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## Optimal Contract Design for Co-development of Companion Diagnostics

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A thesis submitted in partial fulfillment of the requirements for the Master of Science degree in Applied Mathematics

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OPTIMAL CONTRACT DESIGN FOR CO-DEVELOPMENT OF  
COMPANION DIAGNOSTICS

(Thesis format: Monograph)

by

Rodney Tembo

Graduate Program in Applied Mathematics

A thesis submitted in partial fulfillment  
of the requirements for the degree of  
Masters of Science

The School of Graduate and Postdoctoral Studies  
The University of Western Ontario  
London, Ontario, Canada

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## Abstract

As the number of new drugs requiring companion diagnostics rises, more and more partnerships are formed between drug and diagnostics manufacturers to develop the necessary companion diagnostic. An increasingly significant issue is that of the optimal revenue/profit sharing or compensation schemes for such partnerships. We investigate the structure of an optimal compensation scheme under a scenario where a large pharmaceutical firm that is developing a drug intends to partner with a smaller diagnostics firm to develop a companion diagnostic test for the drug. We describe an optimal contract as one that maximizes the pharmaceutical firm's expected profits while offering enough incentives for the diagnostics firm to accept the contract and then work at an effort level that is preferred by the pharmaceutical firm. We formulate the problem of determining the optimal contract as an instance of the Principal-Agent problem. We then present a numerical approach for solving the problem.

**Keywords:** Companion Diagnostics, Optimization, Principal Agent Problem, Moral Hazard

## Acknowledgements

I feel very fortunate to have had two brilliant and wonderful people as my supervisors, Dr Greg Zaric and Dr Matt Davison. I thank them sincerely for their guidance and for affording me this opportunity to be a part of the Applied Math department at Western. I would also like to thank Audrey and Cinthia who were very instrumental in making my stay in the department pleasant. My gratitude also goes to the faculty and graduate students of the Applied Math Department for providing such a great environment. Lastly, I would like to express my thanks to my family and friends who have been an excellent source of motivation.

# Contents

<b>Abstract</b>	<b>ii</b>
<b>Acknowledgements</b>	<b>iii</b>
<b>List of Figures</b>	<b>viii</b>
<b>List of Tables</b>	<b>x</b>
<b>List of Appendices</b>	<b>xii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Introduction to Companion Diagnostics . . . . .	1
1.2 The Value of Patient Stratification . . . . .	2
1.3 The Pharmaceutical Industry and Companion Diagnostics . . . . .	3
1.3.1 Development of Companion Diagnostics . . . . .	4
1.4 Outline of the Thesis . . . . .	5
<b>2 The Principal-Agent Problem</b>	<b>7</b>
2.1 The Partnership Contract as a Principal-Agent Problem . . . . .	7
2.2 The Principal-Agent Problem with Moral Hazard . . . . .	9
2.3 Solutions to the Principal-Agent Problem . . . . .	10
<b>3 A Basic Principal-Agent Model</b>	<b>15</b>
3.1 Analytic Solution to a Basic Principal-Agent Model . . . . .	15
3.1.1 Two Actions, Two Outcomes Model . . . . .	16
3.2 The Two Step Approach . . . . .	17

3.2.1	Inducing High Effort . . . . .	17
	Investigating the Problem Structure . . . . .	18
	First-Best Solution . . . . .	20
	Second-Best Solution . . . . .	21
3.2.2	Inducing Low Effort . . . . .	23
	First-Best Solution . . . . .	24
	Second-Best Solution . . . . .	24
3.2.3	The Effort Level To Implement . . . . .	25
<b>4</b>	<b>Numerical Solution To The Basic Problem</b>	<b>26</b>
4.1	Numerical Solution to the Basic Principal-Agent Model . . . . .	26
4.1.1	Second Best Case: Analytical Solution (for assessing numerical ap- proach) . . . . .	27
4.1.2	Verifying the Solution Graphically . . . . .	28
4.1.3	Solving the Problem Numerically . . . . .	30
	Initialization . . . . .	33
	Gradient Search . . . . .	33
	Summary of Procedure . . . . .	36
4.1.4	Numerical Results . . . . .	37
4.2	Conclusion . . . . .	39
<b>5</b>	<b>The Co-Development Model</b>	<b>41</b>
5.1	Mathematical Formulation of Co-development Contract . . . . .	41
5.1.1	Determining the agent's reservation utility, $U_r$ . . . . .	43
	An upper bound on the agent's reservation utility, $U_{max}$ . . . . .	44
	A lower bound on the agent's reservation utility, $U_{min}$ . . . . .	45
5.2	Solutions to the Co-development Model . . . . .	46
5.2.1	First Best Solution . . . . .	46
5.2.2	Second Best Solution . . . . .	48
5.3	Conclusion . . . . .	50

<b>6</b>	<b>Numerically Solving for Optimal Co-development Contracts</b>	<b>51</b>
6.1	Setting Parameter Values . . . . .	51
6.1.1	The Numerical Approach . . . . .	53
	Summary of Procedure . . . . .	57
6.2	Numerical Results . . . . .	58
6.2.1	Analysis of the numerical results . . . . .	62
6.3	Optimal Contracts with $m_a = 0$ . . . . .	65
6.4	Conclusion . . . . .	69
<b>7</b>	<b>Co-development Model Extensions</b>	<b>71</b>
7.1	Variable Borrowing Cost . . . . .	71
7.1.1	Revised Model C . . . . .	72
7.1.2	Revised Model A . . . . .	73
7.2	Solutions for contracts where borrowing costs are not fixed . . . . .	75
7.3	Numerical Results for different functions describing probability of success . . . . .	78
7.4	Conclusion . . . . .	83
<b>8</b>	<b>Conclusion</b>	<b>85</b>
8.1	Discussion . . . . .	85
8.1.1	Future Work . . . . .	86
	<b>References</b>	<b>87</b>
<b>A</b>	<b>Variable and Symbol Definitions</b>	<b>92</b>
<b>B</b>	<b>Proof of Proposition 1</b>	<b>95</b>
<b>C</b>	<b>Proof of Proposition 2</b>	<b>98</b>
<b>D</b>	<b>Proof of Proposition 4</b>	<b>100</b>
	Determining the optimal values of $m_a, m_p, r$ and $f_a$ . . . . .	101
<b>E</b>	<b>C++ Code for solving Co-Development Model</b>	<b>109</b>
E.1	Header Files . . . . .	109

E.1.1	MyFxns.h . . . . .	109
E.1.2	MatlabFxns.h . . . . .	111
E.2	CPP Files . . . . .	111
E.2.1	Main.cpp . . . . .	111
E.2.2	MatlabFxns.cpp . . . . .	114
E.2.3	MyFxns.cpp . . . . .	115
E.2.4	search.cpp . . . . .	120
E.2.5	Numerical.cpp . . . . .	125
E.2.6	ContractFinder . . . . .	130
<b>Curriculum Vitae</b>		<b>134</b>



# List of Figures

2.1	Limitations of the first order approach . . . . .	13
3.1	The IR and IC constraints and the Principal's wage bill on $w_l-w_h$ plane. . . . .	19
4.1	The expected profit and corresponding optimal value of $w_h$ across values of $w_l$ for inducing high effort. . . . .	30
4.2	The expected profit and corresponding optimal value of $w_h$ across values of $w_l$ for inducing low effort . . . . .	31
4.3	The feasible points for inducing high effort . . . . .	34
4.4	The feasible points for inducing low effort . . . . .	35
4.5	A sample of initial and final points obtained from implementing our search method where we assume the principal wishes to induce high effort. . . . .	39
4.6	A surface plot of the principal's expected profit from inducing high effort. . . . .	40
6.1	Plots of the agent's reservation utility, the principal's expected profit and agent's expected utility as functions of agent's effort, $f_a$ , given the values of $m_a, m_p$ and $r$ obtained analytically. . . . .	66
6.2	Plots of the agent's reservation utility; the principal's expected profit and agent's expected utility as functions of agent's effort, $f_a$ , given the values of $m_a, m_p$ and $r$ obtained numerically with $f_a$ fixed at 0. . . . .	66
6.3	Plots of the agent's reservation utility; the principal's expected profit and agent's expected utility as functions of agent's effort, $f_a$ , given the values of $m_a, m_p$ and $r$ obtained numerically with $\frac{\partial E[U_A]}{\partial f_a}$ fixed at 0. . . . .	67

6.4 Plots of the agent’s reservation utility; the principal’s expected profit and agent’s expected utility as functions of agent’s effort,  $f_a$ , given the values of  $m_a$ ,  $m_p$  and  $r$  obtained numerically with  $f_a$  fixed at  $f_L$ . . . . . 67

7.1 The general shapes of the different functions describing probability of success. . 80

# List of Tables

1.1	Possible Patient Outcomes From Drug Therapy . . . . .	2
2.1	Examples of problems that can be modeled with the Principal Agent framework	8
4.1	Solutions obtained from 5 random initial points denoted $(w_l^{(0)}, w_h^{(0)})$ . . . . .	38
4.2	Solutions obtained from 5 random initial points, $(w_l^{(0)}, w_h^{(0)})$ . . . . .	38
5.1	Optimal Solutions to Co-Development Model (Second Best Solution) . . . . .	49
6.1	A summary of our experimental parameters. . . . .	52
6.2	Numerical Results : $f_L = 5250; U_r = 23,250,000$ . . . . .	59
6.3	Numerical Results : $f_L = 40,000; U_r = 1,750,000$ . . . . .	59
6.4	Numerical Results : $f_L = 10,250; U_r = 23,250,000$ . . . . .	60
6.5	Numerical Results : $f_L = 40,000; U_r = 6,000,000$ . . . . .	60
6.6	Numerical Results : $f_L = 30,000; U_r = 4,750,000$ . . . . .	60
6.7	Numerical Results : $f_L = 20,000; U_r = 2800$ . . . . .	61
6.8	Numerical Results : $f_L = 22,000; U_r = 33,000,000$ . . . . .	61
6.9	Numerical Results : $f_L = 10,250; U_r = 23,250,000; m_a = 0$ . . . . .	68
6.10	Numerical Results : $f_L = 40,000; U_r = 1,750,000; m_a = 0$ . . . . .	68
6.11	Numerical Results : $f_L = 30,000; U_r = 4,750,000; m_a = 0$ . . . . .	68
7.1	Variable $b_a$ : $f_L = 5250; U_r = 23,250,000; b_p = 1.05$ . . . . .	76
7.2	Variable $b_a$ : $f_L = 40,000; U_r = 1,750,000; c_h = 200; b_p = 1.05$ . . . . .	77
7.3	Variable $b_a$ : $f_L = 40,000; U_r = 6,000,000; c_h = 200; b_p = 1.05$ . . . . .	77
7.4	Different $p(f_a, m_a, m_p)$ : $f_L = 5250; U_r = 23,250,000; b_p = 1.05$ . . . . .	81
7.5	Different $p(f_a, m_a, m_p)$ : $f_L = 40,000; U_r = 1,750,000; c_h = 200; b_p = 1.05$ . . . . .	81
7.6	Different $p(f_a, m_a, m_p)$ : $f_L = 40,000; U_r = 6,000,000; c_h = 200; b_p = 1.05$ . . . . .	82

7.7 Different  $p(f_a, m_a, m_p)$ :  $f_L = 30,000$ ;  $U_r = 4,750,000$  . . . . . 82

A.1 Definitions of Variables and Symbols . . . . . 93

# List of Appendices

Appendix A Variable and Symbol Definitions . . . . .	92
Appendix B Proof of Proposition 1 . . . . .	95
Appendix C Proof of Proposition 2 . . . . .	98
Appendix D Proof of Proposition 4 . . . . .	100

# Chapter 1

## Introduction

### 1.1 Introduction to Companion Diagnostics

In recent years, the pharmaceutical and medical fields have witnessed a significant gain of momentum in the drive towards personalized medicine. In contrast to standard clinical practice which is primarily empirical, personalized medicine<sup>1</sup> entails using clinical biomarkers<sup>2</sup> or diagnostic tests to actively seek and match specific patient population characteristics with specific therapies [1], specifically drugs. For a given personalized drug, the biomarkers, presented in the form of approved companion diagnostic tests, typically identify patients who fall into one of the possible scenarios:

- the drug is expected to have improved effectiveness or safety in the patient (I,III)
- the drug is expected to have minimal or no effect in the patient (II,IV)
- the drug may cause serious side effects in the patient (I,II)
- the patient is likely to experience no or reduced side effects (III,IV) [1, 2, 4]

Table 1.1 below provides a visual representation of these possible scenarios.

