_J \Phi_J^r \frac{\partial}{\partial A_J^r} + \sum_{l=1}^p \sum_K \Lambda_K^l \frac{\partial}{\partial Q_K^l}.$$

For this example, there is only one odd dependent variable. Hence, in this superprolongation formula, there is no middle term and $p = 1$. The multi-index $K = \{(1;), (2;)\}$. Then the second order superprolongation is

$$\text{pr}^{(2)}V = V + \Lambda^x \frac{\partial}{\partial Q_x} + \Lambda^{xx} \frac{\partial}{\partial Q_{xx}},$$

where

$$\Lambda^x = D_x(\Lambda - \Xi Q_x) + \Xi Q_{xx}$$

and

$$\Lambda^{xx} = D_{xx}(\Lambda - \Xi Q_x) + \Xi Q_{xxx}, \quad (3.10)$$

by the coefficient formula (3.7).

Applying $\text{pr}^{(2)}V$ to both sides of (3.8) yields

$$\Lambda^{xx} = 0.$$

Hence we have

$$D_{xx}(\Lambda - \Xi Q_x) + \Xi Q_{xxx} = 0 \quad (3.11)$$

by equation (3.10). Expanding (3.11), one has

$$\Lambda_{xx} + (2\Lambda_{xQ} - \Xi_{xx})Q_x = 0 \quad (3.12)$$

Equating the coefficients of the monomial Q_x and 1 to zero decomposes (3.12) into the equivalent system

$$\begin{cases} \Lambda_{xx} = 0, \\ 2\Lambda_{xQ} - \Xi_{xx} = 0. \end{cases} \quad (3.13)$$

Also, two other determining equations

$$\Xi_{QQ} = 0, \quad \text{and} \quad \Lambda_{QQ} = 0$$

hold since Q is an odd variable. The defining system for (3.8) is

$$\begin{cases} \Lambda_{xx} = 0, \\ 2\Lambda_{xQ} - \Xi_{xx} = 0, \\ \Xi_{QQ} = 0, \\ \Lambda_{QQ} = 0. \end{cases} \quad (3.14)$$

The next goal is to find the Lie superalgebra structure resulting from the solution space of the defining system (3.14).

The Gauss-like (or differential Gröbner) reduction procedure we apply to (3.14) requires that we define an ordering on all derivatives of the infinitesimals Ξ and Λ .

To illustrate general features of such orderings, let $\Delta^1 = \Xi$ and $\Delta^2 = \Lambda$. Setting

$\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_{\geq 0}^2$, the first entry of α or β is the order of the derivatives of Δ^1 or Δ^2 with respect to x . The second entry is the order of the derivatives of Δ^1 and Δ^2 with respect to Q . For example, the derivative Λ_{xx} is denoted by $\Delta_{(2,0)}^2$ and Ξ_{xQ} is denoted by $\Delta_{(1,1)}^1$.

Now define a total ordering on all the Δ_α^i , $i = 1, 2$. There are two cases $|\alpha| = |\beta|$ and $|\alpha| < |\beta|$. If $|\alpha| = |\beta|$ then

$$\Delta_\alpha^i \prec \Delta_\beta^i \Leftrightarrow \alpha \prec_{lex} \beta, \quad i = 1, 2$$

If $|\alpha| < |\beta|$ then

$$\Delta_\alpha^i \prec \Delta_\beta^j, \quad i, j = 1, 2.$$

Applying this to our simple example (3.8) yields the ordering

$$\Xi \prec \Lambda \prec \Xi_x \prec \Xi_Q \prec \Lambda_x \prec \Lambda_Q \prec \Xi_{xx} \prec \Xi_{xQ} \prec \Xi_{QQ} \prec \Lambda_{xx} \prec \Lambda_{xQ} \prec \Lambda_{QQ} \prec \dots \quad (3.15)$$

This ordering is a particular case of the class of *lex* orderings given in the following definition.

Definition 3.2.1 (Lexicographic Order). *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_{\geq 0}^n$. We say $\alpha \succ_{lex} \beta$ if, in the vector difference $\alpha - \beta \in \mathbb{Z}_{\geq 0}^n$, the most left nonzero entry is positive.*

Here are some examples:

- a. $(1, 2, 0) \succ_{lex} (0, 3, 4)$ since $\alpha - \beta = (1, -1, -4)$;
- b. $(3, 2, 4) \succ_{lex} (3, 2, 1)$ since $\alpha - \beta = (0, 0, 3)$;
- c. for n -tuples, $(1, 0, \dots, 0) \succ_{lex} (0, 1, 0, \dots, 0) \succ_{lex} \dots \succ_{lex} (0, \dots, 0, 1)$.

Returning to example (3.8), let us solve its the defining system (3.14). For this defining system, we use the ordering (3.15). In the second equation

$$2\Lambda_{xQ} - \Xi_{xx} = 0$$

$\Lambda_{xQ} \succ \Xi_{xx}$ so rewriting this equation in solved form with respect to the ordering (3.15) yields

$$\Lambda_{xQ} = \frac{1}{2}\Xi_{xx}, \quad (3.16)$$

where Λ_{xQ} is called the leading term of (3.16). We cancel the leading terms of the equations via integrability conditions as follows

$$\begin{cases} \Lambda_{xx} = 0, \\ \Lambda_{xQ} = \frac{1}{2}\Xi_{xx}, \end{cases} \Rightarrow \begin{cases} \Lambda_{xxQ} = 0, \\ \Lambda_{xQx} = \frac{1}{2}\Xi_{xxx}, \end{cases} \Rightarrow \Xi_{xxx} = 0,$$

and

$$\begin{cases} \Lambda_{xQ} = \frac{1}{2}\Xi_{xx}, \\ \Lambda_{QQ} = 0, \end{cases} \Rightarrow \begin{cases} \Lambda_{xQQ} = \frac{1}{2}\Xi_{xxQ}, \\ \Lambda_{xQQ} = 0, \end{cases} \Rightarrow \Xi_{xxQ} = 0.$$

We get two new equations

$$\begin{aligned} \Xi_{xxx} &= 0, \\ \Xi_{xxQ} &= 0. \end{aligned}$$

By adjoining these two equations to the defining system (3.14) , we finally get all 6 determining equations

$$\begin{cases} \Lambda_{xx} = 0, \\ \Lambda_{xQ} = \frac{1}{2}\Xi_{xx}, \\ \Xi_{QQ} = 0, \\ \Lambda_{QQ} = 0, \\ \Xi_{xxx} = 0, \\ \Xi_{xxQ} = 0. \end{cases} \quad (3.17)$$

The solution of the system by integration is elementary. First, the equations $\Xi_{QQ} = 0$ and $\Lambda_{QQ} = 0$ show that Λ and Ξ are linear in the variable Q . Suppose that

$$\Lambda = f_1(x)Q + f_2(x) \quad (3.18)$$

and

$$\Xi = g_1(x)Q + g_2(x), \quad (3.19)$$

where $f_1(x), g_2(x)$ are even functions with respect to x and $f_2(x), g_1(x)$ are odd functions with respect to x which are all to be determined. The equation $\Lambda_{xx} = 0$ implies

$$f_1''(x)Q + f_2''(x) = 0$$

which requires

$$f_1''(x) = 0 \quad \text{and} \quad f_2''(x) = 0.$$

Thus

$$f_1(x) = c_1x + c_2 \quad \text{and} \quad f_2(x) = \alpha_1x + \alpha_2,$$

where c_1, c_2 are two even constants and α_1, α_2 are odd constants. Hence we have

$$\Lambda = (c_1x + c_2)Q + \alpha_1x + \alpha_2$$

which implies

$$\Lambda_{xQ} = c_1.$$

The second equation $\Lambda_{xQ} = \frac{1}{2}\Xi_{xx}$ in (3.17) requires that

$$g_1''(x)Q + g_2''(x) = 2c_1$$

and

$$g_1''(x) = 0 \quad \text{and} \quad g_2''(x) = 2c_1.$$

Then

$$g_1(x) = \alpha_3x + \alpha_4 \quad \text{and} \quad g_2(x) = c_1x^2 + c_3x + c_4,$$

where α_3, α_4 are two odd constants and c_3, c_4 are two even constants. We conclude that the most general infinitesimal symmetry of the second order super differential equation

has vector field coefficient functions

$$\begin{aligned}\Lambda &= (c_1x + c_2)Q + \alpha_1x + \alpha_2, \\ \Xi &= (\alpha_3x + \alpha_4)Q + c_1x^2 + c_3x + c_4,\end{aligned}$$

where c_1, \dots, c_4 are arbitrary even constants and $\alpha_1, \dots, \alpha_4$ are arbitrary odd constants. Thus the Lie superalgebra of infinitesimal symmetries of the (3.8) is spanned by the eight basis generators

$$\begin{aligned}L_1 &= \partial_x, \\ L_2 &= x\partial_x, \\ L_3 &= Q\partial_Q, \\ L_4 &= x^2\partial_x + xQ\partial_Q, \\ L_5 &= \partial_Q, \\ L_6 &= x\partial_Q, \\ L_7 &= Q\partial_x, \\ L_8 &= xQ\partial_x,\end{aligned}$$

where L_1, \dots, L_4 are four even basis generators and L_5, \dots, L_8 are four odd basis generators.

Suppose that $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$. The the even basis L_1, \dots, L_4 generate the Lie algebra \mathcal{H}_0 . As is well-known, it is a natural subalgebra of \mathcal{H} and the commutator table of Lie algebra is anti-symmetric and the diagonal entries are zero. The resulting supercommutators

$[L_i, L_j]$, where $i \leq j$ and $i = 1, \dots, 4, j = 2, \dots, 4$ are

$$[L_1, L_2] = \partial_x(x\partial_x) - (-1)^{0 \cdot 0}x\partial_x(\partial_x) = \partial_x = L_1,$$

$$[L_1, L_3] = \partial_x(Q\partial_Q) - (-1)^{0 \cdot 0}Q\partial_Q(\partial_x) = 0,$$

$$\begin{aligned} [L_1, L_4] &= [\partial_x, x^2\partial_x + xQ\partial_Q] = [\partial_x, x^2\partial_x] + [\partial_x, xQ\partial_Q] \\ &= \partial_x(x^2\partial_x) - (-1)^{0 \cdot 0}x^2\partial_x(\partial_x) + \partial_x(xQ\partial_Q) - (-1)^{0 \cdot 0}xQ\partial_Q(\partial_x) \\ &= 2x\partial_x + Q\partial_Q = 2L_2 + L_3, \end{aligned}$$

$$[L_2, L_3] = x\partial_x(Q\partial_Q) - (-1)^{0 \cdot 0}Q\partial_Q(x\partial_x) = 0,$$

$$\begin{aligned} [L_2, L_4] &= [x\partial_x, x^2\partial_x + xQ\partial_Q] = [x\partial_x, x^2\partial_x] + [x\partial_x, xQ\partial_Q] \\ &= x\partial_x(x^2\partial_x) - (-1)^{0 \cdot 0}x^2\partial_x(x\partial_x) + x\partial_x(xQ\partial_Q) - (-1)^{0 \cdot 0}xQ\partial_Q(x\partial_x) \\ &= x^2\partial_x + xQ\partial_Q = L_4, \end{aligned}$$

$$\begin{aligned} [L_3, L_4] &= [Q\partial_Q, x^2\partial_x + xQ\partial_Q] = [Q\partial_Q, x^2\partial_x] + [Q\partial_Q, xQ\partial_Q] \\ &= Q\partial_Q(x^2\partial_x) - (-1)^{0 \cdot 0}x^2\partial_x(Q)\partial_Q + Q\partial_Q(xQ)\partial_Q - (-1)^{0 \cdot 0}xQ\partial_Q(Q\partial_Q) \\ &= Qx\partial_Q - Qx\partial_Q = 0. \end{aligned}$$

It is easy to see that $[\mathcal{H}_0, \mathcal{H}_0] \subseteq \mathcal{H}_0$ which is just the left-upper part of the Table 3.2.1.

Then one computes $[\mathcal{H}_0, \mathcal{H}_1]$,

$$\begin{aligned}
[L_1, L_5] &= \partial_x(\partial_Q) - (-1)^{0 \cdot 1} \partial_Q(\partial_x) = 0, \\
[L_1, L_6] &= \partial_x(x\partial_Q) - (-1)^{0 \cdot 1} x\partial_Q(\partial_x) = \partial_Q = L_5, \\
[L_1, L_7] &= \partial_x(Q\partial_x) - (-1)^{0 \cdot 1} Q\partial_x(\partial_x) = 0, \\
[L_1, L_8] &= \partial_x(xQ\partial_x) - (-1)^{0 \cdot 1} xQ\partial_x(\partial_x) = Q\partial_x = L_7, \\
[L_2, L_5] &= x\partial_x(\partial_Q) - (-1)^{0 \cdot 1} \partial_Q(x\partial_x) = 0, \\
[L_2, L_6] &= x\partial_x(x\partial_Q) - (-1)^{0 \cdot 1} x\partial_Q(x\partial_x) = x\partial_Q = L_6, \\
[L_2, L_7] &= x\partial_x(Q\partial_x) - (-1)^{0 \cdot 1} Q\partial_x(x\partial_x) = -Q\partial_x = -L_7, \\
[L_2, L_8] &= x\partial_x(xQ)\partial_x - (-1)^{0 \cdot 1} xQ\partial_x(x)\partial_x \\
&= xQ\partial_x - xQ\partial_x = 0, \\
[L_3, L_5] &= Q\partial_Q(1)\partial_Q - (-1)^{0 \cdot 1} \partial_Q(Q)\partial_Q = -Q\partial_Q = -L_5, \\
[L_3, L_6] &= Q\partial_Q(x)\partial_Q - (-1)^{0 \cdot 1} x\partial_Q(Q)\partial_Q = -x\partial_Q = -L_6, \\
[L_3, L_7] &= Q\partial_Q(Q)\partial_x - (-1)^{0 \cdot 1} Q\partial_x(Q)\partial_Q = Q\partial_x = L_7, \\
[L_3, L_8] &= Q\partial_Q(xQ)\partial_x - (-1)^{0 \cdot 1} xQ\partial_x(Q)\partial_Q \\
&= Qx\partial_x = xQ\partial_x = L_8, \\
[L_4, L_5] &= [x^2\partial_x + xQ\partial_Q, \partial_Q] = [x^2\partial_x, \partial_Q] + [xQ\partial_Q, \partial_Q] \\
&= x^2\partial_x(1)\partial_Q - (-1)^{0 \cdot 1} \partial_Q(x^2)\partial_x + xQ\partial_Q(1)\partial_Q - (-1)^{0 \cdot 1} \partial_Q(xQ)\partial_Q \\
&= -x\partial_Q = -L_6, \\
[L_4, L_6] &= [x^2\partial_x + xQ\partial_Q, x\partial_Q] = [x^2\partial_x, x\partial_Q] + [xQ\partial_Q, x\partial_Q] \\
&= x^2\partial_x(x)\partial_Q - (-1)^{0 \cdot 1} x\partial_Q(x^2)\partial_x + xQ\partial_Q(x)\partial_Q - (-1)^{0 \cdot 1} x\partial_Q(xQ)\partial_Q \\
&= x^2\partial_Q - x^2\partial_Q = 0, \\
[L_4, L_7] &= [x^2\partial_x + xQ\partial_Q, Q\partial_x] = [x^2\partial_x, Q\partial_x] + [xQ\partial_Q, Q\partial_x] \\
&= x^2\partial_x(Q)\partial_x - (-1)^{0 \cdot 1} Q\partial_x(x^2)\partial_x + xQ\partial_Q(Q)\partial_x - (-1)^{0 \cdot 1} Q\partial_x(xQ)\partial_Q \\
&= -2Qx\partial_x + xQ\partial_x = -xQ\partial_x = -L_8, \\
[L_4, L_8] &= [x^2\partial_x + xQ\partial_Q, Q\partial_x] = [x^2\partial_x, Q\partial_x] + [xQ\partial_Q, Q\partial_x] \\
&= x^2\partial_x(Q)\partial_x - (-1)^{0 \cdot 1} Q\partial_x(x^2)\partial_x + xQ\partial_Q(Q)\partial_x - (-1)^{0 \cdot 1} Q\partial_x(xQ)\partial_Q \\
&= -2Qx\partial_x + xQ\partial_x = -xQ\partial_x = -L_8.
\end{aligned}$$

This implies $[\mathcal{H}_0, \mathcal{H}_1] \subseteq \mathcal{H}_1$. This is the right-upper part of the Table 3.2.1. The left-lower part of Table 3.2.1 is obtained by anti-symmetry. The last part is determined by computing $[\mathcal{H}_1, \mathcal{H}_1]$. This yields the commutators:

$$\begin{aligned}
[L_5, L_5] &= \partial_Q(1)\partial_Q - (-1)^{1\cdot 1}\partial_Q(1)\partial_Q = 0, \\
[L_5, L_6] &= \partial_Q(x)\partial_Q - (-1)^{1\cdot 1}x\partial_Q(1)\partial_Q = 0, \\
[L_5, L_7] &= \partial_Q(Q)\partial_x - (-1)^{1\cdot 1}Q\partial_x(1)\partial_Q = \partial_x = L_1, \\
[L_5, L_8] &= \partial_Q(xQ)\partial_x - (-1)^{1\cdot 1}xQ\partial_x(1)\partial_Q = x\partial_x = L_2, \\
[L_6, L_6] &= x\partial_Q(x)\partial_Q - (-1)^{1\cdot 1}x\partial_Q(x)\partial_Q = 0, \\
[L_6, L_7] &= x\partial_Q(Q)\partial_x - (-1)^{1\cdot 1}Q\partial_x(x)\partial_Q \\
&= x\partial_x + Q\partial_Q = L_2 + L_3, \\
[L_6, L_8] &= x\partial_Q(xQ)\partial_x - (-1)^{1\cdot 1}xQ\partial_x(x)\partial_Q \\
&= x^2\partial_x + xQ\partial_Q = L_4, \\
[L_7, L_7] &= Q\partial_x(Q)\partial_x - (-1)^{1\cdot 1}Q\partial_x(Q)\partial_x = 0, \\
[L_7, L_8] &= Q\partial_x(xQ)\partial_x - (-1)^{1\cdot 1}xQ\partial_x(Q)\partial_x = 0, \\
[L_8, L_8] &= xQ\partial_x(xQ)\partial_x - (-1)^{1\cdot 1}xQ\partial_x(xQ)\partial_x = 0.
\end{aligned}$$

The above commutators are consistent with the property of Lie superalgebra, $[\mathcal{H}_1, \mathcal{H}_1] \subseteq \mathcal{H}_0$. Note that the Lie superbracket of same odd operator is not necessarily zero. Hence, for this example, one needs to compute $[L_i, L_i]$, where $i = 5, \dots, 8$.

Finally, the complete list of supercommutation relations is given by Table 3.2.1.

3.2.2 Supersymmetries of Super KdV equation

This work is concerned with the study of the supersymmetric Korteweg-de Vries (sKdV) equation:

$$Q_t = Q_{xxx} - a\theta QQ_{xx} + aQQ\theta_x + (6 - 3a)Q_\theta Q_x, \quad (3.20)$$

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
L_1	0	L_1	0	$2L_2 + L_3$	0	L_5	0	L_7
L_2	$-L_1$	0	0	L_4	0	L_6	$-L_7$	0
L_3	0	0	0	0	$-L_5$	$-L_6$	L_7	L_8
L_4	$-2L_2 - L_3$	$-L_4$	0	0	$-L_6$	0	$-L_8$	0
L_5	0	0	L_5	L_6	0	0	L_1	L_2
L_6	$-L_5$	$-L_6$	L_6	0	0	0	$L_2 + L_3$	L_4
L_7	0	L_7	$-L_7$	L_8	L_1	$L_2 + L_3$	0	0
L_8	$-L_7$	0	$-L_8$	0	L_2	L_4	0	0

Table 3.2.1: Supercommutator table of the defining system of $Q_{xx} = 0$.

which is integrable when $a = 3$. Notice that there are two even independent variables, x and t , one odd independent variable, θ , and one odd dependent variable, Q .

We consider the following infinitesimals

$$\begin{aligned}
\bar{x} &= x + \varepsilon \Xi^1(x, t, \theta, Q) + O(\varepsilon^2), \\
\bar{t} &= t + \varepsilon \Xi^2(x, t, \theta, Q) + O(\varepsilon^2), \\
\bar{\theta} &= \theta + \varepsilon \Gamma(x, t, \theta, Q) + O(\varepsilon^2), \\
\bar{Q} &= Q + \varepsilon \Lambda(x, t, \theta, Q) + O(\varepsilon^2),
\end{aligned}$$

where $\Xi^1(x, t, \theta, Q)$ and $\Xi^2(x, t, \theta, Q)$ are even functions, while $\Gamma(x, t, \theta, Q)$ and $\Lambda(x, t, \theta, Q)$ are odd functions.

