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Thermal Convection in Non-Fourier Fluids and Application to Liquid Helium II

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Abstract

This thesis examines different conditions for which non-Fourier effects can be significant in the flow of fluids. Non-Fourier fluids of dual-phase-lagging type (DPL) possess a relaxation time and a retardation time, reflecting the delay in the response of the heat flux and the temperature gradient with respect to one another. For non-Fourier fluids of single-phase-lagging type (SPL) the retardation time is zero. Non-Fourier fluids span a wide range of applications, including liquid helium, nanofluids and rarefied gases. The parallels between non-Fourier fluids and polymeric solutions are established.

The instability of steady natural convection of a thin layer of non-Fourier fluid (SPL) between two horizontal (and vertical) surfaces maintained at different temperatures is studied. The SPL model is particularly relevant to liquid helium II, and nanofluids with high nanoparticle concentration. Linear stability analysis is employed to obtain the critical state parameters such as critical Rayleigh (Grashof) numbers. In both cases, as the fluid becomes more non-Fourier, oscillatory convection increasingly becomes the mode of preference, compared to both conduction and stationary convection. Critical Rayleigh (Grashof) number decreases for fluids with higher non-Fourier levels.

By invoking the role of the eigenvectors to detect and quantify short-time behaviour, transient growth of energy of disturbances is studied. The energy of the perturbations is introduced in terms of the primary variables as a disturbance measure in order to quantify the size of the disturbance. It is found that nonlinearities are not required for the energy growth, and a significant energy growth can be observed even if the flow is stable.

The post-critical convective state for Rayleigh-Benard convection is studied using a nonlinear spectral-amplitude-perturbation approach in a fluid layer heated from below. In the spectral method the flow and temperature fields are expanded periodically along the layer and orthonormal shape functions are used in the transverse direction. A combined amplitude-perturbation approach is developed to solve the nonlinear spectral system in the post critical range, even far from the linear stability threshold. Also, to leading order, the Lorenz model is
recovered. Comparison with experimental results is made and a very good qualitative agreement is obtained.

**Keywords**

Rayleigh-Benard convection, vertical slot convection, hyperbolic heat conduction, non-Fourier heat transfer, linear stability analysis, transient energy growth, second-sound wave, spectral method, the Lorenz model, liquid helium II, nanofluids, rarefied gases.
Co-Authorship Statement

The following thesis contains materials that are published, or submitted, or finalized for submission for publication in technical journals as listed below.

**Chapter 2:** On non-Fourier Effects in Macro- and Micro-scale non-Isothermal Flow of Liquids and Gases
Authors: M. Niknami, R. E. Khayat, D. F. Stranges, R. M. H. Khorasany
Status: A version of this chapter is submitted to "International Journal of Thermal Science".

Section 2.5 is written and review by all authors. Section 2.6 is written by D. F. Stranges and reviewed by M. Niknami and R. E. Khayat. The other sections are written and reviewed by M. Niknami and R. E. Khayat.

**Chapter 3:** Energy Growth of Disturbances in a non-Fourier Fluid
Authors: M. Niknami, R. E. Khayat

All the research, developmental and simulation work presented in this chapter was performed by M. Niknami. The manuscript was written by M. Niknami and reviewed by R. E. Khayat.

**Chapter 4:** Thermal Convection in non-Fourier Fluids and Application to Liquid Helium II
Authors: M. Niknami, R. E. Khayat
Status: A version of this chapter will be submitted to Journal of "Physics of Fluids".

All the research, developmental and simulation work presented in this chapter was performed by M. Niknami. The manuscript was written by M. Niknami and reviewed by R. E. Khayat.

**Chapter 5:** A Combined Spectral- Amplitude Approach for Systematic Mode Selection in Thermal Convection
Authors: M. Niknami, Z. U. Ahmed, B. Albaalbaki R. E. Khayat

All the research, developmental and simulation work presented in this chapter was performed by M. Niknami. The manuscript was written by M. Niknami and reviewed by Z. U. Ahmed, B. Albaalbaki and R. E. Khayat.
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List of Abbreviations, Symbols, Nomenclature

Abbreviations

LHe II Liquid Helium II
DPL Dual Phase Lag
MD Molecular Dynamics
NF NanoFluid
NP NanoParticle
SPL Single Phase Lag

Nomenclature used in Chapter 1

C Cattaneo number
$C_T$ Heat wave velocity
$c_p$ Specific heat capacity at constant pressure
$c_v$ Specific heat capacity at constant volume
D Length scale
E Elasticity number
K Thermal conductivity
P Pressure
Pr Prandtl number
$q$ Dimensional heat flux vector
$r$ Position vector
S Entropy per unit mass
T Temperature
$T_{H-T_C}$ Temperature difference
$t$ Time
$t_p$ The time at which the temperature of the embedded thermocouples changes from the initial temperature
$v_n$ Velocity of normal component
$v_s$ Velocity of superfluid component
$x_p$ Position of the thermocouple of experiment in Figure 1.1

Greek letters

$\rho$ Density
$\rho_s$ Density of the superfluid component
$\rho_n$ Density of the normal component
$\kappa$ Thermal diffusivity
$\tau$ Relaxation time
Dimensionless temperature

Viscosity of the normal component

**Nomenclature used in Chapter 2**

- \(a\) Constant (2.1 < \(a\) < 3)
- \(a_v\) Interfacial area per unit volume
- \(C\) Cattaneo number
- \(\bar{c}\) Mean molecular speed
- \(c_s\) Sound speed
- \(c_p\) Specific heat at constant pressure
- \(D\) Length scale, or typical gap distance
- \(d\) Atomic diameter
- \(d_{NP}\) Nanoparticle diameter
- \(E\) Elasticity number
- \(E_b\) Bulk modulus of elasticity
- \(Ec\) Eckert number
- \(G\) Pressure gradient
- \(Gr\) Grashof number
- \(g\) Gravitational acceleration
- \(j = \nabla \cdot q\)
- \(K\) Thermal conductivity
- \(K' = K - K_F\) Change in the thermal conductivity of the solvent
- \(K_F\) Thermal conductivity of Fourier fluid
- \(Kn\) Knudsen number
- \(K_{NP}\) Thermal conductivity of nanoparticles
- \(K_{NP/F}\) Coupling term arising from the averaging process
- \(k_B\) Boltzmann constant
- \(m\) Mass of a single molecule
- \(n\) Number density of atoms
- \(P\) Pressure
- \(Pr\) Prandtl number
- \(Q\) Heat flux vector
- \(Q'\) Maxwell-Cattaneo heat conduction
- \(q\) Dimensionless heat flux vector
- \(q_x\) Streamwise heat flux component
- \(q_y\) Transverse heat flux component
- \(r_s\) Wigner-Seitz radius
- \(Ra\) Rayleigh number
- \(Ra_c\) Critical Rayleigh number
- \(S\) Dimensionless retardation time for nanofluids
T  Temperature
\( t \)  Time
\( u \)  Streamwise velocity component
\( v \)  Dimensionless velocity vector

Greek letters
\( \Delta \)  Laplacian operator
\( \nabla \)  Gradient operator
\( \tau \)  Relaxation time
\( \kappa \)  Thermal diffusivity
\( \delta(\cdot) \)  Li-type derivative
\( \frac{\delta t}{\delta t} \)  Dynamic viscosity
\( \delta T \)  Temperature difference
\( \alpha_T \)  Thermal expansion coefficient
\( \nu \)  Kinematic viscosity
\( \sigma \)  Stress tensor
\( \theta \)  Dimensionless temperature
\( \zeta \)  Adiabatic index
\( \lambda \)  Mean free path
\( \sigma_0 \)  Collision cross section between two atoms
\( \tau_Q \)  Relaxation time for NFs
\( \tau_T \)  Retardation time for NFs
\( \gamma \)  Thermal conductivity ratio of the solvent to that of the NF solution
\( \delta_F \)  Mean interaction length in the base fluid phase
\( \delta_{NP} \)  Mean interaction length in the NP phase
\( \phi \)  Nanoparticle concentration
\( \Sigma' \)  Upper-convective Maxwell stress
\( \Sigma \)  Stress of the solution
\( \mu_N \)  Newtonian viscosity

Nomenclature shared in Chapters 3 and 4

C  Cattaneo number
\( C_H \)  Threshold Cattaneo number
\( c_p \)  Specific heat at constant pressure
\( g \)  Gravity acceleration
K  Thermal conductivity
KE  kinetic energy of the perturbations
\( P \)  Dimensional pressure
\( p \)  Dimensionless pressure deviations from the base state
Pr  Prandtl number
Q  Dimensional heat flux vector
q  Dimensionless heat flux vector
s  Time evolution
T  Temperature
<> Integration over volume cell

**Greek letters**

\( \alpha_t \)  Coefficient of volume expansion
\( \delta T \)  The temperature difference
\( \kappa \)  Thermal diffusivity
\( \Omega \)  Cell volume
\( \omega \)  Frequency
\( \tau \)  Relaxation time
\( \nabla \)  Gradient operator
\( \Delta \)  Laplacian operator
\( \rho \)  Density
\( \mu \)  Dynamic viscosity
\( \theta \)  Dimensionless temperature deviation from the base state

**Nomenclature used in Chapter 3**

D  Length scale (distance between plates)
dKE\(_R\)/dt  Weighted kinetic energy rate
e\(_z\)  Unit vector in the y direction
E  Energy norm
\( \{ \mathbf{E}_j \} \)  Eigenvectors
G  Maximum possible energy growth
k  Wavenumber in the x direction
n  Mode number
Ra  Rayleigh number
\( \{ s_{j} \} \)  Eigenvalues
sup  Supremum
\( \mathbf{V}(U,W) \)  Dimensional velocity vector
\( \mathbf{v}(u,w) \)  Dimensionless velocity vector

**Subscripts and superscripts**

B  Base fluid
i  Intersection
Nomenclature used in Chapter 4

\begin{itemize}
  \item \(c_s\) \quad \text{Sound velocity}
  \item \(c_T\) \quad \text{Dimensionless heat wave propagation speed}
  \item \(c_v\) \quad \text{Specific heat at constant volume}
  \item \(e_y\) \quad \text{Unit vector in the y direction}
  \item \(Gr\) \quad \text{Grashof number}
  \item \(h\) \quad \text{Length scale (distance between plates)}
  \item \(q\) \quad \text{Wavenumber in the x direction}
  \item \(S\) \quad \text{Entropy OR Dimensionless retardation time (when it is noted in the text)}
  \item \(t_B\) \quad \text{Dimensional characteristic time of buoyancy}
  \item \(t_s\) \quad \text{Dimensional characteristic time of the second-sound wave}
  \item \(t_k\) \quad \text{Dimensional characteristic time of thermal diffusion}
  \item \(t_v\) \quad \text{Dimensional characteristic time of viscous dissipation}
  \item \(u_T\) \quad \text{Dimensional second-sound velocity}
  \item \(V(U,V)\) \quad \text{Dimensional velocity vector}
  \item \(v(u, v)\) \quad \text{Dimensionless velocity vector}
\end{itemize}

**Greek letters**

\begin{itemize}
  \item \(\tau_R\) \quad \text{Period of R-processes}
  \item \(\tau_N\) \quad \text{Period of N-processes}
  \item \(\hbar\) \quad \text{Planck’s constant}
  \item \(\rho_s\) \quad \text{Density of the superfluid component}
  \item \(\rho_n\) \quad \text{Density of the normal component}
\end{itemize}

**Subscripts and superscripts**

\begin{itemize}
  \item \(L, R\) \quad \text{Left and Right planes}
  \item \(x, y\) \quad \text{Derivative wrt x, y directions}
  \item \(t\) \quad \text{Derivative wrt time}
\end{itemize}
Nomenclature used in Chapter 5

\( c_p \)  Specific heat at constant pressure

\( D \)  Distance between planes

\( e_z \)  Unit vector in the \( z \) direction

\( g \)  Gravity acceleration

\( K \)  Thermal conductivity

\( k \)  Wavenumber in the \( x \) direction

\( Lcp \)  Linear matrix operator

\( Ncp \)  Non-Linear matrix operator

\( Nu \)  Nusselt number

\( p \)  Dimensionless pressure deviations from the base state

\( Pr \)  Prandtl number

\( Ra \)  Rayleigh number

\( Ra_{cr} \)  Critical Rayleigh number

\( T \)  Temperature

\( u_{cp} \)  Perturbation from the base state

\( v(u,w) \)  Velocity vector

Greek letters

\( \alpha_T \)  Coefficient of volume expansion

\( \delta T \)  The temperature difference

\( \kappa \)  Thermal diffusivity

\( \varepsilon \)  Expansion parameter

\( \nabla \)  Gradient operator

\( \Delta \)  Laplacian operator

\( \rho \)  Density

\( \mu \)  Dynamic viscosity

\( \theta \)  Dimensionless temperature deviation from the base state

\( \psi \)  Streamfunction

Subscripts and superscripts

\( 0 \)  Reference point

\( s \)  Base flow solution
Chapter 1

1 Introduction

1.1 Objectives

This thesis is presented as a collection of manuscripts submitted to technical journals. The link among these studies is established by drawing attention in this chapter to the common contexts of Fourier heat transfer versus non-Fourier heat transfer. Because of its simplicity, the non-Fourier effects are discussed for conduction in solids first, which is followed by discussion of non-Fourier effects in fluids. The main questions to be addressed in this thesis are:

- The concept of finite speed of heat conduction (versus infinite speed of heat conduction),
- The typical situations (specially in fluids) where non-Fourier effects are important,
- Obtaining an expression to provide us with a rough estimation of the level of non-Fourier effects for fluids,
- Determination of the factors that can affect the non-Fourier character of a fluid,
- The influence of the non-Fourier character of a fluid on the onset of thermal convection (stability picture),
- The deviation from the onset of convection in ordinary (Fourier) fluids in two different configurations (convection between two horizontal and two vertical plates).

1.2 Why non-Fourier heat conduction?

Fourier equation of heat conduction results in a parabolic equation for the temperature field when combined with the conservation of energy law. The parabolic temperature equation implies an infinite speed for heat propagation, which is in good agreement with experimental data in many situations. However, there are some cases in which the Fourier equation is not adequate to describe the heat conduction process. More precisely, Fourier
equation of heat conduction is diffusive and cannot predict the finite temperature propagation speed in transient situations. The Cattaneo equation [1] was the first explicit mathematical correction of the propagation speed defect in the Fourier equation, which allows for the existence of heat waves propagate at finite speeds. In the Cattaneo equation (1.1), a new time constant (known as the relaxation time) is introduced in the transient term that is added to the conventional Fourier equation:

\[
q(r,t) + \tau \frac{\partial q(r,t)}{\partial t} = -K\nabla T(r,t),
\]

where \(q, r, t, T\) are the heat flux vector, the position vector, the time, and the temperature, respectively. \(K\) represents the thermal conductivity, \(\tau\) denotes the relaxation time, and \(\nabla\) is the gradient operator. The relaxation time, \(\tau\), represents the time lag for the beginning of heat flow after a temperature gradient is applied on the medium. In other words, heat flow grows (ceases) gradually with a relaxation time \(\tau\) (not immediately) after a temperature gradient is imposed (removed). The thermal relaxation time is on the order of \(10^{-12}\) to \(10^{-15}\) seconds for many applications [2] and consequently, the Cattaneo equation (1.1) reduces to the classical Fourier model. However, in some situations the relaxation time (or the non-Fourier character of the material) is noticeable such as heat transfer in processed meat [3], drying sand [4] and skin burns [5]. Heat wave was first observed experimentally in a solid sample (in solid \(^3\)Helium) [6]. Peshkov [7] also suggested that heat waves might be observed in lattice crystal materials, which was followed by several studies to observe the heat wave phenomena in different crystals of a solid: in LiF crystal [8], in pure alkali halide crystals such as NaF and NaI [9,10], in sapphire [11], and in Al\(_2\)O\(_3\) crystal [12].

Incorporating equation (1.1) together with the energy equation (in the case of no heat source and a one-dimensional sense) as

\[
\rho c_p \frac{\partial T(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x},
\]

(where \(\rho, c_p, \text{ and } x\) are the density, the heat capacity at constant pressure, and the position, respectively) leads to a hyperbolic equation for the temperature field
\[
\frac{\partial^2 T}{\partial t^2} - \frac{\kappa}{\tau} \frac{\partial^2 T}{\partial x^2} + \frac{1}{\tau} \frac{\partial T}{\partial t} = 0.
\]  (1.3)

In this equation, \( \kappa \) is the thermal diffusivity (\( \kappa = \frac{K}{\rho c_p} \)).

One of the experimental studies to indicate the existence of finite heat propagation speed was performed by Mitra et al. (1993) [3]. In this case, two identical processed meat samples (the thickness of each sample is approximately 20 mm) are brought in contact (Figure 1.1), where a layer of grease is applied at the interface. Grease has a high thermal conductivity and eliminates the effect of thermal contact resistance. The temperature of samples is kept at 8.2 °C (refrigerated) and 23.1 °C (room temperature), respectively. Two thermocouples are imbedded at each sample at distances of 6.3 and 6.6 mm from the surface of contact and one thermocouple is placed at the interface. When the samples are place in contact (time equal to zero), the temperatures of the thermocouples are recorded. To find the heat propagation velocity, the distance of thermocouple from the reference (contact surface) is divided by the time at which the temperature of the embedded thermocouples changes from the initial temperature (\( t_p \) in Figure 1.2). Figure 1.2 shows the variation of dimensionless temperature recorded by the thermocouple at \( x = 26.6 \) mm in the cold sample. The results of this experiment show that \( t_p \) for the thermocouple located at \( x = 26.3 \) mm is smaller than \( t_p \) for the thermocouple located at \( x = 26.6 \) mm (Figure 1.2), which implies that the time at which the temperature of the embedded thermocouples changes from the initial temperature increases with distance.

![Figure 1.1: Schematic of the experimental setup to show the non-Fourier effects in heat conduction.](image)
This observation shows the wave nature of heat, where unlike the prediction of Fourier equation, heat is not felt instantly at distant points and heat waves take a finite time to reach a particular point inside the meat sample.

Figure 1.2: Experimental non-dimensional temperature vs. time for the experiment of Figure 1.1 (for the thermocouple located at x = 26.6 mm) (from [13]).

In order to show the finite speed of heat propagation, non-Fourier conduction heat transfer for the experiment of Figure 1.1 is studied with the following set of initial and boundary conditions

\[
\begin{align*}
T(x < D/2, t = 0) &= T_{ri} \cos \left( \frac{\pi x}{D} \right), \quad T(x > D/2, t = 0) = T_{ci}, \\
T_t(x < D/2, t = 0) &= -T_{ri} \frac{\pi^2}{D^2} \cos \left( \frac{\pi x}{D} \right), \quad T_t(x > D/2, t = 0) = 0, \\
T(x = 0, t) &= T_{ri}, \quad T(x = D, t) = T_{ci}.
\end{align*}
\]

In this case, a pulse is applied on the small region of the sample (x < D/2). The problem is conveniently cast in dimensionless form by taking the D, D^2/κ to be, respectively, the typical length, and time scales (κ ≈ 0.0014 cm^2/s [13]). The dimensionless temperature, \( \theta \)
is given by \( \theta = (T - T_{ci})/(T_{ref} - T_{ci}) \), where \( T_{ref} = (T_{ri} + T_{ci})/2 \). Using the above mentioned scales, the dimensionless Cattaneo equation (1.3) becomes

\[
C \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} = 0.
\] (1.5)

Similarly, the set of dimensionless initial and boundary conditions are obtained as

\[
\begin{align*}
\theta(x < 0.5, t = 0) &= 2 \cos(\pi x), & \theta(x > 0.5, t = 0) &= 0, \\
\theta_t(x < 0.5, t = 0) &= -2\pi^2 \cos(\pi x), & \theta_t(x > 0.5, t = 0) &= 0, \\
\theta(x = 0, t) &= 2, & \theta(x = 1, t) &= 0.
\end{align*}
\] (1.6)

Note that the non-dimensional number, \( C \), appearing in (1.5) is the Cattaneo number and is given by \( C = \tau\kappa/D^2 \). The initial temperature profile (as a function of \( x \)) is illustrated in Figure 1.3. There is a difference between the initial experimental and analytical temperature profiles in the hot sample. The initial experimental temperature profile makes the mathematical solution more complex, whereas the physics behind the experiment can be explained by the initial analytical temperature profile as well. This problem will provide us with a good insight into the hyperbolic heat conduction and the finite speed of heat propagation.
Using the method of separation of variables, the heat equation (1.5) for the given initial conditions and boundary conditions (1.6) can be solved. To recover the experimental results of Figure 1.2, the wave velocity is obtained by dividing the experimental values of \( x_p \) (= 26.6 mm) and \( t_p \) (≈ 67 secs). Then, the value of \( C \) in (1.5) is increased gradually from zero until the same \( t_p \) is obtained in the temperature profile as in Figure 1.2 for the same position (\( x = 26.6 \) mm). The matches \( C \) value with the experimental results of Figure (1.2) is approximately 0.0014. The relaxation time can be obtained from \( \tau = \frac{CD^2}{\kappa} \) as 16 seconds (approximately), which can be confirmed from [3]. The analytical results are shown in Figure 1.4, where the variation of non-dimensional temperature vs. time is illustrated at two points (\( x_p = 26.3 \) mm and 26.6 mm). The prediction of Fourier equation of heat conduction (equation (1.5) with \( C = 0 \)) is also included in this figure. Based on the Fourier’s law of heat conduction (\( C = 0 \)), the temperature pulse propagates with an infinite speed through the entire sample. In other
words, the pulse is felt instantly everywhere, though with exponentially small amplitudes at distant points. As C increases, the finite speed of the temperature propagation results in a time lag (although very small for small C) between the time the initial disturbance is applied (t = 0) and the time the disturbance is felt at distant points. The time lag reflects the presence of non-Fourier effects. From Figure (1.4), it can be seen that the time lag increases with the distance (x) from the heat source (initial temperature pulse), which can be confirmed from the experiment [3], as explained before. Interestingly, after the steady state condition is established, the Cattaneo equation (1.5) and the Fourier equation (1.5 with C = 0) predict the same temperature (large time limit in the figure). Consequently, regarding solids and based on the Cattaneo equation (1.5), non-Fourier effects are only important in the transient time, and once the steady state condition is established, non-Fourier effects vanish. In contrast, as will be shown in this thesis, non-Fourier effects could become important in fluids even for the base flow solution (the steady state condition).

Figure 1.4: Analytical results of non-dimensional temperature vs. time for the experiment of Figure 1.1, at x = 26.3 mm and 26.6 mm (C = 0.0014).
The similarity of the experimental results in Figure 1.2 and the theoretical results based on the Cattaneo equation as shown in Figure 1.4 suggests that the Cattaneo equation of heat conduction enables us to capture the transient behaviour of heat conduction in a sample with non-homogenous inner structure (for instance processed meat) more accurately (compared to the Fourier equation). The obtained relaxation time for the processed meat is of the order of 16 s. Hence, this example suggests that the finite speed of heat propagation could be observed for a material with non-homogeneous inner structure even at room temperature conditions [14]. However, the mechanisms of heat conduction in materials with non-homogeneous inner structures are not clearly understood to date [14].

The variation of non-dimensional temperature against time at different distances is shown in Figure 1.5. For points with \( x < 0.5 \), which are initially affected by the temperature pulse, the temperature profile does not display a monotonic behaviour with respect to time. More precisely, the temperature decreases at the beginning (small time) to find a minimum, which is followed by an increasing trend until the steady state condition is reached. For points with \( x > 0.5 \), which are not initially affected by the temperature pulse, a time lag (delay) is observed initially. The longer the distance from the heat source, the more time is needed for the temperature pulse to be felt. During the time lag, the temperature is zero, and once the pulse is felt at a given point (with \( x > 0.5 \)), the temperature increases monotonically until the steady state condition is obtained. In Figure 1.6, the temperature profile through the entire thickness of the sample is illustrated (at different times). The initial temperature profile is kept here for reference. Note that how the temperature evolves from the initial distribution to the final steady state condition. Recall that the steady state temperature profile is the same as that obtained from the Fourier equation, where a linear temperature distribution is predicted between the two boundaries.
Figure 1.5: Variation of temperature against time at different positions (C=0.0014).

Figure 1.6: Variation of temperature against position at different times (C=0.0014).
Up to this point, the Cattaneo equation for processed meat of thickness 40 mm (totally) is found to be 0.0014, which is confirmed by the experimental results [3]. Although the time lag is observed in the results for \( C = 0.0014 \), the wavy response of temperature profiles is not very apparent. To illustrate the wave nature of heat propagation, \( C \) is increased to 0.14. Recall that \( C = \tau \kappa / D^2 \), where \( \tau \) and \( \kappa \) are the material properties and cannot be changed (at a given temperature). However, if the thickness of the processed meat sample is decreased from 40 mm to 4 mm, then \( C = 0.14 \) can be obtained. As expected, a wavy temperature profile is obtained with respect to time at different positions (Figure 1.7). Note that the steady state condition is established at a time much larger compared to that of Figure 1.5 (for \( C = 0.0014 \)). Moreover, the oscillations in the temperature distribution profiles through the entire sample are apparent in Figure 1.8. Again, note that the evolution time from the initial temperature distribution to the steady state condition increases significantly compared to that of Figure 1.6 (\( C = 0.0014 \)). The appeared fluctuations in the temperature profiles suggest the propagation of heat as a wave. However, because of the thermal diffusion effect, these oscillations are damped, and after some time (which depends on the Cattaneo number), the steady state condition is reached. Based on the Fourier’s law of heat conduction, unless a periodic heat flux (or temperature) is applied on the boundary(s) of the sample (Figure 1.1), no fluctuations can be observed in the temperature profiles (inset of Figure 1.8).

It might be of importance to see what would happen if heat propagates as a wave without any diffusion. In this case the diffusion term in (1.5) is zero

\[
\frac{\partial^2 T}{\partial t^2} = \frac{1}{C} \frac{\partial^2 T}{\partial x^2},
\] (1.7a)

and the heat wave equation (without diffusion term) is obtained. Heat wave equation (1.7a) is analogous to the well-known wave equation, and as a result, the dimensionless heat wave velocity, \( C_T \), can be found as

\[
C_T = \sqrt{\frac{1}{C}}.
\] (1.7b)
If the initial conditions are given as

\[
T(x, t = 0) = F(x), \quad \frac{\partial T}{\partial t}(x, t = 0) = G(x), \quad (1.8a)
\]

where \(F\) and \(G\) are arbitrarily functions, then the solution of the heat wave equation (1.7a) subjected to the initial conditions (1.8a) is

\[
T(x, t) = \frac{1}{2} \left[ F(x - \frac{\sqrt{1/C}}{C} t) + F(x + \frac{\sqrt{1/C}}{C} t) \right] + \frac{1}{2} \int_{x - \frac{\sqrt{1/C}}{C} t}^{x + \frac{\sqrt{1/C}}{C} t} G(s) ds, \quad (1.8b)
\]

which clearly shows the wavy shape of the temperature variation with time and position. However, the steady heat conduction can never be established if the heat wave equation (1.7a) is used, which is not realistic. Moreover, in the derivation of the heat wave equation, it was assumed that wave pulses are propagated without attenuation (diffusion term is neglected), which is physically impossible. Therefore, the diffusion always persists though it is very small (at low temperature applications). Consequently, Cattaneo equation of heat conduction (1.5a) has several desirable properties. In contrast to the Fourier equation of heat conduction, Cattaneo equation of heat conduction transmits waves of heat with a finite speed of \(\sqrt{1/C}\). Moreover, the heat waves are attenuated as a result of diffusion, and unlike the pure heat wave equation (1.7a), steady heat condition can be obtained.
Figure 1.7: Variation of temperature against time at different positions (C=0.14).

Figure 1.8: Variation of temperature against position at different times for C = 0.14, and the inset is for C = 0.
1.2.1 Other experimental observations of non-Fourier effects in solids

In another experimental study, using the pico-second and femto-second laser techniques, very fast physical processes are investigated [15]. Since the electron relaxation time in metals is $O(10^{-14} \text{s})$, the pump-probe technique is the only possible way to achieve a high time resolution at femtoseconds. As shown in Figure 1 in [15], a femto-second laser is split into a pump beam (to heat the sample) and a probe beam (to monitor the change of thermoreflectance). It is known that the change of thermoreflectance is proportional to the change in temperature. Figure 2 in [15] shows the back-probe data for Au films of 500, 1000, 2000, and 3000 Å thick. As the sample thickness increases, the delay time of the rise of the reflectivity change increases. Similar to the experimental results of Figure 1.7, the increase in time lag occurs as a result of the finite time, or equivalently the finite speed, needed for heat to propagate through the sample. Note, however, that the measured delay is very short (it takes only $\approx 100 \text{ fs}$ for heat to travel 1000 Å).

The applications of Micro-Electro-Mechanical Systems (MEMS) increased sharply over the last decade including industrial and medical fields [16]. In many microelectronic and photoelectronic devices, nano-scale dielectric films are important components [17]. In the design process of these small-scale devices, better understanding of the governing physical laws is crucial. Consequently, the study of thermal conduction in nano-scale dielectric films is an important step. As the definition of Cattaneo number ($C = \tau \kappa / D^2$) implies, non-Fourier effects could become noticeable for the small-scale applications (small D). In other words, the deviation of transient heat conduction from Fourier’s law could become significant when the Cattaneo number increases (as a result of decrease in D). Classical molecular dynamics (MD) simulations method has been extensively used to study the thermal conduction related problems such as nano-scale thermal conduction. By solving Newton equation of motion for all particles, the positions trajectories of a system of particles are predicted. Using MD method, it is shown [18] that for the heat conduction in carbon nanotubes, a distinct amount of heat is transported in a wave-like form, which implies that non-Fourier effects are noticeable. In a recent study [17], by using the classical MD method, unsteady thermal conduction in argon thin films is analyzed, where the thin film is subjected to a temperature increase at one surface. Similarly to what was
observed in Figure 1.2, the thermal conduction is found to have a wave nature with temperature oscillations at a given point and time. Moreover, it is shown [17] that the time needed for the film temperature distribution to reach steady state condition is much larger than that predicted by Fourier’s law. For instance, for a film thickness of 15 nm, the time needed to reach steady state condition based on the Fourier’s law is 19 ps, whereas the prediction from MD method is 170 ps. The results in [17] shows that as the relaxation time (or equivalently C) increases, the time needed to reach steady state temperature profile increases. Therefore, MD method shows that non-Fourier effects become important in nano-scale argon films.

1.3 Thermal convection in fluids with non-Fourier effects

In this thesis, thermal convection is studied in two different configurations: the Rayleigh-Benard convection and convection in the vertical slot. Rayleigh-Benard convection involves a fluid layer confined between two horizontal plates. The plates are perfect heat conductors, which enables one to assume that the temperature can be kept constant along the plate. The plates are also infinite in extent and the temperature of the lower plate is more than that of the upper. The fluid remains motionless when the temperature difference between two plates is less than a critical value. This state is called conduction where the fluid is at rest and a linear temperature profile is obtained interpolating the temperatures of upper and lower plates. However, because of thermal expansion, the fluid near the hot plate (lower) is lighter and tends to move up, which makes the situation intrinsically instable. As the temperature difference increases, the buoyancy force increases to the point that it can overcome the dissipative effects of viscous force and thermal diffusion (Figure 1.9a). Clearly, because of mass conservation, the fluid layer close to the hot plate cannot rise entirely, since there is no place for the fluid layer above it to go. Therefore, instability at a finite wavelength is encountered and convection rolls emerge (Figure 1.9b).
Figure 1.9: Convective motion in a layer of fluid heated from below (from [19]) (a) and Convective rolls (from [19]) (b).

For typical fluids (such as water), when the convection starts and the fluid begins to move, rolls, which are stationary and aligned parallel to the shorter horizontal side [20], [21], emerge. However, higher temperature difference results in an unstable stationary roll pattern and a time-dependent flow emerges [22]. From linear stability theory, it is expected that overstability (transient oscillatory flow) occurs at the onset, if two opposing effects exist [23]. For instance, a destabilizing temperature gradient in conjunction with rotation [24], or a stabilizing solute gradient [25-28], and thermo-solutal convection to overcome the principle of the exchange of stabilities [23]. In all mentioned cases, convection occurs at the onset in the form of oscillatory rolls (not stationary), which means that the roll speed varies with time.

For a viscoelastic fluid with sufficiently large elasticity and large Prandtl number ($Pr = \nu/\kappa$ and $\nu$ denotes the kinematic viscosity), overstability is expected to occur [29]. In this case, there is a competition between the processes of viscous (stress) relaxation and thermal diffusion, which are the two opposing factors acting on an element of fluid. In addition, the critical Rayleigh number for the onset of oscillatory convection is found to decrease as a function of increasing elasticity number ($E = \tau\nu/D^2$), which has the similar role as the Cattaneo number in the current study. Similarly, it is reported [30] that the oscillatory threshold (for $Pr = 10$) is less than the threshold for the onset of stationary Rayleigh-Benard convection for large elasticity numbers. Consequently, it is expected
that sufficiently viscoelastic fluids exhibit oscillatory convection at the onset. As it will be shown in Chapter 2, there is a parallel between fluids with non-Fourier effects and the viscoelastic fluids. For non-Fourier fluids, thermal relaxation process is important, which is analogous to viscous relaxation process in viscoelastic fluids. Consequently, it is anticipated to observe oscillatory convection (overstability) at the onset for a fluid with high non-Fourier character. Moreover, the oscillatory threshold is expected to be less than the threshold for the onset of stationary Rayleigh-Benard convection for sufficiently large Cattaneo numbers (C).

Non-Fourier effects can be significant in liquids. In this case, non-Fourier effects (second-sound waves) were studied both experimentally [7,31] and numerically [32] in liquid helium at low temperatures (T < 2.17 K). As mentioned previously, the second-sound is a heat transfer mechanism describing the propagation of heat as a wave. In addition, nanofluids also exhibit non-Fourier effects [33].

1.3.1 Liquid Helium

Helium has two common isotopes: $^4$He with mass number 4, which is the most abundant isotope and from now it will be denoted as He, and $^3$He with mass number 3. The boiling point of Liquid He (LHe) is 4.22 K (at atmospheric pressure), and when LHe is cooled below 2.17 K, it has a very intricate behaviour. For simplicity, the liquid Helium above 2.17 K is denoted as LHe I and the liquid Helium below 2.17 K is denoted as LHe II. The thermal conductivity of LHe II is orders of magnitudes larger than that of LHe I. Viscosity is also found to be particularly puzzling. If the capillary flow method is used to measure viscosity, LHe II shows no viscous behaviour and the viscosity is nearly zero, which explains the reason of formation of a thin film of LHe II that gradually fills a test tube lowered partly into a bath of LHe II (LHe II flows without friction up the tube’s wall). Therefore, LHe II has the capability of flowing through narrow channels without observable pressure drop [34]. On the other hand, if a set of oscillating discs is immersed in LHe II, and the viscosity is measured by the damping produced on the discs, then the viscosity of LHe II is found to be comparable to that of LHe I (about one-tenth that of air). Similarly, the oscillations of a torsion pendulum in LHe II gradually decays, which
proves the presence of viscous effects. This phenomenon is called the viscosity paradox in LHe II [34].

The thermo-mechanical effect (or fountain effect) is another complex phenomenon. In this case, as shown in Figure 1.10, two cylinders are connected at the bottom by a very thin tube that can block any viscous fluid (superleak). The two containers are initially filled with LHe II at the same temperature (the liquid levels are initially the same in both containers). When the temperature of one container increases, a rise in the liquid level (pressure rise) of that container is observed. The superfluid component of LHe II flows towards areas that the LHe II is heated [35] to cool that area and restores the uniform mixture of normal and superfluid. Therefore, the superfluid component flows into the hot container through the superleak, which can be made of porous materials. In contrast, since the normal component in the hot container is viscous, it cannot flow through the superleak. Consequently, the liquid level in the hot container increases. This hydrostatic pressure difference (or equivalently the difference between liquid levels in two containers) is proportional to the temperature difference ($T_H - T_C$), where $T_H$ ($T_L$) is the temperature of the hot (cold) container.

![Figure 1.10: Thermo-mechanical effect (fountain pressure) in LHe II.](image)

To tackle the above mentioned issues, Tisza proposed in 1938 [36] to model LHe II as comprising two components (the two-fluid model) with different physical properties. The first component has zero viscosity, and is called the superfluid component, which is responsible for the flow through narrow channels without any observable pressure drop. The second component of LHe II has a non-zero viscosity, and is called the normal component, which is responsible for the damping observed in the oscillating discs.
experiment. The densities of the superfluid and normal components are denoted by \( \rho_s \) and \( \rho_n \), respectively. The total density \( \rho \) of LHe II is then given as \( \rho = \rho_s + \rho_n \). At absolute zero, the ratio \( \rho_s/\rho \) approaches one, which implies that LHe II behaves like a pure superfluid. In contrast, at the transition temperature (2.17 K) \( \rho_s/\rho \) approaches zero, which means that LHe II behaves like a normal fluid. It was also suggested [36] that the superfluid component has no entropy and as a result, the normal component is responsible to carry the total entropy.

As mentioned above, the pressure difference is proportional to the temperature difference \((T_H - T_C)\). More precisely, it can be shown that the pressure difference is equal to \( \rho S(T_H - T_C) \), where \( S \) is the entropy per unit mass [37].

Imagine that the superleak is removed. Similar to the Poiseuille's flow through a wide channel, the normal component flows through the connecting tube (Figure 1.10) as a result of pressure difference. Then, because of mass conservation, a counterflow of the superfluid takes place from the cold to the warm container. The counterflow of superfluid component and normal component of liquid helium is somewhat analogous to convection currents in a fluid. Consequently, the conduction process in LHe II (based on the two-fluid model) is similar to the convection process in ordinary fluids, which can explain the significant thermal conductivity of LHe II compared to ordinary liquids.