The use of companion diagnostic tests to prescribe treatment promises some considerable clinical and/or economic benefits to the key stakeholders within the health care system. To

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<sup>1</sup>Other alternative terms frequently used in literature include ‘stratified medicine’, ‘pharmacogenomics’ and ‘targeted therapy’

<sup>2</sup>A characteristic that is objectively measured and evaluated as an indicator of specific biological processes, pathogenic processes or pharmacological responses to a therapeutic intervention [1].

Table 1.1: Possible Patient Outcomes From Drug Therapy

Benefit	No Benefit	
I	II	Side Effects
III	IV	No Side Effects

the patient and the physician, personalized medicine as guided by companion diagnostics is clearly beneficial in that it leads to better patient outcomes through improved efficacy or reduced side-effects [2]. Formulary<sup>3</sup> managers and third party payers<sup>4</sup> such as insurers, who may be particularly concerned about the cost-effectiveness of a treatment, will benefit from the additional insight by directing resources towards more valuable treatments while spending less on ineffective treatments [1].

While personalized medicine does restrict the potential market size of a drug, it has since been observed that actual revenues might increase since such medicines tend to experience faster and wider adoption due to their significantly improved performance over standard treatments [1]. This implies that drug manufacturers can therefore adopt premium pricing strategies based on the added value gained by use of personalized medicine. Even payers such as insurance companies are willing to pay a higher price for therapy when diagnostics are factored in [7].

## 1.2 The Value of Patient Stratification

A striking and motivating example of the significance of guiding treatment through diagnostic testing is the case of the non-small cell lung cancer (NSCLC) drug crizotinib. Research had indicated that about 5% of NSCLC patients had what is referred to as the *EML4* – *ALK* mutation [9]. When treated with crizotinib, these patients were generally observed to have a markedly

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<sup>3</sup>A list of drugs covered by a particular drug benefit plan. Formularies are based on evaluations of efficacy, safety and cost-effectiveness of drugs [5]

<sup>4</sup>An institution or company that is or may be liable to reimburse health care providers for services given to a patient [6]

different response rate<sup>5</sup> of at least 50% as compared to the 10% response rate observed with traditional treatment [9]. Clearly, the ability to identify patients beforehand can reduce clinical uncertainty and thus provide to both the patient and physician additional benefits in the form of what is termed “the value of knowing”[9]. According to the Academy of Medical Sciences, beyond the benefits obtained at the patient level, the use of personalized medicine can significantly improve allocation of limited health care resources by “focusing treatment on those with a higher probability of responding” [9]. Another major benefit, and perhaps the most significant one, is the potential savings that could result from minimizing the incidence of adverse reactions to drugs. In the United States alone, the annual cost of adverse events has been estimated to be about \$177 billion [10].

### **1.3 The Pharmaceutical Industry and Companion Diagnostics**

The traditional pharmaceutical industry business model has largely been the mass-market blockbuster drug model. However, both industry insiders and analysts agree that the blockbuster model is on the decline and the industry is inevitably being pushed toward a focus on what are termed “mini-busters” or “niche-busters” [11, 12]. Blockbuster drugs may gradually no longer be the preferred treatment with a stronger bias toward more targeted treatment alternatives having the benefits outlined in the previous section [12, 13]. The development of a personalized therapeutic drug will in many instances need to be complemented by the development of a companion diagnostic necessary for identifying members of the patient population subgroup for whom the drug is appropriate. While we have so far highlighted the advantages of using companion diagnostics from the patient and payer’s perspective, we now briefly consider how companion diagnostics can benefit the pharmaceutical industry from a business perspective.

We first consider the nature of revenues that a targeted drug manufacturer may anticipate. Currently most therapeutic drugs are administered empirically with the expectation that the

---

<sup>5</sup>The percentage of patients whose cancer shrinks or disappears after treatment [8]



drug will be beneficial for only some of the relevant patients [1, 13]. Referring back to Table 1.1, this means that a patient receiving the drug could well fall into any of the four categories in the table. With the use of companion diagnostics the goal is often to restrict the prescription of a drug to only those patients who are likely to respond well (*specifically, mostly patients in category III of Table 1.1*) [1, 13]. This therefore means that the introduction of a companion diagnostic will inevitably lead to a smaller potential market as fewer patients are subsequently given the drug after being tested. However, it has since been observed that even with the narrowed potential market, actual revenues of these niche drugs could match blockbuster revenues. Drug manufacturers are often also able to adopt premium pricing strategies based on the added value gained by use of personalized medicine [1].

The use of companion diagnostics can also greatly improve the development process of the companion drug [2]. It has been estimated that the use of a companion diagnostic during the drug development process can reduce both the time and cost of the entire process with potential savings of as much as 60% of the development costs [12]. While most drugs take 10 to 15 years to reach the market, vemurafenib did so in less than 6 years after the drug manufacturer, Roche, partnered with a diagnostics company to identify patients with greater likelihood of responding positively to the drug [3]. Apart from the fact that use of a companion diagnostic can improve the likelihood and speed of regulatory approval, in some cases the presentation of a companion diagnostic has been the only way some drugs have been able to obtain regulatory approval while lack of a companion diagnostic has also resulted in some therapies being rejected by the regulatory authorities [3, 11, 14].

### **1.3.1 Development of Companion Diagnostics**

Pharmaceutical firms generally have three main options through which they acquire companion diagnostics: the drug manufacturer may opt to either develop the test in-house; or to merge with or acquire a diagnostics firm; or to enter into a development partnership with an external party such as a specialist biotechnology lab [11]. However, very few pharmaceutical companies have a diagnostics division. As a result, the most commonly taken route to developing a companion

diagnostic has been the formation of development partnerships between the pharmaceutical firm and another external party, typically a smaller lab or diagnostics company [2, 11]. In one study of 31 companion diagnostics projects announced in the ten year period from 2001 to 2010, only two projects involved in-house development while the rest involved partnerships between the drug manufacturer and a diagnostics firm [13].

Though they are clearly related industries, the pharmaceutical and diagnostics industries operate along very different paths with “different development timelines, product lifecycles, return on investment, customers and regulations” [15]. Co-development of companion diagnostics has itself been a relatively rare occurrence in the pharmaceutical industry [15]. This is likely a significant reason why only a few pharmaceutical firms have an existing or permanent diagnostics division. In most instances, the need for developing a companion diagnostic targeting a specific biomarker only emerges in the later phases of drug development. In some cases, this has been as late as the phase 3 clinical trials [15]. In such instances, instead of attempting to develop the diagnostic itself, the pharmaceutical company may opt to immediately outsource test development to a diagnostic company that has since successfully developed related tests that suit its new drug [16]. Consequently, the pharmaceutical companies lack expertise in discovery, development and marketing of companion diagnostics [13, 17]. Even when it may have the know-how, the pharmaceutical company may also need to consider manufacturing scale-up to facilitate production development of the test [17]. These factors contribute to the drug manufacturers choosing to partner with the experienced diagnostics firms for co-development.

## **1.4 Outline of the Thesis**

As development of drugs requiring companion diagnostics continues to pick up pace and more and more partnerships are formed between drug and diagnostics manufacturers, an increasingly significant issue is that of the optimal revenue/profit sharing or compensation schemes for such partnerships [2, 11, 18]. In this thesis, we investigate the structure of an optimal compensation scheme under a scenario where a large pharmaceutical firm that is developing a drug intends

to form a partnership with a smaller companion diagnostics firm to develop a companion diagnostic test for the drug.

In chapter 2, we introduce the modeling framework referred to as the Principal-Agent (PA) problem and describe how this framework relates to employer-employee type relationships, and in particular the contractual arrangement between the drug manufacturer and diagnostic manufacturer. In the third chapter, we analyze and solve a basic form of the PA problem. We then develop a numerical algorithm for solving the basic PA problem in chapter 4. We are able to assess the performance of the numerical method by comparing its results to those obtained in chapter 3. This numerical approach to solving the principal-agent problem is one of the novel contributions of this thesis.

In chapter 5 we formulate and solve a mathematical model of the problem the pharmaceutical firm faces while designing a contract to offer the companion diagnostic manufacturer. Solutions to the problem describe optimal contract structures from the pharmaceutical firm's perspective. The model formulation and solution approach we use in chapter 5 is based on previous work in [35].

The main contributions of this thesis are contained in chapters 6 and 7. In chapter 6, we develop a new numerical approach to solving the co-development model based on the numerical work done in chapter 4. This numerical approach allows us to solve for problems with some extensions beyond the original co-development model introduced in chapter 5. We present two new extensions to the co-development model in chapter 7. We then solve instances of this modified co-development model using the numerical method. Chapter 8 contains a brief summary of the work done in this thesis and serves as the conclusion.

# Chapter 2

## The Principal-Agent Problem

### 2.1 The Partnership Contract as a Principal-Agent Problem

As mentioned in section 1.1, one of the most common approaches drug manufacturers use to obtain companion diagnostics is entering into a development alliance with another firm. The pharmaceutical firm seeks a suitable biotech firm with which to enter into a contractual agreement to develop the companion diagnostic. Since a primary goal of the pharmaceutical firm is to induce the biotech firm to act on its behalf, the contractual relationship between the drug manufacturer and the diagnostic manufacturer can be analyzed as a Principal-Agent problem where the pharmaceutical firm is the principal and the diagnostics firm is the agent.

The Principal-Agent framework is a method that has been used frequently to model and analyze a diverse array of contractual relationships between two parties where one party (the principal) hires or desires the other party (the agent) to perform some task. The key objective in the framework is to determine the optimal incentive scheme or contract that will induce the agent to behave or work in a manner that maximizes the likelihood of an outcome desired by the principal. Some of the most well known and earliest uses of the model in literature appear in insurance related settings where an insurer (the principal) wishes to develop a contract that guards against less careful behavior by the insured (the agent) [19, 20, 21, 22].

The insurance setting and our co-development problem are only two examples of con-

tractual arrangements to which the principal-agent framework can be applied. We provide examples of other scenarios where the model can or has been applied in table 2.1.

Table 2.1: Examples of problems that can be modeled with the Principal Agent framework

<b>Principal</b>	<b>Agent</b>	<b>Problem</b>
Employer	Employee	To induce employee to work harder to increase employer's profits
Plaintiff	Lawyer	To induce the attorney to put more effort on a plaintiff's case
An individual seeking insurance coverage	Insurance broker	To induce the broker to search for the best policy [23]
Car Dealership	Car Salesman	Inducing the salesman to expend more effort to sell cars [24]

Principal-agent problems are generally divided into two distinct classes namely adverse selection problems<sup>1</sup> and moral hazard problems<sup>2</sup> [19]. Adverse selection represents the situation where an agent has private information about his “inability or unwillingness” to complete the principal's task [19]. In the case of moral hazard, it is assumed that both principal and agent are aware of the agent's true capabilities before signing of the contract. The principal's problem in this case is to ensure that the agent chooses a particular action or level of effort once hired. The agent's effort or actions is often described as either observable or unobservable. If observable, then the principal is assumed to be have the ability to observe how well the agent actually works once the contract is signed. If effort is unobservable, then it means it is too costly or impractical for the principal to fully monitor and observe what exactly the agent does [25].

---

<sup>1</sup>also called hidden information problems or asymmetric information problems

<sup>2</sup>also called hidden action problems

## 2.2 The Principal-Agent Problem with Moral Hazard

We focus on the moral hazard problem since the co-development model we will formulate is of this type. The moral hazard principal-agent framework is a standard approach adopted in studying contractual relationships between two parties where one party (the principal) seeks to induce another party (the agent) to perform an action,  $a$ , that is in the best interest of the principal. The principal offers the agent a contract which specifies the agent's payoff,  $w(x)$ , from accepting and fulfilling the contract requirements where  $x$  represents observed output. The output,  $x$  is assumed to be a random variable with support  $\chi = [x, \bar{x}]$ , the distribution function  $P(x|a)$  parameterized by  $a$  and the density function  $p(x|a)$ . While both parties do have a utility that is increasing in the payoff amount and wish to maximize their utility, effort is costly to the agent, hence the agent generally tries to minimize effort. An important aspect of the problem is therefore that the agent's actions that may be in the best interest of the agent, may not necessarily be in the best interest of the principal.

