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Rationality of the spectral action for Robertson-Walker metrics and the geometry of the determinant line bundle for the noncommutative two torus

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics

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Rationality of the spectral action for Robertson-Walker metrics and the geometry of the determinant line bundle for the noncommutative two torus

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by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
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Abstract

In noncommutative geometry, the geometry of a space is given via a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. Geometric information, in this approach, is encoded in the spectrum of D and to extract them, one should study spectral functions such as the heat trace $\mathrm{Tr}(e^{-tD^2})$, the spectral zeta function $\mathrm{Tr}(|D|^{-s})$ and the spectral action functional, $\mathrm{Tr} f(D/\Lambda)$.

The main focus of this thesis is on the methods and tools that can be used to extract the spectral information. Applying the pseudodifferential calculus and the heat trace techniques, in addition to computing the newer terms, we prove the rationality of the spectral action of the Robertson-Walker metrics, which was conjectured by Chamseddine and Connes. In the second part, we define the canonical trace for Connes' pseudodifferential calculus on the noncommutative torus and use it to compute the curvature of the determinant line bundle for the noncommutative torus. In the last chapter, the Euler-Maclaurin summation formula is used to compute the spectral action of a Dirac operator (with torsion) on the Berger spheres $\mathbb{S}^3(T)$.

Keywords: Robertson-Walker metrics, Dirac operator, Spectral action, Heat kernel, Local invariants, Pseudodifferential calculus, Determinant line bundle, Spectral triple, Euler-Maclaurin summation formula

Co-Authorship

This thesis incorporates material that is result of joint research, as follows:

- Chapter 2 is based on the paper
  Rationality of Spectral Action for Robertson-Walker Metrics, JHEP 12 (2014) 064, arXiv:1407.5972.
  which is the outcome of a joint research undertaken in collaboration with Dr. Farzad Fathizadeh under the supervision of Professor Masoud Khalkhali.

- Chapter 3 is also based on the paper
  The Curvature of the Determinant Line Bundle on the Noncommutative Two Torus, *arXiv:1410.0475 [math.QA]*, Oct 2014.
  which is the outcome of a joint research undertaken in collaboration with Ali Fathi under the supervision of Professor Masoud Khalkhali.

To my wife, Akram,

for letting me into her wonderful dreams.

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Contents

Preface

Noncommutative geometry is a rapidly developing field with extensive applications in other fields of modern mathematics as well as physics. In this new paradigm of geometry, proposed by the great Fields Medalist Alain Connes, the metric $g_{\mu\nu}$ is exchanged for the Dirac operator D. Geometric information, in this approach, is encoded in the spectrum of D. To extract this information one should study a spectral function like the spectral action

$$\mathrm{Tr} f(D/\Lambda),$$

where f is an even positive real-valued function and $\Lambda > 0$ is the mass scale. An outstanding feature of the spectral action defined for noncommutative geometries is that it derives the Lagrangian of the physical models from simple noncommutative geometric data (see section 1.5.1). Moreover, its asymptotic expansion as $\Lambda \to \infty$ is very related to the asymptotic expansion of the heat trace of D^2 which, in the classical case, has the following form:

$$\mathrm{Tr}(e^{-tD^2}) \sim t^{-\dim(M)/2} \sum_{n\geq 0} a_{2n}(D^2) t^n \qquad (t \to 0).$$

The constants $a_{2n}(D^2)$ can be written as $a_{2n}(D^2) = \int_M a_n(x, P) \mathrm{dvol}_g$, where $a_{2n}(x, D^2)$ are local invariants of the jets of the total symbol of D^2. Universal local formulas for heat trace asymptotics of a Laplace type operator are a robust tool to compute these local invariants. However, these formulas are available only up to a_{10}, so developing more tools and methods is necessarily. The main focus of this thesis is on the study of the spectral invariants of spaces, either commutative or noncommutative, and the related tools and methods.

While full of fresh ideas and new paradigms, noncommutative geometry is rooted in the heart of modern mathematics of the 20^{th} century such as index theory, spectral geometry and spin geometry. In the first half of the first chapter, we explore the main ideas from the classical theories and their related tools which play a role in the developments of noncommutative geometry, especially the spectral aspects. This part includes a quick review of pseudodifferential operators, spin geometry and spectral geometry. The second half of this chapter is devoted to the basics of the theory of spectral triples and axioms of noncommutative geometry, as well as the developments of the notions of action and symmetries in noncommutative geometry.

In chapter 2, we use pseudodifferential calculus and heat kernel techniques to prove a conjecture by Chamseddine and Connes on rationality of the coefficients of the terms a_{2n} in the expansion of the spectral action of Robertson-Walker metrics. The (Euclidean) Robertson-Walker metric with the cosmic scale factor $a(t)$ is given by

$$ds^2 = dt^2 + a^2\,(t)\,d\sigma^2,$$

where $d\sigma^2$ is the round metric on the 3-sphere $\mathbb{S}^3$. A detailed study of the spectral action for the Robertson-Walker metrics was initiated by Chamseddine and Connes, where by devising a direct method based on the Euler-Maclaurin formula and the Feynman-Kac formula, the terms up to a_{10} in the expansion are computed. Here, a_{2n} denotes $\int_{S^3_{a(t)}} a_{2n}(x, D^2)\mathrm{dvol}$ and depends only on $a(t)$ and its derivatives. They conjectured that a_{2n} are rational polynomials in $a(t)$ and its derivatives divided by some power of $a(t)$. We used pseudodifferential calculus and heat kernel techniques to prove that the term a_{2n} in the expansion of the spectral action for the Robertson-Walker metric is of the form

$$\frac{1}{a(t)^{2n-3}}\,Q_{2n}\left(a(t), a'(t), \ldots, a^{(2n)}(t)\right),$$

where Q_{2n} is a polynomial with rational coefficients.

Two chief players in the proof of this theorem are the recursive formula of $a_n(x, D^2)$, which we derived from the recursive formula for the symbol of the parametrix, and the symmetries of the metric, which were employed in terms of the Killing vector fields. We also compute the terms up to a_{12} in the expansion of the spectral action by our method and find a formula for the coefficient of the term with the highest derivative of $a(t)$ in a_{2n}.

In the third chapter, the curvature of the determinant line bundle on a family of Dirac operators for a noncommutative two torus is computed. Quillen introduced the determinant line bundle on the space of Fredholm operators and showed that it is a holomorphic line bundle. He endows the determinant line bundle $\mathcal{L}$, pulled back on the space of all Cauchy-Riemann operators on a smooth vector bundle over a Riemann surface, by a Hermitian metric using the zeta regularized determinant of Laplacians. On the open set of invertible operators, each fiber of $\mathcal{L}$ is canonically isomorphic to $\mathbb{C}$ and the nonzero holomorphic section $\sigma = 1$ gives a trivialization. The norm of this section on the fiber of the invertible Cauchy-Riemann operator D is given by

$$\|\sigma(D)\|^2 = e^{-\zeta'_{\Delta}(0)},$$

where ζ_Δ is the spectral zeta function of the Laplacian $\Delta = D^*D$. Quillen studies the geometry of the line bundle $\mathcal{L}$ and he computes the curvature of the metric.

The noncommutative torus is an example of a noncommutative Riemann surface. We investigated the curvature of the determinant line bundle over a family of Cauchy-Riemann operators on the noncommutative two torus $\mathcal{A}_\theta$ with a fixed complex structure. To study the geometry of the determinant line bundle on this family, we had to inevitably develop new tools and use new techniques that are applicable in the noncommutative setting. To this end a version of the canonical trace of Kontsevich-Vishik is developed for the algebra of pseudodifferential operators on the noncommutative two torus. Using the calculus of symbols and the canonical trace we computed the curvature of the determinant line bundle, which is the second variation of $\log\det(\Delta)$ and is given by

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{4\pi\Im(\tau)}\varphi_0\left(\delta_w D(\delta_w D)^*\right).$$

The calculus of symbols and the canonical trace allow us to bypass local calculations involving Green functions in Quillen's work, which are not applicable in the noncommutative case.

Unlike the previous chapters, in which local computations are used to compute the spectral invariants, in the last chapter, which is an ongoing project, we use the Euler-Maclaurin summation formula to compute the asymptotic expansion of the spectral action of the operator $D' = D + T/2$, where D is the Dirac operator on the Berger sphere $\mathbb{S}^3(T)$. This method is useful when the full spectrum of the operator is known. By the Euler-MacLaurin formula the full asymptotic expansions of the spectral action $f(D'^2/\Lambda^2)$ and its heat trace $\mathrm{Tr}(e^{-tD'^2})$ are derived.

Chapter 1

A Prelude to Noncommutative Geometry

Noncommutative geometry is a rapidly developing field with extensive applications in other fields of modern mathematics as well as physics. While full of fresh ideas and new paradigms, it is rooted in the heart of modern mathematics of the 20^{th} century such as index theory, spectral geometry and spin geometry.

Our aim in this chapter will be to explore the main ideas from the classical theories and their related tools which play a role in developments of noncommutative geometry, especially the spectral aspects. The first half of this chapter includes a very quick review of such classical topics and consists of three sections: Pseudodifferential Operators, Spin Geometry and Spectral Geometry. The main focus will be on the tools and results which will lead to new concept in the second half of the chapter.

In the second half we will recall the notions of noncommutative spaces and their application to physics. Section 1.4 includes the basics of the theory of spectral triples and axioms of noncommutative spin geometry. The last section is devoted to the developments of the notions of action and symmetries in noncommutative geometry and how unified theories like Einstein-Yang-Mills theory can be produced through noncommutative spaces.

1.1 Pseudodifferential Operators

With all its great features, the algebra of differential operators fails to deliver important concepts like negative or non-integer order differentiation, which if existed would be very useful in solving partial differential equations. This shortfall can be remedied by introducing pseudodifferential operators and the calculus of their symbols. However, while one loses the local property, the symbol calculus survives and the theory works very well especially with the spectral theory of operators.

In this section we will review the theory of pseudodifferential operators . The main references for this section are [18, 27, 34].

1.1.1 Basics of the Theory

The concept of pseudodifferential operators emerges out of the following property of the Fourier transformation:

$$\mathcal{F}(f')(\xi) = i\xi\mathcal{F}(f)(\xi).$$

This leads to a new way to differentiate functions using the Fourier transform, given by

$$f'(x) = \mathcal{F}^{-1}\Big(i\xi\mathcal{F}(f)(\xi)\Big)(x). \tag{1.1}$$

If we replace ξ by a polynomial, $p(x,\xi) = \sum a_\alpha(x)\xi^\alpha$, in ξ with coefficients depending on x, then (1.1) will define the following differential operator.

$$p(x,D) = \mathcal{F}^{-1}\Big(p(x,\xi)\mathcal{F}(f)(\xi)\Big)(x),$$

where $D = \frac{1}{i}\frac{d}{dx}$. Now one can exchange the polynomial $p(x,\xi)$ with a general function in (x,ξ) with the right growth rate. This is how a general pseudodifferential operator is constructed.

Definition 1.1. Let U be an open subset of $\mathbb{R}^m$ with compact closure. A smooth function $\sigma : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{C}$ is called a symbol of order d on U, denoted by $\sigma \in \mathcal{S}^d(U)$, if its x-support is inside U, and for any non-negative integer multi-indices α, β there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha D_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{d-|\beta|}.$$

Here $\partial_x^\alpha = \partial_{x^1}^{\alpha_1}\cdots\partial_{x^m}^{\alpha_m}$, $D_\xi^\beta = \frac{1}{i^{|\beta|}}\partial_{\xi^1}^{\beta_1}\cdots\partial_{\xi^m}^{\beta_m}$ and $|\beta| = \beta_1 + \cdots + \beta_m$.

For any symbol σ we assign an operator $P_\sigma : C_0^\infty(U) \to C_0^\infty(U)$ given by

$$P_\sigma(f)(x) = \mathcal{F}^{-1}\Big(\sigma(x,\xi)\mathcal{F}(f)(\xi)\Big)(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\mathcal{F}(f)(\xi)d\xi.$$

Here $d\xi = \frac{1}{(2\pi)^m} d_L\xi$, where $d_L\xi$ is the Lebesgue measure on $\mathbb{R}^m$.

Multiplication of two pseudodifferential operators P, Q gives another pseudodifferential operator whose symbol is given by

$$\sigma(PQ) \sim \sum_\alpha \partial_x^\alpha \sigma(P) D_\xi^\alpha \sigma(Q)/\alpha!,$$

where the symbol of a pseudodifferential operator P is denoted by $\sigma(P)$. The equivalence relation $\sim$ on the symbols is defined as

$$\sigma \sim \sigma' \Leftrightarrow \sigma - \sigma' \in \mathcal{S}^{-\infty}.$$

Here $\mathcal{S}^{-\infty} = \cap_k \mathcal{S}^k$.

For a symbol $\sigma \in \mathcal{S}^d$ we define the principal symbol σ_L to be the class of σ in the quotient space $\mathcal{S}^d/\mathcal{S}^{d-1}$. The symbol multiplication for the principal symbol is the function multiplication

$$\sigma_L(PQ) = \sigma_L(P)\sigma_L(Q).$$

The theory of pseudodifferential operators acting on vector valued functions $f \in C^\infty(U, \mathbb{C}^m)$ can now be developed. Symbols in this case are matrix-valued symbols $\sigma(x,\xi) \in M_n(\mathbb{C})$. Pseudodifferential operators on a vector bundle V over a manifold M is defined as below.

Definition 1.2. A linear operator $P : C^\infty(V) \to C^\infty(V)$ is a pseudodifferential operator of order d, denoted by $P \in \Psi^d(M, V)$, if for any chart of M which is a trivialization for V as well, i.e. $V|_U \simeq U \times \mathbb{C}^n$, and for any $\psi, \varphi \in C_0^\infty(M)$, the localized operator

$$\varphi P \psi : C^\infty(U, \mathbb{C}^n) \to C^\infty(U, \mathbb{C}^n) \tag{1.2}$$

is a pseudodifferential operator of order d on U acting on $C^\infty(U, \mathbb{C}^n)$.

In any coordinate chart, we define $\sigma(P)$ to be the symbol of the operator $\varphi P \varphi$ on $\varphi = 1$. The leading symbol is invariantly defined on T^*M, but the total symbol changes with the change of coordinates.

Let us equip V with a Hermitian product $(\cdot,\cdot)$ and fix a Riemannian metric g on M. Then we can define an inner product on $C^\infty(V)$ by

$$\langle \xi, \eta \rangle := \int_M (\xi, \eta) \mathrm{dvol}_g. \tag{1.3}$$

Pseudodifferential operators are densely defined unbounded operators on $\mathcal{H} = L^2(M, V)$. The following theorem determines when a pseudodifferential operator is in the important classes of the operators on $\mathcal{H}$. For a proof see e.g. [18]

Theorem 1.3. *Suppose (M, g) is a closed manifold and V a Hermitian vector bundle over M. Let $P \in \Psi^d(M, V)$; then*

1. *If $d \leq 0$, then P is a bounded operator, i.e., $\Psi^{\leq 0}(M, V) \subset B(\mathcal{H})$.*
2. *If $d < 0$, then P is a compact operator, i.e., $\Psi^{<0}(M, V) \subset \mathcal{K}(\mathcal{H})$.*
3. *If $d \leq -m$, then P is a Dixmier class operator, i.e., $\Psi^{\leq -m}(M, V) \subset \mathcal{L}^{1,\infty}(\mathcal{H})$.*
4. *If $d < -m$, then P is a trace class operator, i.e., $\Psi^{<-m}(M, V) \subset \mathcal{L}^1(\mathcal{H})$.*

Another important class of operators is the class of Fredholm operators, which are the topic of study in index theory. The pseudodifferential operators that give rise to Fredholm operators are called elliptic operators.

Definition 1.4. A symbol $\sigma \in \mathcal{S}^d(U)$ is called elliptic on $U_1 \subset \overline{U_1} \subset U$ if there is an open subset U_2 with $\overline{U_1} \subset U_2 \subset \overline{U_2} \subset U$ such that there exists a $\sigma' \in \mathcal{S}^{-d}$ such that

$$\sigma\sigma' - I \in \mathcal{S}^{-\infty}(U_2) \text{ and } \sigma'\sigma - I \in \mathcal{S}^{-\infty}(U_2).$$

An operator $P \in \Psi^d(M, V)$ is called elliptic if the symbol of localized operators (1.2) are elliptic in $\varphi\psi(x) \neq 0$.

If $P \in \Psi^d(M, V)$ is an elliptic operator, then there exists $Q \in \Psi^{-d}(M, V)$ so that

$$PQ - I \text{ and } QP - I \in \Psi^{-\infty}(M, V).$$

The operator Q is called a parametrix of P. Note that by Theorem 1.3 the operator $QP - I$ is compact, so P is invertible in the Calkin algebra and therefore a Fredholm operator.

Remark 1.5. The spectrum of a positive order elliptic operator P is a set of discrete eigenvalues tending to infinity. This is a consequence of the spectral theorem of compact operators applied to the resolvent of P, which by Theorem 1.3, is compact.

1.1.2 Traces on Pseudodifferential Algebra

As we mentioned in the previous section, if the order of a pseudodifferential operator P is less than $-\dim M$, then P is of trace class. The value of its trace can be computed by expressing the operator as an integral operator with the kernel written in terms of the symbol. By integrating the kernel along the diagonal, one gets

$$\mathrm{Tr}(P) = \int_M \int_{T_x^*M} \mathrm{tr}(\sigma(x,\xi)) d\xi \mathrm{dvol}_g. \tag{1.4}$$

Here tr inside the integral denotes the usual matrix trace. To study other traces on pseudodifferential operators we have to introduce a new class of symbols.

Definition 1.6. Let $\sigma : U \times \mathbb{R}^m \to \mathbb{C}$ be a smooth map such that for any N and each $0 \leq j \leq N$, there exists $\sigma_{\alpha-j}$ positive homogeneous of degree $\alpha - j$, and a symbol $\sigma^N \in \mathcal{S}^{\Re(d)-N-1}(M,V)$ such that

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^m. \tag{1.5}$$

Here, χ is a smooth cut-off function on $\mathbb{R}^m$ which is zero on a small ball around the origin and one outside the unit ball. The map σ is called a classical symbol of order $\alpha \in \mathbb{C}$ and the set of all classical symbols is denoted by $\Psi^{\alpha}_{cl}(M,V)$.

Note that a classical symbol of order α is obviously a symbol of order $\Re(\alpha)$. There is a more general class of symbols called log–polyhomogenous symbols in which terms of the form $\xi^k \log^l |\xi|$ are also present. Most of the theory that will be reviewed in the next section is true for their case but we won't dwell on it here. For a detailed discussion of traces on classical pseudodifferential operators on manifolds we refer the reader to [28, 29, 31] and the references therein.

1.1.2.1 Wodzicki Residue

M. Wodzicki in [37] defined a trace functional on the algebra of classical pseudodifferential operators on M, and proved that it is the only non-trivial trace. This functional on pseudodifferential operators of order $-m$ was discovered independently by Guillemin. [21] In the following we will review this trace and Connes' trace formula, which establishes a deep relationship between the Wodzicki residue and the Dixmier trace.

For a classical pseudodifferential operator P with symbol σ on a vector bundle V, we define the density

$$\mathrm{res}_x(P) = \int_{S_x^*M} \mathrm{tr}(\sigma_{-m}(x,\xi))d_S\xi.$$

Here, $d_S\xi$ denotes the normalized Lebesgue measure $d\xi$ restricted on the unit sphere $S_x^*M = \{|\xi| = 1; \xi \in T_x^*M\}$. Though the symbol $\sigma(x,\xi)$ of P depends on the choice of local coordinates, $\mathrm{res}_x(P)$ is a well-defined density.

Definition 1.7. The Wodzicki residue of P, denoted by $\mathrm{Res}(P)$, is given by

$$\mathrm{Res}(P) = \int_M \mathrm{res}_x(P)(x)dx.$$

A trace formula similar to (1.4) was proven by Connes, in which the left hand side is replaced by the Dixmier trace. This trace is defined on the Dixmier ideal

$$\mathcal{L}^{(1,\infty)} = \Big\{T \in \mathcal{K}(\mathcal{H}); \sum_{n=1}^{N} \mu_n(T) = O(\log N)\Big\},$$

where $\mu_n(T)$, called characteristic values of T, are the eigenvalues of $|T| = (T^*T)^{1/2}$ listed in decreasing order. For any positive scale invariant generalized limit $\lim_\omega$ on the space $\ell^\infty(\mathbb{N})$, there is a positive functional Tr_ω whose value on a positive operator $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ is given by

$$\mathrm{Tr}_\omega(T) = \lim{}_\omega \frac{1}{\log N}\sum_{n=1}^{N} \mu_n(T).$$

The positive functional Tr_ω extends to $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ by linearity. For detailed discussion see e.g. [9].

Theorem 1.8. *[8] Let M be a compact m-dimensional manifold, V a complex vector bundle on M, and P a pseudodifferential operator of order $-m$ acting on sections of*

V. Then the corresponding operator P in $\mathcal{H} = L^2(M,V)$ belongs to the Dixmier ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$. Moreover, the Dixmier trace of P is independent of ω and

$$\mathrm{Tr}_\omega(P) = \frac{1}{m}\mathrm{Res}(P).$$

1.1.2.2 The Canonical Trace

The integral on the right hand side of equation (1.4) diverges if the order of the operator P is not less than $-m$. This phenomenon is known as ultraviolet divergence in physics. Kontsevich and Vishik used Hadamard regularization, based on the concept of the finite part of the integral, to regularize this divergent integral [25].

Given a classical symbol σ, with the expansion given by (3.2), for any fixed $x \in M$ the map $R \mapsto \int_{B(0,R)} \sigma(x,\xi)d\xi$ has an asymptotic expansion as $R \to \infty$ of the following form:

$$\int_{B_0(R)} \mathrm{tr}(\sigma(x,\xi))d\xi \sim_{R\to\infty} c(\sigma_x) + \mathrm{res}_x(\sigma)\log R + \sum_{\substack{j=0 \\ j \neq \alpha+m}}^{\infty} R^{\alpha+m-j} c_j(\sigma_x),$$

where $c(\sigma_x)$ and $c_j(\sigma_x)$ are constants that are determined by the symbol at x

Definition 1.9. For a classical operator with symbol $\sigma \in \Psi^\alpha(M,V)$, the constant $c(\sigma_x)$ is called the finite part of the integral at x and we denote it by

$$⨍ (\sigma(x,\xi))d\xi.$$

The canonical trace of σ is then defined as

$$\mathrm{TR}(P) := \int_M ⨍ \sigma(x,\xi)d\xi \mathrm{dvol}. \tag{1.6}$$

It is evident that if $\Re(\alpha) < -m$ then

$$⨍ \sigma(x,\xi)d\xi = \int_{T_x^*M} \sigma(x,\xi)d\xi \quad \forall x \in M.$$

Hence, $\mathrm{TR}(P) = \mathrm{Tr}(P)$. Upon further investigation of the properties of TR one obtains the following fundamental theorem.

Theorem 1.10. *[25] The linear functional $TR(P)$ on classical pseudodifferential operators of orders from $\alpha_0 + \mathbb{Z}$, $\alpha_0 \in \mathbb{C}\backslash\mathbb{Z}$, in the case of a closed M has the following properties.*

1. *It coincides with the usual trace* $\mathrm{Tr}(A)$ *in* $L^2(M, V)$ *for* $\Re(\mathrm{ord}A) < -m$.
2. *It is a trace type functional, i.e.,* $TR([B, C]) = 0$ *for* $\mathrm{ord}B + \mathrm{ord}C \in \alpha_0 + \mathbb{Z}$.
3. *For any holomorphic family $A(z)$ of classical pseudodifferential operators where $z \in U \subset \mathbb{C}$, and non–constant affine order* $\mathrm{ord}A(z) = \alpha(z)$, *the function* $\mathrm{TR}(A(z))$ *is meromorphic with no more than simple poles at $z = n \in U \cap \mathbb{Z} \cap [-m, \infty)$ and with residues*
$$\mathrm{Res}_{z=n}\mathrm{TR}(A(z)) = -\frac{1}{\alpha'(n)}\mathrm{Res}(A(n)). \tag{1.7}$$

For more general holomorphic families of operators the higher order terms of the Laurent expansion of $\mathrm{TR}(A(z))$ around any pole are computed in [32].

1.2 Spin Geometry

Spin geometry plays an increasingly important role in different areas of modern mathematics and physics. On spin manifolds we can produce a globally defined first order elliptic operator, called the Dirac operator, canonically associated to its underlying geometry. The study of Dirac operators was initiated by Paul Dirac in physics in the late 1920s. Later, Sir Michael Francis Atiyah and Isadore Singer established a strong mathematical foundation for the theory of Dirac operators and used it in index theory. In this section we will review the basics of spin manifolds and the Dirac operator. References for this section are [17, 26].

1.2.1 Clifford Algebras and Spin Groups

To define a spin manifold, we first recall spin groups and their representations.

Definition 1.11. The universal covering group of the special orthogonal group $\mathrm{SO}(m)$, $m > 2$, is called the spin group and we denote it by $\mathrm{Spin}(m)$.

For example, $\mathrm{Spin}(3) = \mathrm{SU}(2)$. The covering homomorphism, $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is given by the adjoint representation of $\mathrm{SU}(2)$ on its Lie algebra $\mathfrak{su}(2)$.

Since for $m > 2$, $\pi_1(\mathrm{SO}(m)) = \mathbb{Z}_2$, the covering group $\mathrm{Spin}(m)$ is a double cover of $\mathrm{SO}(m)$ and we have the following exact sequence of multiplicative groups:

$$1 \to \{\pm 1\} \to \mathrm{Spin}(m) \xrightarrow{\rho} \mathrm{SO}(m) \to 1.$$

Any representation of the special orthogonal group $\pi : \mathrm{SO}(m) \to \mathrm{Aut}(W)$ lifts to a representation of the spin group given by $\pi \circ \rho : \mathrm{Spin}(m) \to \mathrm{Aut}(W)$. However, there are representations of $\mathrm{Spin}(m)$ that are not constructed this way. These representations, unlike the lifted representations from $\mathrm{SO}(m)$, have different values for $1, -1 \in \mathrm{Spin}(m)$. One way to construct such representations is to consider the Clifford algebras.

Definition 1.12. Let W be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$ and B be a nondegenerate symmetric bilinear form on W. The Clifford algebra $\mathrm{Cl}(W, B)$ is the quotient K-algebra defined by

$$\mathrm{Cl}(W, g) = \mathcal{T}(W)/\mathcal{I}_B(W),$$

where $\mathcal{T}(W) = \sum_{r=0}^{\infty} W^{\otimes r}$ is the tensor algebra and $\mathcal{I}_B(W)$ is the ideal generated by elements of the form $v \otimes w + w \otimes v + 2B(v, w)$.

The Clifford algebra is a finite dimensional unital $\mathbb{Z}_2$-graded algebra containing W with the multiplicative property

$$v \cdot w + w \cdot v = -2B(v, w).$$

The even part of Clifford algebra, denoted by $\mathrm{Cl}^0(W, B)$, is the subspace formed by the even number of elements of w, and a similar definition holds for the the odd part $\mathrm{Cl}^1(V, B)$.

We denote the Clifford algebra for $\mathbb{R}^m$ and $\mathbb{C}^m$ with the standard positive definite form respectively by Cl_m and $\mathbb{C}\mathrm{l}_m$. The real and complex Clifford algebras are related to each other by the fact that Cl and $\otimes\mathbb{C}$ (as functors) commute. In other words,

$$\mathrm{Cl}_m \otimes \mathbb{C} = \mathbb{C}\mathrm{l}_m. \tag{1.8}$$

Moreover, Clifford algebras for different dimensions are related by the following periodicities in the real and complex cases,

$$\mathrm{Cl}_{m+8} = \mathrm{Cl}_m \otimes M_{16}(\mathbb{R}) \quad \text{and} \quad \mathbb{C}\mathrm{l}_{m+2} = \mathbb{C}\mathrm{l}_m \otimes M_2(\mathbb{C}).$$

Hence, by knowing only the first eight real Clifford algebras and two complex ones, which are given in the following tables, we can construct all Clifford algebras of all dimensions.

$\mathrm{Cl}_{1,0}$	$\mathbb{C}$
$\mathrm{Cl}_{2,0}$	$\mathbb{H}$
$\mathrm{Cl}_{3,0}$	$\mathbb{H} \oplus \mathbb{H}$
$\mathrm{Cl}_{4,0}$	$M_2(\mathbb{H})$
$\mathrm{Cl}_{5,0}$	$M_4(\mathbb{C})$
$\mathrm{Cl}_{6,0}$	$M_8(\mathbb{R})$
$\mathrm{Cl}_{7,0}$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
$\mathrm{Cl}_{8,0}$	$M_{16}(\mathbb{R})$

$\mathbb{C}\mathrm{l}_1$	$\mathbb{C} \oplus \mathbb{C}$
$\mathbb{C}\mathrm{l}_2$	$M_2(\mathbb{C})$

The representation of the spin group is related to the Clifford algebra because the group $\mathrm{Spin}(m)$ can be realized as a subgroup of invertible elements of the Clifford algebra Cl_m as follows:

$$\mathrm{Spin}(m) = \left\{ x_1.x_2.\cdots.x_{2k} | x_l \in S^{m-1} \right\} \subset \mathrm{Cl}_m^0 \subset \mathrm{Cl}_m.$$

Since $\mathrm{Cl}_m \subset \mathbb{C}\mathrm{l}_m$, $\mathrm{Spin}(m)$ is also a subgroup of invertible elements of $\mathbb{C}\mathrm{l}_m$. This inclusion induces new representations of $\mathrm{Spin}(m)$ by restricting any algebra representation of Cl_m or $\mathbb{C}\mathrm{l}_m$ to $\mathrm{Spin}(m)$.

Definition 1.13. The real spinor representation of $\mathrm{Spin}(m)$ is

$$\Delta_m : \mathrm{Spin}(m) \to \mathrm{GL}(S_m),$$

given by restricting an irreducible real representation $\mathrm{Cl}_m \to \mathrm{Hom}_{\mathbb{R}}(S_m, S_m)$ to $\mathrm{Spin}(m) \to \mathrm{Cl}_m$. Moreover, the complex spinor representation of $\mathrm{Spin}(m)$ is the the homomorphism

$$\Delta_m^{\mathbb{C}} : \mathrm{Spin}(m) \to \mathrm{GL}_{\mathbb{C}}(\mathbb{S}_m),$$

given by restricting an irreducible complex representation $\mathbb{C}\mathrm{l}_m \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{S}_m, \mathbb{S}_m)$ to $\mathrm{Spin}(m) \subset \mathrm{Cl}_m \subset \mathbb{C}\mathrm{l}_m$.