Let

$$\mathbf{v} = \Xi^1(x, t, \theta, Q) \frac{\partial}{\partial x} + \Xi^2(x, t, \theta, Q) \frac{\partial}{\partial t} + \Gamma^1(x, t, \theta, Q) \frac{\partial}{\partial \theta} + \Lambda^1(x, t, \theta, Q) \frac{\partial}{\partial Q}.$$

We wish to determine all possible vector field coefficient functions $\Xi^1(x, t, \theta, Q)$, $\Xi^2(x, t, \theta, Q)$,

$\Gamma(x, t, \theta, Q)$ and $\Lambda(x, t, \theta, Q)$ so that the corresponding one-parameter group is a super-symmetry group of the super KdV equation. Compute the third prolongation

$$\begin{aligned}
\text{pr}^{(3)}\mathbf{v} = & \mathbf{v} + \Lambda^x \frac{\partial}{\partial Q_x} + \Lambda^t \frac{\partial}{\partial Q_t} + \Lambda^\theta \frac{\partial}{\partial Q_\theta} \\
& + \Lambda^{xx} \frac{\partial}{\partial Q_{xx}} + \Lambda^{xt} \frac{\partial}{\partial Q_{xt}} + \Lambda^{x\theta} \frac{\partial}{\partial Q_{x\theta}} \\
& + \Lambda^{tt} \frac{\partial}{\partial Q_{tt}} + \Lambda^{t\theta} \frac{\partial}{\partial Q_{t\theta}} + \Lambda^{\theta\theta} \frac{\partial}{\partial Q_{\theta\theta}} \\
& + \Lambda^{xxx} \frac{\partial}{\partial Q_{xxx}} + \Lambda^{xxt} \frac{\partial}{\partial Q_{xxt}} + \Lambda^{xx\theta} \frac{\partial}{\partial Q_{xx\theta}} \\
& + \Lambda^{xtt} \frac{\partial}{\partial Q_{xtt}} + \Lambda^{xt\theta} \frac{\partial}{\partial Q_{xt\theta}} + \Lambda^{x\theta\theta} \frac{\partial}{\partial Q_{x\theta\theta}} \\
& + \Lambda^{ttt} \frac{\partial}{\partial Q_{ttt}} + \Lambda^{tt\theta} \frac{\partial}{\partial Q_{tt\theta}} + \Lambda^{t\theta\theta} \frac{\partial}{\partial Q_{t\theta\theta}} + \Lambda^{\theta\theta\theta} \frac{\partial}{\partial Q_{\theta\theta\theta}}.
\end{aligned}$$

Applying $\text{pr}^{(3)}\mathbf{v}$ on both sides of (3.9) yields

$$\Lambda^t = \Lambda^{xxx} - a\Gamma Q Q_{xx} - a\theta\Lambda Q_{xx} - a\theta Q\Lambda^{xx} + a\Lambda Q_{\theta x} + aQ\Lambda^{\theta x} + (6-3a)(\Lambda^\theta Q_x + Q_\theta\Lambda^x). \quad (3.21)$$

Here

$$\begin{aligned}
\Lambda^x &= D_x(\Lambda - \Xi^1 Q_x - \Xi^2 Q_t - \Gamma Q_\theta) + \Xi^1 Q_{xx} + \Xi^2 Q_{xt} + \Gamma Q_{x\theta}, \\
\Lambda^{xx} &= (D_x)^2(\Lambda - \Xi^1 Q_x - \Xi^2 Q_t - \Gamma Q_\theta) + \Xi^1 Q_{xxx} + \Xi^2 Q_{xxt} + \Gamma Q_{xx\theta}, \\
\Lambda^{xxx} &= (D_x)^3(\Lambda - \Xi^1 Q_x - \Xi^2 Q_t - \Gamma Q_\theta) + \Xi^1 Q_{xxxx} + \Xi^2 Q_{xxx t} + \Gamma Q_{xxx\theta}, \\
\Lambda^t &= D_t(\Lambda - \Xi^1 Q_x - \Xi^2 Q_t - \Gamma Q_\theta) + \Xi^1 Q_{tx} + \Xi^2 Q_{tt} + \Gamma Q_{t\theta}, \\
\Lambda^{\theta x} &= D_x D_\theta(\Lambda - \Xi^1 Q_x - \Xi^2 Q_t - \Gamma Q_\theta) + \Xi^1 Q_{\theta xx} + \Xi^2 Q_{\theta xt} + \Gamma Q_{\theta x\theta}.
\end{aligned}$$

Plugging $\Lambda^x, \Lambda^{xx}, \Lambda^{xxx}, \Lambda^t$, and $\Lambda^{\theta x}$ into (3.21) and then replacing Q_{xxx} by

$$Q_t + a\theta Q Q_{xx} - aQ Q_{\theta x} - (6-3a)Q_\theta Q_x$$

yields a large symmetry defining system.

Equating the coefficients of the various monomials in the first, second and third order

partial derivatives of Q to zero, we find 28 determining equations for the symmetry group of super KdV equation which are shown in Table 3.2.2.

After simplifying those 28 determining equations, we get a simpler equivalent system of 7 determining equations

$$\Lambda_t - \Lambda_{xxx} - aQ\Lambda_{x\theta} + a\theta Q\Lambda_{xx} = 0, \quad (3.22)$$

$$\Gamma_t + aQ\Lambda_{xQ} - (3a - 6)\Lambda_x = 0, \quad (3.23)$$

$$3\Xi_x^1 - \Xi_t^2 = 0, \quad (3.24)$$

$$2aQ\Xi_x^1 - aQ\Gamma_\theta + a\Lambda = 0, \quad (3.25)$$

$$a\theta Q\Xi_x^1 - aQ\Gamma + a\theta\Lambda + 3\Xi_{xx}^1 - 3\Lambda_{xQ} + aQ\Xi_\theta = 0, \quad (3.26)$$

$$(3a - 6)\Lambda_\theta + 2a\theta Q\Lambda_{xQ} - a\theta Q\Xi_{xx}^1 - \Xi_t^1 - 3\Lambda_{xxQ} + \Xi_{xxx}^1 + aQ\Lambda_{\theta Q} + aQ\Xi_{x\theta}^1 = 0, \quad (3.27)$$

$$(3a - 6)\Lambda_Q - (3a - 6)\Gamma_\theta + (6a - 12)\Xi_x^1 = 0, \quad (3.28)$$

where $\Xi^1 = \Xi^1(x, t, \theta)$, $\Xi^2 = \Xi^2(t)$, $\Gamma = \Gamma(t, \theta)$, and $\Lambda = \Lambda(x, t, \theta, Q)$.

The solution of the determining equations is elementary. First, (3.24) implies

$$\Xi_{xx}^1 = 0.$$

So suppose that

$$\Xi^1 = f_1(t, \theta)x + f_2(t, \theta). \quad (3.29)$$

Substituting (3.29) into (3.24) yields

$$f_1(t, \theta) = \frac{1}{3}\Xi_t^2(t),$$

which implies that f_1 does not depend on θ . Hence we have

$$\Xi^1 = \frac{1}{3}\Xi_t^2(t)x + f_2(t, \theta). \quad (3.30)$$

Suppose

$$\Gamma(t, \theta) = h_1(t)\theta + h_2(t) \quad (3.31)$$

and

$$\Lambda(x, t, \theta, Q) = g_1(x, t, \theta)Q + g_2(x, t, \theta). \quad (3.32)$$

By multiplying Q to both sides of (3.25) with Q , we have

$$Q\Lambda = 0. \quad (3.33)$$

By substituting (3.32) into (3.33), this yields $g_2(x, t, \theta) = 0$. Hence

$$\Lambda(x, t, \theta, Q) = g_1(x, t, \theta)Q. \quad (3.34)$$

By substituting (3.30), (3.31) and (3.34) into (3.25), we get $g_1(x, t, \theta) = h_1(t) - \frac{2}{3}\Xi_t^2(t)$.

Hence we have

$$\Lambda(x, t, \theta, Q) = \left(h_1(t) - \frac{2}{3}\Xi_t^2(t) \right) Q, \quad (3.35)$$

which means that $\Lambda(x, t, \theta, Q)$ does not depend on x and θ . Hence (3.22) requires that $\Lambda_t = 0$, which implies that $\Lambda(x, t, \theta, Q)$ only depends on Q . Therefore let $h_1(t)$ be an even constant c_1 and $\Xi_t^2(t) = c_2$. Also, (3.23) requires $\Gamma_t = 0$, which implies that Γ is a function of t only. Hence $\Gamma = c_1\theta + \alpha_1$ where α_1 is an odd constant. So far we have

$$\Xi^1 = \frac{1}{3}c_2x + f_2(t, \theta), \quad (3.36)$$

$$\Xi^2 = c_2t + c_3, \quad (3.37)$$

where c_3 is an arbitrary even constant, and

$$\Gamma = c_1\theta + \alpha_1, \quad (3.38)$$

$$\Lambda = \left(c_1 - \frac{2}{3}c_2 \right) Q. \quad (3.39)$$

Equation (3.27) requires that $\Xi_t^1 = 0$ which implies that Ξ^1 dose not depend on t . Hence

$f_2(t, \theta) = \alpha_2\theta + c_4$, where α_2 is an arbitrary odd constant and c_4 is an arbitrary even constant. Rewrite

$$\Xi^1 = \frac{1}{3}c_2x + \alpha_2\theta + c_4. \quad (3.40)$$

Substituting (3.40), (3.38), and (3.39) into (3.26) yields $c_1 = \frac{1}{6}c_2$ and $\alpha_2 = -\alpha_1$. Hence the infinitesimals are

$$\Xi^1 = \frac{1}{3}c_2x - \alpha_1\theta + c_4, \quad (3.41)$$

$$\Xi^2 = c_2t + c_3, \quad (3.42)$$

$$\Gamma = \frac{1}{6}c_2\theta + \alpha_1, \quad (3.43)$$

$$\Lambda = -\frac{1}{2}c_2Q. \quad (3.44)$$

The above solutions of Ξ^1 , Ξ^2 , Γ and Λ give us

$$\begin{aligned} \mathbf{v} &= \left(\frac{1}{3}c_2x - \alpha_1\theta + c_4\right)\frac{\partial}{\partial x} + (c_2t + c_3)\frac{\partial}{\partial t} + \left(\frac{1}{6}c_2\theta + \alpha_1\right)\frac{\partial}{\partial \theta} + \left(-\frac{1}{2}c_2Q\right)\frac{\partial}{\partial Q} \\ &= c_2\left(\frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{1}{6}\theta\frac{\partial}{\partial \theta} - \frac{1}{2}Q\frac{\partial}{\partial Q}\right) + c_3\frac{\partial}{\partial t} + c_4\frac{\partial}{\partial x} + \alpha_1\left(-\theta\frac{\partial}{\partial x} + \frac{\partial}{\partial \theta}\right), \end{aligned}$$

which leads a (3|1)-dimensional superalgebra. This means that it is generated by 3 even generator L_1, L_2, L_3 and one odd generator L_4 given by

$$L_1 = \partial_x,$$

$$L_2 = \partial_t,$$

$$L_3 = \frac{1}{3}x\partial_x + t\partial_t + \frac{1}{6}\theta\partial_\theta - \frac{1}{2}Q\partial_Q,$$

$$L_4 = \partial_\theta - \theta\partial_x.$$

By the definition of Lie superbracket (2.10), we have

$$\begin{aligned}
[L_1, L_2] &= \partial_x(1)\partial_t - (-1)^{0\cdot 0}\partial_t(1)\partial_x = 0, \\
[L_1, L_3] &= [\partial_x, \frac{1}{3}x\partial_x + t\partial_t + \frac{1}{6}\theta\partial_\theta - \frac{1}{2}Q\partial_Q] \\
&= \frac{1}{3}[\partial_x, x\partial_x] + [\partial_x, t\partial_t] + \frac{1}{6}[\partial_x, \theta\partial_\theta] + -\frac{1}{2}[\partial_x, Q\partial_Q] \\
&= \frac{1}{3}(\partial_x(x)\partial_x - (-1)^{0\cdot 0}x\partial_x(1)\partial_x) + \partial_x(1)\partial_t - (-1)^{0\cdot 0}\partial_t(1)\partial_x \\
&\quad + \frac{1}{6}(\partial_x(\theta)\partial_\theta - (-1)^{0\cdot 0}\theta\partial_\theta(1)\partial_x) - \frac{1}{2}(\partial_x(Q)\partial_Q - (-1)^{0\cdot 0}Q\partial_Q(1)\partial_x) \\
&= \frac{1}{3}\partial_x = \frac{1}{3}L_1 \\
[L_1, L_4] &= [\partial_x, \partial_\theta - \theta\partial_x] = [\partial_x, \partial_\theta] - [\partial_x, \theta\partial_x] \\
&= \partial_x(1)\partial_\theta - (-1)^{0\cdot 1}\partial_\theta(1)\partial_x + \partial_x(\theta)\partial_x - (-1)^{0\cdot 1}\theta\partial_x(1)\partial_x = 0, \\
[L_4, L_4] &= [\partial_\theta - \theta\partial_x, \partial_\theta - \theta\partial_x] \\
&= [\partial_\theta, \partial_\theta] - [\partial_\theta, \theta\partial_x] - [\theta\partial_x, \partial_\theta] + [\theta\partial_x, \theta\partial_x] \\
&= \partial_x(1)\partial_t - (-1)^{1\cdot 1}\partial_t(1)\partial_x - (\partial_\theta(\theta)\partial_x - (-1)^{1\cdot 1}\theta\partial_x(1)\partial_\theta) \\
&= -(\theta\partial_x(1)\partial_\theta - (-1)^{1\cdot 1}\partial_\theta(\theta)\partial_x) + (\theta\partial_x(\theta)\partial_x - (-1)^{1\cdot 1}\theta\partial_x(\theta)\partial_x) \\
&= -\partial_x - \partial_x = -2L_1.
\end{aligned}$$

In summary, the supercommutator table of the Lie superalgebra is given by Table 3.2.3.

Monomials	Coefficient
$Q_{x\theta}Q_x$	$-2a\theta Q\Gamma_Q + 6\Gamma_{xQ} = 0$
$Q_\theta Q_x$	$-2a\theta Q\Gamma_{xQ} + (3a - 6)\Lambda_Q - (3a - 6)\Gamma_\theta$ $+ (6a - 12)\Xi_x^1 + aQ\Xi_{xQ}^1 - aQ\Gamma_{\theta Q} + 3\Gamma_{xxQ} = 0$
Q_x	$(3a - 6)\Lambda_\theta + 2a\theta Q\Lambda_{xQ} - a\theta Q\Xi_{xx}^1 - \Xi_t^1$ $- 3\Lambda_{xxQ} + \Xi_{xxx}^1 + aQ\Lambda_{\theta Q} + aQ\Xi_{x\theta}^1 = 0$
Q_θ	$(3a - 6)\Lambda_\theta - a\theta Q\Gamma_{xx} - \Gamma_t + \Gamma_{xxx} - aQ\Lambda_{xQ} + aQ\Gamma_{x\theta} = 0$
$Q_{xx\theta}Q_x$	$3\Gamma_Q = 0$
$Q_x Q_{xx}$	$-3\Xi_{xQ}^1 - a\theta Q\Xi_Q^1 = 0$
$Q_{x\theta}Q_{xx}$	$3\Gamma_Q = 0$
$Q_t Q_x$	$3\Xi_{xxQ}^2 - 2a\theta Q\Xi_{xQ}^2 - (3a - 6)\Xi_\theta^2 + aQ\Xi_{\theta Q}^2 + 3\Xi_Q^1 = 0$
1	$-aQ\Lambda_{x\theta} - \Lambda_{xxx} + \Lambda_t + a\theta Q\Lambda_{xx} = 0$
$Q_t Q_{xx}$	$3\Xi_{xQ}^2 = 0$
$Q_x Q_{tx}$	$2a\theta Q\Xi_Q^2 - 6\Xi_{xQ}^2 = 0$
$(Q_\theta)^2 Q_x$	$(3a - 6)\Gamma_\theta = 0$
$Q_\theta Q_t Q_x$	$-(3a - 6)\Xi_Q^2 = 0$
$Q_{x\theta}$	$2aQ\Xi_Q^1 - 2a\theta Q\Gamma_x + 3\Gamma_{xx} - 2aQ\Xi_x^1 + aQ\Gamma_\theta - a\Lambda = 0$
Q_{xx}	$a\theta Q\Xi_x^1 - aQ\Gamma + a\theta\Lambda + 3\Xi_{xx}^1 - 3\Lambda_{xQ} + aQ\Xi_\theta^1 = 0$
Q_t	$-a\theta Q\Xi_{xx}^2 + \Xi_x^1 - \Xi_t^2 + \Xi_{xxx}^2 + aQ\Xi_{x\theta}^2 = 0$
Q_{tx}	$-2a\theta Q\Xi_x^2 + 3\Xi_{xx}^2 + aQ\Xi_\theta^2 = 0$
$Q_\theta Q_{xx}$	$aQ\Xi_Q^1 + 3\Gamma_{xQ} = 0$
$Q_\theta Q_t$	$aQ\Xi_{xQ}^2 - (3a - 6)\Xi_x^2 = 0$
$Q_{t\theta}Q_x$	$aQ\Xi_Q^2 = 0$
$Q_\theta Q_{tx}$	$aQ\Xi_Q^2 = 0$
$Q_\theta Q_{x\theta}$	$aQ\Gamma_Q = 0$
$(Q_\theta)^2$	$(3a - 6)\Gamma_x - aQ\Gamma_{xQ} = 0$
$Q_{xx}Q_{tx}$	$3\Xi_Q^2 = 0$
Q_{txx}	$3\Xi_x^2 = 0$
$Q_{xx\theta}$	$3\Gamma_x = 0$
$Q_{t\theta}$	$aQ\Xi_x^2 = 0$
$Q_x Q_{txx}$	$3\Xi_Q^2 = 0$

Table 3.2.2: Unsimplified determining system of 28 equations for the super KdV.

	L_1	L_2	L_3	L_4
L_1	0	0	$\frac{1}{3}L_1$	0
L_2	0	0	L_2	0
L_3	$-\frac{1}{3}L_1$	$-L_2$	0	$-\frac{1}{6}L_4$
L_4	0	0	$\frac{1}{6}L_4$	$-2L_1$

Table 3.2.3: Supercommutator table of the defining system of the super KdV.

Chapter 4

Algorithmic determination of structure of supersymmetry algebras of super DEs

The main results of this thesis will be given in this chapter, which are two theorems and two algorithms. The theorems are the Existence and Uniqueness Theorem in Section 4.4.2 and Structure Constants Theorem 4.5.1 and its proof in Section 4.5. The algorithms are the MONO Expansion Algorithm 4.2.3 in Section 4.2 and the Structure Constants Algorithm 4.5.2 in Section 4.5. Both algorithms are illustrated by application to simple example $Q_{xx} = 0$ and to more complicated example of the super KdV equation.

In Chapter 3, we determine the structure constants for both these examples by integrating their defining equations. But solving differential equations and super differential equations by integration is not guaranteed to always be successful. Reid and his collaborators developed a method to find the structure constants without integrations for the usual non-Grassmannian case. In this chapter, inspired by their method, we will develop an algorithm to find structure constants for finite-dimensional supersymmetries without integrations.

4.1 Regular super differential equations

The definition of super differential equation has been introduced in Chapter 2. Now we give the definition of a regular/irregular super differential equations and a regular/irregular system of super differential equations.

Definition 4.1.1. *Suppose that $\phi = 0$ is a super differential equation which has highest derivative v with respect to a ranking \succ . Then ϕ is regular with respect \succ if $\frac{\partial\phi}{\partial v}$ is even. Otherwise ϕ is called an irregular differential equation with respect to \succ .*

Definition 4.1.2. *Suppose that \succ is a ranking. If a super differential equation system does not contain any irregular super differential equations with respect to \succ , then it is a regular system of super differential equations with respect to \succ . Otherwise, it is an irregular system of super differential equations with respect to \succ .*

Irregular super differential equations are not trivial in super calculations. For example, consider the super differential equation

$$\phi = Q \cdot v - w = 0,$$

where v denotes the highest derivative of the super differential equation under a certain ranking \succ and Q is odd. Then $\partial\phi/\partial v = Q$ is odd. By Definition 4.1.1, it is not a regular differential equation, and it is an irregular super differential equation. It can not be written in solved form since the coefficient Q can not be divided to the right hand side of the equation. Although an obvious phenomenon, it is critical to the theory underlying algorithms which will be constructed later in this chapter. Hence, we need a way to change irregular super differential equation systems to regular ones. We show how to do this in the next section.

4.2 MONO expansion algorithm

4.2.1 MONO expansion

MONO expansion is a method for converting irregular super differential systems into regular ones.

We define odd variable monomials and then introduce MONO expansion which decomposes expansions into independent odd variable monomials. For a super function $F(X, \Theta)$, where $X = (x_1, \dots, x_m)$ are even independent variables and $\Theta = (\theta_1, \dots, \theta_n)$ are odd independent variables, the odd variable monomials are all the linearly independent products consisting of some or all of the θ_i 's. For example, if $n = 2$, there are three odd variable monomials θ_1, θ_2 and $\theta_1\theta_2$ and they are linearly independent. For n odd variables, there are $2^n - 1$ linearly independent odd variable monomials. In fact, when one decomposes a super function by its odd variable monomials, we actually decompose the super function by odd variable monomials and the unit 1 (which is also linearly independent of odd variable monomials).

The decomposition by odd variable monomials follows from the definition of a Grassmann algebra. According to Definition 2.2.2, two independent odd variables θ_1 and θ_2 generate a 4 - dimensional Grassmann algebra with basis $\{1, \theta_1, \theta_2, \theta_1\theta_2\}$. For example, if $F(x, \theta_1, \theta_2)$ is an even super function then

$$F(x, \theta_1, \theta_2) = f_1(x) + g_1(x)\theta_1 + g_2(x)\theta_2 + f_2(x)\theta_1\theta_2, \quad (4.1)$$

where the $f_i(x)$ are even functions and the $g_i(x)$ are odd functions for $i = 1, 2$. Similarly, the MONO expansion of an odd super function $G(x, \theta_1, \theta_2)$ can be written as

$$G(x, \theta_1, \theta_2) = \tilde{g}_1(x) + \tilde{f}_1(x)\theta_1 + \tilde{f}_2(x)\theta_2 + \tilde{g}_2(x)\theta_1\theta_2, \quad (4.2)$$

where $\tilde{f}_i(x)$ are even functions and $\tilde{g}_i(x)$ are odd functions, $i = 1, 2$. The odd variables

θ_1, θ_2 and θ_3 generate an 8 - dimensional Grassmann algebra with basis

$$\{1, \theta_1, \theta_2, \theta_3, \theta_1\theta_2, \theta_1\theta_3, \theta_2\theta_3, \theta_1\theta_2\theta_3\}.$$

More generally, consider a super function $F(X, \Theta)$, where $X = (x_1, \dots, x_m)$ are even independent variables and $\Theta = (\theta_1, \dots, \theta_n)$ are odd independent variables. Suppose that $\omega_1, \dots, \omega_{2^n-1}$ are the odd variable monomials generated by the odd independent variables $\theta_1, \dots, \theta_n$. Then $F(X, \Theta)$ can be expanded as

$$F(X, \Theta) = f_0(X) + \sum_{i=1}^{2^n-1} f_i(X)\omega_i, \quad (4.3)$$

in term of functions f_j . This is called the MONO expansion of F .

One of the advantages of MONO expansion is giving the odd dependencies of a super function explicitly. The other advantage is more important. Substituting the MONO expansions into the original system and taking coefficients of odd variable monomials yields a new regular system. The new system does not depend on odd variables. Figure 4.2.1 shows this procedure.