The principal's problem is therefore to design a contract which maximizes her utility,  $U_p(\cdot)$ , while also offering the agent enough incentives or wages,  $w$ , to ensure the agent accepts the contract and subsequently implements the optimal action,  $a$ , from the principal's perspective. The agent himself is a utility maximizer with a utility function,  $U_A(\cdot)$ . We also assume that the agent has a minimal level of utility,  $U_r$ , that he is willing to accept before he can agree to work for the principal. A generic formulation of the principal's problem can therefore be described by the following program:

$$\max_w U_p(w, a^*) \quad (2.1)$$

subject to

$$U_A(a^*|w) \geq U_r \quad (2.2)$$

$$a^* = \arg \max_a U_A(a|w). \quad (2.3)$$

By offering a reward which is greater than or equal to the agent's reservation utility, inequality (2.2) represents the guarantee that the contract will be acceptable to the agent by offering incentives that are at least as good as the best outside option available to the agent. This con-

straint is referred to as the participation constraint or individual rationality (IR) constraint. The principal also tries to design a contract such that the utility maximizing effort level from the perspective of the principal coincides with the effort level that maximizes the agent's utility. This requirement is referred to as the incentive compatibility (IC) constraint and is represented above by expression (2.3).

## 2.3 Solutions to the Principal-Agent Problem

Successful development of analytical solution concepts for the principal-agent problem have largely been limited by the complexity of the incentive compatibility constraint [24]. This is because the incentive compatibility constraint embeds an optimization problem within an optimization problem. The problem can therefore be described as a bilevel programming problem where the optimization of the principal's utility function constitutes the upper-level problem and the incentive compatibility constraint (optimizing the agent's utility function) is the lower level problem. In order to make the problem more mathematically tractable, different authors have applied various simplifying assumptions to the problem. We briefly survey some of those approaches.

### Separable Utility Function for the Agent

A common feature among many proposed solution concepts is the use of the assumption that the agent has a utility function that is multiplicatively or additively separable in compensation and effort [24, 25, 26, 27, 28, 29]. The agent's utility function can therefore be written as  $U_A(w, a) = K(a)V(w) - C(a)$  where  $K(a)$  is strictly positive,  $V(w)$  is strictly increasing and  $C(a)$  is non-negative. Often, the agent's utility function is left in the form  $V(w) - C(a)$  which is equivalent to setting  $K \equiv 1$ .

### The First-Best Solution

By assuming that the principal is able to observe the agent's actions, the principal-agent problem simplifies significantly. When the principal can observe the agent's performance, the incen-

tive compatibility constraint becomes unnecessary as the principal can simply offer a contract which specifies exactly how the agent should work and the penalties associated with deviation from the contracted effort. Once an agent accepts the contract, he cannot deviate from the contracted action since the principal is watching. The principal therefore only needs to make a contract that is acceptable to the agent. The resulting generic problem the principal tries to solve becomes:

$$\max_w U_P(w, a^*) \quad (2.4)$$

subject to

$$U_A(a^*|w) \geq U_r. \quad (2.5)$$

The optimal solution to this problem where effort is observable is referred to as the first-best solution. While the first-best solution only solves a more idealized problem, it can be used as a benchmark solution for the complete principal-agent problem.

To solve some aspects of our problem, in this thesis we also use an approach that is quite similar to the first-best problem. We call this the co-ordinated problem. In the co-ordinated problem, we assume that the principal and the agent are actually acting together as one unit with no conflict in objectives. In this case we solve the problem as a first-best problem but without need to consider both the incentive compatibility and individual rationality constraints.

When effort is unobservable, both the individual rationality constrain and the incentive compatibility constraint are retained. The solution to this principal-agent problem where effort is unobservable is referred to as the second-best solution. The net profit for the principal under the second-best scenario is generally always less than or equal to the net profit under the first-best scenario [29]. The difference between the first-best and second-best net profits represent the principal's loss from being unable to observe the agent's actions. However, when the agent's utility function has certain properties, the principal's net profits under the first-best and second-best scenario are equal [29]. An example of such a utility function is the separable function  $U_A(w, a) = K(a)V(w) - C(a)$  where  $V(w)$  is linear and strictly increasing,  $K(a)$  is strictly positive and  $C(a)$  is non-negative.



### The First Order Approach

A now standard simplifying approach adopted by many authors to deal with the incentive compatibility constraint has been to use what is called the first order approach. With the first order approach, instead of strictly requiring that the agent choose a utility maximizing level of effort, the IC constraint is relaxed to require only that the agent choose an effort level that is a stationary point on his utility function. The principal is assumed to be risk neutral and the agent is assumed to have an additively separable utility function  $U_A(a|w) = V(w) - C(a)$ . Under these assumptions the resulting PA problem to solve becomes :

$$\max_w \int (x - w(x)) p(x|a) dx \quad (2.6)$$

subject to

$$\int [V(w(x)) - C(a)] p(x|a) dx \geq U_r \quad (2.7)$$

$$\int V(w(x)) \frac{\partial}{\partial a} p(x|a) dx - \frac{\partial}{\partial a} C(a) = 0. \quad (2.8)$$

Under this method if  $(w, a)$  solves the relaxed problem, then there should exist two numbers  $\lambda$  and  $\delta$  such that

$$\frac{1}{U'_A(w)} = \lambda + \delta \frac{\frac{\partial}{\partial a} p(x|a)}{p(x|a)} \quad (2.9)$$

where  $\delta$  solves the equation :

$$\int (x - w(x)) \frac{\partial}{\partial a} p(x|a) dx + \delta \left[ \int V(w(x)) \frac{\partial^2}{\partial a^2} p(x|a) dx - \frac{\partial^2}{\partial a^2} C(a) \right] = 0 \quad [25, 30]. \quad (2.10)$$

While many have used the first order approach, other authors have sought to highlight that the approach is generally not valid and only works under certain conditions [27, 30]. We use Figure 2.1 to show the shortcomings of the first order approach. This type of example has been illustrated multiple times in the literature [19, 22, 29, 30] and we present, with slight alterations, the specific example given in [30]. We only present a generic example without numerical values but more detailed examples can be found in [19] and [22].

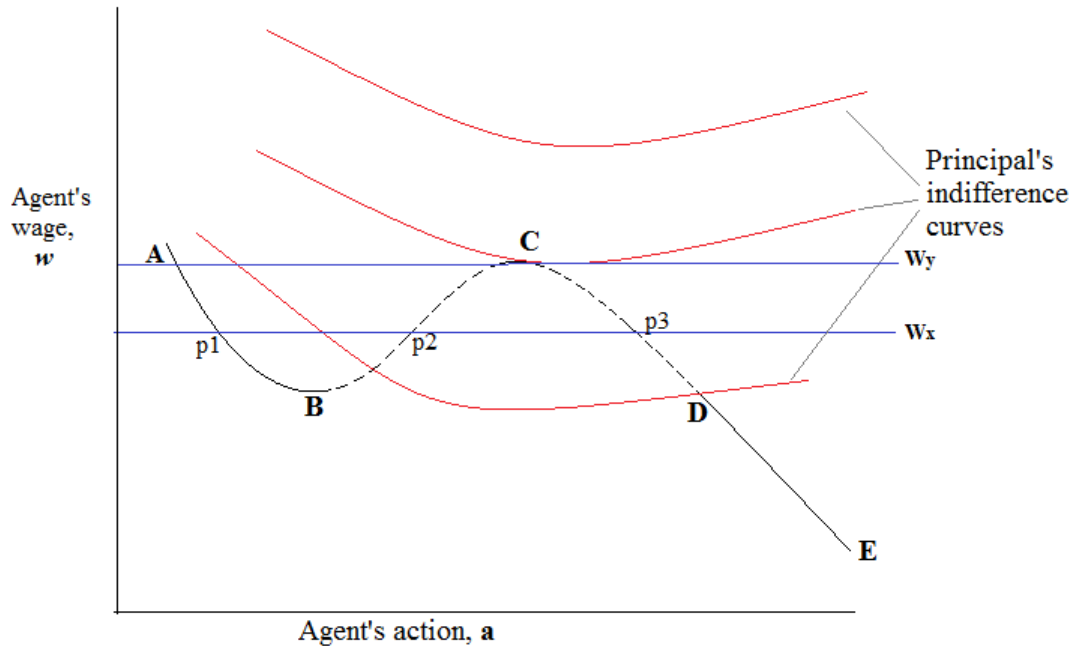


Figure 2.1: Limitations of the first order approach

Now suppose the higher the principal's indifference curve<sup>3</sup> is, the higher her utility. The principal therefore desires  $(w, a)$  that lies on the highest feasible indifference curve. She will attempt to induce an action  $a$  by offering the agent wage  $w$ . Now, suppose curve  $ABCDE$  shows the stationary points of the agent's utility function. These would be the values of  $w$  and  $a$  that satisfy the first order condition described by (2.8). For a given wage,  $w$ , we can see the corresponding actions,  $a$ , at which the first order condition is satisfied. The agent is assumed to strictly prefer lower actions hence the global maxima for the agent lie on  $AB$  and  $DE$ . For example, offered the wage amount  $w_x$  (indicated in the figure), the agent has three possible points ( $p1$ ,  $p2$  and  $p3$ ) at which the first order condition is satisfied. However, the agent would never pick  $p2$  or  $p3$ . Since the agent strictly prefers lower effort, his best response is at  $p1$  which is the lowest of the actions for which he can obtain payment  $w_x$ . This same issue would arise for any value of  $w$  that corresponds to points on  $BCD$ . However, since no points on  $BCD$  would ever be picked by the agent, the best the principal can do to maximize her utility is settle

<sup>3</sup>a plot of all the combinations of  $w$  and  $a$  which produce the exact same amount of utility for the principal (contour surfaces of the principal's utility)

for a point on  $D$  which is clearly not on the highest feasible indifference curve. By using the first order approach, one would most likely arrive at the solution that lies on the point  $C$  which we have just demonstrated to be invalid. Offered wage  $w_y$  which passes through  $C$ , the agent will be faced with two choices of actions. Since he prefers lower action, he will respond to  $w_y$  with an action on  $AB$  and not at  $C$  as the principal intended.

Having highlighted issues like those raised and illustrated in Figure 2.1, some authors have been able to provide conditions that are necessary for the first order approach to be valid. The first order approach is shown to be valid when  $p(x|a)$  satisfies the monotone likelihood ratio condition (MLRC) and the convexity of distribution function condition (CDFC) [29, 30]. The agent's effort is said to satisfy MLRC if  $\frac{\partial}{\partial x} \left( \frac{\frac{\partial}{\partial a} p(x|a)}{p(x|a)} \right) \geq 0 \quad \forall x$  and  $\frac{\partial}{\partial x} \left( \frac{\frac{\partial}{\partial a} p(x|a)}{p(x|a)} \right) > 0$  for at least some values of  $x$  [23, 30]. CDFC is satisfied if  $\frac{\partial^2}{\partial a^2} P(x|a) \geq 0 \quad \forall x$ . We begin our own investigation of solution concepts for the problem in the next section by looking at a version of the simplest form of the principal-agent problem with unobservable effort.

# Chapter 3

## A Basic Principal-Agent Model

### 3.1 Analytic Solution to a Basic Principal-Agent Model

We consider a basic contracting scenario where a risk neutral principal (she) hires a risk averse agent (he), who in turn chooses an effort level. The principal only cares about the output, with the goal of maximizing her payoff from the observed output. We assume the principal has a diverse enough portfolio of projects to counter the risk associated with the project for which she intends to hire an agent. It is therefore reasonable to assume a risk neutral principal.

Output will be dependent on the agent's effort level and a random state of nature. The higher the level of effort the agent expends, the higher the expected output. However, a high output is not guaranteed due to the random state of nature. For the agent, effort is costly, hence the agent has an incentive to expend as little effort as possible. The agent's effort level is not observable (or too costly to observe) by the principal. As a result, the principal proposes a compensation scheme that is based on the observed output, which is only a noisy signal of the agent's effort level. In response, the agent chooses his optimal effort level based on the proposed compensation scheme.

### 3.1.1 Two Actions, Two Outcomes Model

We adopt a model setup and solution approach parallel to those presented in [19, 31]. Our model assumes that the agent only has two possible effort levels, there are two states of nature and only two possible output levels. Properties of the model are as follows :

- two possible output levels - high output,  $\pi_h$  and low output,  $\pi_l$  with  $\pi_l < \pi_h$ .
- agent has two possible actions,  $a \in \{a_l, a_h\}$  where  $a_h$  denotes high effort and  $a_l$  denotes low effort.
- with high effort level, the probability of high output is  $p(\pi_h|a_h) = p_h$  while with low effort level, the probability of high output is  $p(\pi_h|a_l) = p_l < p_h$ .
- the principal is “risk neutral” and so seeks only to maximize her expected payoff.
- the principal proposes a contract that pays the agent a wage  $w(\pi_h) = w_h$  if the observed output is  $\pi_h$ , and pays  $w(\pi_l) = w_l$  otherwise
- the principal is risk neutral such that her utility function satisfies  $U_p(x) = x$ .
- the agent is a utility maximizer with a positive differentiable utility function,  $U_A(\cdot)$ .
- the agent is risk averse with decreasing absolute risk aversion, hence his utility function satisfies  $U'_A > 0$ ,  $U''_A < 0$ .
- the agent incurs a cost  $C(a_l) = C_l \geq 0$  when he applies low effort, and  $C(a_h) = C_h > C_l$  for high effort where  $C(\cdot)$  denotes the agent’s cost function.