In the cases where there is more than one irreducible representation for the Clifford algebras, i.e. $m = 4k+3$ for Cl_m and $m = 2k+1$ for $\mathbb{C}\mathrm{l}_m$, the spinor representation is independent of the irreducible representation used. However, the real or complex spinor representation is not necessarily an irreducible representation of the spin group. Indeed, the complex spinor decomposes into two inequivalent irreducible representations of $\mathrm{Spin}(m)$ if m is even. This decomposition is given by multiplication by the *complex volume element*

$$\omega = i^{\left[\frac{m+1}{2}\right]} e_1 \cdot e_2 \cdot \dots \cdot e_m,$$

where $\{e_j\}$ is an oriented orthonormal basis for $\mathbb{R}^m$. It is easy to check that this is not the identity map but it is an idempotent , i.e. $\omega^2 = 1$. Since ω commutes with the elements of $\mathrm{Spin}(m) \subset \mathbb{C}\mathrm{l}^0_m$, the representation decomposes into irreducible representations denoted by $\Delta^{\mathbb{C}\pm}_m$.

1.2.2 Spin Manifolds

Any Riemannian metric g on a closed oriented manifold M of dimension m defines a principal $\mathrm{SO}(m)$ bundle $P_{SO}(M, g)$, the bundle of oriented orthonormal frames, such that TM can be constructed as its associated vector bundle, i.e.

$$TM = P_{\mathrm{SO}}(M, g) \times_\pi \mathbb{R}^m.$$

Here, π is the standard representation of $\mathrm{SO}(m)$ on $\mathbb{R}^m$. Now, one can wonder if we can find a principal spin bundle such TM is its associated vector bundle. It turns out that this is not possible for every manifold, and we have the following definition.

Definition 1.14. An oriented manifold M is a spin manifold if there exists a principal $\mathrm{Spin}(m)$ bundle P_{Spin} such that

$$TM = P_{\mathrm{Spin}} \times_\pi \mathbb{R}^m, \tag{1.9}$$

where π is the standard representation of $\mathrm{SO}(m)$ on $\mathbb{R}^m$ lifted to a representation of $\mathrm{Spin}(m)$.

If M is a spin manifold for any Riemannian metric g on M, there exists a compatible spin structure $P_{\mathrm{Spin}}(M,g)$ and a map $p : P_{\mathrm{Spin}}(M,g) \to P_{\mathrm{SO}}(M,g)$ such that[1]

$$p(a\ g) = \pi(a)\ \rho(g) \quad a \in P_{\mathrm{Spin}}(M,g),\, g \in \mathrm{Spin}(m).$$

One geometric importance of spin manifolds is that we can construct a new (complex) vector bundle on M which is completely determined by the geometry of the manifold. The (complex) spinor bundle is the associated vector bundle defined by the complex spinor representation,

$$S = P_{\mathrm{Spin}}(M,g) \times_{\Delta_m^{\mathbb{C}}} \mathbb{S}_m.$$

One can use the real spinor representation to produce the real spinor bundle. Since we are interested in working with complex Hilbert spaces we will only consider the complex case.

At each point $x \in M$, $\mathbb{C}\mathrm{l}(T_xM, g_x)$ is represented on S_x. This module structure is a smooth global structure. In other words, the space of sections of the spinor bundle, $C^\infty(M,S)$, is a $C^\infty(\mathbb{C}\mathrm{l}(TM,g))$-module. Note that there is a canonical isomorphism $TM \simeq T^*M$ for Riemannian manifolds. This isomorphism induces a canonical isomorphism on the Clifford algebras $\mathbb{C}\mathrm{l}(TM,g)$ and $\mathbb{C}\mathrm{l}(T^*M)$. We will frequently use this isomorphism and we will denote $\mathbb{C}\mathrm{l}(T^*M, g^{-1})$ by $\mathbb{C}\mathrm{l}(M)$. A consequence of considering this isomorphism is that 1-forms $\alpha \in \Omega^1(M) = C^\infty(T^*M) \subset \mathbb{C}\mathrm{l}(M)$ can act on spinors by the Clifford action.

1.2.3 The Dirac Operator

The derivative of the covering homomorphism ρ at the identity of $\mathrm{Spin}(m)$ defines a Lie algebra isomorphism $\rho' : \mathfrak{so}(m) \to \mathfrak{spin}(m)$ which is explicitly given by

$$\rho'(A) = \frac{1}{4}\sum_{i,j} \langle Ae_j, e_k\rangle e_j \cdot e_k, \tag{1.10}$$

where $\{e_j\}$ is an oriented orthonormal basis of $\mathbb{R}^m$. Here the Lie algebra $\mathfrak{spin}(m)$ is identified by $(\Lambda^2\mathbb{R}^m, [\ ,\ ])$ as a sub Lie algebra of $(\mathrm{Cl}_m, [,])$ with Lie bracket given by

[1] One can compare this with the orientability of a manifold which is a topological property. If a manifold is orientable, for any metric g one can find a compatible orientation given by the volume form dvol_g.

$[x, y] = x \cdot y - y \cdot x$. Such an identification is possible due to the inclusion $\mathrm{Spin}(m) \subset \mathrm{Cl}_m^\times$ and the fact that $(\mathrm{Cl}_m, [\ ,\ ])$ is the Lie algebra of $\mathrm{Cl}_m^\times$.

Using this isomorphism, any connection on $P_{\mathrm{SO}}(M, g)$(equivalently any metric connection on TM), lifts to a connection on $P_{\mathrm{Spin}}(M, g)$. In particular, the Levi-Civita connection, the unique torsion-free metric connection, lifts to the spinor bundle. This is called the *spin connection* and we denote it by ∇^S.

Finally, we can define the Dirac operator on spinors by

$$D(\psi)(x) = c(dx^j)\nabla_{\partial_j}\psi(x), \quad \psi \in C^\infty(S).$$

The above definition is independent of the choice of coordinate chart. Indeed, it can be defined using any frame $\{e_j\}$ with coframe $\{e^j\}$ as $c(e^j)\nabla^S_{e_j}$.

The Dirac operator is an elliptic differential operator with symbol $\sigma(x, \xi) = c(i\xi)$.

1.2.4 Spin^c Manifolds

While having a real spinor bundle on M is equivalent to M be a spin manifold, having a complex spinor bundle is a weaker condition. In other words, there are non spin manifolds which admit complex spinor bundles. This is a consequence of a fact that the complex spinor representation $\Delta_m^{\mathbb{C}}$ is a representation of a larger subgroup of $\mathbb{C}\mathrm{l}_m^\times$, denoted by $\mathrm{Spin}^{\mathbb{C}}(m)$ than $\mathrm{Spin}(m)$. This group is generated by $\mathrm{Spin}(m)$ and $\mathrm{U}(1)$ as subgroups of $\mathbb{C}\mathrm{l}_m$ and it is of the form

$$\mathrm{Spin}^{\mathbb{C}}(m) = \mathrm{Spin}(m) \times_{\mathbb{Z}_2} \mathrm{U}(1).$$

We have the following exact sequence

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}^{\mathbb{C}}(m) \xrightarrow{\rho^{\mathbb{C}}} \mathrm{SO}(m) \to 1.$$

A manifold M is called a *spin^c manifold* if

$$TM = P_{\mathrm{Spin}^{\mathbb{C}}} \times_\pi \mathbb{R}^n.$$

The representation π is the standard representation of $\mathrm{SO}(m)$ lifted to $\mathrm{Spin}^{\mathbb{C}}(m)$ by $\rho^{\mathbb{C}}$. The compatible $\mathrm{Spin}^{\mathbb{C}}(m)$ structure on (M, g) is defined similarly to the compatible spin structure.

In spite of the historical development of spin geometry, in which the notion of spin structure appeared before the spin^c structure, the algebraic formulation of these structures started with spin^c manifolds. Plymen in [33] showed that an oriented manifold is spin^c if $C^\infty(M)$ and $\mathbb{C}l(M)$ are Morita equivalent with a Morita equivalence bimodule S. In this picture a spin manifold is a spin^c manifold with a real or quaternionic structure (depending on the dimension of the manifold) on S with a specific commutation relation which comes from the following theorem.

Theorem 1.15. *(see e.g. [36]) There is an antilinear map J_m on $\mathbb{S}_m$, called charge conjugate, with the following properties:*

- *J_m is either real or complex structure, i.e. $J_m^2 = 1$ or $J_m^2 = -1$ respectively.*
- $J_m(x \cdot \psi) = \pm x \cdot J_m(\psi), \quad x \in \mathbb{R}^m,\ \psi \in \mathbb{S}_m.$
- $Cl_m^0 = \begin{cases} \{x \in \mathbb{C}l_m^0 \mid [J_m, \pi(x)] = 0\} & m = 2k+1 \\ \{x \in \mathbb{C}l_n^0 \mid [J_m, \pi(x)] = 0, [\omega, \pi(x)] = 0\}, & m = 2k \end{cases}$

The exact signs are given in the following table.

J_m	real structure	quaternionic structure
commutes with Clifford multiplication	$m = 0, 6, 7 \bmod 8$	$m = 2, 3, 4 \bmod 8$
anti-commutes with Clifford multiplication	$m = 1 \bmod 8$	$m = 5 \bmod 8$

Note that the map J_m commutes with the even part of Cl_m and thus with the elements of $\mathrm{Spin}(m)$.

An important point is that although there is a complex spinor bundle on spin^c manifolds, we cannot always construct a geometric Dirac operator on S. The reason is hidden in the fact that

$$\mathfrak{spin}^{\mathbb{C}}(m) = \mathfrak{so}(m) \oplus i\mathbb{C}.$$

This means that the metric connections on TM cannot completely determine a connection on the spinor bundle. To do so, one needs a connection on the canonical line bundle, which is a line bundle that can be assigned to each spin^c structure. Since the canonical line bundle usually doesn't admit a geometric connection we need to add an extra piece of information by fixing a connection on this line bundle. In some cases, e.g. Kähler manifolds, there is a geometric connection on the canonical line bundle. Hence we can construct a connection on the spinor bundle, and as such, a geometric Dirac operator exists in this case.

1.3 Spectral Geometry

The main goal of spectral geometry is to study the spectrum of natural operators that can be constructed on a Riemannian manifold (M, g). The topic originated by studying the spectrum of the scalar Laplace operator on a bounded domain $\Omega \subset \mathbb{R}^m$. The earliest result in this regard was what we now refer to as "Weyl's law".

Theorem 1.16. *(***Weyl's law***)*
For a bounded domain $\Omega \subset \mathbb{R}^m$, the Dirichlet eigenvalue counting function $N(\lambda)$, which counts the number of Dirichlet eigenvalues (counting their multiplicities) less than or equal to λ, satisfies

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{m/2}} = (2\pi)^{-m} B_m \mathrm{vol}(\Omega)$$

where B_m is the volume of the unit ball in $\mathbb{R}^m$.

The eigenvalue counting function $N(\lambda)$ is an example of a *spectral function*. That is a function that depend only on the spectrum of the operator under investigation. In this section we shall discuss two other important spectral functions – namely the trace of heat kernel and the spectral zeta function.

In addition, the scalar Laplacian is not the only natural differential operator with interesting spectrum. The natural differential operators in which we are interested are the positive Laplace type operators. This class includes the square of Dirac type operators, which play a very important role in noncommutative geometry.

As we will see in this section, many geometrical properties and quantities, like dimension, volume and scalar curvature, are reflected in the spectrum of these operators. These quantities are so fundamental to the geometry of M that it is natural to investigate

whether or not the geometry of M can be completely determined by spectrum of such operators. This question was rephrased by Kac in the following clever form

Can one hear the shape of a drum?

Although many counterexamples for this question are discovered, starting with an example by Milnor [30], spectral geometry is still a fast growing field of research, especially in its applications in physics.

1.3.1 Laplace Type and Dirac Type Operators

Suppose that (M, g) is a closed Riemannian manifold[2] with $\dim M = m$ and V is a smooth Hermitian vector bundle on M.

Definition 1.17. A second order differential operator $P : C^\infty(V) \to C^\infty(V)$ is called a Laplace type operator if the leading symbol is given by the metric tensor.

A Laplace type operator, in a coordinate chart, can be written as

$$P = -\left(g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + A^k \frac{\partial}{\partial x_k} + B \right), \tag{1.11}$$

where the A^k, B are endomorphisms of the bundle V. Scalar Laplacian and Laplacians on forms are examples of Laplace type operators. However, the interesting cases are Laplace type operators which are the square of a first order operator. In what follows we briefly cover the theory of such operators. For an extensive treatment of the subject we refer the reader to [3].

Definition 1.18. A differential operator $D : C^\infty(V) \to C^\infty(V)$ is called a *Dirac type* operator if D^2 is a Laplace type operator.

One can easily see that a Dirac type operator has to be a first order operator and in a trivialization it is of the form $a^k \partial_k + b$ where $a^k(x)$ and $b(x) \in \mathrm{End}(V_x)$ for any $x \in M$. Symbol calculation shows that $\{a^k(x)\}$ implies the Clifford commutation relation, i.e.,

$$a^k(x)a^l(x) + a^l(x)a^k(x) = -2g^{kl}(x).$$

[2] The closedness condition can be omitted for most of the results.

From this one deduces that $\mathbb{C}\mathrm{l}(T_x^*M, g^{-1}) \subset End(V_x)$ and hence $C^\infty(V)$ is a $\mathbb{C}\mathrm{l}(M) = C^\infty(\mathbb{C}\mathrm{l}(T^*M))$-module. Such a vector bundle V is called a *Clifford module.*

The first example of a Clifford module which exists on any manifold is the exterior algebra bundle $\Lambda^\bullet(M)$ with Clifford module structure given by

$$c(e)(\xi) = e \wedge (\xi) - \iota(e)(\xi), \quad \xi \in \Lambda^\bullet T_x^*M.$$

The spinor bundle on a spin manifold is another important example of a Clifford module, which was discussed in the previous section. In general, any associated vector bundle of the form

$$P_{\mathfrak{spin}}(M, g) \times_\kappa W, \tag{1.12}$$

with the representation $\kappa : \mathrm{Spin}(m) \to Aut(W)$ induced from a representation (not necessarily an irreducible one) of the Clifford algebra $\mathbb{C}\mathrm{l}(m)$, is a Clifford module.

While we can construct Dirac type operators on any Clifford module using a partition of unity, we are interested in a construction which uses connections on V to construct a Dirac type operator.

Definition 1.19. A connection ∇^V on V is called a Clifford connection if it fulfills the following Leibniz rule with respect to Clifford multiplication:

$$[\nabla_X^V, c(\theta)] = c(\nabla_X \theta), \quad \theta \in \Omega(M),\ X \in C^\infty(TM).$$

Here, ∇ is the Levi-Civita connection on the cotangent bundle.

To any Clifford connection ∇^V, we can assign a Dirac operator that at each point $x \in M$, is defined as

$$D = c(dx^j)\nabla_{\partial_j}^V, \tag{1.13}$$

where $\{e_i\}$ a basis for T_xM and $\{e^i\}$ its dual basis. The formula is independent of the choice of the basis $\{e_i\}$ and D is a globally well-defined operator. The spin connection is an example of Clifford connection and the Dirac operator D is the Dirac type operator defined by ∇^S.

We want to study Dirac type operators as unbounded operators. To this end we require the Clifford modules to be equipped with an inner product compatible with the

Clifford action

$$\langle e \cdot v, e \cdot w \rangle_x = \langle v, w \rangle, \quad v, w \in V_x, e \in S_x^* M, \tag{1.14}$$

and the Clifford connection to be compatible with this inner product. Such a Clifford module is called *Dirac bundle*. The Dirac operator defined on a Dirac bundle V is formally self-adjoint on $L^2(M, V)$. The spin bundle with the spin connection and the inner product coming from the spinor representation is an example of a Dirac bundle. There is a procedure to produce new Dirac bundles out of the spin bundle, (or in general from any other Dirac bundle). Let V be a Hermitian vector bundle with a Hermitian connection ∇. Then $S \otimes V$ is a Clifford module. One can define the *twisted connection*

$$\nabla^S \otimes 1 + 1 \otimes \nabla. \tag{1.15}$$

We denote the corresponding Dirac operator by D_∇ and call it the twisted Dirac operator.

1.3.2 Spectral Functions and Spectral Invariants

The study of the spectrum of a Laplace type operator P is usually done via functions defined from the spectrum of the operator. Two important spectral functions which have a central role in this dissertation are the heat kernel trace and the zeta function. Both of these functions are the generating functions of the spectrum, so by considering them, we don't lose any spectral information. Unlike the eigenvalue counting function, these functions can be expressed in terms of the trace of a function of P, an idea that will appear again in the next section in terms of the spectral action functional.

For any positive elliptic (differential) operator P, the heat flow operator e^{-tP}, for any $t > 0$, is an infinitely smoothing pseudodifferential operator , and therefore a trace class operator on $L^2(M, V)$. The heat kernel trace in terms of the eigenvalues $\{\lambda_i\}$ of the operator is given by

$$\mathrm{Tr}(e^{-tP}) = \sum_i e^{-t\lambda_i}.$$

This function has a singularity at $t = 0$ and its asymptotic expansion as $t \to 0^+$ is of the form [18]

$$\mathrm{Tr}(e^{-tP}) \sim_{t\to 0^+} \sum a_n(P) t^{\frac{n-m}{d}}, \tag{1.16}$$

where d is the order of the operator and the constants $a_n(P)$ can be written as follows:

$$a_n(P) = \int_M a_n(x, P)\text{dvol}. \tag{1.17}$$

Here, $a_n(x, P)$ are local invariants of the jets of the total symbol of P and vanish if n is odd. If P is a Laplace type operator, then the local terms contain geometrical information about M. To extract this information, we first need a lemma, which is the analogue of the basic algebraic operation of completing square of a quadratic polynomial.

Lemma 1.20. *[18] Let P be a Laplace type operator. Then there exists a unique connection ∇ on the vector bundle V and an endomorphism $E \in \mathrm{End}(V)$ such that*

$$P = \nabla^*\nabla - E. \tag{1.18}$$

Here $\nabla^*\nabla$ is the connection Laplacian which is locally given by $-g^{ij}\nabla_{\partial_i}\nabla_{\partial_j}$.

The endomorphism E for the square of the Dirac type operator of a Clifford connection ∇ on a Clifford module V is given by the generalized Lichnerowicz formula. First, one can prove, see e.g. [3, Proposition 3.43], that the curvature $\nabla^2 \in \Omega^2(M, \mathrm{End}(V))$ of the Clifford connection ∇ decomposes under the isomorphism $\mathrm{End}(V) \simeq \mathbb{C}\mathrm{l}(M) \otimes \mathrm{End}_{\mathbb{C}\mathrm{l}(M)}$ as

$$R^V + F^{V/S}. \tag{1.19}$$

Here, $R^V \in \Omega^2(M, \mathbb{C}\mathrm{l}(M)) \subset \Omega^2(M, \mathrm{End}(V))$ is the action of the Riemann curvature acting on V by

$$R^V(e_i, e_j) = \frac{1}{4}\sum_{k,l}\langle R(e_i, e_j)e_k, e_l\rangle c(e^k)c(e^l).$$

The endomorphism $F^{V/S} \in \Omega^2(M, \mathrm{End}_{\mathbb{C}\mathrm{l}(M)}(V))$ is called the *twisting curvature* of the Clifford module V.

Theorem 1.21. *[3, Theorem 3.53] Let ∇ be a Clifford connection on the Clifford module V. Then*

$$D^2 = \nabla^*\nabla + c(F^{V/S}) - \frac{R}{4}.$$

For a twisted connection given in (1.15) the twisting curvature is equal to

$$c(F^{V/S}) = \frac{1}{2}c(e^k)c(e^l) \otimes F_{kl} \tag{1.20}$$

where F is the curvature two form of ∇. Thus, the twisting curvature of the spin connection vanishes and we have the Lichnerowicz formula

$$D^2 = \left(\nabla^S\right)^* \nabla^S - \frac{1}{4}R. \tag{1.21}$$

The local formulas for a Laplace type operator are given by the following theorem.

Theorem 1.22. *[18] For a Laplace type operator P in (1.18), the invariants of the heat equation $a_n(x,P)$ is given by*[3]

$$\begin{aligned}
a_0(x,P) &= (4\pi)^{-m/2}\,\mathrm{tr}(\mathrm{Id}).\\
a_2(x,P) &= (4\pi)^{-m/2}\,\mathrm{tr}\big(E - \frac{1}{6}R\mathrm{Id}\big).\\
a_4(x,P) &= \frac{(4\pi)^{-m/2}}{360}\mathrm{tr}\Big(\big(-12R_{;kk} + 5R^2 - 2R_{jk}R_{jk} + 2R_{ijkl}R_{ijkl}\big)\mathrm{Id} - 60RE\\
&\qquad +180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}\Big).
\end{aligned}$$

$$\begin{aligned}
a_6(x,P) = (4\pi)^{-\frac{m}{2}}\,\mathrm{tr}\Big\{&\frac{1}{7!}\Big(-18R_{;kkll} + 17R_{;k}R_{;k} - 2R_{jk;l}R_{jk;l} - 4R_{jk;l}R_{jl;k}\\
&\quad + 9R_{ijku;l}R_{ijku;l} + 28RR_{;ll} - 8R_{jk}R_{jk;ll} + 24R_{jk}R_{jl;kl}\\
&\quad + 12R_{ijkl}R_{ijkl;uu}\Big)\mathrm{Id}\\
&+\frac{1}{9\cdot 7!}\Big(-35R^3 + 42RR_{lp}R_{lp} - 42RR_{klpq}R_{klpq} + 208R_{jk}R_{jl}R_{kl}\\
&\quad - 192R_{jk}R_{ul}R_{jukl} + 48R_{jk}R_{julp}R_{kulp} - 44R_{ijku}R_{ijlp}R_{kulp}\\
&\quad - 80R_{ijku}R_{ilkp}R_{jlup}\Big)\mathrm{Id}\\
&+\frac{1}{360}\Big(8\Omega_{ij;k}\Omega_{ij;k} + 2\Omega_{ij;j}\Omega_{ik;k} + 12\Omega_{ij}\Omega_{ij;kk} - 12\Omega_{ij}\Omega_{jk}\Omega_{ki}\\
&\quad - 6R_{ijkl}\Omega_{ij}\Omega_{kl} + 4R_{jk}\Omega_{jl}\Omega_{kl} - 5R\Omega_{kl}\Omega_{kl}\Big)\\
&+\frac{1}{360}\Big(6E_{;iijj} + 60EE_{;ii} + 30E_{;i}E_{;i} + 60E^3 + 30E\Omega_{ij}\Omega_{ij} - 10RE_{;kk}\\
&\quad - 4R_{jk}E_{;jk} - 12R_{;k}E_{;k} - 30RE^2 - 12R_{;kk}E + 5R^2E\\
&\quad - 2R_{jk}R_{jk}E + 2R_{ijkl}R_{ijkl}E\Big)\Big\}.
\end{aligned}$$

[3] We use the convention

$$\left[\nabla_{\partial/\partial x_i}, \nabla_{\partial/\partial x_j}\right]\partial/\partial x_k = R_{ijk}{}^l\partial/\partial x_l,$$

for the Riemann curvature tensor and its components. Moreover, $R_{ijkl} = g_{ln}R_{ijk}{}^n$ and the Ricci and scalar curvatures are given by

$$R_{jk} = R_{iji}{}^k \text{ and } R = g^{ii}R_{ii}.$$

Here, all the tensors are written in a normal coordinates passing through the base point x. Also, Ω is the curvature two form of the connection given in (1.18).

The higher order coefficients are more complicated and cumbersome. They are only available up to a_{10} and usually their computations requires newer techniques. Avramidi computed a_8 using covariant techniques [2] and van de Ven [35] gives formulas up to a_{10} applying various differential techniques. van de Ven employs a new notation, free of space-time indices notation, which makes the format of his formulas different than the one presented here.

By the local formula given in the above theorem, it is easy to see that

$$a_0(P) = \frac{\mathrm{rank}(V)\mathrm{vol}(M)}{(4\pi)^{m/2}}.$$

Using Karamata's theorem, the Weyl's law for general closed manifolds can be proven using the heat kernel asymptotic expansion.

Corollary 1.23. *[3, Corollary 2.43] The eigenvalue counting function $N(\lambda)$ of P satisfies the following*

$$N(\lambda) \sim \frac{\mathrm{rank}(V)\mathrm{vol}(M)}{(4\pi)^{m/2}\Gamma(m/2+1)}\lambda^{m/2} \quad \lambda \to \infty.$$

Remark 1.24. The integral of the divergence terms in the spectral invariants, e.g. $R_{;kk}$ or $E_{;kk}$, vanishes when M is a closed manifold. However, keeping them is important when we want to localize the heat kernel $\mathrm{Tr}(Fe^{-tP})$ by an endomorphism $F \in C^\infty(\mathrm{End}(V))$. The endomorphism F, called the *smearing endomorphism*, does not change the form of the heat kernel asymptotic expansion, i.e. similar to (1.16) we have

$$\mathrm{Tr}(Fe^{-tP}) \sim_{t\to 0} \sum a_n(F,P)t^{\frac{n-m}{d}}. \tag{1.22}$$

The coefficients are given by

$$a(x,F,P) = \mathrm{tr}(Fe_n(x,P)). \tag{1.23}$$

The endomorphism valued functions $e_n(x,P)$ are those function whose trace give the spectral invariants, i.e. $a_n(x,P) = \mathrm{tr}(e_n(x,P))$.

The other spectral function assigned to a positive elliptic operator P is the *spectral zeta function* defined by

$$\zeta(s,P) = \mathrm{Tr}(P^{-s}), \quad \Re(s) >> 0, \tag{1.24}$$

whose smeared version is $\zeta(s,F,P) = \mathrm{Tr}(FP^{-s})$. Knowing the format of the heat kernel and using the Mellin transform

$$\zeta(s,P) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\mathrm{Tr}(e^{-tP}) - \dim\ker(P) \right) dt,$$

it can be shown that the zeta function extends to a meromorphic function with simple poles. Its values, residues and derivatives have important applications in theoretical and mathematical physics. These values are related to the coefficients of the asymptotic expansion (1.22) by the following proposition.

Proposition 1.25. *(cf. [19]) Let P be an elliptic d^{th} order positive partial differential operator. Then the zeta function $\zeta(s,F,P)$ has a meromorphic extension to $\mathbb{C}$ with possible simple poles at $s = (m-n)/d$ for $n = 0,1,2,\cdots$. Furthermore,*

$$a'_n(F,P) = \mathrm{Res}_{s=\frac{m-n}{d}} \Big(\Gamma(s)\zeta(s,F,P) \Big), \tag{1.25}$$

where

$$a'_n(F,P) = \begin{cases} a_m(F,P) - \dim\ker(P) & \text{if } n = m \\ a_n(F,P) & \text{if } n \neq m \end{cases}.$$

By (1.23) and (1.25), for a Laplace type positive operator P and a smearing endomorphism F, we have

$$\int_M \mathrm{tr}(F)\mathrm{dvol}_g = (4\pi)^{m/2}\Gamma(m/2)\mathrm{Res}_{s=m/2}\zeta(s,F,P). \tag{1.26}$$

All the residues of the zeta function can be expressed as the Wodzicki residue of some power of P, and this justifies the use of the word residue in the Wodzicki residue. To see this, we note that the family $\{FP^{-s}\}$ is a holomorphic family of classical pseudodifferential operators . By Theorem 1.10 and the uniqueness of analytic continuation, the following equality holds:

$$\zeta(s,F,P) = \mathrm{TR}(FP^{-s}).$$

Substituting the order function $\alpha(z) = -z$ of this family into equation (1.7) gives us

$$\mathrm{Res}(FP^{(n-m)/2}) = \mathrm{Res}_{s=\frac{m-n}{2}}\zeta(s, F, P), \quad n < m. \tag{1.27}$$

As a result, by Connes' trace formula, the residue of the highest pole $s = m/2$ is given by the Dixmier trace and we can rewrite (1.26) as follows:

$$\int \mathrm{tr}(F)\mathrm{dvol}_g = m(4\pi)^{m/2}\Gamma(m/2)\mathrm{Tr}_\omega(FP^{-m/2}). \tag{1.28}$$

1.4 Noncommutative Riemannian Geometry

Inspired by the spectral properties of Dirac type operators on closed manifolds and their geometric implications, Alain Connes defines elements of his noncommutative Riemannian geometry in [11, 12]. In his completely new approach, the role of metric $g_{\mu\nu}$, which defines the geometry of the space, is played by a Dirac operator. In this approach, the geometry is given by a positive operator D with a discrete spectrum and a $*$-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$. The triple $(\mathcal{A}, \mathcal{H}, D)$, which is called a *spectral triple*, encodes the spectral information of an abstract Dirac type operator D on a manifold with function algebra $\mathcal{A}$. While we can construct a spectral triple for any spin manifold which recovers the geometry of the manifold completely, nothing prevents us from assigning a spectral triple to a noncommutative algebra. In this regard, one can consider this approach as an extension of geometry to noncommutative spaces.

We shall review the elements of noncommutative Riemannian geometry in this section and main references for this section are [16, 20].

1.4.1 Spectral Triples

Definition 1.26. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by a unital $*$-algebra $\mathcal{A}$ which is represented as bounded operators of the Hilbert space $\mathcal{H}$ and a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$ with compact resolvent and bounded commutators $[D, a] \in B(\mathcal{H})$ for any $a \in \mathcal{A}$.
A spectral triple is called even if there is a $\mathbb{Z}_2$-grading γ such that $a\gamma = \gamma a$ for any $a \in \mathcal{A}$ and $\gamma D = -D\gamma$.

Let M be a closed oriented Riemannian manifold, V a Clifford module and D a self-adjoint Dirac type operator on V. Since D is an elliptic operator, it has compact resolvent as a densely defined (unbounded) operator on $\mathcal{H} = L^2(M, V)$. On the other hand, $[D, f]$ is a differential operator of order zero, so it is a bounded operator on $\mathcal{H}$ and is given by the endomorphism $c(df)$. If the Clliford module is the spinor bundle S on a spin manifold and D is the Dirac operator on the spinor bundle, then the spectral triple $(C^\infty(M), L^2(M, S), D)$ is called the *canonical spectral triple*. One can recover the geodesic distance on the spin manifold in the following algebraic way (see e.g. [11]):

$$d(x, y) = \sup\{|f(x) - f(y)| : \|[D, f]\| \leq 1\}. \tag{1.29}$$

This, in fact, guarantees that by considering the canonical spectral triple, no geometric information of M is lost.

If the dimension of M is even, the map defined by complex volume element on the spin bundle is the $\mathbb{Z}_2$-grading of the canonical spectral triple which makes it an even spectral triple.