The equation for the conservation of mass is

\[
\frac{\partial \rho}{\partial t} + \rho_s \nabla \cdot \mathbf{v}_s + \rho_n \nabla \cdot \mathbf{v}_n = 0 ,
\]

where \( \mathbf{v}_s \) and \( \mathbf{v}_n \) are the velocities of superfluid and normal components, respectively. It should be noted that the flow of the normal component and the counterflow of the superfluid component occurs immediately after a temperature difference is applied between the two containers. However, the total mass flow rate in the connecting tube is zero, since the flow of the normal component is cancelled by the counterflow of the superfluid component \((\rho \mathbf{v} = \rho \mathbf{v}_s + \rho \mathbf{v}_n = 0)\). On the other hand, the entropy (and hence heat) is carried away from the warm to the cold end by the normal component. In fact,
entropy flows from one place to another (not appear or disappear). For the two-fluid system, the equation of motion for the normal component is

\[
\rho_n \frac{\partial v_n}{\partial t} = -\frac{\rho_n}{\rho} \nabla P + \rho_s S V + \eta \nabla^2 v_n ,
\] (1.9b)

where \( P \) is the pressure and \( \eta \) is the viscosity of the normal component. The equation of motion for the superfluid component is

\[
\rho_s \frac{\partial v_s}{\partial t} = -\frac{\rho_s}{\rho} \nabla P - \rho_s S V ,
\] (1.9c)

Equations (1.9a-c) along with the equation for the conservation of entropy admit two different wave equations (after some manipulation) [34]: the first-sound, a pressure wave and the second-sound, an entropy wave. The dimensional second-sound wave velocity, \( C_T \), is obtained as [34,36]

\[
C_T^2 = \frac{\rho_s}{\rho_n} \frac{T S^2}{c_v} ,
\] (1.10)

where \( c_v \) is the specific heat at constant volume of the liquid. Note that \( C_T \) is different from the first-sound wave velocity.

**1.3.2 Experimental detection of second-sound waves in liquids**

The first experimental evidence of the second-sound is reported by Peskov [7]. The experimental set-up (as shown in Figure 1.11) consists of a cylindrical tube (which is closed at one end and contains LHe II), a heater and a moveable sensitive temperature sensor.
Figure 1.11: Experimental setup of Peshkov for measuring velocity of second-sound in LHe II.

In this experiment, a pulse of current is sent through the heater, which in turn generates a pulse of temperature near the heater. A pulse of heat in an ordinary fluid (such as water) diffuses, with the temperature reducing monotonically. On the contrary, if a source of heat is pulsed in time in LHe II, the heat does not simply diffuse out from the source as it would in a normal fluid. Instead, it travels as a wave, with the speed of second-sound. Therefore, the pulse reaches the sensitive thermometer after traveling in the liquid with a finite speed. By measuring the time needed for the pulse to travel from the transmitter to the receiver, the second-sound velocity can be obtained easily. For instance, for $D = 9 \text{ cm}$ and $T \approx 1.6 \text{ K}$ [38], for the input heat pulse of Figure 1.12 (top) with the length of 1 ms (from the heater), the output is recorded as Figure 1.12 (bottom) (in the thermometer) with some noise, which are not shown in the figure. Note that the input pulse from the heater maintains its shape until it is received by the thermometer. Moreover, the length of the output is 1 ms, which is equal to the length of the input pulse. Interestingly, there is a delay of 4.5 ms between the time of sending the input pulse and the time the output pulse is recorded. Therefore, the velocity of the second sound wave can be calculated as $20 \text{ m/s}$ at $T \approx 1.6 \text{ K}$ [38], which can be confirmed from Figure 1 in [40]. In reality, detection of the second-sound becomes very difficult for temperatures below 0.4 K [37].
Figure 1.12: Typical experimental observation of the second-sound wave in LHe II at $T \approx 1.6 \, \text{K}$, input to the heater (top) and output recorded at the thermometer (bottom).

It is worthwhile to mention that if the experiment is repeated for an ordinary fluid, a fluid without non-Fourier effects ($C = 0$), the observations are totally different than those reported for the LHe II. In this case, as a result of diffusion, the waves are rapidly attenuated and the recorded temperature pulse by the detector has no similarity to the shape of the initial temperature pulse.

In contrast to an ordinary liquid, when the heat pulse is applied to LHe II, the original temperature pulse maintains its shape until it is received by the detector. Therefore, the velocity of the second-sound wave is independent of the frequency of the heat source. Note that since the diffusion effect is very small in LHe II, the attenuation is also very weak. Interestingly, the independency of the second-sound wave velocity from the frequency of the heat source was seen in equation (1.7b) for solids, where the second-sound wave (heat wave) velocity is found to be $\sqrt{1/C}$. Note that $C$ is the Cattaneo number and has no dependency on the frequency of oscillations. However, the validity of $C_T = \sqrt{1/C}$ for the second-sound velocity in fluids is not shown yet.

1.3.3 Nanofluids

These fluids are solutions comprised of two main components: a base fluid solvent and nanoparticles (NPs). Base fluid can be water and organic fluids such as ethanol and
ethylene glycol. The materials that have been utilized as NPs include oxides of aluminum and silicon, as well as metals such as copper and gold with small volume fraction (1-5%) and size of O(1-100nm). The exciting feature of nanofluids (NFs) is their allowance for substantial enhancement in conductive heat transfer, as much as 40 percent increase in thermal conductivity [41,42] despite the low volume fraction of the NPs. This makes NFs extremely valuable, especially in processes where cooling is of primary concern, and thus there is an increasing focus on the convective properties of NFs in the literature. The presence of flow is expected to lead to complex and rich physical behaviour [43]. One advantage that a fluid containing NPs has over its milliparticle and microparticle counterparts is the small size of the NPs, which may be on the same order of magnitude as the molecules in the base fluid. This allows the solution to exist in a very stable manner without the occurrence of gravitational settling or particle agglomeration [41], [44]. If the fluid in a cooling process has improved thermal properties, then the workload of other driving components in the system can be reduced. Better thermal conductivity and heat transfer coefficients would allow systems involving microelectronics to run with increased power, while still maintaining appropriate operating temperatures, furthering the processing capabilities. The potential positive impact of NFs in many applications is very promising.

Recently, there is an increasing focus on the convective properties of nanofluids in the literature. The presence of convection term is expected to lead to complex physical behaviour [43]. Forced convection heat transfer in nanofluids has been investigated by several authors [41,44,45], whereas, not so much emphasis has been placed on the natural convection. In this case, Tzou [46] investigated the linear thermal stability of Rayleigh-Benard convection using a two-phase system approach, and found that the onset of convection occurs at a lower Rayleigh number compared with the base fluid. A similar stability analysis was carried out later by Nield & Kuznetsov [47], which confirms Tzou’s findings and predicts the possibility of oscillatory convection at instability, depending on the boundary conditions imposed. In a recent study [48], it is shown that the presence of nanoelements leads to dual-phase-lagging (DPL) heat conduction, which is non-Fourier in nature, in nanofluids. In the DPL model, fluids possess a relaxation time and a retardation time, reflecting the delay in the response of the heat flux and the
temperature gradient with respect to one another. If the retardation time is zero, the DPL model reduces to the single-phase-lag (SPL) model. The non-Fourier character of NFs will be explained in details in Chapter 2.

1.4 Motivations

Non-Fourier effects can be observed in several practical situations such as heat transfer in liquid helium II (LHe II) and nanofluids (NFs). Liquid helium can be used to cool down the superconductive magnets coil in MRI scanners to a temperature below 10 K [49]. As mentioned before, at temperatures below 2.17 K, the superfluid component of LHe II has a very large thermal conductivity. The superconductivity effect occurs in various materials subjected to extremely low temperatures. Superconductivity is a state of material in which there is no electrical resistance for an electrical current to flow through an electrical conductor (zero losses). Liquid helium is the only known material that can promote superconductivity in metal alloys [49]. By using a superconducting magnet and many coils of wires, a very large magnetic field is generated in MRI machines. Using the superconductivity effect that reduces the resistance in the wires, a large magnetic field can be maintained in the wires [50]. Also, the liquid helium can be used in a superfluid pump, which works on the principles explained for Figure 1.10 [51]. In this case, a temperature difference as small as 0.1 K creates enough pressures to pump the liquid for 2 meter height (according to the application requirements). Using the two fluid model, the propagation of heat as a wave can be explained in LHe II. In this model, LHe II is assumed to comprise a normal component and a superfluid component. However, these components are chemically indistinguishable and LHe II cannot be separated into its components by any method. Recall that the conduction process in LHe II (based on the two fluid model) is similar to convection process in ordinary fluids, which explains the reason of significant thermal conductivity of LHe II. In this work, the thermal convection of a non-Fourier fluid, such as LHe II, will be studied based on the Navier-Stokes equations for a homogeneous solution, where the Fourier equation of heat conduction will be replaced by the Cattaneo equation (1.1). The current approach is mathematically more manageable and can be used as an alternative to the two fluid model to study the heat transfer in LHe II. Using the current approach, the stability picture of LHe II can be
obtained, which enables us to predict the occurrence of second-sound wave in LHe II and also to find the velocity of these waves as a function of Cattaneo number.

In addition to LHe II, nanofluids also exhibit non-Fourier effects [33]. Owing to the enhanced thermal conductivities of NFs, they can be used in a wide range of engineering applications such as medical arena [52], automotive industry [53,54], and computers and electronic systems [55]. Regarding the thermal convection, NFs can be used in the solar collectors [56]. In contrast to conventional heat transfer fluids (such as water, ethylene glycol), NFs are not transparent to solar radiant energy, and as a result, they absorb the solar irradiance passing through them, significantly. It is shown that the efficiency of the collector increases by using NFs [56].

NFs can also be used as smart fluids. This application originates from the discovery of bistable heat transfer in suspensions of highly thermophilic nanoparticles (NPs) [57]. In such NFs, depending on the initial distribution of the NPs inside the carrier liquid, the heat transfer can occur in two different ways (conduction or convection). When the initial distribution of NPs is homogenous in the entire carrier liquid, convection occurs and a significant higher heat flux is obtained compared to that of in the conduction regime. Recently, it is shown [58] that the transition from one regime to the other can be achieved without performing any mechanical manipulations of the sample. In other words, the heat valve system (which can control the flow of heat) is working by exploiting the thermophilic nature of the NPs to alter their spatial distribution (Figure 1.13).

![Figure 1.13: Nanofluid with highly thermophilic nanoparticles: Conduction state (left) and Convection state (right).](image)

The linear stability of Rayleigh-Benard convection in NFs has been studied by several authors using a two-phase system approach. However, one of the major drawbacks of the two-phase approach is the nature of the boundary conditions for the concentration of NPs. So far, the unequal NP concentrations at the top and bottom boundaries have been used,
whereas NFs are quite homogeneous solutions. To examine the heat transfer in a NF, an alternative approach to the two-phase model is adopted in this study. Considering the fluid as a homogeneous solution, and accounting for the finite thermal relaxation time resulting from the addition of NPs to the base fluid, enables us to obtain the stability picture of NFs, and to see whether the convection is stationary or oscillatory at the onset, depending on the level of non-Fourier effects.

1.5 Outline

This thesis is organized into six chapters. The objectives and motivations as well as the research background and relative important fundamental theories are addressed in Chapter 1. In Chapter 2, some typical situations that can display noticeable non-Fourier effects are discussed. An expression for the estimation of the relaxation time in liquids is obtained, which also shows the relative importance of the factors that can change the relaxation time. The non-Fourier character of heat transfer in nanofluids is shown in this chapter, and expressions for the estimation of relaxation times in nanofluids are obtained. Moreover, it is shown in this chapter that unlike solids, where non-Fourier effects are present only in the transient heat conduction, the base flow solution in fluids might be affected by the non-Fourier character of fluids.

In Chapter 3, the linear stability for the Rayleigh-Benard convection of a thin layer of a non-Fourier fluid is studied. Also, energy growth for small two-dimensional perturbations to Rayleigh-Benard problem for a non-Fourier fluid is investigated. In Chapter 4, the onset of thermal convection of a thin layer of non-Fourier fluid confined between two vertical plates is investigated. In addition, the connection between the results of thermal convection for a non-Fourier fluid (based on the linear stability analysis) and heat transfer in liquid helium II (based on the two fluid model) is established. In Chapter 5, a combined spectral/amplitude/perturbation approach is used to study nonlinear natural convection of a Fourier fluid. In this case, the required modes (to different orders) to accurately capture the convection even far from the threshold is identified systematically and the presence/absence of each mode is justified. Chapter 6 summarizes the main conclusions together with suggestions for the future works.
References


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Chapter 2

2 On non-Fourier effects in macro- and micro-scale non-isothermal flow of liquids and gases

2.1 Introduction

Heat transfer is commonly governed by the classical Fourier’s constitutive law. When used in combination with the First Law of Thermodynamics, Fourier’s law assumes an infinite speed of heat propagation, and a parabolic equation for the temperature field. In reality, a disturbance wave in the temperature will travel at a finite speed since it is transferred by molecular interaction [1]. This behaviour is characterized by the Maxwell-Cattaneo equation, which adds a transient term multiplied by the thermal relaxation time of the medium. This is the time required for the heat flux to reach a new steady state following a perturbation to the temperature gradient, establishing a hyperbolic heat (wave) response. Non-Fourier effects have long been recognized to emerge in the form of second-sound thermal waves in low-temperature liquids [2,3], but are increasingly observed in a variety of phenomena involving ultrafast heating and complex fluids. Most common practical problems involve materials with relaxation times on the order of pico- or $10^{-12}$ seconds [4], and thus, the Maxwell-Cattaneo equation collapses onto the classical Fourier model. However, physical media exist in which the relaxation time may not be considered negligible. Noteworthy examples include some acids and biological tissues, which may possess relaxation times between 1-100 seconds [4]. It is important to realize that ‘large’ relaxation times should also be identified relative to the rate of heating. Processes such as laser pulse heating introduce a large quantity of energy over a small period of time, which are comparable to the relaxation time of the medium. Letfullin, et al. [5] point out that, in some applications, the duration of the laser pulse can be measured on the scale of femto- or $10^{-15}$ seconds, of the order of the relaxation time of electrons [6]. Thus, when the rate of heating is of the same order of magnitude as the relaxation time, non-Fourier effects must be accounted for. The fact that the importance of non-Fourier effects is not solely embedded in the absolute value of the relaxation time is, perhaps,

\[1\] A version of this chapter is submitted to the International Journal of Thermal Science.
best illustrated using generalized non-dimensional arguments. More specifically, non-Fourier effects become significant whenever the relaxation time is of the same order as the thermal diffusion time. Thus, if \( D \) is the length scale, or typical gap width as in the present problem, and \( \tau \) and \( \kappa \) are, respectively, the relaxation time and thermal diffusivity of the fluid, then the dimensionless relaxation time is given by the Cattaneo number, \( C = \frac{\tau \kappa}{D^2} \). Consequently, \( C \) increases relatively rapidly as \( D \) decreases, and non-Fourier effects are expected to be significant for a very small gap, as in the convection and flow in micro- and nano-devices [7-10] .

It has long been realized that adding the partial time derivative does not completely solve the problem of instantaneous thermal relaxation [11-13]. The Maxwell-Cattaneo equation is not a frame-invariant constitutive relation and, as such, is restricted to non-deformable media. Several objective derivatives have been applied to remedy this situation. However, they each have had their own shortcomings. The most promising modification is that of Christov [14], which was recently revisited by Khayat & Ostoja-Starzewski [15], whose use of the Oldroyds’ upper-convected derivative leads to the frame indifferent Maxwell-Cattaneo equation. Coupled with the energy equation, this constitutive equation can also yield a single equation for the temperature field, an advantage that other invariant formulations do not possess [14]. This equation replaces Fourier’s law for fluid flow whenever the relaxation time is relevant, and collapses back to Fourier’s law whenever it is not.

Currently, there is limited literature available on non-Fourier convection, and much of it pertains only to thermal instability [11,16,17]. However, this is rapidly changing with the advent of fast processes and the emergence of nanofluids (NFs), which comprise a solvent base of an ordinary liquid and nanoparticle (NP) solute. An alternative approach to the two-phase model is adopted to examine the heat transfer in a NF, and non-Fourier fluids in general, by considering the fluid as a homogeneous solution, and accounting for the finite thermal relaxation time resulting from the addition of NPs to the base fluid. This is a similar approach to that used in modelling viscoelastic fluids with colloidal and
particle suspensions: the addition of the polymer to a base solvent leads to a finite relaxation time of the stress [18]. The non-Fourier character of NFs has been recognized in the literature [19-21]. Wang & Wei [20] established the equivalence between the conduction of a two-phase Fourier solution and that of a homogeneous dual-phase-lagging (DPL) fluid. The resulting DPL constitutive equation for heat conduction is of the non-Fourier type, relating the temperature gradient to the heat flux with lagging relaxation and retardation times. The connection between the two-phase model and non-Fourier effect was also recognized in other systems. See, for instance, Donnelly [48] on the two-fluid theory and second-sound in liquid helium. The equivalence between the two-phase and DPL configurations is fundamentally significant, as it offers an alternative approach to the traditional two-phase formulations, where possible advantages of the DPL approach include: easier implementation of measureable/controllable initial and boundary conditions in heat flux and temperature, reduction in the number of fluid parameters, more manageable analytical steady solution for the base state, and a wider range of applications including the non-Fourier heat transfer of single-phase-lagging (SPL) fluids for fast heating processes. The DPL approach will be revisited in section 5, to include convective effects, generalizing the existing SPL model [14,15]. Although the DPL formulation applies to a variety of phenomena [12,22,23], the emphasis in this survey will be on the non-isothermal flow of NFs.

2.2 Kinetic theory basis for non-Fourier heat transfer and governing equations

One of the main objectives of kinetic theory is to describe the macroscopic properties of fluids from microscopic or molecular considerations like the mass, velocity, kinetic energy as well as internal degrees of freedom of molecules, and interaction forces between the molecules. The theory ultimately provides macroscopic conservation and constitutive equations that govern the flow and thermal field variables such as density, velocity, pressure, temperature, stress and heat flux. Explicit expressions of fluid properties such as viscosity and thermal conductivity can also be obtained. While conservation equations are straightforward to establish, and are valid for any fluid,
constitutive equations for stress and heat flux remain elusive for complex fluids and non-isothermal flow under non-ordinary conditions. The role of kinetic theory in establishing appropriate constitutive equations for gases and liquids will be reviewed.

2.2.1 Constitutive equation for heat

When the thermal relaxation time $\tau$ for a solid is comparable to the thermal process time, Fourier’s law ceases to be valid. In this case, the constitutive equation for the heat flux, $Q$, is the Maxwell-Cattaneo or Cattaneo-Vernot equation, which must be used, namely [24]:

$$\tau \frac{\partial Q}{\partial t} + Q = -KVT,$$  \hspace{1cm} (2.2.1)

where $T$ is the temperature, $K$ is the thermal conductivity and $\nabla$ is the gradient operator. For a moving fluid, however, this equation needs to be reformulated and rendered objective (frame indifferent). This was achieved by Christov [14], and later revisited by Khayat & Ostoja-Starzewski [15]. The resulting equation is simply reproduced here as

$$\tau \frac{\delta Q}{\delta t} + Q = -KVT,$$ \hspace{1cm} (2.2.2)

where the Jaumann or Li-type derivative is given by

$$\frac{\delta}{\delta t} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \left( \frac{\partial}{\partial t} \right) - \left( \frac{\partial}{\partial t} \right) \mathbf{V} \cdot \nabla - \left( \frac{\partial}{\partial t} \right) \nabla \cdot \mathbf{V},$$  \hspace{1cm} (2.2.3)

where $\mathbf{V}$ is the velocity. It is not difficult to see that this equation relates the temperature gradient at a material point $X$ and time $t$ to the heat flux vector at the same point at time $t + \tau$ for a medium of thermal conductivity $k$. The relation reads:

$$Q(X, t + \tau) = -KVT(X, t),$$  \hspace{1cm} (2.2.4)
and equation (2.2.1) is recovered by Taylor expansion of (2.2.4) for small relaxation time. Note that equation (2.2.1) is hyperbolic. It is worth noting that other two-phase systems do reflect the phase-lagging character exhibited in (2.2.1). One such connection is the case of a hot jet penetrating a porous medium [25] or the flow through porous layers [26]. In this regard, equations (2.1a and b) of [25] are comparable to equations (4) and (5) of [20]. Also, in this case, the former equations switch from parabolic to hyperbolic for high jet penetration velocity.

2.2.2 Conservation and constitutive equations

Both conservation and constitutive equations can have their basis in kinetic theory. The theory is particularly well established for rarefied monatomic gases, but lately has been generalized to include dense gases and liquids. Its foundations were established in 1867 by James Clerk Maxwell (1831-1879) who proposed a general transport equation for arbitrary macroscopic quantities associated with mean values of microscopic quantities (Maxwell 1867). This equation of transport relates the time evolution of a macroscopic quantity with the motion of the molecules, collision between the molecules and action of external forces. The theory was valid for any molecular interaction potential, but the kinetic theory of gases gained a new impulse in 1872 with the work by Ludwig Eduard Boltzmann (1844-1906), who proposed an integro-differential equation – the Boltzmann equation – which represents the evolution of the velocity distribution function in the phase space spanned by the coordinates and velocities of the molecules [28]. Further advances were due to Sydney Chapman (1888-1970) and David Enskog (1884-1947) who calculated independently and by different methods the transport coefficients for gases whose molecules interact according to any kind of spherically symmetric potential function [29,30]. Another method derived from the Boltzmann equation was proposed in 1949 by Harold Grad (1923-1986) who expanded the distribution function in terms of tensorial Hermite polynomials and introduced balance equations corresponding to higher order moments of the distribution function [31].
The extension of kinetic theory to describe liquids involves daunting difficulties associated with multi-body molecular interactions and additional degrees of freedom. Nevertheless, with the proper development of collision integrals based on suitable interaction potential, Boltzmann equation was applied to liquids [32-35]. In this case, Grad’s 13-moment method can be applied by projecting the extended Boltzmann equation for liquids.

Before pursuing our discussion on non-Fourier effects, it is helpful to recall the governing equations for incompressible non-isothermal flow of a fluid of density \( \rho \), viscosity \( \mu \) and thermal diffusivity \( \kappa \). The conservation and constitutive equations will be cast in dimensionless form, with \( D, U, \frac{D}{U} \) and \( \mu \frac{U}{D} \) being the reference length, velocity, time and stress, respectively. The current paper is mainly concerned with closely coupled flow and heat transfer problems. In this case, a suitable velocity scale is taken as \( U = \frac{K}{D} \). For a general fluid, the equations for conservation of linear momentum and energy read, respectively [36]:

\[
\begin{align*}
Pr^{-1} (v_t + \nu \cdot \nabla v) &= -\nabla p + \nabla \cdot \sigma, \\
\theta_t + \nu \cdot \nabla \theta &= -j + Ec \nabla \cdot \sigma, 
\end{align*}
\]

where \( j = \nabla \cdot \mathbf{q} \). The Prandtl and Eckert numbers are given by \( Pr = \frac{\nu}{\kappa} \) and \( Ec = \frac{\mu D^2}{K \Delta T} \), respectively, \( \Delta T \) being a typical temperature differential of the process. Here \( \nu \) is the kinematic viscosity and \( \kappa = \frac{K}{\rho c_p} \) is the thermal diffusivity. In this work, Newtonian non-Fourier fluids will be considered. It is not unconceivable that most non-Fourier fluids may exhibit a non-Newtonian character. However, in order to keep the discussion tractable, the scope of the paper will be restricted to Newtonian fluids. In this case, \( \sigma = \nabla \nu + \nabla \nu^t \), and one recovers the Navier-Stokes equations. For an incompressible flow, adding the continuity equation, and equations (2.2.5) and (2.2.6) reduce to
\( \nabla \cdot \mathbf{v} = 0 \), \hspace{1cm} (2.2.7)

\( \Pr^{-1} (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \Delta \mathbf{v} \), \hspace{1cm} (2.2.8)

\( \theta_t + \mathbf{v} \cdot \nabla \theta = -j + Ec \nabla \mathbf{v} \cdot \nabla \mathbf{v} \), \hspace{1cm} (2.2.9)

As to the constitutive equation for heat, the formulation can be simplified by casting the constitutive equation for heat flux in terms of the scalar variable \( j \). Thus, upon taking the divergence of equation (2.2.9), and recalling again the identity \( \nabla \cdot (\mathbf{a} \cdot \nabla \mathbf{b}) = \nabla \mathbf{a} : \nabla \mathbf{b} + \mathbf{a} \cdot \nabla (\nabla \cdot \mathbf{b}) \), one obtains the following constitutive equation for \( j \):

\[ C(\mathbf{j}_t + \mathbf{v} \cdot \nabla \mathbf{j}) + j = -\Delta \theta . \] \hspace{1cm} (2.2.10)

The following non-dimensional groups have been introduced, namely, the Cattaneo number, \( C \), given as

\[ C = \frac{\tau \kappa}{D^2} . \] \hspace{1cm} (2.2.11)

Depending on the boundary conditions, the heat flux vector, \( \mathbf{q} \), may or may not be obtained separately. If the boundary conditions are imposed on the temperature and \( j \), then \( \mathbf{q} \) can be calculated by solving

\[ C \left( \frac{\partial \mathbf{q}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{v} \right) + \mathbf{q} = -\nabla \theta , \] \hspace{1cm} (2.2.12)

once \( \mathbf{v}, p, \theta \) and \( j \) are determined. If, on other hand, the heat flux vector is imposed at the boundary, then equations must be solved simultaneously. The Fourier model is recovered upon setting \( C = 0 \) (zero relaxation time). In this case, \( \mathbf{q} = -\nabla \theta \) and \( j = -\Delta \theta \).

### 2.3 Non-Fourier effect in liquids

To the authors’ knowledge, there is no established relation to estimate the relaxation time
for liquids. In contrast, there are proposed relations for the relaxation time for solids and rarefied gases. A quantitative estimate of the relaxation time, $\tau$, in terms of measurable macroscopic parameters has been obtained for solids as [37]

$$\tau_{\text{solid}} = \frac{3\kappa}{c_s^2},$$  \hspace{1cm} (2.3.1)

where $\kappa$ is the thermal diffusivity of the sample and $c_s$ represents the sound speed. On the other hand, for a rarefied gas, the relaxation time can be found from [36] as

$$\tau_{\text{rarefied gas}} = \frac{1}{\Pr} \frac{\mu}{P},$$ \hspace{1cm} (2.3.2)

where $\mu$ is the viscosity, $P$ is the mean pressure of the gas and $\Pr$ is the Prandtl number, which is approximately $2/3$ (Struchtrup & Taheri 2011). For a rarefied gas the mean pressure is given as [38]

$$P = \frac{1}{3} \rho \bar{c}^2,$$ \hspace{1cm} (2.3.3)

where $\rho$ is the density and $\bar{c}$ is the mean molecular speed. Combining (2.3.2) and (2.3.3), one obtains

$$\tau_{\text{rarefied gas}} = \frac{1}{3\Pr} \frac{\nu}{\bar{c}^2},$$ \hspace{1cm} (2.3.4)

where $\nu$ is the kinematic viscosity. Equation (2.3.4) can be further simplified by using $\Pr = \nu/\kappa$ as

$$\tau_{\text{rarefied gas}} = \frac{3\kappa}{\bar{c}^2}.$$ \hspace{1cm} (2.3.5)

The mean molecular speed in a rarefied gas is

$$\bar{c} = \sqrt{\frac{8 k_B T}{\pi m}},$$ \hspace{1cm} (2.3.6)
where \( k_B \) is the Boltzmann constant, \( T \) is the absolute temperature and \( m \) is the mass of a single molecule. On the other hand, the speed of sound in an ideal gas is given by

\[
c_s = \sqrt{\frac{\zeta k_B T}{m}},
\]

where \( \zeta \) is the adiabatic index, which, from kinetic theory, is approximately 5/3 for monatomic gases. Note that the rarefied gases are considered as monatomic ideal gases [36]. From (2.3.6) and (2.3.7) one can conclude that \( \bar{c} \approx 1.2c_s \). Therefore, the relaxation time for a rarefied gas can be approximately expressed as

\[
\tau_{\text{rarefied gas}} \approx \frac{2.1\kappa}{c_s^2}.
\]

Interestingly, the relaxation time for solids, equation (2.3.1), and for a rarefied gas, equation (2.3.8), are similar. Consequently, we can conclude that the values of relaxation time for ordinary homogenous liquids are between those of solids and rarefied gases. In other words, the following relation can be used to estimate the order of magnitude of relaxation time in liquids

\[
\tau_{\text{liquid}} \approx \frac{a\kappa}{c_s^2},
\]

where it can be seen from (2.3.5) and (2.3.8) that \( 2.1 < a < 3 \). The validity of (2.3.9) for liquids is checked by comparing the relaxation time obtained from (2.3.9) with the available relaxation time for liquid Benzene, which is equal to \( 1.22 \times 10^{-13} \) s [39]. Thermal diffusivity of Benzene at 20°C is \( 9.5 \times 10^{-8} \) m²/s (based on \( K = 0.144 \) W/mK, \( \rho = 879 \) kg/m³ and \( c_p = 1.729 \) kJ/kg K). The speed of sound in Benzene is approximately \( c_s = 1300 \) m/s [40]. Therefore, based on the values of \( a=2.1 \) and 3, one can estimate the relaxation time for Benzene to be \( 1.16 \times 10^{-13} \leq \tau_{\text{Benzene}} \leq 1.66 \times 10^{-13} \) s. The obtained relaxation time for Benzene based on equation (3.9) is in an excellent agreement with the reported value of [39].
Recall that the Cattaneo number, which reflects the level of non-Fourier effects, is defined as \( C = \frac{\tau \kappa}{D^2} \), where \( D \) is a typical length scale. Replacing \( \tau \) in the Cattaneo number with \( \tau_{\text{rarefied gas}} \) from (3.5), and using \( \nu = \frac{1}{3} \lambda \bar{c} \), which relates the kinematic viscosity to the mean free path (\( \lambda \)) [38], enables us to express the Cattaneo number for a rarefied gas as

\[
C_{\text{rarefied gas}} = \frac{3\kappa^2}{c^2D^2} = \frac{Kn^2}{3Pr^2} . \tag{2.3.10}
\]

In (2.3.10), \( Kn = \lambda/D \) is the Knudsen number. It is predicted from (3.10) that non-Fourier effects are considerably large for rarefied gases in micro/nano-scale configurations. Similarly, the Cattaneo number for liquids with high thermal diffusivity can be obtained from

\[
C_{\text{liquid}} = \frac{ak^2}{c_s^2D^2} . \tag{2.3.11}
\]

It is expected from equation (2.3.11) that non-Fourier effects become important for liquids with high thermal diffusivity such as liquid metals, liquids with low sound speed such as liquid helium, and small-scale (micro/nano) configurations. These situations are discussed in the following subsections.

2.3.1 Liquid metals

Liquid metals such as mercury and molten gallium have a thermal diffusivity which is considerably larger than that of typical liquids such as water (the reader is referred to Table 2.1). Equation (2.3.11) provides a range for the relaxation time, where the lower (upper) limit is denoted as \( \tau_{\text{min}} (\tau_{\text{max}}) \) in Table 2.1 and all properties are calculated at 20°C. \( \kappa, Pr \) and \( c_s \) values are from [40].
Table 2.1: Relaxation time for water, mercury and molten gallium

<table>
<thead>
<tr>
<th>Liquid</th>
<th>κ (m²/s)</th>
<th>Pr</th>
<th>cₐ (m/s)</th>
<th>τₘᵢₙ (ps)</th>
<th>τₘₐₓ (ps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>1.38×10⁻⁷</td>
<td>7</td>
<td>1482</td>
<td>0.13</td>
<td>0.19</td>
</tr>
<tr>
<td>Mercury</td>
<td>0.45×10⁻⁶</td>
<td>0.025</td>
<td>1449</td>
<td>4.5</td>
<td>6.4</td>
</tr>
<tr>
<td>Molten gallium</td>
<td>2×10⁻⁵</td>
<td>0.023</td>
<td>2870</td>
<td>5.1</td>
<td>7.3</td>
</tr>
</tbody>
</table>

As expected, a liquid with a larger thermal diffusivity has a larger relaxation time. For instance, the relaxation time for molten gallium is approximately 40 times larger than that of water due to its larger κ. Note that the Cattaneo number in (2.3.11) is proportional to κ², which signals the importance of non-Fourier effects for liquids with higher thermal diffusivity. The Cattaneo number based on equation (2.3.11) is calculated for some liquids in Table 2 and all properties are calculated at 20°C.

Table 2.2: Cattaneo number for water, mercury and molten gallium for different D

<table>
<thead>
<tr>
<th>Liquid</th>
<th>Cattaneo number (C)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>D=10⁻⁵ m</td>
</tr>
<tr>
<td>Water</td>
<td>2.73×10⁻¹⁰</td>
</tr>
<tr>
<td>Mercury</td>
<td>2.90×10⁻⁷</td>
</tr>
<tr>
<td>Molten gallium</td>
<td>1.46×10⁻⁶</td>
</tr>
</tbody>
</table>

It should be noted that the C values in Table 2.2 are calculated for a = 3 in equation (2.3.11). The relaxation time for the molten gallium is approximately 5500 times larger than that of water. Consequently, liquids with large thermal diffusivity could be good candidates for studying non-Fourier behaviour. It is worth noting that for the length
scales to $O(\text{mm})$, the Cattaneo number is very small. However, as the length scale decreases, the Cattaneo number increases at a rate proportional to $D^2$.

One way to study the non-Fourier character of a fluid is to examine the natural convection of that fluid. The Rayleigh-Benard configuration is defined as a thin layer of the liquid confined between two infinite horizontal plates maintained at fixed temperatures. The lower plate is maintained at a temperature difference $\delta T$ higher than the upper plate. This problem has been studied widely in the literature both experimentally and numerically as it is a convenient way of analyzing new physics. If $\delta T$ is low enough, viscous effects keep the fluid layer motionless despite the fact that, under the action of gravity, the lower layers are lighter than the upper fluid layers due to thermal expansion. In this steady conduction state a linear temperature profile develops between the upper and lower boundaries. As the temperature difference between the plates increases through a critical limit, the buoyancy effects overcome the viscous effects and some form of convection sets in. This critical limit is typically described by the non-dimensional Rayleigh number which is defined as

$$Ra = \frac{\delta T g \alpha_T D^3}{\nu \kappa},$$

(2.3.12)

where $g$ is the gravitational acceleration acting downward and $\alpha_T$ is the thermal expansion coefficient. In the realistic case of two rigid plates, where the fluid velocity is zero at the boundary, the critical Rayleigh number, $Ra_c \approx 1708$ for a Newtonian, Fourier fluid.

For a non-Fourier fluid, similar to a Fourier fluid, conduction is lost to convection once the critical value of the Rayleigh number is exceeded. However, in contrast to a Fourier fluid, and similar to a viscoelastic fluid [41], non-Fourier conduction can be lost to steady or oscillatory convection, depending on the flow parameters [42,43]. For small relaxation time (small $C$), the marginal stability curve consists of a Fourier branch (for low wavenumber), leading to stationary convection, and an oscillatory branch. Both stationary and oscillatory convection become equally probable as $C$ reaches a critical
value, $C_\text{H}$. For $C > C_\text{H}$, only oscillatory convection is predicted, at a Rayleigh number decreasing with $C$ [42,43].

Experimentally, oscillatory convection in liquid metals has been detected, and the predicted non-Fourier character of the liquid may be responsible for this behaviour. For example, for the Rayleigh-Benard configuration with molten gallium as the working fluid, the convection is observed to be oscillatory at the onset [44,45]. Yamanaka et al. [44] investigated the natural convection in a bounded horizontal layer of gallium ($Pr = 0.023$) heated from below. The horizontal length and depth of fluid layer were 100 mm and 10 mm, respectively, with plexiglass side walls. Their experimental measurements revealed that the natural convection is oscillatory at the onset. Aurnou & Olson [45] studied the Rayleigh-Benard convection of molten gallium for three subcritical experimental conditions, corresponding to $Ra = 0, 800$ and 1275 ($< Ra_{cr}=1708$). Additional experiments with Rayleigh numbers greater than 2000 revealed time-dependent fluctuations in the temperature records. They obtained three different estimates of the critical Rayleigh number, which indicates that the supercritical convection is oscillatory.

### 2.3.2 Liquid helium

Experimental results show the presence of the second-sound wave (thermal wave) in superfluid helium (He II) at low temperatures ($T < 2.2$ K) [46,47]. Consequently, it is anticipated that liquid helium has a non-Fourier character. As mentioned before, equations (2.3.9) and (2.3.11) can be used for ordinary liquids, and thus may not be valid for the liquid helium. However, we can estimate the non-Fourier character of liquid helium in comparison with other liquids based on these expressions. In contrast to liquid metals, which have high thermal diffusivities, the thermal diffusivity of liquid helium at 2.5 K is only $2.8 \times 10^{-8}$ m$^2$/s [48], which is orders of magnitude lower than that of liquid metals.

The factor which leads to the non-Fourier character of liquid helium is the very slow speed of sound in liquid helium, which can make the Cattaneo number (equation (2.3.11)) considerably large. For instance, the sound speed in liquid helium at 4.1 K is 183 m/s
[40], which is approximately 0.06 of the sound speed in molten gallium (at 20°C). Moreover, the Cattaneo number is proportional to $1/c_s^2$, which makes $C$ even bigger for the liquid helium. The Cattaneo number for the liquid helium at 4.1 K and for $D = 1\mu$m is $O(10^{-5})$, which is close to $C$ values for liquid metals and much larger than that of water.