#### Solution Approach (Two Step Process)

The principal’s intention is to determine what the optimal values of  $w_h$  and  $w_l$  are so as to maximize her payoff. A two step process which is similar to that suggested in [29] can be used to determine the optimal solution for this problem. In the first step, the principal separately determines the optimal contract (compensation schedule) for each effort level. In the second step, she then compares her net profits from each of the contracts to determine the optimal effort level. We solve the first step of the problem for the principal in the next section. Given the values of  $p_l$ ,  $p_h$ ,  $\pi_l$ ,  $\pi_h$  and the solutions for inducing high and low effort, the principal can then decide which effort level she prefers to implement based on the net profits.

## 3.2 The Two Step Approach

In the first step of our solution approach, we determine separately the optimal wages to offer the agent for inducing high effort and inducing low effort. Of the two effort levels, the principal will then offer a contract that induces an effort level that yields the greater profit. We begin by determining the optimal contract for inducing high effort.

### 3.2.1 Inducing High Effort

Assuming the principal prefers high effort, the principal's objective function is therefore

$$\max_{w_l, w_h} E[\Pi] = p_h \cdot (\pi_h - w_h) + (1 - p_h) \cdot (\pi_l - w_l). \quad (3.1)$$

Maximizing (3.1) is equivalent to minimizing the expected amount the principal has to pay the agent :

$$\min_{w_l, w_h} p_h \cdot w_h + (1 - p_h) \cdot w_l. \quad (3.2)$$

The principal therefore has linear iso-cost lines<sup>1</sup> of the form  $Z = p_h \cdot w_h + (1 - p_h) \cdot w_l$  where  $Z$  is a constant.

Since the agent's wage is based on observed output, if he uses high effort, then the probability of a high output,  $\pi_h$  which results in wage  $w_h$  is  $p_h$  and the probability of low output which results in low wage  $w_l$  is  $1 - p_h$ . The agent's expected utility given that he expends high effort is therefore :

$$U^h = p_h \cdot U_A(w_h) + (1 - p_h) \cdot U_A(w_l) - C_h. \quad (3.3)$$

The corresponding expected utility for the agent given that he uses low effort is therefore :

$$U^l = p_l \cdot U_A(w_h) + (1 - p_l) \cdot U_A(w_l) - C_l. \quad (3.4)$$

---

<sup>1</sup>a plot of all the combinations of  $w_l$  and  $w_h$  that cost the same total amount,  $Z$ , to the principal where  $Z$  is some fixed value

In order to induce the agent to work hard (high effort) instead of applying low effort, the principal has to include the incentive compatibility constraint. This can be represented as

$$a^* = \arg \max_a p(\pi_h|a) \cdot U_A(w_h|a) + p(\pi_l|a) \cdot U_A(w_l|a) - C(a). \quad (3.5)$$

The agent only has two possible actions and the principal wishes to induce high effort while satisfying (3.5). To accomplish this, the principal then only has to ensure that  $U^h \geq U^l$  so that the agent maximizes his utility by choosing high effort. This requirement therefore represents the incentive compatibility (IC) constraint for this model. The constraint  $U^h \geq U^l$  implies

$$(p_h - p_l) \cdot [U_A(w_h) - U_A(w_l)] \geq C_h - C_l. \quad \text{IC} \quad (3.6)$$

We assume that the agent has some reservation utility,  $U_r \geq 0$ . The agent will only accept the contract offered by the principal if his expected utility from the contract is at least as high as his reservation utility. This requirement is the individual rationality (IR) constraint. In the case where the agent chooses high effort, the IR constraint is

$$p_h \cdot U_A(w_h) + (1 - p_h) \cdot U_A(w_l) - C_h \geq U_r. \quad \text{IR} \quad (3.7)$$

### Investigating the Problem Structure

To solve this problem we consider  $w_h$  as a differentiable function of  $w_l$  and search for the optimal solutions on the  $w_l - w_h$  plane. We consider the properties of the principal's wage bill first. On the  $w_l - w_h$  plane, we note that the principal has linear iso-cost lines of the form  $w_h = Z - \frac{1 - p_h}{p_h} w_l$ . Each iso-cost line represents different pairs of  $w_h$  and  $w_l$  values that result in the same expected cost,  $Z$ , to the principal. The iso-cost lines are therefore downward sloping with slope  $-\left(\frac{1 - p_h}{p_h}\right)$  as seen in Figure 3.1 on page 19.

We now investigate the structure of the agent's individual rationality constraint, (3.7). Differentiating (3.7) with respect to  $w_l$  we obtain

$$\frac{dw_h}{dw_l} = -\frac{1 - p_h}{p_h} \cdot \frac{U'_A(w_l)}{U'_A(w_h)}. \quad (3.8)$$

Since  $U'_A > 0$  and  $0 < p_h < 1$ , it follows from (3.8) that  $\frac{dw_h}{dw_l} < 0$  hence the IR constraint is decreasing. We differentiate constraint (3.7) twice to determine its concavity. The result

obtained is

$$\frac{d^2 w_h}{dw_l^2} = -\frac{1-p_h}{p_h} \left[ \frac{U_A''(w_l)}{U_A'(w_h)} - \frac{U_A'(w_l)U_A''(w_h)}{[U_A'(w_h)]^2} \cdot \frac{dw_h}{dw_l} \right]. \quad (3.9)$$

Since  $U_A' > 0$ ,  $U_A'' < 0$  and  $\frac{dw_h}{dw_l} < 0$  we therefore have  $\frac{d^2 w_h}{dw_l^2} > 0$  which implies that the IR constraint is convex and decreasing (Figure 3.1).

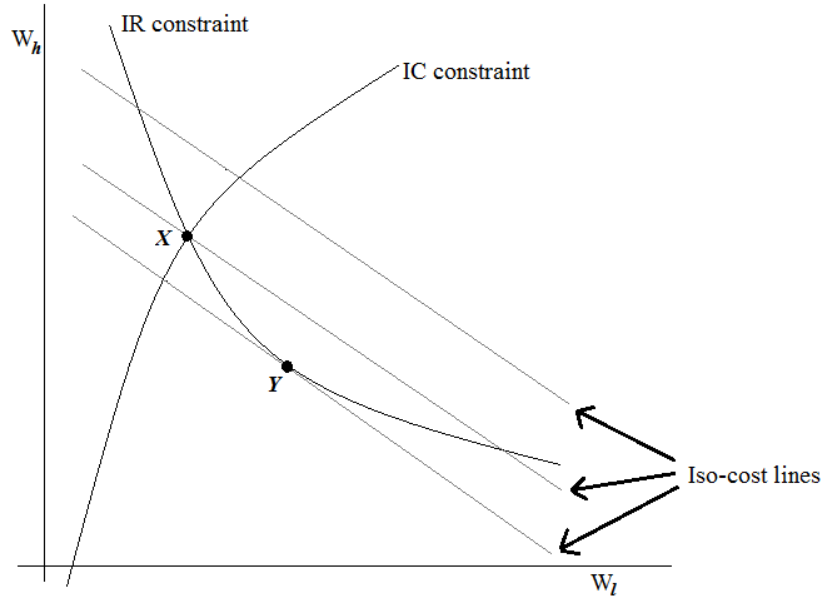


Figure 3.1: The IR and IC constraints and the Principal's wage bill on  $w_l$ - $w_h$  plane.

(By moving downwards and/or to the left, the principal moves on to a lower iso-cost line)

To investigate the form of the incentive compatibility constraint we differentiate (3.6) and get

$$\frac{dw_h}{dw_l} = \frac{U_A'(w_l)}{U_A'(w_h)}. \quad (3.10)$$

Since  $w_l < w_h$  and  $U_A'' < 0$  (i.e.  $U_A'$  is decreasing),  $U_A'(w_l) > U_A'(w_h)$  hence  $\frac{dw_h}{dw_l} = \frac{U_A'(w_l)}{U_A'(w_h)} > 1 > 0$ . The IC constraint is therefore increasing. Differentiating the IC a second time to



determine the concavity of the constraint yields :

$$\frac{d^2 w_h}{dw_l^2} = \frac{U_A''(w_l)}{U_A'(w_h)} - \frac{U_A'(w_l) \cdot U_A''(w_h)}{[U_A'(w_h)]^2} \cdot \frac{dw_h}{dw_l} \quad (3.11)$$

$$= \frac{1}{U_A'(w_h)} \left[ U_A''(w_l) - U_A''(w_h) \left( \frac{dw_h}{dw_l} \right)^2 \right]. \quad (3.12)$$

To determine whether  $\frac{d^2 w_h}{dw_l^2}$  is positive or negative, we note that  $\frac{1}{U_A'(w_h)}$  is positive since  $U_A' > 0$ .  $\frac{dw_h}{dw_l} > 1$  implies that  $\left( \frac{dw_h}{dw_l} \right)^2 > 1$ . In contract theory literature, a now common measure of risk aversion is called the Arrow-Pratt measure. The Arrow-Pratt measure of absolute risk aversion,  $A$ , is defined as  $A = \frac{-U_A''}{U_A'}$  [32]. Since the agent has decreasing absolute risk aversion,  $A' < 0$  which only holds if  $U_A''' > 0$ .  $U_A''' > 0$  implies that  $U_A''(w_l) < U_A''(w_h)$ . It follows therefore that  $\frac{d^2 w_h}{dw_l^2} < 0$  hence the IR constraint is increasing and concave in Figure 3.1.

### First-Best Solution

Before considering the case with unobservable effort, we begin by deriving a simpler first-best solution of the problem. In the first-best case, we assume effort is observable and the principal does not have to consider the IC constraint. We recall that we transformed the principal's problem from maximizing expected profit to minimizing the expected cost of paying the agent. Since the principal wishes to minimize her wage bill, at optimality the IR constraint must be binding. To see this, we consider the following proof by contradiction.

Suppose the solution  $(w_l^*, w_h^*)$  is an optimal solution to the problem and the IR constraint is not binding at optimality. The IR constraint, (3.7), is thus only satisfied as a strict inequality. The optimal solution therefore satisfies

$$p_h \cdot U_A(w_h^*) + (1 - p_h) \cdot U_A(w_l^*) - C_h > U_r. \quad (3.13)$$

The inequality above implies that there exists some  $\epsilon > 0$  such that

$$p_h \cdot [U_A(w_h^*) - \epsilon] + (1 - p_h) \cdot [U_A(w_l^*) - \epsilon] - C_h \geq U_r. \quad (3.14)$$

Now let  $\tilde{w}_h = U_A^{-1}(U_A(w_h^*) - \epsilon)$  and  $\tilde{w}_l = U_A^{-1}(U_A(w_l^*) - \epsilon)$ . Since  $U_A$  is increasing in wages, we know therefore that  $\tilde{w}_l < w_l^*$  and  $\tilde{w}_h < w_h^*$ . It follows then that the solution  $(\tilde{w}_l, \tilde{w}_h)$  is a lower cost contract and thus results in higher profit for the principal than  $(w_l^*, w_h^*)$ . Clearly  $(\tilde{w}_l, \tilde{w}_h)$  also satisfies the IR constraint. This violates the original assumption that  $(w_l^*, w_h^*)$  was optimal (profit maximizing) for the principal. We can conclude that at optimality, the individual rationality constraint is binding. This argument holds true even in the second-best solution.

To minimize her cost while satisfying the IR constraint, the principal can pick the point of tangency between the IR constraint and an iso-cost line (point **Y** in Figure 3.1). Point **Y** therefore corresponds to what would be the first-best solution of this problem. Now we recall that the slope of any iso-cost line is  $-\left(\frac{1-p_h}{p_h}\right)$  and that of the IR constraint is  $-\left(\frac{1-p_h}{p_h}\right) \cdot \frac{U'_A(w_l)}{U'_A(w_h)}$ . At the point of tangency the slopes should be equal hence  $\frac{U'_A(w_l)}{U'_A(w_h)}$  is equal to 1. Since  $U''_A < 0$ , this is only possible if the agent's wage for low effort is equal to the wage for high effort i.e.  $w^{FB} = w_l = w_h$ , where  $w^{FB}$  represents the first best wage. We can then use the binding IR constraint to solve for the optimal wage in the first-best scenario and get  $w^{FB} = U_A^{-1}(U_r + C_h)$ . This therefore means that, under the first-best scenario, the optimal solution would be to offer the agent a fixed wage that is independent of output. This agrees with the intuition that if the principal can fully observe what the agent is doing, beyond making him accept the contract, there is no reason for her to offer the agent a contract with variable incentives to induce him to work at the agreed effort level.