1.4.2 Spectral Dimension and Integral

The condition that the operator D of a spectral triple has compact resolvent implies that its spectrum consists of a discrete set of eigenvalues 1.5, and if the Hilbert space is infinite dimensional, $\mu_n(D) \to \infty$. The form in which this sequence tends to infinity has several geometric implications.

Definition 1.27. A spectral triple is finitely summable when the resolvent of D has characteristic values $\mu_n = O(n^{-\alpha})$ for some $\alpha > 0$. Moreover, a finitely summable spectral triple is of metric dimension m if $\mu_n(D)$ is of order $n^{1/m}$.

By Weyl's law, Corollary 1.23, any spectral triple defined by a Dirac type operator on a manifold M is finitely summable and its spectral dimension is equal to the dimension of M.

Spectral dimension can be a non-integer number. By this possibility, noncommutative geometry can give a geometric interpretation to dimensional regularisation, which is used in modern quantum field theory. In [16, 1.19.2], Connes and Marcolli constructed spectral triples with spectral dimension for any $z \in (0, \infty)$.

For a finitely summable spectral triple, with α given in the definition, the operator $|D|^{-s}$ is trace class for all $\Re(s) > \frac{1}{\alpha}$. Therefore, $\zeta(s, |D|) = \mathrm{Tr}(|D|^{-s})$ is a well defined holomorphic function on the right half plane $\Re(s) > \frac{1}{\alpha}$. Moreover, if it has spectral dimension equal to m, then the first pole of the function will be at $s = m$ and the residue at this point is given by [10, Proposition IV.2.β.2]

$$\mathrm{Res}_{s=m}\zeta(s, |D|) = \lim_{N\to\infty} \frac{\sum_{n=0}^{N-1} (\mu_n(D))^{-m}}{\log(N)} = \mathrm{Tr}_\omega(|D|^{-m}). \tag{1.30}$$

The above equality is the counterpart of the classical one (1.28). It suggests that on a spectral triple with spectral dimension m we can define a noncommutative integral by the trace on $\mathcal{A}$ defined by

$$a \to \mathrm{Tr}_\omega(a|D|^{-m}), \quad a \in \mathcal{A}. \tag{1.31}$$

Note that equation (1.28) is true for a more general smearing endomorphism $F \in C^\infty(\mathrm{End}(V))$. Inspired by the fact that $[D, f] = c(df) \in \mathbb{C}\mathrm{l}(M) \subset C^\infty(\mathrm{End}(V))$, we can define the algebra of endomorphisms that can smear out the zeta function to be the algebra $\mathcal{B}$ generated by $\mathcal{A}$ and $[D, a]$ for all $a \in \mathcal{A}$. The problem is that the functional (1.31) is not in general a trace on $\mathcal{B}$. A sufficient condition is that any $b \in \mathcal{B}$ be in the domain of the derivation $\delta = [|D|, \cdot]$, that is,

$$\delta(b) = [|D|, b] \in B(\mathcal{H}), \quad b \in \mathcal{B}.$$

Definition 1.28. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called regular if

$$\mathcal{B} \subset \mathrm{Dom}^\infty\delta = \bigcap_k \mathrm{Dom}\ \delta^k.$$

The spectral triple coming from a Dirac type operator on a manifold is regular. This result follows from pseudodifferential theory on M. Note that $\sigma_L(|D|)$ commutes with $\sigma_L([D, a])$, which implies that $\delta([D, a])$ is a zero order pseudodifferential operator and thus is bounded. A similar argument works for $\delta^k([D, a])$.

For regular and finitely summable spectral triples all the spectral zeta functions

$$\zeta(s, b, |D|) := \mathrm{Tr}(b|D|^{-s}), \quad b \in \mathcal{B}, \tag{1.32}$$

are holomorphic on $\Re(s) > 1/m$. As we saw before in classical cases, the poles of the zeta functions $\mathrm{Tr}(FP^{-s})$ contain important geometrical information. Another notion of dimension in noncommutative geometry emerges out of this concept. Instead of taking into account only the top pole, which gives the spectral dimension, this new notion is defined to be a subset of the complex plane consisting of all singularities of the zeta functions (1.32).

Definition 1.29. [16] Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable regular spectral triple. The dimension spectrum is the subset $\Pi = \{z \in \mathbb{C}, \Re(z) \geq 0\}$ of singularities of the analytic function $\zeta(z, b, |D|)$ for all $b \in \mathcal{B}$. We say that the dimension spectrum is simple when these spectral functions have at most simple poles.

Based on (1.27), then one can define the Wodzicki residue on the algebra generated by $\mathcal{B}$ and all powers of $|D|$ by the following equality which defines a trace [16, Theorem 1.134]

$$⨍ T := \mathrm{Res}_{s=0}\mathrm{Tr}(T|D|^{-s}).$$

1.4.3 Real Structure

A real structure for spectral triples, introduced in [11], has three different origins. On one hand, it is motivated by the modular conjugate in the Tomita-Takasaki theory. On the other hand, it is a way to formulate a spin structure on manifolds. Finally, a real spectral triple on $\mathcal{A}$ determines a class in the KO-homology of $\mathcal{A}$.

Definition 1.30. A real structure of KO-dimension m mod 8 on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an anti-linear isometry J such that

$$J^2 = \epsilon \text{ and } JD = \epsilon' DJ,$$

where $\epsilon, \epsilon' \in \{\pm 1\}$ are given by Table 1.1. Moreover, for any $a, b \in \mathcal{A}$ we have

- Order zero condition $[a, Jb^*J^{-1}] = 0$,
- Order one condition $[[D, a], Jb^*J^{-1}] = 0$.

A spectral triple with a real structure is called a real spectral triple.

n mod 8	0	1	2	3	4	5	6	7
$J^2 = \epsilon$	1	1	-1	-1	-1	-1	1	1
$JD = \epsilon' DJ$	1	-1	1	1	1	-1	1	1
$J\gamma = \epsilon'' \gamma J$	1		-1		1		-1	

TABLE 1.1: KO-dimension of a real structure

The charge conjugate on spinors is the real structure on the canonical spectral triples. In this case the second and third rows of Table 1.1 result from Theorem 1.15. Moreover, J_m commutes with any function $f \in C^\infty(M) \subset \mathbb{C}l(m)$, hence $Jf^*J^{-1} = f$. Thus, by commutativity of $C^\infty(M)$, we have both the order zero and the order one conditions for free.

The key role of a real structure is in defining a right $\mathcal{A}$-module structure for $\mathcal{H}$ by

$$\xi b := Jb^*J^{-1}\xi.$$

The order zero condition specifies that left and right multiplication commute, hence $\mathcal{H}$ is an $\mathcal{A}$-bimodule. In addition, the order one condition (together with the order zero condition) makes $\mathcal{H}$ an $\Omega^1_D - \mathcal{A}$ bimodule. For a real spectral triple one can define the adjoint action of the unitary group of the algebra $\mathcal{A}$ on the Hilbert space as

$$Ad(u)(\xi) = u\xi u^* = JuJ^{-1}u\xi, \quad \xi \in \mathcal{H}, u \in \mathcal{U}(\mathcal{A}).$$

1.4.4 Reconstruction Theorem

The spectral characterization of manifolds, as a more elaborated version of its topological counterpart, i.e. Gelfand-Naimark theorem, was discussed in [12], and axioms for noncommutative spin geometry were also introduced. It was claimed that a commutative spectral triple satisfying these axioms is (equivalent to) the canonical spectral triple of a spin manifold. The complete proof of this claim was given recently by Connes [14].

The axioms for commutative spin geometry are as follows (for the general version see [12]).

1. *Dimension:* the spectral dimension is a non-negative integer m.

2. *First order condition:* For the commutative case this property can be considered without having a real structure and it reads as follows:

$$[[D, f], h] = 0, \qquad f, h \in \mathcal{A}.$$

3. *Regularity:* $\mathcal{B} \subset \mathrm{Dom}^\infty \delta$.

4. *Orientability:* There exists a Hochschild cycle $c = a^0 \otimes a^1 \otimes \cdots a^m \in Z_m(\mathcal{A}, \mathcal{A})$ such that

$$\pi_D(c) := [D, a^0][D, a^1] \cdots [D, a^m] = \begin{cases} \gamma & \text{if } m = 2k, \\ I & \text{if } m = 2k+1. \end{cases}$$

5. *Finiteness and absolute continuity:* Viewed as an $\mathcal{A}$-module the space $\mathcal{H}^\infty = \bigcap_k \mathrm{Dom} D^k$ is finite and projective. Moreover, the following equality defines a Hermitian structure $(\ , \ )$ on this module:

$$\langle \xi, a\eta \rangle = \fint a(\xi, \eta)|D|^{-m} \quad \forall a \in \mathcal{A}, \xi, \eta \in \mathcal{H}^\infty.$$

6. *Poincaré duality:* The intersection form $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$ is invertible.

7. *Reality:* There is a real structure on the spectral triple.

In [12], Connes proves that these axioms on a spectral triple whose algebra is the space of smooth functions of a smooth manifold M defines a Riemannian metric g such that the distance formula (1.29) gives the geodesic distance of (M, g). Moreover, the unique minimizer of the functional $\fint D^{2-m}$ on the space of all such spectral triples fixes a compatible spin structure on (M, g). The canonical spectral triple associated to this spin structure is unitary equivalent to the original spectral triple.

The remaining part of the proof was to show how a smooth structure on M can be defined using these axioms. To construct only the smooth structure a fewer number of axioms is needed as shown in the following theorem.

Theorem 1.31. *[14] Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, with $\mathcal{A}$ commutative, fulfilling the first five conditions in a slightly stronger form, i.e. we assume that*

- *The regularity holds for all A-endomorphisms of* $\mathrm{Dom} D^k$,

- *The Hochschild cycle c is antisymmetric.*

Then there exists a compact oriented smooth manifold M such that $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M.

A variant of this theorem is also proved for spin^c manifolds [14, Theorem 11.5]. In the characterizing theorem of spin^c manifolds, a weaker form of the Poincaré duality condition is assumed, in addition to the assumptions of the above theorem.

1.5 Action Functional in Noncommutative Geometry

From the beginning, noncommutative geometry was believed to have deep applications in physics. Searching for the right formulation of action, as the innermost notion of modern physics, in noncommutative geometry started from the very early stages [8] and later evolved to what now is called the spectral action functional. In this section we will discuss the three main milestones achieved toward the final formulation of the spectral principal and spectral action introduced in [5]. There is a huge list of references for this topic, among which we suggest [16] and [36].

1.5.1 Action Functional in Noncommutative Geometry

The first formulation of the action for noncommutative geometry was a spectral formulation of the Yang-Mills action [8]. By (1.28) we know that for a Hermitian connection ∇ on a Hermitian vector bundle V over a spin manifold M we have

$$YM(\nabla) = c\mathrm{Tr}_\omega(\overline{F}_{ij}F_{ij}|D|^{-m}).$$

Here F is the curvature 2-form of ∇. To generalize this formula for any spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we have to make sense of forms (at least up to 2-forms) as operators on the Hilbert space and the associated inner product.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. The reduced universal differential graded algebra $\Omega^*\mathcal{A}$ over $\mathcal{A}$ is $\bigoplus \Omega^k\mathcal{A}$ where $\Omega^k\mathcal{A} = \{\sum a^0 da^1 \dots da^k;\ a^j \in \mathcal{A}\}$. The differential map $d : \Omega^k\mathcal{A} \to \Omega^{k+1}\mathcal{A}$ is defined on monomials by $d(a^0 da^1 \dots da^k) = da^0 da^1 \dots da^k$ and

extends uniquely on $\Omega^*\mathcal{A}$ via the following properties

$$\begin{aligned} d^2 &= 0, \\ d(\omega_1\omega_2) &= d(\omega_1)\omega_2 + (-1)^{\deg \omega_1}\omega_1(d\omega_2), \\ (da)^* &= -da^*. \end{aligned}$$

The map

$$\pi(a^0 da^1 \dots da^k) = a^0[D, a^1]\dots[D, a^k]$$

represents this algebra on $\mathcal{H}$. However, there is an ambiguity if we define $da = [D, a]$ which can be solved by considering the quotient algebra

$$\Omega_D^*\mathcal{A} = \Omega^*\mathcal{A}/(\ker\pi + d\ker\pi).$$

Note that π defines an isomorphism between Ω_D^k and the subalgebra

$$\pi(\Omega^k\mathcal{A})/\pi\big(d(\ker\pi \cap \Omega^k\mathcal{A})\big).$$

For the canonical spectral triple, Ω_D^* is isomorphic to de Rham algebra of forms on the manifold by the isomorphism

$$f^0 df^1 \dots df^k \mapsto f^0 df^1 \cdot df^2 \cdot \dots \cdot df^k,$$

where df^k is considered as a section of the Clifford bundle. Moreover, the inner product

$$\langle T_1, T_2\rangle = \mathrm{Tr}_\omega(T_2^* T_1 |D|^{-m}), \quad T_1, T_2 \in \pi(\Omega^k\mathcal{A}),$$

on $\Omega^k\mathcal{A}$, induces an inner product on Ω_D^k as a quotient space. This inner product, under the above isomorphism, is equal to a constant multiple of the inner product on the k-forms, defined by $\langle\omega_1, \omega_2\rangle_k = \int_M \omega_1 \wedge *\omega_2$. In other words,

$$\|\omega\|_k^2 = \inf\{\mathrm{Tr}_\omega(\alpha^*\alpha|D|^{-m});\ \pi_D(\alpha) = \omega\}.$$

Theorem 1.32. *[10] The functional* $YM(A) = \mathrm{Tr}_\omega\Big((dA + A^2)^*(dA + A^2)|D|^{-m}\Big)$ *is positive, quartic and invariant under gauge transformations,i.e.,*

$$\gamma_u(A) = udu^* + uAu^* \quad \forall u \in \mathcal{U}(\mathcal{A}). \tag{1.33}$$

The functional

$$I(\alpha) = Tr_\omega(\theta^2 |D|^{-m}), \tag{1.34}$$

with $\theta = \pi(d\alpha + \alpha^2)$*, is positive, quartic and gauge invariant on* $\{\alpha \in \Omega^1\mathcal{A};\ \alpha = \alpha^*\}$*. Moreover, one has*[4]

$$YM(A) = \inf_\alpha \{I(\alpha);\ \pi(\alpha) = A\}. \tag{1.35}$$

This formulation of the Yang-Mills action applied to canonical spectral triples gives the the classical Yang-Mills action. On the other hand, on a toy model,

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix};\ a, b \in \mathbb{C} \right\},\ \mathcal{H} = \mathbb{C}^2,\ D = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix},$$

produces a typical expression of the Higgs potential given by $(|\varphi|^2+1)^2$ [10]. The following quotation from D. Kastler best explains how this can improve our understanding about the puzzling piece, i.e. the Higgs boson, of the standard model.

> This at once enlightens our physical picture: the world is two-sheeted, the mysterious Higgs is nothing but a gauge boson, however needing noncommutative geometry to be recognized as such because the corresponding potential is not a connection within the realm of classical differential geometry, but a *discrete connection* (so-to-speak with parallel transport jumping from one world-sheet to the other). [24, pp. 3869]

By constructing a spectral triple on $C^\infty(M) \otimes \mathcal{A}_F$, where $\mathcal{A}_f$ is a finite dimensional algebra, and using (1.35), Connes and Lott [15] were able to produce the bosonic electroweak sector of the standard model.

The action of general relativity was not included in the Connes-Lott model. The first step to include gravity sector in the spectral theory, is to reproduce the Einstein-Hilbert action from the spectrum of the Dirac operator. It was proposed by Connes and shown in detail by Kastler in [23] for dimension $m = 4$ and for the more general case $m \geq 4$ by Kalau and Walze in [22] that the Einstein-Hilbert action is given by a multiple of the Wodzicki residue of D^{-m+2}. Unlike [22, 23] in which the Wodzicki residue is computed

[4] Originally [8], the map $da \mapsto i[D|D|^{-1}, a]$ was used to quantize one forms. In this approach one forms were presented by elements of the ideal $\mathcal{L}^{m+}$ and the Yang-Mills action on dimension $m = 4$ was given by $\inf \mathrm{Tr}_\omega(\theta^2) = YM(\alpha)$. On the higher dimensions, it changes to $\mathrm{Tr}_\omega(\theta^{m/2})$ which is not quadratic in θ anymore.

by explicitly computing $\sigma_{-4}(D^{-m+2})$, we will use the local invariant formula and prove the theorem. A similar proof can be found in [1].

Theorem 1.33. *Let (M, g) be a Riemannian manifold of dimension $m > 2$ and D be a Dirac type operator defined by a Clifford connection ∇ on a Clifford module V. Then*

$$\mathrm{Res}(D^{-m+2}) = \frac{\mathrm{rank}(V)}{6(4\pi)^{m/2}\Gamma(\frac{m-2}{2})} I_{EH}(g), \tag{1.36}$$

where $I_{EH}(g) = \int_M R_g \mathrm{dvol}_g$ is the Einstein-Hilbert action on the metric g.

Proof. By the generalized Lichnerowicz formula given by Theorem 1.21 we have

$$D^2 = \nabla^*\nabla + c(F^{V/S}) - \frac{R}{4},$$

where $F^{V/S} \in \Omega^2(M, End_{\mathbb{C}l(M)}(V))$. That is, any element of the Clifford algebra, in particular, any one form, commutes with $F^{V/S}$. This implies that for any $i \neq j$, we have

$$\mathrm{tr}(c(e^i)c(e^j)F_{ij}^{V/S}) = \mathrm{tr}(c(e^j)F_{ij}^{V/S}c(e^i)) = -\mathrm{tr}(c(e^i)c(e^j)F_{ij}^{V/S}).$$

Hence $c(F^{V/S})$ is traceless and the second heat invariant $a_2(x, D^2)$ is given by

$$a_2(x, D^2) = (4\pi)^{m/2}\mathrm{tr}\Big(- R/6 - \frac{1}{2}c(F^{V/S}) + \frac{R}{4}\Big) = \frac{\mathrm{rank}(V)}{12(4\pi)^{m/2}} R. \tag{1.37}$$

Therefore, $a_2(D^2) = \frac{\mathrm{rank}(V)}{12(4\pi)^{m/2}} I_{EH}(g)$. On the other hand, by (1.25) and (1.27) we have

$$\begin{aligned}
\mathrm{Res}(D^{-(m-2)}) &= 2\mathrm{Res}_{s=\frac{m-2}{2}}\zeta(s, D^2) \\
&= \frac{2}{\Gamma(\frac{m-2}{2})} a_2(D^2) \\
&= \frac{\mathrm{rank}(V)}{6(4\pi)^{m/2}\Gamma(\frac{m-2}{2})} I_{EH}(g).
\end{aligned}$$

□

Note that (1.36) is true for any Dirac operator defined by a Clifford connection.

1.5.2 Spectral Action and Einstein-Yang-Mills System

Despite the fact that both the Yang-Mills (1.34) and Einstein-Hilbert (1.36) actions are formulated spectrally, they are of a different computational nature. In (1.34) we are computing the first coefficient of the smeared heat kernel $\mathrm{Tr}(\theta^2 e^{-tD^2})$, in which, other than the Dirac operator D, we computed θ^2 by hand and plugged it into the formula. In turn, equation (1.36), for which we computed the second heat kernel coefficient, only needs a Dirac operator. The next coefficient of the heat kernel of D^2 have terms like $\Omega_{kl}\Omega_{kl}$, which are similar to the Yang-Mills Lagrangian. Indeed, in addition to $a_2(D^2_\nabla)$ producing the Einstein-Hilbert action of the metric g, we will show that $a_4(D^2_\nabla)$ contains the Yang-Mills action of ∇.

First, Note that the endomorphism E of D^2_∇ and the curvature Ω of twisted connection is given by

$$E = \frac{1}{4}R - \frac{1}{2}c(e^k)c(e^l) \otimes F_{kl}, \tag{1.38}$$

$$\Omega_{ij} = \frac{1}{4}R_{ijkl}c(e^k)c(e^l) \otimes \mathrm{Id}_S + \mathrm{Id}_V \otimes F_{ij}. \tag{1.39}$$

To rewrite $a_4(x, D^2_\nabla)$ in terms of the Riemann curvature tensor and the curvature F of ∇, we need the following computations:

$$\begin{aligned}
\mathrm{tr}(RE) &= \mathrm{rank}(V)2^{[m/2]-2}R^2, \\
\mathrm{tr}(E^2) &= \mathrm{rank}(V)2^{[m/2]-4}R^2 - 2^{[m/2]-1}\mathrm{tr}(F_{kl}\overline{F}_{kl}), \\
\mathrm{tr}(\Omega_{ij}\Omega_{ij}) &= \mathrm{rank}(V)2^{[m/2]-2}R_{ijkl}R_{ijkl} - 2^{[m/2]}\mathrm{tr}(F_{ij}\overline{F}_{ij}).
\end{aligned}$$

Note that $2^{[m/2]}$ is the dimension of the spinor bundle and we have used the property $\overline{F}_{ij} = F_{ji} = -F_{ij}$. By substituting these into the formula of $a_4(x, D^2)$ at dimension $m = 4$ we find that

$$\begin{aligned}
a_4(x, D^2) &= \frac{\mathrm{rank}(V)}{16\pi^2}\Big(- \frac{1}{20}C_{ijkl}C_{ijkl} + \frac{11}{360}E_4\Big) + \frac{1}{24}\mathrm{tr}(F_{ij}\overline{F}_{ij}) \\
&\quad + \frac{\mathrm{rank}(V)}{360(16\pi^2)}\Big(- 48R_{;kk} + 60\mathrm{tr}(E_{;kk})\Big).
\end{aligned} \tag{1.40}$$

Here, C_{ijkl} is the Weyl curvature and $E_4 \mathrm{dvol}_g = \frac{1}{2}(R^2 - 4R_{ij}R_{ij} + R_{ijkl}R_{ijkl})\mathrm{dvol}_g$ is the Pfaffian (or Euler) form. By the Chern-Gauss-Bonnet theorem we have

$$\chi(M) = \frac{1}{16\pi^2}\int_M E_4 \mathrm{dvol}_g.$$

Now, both actions are presented spectrally, and unlike the first formulation of the Yang-Mills action, we can have them simultaneously by considering the asymptotic expansion of a function such as $\mathrm{Tr}(e^{-D^2/\Lambda^2})$. This guides us toward a very strong hypothesis called the spectral action principle introduced by Connes and Chamseddine [5]:

The physical action only depends on the spectrum of the Dirac operator of the spectral triple that models the theory.

Imposing the condition that the action has to add up when evaluated on the direct sum of the geometric spaces, one can see that the fundamental action functional has to be of the form

$$\mathrm{Tr}(f(D/\Lambda)), \tag{1.41}$$

where f is a positive function of real variable and Λ is the mass scale. As the following Theorem shows, f plays a small role in this action.

Theorem 1.34. *[16] Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with $\ker D = \{0\}$, fulfilling*

$$\mathrm{Tr}(e^{-tD^2}) \sim \sum_\alpha a_\alpha t^\alpha.$$

Then the spectral action (1.41) can be expanded in powers of the scale Λ in the form

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta\in\Pi} f_\beta \Lambda^\beta ⨍ |D|^{-\beta} + f(0)\zeta(0, D) + \dots, \tag{1.42}$$

with the summation over the dimension spectrum Π. Here the function f only appears through the scalars $f_\beta = \int_0^\infty f(v)v^{\beta-1}dv$. The terms involving negative powers of Λ involve the full Taylor expansion of f at 0.

If the spectral triple is given by a twisted Dirac type operator on a 4 dimensional manifold, formula (1.42) can be rewritten as

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0(D^2) + 2\Lambda^2 f_2 a_2(D^2) + f_0 a_4(D^2) + \cdots + \Lambda^{-2k} f_{-2k} a_{4+2k}(D^2) + \dots. \tag{1.43}$$

This, together with (1.40) and (1.36), gives rise to the following spectral formula for the *Einstein-Yang-Mills* action [16, Theorem 1.158]:

$$\mathrm{Tr}(f(D_\nabla/\Lambda)) \sim \frac{1}{4\pi^2}\int_M \mathcal{L}(g_{\mu\nu}, A)\mathrm{dvol}_g, \tag{1.44}$$

where $\mathcal{L}(g_{\mu\nu}, A)$ is the Lagrangian given by

$$\mathcal{L}(g_{\mu\nu}, A) = \tag{1.45}$$
$$2\mathrm{rank}(V)\Lambda^4 f_4 + \frac{\mathrm{rank}(V)}{6}\Lambda^2 f_2 R + \frac{f(0)}{6}\mathrm{tr}(F_{ij}\overline{F}_{ij}) - \frac{\mathrm{rank}(V)f(0)}{80}C_{ijkl}C^{ijkl}$$

modulo topological, i.e. $\chi(M)$, and boundary, e.g. $\int_M R_{;kk}\mathrm{dvol}_g$, terms.

This procedure enables us to create a Lagrangian on geometries defined by spectral triples. The missing point, however, is how to twist an abstract Dirac operator with a connection in this new setting.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and $\mathcal{A}$ be Morita equivalent to $\mathcal{B}$, i.e. there is a finite projective (right) module $\mathcal{E}$ such that $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$. If we fix a Hermitian connection, $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ on $\mathcal{E}$, we can define a spectral triple on $\mathcal{B}$ with Hilbert space $\mathcal{H}' = \mathcal{E} \otimes \mathcal{H}$ and its Dirac operator, D', is given by

$$D'(\xi \otimes \eta) = \xi \otimes D\eta + \nabla(\xi)\eta.$$

Any algebra is Morita self-equivalent with $\mathcal{E} = \mathcal{A}$ and a connection on $\mathcal{A}$ is determined by its value at the identity, i.e. $\omega = \nabla(1)$. We call ω an *inner fluctuation*. The Dirac operator D', which is called *fluctuated Dirac operator*, is given by $D_\omega = D + \omega$. In the case of real spectral triples, the above construction produces fluctuated Dirac operators of the form

$$D_\omega = D + \omega + \epsilon' J\omega J^{-1}.$$

More details of this construction and also all the entries of the following table can be found in [36]. Note that the last row implies that the spectral action is invariant under the local gauge transformations

Gauge fields	Vector potential A	Inner fluctuations ω
Twisted Dirac operator	D_∇	$D_\omega = D + \omega$
Local gauge group	$\mathcal{G} = C^\infty(M, G)$	$\mathcal{U}(\mathcal{A})$
Action on fields	$A'_\mu = gA_\mu g^{-1} + (\partial_\mu g)g^{-1}$	$\omega^u = u\omega u^* + u[D, u^*]$
Action on Dirac operators	$g^{-1}D_\nabla g$	$D_{\omega^u} = UDU^*$ $U = \pi(u)J\pi(u)J^{-1}$

1.5.3 Symmetries and Standard Model Through Noncommutative Geometry

The action of the standard model I_{SM} together with the Einstein-Hilbert action I_{EH} encodes the physics of the low energy. The main difference of these two actions, besides the difference of the fields involved in each one, is the symmetries that each one is required to satisfy. The symmetries of Einstein-Hilbert action is the diffeomorphism group $\mathrm{Diff}(M)$, which is exactly the group $\mathrm{Aut}(C^\infty(M))$. The group of symmetries of I_{SM} is the local gauge group $\mathcal{U} = C^\infty(M, \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))$. Hence, a unified theory should have a symmetry group G which is the semidirect product of $\mathrm{Diff}(M)$ and $\mathcal{U}$ coming from the following exact sequence of groups

$$1 \to \mathcal{U} \to G \to \mathrm{Diff}(M) \to 1. \tag{1.46}$$

To have a geometric theory that contains both general relativity and standard model, beside $\mathrm{Diff}(M)$, one needs to deal with $\mathcal{U}$ as geometric symmetries as well. In other words, we want to have an algebra of fields $\mathcal{A}$ such that $G = \mathrm{Aut}(\mathcal{A})$. Note that $\mathrm{Aut}(A)$, similar to G, is a semidirect product of the inner and outer automorphisms of $\mathcal{A}$ and is given by the following exact sequence:

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1. \tag{1.47}$$

If $\mathcal{A}$ is commutative then $\mathrm{Inn}(\mathcal{A})$ is trivial. Hence, an algebra $\mathcal{A}$ with symmetry group equal to G cannot be commutative. Connes and Chamseddine [4, 6, 7, 13] were able to find a finite dimensional algebra $\mathcal{A}_F$ such that $\mathcal{A} = C^\infty(M) \times \mathcal{A}_F$ has symmetry group equal to G. Moreover, with the tools of noncommutative geometry one can define a geometry on $\mathcal{A}$ by considering a spectral triple of the following form:

$$(\mathcal{A}, L^2(M, S) \otimes \mathcal{H}_F, D = D \otimes 1 + \gamma^5 \otimes D_F).$$

Here $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, where γ^j are 4×4 gamma matrices. The finite dimensional Dirac operator D_F contains all coupling constants of the standard model and the spectral action on a fluctuated Dirac operator, $\mathrm{Tr}\big(f(D_\omega/\Lambda)\big)$, produces the bosonic part of the action functional $I_{EH} + I_{SM}$ and the complete action is given by

$$\mathrm{Tr}\big(f(D_\omega/\Lambda)\big) + \frac{1}{2}\langle D_\omega\psi, J\psi\rangle.$$

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Chapter 2

Rationality of Spectral Action for Robertson-Walker Metrics

2.1 Introduction

Noncommutative geometry in the sense of Alain Connes [11] has provided a paradigm for geometry in the noncommutative setting based on spectral data. This generalizes Riemannian geometry [14] and incorporates physical models of elementary particle physics [5, 7, 10, 12, 13, 15, 19, 32–34]. An outstanding feature of the spectral action defined for noncommutative geometries is that it derives the Lagrangian of the physical models from simple noncommutative geometric data [4, 10, 13]. Thus, various methods have been developed for computing the terms in the expansion in the energy scale Λ of the spectral action [3, 6, 8, 9, 20, 21]. Potential applications of noncommutative geometry in cosmology have recently been carried out in [16, 22, 25–31].