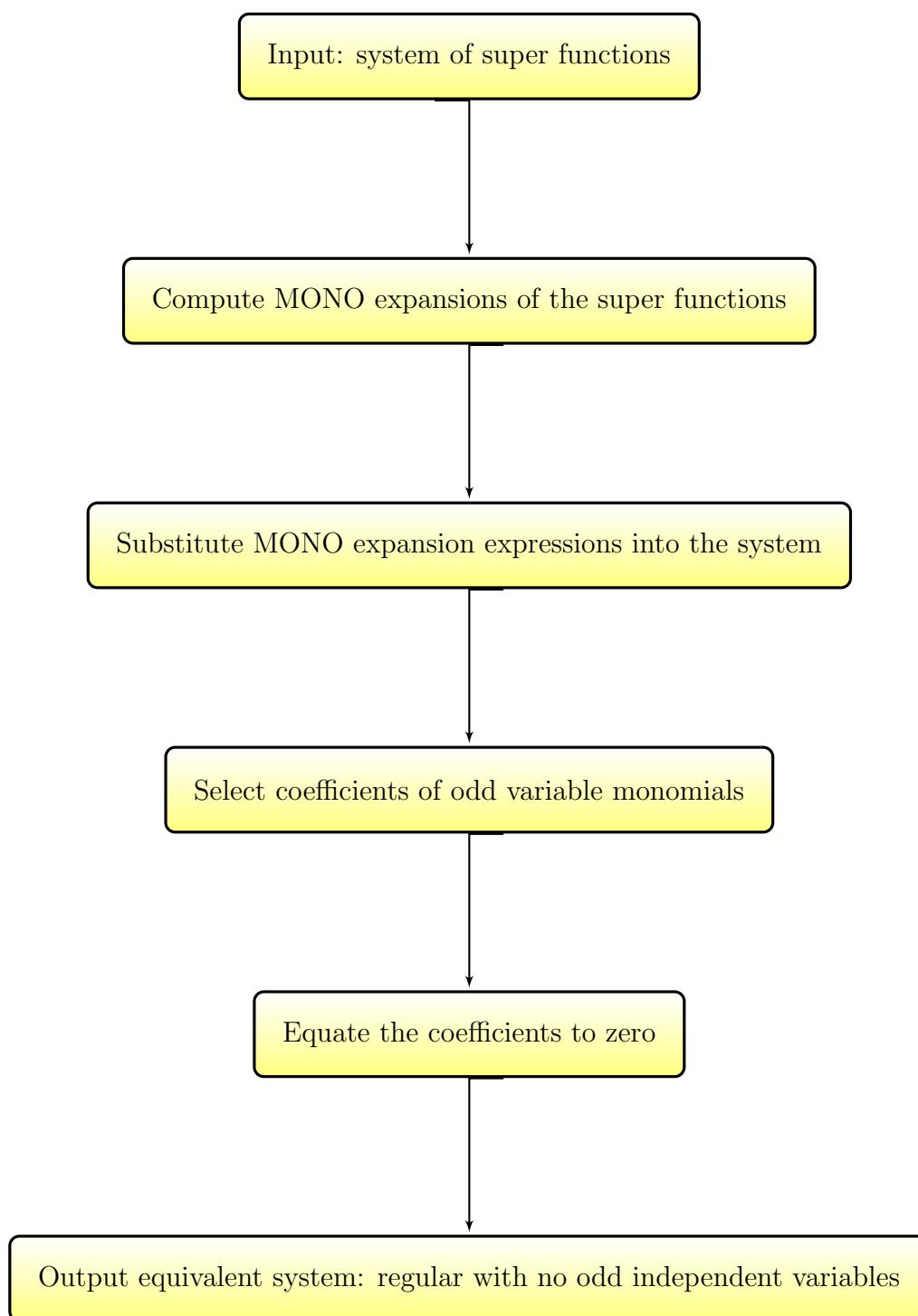


Figure 4.2.1: The MONO expansion procedure

4.2.2 Two commutative Maple commands

The MAPLE commands `rifsimp` and `initialdata` can be used to simplify overdetermined systems of PDEs or ODEs. They are designed for the commutative case. The underlying idea of `rifsimp` is similar to Gröbner bases. Under a given ordering, the calculation of dividing the coefficient of the leading term and moving it to the right hand side of the equation is a fundamental operation and is always possible in commutative case. But in our case, if the coefficient of the leading term is odd, we can not divide by it. So we need certain assumptions to be able to apply the commutative algorithm `rifsimp` to non-commutative calculations.

4.2.3 MONO expansion algorithm

MONO is the Maple procedure for exacting the MONO expansion of super functions [See Appendix A.1].

Algorithm 1 MONO expansion algorithm

Input: Defining system \mathcal{S} .

1. Decompose each of the infinitesimals by **MONO** expansion.
2. Substitute them into the input system \mathcal{S} .
3. Equating all the coefficients of independent odd variable monomials to zero forms the new defining system \mathcal{S}_{red} .
4. Send \mathcal{S}_{red} to the commutative Maple commands `rifsimp` and `initialdata`.

Output: Return \mathcal{S}_{red} in `rifsimp` form and the size of the symmetry algebra.

One immediately has the following observations.

Remark 4.2.1.

1. *This algorithm gives the details of the method outlined in Figure 4.2.1.*
2. *The new system in step 3 is regular and it does not depend on odd variables.*

3. Suppose that \mathcal{H} is the supersymmetry algebra. Then

$$\text{the size of the symmetry algebra output} = \dim\mathcal{H}_0 + \dim\mathcal{H}_1.$$

Definition 4.2.2 (Reduced defining system). *The new defining system generated at step 3 in Algorithm 4.2.3 is called the reduced defining system.*

As Remark 4.2.1 mentioned, the reduced defining system is regular and has no odd independent variables. For calculational convenience, we can assume without loss of generality that the reduced defining system is monic.

4.3 MONO expansion applications

4.3.1 Applying MONO expansion

We first apply MONO expansion to general super differential equation system. Then we apply it to specific examples.

For a super differential equation system

$$\Delta(X, \Theta, A^{(k_1)}, Q^{(k_2)}) = 0,$$

the corresponding infinitesimals

$$\Xi^i(X, \Theta, A, Q), \Gamma^j(X, \Theta, A, Q), \Phi^l(X, \Theta, A, Q) \text{ and } \Lambda^k(X, \Theta, A, Q)$$

are the infinitesimals of the transformations for both independent and dependent (even and odd) variables, X, Θ, A and Q . To solve the determining equations means finding Ξ^i, Γ^j, Φ^l and Λ^k . In the last chapter, we solved for the infinitesimals by integration. In this chapter, we will avoid integration and solve the same determining system by a new method. We first decompose the unknowns (infinitesimals) by MONO expansion. In the determining equations, there are $n + p$ odd independent variables $\theta_1, \dots, \theta_n$ and Q^1, \dots, Q^p . We decompose Ξ^i, Γ^j, Φ^l and Λ^k with respect to all the monomials of those

$n + p$ odd independent variables $\theta_1, \dots, \theta_n$ and Q^1, \dots, Q^p . Hence we have 2^{n+p-1} linearly independent monomials denoted by $\omega_1, \dots, \omega_s$, where $s = 2^{n+p-1}$. For convenience in what follows, we suppose that $\omega_0 = 1$. Hence, the decompositions of the infinitesimals are

$$\begin{aligned}\Xi^i &= P_{1_0}^i(X, A) + P_{1_1}^i(X, A)\omega_1 + \dots + P_{1_s}^i(X, A)\omega_s = \sum_{\mu=0}^s P_{1_\mu}^i \omega_\mu, \\ \Gamma^j &= P_{2_0}^j(X, A) + P_{2_1}^j(X, A)\omega_1 + \dots + P_{2_s}^j(X, A)\omega_s = \sum_{\mu=0}^s P_{2_\mu}^j \omega_\mu, \\ \Phi^l &= P_{3_0}^l(X, A) + P_{3_1}^l(X, A)\omega_1 + \dots + P_{3_s}^l(X, A)\omega_s = \sum_{\mu=0}^s P_{3_\mu}^l \omega_\mu, \\ \Lambda^k &= P_{4_0}^k(X, A) + P_{4_1}^k(X, A)\omega_1 + \dots + P_{4_s}^k(X, A)\omega_s = \sum_{\mu=0}^s P_{4_\mu}^k \omega_\mu,\end{aligned}$$

where $P_{1_\mu}^i(X, A), \dots, P_{4_\mu}^k(X, A)$ are super functions only depending on even variables X and A .

Remark 4.3.1. We use $P_{1_\mu}^i(X, A), \dots, P_{4_\mu}^k(X, A)$ to denote the coefficients of the odd monomials. In particular expansions, we will use PE 's and PO 's to denote even and odd coefficients respectively.

We substitute these expansions into the determining system. Then we get a new system without any odd independent variables by considering the even or odd parity in each super differential equation in the determining system. The advantage of MONO expansion is to eliminate dependence on odd variables in the system. This often considerably simplifies the system. The following example illustrates this simplification.

Example 4.3.2. Let $\Xi_x(x, \theta, Q) = 0$ be the determining equation, where Ξ is the even dependant variable which depends on an even independent variable, x , and two odd independent variables, θ and Q . The MONO expansion of $\Xi(x, \theta, Q)$ is

$$\Xi(x, \theta_1, \theta_2) = PE_1(x) + PO_1(x)\theta_1 + PO_2(x)\theta_2 + PE_2(x)\theta_1\theta_2.$$

By substitution, one gets the new system

$$\begin{aligned}\frac{dPE_1(x)}{dx} &= 0, \\ \frac{dPO_1(x)}{dx} &= 0, \\ \frac{dPO_2(x)}{dx} &= 0, \\ \frac{dPE_2(x)}{dx} &= 0.\end{aligned}$$

Although there are more equations in the system, they are trivially solvable and do not depend on odd independent variables.

4.3.2 Applying the MONO algorithm to examples

Let us apply MONO to two examples which were introduced in the previous chapter.

Example 4.3.3. Apply the MONO algorithm to $Q_{xx} = 0$, which was presented in Section 3.2.1.

The input is its defining system (3.13).

Input: Defining system

$$\mathcal{S} = \begin{cases} \Lambda(x, Q)_{xx} = 0, \\ -\Xi(x, Q)_{xx} + 2\Lambda(x, Q)_{xQ} = 0. \end{cases} \quad (4.4)$$

1. The MONO expansion of infinitesimals are

$$\begin{cases} \Xi(x, Q) = f_1(x) + g_1(x) * Q, \\ \Lambda(x, Q) = g_2(x) + f_2(x) * Q. \end{cases} \quad (4.5)$$

2. Substituting the expansions (4.5) into the input system (4.4), we get

$$\begin{cases} g_{2xx} + f_{2xx} * Q = 0, \\ -f_{1xx} + Q * g_{1xx} + 2f_{2x} = 0. \end{cases} \quad (4.6)$$

3. Equate the coefficients of odd variable monomials, 1 and Q , to be zero. The defining system is now

$$\mathcal{S}_{\text{red}} = \begin{cases} g_{2xx} = 0, \\ f_{2xx} = 0, \\ f_{1xx} = 2f_{2x}, \\ g_{1xx} = 0. \end{cases} \quad (4.7)$$

4. Send (4.7) to the commutative Maple commands `rifsimp` and `initialdata`.

The following figure is the detailed Maple output.

```
read "/Users/liux0578/Documents/Study/Meeting-4/SymmetryPrograms/QxxVer1.mpl"
[anticommutativeprefix = {Lambda, PO, Q, Xi}]
d^2
dx^2 Q(x)
[Q(x)]
Q_x,x
[Xi(x, Q), Lambda(x, Q)]
Lambda_x,x + 2*Lambda_x,Q*Q_x - Xi_x,x*Q_x + Q_x,x*(Lambda_Q - Xi_x) - Xi_x*Q_x,x + Q_x*Q_x,x*Xi_Q
Q_x,x
Lambda_x,x + 2*Lambda_x,Q*Q_x - Xi_x,x*Q_x
{Lambda_x,Q*Lambda_x,x*Q_x, Xi_x,x}
[Q_x], []
[Lambda_x,x = 0, -Xi_x,x + 2*Lambda_x,Q = 0]
[d^2/dx^2 Lambda(x, Q) = 0, -(d^2/dx^2 Xi(x, Q)) + 2*(d^2/dx^2 Lambda(x, Q)) = 0]
[Xi(x, Q) = PEI_1(x) - Q*POI_1(x), Lambda(x, Q) = PO2_1(x) + PE2_1(x)*Q]
{Q, x, Q*POI_1(x), Lambda(x, Q), Xi(x, Q), PEI_1(x), PE2_1(x), POI_1(x), PO2_1(x)}
[PEI_1(x), PE2_1(x), POI_1(x), PO2_1(x)]
[d^2/dx^2 PO2_1(x) + (d^2/dx^2 PE2_1(x))*Q = 0, -(d^2/dx^2 PEI_1(x)) + Q*(d^2/dx^2 POI_1(x)) + 2*(d/dx PE2_1(x)) = 0]
[PO2_1,x,x + PE2_1,x,x*Q = 0, -PEI_1,x,x + Q*POI_1,x,x + 2*PE2_1,x = 0]
[PO2_1,x,x = 0, PE2_1,x,x = 0, 2*PE2_1,x - PEI_1,x,x = 0, POI_1,x,x = 0]
[d^2/dx^2 PO2_1(x) = 0, d^2/dx^2 PE2_1(x) = 0, 2*(d/dx PE2_1(x)) - (d^2/dx^2 PEI_1(x)) = 0, d^2/dx^2 POI_1(x) = 0]
table([Solved = [d^2/dx^2 PEI_1(x) = 2*(d/dx PE2_1(x)), d^2/dx^2 PE2_1(x) = 0, d^2/dx^2 POI_1(x) = 0, d^2/dx^2 PO2_1(x) = 0]])
table([Infinite = [], Finite = [PEI_1(x_0) = _C1, D(PEI_1)(x_0) = _C2, PE2_1(x_0) = _C3, D(PE2_1)(x_0) = _C4, POI_1(x_0) = _C5, D(POI_1)(x_0) = _C6, PO2_1(x_0) = _C7, D(PO2_1)(x_0) = _C8]])
```

Figure 4.3.1: Maple output of the MONO algorithm applied to $Q_{xx} = 0$.

In Figure 4.3.1, the Maple output from initial data yields 8 initial conditions of the form

$$PE1_1(x_0) = _C1, D(PE1_1)(x_0) = _C2, PE2_1(x_0) = _C3, D(PE2_1)(x_0) = _C4,$$

$$PO1_1(x_0) = _C5, D(PO1_1)(x_0) = _C6, PO2_1(x_0) = _C7, D(PO2_1)(x_0) = _C8,$$

so there is an 8-dimensional symmetry group. Since Maple commands, `rifsimp` and `initialdata` are designed for commutative calculations, it treats input as commutative calculations. In fact, it is a $4 | 4$ - dimensional supersymmetry group. The dimension 8 returned by Maple output is the sum of even dimension and odd dimension of a finite supersymmetry group.

By looking at the following diagram, one can easily graphically determine the dimension of the supersymmetry group.

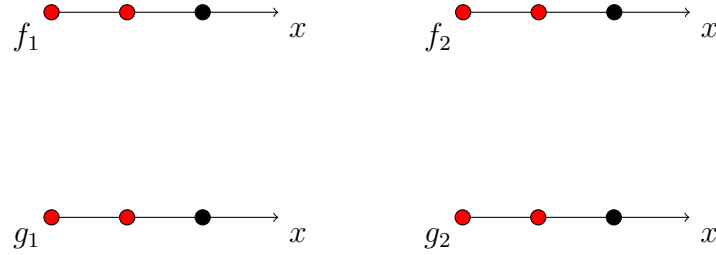


Figure 4.3.2: Dimension analysis diagram for the defining system of $Q_{xx} = 0$.

The number of red dots equal the dimension of the supersymmetry group. The parametric derivatives are the set

$$\{f_1, f_{1x}, f_2, f_{2x}, g_1, g_{1x}, g_2, g_{2x}\}.$$

In this set, the first four parametric derivatives are even and the other four are odd. Hence it is a $4 | 4$ - dimensional supersymmetry group. For analogous diagrams for Gröbner bases of polynomials, see [28].

The input for this example is already regular. In the same way the MONO expansion algorithm can be applied to the reduced defining system and the dimension of its supersymmetry group obtained. The next example shows the wonderful power of MONO expansion for irregular systems.

Example 4.3.4. *Apply the MONO algorithm to the defining system of the super KdV equation (3.9).*

The second example is the super KdV equation. Let us look at the simplified defining system of the super KdV equation (3.22-3.28), which is also the input system:

$$\begin{aligned}
\Xi_{xx}^1 &= 0, \\
\Xi_{x\theta}^1 &= 0, \\
\Xi_t^1 &= 0, \\
\Xi_Q^1 &= 0, \\
\Xi_x^2 &= 0, \\
\Xi_t^2 &= 3\Xi_x^1, \\
\Xi_\theta^2 &= 0, \\
\Xi_Q^2 &= 0, \\
\Gamma_x &= 0, \\
\Gamma_t &= 0, \\
\Gamma_\theta &= \Lambda + 2Q\Xi_x^1, \\
\Gamma_Q &= 0, \\
\Lambda_x &= 0, \\
\Lambda_t &= 0, \\
\Lambda_\theta &= 0, \\
\Lambda_Q &= -2\Xi_x^1 + \Gamma_\theta \\
\underline{Q\Xi_\theta^1} &= \underline{Q\Gamma - \theta\Lambda - \theta Q\Xi_x^1}, \text{ irregular.}
\end{aligned}$$

The last super differential equation of the system above

$$Q\Xi_\theta^1 = Q\Gamma - \theta\Lambda - \theta Q\Xi_x^1 \quad (4.8)$$

is irregular if Ξ_θ^1 is the highest derivative. If one changes the ordering of the super derivatives to make Ξ_x^1 the highest derivative, then the coefficient of Ξ_x^1 is θQ , which is an even coefficient. They can be written in solved form

$$\Xi_x^1 = \frac{1}{\theta Q}(Q\Gamma - \theta\Lambda - Q\Xi_\theta^1). \quad (4.9)$$

But there is still no better way to solve (4.9) with a quantity like $\frac{1}{\theta Q}$. This emphasizes the advantages of the MONO expansion algorithm which we now apply.

Step 1: The MONO expansions of the infinitesimals are

$$\begin{pmatrix} \Xi^1 \\ \Xi^2 \\ \Gamma \\ \Lambda \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & f_{12} \\ g_{21} & g_{22} & f_{22} \\ f_{31} & f_{32} & g_{32} \\ f_{41} & f_{42} & g_{42} \end{pmatrix}_{(x,t)} \begin{pmatrix} \theta \\ Q \\ \theta Q \end{pmatrix} + \begin{pmatrix} f_{11} \\ f_{21} \\ g_{31} \\ g_{41} \end{pmatrix}_{(x,t)}, \quad (4.10)$$

where $f_{11}, f_{12}, f_{21}, f_{22}, f_{31}, f_{32}, f_{41}, f_{42}$ are even functions depending on (x, t) , and $g_{11}, g_{12}, g_{21}, g_{22}, g_{31}, g_{32}, g_{41}, g_{42}$ are odd functions also depending on (x, t) .

Step 2 and 3: By substituting the expansion back to the input defining system and equating the coefficients of odd variable monomials for each equation, one forms the reduced defining system

$$\begin{aligned} (g_{11})_x &= (g_{11})_t = g_{12} = f_{12} = 0, (f_{11})_x = 2f_{31}, (f_{11})_t = 0; \\ g_{21} &= g_{22} = f_{22} = 0, (f_{21})_x = 0, (f_{21})_t = 6f_{31}; \\ (f_{31})_x &= (f_{31})_t = f_{32} = g_{32} = 0, g_{31} = -g_{11}; \\ f_{41} &= g_{42} = 0, f_{42} = -3f_{31}, g_{41} = 0. \end{aligned} \quad (4.11)$$

This reduced defining system is regular. Therefore, it can be sent to the Maple commu-

tative commands `rifsimp` and `initialdata`.

The Maple output shows that the super KdV equation has a 4 - dimensional symmetry group. Actually it is a 3 | 1 - dimensional supersymmetry group. As in the previous example, the diagram helps us to determin the dimension of the supersymmetry group directly.

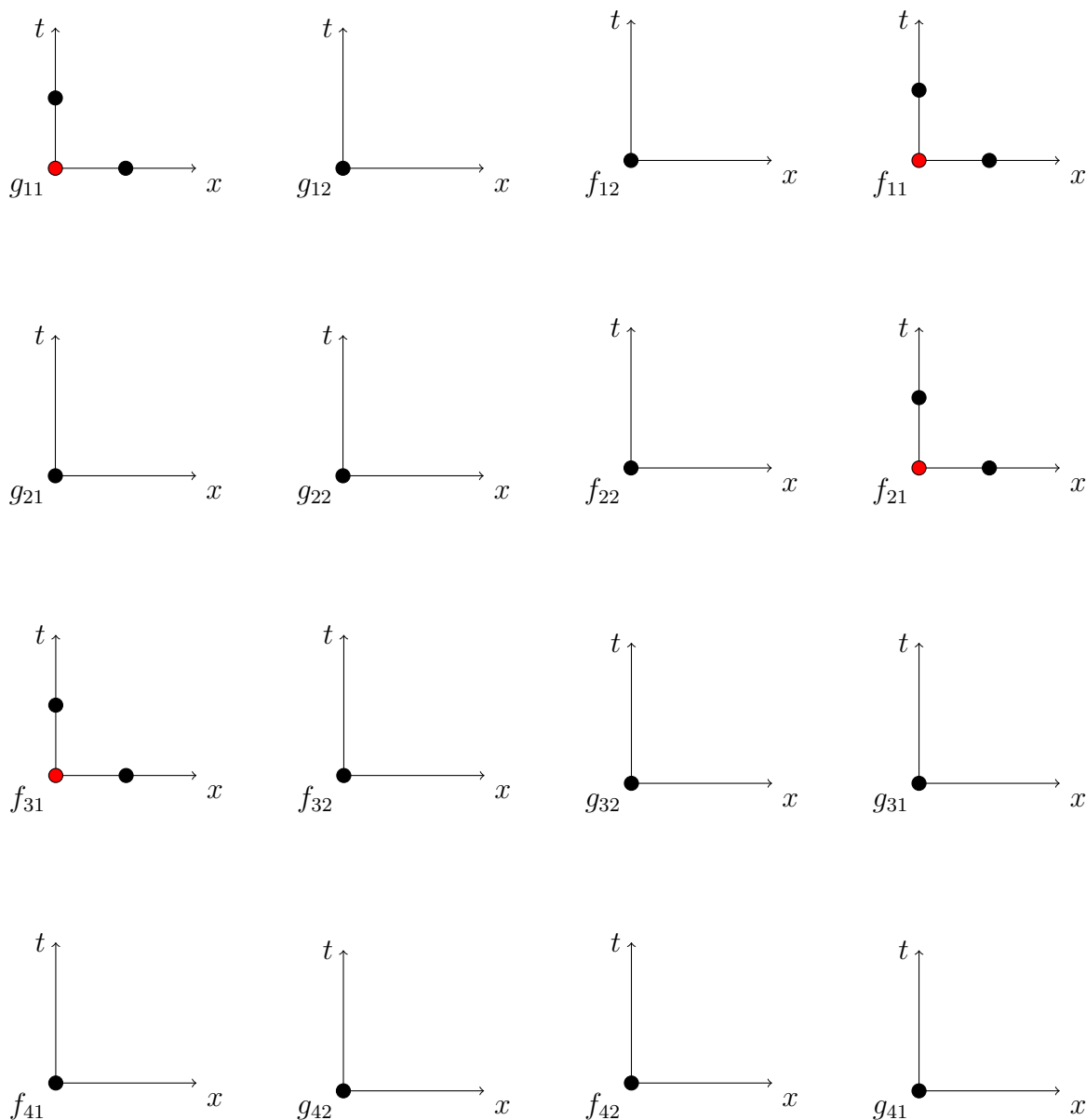


Figure 4.3.3: Dimension analysis diagram for the defining system of the super KdV.