### Second-Best Solution

In the second best solution, we assume the principal cannot observe the agent's effort hence the incentive compatibility constraint has to be considered. We thus consider an optimal solution as one which minimizes the principal's wage bill while satisfying the agent's individual rationality and incentive compatibility constraints.

We recall from the argument made in the first-best solution that at optimality, the IR constraint must be binding. However, unlike in the first-best solution the principal cannot simply

look for the point of tangency between the IR constraint and an iso-cost line. This is because she now has to consider the IC constraint. We highlight that a binding IR constraint is only a property of this model. In general, IR is not binding in the second-best case. Looking at the IC constraint, (3.6), we have

$$(p_h - p_l) [U_A(w_h) - U_A(w_l)] \geq C_h - C_l \quad (3.15)$$

$$\Rightarrow U_A(w_h) \geq U_A(w_l) + \omega \quad \text{where } \omega = \frac{C_h - C_l}{p_h - p_l} > 0. \quad (3.16)$$

We observe that since  $\omega > 0$ , it means  $U_A(w_h) > U_A(w_l)$  hence  $w_h > w_l$  since  $U'_A > 0$ . Therefore in this case we know the solution that satisfies the second-best solution is different from that of the first-best solution. In this case, the agent's wage for high output,  $w_h$  is strictly greater than that for low output,  $w_l$ .  $w_h > w_l$  also implies that the optimal solution must lie above and to the left of point **Y**, the first-best solution on the IR constraint. As we would expect of a second-best solution, the solution will therefore lie on a higher iso-cost line than the first-best solution (i.e. the second-best solution is more costly to the principal than the first-best.) In order to satisfy the IC constraint, the principal needs to set the values of  $w_h$  and  $w_l$  that satisfy (3.16). We note that from the principal's perspective, for any  $w_l$ , the cost-minimizing value of  $w_h$  will thus be the value of  $w_h$  that satisfies  $U_A(w_h) = U_A(w_l) + \omega$ . At optimality, the IC constraint must therefore be binding. It follows then that the optimal solution lies at the point **X** in Figure 3.1. This is where the IC constraint intersects with the IR constraint.

To determine the optimal values of  $w_h$  and  $w_l$ , we start by solving for  $U_A(w_l)$  in the binding IR constraint to get

$$U_A(w_l) = \frac{1}{1 - p_h} [U_r + C_h - p_h U_A(w_h)]. \quad (3.17)$$

From the binding IC constraint we have

$$U_A(w_h) - U_A(w_l) = \omega. \quad (3.18)$$

Combining (3.17) and (3.18), we obtain

$$U_A(w_h) - \frac{1}{1 - p_h} [U_r + C_h - p_h U_A(w_h)] = \omega \quad (3.19)$$

$$\Rightarrow U_A(w_h) = U_r + C_h + (1 - p_h)\omega \quad (3.20)$$

$$\therefore w_h^* = U_A^{-1}(U_r + C_h + (1 - p_h)\omega) \quad (3.21)$$

where  $w_i^*$  denotes the optimal value of  $w_i$ .

From (3.18) and (3.20) we can then find  $w_l^*$  :

$$U_A(w_l) = U_A(w_h) - \omega \quad (3.22)$$

$$= U_r + C_h - p_h \omega \quad (3.23)$$

$$\therefore w_l^* = U_A^{-1}(U_r + C_h - p_h \omega). \quad (3.24)$$

Comparing the optimal wages for high output and low output, we observe that there is a common component to both wages, which is  $U_r + C_h$ . We recall that the first-best solution was found to be  $w^{FB} = U_A^{-1}(U_r + C_h)$ . Unlike in the first-best case, now the agent bears some level of risk which is related to the probability of high output. Relative to the first-best scenario, the agent faces a penalty  $p_h \cdot \omega$  in the event of low output, while he stands to obtain a premium amount  $(1 - p_h) \cdot \omega$  in the event of high output. We observe that the greater the likelihood of a high output given a high effort, the greater the penalty the agent faces if a low output is realized. However, the smaller the probability of high output, the higher the premium over the  $U_r + C_h$ . This suggests that the greater the likelihood of failure given high effort, the greater the agent's payment. As we would expect this means the risk averse agent should receive higher incentives to accept bearing risk in a situation where the likelihood of failure is greater.

### 3.2.2 Inducing Low Effort

We suppose now that the principal prefers that the agent use low effort. The solution to this problem can be derived through carrying out a process similar to the one used for inducing high effort. We describe the solutions in the next section. We denote the agent's wage given low output as  $\hat{w}_l$  and his wage given high output as  $\hat{w}_h$ .

### First-Best Solution

As in the previous case where the principal wanted to induce high effort, we find that when the agent has observable effort, the optimal wages are independent of output. The optimal wages for the agent should therefore be fixed under high or low output and thus satisfy  $\hat{w}^{FB} = \hat{w}_l = \hat{w}_h$ . From the binding individual rationality constraint we can then solve for  $\hat{w}^{FB}$  to get  $\hat{w}^{FB} = U_A^{-1}(U_r + C_l)$ .

### Second-Best Solution

We recall that the difference between the first-best and second-best problem is the presence of the incentive compatibility constraint in the latter. The agent wishes to maximize his utility hence, in order for the incentive compatibility constraint to be satisfied, the principal should propose a wage schedule such that the agent's expected utility from using low effort is not lower than from using high effort. Suppose that the principal offers the agent wages that are independent of output, then from the first-best scenario a candidate solution to consider is  $\hat{w}^{SB} = \hat{w}_l = \hat{w}_h = U_A^{-1}(U_r + C_l)$ . We first verify whether the solution satisfies the individual rationality constraints and the incentive compatibility constraints for the agent. If these conditions are met, we then only need to check that the solution maximizes the principal's profit.

Since the solution  $\hat{w}^{SB} = U_A^{-1}(U_r + C_l)$  is directly derived from a binding individual rationality constraint, it follows that  $\hat{w}^{SB}$  clearly satisfies the constraint. We recall that under the second-best scenario, the agent's actions cannot be observed by the principal. Since the principal wishes to induce low effort, incentive compatibility requires that choosing low effort,  $a_l$  be optimal for the agent. Presented with a fixed wage that is independent of output, the agent will choose to implement the less costly effort,  $a_l$ , in order to maximize his utility. Thus the incentive compatibility constraint is satisfied in this case. We now only need to check that  $\hat{w}^{SB} = U_A^{-1}(U_r + C_l)$  is the profit maximizing (cost minimizing) solution in the second-best case. We know that the second-best solution can only yield a profit that is at most equal to that of the first-best solution (the first-best profit is an upper bound for the second-best profit).  $\hat{w}^{SB} = \hat{w}^{FB} = U_A^{-1}(U_r + C_l)$  is therefore the profit maximizing wage for the principal.

### **3.2.3 The Effort Level To Implement**

Given the two optimal contracts for inducing low effort and high effort, the principal then has to calculate her expected profits. If her profit from inducing high effort is higher than that from low effort, then she offers the agent a wage schedule that induces high effort. If not, the principal offers the agent a contract to induce low effort. Should neither contract offer a positive profit, the principal can choose to not offer any contract at all.

# Chapter 4

## Numerical Solution To The Basic Problem

### 4.1 Numerical Solution to the Basic Principal-Agent Model

Having solved the basic model analytically, we now consider how the same model can be solved numerically. Solving this basic problem numerically will be instructive for developing techniques for the more advanced principal-agent problems considered later in the thesis. We intend to develop a method that can easily be adopted to solve the more advanced problems. We consider again our basic two actions, two outcomes model but now with the following parameters:

- two possible output levels - high output,  $\pi = 2$  and low output,  $\pi = 1$ .
- agent has two possible actions - high effort,  $a = 1$  and low effort,  $a = 0$ .
- with high effort level, the probability of high output is  $p(\pi = 2|a = \text{high}) = 0.8$ . while with low effort level, the probability of high output is  $p(\pi = 2|a = \text{low}) = 0.4$ .
- the principal has a large number of these projects and so is “risk neutral” with a utility function  $U_P(x) = x$  and so seeks only to maximize her expected payoff.
- the agent is a utility maximizer with a log utility function,  $U_A(\cdot) = \ln(\cdot)$ .
- the principal proposes a contract that pays the agent a wage  $w(2) = w_h$  if the observed output is  $\pi_h = 2$ , and pays  $w(1) = w_l$  otherwise.
- the agent incurs a cost  $C_h = 0.2$  when he applies high effort, and  $C_l = 0$  for low effort.
- the agent has reservation utility  $U_r = 0$ .

Before we solve the optimal wage schedule for this model numerically, we apply the same analytical solution approach as used in the last section for the basic two actions, two output model. We can then use the exact results obtained this way to assess the performance of our numerical approach. We will focus on the second-best solution of the problem.

### 4.1.1 Second Best Case: Analytical Solution (for assessing numerical approach)

#### Inducing High Effort

The optimal wage schedule for inducing high effort is the solution to the program:

$$\max_{w_l, w_h} E[\Pi] = 0.8 \cdot (2 - w_h) + 0.2 \cdot (1 - w_l) \quad (4.1)$$

subject to

$$0.8 \cdot \ln(w_h) + 0.2 \cdot \ln(w_l) - 0.2 \geq 0 \quad \text{IR} \quad (4.2)$$

$$0.8 \cdot \ln(w_h) + 0.2 \cdot \ln(w_l) - 0.2 \geq 0.4 \cdot \ln(w_h) + 0.6 \cdot \ln(w_l). \quad \text{IC} \quad (4.3)$$

Applying the exact analytic approach as used previously but with the parameters and utility functions as described above, we can find the optimal wages for inducing high effort. The solution is found to be

$$\begin{aligned} w_l^* &= U_A^{-1} \left( U_R + C_h - p_h \cdot \frac{C_h - C_l}{p_h - p_l} \right) \\ &= e^{(0.2 - 0.8 \cdot 0.2 / 0.4)} = e^{-0.2} \\ &\approx 0.8187 \end{aligned}$$

$$\begin{aligned} w_h^* &= U_A^{-1} \left( U_R + C_h + (1 - p_h) \cdot \frac{C_h - C_l}{p_h - p_l} \right) \\ &= e^{(0.2 + 0.2 \cdot 0.2 / 0.4)} = e^{0.3} \\ &\approx 1.3498. \end{aligned}$$



The resulting expected net profit for the principal for inducing high effort is therefore:

$$\begin{aligned} E[\Pi] &= 0.8 \cdot (2 - e^{0.3}) + 0.2 \cdot (1 - e^{-0.2}) \\ &\approx 0.5564. \end{aligned}$$

### Inducing Low Effort

The corresponding wage schedule for inducing low effort can be obtained from solving the program:

$$\max_{w_l, w_h} E[\Pi] = 0.4 \cdot (2 - w_h) + 0.6 \cdot (1 - w_l) \quad (4.4)$$

subject to

$$0.4 \cdot \ln(w_h) + 0.6 \cdot \ln(w_l) \geq 0 \quad \text{IR} \quad (4.5)$$

$$0.4 \cdot \ln(w_h) + 0.6 \cdot \ln(w_l) \geq 0.8 \cdot \ln(w_h) + 0.2 \cdot \ln(w_l) - 0.2. \quad \text{IC} \quad (4.6)$$

The optimal wages for inducing low effort are found to be:

$$\begin{aligned} w_l^* = w_h^* &= U_A^{-1}(U_R + C_l) \\ &= e^0 = 1. \end{aligned}$$

In this case the principal's expected net profit from inducing the agent to pick low effort is

$$\begin{aligned} E[\Pi] &= 0.4 \cdot (2 - 1) + 0.6 \cdot (1 - 1) \\ &= 0.4. \end{aligned}$$

Comparing the principal's net profits, we see that the principal should therefore prefer that the agent apply high effort. The agent is therefore presented with a contract that pays the agent  $w_l = e^{-0.2} \approx 0.8187$  if low output is observed, and  $w_h = e^{0.3} \approx 1.3499$  when high output is observed. This will have the effect of inducing the agent to prefer high as opposed to low effort.