Noncommutative geometric spaces are described by spectral triples $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive algebra represented by bounded operators on a Hilbert space $\mathcal{H}$, and D is an unbounded self-adjoint operator acting in $\mathcal{H}$ [11]. The operator D, which plays the role of the Dirac operator, encodes the metric information and it is further assumed that it has bounded commutators with elements of $\mathcal{A}$. It has been shown that if $\mathcal{A}$ is commutative and the triple satisfies suitable regularity conditions then $\mathcal{A}$ is the algebra of smooth functions on a spin^c manifold M and D is the Dirac operator acting in the Hilbert space of L^2-spinors [14]. In this case, the Seeley-de Witt coefficients $a_n(D^2) = \int_M a_n(x, D^2)\, dv(x)$, which vanish for odd n, appear in a small time asymptotic

expansion of the form

$$\mathrm{Tr}(e^{-sD^2}) \sim s^{-\dim(M)/2} \sum_{n\geq 0} a_{2n}(D^2) s^n \qquad (s \to 0).$$

As noted in (1.17), coefficients $a_{2n}(D^2)$ are of the form $\int_M a_{2n}(x, D^2) \mathrm{dvol}_g$. These coefficients determine the terms in the expansion of the spectral action. That is, there is an expansion of the form

$$\mathrm{Tr} f(D^2/\Lambda^2) \sim \sum_{n\geq 0} f_{2n}\, a_{2n}(D^2/\Lambda^2),$$

where f is a positive even function defined on the real line, and f_{2n} are the moments of the function f [3, 4]. See Theorem 1.145 in [15] for details in a more general setup, namely for spectral triples with simple dimension spectrum.

By devising a direct method based on the Euler-Maclaurin formula and the Feynman-Kac formula, Chamseddine and Connes have initiated in [9] a detailed study of the spectral action for the Robertson-Walker metric with a general cosmic scale factor $a(t)$. They calculated the terms up to a_{10} in the expansion and checked the agreement of the terms up to a_6 against Gilkey's universal formulas [17, 18].

The present paper is intended to compute the term a_{12} in the spectral action for general Robertson-Walker metrics, and to prove the conjecture of Chamseddine and Connes [9] on rationality of the coefficients of the polynomials in $a(t)$ and its derivatives that describe the general terms a_{2n} in the expansion. In passing, we compare the outcome of our computations up to the term a_{10} with the expressions obtained in [9], and confirm their agreement.

In terms of the above aims, explicit formulas for the Dirac operator of the Robertson-Walker metric and its pseudodifferential symbol in Hopf coordinates are derived in §2.2. Following a brief review of the heat kernel method for computing local invariants of elliptic differential operators using pseudodifferential calculus [17], we compute in §2.3 the terms up to a_{10} in the expansion of the spectral action for Robertson-Walker metrics. The outcome of our calculations confirms the expressions obtained in [9]. This forms a check in particular on the validity of a_8 and a_{10}, which as suggested in [9] also, seems necessary due to the high complexity of the formulas. In §2.4, we record the expression for the term a_{12} achieved by a significantly heavier computation, compared to the previous terms. It is checked that the reduction of a_{12} to the round case $a(t) = \sin t$ conforms to

the full expansion obtained in [9] for the round metric by remarkable calculations that are based on the Euler-Maclaurin formula. In order to validate our expression for a_{12}, parallel but completely different computations are performed in spherical coordinates and the final results are confirmed to match precisely with our calculations in Hopf coordinates.

In §2.5, we prove the conjecture made in [9] on rationality of the coefficients appearing in the expressions for the terms of the spectral action for Robertson-Walker metrics. That is, we show that the term a_{2n} is of the form $Q_{2n}\big(a(t), a'(t), \dots, a^{(2n)}(t)\big)/a(t)^{2n-3}$, where Q_{2n} is a polynomial with rational coefficients. Note that, $a_{2n} = \int_{\mathbb{S}^3_{a(t)}} a_{2n}(x, D^2)\text{dvol}$, and it is indeed t-dependent. We also find a formula for the coefficient of the term with the highest derivate of $a(t)$ in a_{2n}. It is known that values of Feynman integrals for quantum gauge theories are closely related to multiple zeta values and periods in general and hence tend to be transcendental numbers [24]. In sharp distinction, the rationality result proved in this paper is valid for all scale factors $a(t)$ in Robertson-Walker metrics. Although it might be exceedingly difficult, it is certainly desirable to find all the terms a_{2n} in the spectral action. The rationality result is a consequence of a certain symmetry in the heat kernel and it is plausible that this symmetry would eventually reveal the full structure of the coefficients a_{2n}. This is a task for a future work. Our main conclusions are summarized in §2.6.

2.2 The Dirac Operator for Robertson-Walker Metrics

According to the spectral action principle [4, 12], the spectral action of any geometry depends on its Dirac operator since the terms in the expansion are determined by the high frequency behavior of the eigenvalues of this operator. For spin manifolds, the explicit computation of the Dirac operator in a coordinate system is most efficiently achieved by writing its formula after lifting the Levi-Civita connection on the cotangent bundle to the spin connection on the spin bundle. In this section, we summarize this formalism and compute the Dirac operator of the Robertson-Walker metric in Hopf coordinates. Throughout this paper we use Einstein's summation convention without any further notice.

2.2.1 Levi-Civita connection.

The spin connection of any spin manifold M is the lift of the Levi-Civita connection for the cotangent bundle T^*M to the spin bundle. Let us, therefore, recall the following recipe for computing the Levi-Civita connection and thereby the spin connection of M. Given an orthonormal frame $\{\theta_\alpha\}$ for the tangent bundle TM and its dual coframe $\{\theta^\alpha\}$, the connection 1-forms ω^α_β of any connection ∇ on T^*M are defined by

$$\nabla\theta^\alpha = \omega^\alpha_\beta\, \theta^\beta.$$

Since the Levi-Civita connection is the unique torsion free connection which is compatible with the metric, its 1-forms are uniquely determined by

$$d\theta^\beta = \omega^\beta_\alpha \wedge \theta^\alpha.$$

This is justified by the fact that the compatibility with metric enforces the relations

$$\omega^\alpha_\beta = -\omega^\beta_\alpha,$$

while, taking advantage of the first Cartan structure equation, the torsion-freeness amounts to the vanishing of

$$T^\alpha = d\theta^\alpha - \omega^\alpha_\beta \wedge \theta^\beta.$$

2.2.2 The spin connection of Robertson-Walker metrics in Hopf coordinates.

The (Euclidean) Robertson-Walker metric with the cosmic scale factor $a(t)$ is given by

$$ds^2 = dt^2 + a^2\,(t)\, d\sigma^2,$$

where $d\sigma^2$ is the round metric on the 3-sphere $\mathbb{S}^3$. It is customary to write this metric in spherical coordinates, however, for our purposes which will be explained below, it is more convenient to use the Hopf coordinates, which parametrize the 3-sphere $S^3 \subset \mathbb{C}^2$ by

$$z_1 = e^{i\phi_1}\sin(\eta), \qquad z_2 = e^{i\phi_2}\cos(\eta),$$

with η ranging in $[0, \pi/2)$ and ϕ_1, ϕ_2 ranging in $[0, 2\pi)$. The Robertson-Walker metric in the coordinate system $x = (t, \eta, \phi_1, \phi_2)$ is thus given by

$$ds^2 = dt^2 + a^2(t)\left(d\eta^2 + \sin^2(\eta) d\phi_1^2 + \cos^2(\eta) d\phi_2^2\right).$$

An orthonormal coframe for ds^2 is then provided by

$$\theta^1 = dt, \qquad \theta^2 = a(t)\, d\eta, \qquad \theta^3 = a(t)\, \sin\eta\, d\phi_1, \qquad \theta^4 = a(t)\, \cos\eta\, d\phi_2.$$

Applying the exterior derivative to these forms, one can easily show that they satisfy the following equations, which determine the connection 1-forms of the Levi-Civita connection:

$$\begin{aligned}
d\theta^1 &= 0, \\
d\theta^2 &= \frac{a'(t)}{a(t)}\, \theta^1 \wedge \theta^2, \\
d\theta^3 &= \frac{a'(t)}{a(t)}\, \theta^1 \wedge \theta^3 + \frac{\cot\eta}{a(t)}\, \theta^2 \wedge \theta^3, \\
d\theta^4 &= \frac{a'(t)}{a(t)}\, \theta^1 \wedge \theta^4 - \frac{\tan\eta}{a(t)}\, \theta^2 \wedge \theta^4.
\end{aligned}$$

We recast the above equations into the matrix of connection 1-forms

$$\omega = \frac{1}{a(t)} \begin{pmatrix} 0 & -a'(t)\, \theta^2 & -a'(t)\, \theta^3 & -a'(t)\, \theta^4 \\ a'(t)\, \theta^2 & 0 & -\cot\eta\, \theta^3 & \tan\eta\, \theta^4 \\ a'(t)\, \theta^3 & \cot\eta\, \theta^3 & 0 & 0 \\ a'(t)\, \theta^4 & -\tan\eta\, \theta^4 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(4),$$

which lifts to the spin bundle using the Lie algebra isomorphism $\mu : \mathfrak{so}(4) \to \mathfrak{spin}(4)$ given by (see [23])

$$\mu(A) = \frac{1}{4} \sum_{\alpha,\beta} \langle A\theta^\alpha, \theta^\beta \rangle c(\theta^\alpha) c(\theta^\beta), \qquad A \in \mathfrak{so}(4).$$

Since $\langle \omega\theta^\alpha, \theta^\beta \rangle = \omega^\alpha_\beta$, the lifted connection $\tilde{\omega}$ is written as

$$\tilde{\omega} = \frac{1}{4} \sum_{\alpha,\beta} \omega^\alpha_\beta c(\theta^\alpha) c(\theta^\beta).$$

In the case of the Robertson-Walker metric we find that

$$\tilde{\omega} = \frac{1}{2a(t)} \left(a'(t)\theta^2\gamma^{12} + a'(t)\theta^3\gamma^{13} + a'(t)\theta^4\gamma^{14} + \cot(\eta)\theta^3\gamma^{23} - \tan(\eta)\theta^4\gamma^{24} \right), \qquad (2.1)$$

where we use the notation $\gamma^{ij} = \gamma^i\gamma^j$ for products of pairs of the gamma matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$, which are respectively written as

$$\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

2.2.3 The Dirac Operator of Robertson-Walker metrics in Hopf coordinates.

Using the expression (2.1) obtained for the spin connection and considering the predual of the orthonormal coframe $\{\theta^\alpha\}$,

$$\theta_1 = \frac{\partial}{\partial t}, \qquad \theta_2 = \frac{1}{a(t)}\frac{\partial}{\partial \eta}, \qquad \theta_3 = \frac{1}{a(t)\,\sin\eta}\frac{\partial}{\partial \phi_1}, \qquad \theta_4 = \frac{1}{a(t)\,\cos\eta}\frac{\partial}{\partial \phi_2},$$

we compute the Dirac operator for the Robertson-Walker metric explicitly:

$$\begin{aligned} D &= c(\theta^\alpha)\nabla_{\theta_\alpha} \\ &= \gamma^\alpha\left(\theta_\alpha + \tilde{\omega}(\theta_\alpha)\right) \\ &= \gamma^1\left(\frac{\partial}{\partial t}\right) + \gamma^2\left(\frac{1}{a}\frac{\partial}{\partial \eta} + \frac{a'}{2a}\gamma^{12}\right) + \gamma^3\left(\frac{1}{a\sin(\eta)}\frac{\partial}{\partial \phi_1} + \frac{a'}{2a}\gamma^{13} + \frac{\cot(\eta)}{2a}\gamma^{23}\right) \\ &\quad + \gamma^4\left(\frac{1}{a\cos(\eta)}\frac{\partial}{\partial \phi_2} + \frac{a'}{2a}\gamma^{14} - \frac{\tan(\eta)}{2a}\gamma^{24}\right) \\ &= \gamma^1\frac{\partial}{\partial t} + \gamma^2\frac{1}{a}\frac{\partial}{\partial \eta} + \gamma^3\frac{1}{a\,\sin\eta}\frac{\partial}{\partial \phi_1} + \gamma^4\frac{1}{a\,\cos\eta}\frac{\partial}{\partial \phi_2} + \frac{3a'}{2a}\gamma^1 + \frac{\cot(2\eta)}{a}\gamma^2. \end{aligned}$$

Thus the pseudodifferential symbol of D is given by

$$\sigma_D(x,\xi) = i\xi_1\gamma^1 + \frac{i\xi_2}{a}\gamma^2 + \frac{i\xi_3}{a\,\sin\eta}\gamma^3 + \frac{i\xi_4}{a\,\cos\eta}\gamma^4 + \frac{3a'}{2a}\gamma^1 + \frac{\cot(2\eta)}{a}\gamma^2.$$

For the purpose of employing pseudodifferential calculus in the sequel to compute the heat coefficients, we record in the following proposition the pseudodifferential symbol of

D^2. This can be achieved by a straightforward computation to find an explicit expression for D^2, or alternatively, one can apply the composition rule for symbols, $\sigma_{P_1P_2}(x,\xi) = \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha \sigma_{P_1}\partial_x^\alpha \sigma_{P_2}$, to the symbol of D.

Proposition 2.1. *The pseudodifferential symbol of D^2, where D is the Dirac operator for the Robertson-Walker metric, is given by*

$$\sigma(D^2) = p_2 + p_1 + p_0,$$

where the homogeneous components p_i of order i are written as

$$\begin{aligned}
p_2 &= \xi_1^2 + \frac{1}{a^2}\xi_2^2 + \frac{1}{a^2\sin^2(\eta)}\xi_3^2 + \frac{1}{a^2\cos^2(\eta)}\xi_4^2, \\
p_1 &= \frac{-3iaa'}{a^2}\xi_1 + \frac{-ia'\gamma^{12} - 2i\cot(2\eta)}{a^2}\xi_2 - \frac{ia'\csc(\eta)\gamma^{13} + i\cot(\eta)\csc(\eta)\gamma^{23}}{a^2}\xi_3 \\
&\quad + \frac{i\tan(\eta)\sec(\eta)\gamma^{24} - ia'\sec(\eta)\gamma^{14}}{a^2}\xi_4, \\
p_0 &= \frac{1}{4a(t)^2}\Big(- 6a(t)a''(t) - 3a'(t)^2 + \csc^2(\eta) + \sec^2(\eta) \\
&\quad + 4 + 2a'(t)(\cot(\eta) - \tan(\eta))\gamma^{12}\Big).
\end{aligned} \tag{2.2}$$

2.3 Terms up to a_{10} and their Agreement with Chamseddine-Connes' Result

The computation of the terms in the expansion of the spectral action for a spin manifold, or equivalently the calculation of the heat coefficients, can be achieved by recursive formulas while working in the heat kernel scheme of local invariants of elliptic differential operators and index theory [17]. Pseudodifferential calculus is an effective tool for dealing with the necessary approximations for deriving the small time asymptotic expansions in which the heat coefficients appear. Universal formulas in terms of the Riemann curvature operator and its contractions and covariant derivatives are written in the literature only for the terms up to a_{10}, namely Gilkey's formulas up to a_6 [17, 18] and the formulas in [1, 2, 35] for a_8 and a_{10}.

2.3.1 Small time heat kernel expansions using pseudodifferential calculus.

In [17], by appealing to the Cauchy integral formula and using pseudodifferential calculus, recursive formulas for the heat coefficients of elliptic differential operators are derived. That is, one writes

$$e^{-sD^2} = -\frac{1}{2\pi i}\int_\gamma e^{-s\lambda}(D^2-\lambda)^{-1}d\lambda,$$

where the contour γ goes around the non-negative real axis in the counterclockwise direction, and one uses pseudodifferential calculus to approximate $(D^2-\lambda)^{-1}$ via the homogeneous terms appearing in the expansion of the symbol of the parametrix of $D^2-\lambda$. Although left and right parametrices have the same homogeneous components, for the purpose of finding recursive formulas for the coefficients appearing in each component, which will be explained shortly, it is more convenient for us to consider the right parametrix $\tilde{R}(\lambda)$. Therefore, the next task is to compute recursively the homogeneous pseudodifferential symbols r_j of order $-2-j$ in the expansion of $\sigma(\tilde{R}(\lambda))$. Using the calculus of symbols, with the crucial nuance that λ is considered to be of order 2, one finds that

$$r_0 = (p_2-\lambda)^{-1},$$

and for any $n>1$

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+2-k=n \\ j<n}} \frac{(-i)^{|\alpha|}}{\alpha!} d_\xi^\alpha p_k \, d_x^\alpha r_j. \tag{2.3}$$

We summarize the process of obtaining the heat coefficients by explaining that one then uses these homogeneous terms in the Cauchy integral formula to approximate the integral kernel of e^{-sD^2}. Integration of the kernel of this operator on the diagonal yields a small time asymptotic expansion of the form

$$\mathrm{Tr}(e^{-sD^2}) \sim \sum_{n=0}^\infty \frac{s^{(n-4)/2}}{16\pi^4}\int \mathrm{tr}(e_n(x))\, dvol_g \qquad (t\to 0),$$

where

$$e_n(x)\sqrt{\det g} = \frac{-1}{2\pi i}\int\int_\gamma e^{-\lambda} r_n(x,\xi,\lambda)\, d\lambda\, d\xi. \tag{2.4}$$

For detailed discussions, we refer the reader to [17].

It is clear from (2.2) that cross derivatives of p_2 vanish and $d_\xi^\alpha p_k = 0$ if $|\alpha| > k$. Furthermore, $\frac{\partial}{\partial \phi_k} r_n = 0$ for $n \geq 0$, and the summation (2.3) is written as

$$\begin{aligned} r_n \;&=\; -r_0\, p_0\, r_{n-2} - r_0\, p_1\, r_{n-1} + i r_0 \frac{\partial}{\partial \xi_1} p_1 \frac{\partial}{\partial t} r_{n-2} + i r_0 \frac{\partial}{\partial \xi_2} p_1 \frac{\partial}{\partial \eta} r_{n-2} \\ &\quad + i r_0 \frac{\partial}{\partial \xi_1} p_2 \frac{\partial}{\partial t} r_{n-1} + i r_0 \frac{\partial}{\partial \xi_2} p_2 \frac{\partial}{\partial \eta} r_{n-1} + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_1^2} p_2 \frac{\partial^2}{\partial t^2} r_{n-2} \\ &\quad + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_2^2} p_2 \frac{\partial^2}{\partial \eta^2} r_{n-2}. \end{aligned} \tag{2.5}$$

Using induction, we find that

$$r_n = \sum_{\substack{2j-2-|\alpha| = n \\ n/2+1 \leq j \leq 2n+1}} r_{n,j,\alpha}(x)\, r_0^j\, \xi^\alpha. \tag{2.6}$$

For example, one can see that for $n = 0$ the only non-zero $r_{0,j,\alpha}$ is $r_{0,1,\mathbf{0}} = 1$, and for $n = 1$ the non-vanishing terms are

$$r_{1,2,\mathbf{e}_k} = \frac{\partial p_1}{\partial \xi_k}, \qquad r_{1,3,2\mathbf{e}_l+\mathbf{e}_k} = -2i g^{kk} \frac{\partial g^{ll}}{\partial x_k},$$

where $\mathbf{e}_j$ denotes the j-th standard unit vector in $\mathbb{R}^4$.

It then follows from the equations (2.4), (2.5) and (2.6) that

$$\begin{aligned} e_n(x)\, a(t)^3 \sin(\eta)\cos(\eta) &= \frac{-1}{2\pi i} \int_{\mathbb{R}^4} \int_\gamma e^{-s\lambda} r_n(x,\xi,\lambda)\, d\lambda\, d\xi \\ &= \sum r_{n,j,\alpha}(x) \int_{\mathbb{R}^4} \xi^\alpha \frac{-1}{2\pi i} \int_\gamma e^{-s\lambda} r_0^j\, d\lambda\, d\xi \\ &= \sum \frac{c_\alpha}{(j-1)!} r_{n,j,\alpha}\, a(t)^{\alpha_2+\alpha_3+\alpha_4+3} \sin(\eta)^{\alpha_3+1} \cos(\eta)^{\alpha_4+1}, \end{aligned} \tag{2.7}$$

where

$$c_\alpha = \prod_k \Gamma\left(\frac{\alpha_k+1}{2}\right) \frac{(-1)^{\alpha_k}+1}{2}.$$

It is straightforward to justify the latter using these identities:

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma e^{-\lambda} r_0^j d\lambda \;&=\; (-1)^j \frac{(-1)^{j-1}}{(j-1)!} e^{-||\xi||^2} = \frac{-1}{(j-1)!} \prod_{k=1}^4 e^{-g^{kk}\xi_k^2}, \\ \int_{\mathbb{R}} x^n e^{-bx^2} dx \;&=\; \frac{1}{2}\left((-1)^n + 1\right) b^{-\frac{n}{2}-\frac{1}{2}} \Gamma\left(\frac{n+1}{2}\right). \end{aligned}$$

A key point that facilitates our calculations and the proof of our main theorem presented in §2.5.1 is the derivation of recursive formulas for the coefficients $r_{n,j,\alpha}$ as follows. By substitution of (2.6) into (2.5) we find a recursive formula of the form

$$\begin{aligned}
r_{n,j,\alpha} = & -p_0 r_{n-2,j-1,\alpha} - \sum_k \frac{\partial p_1}{\partial \xi_k} r_{n-1,j-1,\alpha-\mathbf{e}_k} \\
& + i \sum_k \frac{\partial p_1}{\partial \xi_k} \frac{\partial}{\partial x_k} r_{n-2,j-1,\alpha} + i(2-j) \sum_{k,l} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial p_1}{\partial \xi_k} r_{n-2,j-2,\alpha-2\mathbf{e}_l} \\
& + 2i \sum_k g^{kk} \frac{\partial}{\partial x_k} r_{n-1,j-1,\alpha-\mathbf{e}_k} + i(4-2j) \sum_{k,l} g^{kk} \frac{\partial g^{ll}}{\partial x_k} r_{n-1,j-2,\alpha-2\mathbf{e}_l-\mathbf{e}_k} \\
& + \sum_k g^{kk} \frac{\partial^2}{\partial x_k^2} r_{n-2,j-1,\alpha} + (4-2j) \sum_{k,l} g^{kk} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial}{\partial x_k} r_{n-2,j-2,\alpha-2\mathbf{e}_l} \\
& + (2-j) \sum_{k,l} g^{kk} \frac{\partial^2 g^{ll}}{\partial x_k^2} r_{n-2,j-2,\alpha-2\mathbf{e}_l} \\
& + (3-j)(2-j) \sum_{k,l,l'} g^{kk} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial g^{l'l'}}{\partial x_k} r_{n-2,j-3,\alpha-2\mathbf{e}_l-2\mathbf{e}_{l'}}.
\end{aligned} \tag{2.8}$$

It is undeniable that the mechanism described above for computing the heat coefficients involves heavy computations which need to be overcome by computer programming. Calculating explicitly the functions $e_n(x)$, $n = 0, 2, \ldots, 12$, and computing their integrals over $\mathbb{S}_a^3$ with computer assistance, we find the explicit polynomials in $a(t)$ and its derivatives recorded in the sequel, which describe the corresponding terms in the expansion of the spectral action for the Robertson-Walker metric. That is, each function a_n recorded below is the outcome of

$$\begin{aligned}
a_n &= \frac{1}{16\pi^4} \int_{\mathbb{S}_a^3} \mathrm{tr}(e_n)\, dvol_g \\
&= \frac{1}{16\pi^4} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \mathrm{tr}(e_n)\, a(t)^3 \sin(\eta) \cos(\eta)\, d\eta\, d\phi_1\, d\phi_2.
\end{aligned}$$

2.3.2 The terms up to a_6

These terms were computed in [9] by their direct method, which is based on the Euler-Maclaurin summation formula and the Feynman-Kac formula, and they were checked by Gilkey's universal formulas. Our computations based on the method explained in the previous subsection also gives the same result.

The first term, whose integral up to a universal factor gives the volume, is given by

$$a_0 = \frac{a(t)^3}{2}.$$

Since the latter appears as the leading term in the small time asymptotic expansion of the heat kernel it is related to Weyl's law, which reads the volume from the asymptotic distribution of the eigenvalues of D^2. The next term, which is related to the scalar curvature, has the expression

$$a_2 = \frac{1}{4}a(t)\left(a(t)a''(t) + a'(t)^2 - 1\right).$$

The term after, whose integral is topological, is related to the Gauss-Bonnet term (cf. [9]) and is written as

$$a_4 = \frac{1}{120}\Big(3a^{(4)}(t)a(t)^2 + 3a(t)a''(t)^2 - 5a''(t) + 9a^{(3)}(t)a(t)a'(t) - 4a'(t)^2a''(t)\Big).$$

The term a_6, which is the last term for which Gilkey's universal formulas are written, is given by

$$a_6 = \tfrac{1}{5040a(t)^2}\Big(9a^{(6)}(t)a(t)^4 - 21a^{(4)}(t)a(t)^2 - 3a^{(3)}(t)^2a(t)^3 - 56a(t)^2a''(t)^3 + 42a(t)a''(t)^2 + 36a^{(5)}(t)a(t)^3a'(t) + 6a^{(4)}(t)a(t)^3a''(t) - 42a^{(4)}(t)a(t)^2a'(t)^2 + 60a^{(3)}(t)a(t)a'(t)^3 + 21a^{(3)}(t)a(t)a'(t) + 240a(t)a'(t)^2a''(t)^2 - 60a'(t)^4a''(t) - 21a'(t)^2a''(t) - 252a^{(3)}(t)a(t)^2a'(t)a''(t)\Big).$$

2.3.3 The terms a_8 and a_{10}

These terms were computed by Chamseddine and Connes in [9] using their direct method. In order to form a check on the final formulas, they have suggested to use the universal formulas of [1, 2, 35] to calculate these terms and compare the results. As mentioned earlier, Gilkey's universal formulas were used in [9] to check the terms up to a_6, however, they are written in the literature only up to a_6 and become rather complicated even for this term.

In this subsection, we pursue the computation of the terms a_8 and a_{10} in the expansion of the spectral action for Robertson-Walker metrics by continuing to employ pseudodifferential calculus, as presented in §2.3.1, and check that the final formulas agree with the result in [9]. The final formulas for a_8 and a_{10} are the following expressions:

$$a_8 =$$

$$-\frac{1}{10080a(t)^4}\Big(-a^{(8)}(t)a(t)^6+3a^{(6)}(t)a(t)^4+13a^{(4)}(t)^2a(t)^5-24a^{(3)}(t)^2a(t)^3-114a(t)^3a''(t)^4+$$
$$43a(t)^2a''(t)^3-5a^{(7)}(t)a(t)^5a'(t)+2a^{(6)}(t)a(t)^5a''(t)+9a^{(6)}(t)a(t)^4a'(t)^2+$$
$$16a^{(3)}(t)a^{(5)}(t)a(t)^5-24a^{(5)}(t)a(t)^3a'(t)^3-6a^{(5)}(t)a(t)^3a'(t)+69a^{(4)}(t)a(t)^4a''(t)^2-$$
$$36a^{(4)}(t)a(t)^3a''(t)+60a^{(4)}(t)a(t)^2a'(t)^4+15a^{(4)}(t)a(t)^2a'(t)^2+90a^{(3)}(t)^2a(t)^4a''(t)-$$
$$216a^{(3)}(t)^2a(t)^3a'(t)^2-108a^{(3)}(t)a(t)a'(t)^5-27a^{(3)}(t)a(t)a'(t)^3+801a(t)^2a'(t)^2a''(t)^3-$$
$$588a(t)a'(t)^4a''(t)^2-87a(t)a'(t)^2a''(t)^2+108a'(t)^6a''(t)+27a'(t)^4a''(t)+$$
$$78a^{(5)}(t)a(t)^4a'(t)a''(t)+132a^{(3)}(t)a^{(4)}(t)a(t)^4a'(t)-312a^{(4)}(t)a(t)^3a'(t)^2a''(t)-$$
$$819a^{(3)}(t)a(t)^3a'(t)a''(t)^2+768a^{(3)}(t)a(t)^2a'(t)^3a''(t)+102a^{(3)}(t)a(t)^2a'(t)a''(t)\Big),$$

and

$$a_{10} =$$

$$\frac{1}{665280a(t)^6}\Big(3a^{(10)}(t)a(t)^8-222a^{(5)}(t)^2a(t)^7-348a^{(4)}(t)a^{(6)}(t)a(t)^7-147a^{(3)}(t)a^{(7)}(t)a(t)^7-$$
$$18a''(t)a^{(8)}(t)a(t)^7+18a'(t)a^{(9)}(t)a(t)^7-482a''(t)a^{(4)}(t)^2a(t)^6-331a^{(3)}(t)^2a^{(4)}(t)a(t)^6-$$
$$1110a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6-1556a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6-448a''(t)^2a^{(6)}(t)a(t)^6-$$
$$1074a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6-476a'(t)a''(t)a^{(7)}(t)a(t)^6-43a'(t)^2a^{(8)}(t)a(t)^6-11a^{(8)}(t)a(t)^6+$$
$$8943a'(t)a^{(3)}(t)^3a(t)^5+21846a''(t)^2a^{(3)}(t)^2a(t)^5+4092a'(t)^2a^{(4)}(t)^2a(t)^5+396a^{(4)}(t)^2a(t)^5+$$
$$10560a''(t)^3a^{(4)}(t)a(t)^5+39402a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5+11352a'(t)a''(t)^2a^{(5)}(t)a(t)^5+$$
$$6336a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5+594a^{(3)}(t)a^{(5)}(t)a(t)^5+2904a'(t)^2a''(t)a^{(6)}(t)a(t)^5+$$
$$264a''(t)a^{(6)}(t)a(t)^5+165a'(t)^3a^{(7)}(t)a(t)^5+33a'(t)a^{(7)}(t)a(t)^5-10338a''(t)^5a(t)^4-$$
$$95919a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4-3729a''(t)a^{(3)}(t)^2a(t)^4-117600a'(t)a''(t)^3a^{(3)}(t)a(t)^4-$$
$$68664a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4-2772a''(t)^2a^{(4)}(t)a(t)^4-23976a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4-$$
$$2640a'(t)a^{(3)}(t)a^{(4)}(t)a(t)^4-12762a'(t)^3a''(t)a^{(5)}(t)a(t)^4-1386a'(t)a''(t)a^{(5)}(t)a(t)^4-$$
$$651a'(t)^4a^{(6)}(t)a(t)^4-132a'(t)^2a^{(6)}(t)a(t)^4+111378a'(t)^2a''(t)^4a(t)^3+2354a''(t)^4a(t)^3+$$
$$31344a'(t)^4a^{(3)}(t)^2a(t)^3+3729a'(t)^2a^{(3)}(t)^2a(t)^3+236706a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3+$$
$$13926a'(t)a''(t)^2a^{(3)}(t)a(t)^3+43320a'(t)^4a''(t)a^{(4)}(t)a(t)^3+5214a'(t)^2a''(t)a^{(4)}(t)a(t)^3+$$
$$2238a'(t)^5a^{(5)}(t)a(t)^3+462a'(t)^3a^{(5)}(t)a(t)^3-162162a'(t)^4a''(t)^3a(t)^2-$$
$$11880a'(t)^2a''(t)^3a(t)^2-103884a'(t)^5a''(t)a^{(3)}(t)a(t)^2-13332a'(t)^3a''(t)a^{(3)}(t)a(t)^2-$$
$$6138a'(t)^6a^{(4)}(t)a(t)^2-1287a'(t)^4a^{(4)}(t)a(t)^2+76440a'(t)^6a''(t)^2a(t)+10428a'(t)^4a''(t)^2a(t)+$$
$$11700a'(t)^7a^{(3)}(t)a(t)+2475a'(t)^5a^{(3)}(t)a(t)-11700a'(t)^8a''(t)-2475a'(t)^6a''(t)\Big).$$

2.4 Computation of the Term a_{12} in the Expansion of the Spectral Action

We pursue the computation of the term a_{12} in the expansion of the spectral action for Robertson-Walker metrics by employing pseudodifferential calculus to find the term r_{12} for the parametrix of $\lambda - D^2$, which is homogeneous of order -14, and by performing the appropriate integrations. Since there is no universal formula in the literature for this term, we have performed two heavy computations, one in Hopf coordinates and the other in spherical coordinates, to form a check on the validity of the outcome of our calculations. Another efficient way of computing the term a_{12} is to use the direct method of [9].