It is easy to see that there are four red dots corresponding to 4 parametric derivatives

implying that it is a 4 dimensional symmetry group. The parametric derivatives are

$$g_{11}, f_{11}, f_{21}, f_{31}.$$

Among them, g_{11} is odd and other three functions are even. Hence, in more detail, the supersymmetry group is a $3 | 1$ - dimensional supersymmetry group.

4.4 Existence and uniqueness theorem

From now on, we are working on the reduced defining system which is the output system of the MONO expansion algorithm. The reduced defining system is regular (and monic) and does not have any odd independent variables.

Rust, Reid and Wittkopf [18] have proved the existence and uniqueness theorems for formal power series solutions of analytic differential systems. We adapt their result to the non-commutative case.

4.4.1 Super initial data mapping and super Riquier bases

We extend the definition of the initial data mapping in the commutative case [18] to super initial data mapping sID :

$$\begin{aligned} \text{sID} : \{x\} \cup \text{EvenPar}(\mathcal{S}_{\text{red}}) &\rightarrow \mathbb{F} \\ \text{OddPar}(\mathcal{S}_{\text{red}}) &\rightarrow \mathbb{G} \quad (\text{Grassmann numbers}). \end{aligned}$$

For $x^0 \in \mathbb{F}^n$, we say that sID is a specification of super initial data at x^0 if $\text{sID}(x) = x^0$.

For $\text{sID}(x) = x^0$, we mean

$$(\text{sID}(x_1), \text{sID}(x_2), \dots, \text{sID}(x_n)) = x^0.$$

This is a well-defined mapping. For any f in \mathcal{S}_{red} , evaluating f is

$$\text{sID}(f) = f(\text{sID}(X), \text{sID}(\text{Par}(\mathcal{S}_{\text{red}})), \text{Prin}(\mathcal{S}_{\text{red}})).$$

Riquier bases are the differential analogs of Gröbner bases. Since we are working on reduced defining systems, the super Riquier bases in this thesis differ from Riquier bases only in that they involve odd dependent variables.

4.4.2 Existence and uniqueness theorem

Since we have already reduced the system to be regular with no odd independent variables, the underlying Riquier bases theory is the same as the commutative case. We adapt the existence and uniqueness theorem in [18] to the non-commutative case as the following:

Theorem 4.4.1. *Suppose that \mathcal{M} is a super Riquier basis with respect to ranking \succ . Fix $x^0 \in \mathbb{F}^n$. Let sID be a specification of initial data for \mathcal{M} at x^0 such that $\text{sID}(f)$ is well-defined for all $f \in \mathcal{M}$. Then there is an unique solution*

$$\begin{aligned} u(x) &\in \mathbb{F}[[x - x^0]]^n, & \text{if } u(x) \text{ is even;} \\ u(x) &\in \mathbb{G}[[x - x^0]]^n, & \text{if } u(x) \text{ is odd,} \end{aligned}$$

to \mathcal{M} at x^0 such that $D_\alpha u^i(x^0) = \text{sID}(\delta_\alpha^i)$ for all $\delta_\alpha^i \in \text{Par}\mathcal{M}$.

4.5 Structure constants algorithm

4.5.1 Structure constants algorithm theorem and proof

Theorem 4.5.1. *Suppose that \mathcal{S} is a finite defining system with m_1 even infinitesimals and m_2 odd infinitesimals. Suppose that \mathcal{S}_{red} is the reduced defining system of \mathcal{S} and has $d_1(< \infty)$ even parametric derivatives and $d_2(< \infty)$ odd parametric derivatives. Then the structure constants can be algorithmically determined.*

Proof : For any finite defining system \mathcal{S} , one can write two supervector fields L_i and L_j as:

$$\begin{aligned} L_i &= \sum_{\ell_1=1}^{m_1} \text{EvenInf}_{\ell_1}^i \partial_{\text{EvenVar}_{\ell_1}} + \sum_{\ell_2=1}^{m_2} \text{OddInf}_{\ell_2}^i \partial_{\text{OddVar}_{\ell_2}}, \\ L_j &= \sum_{\ell_1=1}^{m_1} \text{EvenInf}_{\ell_1}^j \partial_{\text{EvenVar}_{\ell_1}} + \sum_{\ell_2=1}^{m_2} \text{OddInf}_{\ell_2}^j \partial_{\text{OddVar}_{\ell_2}}, \end{aligned}$$

where $1 \leq i, j \leq d_1 + d_2$.

On one hand, by computing the Lie superbracket of L_i and L_j and by the closure property of Lie superalgebra, we have

$$[L_i, L_j] = \sum_{\ell_1=1}^{m_1} A^{\ell_1} \partial_{\text{EvenVar}_{\ell_1}} + \sum_{\ell_2=1}^{m_2} B^{\ell_2} \partial_{\text{OddVar}_{\ell_2}}. \quad (4.12)$$

On the other hand, by the definition of the structure constants, we have

$$\begin{aligned} [L_i, L_j] &= \sum_{k=1}^{d_1+d_2} c_{ij}^k L_k \\ &= \sum_{k=1}^{d_1+d_2} c_{ij}^k \left(\sum_{\ell_1=1}^{m_1} \text{EvenInf}_{\ell_1}^k \partial_{\text{EvenVar}_{\ell_1}} + \sum_{\ell_2=1}^{m_2} \text{OddInf}_{\ell_2}^k \partial_{\text{OddVar}_{\ell_2}} \right) \\ &= \sum_{\ell_1=1}^{m_1} \left(\sum_{k=1}^{d_1+d_2} c_{ij}^k \text{EvenInf}_{\ell_1}^k \right) \partial_{\text{EvenVar}_{\ell_1}} + \sum_{\ell_2=1}^{m_2} \left(\sum_{k=1}^{d_1+d_2} c_{ij}^k \text{OddInf}_{\ell_2}^k \right) \partial_{\text{OddVar}_{\ell_2}}. \end{aligned} \quad (4.13)$$

By equating the coefficients of the same operators in (4.12) and (4.13), we obtain a system with $m_1 + m_2$ equations of the form

$$m_1 \text{ equations} \quad \left\{ \begin{array}{l} \sum_{k=1}^{d_1+d_2} c_{ij}^k \text{EvenInf}_1^k = A^1, \\ \dots, \\ \sum_{k=1}^{d_1+d_2} c_{ij}^k \text{EvenInf}_{m_1}^k = A^{m_1}, \end{array} \right. \quad (4.14)$$

$$m_2 \text{ equations } \left\{ \begin{array}{l} \sum_{k=1}^{d_1+d_2} c_{ij}^k \text{OddInf}_1^k = B^1, \\ \dots, \\ \sum_{k=1}^{d_1+d_2} c_{ij}^k \text{OddInf}_{m_2}^k = B^{m_2}. \end{array} \right. \quad (4.15)$$

Since $\text{Par}(\mathcal{S})$ consists of some (or all) infinitesimals and their derivatives, one can always differentiate the above systems (4.14) and (4.15) with respect to $\text{Par}(\mathcal{S})$. Suppose that $\text{Par}(\mathcal{S}) = \{Par_1, \dots, Par_{d_1+d_2}\}$. After differentiation, one gets a new system consisting of all parametric derivatives of \mathcal{S} :

$$\left\{ \begin{array}{l} \sum_{k=1}^{d_1+d_2} c_{ij}^k Par_1^k = C^1, \\ \dots, \\ \sum_{k=1}^{d_1+d_2} c_{ij}^k Par_{d_1+d_2}^k = C^{d_1+d_2}, \end{array} \right. \quad (4.16)$$

where $C^1, \dots, C^{d_1+d_2}$ are the derivatives of some (or all) of the A^1, \dots, A^{m_1} and B_1, \dots, B^{m_2} . Substituting the MONO expansion expressions for the infinitesimals of \mathcal{S} and then computing coefficients of odd variable monomials for each equation, we get a new system containing all the parametric derivatives of \mathcal{S}_{red} . Suppose that $\mathcal{S}_{\text{red}} = \{\widehat{Par}_1, \dots, \widehat{Par}_{d_1+d_2}\}$. Then the system (4.25) becomes

$$\left\{ \begin{array}{l} \sum_{k=1}^{d_1+d_2} c_{ij}^k \widehat{Par}_1^k = \widehat{C}^1, \\ \dots, \\ \sum_{k=1}^{d_1+d_2} c_{ij}^k \widehat{Par}_{d_1+d_2}^k = \widehat{C}^{d_1+d_2}. \end{array} \right. \quad (4.17)$$

Next we list all the elements in $\text{Par}(\mathcal{S}_{\text{red}}) = \{\widehat{Par}_1, \widehat{Par}_2, \dots, \widehat{Par}_{d_1+d_2}\}$ in a certain order, say δ , for example,

$$\{\text{Even}\widehat{Par}_1, \dots, \text{Even}\widehat{Par}_{d_1}, \text{Odd}\widehat{Par}_1, \dots, \text{Odd}\widehat{Par}_{d_2}\}.$$

Most importantly, even parameters are listed before odd parameters. Note that according to convention for Lie superalgebra commutator tables, we always put even parametric derivatives before odd parametric derivatives. That yields a Lie superalgebra commutator table with even basis elements listed followed by odd basis elements. Rearranging the

equations in the system (4.17) in the same order δ , one has

$$\left\{ \begin{array}{l} \sum_{k=1}^{m_1+m_2} c_{ij}^k \widehat{\text{EvenPar}}_1^k = \tilde{C}^1, \\ \dots, \\ \sum_{k=1}^{m_1+m_2} c_{ij}^k \widehat{\text{EvenPar}}_{d_1}^k = \tilde{C}^{d_1}, \\ \sum_{k=1}^{m_1+m_2} c_{ij}^k \widehat{\text{OddPar}}_1^k = \tilde{C}^{d_1+1}, \\ \dots, \\ \sum_{k=1}^{m_1+m_2} c_{ij}^k \widehat{\text{OddPar}}_{d_2}^k = \tilde{C}^{d_1+d_2}, \end{array} \right. \quad (4.18)$$

where $\{\tilde{C}^1, \dots, \tilde{C}^{d_1+d_2}\} = \{\hat{C}^1, \dots, \hat{C}^{d_1+d_2}\}$ in sense of sets.

We then simplify \tilde{C}^1 to $\tilde{C}^{d_1+d_2}$ modulo the rifsimp form of \mathcal{S}_{red} and denote the simplified forms with same notation, \tilde{C}^1 to $\tilde{C}^{d_1+d_2}$. Hence, there are only parametric derivatives of \mathcal{S}_{red} appearing in each expression for \tilde{C}^1 to $\tilde{C}^{d_1+d_2}$. According to the properties of Lie superbrackets, the parametric derivatives must always appear in bilinear pairs associated coefficient in \mathbb{F} , for example,

$$a(\widehat{\text{Par}}_1^i \widehat{\text{Par}}_2^j - \widehat{\text{Par}}_1^j \widehat{\text{Par}}_2^i). \quad (4.19)$$

By the super Riquier existence and uniqueness theorem, providing initial data for the parametric derivatives of \mathcal{S}_{red} uniquely determines the structure constants. For example, providing two copies of initial data $a_1, \dots, a_{d_1+d_2}$ and $b_1, \dots, b_{d_1+d_2}$ for all parametric derivatives of \mathcal{S}_{red} of system (4.18), the bilinear pair of parametric derivatives (4.19) becomes

$$a(a_1 b_2 - b_1 a_2). \quad (4.20)$$

Equation (4.20) implies

$$c_{12}^j = a,$$

if (4.19) appears in the j -th equation in the system (4.27). Similarly, we are able to read off all the nonzero structure constants from every initial data pair of form (4.20).

4.5.2 Structure constant algorithm

Now we put Theorem 4.5.1 and its proof in the previous section into the form of an algorithm.

Input: \mathcal{S} , $\text{Par}(\mathcal{S})$, $\text{EvenInf}(\mathcal{S})$, $\text{OddInf}(\mathcal{S})$; \mathcal{S}_{red} , $\text{Par}(\mathcal{S}_{\text{red}})$.

1. Write two supersymmetry vector fields L_i, L_j , where

$$L_i = \sum_{\ell_1=1}^{m_1} \Phi_{\ell_1}^i \partial_{X_{\ell_1}} + \sum_{\ell_2=1}^{m_2} \Psi_{\ell_2}^i \partial_{Y_{\ell_2}}$$

and L_j has the same form.

2. Take their Lie superbracket $[L_i, L_j]$ to yield

$$[L_i, L_j] = \sum_{\ell_1=1}^{m_1} A^{\ell_1} \partial_{X_{\ell_1}} + \sum_{\ell_2=1}^{m_2} B^{\ell_2} \partial_{Y_{\ell_2}}. \quad (4.21)$$

3. By the definition of structure constants we have

$$\begin{aligned} [L_i, L_j] &= \sum_{k=1}^{d_1+d_2} c_{ij}^k L_k \\ &= \sum_{\ell_1=1}^{m_1} \left(\sum_{k=1}^{d_1+d_2} c_{ij}^k \Phi_{\ell_1}^k \right) \partial_{X_{\ell_1}} + \sum_{\ell_2=1}^{m_2} \left(\sum_{k=1}^{d_1+d_2} c_{ij}^k \Psi_{\ell_2}^k \right) \partial_{Y_{\ell_2}}. \end{aligned} \quad (4.22)$$

4. The equations in (4.21) and (4.22) form a linear system with m_1+m_2 equations.

$$m_1 \text{ equations} \begin{cases} \sum_{k=1}^{d_1+d_2} c_{ij}^k \Phi_1^k = A^1, \\ \dots, \\ \sum_{k=1}^{d_1+d_2} c_{ij}^k \Phi_{m_1}^k = A^{m_1}, \end{cases} \quad (4.23)$$

$$m_2 \text{ equations} \begin{cases} \sum_{k=1}^{d_1+d_2} c_{ij}^k \Psi_1^k = B^1, \\ \dots, \\ \sum_{k=1}^{d_1+d_2} c_{ij}^k \Psi_{m_2}^k = B^{m_2}. \end{cases} \quad (4.24)$$

5. Differentiate (4.23) and (4.24) w.r.t. $\text{Par}(\mathcal{S}) = \{P_1, \dots, P_{d_1+d_2}\}$ to obtain:

$$\begin{cases} \sum_{k=1}^{d_1+d_2} c_{ij}^k P_1^k = C^1, \\ \dots, \\ \sum_{k=1}^{d_1+d_2} c_{ij}^k P_{d_1+d_2}^k = C^{d_1+d_2}, \end{cases} \quad (4.25)$$

6. Substitute the MONO expansions of $\text{EvenInf}(\mathcal{S})$ and $\text{OddInf}(\mathcal{S})$ in step 5 and equate the coefficients of odd variable monomials to obtain

$$\begin{cases} \sum_{k=1}^{d_1+d_2} c_{ij}^k \hat{P}_1^k = \hat{C}^1, \\ \dots, \\ \dots, \\ \sum_{k=1}^{d_1+d_2} c_{ij}^k \hat{P}_{d_1+d_2}^k = \hat{C}^{d_1+d_2}, \end{cases} \quad (4.26)$$

where $\text{Par}(\mathcal{S}_{\text{red}}) = \{\hat{P}_1, \dots, \hat{P}_{d_1+d_2}\}$

7. Select an order δ on the elements of $\text{Par}(\mathcal{S}_{\text{red}})$:

$$\underbrace{\{\tilde{P}_1, \dots, \tilde{P}_{d_1}\}}_{\text{even}}, \underbrace{\{\tilde{P}_{d_1+1}, \dots, \tilde{P}_{d_1+d_2}\}}_{\text{odd}},$$

and rearrange (4.26) in the order δ :

$$\begin{cases} \sum_{k=1}^{m_1+m_2} c_{ij}^k \tilde{P}_1^k = \tilde{C}^1, \\ \dots, \\ \sum_{k=1}^{m_1+m_2} c_{ij}^k \tilde{P}_{d_1}^k = \tilde{C}^{d_1}, \\ \sum_{k=1}^{m_1+m_2} c_{ij}^k \tilde{P}_{d_1+1}^k = \tilde{C}^{d_1+1}, \\ \dots, \\ \sum_{k=1}^{m_1+m_2} c_{ij}^k \tilde{P}_{d_1+d_2}^k = \tilde{C}^{d_1+d_2}. \end{cases} \quad (4.27)$$

8. Provide two copies of initial data $a_1, \dots, a_{d_1+d_2}$ and $b_1, \dots, b_{d_1+d_2}$ for the $\text{Par}(\mathcal{S}_{\text{red}})$ in the given order and read-off the nonzero structure constants as coefficients.

Output: Nonzero structure constants c_{ij}^k 's and the supercommutator table.

4.5.3 Applications of the structure constants algorithm

We apply the structure constant algorithm to the defining system of the example $Q_{xx} = 0$ and the super KdV equation.

Example 4.5.2. *Apply the structure constant algorithm to example the $Q_{xx} = 0$. This example does not contain any irregular super differential equation.*

Input: \mathcal{S} in standard form

$$\mathcal{S} = \{\Lambda_{xx} = 0, \Lambda_{xQ} = \frac{1}{2}\Xi_{xx}, \Xi_{QQ} = 0, \Lambda_{QQ} = 0, \Xi_{xxx} = 0, \Xi_{xxQ} = 0\},$$

$$m_1 = 1, m_2 = 1, \{\Xi, \Xi_x, \Lambda_Q, \Xi_{xx}, \Lambda, \Xi_Q, \Lambda_x, \Xi_{xQ}\}, d_1 = 4, d_2 = 4.$$

1. Write two general supersymmetry vector fields L_i, L_j ,

$$L_i = \Xi^i \partial_x + \Lambda^i \partial_Q \quad \text{and} \quad L_j = \Xi^j \partial_x + \Lambda^j \partial_Q.$$

2. Take their Lie superbracket

$$\begin{aligned} [L_i, L_j] &= \overbrace{(\Xi^i \Xi_x^j - \Xi^j \Xi_x^i + \Lambda^i \Xi_Q^j - \Lambda^j \Xi_Q^i)}^A \partial_x \\ &\quad + \overbrace{(\Xi^i \Lambda_x^j - \Xi^j \Lambda_x^i + \Lambda^i \Lambda_Q^j - \Lambda^j \Lambda_Q^i)}^B \partial_Q. \end{aligned}$$

3. Expand the commutator in terms of structure constants

$$\begin{aligned} [L_i, L_j] &= \sum_{k=1}^8 c_{ij}^k L_k \\ &= \left(\sum_{k=1}^8 c_{ij}^k \Xi^k \right) \partial_x + \left(\sum_{k=1}^8 c_{ij}^k \Lambda^k \right) \partial_Q. \end{aligned}$$

4. The results of step 2 and 3 yield a linear system with 1 + 1 equations:

$$\sum_{k=1}^8 c_{ij}^k \Xi^k = \Xi^i \Xi_x^j - \Xi^j \Xi_x^i + \Lambda^i \Xi_Q^j - \Lambda^j \Xi_Q^i, \quad (4.28)$$

$$\sum_{k=1}^8 c_{ij}^k \Lambda^k = \Xi^i \Lambda_x^j - \Xi^j \Lambda_x^i + \Lambda^i \Lambda_Q^j - \Lambda^j \Lambda_Q^i. \quad (4.29)$$

5. Arrange the parametric derivatives in the order δ :

$$\{\Xi, \Xi_x, \Lambda_Q, \Xi_{xx}, \Lambda, \Xi_Q, \Lambda_x, \Xi_{xQ}\}.$$

6. Skipped. Since input system is regular, it is not necessary to do the MONO expansion.

7. Keep differentiating the two equation in step 5 w.r.t. parametric derivatives until we have 8 equations. Then simplify them with respect to the modulo

parametric derivatives and arrange them in the order δ ,

$$\begin{aligned}
\sum_{k=1}^8 c_{ij}^k \Xi^k &= (\Xi^i \Xi_x^j - \Xi^j \Xi_x^i) + (\Lambda^i \Xi_Q^j - \Lambda^j \Xi_Q^i), \\
\sum_{k=1}^8 c_{ij}^k \Xi_x^k &= (\Xi^i \Xi_{xx}^j - \Xi^j \Xi_{xx}^i) + (\Xi_Q^i \Lambda_x^j - \Xi_Q^j \Lambda_x^i) + (\Lambda^i \Xi_{xQ}^j - \Lambda^j \Xi_{xQ}^i), \\
\sum_{k=1}^8 c_{ij}^k \Lambda_Q^k &= (\Xi_Q^i \Lambda_x^j - \Xi_Q^j \Lambda_x^i) + \frac{1}{2}(\Xi^i \Xi_{xx}^j - \Xi^j \Xi_{xx}^i) + (\Lambda_Q^i \Lambda_Q^j - \Lambda_Q^j \Lambda_Q^i), \\
\sum_{k=1}^8 c_{ij}^k \Xi_{xx}^k &= (\Xi_x^i \Xi_{xx}^j - \Xi_x^j \Xi_{xx}^i) + 2(\Lambda_x^i \Xi_{xQ}^j - \Lambda_x^j \Xi_{xQ}^i), \\
\sum_{k=1}^8 c_{ij}^k \Lambda^k &= (\Xi^i \Lambda_x^j - \Xi^j \Lambda_x^i) + (\Lambda^i \Lambda_Q^j - \Lambda^j \Lambda_Q^i), \\
\sum_{k=1}^8 c_{ij}^k \Xi_Q^k &= (\Xi_Q^i \Xi_x^j - \Xi_Q^j \Xi_x^i) + (\Xi^i \Xi_{xQ}^j - \Xi^j \Xi_{xQ}^i) + (\Lambda_Q^i \Xi_Q^j - \Lambda_Q^j \Xi_Q^i), \\
\sum_{k=1}^8 c_{ij}^k \Lambda_x^k &= (\Xi_x^i \Lambda_x^j - \Xi_x^j \Lambda_x^i) + (\Lambda_x^i \Lambda_Q^j - \Lambda_x^j \Lambda_Q^i) + \frac{1}{2}(\Lambda^i \Xi_{xx}^j - \Lambda^j \Xi_{xx}^i), \\
\sum_{k=1}^8 c_{ij}^k \Xi_{xQ}^k &= -\frac{1}{2}(\Xi_{xx}^i \Xi_Q^j - \Xi_{xx}^j \Xi_Q^i) + (\Lambda_Q^i \Xi_{xQ}^j - \Lambda_Q^j \Xi_{xQ}^i).
\end{aligned}$$