### 4.1.2 Verifying the Solution Graphically

Due to the relatively simple structure of the principal's profit function and the agent's constraints (IC and IR) we can deduce some properties of the relationship between any value of  $w_l$

and the corresponding (optimal) value of  $w_h$  given  $w_l$ . This can then enable us to analyze the problem graphically and hence serve as a verification of the analytical results.

### Inducing High Effort

We recall that for this specific type of problem, maximizing the principal's expected profit (4.1) is equivalent to minimizing the total expected wage bill. Both the IR constraint (4.2) and the IC constraint (4.3) can be rearranged to express a relationship between any  $w_l$  value and a feasible value of  $w_h$ . The resulting program from applying these modifications is:

$$\max_{w_l, w_h} E[\Pi] = 0.8 \cdot (2 - w_h) + 0.2 \cdot (1 - w_l) \quad (4.7)$$

subject to

$$w_h \geq e^{0.25 - 0.25 \cdot \ln(w_l)} = \left(\frac{e}{w_l}\right)^{0.25} \quad \text{IR} \quad (4.8)$$

$$w_h \geq e^{0.5 + \ln(w_l)} = e^{0.5} w_l. \quad \text{IC} \quad (4.9)$$

Given any value of  $w_l$ , the least costly amount (profit maximizing value) of  $w_h$  that satisfies both (4.8) and (4.9) is therefore  $w_h = \max\{ (e/w_l)^{0.25}, e^{0.5} w_l \}$ . We can then plot the value of the principal's expected profit across different values of  $w_l$  in order to determine the profit maximizing values of  $w_l$  and  $w_h$ . Figure 4.1 on page 30 shows the plot for inducing high effort where we indeed observe that the optimal value of  $w_l \approx 0.82$  and hence  $w_h \approx 1.35$ .

### Inducing Low Effort

We can carry out the exact same process as above for implementing low effort. The problem becomes:

$$\max_{w_l, w_h} Z = 0.4 \cdot (2 - w_h) + 0.6 \cdot (1 - w_l) \quad (4.10)$$

subject to

$$w_h \geq w_l^{-1.5} \quad \text{IR} \quad (4.11)$$

$$w_h \leq e^{0.5} w_l. \quad \text{IC} \quad (4.12)$$

We note that satisfying (4.11) and (4.12) implies that  $w_l^{-1.5} \leq w_h \leq e^{0.5} w_l$ . Within this region of the feasible values of  $w_h$  the cost minimizing value of  $w_h = w_l^{-1.5}$ . It is then apparent that

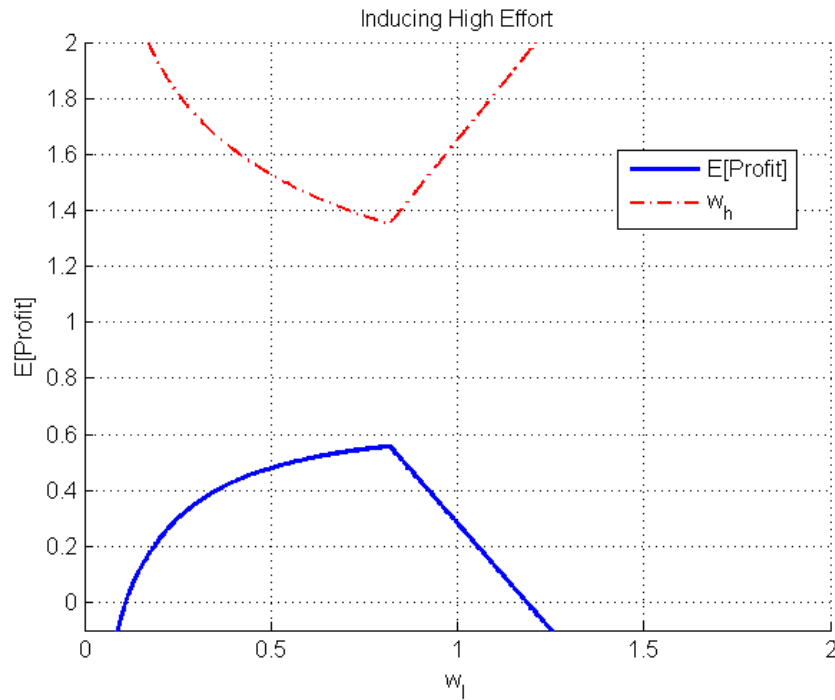


Figure 4.1: The expected profit and corresponding optimal value of  $w_h$  across values of  $w_l$  for inducing high effort.

we only need to search for an optimal solution in the region of  $w_l$  where  $w_l^{-1.5} \leq e^{0.5} \cdot w_l \Rightarrow w_l \geq e^{-0.2}$ . The plot of the principal's expected profit across different values of  $w_l$  is provided in Figure 4.2 on page 31. As expected the observed profit maximizing value of  $w_l$  is consistent with the previously obtained result of  $w_l = w_h = 1$  and a resulting expected net profit of 0.4.

### 4.1.3 Solving the Problem Numerically

For our numerical approach, we implement a strategy that is identical to the Sequential Unconstrained Minimization Technique (SUMT) Algorithm [33]. The SUMT works by transforming the constrained minimization problem :

$$(A) : \min f(x)$$

$$\text{subject to } g_i(x) \geq 0 \quad (i = 1, 2, \dots, m)$$

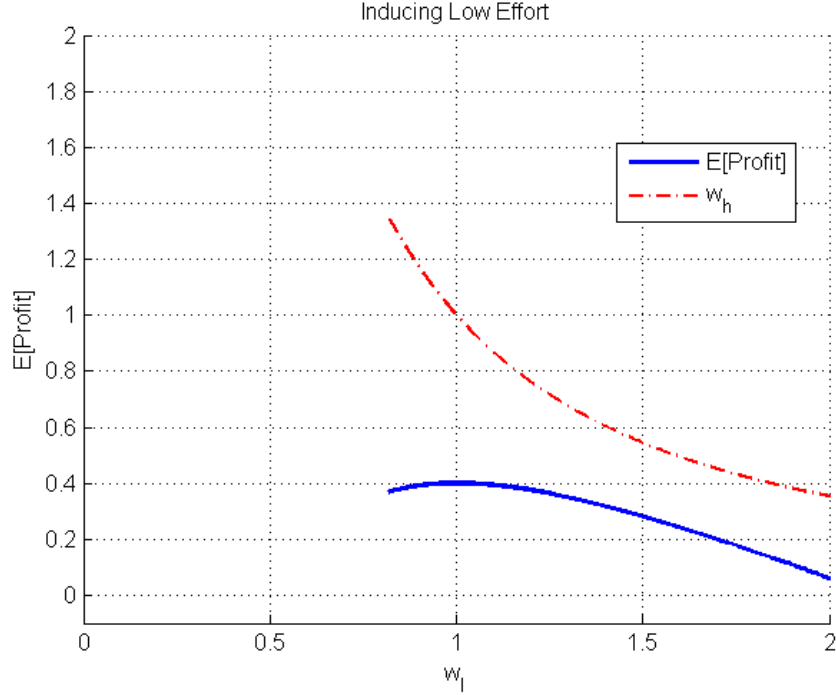


Figure 4.2: The expected profit and corresponding optimal value of  $w_h$  across values of  $w_l$  for inducing low effort

into the unconstrained problem :

$$(B) : \min Y(x, \Lambda_k) = f(x) + \Lambda_k \sum_{i=1}^{i=m} \frac{1}{g_i(x)}.$$

The method minimizes  $Y(x, \Lambda_k)$  within the interior feasible region over a decreasing sequence,  $\{\Lambda_k\}$ , where  $\Lambda_k > 0 \forall k$ . As  $\Lambda_k \rightarrow 0$ , the sequence of solutions to (B) converge to the optimal solution of (A).

The key feature of the SUMT which we adopt is the transformation of a constrained problem into an unconstrained problem. However in our case, instead of minimizing the objective function, we implement the algorithm as a maximization problem as proposed in [34]. This technique enables us to solve the originally constrained problem as an unconstrained maximization problem which is relatively less complicated to solve. To solve the unconstrained optimization problem, we use a gradient search approach.

Before we can introduce the new optimization problems, we define or recall the following functions and symbols:

- $\Pi_h(w_l, w_h)$  and  $\Pi_l(w_l, w_h)$  represent the principal's expected profit functions given high and low effort respectively.
- $L_h(w_l, w_h)$  and  $L_l(w_l, w_h)$  are barrier functions obtained from the original constraints.
- $\Lambda$  is a constant that regulates the overall weighting of the barrier functions relative to  $\Pi_h(w_l, w_h)$  and  $\Pi_l(w_l, w_h)$ .
- $U_r$  is the agent's reservation utility.
- $U_h(w_l, w_h)$  and  $U_l(w_l, w_h)$  represent the agent's expected utility from implementing high and low effort respectively.

The new objective functions that incorporate both the original constraints and objective functions are

$$Y_h(w_l, w_h) = \Pi_h(w_l, w_h) - \Lambda \cdot L_h(w_l, w_h) \quad (4.13)$$

$$= \Pi_h(w_l, w_h) - \Lambda \cdot \left( \frac{1}{U_r - U_h(w_l, w_h)} + \frac{1}{U_l(w_l, w_h) - U_h(w_l, w_h)} + \frac{1}{w_h} + \frac{1}{w_l} \right) \quad (4.14)$$

and

$$Y_l(w_l, w_h) = \Pi_l(w_l, w_h) - \Lambda \cdot L_l(w_l, w_h) \quad (4.15)$$

$$= \Pi_l(w_l, w_h) - \Lambda \cdot \left( \frac{1}{U_r - U_l(w_l, w_h)} + \frac{1}{U_h(w_l, w_h) - U_l(w_l, w_h)} + \frac{1}{w_h} + \frac{1}{w_l} \right) \quad (4.16)$$

where  $Y_h(w_l, w_h)$  is the function we maximize to determine the optimal wage schedule for high effort and  $Y_l(w_l, w_h)$  is the corresponding function for low effort.

Rewriting the problem in the form of (4.14) or (4.16) implies that we have effectively reduced the multiple constraint problem into a single unconstrained objective function. The effect of the rational expressions in the second part of both  $L_h(w_l, w_h)$  and  $L_l(w_l, w_h)$  is to encourage the solutions obtained from optimizing (4.14) and (4.16) to stay within the feasible region that satisfies all original constraints. The functions  $Y_l(\cdot)$  and  $Y_h(\cdot)$  are then optimized across a sequence of successively smaller values of  $\Lambda$  up to a desired number of iterations. Given a fixed

$\Lambda$  and an initial feasible trial solution  $\mathbf{w}^{(0)} = (w_l^{(0)}, w_h^{(0)})$ , we implement a gradient search procedure (steepest ascent) for maximizing the objective functions ( $Y_l(\cdot)$  and  $Y_h(\cdot)$ ). We explain the full sequence of steps for finding the optimal solutions. We focus on solving the problem for inducing high effort. The optimal contract schedule for inducing low effort can be obtained analogously.

The following sequence of steps will be carried out for different values of  $\Lambda$  and initial feasible solution. We consider a feasible solution as one that satisfies the individual rationality, incentive compatibility and non-negativity constraints.

### Initialization

Set an initial value of  $\Lambda$  and pick an initial feasible point  $\mathbf{w}^{(0)} = (w_l^{(0)}, w_h^{(0)})$ . The algorithm will not produce a reasonable result if the chosen initial point is not feasible. One way to determine feasible points that satisfy all constraints is to create a contour plot indicating feasible and non-feasible points. Figures 4.3 and 4.4 are examples of such a plot.

### Gradient Search

Before we explain our approach, we highlight that the index  $k$  (superscript) is used to denote iteration across values of  $\mathbf{w} = (w_l, w_h)$ . Later the index  $j$  (subscript) will denote iteration across different values of  $\tilde{\lambda}$  for a given point,  $\mathbf{w}^{(k)} = (w_l^{(k)}, w_h^{(k)})$ .

Given a point  $\mathbf{w}^{(k)}$  obtained after the  $k$ -th iteration, we determine the non-negative number  $\tilde{\lambda} = \lambda^{(k)}$  that maximizes  $Y_h(\mathbf{w}^{(k+1)})$  where  $\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + \lambda^{(k)} \cdot \nabla Y_h(\mathbf{w}^{(k)})$ . Finding the optimal  $\lambda^{(k)}$  is another optimization problem but only for one variable. We use an iterative search procedure to determine the optimal value of  $\tilde{\lambda}$ . We outline the steps for finding this value below.