### 2.4 Effect of temperature on the Cattaneo number

From (2.3.7), it is anticipated that the Cattaneo number depends on the temperature. This can be clearly seen for rarefied gases, whereas for the liquids, it is not obvious. The influence of the temperature on the Cattaneo number, or equivalently the relaxation time, will be studied next. For a rarefied gas, the Cattaneo number is proportional to $\lambda^2$. As mentioned before, the Prandtl number is approximately $2/3$. Consequently, the Cattaneo number for a rarefied gas can be obtained from

\[
C \approx \frac{3\lambda^2}{4D^2}. \tag{2.4.1}\]

The mean free path can be obtained from

\[
\lambda \approx \frac{1}{\sqrt{2n\sigma_0}}, \tag{2.4.2}\]

where $n$ is the number density, and $\sigma_0$ is the collision cross section between two atoms:

\[
n = \frac{P}{k_B T}, \quad \sigma_0 = \pi d^2 \tag{2.4.3}\]

In (2.4.3), $d$ represents the atomic diameter, which might have a typical value of $2 \times 10^{-8}$ cm [38]. Combining (2.4.2) and (2.4.3), one obtains the mean free path as a function of temperature as

\[
\lambda \approx \frac{k_B T}{\sqrt{2\pi d^2 P}} \tag{2.4.4}\]

For atmospheric pressure (101.32 kPa) and room temperature (300 °K), the mean free
path is equal to $\lambda \approx 3 \times 10^{-7}$ m. Thus, from (2.4.1) the Cattaneo number for $D = 1$ μm is calculated as $C = 0.0675$, which is very large compared to values for liquids in Table 2.2 for similar $D$. Moreover, it is not difficult to see from (2.4.1) and (2.4.4) that the Cattaneo number for a rarefied gas at a given pressure is proportional to $T^2$, where $T$ is the absolute temperature.

In contrast to rarefied gases, the dependence of the Cattaneo number on temperature is not obvious for liquids. The reason is that the sound velocity in liquids has no direct relation to the temperature. In this case, the sound velocity in liquids can be expressed as

$$c_s = \sqrt{\frac{E}{\rho}},$$

(2.4.5)

where $E$ is the bulk modulus of elasticity. Clearly, the density, $\rho$, is a function of temperature and its values at different temperatures can be obtained from tables, or it can be assumed to depend on the temperature, $T$, following

$$\rho = \rho_0 \left[1 - \alpha_T (T - T_0)\right],$$

(2.4.6)

where $\alpha_T$ is the coefficient of volume expansion and $\rho_0$ is the density of the fluid at $T_0$. The bulk modulus of elasticity, $E_b$, for liquids is defined as $E_b = \rho \frac{dP}{d\rho}$, where $P$ is the pressure. The inverse of $E_b$ is defined as the compressibility of the liquid. In the temperature range of 0-100°C, the compressibility is found to be approximately unchanged [49]. Therefore, the bulk modulus of elasticity for ordinary liquids is not affected by temperature. Consequently, to a good level of approximation, the effect of temperature on the sound velocity in liquids can be studied solely from the density dependence on the temperature.
Figure 2.1: Influence of temperature on the Cattaneo number.

The variation of Cattaneo number for mercury, which is shown in section 2.3 to have high non-Fourier character, water and a rarefied gas versus temperature is illustrated in Figure 2.1. For better comparison, all Cattaneo numbers are divided by the Cattaneo number of the corresponding fluid at T=20°C. All curves demonstrate monotonic increase with respect to temperature. However, the Cattaneo number for water increases with a slower rate compared to others. As anticipated, since the mean free path of a rarefied gas increases linearly with temperature (2.4.4), the Cattaneo number for a rarefied gas increases quadratically with temperature. Similarly, the variation of Cattaneo number for mercury and water versus temperature are assumed to follow a quadratic trend (for 20°C < T < 100°C). The quadratic equations that describe the variation of Cattaneo number against the temperature are

\[
C_{\text{water}} = C_{\text{water} \ 20^\circ C} \left( 3.95 \times 10^{-5} T^2 - 1.32 \times 10^{-3} T + 1.01 \right)
\]

\[
C_{\text{mercury}} = C_{\text{mercury} \ 20^\circ C} \left( 9.24 \times 10^{-6} T^2 + 2.61 \times 10^{-3} T + 0.94 \right)
\]

\[
C_{\text{rarefied gas}} = C_{\text{rarefied gas} \ 20^\circ C} \left( 1.16 \times 10^{-5} T^2 + 6.36 \times 10^{-3} T + 0.87 \right),
\]

(2.4.7)
where C values at T=20°C are given in Table 2. It should be noted that (2.4.7) are valid only for 20°C < T < 100°C.

2.5 Non-Fourier character of nanofluids

Nanofluids (NFs) are solutions consisting of a base fluid solvent, containing a small volume fraction (1-5%) of nanoparticles (NPs) of size of O(1-100nm). The most commonly used base fluids are water and organic fluids such as ethanol and ethylene glycol. The materials that have been utilized as NPs include oxides of aluminum and silicon, as well as metals such as copper and gold. Diamonds and nanotubes have also been widely experimented with [50,51]. The exciting feature of NFs is their allowance for substantial enhancement in conductive heat transfer, as much as 40 percent increase in thermal conductivity [50,52,53] despite the low volume fraction of the NPs. This makes NFs extremely valuable, especially in processes where cooling is of primary concern, and thus there is an increasing focus on the convective properties of NFs in the literature. The presence of flow is expected to lead to complex and rich physical behaviour [54]. One advantage that a fluid containing NPs has over its milliparticle and microparticle counterparts is the small size of the NPs, which may be on the same order of magnitude as the molecules in the base fluid. This allows the solution to exist in a very stable manner without the occurrence of gravitational settling or particle agglomeration [50], [55]. If the fluid in a cooling process has improved thermal properties, then the workload of other driving components in the system can be reduced. Better thermal conductivity and heat transfer coefficients would allow systems involving microelectronics to run with increased power, while still maintaining appropriate operating temperatures, furthering the processing capabilities. The potential positive impact of NFs in many applications is very promising.

Earlier attempts to examine NFs were based on modelling the overall change in the thermophysical properties, such as thermal conductivity and viscosity of the NF, as a result of the addition of NPs [56]. More recently, extensive efforts have been devoted to understand heat transfer enhancement during conduction [55]. Relative to conduction, the convection of NFs has received little attention. So far, emphasis has been on forced
convection (see, for instance, 55,57), and not so much on natural convection, despite the rich dynamics expected for a NF. Buongiorno [50] examined several potential factors that could be responsible for the observed heat transfer enhancement in forced convection, namely, the mechanisms behind slip (the difference between the absolute velocity of a NP and the velocity of the base fluid). The potential mechanisms for slip include inertia, gravity settling, diffusiophoresis, the Magnus effect, fluid drainage, Brownian diffusion and thermophoresis. However, Brownian motion and thermophoresis seem to be the only significant factors that could properly explain the unexpected heat transfer capabilities of NFs observed in experiment [50]. The Brownian motion, in this case, is the induced random drifting caused by the collisions of NPs with the molecules in the base fluid, whereas thermophoresis is a diffusive effect that causes particles to move as a result of a temperature gradient within the fluid.

Given the small spatio-temporal scales involved in heat transfer enhancement in NFs, resulting from the addition of NPs to the base fluid, it is not surprising that non-Fourier effects are envisaged as being behind this enhancement. That this may be the case is suggested at both the theoretical [20,53,58] and the experimental [21] levels. As mentioned above, Wang & Wei [20] established the equivalence between DPL and two-phase systems for heat conduction of NFs, by assuming each of the base solvent and NP phases to obey Fourier’s law. Following Quintard & Whitaker [60] and accounting for interfacial interaction, the energy equation in each phase is then integrated over a representative volume element, yielding averaged coupled equations in terms of an effective thermal conductivity tensor, the NP concentration, the film heat transfer coefficient, $h$, the interfacial area per unit volume, $a_v$, and the base fluid and NP properties. The two averaged equations are then combined to obtain an equation for heat conduction of the NF, of the DPL type (as opposed to the SPL Maxwell-Cattaneo equation), involving both a relaxation time and a retardation time, which can be expressed in terms of the properties of the original two-phase system (see below). The DPL model for heat conduction is now generalized for thermal convection.
2.5.1 Constitutive equation for nanofluids

Following the formulation for viscoelastic constitutive equations for polymeric solutions [18], consider a NF solution of thermal conductivity $K$, comprising a solvent Fourier fluid of thermal conductivity $K_F$ and a solute of NPs of thermal conductivity $K_{NP}$. The addition of NPs causes a change (normally increase) in the thermal conductivity of the solvent given by

$$K' = K - K_F. \quad (2.5.1)$$

In this case, the heat flux of the solution may be written as a superposition of Fourier and non-Fourier contributions:

$$Q = Q_F + Q', \quad Q_F = -K_F \nabla T. \quad (2.5.2)$$

The most commonly used constitutive equation for $Q'$ is the Maxwell-Cattaneo equation for heat conduction [1]. For a moving fluid, however, this equation needs to be reformulated and rendered objective (frame indifferent). This was achieved by Christov [14], and later revisited by Khayat & Ostoja-Starzewski [15]. The resulting equation is simply reproduced here:

$$\tau_Q \frac{\delta Q'}{\delta t} + Q' = -K' \nabla T, \quad (2.5.3)$$

where $\tau_Q$ is the relaxation time, and the Jaumann or Li type derivative is given by

$$\frac{\delta}{\delta t} \equiv \frac{\partial}{\partial t} + \nabla \cdot \left( - \frac{\partial}{\partial t} - \nabla \right) \nabla - \left( \frac{\partial}{\partial t} \right) \nabla \cdot \nabla. \quad (2.5.4)$$

In this case, the constitutive equation for the solution heat flux is obtained from (2.5.2) and (2.5.3) to read
\[
\tau_Q \frac{\delta Q}{\delta t} + Q = -KV_T - \tau_Q K_F \frac{\delta V_T}{\delta t} = -K \left( V_T + \tau_T \frac{\delta V_T}{\delta t} \right),
\]  
(2.5.5)

where

\[
\tau_T = \tau_Q \frac{K_F}{K} = \tau_Q \gamma,
\]  
(2.5.6)

is the retardation time, expressed as the product of the relaxation time and \( \gamma \), the thermal conductivity ratio of the solvent to that of the NF solution. Relation (2.5.6) is the same as that relating the relaxation and retardation times in terms of the solvent-to-solution viscosity ratio of purely elastic (non-shear thinning) Boger fluids (see, for instance, Park & Lee [61] in the context of thermal convection). For pure heat conduction, equation (2.5.5) reduces to the dual-phase-lagging model [20,58]:

\[
\tau_Q \frac{\partial Q}{\partial t} + Q = -K \left( V_T + \tau_T \frac{\partial V_T}{\partial t} \right).
\]  
(2.5.7)

It is not difficult to see that this equation relates the temperature gradient at a material point \( x \) and time \( t + \tau_T \) to the heat flux vector at the same point at time \( t + \tau_Q \) for a medium of thermal conductivity \( k \). The relation reads:

\[
Q(x, t + \tau_Q) = -KV_T(x, t + \tau_T),
\]  
(2.5.8)

and equation (2.5.7) is recovered by Taylor expansion of (2.5.8) for small relaxation and retardation times. Note that equation (2.5.7) is parabolic unless the retardation time vanishes, in which case it becomes hyperbolic. It is worth noting that other two-phase systems do reflect the dual-phase-lagging character exhibited in (2.5.7) and (2.5.8). One such connection is the case of a hot jet penetrating a porous medium [25] or the flow through porous layers [26]. In this regard, equations (2.1a and b) of [25] are comparable to equations (4) and (5) of [20]. Also, in this case, the former equations switch from parabolic to hyperbolic for high jet penetration velocity.
In dimensionless form:

\[
C (j_t + \mathbf{v} \cdot \nabla j) + j = -\Delta \theta - S (\Delta \theta_t + \mathbf{v} \cdot \nabla \Delta \theta) \tag{2.5.9}
\]

The following non-dimensional groups have been introduced, namely, the Cattaneo number, \( C \), and dimensionless retardation time, \( S \), given, respectively, as

\[
C = \frac{\tau Q K}{D^2}, \quad S = \frac{K_F}{K}. \tag{2.5.10}
\]

Clearly, \( C \) and \( S \) are not entirely unrelated. In fact, they are both related directly to the NP concentration. Since \( S = \frac{K_F}{C} \), \( S \) must always be smaller than \( C \) for a NF. In this work, although \( C \) and \( S \) involve other parameters, they will be interpreted as reflecting the effect of NP concentration (see Figure 2.2). Note that

\[
S = \gamma C, \quad \gamma = \frac{K_F}{K} < 1. \tag{2.5.11}
\]

Depending on the boundary conditions, the heat flux vector, \( \mathbf{q} \), may or may not be obtained separately. If the boundary conditions are imposed on the temperature and \( j \), then \( \mathbf{q} \) can be calculated by solving

\[
C \left( \frac{\partial \mathbf{q}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{v} \right) + \mathbf{q} = -\nabla \theta - S \left( \frac{\partial \nabla \theta}{\partial t} + \mathbf{v} \cdot \nabla \nabla \theta - \nabla \theta \cdot \nabla \mathbf{v} \right) \tag{2.5.12}
\]

once \( \mathbf{v}, p, \theta \) and \( j \) are determined. If, on other hand, the heat flux vector is imposed at the boundary, then all equations must be solved simultaneously. The Fourier model is recovered upon setting \( C = 0 \) (zero relaxation time) in (2.5.9) and (2.5.12). Moreover, the Fourier limit is also recovered upon setting \( \gamma = 1 \) (thermal conductivity of the NF solution is the same as that of the base fluid solvent). In this case, \( \mathbf{q} = -\nabla \theta \) and \( j = -\Delta \theta \) satisfy equations (2.5.9) and (2.5.12), respectively.
2.5.2 Relaxation and retardation times for nanofluids

For a NF, it is generally observed that the thermal conductivity increases relative to that of the base fluid. In this case, expression (2.5.6) suggests that the retardation time should be smaller than the relaxation time for a NF. Thus, it will be assumed that

$$\tau_T < \tau_Q. \quad (2.5.13)$$

The inequality is in agreement with the observation made by Wang & Wei [20]. They showed that the relaxation and retardation times can be related to the properties of the two-phase system, of the Fourier solvent phase (F) and nanoparticle phase (NP), with corresponding concentrations $1 - \phi$ and $\phi$, respectively. The relations are rewritten here as:

$$\tau_Q = \frac{\phi (\rho c)_N (1-\phi) (\rho c)_F}{ha_v \left[ \phi (\rho c)_N + (1-\phi) (\rho c)_F \right]}, \quad \tau_T = \frac{\phi (\rho c)_N K_F + (1-\phi) (\rho c)_F K_{NP}}{ha_v (K_{NP} + K_F + 2K_{NP/F})}. \quad (2.5.14)$$

Here $\rho$, $c$ and $K$ are the density, specific heat and thermal conductivity of each phase, respectively, and $K_{NP/F}$ is a coupling term arising from the averaging process [20,23] [20,63]. Recall that $h$ is the film heat transfer coefficient, and $a_v = \frac{2}{\delta_F + \delta_{NP}}$ is the interfacial area per unit volume, where $\delta_F$ and $\delta_{NP}$ are mean interaction length in the base fluid and NP phase, which in the present case, are identified as the mean inter-particle distance and particle diameter, $d_{NP}$, respectively. Thus, the relaxation time can be explicitly determined in terms of NP concentration from (2.5.14) once $a_v$ is evaluated.

The interfacial area per unit volume may be estimated as is customarily done in condensed matter physics, by recalling the radius of a sphere of solvent whose volume is equal to the mean volume per NP, the Wigner-Seitz radius, $r_s$ (see, for instance, page 123 in [62]). In a 3-D system with $N$ particles in a volume $V$, the radius is defined by
The retardation time is determined from (2.5.6) once the thermal conductivity of the NF is estimated. Generally, the retardation increases with NP concentration and vanishes in the absence of NP (see below). In this case, $\tau_Q, \tau_T \to 0$ as $\phi \to 0$, and the Fourier limit is recovered as expected. Interestingly, $k_{NF/F}$ can be estimated by combining (2.5.6) and (2.5.10). The retardation time can be estimated from (2.5.6) once the thermal conductivity of the NF is known and the relaxation time is evaluated from (2.5.10). Here, the dependence of the thermal conductivity on NP concentration for Al$_2$O$_3$, TiO$_2$ and CuO are determined following [50,56,63], respectively:

\[
\frac{K}{K_F} = 1 + 7.47\phi \quad (\text{Al}_2\text{O}_3) \tag{2.5.16a}
\]

\[
\frac{K}{K_F} = 1 + 2.92\phi - 11.99\phi^2 \quad (\text{TiO}_2) \tag{2.5.16b}
\]

\[
\frac{K}{K_F} = \frac{K_{NP} + 2K_F + 2(K_{NP} - K_F) + 1.33\phi}{K_{NP} + 2K_F - (K_{NP} - K_F) + 1.33\phi} \quad (\text{CuO}) \tag{2.5.16c}
\]
Figure 2.2: Influence of the NP concentration on the relaxation (top) and retardation times (bottom) for Al$_2$O$_3$, TiO$_2$ and CuO. Insets show plots on log-log scale.

Obviously, thermal conductivity of NPs affects the relaxation and retardation times. Figure 2.2 illustrates the dependence of the relaxation and the retardation times on NP concentration. In this case, the NF solution comprises water, as a base (Fourier) fluid solvent, and Al$_2$O$_3$, TiO$_2$ and CuO, as solute NPs. Figure 2.2 suggests that Al$_2$O$_3$ has the highest relaxation and retardation times. The main reason is the higher thermal conductivity of Al$_2$O$_3$ compared to the other NPs. Surprisingly, Figure 2.2 shows that
despite of lower thermal conductivity of TiO$_2$ compare to CuO, TiO$_2$ has higher relaxation and retardation times than CuO, as a result of the dependence of time and phase lags on specific heat and density of NPs. In other words, time and phase lags intensively increase with specific heat and density of NPs. In Figure 2.2, Al$_2$O$_3$ has higher relaxation and retardation times because of its higher specific heat and density (and thermal conductivity). Obviously the ability to store the heat of NPs is a considerable factor for induced time and phase lags and it is even more important than thermal conductivity. Note that the $\rho c$ term for Al$_2$O$_3$, CuO and TiO$_2$ are $3.5 \times 10^6$, $2.4 \times 10^6$ and $2.9 \times 10^6$, respectively. Both times or phase lags increase with concentration, with the retardation time remaining always lower than the relaxation time ($\gamma < 1$). The inset displays an essentially power-law growth for both times, particularly at low concentration: $\tau_Q, \tau_T \sim \phi^{3.5}$. Using (2.5.6), (2.5.10) and (2.5.14) one obtains

$$K_{NP/F} = \left(\frac{K}{K_F} - 1\right)\left(K_{NP} + K_F\right) + \left(1 - \phi\right)\frac{(\rho c)}{K_F}\phi K_{NP} + \frac{\phi (\rho c)_{NP}}{1 - \phi\left(\rho c\right)_F}K$$

**Figure 2.3:** Dependence of the coupled thermal conductivity on NP concentration for Al$_2$O$_3$, TiO$_2$ and CuO in water.
The influence of the NP concentration on the coupled thermal conductivity for Al₂O₃, TiO₂ and CuO in the water is illustrated in Figure 2.3. From (2.5.17), definitely, conductivity of NPs is the most important parameter in the coupled thermal conductivity and as Figure 2.3 illustrates, it increases by thermal conductivity of NPs. Note that thermal conductivity of Al₂O₃, CuO and TiO₂ are 40, 20 and 11.7 W/m K respectively. The coupled thermal conductivity is high at low NP concentration, decreases monotonically with NP concentration and eventually vanishes. It means that the thermal resistance, which grows with the NP concentration, should be considered especially at higher NP concentrations. Finally, for a NF solution of water and TiO₂ NPs, the relaxation time is obtained as per the procedure in this section. In this case, 
\[ \tau_Q = 2.05 \times 10^{-4} \text{s} \] for NP concentration of 7%, and can lead to a non-negligible Cattaneo number for a moderately small gap: \( C = 0.003 \) for \( D = 0.1 \text{ mm} \). The corresponding retardation time is \( \tau_T = 1.775 \times 10^{-4} \text{s} \).

### 2.5.3 Parallel with viscoelasticity

For heat transfer phenomena, the dimensionless relaxation time is given by the Cattaneo number: \( C = \frac{\tau_Q \kappa}{D^2} \), which should be compared with the elasticity number, \( E = \frac{\tau_S \kappa}{D^2} \), for a polymeric solution with hydrodynamic relaxation time, \( \tau_S \) for stress. Thus, both \( C \) and \( E \) are the ratios of a relaxation time over a thermal diffusion time. Following the formulation for viscoelastic constitutive equations for polymeric solutions [18], consider a polymeric solution of viscosity \( \mu \), comprising a Newtonian solvent of viscosity \( \mu_N \) and a polymeric solute. The addition of the polymer causes a change (normally increase) in the viscosity of the solvent given by

\[ \mu' = \mu - \mu_N. \quad (2.5.18) \]

In this case, the stress of the solution may be written as a superposition of Newtonian and non-Newtonian contributions:
\[ \mathbf{\Sigma} = \mathbf{\Sigma}_N + \mathbf{\Sigma}', \quad \mathbf{\Sigma}_N = \mu_N \left( \nabla \mathbf{V} + (\nabla \mathbf{V})^T \right). \]  

(2.5.19)

The most commonly used constitutive equation for \( \mathbf{\Sigma}' \) is the upper-convective Maxwell equation [18]:

\[ \tau_S \frac{\delta \mathbf{\Sigma}}{\delta t} + \mathbf{\Sigma} = \mu \left( \mathbf{\Gamma} + \tau_R \frac{\delta \mathbf{\Gamma}}{\delta t} \right), \]  

(2.5.20)

where \( \mathbf{\Gamma} = \nabla \mathbf{V} + (\nabla \mathbf{V})^T \), and the upper convective derivative is now given by

\[ \frac{\delta(\cdot)}{\delta t} = \frac{\partial(\cdot)}{\partial t} + \mathbf{V} \cdot \nabla(\cdot) - (\cdot) \cdot \nabla \mathbf{V} = \nabla (\cdot) \cdot (\nabla \mathbf{V}). \]  

(2.5.21)

Note that the operator in (2.5.21) is not the same as in (2.2.3). In this case, the retardation time for stress is given by \( \tau_R = \tau_S \frac{\mu_N}{\mu} \), expressed as the product of the relaxation time and the solvent-to-solution viscosity ratio. Equation (2.5.20) corresponds to purely elastic (non-shear thinning) Boger fluids [18]. Generally, \( \frac{\mu_N}{\mu} < 1 \) or \( \tau_R < \tau_S \).

### 2.6 Microscale thermal convection

As mentioned in section 3, one way to study the non-Fourier character of a fluid is to examine the natural convection of that fluid. At small length scale, the natural convection around MEMS devices has been explained in terms of non-Fourier behaviour [64-66], especially since the significance of nano-structured devices has grown. It has been suggested that conventional models may not be capable of estimating heat transfer in microscale flows, a point that has been observed experimentally [65]. Also, the efficiency of applications making use of Polymerase Chain Reactions (PCR) could be enhanced by further characterization of the small-scale convective flow fields [67]. PCR utilizes Rayleigh-Benard convection at a scale on the order of \( 10^{-3} \) m where non-Fourier effects could be significant.
Experimentally, for natural convection around MEMS devices, it appears that convection may set in at a Rayleigh number much lower than previously expected [65]. This trend has also been theoretically predicted [42,43] where non-Fourier effects were considered in the linear stability analysis. If the value of the Cattaneo number, $C$, becomes large enough (roughly of order $10^{-2}$), which is a plausible value in small scale systems, then instability of the conduction state is predicted at smaller $Ra$ than for a Fourier fluid.

Furthermore, the dimensional analysis of Guo & Li [65] can provide some insight. The Grashof number is defined as

$$\text{Gr} = \frac{g\beta \Delta T D^3}{\nu^2} = \frac{Ra}{Pr}. \quad (2.6.1)$$

For natural convection on macroscopic scales, fluid inertia balances the buoyancy force, in which case it can be shown by dimensional analysis that the Nusselt number becomes

$$\text{Nu} \sim \text{Gr}^{0.5} Pr. \quad (2.6.2)$$

As the length scale of the problem decreases, viscous forces dominate inertia and balance the buoyancy force. Again from dimensional analysis,

$$\text{Nu} \sim \text{Gr} \Pr. \quad (2.6.3)$$

This analysis predicts that $\text{Nu}$ should increase for microscale heat transfer at a given value of Gr. Additionally, as the length scale decreases, the Knudsen number of common gases becomes significant and these gases are considered to be rarefied. Heat transport properties of rarefied gases have been considered for many years [68,69]. The Table below shows the estimated Cattaneo number of some common gases at $20^\circ C$ and standard atmospheric pressure using $\text{Pr} = 2/3$. 


Table 2.3: Cattaneo number for some gases

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Cattaneo Number (C)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>D=10⁻² m</td>
</tr>
<tr>
<td>Air</td>
<td>3.47×10⁻⁵</td>
</tr>
<tr>
<td>Argon</td>
<td>3.89×10⁻⁵</td>
</tr>
<tr>
<td>CO₂</td>
<td>1.52×10⁻⁵</td>
</tr>
<tr>
<td>Hydrogen</td>
<td>1.17×10⁻⁴</td>
</tr>
<tr>
<td>Water Vapour</td>
<td>1.32×10⁻⁵</td>
</tr>
<tr>
<td>Helium</td>
<td>2.88×10⁻⁴</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>3.37×10⁻⁵</td>
</tr>
<tr>
<td>Neon</td>
<td>1.47×10⁻⁴</td>
</tr>
<tr>
<td>Oxygen</td>
<td>3.89×10⁻⁵</td>
</tr>
</tbody>
</table>

Even at standard pressures, the length scale of many practical problems is close to the order of magnitude of the mean free path. Gases that would only be considered rarefied (high Knudsen number) due to high vacuum can now enter the rarefied regime due to the length scales as large as 10⁻⁶ [70]. Here, the Cattaneo number is approximately 10⁻², which, as stated above, can lead to significant non-Fourier effects. As a result, convection around MEMS devices should again be expected to exhibit strong non-Fourier effects as predicted by equation (2.3.10).

2.7 Non-isothermal non-Fourier simple flows

Although non-Fourier heat transfer has been extensively covered in the literature, most applications were limited to transient heat conduction since non-Fourier effects do not manifest themselves in steady heat conduction. They do, however, appear under steady conditions when flow is present. This is one of many interesting phenomena, which do not arise in standard hydrodynamics such as the emergence of heat flux in flow direction without temperature gradient [71-75].
As an illustration consider the pressure-driven (Poiseuille) flow between two flat plates maintained at two different temperatures in the (x, y) plane, with the plates coinciding with the y = 0 and 1 lines. In this case, the streamwise velocity component, u, is obtained from equation (2.2.8) in terms of the (dimensionless) pressure gradient, \( G = \frac{dp}{dx} \), as

\[
u(y) = -\frac{G}{2} y(1-y), \tag{2.7.1}\]

whereas j is obtained from equation (2.2.8) as

\[
j(y) = \frac{dq_y}{dy} = Ec \left(\frac{du}{dy}\right)^2 = Ec \frac{G^2}{4} (1-2y)^2. \tag{2.7.2}\]

Two additional equations are provided from the constitutive equation (2.2.10) for heat, namely

\[
-C q_y \frac{du}{dy} + q_x = S \frac{d\theta}{dy} \frac{du}{dy}, \tag{2.7.3}\]

\[
q_y = -\frac{d\theta}{dy}. \tag{2.7.4}\]

In this case,

\[
q_x = C \left( q_y + \gamma \frac{d\theta}{dy} \right) \frac{du}{dy} = C (\gamma - 1) \frac{d\theta}{dy} \frac{du}{dy}. \tag{2.7.5}\]

Combining (2.7.2) and (2.7.4) leads to an equation for the temperature, namely

\[
\frac{d^2 \theta}{dy^2} = -Ec \frac{G^2}{4} (1-2y)^2, \tag{2.7.6}\]

which can be easily integrated subject to boundary conditions. For instance, if the temperature is equally fixed at the two channel walls, the profile is symmetric, and
\[
\frac{d\theta}{dy} = Ec \frac{G^2}{24} (1 - 2y)^3. \quad (2.7.7)
\]

Finally, from (2.7.5),

\[
q_x = C(\gamma - 1) Ec \frac{G^3}{48} (1 - 2y)^4. \quad (2.7.8)
\]

In this case, the transverse heat flux component, \( q_y = -Ec \frac{G^2}{24} (1 - 2y)^3 \). Clearly \( q_x \) vanishes in the Fourier limit, and thus constitutes a reliable means to assess non-Fourier effect in fluid flow.

### 2.8 Concluding remarks

This survey examines conditions and configurations where non-Fourier effects can be significant for gases and liquids. The formulation for non-isothermal flow is based on the generalized Cattaneo-Vernotte equation for single-phase-lagging (SPL) fluids with only a relaxation time and dual-phase-lagging (DPL) fluids with additional retardation time. The relaxation time and retardation time reflect the delay in the response of the heat flux and the temperature gradient with respect to one another, respectively. The limit of a single-phase-lagging (SPL) fluid is recovered upon setting the retardation time to zero. The SPL model is particularly relevant to low-temperature liquids or to fast heat transfer processes.

The relevance of the DPL model to nanofluids (NFs) has recently been recognized in the literature, and is emphasized here. The equivalence between the two-phase and DPL models allows the expression of the relaxation time in terms of the nanoparticle (NP) concentration (Wang & Wei 2009). The retardation-to-relaxation time ratio is then found to be equal to the NF solution-to-solvent thermal conductivity ratio, \( \gamma \). The parallels between NFs and polymeric solutions of the Boger type, obeying the Oldroyd-B constitutive equation for stress are established throughout the paper. Similar to viscoelastic fluids, the constitutive equation for heat flux used in the present analysis is frame invariant. For non-isothermal flow, the presence of non-Fourier effects leads to the presence of a heat flux component in the streamwise direction, otherwise absent for
Fourier fluids, despite the absence of streamwise temperature gradient. This is again similar to viscoelastic flow where a streamwise normal stress component emerges.

References


Chapter 3

3 Linear stability analysis of convection in a non-Fourier fluid

3.1 Introduction

Generally, conduction is considered to be a heat transfer process in which heat is transported in a diffusive way, and is described by Fourier’s law. When combined with the conservation of energy, Fourier’s law results in a parabolic equation for the temperature field. It means that if a sample is subjected to a thermal disturbance, the disturbance is felt instantaneously at all points of the sample. Fundamentally, Fourier’s law is not realistic since a disturbance in the temperature will travel at a finite speed as it is transferred by molecular interaction [1]. This behaviour is characterized by the Maxwell-Cattaneo or Cattaneo-Vernotte (C-V) equation proposed by Cattaneo [2] and Vernotte [3], which includes a transient term accounting for the finite thermal relaxation time of the medium. This is the time required for the heat flux to relax to a new (stable) steady state following a perturbation in the temperature gradient, establishing a hyperbolic heat (second-sound wave) response. It is recalled that what is meant by the first sound wave is a propagated disturbance of pressure (or density) through a continuous medium such as air or water. Hearing is the sense of first sound wave perception. The second-sound is a heat transfer mechanism describing the propagation of heat as a temperature wave.

In practice, most heat transfer problems involve materials with relaxation times on the order of pico- or $10^{-12}$ seconds [4]. In this case, the C-V equation collapses onto the classical Fourier model. However, the recent interest in second-sound is due to its

\[ \text{[A version of this chapter has been published as – Niknami, M. & Khayat, R. E. 2013 Energy growth of disturbances in a non-Fourier fluid, International Journal of Heat and Mass Transfer, 67 613–626]} \]
potential application in some situations such as the heat transfer in drying sand [5],
cooling or heating in stars [6], and in skin burns [7], where the Fourier’s law is not
adequate to describe the heat transfer process. Second-sound could also be used in
modeling of heat transport in a nuclear fuel rod in a light water reactor [8], and in phase
changes [9]. Different models exist to describe non-Fourier heat conduction.
Experimental evidence of the wave nature of heat propagation in processed meat was
demonstrated by Mitra et al. [10]. The value of the thermal relaxation time of processed
bologna meat was found to be on the order of 15s, which obviously requires the use of
the hyperbolic heat conduction model for such a biological material. Antaki [11] used a
Dual-Phase-Lag (DPL) model for non-Fourier heat conduction to offer a new
interpretation for the experimental evidence in the experiments of Mitra et al. [10] with
processed meat. In the DPL model, materials possess a relaxation time and a retardation
time, reflecting the delay in the response of the heat flux and the temperature gradient
with respect to one another. Unlike the DPL model, in the Single-Phase-Lag model
(SPL), materials possess only a relaxation time (the retardation time is zero in this case).
Xu et al. [12] developed a computational approach to examine the non-Fourier heat
transfer process in skin tissue. They employed the DPL model to study bioheat transfer.
Non-Fourier effect has been examined in other applications such as welding [13], laser
industry [14,15], biothermomechanics of skin [16], and bio-heat transfer during magnetic
hyperthermia treatment [17].

Using analytical [18,19] and numerical approaches [17,20-30], non-Fourier effects have
been investigated in different geometries such as irregular geometries [21], fins [31],
-crack tip [32], slabs [33,34], spherical [35] and cylindrical geometry [36]. Also, different
boundary conditions are studied such as on-off heat flux boundary condition [37],
periodic surface thermal disturbance [19], and axisymmetric surface sources [38].

Regarding non-Fourier heat conduction in fluids, the first experimental evidence was the
detection of the second-sound wave (thermal wave) in superfluid helium (He II) at low
temperatures (T < 2.2 K) [39]. Superfluid helium exhibits outstanding properties such as
an extremely high thermal conductivity and the ability to flow through extremely narrow
channels without noticeable pressure loss. Shimazaki et al. [40] investigated
experimentally transient heat transport phenomena in He II. Their results show that at the lower temperatures, \( T < 1.9 \, \text{K} \), the transported energy in the second-sound mode becomes more efficient (low temperature phenomenon). Zhang [41] studied numerically the transient thermal wave heat transfer in He II, using the classical C-V thermal wave equation.

Using Maxwell-Cattaneo (or Maxwell-Cattaneo-Fox) heat law, Straughan et al. [42] investigated convective stability in the Benard problem. The problem was modeled using the Jaumann derivative of Fox [43] in the constitutive heat equation, and adopting the Boussinesq approximation in the buoyancy term in the momentum equation. Puri et al. [44] studied non-Fourier heat conduction using the MCF model for the Stokes’ first and second problems. The finite thermal relaxation time was found to affect both the temperature and the velocity fields, but this influence is not always consistent. It tends to increase the amplitude of both of these fields under some cases and to decrease it under other cases. Later, Puri et al. [45] used the Maxwell-Cattaneo-Fox (MCF) model to analyze the non-Fourier heat conduction effects in Stokes' first problem for a dipolar fluid. They found that increasing the relaxation time reduces the velocity in heating and increases the velocity in cooling. The temperature predicted by the MCF model is greater than that predicted by Fourier’s law. Note that dipolar fluids are special cases of non-Newtonian fluids with deformable microstructure, consisting of such entities as bubbles, atoms, particulate matter, ions or other suspended bodies.

More recently, Ibrahem et al. [46] studied the nonclassical heat conduction effects in Stokes’ second problem of a micropolar fluid, by examining the influences of the thermal relaxation time on angular velocity, velocity field, and temperature. Micropolar fluids possess a microstructure that renders the stress tensor non-symmetric. These fluids consist of randomly oriented particles suspended in a viscous medium such as dust, dirt, ice or raindrops, or other additives. Pranesh et al. [47] studied the Rayleigh-Benard magneto convection in a micropolar fluid. Using Cattaneo law, Pranesh analyzed the onset of convection. The classical Fourier’s law overpredicts the critical Rayleigh number compared to that predicted by the non-classical hyperbolic law.
In addition, non-Fourier effect can be important in nanofluids [48]. These fluids are solutions consisting of a base fluid solvent, containing a small volume fraction (1-5%) of nanoparticles (NPs) of size of O(1-100nm). The base fluids can be water and organic fluids such as ethanol and ethylene glycol. The NPs can be the oxides of aluminum and silicon, as well as metals such as copper and gold [49,50]. NFs allow for substantial enhancement in conductive heat transfer, as much as 40 percent increase in thermal conductivity [49,51,52] despite the low volume fraction of the NPs. On the other hand, conventional particle-liquid suspensions require high concentrations (>10%) of particles to achieve such enhancement. In practice, rheological and stability problems precluded the widespread application of high concentration suspensions. This makes NFs a valuable fluid for different industrial applications, especially in processes where cooling is of primary concern. Recently, there is an increasing focus on the convective properties of NFs in the literature. The presence of convection term is expected to lead to complex physical behaviour [53]. One advantage that a fluid containing NPs has over its milliparticle and microparticle counterparts is the small size of the NPs. These NPs may be on the same order of magnitude as the molecules in the base fluid. The small size of NPs provides the possibility to have the solution to exist in a very stable manner without the occurrence of gravitational settling or particle agglomeration [49,54].

Heat transfer in nanofluids has been investigated by using different approaches. Xuan [55] developed the dispersion model of nanofluids based on two facts, which would contribute to the significant heat transfer enhancement of nanofluids. First is the enhanced thermal conductivity of nanofluids, and the second is the irregular movement of nanoparticles (thermal dispersion), which can accelerate the energy exchange rate between nanoparticles and the base fluid molecules. Buongiorno [49] examined several potential factors that could be the reason of the observed enhancement in heat transfer in forced convection, namely, the mechanisms behind slip (the difference between the absolute velocity of a NP and the velocity of the base fluid). The potential mechanisms for slip include inertia, gravity settling, diffusiophoresis, the Magnus effect, fluid drainage, Brownian diffusion and thermophoresis. However, Buongiorno found that Brownian motion and thermophoresis seem to be the only significant factors that could properly explain the unexpected heat transfer capabilities of NFs. It should be noted that
the Brownian motion is the induced random drifting caused by the collisions of NPs with the molecules in the base fluid, whereas thermophoresis is a diffusive effect that causes particles to move as a result of a temperature gradient within the fluid. Buonjiorno used a four-equation model for the conservation laws of nanofluids (two mass equations, the momentum equation, and the energy equation). It is found that the energy equation for a NF has the same form as the base fluid, and Brownian motion and thermophoresis do not contribute in the heat transfer equation. On the other hand, the observations showed a strong dependency of thermal properties of NFs on the volume concentration of nanoparticles. Consequently, all conservation equations should be considered simultaneously to solve.