2.4.1 The result of the computation in Hopf coordinates.

Continuing the recursive procedure commenced in the previous section and exploiting computer assistance, while the calculation becomes significantly heavier for the term a_{12}, we find the following expression:

$$a_{12} =$$

$$\frac{1}{17297280a(t)^8}\Big(3a^{(12)}(t)a(t)^{10} - 1057a^{(6)}(t)^2a(t)^9 - 1747a^{(5)}(t)a^{(7)}(t)a(t)^9 - 970a^{(4)}(t)a^{(8)}(t)a(t)^9 -$$
$$317a^{(3)}(t)a^{(9)}(t)a(t)^9 - 34a''(t)a^{(10)}(t)a(t)^9 + 21a'(t)a^{(11)}(t)a(t)^9 + 5001a^{(4)}(t)^3a(t)^8 +$$
$$2419a''(t)a^{(5)}(t)^2a(t)^8 + 19174a^{(3)}(t)a^{(4)}(t)a^{(5)}(t)a(t)^8 + 4086a^{(3)}(t)^2a^{(6)}(t)a(t)^8 +$$
$$2970a''(t)a^{(4)}(t)a^{(6)}(t)a(t)^8 - 5520a'(t)a^{(5)}(t)a^{(6)}(t)a(t)^8 - 511a''(t)a^{(3)}(t)a^{(7)}(t)a(t)^8 -$$
$$4175a'(t)a^{(4)}(t)a^{(7)}(t)a(t)^8 - 745a''(t)^2a^{(8)}(t)a(t)^8 - 2289a'(t)a^{(3)}(t)a^{(8)}(t)a(t)^8 -$$
$$828a'(t)a''(t)a^{(9)}(t)a(t)^8 - 62a'(t)^2a^{(10)}(t)a(t)^8 - 13a^{(10)}(t)a(t)^8 + 45480a^{(3)}(t)^4a(t)^7 +$$
$$152962a''(t)^2a^{(4)}(t)^2a(t)^7 + 203971a'(t)a^{(3)}(t)a^{(4)}(t)^2a(t)^7 + 21369a'(t)^2a^{(5)}(t)^2a(t)^7 +$$
$$1885a^{(5)}(t)^2a(t)^7 + 410230a''(t)a^{(3)}(t)^2a^{(4)}(t)a(t)^7 + 163832a'(t)a^{(3)}(t)^2a^{(5)}(t)a(t)^7 +$$
$$250584a''(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^7 + 244006a'(t)a''(t)a^{(4)}(t)a^{(5)}(t)a(t)^7 + 42440a''(t)^3a^{(6)}(t)a(t)^7 +$$
$$163390a'(t)a''(t)a^{(3)}(t)a^{(6)}(t)a(t)^7 + 35550a'(t)^2a^{(4)}(t)a^{(6)}(t)a(t)^7 + 3094a^{(4)}(t)a^{(6)}(t)a(t)^7 +$$
$$34351a'(t)a''(t)^2a^{(7)}(t)a(t)^7 + 19733a'(t)^2a^{(3)}(t)a^{(7)}(t)a(t)^7 + 1625a^{(3)}(t)a^{(7)}(t)a(t)^7 +$$
$$6784a'(t)^2a''(t)a^{(8)}(t)a(t)^7 + 520a''(t)a^{(8)}(t)a(t)^7 + 308a'(t)^3a^{(9)}(t)a(t)^7 + 52a'(t)a^{(9)}(t)a(t)^7 -$$
$$2056720a'(t)a''(t)a^{(3)}(t)^3a(t)^6 - 1790580a''(t)^3a^{(3)}(t)^2a(t)^6 - 900272a'(t)^2a''(t)a^{(4)}(t)^2a(t)^6 -$$
$$31889a''(t)a^{(4)}(t)^2a(t)^6 - 643407a''(t)^4a^{(4)}(t)a(t)^6 - 1251548a'(t)^2a^{(3)}(t)^2a^{(4)}(t)a(t)^6 -$$
$$43758a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 4452042a'(t)a''(t)^2a^{(3)}(t)a^{(4)}(t)a(t)^6 -$$
$$836214a'(t)a''(t)^3a^{(5)}(t)a(t)^6 - 1400104a'(t)^2a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 -$$

$48620a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 181966a'(t)^3a^{(4)}(t)a^{(5)}(t)a(t)^6 - 18018a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 319996a'(t)^2a''(t)^2a^{(6)}(t)a(t)^6 - 11011a''(t)^2a^{(6)}(t)a(t)^6 - 115062a'(t)^3a^{(3)}(t)a^{(6)}(t)a(t)^6 - 11154a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 42764a'(t)^3a''(t)a^{(7)}(t)a(t)^6 - 4004a'(t)a''(t)a^{(7)}(t)a(t)^6 - 1649a'(t)^4a^{(8)}(t)a(t)^6 - 286a'(t)^2a^{(8)}(t)a(t)^6 + 460769a''(t)^6a(t)^5 + 1661518a'(t)^3a^{(3)}(t)^3a(t)^5 + 83486a'(t)a^{(3)}(t)^3a(t)^5 + 13383328a'(t)^2a''(t)^2a^{(3)}(t)^2a(t)^5 + 222092a''(t)^2a^{(3)}(t)^2a(t)^5 + 342883a'(t)^4a^{(4)}(t)^2a(t)^5 + 36218a'(t)^2a^{(4)}(t)^2a(t)^5 + 7922361a'(t)a''(t)^4a^{(3)}(t)a(t)^5 + 6367314a'(t)^2a''(t)^3a^{(4)}(t)a(t)^5 + 109330a''(t)^3a^{(4)}(t)a(t)^5 + 7065862a'(t)^3a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 360386a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 1918386a'(t)^3a''(t)^2a^{(5)}(t)a(t)^5 + 98592a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + 524802a'(t)^4a^{(3)}(t)a^{(5)}(t)a(t)^5 + 55146a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 226014a'(t)^4a''(t)a^{(6)}(t)a(t)^5 + 23712a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 8283a'(t)^5a^{(7)}(t)a(t)^5 + 1482a'(t)^3a^{(7)}(t)a(t)^5 - 7346958a'(t)^2a''(t)^5a(t)^4 - 72761a''(t)^5a(t)^4 - 11745252a'(t)^4a''(t)a^{(3)}(t)^2a(t)^4 - 725712a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 27707028a'(t)^3a''(t)^3a^{(3)}(t)a(t)^4 - 819520a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 8247105a'(t)^4a''(t)^2a^{(4)}(t)a(t)^4 - 520260a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 1848228a'(t)^5a^{(3)}(t)a^{(4)}(t)a(t)^4 - 205296a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - 973482a'(t)^5a''(t)a^{(5)}(t)a(t)^4 - 110136a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 36723a'(t)^6a^{(6)}(t)a(t)^4 - 6747a'(t)^4a^{(6)}(t)a(t)^4 + 17816751a'(t)^4a''(t)^4a(t)^3 + 721058a'(t)^2a''(t)^4a(t)^3 + 2352624a'(t)^6a^{(3)}(t)^2a(t)^3 + 274170a'(t)^4a^{(3)}(t)^2a(t)^3 + 24583191a'(t)^5a''(t)^2a^{(3)}(t)a(t)^3 + 1771146a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 3256248a'(t)^6a''(t)a^{(4)}(t)a(t)^3 + 389376a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 135300a'(t)^7a^{(5)}(t)a(t)^3 + 25350a'(t)^5a^{(5)}(t)a(t)^3 - 15430357a'(t)^6a''(t)^3a(t)^2 - 1252745a'(t)^4a''(t)^3a(t)^2 - 7747848a'(t)^7a''(t)a^{(3)}(t)a(t)^2 - 967590a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 385200a'(t)^8a^{(4)}(t)a(t)^2 - 73125a'(t)^6a^{(4)}(t)a(t)^2 + 5645124a'(t)^8a''(t)^2a(t) + 741195a'(t)^6a''(t)^2a(t) + 749700a'(t)^9a^{(3)}(t)a(t) + 143325a'(t)^7a^{(3)}(t)a(t) - 749700a'(t)^{10}a''(t) - 143325a'(t)^8a''(t))\Big).$

2.4.2 Agreement of the result with computations in spherical coordinates.

Taking a similar route as in §2.2, we explicitly write the Dirac operator for the Roberson-Walker metric in spherical coordinates

$$ds^2 = dt^2 + a^2\,(t)\left(d\chi^2 + \sin^2(\chi)\left(d\theta^2 + \sin^2(\theta)\,d\varphi^2\right)\right).$$

Using the computations carried out in [9] with the orthonormal coframe

$$dt, \qquad a(t)\,d\chi, \qquad a(t)\,\sin\chi\,d\theta, \qquad a(t)\,\sin\chi\,\sin\theta\,d\varphi,$$

the corresponding matrix of connection 1-forms for the Levi-Civita connection is written as

$$\begin{pmatrix} 0 & -a'(t)d\chi & -a'(t)\sin(\chi)d\theta & -a'(t)\sin(\chi)\sin(\theta)d\varphi \\ a'(t)d\chi & 0 & -\cos(\chi)d\theta & -\cos(\chi)\sin(\theta)d\varphi \\ a'(t)\sin(\chi)d\theta & \cos(\chi)d\theta & 0 & -\cos(\theta)d\varphi \\ a'(t)\sin(\chi)\sin(\theta)d\varphi & \cos(\chi)\sin(\theta)d\varphi & \cos(\theta)d\varphi & 0 \end{pmatrix}.$$

Lifting to the spin bundle by means of the Lie algebra isomorphism $\mu : \mathfrak{so}(4) \to \mathfrak{spin}(4)$ and writing the formula for the Dirac operator yield the following expression for this operator expressed in spherical coordiantes:

$$\begin{aligned} D &= \gamma^1\frac{\partial}{\partial t} + \gamma^2\frac{1}{a}\frac{\partial}{\partial\chi} + \gamma^3\frac{1}{a\,\sin\chi}\frac{\partial}{\partial\theta} + \gamma^4\frac{1}{a\,\sin\chi\,\sin\theta}\frac{\partial}{\partial\varphi} \\ &\quad + \frac{3a'}{2a}\gamma^1 + \frac{\cot(\chi)}{a}\gamma^2 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^3. \end{aligned}$$

Thus the pseudodifferential symbol of D is given by

$$\begin{aligned} \sigma_D(x,\xi) &= i\gamma^1\xi_1 + \frac{i}{a}\gamma^2\xi_2 + \frac{i}{a\sin(\chi)}\gamma^3\xi_3 + \frac{i}{a\sin(\chi)\sin(\theta)}\gamma^4\xi_4 \\ &\quad + \frac{3a'}{2a}\gamma^1 + \frac{\cot(\chi)}{a}\gamma^2 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^3. \end{aligned}$$

Accordingly, the symbol of D^2 is the sum $p_2' + p_1' + p_0'$ of three homogeneous components

$$\begin{aligned} p_2' &= \xi_1^2 + \frac{1}{a(t)^2}\xi_2^2 + \frac{1}{a(t)^2\sin^2(\chi)}\xi_3^2 + \frac{1}{a(t)^2\sin^2(\theta)\sin^2(\chi)}\xi_4^2, \\ p_1' &= -\frac{3ia'(t)}{a(t)}\xi_1 - \frac{i}{a(t)^2}\left(\gamma^{12}a'(t) + 2\cot(\chi)\right)\xi_2 \\ &\quad -\frac{i}{a(t)^2}\left(\gamma^{13}\csc(\chi)a'(t) + \cot(\theta)\csc^2(\chi) + \gamma^{23}\cot(\chi)\csc(\chi)\right)\xi_3 \\ &\quad -\frac{i}{a(t)^2}(\csc(\theta)\csc(\chi)a'(t)\gamma^{14} + \cot(\theta)\csc(\theta)\csc^2(\chi)\gamma^{34} \\ &\quad + \csc(\theta)\cot(\chi)\csc(\chi)\gamma^{24})\xi_4, \\ p_0' &= \frac{1}{8a(t)^2}\big(-12a(t)a''(t) - 6a'(t)^2 + 3\csc^2(\theta)\csc^2(\chi) - \cot^2(\theta)\csc^2(\chi) + \\ &\quad 4i\cot(\theta)\cot(\chi)\csc(\chi) - 4i\cot(\theta)\cot(\chi)\csc(\chi) - 4\cot^2(\chi) + 5\csc^2(\chi) + 4\big) \\ &\quad -\frac{(\cot(\theta)\csc(\chi)a'(t))}{2a(t)^2}\gamma^{13} - \frac{(\cot(\chi)a'(t))}{a(t)^2}\gamma^{12} - \frac{(\cot(\theta)\cot(\chi)\csc(\chi))}{2a(t)^2}\gamma^{23}. \end{aligned}$$

We have performed the computation of the heat coefficients up to the term a_{12} using

the latter symbols and have checked the agreement of the result with the computations in Hopf coordinates, presented in the previous subsections. This is in particular of great importance for the term a_{12}, since it ensures the validity of our computations performed in two different coordinates.

2.4.3 Agreement with the full expansion for the round metric.

We first recall the full expansion for the spectral action for the round metric, namely the case $a(t) = \sin(t)$, worked out in [9]. Then we show that the term a_{12} presented in §2.4.1 reduces correctly to the round case.

The method devised in [9] has wide applicability in the spectral action computations since it can be used for the cases when the eigenvalues of the square of the Dirac operator have a polynomial expression while their multiplicities are also given by polynomials. In the case of the round metric on $\mathbb{S}^4$, after remarkable computations based on the Euler-Maclaurin formula, this method leads to the following expression with control over the remainder term [9]:

$$\begin{aligned}\frac{3}{4}\text{Trace}(f(tD^2)) &= \int_0^\infty f(tx^2)(x^3 - x)dx + \frac{11f(0)}{120} - \frac{31f'(0)t}{2520} + \frac{41f''(0)t^2}{10080} \\ &\quad - \frac{31f^{(3)}(0)t^3}{15840} + \frac{10331f^{(4)}(0)t^4}{8648640} - \frac{3421f^{(5)}(0)t^5}{3931200} + \cdots + R_m.\end{aligned}$$

This implies that the term a_{12} in the expansion of the spectral action for the round metric is equal to $\frac{10331}{6486480}$. To check our calculations against this result, we find that for $a(t) = \sin(t)$ the expression for $a_{12}(t)$ reduces to $\frac{10331\sin^3(t)}{8648640}$, and hence

$$a_{12} = \int_0^\pi a_{12}(\mathbb{S}^4)\,dt = \frac{4}{3}\frac{10331}{8648640} = \frac{10331}{6486480},$$

which is in complete agreement with the result in [9], mentioned above.

2.5 Chameseddine-Connes' Conjecture

In this section we prove a conjecture of Chamseddine and Connes from [9]. More precisely, we show that the term a_{2n} in the asymptotic expansion of the spectral action for Robertson-Walker metrics is, up to multiplication by $a(t)^{3-2n}$, of the form $Q_{2n}(a, a', \ldots, a^{(2n)})$, where Q_{2n} is a polynomial with rational coefficients.

2.5.1 Proof of rationality of the coefficients in the expressions for a_{2n}

A crucial point that enables us to furnish the proof of our main theorem, namely the proof of the conjecture mentioned above, is the independence of the integral kernel of the heat operator of the Dirac operator of the Robertson-Walker metric from the variables ϕ_1, ϕ_2, η. Note that since the symbol and the metric are independent of ϕ_1, ϕ_2, the computations involved in the symbol calculus clearly imply the independence of the terms e_n from these variables. However, the independence of e_n from η is not evident, which is proved as follows.

Lemma 2.2. *The heat kernel $k(s, x, x)$ for the Robertson-Walker metric is independent of ϕ_1, ϕ_2, η.*

Proof. The round metric on $\mathbb{S}^3$ is the bi-invariant metric on SU(2) induced from the Killing form of its Lie algebra $\mathfrak{su}(2)$. The corresponding Levi-Civita connection restricted to the left invariant vector fields is given by $\frac{1}{2}[X, Y]$, and to the right invariant vector fields by $\frac{-1}{2}[X, Y]$. Since the Killing form is ad-invariant, we have

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0, \qquad X, Y, Z \in \mathfrak{su}(2),$$

which implies that in terms of the connection on left (right) invariant vector fields X, Y, Z, it can be written as

$$\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = 0. \tag{2.9}$$

Considering the fact that $\nabla X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is an endomorphism of the tangent bundle, the latter identity holds for any $Y, Z \in \mathfrak{X}(M)$. Therefore, the equation (2.9) is the Killing equation and shows that any left and right invariant vector field on SU(2) is a Killing vector field.

By direct computation in Hopf coordinates, we find the following vector fields which respectively form bases for left and right invariant vector fields on SU(2):

$$
\begin{aligned}
X_1^L &= \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}, \\
X_2^L &= \sin(\phi_1+\phi_2)\frac{\partial}{\partial \eta} + \cot(\eta)\cos(\phi_1+\phi_2)\frac{\partial}{\partial \phi_1} - \tan(\eta)\cos(\phi_1+\phi_2)\frac{\partial}{\partial \phi_2}, \\
X_3^L &= \cos(\phi_1+\phi_2)\frac{\partial}{\partial \eta} - \cot(\eta)\sin(\phi_1+\phi_2)\frac{\partial}{\partial \phi_1} + \tan(\eta)\sin(\phi_1+\phi_2)\frac{\partial}{\partial \phi_2}, \\
X_1^R &= -\frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}, \\
X_2^R &= -\sin(\phi_1-\phi_2)\frac{\partial}{\partial \eta} - \cot(\eta)\cos(\phi_1-\phi_2)\frac{\partial}{\partial \phi_1} - \tan(\eta)\cos(\phi_1-\phi_2)\frac{\partial}{\partial \phi_2}, \\
X_3^R &= \cos(\phi_1-\phi_2)\frac{\partial}{\partial \eta} - \cot(\eta)\sin(\phi_1-\phi_2)\frac{\partial}{\partial \phi_1} - \tan(\eta)\sin(\phi_1-\phi_2)\frac{\partial}{\partial \phi_2}.
\end{aligned}
$$

One can check that these vector fields are indeed Killing vector fields for the Robertson-Walker metrics on the four dimensional space. Thus, for any isometry invariant function f we have:

$$
\begin{aligned}
\frac{\partial}{\partial \phi_1} f &= \frac{1}{2}(X_1^L - X_1^R) f = 0, \\
\frac{\partial}{\partial \phi_2} f &= \frac{1}{2}(X_1^L + X_1^R) f = 0, \\
\frac{\partial}{\partial \eta} f &= (\sin(\phi_1+\phi_2) X_2^L + \cos(\phi_1+\phi_2) X_3^L) f = 0.
\end{aligned}
$$

In particular, the heat kernel restricted to the diagonal, $k(s,x,x)$, is independent of ϕ_1, ϕ_2, η, and so are the coefficient functions e_n in its asymptotic expansion. $\square$

We stress that although $e_n(x)$ is independent of η, ϕ_1, ϕ_2, its components denoted by $e_{n,j,\alpha}$ in the proof of the following theorem are not necessarily independent of these variables.

Theorem 2.3. *The term a_{2n} in the expansion of the spectral action for the Robertson-Walker metric with cosmic scale factor $a(t)$ is of the form*

$$
\frac{1}{a(t)^{2n-3}} Q_{2n}\left(a(t), a'(t), \dots, a^{(2n)}(t)\right),
$$

where Q_{2n} is a polynomial with rational coefficients.

Proof. Using (2.7) we can write

$$e_n = \sum_{\substack{2j-2-|\alpha| = n \\ n/2+1 \leq j \leq 2n+1}} c_\alpha \, e_{n,j,\alpha}, \tag{2.10}$$

where

$$e_{n,j,\alpha} = \frac{1}{(j-1)!} r_{n,j,\alpha} \, a(t)^{\alpha_2+\alpha_3+\alpha_4} \sin(\eta)^{\alpha_3} \cos(\eta)^{\alpha_4}.$$

The recursive equation (2.8) implies that

$$e_{n,j,\alpha} = \tag{2.11}$$

$$\begin{aligned}
&\tfrac{1}{(j-1)a(t)}\Big((\gamma^{14}a'(t) - \tan(\eta)\gamma^{24})e_{n-1,j-1,\alpha-\mathbf{e}_4} + (\gamma^{13}a'(t) + \cot(\eta)\gamma^{23})e_{n-1,j-1,\alpha-\mathbf{e}_3} + (\gamma^{12}a'(t) + \\
&1((2\alpha_4-1)\tan(\eta) + (1-2\alpha_3)\cot(\eta)))e_{n-1,j-1,\alpha-\mathbf{e}_2} + 4a'(t)e_{n-1,j-2,\alpha-\mathbf{e}_1-2\mathbf{e}_2} + \\
&4a'(t)e_{n-1,j-2,\alpha-\mathbf{e}_1-2\mathbf{e}_3} + 4a'(t)e_{n-1,j-2,\alpha-\mathbf{e}_1-2\mathbf{e}_4} + (-2\alpha_2 - 2\alpha_3 - 2\alpha_4 + 3)a'(t)e_{n-1,j-1,\alpha-\mathbf{e}_1} + \\
&2a(t)\tfrac{\partial}{\partial t}e_{n-1,j-1,\alpha-\mathbf{e}_1} - 4\tan(\eta)e_{n-1,j-2,\alpha-\mathbf{e}_2-2\mathbf{e}_4} + 4\cot(\eta)e_{n-1,j-2,\alpha-\mathbf{e}_2-2\mathbf{e}_3} + 2\tfrac{\partial}{\partial \eta}e_{n-1,j-1,\alpha-\mathbf{e}_2}\Big) \\
&+ \tfrac{1}{(j-1)a(t)^2}\Big(a(t)^2\tfrac{\partial^2}{\partial t^2}e_{n-2,j-1,\alpha} + 4a'(t)a(t)\tfrac{\partial}{\partial t}e_{n-2,j-2,\alpha-2\mathbf{e}_2} + 4a'(t)a(t)\tfrac{\partial}{\partial t}e_{n-2,j-2,\alpha-2\mathbf{e}_3} + \\
&4a'(t)a(t)\tfrac{\partial}{\partial t}e_{n-2,j-2,\alpha-2\mathbf{e}_4} + (-2\alpha_2 - 2\alpha_3 - 2\alpha_4 + 3)a'(t)a(t)\tfrac{\partial}{\partial t}e_{n-2,j-1,\alpha} + \\
&4a'(t)^2e_{n-2,j-3,\alpha-4\mathbf{e}_2} + 8a'(t)^2e_{n-2,j-3,\alpha-2\mathbf{e}_2-2\mathbf{e}_3} + 8a'(t)^2e_{n-2,j-3,\alpha-2\mathbf{e}_2-2\mathbf{e}_4} + \\
&4\cot(\eta)\tfrac{\partial}{\partial \eta}e_{n-2,j-2,\alpha-2\mathbf{e}_3} - 4\tan(\eta)\tfrac{\partial}{\partial \eta}e_{n-2,j-2,\alpha-2\mathbf{e}_4} + \tfrac{\partial^2}{\partial \eta^2}e_{n-2,j-1,\alpha} + \big(2\cot(\eta)\gamma^{12}a'(t) + \\
&(-4(\alpha_2+\alpha_3+\alpha_4-2)a'(t)^2 + 4(-(\alpha_3-1)\csc^2(\eta) + \alpha_3 + \alpha_4 - 2) + 2a(t)a''(t))\big)e_{n-2,j-2,\alpha-2\mathbf{e}_3} + \\
&\big((\cot(\eta)(1-2\alpha_3) + (2\alpha_4-1)\tan(\eta)) + \gamma^{12}a'(t)\big)\tfrac{\partial}{\partial \eta}e_{n-2,j-1,\alpha} + \big((-4(\alpha_2+\alpha_3+\alpha_4-2)a'(t)^2 + \\
&4(-(\alpha_4-1)\sec^2(\eta) + \alpha_3 + \alpha_4 - 2) + 2a(t)a''(t)) - 2\gamma^{12}\tan(\eta)a'(t)\big)e_{n-2,j-2,\alpha-2\mathbf{e}_4} + 8(a'(t)^2 - \\
&1)e_{n-2,j-3,\alpha-2\mathbf{e}_4} + 4(\cot^2(\eta) + a'(t)^2)e_{n-2,j-3,\alpha-4\mathbf{e}_3} + 4(\tan^2(\eta) + a'(t)^2)e_{n-2,j-3,\alpha-4\mathbf{e}_4} + \\
&(2a(t)a''(t) - 4(\alpha_2+\alpha_3+\alpha_4-2)a'(t)^2)e_{n-2,j-2,\alpha-2\mathbf{e}_2} + \big(\tfrac{1}{2}(\cot(\eta)(1-2\alpha_3) + (2\alpha_4 - \\
&1)\tan(\eta))\gamma^{12}a'(t) + \tfrac{1}{4}((4\alpha_3^2-1)\csc^2(\eta) - 4(\alpha_3+\alpha_4-1)^2 + (2\alpha_2+2\alpha_3+2\alpha_4-3)(2\alpha_2+2\alpha_3 + \\
&2\alpha_4-1)a'(t)^2 + \sec^2(\eta)(4\alpha_4^2-1) - 2(2\alpha_2+2\alpha_3+2\alpha_4-3)a(t)a''(t))\big)e_{n-2,j-1,\alpha}\Big).
\end{aligned}$$

The functions associated with the initial indices are:

$$\begin{aligned}
&e_{0,1,0,0,0,0} = 1, \qquad e_{1,2,1,0,0,0} = \frac{3ia'(t)}{a(t)}, \qquad e_{1,3,1,2,0,0} = \frac{2ia'(t)}{a(t)}, \\
&e_{1,3,1,0,2,0} = \frac{2ia'(t)}{a(t)}, \qquad e_{1,3,1,0,0,2} = \frac{2ia'(t)}{a(t)}, \qquad e_{1,3,0,1,0,2} = -\frac{(2i)\tan(\eta)}{a(t)}, \\
&e_{1,3,0,1,2,0} = \frac{(2i)\cot(\eta)}{a(t)}, \qquad e_{1,2,0,0,1,0} = \frac{i\gamma^{13}a'(t)}{a(t)} + \frac{i\gamma^{23}\cot(\eta)}{a(t)}, \\
&e_{1,2,0,0,0,1} = \frac{i\gamma^{14}a'(t)}{a(t)} - \frac{i\gamma^{24}\tan(\eta)}{a(t)}, \qquad e_{1,2,0,1,0,0} = \frac{2i\cot(2\eta)}{a(t)} + \frac{i\gamma^{12}a'(t)}{a(t)}.
\end{aligned}$$

It is then apparent that e_0 and e_1 are, respectively, a polynomial in $a(t)$, and a polynomial in $a(t)$ and $a'(t)$, divided by some powers of $a(t)$. Thus, it follows from the above recursive formula that all $e_{n,j,\alpha}$ are of this form. Accordingly, we have

$$e_n = \frac{P_n}{a(t)^{d_n}},$$

where P_n is a polynomial in $a(t)$ and its derivatives with matrix coefficients. Writing $e_{n,j,\alpha} = P_{n,j,\alpha}/a(t)^{d_n}$, we obtain $d_n = \max\{d_{n-1}+1, d_{n-2}+2\}$. Starting with $d_0 = 0$, $d_1 = -1$, and following to obtain $d_n = n$, we conclude that

$$e_{n,j,\alpha} = \frac{1}{a^n(t)} P_{n,j,\alpha}(a(t), \ldots, a^{(n)}(t)),$$

where $P_{n,j,\alpha}$ is a polynomial whose coefficients are matrices with entries in the algebra generated by $\sin(\eta), \csc(\eta), \cos(\eta), \sec(\eta)$ and rational numbers.

In the calculation of the even terms a_{2n}, only even α_k have contributions in the summation (2.10). This implies that the corresponding c_α is a rational multiple of π^2 and P_{2n} is a polynomial with rational matrix coefficients, which is independent of variables η, ϕ_1, ϕ_2 by Lemma 2.2. Hence

$$a_{2n} = \frac{1}{16\pi^4} \int_{\mathbb{S}^3_a} \mathrm{tr}(e_{2n})\, dvol_g = \frac{2\pi^2 a(t)^3}{16\pi^4}\, \mathrm{tr}\Big(\frac{P_{2n}}{a(t)^{2n}}\Big) = \frac{Q_{2n}}{a(t)^{n-3}},$$

where Q_{2n} is a polynomial in $a(t), a'(t), \ldots, a^{(2n)}(t)$ with rational coefficients. □

The polynomials $P_{n,j,\alpha}$ also satisfy recursive relations that illuminate interesting features about their structure.

Proposition 2.4. *Each $P_{n,j,\alpha}$ is a finite sum of the form*

$$\sum c_k\, a(t)^{k_0} a'(t)^{k_1} \cdots a^{(n)}(t)^{k_n},$$

where each c_k is a matrix of functions that are independent from the variable t, and $\sum_{j=0}^n k_j = \sum_{j=0}^n j k_j = l$, for some $0 \le l \le n$.