We note at this point that if the maximization problem is not convex, depending on the chosen initial point this algorithm may only converge to a local maximum instead of the global maximum. In order to minimize the likelihood of landing on a local extreme point, one should

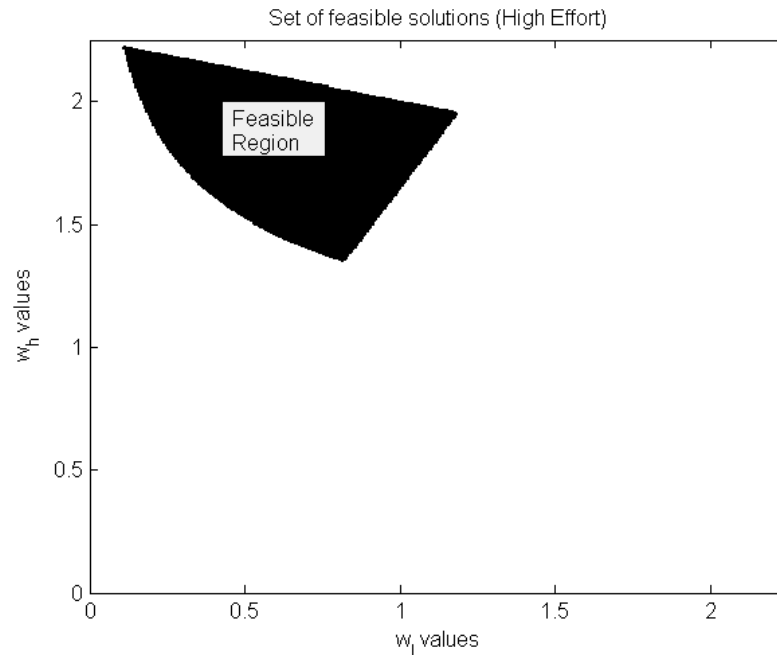


Figure 4.3: The feasible points for inducing high effort

run this algorithm multiple times from different initial points to increase the likelihood of finding the global maximum.

### Searching for Best Value of $\tilde{\lambda}$

We carry out an iterative process to determine the optimal  $\tilde{\lambda} = \lambda^{(k)}$  to permit transition from the  $k$ -th solution  $\mathbf{w}^{(k)}$  to the  $(k + 1)$ -th solution. The optimal value of  $\tilde{\lambda}$  should also ensure that any candidate solutions we generate do not violate the non-negativity requirement for  $\mathbf{w}^{(k+1)}$ . At each iteration a new value of  $\tilde{\lambda}$  is obtained according to a specific set of rules which we outline below.

#### Reasonable Bounds for $\tilde{\lambda}$

Before we begin our search for  $\lambda^{(k)}$  we need to establish reasonable bounds within which we can search for this value. We can easily estimate these bounds from the previously plotted feasible region. In this case we note that the feasible values of  $w_l$  fall within the interval  $[0.11, 2.12]$  and those of  $w_h$  fall within  $[1.35, 2.22]$ . This therefore means that given

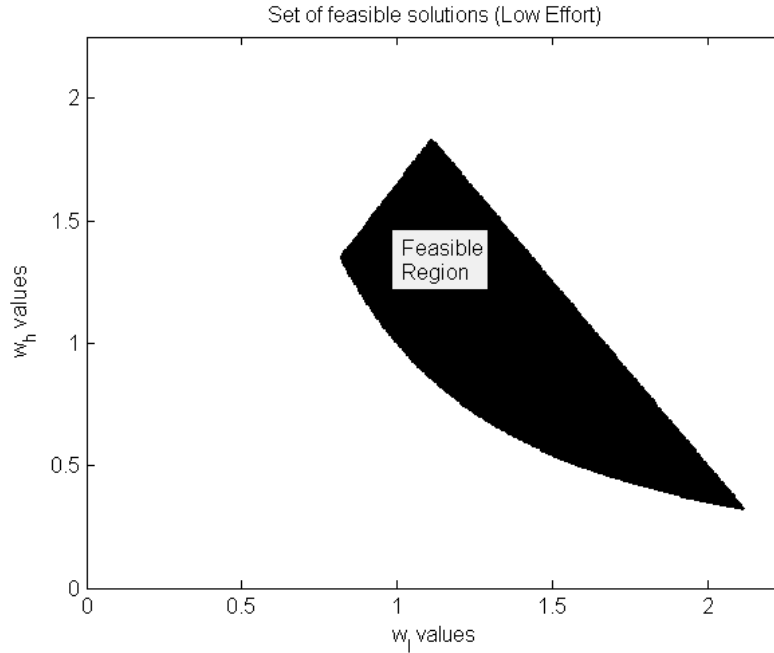


Figure 4.4: The feasible points for inducing low effort

$\mathbf{w}^{(k)}$ , the value of  $\tilde{\lambda}^{(k)}$  should satisfy both  $1.35 \leq w_h^{(k)} + \tilde{\lambda}^{(k)} \cdot \frac{\partial}{\partial w_h} Y_h(w_l^{(k)}, w_h^{(k)}) \leq 2.22$  and  $0.11 \leq w_l^{(k)} + \tilde{\lambda}^{(k)} \cdot \frac{\partial}{\partial w_l} Y_h(w_l^{(k)}, w_h^{(k)}) \leq 1.182$ . We know  $\tilde{\lambda}^{(k)} \geq 0$  hence we only need to deduce the upper bound for  $\tilde{\lambda}^{(k)}$  from the above inequalities. We let  $\tilde{\lambda}_u^{(k)}$  represent this upper bound.

#### Search Procedure

Having established the upper bound of  $\tilde{\lambda}^{(k)}$ , we pick  $N$  equally spaced points in the interval 0 and  $\tilde{\lambda}_u^{(k)}$  inclusive. Each of these  $N$  points is considered as a trial value for  $\tilde{\lambda}^{(k)}$ .  $N$  calculations of  $Y = Y_h(\mathbf{w}^{(k)} + \tilde{\lambda} \nabla Y_h(\mathbf{w}^{(k)}))$  are then made, one using each of the trial values of  $\tilde{\lambda}^{(k)}$ . Suppose the  $n$ -th candidate value of  $\tilde{\lambda}^{(k)}$ , ( $n < N$ ), produces the largest value of  $Y$ . A narrower and more refined interval of  $N$  points is formed between the  $n$ -th and the  $(n + 1)$ -th trial values of  $\tilde{\lambda}^{(k)}$ . This process of searching within more refined search intervals can be repeated a desired number of times assuming in each case that  $n < N$ . Ultimately,  $\tilde{\lambda}^{(k)}$  is set equal to the  $n$ -th trial value that produces the largest value of  $Y$  among the last  $N$  generated candidates. At any iteration, if the  $N$ -th trial value of  $\tilde{\lambda}^{(k)}$  produces the largest value of  $Y$ , then  $\tilde{\lambda}^{(k)}$  is set equal to the  $N$ -th trial value and the search is terminated at this point. Once  $\tilde{\lambda}^{(k)}$  is determined,  $\mathbf{w}^{(k+1)}$  is



set equal to  $\mathbf{w}^{(k)} + \tilde{\lambda}^{(k)} \cdot \nabla Y_h(\mathbf{w}^{(k)})$ .

#### Checking candidate solutions for constraint violation

We recall that at each iteration, the search for  $\tilde{\lambda}^{(k)}$  was carried out within what were only assumed to be reasonable bounds. This implies that some of the values candidate values of  $\tilde{\lambda}^{(k)}$  may actually violate the IR or IC constraints. In order to avoid obtaining solutions that violate the constraints, at each iteration when the  $N$  candidate values of  $\tilde{\lambda}^{(k)}$  are calculated, the corresponding candidate values of  $\mathbf{w}^{(k+1)}$  must immediately be checked against the constraints. If any of these violates a constraint, the corresponding value of  $Y$  is set to a large negative value so that it is never carried over into the next iteration.

#### **Summary of Procedure**

We conclude this section with a brief review of our method. A key aspect of the method involves transforming the original problem into an unconstrained optimization problem. The transformation uses a penalty function multiplied by some positive number,  $\Lambda$ , to represent the original constraints. We use a sequence of successively smaller values of  $\Lambda$  and for each of these values of  $\Lambda$  we perform the following steps :

1. Transform the multiple constraint Principal-Agent problem into an unconstrained maximization problem.
2. Identify a region within which all feasible candidate solutions are contained. Feasible candidate solutions should satisfy both IR and IC.
3. Identify a feasible initial trial solution.
4. Determine the direction of steepest ascent from the current trial solution .
5. Move as far as feasible in the direction of steepest ascent (a feasible move is one that yields a trial solution that satisfies the IR and IC constraints).
6. Once at the next trial solution, return to step 4 and repeat until a desired number of iterations is reached or until the level of improvement in the objective value is sufficiently small.

As  $\Lambda \rightarrow 0$ , the obtained solutions should converge to some point which we then take as the optimal solution.

#### 4.1.4 Numerical Results

For our numerical study, we optimized the functions  $Y_h(w_l, w_h)$  and  $Y_l(w_l, w_h)$  using successively smaller values of  $\Lambda$ . Our arbitrarily chosen sequence of  $\Lambda$  values was  $\{100, 1, 0.01, 10^{-3}, 10^{-5}\}$ . We observed that for this problem as we successively changed the values of  $\Lambda$  there was no apparent improvement in results. The method we outlined above produced results that were acceptably consistent with those obtained analytically after running the algorithm a number of times from different initial trial solutions. We recall that our method works by picking a random but feasible initial point. From this initial point, the method then searches for a point that maximizes the objective function.

In tables 4.1 and 4.2 below, we show a few random initial points and the final point to which the method converges. The initial points are denoted by  $(w_l^{(0)}, w_h^{(0)})$  and the corresponding final points by  $(w_l^*, w_h^*)$ . The first set of numbers shows results for the case where we assume the principal wishes to induce high effort. The second set of results is for the case where we assume the principal wished to induce low effort. The table shows the results obtained by applying the method outlined above with  $\Lambda = 1$  from 5 randomly generated initial points. The highest expected profit is indicated in bold font.

We observe that the best results (from the 5 random initial points) obtained from this approach are comparable to the exact solutions previously determined. The best numerically obtained net profit for inducing low effort was found to be approximately 0.3978 which is close to the exact solution of 0.4. The best numerically obtained net profit for inducing high effort was found to be approximately 0.5503 which is very close to the exact solution which is  $1.8 - 0.8e^{0.3} - 0.2e^{-0.2} \approx 0.5564$ . We do however note that the proximity of the numerical solution to that of the exact solution is heavily dependent on the initial point chosen, hence to improve our results it would be necessary to recalculate the problem from many more initial

Table 4.1: Solutions obtained from 5 random initial points denoted  $(w_l^{(0)}, w_h^{(0)})$ 

<b>Inducing High Effort</b>				
$w_l^{(0)}$	$w_h^{(0)}$	$w_l^*$	$w_h^*$	<b>Exp. Profit</b>
0.5353	1.8188	0.4883	1.5360	0.4735
0.5570	1.9767	0.7755	1.3683	<b>0.5503</b>
0.5594	1.9461	0.7203	1.3938	0.5409
0.3292	2.1140	0.2178	1.8795	0.2528
0.5574	1.8361	0.5544	1.4880	0.4987

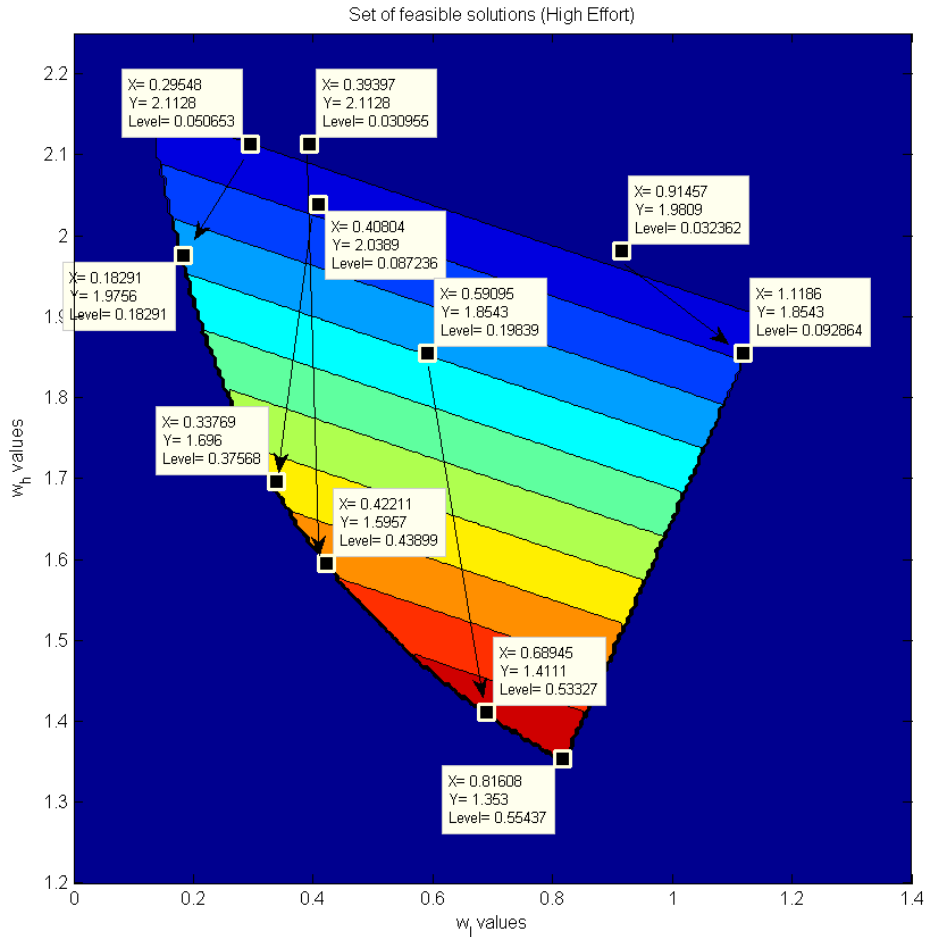
Table 4.2: Solutions obtained from 5 random initial points,  $(w_l^{(0)}, w_h^{(0)})$ 