So far, emphasis has been on forced convection [54,56], and not so much on natural convection. In the case of natural convection, Tzou [57] investigated the linear thermal stability of Rayleigh-Benard convection using a two-phase system approach, incorporating the effects of Brownian motion and thermophoresis. Tzou [57] showed that the ability of the fluid to transfer heat increases by adding NPs to the base fluid. Tzou [57] found that the onset of convection occurs at a Rayleigh number lower by one or two orders of magnitude compared to the base fluid. A similar stability analysis was carried out later by Nield & Kuznetsov [58]. Their analysis confirms Tzou’s findings and suggests the possibility of oscillatory convection at instability, depending on the boundary conditions imposed. In fact, one of the major drawbacks of the two-phase approach for NFs is the nature of the boundary conditions for the concentration of NPs.

In reality, NFs are quite homogeneous solutions. Consequently, it is expected that the NF behaves more like a pure fluid rather than a solid/fluid mixture. Vadasz et al. [48] questioned the basis on which experimental measurements on the thermal conductivity of nanofluids usually use theory that is based on the classical Fourier’s law of heat conduction. They argued that non-Fourier effects should be considered when interpreting experimental results. They used the Cattaneo equation to investigate heat transfer in a slab. They compared the experiments by employing both the Cattaneo and Fourier theories of heat conduction. Interestingly, they found that the validity of either one of the models (Cattaneo and Fourier) as the correct one cannot be confirmed. In addition, Wang
et al. [59] showed that the presence of nanoelements leads to non-Fourier heat conduction in nanofluids, which suggests that nanofluids could be a good candidate to study non-Fourier heat conduction. They showed that the presence of nanoelements leads to the dual-phase-lagging heat conduction in nanofluids. As explained earlier, DPL fluids possess a relaxation time and a retardation time, reflecting the delay in the response of the heat flux and the temperature gradient with respect to one another. Straughan [60] studied nanofluid flow between two horizontal plates, and assumed that nanofluids follow the SPL formulation. Note that the single-phase-lag fluids possess only the relaxation time and the retardation time is zero. Similarly, an alternative approach to the two-phase model is adopted in this study to examine the energy growth of disturbances in a NF, and non-Fourier fluids in general, by considering the fluid as a homogeneous solution, and accounting for the finite thermal relaxation time resulting from the addition of NPs to the base fluid.

It has long been realized that adding the partial time derivative does not completely solve the problem of instantaneous thermal relaxation [61-63]. The Maxwell-Cattaneo equation is restricted to non-deformable media, and is not a frame-invariant constitutive relation. Several objective derivatives have been examined to solve this problem. However, they each have had their own shortcomings. The most promising modification is that of Christov [64], which was recently revisited by Khayat & Ostoja-Starzewski [65]. He used the Oldroyds’ upper-convected derivative, which leads to the frame indifferent Maxwell-Cattaneo equation. Coupled with the energy equation, this constitutive equation can also yield a single equation for the temperature field. This equation replaces Fourier’s law for fluid flow whenever the relaxation time is relevant, and collapses back to Fourier’s law whenever it is not.

To investigate the long-term stability of flow, the disturbance is assumed to behave exponentially with time, \( \exp(st) \), where \( s \) dictates the time evolution of the disturbance and \( t \) denotes time. To quantify the short-time behavior of the disturbance, the change in the energy growth of perturbations with respect to time could be studied. To find the energy growth as a function of time, an initial value problem, which results from the substitution of perturbations in corresponding conservation laws, should be solved.
Homsy [66] used the energy method to discuss the stability of time-dependent diffusive temperature profiles in fluid layers subject to impulsive changes in surface temperature. Reddy et al. [67] investigated the energy growth in viscous channel flows and found that the expansion coefficients of an initial perturbation with zero normal vorticity can be relatively large, and such perturbations can lead to large growth in the amplitude of the normal vorticity. Gustavsson [68] investigated the energy growth of perturbations in Poiseuille flow. He found that the energy in the normal vorticity could grow to be $O(1000)$ at subcritical Reynolds numbers. He used the complete solution of the linear initial value problem in the plane Poiseuille flow for his study. Joo et al. [69] studied the linear stability of the viscoelastic flow of an Oldroyd-B fluid between rotating cylinders with a pressure gradient. Using energy analysis, he found the relative importance of involved terms. López [70] studied numerically the pipe flow of purely viscous shear-thinning fluids. Using the growth function, he stated the initial value problem for the perturbation field.

The paper is organized as follows. The single-phase-lag approach is the premise of the current work, and is revisited in section 3.2, to include convective effects, generalizing the existing single-phase-lag model [64,65]. In section 3.3, the linear stability is examined. In section 3.4, the mechanism of instability is investigated through an examination of the disturbance-energy equation. The emphasis in this work will be on the energy analysis of perturbations, which is explained later in section 3.5. The method for computing transient energy growth is based on the eigenfunction expansion. Finally, concluding remarks are given in section 3.6.

### 3.2 Governing equations and boundary conditions

Consider a thin layer of a Newtonian non-Fourier liquid confined between the $(X, Z)$ planes at $Z=0$ and the plane $Z=D$, maintained at fixed temperatures $T_0 + \delta T$ and $T_0$, respectively. The fluid layer is assumed to be of infinite horizontal extent. Convection emerges when the buoyancy effect exceeds a critical threshold relative to the viscous effect. The gravity acceleration vector is given by $g = -g e_z$, where $e_z$ is the unit vector in the $Z$ direction. The fluid density, $\rho$, is assumed to depend on the temperature, $T$, following
\[ \rho = \rho_0 \left[ 1 - \alpha_T (T - T_0) \right], \]  
\[ \text{(3.2.1)} \]

where \( \alpha_T \) is the coefficient of volume expansion and \( \rho_0 \) is the mass density of the fluid at \( T_0 \). The fluid is assumed to be incompressible, of specific heat at constant pressure \( C_p \), thermal conductivity \( K \) and viscosity \( \mu \). The general governing equations for a non-Fourier fluid comprise the conservation of mass, linear momentum and energy, as well as the constitutive equation for the heat flux. In this case, the conservation equations are given by

\[ \nabla \cdot \mathbf{V} = 0, \quad \text{(3.2.2a)} \]

\[ \rho_0 (\mathbf{V}_t + \mathbf{V} \cdot \nabla \mathbf{V}) = -\nabla P - \rho \mathbf{g} e_z + \mu \nabla^2 \mathbf{V}, \quad \text{(3.2.2b)} \]

\[ \rho_0 \varepsilon_p (T_t + \mathbf{V} \cdot \nabla T) = -\nabla \cdot \mathbf{Q}, \quad \text{(3.2.2c)} \]

where \( \nabla \) and \( \nabla^2 \) are the gradient and Laplacian operators, respectively, and a subscript denotes partial differentiation. Here \( \mathbf{V} = (U, W) \) is the velocity vector, \( P \) is the pressure, \( T \) is the temperature and \( \mathbf{Q} \) is the heat flux vector. The boundary conditions at the lower and upper surfaces are taken to correspond to free-free conditions. In this case

\[ \mathbf{V}(X, Z = 0, t) \cdot e_z = \mathbf{V}(X, Z = D, t) \cdot e_z = 0, \quad \text{(3.2.3a)} \]

\[ \mathbf{V}_{zz}(X, Z = 0, t) \cdot e_z = \mathbf{V}_{zz}(X, Z = D, t) \cdot e_z = 0, \quad \text{(3.2.3b)} \]

\[ T(X, Z = 0, t) = T_0 + \delta T, \quad T(X, Z = D, t) = T_0. \quad \text{(3.2.3c)} \]

Note that the Boussinesq approximation, which states that the effect of compressibility is negligible everywhere in the conservation equations except in the buoyancy term, is assumed to hold. Given the small spatio-temporal scales involved in heat transfer enhancement in NFs, resulting from the addition of NPs to the base fluid, it is not surprising that non-Fourier effects are envisaged as being behind this enhancement. The most commonly used constitutive equation for \( \mathbf{Q} \) is the Maxwell-Cattaneo equation for heat conduction [2]. For a moving fluid, however, this equation needs to be reformulated
and rendered objective (frame indifferent). This was achieved by Christov [64], and later revisited by Khayat & Ostoja-Starzewski [65]. The resulting equation is simply reproduced here:

\[ \tau \frac{\delta Q}{\delta t} + Q = -KV T \]  

(3.2.4)

where \( \tau \) is the relaxation time, and the Li type derivative is given by

\[ \frac{\delta}{\delta t} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \left( \frac{\partial}{\partial t} + \mathbf{V} \right) - \mathbf{V} - \left( \mathbf{V} \right) \mathbf{V} \cdot \mathbf{V} \]  

(3.2.5)

In this case, the constitutive equation for the solution heat flux is obtained from (3.2.2) and (3.2.4-5) to read

\[ \tau \left( Q_t + V \cdot \nabla Q - Q \cdot \nabla V \right) = -Q - KV T, \]  

(3.2.6)

For pure heat conduction, equation (3.2.6) reduces to

\[ \tau \frac{\delta Q}{\delta t} + Q = -KV T. \]  

(3.2.7)

It is not difficult to see that this equation relates the temperature gradient at a material point \( \mathbf{x} \) and time \( t \) to the heat flux vector at the same point at time \( t + \tau \) for a medium of thermal conductivity \( K \). The relation reads:

\[ \mathbf{Q}(x, t + \tau) = -KV T(x, t), \]  

(3.2.8)

and equation (3.2.7) is recovered by Taylor expansion of (3.2.8) for small relaxation time. The base state corresponds to stationary heat conduction, which remains the same as for a Fourier fluid since both transient and upper convective terms in (3.2.6) vanish in this case. Consequently, the temperature, pressure gradient and heat flux for the conductive state are given by

\[ T_B = -\left( \frac{Z}{D} \right) \delta T + T_0 + \delta T, \]

\[ \frac{dP}{dZ} = -\rho_0 \left[ 1 - \alpha_T \delta T(1 - Z/D) \right] g, \]  

(3.2.9)
\[ Q_B = \left( 0, K \frac{\delta T}{D} \right), \]

respectively. The problem is conveniently cast in dimensionless form by taking the length, time and velocity scales as \( D, \frac{D^2}{\kappa} \) and \( \frac{\kappa}{D} \) respectively, where \( \kappa = \frac{K}{\rho_0 c_p} \) is the thermal diffusivity. Let \( p = \frac{D^2}{\kappa \mu} (P - P_B) \) and \( \theta = \frac{T - T_B}{\delta T} \) be the dimensionless pressure and temperature deviations from the base (conductive) state. In this case, the dimensionless equations are given by

\[ \nabla \cdot \mathbf{v} = 0, \quad (3.2.10a) \]

\[ \text{Pr}^{-1} \left( \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \text{Ra} \theta \mathbf{e}_z + \Delta \mathbf{v}, \quad (3.2.10b) \]

\[ \theta_t + \mathbf{v} \cdot \nabla \theta = -j + \mathbf{w}, \quad (3.2.10c) \]

\[ C \left( \mathbf{q}_t - \mathbf{v}_Z + \mathbf{v} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{v} \right) = -\mathbf{q} - \nabla \theta, \quad (3.2.10d) \]

where \( \mathbf{v} = (u, w) \) and \( \mathbf{q} \) are the dimensionless velocity and heat flux vectors, respectively. Here \( j = \nabla \cdot \mathbf{q} \). The following non-dimensional groups have been introduced, namely, the Prandtl number, the Cattaneo number and the Rayleigh number, respectively, given by

\[ \text{Pr} = \frac{\nu}{\kappa}, \quad C = \frac{\tau \kappa}{D^2}, \quad \text{Ra} = \frac{\delta T \alpha T g D^3}{\nu \kappa}. \]

The problem can be simplified by casting the constitutive equation for heat flux in terms of the scalar variable \( j \). Thus, upon taking the divergence of equation (3.2.10d), and recalling again the identity \( \nabla \cdot (\mathbf{a} \cdot \nabla \mathbf{b}) = \nabla \mathbf{a} : \nabla \mathbf{b} + \mathbf{a} \cdot \nabla (\nabla \cdot \mathbf{b}) \), one obtains the following constitutive equation for \( j \):

\[ C (j_t + \mathbf{v} \cdot \nabla j) + j = -\Delta \theta, \quad (3.2.11) \]
where the continuity equation (3.2.10a) is used. Finally, Fourier model is recovered upon setting \( C = 0 \) (zero relaxation time) in (3.2.11d). The boundary conditions (3.2.3) reduce to

\[
\begin{align*}
wx(x, z=0, t) &= u_z(x, z=0, t) = u_z(x, z=1, t) \\
&= \theta(x, z=0, t) = \theta(x, z=1, t) = 0,
\end{align*}
\]

which must be used to solve equations (3.2.10a-c) and (3.2.11).

### 3.3 Linear stability analysis

Comparable to a Fourier fluid, the conduction of a non-Fourier fluid is lost to convection once a critical value of the Rayleigh number, \( Ra_c \), is exceeded. However, in contrast to a Fourier fluid, non-Fourier conduction can be lost to steady or oscillatory convection, depending on the flow parameters. The stability of the conduction state is examined to a small (infinitesimal) perturbation of the form

\[
v(x, z, t) = V(z)e^{st + ikx}, \quad \theta(x, z, t) = \Theta(z)e^{st + ikx},
\]

\[
p(x, z, t) = P(z)e^{st + ikx}, \quad j(x, z, t) = J(z)e^{st + ikx},
\]

where \( k \) is the perturbation wavenumber in the \( x \) direction, and \( s \) dictates the time evolution of the disturbance. Thus, the conduction/base state is stable (unstable) if the real part of \( s \) is negative (positive). Following the standard procedure in linear stability analysis, the \( z \)-dependent eigenvector components are governed by

\[
\begin{align*}
&ikU + DW = 0, \\
&\Pr^{-1}sU = -ikP + (D^2 - k^2)U, \\
&\Pr^{-1}sW = -DP + (D^2 - k^2)W + Ra\Theta, \\
&s\Theta = -J + W, \\
&(Cs + 1)J = -\left(D^2 - k^2\right)\Theta,
\end{align*}
\]
where $D = d/dz$. In this case, taking $W = \sin(n\pi z)$, and eliminating $U$, $P$, $\Theta$ and $J$, lead to the following dispersion relation:

$$s^3 + \left(Pr\beta_n + \frac{1}{C}\right)s^2 + \left(\beta_n Pr - \frac{1}{C} - \frac{k^2RaPr}{\beta_n}\right)s + \frac{Pr}{C\beta_n} \left(\beta_n^3 - k^2Ra\right) = 0,$$

(3.3.3)

where $\beta_n = k^2 + n^2\pi^2$, and $n$ is the mode number. In contrast to a Fourier fluid, the presence of the cubic term in (3.3.3) hints to the possibility of stationary or oscillatory convection. Thus, the stability picture depends on the relaxation time, $Pr$ and $Ra$. Finally, it is helpful to list the eigenvector components, which will be used later, and take the form:

$$U = -\frac{in\pi}{k}\cos(n\pi z),\ W = \sin(n\pi z),\ P = \left[\frac{Ra}{\beta_n}\left(\frac{Cs + 1}{Cs^2 + s + \beta_n}\right) - 2\right]n\pi\cos(n\pi z),$$

$$\Theta = \left(\frac{Cs + 1}{Cs^2 + s + \beta_n}\right)\sin(n\pi z),\ J = \left(\frac{1}{Cs^2 + s + \beta_n}\right)\beta_n\sin(n\pi z).$$

(3.3.4)

It is well established that the $n = 1$ mode is the most dangerous mode for a Fourier fluid. Also, the $n = 1$ mode turns out to be the most dangerous mode for both stationary and oscillatory convection. Consequently, only the $n = 1$ mode will be considered hereafter, except in section 5 where higher-order modes will be needed. For steady convection, one recovers the same critical Rayleigh number, $Ra_c$, as a Fourier fluid, namely

$$Ra_c = Ra_{cF} = \frac{\beta^3}{k^2}.$$  

(3.3.5)

In this case, $Ra_c$ displaying a minimum, $Ra_m = \frac{27\pi^4}{4}$ at $k_m = \frac{\pi}{\sqrt{2}}$. An important quantity in this analysis is the minimum critical Rayleigh number, $Ra_m$, as it reflects the Rayleigh number (temperature difference) for the observation of convection in reality. For oscillatory convection, the corresponding neutral curves are obtained upon setting $s = i\omega$ in (3.3.3), $\omega_c$ being the frequency, and separating real and imaginary parts to give
\[ \text{Ra}_c = \frac{\beta C \text{Pr}^2 + \text{Pr} + 1}{\beta^2 C^2 \text{Pr}^2} \text{Ra}_c \text{F}, \quad \omega_c = \frac{1}{C} \sqrt{\frac{C \Pr \beta - \text{Pr} - 1}{\text{Pr}}}. \quad (3.3.6) \]

Clearly, oscillatory convection is possible only if

\[ k > k_i = \sqrt{\frac{1 + \text{Pr}}{\text{Pr}C} - \pi^2}. \quad (3.3.7) \]

This criterion constrains the range of wavenumber for oscillatory convection. Thus, it is anticipated that each marginal stability curve in the Ra-k plane comprises two distinct branches: a Fourier branch, corresponding to steady convection, for \( k < k_i \), and a non-Fourier branch, corresponding to oscillatory convection, for \( k > k_i \), where \( k_i \) is the wavenumber at which the two branches intersect. It is not difficult to verify that substituting (3.3.7) into (3.3.6) leads to \( \text{Ra}_c(k=k_i) \equiv \text{Ra}_i = \text{Ra}_c \text{F}(k=k_i) \). Expression (3.3.7) suggests that a limit, \( C_\infty \), of \( C \) exists for which \( k_i = 0 \).

The influence of the Cattaneo number on the overall marginal stability picture is illustrated in Figure 3.1, where the marginal stability curves are plotted against the wavenumber for \( \text{Pr} = 10 \). At the center, is the marginal stability curve for a Fourier fluid \( (C = 0) \). In this case, there is an exchange of stability between pure conduction and stationary convection for any wavenumber. For relatively small \( C > 0 \) (for instance, \( C=0.05 \) in Figure 3.1), each non-Fourier curve comprises a stationary branch, also part of the Fourier curve for \( k < k_i \), and an oscillatory convective branch (overstability) for \( k > k_i \). The weakly non-Fourier regime is taken to correspond to \( C < C_H \). Here, \( C_H = 0.0655 \) is the level of non-Fourier threshold below which stationary convection is predicted to be observed first in reality. Note, however, that the oscillatory branch remains present for \( k > k_i \). The oscillatory branch is expected to play an increasingly dominant role as \( C \) increases from zero in the post-critical range of Rayleigh number as a result of nonlinear modal interaction from different wavenumbers. As \( C \) increases, \( k_i \) moves to the left, reducing the range for the stationary convection, and, simultaneously, lowering the
oscillatory branch until the minimum Rayleigh number of this branch becomes equal to
$$Ra_{mF} = \frac{27\pi^4}{4}, \text{ at } C = C_H.$$ 

For a strongly non-Fourier fluid ($C = 0.1 > C_H$), oscillatory convection is predicted to occur at $Ra_c = Ra_{m}$, with the steady convection range ($k < k_i$) diminishing as $C$ increases, to eventually disappear for $C > C_\infty = -\frac{11}{10\pi^2} = 0.111$. In this range, oscillatory convection is predicted to be observed first for any wavenumber. The conductive state loses its stability to the oscillatory convection at a Rayleigh number that is increasingly smaller than $Ra_{mF}$ as $C$ increases. Equation (3.3.6) suggests that the critical Rayleigh number vanishes at large $C$, indicating that strongly non-Fourier fluids (for instance, for nanofluids with a high nanoparticle concentration) exhibit spontaneous convection with no prior conduction. Note that the wavenumber, $k_m$, corresponding to the onset of oscillatory convection is always larger than $k_{mF}$. Thus, the decrease in roll size makes it more difficult to detect the convective roll pattern for more non-Fourier fluids.

The linear stability picture suggests that different regimes are possible for a non-Fourier fluid. However, for a given wavenumber and Rayleigh number, it is not obvious if the convection is stationary or oscillatory. By finding the roots of the dispersion relation, (3.3.3), and plotting the marginal oscillatory curve, the mode of convection (stationary/oscillatory) is revealed. The marginal oscillatory curves are illustrated in Figures 3.2 (top) and 3.2 (bottom) for Cattaneo numbers 0.05 and 0.1, respectively. Note that Figure 3.2 is plotted for $Pr = 10$. As explained earlier, for $C < C_H (= 0.0655)$, each non-Fourier curve comprises a stationary branch, which is a part of the Fourier curve, for $k < k_i$, and an oscillatory convective branch, for $k > k_i$. Interestingly, the marginal oscillatory curve intersects the existing two branches (Fourier/non-Fourier) at the same point ($k_i, Ra_i$), regardless of the value of Cattaneo number.
Figure 3.1: Influence of the Cattaneo number on the marginal stability curves in the Ra-k plane (Pr = 10).

Considering Figure 3.2, the stability mode (stationary/oscillatory) can be predicted as follows: above the marginal oscillatory curve, the mode is stationary, regardless of the stability condition (stable/unstable). If the Rayleigh number is higher than the critical Rayleigh number, the convection mode is stationary. If the Rayleigh number is lower than the critical Rayleigh number, the conduction mode is stationary. To observe the oscillatory mode, the Rayleigh number should be lower than the marginal oscillatory Rayleigh number, regardless of the stability condition (stable/unstable). If the Rayleigh number is higher than the critical Rayleigh number, the convection mode is oscillatory. If the Rayleigh number is lower than the critical Rayleigh number, the oscillatory mode is expected to be observed. In addition, from mathematical view-point, Figures 3.2 implies that above the marginal oscillatory curve, all roots of the dispersion relation, (3.3.3), are real and below the marginal oscillatory curve, there is one real root and a pair of complex conjugate roots.
When $C = 0.05$ ($< C_H = 0.0655$), four different zones can be distinguished in Figure 3.2 (top). In this case, the critical Rayleigh number is 657, and the marginal oscillatory Rayleigh number is found to be 620. At the top is the 1st zone that indicates the presence of the stationary convective mode. The 2nd zone represents the oscillatory convection zone. At the bottom is the 3rd zone that shows the presence of the oscillatory stable mode, whereas the 4th zone (at the center) shows the presence of the stationary stable mode. In contrast to Figure 3.2 (top), three different zones can be distinguished in Figure 3.2 (bottom) for $C = 0.1$ ($> C_H = 0.0655$). In this case, the critical Rayleigh number is 470, and the marginal oscillatory Rayleigh number is found to be 640. At the top is the 1st zone that represents the stationary convection zone. The 2nd zone (at the center) represents the oscillatory convection zone. Finally, the 3rd zone indicates the presence of the oscillatory stable mode.
Figure 3.2: C=0.05 < C_H, Pr=10, Region 1: Stationary Unstable, 2: Oscillatory Unstable, 3: Oscillatory Stable, 4: Stationary Stable (top) and C=0.1 > C_H, Pr=10, Region 1: Stationary Unstable, 2: Oscillatory Unstable, 3: Oscillatory Stable (bottom).

3.4 Energy Analysis

In this section, the mechanism of the instability is investigated through an examination of the disturbance-energy equation. In this method, the energy transfer between mean flow
and disturbance flow is considered by evaluating the energy balance in the flow. The kinetic energy of the perturbations is defined as:

\[
\text{KE} = \int_{\Omega} \frac{1}{2} v_i^2 \, dv = \int_{\Omega} \frac{1}{2} (u^2 + w^2) \, dv ,
\]

(3.4.1)

where \( v_i \) is the perturbation velocity and \( \Omega \) is the cell volume which is defined as \( \Omega: 0 < z < 1, 0 < x < 2\pi/k \). The energy balance rate is then obtained by taking the derivative of (3.4.1) with respect to time. Multiplying the linearized disturbance momentum equation by the perturbation velocity, \( v_i \), the kinetic energy rate reads as

\[
\frac{d\text{KE}}{dt} = \int_{\Omega} v_i \frac{\partial v_i}{\partial t} \, dv = \int_{\Omega} \left( u \frac{\partial u}{\partial t} + w \frac{\partial w}{\partial t} \right) \, dv .
\]

(3.4.2)

Substituting \( \frac{\partial u}{\partial t} \) and \( \frac{\partial w}{\partial t} \) from (3.2.10b) into (3.4.2) and rearranging the terms, the kinetic energy rate reads as

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} (u^2 + w^2) \right] = u \frac{\partial u}{\partial t} + w \frac{\partial w}{\partial t} = \text{Pr \, Ra} \, w \, \theta - \text{Pr} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right].
\]

(3.4.3)

Following (3.4.2), the kinetic energy rate is obtained by integrating (3.4.3) over the volume, \( \Omega \). Thus, the final kinetic energy rate equation becomes:

\[
\frac{1}{\text{Pr}} \frac{d\text{KE}}{dt} = \frac{1}{\text{Pr}} \frac{\partial}{\partial t} \frac{1}{2} \left( u^2 + w^2 \right) = \text{Ra} \, \left( w \, \theta \right) - \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right) ,
\]

(3.4.4)

where the brackets define the integral over the volume, \( \Omega \). Note that some terms vanish after the integration, e.g., the pressure gradient. If \( \frac{d\text{KE}}{dt} < 0 \) for any time \( t > 0 \), it implies
that $\text{KE}(t) < \text{KE}(0)$. Since the kinetic energy within any system is always positive, $K(t)$ tends to zero at large $t$. This plausibly implies that, $\mathbf{v}(x,z,t) \to 0$ as $t \to \infty$ for all $x, z \in \Omega$. In other words, the perturbed velocity approaches zero (decays with time), or equivalently, the flow is stable. Clearly, $-\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\right)^2$ has a negative value for all perturbations, which confirms that viscosity always dissipates energy. In contrast to the viscous term, the sign of the buoyancy term, $\text{Ra}(\text{w}\theta)$, is not obvious. Different signs are possible depending on the values of the velocity ($w$) and the temperature ($\theta$). Thus, it is expected that the kinetic energy rate, (3.4.4), changes by changing the ratio of the buoyancy over viscous term.

Interestingly, there is not any contribution from the mean flow parameters in the kinetic energy rate relation, (3.4.4). In other words, the natural tendency of the flow causes the cells to take energy out of the perturbed velocity and temperature. Furthermore, the pressure gradient term is entirely absent from the energy balance within the system. Apparently, there are two main factors which can change the kinetic energy rate: buoyancy and viscous effect. Hence, the weighted kinetic energy rate, $\frac{d\text{KE}_R}{dt}$, is defined as

$$
\text{Weighted}(\frac{d\text{KE}}{dt}) = \frac{d\text{KE}_R}{dt} = \text{Pr} \frac{\text{Ra}(\text{w}\theta)}{\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\right)^2} - \text{Pr}, \quad (3.4.5)
$$

to show the relative importance of these factors. Here $u$, $w$ and $\theta$ can be obtained from (3.3.1) and (3.3.4). In Figure 3.3, the weighted kinetic energy rate is shown against the Rayleigh number for different Cattaneo numbers. In this figure, $\text{Pr} = 10$ and the Rayleigh number changes from 100 to 1000. The exact ratio of the buoyancy and viscous terms can be found directly by using (3.4.5). From Figure 3.3, it is found that the ratio of the buoyancy and viscous terms changes from 0.9 (<1) to 1.1 (>1) depending on the value of the Cattaneo number. In other words, the dominance of the buoyancy and viscous terms
is replaced by each other. By increasing the Cattaneo number, the critical Rayleigh number decreases. Note that at the critical Rayleigh number, the weighted kinetic energy rate is equal to zero, or equivalently the ratio of the buoyancy term over viscous term is one. When the Rayleigh number is less than the critical Rayleigh number, the ratio of the buoyancy term over the viscous term is less than one, which signals that the viscous term plays more important role compared to the buoyancy term. For the neutral stable state, the ratio is equal to one, which shows that at this point the buoyancy and the viscous terms balance each other. For Rayleigh numbers larger than the critical Rayleigh number, the ratio is more than one, which signals the dominance of the buoyancy hereafter. As the Rayleigh number increases, the dominance of the buoyancy term becomes more significant, and the ratio (3.4.5) becomes larger.

For a Fourier fluid (C = 0), Figure 3.3 indicates a monotonic increase in the weighted kinetic energy rate curve with respect to the Rayleigh number. As the Cattaneo number increases, the monotonic increase does not persist anymore. Unlike the Fourier case, a break-point is observed in the weighted kinetic energy rate curves for the non-Fourier (C ≠ 0 in Figure 3.3) fluids. At this point, a discontinuity is observed in the slope of the weighted kinetic energy rate curve with respect to Ra. The break-point signals the possibility of the mode change for the non-Fourier fluids. Using mathematical analysis, it is found that the break-point in the curves shows a transition from the oscillatory mode to the stationary mode. In other words, for relatively low Rayleigh numbers (less than the critical Rayleigh number), two of the three roots of the dispersion relation (3.3.3) have complex conjugate values. The presence of the complex conjugate roots leads to the oscillatory mode. On the other hand, for relatively large Rayleigh numbers (more than the critical Rayleigh number) all three roots are real, which predicts the stationary mode for the flow. For the Fourier fluid, the dispersion relation (3.3.3) reduces to a 2nd order polynomial with a discriminant always positive (with corresponding two real roots). This is confirmed from Figure 3.3, where for the Fourier fluid the break-point is not observed in the weighted kinetic energy rate curve. Thus, the transition does not occur in the case of the Fourier fluid, and the stability mode is always stationary.
Figure 3.3 confirms the results of the linear stability analysis, which are explained in section 3.3. For all Cattaneo numbers, as Ra increases, the buoyancy term plays more important role. The kinetic energy is actually supplied from the interaction of the perturbed velocity and temperature. Considering $u$, $w$ and $\theta$, the only variable that reflects the contribution of the Cattaneo number, or equivalently the non-Fourier effect, is the temperature $\theta$. From (3.4.5), it is found that the viscous term (denominator) does not change by changing the Cattaneo number (the non-Fourier effect), whereas it changes the buoyancy term via the temperature. For the neutral stable point, the weighted kinetic energy rate approaches zero. In other words, the kinetic energy remains constant with time, and the perturbed velocity and temperature neither grows nor decays with time.

As explained in section 3.3, Figure 3.1 shows that for certain values of the Cattaneo number ($C < C_h$), the critical Rayleigh number is equal to the critical Rayleigh number for the Fourier fluid, which is known to be 657. As the Cattaneo number exceeds $C_h = 0.0655$ (for instance, $C = 0.1$), the critical Rayleigh number decreases, which is
confirmed from Figure 3.3. For example, when \( C = 0.1 \), the weighted kinetic energy rate is zero at a lower Rayleigh number (\( Ra = 470 \)) compared to the critical Rayleigh number for a non-Fourier fluid (\( Ra = 657 \)). More findings from Figure 3.3 will be discussed later in section 3.5. The definition of the Rayleigh number suggests that any increase in the temperature difference between two plates increases the Rayleigh number. Referring to Figure 3.1, as the Cattaneo number increases the onset of convection occurs at a lower Rayleigh number. Thus, by increasing the Cattaneo number, the onset of convection occurs at a lower temperature difference. Additionally, it is well-known that by increasing the temperature difference between two plates, the role of the buoyancy term will be more significant. Thus, by increasing the Cattaneo number, the onset of convection occurs at a lower temperature difference between two plates.

3.5 Energy growth of perturbations

Linear stability analysis investigates the growth/decay of small perturbations by linearizing the Navier-Stokes equations, yielding the dispersion relation (3.3.3). To determine the stability, the roots of the dispersion relation are to be found. If there is an eigenvalue (root of the dispersion relation) with positive real part, exponentially growing perturbed variables (velocity and temperature) are expected to be observed. In this case, the flow is linearly unstable. Thus, linear stability analysis gives conditions for the exponential instability. To describe the time behavior of the general initial value problems, eigenvectors are as important as eigenvalues. By studying eigenvalues only, it is impossible to find the full stability picture of the flow. The objective of this section is to investigate the initial value problem, and to examine its potential to exhibit and quantify the short time behavior of the energy growth. Following Schmid & Henningson (2001) [71], the energy of perturbations is introduced in terms of primary variables as a disturbance measure in order to quantify the disturbance size. In order to obtain the stability picture, the evolution of the initial conditions in the pre- and post-critical range of the Rayleigh number is monitored. The primary variables are the perturbed velocity and the perturbed temperature, which are given by
To measure the growth, a physically relevant quantity is the energy norm. Then, the energy of the disturbances can be defined as (3.5.2), which is formed by adding the kinetic energy and the thermal energy of the perturbations to read

\[
E(t) = \int \frac{1}{2} \left( |u(x, z, t)|^2 + |w(x, z, t)|^2 + \Pr |\theta(x, z, t)|^2 \right) dv = \\
\int_0^{2\pi/k} \cos(kx)^2 dx \int_0^1 \left( |\tilde{u}(z, t)|^2 + |\tilde{w}(z, t)|^2 + \Pr |\tilde{\theta}(z, t)|^2 \right) dz = \\
\frac{\pi}{k} \left( |\tilde{u}(z, t)|^2 + |\tilde{w}(z, t)|^2 + \Pr |\tilde{\theta}(z, t)|^2 \right),
\]

where the brackets denote the integration over \( z \in [0, 1] \). Note that the definition of the energy in (3.5.2) is reminiscent to the energy definition used in the equation (3.59) of Straughan [72]. The maximum possible energy growth, which is a function of time, is introduced as

\[
G(t) = \sup_{E(t=0) \neq 0} \frac{E(t)}{E(t=0)}. 
\]

\( G(t) \) measures the greatest possible growth in the energy of an initial perturbation at time \( t \). In order to compute \( G(t) \), let \( \{s_j\} \) denote the eigenvalues of the system and \( \{E_j\} \) denote the corresponding eigenvectors, namely

\[
E_j(z) = \begin{pmatrix}
U_j(z) \\
W_j(z) \\
\sqrt{\Pr} \Theta_j(z)
\end{pmatrix},
\]

(3.5.4a)
where \( U, W \) and \( \Theta \) are components of \( E_j \) and read as:

\[
\begin{align*}
U_1(z) &= \frac{i\pi}{k}\cos(\pi z), \quad U_2(z) = \frac{i\pi}{k}\cos(\pi z), \quad U_3(z) = \frac{i\pi}{k}\cos(\pi z), \\
U_4(z) &= \frac{2i\pi}{k}\cos(2\pi z), \quad U_5(z) = \frac{2i\pi}{k}\cos(2\pi z), \quad U_6(z) = \frac{2i\pi}{k}\cos(2\pi z), \ldots
\end{align*}
\]

\[
\begin{align*}
W_1(z) &= \sin(\pi z), \quad W_2(z) = \sin(\pi z), \quad W_3(z) = \sin(\pi z), \\
W_4(z) &= \sin(2\pi z), \quad W_5(z) = \sin(2\pi z), \quad W_6(z) = \sin(2\pi z), \ldots
\end{align*}
\]  \hspace{1cm} (3.5.4b)

\[
\begin{align*}
\Theta_1(z) &= \frac{Cs_1 + 1}{Cs_1^2 + s_1 + \beta_1}\sin(\pi z), \quad \Theta_2(z) = \frac{Cs_2 + 1}{Cs_2^2 + s_2 + \beta_1}\sin(\pi z), \quad \Theta_3(z) = \frac{Cs_3 + 1}{Cs_3^2 + s_3 + \beta_1}\sin(\pi z), \\
\Theta_4(z) &= \frac{Cs_4 + 1}{Cs_4^2 + s_4 + \beta_2}\sin(2\pi z), \quad \Theta_5(z) = \frac{Cs_5 + 1}{Cs_5^2 + s_5 + \beta_2}\sin(2\pi z), \quad \Theta_6(z) = \frac{Cs_6 + 1}{Cs_6^2 + s_6 + \beta_2}\sin(2\pi z), \ldots
\end{align*}
\]

In general, the eigenfunctions form a complete set as a basis. Thus, the solution of (3.3.3) can be expanded in terms of the eigenfunctions. Taking the eigenvectors as the basis and expanding, \( \hat{E}(z, t) \) reads

\[
\hat{E}(z, t) = \begin{pmatrix}
\hat{u}(z, t) \\
\hat{w}(z, t) \\
\sqrt{Pr_0}(z, t)
\end{pmatrix} = \sum_{j=1}^{\infty} \kappa_j(0) \exp \left( s_j t \right) E_j(z)
\]

\[
= \sum_{j=1}^{\infty} \kappa_j(t) \begin{pmatrix}
U_j(z) \\
W_j(z) \\
\sqrt{Pr_0} \Theta_j(z)
\end{pmatrix}.
\]  \hspace{1cm} (3.5.4c)

Clearly, the coefficients \( \kappa_j(t) \) depend on the initial conditions, \( \kappa_0(t) \). In practice, terms for large \( j \) are negligible for \( t > 0 \), and only a relatively small number of terms, \( N \), are needed. Thus, for each time \( t > 0 \), the maximum energy growth becomes:
\[
\hat{E}(z,t) = \begin{pmatrix}
\hat{u}(z,t) \\
\hat{w}(z,t) \\
\sqrt{\Pr \tilde{\theta}(z,t)}
\end{pmatrix}
= \sum_{j=1}^{N} \kappa_j(t) \begin{pmatrix}
U_j(z) \\
W_j(z) \\
\sqrt{\Pr \tilde{\theta}_j(z)}
\end{pmatrix}.
\tag{3.5.5}
\]

The eigenvalues are found as the roots of the dispersion relation, (3.3.3). It should be noted that \( s_j \)'s are functions of the mode number, \( n \). For each time \( t > 0 \), the maximum energy growth at time \( t \), \( G(t) \), is defined as

\[
G(t) = \sup_{E(z,0) \neq 0} \frac{\| \hat{E}(z,t) \|^2}{\| \hat{E}(z,0) \|^2} = \sup_{E(z,0) \neq 0} \frac{\langle \hat{E}(z,t), \hat{E}(z,t) \rangle}{\langle \hat{E}(z,0), \hat{E}(z,0) \rangle},
\tag{3.5.6}
\]

where the comma denotes the inner product of two complex vectors. After taking \( \kappa(t) = (\kappa_1(t), \kappa_2(t), ..., \kappa_3(t))^T \), \( \kappa(t) \) is expressed as

\[
\kappa(t) = \exp(S_{\kappa} t) \cdot \kappa(0),
\tag{3.5.7}
\]

where \( S_{\kappa} \) is a matrix of dimension \( K \), with the \( K \) eigenvalues on the diagonal. The matrix exponential \( \exp(S_{\kappa} t) \) has the exponential terms \( \exp(s_j t) \) on the diagonal,

\[
\exp(S_{\kappa} t) = \begin{bmatrix}
e^{s_1 t} & \ldots & 0 \\
\ldots & \ldots & \ldots \\
0 & \ldots & e^{s_K t}
\end{bmatrix}.
\]

Considering (3.5.6), the inner product of two vectors is given by

\[
\langle \hat{E}(z,t), \hat{E}(z,t) \rangle = \sum_{i=1}^{K} \sum_{j=1}^{K} \kappa_j(t) \kappa_i(t) \langle E_j(z), E_i(z) \rangle = \sum_{i=1}^{K} \sum_{j=1}^{K} \kappa_j(t) \kappa_i(t) A_{ji},
\tag{3.5.8}
\]

where \( A \) is a matrix defined in terms of the inner product of the eigenvectors as
\[ A_{ji} = \left( E_j(z), E_i(z) \right) = \left( U_j, U_i \right) + \left( W_j, W_i \right) + \text{Pr} \left( \Theta_j, \Theta_i \right) = \]
\[ \frac{1}{2} \int_0^1 U_i^*(z) U_j(z) dz + \frac{1}{2} \int_0^1 W_i^*(z) W_j(z) dz + \text{Pr} \int_0^1 \Theta_i^*(z) \Theta_j(z) dz , \]  

where \(^*\) shows the conjugate of the corresponding variable. The matrix \( A \) is Hermitian and can be decomposed in the form \( A = F^* \cdot F \), where \( F^* \) is the conjugate transpose of \( F \). It follows that

\[ \| E(z, t) \|^2 = \kappa^* (t) \cdot F^* \cdot F \cdot \kappa (t) = \| F \cdot \kappa (t) \|^2_2 , \]

where the subscript 2 denotes the 2-norm (norm). Hence, upon using (3.5.10), (3.5.6) becomes:

\[ G(t) = \sup_{\kappa(0) \neq 0} \frac{\| F \cdot \kappa (t) \|^2_2}{\| F \cdot \kappa(0) \|^2_2} = \sup_{\kappa(0) \neq 0} \frac{\| F \cdot \exp(S t) \cdot \kappa(0) \|^2_2}{\| F \cdot \kappa(0) \|^2_2} = \| F \cdot \exp(S t) \cdot F^{-1} \|^2_2 . \]

\[ (3.5.11) \]

A symmetric matrix (Hermitian matrix is a symmetric matrix) can be decomposed as \( A = U^T \cdot D \cdot U \), where \( D \) is the diagonal matrix formed by the eigenvalues, and \( U \) is the unitarian matrix of the eigenvectors. Therefore,

\[ A = U^T \cdot D \cdot U = U^T \cdot \sqrt{D}^T \cdot \sqrt{D} \cdot U = \left( \sqrt{D} \cdot U \right)^T \cdot \left( \sqrt{D} \cdot U \right) , \]

where \( \sqrt{D} \) is the diagonal matrix formed by the square root of the eigenvalues on its diagonal. Clearly, since \( \sqrt{D} \) is diagonal, \( \sqrt{D}^T \) is equal to \( \sqrt{D} \). In this case, \( F \) is found as

\[ F = \sqrt{D} \cdot U . \]

\[ (3.5.13) \]
The 2-norm of any matrix can be determined using singular value decomposition method (SVD). Hence, by using the singular value decomposition method, G(t) can be calculated. Clearly, to compute the growth function, G(t), the eigenvalues \( \{ \lambda_j \} \) and the vector eigenfunctions \( \{ E_j \} \) are needed to be determined. In this section, two-dimensional Rayleigh-Benard flow with the wavenumber \( k = 2.2 \) and \( Pr = 10 \) is examined. The computation is performed for different Cattaneo numbers \( C = 0.0001, 0.01, 0.05 \) and 0.1. In addition, the role of changing the Rayleigh number is investigated. It is found that G(t) changes by less than 1% if the number of eigenvalues (N) is changed from 60 to 90 (or equivalently changing the modenumber, n, from 20 to 30). Therefore, a value of N = 90 is sufficient to ensure numerical accuracy in the results.