Proof. This follows from an algebraically lengthy recursive formula for $P_{n,j,\alpha}$ which stems from the equation (2.8), similar to the recursive formula for $e_{n,j,\alpha}$ in the proof of Theorem 2.3. In addition, one needs to find the following initial cases:

$$P_{0,1,0,0,0,0}=I, \qquad P_{1,2,1,0,0,0}=3ia'(t), \quad P_{1,2,0,0,1,0}=i\gamma^{13}a'(t)+i\gamma^{23}\cot(\eta),$$
$$P_{1,2,0,0,0,1}=i\gamma^{14}a'(t)-i\gamma^{24}\tan(\eta), \qquad P_{1,2,0,1,0,0}=2i\cot(2\eta)+i\gamma^{12}a'(t),$$
$$P_{1,3,0,1,0,2}=-2i\tan(\eta), \qquad P_{1,3,0,1,2,0}=2i\cot(\eta), \qquad P_{1,3,1,2,0,0}=2ia'(t),$$
$$P_{1,3,1,0,2,0}=2ia'(t), \qquad P_{1,3,1,0,0,2}=2ia'(t).$$

□

2.5.2 A recursive formula for the coefficient of the highest order term in a_{2n}

The highest derivative of the cosmic scale factor $a(t)$ in the expression for a_n is seen in the term $a(t)^{n-1}a^{(n)}(t)$, which has a rational coefficient based on Theorem 2.3. Let us denote the coefficient of $a(t)^{n-1}a^{(n)}(t)$ in a_n by h_n. Since the coefficients h_n are limited to satisfy the recursive relations derived in the proof of the following proposition, one can find the following closed formula for these coefficients.

Proposition 2.5. *The coefficient h_n of $a(t)^{n-1}a^{(n)}(t)$ in a_n is equal to*

$$\sum_{\substack{[n/2]+1\leq j\leq 2n+1\\ 0\leq k\leq j-n/2-1}} \Gamma\left(\frac{2k+1}{2}\right)H_{n,j,2k},$$

where, starting from

$$H_{1,2,1}=H_{1,3,1}=\frac{3i}{2\sqrt{\pi}}, \qquad H_{2,4,2}=-\frac{1}{\sqrt{\pi}},$$
$$H_{2,3,0}=H_{2,2,0}=\frac{3}{4\sqrt{\pi}}, \qquad H_{2,3,2}=-\frac{3}{2\sqrt{\pi}},$$

the quantities $H_{n,j,\alpha}$ are computed recursively by

$$H_{n,j,\alpha}=\frac{1}{j-1}\left(H_{n-2,j-1,\alpha}+2iH_{n-1,j-1,\alpha-1}\right).$$

Proof. It follows from Proposition 2.4 that the highest derivative of $a(t)$ in a_n appears in the term $a(t)^{n-1}a^{(n)}(t)$. By a careful analysis of the equation (2.11) we find that only

the terms

$$\frac{1}{j-1}\Big(a(t)^2\frac{\partial^2}{\partial t^2}P_{n-2,j-1,\alpha}+2ia(t)\frac{\partial}{\partial t}P_{n-1,j-1,\alpha-\mathbf{e}_1}\Big)$$

contribute to its recursive formula. Denoting the corresponding monomial in $P_{n,j,\alpha}$ by $H_{n,j,\alpha}a(t)^{n-1}a^{(n)}(t)$ and substituting it into the above formula we obtain the equation

$$H_{n,j,\alpha}=\frac{1}{j-1}(H_{n-2,j-1,\alpha}+2iH_{n-1,j-1,\alpha-\mathbf{e}_1}),$$

for any $n>2$. Denoting

$$H_{n,j,\alpha_1}=\sum\prod_{k=2}^{4}\Gamma\left(\frac{\alpha_k+1}{2}\right)\frac{(-1)^{\alpha_k}+1}{2}\,\mathrm{tr}\left(\frac{1}{(2\pi)^2}\int_0^{\pi/2}H_{n,j,\alpha_1,\alpha_2,\alpha_3,\alpha_4}d\eta\right),$$

the recursive formula converts to

$$H_{n,j,\alpha}=\frac{1}{j-1}(H_{n-2,j-1,\alpha}+2iH_{n-1,j-1,\alpha-1}).$$

Thus, the coefficient of $a(t)^{n-1}a^{(n)}(t)$ in a_n is given by the above expression. □

Using the above proposition we find that:

$$h_2=\frac{1}{4},\qquad h_4=\frac{1}{40},\qquad h_6=\frac{1}{560},\qquad h_8=\frac{1}{10080},\qquad h_{10}=\frac{1}{221760},$$
$$h_{12}=\frac{1}{5765760},\qquad h_{14}=\frac{1}{172972800},\qquad h_{16}=\frac{1}{5881075200},$$
$$h_{18}=\frac{1}{223480857600},\qquad h_{20}=\frac{1}{9386196019200}.$$

2.6 Conclusions

Pseudodifferential calculus is an effective tool for applying heat kernel methods to compute the terms in the expansion of a spectral action. We have used this technique to derive the terms up to a_{12} in the expansion of the spectral action for the Robertson-Walker metric on a 4-dimensional geometry with a general cosmic scale factor $a(t)$. Performing the computations in Hopf coordinates, which reflects the symmetry of the space more conveniently at least from a technical point of view, we proved the independence of the integral kernel of the corresponding heat operator from three coordiantes of the space. This allowed us to furnish the proof of the conjecture of Chamseddine and Connes on

rationality of the coefficients of the polynomials in $a(t)$ and its derivatives that describe the general terms a_{2n} in the expansion.

The terms up to a_{10} were previously computed in [9] using their direct method, where the terms up to a_6 were checked against Gilkey's universal formulas [17, 18]. The outcome of our computations confirms the previously computed terms. Thus, we have formed a check on the terms a_8 and a_{10}. In order to confirm our calculation for the term a_{12}, we have performed a completely different computation in spherical coordinates and checked its agreement with our calculation in Hopf coordinates. It is worth emphasizing that the high complexity of the computations, which is overcome by computer assistance, raises the need to derive the expressions at least in two different ways to ensure their validity.

We have found a formula for the coefficient of the term with the highest derivative of $a(t)$ in a_{2n} for all n and make the following observation. The polynomials Q_{2n} in $a_{2n} = Q_{2n}\left(a(t), a'(t), \dots, a^{(2n)}(t)\right)/a(t)^{2n-3}$ are of the following form up to Q_{12}:

$$Q_{2n}(x_0, x_1, \dots, x_{2n}) = \sum c_k\, x_0^{k_0} x_1^{k_1} \cdots x_{2n}^{k_{2n}}, \qquad c_k \neq 0,$$

where the summation is over all tuples of non-negative integers $k = (k_0, k_1, \dots, k_{2n})$ such that either $\sum k_j = 2n$ while $\sum jk_j = 2n$, or $\sum k_j = 2n-2$ while $\sum jk_j = 2n-2$. This provides enough evidence and hope to shed more light on general structure of the terms a_{2n} by further investigations, which are under way.

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Chapter 3

The Curvature of the Determinant Line Bundle on the Noncommutative Two Torus

3.1 Introduction

In this paper we compute the curvature of the determinant line bundle associated to a family of Dirac operators on the noncommutative two torus. Following Quillen's pioneering work [23], and using zeta regularized determinants, one can endow the determinant line bundle over the space of Dirac operators on the noncommutative two torus with a natural Hermitian metric. Our result computes the curvature of the associated Chern connection on this holomorphic line bundle. In the noncommutative case the method of proof applied in [23] does not work and we had to use a different strategy. To this end we found it very useful to extend the canonical trace of Kontsevich-Vishik [16] to the algebra of pseudodifferential operators on the noncommutative two torus.

This paper is organized as follows. In Section 2 we review some standard facts about Quillen's determinant line bundle on the space of Fredholm operators from [23], and about noncommutative two torus that we need in this paper. In Section 3 we develop the tools that are needed in our computation of the curvature of the determinant line bundle in the noncommutative case. We recall Connes' pseudodifferential calculus and define an analogue of the Kontsevich-Vishik trace for classical pseudodifferential symbols

on the noncommutative torus. A similar construction of the canonical trace can be found in [20], where one works with the algebra of toroidal symbols. Section 4 is devoted to Cauchy-Riemann operators on $\mathcal{A}_\theta$ with a fixed complex structure. This is the family of elliptic operators that we want to study its determinant line bundle. In Section 5 using the calculus of symbols and the canonical trace we compute the curvature of determinant line bundle. Calculus of symbols and the canonical trace allow us to bypass local calculations involving Green functions in [23], which is not applicable in our noncommutative case.

The study of the conformal and complex geometry of the noncommutative two torus started with the seminal work [7] (cf. also [5] for a preliminary version) where a Gauss-Bonnet theorem is proved for a noncommutative two torus equipped with a conformally perturbed metric. This result was refined and extended in [10] where the Gauss-Bonnet theorem was proved for metrics in all translation invariant conformal structures. The problem of computing the scalar curvature of the curved noncommutative two torus was fully settled in [6], and, independently, in [11], and in [12] for the four dimensional case. Other related works include [1, 8, 9, 15, 18, 24].

3.2 Preliminaries

In this section we recall the definition of Quillen's determinant line bundle over the space of Fredholm operators. We also recall some basic notions about noncommutative torus that we need in this paper.

3.2.1 The determinant line bundle

Unless otherwise stated, in this paper by a Hilbert space we mean a separable infinite dimensional Hilbert space over the field of complex numbers. Let $\mathcal{F} = \mathrm{Fred}(\mathcal{H}_0, \mathcal{H}_1)$ denote the set of Fredholm operators between Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. It is an open subset, with respect to norm topology, in the complex Banach space of all bounded linear operators between $\mathcal{H}_0$ and $\mathcal{H}_1$. The index map $\mathit{index} : \mathcal{F} \to \mathbb{Z}$ is a homotopy invariant and in fact defines a bijection between connected components of $\mathcal{F}$ and the set of integers $\mathbb{Z}$.

It is well known that $\mathcal{F}$ is a classifying space for K-theory: for any compact space X we have a natural ring isomorphism

$$K^0(X) = [X, \mathcal{F}]$$

between the K-theory of X and the set of homotopy classes of continuous maps from X to $\mathcal{F}$. In other words, homotopy classes of continuous families of Fredholm operators parametrized by X determine the K-theory of X. It thus follows that $\mathcal{F}$ is homotopy equivalent to $\mathbb{Z} \times BU$, the latter being also a classifying space for K-theory. Let $\mathcal{F}_0$ denote the set of Fredholm operators with index zero. By Bott periodicity, $\pi_{2j}(\mathcal{F}) \cong \mathbb{Z}$ and $\pi_{2j+1}(\mathcal{F}) = \{0\}$ for $j \geq 0$. So by Hurewicz's theorem, $H^2(\mathcal{F}_0, Z) \cong \mathbb{Z}$. Now the determinant line bundle DET defined below has the property that its first Chern class, $c_1(\text{DET})$, is a generator of $H^2(\mathcal{F}_0, \mathbb{Z}) \cong \mathbb{Z}$. We refer to [2, 25] and references therein for details.

In [23] Quillen defines a line bundle DET $\to \mathcal{F}$ such that for any $T \in \mathcal{F}$

$$\text{DET}_T = \Lambda^{max}(\ker(T))^* \otimes \Lambda^{max}(\text{coker}(T)).$$

This is remarkable if we notice that $\ker(T)$ and $\text{coker}(T)$ are not vector bundles due to discontinuities in their dimensions as T varies within $\mathcal{F}$. Let us briefly recall the construction of this determinant line bundle DET. For each finite dimensional subspace F of $\mathcal{H}_1$ let $U_F = \{T \in \mathcal{F}_1 : \text{Im}(T) + F = \mathcal{H}_1\}$ denote the set of Fredholm operators whose range is transversal to F. It is an open subset of $\mathcal{F}$ and we have an open cover $\mathcal{F} = \bigcup U_F$.

For $T \in U_F$, the exact sequence

$$0 \to \ker(T) \to T^{-1}F \xrightarrow{T} F \to \text{coker}(T) \to 0 \tag{3.1}$$

shows that the rank of $T^{-1}F$ is constant when T varies within a continuous family in U_F. Thus we can define a vector bundle $\mathcal{E}^F \to U_F$ by setting $\mathcal{E}^F_T = T^{-1}F$. We can then define a line bundle $\text{DET}^F \to U_F$ by setting

$$\text{DET}^F_T = \Lambda^{max}(T^{-1}F)^* \otimes \Lambda^{max} F.$$

We can use the inner products on $\mathcal{H}_0$ and $\mathcal{H}_1$ to split the above exact sequence (3.1) canonically and get a canonical isomorphism $\ker(T) \oplus F \cong T^{-1}F \oplus \mathrm{coker}(T)$. Therefore

$$\Lambda^{max}(\ker(T))^* \otimes \Lambda^{max}(\mathrm{coker}(T)) \cong \Lambda^{max}(T^{-1}F)^* \otimes \Lambda^{max}F.$$

Now over each member of the cover U_F a line bundle $\mathrm{DET}^F \to U_F$ is defined. Next one shows that over intersections $U_{F_1} \cap U_{F_2}$ there is an isomorphism $\mathrm{DET}^{F_1} \to \mathrm{DET}^{F_2}$ and moreover the isomorphisms satisfy a cocycle condition over triple intersections $U_{F_1} \cap U_{F_2} \cap U_{F_3}$. This shows that the line bundles $\mathrm{DET}^F \to U_F$ glue together to define a line bundle over $\mathcal{F}$. It is further shown in [23] that this line bundle is holomorphic as a bundle over an open subset of a complex Banach space.

It is tempting to think that since $c_1(\mathrm{DET})$ is the generator of $H^2(\mathcal{F}_0, \mathbb{Z}) \cong \mathbb{Z}$, there might exits a natural Hermitian metric on DET whose curvature 2-form would be a representative of this generator. One problem is that the induced metric from $\ker(T)$ and $\ker(T^*)$ on DET is not even continuous. In [23] Quillen shows that for families of Cauchy-Riemann operators on a Riemann surface one can correct the Hilbert space metric by multiplying it by zeta regularized determinant and in this way one obtains a smooth Hermitian metric on the induced determinant line bundle. In Section 5 we describe a similar construction for noncommutative two torus.

3.2.2 Noncommutative two torus

For $\theta \in \mathbb{R}$, the noncommutative two torus A_θ is by definition the universal unital C^*-algebra generated by two unitaries U, V satisfying

$$VU = e^{2\pi i\theta}UV.$$

There is a continuous action of $\mathbb{T}^2$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, on A_θ by C^*-algebra automorphisms $\{\alpha_s\}$, $s \in \mathbb{R}^2$, defined by

$$\alpha_s(U^mV^n) = e^{is.(m,n)}U^mV^n.$$

The space of smooth elements for this action will be denoted by A_θ^∞. It is a dense subalgebra of A_θ which can be alternatively described as the algebra of elements in A_θ

whose (noncommutative) Fourier expansion has rapidly decreasing coefficients:

$$A_\theta^\infty = \left\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n : a_{m,n} \in \mathcal{S}(\mathbb{Z}^2) \right\}.$$

There is a normalized, faithful and positive, trace φ_0 on A_θ whose restriction on smooth elements is given by

$$\varphi_0\big(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\big) = a_{0,0}.$$

The algebra A_θ^∞ is equipped with the derivations δ_1, $\delta_2 : A_\theta^\infty \to A_\theta^\infty$, uniquely defined by the relations

$$\delta_1(U) = U,\ \delta_1(V) = 0, \quad \delta_2(U) = 0,\ \delta_2(V) = V.$$

We have $\delta_j(a^*) = -\delta_j(a)^*$ for $j = 1, 2$ and all $a \in A_\theta^\infty$. Moreover, the analogue of the integration by parts formula in this setting is given by:

$$\varphi_0(a\delta_j(b)) = -\varphi_0(\delta_j(a)b), \ \ \forall a, b \in A_\theta^\infty.$$

We apply the GNS construction to A_θ. The state φ_0 defines an inner product

$$\langle a, b\rangle = \varphi_0(b^*a), \ \ a, b \in A_\theta,$$

and a pre-Hilbert structure on A_θ. After completion we obtain a Hilbert space denoted $\mathcal{H}_0$. The derivations δ_1, δ_2, as densely defined unbounded operators on $\mathcal{H}_0$, are formally selfadjoint and have unique extensions to selfadjoint operators.

We introduce a complex structure associated with a complex number $\tau = \tau_1 + i\tau_2$, $\tau_2 > 0$, by defining

$$\bar\partial = \delta_1 + \tau\delta_2, \ \ \bar\partial^* = \delta_1 + \bar\tau\delta_2.$$

Note that $\bar\partial$ is an unbounded operator on $\mathcal{H}_0$ and $\bar\partial^*$ is its formal adjoint. The analogue of the space of anti-holomorphic 1-forms on the ordinary two torus is defined to be

$$\Omega_\theta^{0,1} = \left\{ \sum a\bar\partial b\ , a, b \in A_\theta^\infty \right\}.$$

Using the induced inner product from ψ, one can turn $\Omega_\theta^{0,1}$ into a Hilbert space which we denote by $\mathcal{H}^{0,1}$.

3.3 The canonical trace and noncommutative residue

In this section we define an analogue of the canonical trace of Kontsevich and Vishik [16] for the noncommutative torus. Let us first recall the algebra of pseudodifferential symbols on the noncommutative torus [3, 7].

3.3.1 Pseudodifferential calculus on $\mathcal{A}_\theta$

Using operator valued symbols, one can define an algebra of pseudodifferential operators on A_θ^∞. We shall use the notation $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial \xi_2^{\alpha_2}}$, and $\delta^\alpha = \delta_1^{\alpha_1} \delta_2^{\alpha_2}$, for a multi-index $\alpha = (\alpha_1, \alpha_2)$.

Definition 3.1. For a real number m, a smooth map $\sigma : \mathbb{R}^2 \to A_\theta^\infty$ is said to be a symbol of order m, if for all non-negative integers i_1, i_2, j_1, j_2,

$$||\delta^{(i_1,i_2)} \partial^{(j_1,j_2)} \sigma(\xi)|| \le c(1 + |\xi|)^{m-j_1-j_2},$$

where c is a constant, and if there exists a smooth map $k : \mathbb{R}^2 \to A_\theta^\infty$ such that

$$\lim_{\lambda \to \infty} \lambda^{-m} \sigma(\lambda \xi_1, \lambda \xi_2) = k(\xi_1, \xi_2).$$

The space of symbols of order m is denoted by $\mathcal{S}^m(\mathcal{A}_\theta)$.

Definition 3.2. To a symbol σ of order m, one can associate an operator on A_θ^∞, denoted by P_σ, given by

$$P_\sigma(a) = \int\int e^{-is\cdot\xi} \sigma(\xi) \alpha_s(a)\, ds\, d\xi.$$

Here, $d\xi = (2\pi)^{-2} d_L \xi$ where $d_L \xi$ is the Lebesgue measure on $\mathbb{R}^2$. The operator P_σ is said to be a pseudodifferential operator of order m.

For example, the differential operator $\sum_{j_1+j_2 \le m} a_{j_1,j_2} \delta^{(j_1,j_2)}$ is associated with the symbol $\sum_{j_1+j_2 \le m} a_{j_1,j_2} \xi_1^{j_1} \xi_2^{j_2}$ via the above formula.

Two symbols σ, $\sigma' \in \mathcal{S}^m(\mathcal{A}_\theta)$ are said to be equivalent if and only if $\sigma - \sigma' \in \mathcal{S}^n(\mathcal{A}_\theta)$ for all integers n. The equivalence of the symbols will be denoted by $\sigma \sim \sigma'$.

Let P and Q be pseudodifferential operators with the symbols σ and σ' respectively. Then the adjoint P^* and the product PQ are pseudodifferential operators with the following symbols

$$\sigma(P^*) \sim \sum_{\ell=(\ell_1,\ell_2)\geq 0} \frac{1}{\ell!}\partial^\ell \delta^\ell (\sigma(\xi))^*,$$

$$\sigma(PQ) \sim \sum_{\ell=(\ell_1,\ell_2)\geq 0} \frac{1}{\ell!}\partial^\ell (\sigma(\xi))\delta^\ell(\sigma'(\xi)).$$

Definition 3.3. A symbol $\sigma \in \mathcal{S}^m(\mathcal{A}_\theta)$ is called elliptic if $\sigma(\xi)$ is invertible for $\xi \neq 0$, and for some c

$$||\sigma(\xi)^{-1}|| \leq c(1+|\xi|)^{-m},$$

for large enough $|\xi|$.

A smooth map $\sigma : \mathbb{R}^2 \to \mathcal{A}_\theta$ is called a classical symbol of order $\alpha \in \mathbb{C}$ if for any N and each $0 \leq j \leq N$ there exist $\sigma_{\alpha-j} : \mathbb{R}^2\backslash\{0\} \to \mathcal{A}_\theta$ positive homogeneous of degree $\alpha - j$, and a symbol $\sigma^N \in \mathcal{S}^{\Re(\alpha)-N-1}(\mathcal{A}_\theta)$, such that

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2. \tag{3.2}$$

Here χ is a smooth cut off function on $\mathbb{R}^2$ which is equal to zero on a small ball around the origin, and is equal to one outside the unit ball. It can be shown that the homogeneous terms in the expansion are uniquely determined by σ. We denote the set of classical symbols of order α by $\mathcal{S}^\alpha_{cl}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi^\alpha_{cl}(\mathcal{A}_\theta)$.

The space of classical symbols $\mathcal{S}_{cl}(\mathcal{A}_\theta)$ is equipped with a Fréchet topology induced by the semi-norms

$$p_{\alpha,\beta}(\sigma) = \sup_{\xi\in\mathbb{R}^2} (1+|\xi|)^{-m+|\beta|}||\delta^\alpha \partial^\beta \sigma(\xi)||. \tag{3.3}$$

The analogue of the Wodzicki residue for classical pseudodifferential operators on the noncommutative torus is defined in [13].

Definition 3.4. The Wodzicki residue of a classical pseudodifferential operator P_σ is defined as

$$\mathrm{Res}(P_\sigma) = \varphi_0\left(\mathrm{res}(P_\sigma)\right),$$

where $\mathrm{res}(P_\sigma) := \int_{|\xi|=1} \sigma_{-2}(\xi)d\xi$.

It is evident from its definition that Wodzicki residue vanishes on differential operators and on non-integer order classical pseudodifferential operators.

3.3.2 The canonical trace

In what follows, we define the analogue of Kontsevich-Vishik trace [16] on non-integer order pseudodifferential operators on the noncommutative torus. For an alternative approach based on toroidal noncommutative symbols see [20]. For a thorough review of the theory in the classical case we refer to [19, 22]. First we show the existence of the so called cut-off integral for classical symbols.

Proposition 3.5. *Let $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ and $B(R)$ be the ball of radius R around the origin. One has the following asymptotic expansion*

$$\int_{B(R)} \sigma(\xi)d\xi \sim_{R\to\infty} \sum_{j=0,\alpha-j+2\neq 0}^{\infty} \alpha_j(\sigma)R^{\alpha-j+2} + \beta(\sigma)\log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi)d\xi$ and the constant term in the expansion, $c(\sigma)$, is given by

$$\int_{\mathbb{R}^n} \sigma^N + \sum_{j=0}^{N} \int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi - \sum_{j=0,\alpha-j+2\neq 0}^{N} \frac{1}{\alpha-j+2} \int_{|\xi|=1} \sigma_{\alpha-j}(\omega)d\omega. \tag{3.4}$$

Here we have used the notation of (3.2).

Proof. First, we write $\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi)$ with large enough N, so that σ^N is integrable. Then we have,

$$\int_{B(R)} \sigma(\xi)d\xi = \sum_{j=0}^{N} \int_{B(R)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi + \int_{B(R)} \sigma^N(\xi)d\xi. \tag{3.5}$$

For $N > \alpha + 1$, $\sigma^N \in \mathcal{L}^1(\mathbb{R}^2, \mathcal{A}_\theta)$, so

$$\int_{B(R)} \sigma^N(\xi)d\xi \to \int_{\mathbb{R}^2} \sigma^N(\xi)d\xi, \quad R\to\infty.$$

Now for each $j \leq N$ we have

$$\int_{B(R)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi + \int_{B(R)\setminus B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi.$$

Obviously $\int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi < \infty$ and by using polar coordinates $\xi = r\omega$, and homogeneity of $\sigma_{\alpha-j}$, we have

$$\int_{B(R)\backslash B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \int_1^R r^{\alpha-j+2-1}dr \int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi. \tag{3.6}$$

Note that the cut-off function is equal to one on the set $\mathbb{R}^2\backslash B(1)$. For the term with $\alpha - j = -2$ one has

$$\int_{B(R)\backslash B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \log R \int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi.$$

The terms with $\alpha - j \neq -2$ will give us the following:

$$\begin{aligned} \int_{B(R)\backslash B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi &= \\ \frac{R^{\alpha-j+2}}{m-j+2}\int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi &- \frac{1}{\alpha-j+2}\int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi. \end{aligned} \tag{3.7}$$

Adding all the constant terms in (3.5)-(3.7), we get the constant term given in (3.4). □

Definition 3.6. The cut-off integral of a symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi)d\xi$.

Remark 3.7. Two remarks are in order here. First note that the cut-off integral of a symbol is independent of the choice of N. Second, it is also independent of the choice of the cut-off function χ.

We now give the definition of the canonical trace for classical pseudodifferential operators on $\mathcal{A}_\theta$.

Definition 3.8. The canonical trace of a classical pseudodifferential operator $P \in \Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$ of non-integral order α is defined as

$$\mathrm{TR}(P) := \varphi_0\left(⨍ \sigma_P(\xi)d\xi\right).$$

In the following, we establish the relation between the TR-functional and the usual trace on trace-class pseudodifferential operators. Note that any pseudodifferential operator P of order less that -2, is a trace-class operator on $\mathcal{H}_0$ and its trace is given by

$$\mathrm{Tr}(P) = \varphi_0\left(\int_{\mathbb{R}^2} \sigma_P(\xi)d\xi\right).$$

On the other hand, for such operator the symbol is integrable and we have

$$⨍ \sigma_P(\xi) = \int_{\mathbb{R}^2} \sigma_P(\xi) d\xi. \tag{3.8}$$

Therefore, the TR-functional and operator trace coincide on classical pseudodifferential operators of order less than -2.

Next, we show that the TR-functional is in fact an analytic continuation of the operator trace and using this fact we can prove that it is actually a trace.

Definition 3.9. A family of symbols $\sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta)$, parametrized by $z \in W \subset \mathbb{C}$, is called a holomorphic family if

i) The map $z \mapsto \alpha(z)$ is holomorphic.

ii) The map $z \mapsto \sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta)$ is a holomorphic map from W to the Fréchet space $\mathcal{S}_{cl}(\mathcal{A}_\theta^n)$.

iii) The map $z \mapsto \sigma(z)_{\alpha(z)-j}$ is holomorphic for any j, where

$$\sigma(z)(\xi) \sim \sum_j \chi(\xi)\sigma(z)_{\alpha(z)-j}(\xi) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta). \tag{3.9}$$

iv) The bounds of the asymptotic expansion of $\sigma(z)$ are locally uniform with respect to z, i.e, for any $N \geq 1$ and compact subset $K \subset W$, there exists a constant $C_{N,K,\alpha,\beta}$ such that for all multi-indices α, β we have

$$\left\| \delta^\alpha \partial^\beta \left(\sigma(z) - \sum_{j<N} \chi \sigma(z)_{\alpha(z)-j} \right)(\xi) \right\| < C_{N,K,\alpha,\beta} |\xi|^{\Re(\alpha(z))-N-|\beta|}.$$

A family $\{P_z\} \in \Psi_{cl}(\mathcal{A}_\theta)$ is called holomorphic if $P_z = P_{\sigma(z)}$ for a holomorphic family of symbols $\{\sigma(z)\}$.

The following Proposition is an analogue of a result of Kontsevich and Vishik[16], for pseudodifferential calculus on noncommutative tori.

Proposition 3.10. *Given a holomorphic family $\sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta)$, $z \in W \subset \mathbb{C}$, the map*

$$z \mapsto ⨍ \sigma(z)(\xi) d\xi,$$

is meromorphic with at most simple poles located in

$$P = \{z_0 \in W; \ \alpha(z_0) \in \mathbb{Z} \cap [-2, +\infty]\}.$$

The residues at poles are given by

$$\mathrm{Res}_{z=z_0} ⨍ \sigma(z)(\xi)d\xi = -\frac{1}{\alpha'(z_0)} \int_{|\xi|=1} \sigma(z_0)_{-2} d\xi.$$

Proof. By definition, one can write $\sigma(z) = \sum_{j=0}^{N} \chi(\xi)\sigma(z)_{\alpha(z)-i}(\xi) + \sigma(z)^N(\xi)$, and by Proposition 3.5 we have,

$$\begin{aligned} ⨍ \sigma(z)(\xi)d\xi = &\int_{\mathbb{R}^2} \sigma(z)^N(\xi)d\xi + \sum_{j=0}^{N} \int_{B(1)} \chi(\xi)\sigma(z)_{\alpha(z)-j}(\xi) \\ &- \sum_{j=0}^{N} \frac{1}{\alpha(z)+2-j} \int_{|\xi|=1} \sigma(z)_{\alpha(z)-j}(\xi)d\xi. \end{aligned}$$

Now suppose $\alpha(z_0) + 2 - j_0 = 0$. By holomorphicity of $\sigma(z)$, we have $\alpha(z) - \alpha(z_0) = \alpha'(z_0)(z - z_0) + o(z - z_0)$. Hence

$$\mathrm{Res}_{z=z_0} ⨍ \sigma(z) = \frac{-1}{\alpha'(z_0)} \int_{|\xi|=1} \sigma(z_0)_{-2}(\xi)d\xi.$$

□

Corollary 3.11. *The functional* TR *is the analytic continuation of the ordinary trace on trace-class pseudodifferential operators.*

Proof. First observe that, by the above result, for a non-integer order holomorphic family of symbols $\sigma(z)$, the map $z \mapsto ⨍ \sigma(z)(\xi)d\xi$ is holomorphic. Hence, the map $\sigma \mapsto ⨍ \sigma(\xi)d\xi$ is the unique analytic continuation of the map $\sigma \mapsto \int_{\mathbb{R}^2} \sigma(\xi)d\xi$ from $\mathcal{S}_{cl}^{<-2}(\mathcal{A}_\theta)$ to $\mathcal{S}_{cl}^{\notin\mathbb{Z}}(\mathcal{A}_\theta)$. By (3.8) we have the result. □

Let $Q \in \Psi_{cl}^{q}(\mathcal{A}_\theta)$ be a positive elliptic pseudodifferential operator of order $q > 0$. The complex power of such an operator, Q_ϕ^z, for $\Re(z) < 0$ can be defined by the following Cauchy integral formula.

$$Q_\phi^z = \frac{i}{2\pi} \int_{C_\phi} \lambda_\phi^z (Q - \lambda)^{-1} d\lambda. \tag{3.10}$$

Here λ_ϕ^z is the complex power with branch cut $L_\phi = \{re^{i\phi}, r \geq 0\}$ and C_ϕ is a contour around the spectrum of Q such that

$$C_\phi \cap \mathrm{spec}(Q)\backslash\{0\} = \emptyset, \qquad L_\phi \cap C_\phi = \emptyset,$$

$$C_\phi \cap \{\mathrm{spec}(\sigma(Q)^L(\xi)),\, \xi \neq 0\} = \emptyset.$$

In general an operator for which one can find a ray L_ϕ with the above property, is called an admissible operator with the spectral cut L_ϕ. Positive elliptic operators are admissible and we take the ray L_π as the spectral cut, and in this case we drop the index ϕ and write Q^z.