8. Provide two copies of initial data a_1, \dots, a_8 and b_1, \dots, b_8 and substitute them in

step 7,

$$\begin{aligned}
\sum_{k=1}^8 c_{ij}^k \Xi^k &= \underbrace{(\Xi^i \Xi_x^j - \Xi^j \Xi_x^i)}_{a_1 b_2 - b_1 a_2} + \underbrace{(\Lambda^i \Xi_Q^j - \Lambda^j \Xi_Q^i)}_{a_5 b_6 - b_5 a_6} \\
&\longrightarrow c_{12}^1 = 1, c_{56}^1 = 1, \\
\sum_{k=1}^8 c_{ij}^k \Xi_x^k &= \underbrace{(\Xi^i \Xi_{xx}^j - \Xi^j \Xi_{xx}^i)}_{a_1 b_4 - b_1 a_4} + \underbrace{\Xi_Q^i \Lambda_x^j - \Xi_Q^j \Lambda_x^i}_{a_6 b_7 - b_6 a_7} + \underbrace{\Lambda^i \Xi_{xQ}^j - \Lambda^j \Xi_{xQ}^i}_{a_5 b_8 - b_5 a_8} \\
&\longrightarrow c_{14}^2 = 1, c_{67}^2 = 1, c_{58}^2 = 1, \\
\sum_{k=1}^8 c_{ij}^k \Lambda_Q^k &= \underbrace{(\Xi_Q^i \Lambda_x^j - \Xi_Q^j \Lambda_x^i)}_{a_6 b_7 - b_6 a_7} + \frac{1}{2} \underbrace{(\Xi^i \Xi_{xx}^j - \Xi^j \Xi_{xx}^i)}_{a_1 b_4 - b_1 a_4} \\
&\longrightarrow c_{67}^3 = 1, c_{14}^3 = 1/2, \\
\sum_{k=1}^8 c_{ij}^k \Xi_{xx}^k &= \underbrace{(\Xi_x^i \Xi_{xx}^j - \Xi_x^j \Xi_{xx}^i)}_{a_2 b_4 - b_2 a_4} + 2 \underbrace{(\Lambda_x^i \Xi_{xQ}^j - \Lambda_x^j \Xi_{xQ}^i)}_{a_7 b_8 - b_7 a_8} \\
&\longrightarrow c_{24}^4 = 1, c_{78}^4 = 2, \\
\sum_{k=1}^8 c_{ij}^k \Lambda^k &= \underbrace{(\Xi^i \Lambda_x^j - \Xi^j \Lambda_x^i)}_{a_1 b_7 - b_1 a_7} + \underbrace{(\Lambda^i \Lambda_Q^j - \Lambda^j \Lambda_Q^i)}_{a_5 b_3 - b_5 a_3}, \\
&\longrightarrow c_{17}^5 = 1, c_{53}^5 = 1, \\
\sum_{k=1}^8 c_{ij}^k \Xi_Q^k &= \underbrace{(\Xi_Q^i \Xi_x^j - \Xi_Q^j \Xi_x^i)}_{a_6 b_2 - b_6 a_2} + \underbrace{(\Xi^i \Xi_{xQ}^j - \Xi^j \Xi_{xQ}^i)}_{a_1 b_8 - b_1 a_8} + \underbrace{(\Lambda_Q^i \Xi_Q^j - \Lambda_Q^j \Xi_Q^i)}_{a_3 b_6 - b_3 a_6}, \\
&\longrightarrow c_{62}^6 = 1, c_{18}^6 = 1, c_{36}^6 = 1, \\
\sum_{k=1}^8 c_{ij}^k \Lambda_x^k &= \underbrace{(\Xi_x^i \Lambda_x^j - \Xi_x^j \Lambda_x^i)}_{a_2 b_7 - b_2 a_7} + \underbrace{(\Lambda_x^i \Lambda_Q^j - \Lambda_x^j \Lambda_Q^i)}_{a_7 b_3 - b_7 a_3} + \frac{1}{2} \underbrace{(\Lambda^i \Xi_{xx}^j - \Lambda^j \Xi_{xx}^i)}_{a_5 b_4 - b_5 a_4}, \\
&\longrightarrow c_{27}^7 = 1, c_{73}^7 = 1, c_{54}^7 = 1/2, \\
\sum_{k=1}^8 c_{ij}^k \Xi_{xQ}^k &= -\frac{1}{2} \underbrace{(\Xi_{xx}^i \Xi_Q^j - \Xi_{xx}^j \Xi_Q^i)}_{a_4 b_6 - b_4 a_6} + \underbrace{(\Lambda_Q^i \Xi_{xQ}^j - \Lambda_Q^j \Xi_{xQ}^i)}_{a_3 b_8 - b_3 a_8}, \\
&\longrightarrow c_{46}^8 = -1/2, c_{38}^8 = 1.
\end{aligned}$$

Output: Read off the all nonzero structure constansts c_{ij}^k ' s. They are $c_{12}^1 = 1, c_{56}^1 = 1, c_{14}^2 = 1, c_{67}^2 = 1, c_{58}^2 = 1, c_{67}^3 = 1, c_{14}^3 = 1/2, c_{24}^4 = 1, c_{78}^4 = 2, c_{17}^5 = 1, c_{53}^5 = 1, c_{62}^6 = 1, c_{18}^6 = 1, c_{36}^6 = 1, c_{27}^7 = 1, c_{73}^7 = 1, c_{54}^7 = 1/2, c_{46}^8 = -1/2, c_{38}^8 = 1.$

The structure constants computation above implies that

$$[L_1, L_2] = c_{12}^1 L_1 + c_{12}^2 L_2 + c_{12}^3 L_3 + c_{12}^4 L_4 + c_{12}^5 L_5 + c_{12}^6 L_6 + c_{12}^7 L_7 + c_{12}^8 L_8 = L_1,$$

$$[L_1, L_3] = 0,$$

$$\begin{aligned} [L_1, L_4] &= c_{14}^1 L_1 + c_{14}^2 L_2 + c_{14}^3 L_3 + c_{14}^4 L_4 + c_{14}^5 L_5 + c_{14}^6 L_6 + c_{14}^7 L_7 + c_{14}^8 L_8 \\ &= L_2 + \frac{1}{2} L_3, \end{aligned}$$

$$[L_1, L_5] = 0,$$

$$[L_1, L_6] = 0,$$

$$[L_1, L_7] = c_{17}^1 L_1 + c_{17}^2 L_2 + c_{17}^3 L_3 + c_{17}^4 L_4 + c_{17}^5 L_5 + c_{17}^6 L_6 + c_{17}^7 L_7 + c_{17}^8 L_8 = L_5,$$

$$[L_1, L_8] = c_{18}^1 L_1 + c_{18}^2 L_2 + c_{18}^3 L_3 + c_{18}^4 L_4 + c_{18}^5 L_5 + c_{18}^6 L_6 + c_{18}^7 L_7 + c_{18}^8 L_8 = L_6,$$

$$[L_2, L_3] = 0,$$

$$[L_2, L_4] = c_{24}^1 L_1 + c_{24}^2 L_2 + c_{24}^3 L_3 + c_{24}^4 L_4 + c_{24}^5 L_5 + c_{24}^6 L_6 + c_{24}^7 L_7 + c_{24}^8 L_8 = L_4,$$

$$[L_2, L_5] = 0,$$

$$[L_2, L_6] = c_{26}^1 L_1 + c_{26}^2 L_2 + c_{26}^3 L_3 + c_{26}^4 L_4 + c_{26}^5 L_5 + c_{26}^6 L_6 + c_{26}^7 L_7 + c_{26}^8 L_8 = -L_6,$$

$$[L_2, L_7] = c_{27}^1 L_1 + c_{27}^2 L_2 + c_{27}^3 L_3 + c_{27}^4 L_4 + c_{27}^5 L_5 + c_{27}^6 L_6 + c_{27}^7 L_7 + c_{27}^8 L_8 = L_7,$$

$$[L_2, L_8] = 0,$$

$$[L_3, L_4] = 0,$$

$$[L_3, L_5] = c_{35}^1 L_1 + c_{35}^2 L_2 + c_{35}^3 L_3 + c_{35}^4 L_4 + c_{35}^5 L_5 + c_{35}^6 L_6 + c_{35}^7 L_7 + c_{35}^8 L_8 = -L_5,$$

$$[L_3, L_6] = c_{36}^1 L_1 + c_{36}^2 L_2 + c_{36}^3 L_3 + c_{36}^4 L_4 + c_{36}^5 L_5 + c_{36}^6 L_6 + c_{36}^7 L_7 + c_{36}^8 L_8 = L_6,$$

$$[L_3, L_7] = c_{37}^1 L_1 + c_{37}^2 L_2 + c_{37}^3 L_3 + c_{37}^4 L_4 + c_{37}^5 L_5 + c_{37}^6 L_6 + c_{37}^7 L_7 + c_{37}^8 L_8 = -L_7,$$

$$[L_3, L_8] = c_{38}^1 L_1 + c_{38}^2 L_2 + c_{38}^3 L_3 + c_{38}^4 L_4 + c_{38}^5 L_5 + c_{38}^6 L_6 + c_{38}^7 L_7 + c_{38}^8 L_8 = L_8,$$

$$[L_4, L_5] = c_{45}^1 L_1 + c_{45}^2 L_2 + c_{45}^3 L_3 + c_{45}^4 L_4 + c_{45}^5 L_5 + c_{45}^6 L_6 + c_{45}^7 L_7 + c_{45}^8 L_8 = -\frac{1}{2} L_7,$$

$$[L_4, L_6] = c_{46}^1 L_1 + c_{46}^2 L_2 + c_{46}^3 L_3 + c_{46}^4 L_4 + c_{46}^5 L_5 + c_{46}^6 L_6 + c_{46}^7 L_7 + c_{46}^8 L_8 = -\frac{1}{2} L_8,$$

$$[L_4, L_7] = 0,$$

$$[L_4, L_8] = 0,$$

$$[L_5, L_5] = 0,$$

$$\begin{aligned}
[L_5, L_6] &= c_{56}^1 L_1 + c_{56}^2 L_2 + c_{56}^3 L_3 + c_{56}^4 L_4 + c_{56}^5 L_5 + c_{56}^6 L_6 + c_{56}^7 L_7 + c_{56}^8 L_8 = L_1, \\
[L_5, L_7] &= 0, \\
[L_5, L_8] &= c_{58}^1 L_1 + c_{58}^2 L_2 + c_{58}^3 L_3 + c_{58}^4 L_4 + c_{58}^5 L_5 + c_{58}^6 L_6 + c_{58}^7 L_7 + c_{58}^8 L_8 = L_2, \\
[L_6, L_6] &= 0, \\
[L_6, L_7] &= c_{67}^1 L_1 + c_{67}^2 L_2 + c_{67}^3 L_3 + c_{67}^4 L_4 + c_{67}^5 L_5 + c_{67}^6 L_6 + c_{67}^7 L_7 + c_{67}^8 L_8 = L_1 + L_2, \\
[L_6, L_8] &= 0, \\
[L_7, L_7] &= 0, \\
[L_7, L_8] &= c_{78}^1 L_1 + c_{78}^2 L_2 + c_{78}^3 L_3 + c_{78}^4 L_4 + c_{78}^5 L_5 + c_{78}^6 L_6 + c_{78}^7 L_7 + c_{78}^8 L_8 = 2L_4, \\
[L_8, L_8] &= 0.
\end{aligned}$$

The supercommutator table is

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
L_1	0	L_1	0	$L_2 + 1/2L_3$	0	0	L_5	L_6
L_2	$-L_1$	0	0	L_4	0	$-L_6$	L_7	0
L_3	0	0	0	0	$-L_5$	L_6	$-L_7$	L_8
L_4	$-L_2 - 1/2L_3$	$-L_4$	0	0	$-1/2L_7$	$-1/2L_8$	0	0
L_5	0	0	L_5	$1/2L_7$	0	L_1	0	L_2
L_6	0	L_6	$-L_6$	$1/2L_8$	L_1	0	$L_2 + L_3$	0
L_7	$-L_5$	$-L_7$	L_7	0	0	$L_2 + L_3$	0	$2L_4$
L_8	$-L_6$	0	$-L_8$	0	L_2	0	$2L_4$	0

Table 4.5.1: Supercommutator table for the defining system of $Q_{xx} = 0$.

Secondly, let us find supercommutator table for the defining system of the super KdV equation by applying the Structure Constants Algorithm.

Example 4.5.3. *Apply the structure constant algorithm to the defining system of the super KdV equation system.*

Recall that in Section 4.3.2 we obtained the reduced defining system of the super KdV equation (4.11), which is

$$\begin{aligned}(g_{11})_x &= (g_{11})_t = g_{12} = f_{12} = 0, (f_{11})_x = 2f_{31}, (f_{11})_t = 0; \\ g_{21} &= g_{22} = f_{22} = 0, (f_{21})_x = 0, (f_{21})_t = 6f_{31}; \\ (f_{31})_x &= (f_{31})_t = f_{32} = g_{32} = 0, g_{31} = -g_{11}; \\ f_{41} &= g_{42} = 0, f_{42} = -3f_{31}, g_{41} = 0.\end{aligned}$$

Then we have a simplified version of the infinitesimals Ξ^1 , Ξ^2 , Γ , Λ as the following:

$$\begin{aligned}\Xi^1 &= g_{11}\theta + f_{11}, \\ \Xi^2 &= f_{21}, \\ \Gamma &= f_{31}\theta - g_{11}, \\ \Lambda &= -3f_{31}Q.\end{aligned}$$

Now we follow the steps of the Structure Constants Algorithm.

Input:

1. Write down two supersymmetry vector fields L_i, L_j ,

$$\begin{aligned}L_i &= \Xi^{1i}\partial_x + \Xi^{2i}\partial_t + \Gamma^i\partial_\theta + \Lambda^i\partial_Q, \\ L_j &= \Xi^{1j}\partial_x + \Xi^{2j}\partial_t + \Gamma^j\partial_\theta + \Lambda^j\partial_Q.\end{aligned}$$

2. Work out their Lie superbracket

$$\begin{aligned}[L_i, L_j] &= [\Xi^{1i}\partial_x + \Xi^{2i}\partial_t + \Gamma^i\partial_\theta + \Lambda^i\partial_Q, \Xi^{1j}\partial_x + \Xi^{2j}\partial_t + \Gamma^j\partial_\theta + \Lambda^j\partial_Q] \\ &= ((g_{11}^j f_{31}^j - g_{11}^j f_{31}^i)\theta + 2(f_{11}^i f_{31}^j - f_{11}^j f_{31}^i) + (g_{11}^i g_{11}^j - g_{11}^j g_{11}^i))\partial_x \\ &\quad + 6(f_{21}^i f_{31}^j - f_{21}^j f_{31}^i)\partial_t - (g_{11}^i f_{31}^j - g_{11}^j f_{31}^i)\partial_\theta.\end{aligned}$$

3. Expand the supercommutations in terms of structure constants

$$\begin{aligned}
[L_i, L_j] &= \sum_{k=1}^4 C_{ij}^k L_k \\
&= \sum_{k=1}^4 C_{ij}^k (\Xi^{1k} \partial_x + \Xi^{2k} \partial_t + \Gamma^k \partial_\theta + \Lambda^k \partial_Q) \\
&= \sum_{k=1}^4 C_{ij}^k (g_{11}^k \theta + f_{11}^k) \partial_x + \sum_{k=1}^4 C_{ij}^k f_{21}^k \partial_t \\
&\quad + \sum_{k=1}^4 C_{ij}^k (f_{31}^k \theta - g_{11}^k) \partial_\theta + \sum_{k=1}^4 C_{ij}^k (-3f_{31}^k Q) \partial_Q.
\end{aligned}$$

4. Computing and equating the coefficients for the same operator in (4.30) and (4.30) yields a linear system with 2 + 2 equations:

$$\begin{aligned}
\sum_{k=1}^4 C_{ij}^k (g_{11}^k \theta + f_{11}^k) &= (g_{11}^j f_{31}^j - g_{11}^j f_{31}^i) \theta \\
&\quad + 2(f_{11}^i f_{31}^j - f_{11}^j f_{31}^i) + (g_{11}^i g_{11}^j - g_{11}^j g_{11}^i). \tag{4.30}
\end{aligned}$$

$$\sum_{k=1}^4 C_{ij}^k f_{21}^k = 6(f_{21}^i f_{31}^j - f_{21}^j f_{31}^i), \tag{4.31}$$

$$\sum_{k=1}^4 C_{ij}^k (f_{31}^k \theta - g_{11}^k) = g_{11}^i f_{31}^j - g_{11}^j f_{31}^i, \tag{4.32}$$

$$\sum_{k=1}^4 C_{ij}^k (-3f_{31}^k Q) = 0. \tag{4.33}$$

Old variable θ remains in the system (4.31-4.33). Continue to equate coefficients of odd variable monomials for (4.31) and (4.33) until we get a new system

$$\left\{ \begin{array}{l}
\sum_{k=1}^4 C_{ij}^k g_{11}^k = g_{11}^i f_{31}^j - g_{11}^j f_{31}^i, \\
\sum_{k=1}^4 C_{ij}^k f_{11}^k = 2(f_{11}^i f_{31}^j - f_{11}^j f_{31}^i) + (g_{11}^i g_{11}^j - g_{11}^j g_{11}^i), \\
\sum_{k=1}^4 C_{ij}^k f_{21}^k = 6(f_{21}^i f_{31}^j - f_{21}^j f_{31}^i), \\
\sum_{k=1}^4 C_{ij}^k f_{31}^k = 0,
\end{array} \right. \tag{4.34}$$

without any odd independent variables.

5. Skipped. Since the new system (4.35) contains purely parametric derivatives, one does not need to differentiate it with respect to $\text{Par}(\mathcal{S})$.
6. Skipped.
7. Set an order *delta* for parametric derivatives, $\{f_{11}, f_{21}, f_{31}, g_{11}\}$ and rearrange the linear system in the given order *delta*:

$$\left\{ \begin{array}{l} \sum_{k=1}^4 C_{ij}^k f_{11}^k = 2(f_{11}^i f_{31}^j - f_{11}^j f_{31}^i) + (g_{11}^i g_{11}^j - g_{11}^j g_{11}^i), \\ \sum_{k=1}^4 C_{ij}^k f_{21}^k = 6(f_{21}^i f_{31}^j - f_{21}^j f_{31}^i), \\ \sum_{k=1}^4 C_{ij}^k f_{31}^k = 0, \\ \sum_{k=1}^4 C_{ij}^k g_{11}^k = g_{11}^i f_{31}^j - g_{11}^j f_{31}^i. \end{array} \right. \quad (4.35)$$

8. Provide two copies of initial data $\{a_1, \dots, a_4\}$ and $\{b_1, \dots, b_4\}$ to the parametric derivatives $\{f_{11}, f_{21}, f_{31}, g_{11}\}$. All the nonzero structure constants are

$$\begin{aligned} \sum_{k=1}^4 C_{ij}^k f_{11}^k &= 2 \underbrace{(f_{11}^i f_{31}^j - f_{11}^j f_{31}^i)}_{a_1 b_3 - b_1 a_3} + \underbrace{(g_{11}^i g_{11}^j - g_{11}^j g_{11}^i)}_{a_4 b_4 - b_4 a_4}, \\ &\longrightarrow c_{13}^1 = 2, c_{44}^1 = 1, \\ \sum_{k=1}^4 C_{ij}^k f_{21}^k &= 6 \underbrace{(f_{21}^i f_{31}^j - f_{21}^j f_{31}^i)}_{a_2 b_3 - b_2 a_3}, \\ &\longrightarrow c_{23}^2 = 6, \\ \sum_{k=1}^4 C_{ij}^k g_{11}^k &= \underbrace{(g_{11}^i f_{31}^j - g_{11}^j f_{31}^i)}_{a_4 b_3 - b_4 a_3}. \\ &\longrightarrow c_{43}^4 = 1. \end{aligned}$$

Output: Read off the all nonzero structure constants c_{ij}^k 's. They are $c_{13}^1 = 2, c_{44}^1 = 1, c_{23}^2 = 6, c_{43}^4 = 1$.

Therefore, by the definition of structure constants, one has

$$[L_1, L_2] = c_{12}^1 L_1 + c_{12}^2 L_2 + c_{12}^3 L_3 + c_{12}^4 L_4 = 0,$$

$$[L_1, L_3] = c_{13}^1 L_1 + c_{13}^2 L_2 + c_{13}^3 L_3 + c_{13}^4 L_4 = 2L_1,$$

$$\begin{aligned}
[L_1, L_4] &= c_{14}^1 L_1 + c_{14}^2 L_2 + c_{14}^3 L_3 + c_{14}^4 L_4 = 0, \\
[L_2, L_3] &= c_{23}^1 L_1 + c_{23}^2 L_2 + c_{23}^3 L_3 + c_{23}^4 L_4 = 6L_2, \\
[L_2, L_4] &= c_{24}^1 L_1 + c_{24}^2 L_2 + c_{24}^3 L_3 + c_{24}^4 L_4 = 0, \\
[L_3, L_4] &= c_{34}^1 L_1 + c_{34}^2 L_2 + c_{34}^3 L_3 + c_{34}^4 L_4 = -L_4, \\
[L_4, L_4] &= c_{44}^1 L_1 + c_{44}^2 L_2 + c_{44}^3 L_3 + c_{44}^4 L_4 = L_1,
\end{aligned}$$

and the supercommutator table is given by

	L_1	L_2	L_3	L_4
L_1	0	0	$2L_1$	0
L_2	0	0	$6L_2$	0
L_3	$-2L_1$	$-6L_2$	0	$-L_4$
L_4	0	0	L_4	L_1

Table 4.5.2: Supercommutator table for the defining system of the super KdV equation obtained by the Structure Constant Algorithm.

Chapter 5

Supersymmetry for a class of super Lagrangians

In this chapter, we find nontrivial hidden supersymmetry for the Euler-Lagrange equations of

$$L = \frac{1}{2}\phi_\mu^2 + \frac{1}{2}(\bar{\psi}\gamma^\mu\psi_\mu - \bar{\psi}_\mu\gamma^\mu\psi) + F(\phi, \psi, \bar{\psi}), \quad \mu = 1, \dots, d, \quad (5.1)$$

when the dimension $d = 2$.

Even though $d = 2$ has only 2 space variables x_1 and x_2 , hand calculation of the supersymmetry defining system of the Euler-Lagrange equations is difficult. Edgardo Cheb-Terrab implemented `DeterminingPDE` in the `PDEtools` package in Maple to compute the defining systems satisfied by the infinitesimals of Lie symmetry groups of differential equations. In 2011, he upgraded `DeterminingPDE` to be compatible with the `Physics` package to deal with odd quantities.

In this chapter, we use `DeterminingPDE` to generate the defining system of the Euler-Lagrange equations of (5.1). Then we apply MONO expansion to those infinitesimals having odd independent variables. We also apply MONO expansion to the potential super function $F(\phi, \psi, \bar{\psi})$. By substituting these MONO expansions in the defining system and taking the coefficients of odd variable monomials, the reduced defining system is formed. The reduced defining system is then sent to the Maple commutative commands `rifsimp`, `initialdata` and `caseplot`.