Results for the growth function, G(t), are illustrated in Figures 3.4-3.7 for different Cattaneo numbers. It should be noted that for all figures the wavenumber is \( k = 2.2 \) and \( Pr = 10 \). Each figure typically shows the variation of G(t) with respect to time in the pre- and post-critical ranges of Rayleigh number. A significant energy growth is observed for all Rayleigh numbers, even if the flow is stable. Typically, the growth function reaches a maximum. After the initial transient maximum, the behavior of the growth function is governed by the eigenvalues.

G(t) for a fluid with very small non-Fourier level (\( C = 0.0001 \)) is shown in Figure 3.4, which depicts a very sharp maximum for both pre- and post-critical ranges. Recall that the critical Rayleigh number is 657 (see Figure 3.1). The sharp maximum occurs after a very short period of time, \( t \sim 0.001 \), and is replaced by a monotonic decay in the pre-critical Ra range. Interestingly, as the Rayleigh number increases, the time at which \( G_{\text{max}} \) occurs remains roughly the same for all Rayleigh numbers. In addition, as the Rayleigh number increases, trend of G(t) with respect to time remains nearly the same for all pre-critical Rayleigh numbers. The only noticeable change is the maximum of G(t), which increases as the Rayleigh number increases. For post-critical Rayleigh numbers (for instance, \( \text{Ra}=800 \)), the maximum is followed by a minimum before G(t) increases monotonically with time.
In Figure 3.5, $G(t)$ is shown for a weakly non-Fourier fluid, with $C = 0.01$. In this case, the critical Rayleigh number remains equal to 657, as for a Fourier fluid (refer again to Figure 3.1 for $C = 0.01 < C_H = 0.0655$). Figure 3.5 depicts a maximum for the stable condition ($Ra < 657$). As the Rayleigh number exceeds the critical value ($Ra = 657$), the fluid experiences convection. The maximum vanishes for post-critical Rayleigh numbers ($Ra > 657$). These observations are confirmed from Figure 3.3, where the weighted kinetic energy rate curve (for $C = 0.01$) intersects the neutral line at $Ra = 657$. The neutral line is the horizontal line represents the zero value for the weighted kinetic energy rate. For Rayleigh numbers lower than 657, the flow is stable and as the Rayleigh number exceeds this value, the flow will be unstable. For pre-critical Rayleigh numbers ($Ra < 657$), the maximum of $G(t)$ occurs after a short period of time, $t \sim 0.02$. This maximum is followed by a gradual decay to one. In this case, the maximum of $G(t)$ is not as prominent as the maximum for the Fourier case. As the Rayleigh number increases, $G_{\text{max}}$ increases, however, the time at which $G_{\text{max}}$ occurs, remains roughly constant for all Rayleigh numbers. As the Rayleigh number increases to 800 ($> 657$), the maximum vanishes. In this case, $G(t)$ increases with respect to time, however, the sharp initial growth rate is slightly decreased after a very short period of time ($\sim 0.01$).
Figure 3.4: Energy growth function $G(t)$ for a non-Fourier fluid with very small non-Fourier effects ($C=0.0001$) with $Pr=10$, $k=2.2$.

Figure 3.5: Energy growth function $G(t)$ for a weakly non-Fourier fluid ($C=0.01$) with $Pr=10$, $k=2.2$. 
In Figure 3.6, G(t) is shown for a moderately strong non-Fourier fluid, with \( C = 0.05 \). Referring to Figure 3.1, when \( C = 0.05 < C_H = 0.0655 \), the critical Rayleigh number is 657. Thus, as the Rayleigh number exceeds 657, the fluid experiences convection. This is confirmed from Figure 3.3, where the weighted kinetic energy rate curve (for \( C = 0.05 \)) intersects the neutral line at \( Ra = 657 \). In addition, Figure 3.3 shows that the oscillatory mode is replaced by the stationary mode at the Rayleigh number near 620 (break-point). Figure 3.6 confirms the results obtained in sections 3.3 and 4 (Figures 3.1-3.3). It depicts a maximum for the variation of \( G(t) \) with respect to time for the stable conduction (\( Ra < 657 \)). For the oscillatory mode (for instance, \( Ra = 400 < 620 \)) and for large t, Figure 3.6 demonstrates a weak oscillatory pattern for the variation of \( G(t) \) with respect to time. Note that, these oscillations are weak compared to the maximum of the growth function, \( G_{\text{max}} \).

For \( Ra = 640 (> 620 \text{ and } < 657) \), the maximum is more localized and stronger. For \( Ra = 660 (> 657) \) and small t, the growth function is initially qualitatively the same as the pre-critical Rayleigh number, \( Ra = 640 \). In this initial transient phase, the behavior of \( G(t) \) does not depend on the stability/instability condition of the conduction state. The stability of the conduction state is only revealed once the maximum is exceeded, for large t. In addition, for \( Ra = 660 \) the maximum is followed by a minimum before \( G(t) \) increases monotonically with time. For Rayleigh numbers 640 and 660, the variation of \( G(t) \) against time is reminiscent of that encountered for the stability of channel flow. The reader is referred to Figure 9 in Reddy & Henningson [67], and Figure 4.3a in Schmid & Henningson [71]. Their results show that after a transient maximum for the unstable condition case, a minimum is observed before \( G(t) \) increases monotonically with time. In contrast to the case of \( Ra = 660 \), as the Rayleigh number increases to 800, both the maximum and the minimum vanish. In this case, \( G(t) \) increases monotonically with respect to time. However, the sharp initial growth rate is slightly decreased after a short period of time (~ 0.01). As the Rayleigh number increases from 200 to 400, the time at which \( G_{\text{max}} \) occurs, increases. When the Rayleigh number is equal to 640, this time decreases, and it remains roughly constant for higher Rayleigh numbers.
**Figure 3.6:** Energy growth function $G(t)$ for a moderately non-Fourier fluid ($C=0.05$) with $Pr=10$, $k=2.2$.

In Figure 3.7a, $G(t)$ is shown with respect to time for a strongly non-Fourier fluid, with $C=0.1$. Referring to Figure 3.1, when the Cattaneo number is equal to 0.1, the critical Rayleigh number is 470. This is confirmed from Figure 3.3, where the weighted kinetic energy rate curve (for $C = 0.1$) intersects the neutral line at $Ra = 470$. For Rayleigh numbers lower than 470, the conduction state persists and as the Rayleigh number exceeds this critical value, conduction is lost to oscillatory convection. In addition, Figure 3.3 shows that the oscillatory mode is replaced by the stationary mode when the Rayleigh number exceeds $Ra = 640$, where a break-point is predicted at this Rayleigh number (signaling the discontinuous change in slope). For the oscillatory mode ($Ra < 640$), Figure 3.7a indicates an oscillatory pattern for the growth function. Additionally, for $Ra < 470$ ($Ra = 400$ in Figure 3.7a), $G(t)$ exhibits decaying oscillatory behavior, whereas for $470 < Ra < 640$ ($Ra = 600$ in Figure 3.7a), $G(t)$ exhibits amplifying oscillatory behavior. Referring to Figure 3.7b, which is a blowup of the initial stages, the sharp initial transient growth for the conduction state (for instance, for $Ra = 400 < 470$) occurs at the very early
stage. More specifically, Figure 3.7b reflects a rather sharp change in the slope of $G(t)$ instead of the maximum predicted in the early stage for a weakly Fourier fluid (see Figure 3.5). This sharp initial change of $G(t)$ is reminiscence of the rapid initial transient growth of $G(t)$ at high Rayleigh number when no maximum occurs (see curve corresponding to $Ra = 800$ in Figure 3.5). However, for $Ra < 470$ ($Ra = 400$ in Figure 3.7a) this sharp growth is followed by decaying oscillation. In contrast, for $470 < Ra < 640$ ($Ra = 600$ in Figure 3.7a) the sharp growth is followed by amplifying oscillation. Referring to Figure 3.7a for stable conduction ($Ra = 400 < 470$), the maximum of $G(t)$ is found to be the amplitude of the first peak. Consider next the convection for a strongly non-Fourier fluid ($C = 0.1$) at a relatively high Rayleigh number ($Ra = 800 > 640$ in Figure 3.7). In this case, convection is stationary for any time. Figure 3.1 confirms that, for $C = 0.1$ and a $Ra > 640$, the oscillatory mode is replaced by the stationary mode. $G(t)$ does not display a maximum, and increases monotonically with respect to time. However, the sharp initial growth rate is slightly decreased after a short period of time.

From Figures 3.4-3.7, it is found that the initial transient growth might be as large as $O(100)$. Interestingly, this growth occurs in the absence of non-linear effects and can be explained by the non-orthogonality of the governing eigenfunctions. Considering the eigenfunctions (3.5.4b) and their inner product (3.5.8), for all Rayleigh numbers and Cattaneo numbers, the non-orthogonality of the eigenfunctions persists. Therefore, regardless of the values of the Cattaneo number and the Rayleigh number, the maximum of $G(t)$ always exists for stable conduction (when the Rayleigh number is less than the critical value). Eventually, the perturbations in velocity and temperature approach zero for the stable flow, or equivalently, $G(t)$ approaches zero. Interestingly, even for a fluid with very small non-Fourier level (Figure 3.4), the non-orthogonality of the eigenfunctions persists and generates a transient maximum for the energy growth of perturbations.
Figure 3.7: (a and b). Energy growth function $G(t)$ for a strongly non-Fourier fluid (C=0.1) with $Pr=10$, $k=2.2$. 
Mathematically, the non-orthogonality of the eigenfunctions results in the presence of non-zero off-diagonal terms in the matrix $A$ (3.5.12). A fundamental implication of the non-zero off-diagonal terms in the matrix $A$ is the existence of substantial transient energy growth for small perturbations, which occurs in the absence of non-linearities. Note that for a Fourier fluid, the Cattaneo number is zero ($C = 0$) in the corresponding eigenfunctions (3.3.4). Finally, the maximum of $G(t)$ confirms that degeneracies are not necessary for the energy growth to occur. At $t = 0$ large terms are canceled, whereas at moderate $t > 0$ the energy of perturbations (3.5.4c) is the sum of large terms, even if all exponential terms decay. Thus, it is possible that the energy of perturbations at time $t$ becomes larger than the initial energy.

3.6 Conclusion

This study examines the natural convection of non-Fourier fluids of the single-phase-lagging (SPL) type. These fluids possess a relaxation time, reflecting the delay in the response of the heat flux and the temperature gradient with respect to one another. Linear stability analysis indicates that, in contrast to ordinary fluids, a SPL fluid can lose its conductive mode to stationary or oscillatory convection. For small relaxation time (small Cattaneo number, $C$), the neutral stability curve comprises a Fourier branch ($k < k_i$) and an oscillatory branch ($k > k_i$). As $C$ increases and reaches a critical value, $C_H$, both stationary and oscillatory convection become equally probable. It confirms the existence of the bistable mode observed in experiment [73]. For $C > C_H$, only oscillatory convection is predicted, at a Rayleigh number decreasing with $C$ (see Figures 3.1 and 3.2). Thus, oscillatory convection increasingly becomes the mode of preference, compared to both conduction and stationary convection. In fact, it is found that, for strongly non-Fourier fluids, oscillatory convection becomes spontaneously observed with no prior conduction.

In addition, the mechanism of instability is investigated through an examination of the disturbance-energy equation. It is found that the instability mechanism is associated with the coupling of the perturbation velocity and the perturbation temperature. Also, energy growth for small two-dimensional perturbations to Rayleigh-Benard problem for a non-Fourier fluid has been investigated. The eigenfunctions found to be non-orthogonal,
which implies that there can be substantial transient growth in the energy of small perturbations. The transient growth occurs even if the Rayleigh number is less than the critical value (for the stable conduction state). The growth function G(t) has been calculated, which gives the maximum potential growth for small perturbations. This growth occurs in the absence of nonlinearities. Finally, the behavior of the energy growth also reflects the distinct character of non-Fourier convection.

References


Chapter 4

4 Thermal convection in non-Fourier fluids and application to Helium II

4.1 Introduction

Heat conduction is conventionally governed by the classical Fourier equation. When the Fourier equation is combined with the conservation of energy law, a parabolic equation for the temperature field is resulted. From a physical point of view, Fourier’s law is not realistic since it predicts that a disturbance wave in the temperature field will travel at an infinite speed [1]. The finite speed of the disturbance wave in the temperature field is ensured by using the Maxwell-Cattaneo or Cattaneo-Vernotte (C-V) equation proposed by Cattaneo [2] and Vernotte [3]. This model adds a new transient term to the conventional Fourier equation. The new transient term is multiplied by a time constant, or the thermal relaxation time, which reflects the time required for the heat flux to relax to a new (stable) steady state following a perturbation in the temperature gradient. Since most materials have thermal relaxation times on the order of pico- or $10^{-12}$ seconds [4], the C-V equation collapses onto the classical Fourier model. However, there are some situations in which the Fourier’s law is not adequate to describe the heat transfer process [5-9]. For instance, the thermal relaxation time for processed Bologna meat was found to be of O(10s) [10]. Many studies can be found in the literature that examines the non-Fourier heat conduction in different conditions using both analytical and numerical approaches [11-18]. For instance, Antaki [18] offered a new interpretation for the experimental evidence of the hyperbolic (non-Fourier) heat conduction equation in the experiments with processed meat [10].

The importance of non-Fourier effects could not be realized solely through the absolute value of the relaxation time. For instance, in some processes such as laser pulse heating, a large quantity of energy is released over a very short period of time, which is comparable

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1 A version of this chapter will be submitted to the Journal of Physics of Fluids
to the relaxation time of the medium. It is expected that when the rate of heating is of the same order of magnitude as the relaxation time, non-Fourier effects become important [19]. More generally, non-Fourier effects should be accounted for if the relaxation time is of the same order as the thermal diffusion time. Using dimensionless relaxation time, also known as the Cattaneo number, can best illustrate the importance of non-Fourier effects. The Cattaneo number is defined as $C = \frac{\tau\kappa}{h^2}$, where $h$ is the typical length scale, $\tau$ and $\kappa$ are the relaxation time and thermal diffusivity of the medium, respectively. Clearly, the Cattaneo number suggests that non-Fourier effects are significant for a very small gap, as in the convection and flow in micro- and nano-devices [20-22], or when $'\tau\kappa'$ is large enough.

Non-Fourier effects can also be important in the heat conduction/convection of fluids. In this case, the first experimental evidence was the detection of the thermal wave in low temperature ($T < 2.2$ K) superfluid helium (He II) [23]. Several studies investigated the non-Fourier heat conduction in the liquid helium [24-26]. Nanofluids are also a good candidate to study the non-Fourier effects [27,28]. In this case, Wang et al. [28] showed that the presence of nanoelements leads to the dual-phase-lagging heat conduction in nanofluids. In the Dual-Phase-Lag model (DPL), materials possess a relaxation time and a retardation time, reflecting the delay in the response of the heat flux and the temperature gradient with respect to one another. Recently, the natural convection of a non-Fourier fluid contained between two horizontal planes (Rayleigh-Benard problem) was studied [29,30]. The Single-Phase-Lag (SPL) model for the heat equation was used, where unlike the DPL model, materials possess only a relaxation time (the retardation time is zero). The SPL model is particularly relevant to low-temperature liquids or to fast heat transfer processes, and also to nanofluids with high nanoparticle concentration.

The relaxation time for ordinary homogenous liquids can be roughly estimated from $\tau_{\text{liquid}} \approx \frac{a\kappa}{c_s^2}$, where $c_s$ is the speed of sound and $2.1 < a < 3$. In the derivation of this relation, it was assumed that the relaxation time value for liquids is between those of solids ($a=3$ [31]) and rarefied gases ($a=2.1$[32]). The validity of the proposed relaxation time for liquids is checked by comparing the obtained results with the available
relaxation time for liquid Benzene [33], where an excellent agreement was observed. Based on this relation, the relaxation time in the liquids can be noticeable for liquids with high thermal diffusivity and for the liquids with the low propagation speed of sound. Among the liquids with high thermal diffusivity, liquid metals such as mercury or molten gallium can be mentioned. On the other hand, liquid helium has very low sound speed compared with the ordinary liquids.

In this paper, the natural convection of a non-Fourier liquid confined between two vertical planes is studied. The two planes are maintained at constant, but different temperatures. Natural convection in a vertical slot and tall annular enclosures (with the inner wall heated and the outer one cooled) has long been studied experimentally and numerically [34-45]. In this case, unlike the Rayleigh-Benard problem, the fluid in the conduction state (when heat is transferred between the vertical walls by conduction only) is not motionless. Since the Maxwell-Cattaneo equation is restricted to non-deformable media, and is not a frame-invariant constitutive relation, the Oldroyds’ upper-convected derivative, which leads to a frame indifferent Maxwell-Cattaneo equation is used. This was first proposed by Christov [46], and was revisited by Khayat & Ostoja-Starzewski [47]. In addition, the SPL model is used to study the non-Fourier effects. The presence of base flow velocity profile is expected to lead to a more complex physical behaviour compared with what was observed in the Rayleigh-Benard problem [29]. The paper is organized as follows: In section 4.2, the Rayleigh-Benard convection for a non-Fourier fluid is revisited and the non-Fourier character of liquid helium II is shown. The physical origin of observing heat waves in the liquid helium is also explained in this section. In section 4.3 the problem is introduced, and the governing equations and boundary conditions are derived. Section 4.4 covers the linear stability analysis. The results are presented in section 4.5, which is followed by discussion in section 4.6. Finally, concluding remarks are given in section 4.7.
4.2 Rayleigh-Benard convection for a non-Fourier fluid

Linear stability analysis of Rayleigh-Benard convection for a non-Fourier fluid [30] indicates that, in contrast to ordinary fluids, a Single-Phase-Lag (SPL) fluid can lose its conductive mode to stationary or oscillatory convection. For steady convection, the same critical Rayleigh number, $Ra_c$, as a Fourier fluid, is recovered. For oscillatory convection, the critical Rayleigh number and the neutral frequency, $\omega_c$, are found as

$$Ra_c = \frac{\beta C Pr^2 + Pr + 1}{\beta^2 C^2 Pr^2} Ra_{cF} \quad \text{and} \quad \omega_c = \frac{1}{C} \sqrt{\frac{C Pr \beta - Pr - 1}{Pr}},$$

(4.2.1)

respectively. In (4.2.1), $\beta = q^2 + \pi^2$, $q$ is the wavenumber, $Pr$ is the Prandtl number, $C$ is the Cattaneo number and $Ra_{cF}$ is the critical Rayleigh number for a Fourier fluid ($C = 0$). One way to study the non-Fourier character of a fluid is to examine the natural convection of that fluid. For a non-Fourier fluid, similar to a Fourier fluid ($C = 0$), conduction is lost to convection once the critical value of the Rayleigh number is exceeded. However, in contrast to a Fourier fluid, and similar to a viscoelastic fluid [48], non-Fourier conduction can be lost to steady or oscillatory convection, depending on the flow parameters. For very small relaxation time (small $C$), the marginal stability curve consists of a Fourier branch (for low wavenumber), leading to stationary convection, and an oscillatory branch (for very large wavenumber). Both stationary and oscillatory convection become equally probable as $C$ reaches a critical value, $C_H (= 0.11$ for $Pr = 1$). Interestingly, for $C > C_H$, only oscillatory convection is predicted (for all wavenumbers), at a Rayleigh number decreasing with $C$.

A dimensional argument in terms of characteristic times can be used to study the onset of convection. A fluid element subjected to temperature fluctuation experiences buoyancy, which is destabilizing. The characteristic buoyancy time, $t_B$, can be obtained from

$$\alpha_T g \delta T = \frac{h}{t_B^2},$$

which is the acceleration due to the buoyancy. $\alpha_T$ is the coefficient of volume expansion, $g$ is the gravitational acceleration, $\delta T$ is the temperature difference and $h$ is the length scale (distance between plates). The second destabilization process is
the thermal relaxation (second-sound wave), which has the same characteristic time as the second-sound wave propagation. The dimensional characteristic time of the second-sound wave can be obtained analytically for Rayleigh-Benard convection as

\[ t_s = \omega_c^{-1} = \frac{h^2}{\kappa} = \frac{h^2}{\kappa} \sqrt{\frac{Pr}{C Pr \beta - Pr - 1}}. \]

Two stabilizing factors are the dissipative processes tend to maintain the fluid in its initial rest state: viscous dissipation and thermal diffusion. Thermal diffusion tries to remove heat from the warmer element and as a result, the element experiences less buoyancy. The characteristic time of viscous dissipation and thermal diffusion are \( t_v = \frac{h^2}{\nu} \) and \( t_\kappa = \frac{h^2}{\kappa} \), respectively. For a Fourier fluid \( (C = 0) \), there is no oscillatory branch in the stability curve (Figure 4.1). In this case, the competition is between three factors: buoyancy, viscous dissipation and thermal diffusion. The instability occurs if the fluid element is accelerated sufficiently (by buoyancy) to overcome stabilizing processes (viscous dissipation and thermal diffusion).

One can obtain the Rayleigh number by setting \( t_B^2 = t_v t_\kappa \).

For a non-Fourier fluid \( (C \neq 0) \), there are two destabilizing factors: buoyancy and thermal relaxation (second-sound wave). Consequently, it is expected that for a non-Fourier fluid the onset of convection occurs at a lower Rayleigh number compared to that of for a Fourier fluid. Any perturbation in the non-Fourier fluid \( (C \neq 0) \) leads to the appearance of second-sound wave. However, when \( C \) is very small, the diffusion process is dominant against the thermal relaxation process and heat waves (the second-sound waves) are damped quickly and no oscillatory behaviour with respect to time is observed (for ordinary wavenumbers). Note that even for a very small Cattaneo number, an oscillatory branch is observed for very high wavenumbers for the Rayleigh-Benard configuration (Figure 4.1). This means that the second-sound waves have destabilizing effect, and even for a very small \( C \), waves (that are observed at very high wave numbers) reduce the region of stable conduction. In this case (very small \( C \)), it can be seen from (4.1.1) that the frequency of oscillations is very high.
As C increases, the thermal relaxation process will be more effective until it can balance the thermal diffusion effects. At this value of C (\(=C_{nf}=0.11\)), both stationary and oscillatory convection become equally probable. For a fluid with a high non-Fourier character (high C), the thermal relaxation process is dominant and the oscillations (heat waves) are observed at the critical point. Interestingly, the neutral frequency is proportional to the second-sound wave velocity (the initiation of heat waves becomes possible at the critical point). In fact, \(C_T = \frac{\alpha c}{q}\), where \(c_T\) denotes the heat wave propagation speed. Therefore, the heat wave velocity in a layer of non-Fourier fluid confined between two horizontal plates (with free-free boundary conditions) can be obtained as

\[
C_T = \frac{1}{qC} \sqrt{\frac{C Pr \beta - Pr - 1}{Pr}}. \tag{4.2.2}
\]
Note that second-sound velocity in solids \( (C_T = \sqrt{1/C}) \) can be recovered from (4.2.2) when the wavenumber \( (q) \) is very large. Recall that the stability picture (Figure 4.1) shows that convection becomes increasingly more difficult as \( q \) increases. In the case of a fluid with a high non-Fourier character (very large \( C \)), the heat wave propagation velocity behaves like \( C_T \sim 1/\sqrt{C} \). Moreover, this suggests that oscillatory rolls may not be detectable in reality for large \( C \).

### 4.2.1 Non-Fourier character of Liquid Helium II

As mentioned previously, heat waves are observed in liquid helium II (LHe II) both experimentally and numerically. One method to justify the propagation of heat as a wave in LHe II is using the two fluid model, where LHe II is assumed to comprise a normal component and a superfluid component [26]. The dimensional second-sound wave velocity in LHe II based on the two fluid model is obtained as

\[
 u_T^2 = \frac{\rho_s}{\rho_n} \frac{TS^2}{c_v}, \quad (4.2.3)
\]

where \( \rho_s \) and \( \rho_n \) are the density of the superfluid component and the normal component, respectively, \( c_v \) is the specific heat at constant volume of the liquid, \( T \) is the temperature, and \( S \) is the entropy per unit mass. The experimental values for dimensional velocity of second-sound in helium II is shown in Figure 4.1. From this figure, which is based on the two fluid model, and using (4.2.2), one can obtain the Cattaneo number for LHe II at different temperatures (below 2.17 K). In this case, the dimensional velocity of the second-sound in LHe II can be approximated from (4.2.2) as (for large \( C \))

\[
 u_T = \frac{\kappa}{h} C_T \sim \frac{\kappa}{h} \frac{1}{q} \sqrt{\frac{\beta}{C}}, \quad (4.2.4)
\]

where \( u_T \) is the dimensional second-sound velocity. Therefore, the Cattaneo number is found explicitly as
\[ C \sim \frac{\beta}{q^2 u_T^2 h^2}, \]  \hspace{1cm} (4.2.5)

where \( \beta = q^2 + \pi^2 \). At the critical point (Figure 4.1, large C), \( q_c \) is approximately equal to 3.1. Therefore, the Cattaneo number at the critical point (which is the most probable point to occur in reality) can be approximated as

\[ C \sim 2 \frac{\kappa^2}{u_T^2 h^2}, \]  \hspace{1cm} (4.2.6)

Note that (4.2.6) is reminiscent of what is obtained for the Cattaneo number for liquids as

\[ C_{\text{liquid}} \sim \frac{ak^2}{c_s^2 h^2}, \] where \( 2.1 < a < 3 \), and \( c_s \) is the sound velocity. By comparing \( C \) for LHe II from (4.2.6) and \( C_{\text{liquid}} \), it can be concluded that \( c_s > u_T \), which means that the sound speed in LHe II is larger than the second-sound speed, which can be confirmed from Figure 5 in [26]. Interestingly, by comparing (4.2.6) and the definition of Cattaneo number \( \left( C = \frac{\tau \kappa}{h^2} \right) \), the dimensional second-sound velocity in LHe II is found to be proportional to \( \sqrt{\kappa/\tau} \), which is similar to dimensional heat wave velocity in solids [1], [51]. Moreover, the relaxation time for LHe II can be obtained as

\[ \tau \sim \frac{2\kappa}{u_T^2}, \] \hspace{1cm} (4.2.7)

where from Figure 4.1 in [49] and Figure 1 in [50], \( u_T \) for \( 1 \text{ K} < T < 2 \text{ K} \) is approximately 20 m/s. Therefore at this temperature range, \( \tau(s) \approx 0.005\kappa(\text{m}^2/\text{s}) \). For instance, the thermal diffusivity of LHe II at 1.7 K is approximately 0.630 m²/s (values of \( K, \rho \) and \( c_p \) are obtained from [52]). Therefore, \( \tau \approx 0.003 \text{ s} \), which is very large compared to the relaxation time values for ordinary fluids. Then, the Cattaneo number \( (C) \) is obtained as \( C = 0.0019/h^2 \) (m), where \( C \) values are given in Table 4.1 for some typical gap width \( (h) \).
The Cattaneo number based on $C_{\text{liquid}} \sim \frac{ak^2}{c_s^2 h^2}$ and $a = 3$ is calculated for water in Table 4.1.

**Table 4.1**: Cattaneo number for liquid helium (at 1.7 K) and water (at 20°c) for different $h$

<table>
<thead>
<tr>
<th>Liquid</th>
<th>Cattaneo number (C)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>h=10 cm</td>
</tr>
<tr>
<td>Liquid Helium</td>
<td>0.19</td>
</tr>
<tr>
<td>Water</td>
<td>$2.73 \times 10^{-18}$</td>
</tr>
</tbody>
</table>

For Rayleigh-Benard convection, it can be seen from (4.2.1) that the critical Rayleigh number for the oscillatory convection is lower than the Fourier level (for sufficiently large Cattaneo numbers). In this case, the critical Rayleigh number diminishes like $Ra_c \sim 1/C$, suggesting that convection may be spontaneously observed (without a conduction phase), for any temperature differential, for large $C$. The counterflow of superfluid component and normal component of liquid helium is somewhat analogous to convection currents in a fluid. Consequently, the conduction process in LHe II (based on the two fluid model) is similar to convection in ordinary fluids, which can explain the significant thermal conductivity of LHe II compared to ordinary liquids. Therefore, the results of non-Fourier analysis can be confirmed from the two fluid model, where in both cases, the thermal convection can occur without a conduction phase (for a very large C). Note that the components in the two fluid model are chemically indistinguishable, and LHe II cannot be separated into its components by any method. Using non-Fourier assumption for the heat flux, the fluid is assumed to be a homogenous single phase. In this case, the convective rolls can be observed (similar to ordinary fluids), where the frequency of oscillations is proportional to the second-sound wave velocity. To summarize, the oscillatory convection (overstability) is expected to be observed at the
onset for a fluid with high non-Fourier character (large Cattaneo number) with the oscillatory threshold smaller than the threshold for the onset of stationary Rayleigh-Benard convection.

4.2.2 Physical origin of heat waves in Liquid Helium II

The inner structure of LHe II (at very low temperatures, $T < 1$ K) can be described by the phonon model. In the phonon model, the sample is considered as a box containing a gas of phonons [53,54]. A phonon is a quantum mechanical description of an elementary vibrational motion in which a lattice of atoms or molecules uniformly oscillates at a single frequency. The phonons represent the eigen-vibrations of the atoms. This implies that a phonon of momentum $\hbar k$ and energy $\hbar \omega$ corresponds to vibrations of frequency $\omega$ with the wave vector $k$, where $\hbar$ denotes the Planck’s constant. Therefore, by using the phonon concept, the transport processes in gases with real particles is analogous to the transport processes in gas of phonons. However, there are several important differences between the above mentioned gases (phonons and real particles). In case of gas of real particles, annihilation or creation of particles cannot happen, whereas phonons may be annihilated or created. This is the reason that phonons are called quasi-particles. The number density of phonons at a given time and position is a function of the local temperature at the same time and position. Moreover, momentum may not be conserved in phonon interactions (quasi-momentum). However, similar to real gas particles, energy is always conserved in phonon interactions. The phonons interactions are of two main types: Normal (N-) processes and the Resistive (R-) processes. In the N-process, unlike the R-process, the momentum of phonon is conserved. Fourier heat conduction is known to be due to R-process. Interestingly, it is found the thermal conductivity is proportional to $\tau_R$, where $1/\tau_R$ ($1/\tau_N$) is the frequency of R-processes (N-processes). Therefore, as the frequency of R-processes increases, the thermal conductivity decreases. If there are many R-processes and only few N-processes ($\tau_R \rightarrow 0$, $\tau_N \rightarrow \infty$), the conduction heat transfer occurs by diffusive processes. In this case, the Fourier’s law is sufficient for the description of heat conduction and energy is mainly transported by diffusion. Note that since a large number of R-processes take place for diffusion, quasi-momentum is not conserved in the phonon interactions. On the other
hand, if there are many N-processes and only few R-processes (\(\tau_{R} \to \infty, \tau_{N} \to 0\)), the energy is transported as a result of heat waves. In other words, the quasi-momentum is observed in the phonon interactions, which gives rise to a wave like energy transport. Recall that the energy and momentum is conserved in collisions between real particles when the sound is transmitted. In an analogy to the sound wave, the process of wave like energy transport is called the second-sound. The second-sound can be observed in LHe II (and also in crystals) at very low temperatures (near the absolute zero). In this case, the Fourier equation is not accurate enough to describe the heat conduction process, since the diffusion is very smaller compared to wave-like energy transportation. Unlike the energy transport by the second-sound, when the diffusion is the dominant process (many R-processes), the wave like energy transportation (second-sound) is damped almost instantly and conduction occurs as a result of diffusion process. It is worthwhile to mention that there is also another way of energy transport in a phonon gas called ballistic [53]. In this case, there is no interaction between phonons in their way travelling through the medium (\(\tau_{R} \to \infty, \tau_{N} \to \infty\)). However, ballistic phonons are more important in crystals.

4.3 Governing equations and boundary conditions

Consider a two-dimensional thin layer of a non-Fourier fluid (Newtonian) confined between two vertical planes at \(X = -h/2\) and \(X = h/2\), maintained at fixed temperatures \(T_{L}\) and \(T_{R}\) (\(T_{R}\) is always larger than \(T_{L}\)), respectively. The physical configuration is described by a Cartesian coordinate system (X, Y) with the Y axis in the vertical direction. Here \(g = -ge_{y}\), where \(g\) is the gravitational acceleration and \(e_{y}\) is the unit vector in the vertical direction. The fluid layer is assumed to be of infinite vertical extent. In this study, the Boussinesq approximation is assumed, which states that the effect of compressibility is negligible everywhere in the conservation equations except in the buoyancy term. In this case, the density \(\rho\) is assumed to depend on the temperature \(T\), following:

\[
\rho = \rho_{0} \left(1 - \alpha_{T} (T - T_{0})\right),
\]

(4.3.1)
where $\rho_0$ is the mass density of the fluid at $T_0$ and $\alpha_T$ is the coefficient of volume expansion. The fluid is assumed to be incompressible, of specific heat at constant pressure $C_p$, thermal conductivity $K$ and viscosity $\mu$. The general governing equations for a non-Fourier fluid comprise the conservation of mass, linear momentum and energy, as well as the constitutive equation for the heat flux. In this case, the conservation equations are given by

$$\nabla \cdot \mathbf{V} = 0, \quad (4.3.2a)$$

$$\rho_0 (V_t + \mathbf{V} \cdot \nabla \mathbf{V}) = -\nabla P - \rho g e_y + \mu \Delta \mathbf{V}, \quad (4.3.2b)$$

$$\rho_0 c_p (T_t + \mathbf{V} \cdot \nabla T) = -\nabla \cdot \mathbf{Q}, \quad (4.3.2c)$$

where $\nabla$ and $\Delta$ are the gradient and Laplacian operators, respectively, and a subscript denotes partial differentiation. Here $\mathbf{V} = (U, V)$ is the velocity vector, $P$ is the pressure, $T$ is the temperature and $\mathbf{Q}$ is the heat flux vector. In this work, the heat flux is assumed to be governed by the single-phase-lagging (SPL) model as

$$\tau \frac{\delta \mathbf{Q}}{\delta t} + \mathbf{Q} = -K \nabla T, \quad (4.3.2d)$$

where $\tau$ is the relaxation time. Here the Jaumann or Li type derivative is given in [47]

$$\frac{\delta (\cdot)}{\delta t} = \frac{\partial (\cdot)}{\partial t} + \mathbf{V} \cdot \nabla (\cdot) - (\cdot) \cdot \nabla \mathbf{V} - (\cdot) \nabla \cdot \mathbf{V}. \quad (4.3.2e)$$

The boundary conditions at the left and right planes are taken as

$$U(X = \pm h/2, Y) = 0, \quad V(X = \pm h/2, Y) = 0,$$

$$T(X = -h/2, Y) = T_L, \quad T(X = h/2, Y) = T_R.$$

The problem is conveniently cast in dimensionless form by taking the $h$, $h^2/\nu$, $\rho_0 \alpha_T (T_R - T_L)gh^2/\mu$, $\rho_0 \alpha_T (T_R - T_L)gh$, $T_R - T_L$ and $k(T_R - T_L)/h$ to be,
respectively, the typical length, time, velocity, pressure, temperature and heat flux scales. u and v are the dimensionless velocity components in x and y direction, respectively. p is the dimensionless pressure and \( \theta = (T - T_{\text{Ref}})/(T_R - T_L) \) is the dimensionless temperature deviation from a reference temperature defined as \( T_{\text{Ref}} = (T_R + T_L)/2 \). q_x and q_y are the dimensionless heat flux components in x and y directions, respectively.