To extend (3.10) to $\Re(z) > 0$ we choose a positive integer such that $\Re(z) < k$ and define

$$Q_\phi^z := Q^k Q_\phi^{z-k}.$$

It can be proved that this definition is independent of the choice of k.

Corollary 3.12. *Let $A \in \Psi_{cl}^{\alpha}(\mathcal{A}_\theta)$ be of order $\alpha \in \mathbb{Z}$ and let Q be a positive elliptic classical pseudodifferential operator of positive order q. We have*

$$\mathrm{Res}_{z=0}\mathrm{TR}(AQ^{-z}) = \frac{1}{q}\mathrm{Res}(A).$$

Proof. For the holomorphic family $\sigma(z) = \sigma(AQ^{-z})$, $z = 0$ is a pole for the map $z \mapsto \fint \sigma(z)(\xi)d\xi$ whose residue is given by

$$\mathrm{Res}_{z=0}\left(z \mapsto \fint \sigma(z)(\xi)d\xi\right) = -\frac{1}{\alpha'(0)}\int_{|\xi|=1}\sigma_{-2}(0)d\xi = -\frac{1}{\alpha'(0)}\mathrm{res}(A).$$

Taking trace on both sides gives the result. □

Now we can prove the trace property of TR-functional.

Proposition 3.13. *We have* $\mathrm{TR}(AB) = \mathrm{TR}(BA)$ *for any* $A, B \in \Psi_{cl}(\mathcal{A}_\theta)$*, provided that* $ord(A) + ord(B) \notin \mathbb{Z}$.

Proof. Consider the families $A_z = AQ^z$ and $B_z = BQ^z$ where Q is an injective positive elliptic classical operator of order $q > 0$. For $\Re(z) << 0$, the two families are trace class and $\mathrm{Tr}(A_zB_z) = \mathrm{Tr}(B_zA_z)$. By the uniqueness of the analytic continuation, we have

$$\mathrm{TR}(A_zB_z) = \mathrm{TR}(B_zA_z),$$

for those z for which $2qz + \mathrm{ord}(A) + \mathrm{ord}(B) \notin \mathbb{Z}$. At $z = 0$, we obtain $\mathrm{Tr}(AB) = \mathrm{TR}(BA)$. $\square$

3.3.3 Log-polyhomogeneous symbols

Proposition 3.10 can be extended and one can explicitly write down the Laurent expansion of the cut-off integral around each of the poles. The terms of the Laurent expansion involve residue densities of z-derivatives of the holomorphic family. In general, z-derivatives of a classical holomorphic family of symbols is not classical anymore and therefore we introduce log-polyhomogeneous symbols which include the z-derivatives of the symbols of the holomorphic family $\sigma(AQ^{-z})$.

Definition 3.14. A symbol σ is called a log-polyhomogeneous symbol if it has the following form

$$\sigma(\xi) \sim \sum_{j \geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0, \tag{3.11}$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

An important example of an operator with such a symbol is $\log Q$ where $Q \in \Psi^q_{cl}(\mathcal{A}_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$. The logarithm of Q can be defined by

$$\log Q = Q \left.\frac{d}{dz}\right|_{z=0} Q^{z-1} = Q \left.\frac{d}{dz}\right|_{z=0} \frac{i}{2\pi} \int_C \lambda^{z-1}(Q-\lambda)^{-1} d\lambda.$$

It is a pseudodifferential operator with symbol

$$\sigma(\log Q) \sim \sigma(Q) \star \sigma\Big(\left.\frac{d}{dz}\right|_{z=0} Q^{z-1} \Big), \tag{3.12}$$

where $\star$ denotes the products of the pseudodifferential symbols. Using symbol calculus and homogeneity properties, we can show that (3.12) is a log-homogeneous symbol of the form

$$\sigma(\log Q)(\xi) = 2 \log |\xi| I + \sigma_{cl}(\log Q)(\xi),$$

where $\sigma_{cl}(\log Q)$ is a classical symbol of order zero. This symbol can be computed using the homogeneous parts of the classical symbol $\sigma(Q^z) = \sum_{j=0}^{\infty} b(z)_{2z-j}(\xi)$ and it is given

by the following formula (see e.g. [19]).

$$\sigma_{cl}(\log Q)(\xi) = \tag{3.13}$$
$$\sum_{k=0}^{\infty} \sum_{i+j+|\alpha|=k} \frac{1}{\alpha!} \partial^{\alpha} \sigma_{2-i}(Q) \delta^{\alpha} \left[|\xi|^{-2-j} \left. \frac{d}{dz} \right|_{z=0} b(z-1)_{2z-2-j} \left(\xi/|\xi| \right) \right].$$

The Wodzicki residue can also be extended to this class of pseudodifferential operators [17]. For an operator A with log-polyhomogeneous symbol as (3.11) it can be defined by

$$\operatorname{res}(A) = \int_{|\xi|=1} \sigma_{-2,0}(\xi) d\xi.$$

By adapting the proof of Theorem 1.13 in [22] to the noncommutative case, we have the following theorem which is written only for the families of the form $\sigma(AQ^{-z})$ which we will use in Section 3.5.

Proposition 3.15. *Let $A \in \Psi_{cl}^{\alpha}(\mathcal{A}_\theta)$ and Q be a positive , in general an admissible, elliptic pseudodifferential operator of positive order q. If $\alpha \in P$ then 0 is a possible simple pole for the function $z \mapsto \mathrm{TR}(AQ^{-z})$ with the following Laurent expansion around zero.*

$$\begin{aligned}
\mathrm{TR}(AQ^{-z}) = {} & \frac{1}{q} \operatorname{Res}(A) \frac{1}{z} \\
& + \varphi_0 \left(⨍ \sigma(A) - \frac{1}{q} \operatorname{res}(A \log Q) \right) - \operatorname{Tr}(A \Pi_Q) \\
& + \sum_{k=1}^{K} (-1)^k \frac{(z)^k}{k!} \\
& \times \left(\varphi_0 \left(⨍ \sigma(A (\log Q)^k) d\xi - \frac{1}{q(k+1)} \operatorname{res}(A (\log Q)^{k+1}) \right) - \operatorname{Tr}(A \log^k Q \Pi_Q) \right) \\
& + o(z^K).
\end{aligned}$$

Where Π_Q is the projection on the kernel of Q.

For operators A and Q as in the previous Proposition, we define a zeta function by

$$\zeta(A, Q, z) = \mathrm{TR}(AQ^{-z}). \tag{3.14}$$

By Corollary 3.11, it is obvious that $\zeta(A, Q, z)$ is the analytic continuation of the zeta function $\mathrm{Tr}(AQ^{-z})$ defined by the regular trace only for $\Re(z) >> 0$.

Remark 3.16. If A is a differential operator, the zeta function (3.14) is holomorphic at $z = 0$ with the value equal to

$$\varphi_0\left(⨍\sigma(A) - \frac{1}{q}\mathrm{res}(A\log Q)\right) - \mathrm{Tr}(A\Pi_Q).$$

3.4 Cauchy-Riemann operators on noncommutative tori

In [23], Quillen studies the geometry of the determinant line bundle on the space of all Cauchy-Riemann operators on a smooth vector bundle on a closed Riemann surface. To investigate the same notion on noncommutative tori, we first briefly recall some basic facts in the classical case on how Cauchy-Riemann operators are related to Dirac operators and spectral triples. Then by analogy we define our Cauchy-Riemann operator on $\mathcal{A}_\theta$, and consider the spectral triples defined by them.

Let M be a compact complex manifold and V be a smooth complex vector bundle on M. Let $\Omega^{p,q}(M,V)$ denote the space of (p,q) forms on M with coefficients in V. A $\bar{\partial}$-flat connection on V is a $\mathbb{C}$-linear map $D : \Omega^{0,0}(M,V) \to \Omega^{0,1}(M,V)$, such that for any $f \in C^\infty(M)$ and $u \in \Omega^{0,0}(M,V)$,

$$D(fu) = (\bar{\partial}f)\otimes u + fDu, \tag{3.15}$$

and $D^2 = 0$. Here to define D^2, note that any $\bar{\partial}$-connection as above has a unique extension to an operator $D : \Omega^{p,q}(M,V) \to \Omega^{p,q+1}(M,V)$, defined by

$$D(\alpha\otimes\beta) = \bar{\partial}\alpha\otimes u + (-1)^{p+q}\alpha\wedge Du, \quad \alpha\in\Omega^{p,q}(M),\ u \in C^\infty(V).$$

We refer to $\bar{\partial}$-flat connections as Cauchy-Riemann operators. A holomorphic vector bundle V has a canonical Cauchy-Riemann operator $\bar{\partial}_V : \Omega^0(M,V) \to \Omega^{0,1}(M,V)$, whose extension to $\Omega^{0,*}(M,V)$ forms the Dolbeault complex of M with coefficients in V. In fact there is a one-one correspondence between Cauchy-Riemann operators on V up to (gauge) equivalence, and holomorphic structures on V. We denote by $\mathcal{A}$ the set of all Cauchy-Riemann operators on V.

Any holomorphic structure on a Hermitian vector bundle V determines a unique Hermitian connection, called the Chern connection, whose projection on $(0,1)$-forms, $\nabla^{0,1}(M,V)$, is the Cauchy-Riemann operator coming from the holomorphic structure.

Now, if M is a Kähler manifold, the tensor product of the Levi-Civita connection for M with the Chern connection on V defines a Clifford connection on the Clifford module $(\Lambda^{0,+} \oplus \Lambda^{0,-}) \otimes V$ and the operator $D_0 = \sqrt{2}(\bar{\partial}_V + \bar{\partial}_V^*)$ is the associated Dirac operator (see e.g. [14]). Any other Dirac operator on the Clifford module $(\Lambda^{0,+} \oplus \Lambda^{0,-}) \otimes V$ is of the form $D_0 + A$ where A is the connection one form of a Hermitian connection. This connection need not be a Chern connection. However, on a Riemann surface (with a Riemannian metric compatible with its complex structure) any Hermitian connection on a smooth Hermitian vector bundle is the Chern connection of a holomorphic structure on V. Therefore, the positive part of any Dirac operator on $(\Lambda^{0,0} \oplus \Lambda^{0,1}) \otimes V$ is a Cauchy-Riemann operator, and this gives a one to one correspondence between all Dirac operators and the set of all Cauchy-Riemann operators.

Next we define the analogue of Cauchy-Riemann operators for the noncommutative torus. First, following [7, 10], we fix a complex structure on $\mathcal{A}_\theta$ by a complex number τ in the upper half plane and construct the spectral triple

$$(\mathcal{A}_\theta, \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, D_0 = \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix}), \tag{3.16}$$

where $\bar{\partial} : \mathcal{A}_\theta \to \mathcal{A}_\theta$ is given by $\bar{\partial} = \delta_1 + \tau\delta_2$. The Hilbert space $\mathcal{H}_0$ is obtained by GNS construction from $\mathcal{A}_\theta$ using the trace φ_0 and $\bar{\partial}^*$ is the adjoint of the operator $\bar{\partial}$.

As in the classical case, we define our Cauchy-Riemann operators on $\mathcal{A}_\theta$ as the positive part of twisted Dirac operators. All such operators define spectral triples of the form

$$(\mathcal{A}_\theta, \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, D_A = \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix}),$$

where $\alpha \in \mathcal{A}_\theta$ is the positive part of a selfadjoint element

$$A = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix} \in \Omega^1_{D_0}(\mathcal{A}_\theta).$$

We recall that $\Omega^1_{D_0}(\mathcal{A}_\theta)$ is the space of quantized one forms consisting of the elements $\sum a_i[D_0, b_i]$ where $a_i, b_i \in \mathcal{A}_\theta$ [4]. Note that the in this case, the space $\mathcal{A}$ of Cauchy-Riemann operators is the space of $(0,1)$-forms on $\mathcal{A}_\theta$.

We should mention that in the noncommutative case, in the work of Chakraborty and Mathai [2] a general family of spectral triples is considered and, under suitable regularity

conditions, a determinant line bundle is defined for such families. The curvature of the determinant line bundle however is not computed and that is the main object of study in the present paper, as well as in [23].

3.5 The curvature of the determinant line bundle for $\mathcal{A}_\theta$

For any $\alpha \in \mathcal{A}$, the Cauchy-Riemann operator

$$\bar{\partial}_\alpha = \bar{\partial} + \alpha : \mathcal{H}_0 \to \mathcal{H}^{0,1}$$

is a Fredholm operator. We pull back the determinant line bundle DET on the space of Fredholm operators $\mathrm{Fred}(\mathcal{H}_0, \mathcal{H}^{0,1})$, to get a line bundle $\mathcal{L}$ on $\mathcal{A}$. Following Quillen [23], we define a Hermitian metric on $\mathcal{L}$ and compute its curvature in this section. Let us define a metric on the fiber

$$\mathcal{L}_\alpha = \Lambda^{max}(\ker \bar{\partial}_\alpha)^* \otimes \Lambda^{max}(\ker \bar{\partial}_\alpha^*).$$

as the product of the induced metrics on $\Lambda^{max}(\ker \bar{\partial}_\alpha)^*$, $\Lambda^{max}(\ker \bar{\partial}_\alpha^*)$, with the zeta regularized determinant $e^{-\zeta'_{\Delta_\alpha}(0)}$. Here we define the Laplacian as $\Delta_\alpha = \bar{\partial}_\alpha^* \bar{\partial}_\alpha : \mathcal{H}_0 \to \mathcal{H}_0$, and its zeta function by

$$\zeta(z) = \mathrm{TR}(\Delta_\alpha^{-z}).$$

It is a meromorphic function and by Remark 3.16 it is regular at $z = 0$. Similar proof as in [23] shows that this defines a smooth Hermitian metric on $\mathcal{L}$.

On the open set of invertible operators each fiber of $\mathcal{L}$ is canonically isomorphic to $\mathbb{C}$ and the nonzero holomorphic section $\sigma = 1$ gives a trivialization. Also, according to the definition of the Hermitian metric, the norm of this section is given by

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}. \tag{3.17}$$

3.5.1 Variations of LogDet and curvature form

We begin by explaining the motivation behind the computations of Quillen in [23]. Recall that a holomorphic line bundle equipped with a Hermitian inner product has a canonical

connection compatible with the two structures. This is also known as the Chern connection. The curvature form of this connection is computed by $\bar{\partial}\partial \log \|\sigma\|^2$, where σ is any non-zero local holomorphic section.

In our case we will proceed by analogy and compute the second variation $\bar{\partial}\partial \log \|\sigma\|^2$ on the open set of invertible index zero Cauchy-Riemann operators. Let us consider a holomorphic family of invertible index zero Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$, where α_w depends holomorphically on the complex variable w and compute

$$\delta_{\bar{w}}\delta_w \zeta'_\Delta(0).$$

One has the following first variational formula,

$$\delta_w \zeta(z) = \delta_w \mathrm{TR}(\Delta^{-z}) = \mathrm{TR}(\delta_w \Delta^{-z}) = -z\mathrm{TR}(\delta_w \Delta \Delta^{-z-1}),$$

where in the second equality we were able to change the order of δ_w and TR because of the uniformity condition in the definition of holomorphic families (cf. [21]).

Note that, although $\mathrm{TR}(\Delta^{-z})$ is regular at $z = 0$, $\mathrm{TR}(\delta_w \Delta\Delta^{-z-1})$ might have a pole at $z = 0$ since $\delta_w \Delta\Delta^{-z-1}|_{z=0} = \delta_w \Delta\Delta^{-1}$ is not a differential operator any more and may have non-zero residue. Around $z = 0$ one has the following Laurent expansion:

$$-z\mathrm{TR}(\delta_w \Delta\Delta^{-z-1}) = -z(\frac{a_{-1}}{z} + a_0 + a_1 z + \cdots).$$

Hence,

$$\delta_w \zeta(z)|_{z=0} = -a_{-1}, \qquad \frac{d}{dz}\delta_w \zeta(z)\Big|_{z=0} = -a_0.$$

Using Proposition 3.15 we have

$$\delta_w \zeta'(0) = \frac{d}{dz}\delta_w \zeta(z)\Big|_{z=0} = -\varphi_0 \left(⨍ \sigma(\delta_w \Delta\Delta^{-1}) - \frac{1}{2}\mathrm{res}_x(\delta_w \Delta\Delta^{-1} \log \Delta) \right).$$

To compute the right hand side of the above equality, we need to note that since D_w depends holomorphically on w, $\delta_w D^* = 0$ and hence

$$\delta_w \Delta = \delta_w D^* D + D^* \delta_w D = D^* \delta_w D.$$

Since $\delta_w D$ is a zero order differential operator, we have

$$\begin{aligned}
\delta_w \zeta'(0) &= -\varphi_0 \left(\fint \sigma(D^* \delta_w D \Delta^{-1}) - \frac{1}{2}\mathrm{res}(D^* \delta_w D \Delta^{-1} \log \Delta) \right) \\
&= -\varphi_0 \left(\fint \sigma(\delta_w D \Delta^{-1} D^*) - \frac{1}{2}\mathrm{res}(\delta_w D \log \Delta\, \Delta^{-1} D^*) \right) \\
&= -\varphi_0 \left(\delta_w D \left(\fint \sigma(D^{-1}) - \frac{1}{2}\mathrm{res}(\log \Delta\, D^{-1}) \right) \right) \\
&= -\varphi_0 \left(\delta_w D\, J \right),
\end{aligned}$$

where

$$J = \fint \sigma(D^{-1}) - \frac{1}{2}\mathrm{res}(\log(\Delta) D^{-1}).$$

The reader can compare this to the term J in Quillen's computations [23].

Now we compute the second variation $\delta_{\bar{w}}\delta_w \zeta'(0)$. Since D_w is holomorphic we have

$$\delta_{\bar{w}}\delta_w \zeta'(0) = -\varphi_0 \left(\delta_w D \delta_{\bar{w}} J \right).$$

Next we compute the variation $\delta_{\bar{w}} J$. Note that since D_w is invertible, D_w^{-1} is also holomorphic and hence $\delta_{\bar{w}} \fint \sigma(D^{-1}) = 0$. Therefore

$$\delta_{\bar{w}} J = \delta_{\bar{w}} \left(\fint \sigma(D^{-1}) - \frac{1}{2}\mathrm{res}(\log \Delta\, D^{-1}) \right) = -\frac{1}{2}\delta_{\bar{w}}\mathrm{res}(\log \Delta\, D^{-1}).$$

Thus, we have shown that

Lemma 3.17. *For the holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ reads:*

$$\delta_{\bar{w}}\delta_w \zeta'(0) = \frac{1}{2}\varphi_0 \left(\delta_w D \delta_{\bar{w}} \mathrm{res}(\log \Delta\, D^{-1}) \right).$$

□

Our next goal is to compute $\delta_{\bar{w}}\mathrm{res}(\log \Delta\, D^{-1})$. This combined with the above lemma shows that the curvature form of the determinant line bundle equals the Kähler form on the space of connections.

Lemma 3.18. *With above definitions and notations, we have*

$$\sigma_{-2,0}(\log\Delta\, D^{-1}) = \frac{(\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2}{(\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)(\xi_1+\tau\xi_2)}$$

$$-\log\left(\frac{\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2}{|\xi|^2}\right)\frac{\alpha}{\xi_1+\tau\xi_2},$$

and

$$\delta_{\bar{w}}\mathrm{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Proof. By writing down the homogeneous terms in the expansion of $\sigma_{\bullet,0}(\log\Delta)$ and $\sigma(D^{-1})$ and using the product formula of the symbols we see that

$$\sigma_{-2,0}(\log\Delta D^{-1}) \sim \sigma_{-1,0}(\log\Delta)\sigma_{-1}(D^{-1}) + \sigma_{0,0}(\log\Delta)\sigma_{-2}(D^{-1}).$$

Starting with the symbol of Δ, we have

$$\sigma(\Delta) = \xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2 + (\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2 + \bar{\partial}^*(\alpha).$$

Then, the homogeneous parts of $\sigma((\lambda-\Delta)^{-1}) = \sum_j b_{-2-j}$ is given by the following recursive formula

$$b_{-2} = (\lambda-\sigma_2(\Delta))^{-1},$$
$$b_{-2-j} = -b_{-2}\sum_{k+l+|\gamma|=j,\, l<j} \partial^\gamma\sigma_{2-k}(\Delta)\delta^\gamma b_{-2-l}/\gamma!,$$

which gives us

$$b_{-2} = \frac{1}{\lambda-(\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)},$$

and

$$b_{-3} = \frac{1}{(\lambda-(\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2))^2}\left((\alpha+\alpha^*)\xi_1+(\bar{\tau}\alpha+\tau\alpha^*)\xi_2\right).$$

Also, Δ^z is a classical operator defined by

$$\Delta^z = \frac{1}{2\pi i}\int_C \lambda^z(\lambda-\Delta)^{-1}d\lambda,$$

with the homogeneous parts of the symbol given by

$$b(z)_{2z-j} := \sigma_{2z-j}(\Delta^z) = \frac{1}{2\pi i}\int_C \lambda^z b_{-2-j} d\lambda.$$

Hence we have

$$b(z)_{2z} = \frac{1}{2\pi i}\int_C \lambda^z \frac{1}{\lambda - (\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)} d\lambda$$

$$= (\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^z$$

$$b(z)_{2z-1} = \frac{1}{2\pi i}\int_C \lambda^z \frac{((\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha + \tau\alpha^*)\xi_2)}{(\lambda - (\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2))^2} d\lambda$$

$$= z(\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2))^{z-1}\left((\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha + \tau\alpha^*)\xi_2\right).$$

Using (3.13) and what we have computed up to here, it is clear that

$$\sigma_{0,0}(\log\Delta)(\xi) = \sigma_2(\Delta)|\xi|^{-2} \left.\frac{d}{dz}\right|_{z=0} b(z-1)_{2z-2}\left(\xi/|\xi|\right)$$

$$= \sigma_2(\Delta)|\xi|^{-2} \left.\frac{d}{dz}\right|_{z=0} ((\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)/|\xi|^2)^{z-1}$$

$$= \log((\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)/|\xi|^2).$$

Note that the above term is homogeneous of order zero in ξ.

$$
\begin{aligned}
&\sigma_{-1,0}(\log\Delta)(\xi)\\
&= \sum_{i+j+|\alpha|=1} \frac{1}{\alpha!}\partial^\alpha \sigma_{2-i}(\Delta)\delta^\alpha |\xi|^{-2-j}\ \frac{d}{dz}\bigg|_{z=0} b(z-1)_{2z-2-j}\left(\xi/|\xi|\right)\\
&= \sigma_2(\Delta)|\xi|^{-3}\ \frac{d}{dz}\bigg|_{z=0} b(z-1)_{2z-3}\left(\xi/|\xi|\right)\\
&+ \sigma_1(\Delta)|\xi|^{-2}\ \frac{d}{dz}\bigg|_{z=0} b(z-1)_{2z-2}\left(\xi/|\xi|\right)\\
&= \frac{1-\log(\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)/|\xi|^2)}{(\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)}\left[(\alpha+\alpha^*)\xi_1+(\bar\tau\alpha+\tau\alpha^*)\xi_2\right]\\
&+ \frac{\log(\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)/|\xi|^2)}{\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2}\left[(\alpha+\alpha^*)\xi_1+(\bar\tau\alpha+\tau\alpha^*)\xi_2\right]\\
&= (\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)^{-1}\left[(\alpha+\alpha^*)\xi_1+(\bar\tau\alpha+\tau\alpha^*)\xi_2\right].
\end{aligned}
$$

Next we compute the symbol of D^{-1}. The symbol of D reads

$$\sigma(D)=\xi_1+\tau\xi_2+\alpha.$$

We need to compute the homogeneous parts of order -1 and -2 of D^{-1}. By using recursive formula for the symbol of the inverse we get:

$$
\begin{aligned}
\sigma_{-1}(D^{-1}) &= \sigma_1(D)^{-1} = (\xi_1+\tau\xi_2)^{-1}\\
\sigma_{-2}(D^{-1}) &= -\sigma_{-1}(D^{-1})\sum_{k+|\gamma|=1}\partial^\gamma\sigma_{1-k}(D)\delta^\gamma\sigma_{-1}(D^{-1})/\gamma!\\
&= -\sigma_{-1}(D^{-1})^2\sigma_0(D)\\
&= -(\xi_1+\tau\xi_2)^{-2}\alpha.
\end{aligned}
$$

Finally, we have

$$\begin{aligned}\sigma_{-2,0}(\log\Delta\, D^{-1}) &= \sigma_{-1,0}(\log\Delta)\sigma_{-1}(D^{-1}) + \sigma_{0,0}(\log\Delta)\sigma_{-2}(D^{-1}) \\ &= (\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}(\xi_1+\tau\xi_2)^{-1}\left[(\alpha+\alpha^*)\xi_1 + (\bar\tau\alpha+\tau\alpha^*)\xi_2\right] \\ &- \log((\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)/|\xi|^2)(\xi_1+\tau\xi_2)^{-2}\alpha.\end{aligned}$$

Therefore, we compute the variation:

$$\begin{aligned}\delta_{\bar w}\sigma_{-2,0}(\log\Delta\, D^{-1}) &= (\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}\left[(\delta_{\bar w}\alpha^*)\xi_1 + (\tau\delta_{\bar w}\alpha^*)\xi_2\right](\xi_1+\tau\xi_2)^{-1} \\ &= (\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}(\delta_{\bar w}\alpha^*) \\ &= (\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}(\delta_w D)^*. \end{aligned} \tag{3.18}$$

In order to compute the variation of the residue density, we need to integrate (3.18) with respect to ξ variable:

$$\delta_{\bar w}\mathrm{res}(\log(\Delta)D^{-1}) = \int_{|\xi|=1}(\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}(\delta_w D)^* d\xi = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Note that we have used the normalized Lebesgue measure in the last integral (see (3.2)).

□

We record the main result of this paper in the following theorem. It computes the curvature of the determinant line bundle in terms of the natural Kähler form on the space of connections.

Theorem 3.19. *The curvature of the determinant line bundle for the noncommutative two torus is given by*

$$\delta_{\bar w}\delta_w\zeta'(0) = \frac{1}{4\pi\Im(\tau)}\varphi_0\left(\delta_w D(\delta_w D)^*\right). \tag{3.19}$$

□

Remark 3.20. In order to recover the classical result of Quillen for $\theta = 0$, we have to take into account the change of the volume form due to a change of the metric. This means we have to multiply the above result by $\Im(\tau)$.

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Chapter 4

Spectral action of the Berger spheres $\mathbb{S}^3(T)$

4.1 Introduction

The spectral action is a tool to extract geometric information of a spectral triple, which is a generalization of space in the sense of noncommutative geometry. The spectral action is an effective method to find spectral formulation of the action functional for physical models through the geometry of a carefully chosen spectral triple [4, 7]. The spectral action of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is defined by

$$\mathrm{Tr} f(D/\Lambda), \tag{4.1}$$

where f is an even positive real valued function and Λ is a positive number which is called the mass scale. The asymptotic expansion of the spectral action as $\Lambda \to \infty$ is closely related to the asymptotic expansion of the heat trace $\mathrm{Tr}(e^{-tD^2})$ [8, Theorem 1.145].

There are different techniques to compute the asymptotic expansion of the spectral action. The universal formula for the heat trace coefficients is an effective method. However, the formulas are only for the Laplace type operators and computed up to the tenth term [10, 11, 20]. The method of pseudodifferential operators, with all its difficulties in the computations, is another method that can be applied for any positive operator with positive principal symbol and also can compute higher asymptotic terms.

Moreover, this method is effective when one wants to prove a general fact about the terms (see e.g. [9]).

To apply the above methods there is no need to compute even a single eigenvalue of D, knowing the symbol of the operator in local charts is enough. For the cases that the explicit spectrum of the Dirac operator is available, one can apply numeric formulas to approximate $\sum_{\lambda\in\mathrm{spec}D} f(\lambda/\Lambda)$ in terms of powers of Λ. In [5], the Poisson summation formula is used to compute the spectral action of the round 3-sphere. The spectral action of the homogeneous space $\mathrm{SU}(2)/\Gamma$, where Γ is a finite subgroup of $\mathrm{SU}(2)$, is also computed by the Poisson summation formula in [19].

The Euler-Maclaurin summation is another important tool from numerical analysis that can be used in spectral action computations. This formula was discovered independently by Euler and Maclaurin in the eighteenth century. Further historical notes can be find in [16, 17].

Let $f \in C^{2m}([a,b])$; then

$$\begin{aligned} \sum_{n=a}^{b} g(n) = \int_a^b g(x)\,dx + \frac{1}{2}\left(g(b)+g(a)\right) \quad &+ \quad \sum_{j=2}^{m} \frac{B_j}{j!}\left(g^{(j-1)}(b) - g^{(j-1)}(a)\right) \\ &- \quad R_m(g,a,b), \end{aligned} \tag{4.2}$$

where the remainder is given by $R_m(g,a,b) = \frac{(-1)^m}{m!}\int_a^b B_m(\{x\})g^{(m)}(x)dx$ and $\{x\}$ denotes the fractional part of x, i.e., $x-[x]$. Here, $B_m(x)$ are the Bernoulli polynomials which are defined by the coefficients of the following power series:

$$\frac{ze^{zx}}{e^z-1} = \sum_{m=0}^{\infty} \frac{B_m(x)z^n}{m!}.$$

The Bernoulli number B_m is the value of the Bernoulli polynomial $B_m(x)$ at $x=0$. In this notation, $B_1 = -\frac{1}{2}$ and $B_{2n+1} = 0$ for any $n \geq 1$ and

$$B_2 = \frac{1}{6}, B_4 = \frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, \cdots.$$

The spectral action of the 4-sphere and SU(3) are computed using the Euler-Maclaurin summation formula in [6] and [15], respectively. Both the Poisson and Euler-Maclaurin summation formulas are very effective methods in computing the non-perturbative version of the spectral action, and the forms of the eigenvalues are important in determining

which method can be used. For instance, if the eigenvalues of the Dirac operator, $\pm h(k)$, and their multiplicities, $P(k)$, are polynomials in k, then the Euler-Maclaurin formula can be applied to the summation $\sum_k P(k) f(h(k)^2/\Lambda^2)$ to compute the spectral action.

In this work we use the Euler-Maclaurin formula to compute the spectral action of a Dirac operator $D' = D + T/2$, where D is the Dirac operator of the Berger sphere $\mathbb{S}^3(T)$.