All cases have the obvious translation symmetry in x_1 and x_2 . One extreme case is when the potential $F(\phi, \psi, \bar{\psi}) = 0$. This is infinite dimensional supersymmetry group and is solvable. The other extreme occurs when the potential $F(\phi, \psi, \bar{\psi})$ is nonzero and its highest order term does not vanish. With the help of `rifsimp` and one of its features `casesplit`, Maple splits this case into thousands of subcases. Digging into such big data, we classify the subcases into two sets, those with finite dimension and those with infinite dimension. In particular, we show that the finite cases with nontrivial term in $F(\phi, \psi, \bar{\psi})$ have supersymmetry groups of maximal finite dimension 5. As a result, we find a non-trivial hidden supersymmetry for such 5-dimensional cases. We verify that it leaves the defining system of the Euler-Lagrange equations invariant.

5.1 Lagrangian and Euler-Lagrange equations

We study supersymmetries of a higher dimensional version of a popular Lagrangian model [8] in supersymmetric quantum mechanics

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{1}{2}\left(\frac{dW}{dx}\right)^2 + \bar{\psi}\psi\frac{d^2W}{dx^2}, \quad (5.2)$$

where $x(t)$ is real scalar field, $\psi(t)$ is complex Grassmann field and $\bar{\psi}(t)$ is its complex conjugate. These fields are functions of space time coordinates t . The model (5.2) is a super ODE model.

We want to show how to find the symmetry and invariants which will reduce a super PDE system to a super ODE system. Hence, we generalize the super ODE model (5.2) to a super PDE model by generalizing this model to two dimensions x_1 and x_2 . To improve the possibility of supersymmetries, we provide the new PDE model with a general potential $F(\phi, \psi, \bar{\psi})$. Our generalization of (5.2) is

$$L = \frac{1}{2}\phi_\mu^2 + \frac{1}{2}(\bar{\psi}\gamma^\mu\psi_\mu - \bar{\psi}_\mu\gamma^\mu\psi) + F(\phi, \psi, \bar{\psi}), \quad \mu = 1, \dots, d, \quad (5.3)$$

where we use the Einstein summation convention over repeated indices, and $I = \sqrt{-1}$.

In particular, we focus on the simple case when $d = 2$ as our main task. Writing this Lagrangian in detail gives

$$\begin{aligned}
L = & \frac{1}{2}(-\phi_{x_1}^2 + \phi_{x_2}^2) \\
& + \frac{1}{2}(\psi_{1x_1}\bar{\psi}_2 - \psi_{2x_1}\bar{\psi}_1 - \psi_{1x_2}\bar{\psi}_2 - \psi_{2x_2}\bar{\psi}_1 \\
& + \bar{\psi}_{2x_1}\psi_1 - \bar{\psi}_{1x_1}\psi_2 - \bar{\psi}_{2x_2}\psi_1 - \bar{\psi}_{1x_2}\psi_2) \\
& + F(\phi, \psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2),
\end{aligned} \tag{5.4}$$

where x_1, x_2 are even independent variables, ϕ is an even dependent variables, and $\psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2$ are odd dependent variables. Note that x_j appearing as a subscripts means partial derivative with respect to x_j .

The Euler-Lagrange equations of the given Lagrangian (5.4) are obtained as

$$\begin{aligned}
\phi_{x_2, x_2} &= \phi_{x_1, x_1} + F_\phi, \\
(\psi_1)_{x_2} &= (\psi_1)_{x_1} + \mathbf{I}F_{\bar{\psi}_2}, \\
(\psi_2)_{x_2} &= -(\psi_2)_{x_1} + \mathbf{I}F_{\bar{\psi}_1}, \\
(\bar{\psi}_1)_{x_2} &= -(\bar{\psi}_1)_{x_1} + \mathbf{I}F_{\psi_2}, \\
(\bar{\psi}_2)_{x_2} &= (\bar{\psi}_2)_{x_1} + \mathbf{I}F_{\psi_1},
\end{aligned} \tag{5.5}$$

by using the multi-variable Euler-Lagrange formula. We have programmed this formula as a Maple procedure called `EL`. Maple code is provided in Appendix A.2. We used `EL` to generate the Euler-Lagrange equations of (5.4). These equations are put in the solved form as is (5.5). So the integrability conditions of the Euler-Lagrange system are satisfied and no hidden supersymmetries missed. In each equation in (5.5), the partial derivative of x_2 is considered to be the leading derivative.

5.2 Determining equations generated by Maple

In this section, we will demonstrate how to use the MAPLE command `DeterminingPDE` to help us get the determining equations of the supersymmetries.

5.2.1 One problem: the conjugates

Before sending the Euler-Lagrange system to MAPLE, we need some clarification while dealing with ψ_j and $\bar{\psi}_j$, where $j = 1, 2$. In fact, ψ_j and $\bar{\psi}_j$ are related by conjugation. So the Lagrangian is not a superanalytic functions since it depends on conjugates. But supersymmetry is a superanalytic theory. So as is usual in the non-Grassmannian case, we need to embed all non-superanalytic equations in a superanalytic formulation. We do this simply by introducing new variables ω_j to replace $\bar{\psi}_j$, where $j = 1, 2$,

$$\bar{\psi}_1 \rightarrow \omega_1 \quad \text{and} \quad \bar{\psi}_2 \rightarrow \omega_2. \quad (5.6)$$

We seek superanalytic symmetries as superanalytic transformations of $(x_1, x_2, \phi, \psi_1, \psi_2, \omega_1, \omega_2)$.

5.2.2 Maple demonstration

1. Load `Physics`, `DEtools` and `PDEtools` packages.

```
with(Physics);
with(PDEtools);
with(DEtools);
```

2. Declare odd variables.

```
Physics[Setup](anticommutativeprefix={psi, omega, Lambda, Omega},
mathematicalnontation=true):
```

Here `Lambda` and `Omega` are infinitesimal names corresponding to `psi` and `omega`. Note that once a name, for instance, `psi`, is declared as an odd variable, the sub-

5.2.3 Determining system

By examining the determining system obtained in last section, we can narrow down the dependencies of the infinitesimals. In the determining system, we have the following simple PDEs,

$$\begin{aligned}(\xi_1)_\phi &= 0, \\(\xi_1)_{\psi_1} &= 0, \\(\xi_1)_{\psi_2} &= 0, \\(\xi_1)_{\omega_1} &= 0, \\(\xi_1)_{\omega_2} &= 0.\end{aligned}$$

This implies that ξ_1 does not depend on $\phi, \psi_1, \psi_2, \omega_1, \omega_2$. So $\xi_1(x_1, x_2, \phi, \psi_1, \psi_2, \omega_1, \omega_2)$ can be narrowed down to $\xi_1(x_1, x_2)$. Similarly, we also narrow the dependencies of other infinitesimals:

$$\begin{aligned}\xi_2(x_1, x_2, \phi, \psi_1, \psi_2, \omega_1, \omega_2) &\rightarrow \xi_2(x_1, x_2), \\ \Xi(x_1, x_2, \phi, \psi_1, \psi_2, \omega_1, \omega_2) &\rightarrow \Xi(x_1, x_2, \phi), \\ \Lambda_1(x_1, x_2, \phi, \psi_1, \psi_2, \omega_1, \omega_2) &\rightarrow \Lambda_1(x_1, x_2, \psi_1, \omega_2), \\ \Lambda_2(x_1, x_2, \phi, \psi_1, \psi_2, \omega_1, \omega_2) &\rightarrow \Lambda_2(x_1, x_2, \psi_2, \omega_1), \\ \Omega_1(x_1, x_2, \phi, \psi_1, \psi_2, \omega_1, \omega_2) &\rightarrow \Omega_1(x_1, x_2, \psi_2, \omega_1), \\ \Omega_2(x_1, x_2, \phi, \psi_1, \psi_2, \omega_1, \omega_2) &\rightarrow \Omega_2(x_1, x_2, \psi_1, \omega_2).\end{aligned}$$

The dependencies above simplify the determining system to 13 super differential equations,

$$\begin{aligned}\Xi_{\phi, \phi} &= 0, \\ 2(\xi_1)_{x_1} - 2(\xi_2)_{x_2} &= 0,\end{aligned}$$

$$\begin{aligned}
2(\xi_2)_{x_1} - 2(\xi_1)_{x_2} &= 0, \\
2\Xi_{x_2,\phi} + (\xi_2)_{x_1,x_1} - (\xi_2)_{x_2,x_2} &= 0, \\
(\xi_1)_{x_1,x_1} - (\xi_1)_{x_2,x_2} - 2\Xi_{x_1,\phi} &= 0, \\
(\xi_2)_{x_1} - (\xi_1)_{x_1} + (\xi_2)_{x_2} - (\xi_1)_{x_2} &= 0, \\
(\xi_2)_{x_1} + (\xi_1)_{x_1} - (\xi_2)_{x_2} - (\xi_1)_{x_2} &= 0, \\
\Xi_\phi F_\phi + F_{\omega_2,\phi}\Omega_2 + F_{\psi_1,\phi}\Lambda_1 - 2F_\phi(\xi_2)_{x_2} + F_{\omega_1,\phi}\Omega_1 - \Xi_{x_1,x_1} \\
&\quad + F_{\psi_2,\phi}\Lambda_2 + \Xi_{x_2,x_2} - \Xi F_{\phi,\phi} = 0, \\
-IF_{\psi_2,\omega_2}\Lambda_2 - I\xi_{2x_2}F_{\omega_2} - (\Lambda_1)_{x_1} + I(\Lambda_1)_{\psi_1}F_{\omega_2} + I(\Lambda_1)_{\omega_2}F_{\psi_1} \\
&\quad + (\Lambda_1)_{x_2} - IF_{\psi_1,\omega_2}\Lambda_1 + IF_{\omega_2,\omega_1}\Omega_1 + I(\xi_2)_{x_1}F_{\omega_2} - I\Xi F_{\omega_2,\phi} = 0, \\
IF_{\psi_1,\omega_2}\Omega_2 - IF_{\psi_2,\psi_1}\Lambda_2 + I(\xi_2)_{x_1}F_{\psi_1} - I\Xi F_{\psi_1,\phi} + I(\Omega_2)_{\omega_2}F_{\psi_1} - (\Omega_2)_{x_1} \\
&\quad + (\Omega_2)_{x_2} + I(\Omega_2)_{\psi_1}F_{\omega_2} + IF_{\psi_1,\omega_1}\Omega_1 - I(\xi_2)_{x_1}F_{\psi_1} = 0, \\
-I\Xi F_{\psi_2,\phi} + (\Omega_1)_{x_1} - I(\xi_2)_{x_1}F_{\psi_2} + I(\Omega_1)_{\psi_2}F_{\omega_1} - I(\xi_2)_{x_2}F_{\psi_2} + (\Omega_1)_{x_2} \\
&\quad + IF_{\psi_2,\psi_1}\Lambda_1 + IF_{\psi_2,\omega_1}\Omega_1 + IF_{\psi_2,\omega_2}\Omega_2 + I(\Omega_1)_{\omega_1}F_{\psi_2} = 0, \\
-I(\xi_2)_{x_1}F_{\omega_1} - IF_{\psi_2,\omega_1}\Lambda_2 - I\Xi F_{\omega_1,\phi} - IF_{\omega_2,\omega_1}\Omega_2 - I(\xi_2)_{x_2}F_{\omega_1} + I(\Lambda_2)_{\psi_2}F_{\omega_1} \\
&\quad + (\Lambda_2)_{x_1} + (\Lambda_2)_{x_2} + I(\Lambda_2)_{\omega_1}F_{\psi_2} - IF_{\psi_1,\omega_1}\Lambda_1 = 0.
\end{aligned} \tag{5.7}$$

Note that $I = \sqrt{-1}$.

5.3 Reduced defining system

MONO expansion has been introduced in Section 4.2. To decompose the calculation which contains odd variables, we apply MONO expansion to decompose the infinitesimals and potential F in the system (5.8) with respect to their odd variable monomials. Three even infinitesimals ξ_1, ξ_2 and ϕ do not depend on any odd variables. So they do not need to be decomposed by MONO expansion. We apply MONO expansion to the remaining odd infinitesimals $\Lambda_1, \Lambda_2, \Omega_1$ and Ω_2 and obtain

$$\Lambda_1(x_1, x_2, \psi_1, \omega_2) = PO1_1(x_1, x_2) + \psi_1\omega_2PO1_2(x_1, x_2)$$

$$+PE1_1(x_1, x_2)\omega_2 + PE1_2(x_1, x_2)\psi_1, \quad (5.8)$$

$$\begin{aligned} \Lambda_2(x_1, x_2, \psi_1, \omega_2) &= PO2_1(x_1, x_2) + \psi_2\omega_1 PO2_2(x_1, x_2) \\ &+ PE2_1(x_1, x_2)\omega_1 + PE2_2(x_1, x_2)\psi_2, \end{aligned} \quad (5.9)$$

$$\begin{aligned} \Omega_1(x_1, x_2, \psi_1, \omega_2) &= PO3_1(x_1, x_2) + \psi_2\omega_1 PO3_2(x_1, x_2) \\ &+ PE3_1(x_1, x_2)\omega_1 + PE3_2(x_1, x_2)\psi_2, \end{aligned} \quad (5.10)$$

$$\begin{aligned} \Omega_2(x_1, x_2, \psi_1, \omega_2) &= PO4_1(x_1, x_2) + \psi_1\omega_2 PO4_2(x_1, x_2) \\ &+ PE4_1(x_1, x_2)\omega_2 + PE4_2(x_1, x_2)\psi_1. \end{aligned} \quad (5.11)$$

Applying the same decomposition to $F(\phi, \psi_1, \psi_2, \omega_1, \omega_2)$, we obtain its MONO expansions:

$$\begin{aligned} F(\phi, \psi_1, \psi_2, \omega_1, \omega_2) &= PE_1(\phi) + PE_2(\phi)\omega_1\omega_2 + PE_3(\phi)\psi_2\omega_2 + PE_4(\phi)\psi_2\omega_1 \\ &+ PE_5(\phi)\psi_1\omega_2 + PE_6(\phi)\psi_1\omega_1 + PE_7(\phi)\psi_1\psi_2 \\ &+ PE_8(\phi)\psi_1\psi_2\omega_1\omega_2 \\ &+ PO_1(\phi)\omega_2 + PO_2(\phi)\omega_1 + PO_3(\phi)\psi_2 + PO_4(\phi)\omega_1\omega_2\psi_2 \\ &+ PO_5(\phi)\psi_1 + PO_6(\phi)\omega_1\omega_2\psi_1 + PO_7(\phi)\omega_2\psi_1\psi_2 \\ &+ PO_8(\phi)\omega_1\psi_1\psi_2. \end{aligned}$$

Since ψ_1, ψ_2, ω_1 and ω_2 always appear in pairs in F , the following assumption is made: suppose that

$$PO_j(\phi) = 0, \quad i = 1, \dots, 8.$$

Hence $F(\phi, \psi_1, \psi_2, \omega_1, \omega_2)$ has only even components

$$\begin{aligned} F(\phi, \psi_1, \psi_2, \omega_1, \omega_2) &= PE_1(\phi) + PE_2(\phi)\omega_1\omega_2 + PE_3(\phi)\psi_2\omega_2 + PE_4(\phi)\psi_2\omega_1 \\ &+ PE_5(\phi)\psi_1\omega_2 + PE_6(\phi)\psi_1\omega_1 + PE_7(\phi)\psi_1\psi_2 \\ &+ PE_8(\phi)\psi_1\psi_2\omega_1\omega_2. \end{aligned} \quad (5.12)$$

Then we substitute the expansions (5.8)-(5.11) and (5.12) back into the simplified determining system (5.8). Taking all the coefficients of the odd variable monomials forms the final version of the determining system, that is, the reduced defining system.

In the reduced defining system, the unknowns are

$$PE1_1(x_1, x_2), PE1_2(x_1, x_2), PO1_1(x_1, x_2), PO1_2(x_1, x_2),$$

$$PE2_1(x_1, x_2), PE2_2(x_1, x_2), PO2_1(x_1, x_2), PO2_2(x_1, x_2),$$

$$PE3_1(x_1, x_2), PE3_2(x_1, x_2), PO3_1(x_1, x_2), PO3_2(x_1, x_2),$$

$$PE4_1(x_1, x_2), PE4_2(x_1, x_2), PO4_1(x_1, x_2), PO4_2(x_1, x_2),$$

and

$$\xi_1(x_1, x_2), \xi_2(x_1, x_2), \Xi(x_1, x_2, \phi).$$

Next, we are going to use `rifsimp` and `initialdata` to help us to get the all the supersymmetry cases.

5.4 Supersymmetry analysis

In this section, we will give detailed supersymmetry analysis for two extreme cases. One is when $F = 0$. The other one is when F is non-trivial enough.

5.4.1 Generic case

Recall the Euler-Lagrange equations (5.5) ,

$$\begin{aligned} \phi_{x_2, x_2} &= \phi_{x_1, x_1} + F_\phi, \\ (\psi_1)_{x_2} &= (\psi_1)_{x_1} + \mathbf{I}F_{\bar{\psi}_2}, \\ (\psi_2)_{x_2} &= -(\psi_2)_{x_1} + \mathbf{I}F_{\bar{\psi}_1}, \\ (\bar{\psi}_1)_{x_2} &= -(\bar{\psi}_1)_{x_1} + \mathbf{I}F_{\psi_2}, \end{aligned}$$

$$(\bar{\psi}_2)_{x_2} = (\bar{\psi}_2)_{x_1} + \mathbf{I}F_{\psi_1}.$$

It is easy to see that the Euler-Lagrange system always has two translation symmetries in x_1 and x_2 for any form of F . Indeed, there are cases for which there are only these 2 symmetries. This is what we call the generic case. We are interested in looking for other supersymmetries rather than these two obvious translation symmetries.

5.4.2 Symmetry analysis of $F=0$ case

When $F = 0$, the Euler-Lagrange system is

$$\begin{aligned}\phi_{x_2,x_2} &= \phi_{x_1,x_1}, \\ (\psi_1)_{x_2} &= (\psi_1)_{x_1}, \\ (\psi_2)_{x_2} &= -(\psi_2)_{x_1}, \\ (\omega_1)_{x_2} &= -(\omega_1)_{x_1}, \\ (\omega_2)_{x_2} &= (\omega_2)_{x_1}.\end{aligned}$$

The general solution of

$$\begin{aligned}\phi &= f(x_1 - x_2) + g(x_1 + x_2), \\ \psi_1 &= f_1(x_1 + x_2), \\ \psi_2 &= f_2(x_1 - x_2), \\ \omega_1 &= h(x_1 - x_2), \\ \omega_2 &= h(x_1 + x_2),\end{aligned}$$

where f, g are arbitrary even analytic functions and f_1, f_2, h_1, h_2 are arbitrary odd super-analytic functions. The corresponding determining system is reduced to

$$\Xi_{\phi,\phi} = 0,$$

$$\begin{aligned}
2(\xi_1)_{x_1} - 2(\xi_2)_{x_2} &= 0, \\
2(\xi_2)_{x_1} - 2(\xi_1)_{x_2} &= 0, \\
2\Xi_{x_2,\phi} + (\xi_2)_{x_1,x_1} - (\xi_2)_{x_2,x_2} &= 0, \\
(\xi_1)_{x_1,x_1} - (\xi_1)_{x_2,x_2} - 2\Xi_{x_1,\phi} &= 0, \\
(\xi_2)_{x_1} - (\xi_1)_{x_1} + (\xi_2)_{x_2} - (\xi_1)_{x_2} &= 0, \\
(\xi_2)_{x_1} + (\xi_1)_{x_1} - (\xi_2)_{x_2} - (\xi_1)_{x_2} &= 0, \\
-\Xi_{x_1,x_1} + \Xi_{x_2,x_2} &= 0, \\
-(\Lambda_1)_{x_1} + (\Lambda_1)_{x_2} &= 0, \\
-(\Omega_2)_{x_1} + (\Omega_2)_{x_2} &= 0, \\
(\Omega_1)_{x_1} + (\Omega_1)_{x_2} &= 0, \\
(\Lambda_2)_{x_1} + (\Lambda_2)_{x_2} &= 0.
\end{aligned}$$

The symmetry defining system is easily solved to show that the original Euler-Lagrange equations admits an ∞ -dimensional supersymmetry group. However this is not interesting since the EL system is trivially solvable. We remark that we found many cases of nontrivial nonlinear F admitting nontrivial ∞ -dimensional supersymmetry groups. It would be interesting to investigate such cases in future research.

5.4.3 Symmetry analysis of one 5-dim case

For the other extreme case, F is as in (5.12). We also add one constraint on F to the determining system

$$PE_8'''(\phi) \neq 0, \quad (5.13)$$

where $PE_8(\phi)$ is the coefficient of the highest order nonlinear term $\psi_1\psi_2\omega_1\omega_2$ in F . The reason is that we want to find some non-trivial supersymmetry of the defining system with nontrivial F . Then we send the reduced defining system and the constraint (5.13) to `rifsimp`:

```

# FinalDet: the defining system of the input super Euler–Lagrange equations;
# AllInfNames: all infinitesimal functions;
msys2 := rifsimp([op(FinalDet), diff(PE[8](phi), phi, phi, phi) ≠ 0],
                AllInfNames, casesplit):
caseplot(msys2, AllInfNames):

```

The Maple output indicates us that there are over 3000 cases. What can we do about this big data? First of all, we split all the cases into two classes. One class is the set of all cases with infinite dimensional supersymmetry groups. The other class is the set of all cases with finite dimensional supersymmetry groups. Out of these, we seek the ones with maximal dimension since they contain more supersymmetries. We found that the maximal dimension is 5. We get 32 five dimensional cases with 5 dimensional supersymmetry groups. In this section, we will pick one of the 32 five dimensional case and give the supersymmetry results in detail for that case.

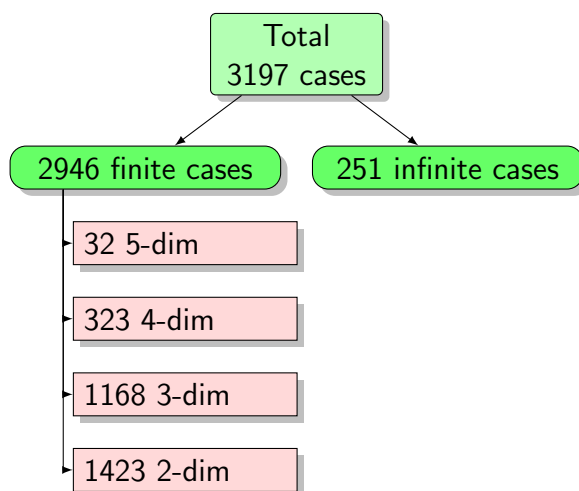


Figure 5.4.1: Case split

The most interesting cases are the ones with more symmetries and in particular those with maximal dimensional supersymmetry groups. As is shown in Figure 5.4.1, there are 32 5-dimensional cases. Among them, Case 1124 has been selected and analyzed. Note that the number of cases can vary if the Maple worksheet is re-executed.