The dimensionless equations for the conservation of mass, momentum and energy are:

\[
\frac{\hat{\partial}u}{\hat{\partial}x} + \frac{\hat{\partial}v}{\hat{\partial}y} = 0, \quad (4.3.3a)
\]

\[
\frac{\hat{\partial}u}{\hat{\partial}t} + Gr \left( u \frac{\hat{\partial}u}{\hat{\partial}x} + v \frac{\hat{\partial}u}{\hat{\partial}y} \right) = -\dfrac{\hat{\partial}p}{\hat{\partial}x} + \left( \frac{\hat{\partial}^2 u}{\hat{\partial}x^2} + \frac{\hat{\partial}^2 u}{\hat{\partial}y^2} \right), \quad (4.3.3b)
\]

\[
\frac{\hat{\partial}v}{\hat{\partial}t} + Gr \left( u \frac{\hat{\partial}v}{\hat{\partial}x} + v \frac{\hat{\partial}v}{\hat{\partial}y} \right) = -\dfrac{\hat{\partial}p}{\hat{\partial}y} + \left( \frac{\hat{\partial}^2 v}{\hat{\partial}x^2} + \frac{\hat{\partial}^2 v}{\hat{\partial}y^2} \right) + \theta, \quad (4.3.3c)
\]

\[
\frac{\hat{\partial}\theta}{\hat{\partial}t} + Gr \left( u \frac{\hat{\partial}\theta}{\hat{\partial}x} + v \frac{\hat{\partial}\theta}{\hat{\partial}y} \right) = -\frac{1}{Pr} \left( \frac{\hat{\partial}q_x}{\hat{\partial}x} + \frac{\hat{\partial}q_y}{\hat{\partial}y} \right), \quad (4.3.3d)
\]

where q_x and q_y are governed by:

\[
C Pr \frac{\hat{\partial}q_x}{\hat{\partial}t} + q_x = -\frac{\hat{\partial}\theta}{\hat{\partial}x}, \quad (4.3.3e)
\]

\[
C Pr \frac{\hat{\partial}q_y}{\hat{\partial}t} + q_y = -\frac{\hat{\partial}\theta}{\hat{\partial}y}. \quad (4.3.3f)
\]

In this case, the constitutive equations for the solution heat flux components are obtained from (4.3.2e) and (4.3.3e, f) to read

\[
C Pr \hat{\partial}q_x + C Pr Gr \left( u \frac{\hat{\partial}q_x}{\hat{\partial}x} + v \frac{\hat{\partial}q_x}{\hat{\partial}y} - q_x \frac{\hat{\partial}u}{\hat{\partial}x} - q_y \frac{\hat{\partial}u}{\hat{\partial}y} \right) + q_x = -\frac{\hat{\partial}\theta}{\hat{\partial}x}, \quad (4.3.3g)
\]
\[
C \text{Pr} \frac{\partial q_y}{\partial t} + C \text{Pr} \text{Gr} \left( u \frac{\partial q_y}{\partial x} + v \frac{\partial q_y}{\partial y} - q_x \frac{\partial v}{\partial x} - q_y \frac{\partial v}{\partial y} \right) + q_y = -\frac{\partial \theta}{\partial y}. \tag{4.3.3h}
\]

The following non-dimensional groups have been introduced, namely, the Grashof number, the Prandtl number and the Cattaneo number, respectively, given by
\[
\text{Gr} = \alpha \frac{(T_R - T_L) h^3}{\nu^2}, \quad \text{Pr} = \mu C_p / \nu = \nu / \kappa, \quad C = \tau \kappa / (\rho_0 C_p)
\]
Here, \( \kappa = K / (\rho_0 C_p) \) is the thermal diffusivity. Equations (4.3 a, b, c, d, g and h) are the governing equations to this problem. The corresponding dimensionless boundary conditions are
\[
u \left( \pm \frac{1}{2} \right) = 0, \quad \theta \left( \pm \frac{1}{2} \right) = \frac{1}{2}, \quad \text{and} \quad \theta \left( \frac{1}{2} \right) = 1,
\tag{4.3.4a}
\]
which must be used to solve equations (4.3.3 a, b, c, d, g and h). Clearly, the Fourier model is recovered upon setting \( C = 0 \) (zero relaxation time) in (4.3.3 g and h).

The base flow solution in the conduction regime is found by assuming the motion to be steady and independent of the vertical coordinate. Under these conditions the equation of continuity is satisfied automatically and conservation of mass is ensured by the integral constraint
\[
\int_{-1/2}^{1/2} \bar{v}(x) \, dx = 0,
\tag{4.3.4b}
\]
where an overbar indicates base flow conditions. The solution of (4.3.3a, b, c, d, g and h) satisfying conservation of mass (4.3.4b) and boundary conditions (4.3.4a) yields
\[
\bar{u} = 0, \quad \bar{v} = -\frac{1}{6} x^3 + \frac{1}{24} x, \quad \bar{\theta} = x,
\tag{4.3.5a}
\]
\[
\bar{q}_x = -1, \quad \bar{q}_y = \frac{1}{24} C \text{Pr} \text{Gr} \left( 12 x^2 - 1 \right).
\tag{4.3.5b}
\]
Interestingly, (4.3.5b) shows that the non-Fourier effects are important even for the base state solution, where the heat flux in the y-direction is proportional to the Cattaneo number. This is in contrast to the Rayleigh-Benard problem for a non-Fourier fluid,
where the base state, corresponds to stationary heat conduction, remains the same as for a Fourier fluid (since both transient and upper convective terms in (4.3.2d) vanish in the Rayleigh-Benard problem). Figure 4.2a illustrates the variation of the base flow velocity in the $y$-direction ($\nabla$) and the temperature ($\bar{\theta}$) against the horizontal coordinate, $x$. It shows that the base flow can be regarded as a single closed cell of fluid descending along the cold wall and rising along the hot wall. In other words, except near the close ends of the slot, the base flow motion is parallel with two vertical walls. This is in contrast to the Rayleigh-Benard problem, where the fluid is at complete rest in the conduction state. In Figure 4.2b, the heat flux in the $y$-direction ($\bar{q}_y$) is shown for some typical values of $C$ and $Pr$. For given Prandtl and Grashof numbers, as Cattaneo number increases, the minimum value for $\bar{q}_y$, which occurs along the centerline ($x = 0$), decreases. Hence, it is expected that the presence of a non-Fourier term in the base flow solution modifies the stability of the base flow, which will be studied in the next sections.

Since non-Fourier effects in solids do not appear in steady heat conduction, most applications of non-Fourier heat transfer in the literature were limited to transient heat conduction. However, when flow is present, non-Fourier effects can appear under steady conditions. This is one of many interesting phenomena, which do not arise in standard hydrodynamics such as the emergence of heat flux in flow direction without temperature gradient [55-59]. In the current problem, the heat flux in the $y$ direction is present ($\bar{q}_y$), even when there is no temperature gradient in that direction ($d\bar{\theta}/dy = 0$). Clearly, for a Fourier fluid ($C = 0$), $\bar{q}_y$ is zero.
Figure 4.2: Base flow velocity and temperature (a) and Base flow heat flux in y direction, Gr = 1000 (b).
4.4 Linear stability analysis

Similarly to a Fourier fluid, the conduction of a non-Fourier fluid is lost to the convection once a critical value of the Grashof number, $Gr_c$, is exceeded for a given wave number, $q$. Although the velocity profile for the steady conduction state for a non-Fourier fluid is the same as that of a Fourier fluid, its stability will be influenced by non-Fourier effects. More specifically, it can be seen from equations (4.3.3e and f) that the Cattaneo number ($C$), which reflects the level of non-Fourier effects, is present in the heat flux equation. Certainly, the linear stability analysis of the conduction state does not allow the determination of the flow pattern beyond the critical threshold, but does indicate the type of flow to be expected (stationary or oscillatory).

The linear stability analysis of the conduction state is the starting point for the treatment of the nonlinear problem in the post-critical range [60]. The perturbation variables are governed by the following linearized set of equations (linearized about the base flow state):

\[
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \quad (4.4.1a)
\]

\[
\frac{\partial u'}{\partial t} + Gr \frac{\partial u'}{\partial y} = -\frac{\partial p'}{\partial x} + \left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right), \quad (4.4.1b)
\]

\[
\frac{\partial v'}{\partial t} + Gr \left(\frac{\partial v'}{\partial x} + \frac{\partial v'}{\partial y}\right) = -\frac{\partial p'}{\partial y} + \left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right) + \theta', \quad (4.4.1c)
\]

\[
Pr \frac{\partial \theta'}{\partial t} + Pr Gr \left(\frac{d\theta}{dx} u' + \frac{\partial \theta}{\partial y}\right) = -\left(\frac{\partial q'_x}{\partial x} + \frac{\partial q'_y}{\partial y}\right), \quad (4.4.1d)
\]

\[
C Pr \frac{\partial q'_x}{\partial t} + C Pr Gr \left(\frac{d\bar{q}_x}{dx} u' + \frac{\partial \bar{q}_x}{\partial y} - \bar{q}_x \frac{\partial u'}{\partial x} - \bar{q}_y \frac{\partial u'}{\partial y}\right) + q'_x = -\frac{\partial \theta'}{\partial x}, \quad (4.4.1e)
\]
\[
C \text{Pr} \frac{\partial q_y'}{\partial t} + C \text{Pr} \text{Gr} \left( \frac{d q_y'}{dx} - v \frac{\partial q_y'}{\partial y} - q_x \frac{\partial q_y'}{\partial x} - \frac{d v}{dx} q_x' - q_y \frac{\partial q_y'}{\partial y} \right) + q_y' = \frac{-\partial \theta'}{\partial y},
\]

(4.4.1f)

with the corresponding boundary conditions:

\[
u^0 \left( \pm \frac{1}{2} \right) = 0, \quad v' \left( \pm \frac{1}{2} \right) = 0, \quad \theta' \left( \pm \frac{1}{2} \right) = 0.
\]

For a Fourier fluid, it has been shown that the critical disturbances are independent of the y-coordinate [41]. Assuming that this assumption is valid for the non-Fourier fluid, the disturbance to the conduction state is of the form:

\[u'(x, y, t) = U(x)e^{st+iq_y}, \quad v'(x, y, t) = V(x)e^{st+iq_y},\]

\[p'(x, y, t) = \Pi(x)e^{st+iq_y}, \quad \theta'(x, y, t) = \Theta(x)e^{st+iq_y},\]

\[q_x'(x, y, t) = Q_X(x)e^{st+iq_y}, \quad q_y'(x, y, t) = Q_Y(x)e^{st+iq_y},\]

(4.4.2)

where \(s\) dictates the time evolution of the disturbance. By substitution of (4.4.2) in (4.4.1) and eliminating \(Q_X\) and \(Q_Y\), the following set of differential equations is obtained:

\[
\frac{dU}{dx} = -iqV,
\]

(4.4.3a)

\[
\frac{d\Pi}{dx} = \left( -q^2 - s - iq \text{Gr} \bar{v} \right) U - iq \frac{dV}{dx},
\]

(4.4.3b)

\[
\frac{d^2 V}{dx^2} = \text{Gr} \frac{dV}{dx} U + \left( q^2 + s + i q \text{Gr} \bar{v} \right) V + iq\pi - \Theta,
\]

(4.4.3c)

\[
\frac{d^2 \Theta}{dx^2} = \text{Pr} \text{Gr} \Lambda \frac{d\Theta}{dx} U + \left( s \text{Pr} \Lambda + i q \text{Pr} \text{Gr} \Lambda \bar{v} + q^2 \right) \Theta,
\]

(4.4.3d)

where \(\Lambda = s C \text{Pr} + i q C \text{Pr} \text{Gr} \bar{v} + 1\) in the \(\Theta\) equation, (4.4.3d). The corresponding boundary conditions are:
\[ U\left(\pm \frac{1}{2}\right) = 0, \quad V\left(\pm \frac{1}{2}\right) = 0, \quad \Theta\left(\pm \frac{1}{2}\right) = 0. \quad (4.4.3e) \]

It should be noted that the Grashof number enters into the whole system of the perturbation equations (4.4.3a-d), whereas the Cattaneo number and the Prandtl number enter only in the temperature perturbation equation (4.4.3d). Equations (4.4.3a-e) constitute a set of linear, ordinary, homogeneous differential equations, which is solved by using a variable order, variable step size finite difference method with a standard subroutine for boundary-value problem such as BVPFD of I.M.S.L. However, this method has the disadvantage that a good initial guess is necessary to ensure convergence.

4.5 Results

In this section, results based on the formulation above are discussed and conditions for both stationary and oscillatory convection are emphasized. The parameters governing this problem are the Grashof number, the Prandtl number and the Cattaneo number. The study is limited primarily to two Prandtl numbers, \( \text{Pr} = 1 \) (low Prandtl number) and 7.5 (intermediate Prandtl number). For \( \text{Pr} = 1 \), the effects of changing the Cattaneo number between 0 and 0.05 are studied, while for \( \text{Pr} = 7.5 \), \( C \) is changed between 0 and 0.005. Although the values of the studied Cattaneo numbers seem not to be high, it covers a wide range of applications such as nanofluids. For instance, the relaxation time for a nanofluid solution of water and TiO\(_2\) nanoparticles based on the paper written by Wang and Wei [28] is equal to \( \tau = 1.875 \times 10^{-4} \) s. In this case, for nanoparticle concentration of 7\%, and a moderately small gap (\( h = 0.1 \) mm) can lead to a non-negligible Cattaneo number \( C = 0.003 \). Recall that decreasing the distance between two vertical walls results in an increase in the Cattaneo number.

4.5.1 Low Prandtl number (\( \text{Pr} = 1 \))

The neutral curves are illustrated in the (Gr-q) plane in Figure 4.3 for different non-Fourier levels (\( 0 \leq C \leq 0.01 \)) or, equivalently, different Cattaneo numbers. Apparently, the shape of marginal stability curves depends on the Cattaneo number. To compare with available results in the natural convection of a Fourier liquid limit, critical Grashof
number, $Gr_c$, was computed at $C = 0$ (and $Pr = 0.72$). The results reported by previous
works [34,41,44,45] are reproduced with a very good agreement. The $C = 0$ curve
illustrates the marginal stability curve for a Fourier fluid, where there is an exchange of
stability between the pure conduction state and stationary convection for any
wavenumber. When the Cattaneo number is 0.005, for a wide range of wavenumbers ($q <
3.8$), the onset of convection occurs at a lower Grashof number compared with the
Fourier fluid, while for higher wavenumbers ($q > 3.8$) the onset of convection is slightly
delayed to a higher Grashof number. As the Cattaneo number increases to 0.008, the
shape of the marginal stability curve will be different from that of a Fourier fluid, which
is evident from Figure 4.3. In this case, the typical parabolic shape expected for the
marginal stability curve gives way to a wavy response. However, the critical Grashof
number, $Gr_c$, which is determined by the minimum of the marginal stability curve, is
smaller compared with that of a Fourier fluid. Finally, when the Cattaneo number
increases to 0.01, the wavy response is more significant. In this case, the neutral curve
has two local minima, one at $q = 2.5$, the other at $q = 3.3$, and the critical Grashof number
is observed at the smaller wavenumber ($q_c = 2.5$). It should be noted that for all Cattaneo
numbers included in Figure 4.3 ($C \leq 0.01$), the marginal stability curves are single-
valued.
Figure 4.3: Influence of the Cattaneo number on the marginal stability curves in the Gr-q plane (Pr = 1, C = 0-0.01).

The marginal stability curve for a non-Fourier fluid with C = 0.02 is illustrated in Figure 4.4, where curves for lower Cattaneo numbers are included for reference. Here, the neutral curve has two local minima (at q = 2.85 and 3.6). However, the critical Grashof number is observed at the smaller wavenumber (q = 2.85). Although more minima can be located for higher wavenumbers, the critical state parameters would not change (Gr<sub>c</sub> = 7344 and q<sub>c</sub> = 2.85). Furthermore, the wavy shape of the marginal stability curve in this figure gives rise to local multiple-valued solutions. For example, if one increases the temperature difference at a constant wavenumber in the neighbourhood of q = 2.7, instability at a lower Grashof number (Gr = 7344) would be encountered first. At an elevated Grashof number (Gr = 8150), or equivalently higher temperature difference, the flow would experience restabilization and this would be followed at higher Grashof number (Gr = 8936) by a second transition to instability. The multiple-valued solutions are observed again around the local minima for higher wavenumbers. This behaviour is
reminiscent of the stability of circular Couette flow with radial heating; the reader is referred to Figure 4 in [34].

**Figure 4.4:** Influence of the Cattaneo number on the marginal stability curves in the Gr-q plane (Pr = 1, C = 0-0.02).

In Figure 4.5, the marginal stability curve is shown for a non-Fourier fluid with C = 0.05. In this case, the critical Grashof number reduces significantly from 7344 for a non-Fourier fluid with C = 0.02 (Figure 4.4) to 5867. More interestingly, an oscillatory convective branch appears here for relatively small wavenumbers (q < 1.9). Recall that the system (4.3.3a-d) contains the base flow velocity profile $\nabla$ and the horizontal temperature gradient profile $d\bar{\theta}/dx$. Consequently, the occurrence of hydrodynamic modes (thermal modes) due to destabilization of the velocity profile (temperature profile) is possible. Apparently, the occurrence of each mode depends on the values of control parameters C, Pr and Gr. The Cattaneo number and the Prandtl number are only present in the temperature perturbation equation (4.3.3d). This implies that for very small Prandtl or Cattaneo numbers, the effect of the temperature disturbance becomes unimportant.
Additionally, the two parameters $C$ and $\text{Pr}$ are multiplied by each other in (4.3.3d), ensuring that when they both have small values, the temperature perturbation equation will be totally unimportant and the hydrodynamic mode, which is related to destabilization of the velocity profile, occurs. The hydrodynamic mode, which is also stationary, was observed for small Cattaneo numbers ($C \leq 0.02$) as shown in Figures 4.3 and 4.5. As the Cattaneo number increases to $C = 0.05$, the oscillatory branch, or equivalently the thermal mode, appears. However, the critical Grashof number ($\text{Gr}_c = 5867$) still belongs to the stationary branch ($q_e = 3.3$), or the hydrodynamic mode. The neutral curves for fluids with $C = 0$ (Fourier) and $C = 0.02$ are included in Figure 4.5 for reference. For higher Cattaneo numbers, oscillatory convection is predicted to occur at a lower Rayleigh number compared with that of a Fourier fluid. In addition, it can be predicted from Figures 4.3 and 4.4 that by increasing the Cattaneo number, the deviation of the neutral curves from the typical expected parabolic shape is more significant. This deviation can be observed in Figure 4.5, where the wavy shape of the natural curve for a non-Fourier fluid with $C = 0.05$ is more noticeable compared with that of a non-Fourier fluid with $C = 0.02$. Similar to Figure 4.4, local multiple-valued solutions can be observed in this figure.

An important quantity in this analysis is the critical Grashof number, which reflects the minimum required temperature difference between the walls for the observation of convection in reality. The variation of the critical Grashof number against the Cattaneo number is illustrated in Figure 4.6. $\text{Gr}_c$ decreases with $C < 0.008$, and as $C$ increases the figure displays a narrow region in which $\text{Gr}_c$ increases locally. However, the value of the local maximum for $\text{Gr}_c$ in the narrow region is still less than the critical Grashof number for a Fourier fluid. The local maximum is followed by a decrease in the critical Grashof number for higher Cattaneo numbers studied in this paper. The decreasing trend for the critical Grashof numbers suggests that rolls emerge more readily for a fluid with more non-Fourier behaviour (higher Cattaneo numbers). The presence of the local maximum can also be confirmed from Figure 4.4, where the inset shows that by increasing the
Cattaneo number from 0 to 0.008, the minimum of the marginal stability curves shifts to the bottom (lower $Gr_c$). As $C$ increases above 0.008, $Gr_c$ shifts to the top.

Figure 4.5: Influence of the Cattaneo number on the marginal stability curves in the Gr-q plane ($Pr = 1, C = 0.05$).

Again, for higher Cattaneo numbers the minimum of the marginal stability curves shifts to the bottom (lower $Gr_c$), and this can be confirmed from Figure 4.5. Figure 4.6, also, illustrates the variation of the critical wavenumber against the Cattaneo number. For $C < 0.008$, the critical wavenumber remains almost constant. As $C$ increases above 0.008, the critical wavenumber is slightly decreased to a minimum at $C = 0.01$ and then gradually increases for higher Cattaneo numbers. The minimum of $q_c$ occurs at the same Cattaneo number as that of the local maximum of $Gr_c$. The occurrence of this minimum can also be confirmed from Figure 4.4, where the inset shows that by incasing the Cattaneo number from 0 to 0.01, the minimum of the marginal stability curves move to the left
(lower $q_c$). As $C$ increases above 0.01, $q_c$ moves to the right (higher $q_c$), which can be confirmed from both Figures 4.2 and 4.3.

**Figure 4.6:** Influence of the Cattaneo number on the critical Grashof number and wavenumber ($Pr=1$).

### 4.5.2 Intermediate Prandtl number ($Pr = 7.5$)

For the second case, $Pr = 7.5$ is selected because it is close to the practical range of Prandtl numbers for most of the water-based nanofluids. Moreover, both buoyancy and shear forces are known to be in competition for the onset of convection at intermediate Prandtl numbers, which leads to a rich physical behaviour. The marginal stability curves are illustrated in the (Gr-q) plane in Figure 4.7 for different non-Fourier levels ($0 \leq C \leq 0.005$). To compare with published results in the natural convection of a Fourier liquid limit, critical Grashof number, $Gr_c$, was computed at $C = 0$ (and $Pr = 7$). The results reported by Bahloul et al. [45] are reproduced with a very good agreement. For the Fourier fluid ($C = 0$), an exchange of stability between the pure conduction state and stationary convection for any wavenumber is observed here. Moreover, for the assumed
Prandtl number \((Pr = 7.5)\), the exchange of stability between the pure conduction state and stationary convection is still present for a non-Fourier fluid with \(C \leq 0.004\). In this range, no new observations are available for the non-Fourier fluid and all marginal stability curves are coincident with the corresponding curve of a Fourier fluid. In other words, there is no contribution of non-Fourier behaviour of the fluid for this range of Cattaneo numbers. Similar results were reported for the case of Rayleigh-Benard convection of a non-Fourier fluid with an intermediate Prandtl number \([29]\). In that case, for very small Cattaneo numbers and also wavenumber less than 4, all stability curves were coincident with that of a Fourier fluid. However, it might be slightly different from what is observed here in the case of a non-Fourier fluid between two vertical walls. In the Rayleigh-Benard problem, even for a very small \(C > 0\), each non-Fourier curve comprises a stationary branch for a wide range of wavenumbers and an oscillatory convective branch for very high Rayleigh and wavenumbers \([29,30]\). It should be noted that since the numerical approaches are used for finding the critical state parameters here, very high Rayleigh numbers and wavenumbers are not studied in the current paper, whereas in the Rayleigh-Benard problem, by using the analytical approach (for free-free boundary conditions) the critical state parameters could be obtained more simply at any given wavenumber.

Additionally, Figure 4.7 displays an oscillatory branch for a non-Fourier fluid with \(C = 0.0045\) (for lower wavenumbers). In this case, the marginal stability curve comprises a stationary branch, which is coincident with the Fourier curve, and an oscillatory convective branch. Again, this is similar to observations for Rayleigh-Benard convection of a non-Fourier fluid. In the Rayleigh-Benard problem, for small Cattaneo numbers stationary convection (stationary branch was also part of the Fourier curve) was predicted to be observed first since the minimum of the oscillatory branch lies above that of the stationary branch (Fourier curve). For higher Cattaneo numbers, oscillatory convection was predicted to occur at a lower Rayleigh number compared with that of a Fourier fluid. In other words, as \(C\) increased the stationary convection regime was diminishing to the point where the oscillatory convection was predicted to be observed first for any wavenumber \([29]\). In the current study, as \(C\) increases to 0.005 the oscillatory branch moves to the right, increasing the wavenumber range over which oscillatory convection
occurs first, and simultaneously lowering the oscillatory branch. It could be expected that by increasing C, the oscillatory convection occurs first at any wavenumber.

Although the observations in Figure 4.7 would be compatible with the results for the Rayleigh-Benard problem of a non-Fourier fluid, some inconsistencies are obvious. For instance, the oscillatory branch for the Rayleigh-Benard problem has the usual parabolic open-loop shape [29,30]. This is in contrast to what observed in Figure 4.7 for C > 0.004, where the parabolic open-loop stationary branch is accompanied by a semi-closed-loop oscillatory branch. This behaviour is reminiscent of that encountered for the stability of circular Couette flow with radial heating; the reader is referred to Figure 4 in [34], Figure 3 the stability of flow inside a cylindrical annulus with radial heating by [44] and Figure 3 in [45]. Other closed-loop stability maps have been reported for convection in an inclined heated box [42] and in a study of the stability of crystal morphology [43]. As another inconsistency, the oscillatory branch appeared first for higher wavenumbers in the Rayleigh-Benard problem and increasing the Cattaneo number resulted in the lowering of the oscillatory branch. On the contrary, in Figure 4.5 the oscillatory branch appears first for lower wavenumbers and then increasing the Cattaneo numbers leads to the lowering of the branch. Therefore, it can be inferred from Figure 1 in [30] that for the Rayleigh-Benard problem, as C increased, the oscillatory branch moved from the top-right of the (Ra_c-k_c) plane to the bottom-left, whereas in Figure 4.7 (of the current paper) the oscillatory branch moves from the top-left of the (Gr_c-q_c) plane to the bottom-right. It should be emphasised again that in both cases, for high Cattaneo numbers, the oscillatory convection is expected to be observed first for any wavenumber.
Figure 4.7: Influence of the Cattaneo number on the marginal stability curves in the Gr-q plane (Pr = 7.5, C = 0-0.005).

The influence of the Cattaneo number on the critical Grashof number is illustrated in Figure 4.8 for Pr = 7.5. Clearly, for C ≤ 0.004, no change can be seen for the variation of Gr_c against the Cattaneo number. However, as C increases, Gr_c decreases sharply as a result of onset of oscillatory convection. Note that the minimum of the stationary branch is still present for C > 0.004, which is the same as that of the Fourier fluid. However, it is much higher compared with the minimum of the oscillatory branch. It should be emphasized that for higher wavenumbers (for C = 0.004, when q > q_c = 2.1), the stationary convection is still expected to be observed first. The variation of Gr_c against the Cattaneo number is very similar to what observed in the stability of the conduction regime of natural convection in a vertical annulus; the reader is referred to Figure 4a in [45], where the variation of the critical Grashof number is illustrated against the Prandtl number. They found that for Pr = 7 and for small radius ratio (η = 0.4), both stationary and oscillatory branches coexisted, while the minimum Grashof number
belonged to the oscillatory curve. More specifically, they found that for a small radius ratio ($\eta = 0.4$) and small Prandtl number ($Pr < 5$), only stationary branch was present and $Gr_c$ remained almost constant in this region. However, as $Pr$ increases above 5 the oscillatory branch was appeared (with a similar form to what observed in Figure 4.5 of the current paper) and $Gr_c$ decreased sharply with increasing the Prandtl number hereafter.

Figure 4.8, also, depicts the variation of the critical wavenumber ($q_c$) against the Cattaneo number for $Pr = 7.5$. As $C$ increases above 0.004, $q_c$ decreases sharply and finds a minimum, which is followed by an increase (but with a lower rate compared with the rate of the sharp decrease). This is also similar to what observed in Figure 4b in [45]. They found that for a small radius ratio ($\eta = 0.4$) and small Prandtl number ($Pr < 5$), $q_c$ remained almost constant. $Gr_c$ decreased sharply with increasing the Prandtl number more than 5. Therefore, the similarity in the results of the current paper and the results of [45] enables us to conclude that there is an analogy between the effect of the Cattaneo number in the current paper and that of the Prandtl number in [45].

![Figure 4.8: Influence of the Cattaneo number on the critical Grashof number and wavenumber (Pr=7.5)](image-url)
4.6 Discussion

In this chapter, the results are obtained for the case of convection of a non-Fourier fluid confined between two vertical walls. Since analytical results could not be obtained for the vertical slot configuration, the stability of conduction state is examined numerically for a fluid with a low Prandtl number (such as liquid helium II) and a fluid with an intermediate Prandtl number (such as nanofluids). In both cases, as the Cattaneo number increases the critical Grashof number shows a decreasing trend (Figures 4.6 and 4.8). In addition, when the Cattaneo number increases, an oscillatory branch is observed in the stability picture, which tries to lower the critical point. In general, it can be concluded that if the onset of convection occurs at a lower Grashof number (Rayleigh number) for the vertical slot (Rayleigh-Benard) configuration compared to that of for a Fourier fluid, it signals the presence of non-Fourier effects. In addition, if the conduction is lost to oscillatory convection, this can be another possible situation where non-Fourier effects are important.

In addition to liquid helium at very low temperatures (see section 4.2) that has small Prandtl number (Pr < 1) (T < 2.17 K), nanofluids have also non-Fourier character (see Chapter 2) (Pr ≈ 7). The relevance of the DPL model to nanofluids (NFs) has recently been recognized in the literature, and is emphasized in Chapter 2. The equivalence between the two-phase and DPL models allows the expression of the relaxation time in terms of the nanoparticle (NP) concentration [28]. Two non-dimensional groups are introduced for NFs in Chapter 2, the Cattaneo number, $C (= \frac{\tau Q K}{\kappa h^2})$, and dimensionless retardation time, $S (= C \frac{K_F}{K})$. C and S are both related directly to the NP concentration. The retardation-to-relaxation time ratio is found to be equal to the NF solution-to-solvent thermal conductivity ratio, $\gamma$. Since $S = \frac{K_F C}{K}$, S must always be smaller than C for a NF.

In this paper, only SPL ($S = 0$) fluids are studied. This limit case has multiple advantages over a fluid with retardation. It serves as reference case for a fluid with strong non-
Fourier character, is mathematically more manageable, and can, in the limit, reflect the behaviour of a NF with very high NP concentration.

As mentioned previously, if the onset of convection occurs at a lower Rayleigh number (Grashof number) for the Rayleigh-Benard configuration (vertical slot) compared to that of for a Fourier fluid, it signals the presence of non-Fourier effects. This phenomenon is observed experimentally [61], where the onset of convection for a nanofluid comprised of water and Al$_2$O$_3$ ($\Phi = 2.72\%$) and confined between two vertical plates occurs at a lower Rayleigh number compared to that of a pure water (Figure 3 in [61] and $\theta$=90°). From Figure 3 in [61], the ratio of critical Rayleigh number for NF over the critical Rayleigh number for pure water can be read as 0.75. Using Figure 4.6, it can be seen that for $C \approx 0.0045$ the ratio of critical Grashof number for a non-Fourier fluid over the critical Grashof number for a Fourier fluid is 0.75. Recall that if the conduction is lost to oscillatory convection, this can also signal the importance of non-Fourier effects. In this case, transient oscillatory convection is observed [62] at the onset for a layer of nanofluid confined between two horizontal plates (Rayleigh-Benard convection). In a recent experimental study, oscillatory convection is observed [63] for nanofluids (between two horizontal planes) with high thermophilic nanoparticles. For nanofluids, the oscillatory convection is resulted from the competition between two opposing factors: buoyancy (destabilizing) and thermophoresis (stabilizing).

The mechanism of the instability can also be studied through an examination of the disturbance-energy equation. The kinetic energy of the perturbations can be defined as:

$$KE = \frac{1}{2} \int_{\Omega} \left( u'^2 + v'^2 \right) dv ,$$  \hspace{0.5cm} (4.6.1)

where $\Omega$ is the cell volume which is defined as $\Omega$: $-1/2 < x < +1/2$, $0 < y < 2\pi/q$. The energy balance rate is then obtained by taking the derivative of (4.6.1) with respect to time. Multiplying the linearized disturbance momentum equation by the perturbation velocity, $v'^i$, the kinetic energy rate reads as
\[
\frac{dKE}{dt} = \int v_i \frac{\partial v'}{\partial t} \, dv = \int \left( u' \frac{\partial u'}{\partial t} + v' \frac{\partial v'}{\partial t} \right) \, dv .
\]  
(4.6.2)

Substituting \( \frac{\partial u'}{\partial t} \) and \( \frac{\partial v'}{\partial t} \) from (4.4.1b-c) into (4.6.2), the equation for the kinetic energy rate of the perturbations is

\[
\frac{dKE}{dt} = -Gr\left( \overline{v_x u' v'} \right) + \langle v' \theta' \rangle - \left( u'_x^2 + u'_y^2 + v'_x^2 + v'_y^2 \right),
\]  
(4.6.3)

with brackets defining the average over one spatial cycle. Unlike the Rayleigh-Benard problem, where the equation for the kinetic energy rate (equation 4.4 in [30]) consists of two terms (buoyancy and dissipation), in (4.6.3) three terms are present. The first term on the RHS, \(-Gr\left( \overline{v_x u' v'} \right)\), reflects the effects of the base flow interaction with the perturbations and is called ‘transfer’ [42]. The second term, \( \langle v' \theta' \rangle \), represents the ‘buoyancy’ and the last term, \(- \left( u'_x^2 + u'_y^2 + v'_x^2 + v'_y^2 \right)\) shows the ‘dissipation’. Some points can be noted about (4.6.3). First, at the critical Grashof number, the kinetic energy rate \( dKE/dt \) is equal to zero. Second, the dissipative term has always negative value, which emphasizes the fact that the viscosity always dissipates energy. In contrast, the values of buoyancy and transfer terms depend on \( \overline{v_x}, u', v' \) and \( \theta' \), which cannot be predicted easily. Third, it can be seen from (4.6.3) that for higher Grashof numbers, the transfer term plays a more important role. It was seen that the critical Grashof number decreases sharply for higher Cattaneo numbers (Figures 4.4 and 4.5). Therefore, as \( C \) increases, it is expected from equation (4.6.3) that the dominance of the transfer term will be replaced by the buoyancy term. The stationary (oscillatory) mode in Figure 4.5 signals the dominance of the transfer (buoyancy) term in (4.6.3).

**4.7 Conclusion**

This study examines the natural convection of non-Fourier fluids of the single-phase-lagging (SPL) type. These fluids possess a relaxation time, reflecting the delay in the response of the heat flux and the temperature gradient with respect to one another. The
SPL model is particularly relevant to low-temperature liquid. Although somewhat unrealistic as a model for nanofluids, this limit case has multiple advantages over a fluid with retardation. It serves as reference case for a fluid with strong non-Fourier character (relaxation time of the same order as the process time), is mathematically more manageable, and can, in the limit, reflect the behaviour of a nanofluid with very high nanoparticle concentration.

For Pr = 1 (small Prandtl number), linear stability analysis indicates a decreasing trend for the critical Grashof number with respect to the Cattaneo number, except for a very small region in which the critical Grashof number increases. In this curve, stability boundaries for higher Cattaneo numbers are multiple-valued. For Pr = 7.5 (intermediate Prandtl number), as Cattaneo number exceeds a certain value (0.004), a SPL fluid can lose its conductive mode to stationary or oscillatory convection. In this case, for Cattaneo numbers less than 0.004, no change is observed in the marginal stability curves compared with that of a Fourier fluid. In contrast, for Cattaneo numbers more than 0.004, the neutral stability curve comprises a Fourier branch (higher wavenumbers) and an oscillatory branch (lower wavenumbers). It is anticipated that oscillatory convection increasingly becomes the mode of preference, compared to both conduction and stationary convection. Moreover, a qualitative similarity is observed between the marginal stability curves here and the results for flow inside a hollow cylinder with a radial temperature gradient [45]. Finally, the mechanism of instability is discussed through an examination of the disturbance-energy equation.

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Chapter 5

5 A combined spectral-amplitude-perturbation approach for systematic mode selection in thermal convection

5.1 Introduction

Rayleigh-Benard (RB) convection, the natural convection of a fluid layer heated from below, is an important part of many physical systems and processes such as weather [1], earth mantle movement [2,3], and oceanic currents [4,5]. Starting with the original work of Bénard (1900)[6], Rayleigh-Benard convection has long been studied experimentally [7-11] and theoretically [12-15]. From the fundamental point of view, Rayleigh-Benard convection is an excellent system to test new approaches for understanding nonlinear dynamics. To describe the nonlinear Rayleigh-Benard convection in the weakly nonlinear methodologies, when the critical condition is only slightly exceeded, linear stability analysis is considered as the first step. Using linear stability analysis, the instability threshold or the critical Rayleigh number is accurately predicted. A review of the linear stability analysis of Rayleigh-Benard convection for different boundary conditions, such as free-free and rigid-rigid is available in the basic literature [16-18]. However, the amplitude of the motion, the preferred pattern and the size of convective cells can only be predicted by considering nonlinear effects. Common nonlinear approaches to predict the convective flow pattern are the low-order dynamical system approach such as the Lorenz model [19], the method of amplitude equations [20-23], and the perturbation approach [24-26].