4.2 Berger Spheres $\mathbb{S}^3(T)$

For $T > 0$, the Berger 3-sphere $\mathbb{S}^3(T)$, introduced by Marcel Berger [3], is a homogeneous Riemannian manifold which is homeomorphic to the 3-sphere $\mathbb{S}^3$ and equipped with a homogeneous Riemannian metric, denoted by g_T. To define the metric g_T, let's first identify $\mathbb{S}^3$ and the Lie group SU(2) by the map

$$(z, w) \in \mathbb{S}^3 \subset \mathbb{C}^2 \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

Then, for a fix parameter $T > 0$, the *Berger metric* g_T is the SU(2)-invariant metric induced from the inner product on $\mathfrak{su}(2)$ with respect to which the following basis is orthonormal:

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{T} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{4.3}$$

The Berger metric g_T on $\mathbb{S}^3 \subset \mathbb{R}^4$ is given by

$$g_T = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta + T^2 \theta^3 \otimes \theta^3. \tag{4.4}$$

Here $\theta^k = Y_k^\flat$, where $\flat$ is the musical isomorphism with respect to the standard round metric on $\mathbb{S}^3$ and the vector fields $\{Y_k\}$ are given by

$$\begin{aligned} Y_1 &= -x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^4} \\ Y_2 &= x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} - x^1 \frac{\partial}{\partial x^4} \\ Y_3 &= -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4} \end{aligned}$$

In the Hopf coordinates, i.e.

$$x^1 = \cos\varphi_1 \sin(\eta), \quad x^2 = \sin\varphi_1 \sin\eta, \quad x^3 = \cos\varphi_2 \sin\eta, \quad x^4 = \sin\varphi_2 \sin\eta,$$

with η ranging in $[0, \pi/2)$ and ϕ_1, ϕ_2 ranging in $[0, 2\pi)$, the direct computation shows that the Berger metric tensor is given by

$$\begin{aligned} g_T &= d\eta^2 + \sin^2\eta(\cos^2\eta + T^2 \sin^2\eta)d\varphi_1^2 + \cos^2\eta(\sin^2\eta + T^2\cos^2\eta)d\varphi_2^2 \\ &+ 2(T^2-1)\sin^2\eta\cos^2\eta d\varphi_1 d\varphi_2. \end{aligned} \tag{4.5}$$

It is easy to check that, for instance, the scalar curvature of $\mathbb{S}^3(T)$ is constant and equal to $R = 8 - 2T^2$, or the volume form is $\mathrm{dvol}_{g_T} = T\sin(\eta)\cos(\eta)d\eta d\varphi_1 d\varphi_2$. Hence, the volume of the Berger sphere $\mathbb{S}^3(T)$ is given by

$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2\pi} T\sin(\eta)\cos(\eta)d\eta d\varphi_1 d\varphi_2 = 2\pi^2 T. \tag{4.6}$$

The spin structures and the Dirac operators on the Berger spheres are studied in [2]. The author uses the representation theory of SU(2) and explicitly computes the eigenvalues of the Dirac operator and their multiplicities, which are given below.

Eigenvalue	Multiplicity	
$-\frac{n+1}{T} - \frac{T}{2}$	$2(n+1)$	$n = 0, 1, 2, \ldots$
$-\frac{T}{2} \pm \left(c(n-2l-1)^2 + (n+1)^2\right)^{1/2}$	$n+1$	$n = 1, 2, 3, \cdots$ $l = 0, 1, \cdots, n-1,$

where $c = \left(\frac{1}{T^2} - 1\right)$. We shall work with $D' = D + T/2$ rather than D itself. The eigenvalues of $(D')^2$ then are given by

Eigenvalue	Multiplicity	
$c(n-2l-1)^2 + (n+1)^2$	$2(n+1)$	$n = 0, 1, 2, \ldots,$ $l = 0, 1, \ldots, n.$

4.3 Spectral action of D'

In this section we apply the Euler-Maclaurin summation formula to compute the asymptotic expansion of the spectral action $\mathrm{Tr}\, f\big((D'/\Lambda)^2\big)$ in powers of Λ. Let k be an even positive Schwartz class function on $\mathbb{R}$. Moreover, let f be a function such that $k(x) = f(x^2)$.

Then we have

$$
\begin{aligned}
\mathrm{Tr}k\left(D'/\Lambda\right) &= \mathrm{Tr}f\big((D'/\Lambda)^2\big)\\
&= \sum_{n=0}^{\infty} 2(n+1)\sum_{l=0}^{n} f\big(\frac{(n+1)^2+c(n-2l-1)^2}{\Lambda^2}\big)\\
&= \sum_{n=0}^{\infty} 2n\sum_{l=0}^{n} f\big(\frac{n^2+c(n-2l)^2}{\Lambda^2}\big) - \sum_{n=0}^{\infty} 2nf\big(\frac{c+1}{\Lambda^2}n^2\big).
\end{aligned}
$$

Let us first define $f_n(x) := f\left(\frac{n^2+c(n-2x)^2}{\Lambda^2}\right)$; then we can approximate $\sum_{l=0}^{n} f_n(l)$, which is the inner summation, by the Euler-Maclaurin formula as follows:

$$
\sum_{l=0}^{n} f\left(\frac{n^2+c(n-2l)^2}{\Lambda^2}\right) =
$$

$$
\int_0^n f_n(x)dx + f_n(0) \quad + \quad \sum_{j=2}^{m} \frac{B_j}{j!}\left(f_n^{(j-1)}(n) - f_n^{(j-1)}(0)\right) - R_m(f_n,0,n).
$$

Note that $f_n(x)$ is the translation of an even function by $\frac{n}{2}$; therefore, for any y we have

$$
f_n^{(2j-1)}(\frac{n}{2}-y) = -f_n^{(2j-1)}(\frac{n}{2}+y).
$$

In particular, $f_n^{(2j-1)}(0) = -f_n^{(2j-1)}(n)$. Therefore, $\mathrm{Tr}f\left((D'/\Lambda)^2\right)$ is given by

$$
\mathrm{Tr}\ f\left((D'/\Lambda)^2\right) =
$$

$$
\sum_{n=0}^{\infty} 2n\int_0^n f_n(x)dx + \sum_{n=0}^{\infty} 2n\sum_{j=2}^{m}\frac{B_j}{j!}\ f_n^{(j-1)}(x)\Big|_{x=n} - \sum_{n=0}^{\infty} 2nR_m(f_n,0,n).
$$

To continue, we fix the following notations:

$$
h_j(y) = \begin{cases}
2y\int_0^y f_y(x)dx & \text{if } j=1,\\[2em]
\frac{2B_j}{j!}2y\frac{d^{j-1}}{dx^{j-1}}\ f_y(x)\big|_{x=y} & \text{if } 2\le j\le m,\\[2em]
2yR_m(f_y,0,y) & \text{if } j=m+1.
\end{cases}
$$

To compute the trace we shall apply the Euler-Maclaurin formula on summations of the form $\sum_{n=0}^{\infty} h_j(n)$.

• **Euler-Maclaurin on $\sum_{n=0}^{\infty} h_1(n)$:** The first summation can be approximated by the Euler-Maclaurin formula as follows.

$$\sum_{n=0}^{\infty} h_1(n) = \int_0^{\infty} h_1(y)dy - \sum_{j=2}^{m} \frac{B_j}{j!} \frac{d^{j-1}}{dx^{j-1}}\, h_1(y)|_{y=0}^{\infty} - R_m(h_1, 0, \infty).$$

Note that $h_1(y)$ is an even function and so its odd derivatives at $x = 0$ vanish. Moreover, $h_1(0) = 0$ and also

$$0 \leq \lim_{y\to\infty} 2y \int_0^y f_y(x)dx \leq \int_0^{\infty} \lim_{y\to\infty} 2yf((y^2 + x^2)/\Lambda^2)dx = 0.$$

Finally, we have

$$\begin{aligned}\int_0^{\infty} h_1(y)dy &= \int_0^{\infty}\int_0^{y} 2yf((y^2 + c(2x-y)^2)/\Lambda^2)dxdy \\ &= \Lambda^2 \int_0^{\infty}\int_0^{\frac{\Lambda}{\sqrt{c+1}}w} 2wf(w^2/\Lambda^2)dxdw \\ &= \frac{2\Lambda^3}{\sqrt{c+1}} \int_0^{\infty} w^2 f(w^2)dw.\end{aligned}$$

We now compute the order of Λ in $R_m(h_1, 0, \infty)$.

$$\begin{aligned}R_m(h_1, 0, \infty) &= \frac{(-1)^m}{m!} \int_0^{\infty} B_m(\{y\})\frac{d^m}{dy^m} h_1(y)dy \\ &= \frac{(-1)^m\Lambda}{m!} \int_0^{\infty} B_m(\{y\})\frac{d^m}{dy^m} 2y \int_0^{y/\Lambda} f(\frac{y^2}{\Lambda^2} + cu^2)dudy \\ &= \frac{(-1)^m}{m!\Lambda^{m-3}} \int_0^{\infty} B_m(\{\Lambda z\})\frac{d^m}{dz^m} 2z \int_0^{z} f(z^2 + cu^2)dudy.\end{aligned} \tag{4.7}$$

Hence $R_m(h_1, 0, \infty) = O(\Lambda^{-m+3})$ and we have the following approximation for $\sum_{n=0}^{\infty} h_1(n)$:

$$\sum_{n=0}^{\infty} h_{-1}(n) = \frac{2\Lambda^3}{\sqrt{c+1}} \int_0^{\infty} w^2 f(w^2)dw + O(\Lambda^{3-m}). \tag{4.8}$$

• **Euler-Maclaurin on $\sum_0^{\infty} h_j(n)$ for $2 \leq j \leq m$:** Since odd Bernoulli numbers vanish, i.e. $B_{2j+1} = 0$, we only need to consider summations of the form $\sum h_{2j}$. Furthermore, functions h_{2j} are even and so any odd derivatives of h_{2j} vanish at $z = 0$. Hence,

Euler-Maclaurin formula gives us the following.

$$\sum_0^\infty h_{2j}(n) = \int_0^\infty h_{2j}(y)dy + R_m(h_{2j}, 0, \infty).$$

The integral term can be computed in terms of derivatives of f at zero as follows:

$$\int_0^\infty h_{2j}(y)dy = \frac{B_{2j}2^{2j}}{(2j)!}\Lambda^{-2j+3}\int_0^\infty 2w\, \frac{d^{2j-1}}{du^{2j-1}} f\left(w^2+cu^2\right)\Big|_{u=w} dw.$$

For example, for $j=1$ we have

$$\frac{d}{du}f\left(w^2+cu^2\right)\Big|_{u=w} = 2cwf'\left((c+1)w^2\right) = \frac{c}{c+1}\frac{d}{dw}f\left((c+1)w^2\right).$$

A computation similar to (4.7) shows that $R_m(h_{2j}, 0, \infty) = O(\Lambda^{-2j-m+3})$. This, together with the above computation, gives us the following approximation:

$$\begin{aligned}\sum_0^\infty h_{2j}(n) &= \frac{B_{2j}2^{2j}}{(2j)!}\Lambda^{-2j+3}\int_0^\infty 2w\, \frac{d^{2j-1}}{du^{2j-1}} f\left(w^2+cu^2\right)\Big|_{u=w} dw \\ &+ O(\Lambda^{-2j-m+3}).\end{aligned} \tag{4.9}$$

• **Euler-Maclaurin formula on $\sum_0^\infty h_{m+1}(n)$:** Now, we apply the Euler-Maclaurin formula to $R_2(h_{m+1}, 0, \infty)$. Via the substitutions $u = \frac{2x-y}{\Lambda}$ and $w = y/\Lambda$, the integral term $\int_0^\infty h_{m+1}(y)dy$ gives us the following:

$$\Lambda^{-m+3}\frac{(-1)^m 2^{m-1}}{m!}\int_0^\infty 2w\int_{-w}^{w} B_m(\{\Lambda(x+w)/2\})\frac{d^m}{du^m}f(w^2+cu^2)dudw.$$

This shows that the order of the integral is $O(\Lambda^{-m+3})$. Moreover, there exists $\epsilon > 0$ such that $R_2(h_{m+1}, 0, \infty) = O\left(\Lambda^{-m+3-\epsilon}\right)$. Combining the above, yields the following approximation formula:

$$\sum_0^\infty h_{m+1}(n) = O\left(\Lambda^{-m+3}\right). \tag{4.10}$$

By equations (4.8), (4.9) and (4.10), for any $m > 2$, we have proven the following theorem.

Theorem 4.1. *The asymptotic expansion of the spectral action of the operator* $D' = D + \frac{T}{2}$ *on the Berger sphere* $\mathbb{S}^3(T)$ *as* $\Lambda \to \infty$ *is given by*

$$\begin{aligned}\mathrm{Tr}\ f\left((D'/\Lambda)^2\right) = \frac{2\Lambda^3}{\sqrt{c+1}}\int_0^\infty w^2 f(w^2)dw + \frac{-4cB_2}{(c+1)^{3/2}}\Lambda\int_0^\infty f\left(y^2\right)dy \\ + \sum_{j=2}^{m}\frac{B_{2j}2^{2j}}{(2j)!}\Lambda^{-2j+3}\int_0^\infty 2w\,\frac{d^{2j-1}}{du^{2j-1}}f\left(w^2+cu^2\right)\Big|_{u=w}dw,\end{aligned} \tag{4.11}$$

where $c = \frac{1}{T^2} - 1$. $\square$

The case $c = 0$, i.e. $T = 1$, gives the round sphere and all the terms of order less that Λ^3 will have zero coefficient. The spectral action is given by

$$\mathrm{Tr}\ f\left((D'/\Lambda)^2\right) = 2\Lambda^3 f_2 + O(\Lambda^{-\infty}).$$

Formula (4.11) is given for more general functions and if we consider $f(x) = e^{-x}$ and set $t = 1/\Lambda^2$, then we can compute the asymptotic expansion of the heat trace of D'^2.

Corollary 4.2. *The asymptotic expansion of the heat trace of the operator* ${D'}^2$ *on the Berger sphere* $\mathbb{S}^3(T)$ *as* $t \to 0^+$ *is given by*

$$t^{-3/2}\frac{\sqrt{\pi}}{2\sqrt{c+1}} - t^{-1/2}\frac{c\sqrt{\pi}}{3(1+c)^{3/2}} + \sum_{j=2}^{\infty} t^{j-3/2}\left(\frac{2^{4j-1}B_{2j}\sqrt{\pi}}{2j!}c^j\frac{d^j}{dc^j}\frac{1}{(1+c)^{1/2}}\right), \tag{4.12}$$

where $c = \frac{1}{T^2} - 1$.

Proof. The coefficient of $t^{-3/2}$ is given by

$$\frac{2}{\sqrt{c+1}}\int_0^\infty w^2 e^{-w^2}dw = \frac{\sqrt{\pi}}{2\sqrt{c+1}}.$$

Moreover, the coefficient of $t^{-1/2}$ is also given by

$$\frac{-4cB_2}{(c+1)^{3/2}}\Lambda\int_0^\infty e^{-y^2}dy = \frac{-c\sqrt{\pi}}{3(c+1)^{3/2}}.$$

For the positive powers of t we can compute all the coefficients as follows. For any $j \geq 2$, we have

$$\begin{aligned}
\int_0^\infty 2w \, \frac{d^{2j-1}}{du^{2j-1}} e^{-(w^2+cu^2)}\Big|_{u=w} dw &= \int_0^\infty 2w \; e^{-w^2} \frac{d^{2j-1}}{du^{2j-1}} e^{-cu^2}\Big|_{u=w} dw \\
&= \int_0^\infty e^{-w^2} \frac{d^{2j}}{dw^{2j}} e^{-cw^2} dw \\
&= \sum_{n=j}^\infty \frac{(-c)^n}{n!} \frac{(2n)!}{(2n-2j)!} \int_0^\infty e^{-w^2} w^{2n-2j} dw,
\end{aligned}$$

Integrating by parts $n-j$-times, we get

$$\begin{aligned}
\int_0^\infty 2we^{-w^2} \frac{d^{2j-1}}{dw^{2j-1}} e^{-w^2} dw &= \sqrt{\pi} 2^{2j-1} c^j \sum_{n=j}^\infty \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{n!(n-j)!} c^{n-j} \\
&= \sqrt{\pi} 2^{2j-1} c^j \frac{d^j}{dc^j} \left(\frac{1}{(1+c)^{1/2}} \right).
\end{aligned}$$

□

4.4 The Heat Trace Coefficients Using the Universal Formulas

In this section, we produce few first coefficients of the heat trace for D'^2 using the universal local formulas. We first show that D' is the Dirac operator of a metric connection with torsion. Then using the the Schrödinger-Lichnerowicz formula given in [18], we find the endomorphism E in the decomposition of D'^2 and plug it in the local formulas.

The theory of G-invariant connections on an induced vector bundle over a homogeneous space is studied in [13, 14]. Also, spin structured and the Dirac operators of such spaces are investigated in [2, 12]. Here, we briefly review the theory for a Lie group G. The set of all G-invariant connections on a vector bundle of the form $G \times V$ is in one to one correspondence with all $\mathbb{R}$-linear maps $\Lambda : \mathfrak{g} \to \mathfrak{gl}(V)$. The correspondence is given by [12]

$$\nabla\varphi = X^i \otimes X_i\varphi + X^i \otimes \Lambda(X_i)\varphi, \tag{4.13}$$

where $\{X_i\}$ is a basis for $\mathfrak{g}$ and $\{X^i\}$ is its dual basis. Moreover, $X_i\varphi$ is the Lie derivative of φ with respect to the G-invariant vector field defined by $X_i \in \mathfrak{g}$, and $\Lambda(X_i)\varphi$ is a smooth function from G to $\mathfrak{g}$ defined by $(\Lambda(X_i)\varphi)(g) = \Lambda(X_i)\varphi(g)$ for $g \in G$. For

instance, the Levi-Civita connection on $TG \simeq G \times \mathfrak{g}$ equipped with the left invariant Riemannian metric g produced by an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is given by (see e.g. [13, Theorem X.3.3])

$$\Lambda(X)Y := (1/2)[X,Y] + U(X,Y) \quad X, Y \in \mathfrak{g},$$

where $U(X,Y) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the bilinear map defined by

$$2\langle U(X,Y), Z\rangle = \langle X, [Z,Y]\rangle + \langle [Z,X], Y\rangle, \quad X, Y, Z \in \mathfrak{g}. \tag{4.14}$$

In general, any $\mathbb{R}$-linear map Λ with the property that $\Lambda(X) \in \mathfrak{so}(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ for any $X \in \mathfrak{g}$, induces a metric connection on TG. The torsion tensor for such a connection at the origin is given by

$$\mathcal{T}(X,Y) = \Lambda(X)(Y) - \Lambda(Y)(X) - [X,Y], \quad X, Y \in \mathfrak{g}.$$

Such a metric connection lifts to a G-invariant connection on the spinor bundle $S = G \times \mathbb{S}_m$ induced by

$$\tilde{\Lambda}(X) \mapsto \frac{1}{4}\sum_{i,j} \langle \Lambda(X)(X_i), X_j\rangle c(X_i)c(X_j), \quad X \in \mathfrak{g},$$

where $\{X_i\}$ is any orthonormal basis for $\mathfrak{g}$ and c denotes the Clifford multiplication. The induced connection $\widetilde{\nabla}$, similar to (4.13), acts on spinor fields $\psi : G \to \mathbb{S}_m$ as follows,

$$(\widetilde{\nabla}\psi)(g) = \sum X^i \otimes \left(X_i\psi(g) + \widetilde{\Lambda}(X_i)\psi(g) \right).$$

The Dirac operator defined by $\widetilde{\nabla}$ is given by

$$D\varphi = \sum_i c(X_i)X_i\psi + \frac{1}{4}\sum_{i,j,k} \langle \Lambda(X_i)X_j, X_k\rangle c(X_i)c(X_j)c(X_k)\psi. \tag{4.15}$$

The Dirac operator D is a formally self-adjoint operator if and only if for any orthonormal basis $\{X_i\}$ for $\mathfrak{g}$, we have (cf. [12])

$$\sum_i \Lambda(X_i)X_i = 0. \tag{4.16}$$

On the Berger sphere $\mathbb{S}^3(T) = (\mathrm{SU}(2), g_T)$, direct computations show that the map given by (4.14), vanishes and thus the Levi-Civita connection is induced by the map

$$\Lambda(X) = \frac{1}{2}[X, \cdot], \quad X \in \mathfrak{g}.$$

Then the following maps define a family of metric connections:

$$\Lambda^t(X) = t[X, \cdot], \quad X \in \mathfrak{g}, \quad t \in \mathbb{R}.$$

The torsion tensor of these connections, ∇^t, is equal to $\mathcal{T}^t(X,Y) = (2t-1)[X,Y]$ and Λ^t satisfies the condition (4.16). Moreover, they lift to a family of connections $\widetilde{\nabla}^t$ on the spinor bundle induced by

$$\tilde{\Lambda}^t(X) = \frac{t}{2}\sum_{i,j}\langle \Lambda(X)X_i, X_j\rangle c(X_i)c(X_j) = 2t\tilde{\Lambda}(X), \quad X \in \mathfrak{g}.$$

Lemma 4.3. *The operator D' is the Dirac operator defined by the connection $\widetilde{\nabla}^{\frac{1}{2+T^2}}$.*

Proof. Equation (4.15) applied on $\widetilde{\nabla}^t$ gives us the formula for the Dirac operator

$$D^t\psi = \sum_i c(X_i)X_i(\psi) + 2tc(X_i)\tilde{\Lambda}(X_i)\psi.$$

By the direct computation we have

$$2tc(X_i)\frac{1}{4}\sum_{i,j,k}\langle \Lambda(X_i)X_j, X_k\rangle c(X_i)c(X_j)c(X_k) = -\frac{t}{4}\left(\frac{8}{T} + 4T\right).$$

Hence,

$$D^t\psi = \sum_i c(X_i)X_i(\psi) - \frac{t}{4}\left(\frac{8}{T} + 4T\right)\psi,$$

which clearly shows that

$$D' = D + T/2 = D^{\frac{1}{2+T^2}}.$$

□

The Schrödinger-Lichnerowicz formula for D^t is given by (see [1, 18])

$$(D^{t/3})^2 = (\widetilde{\nabla}^t)^*\widetilde{\nabla}^t + td\mathcal{T} + \frac{1}{4}R - 2t^2T_0^2,$$

where R is the scalar curvature of the metric and $T_0^2 = \frac{1}{6}\sum_{i,j=1}^{n} ||T(X_i, X_j)||^2$.

In the case of Berger spheres we have $\|T\|^2 = (\frac{2t-1}{6})^2 6 \left(2T + \frac{4}{T}\right)^2 = \frac{2}{3}T^2$, dT vanishes and $R = 8 - 2T^2$. Hence, the endomorphism is given by

$$E = \left(-T^2 + 2\right) I_2.$$

Using the local formula now we have

$$\begin{aligned}
a_0 &= \int_{\mathbb{S}^3} a_0(x, D'^2)\mathrm{dvol}_{g_T} = \frac{2}{(4\pi)^{3/2}}\mathrm{vol}(\mathbb{S}^3(T)) = \frac{\sqrt{\pi}}{2}T,\\
a_2 &= \int_{\mathbb{S}^3} a_2(x, D'^2)\mathrm{dvol}_{g_T} = \frac{2}{(4\pi)^{3/2}}\left(\frac{2-2T^2}{3}\right)\mathrm{vol}(\mathbb{S}^3(T)) = \frac{-\sqrt{\pi}}{3}(T^2-1)T,
\end{aligned}$$

which are equal to the results given by Corollary 4.2.

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Appendix A

Curriculum Vitae

Asghar Ghorbanpour

Curriculum Vitae

Education

2011–Present **Ph.D.**, *The University of Western Ontario (UWO)*, London, ON, Canada.
Supervisor: Masoud Khalkhali

2005–2008 **M.Sc.**, *Sharif University of Technology (SUT)*, Tehran, Iran.
Supervisors: Bijan Z. Zangeneh, Massoud Amini

2001–2005 **B.Sc.**, *Shahid Beheshti University (SBU)*, Tehran, Iran.

Research Interests

- Noncommutative geometry and its application in physics
- Spectral geometry and spectral invariants of noncommutative spaces
- Pseudodifferential operators and their spectral theory
- Mathematical physics

Publications

- A Time Average for Quantum Geodesic on Curved Noncommutative Tori, with A. Fathi and M. Khalkhali, *in preparation*.
- The Curvature of the Determinant Line Bundle on the Noncommutative Two Torus, with A. Fathi and M. Khalkhali, *ArXiv: 1410.0475*, October 2014.
- Rationality of Spectral Action for Robertson-Walker Metrics, with F. Fathizadeh and M. Khalkhali, to appear in JHEP, *ArXiv: 1407.5972*, July 2014.
- Quasilinearization approach for solving Volterra's population model, with K. Parand, M. Ghasemi, S. Rezazadeh, A. Peiravi and A. Golpaygani, *Appl. Comput. Math.*, 9(1):95–103, 2010.

Books and Lecture Series

- *Comprehensive Description of Differential Equations: Preparatory Course for National Entrance Exam in Masters Degree (in Farsi)*, Entesharate Arshad, Tehran, 2009.
- *Differential Equations: Problems of ten years masters enterance exams for science and engeneering students (in Farsi)*, Entesharate Arshad, Theran, 2009.

Talks and Presentations

- On Spectral Action Principle for Robertson-Walker Metrics, 42nd Canadian Annual Symposium on Operator Algebras and Their Applications, Fields Institute, Toronto, Canada, June 2014
- Morse inequalities through spectral geometry, UWO, NCG Seminar, January 2014
- Quantum dynamical systems I: Geodesic flow in NCG, UWO, NCG Seminar, January 2014
- Mathematics and Physics of the Quantum Hall Effect, UWO, NCG Seminar, August 2013
- Applications of the Atiyah-Singer Index Theorem: The Hirzebruch-Riemann-Roch Theorem, UWO, NCG Seminar, April 2013
- An Introduction to Spin Geometry, UWO, NCG Seminar, November-September 2012

- Spectral Zeta Function and Spectral Invariants of Noncommutative 4-Dimensional Tori, Oral Comprehensive Exam, UWO, April 2012
- Selberg Trace Formula and Heisenberg Group, UWO, NCG Seminar, February 2012
- Non-commutative Spin Geometries, 40th Annual Iranian Mathematics Conference, SUT, Tehran, Iran, August 2009

Attended Conferences and Programs

- Non-commutative Geometry and its Applications: Hausdorff Trimester Program, Bonn, Germany, September 1–30, 2014
- 42nd Canadian Annual Symposium on Operator Algebras and Their Applications, Fields Institute, Toronto, June 23–27, 2014
- Focus Program on Noncommutative Geometry and Quantum Groups, Fields Institute, Toronto, June 3–28, 2013
- Canadian Operator Symposium, Queen's University, Kingston, May 21-25, 2012
- The Canadian Mathematical Society (CMS) Winter Meeting, Ryerson University and York University, Toronto, December 10–12, 2011
- Canadian Operator Symposium, University of Victoria, Victoria, May 24–28, 2011
- School of Non-commutative geometry, IPM, Tehran, Iran, September–December 2010
- 40th Annual Iranian Mathematics Conference, SUT, Tehran, Iran, 17–20 August 2009
- The 2nd School & Conference on Non-commutative geometry, IPM, Tehran, Iran, April 19–30, 2009

Awards and Honors

- Science Learning Development Graduate Fellowship (LDGF), UWO, Fall 2013 & Winter 2014
- Western Graduate Research Scholarship (WGRS), UWO, Winter 2011–Fall 2014
- Ranked 7th in the national entrance exam for masters of mathematics, 2005
- Silver medal in the 25th Mathematical Competition of Iranian Mathematical Society, Babolsar, Iran, May 2005
- Ranked 9th in the 9th National Mathematical Olympiad for University Students, Iran, July 2005
- Ranked top undergraduate student of applied mathematics major, SBU, 2004

Teaching Certificates

- Future Professor Workshop Series, Teaching Support Centre, UWO, 2012-2014
- Teaching Assistant Training Program (TATP), Teaching Support Centre, UWO, Summer 2011
- The Language of Teaching in Science, Teaching Support Centre, UWO, Spring 2011
- The Teaching Assistant Training Program, Teaching Support Centre, UWO, Spring 2011
- Teaching in the Canadian Classroom, Teaching Support Centre, UWO, Winter 2011

Teaching Experiences

Lecturer

- Calculus 1, Azad University, Maku, Iran, Fall 2009
- Selected topics in mathematical analysis: Preparatory Course for national Mathematical Olympiad, Department of Mathematics, SBU, Tehran, Iran,Fall 2006

Running Tutorial as Teaching Assistant

- Linear Algebra, Department of Mathematics, UWO, London, Canada, 6 sequential semesters, Fall 2011–Summer 2013
- Mathematical Analysis 2, Department of Mathematics, SBU, Tehran, Iran, Winter 2007
- Elementary mathematical Analysis, Department of Mathematics, SBU, Tehran, Iran, Winter 2007
- Abstract Algebra 1, Department of Mathematics, SBU, Tehran, Iran, Winter 2005
- Mathematical Analysis, Department of Mathematics, SBU,Tehran, Iran, Fall 2004 & Fall 2005

Other Teaching Assistant Experiences

- Calculus 1000, Department of Mathematics, UWO, London, Canada, Winter 2013 & Fall 2014
- Finite Math, Department of Mathematics, UWO, London, Canada, Summer 2011 & Summer 2012
- Calculus 2, Running Online sessions, E-Learning Center of K.N. Toosi University of Technology, Tehran, Iran, Winter 2006
- Calculus 1, Running Online sessions, E-Learning Center of K.N. Toosi University of Technology, Tehran, Iran, Fall 2005

Relevant Experiences

- Volunteer at Western Conference on Science Education, UWO, Summer 2011 & Summer 2013
- Co-organizer of Non-commutative Geometry Seminar, Department of Mathematics, UWO, Fall 2011-Winter 2012
- Multiple Choice Exam Designer and Supervisor of Math group, National Organization for Educational Testing, Tehran, Iran, 2011–2012
- Assistant Coordinator, Noncommutative Geometry School at IPM, Tehran, September–December 2010
- Assistant Coordinator, Office of Evaluation & Supervision, SUT, Tehran, Iran, April 2009–June 2010
- Co-organizer of Math Day for “Daheye Riazi”, SBU, Tehran, Iran, October 31, 2004

Computer skills

- Mathematica
- GAP

References

- **Masoud Khalkhali**, *Professor of Mathematics, UWO*.
- **Martin Pinsonnault**, *Assistant Professor, UWO*.
- **Ján Mináč**, *Professor of Mathematics, UWO*.

Teaching Reference

- **Arash Pourkia**, *Sessional Assistant Professor at Huron University College, UWO*.