For Case 1124, we get the complete solutions of for the supersymmetry infinitesimals:

$$\begin{aligned}
\xi_1(x_1, x_2) &= \frac{(2.C4 C2 - I(x_1 + x_2)(.C1 + .C2))C1 - I C2(x_1 - x_2)(.C1 - .C2)}{2C1 C2} \\
\xi_2(x_1, x_2) &= \frac{(2.C5 C2 - I(x_1 + x_2)(.C1 + .C2))C1 + I C2(x_1 - x_2)(.C1 - .C2)}{2C1 C2} \\
\Xi(x_1, x_2, \phi) &= \frac{I(C1(.C1 + .C2) + C2(.C1 - .C2))(a\phi + b)}{aC1 C2(c2 + d2)} \\
\Lambda_1(x_1, x_2, \psi_1, \omega_2) &= \frac{1}{C1 C2(c2 + d2)}((C1 C2 .C1(c2 + d2)x_1 + C1 C2 .C2(c2 + d2)x_2 \\
&\quad + C1 C2 .C3(c2 + d2) + I c2 C1(.C1 + .C2) \\
&\quad - I d2 C2(.C1 - .C2))\psi_1), \\
\Lambda_2(x_1, x_2, \psi_2, \omega_1) &= -\frac{1}{C1 C2(c2 + d2)}((C1 C2 .C1(c2 + d2)x_1 + C1 C2 .C2(c2 + d2)x_2 \\
&\quad + C1 C2 .C3(c2 + d2) - I .C1 C2(c2 + d2) + I .C2 C2(c2 + d2) \\
&\quad - 2I C1(.C1 + .C2) - 2I C2(.C1 - .C2))\psi_2), \\
\Omega_1(x_1, x_2, \psi_2, \omega_1) &= (.C1 x_1 + .C2 x_2 + .C3)\omega_1, \\
\Omega_2(x_1, x_2, \psi_1, \omega_2) &= -\frac{1}{C1 C2(c2 + d2)}((C1 C2 .C1(c2 + d2)x_1 + C1 C2 .C2(c2 + d2)x_2 \\
&\quad + C1 C2 .C3(c2 + d2) - I C1 d2(.C1 + .C2) - I C2 d2(.C1 - .C2) \\
&\quad - 2I C1(.C1 + .C2) - 2I C2(.C1 - .C2))\omega_2).
\end{aligned}$$

Fot this maximal case we explicitly found F as

$$\begin{aligned}
F(\phi, \psi_1, \psi_2, \omega_1, \omega_2) &= C3 + C4\left(\frac{a\phi + b}{a}\right)^{c2+d2+2} \\
&\quad + d1(a\phi + b)^{d2}\psi_1\psi_2 + c1(a\phi + b)^{c2}\omega_1\omega_2 \\
&\quad - C1 \omega_2\psi_1 - C2 \omega_1\psi_2 \\
&\quad + \frac{4}{(a\phi + b)^2}\omega_1\omega_2\psi_1\psi_2.
\end{aligned}$$

These solutions were also checked by substituting them back into the determining system.

Consider the supersymmetry vector field

$$v = \xi_1\partial_{x_1} + \xi_2\partial_{x_2} + \Xi\partial_\phi + \Lambda_1\partial_{\psi_1} + \Lambda_2\partial_{\psi_2} + \Omega_1\partial_{\omega_1} + \Omega_2\partial_{\omega_2}. \quad (5.14)$$

Since this is a 5-dim case with arbitrary constants $_C1, _C2, \dots, _C5$, the coefficients of the five constants yields the basis of the solution space.

By setting $(_C1, _C2, _C3, _C4, _C5)$ equal to

$$(1, 1, 0, 0, 0),$$

$$(1, -1, 0, 0, 0),$$

$$(0, 0, 1, 0, 0),$$

$$(0, 0, 0, 1, 0),$$

$$(0, 0, 0, 0, 1).$$

in (5.14) yields the basis of supersymmetry operators

$$L_1 = \partial_{x_1},$$

$$L_2 = \partial_{x_2},$$

$$L_3 = \psi_1 \partial_{\psi_1} - \psi_2 \partial_{\psi_2} + \omega_1 \partial_{\omega_1} - \omega_2 \partial_{\omega_2},$$

$$L_4 = -\frac{1}{aC2(c2+d2)}(I a(c2+d2)(x_1+x_2)\partial_{x_1} + I a(c2+d2)(x_1+x_2)\partial_{x_2} \\ - 2I(a\phi+b)\partial_\phi \\ - (aC2(c2+d2)(x_1+x_2) + 2I a c2)\psi_1\partial_{\psi_1} \\ + (aC2(c2+d2)(x_1+x_2) - 4I a)\psi_2\partial_{\psi_2} \\ - aC2(c2+d2)(x_1+x_2)\omega_1\partial_{\omega_1} \\ + (aC2(c2+d2)(x_1+x_2) - 2I a d2 - 4I a)\omega_2\partial_{\omega_2}),$$

$$L_5 = -\frac{1}{aC1(c2+d2)}(I a(c2+d2)(x_1-x_2)\partial_{x_1} - I a(c2+d2)(x_1-x_2)\partial_{x_2} \\ - 2I(a\phi+b)\partial_\phi \\ - (aC1(c2+d2)(x_1-x_2) - 2I a d2)\psi_1\partial_{\psi_1} \\ + (aC1(c2+d2)(x_1-x_2) - 4I a - 2I a d2 - 2I a c2)\psi_2\partial_{\psi_2} \\ - aC1(c2+d2)(x_1-x_2)\omega_1\partial_{\omega_1}$$

$$+(a C1(c2 + d2)(x_1 - x_2) - 2I a d2 - 4I a)\omega_2\partial_{\omega_2}.$$

This yields the commutator table in Table 5.4.1.

	L_1	L_2	L_3	L_4	L_5
L_1	0	0	0	$-\frac{L_1+L_2}{C2} + L_3$	$-\frac{L_1-L_2}{C1} + L_3$
L_2	0	0	0	$-\frac{L_1+L_2}{C2} + L_3$	$\frac{L_1+L_2}{C2} - L_3$
L_3	0	0	0	0	0
L_4	$\frac{L_1+L_2}{C2} - L_3$	$\frac{L_1+L_2}{C2} - L_3$	0	0	0
L_5	$\frac{L_1-L_2}{C1} - L_3$	$-\frac{L_1+L_2}{C2} + L_3$	0	0	0

Table 5.4.1: Supercommutator table for the super Lie algebra defining system of Case 1124.

5.4.4 Finding supersymmetries for a sub-class of (5.5)

In the previous Section 5.4.3, we analyzed the case with

$$\begin{aligned}
F(\phi, \psi_1, \psi_2, \omega_1, \omega_2) &= C3 + C4 \left(\frac{a\phi + b}{a} \right)^{c2+d2+2} \\
&+ d1(a\phi + b)^{d2}\psi_1\psi_2 + c1(a\phi + b)^{c2}\omega_1\omega_2 \\
&- C1 \omega_2\psi_1 - C2 \omega_1\psi_2 \\
&+ \frac{4}{(a\phi + b)^2}\omega_1\omega_2\psi_1\psi_2.
\end{aligned}$$

In this section, our goal is to find at least one non-trivial supersymmetry for the Euler-Lagrange system (5.5) with a sub-class of F above, which is

$$\begin{aligned}
\mathcal{F}(\phi, \psi_1, \psi_2, \omega_1, \omega_2) &= C3 + C4 \phi^{2c+2} + d1 \phi^c\psi_1\psi_2 + c1 \phi^c\omega_1\omega_2 \\
&- C\omega_2\psi_1 - C\omega_1\psi_2 + \frac{4}{\phi^2}\omega_1\omega_2\psi_1\psi_2,
\end{aligned} \tag{5.15}$$

by making the assumptions

$$a = 1, b = 0, c_2 = d_2 = c, C_1 = C_2 = C. \quad (5.16)$$

The first two assumptions are natural and do not make any essential restriction on F . The other two assumptions are a normalization of F in some sense. At the same time, under the assumptions (5.16), the sub-class of the Euler-Lagrange system (5.5) is changed to

$$\begin{aligned} \phi_{x_2, x_2} &= \phi_{x_1, x_1} + \mathcal{F}_\phi, \\ (\psi_1)_{x_2} &= (\psi_1)_{x_1} + \mathbf{I}\mathcal{F}_{\omega_2}, \\ (\psi_2)_{x_2} &= -(\psi_2)_{x_1} + \mathbf{I}\mathcal{F}_{\omega_1}, \\ (\omega_1)_{x_2} &= -(\omega_1)_{x_1} + \mathbf{I}\mathcal{F}_{\psi_2}, \\ (\omega_2)_{x_2} &= (\omega_2)_{x_1} + \mathbf{I}\mathcal{F}_{\psi_1}, \end{aligned} \quad (5.17)$$

and the last two most complicated basis generators of the Lie superalgebra in Table 5.4.1 become

$$\begin{aligned} \mathcal{L}_4 &= -\frac{\mathbf{I}}{C}((x_1 + x_2)\partial_{x_1} + (x_1 + x_2)\partial_{x_2}) + \frac{\mathbf{I}}{C_C}\phi\partial_\phi \\ &\quad (x_1 + x_2)(\psi_1\partial_{\psi_1} - \psi_2\partial_{\psi_2} + \omega_1\partial_{\omega_1} - \omega_2\partial_{\omega_2}) \\ &\quad + \frac{\mathbf{I}}{C}\psi_1\partial_{\psi_1} + \frac{2\mathbf{I}}{C_C}\psi_2\partial_{\psi_2} + \frac{\mathbf{I}(c+2)}{C_C}\omega_2\partial_{\omega_2}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_5 &= -\frac{\mathbf{I}}{C}((x_1 - x_2)\partial_{x_1} - (x_1 - x_2)\partial_{x_2}) + \frac{\mathbf{I}}{C_C}\phi\partial_\phi \\ &\quad (x_1 - x_2)(\psi_1\partial_{\psi_1} - \psi_2\partial_{\psi_2} + \omega_1\partial_{\omega_1} - \omega_2\partial_{\omega_2}) \\ &\quad - \frac{\mathbf{I}}{C}\psi_1\partial_{\psi_1} + \frac{2\mathbf{I}(c+1)}{C_C}\psi_2\partial_{\psi_2} + \frac{\mathbf{I}(c+2)}{C_C}\omega_2\partial_{\omega_2}. \end{aligned}$$

By adding \mathcal{L}_4 and \mathcal{L}_5 , a new independent basis is formed

$$\begin{aligned}\mathcal{L}_4 + \mathcal{L}_5 &= -\frac{2\mathbb{I}}{C}x_1\partial_{x_1} + -\frac{2\mathbb{I}}{C}x_2\partial_{x_2} + \frac{2\mathbb{I}}{Cc}\phi\partial_\phi \\ &\quad + 2x_1(\psi_1\partial_{\psi_1} - \psi_2\partial_{\psi_2} + \omega_1\partial_{\omega_1} - \omega_2\partial_{\omega_2}) \\ &\quad + \frac{2\mathbb{I}(c+2)}{Cc}(\psi_2\partial_{\psi_2} + \omega_2\partial_{\omega_2}).\end{aligned}$$

Then, we can obtain the one parameter supersymmetry differential equations corresponding to $\mathcal{L}_4 + \mathcal{L}_5$:

$$\frac{d\hat{x}_1}{d\epsilon} = -\frac{2\mathbb{I}}{C}\hat{x}_1, \quad \hat{x}_1(0) = x_1; \quad (5.18)$$

$$\frac{d\hat{x}_2}{d\epsilon} = -\frac{2\mathbb{I}}{C}\hat{x}_2, \quad \hat{x}_2(0) = x_2; \quad (5.19)$$

$$\frac{d\hat{\phi}}{d\epsilon} = \frac{2\mathbb{I}}{Cc}\hat{\phi}, \quad \hat{\phi}(0) = \phi; \quad (5.20)$$

$$\frac{d\hat{\psi}_1}{d\epsilon} = 2\hat{x}_1\hat{\psi}_1, \quad \hat{\psi}_1(0) = \psi_1; \quad (5.21)$$

$$\frac{d\hat{\psi}_2}{d\epsilon} = \left(\frac{2\mathbb{I}(c+2)}{Cc} - 2\hat{x}_1\right)\hat{\psi}_2, \quad \hat{\psi}_2(0) = \psi_2; \quad (5.22)$$

$$\frac{d\hat{\omega}_1}{d\epsilon} = 2\hat{x}_1\hat{\omega}_1, \quad \hat{\omega}_1(0) = \omega_1; \quad (5.23)$$

$$\frac{d\hat{\omega}_2}{d\epsilon} = \left(\frac{2\mathbb{I}(c+2)}{Cc} - 2\hat{x}_1\right)\hat{\omega}_2, \quad \hat{\omega}_2(0) = \omega_2. \quad (5.24)$$

By solving the above differential equations with their initial conditions, one has

$$\begin{aligned}\hat{x}_1 &= e^{-\frac{2\mathbb{I}\epsilon}{C}}x_1, \\ \hat{x}_2 &= e^{-\frac{2\mathbb{I}\epsilon}{C}}x_2, \\ \hat{\phi} &= e^{\frac{2\mathbb{I}\epsilon}{Cc}}\phi, \\ \hat{\psi}_1 &= e^{\mathbb{I}Cx_1\left(e^{-\frac{2\mathbb{I}\epsilon}{C}}-1\right)}\psi_1, \\ \hat{\psi}_2 &= e^{\frac{2\mathbb{I}(c+2)\epsilon}{Cc}+\mathbb{I}Cx_1\left(1-e^{-\frac{2\mathbb{I}\epsilon}{C}}\right)}\psi_2, \\ \hat{\omega}_1 &= e^{\mathbb{I}Cx_1\left(e^{-\frac{2\mathbb{I}\epsilon}{C}}-1\right)}\omega_1, \\ \hat{\omega}_2 &= e^{\frac{2\mathbb{I}(c+2)\epsilon}{Cc}+\mathbb{I}Cx_1\left(1-e^{-\frac{2\mathbb{I}\epsilon}{C}}\right)}\omega_2.\end{aligned} \quad (5.25)$$

We checked that the supersymmetry (5.25) leaves the system (5.18) invariant by showing that each of the PDE in (5.18) are left invariant.

We give the details of this check for the first two PDE in (5.18).

- a) We show that the supersymmetry (5.25) leaves the first equation in the sub-class of Euler-Lagrange system (5.18) invariant.
- b) We show that the supersymmetry (5.25) leaves the second equation in the sub-class of Euler-Lagrange system (5.18) invariant.

We now do the details for a). If it does so, one should have

$$\hat{\phi}_{\hat{x}_2, \hat{x}_2} = \hat{\phi}_{\hat{x}_1, \hat{x}_1} + \hat{\mathcal{F}}_{\hat{\phi}}, \quad (5.26)$$

where

$$\hat{\mathcal{F}}_{\hat{\phi}} = C4(2c + 2)\hat{\phi}^{2c+1} + d1 c\hat{\phi}^{c-1}\hat{\psi}_1\hat{\psi}_2 + c1 c\hat{\phi}^{c-1}\hat{\omega}_1\hat{\omega}_2 - \frac{8}{\hat{\phi}^3}\hat{\omega}_1\hat{\omega}_2\hat{\psi}_1\hat{\psi}_2. \quad (5.27)$$

The left hand side of (5.26) is

$$\hat{\phi}_{\hat{x}_2, \hat{x}_2} = e^{\frac{2I\epsilon}{C}} e^{\frac{2I\epsilon}{C}} e^{\frac{2I\epsilon}{C}} \phi_{x_2, x_2} = e^{\frac{2I(2c+1)\epsilon}{C}} \phi_{x_2, x_2}. \quad (5.28)$$

The first term of the right hand side of (5.26) is

$$\hat{\phi}_{\hat{x}_1, \hat{x}_1} = e^{\frac{2I\epsilon}{C}} e^{\frac{2I\epsilon}{C}} e^{\frac{2I\epsilon}{C}} \phi_{x_1, x_1} = e^{\frac{2I(2c+1)\epsilon}{C}} \phi_{x_1, x_1}. \quad (5.29)$$

The second term of the right hand side of (5.26) is

$$\begin{aligned} \hat{\mathcal{F}}_{\hat{\phi}} &= C4(2c + 2)e^{\frac{2I(2c+1)\epsilon}{C}} \phi^{2c+1} \\ &+ d1 ce^{\frac{2I\epsilon}{C}} \phi^{c-1} e^{ICx_1} \left(e^{-\frac{2I\epsilon}{C}} - 1 \right) \psi_1 e^{\frac{2I(c+2)\epsilon}{C} + ICx_1} \left(1 - e^{-\frac{2I\epsilon}{C}} \right) \psi_2 \\ &+ c1 ce^{\frac{2I\epsilon}{C}} \phi^{c-1} e^{ICx_1} \left(e^{-\frac{2I\epsilon}{C}} - 1 \right) \omega_1 e^{\frac{2I(c+2)\epsilon}{C} + ICx_1} \left(1 - e^{-\frac{2I\epsilon}{C}} \right) \omega_2 \\ &- e^{-\frac{6I\epsilon}{C}} \frac{8}{\phi^3} e^{2ICx_1} \left(e^{-\frac{2I\epsilon}{C}} - 1 \right) e^{\frac{4I(c+2)\epsilon}{C} + 2ICx_1} \left(1 - e^{-\frac{2I\epsilon}{C}} \right) \omega_1 \omega_2 \psi_1 \psi_2. \end{aligned}$$

$$\begin{aligned}
&= e^{\frac{2I(2c+1)\epsilon}{Cc}} (C4(2c+2)\phi^{2c+1} + d1 c\phi^{c-1}\psi_1\psi_2 + c1 c\phi^{c-1}\omega_1\omega_2 - \frac{8}{\phi^3}\omega_1\omega_2\psi_1\psi_2) \\
&= e^{\frac{2I(2c+1)\epsilon}{Cc}} \mathcal{F}_\phi.
\end{aligned} \tag{5.30}$$

Therefore, (5.28), (5.29) and (5.30) imply

$$\phi_{x_2, x_2} = \phi_{x_1, x_1} + \mathcal{F}_\phi,$$

which means that the supersymmetry (5.25) leaves the first equation in (5.18) invariant!

Then we show that the supersymmetry (5.25) leaves the second equation in the subclass of Euler-Lagrange system (5.18) invariant. We use the same strategy as it in a). If b) holds, then one has

$$\hat{\psi}_{1\hat{x}_2} = \hat{\psi}_{1\hat{x}_1} + I\hat{\mathcal{F}}_{\hat{\omega}_2}, \tag{5.31}$$

where

$$\hat{\mathcal{F}}_{\hat{\omega}_2} = -c1\hat{\phi}^c\hat{\omega}_1 - C\hat{\psi}_1 - \frac{4}{\hat{\phi}^2}\hat{\omega}_1\hat{\psi}_1\hat{\psi}_2. \tag{5.32}$$

The left hand side of (5.31) is

$$\begin{aligned}
\hat{\psi}_{1\hat{x}_2} &= e^{\frac{2I\epsilon}{C}} e^{ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) \psi_{1x_2} \\
&= e^{\frac{2I\epsilon}{C} + ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) \psi_{1x_2} \\
&= A\psi_{1x_2},
\end{aligned} \tag{5.33}$$

where A is supposed to be $e^{\frac{2I\epsilon}{C} + ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right)$. The first term of the right hand side of (5.31) is

$$\begin{aligned}
\hat{\psi}_{1\hat{x}_1} &= e^{\frac{2I\epsilon}{C}} \left(e^{ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) \psi_1 \right)_{x_1} \\
&= e^{\frac{2I\epsilon}{C}} \left(IC \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) e^{ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) \psi_1 + e^{ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) \psi_{1x_1} \right) \\
&= e^{\frac{2I\epsilon}{C} + ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) IC \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) \psi_1 + e^{\frac{2I\epsilon}{C} + ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) \psi_{1x_1} \\
&= e^{\frac{2I\epsilon}{C} + ICx_1} \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) IC \left(e^{\frac{-2I\epsilon}{C}} - 1 \right) \psi_1 + A\psi_{1x_1}.
\end{aligned} \tag{5.34}$$

The second term of the right hand side of (5.31) is

$$\begin{aligned}
\mathbf{I}\hat{\mathcal{F}}_{\hat{\omega}_2} &= -\mathbf{I} c1 e^{\frac{2\mathbf{I}\epsilon}{c}} e^{\mathbf{I}C x_1 \left(e^{-\frac{2\mathbf{I}\epsilon}{c}} - 1\right)} \phi^c \omega_1 - \mathbf{I} C e^{\mathbf{I}C x_1 \left(e^{-\frac{2\mathbf{I}\epsilon}{c}} - 1\right)} \psi_1 \\
&\quad - \mathbf{I} e^{-\frac{4\mathbf{I}\epsilon}{c}} e^{2\mathbf{I}C x_1 \left(e^{-\frac{2\mathbf{I}\epsilon}{c}} - 1\right)} e^{\frac{2\mathbf{I}(c+2)\epsilon}{c} + \mathbf{I}C x_1 \left(1 - e^{-\frac{2\mathbf{I}\epsilon}{c}}\right)} \frac{4}{\phi^2} \omega_1 \psi_1 \psi_2 \\
&= A(-\mathbf{I} c1 \phi^2 \omega_1) - \mathbf{I} C e^{\mathbf{I}C x_1 \left(e^{-\frac{2\mathbf{I}\epsilon}{c}} - 1\right)} \psi_1 + A\left(-\mathbf{I} \frac{4}{\phi^2} \omega_1 \psi_1 \psi_2\right). \tag{5.35}
\end{aligned}$$

By adding (5.34) and (5.35), the right hand side of (5.31) becomes

$$A\psi_{1x_1} + A(-\mathbf{I} c1 \phi^2 \omega_1) + A(-\mathbf{I} C \psi_1) + A\left(-\mathbf{I} \frac{4}{\phi^2} \omega_1 \psi_1 \psi_2\right) = A(\psi_{1x_1} + \mathbf{I}\mathcal{F}_{\omega_2}). \tag{5.36}$$

Then (5.33) and (5.36) imply that

$$\psi_{1x_2} = \psi_{1x_1} + \mathbf{I}\mathcal{F}_{\omega_2},$$

which means that the supersymmetry (5.25) leaves the second equation in (5.18) invariant!