In the Lorenz model, only three modes in the double Fourier expansions of stream function and temperature variations are considered [18,19]. In contrast, the method of amplitude equations and perturbation approach are not limited to three low-order modes. An interesting review regarding the method of amplitude equations can be found in the paper by Cross and Hohenberg (1993) [27]. As described by Getling (1998) [28]: “In the

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amplitude equation approach, the velocity and temperature disturbances in the slightly post-critical range adopt a spatial dependence that strongly resembles the critical mode. The convective state just above the critical Rayleigh number is described by simply giving the amplitude $A$, corresponding to the projection of the motion onto this mode with, possibly, correction for small adjustment in the shape of rolls.” However, from a theoretical point of view, the original amplitude equation method is valid only close to the critical Rayleigh number found from linear stability analysis [26,29].

Perturbation expansion is another approach to study finite amplitude convection, where nonlinear inertial and thermal convection terms are considered as perturbations of the linear convection problem [24,26]. Existing nonlinear methods to study natural convection are mostly based on arbitrary mode selection [30-32]. Recently, Ahmed et al. (2013) [26] used a spectral approach to identify the required modes to accurately capture the convection, when the deviation from the critical Rayleigh number is small. However, their results are limited to the very narrow Rayleigh number region above criticality. Moreover, some inconsistency is observed in their results, particularly in the variation of the Nusselt number against the Rayleigh number to different orders.

As mentioned earlier, the validity of the method of amplitude equations and perturbation approach is limited to the close neighborhood of the linear threshold (weakly nonlinear convection). Consequently, attempting these methods far from the criticality is not justified. Dauby et al. (2001) [33] showed that the method of amplitude equations could be generalized to provide precise quantitative/qualitative results even far from the linear threshold. Their results were in very good agreement with purely numerical results and experiment. Their work was, however, restricted to a fluid with a large Prandtl number ($\approx 880$). Moreover, a rescaling factor was introduced in the basis functions used in the expansions of the unknown temperature field, whose presence was not completely justified. Dondlinger et al. (2007) [29] revisited the amplitude equations used by [33] and examined the applicability of the method to other situations, with other boundary conditions or with other values of the Prandtl number. They found that in all thermoconvection problems considered, the amplitude equation approach provides very good results, even far from criticality, provided the basis functions are appropriately
chosen. However, the motivation of their work was the experimental evidence of the appearance of second- and third-order harmonics in a thermoconvective roll pattern in the Rayleigh-Benard problem [33], whereas the absence of higher order harmonics was not justified.

In this paper a combined spectral/amplitude/perturbation approach is used to study nonlinear natural convection. In this case, the deviation from the critical Rayleigh number is defined as \( \varepsilon = \frac{Ra - Ra_{\text{cr}}}{Ra} \), which is smaller than 1 for any Rayleigh number. This is in contrast to \( \varepsilon' = \frac{Ra - Ra_{\text{cr}}}{Ra_{\text{cr}}} \), the deviation from the critical Rayleigh number used by Ahmed et al. (2013) [26], which makes their solution limited to a very narrow Rayleigh number region above criticality. Using the expansion parameter \( \varepsilon \), one is able to identify the required modes (to different orders) to accurately capture the convection even far from the threshold. In this approach, non-zero modes are determined systematically and the presence/absence of each mode can be justified.

The paper is organized as follows: In section 5.2 the problem is introduced, and the governing equations and boundary conditions are derived. Next, the nonlinear spectral approach is formulated in section 5.3. In section 5.4, an amplitude/perturbation approach is used to solve the nonlinear equations. The results are discussed in section 5.5. Finally, concluding remarks are given in section 5.6.

5.2 Problem formulation

In this section, the general governing equations and boundary conditions are introduced. In addition, the scaling process and dimensionless similarity groups are explained.

5.2.1 Governing equations and boundary conditions

Consider a two-dimensional thin layer of a Newtonian fluid confined between two horizontal planes. The planes are at \( Z = 0 \) and \( Z = D \) and maintained at fixed temperatures \( T_0 + \delta T \) and \( T_0 \), respectively. The physical configuration is described by a Cartesian coordinate system \((X, Z)\) with the \( Z \) axis in the vertical direction. Here \( \mathbf{g} = - \)
ge_z is the gravitational acceleration vector and e_z is the unit vector in the vertical direction. The fluid layer is assumed to be of infinite horizontal extent. Also, the Boussinesq approximation is assumed in the current paper, which states that the effect of compressibility is negligible everywhere in the conservation equations except in the buoyancy term. In this case, the density ρ is assumed to depend on the temperature T, following:

\[ \rho = \rho_0 \left( 1 - \alpha_T (T - T_0) \right), \]  

(5.2.1)

where \( \rho_0 \) is the mass density of the fluid at \( T_0 \) and \( \alpha_T \) is the coefficient of volume expansion. \( C_p \) is the specific heat of the fluid at constant pressure and \( K \) is the thermal conductivity. For stationary fluid (or in the conduction state), the temperature and pressure are the only non-vanishing variables and are found as \( T_s = -(Z/D)\delta T + T_0 + \delta T \) and \( dP_s / dZ = -\rho_0 \left[ 1 - \alpha_T \delta T (1 - Z/D) \right] g \) (the pressure gradient), respectively.

Let \( D, \ D^2/\kappa \) and \( \kappa/D \) be the typical length, time and velocity scales, respectively. Here \( \kappa = K/(\rho C_p) \) is the thermal diffusivity. Consequently, the non-dimensional equations for the conservation of mass, momentum and energy, for the steady state condition are, respectively,

\[ \nabla \cdot \mathbf{v} = 0, \]  

(5.2.2a)

\[ Pr^{-1} \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + Ra \theta e_z + \Delta \mathbf{v}, \]  

(5.2.2b)

\[ \mathbf{v} \cdot \nabla \theta = \Delta \theta + \mathbf{v} \cdot e_z, \]  

(5.2.2c)

where \( \nabla \) and \( \Delta \) are the gradient and the Laplacian operators, respectively. A subscript after a variable denotes partial differentiation. Here, \( \mathbf{v} = (u, w) \) is the velocity vector and \( p = D^2 (\mu_0 \kappa)^{-1} (P - P_s) \) is the pressure deviation from the conduction state pressure. \( \mu_0 \) is the dynamic viscosity of the fluid at rest, and \( \theta = (T - T_s) / \delta T \) is the departure from the
conduction state temperature (the linear profile). The two similarity groups in the equations, the Rayleigh number and the Prandtl number are given by

\[
\text{Ra} = \frac{\delta T g \alpha_{T} D^{3}}{\nu \kappa}, \quad \text{Pr} = \frac{\nu}{\kappa},
\]

where \( \nu = \mu / \rho \) is the kinematic viscosity. The relevant boundary conditions at the lower and upper surfaces are taken to correspond to the free-free condition. In this case

\[
v(x, z = 0) \cdot e_{z} = v(x, z = 1) \cdot e_{z} = 0,
\]

\[
v_{zz}(x, z = 0) \cdot e_{z} = v_{zz}(x, z = 1) \cdot e_{z} = 0,
\]

\[
\theta(x, z = 0) = \theta(x, z = 1) = 0.
\]

It should be noted that assuming slip condition at both upper and lower boundaries significantly simplifies the solution procedure. Other boundary conditions could have been adopted, such as the rigid-rigid or rigid-free conditions. The free-free conditions are, however, the most commonly used in the literature, which will allow easy reference. However, the flow and heat transfer characteristics of Rayleigh-Benard remain qualitatively the same whether rigid-rigid or free-free conditions are used [28]. Since two-dimensional flow is considered, the stream function \( \psi(x, z) \) is assumed such that

\[
u = \psi_{z}, \quad w = -\psi_{x}.
\]

Using the streamfunction \( \psi(x, z) \), the continuity equation (5.2.2a) is satisfied automatically. Upon taking the curl of (5.2.2b) and eliminating the pressure, one obtains the vorticity equation as

\[
\text{Pr}^{-1} \left( \psi_{z} \psi_{x z z} - \psi_{x} \psi_{z z z} + \psi_{z} \psi_{x x x} - \psi_{x} \psi_{x x z} \right) = \Delta^{2} \psi - \text{Ra} \theta_{x},
\]

\[
\psi_{z} \theta_{x} - \psi_{x} \theta_{z} + \psi_{x} = \Delta \theta,
\]
\[
\psi(x,0) = \psi(x,1) = \psi_{zz}(x,0) = \psi_{zz}(x,1) = 0(x,0) = 0(x,1) = 0.
\] (5.2.6c)

5.3 Spectral solution

In this section, a general spectral approach is presented to find the steady flow and temperature fields. Interestingly, this approach is valid for any post-critical range of the Rayleigh number and can be applied to convection regime even far from the linear stability threshold and strongly nonlinear convection. The stream function and temperature are expanded in terms of periodic functions in the streamwise direction and orthonormal shape function in the transverse direction. Therefore, by assuming periodicity in the \(x\) direction with wavelength \(2\pi/k\), the most general solutions of equations (5.2.6a) and (5.2.6b), is of the form

\[
\psi(x,z) = \sum_{a=-L}^{L} \sum_{m=1}^{M} \psi_{am} e^{iakx} f_m(z), \quad \theta(x,z) = \sum_{a=-L}^{L} \sum_{m=1}^{M} \theta_{am} e^{iakx} g_m(z), \quad (5.3.1)
\]

where \(\psi_{am}\) and \(\theta_{am}\) are constant coefficients, and \(f_m\) and \(g_m\) are appropriately chosen orthonormal polynomials that satisfy the boundary conditions at \(z = 0\) and \(z = 1\). Here, indices \(a, b, c \in [-L, L]\) and \(m, n, p \in [0, M]\), where \(L\) and \(M\) are the number of modes in the \(x\) and \(z\) directions, respectively. Taking advantage of the homogeneity of conditions (5.2.6c) and the invariance under the following symmetry transformation as (5.3.2), solution (5.3.1) can be simplified.

\[
x \rightarrow -x, \quad z \rightarrow z, \quad \psi \rightarrow -\psi, \quad \theta \rightarrow \theta.
\] (5.3.2)

In this case, solution (5.3.1) reduces to

\[
\psi(x,z) = \sum_{a=1}^{L} \sum_{m=1}^{M} \psi_{am} \sin(akx) f_m(z), \quad (5.3.3a)
\]

\[
\theta(x,z) = \sum_{m=1}^{M} \theta_{0m} g_m(z) + \sum_{a=1}^{L} \sum_{m=1}^{M} \theta_{am} \cos(akx) g_m(z). \quad (5.3.3b)
\]
The first step to find the equations governing the coefficients is substituting (5.3.3a) and (5.3.3b) into (5.2.6a) and (5.2.6b). Then, the equation resulted from substitution of (5.3.3a) and (5.3.3b) into (5.2.6a) is multiplied by $\sin(ckx)f_p(z)$. Similarly, the equation resulted from substitution of (5.3.3a) and (5.3.3b) into (5.2.6b) is multiplied by $\cos(ckx)f_p(z)$, where the details are not shown here for brevity. After integrating over the intervals $0 \leq x \leq 2\pi/k$ and $0 \leq z \leq 1$, and taking advantage of shape function orthonormality, one obtains

\[
cR\pi\theta_{cp} = Pr^{-1} \sum_{a,b=1}^{L} \sum_{m,n=1}^{M} D_{abc,mnp} \psi_{am} \psi_{bn} + \sum_{a=1}^{L} \sum_{m=1}^{M} D_{2ac,mp} \psi_{am}, \tag{5.3.4a}
\]

\[
c\pi \sum_{m,n=1}^{M} \left( \langle f_{m}g'_{n}g_{p} \rangle \psi_{cm} \theta_{0n} + \sum_{a=1}^{L} \sum_{b=1}^{M} D_{3abc,mp} \psi_{am} \theta_{bn} + \sum_{a=1}^{L} \sum_{m=1}^{M} D_{4ac,mp} \psi_{am} + \sum_{a=1}^{L} \sum_{m=1}^{M} D_{5ac,mp} \theta_{am} = 0, \tag{5.3.4b}
\]

\[
\sum_{a=1}^{L} \sum_{m,n=1}^{M} a_{n} \left( \langle f_{m}g'_{n}g_{p} \rangle + \langle f_{m}g'_{n}g_{p} \rangle \right) \psi_{am} \theta_{an} + \sum_{m=1}^{M} D_{500,mp} \theta_{0m} = 0. \tag{5.3.4c}
\]

In (5.3.4a), $D_{abc,mnp}$ reduces to (upon letting $S_{a} \equiv \sin(akx)$ and $C_{a} \equiv \cos(akx)$)

\[
D_{abc,mnp} \bigg|_{c>0} = bk \left\langle S_{a}C_{b}S_{c} \right\rangle \left\langle f_{m}^{'e}f_{n} \right\rangle - b^3k^3 \left\langle S_{a}C_{b}S_{c} \right\rangle \left\langle f_{m}^{'e}f_{n}f_{p} \right\rangle + ak \left\langle C_{a}S_{b}S_{c} \right\rangle \left( b^2k^2 \left\langle f_{m}^{'e}f_{n}f_{p} \right\rangle - \left\langle f_{m}^{'e}f_{n}f_{p} \right\rangle \right). \tag{5.3.5a}
\]

Similarly, the other matrix coefficients are given as
\[ D_{2,ac,mp} |_{c > 0} = -\left( S_a S_c \right) \left( a^4 k^4 \langle f_m f_p \rangle - 2a^2 k^2 \langle f_m^\prime f_p \rangle + \langle f_m f_p \rangle \right) \]
\[ D_{3,abc,mnp} = bk \left( S_a S_b C_c \right) \left( f_m^\prime g_n g_p \right) + ak \left( C_a C_b C_c \right) \left( f_m g_n^\prime g_p \right) \]
\[ D_{4,ac,mp} = -ka \left( C_a C_c \right) \left( f_m g_p \right) \]
\[ D_{5,ac,mp} = -\left( C_a C_c \right) \left[ a^2 k^2 \left( g_m g_p \right) - \langle g_m^\prime g_p \rangle \right] \]

Note that \( a, b, c \in [1, L] \) and the brackets \( <> \) denote integration over \( x \) or \( z \). It should be noted that the additional equation (5.3.4c) will make the system of equations easier to manage compared with two equations system. Since the current study focuses on the free-free conditions, the \( z \)-dependent functions are chosen as

\[ f_m(z) = g_m(z) = \sqrt{2} \sin(m\pi z). \quad (5.3.6) \]

However, the general system of equations (5.3.4a-c) is valid for other boundary conditions such as free-rigid or rigid-rigid at \( z = 0 \) and \( z = 1 \). Substituting the \( z \)-dependent functions (5.3.6) into (5.3.4a-c), the generalized system of equations (5.3.4a-c) further simplifies to

\[ \frac{\pi}{k} \beta_{cp}^2 \psi_{cp} + cRa \pi \theta_{cp} = Pr^{-1} \sum_{a,b=1}^{L} \sum_{m,n=1}^{M} F_{1,abc,mnp} \psi_{am} \psi_{bn}, \quad (5.3.7a) \]
\[ c\pi \beta_{cp}^2 \psi_{cp} + \frac{\pi}{k} \beta_{cp} \theta_{cp} = c\pi \sum_{m,n=1}^{M} E_{1,mnp} \psi_{cm} \theta_{0n} + \sum_{a,b=1}^{L} \sum_{m,n=1}^{M} F_{2,abc,mnp} \psi_{am} \theta_{bn}, \quad (5.3.7b) \]
\[ \theta_{0p} = \frac{k}{2\beta_{0p}} \sum_{a=1}^{L} \sum_{m,n=1}^{M} aE_{2,mnp} \psi_{am} \theta_{an}, \quad (5.3.7c) \]

where \( \beta_{cp} = c^2 k^2 + p^2 \pi^2 \). Moreover, the matrices \( F1, F2, E1 \) and \( E2 \) are introduced as

\[ F_{1,abc,mnp} = D_{1,abc,mnp}, \quad F_{2,abc,mnp} = D_{3,abc,mnp}, \quad (5.3.8a) \]
\[ \text{E}_1_{\text{mnp}} = \left< f_m^g g_n^g g_p^g \right>, \quad \text{E}_2_{\text{mnp}} = \left< f_m^g g_n^g g_p^g \right> + \left< f_m^g g_n^g g_p^g \right> . \]  

(5.3.8b)

For instance, upon setting \( a=b=c=m=n=p=1 \) one finds that \( F_{111,111} \) and \( F_{211,111} \) are equal to zero. When \( m=p=1 \) and \( n=2 \), \( E_{121} = -\sqrt{2} \pi \) and in a similar way, \( E_{212} \) is equal to \( \sqrt{2} \pi \). Upon using (5.3.3a), (5.3.3b) and (5.3.6), the velocity components and the temperature deviation are easily deduced as:

\[ u(x,z) = \sqrt{2\pi} \sum_{a=1}^{L} \sum_{m=1}^{M} m\psi_{am} \sin(akx) \cos(m\pi z), \]  

(5.3.9a)

\[ w(x,z) = -\sqrt{2k} \sum_{a=1}^{L} \sum_{m=1}^{M} a\psi_{am} \cos(akx) \sin(m\pi z), \]  

(5.3.9b)

\[ \theta(x,z) = \sqrt{2} \sum_{m=1}^{M} \theta_{0m} \sin(m\pi z) + \sqrt{2} \sum_{a=1}^{L} \sum_{m=1}^{M} \theta_{am} \cos(akx) \sin(m\pi z). \]  

(5.3.9c)

In addition, the Nusselt number is

\[ Nu = 1 - \sqrt{2\pi} \sum_{m=1}^{M} m\theta_{0m} . \]  

(5.3.10)

### 5.4 Solution and mode selection

A combined amplitude-perturbation method that determines systematically the number and type of mode required to each order is discussed in this section.

#### 5.4.1 Combined Spectral-Amplitude-Perturbation approach

The generalized system (5.3.7a-c) can be cast in a matrix form as

\[ \mathbf{L}_{\text{cp}} \mathbf{u}_{\text{cp}} = -\mathbf{N}_{\text{cp}} \begin{bmatrix} \mathbf{u}_{\text{cp}} \end{bmatrix} . \]  

(5.4.1)
where $u_{cp}$ is the perturbation from the base state (in this case, base state is $u_b = 0$), and $L_{cp}$ is the linear matrix operator acting on $u_{cp}$ as:

$$
\begin{pmatrix}
\psi_{cp} \\
\theta_{cp} \\
\theta_{0p}
\end{pmatrix} =
\begin{pmatrix}
-Pr \beta_{cp} & -k Pr c \frac{Ra}{\beta_{cp}} & 0 \\
-ck & -\beta_{cp} & 0 \\
0 & 0 & -\beta_{0p}
\end{pmatrix},
\quad (5.4.2a)
$$

and $N_{cp}$ covers all terms that are nonlinear in $u_{cp}$

$$
N_{cp} \left[ u_{cp} \right] =
\begin{pmatrix}
k \sum_{m,n=1}^{L} E_{1mnp} \psi_{cm} \theta_{0n} \\
-k \sum_{a=1,b=1}^{L} \sum_{m,n=1}^{M} \sum_{a=1}^{L} \sum_{b=1}^{M} F_{1abc,mnp} \psi_{am} \psi_{bn}
\end{pmatrix}.
\quad (5.4.2b)
$$

$E_{1abc,mnp}, E_{2abc,mnp}, F_{1mnp}$ and $F_{2mnp}$ are given in (5.3.8a-b). The current nonlinear formulations (5.4.1) are valid over any range of the parameters, particularly the post-critical range of the Rayleigh number and can be applied to modes with any horizontal wavelength. In this paper, a nonlinear approach, spectral-amplitude-perturbation expansion, is adopted to solve equations. The perturbation parameter in the current problem represents a deviation from the critical Rayleigh number for the onset of convection, and is given by

$$
\varepsilon = \frac{Ra - Ra_{cr}}{Ra},
\quad (5.4.3a)
$$

where $Ra_{cr}$ is the critical Rayleigh number defined as $Ra_{cr} = \beta^3 / k^2$ and $\beta = k^2 + \pi^2$. It is worth noting that another expansion parameter denoted by $\varepsilon' = \frac{Ra - Ra_{cr}}{Ra_{cr}}$ is common
in literature (Kuo, 1961; Ahmed et al., 2013) to treat weakly non-linear solutions via perturbation approaches. An obvious advantage of using of expansion parameter by (5.4.3a) is that it remains less than 1 for any Rayleigh number. The solution so obtained may be valid for a much larger range of temperature differences. Again, it is a well established experimental fact that the amplitude of motion is proportional to \( \sqrt{\varepsilon} \) [18], [28]. Therefore, \( \mathbf{u}_{\text{cp}} \) is expanded as

\[
\mathbf{u}_{\text{cp}} = \sqrt{\varepsilon} \mathbf{u}_{1\text{cp}} + \varepsilon \mathbf{u}_{2\text{cp}} + \varepsilon^{3/2} \mathbf{u}_{3\text{cp}} + \varepsilon^2 \mathbf{u}_{4\text{cp}} + \varepsilon^{5/2} \mathbf{u}_{5\text{cp}} \ldots \quad (5.4.3b)
\]

The unknown stream function and temperature coefficients are formally expanded as

\[
\begin{align*}
\psi_{am} &= \sqrt{\varepsilon} \psi_{1am} + \varepsilon \psi_{2am} + \varepsilon^{3/2} \psi_{3am} + \varepsilon^2 \psi_{4am} + \varepsilon^{5/2} \psi_{5am} \ldots \\ 
\theta_{am} &= \sqrt{\varepsilon} \theta_{1am} + \varepsilon \theta_{2am} + \varepsilon^{3/2} \theta_{3am} + \varepsilon^2 \theta_{4am} + \varepsilon^{5/2} \theta_{5am} \ldots \\ 
\theta_{0m} &= \sqrt{\varepsilon} \theta_{01m} + \varepsilon \theta_{20m} + \varepsilon^{3/2} \theta_{30m} + \varepsilon^2 \theta_{40m} + \varepsilon^{5/2} \theta_{50m} \ldots 
\end{align*}
\]

(5.4.3c)

Upon substituting (5.4.3c) into (5.4.1) and noting

\[
\mathrm{Ra} = \frac{Ra_{cr} \varepsilon^{3/2}}{1-\varepsilon} = Ra_{cr} \left(1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \ldots \right), \quad (5.4.3d)
\]

the terms of equal orders in \( \varepsilon \) are isolated. The methodology of obtaining modes will be discussed in details. Note that only first few expansion coefficients are obtained here to elucidate the way of development of the solution. The linear operator is expanded as

\[
\mathbf{L}_{\text{cp}} = \mathbf{L}_{0\text{cp}} + \varepsilon^{1/2} \mathbf{L}_{1\text{cp}} + \varepsilon \mathbf{L}_{2\text{cp}} + \varepsilon^{3/2} \mathbf{L}_{3\text{cp}} + \varepsilon^2 \mathbf{L}_{4\text{cp}} + \varepsilon^{5/2} \mathbf{L}_{5\text{cp}} \ldots 
\]

(5.4.4a)

In this case, \( \mathbf{L}_{0\text{cp}} \), \( \mathbf{L}_{1\text{cp}} \), \( \mathbf{L}_{2\text{cp}} \), \( \mathbf{L}_{3\text{cp}} \), \( \mathbf{L}_{4\text{cp}} \) and \( \mathbf{L}_{5\text{cp}} \) are
The eigenvalues of the matrix $L_{0cp}$ are

$$
\lambda_{1cp} = -\frac{1}{2} \beta_{cp} (Pr+1) + \frac{1}{2} \sqrt{\beta_{cp}^2 (Pr-1)^2 + 4\epsilon^2 Pr \frac{\beta^3}{\beta_{cp}}},
$$

$$
\lambda_{2cp} = -\frac{1}{2} \beta_{cp} (Pr+1) - \frac{1}{2} \sqrt{\beta_{cp}^2 (Pr-1)^2 + 4\epsilon^2 Pr \frac{\beta^3}{\beta_{cp}}},
$$

$$
\lambda_{3cp} = -\beta_{0p}
$$

and the corresponding eigenvectors read as

$$
e_{0cp} = \delta_{1c} \delta_{1p} \begin{pmatrix} \frac{\beta}{k} \\ 1 \\ 0 \end{pmatrix}, \quad e_{1cp} = \delta_{1c} \delta_{1p} \begin{pmatrix} \frac{Pr \beta}{k} \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_{2cp} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

To leading order, that is to $O(\sqrt{\epsilon})$, $N_{1cp} = 0$. Therefore, (5.4.2a) reduces to

$$
L_{0cp} u_{1cp} = 0.
$$

(5.4.7a)

For equation (5.4.7a) to have a solution, the determinant of $L_{0cp}$ must be zero and thus $c=p=1$. This shows that $u_{1cp}$ is some amplitude of the zero-eigenvalue mode, namely,
\[ u_{1c} = B_0 \delta_1 \delta_1 \begin{pmatrix} -\frac{\beta}{k} \\ 1 \\ 0 \end{pmatrix}, \] (5.4.7b)

which introduces the amplitude function \( B_0 \). Note that \( B_0 \) is the amplitude of a real vector and can be taken as a real function. Note that the term on the RHS represents \( \psi_{11} \) and \( \theta_{11} \). To the next order that is to \( O(\varepsilon) \), one has

\[ L_{cp} u_{cp} = L_{0cp} u_{2cp} + L_{1cp} u_{1cp}, \] (5.4.8a)

where \( L_{1cp} u_{1cp} = 0 \) (since \( L_{1cp} = 0 \)). Noting that \( F_{11c,11p} = F_{21c,11p} = 0 \) and \( E_{21p} = \sqrt{2\pi} \delta_2 \) one has:

\[ N_{2cp} = -\frac{\sqrt{2\pi}}{2} \beta \delta_0 \delta_2 B_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

where the term \( B_0^2 \) is the nonlinear term to \( O(\varepsilon) \) and represents \( \psi_{11} \theta_{11} \) evaluated with the solution \( u_{1cp} \). Therefore,

\[ u_{2cp} = \delta_1 \delta_1 B_1 \begin{pmatrix} -\frac{\beta}{k} \\ 1 \\ 0 \end{pmatrix} - \delta_0 \delta_2 \frac{\sqrt{2\pi}}{2\beta_0^2} B_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \] (5.4.8b)

where the first term introduces the next order correction \( B_1 \) to the amplitude that will be determined at the higher orders. Note that the first term on the RHS represents \( \psi_{21} \) and \( \theta_{21} \), whereas the second term shows \( \theta_{02} \). To the next order, that is to \( O(\varepsilon^{3/2}) \), (5.4.2a) reduces to
\[ L_{cp}u_{cp} = -N_{3cp}, \]

where \( L_{cp}u_{cp} = L_{0cp}u_{3cp} + L_{1cp}u_{2cp} + L_{2cp}u_{1cp} \). Since, \( L_{1cp} = 0 \), (5.4.2a) reads as

\[ L_{0cp}u_{3cp} = -L_{2cp}u_{1cp} - N_{3cp}. \quad (5.4.9a) \]

Noting that \( E_{12p} = \sqrt{2}\pi \left( -\delta_{1p} + \delta_{3p} \right) \), \( N_{3cp} \) is

\[
N_{3cp} = \begin{pmatrix}
0 \\
\delta_{1c} \left( -\delta_{1p} + \delta_{3p} \right) \frac{\pi^2 \beta^2}{\beta_{02}} B0^3 \\
-\delta_{2p} \sqrt{2\pi} \beta B0B1
\end{pmatrix},
\]

where \( B0^3 \) and \( B0B1 \) are nonlinear terms to \( O\left( \varepsilon^{3/2} \right) \), namely \( \psi_{11} \theta_{02} \) and \( \psi_{11} \theta_{01} + \psi_{21} \theta_{11} \) evaluated with the solution \( u_{1cp} \) and \( u_{2cp} \). Now, to solve for \( u_{3cp} \), the RHS of (5.4.9a) must be orthogonal to the zero-eigenvalue eigenvector of the adjoint operator \( \L^* \). Note that for a real matrix, the adjoint is simply the transpose of the matrix. The solvability condition [27] yields the following amplitude equation for \( B0 \)

\[ B0 - \frac{\beta_{11}}{4} B0^3 = 0. \quad (5.4.9b) \]

If \( A0 \) is defined as \( A0 = \frac{\sqrt{\beta}}{2} B0 \), the amplitude equation for \( A0 \) reads as

\[ A0 - A0^3 = 0. \quad (5.4.9c) \]

It is not difficult to see that the Lorenz system is recovered as the leading order solution [27]. Clearly, the solutions of (5.4.9c) are \( A0 = 0 \) and \( A0 = \pm 1 \), which corresponds to the conduction state and the supercritical convective branches, respectively. Therefore, \( u_{1cp} \) for convection state is
\[
\mathbf{u}_{1\text{cp}} &= \begin{pmatrix}
\psi_{1\text{cp}} \\
\theta_{1\text{cp}} \\
\theta_{10p}
\end{pmatrix} = \pm \delta_{1c}\delta_{1p} \begin{pmatrix}
\frac{-\beta}{k} \\
1 \\
0
\end{pmatrix} = \delta_{1c}\delta_{1p} \begin{pmatrix}
\frac{-\beta}{k} \\
1 \\
0
\end{pmatrix}.
\]

Moreover, from (5.4.9a) \( \mathbf{u}_{3\text{cp}} \) is found as

\[
\mathbf{u}_{3\text{cp}} = \delta_{1c}\delta_{1p} \begin{pmatrix}
\frac{-\beta}{k} \\
1 \\
0
\end{pmatrix} - \frac{\beta}{k(Pr+1)} \left( Pr + \frac{\beta}{4} B0^2 \right) B0 \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

\[
+ \delta_{1c}\delta_{3p} \left( \frac{-\beta_1^{23}}{\beta^3 - \beta_1^{33}} \right) \pi^2 \beta^2 B0^3 \begin{pmatrix}
\frac{-\beta^3}{k\beta_1^{23}} \\
1 \\
0
\end{pmatrix} - \delta_{0c}\delta_{2p} \frac{2\sqrt{2}\pi\beta}{2\beta_0^2} B0B1 \begin{pmatrix}
0 \\
1
\end{pmatrix},
\]

which introduces the amplitude correction \( B_2 \). Again, since \( B_2 \) is the amplitude of a real vector, it can be taken as a real function. Also, (5.4.9e) shows that to \( O(\varepsilon^{3/2}) \), \( \mathbf{u}_{313} \) is not simply zero (unlike the lower orders \( \mathbf{u}_{113, u_{213}} \)). Interestingly, \( \psi_{313} \) and \( \theta_{313} \) can be determined explicitly at this order (without need to go to higher orders) as

\[
\begin{aligned}
\theta_{313} &= \left( \frac{-\beta_1^{23}}{\beta^3 - \beta_1^{33}} \right) \pi^2 \beta^2 B0^3 \\
\psi_{313} &= -\frac{\beta^3}{k\beta_1^{23}} \theta_{313}
\end{aligned}
\]

(5.4.9f)

To find \( B_2 \), or equivalently \( \psi_{311} \) and \( \theta_{311} \), the higher order terms should be considered. Also, \( \theta_{30p} \) is found to be
\[ \theta_{0p} = -\delta_2 p \sqrt{2\pi B_0 B_1}. \] (5.4.9g)

where \( B_1 \) is yet to be determined. Upon applying solvability conditions, \( B_1 \) and \( B_2 \) are determined in a similar way when the solutions to \( O(\varepsilon^2) \) and \( O(\varepsilon^{5/2}) \) are considered.

Noting that \( A_1 = \frac{\sqrt{\beta}}{2} B_1 \) and \( A_2 = \frac{\sqrt{\beta}}{2} B_2 \), the following amplitude equations are obtained for \( A_1 \) and \( A_2 \):

\[ (1 - 3A_0^2) A_1 = 0, \] (5.4.10)

\[ (1 - 3A_0^2) A_2 = \frac{2B_2^2 + \beta_{13}^2}{\beta^3 - \beta_{13}^3} A_0^5 + \frac{2}{(\text{Pr}+1)} (\text{Pr}+A_0^2) A_0^3 - A_0, \] (5.4.11a)

where the equation (5.4.10) shows that \( A_1 = 0 \). Recall that \( A_0 \) can be either 0 or \( \pm 1 \). When \( A_0 = 0 \) (the conduction state), \( A_2 \) will be zero, whereas for \( A_0 = \pm 1 \) (convection branches), \( A_2 \) will be equal to

\[ A_2 = \pm \frac{1}{2} \left( 1 + \frac{2B_2^2 + \beta_{13}^2}{\beta^3 - \beta_{13}^3} \right). \] (5.4.11b)

Note that \( \text{Pr} \) does not appear in (5.4.11b). \( u_{3cp} \) can be expressed in terms of \( A_0 \) and \( A_2 \) as

\[ u_{3cp} = \delta_{1c} \delta_{1p} \frac{2}{\sqrt{\beta_{11}}} \begin{bmatrix} A_2 & -\frac{\beta}{k} & 1 \\ 1 & 0 & A_0 \\ 0 & 0 & 0 \end{bmatrix} - 2\delta_{1c} \delta_{3p} \frac{\sqrt{\beta_{13}^2}}{(\beta^3 - \beta_{13}^3)} A_0^3 \begin{bmatrix} \frac{\beta^3}{k \beta_{13}^2} \\ 1 \\ 0 \end{bmatrix}. \] (5.4.11c)

To this order, the non-zero coefficients can be expressed in terms of \( A_0 \) and \( A_2 \) as \( A_1 \) is zero.
\[
\psi_{11} = -2 \frac{\sqrt{\beta}}{k} A_0, \theta_{11} = \frac{2}{\sqrt{\beta}} A_0, \theta_{202} = -\frac{\sqrt{2}}{2\pi} A_0^2,
\]
\[
\psi_{311} = -2 \frac{\sqrt{\beta}}{k} (A_2 + A_0), \theta_{311} = \frac{2}{\sqrt{\beta}} A_2,
\]
\[
\psi_{313} = \frac{2}{k} \left( \frac{\sqrt{\beta} \beta^3}{\beta^3 - \beta_{13}^3} \right) A_0^3, \theta_{313} = -2 \sqrt{\beta} \left( \frac{\beta_{13}^2}{\beta^3 - \beta_{13}^3} \right) A_0^3,
\]
\[
\theta_{402} = -\frac{\sqrt{2}}{2\pi} \left( 2 A_0 A_2 + A_0^2 + \frac{\beta \left( \beta^2 + \beta_{13}^2 \right)}{\beta^3 - \beta_{13}^3} A_0^4 \right), \tag{5.4.11d}
\]
\[
\theta_{404} = \frac{\sqrt{2}}{4\pi} \frac{\beta \left( \beta^2 + \beta_{13}^2 \right)}{\beta^3 - \beta_{13}^3} A_0^4.
\]

Following similar steps, \( u_{4c_p} \), \( u_{5c_p} \) and \( \theta_{60_p} \) coefficients are determined when the solutions to \( O(\epsilon^3) \) and to \( O(\epsilon^{7/2}) \) are considered. Details of which are not explicitly shown here for brevity and the coefficients are listed in the Appendix A. At this stage, it is helpful to give the expressions for the streamfunction and temperature to

\[
\psi(x, z) = \sqrt{2} \left( \epsilon^{1/2} \psi_{11} + \epsilon^{3/2} \psi_{311} + \epsilon^{5/2} \psi_{511} \right) \sin(kx) \sin(\pi z) \\
+ \sqrt{2} \left( \epsilon^{3/2} \psi_{313} + \epsilon^{5/2} \psi_{513} \right) \sin(kx) \sin(3\pi z) \\
+ \sqrt{2} \epsilon^2 \left( \psi_{422} \sin(2\pi z) + \psi_{424} \sin(4\pi z) \right) \sin(2kx) + \\
+ \sqrt{2} \epsilon^5 \left( \psi_{531} \sin(\pi z) + \psi_{533} \sin(3\pi z) + \psi_{535} \sin(5\pi z) \right) \sin(3kx)
\]

and
\[ \theta(x, z) = \sqrt{2} \left( e^{1/2} \theta_{11} + e^{3/2} \theta_{31} + e^{5/2} \theta_{51} \right) \cos(kx) \sin(\pi z) + \sqrt{2} \left( e^{3/2} \theta_{13} + e^{5/2} \theta_{33} \right) \cos(kx) \sin(3\pi z) + \sqrt{2} e^2 \left( \theta_{42} \sin(2\pi z) + \theta_{44} \sin(4\pi z) \right) \cos(2kx) + \sqrt{2} e^{5/2} \left( \theta_{51} \sin(\pi z) + \theta_{53} \sin(3\pi z) + \theta_{55} \sin(5\pi z) \right) \cos(3kx) + \sqrt{2} e^2 \theta_{02} + e^2 \theta_{04} + e^3 \theta_{06} \sin(2\pi z) + \sqrt{2} e^2 \theta_{04} + e^3 \theta_{06} \sin(4\pi z) + \sqrt{2} e^3 \theta_{06} \sin(6\pi z) \] 

\hspace{1cm} \text{(5.4.12b)}

These expressions show the modes required to each desired order. The corresponding Nusselt number is obtained from (5.3.10) as

\[ \text{Nu} = 1 - \sqrt{2} \pi \left( 2 e \theta_{02} + 2 e^2 \left( \theta_{04} + 2 \theta_{04} \right) + 2 e^3 \left( \theta_{06} + 2 \theta_{06} + 3 \theta_{06} \right) \right). \] 

\hspace{1cm} \text{(5.4.13)}

\subsection*{5.4.2 The Lorenz model}

Lorenz arbitrarily limited the expansion of the streamfunction to only one mode. He considered two modes for the temperature deviation. More explicitly, expressions (5.3.3a-b) reduce to

\[ \psi(x, z) = \psi_{11} \sin(kx) \sin(\pi z), \] 

\hspace{1cm} \text{(5.4.14)}

\[ \theta(x, z) = \theta_{02} \sin(2\pi z) + \theta_{11} \cos(kx) \sin(\pi z) . \] 

The corresponding coefficients are found from (5.3.8a-b) upon setting \(a = b = m = n = 1\). It should be noted that \( F_{11,11} = F_{21,11,11} = 0 \), \( E_{121} = -\sqrt{2} \pi \) and \( E_{212} = \sqrt{2} \pi \). Therefore, one has

\[ \psi_{11} = \pm 2 \frac{\sqrt{\beta}}{k} \sqrt{\frac{\varepsilon}{1-\varepsilon}}, \quad \theta_{11} = \frac{2 \sqrt{\varepsilon (1-\varepsilon)}}{\sqrt{\beta}}, \quad \theta_{02} = -\frac{\sqrt{2} \varepsilon}{2\pi} . \] 

\hspace{1cm} \text{(5.4.15a)}

Finally, the Nusselt number is found from (5.3.10) as
\[ \text{Nu} = 1 - 2\sqrt{2\pi\theta_{02}} = 1 + 2\varepsilon. \]  

(5.4.15b)

In order to compare the results of combined spectral-amplitude-perturbation approach with the results of the Lorenz model, it is helpful to examine the Lorenz solution near criticality. Thus, for small \( \varepsilon \), expressions (5.4.15a) are found as

\[ \psi_{11} = \pm 2\frac{\sqrt{\beta}}{k} \sqrt{\varepsilon(1 + \varepsilon)} + O\left(\varepsilon^{3/2}\right), \quad \theta_{11} = m\frac{2}{\sqrt{\beta}} \sqrt{\varepsilon(1-\varepsilon)} + O\left(\varepsilon^{3/2}\right), \theta_{02} = \pm \frac{\sqrt{2}}{2\pi} \varepsilon. \]  

(5.4.16)

### 5.5 Results and discussion

Figure 5.1 shows the variation of the temperature against the horizontal coordinate \( x \) and shows the influence of the leading and higher-order modes on the temperature at \( z = 1/8 \) for \( Pr = 7 \) and \( k = 2.2 \) and \( Ra/Ra_{cr} = 2 \). By comparing the temperature profiles, the effect of higher order modes is found to be negligible especially at the cell center, where the temperature has the maximum value.