Similarly, we can verify that the supersymmetry (5.25) also leaves the remaining three equations in (5.18) invariant. Hence, one can claim that

Theorem 5.4.1. *There exists at least one non-trivial supersymmetry that leaves the Euler-Lagrange system (5.18) invariant.*

□

Next, we are going to find the invariants. By the separation of hat variables and non-hat variables for each symmetry in (5.25), one has

$$\frac{\hat{x}_1}{\hat{x}_2} = \frac{x_1}{x_2}, \tag{5.37}$$

$$\hat{x}_1^{\frac{1}{c}} \hat{\phi} = x_1^{\frac{1}{c}} \phi; \tag{5.38}$$

$$e^{-\mathbf{I}C \hat{x}_1} \hat{\psi}_1 = e^{-\mathbf{I}C x_1} \psi_1, \tag{5.39}$$

$$\hat{x}_1^{\frac{c+2}{c}} e^{\mathbf{I}C \hat{x}_1} \hat{\psi}_2 = x_1^{\frac{c+2}{c}} e^{\mathbf{I}C x_1} \psi_2, \tag{5.40}$$

$$e^{-\mathbf{I}C \hat{x}_1} \hat{\omega}_1 = e^{-\mathbf{I}C x_1} \omega_1, \tag{5.41}$$

$$\hat{x}_1^{\frac{c+2}{c}} e^{\mathbf{I}C \hat{x}_1} \hat{\omega}_2 = x_1^{\frac{c+2}{c}} e^{\mathbf{I}C x_1} \omega_2. \tag{5.42}$$

For each relation, suppose that

$$z = \frac{x_1}{x_2} = \frac{\hat{x}_1}{\hat{x}_2}; \quad (5.43)$$

$$y(z) = x_1^{\frac{1}{c}} \phi = \hat{x}_1^{\frac{1}{c}} \hat{\phi}; \quad (5.44)$$

$$\delta_1(z) = e^{-ICx_1} \psi_1 = e^{-IC\hat{x}_1} \hat{\psi}_1; \quad (5.45)$$

$$\delta_2(z) = x_1^{\frac{c+2}{c}} e^{ICx_1} \psi_2 = \hat{x}_1^{\frac{c+2}{c}} e^{IC\hat{x}_1} \hat{\psi}_2; \quad (5.46)$$

$$\rho_1(z) = e^{-ICx_1} \omega_1 = e^{-IC\hat{x}_1} \hat{\omega}_1; \quad (5.47)$$

$$\rho_2(z) = x_1^{\frac{c+2}{c}} e^{ICx_1} \omega_2 = \hat{x}_1^{\frac{c+2}{c}} e^{IC\hat{x}_1} \hat{\omega}_2. \quad (5.48)$$

Therefore, these new variables $z, y(z), \delta_1(z), \delta_2(z), \rho_1(z)$ and $\rho_2(z)$ are the invariants we were looking for. Note that $\delta_1(z), \delta_2(z), \rho_1(z)$ and $\rho_2(z)$ are odd invariants.

Let us show that they can reduce the original super PDE system (5.18) to super ODE system.

- a) Show that the invariants $z, y(z), \delta_1(z), \delta_2(z), \rho_1(z)$ and $\rho_2(z)$ reduce the first super PDE

$$\phi_{x_2, x_2} = \phi_{x_1, x_1} + \mathcal{F}_\phi \quad (5.49)$$

to a super ODE.

Substitute the invariants to the left hand side of the super PDE (5.49). By (5.45), we have

$$\phi = x_1^{-\frac{1}{c}} y(z). \quad (5.50)$$

Then we have

$$\phi_{x_2} = -x_1^{1-\frac{1}{c}} x_2^{-2} y'(z) \quad (5.51)$$

and

$$\phi_{x_2, x_2} = x_1^{2-\frac{1}{c}} x_2^{-4} y''(z) + 2x_1^{1-\frac{1}{c}} x_2^{-3} y'(z). \quad (5.52)$$

Eliminating x_2 from (5.56) by the relation $x_2 = x_1 z^{-1}$, we have

$$\phi_{x_2, x_2} = x_1^{-\frac{2c+1}{c}} z^4 y''(z) + 2x_1^{-\frac{2c+1}{c}} z^3 y'(z). \quad (5.53)$$

Then substitute the invariants to the first term of the right hand side of the super PDE (5.49). We have

$$\phi_{x_1} = -\frac{1}{c}x_1^{-1-\frac{1}{c}}y(z) + x_1^{-\frac{1}{c}}x_2^{-1}y'(z) \quad (5.54)$$

and

$$\phi_{x_1, x_1} = \frac{1}{c}\left(\frac{1}{c} + 1\right)x_1^{-\frac{2c+1}{c}}y(z) - \frac{2}{c}x_1^{-1-\frac{1}{c}}x_2^{-1}y'(z). \quad (5.55)$$

By the eliminating x_2 , we have

$$\phi_{x_1, x_1} = \frac{1}{c}\left(\frac{1}{c} + 1\right)x_1^{-\frac{2c+1}{c}}y(z) - \frac{2}{c}x_1^{-\frac{2c+1}{c}}zy'(z). \quad (5.56)$$

The second term of the right hand side

$$\begin{aligned} \mathcal{F}_\phi &= C4(2c+2)\phi^{2c+1} + d1 c\phi^{c-1}\psi_1\psi_2 + c1 c\phi^{c-1}\omega_1\omega_2 - \frac{8}{\phi^3}\omega_1\omega_2\psi_1\psi_2 \\ &= C4(2c+2)x_1^{-\frac{2c+1}{c}}y(z)^{2c+1} + d1 cx_1^{-\frac{2c+1}{c}}y(z)^{c-1}\delta_1(z)\delta_2(z) \\ &\quad + c1 cx_1^{-\frac{2c+1}{c}}y(z)^{c-1}\rho_1(z)\rho_2(z) - x_1^{-\frac{2c+1}{c}}\frac{8}{y^3(z)}\rho_1(z)\rho_2(z)\delta_1(z)\delta_2(z). \end{aligned}$$

By the cancelation of the common factor $x_1^{-\frac{2c+1}{c}}$, we finally obtain the super ODE

$$\begin{aligned} (z^4 - z^2)y'' + (2z^3 + \frac{2}{c}z)y' \\ = \frac{1}{c}\left(\frac{1}{c} + 1\right)y + C4(2c+2)y^{2c+1} + d1 cy^{c-1}\delta_1\delta_2 + c1 cy^{c-1}\rho_1\rho_2 - \frac{8}{y^3}\rho_1\rho_2\delta_1\delta_2, \end{aligned}$$

where $y, \delta_1, \delta_2, \rho_1$ and ρ_2 are functions of z .

Similarly, the invariants $z, y(z), \delta_1(z), \delta_2(z), \rho_1(z)$ and $\rho_2(z)$ also reduce the other four super PDE in (5.18)

$$\begin{aligned} (\psi_1)_{x_2} &= (\psi_1)_{x_1} + \mathbf{IF}_{\omega_2}, \\ (\psi_2)_{x_2} &= -(\psi_2)_{x_1} + \mathbf{IF}_{\omega_1}, \\ (\omega_1)_{x_2} &= -(\omega_1)_{x_1} + \mathbf{IF}_{\psi_2}, \\ (\omega_2)_{x_2} &= (\omega_2)_{x_1} + \mathbf{IF}_{\psi_1}, \end{aligned}$$

to super ODEs

$$\begin{aligned}
(z^2 + z)\delta'_1 &= I c1 y^c \rho_1 + \frac{4 I}{y^2} \rho_1 \delta_1 \delta_2; \\
(z^2 - z)\delta'_2 &= -\frac{c+2}{c} \delta_2 - I c1 y^c \rho_2 - \frac{4 I}{y^2} \rho_2 \delta_1 \delta_2; \\
(z^2 - z)\rho'_1 &= I d1 y^c \delta_1 + \frac{4 I}{y^2} \rho_1 \rho_2 \delta; \\
(z^2 + z)\rho'_2 &= \frac{c+2}{c} \rho_2 - I d1 y^c \delta_2 - \frac{4 I}{y^2} \rho_1 \rho_2 \delta_2;
\end{aligned}$$

5.5 Further discussion

The substitution we made in Section 5.2.1 of introducing two new variables ω_1 and ω_2 to replace $\bar{\psi}_1$ and $\bar{\psi}_2$ brings us into a bigger space where ψ_1, ψ_2, ω_1 and ω_2 are treated as four independent variables. For this bigger space, we have shown that we could find hidden superanalytic supersymmetry. The resulting invariants of this supersymmetry reduce the super PDE system to a super ODE system. Future research involves the consequences for the original variables $\bar{\psi}_1, \bar{\psi}_2$. Also, in future work, it is interesting to investigate the infinite dimensional supersymmetry groups for the cases with nontrivial potential F .

Chapter 6

Discussion and future work

The main results of this thesis have been presented in Chapter 3, 4 and 5. A summary and discussion of the main results will be given. Future work will also be discussed.

6.1 Concluding remarks

Symbolic computation research about supersymmetry is a strong and evolving area. Ayari [19, 20], Hussin [20] and Cheb-Terrab [15] have their own symbolic implementations in Maple. Wolf and his collaborators [21, 22, 23, 25, 29] have developed powerful algorithms in REDUCE for computation of polynomials of supersymmetries.

In order to make this thesis self-contained, we first introduced the infinitesimal method for getting the defining system of a given super differential equation in Chapter 3. This method is applied to two examples, the second order super ODE (3.8) and the super KdV equation (3.9). The first example is relatively easy and the second example is more complicated. We obtain the defining system for each example and work out the Lie super-algebra structure (supercommutator table) by heuristic integrals. Part of these works, such as the structure constants of Lie supersymmetry of super KdV example by integration has been done by Ayari [19, 20] Hussin [20]. My work focuses on algorithmic aspects of the reduction of supersymmetry defining system and the algorithmic determination of the structure of the Lie supersymmetry algebra without heuristic integration.

Rather than using heuristic integrals, we contribute an alternative method of the determination of structure constants of Lie supersymmetry of finite dimensional super differential equations. The new method is inspired by I. Lisle and G. Reid's methods for the determination of the structure constants for non-super differential equations. They developed a symbolic algorithm of the determination of the structure constants by inducing a commutator on initial data space, where computations can be done algorithmically via the existence uniqueness theorem. Our algorithm follows the same approach. However, there are technical difficulties that occur in the super case which means the generalization of Lisle and Reid's algorithm is not trivial.

For the super case, there are complications. Under a certain ranking of the derivatives, we define *regular* super differential equations by the parity of the coefficient of the leading term of the given super differential equation. If the coefficient of the leading term is even, then it is a regular super differential equation. A regular super differential equation can be written in solved form with respect to their leading derivative. But an irregular super differential equation can not be written in solved form with respect to their leading derivative since the odd coefficient of the leading derivative is not invertible.

Being able to write a system in solved form is crucial for us. The underlying theory of the Riquier bases (the differential analog of Gröbner bases) depends on inverting coefficients of leading derivatives and computing integrability conditions. We develop the MONO expansion algorithm to address the difficulty of irregular super differential equation systems. MONO expansion decomposes super functions by their odd variable monomials. Figure (4.2.1) outlines the MONO expansion procedure. Irregular super differential equation systems are converted into regular super differential equation systems by the MONO expansion of super functions. Then the coefficients of odd variable monomials are computed for each differential equation in the resulting system. The new system is formed and called the reduced defining system. That system is regular and does not depend on odd variables and more important. In Section 4.3.2, we show that how MONO expansion algorithm reduces the irregular defining system of the super KdV equation to a regular defining system.

Once irregular super differential equations are converted to regular equations, the existence and uniqueness for the non-super case proved by Rust, Reid and Wittkopf [18] can be applied. Under the assurance of the existence and uniqueness theorem, we develop a structure constant determination algorithm by using two copies of initial data of the parametric derivatives. This method of the determination of structure constants is algorithmic and programmable unlike the previous heuristic method based on integrations.

The third contribution shows that how to use the Maple `physics` package to help us to find hidden supersymmetry for a certain class of super Lagrangian models with an unspecified potential function. Firstly, we investigate the defining system of the Euler-Lagrange equations of the given super Lagrangian. In order to apply `rifsimp` and `initialdata` to the defining system, we apply MONO expansion to the infinitesimals and potential function to obtain it in reduced form. Two extreme cases are analyzed in the thesis. One extreme case is the with zero potential. The conclusion is that it is solvable and admits an infinitely dimensional supersymmetry group. The other extreme case is when the potential is nontrivial enough. This is imposed by the constraint that the third order derivative of the coefficient of the highest order term of the potential is nonzero. With the Maple option of `casesplit`, this leads to thousands of cases. The most interesting cases are the maximal finite-dimensional cases. In this thesis, we give a detailed analysis for one of the maximal finite-dimensional cases. The conclusions for this particular case are: the infinitesimals and potential are found explicitly; the supercommutator table is given. For a subclass of this particular case, at least one hidden supersymmetry has been found explicitly. We show that it leaves the Euler-Lagrange system invariant and the Euler-Lagrange PDE system can be reduced to ODE system by this supersymmetry using super invariants.

6.2 Future work

This thesis focuses on finite-dimensional Lie supersymmetry algebras of super differential equations. The algorithm for the determination of structure constants for the finite-dimensional supersymmetry groups is a good foundation for research about the

determination of structure of infinite-dimensional supersymmetry groups. In fact, it is usually impossible to write each the Lie superalgebra generators of infinite dimensional Lie superalgebra explicitly. However, our algorithm for the determination of structure constants does not depend on explicitly obtaining the Lie superalgebra generators. Hence, our algorithm should give a good direction for the determination of structure of infinite-dimensional supersymmetry groups of super differential equations.

Also, recall Figure 5.4.1 in Section 5.4. There are 251 infinite cases which have not been investigate. We will continue our work for finding the interesting hidden supersymmetries for those infinite-dimensional cases.

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Appendix A

Maple coding

Throughout the thesis, Maple is used as a powerful solving tool for generating the defining system and finding the symmetry properties of super differential equations. This is especially true for large systems which are extremely difficult for hand calculations. We provide some Maple procedures for calculations in Chapter 4 and Chapter 5.

A.1 MONO code

In Section 4.3, we introduce MONO expansion for decomposing a super function by its odd variable monomials.

```

Mono := proc(coeffnameE, coeffnameO, coeffindeps, oddvars)
    local T, S, MonoListE, MonoListO, j, k, ff;
    T := combinat:-cartprod([seq([0,1], i=1..nops(oddvars))]);
    S := NULL;
    while not(T[finished]) do
        S := S, T[nextvalue]();
    end do;
    MonoListE := NULL;
    for k from 1 to nops([S]) do
        if type(add(S[k][i], i=1..nops(S[1])), odd)=false then

```

```

        MonoListE:=MonoListE, product(oddvars[i]^(S[k][i]),
                                      i=1..nops(S[1]));
    end if;
end do;
userinfo(2, Mono, 'MonoListE = ', MonoListE);
MonoList0 := NULL;
for k from 1 to nops([S]) do
    if type(add(S[k][i],i=1..nops(S[1])),even)=false then
        MonoList0 := MonoList0,product(oddvars[i]^(S[k][i]),
                                      i=1..nops(S[1]));
    end if;
end do;
userinfo(2, Mono, 'MonoList0 = ', MonoList0);
ff := add(coeffnameE[h](op(coeffindeps))
          *[MonoListE][h], h = 1 .. nops([MonoListE]))
+
add(coeffname0[h](op(coeffindeps))
    *[MonoList0][h], h = 1 .. nops([MonoList0]));
return(ff);
end proc:

```

Glossary:

- Mono: the MONO expansion procedure.
- coeffnameE: even coefficient names in the output expansion.
- coeffname0: odd coefficient names in the output expansion.
- coeffindeps: independences of coeffnameE and coeffname0.
- oddvars: the odd variables of the given super function.
- MonoListE: odd variable monomials with even parity.

- `MonoList0`: odd variable monomials with odd parity.
- `ff`: the mono expansion of the given super function.

Code examples:

1. Recall that in section 5.3, we need to do MONO expansion for odd infinitesimals $\Lambda_1(x_1, x_2, \psi_1, \omega_2)$, $\Lambda_2(x_1, x_2, \psi_2, \omega_1)$, $\Omega_1(x_1, x_2, \psi_2, \omega_1)$ and $\Omega_2(x_1, x_2, \psi_1, \omega_2)$. Then the MONO input is

```

Lambda[1](x[1], x[2], psi[1], omega[2])
    = Mono(P01, PE1, [x[1], x[2]], [psi[1], omega[2]]);
Lambda[2](x[1], x[2], psi[2], omega[1])
    = Mono(P02, PE2, [x[1], x[2]], [psi[2], omega[1]]);
Omega[1](x[1], x[2], psi[2], omega[1])
    = Mono(P03, PE3, [x[1], x[2]], [psi[2], omega[1]]);
Omega[2](x[1], x[2], psi[1], omega[2])
    = Mono(P04, PE4, [x[1], x[2]], [psi[1], omega[2]]);

```

Maple returns us:

$$\begin{aligned}
 \Lambda_1(x_1, x_2, \psi_1, \omega_2) &= PO1_1(x_1, x_2) + \psi_1 \omega_2 PO1_2(x_1, x_2) \\
 &\quad + PE1_1(x_1, x_2) \omega_2 + PE1_2(x_1, x_2) \psi_1, \\
 \Lambda_2(x_1, x_2, \psi_2, \omega_1) &= PO2_1(x_1, x_2) + \psi_2 \omega_1 PO2_2(x_1, x_2) \\
 &\quad + PE2_1(x_1, x_2) \omega_1 + PE2_2(x_1, x_2) \psi_2, \\
 \Omega_1(x_1, x_2, \psi_2, \omega_1) &= PO3_1(x_1, x_2) + \psi_2 \omega_1 PO3_2(x_1, x_2) \\
 &\quad + PE3_1(x_1, x_2) \omega_1 + PE3_2(x_1, x_2) \psi_2, \\
 \Omega_2(x_1, x_2, \psi_1, \omega_2) &= PO4_1(x_1, x_2) + \psi_1 \omega_2 PO4_2(x_1, x_2) \\
 &\quad + PE4_1(x_1, x_2) \omega_2 + PE4_2(x_1, x_2) \psi_1.
 \end{aligned}$$

2. Also in Section 5.3, the MONO expansion for the potential $F(\phi, \psi_1, \psi_2, \omega_1, \omega_2)$ is

```

F(phi, psi[1], psi[2], omega[1], omega[2])
  = Mono(PE, PO, [phi],
          [psi[1], psi[2], omega[1], omega[2]]);

```

The Maple output is

$$\begin{aligned}
F(\phi, \psi_1, \psi_2, \omega_1, \omega_2) = & PE_1(\phi) + PE_2(\phi)\omega_1\omega_2 + PE_3(\phi)\psi_2\omega_2 + PE_4(\phi)\psi_2\omega_1 \\
& + PE_5(\phi)\psi_1\omega_2 + PE_6(\phi)\psi_1\omega_1 + PE_7(\phi)\psi_1\psi_2 \\
& + PE_8(\phi)\psi_1\psi_2\omega_1\omega_2 \\
& + PO_1(\phi)\omega_2 + PO_2(\phi)\omega_1 + PO_3(\phi)\psi_2 + PO_4(\phi)\omega_1\omega_2\psi_2 \\
& + PO_5(\phi)\psi_1 + PO_6(\phi)\omega_1\omega_2\psi_1 + PO_7(\phi)\omega_2\psi_1\psi_2 \\
& + PO_8(\phi)\omega_1\psi_1\psi_2.
\end{aligned}$$

A.2 Euler-Lagrange code

To generate the Euler-Lagrange equation for a give Lagrangian automatically, we have written the Maple procedure EL

```

EL := proc(Lag, t::list, U::list)
    local j, ELeqns, Ut, JU, JUt, JLAG;
    lprint('Indep vars = ', t, 'Dep vars = ', U);
    JU := ToJet(U, U);
    lprint('Jet form dep vars =JU=', JU);
    Ut := [seq(diff(U[j],t), j = 1 .. nops(U) )];
    JLAG:= ToJet(Lag, U);
    lprint('JLAG=', JLAG);
    ELeqns:= NULL;
    for j from 1 to nops(U) do
        ELeqns := ELeqns,diff(JLAG,JU[j]) -
ToJet(add(diff(FromJet(diff(JLAG,(JU[j]))[t[k]]),U),t[k]),

```



```

                                k = 1 .. nops(t)),U);
    end do;
    return [ELeqns];
end proc;

```

Glossary:

- EL: the name of the Euler-Lagrange equation procedure.
- Lag: the given Lagrangian.
- t::list: the list of independent variables in Lag.
- U::list: the list of dependent variables in Lag.
- JU: the jet notation of U.
- Ut: the derivatives of U.
- JLAG: the jet notation of Lag.
- ELeqns: the output Euler-Lagrange equations.

Code example: the input super Lagrangian is

$$\begin{aligned}
 L = & \frac{1}{2}(-\phi_{x_1}^2 + \phi_{x_2}^2) \\
 & + \frac{i}{2}(\psi_{1x_1}\bar{\psi}_2 - \psi_{2x_1}\bar{\psi}_1 - \psi_{1x_2}\bar{\psi}_2 - \psi_{2x_2}\bar{\psi}_1 + \bar{\psi}_{2x_1}\psi_1 - \bar{\psi}_{1x_1}\psi_2 - \bar{\psi}_{2x_2}\psi_1 - \bar{\psi}_{1x_2}\psi_2) \\
 & + F(\phi, \psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2).
 \end{aligned}$$

Define the set of dependent variables first:

```
DepVar := [phi, psi[1], psi[2], omega[1], omega[2]](x[1], x[2]);
```

Then send L to EL:

```
EularLag := EL(L, [x[1], x[2]], DepVar);
```

Maple returns us

```

EulerLag := [diff(F(phi, psi[1], psi[2], omega[1], omega[2]), phi)
+phi[x[1], x[1]]-phi[x[2], x[2]],
-I*omega[2][x[1]]+I*omega[2][x[2]]
+diff(F(phi, psi[1], psi[2], omega[1], omega[2]), psi[1]),
I*omega[1][x[1]]+I*omega[1][x[2]]
+diff(F(phi, psi[1], psi[2], omega[1], omega[2]), psi[2]),
I*psi[2][x[1]]+I*psi[2][x[2]]
+diff(F(phi, psi[1], psi[2], omega[1], omega[2]), omega[1]),
-I*psi[1][x[1]]+I*psi[1][x[2]]
+diff(F(phi, psi[1], psi[2], omega[1], omega[2]), omega[2])],

```

which is Euler-Lagrange equation system (5.5) in Section 5.1.

Vita

Curriculum Vita

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Post-secondary education and degrees:

The University of Western Ontario, London, Ontario, Canada
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Honors and awards:

Graduate Student Teaching Award,
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Related work experience:

Research Assistant (G.R.A), 2009-2015

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Supervisor: Dr. Greg Reid

Research area: Symbolic Computation

Developed new algorithms for super differential equations by Lie supersymmetry methods.

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Teaching first year Calculus, Calculus 1000B

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Various first year and second year Calculus, first year Linear Algebra

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