Figure 5.2 illustrates the variation of the Nusselt number (to different orders) against the Rayleigh number. The Nusselt number based on the Lorenz model (5.4.15b) is also included for reference. The subscript LM denotes the corresponding solution from the Lorenz model. For \( Ra/Ra_{cr} < 1.2 \) (very close to the criticality), results appear to agree well with each other. Thus, the discrepancy among all formulations near criticality is negligible. As \( Ra/Ra_{cr} \) increases, all results show monotonic increase with respect to the Rayleigh number. Therefore, by using the combined spectral-amplitude-perturbation approach, a high level of consistency is observed in the variation of the Nusselt number (to different orders) against the Rayleigh number. It is apparent, however, that the Nusselt number increases if one considers higher order modes. More precisely, \( \text{Nu}_{O(\varepsilon)} < \text{Nu}_{O(\varepsilon^{2})} < \text{Nu}_{O(\varepsilon^{3})} \), where the subscript shows the order of \( \varepsilon \) to which the Nusselt number is expanded.
Unlike the current study, some inconsistency was seen in the results obtained by Ahmed et al. (2013), particularly in the variation of the Nusselt number against the Rayleigh number. For better comparison, the Nusselt number obtained by Ahmed et al. (2013) [26] to $O(\varepsilon^2)$ is shown in Figure 5.2. At the very close neighbor of the critical Rayleigh number ($Ra/Ra_{cr} < 1.2$), the Nusselt number to $O(\varepsilon)$ has the minimum value among all curves. More precisely, $Nu_{O(\varepsilon)} < Nu_{Ahmed\ et\ al.(2013)} < Nu_{O(\varepsilon^2)} < Nu_{O(\varepsilon^3)}$. As $Ra/Ra_{cr}$ increases ($Ra/Ra_{cr} > 1.2$), $Nu_{Ahmed\ et\ al.(2013)}$ to $O(\varepsilon^2)$ persistently has the minimum value compared with those of calculated based on the combined spectral-amplitude-perturbation approach. Additionally, as $Ra/Ra_{cr}$ exceeds 1.6, $Nu_{Ahmed\ et\ al.(2013)}$ does not reflect a realistic behavior hereafter and decreases for higher Rayleigh numbers. The decreasing trend in the Nusselt number continues, where at $Ra/Ra_{cr} = 2.2$, it becomes less than 1. Following this trend, even negative Nusselt numbers are predictable for higher Rayleigh numbers, which is not physically possible. Moreover, Ahmed et al. (2013)
found a linear trend for the variation of the Nusselt number (to $O(\varepsilon)$) with respect to the Rayleigh number, whereas they obtained a nonlinear trend (parabolic) for the results to $O(\varepsilon^2)$. Recall that in their work, the definition of $\varepsilon'$ (equation 5.1 of [26]) implies that the Rayleigh number should be limited to the narrow neighbor of the critical Rayleigh number ($Ra_{cr}$) ($\varepsilon' < 1$ implies that $Ra/Ra_{cr} < 2$). However, as mentioned earlier, even at this range ($Ra/Ra_{cr} < 2$) as $Ra/Ra_{cr}$ increases above 1.6, Nu has a maximum followed by a monotonic decrease, which is not physically possible. Therefore, one of the advantages of using $\varepsilon$ as equation (5.4.3a) instead of $\varepsilon'$ (equation 5.1 of [26]) is the consistency obtained between the results to different orders.

![Figure 5.2: Effect of higher order modes on the Nusselt number (Pr = 7, k = 2.22).](image)

The influence of the Prandtl number on the Nusselt number is shown in Figure 5.3 for $Ra/Ra_{cr} = 2$. To $O(\varepsilon)$ and $O(\varepsilon^2)$, the Nusselt number is constant for any Prandtl number. However, the values of Nusselt numbers to $O(\varepsilon)$ and $O(\varepsilon^2)$ are different,
where the latter is always larger. Considering modes to $O(\varepsilon^3)$, a narrow range appears for very small Prandtl numbers ($Pr < 0.1$), in which the Nusselt number shows dependency on the Prandtl number. In this range, for Prandtl numbers less than 0.02 the Nusselt number is less than 1, which is physically impossible. As the Prandtl number increases above 0.02, the Nusselt number increases monotonically. However, when the Prandtl number exceeds 0.15, which is the case for a wide range of fluids, the Nusselt number remains constant hereafter. Additionally, it can be seen from Figure 5.3 that $Nu_{O(\varepsilon)} < Nu_{O(\varepsilon^2)} < Nu_{O(\varepsilon^3)}$, which is confirmed from Figure 5.2, where the Nusselt number increases considering higher order modes. The variation of the Nusselt number to $O(\varepsilon^3)$ against the Prandtl number is reminiscent of that obtained from the extended Galerkin–Eckhaus method. The reader is referred to Figure 5.4 in the work by Dondlinger et al. (2007) [29], where at small Prandtl numbers, a monotonic increase was observed for the variation of the Nusselt number against the Prandtl number. They found that at higher Prandtl numbers, the effect of the Prandtl number died out and the Nusselt number became constant. The dependency of the Nusselt number on the Prandtl numbers can also be predicted from equations (5.3.7a) and (5.4.1). Note that at lower orders ($< O(\varepsilon^{5/2})$), $Cl_{abc,mnp}$ is always zero and as a result, changing the Prandtl number does not affect the Nusselt number. Considering higher order modes ($O(\varepsilon^{5/2})$ and higher), $Cl_{abc,mnp}$ becomes non-zero for some combinations of $abc,mnp$ and consequently, the Prandtl number can affect the results. However, it should be noted that even for non-zero $Cl_{abc,mnp}$, the RHS of (5.3.7a) is proportional to the inverse of the Prandtl number, which implies that the effect of the Prandtl number is more prominent for smaller values. The current approach suggests that, as $Pr$ exceeds 0.15, the effect of Prandtl number will be negligible.
In order to compare with the results of Ahmed et al. (2013) [26], the variation of the Nusselt number against the Prandtl number, based on their approach, is shown in Figure 5.3. To simplify, only the Nusselt number to $O\left(\varepsilon^3\right)$ is considered in this figure and the lower orders are not shown. In this case, Figure 5.3 shows the same trend for the variation of the Nusselt number against the Prandtl number for both approaches. Again, $\text{Nu}_{\text{Ahmed et al. (2013)}}$ to $O\left(\varepsilon^2\right)$, which is not shown here, is independent of the Prandtl number, whereas to $O\left(\varepsilon^3\right)$, their approach predicts a monotonic increase in the Nusselt number for lower Prandtl numbers. Their formulation predicts that as the Prandtl number exceeds a limit ($\text{Pr} = 0.2$), the Nusselt number becomes constant, but with a higher value compared with the Nusselt numbers calculated in the current paper.

The influence of the leading and higher-order modes on the streamfunction is shown in Figure 5.4, where it depicts the streamfunction against $x$ at $z = 1/8$ for $\text{Pr} = 7$ and $k = 2.2$ and $\text{Ra}/\text{Ra}_{cr} = 2$. Including higher order modes leads to an increase (decrease) in the
maximum (minimum) value of the streamfunction. Moreover, no noticeable change is observed between the streamfunction curves to $O\left(\varepsilon^{3/2}\right)$ and $O\left(\varepsilon^2\right)$, which emphasizes the fact that the non-zero streamfunction coefficients to $O\left(\varepsilon^2\right)$, namely $\psi_{422}$ and $\psi_{424}$, are negligible compared to the lower order coefficients.

Further insight into the flow and temperature fields is inferred from Figures 5.5a and 5.5b, respectively. In Figure 5.5a, $\psi(x, z = 1/8)$ is plotted against $x$ for $Pr = 7$ and $k = 2.2$. It shows that at higher Rayleigh numbers, the maximum (minimum) value of the streamfunction increases (decrease). However, increasing the Rayleigh number has a little influence on the streamfunction at the edges of the cell ($x = 0$ and $2\pi/k$) and the cell center. The temperature profile is illustrated against $x$ for $Pr = 7$ and $k = 2.2$ in Figure 5.5b. In this case, as $Ra/Ra_c$ increases, the maximum value of the temperature decreases, however, unlike the streamfunction, the difference between the maximums at different Rayleigh numbers is not noticeable. It should be noted that the maximum of the temperature occurs at the cell center. Also, the difference between the temperature profiles at different Rayleigh numbers is found to be noticeable at the edges of the cell ($x = 0$ and $2\pi/k$).
In Figure 5.6, the horizontal temperature gradient, \( \partial_x T \), is represented at midheight of the layer as a function of the horizontal coordinate, \( x \). In order to validate our approach, the analytical results are compared with the experimental results obtained by Dauby et al. (2001) [33]. Interestingly, the results in Figure 5.6 show that if modes to \( O(\varepsilon^{5/2}) \) are considered, the qualitative agreement between our model and the experiments of [33] is very good. It means that the second and third order spatial harmonics should be considered if the formulation is used far from the linear threshold. Recall that from Figure 5.1, the effect of higher order modes on the temperature profile is found to be negligible (especially around the cell center), whereas, regarding the temperature gradient, higher order modes will be very important. It should be noted that assuming slip condition at both boundaries (upper and lower) simplifies the solution procedure. However, the flow and heat transfer characteristics of Rayleigh-Benard convection remain qualitatively the same whether rigid-rigid or free-free conditions are used [28].
Figure 5.5: Streamfunction vs. $x$ to the order of $O(\varepsilon^{5/2})$ (a) and Temperature vs. $x$ to the order of $O(\varepsilon^{5/2})$ (Pr = 7, $k$ = 2.22) (b).
5.6 Conclusion

A combined spectral-amplitude-perturbation approach is developed to simulate the post critical convective state for thermo-gravitational instability in a Newtonian fluid layer heated from below. Most of the existing literature for the methodologies to capture nonlinear convection is mostly limited to very close neighbor of the critical Rayleigh number (weakly post critical range of Rayleigh number). Interestingly, the proposed approach can be used even for the Rayleigh numbers far from criticality. The results were compared to experiments and a very good qualitative agreement is observed. Our work provides a systematic mode selection (to different orders) for the convection regime and the appearance of higher order spatial harmonics and the interactions between the different modes can be predicted.

The effects of higher order modes (to $O(\epsilon^{5/2})$) are studied in the current paper, whereas only modes to $O(\epsilon^{3/2})$ were considered in (Ahmed et al., 2013). Considering higher order modes reveals that small Prandtl number (Pr < 0.1) can change (reduce) the Nusselt
number significantly. It is shown that the effect of Prandtl number dies out as it increases (Pr > 0.1), which is the case for most of practical fluids. In addition, calculation of horizontal temperature gradient shows that for a good qualitative agreement with experimental results of Dauby et al. (2001), modes to $O(\varepsilon^{5/2})$ should be considered, where the second and third order spatial harmonics found to play an important role. Moreover, by using the current approach, consistency is observed in the results, particularly in the variation of the Nusselt number against the Rayleigh number to different orders. Finally, the current study suggests that in order to have accuracy to $O(\varepsilon)$, 5 modes must be included. For more accuracy to $O(\varepsilon^{3/2})$, $O(\varepsilon^2)$ and $O(\varepsilon^{5/2})$, one must consider 9, 17 and 29 modes, respectively.

### 5.7 References


Chapter 6

6 Conclusions and Recommendations

6.1 Conclusions

In this thesis, some typical conditions where non-Fourier effects can be significant for solids and fluids are explained. The formulation for non-isothermal flow is based on the generalized Cattaneo-Vernotte equation for single-phase-lagging (SPL) fluids with only a relaxation time and dual-phase-lagging (DPL) fluids with additional retardation time. The relaxation time and retardation time reflect the delay in the response of the heat flux and the temperature gradient with respect to one another, respectively. The limit of a single-phase-lagging (SPL) fluid is recovered upon setting the retardation time to zero. The SPL model is particularly relevant to low-temperature liquid helium or to nanofluids with high nanoparticle concentration.

Fourier equation of heat conduction results in a parabolic equation for temperature field when combined with the conservation of energy law. The parabolic temperature equation implies an infinite speed for heat propagation, which is in good agreement with experimental data in many situations. However, as shown in this thesis, there are some cases in which the Fourier equation is not adequate to describe the heat conduction process. For instance, it is shown that the transient heat conduction in a sample of processed meat cannot be predicted by Fourier’s law. The experiment on the heat conduction in processed meat displays the wave nature of heat, where unlike the prediction of Fourier’s law, heat is not felt instantly at distant points. More precisely, heat waves take a finite time to reach a particular point inside the meat sample. This experimental observation is confirmed by the results from the Cattaneo equation of heat conduction (non-Fourier), where a new time constant (the relaxation time) is introduced in the transient term that is added to the conventional Fourier equation.

The thermal relaxation time is on the order of $10^{-12}$ to $10^{-15}$ seconds for many applications [1], and consequently, the Cattaneo equation reduces to the classical Fourier model. However, in some situations the relaxation time (or the non-Fourier character of the
material) is noticeable such as heat transfer in the processed meat (relaxation time \( \approx 16 \) secs). The dimensionless number that includes the relaxation time is introduced as the Cattaneo number, \( C \), which shows that non-Fourier effects could become important if the sample has large relaxation time and/or large thermal diffusivity. Also, the Cattaneo number suggests that the importance of non-Fourier heat conduction increases with decrease in the length scale (such as nano-scale problems).

It is shown that if a temperature pulse is applied on a limited region of a solid sample (that has a non-Fourier character), for points that are not initially affected by the temperature pulse, a time lag (delay) is needed for the temperature pulse to be felt. The longer the distance from the heat source, the more time is needed for the temperature pulse to be felt. The velocity of heat propagation is found as \( \sqrt{1/C} \), where \( C \) is the Cattaneo number. On the contrary, based on the Fourier’s law of heat conduction (\( C = 0 \)), the temperature pulse propagates with an infinite speed through the entire sample. Therefore, Fourier’s law of heat conduction predicts that the pulse is felt instantly everywhere, though with exponentially small amplitudes at distant points. Moreover, the wavy nature of heat conduction in solids is clearly illustrated in solids with high Cattaneo numbers, where oscillations are observed in the temperature profiles with respect to time and position. However, because of thermal diffusion effect, these oscillations are damped, and after some time (which depends on the Cattaneo number), the steady state condition is reached. Note that based on the Fourier’s law of heat conduction, unless a periodic heat flux (or temperature) is applied on the boundary(s) of the sample, no oscillations can be observed in the temperature profiles. To summary, the Cattaneo equation of heat conduction has several desirable properties. In contrast to the Fourier equation of heat conduction, Cattaneo equation of heat conduction transmits waves of heat with a finite speed of \( \sqrt{1/C} \). Moreover, the heat waves are attenuated as a result of diffusion, and unlike the pure heat wave equation, steady heat condition can be obtained.

It is shown that non-Fourier effects can be significant in liquids as well. In this case, non-Fourier effects (second-sound waves) were studied both experimentally \([2,3]\) and numerically \([4]\) in liquid helium at low temperatures \( T < 2.17 \) K. As mentioned
previously, the second-sound is a heat transfer mechanism describing the propagation of heat as a wave. In addition, nanofluids also exhibit non-Fourier effects [5]. In contrast to an ordinary liquid, when a heat pulse is applied to LHe II, the original temperature pulse maintains its shape until it is received by the detector. Therefore, the velocity of the second sound wave is independent of the frequency, and since the diffusion effect is very small in LHe II, the attenuation is also very weak. The independency from the frequency is also shown for solids, where the second sound wave (heat wave) velocity is found to be $\sqrt{1/C}$, where $C$ is the Cattaneo number and has no dependency on the frequency of oscillations.

Using the two fluid model, the propagation of heat as a wave was explained in LHe II [6], [7]. In this model, LHe II is assumed to comprise the normal and the superfluid components. However, these components are chemically indistinguishable and LHe II cannot be separated into its components by any method. In this thesis, the thermal convection of a non-Fourier fluid, such as LHe II, is studied based on the Navier-Stokes equations, where the Fourier equation of heat conduction is replaced by the Cattaneo equation (in the energy equation). The current approach is mathematically more manageable and can be used as an alternative to the two fluid model to study the heat transfer in LHe II. Using the current approach, the stability picture of a fluid that has a non-Fourier character (such as LHe II) is obtained.

Linear stability analysis shows that for small relaxation time (small C), the neutral stability curve comprises a Fourier branch ($k < k_i$) and an oscillatory branch ($k > k_i$). As C increases and reaches a critical value, $C_H$, both stationary and oscillatory convection become equally probable. It confirms the existence of the bistable mode observed in experiment [8]. The results of the linear stability analysis show that the oscillatory convection (overstability) occurs at the onset for a fluid with high non-Fourier character. In addition, the oscillatory threshold (critical Rayleigh number) for a fluid with sufficiently large non-Fourier character (large Cattaneo number $C > C_H$) is found to be less than the threshold for the onset of stationary Rayleigh-Benard convection for an ordinary fluid. In other words, for $C > C_H$, only oscillatory convection is predicted, at a Rayleigh number decreasing with C. Moreover, the frequency of oscillations is shown to
be proportional to the second-sound velocity in LHe II. The advantage of the current method (compared to the two fluid model) is that there is no need to assume LHe II comprises two components that are chemically indistinguishable. An equation for the prediction of the second-sound wave in LHe II is obtained, which by using the experimental values of the second-sound wave at low temperatures [9], the Cattaneo number (and hence the relaxation time) for the liquid helium II is calculated. Interestingly, the relaxation time of LHe II is found to be very large (O(10^{-2}-10^{-3} \text{secs})) compared to that of the ordinary fluids (such as water). Consequently, unlike the ordinary fluids where very small-scale dimensions are needed to observe the non-Fourier effects, LHe II shows the non-Fourier character even at large scales (O(1 cm)).

As mentioned previously, linear stability analysis of the Rayleigh-Benard problem shows that the critical Rayleigh number for the oscillatory convection is lower than the Fourier level (for sufficiently large Cattaneo numbers $C > C_H$). In this case, the critical Rayleigh number diminishes like $\sim 1/C$, suggesting that convection may be spontaneously observed (without a conduction phase), for any temperature differential, for large $C$. On the other hand, the counterflow of superfluid component and normal component of liquid helium is somewhat analogous to convection currents in a fluid. Consequently, the conduction process in LHe II (based on the two fluid model) is similar to convection in ordinary fluids, which can explain the significant thermal conductivity of LHe II compared to ordinary liquids [7]. Therefore, the results obtained from the non-Fourier heat transfer analysis can be confirmed from the two fluid model, where in both cases, the thermal convection can occur without a conduction phase (for a very large $C$). To summarize, the oscillatory convection (overstability) is expected to be observed at the onset for a fluid with high non-Fourier character with the oscillatory threshold less than the threshold for the onset of stationary Rayleigh-Benard convection for a fluid with sufficiently large non-Fourier character (large Cattaneo number).

It is shown that based on the Cattaneo equation of heat conduction, non-Fourier effects are only important in the transient time for solids, and once the steady state condition is established, non-Fourier effects vanish. Since non-Fourier effects in solids do not appear in steady heat conduction, most applications of non-Fourier heat transfer in the literature
were limited to transient heat conduction. However, when flow is present, non-Fourier effects can appear under steady conditions. This is one of many interesting phenomena, which do not arise in standard hydrodynamics such as the emergence of heat flux in flow direction without temperature gradient [10-14]. For the convection in the vertical slot (Chapter 4), the heat flux in the y direction is present, even when there is no temperature gradient in that direction. Therefore, non-Fourier effects could become important in fluids even for the base flow solution (the steady state condition).

In addition, the mechanism of instability is investigated through an examination of the disturbance-energy equation. It is found that the instability mechanism is associated with the coupling of the perturbation velocity and the perturbation temperature. Also, energy growth for small two-dimensional perturbations to Rayleigh-Benard problem for a non-Fourier fluid has been investigated. The eigenfunctions found to be non-orthogonal, which implies that there can be a substantial transient growth in the energy of small perturbations. The transient growth occurs even for the stable regime. The growth function G(t) has been calculated, which gives the maximum potential growth for small perturbations. This growth occurs in the absence of nonlinearities.

If a layer of non-Fourier fluid is confined between two (infinite) vertical plates, for Pr = 1 (small Prandtl number), linear stability analysis indicates a decreasing trend for the critical Grashof number with respect to the Cattaneo number, except for a very small region in which the critical Grashof number increases. For Pr = 7.5 (intermediate Prandtl number), as Cattaneo number exceeds a certain value (0.004), a SPL fluid can lose its conductive mode to stationary or oscillatory convection. In this case, for Cattaneo numbers less than 0.004, no change is observed in the marginal stability curves compared with that of a Fourier fluid. In contrast, for Cattaneo numbers more than 0.004, the neutral stability curve comprises a Fourier branch (higher wavenumbers) and an oscillatory branch (lower wavenumbers). It is anticipated that oscillatory convection increasingly becomes the mode of preference, compared to both conduction and stationary convection. Moreover, a qualitative similarity is observed between the
marginal stability curves here and the results for flow inside a hollow cylinder with a radial temperature gradient [15].

In addition to liquid helium at very low temperatures (T < 2.17 K), nanofluids have also non-Fourier character (Pr ≈ 7). The relevance of the DPL model to nanofluids (NFs) has recently been recognized in the literature [16], and is emphasized in Chapter 2. The equivalence between the two-phase and DPL models allows the expression of the relaxation time in terms of the nanoparticle (NP) concentration [16]. Two non-dimensional groups are introduced for NFs (in Chapter 2), the Cattaneo number, C, and dimensionless retardation time, S. C and S are both related directly to the NP concentration. The retardation-to-relaxation time ratio is found to be equal to the NF solution-to-solvent thermal conductivity ratio, \( \frac{\gamma}{C} \). Since \( S = \frac{C}{K} \), S must always be smaller than C for a NF. In this thesis, only SPL (S = 0) fluids are studied. This limit case has multiple advantages over a fluid with retardation. It serves as reference case for a fluid with strong non-Fourier character, is mathematically more manageable, and can, in the limit, reflect the behaviour of a NF with very high NP concentration. However, real NFs tend to possess non-zero retardation, which leads to a non-zero minimum critical Rayleigh number for large C, above which convection sets in. In this case, conduction is always present, but its range of temperature differential may decrease significantly with NP concentration.

It is shown in this thesis that if the onset of convection occurs at a lower Rayleigh number (Grashof number) for the Rayleigh-Benard configuration (vertical slot) compared to that of for a Fourier fluid, it signals the possibility of existence of non-Fourier effects. This phenomenon is observed experimentally [17], where the onset of convection for a nanofluid comprised of water and Al_2O_3 (\( \Phi = 2.72\% \)) and confined between two vertical plates occurs at a lower Rayleigh number compared to that of a pure water (Figure 3 in [17] and \( \theta=90^\circ \)). From Figure 3 in [17], the ratio of critical Rayleigh number for NF over the critical Rayleigh number for the pure water can be read as 0.75. Using Figure (4.7), it can be seen that for C ≈ 0.0045 the ratio of critical Grashof number for a non-Fourier fluid over the critical Grashof number for a Fourier fluid is 0.75. Recall that if the
conduction is lost to oscillatory convection, this can also signal the importance of non-Fourier effects. In this case, transient oscillatory convection is observed [8] at the onset for a layer of nanofluid confined between two horizontal plates (Rayleigh-Benard convection). In a recent experimental study, oscillatory convection is observed [18] for nanofluids (between two horizontal planes) with high thermophilic nanoparticles. For nanofluids, the oscillatory convection is resulted from the competition between two opposing factors: buoyancy (destabilizing) and thermophoresis (stabilizing).

Finally, a combined spectral-amplitude-perturbation approach is developed to simulate the post critical convective state for thermo-gravitational instability in a Newtonian/Fourier fluid layer heated from below. Most of the existing literature for the methodologies to capture nonlinear convection is mostly limited to very close neighbor of the critical Rayleigh number (weakly post critical range of Rayleigh number). Interestingly, the proposed approach can be used even for the Rayleigh numbers far from criticality. The results were compared to experiments and a very good qualitative agreement is observed. Our work provides a systematic mode selection (to different orders) for the convection regime and the appearance of higher order spatial harmonics and the interactions between the different modes can be predicted. In addition, calculation of horizontal temperature gradient shows that for a good qualitative agreement with experimental results of Dauby et al. [19], modes to $O(\varepsilon^{5/2})$ should be considered, where the second and third order spatial harmonics found to play an important role. Moreover, by using the current approach, consistency is observed in the results, particularly in the variation of the Nusselt number against the Rayleigh number to different orders.

6.2 Recommendations for future work

Thermal convection of fluids with non-Fourier effects have been studied in this thesis. In order to further advance knowledge in this area, the following directions may be considered for the future work:

i. Only the onset of convection (for non-Fourier fluids) is studied in this thesis. Therefore, it is not clear that if non-Fourier effects enhance the natural convection heat transfer or not. In other words, from the stability picture, nothing can be
concluded for the Rayleigh (Grashof) numbers exceed the critical value. Therefore, the post critical convective state for thermo-gravitational instability in a non-Fourier fluid layer heated from below should be examined to predict the variation of Nusselt number with respect to the Cattaneo number (C). In this thesis, the destabilization effect resulted from the non-Fourier character of a fluid is clearly shown in the stability pictures (provided for both Rayleigh-Benard convection and convection in the vertical slot).

ii. In Chapter 5, a systematic mode selection approach (to different orders) for the convection regime is developed. This chapter was limited to the Fourier fluids (to explain the concept of the approach). It is recommended to apply this approach to the problem of thermal convection of non-Fourier fluids to find the important modes to different orders. Consequently, the post critical convective state for a non-Fourier fluid can be investigated analytically, and the variation of Nusselt number with respect to Cattaneo number can be obtained.

iii. The systematic mode selection approach (Chapter 5) was also limited to the steady state and free-free boundary conditions. It is recommended to extend this approach to the transient problems. In addition, using other polynomials (such as Chandrasekhar polynomials), the approach can be employed for rigid-rigid boundary conditions as well.

iv. In this thesis, the SPL model is used to model the non-Fourier heat transfer in fluids. This model provides us with a good knowledge about the linear stability picture of the conduction state. However, there are other models to describe the non-Fourier heat conduction, which are more complicated. Among these models, DPL model is explained in this thesis (Chapter 2), which introduces two time constants (instead of one time constant) in the energy equation. It is recommended to use other models of non-Fourier heat transfer (such as DPL), and compare the results obtained from different models.

v. Another direction to develop the current study is considering the non-Newtonian effects in thermal convection. The problem of thermal convection for a non-Newtonian fluid is well studied [20], [21]. However, there are some situations where the non-Fourier and non-Newtonian effects are simultaneously present
such as rarefied gases with high Knudsen numbers [22]. Therefore, the interaction between the thermal relaxation and viscous (stress) relaxation should lead to a rich physical problem.

vi. Based on the results obtained in this thesis, an experimental study can be proposed to measure the relaxation time for different fluids.

6.3 References


Appendix A

The constants used in the expressions (5.4.12-5.4.13) of different stream function and temperature coefficients are shown below.

\( \mathcal{O}(\varepsilon^2): \)

\[
\begin{align*}
\mathbf{u}^4_{cp} &= \delta_{ic} \delta_{lp} \frac{2}{\sqrt{\beta_{11}}} \begin{pmatrix}
\frac{\beta_{11}}{k} \\
1 \\
0
\end{pmatrix} \\
+ &2\delta_{2c} \delta_{2p} \psi_{313} \psi_{11} \begin{pmatrix}
-Q_{4222} \\
Q_{4122} \\
0
\end{pmatrix} + \delta_{2c} \delta_{4p} \psi_{313} \psi_{11} \begin{pmatrix}
Q_{4224} \\
-Q_{4124} \\
0
\end{pmatrix} \\
+ &\delta_{0c} \delta_{2p} \frac{\sqrt{2\pi\beta}}{2\beta_{02}} \begin{pmatrix}
1 + \frac{\beta^2}{\beta_{13}^2} \\
0
\end{pmatrix} \begin{pmatrix}
\theta_{313} - 2\theta_{311} - 2\theta_{l11} \\
\theta_{l11}
\end{pmatrix} \\
- &\delta_{0c} \delta_{4p} \frac{\sqrt{2\pi\beta}}{\beta_{04}} \begin{pmatrix}
1 + \frac{\beta^2}{\beta_{13}^2} \\
0
\end{pmatrix} \begin{pmatrix}
\theta_{313} \theta_{l11} \\
0
\end{pmatrix} \\
\end{align*}
\]

(A.1)

where

\[
Q_{42cp} = \frac{\beta_{cp}}{ck} Q_{41cp} + \frac{\sqrt{2\pi k}}{2c\beta} \begin{pmatrix}
1 - \frac{\beta_{13}^2}{\beta^2}
\end{pmatrix}
\]
$$Q41_{cp} = \frac{Pr^{-1} \sqrt{\frac{2\pi}{2c}} (\beta - \beta_{13}) + \sqrt{\frac{2\pi \beta_{cp}^2}{2c^2 \beta}} \left(1 - \frac{\beta_{13}^2}{\beta^2}\right)}{Ra_{cr} - \frac{\beta_{cp}^3}{c^2 k^2}}$$

(A.2)

and

$$\left(1 - 3A0^2\right) A3 = 0$$ which shows that $A3 = 0$.

$O(\epsilon^{5/2})$:

$$u5_{cp} = \delta_{1c} \delta_{1p} \left( \begin{array}{c} \frac{2}{\sqrt{\beta} A4} \frac{-\beta}{k} \\ -1 \\ 0 \end{array} \right) \left( \begin{array}{c} \frac{\theta3_{11} + \theta1_{11}}{0} \end{array} \right)$$

$$-\left(\delta_{1c} \delta_{3p} + \delta_{1c} \delta_{5p} + \delta_{3c} \delta_{1p}\right) \frac{1}{Ra_{cr} - \frac{\beta_{cp}^3}{c^2 k^2}} \psi3_{13}$$

where

$$A4 = -\frac{2k}{\beta \sqrt{\beta}} \psi5_{13} + \frac{1}{\sqrt{\beta}} \theta5_{13} - \frac{2k}{\beta \sqrt{\beta}} R5_{11} \psi3_{13} - \frac{2}{\sqrt{\beta}} \theta3_{11} - \frac{3k}{4\beta \sqrt{\beta}} \psi1_{11} - \frac{2}{\sqrt{\beta}} \theta1_{11}$$
\[
R_{511} = \frac{1}{2} \left\{ -\Pr^{-1} \sqrt{2\pi} \frac{4}{k} \beta_{13} \left( 2(\beta_{22} - \beta_{13})Q_{4222} + \frac{1}{2}(\beta_{24} - \beta_{13})Q_{4224} \right) + \frac{1}{\Delta_{13}} \left( \frac{1}{4} \beta_{13}^4 - \frac{\beta_{13}^2}{\beta^2} - 1 \right) + \frac{\beta_{13}^2}{\beta^2} \\
+ \sqrt{2\pi} \frac{4\beta}{k^2 \Delta_{13}} \left( -2Q_{4122} - \frac{1}{2} Q_{4124} - 2Q_{4222} \frac{\beta_{13}^2}{\beta^3} - \frac{1}{2} Q_{4224} \frac{\beta_{13}^2}{\beta^3} \right) \right\}
\]

\[
R_{5113} = \frac{\beta_{13}^2}{k} + \Pr^{-1} \frac{4\beta}{k^2} \sqrt{2\pi} \left( -2(\beta_{22} - \beta)Q_{4222} - \left( 2\beta_{12} - \frac{1}{2} \beta \right)Q_{4224} \right)
\]

\[
R_{5213} = \frac{\Delta_{13}}{2} \left( 1 - \frac{\beta_{13}^2}{2\beta^2} \right) + \sqrt{2\pi} \frac{4\beta}{k^2} \left( 2Q_{4122} + \frac{2k}{\beta} Q_{4222} + \frac{1}{2} Q_{4124} + \frac{k}{2\beta} Q_{4224} \right)
\]

\[
R_{5115} = \Pr^{-1} \frac{12\sqrt{2}\beta \pi}{k^2} \left( 2\beta_{12} - \frac{1}{2} \beta \right)Q_{4224},
\]

\[
R_{5215} = -\left( 2 + \frac{\beta_{13}^2}{\beta^2} + \frac{6\sqrt{2}\beta \pi}{k^2} \left( Q_{4124} + \frac{k}{\beta} Q_{4224} \right) \right)
\]

\[
R_{5131} = \Pr^{-1} \frac{8\sqrt{2}\beta \pi}{3k^2} (\beta_{22} - \beta)Q_{4222},
\]

\[
R_{5231} = \frac{8\sqrt{2}\beta \pi}{3k^2} \left( -Q_{4122} + \frac{k}{\beta} Q_{4222} \right)
\]

\[
R_{5133} = -\Pr^{-1} \frac{4\sqrt{2}\beta \pi}{k^2} \left( 2\beta_{12} - \frac{1}{2} \beta \right)Q_{4224},
\]

\[
R_{5233} = -\frac{2\sqrt{2}\beta}{k^2} \pi \left( -Q_{4124} + \frac{k}{\beta} Q_{4224} \right)
\]

\[
R_{5135} = \Pr^{-1} \frac{4\sqrt{2}\beta \pi}{3k^2} \left( 2\beta_{12} - \frac{1}{2} \beta \right)Q_{4224},
\]
\[ R_{52}^{35} = \frac{2\sqrt{2}\beta\pi}{3k^2} \left( -Q41_{24} + \frac{k}{\beta} Q42_{24} \right) \]  \quad (A.4)

O(\varepsilon^3):

\[ \theta_{602} = \frac{\sqrt{2}k}{8\pi} \left( \psi_{111} \theta_{511} - \psi_{111} \theta_{513} + \psi_{311} \theta_{311} - \psi_{311} \theta_{313} \right) \]  \quad (A.5)

\[ \theta_{604} = \frac{\sqrt{2}k}{16\pi} \left( \psi_{111} \theta_{513} - \psi_{111} \theta_{515} + \psi_{311} \theta_{313} + \psi_{313} \theta_{311} + \psi_{513} \theta_{011} - \psi_{515} \theta_{011} \right) \]  \quad (A.6)

\[ \theta_{60p} = \frac{\sqrt{2}k}{24\pi} \left( \psi_{111} \theta_{515} + \psi_{313} \theta_{313} + \psi_{515} \theta_{011} \right) \]  \quad (A.7)

\[ Q61_{02} = \frac{\sqrt{2}k}{8\pi} \left( \theta_{51} - \frac{k}{\beta} \psi_{51} - \theta_{53} + \frac{k}{\beta} \psi_{53} - \frac{\theta_{31}}{2} - \frac{k}{\beta} \psi_{31} + \frac{k}{\beta} \psi_{11} \right) \]

\[ Q62_{02} = -\frac{\sqrt{2}k}{8\pi} \left( 1 + \frac{1}{2} \frac{\beta^2}{\beta^2} \right) \theta_{313} \]

\[ Q61_{04} = \frac{\sqrt{2}k}{16\pi} \left( \theta_{51} - \frac{k}{\beta} \psi_{51} - \theta_{53} + \frac{k}{\beta} \psi_{53} + \frac{\theta_{31}}{2} + \frac{k}{2\beta} \psi_{31} \right) \]

\[ Q62_{04} = \frac{\sqrt{2}k}{16\pi} \left( 1 + \frac{1}{2} \frac{\beta^2}{\beta^2} \right) \left( \theta_{31} - \frac{k}{\beta} \psi_{31} \right) \]

\[ Q61_{06} = \frac{\sqrt{2}k}{24\pi} \left( \theta_{51} - \frac{k}{\beta} \psi_{51} \right) \]

\[ Q62_{06} = \frac{\sqrt{2}k}{24\pi} \theta_{313} \]  \quad (A.8)
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