<b>Inducing Low Effort</b>				
$w_l^{(0)}$	$w_h^{(0)}$	$w_l^*$	$w_h^*$	<b>Exp. Profit</b>
1.3563	1.4077	0.9582	1.5798	0.1932
1.3298	1.1336	0.9477	1.0839	<b>0.3978</b>
1.3591	1.3547	0.9167	1.5114	0.2454
1.0778	1.6460	1.0203	1.6822	0.1149
0.8459	1.3324	0.8268	1.3302	0.3718

points. If instead we generate 100 initial trial solutions, we find that the numerical results are even much closer to the exact solution. The optimal solution found for inducing high effort results in an expected profit of 0.5540 for the principal, while that for low effort produces an expected profit of 0.399998.

Figure 4.6 carries a visual sample of random initial points and the eventual ‘best’ solutions it finds after starting at these points. The arrows in the figure point to the eventual best result that the algorithm converges to from a given initial point. In the figure, the quantity  $X = w_l$ ,  $Y = w_h$  and the quantity *Level* refers to the principal’s expected profit.

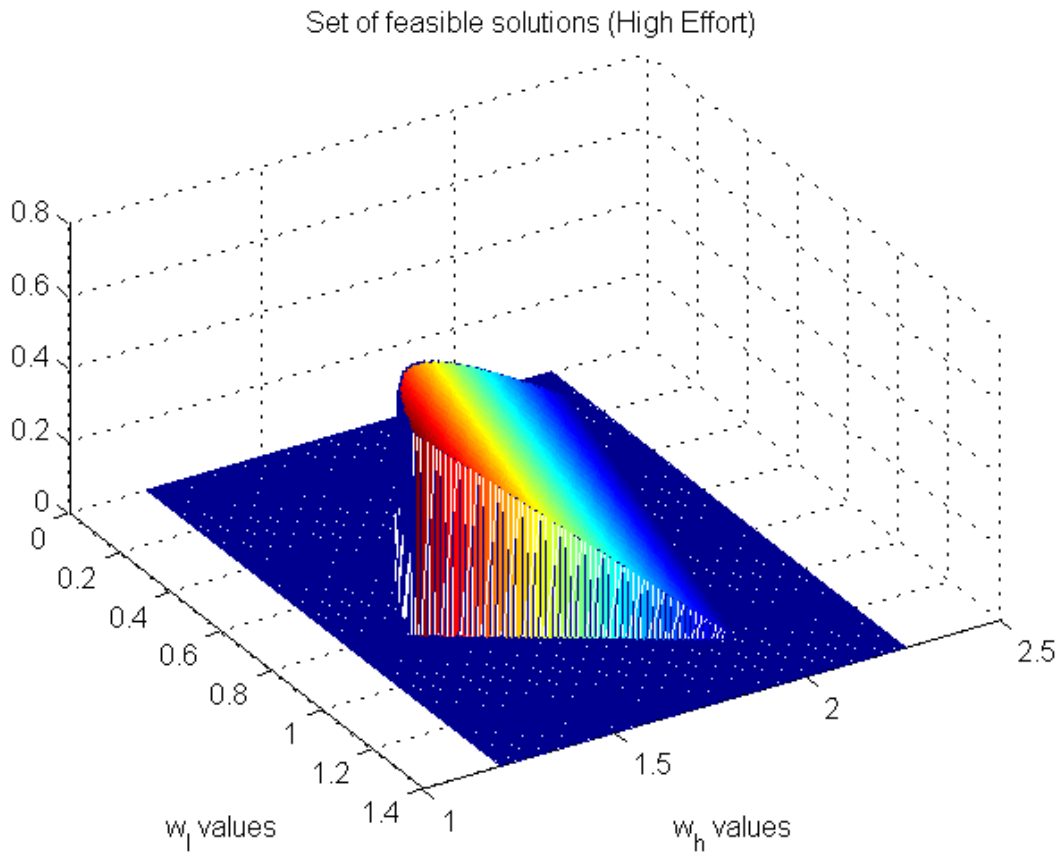
Figure 4.5: A sample of initial and final points obtained from implementing our search method where we assume the principal wishes to induce high effort.



## 4.2 Conclusion

In this and the previous chapter, we have analyzed what is a very basic form of the principal-agent model. We managed to completely solve the problem analytically and then worked on a numerical method for solving the problem. In obtaining both the analytical and numerical results, we adopted a two-step approach where, in the first step, we sought to maximize the principal’s expected profit assuming the agent implemented a specific effort level. In this case we only had to consider two distinct effort levels. In the second step, we then compared the principal’s expected profits from implementing each of the effort levels. The optimal contract

Figure 4.6: A surface plot of the principal's expected profit from inducing high effort.



was then chosen as the one that induced the agent to pick an effort level that would produce higher expected profits for the principal.

Equipped with this guiding solution strategy that has proven viable in the basic case, we can now transition to the more interesting principal-agent model presented in the form of our co-development problem. We begin the next chapter by formulating the model of the co-development partnership.

# Chapter 5

## The Co-Development Model

### 5.1 Mathematical Formulation of Co-development Contract

We adopt the same Principal-Agent problem modeling framework introduced in [35], where the principal (she) is the drug manufacturer and the agent (he) is the companion diagnostics company. The principal proposes a contract which stipulates the amounts of initial investment,  $m_p$  and  $m_a$ , that must be made by the principal and the agent respectively. The contract also specifies the royalty percentage,  $r$ , of total test sales revenue,  $R_T$ , that the agent will receive while the principal collects revenue from sale of the drug,  $R_D$  and the remaining share of test sales,  $(1 - r)R_T$ . The agent then decides whether he should accept the contract proposed by the principal and also the appropriate effort level,  $f_a$ , that he should apply based on the contract specifications.

The level of effort that the agent can exert is bounded above by  $f_L > 0$  and the effort is associated with a unit cost of  $c_a$ . If there is need for additional effort beyond  $f_L$ , or a decision is made to use an external workforce, then the external workforce would have to be hired at a unit cost of  $c_h$ , which is funded using the initial investments,  $m_a$  and  $m_p$ . We assume that  $c_h > c_a$ . Both the principal and the agent borrow the amounts they need to make the initial investment. The principal borrows her investment amount of  $m_p$  at a cost of  $b_p \geq 1$ , while the agent borrows his investment amount  $m_a$  at a cost of  $b_a \geq 1$ . Since the diagnostics company is a smaller firm, we assume it incurs a higher cost of borrowing than the principal hence we

assume  $b_a \geq b_p$ .

The probability of the project being a success is assumed to be a function,  $p(m_a, m_p, f_a)$ , that is dependent on the effort expended in the project and the amount of the initial investment made. If no initial investment is made, and the agent does no work, then the probability of success is zero (i.e.  $p(0, 0, 0) = 0$ ). The probability function is increasing in the variables  $f_a$ ,  $m_a$  and  $m_p$ , however there is a diminishing rate of increase which implies that the probability function is concave. We assume that the probability of success is described by the function :

$$p(f_a, m_a, m_p) = 1 - e^{-k\left(f_a + \frac{m_a + m_p}{c_h}\right)} \quad (5.1)$$

which has the properties just described. There is also another level of uncertainty attached to the project that is a result of factors such as scientific and regulatory uncertainty which are beyond the control of the agent. This uncertainty is modeled using a Bernoulli random variable,  $\theta$ , with  $\mu$  as its probability of success.

The principal's profit function consists of the expected revenue from the sales of the test and drug,  $(1 - r)R_T + R_D$ , less the costs associated with the borrowed initial investment,  $b_p m_p$ . The expected profit is described by the function  $E[\Pi]$  below. The agent's expected utility,  $E[U_A]$ , consists of his share of the expected revenues from the sale of the test,  $rR_T$ , less the repayment value of the previously borrowed investment amount,  $b_a m_a$ , and the cost of his effort,  $c_a f_a$ .

Under co-development, the goal of the pharmaceutical firm (principal) is to set the optimal values of  $m_a$ ,  $m_p$  and  $r$  so as to maximize her expected profit,  $E[\Pi]$  as described in equation (5.2). The principal cannot directly control or observe the agent's effort level,  $f_a$ , hence she must design a contract which will induce the agent to provide the optimal effort level that maximizes her profit. The principal therefore attempts to ensure that the agent will choose the desired effort level,  $f_a$ . The incentive compatibility constraint, (5.3), is added to the model to ensure that the optimal effort level coincides with the highest expected utility that the agent can obtain from participating in the co-development project. The agent will choose the effort level that maximizes his utility, which in turn will also maximize the principal's profit. The

co-development contract should also offer the agent enough incentives to accept the contract voluntarily. If these incentives are not present the agent will not work with or for the principal. This means that the minimal utility gained by the agent for participating in the project should be at least equal to the agent's reservation utility, denoted  $U_r$ . This requirement is referred to as the individual rationality constraint, (5.4). We refer to the model that describes the co-development problem as model  $D$ . This model serves as the base model upon which all further extensions will be based, and can be stated as:

$$(D) \quad \max_{m_a, m_p, r \geq 0} E[\Pi] = \mu \left( 1 - e^{-k(f_a^* + \frac{m_a + m_p}{c_h})} \right) ((1-r)R_T + R_D) - b_p m_p \quad (5.2)$$

s.t.

$$f_a^* = \operatorname{argmax}_{f_a \leq f_L} \left\{ E[U_A] = \mu \left( 1 - e^{-k(f_a + \frac{m_a + m_p}{c_h})} \right) r R_T - c_a f_a - b_a m_a \right\} \quad (5.3)$$

$$U_r \leq \mu \left( 1 - e^{-k(f_a^* + \frac{m_a + m_p}{c_h})} \right) r R_T - c_a f_a^* - b_a m_a. \quad (5.4)$$

To solve this co-development problem, we follow the approach used in [35] where the model was first introduced. We begin by considering the the agent's reservation utility and his optimal effort levels. We then use those results to determine the principal's expected profit.

### 5.1.1 Determining the agent's reservation utility, $U_r$

To determine the agent's reservation utility we consider first obtaining an upper bound,  $U_{max}$  and then a lower bound,  $U_{min}$  for the agent's reservation utility. Two modifications of model  $D$  are used to determine the bounds for  $U_r$ . These are model  $C$  which represents a coordinated problem and model  $A$  which is an agent-only problem.



### An upper bound on the agent's reservation utility, $U_{max}$

A value for the agent's maximum reservation utility,  $U_{max}$ , can be inferred from the solution of the coordinated problem where we assume that all profits generated from the contacts are actually given to the agent, with the principal getting none. The coordinated problem represents a scenario where we view the principal and the agent as a single entity. In this case the two parties do not have to divide the profits between the principal and the agent since we assume all profits are directed to one entity thus the agent's royalty,  $r$ , is not considered. The total revenues for the partnership will in this case consist of all of  $R_T$  and  $R_D$ . The profits are determined by deducting the cost of effort ( $c_a f_a$ ), and the costs and amount of borrowed capital for both the principal and the agent ( $b_p m_p + b_a m_a$ ).

Since under this model we assume that the agent and the principal are in partnership, we need not consider the incentive related constraints necessary to control the agent's actions. In this case we do not consider individual rationality and incentive compatibility. The problem can therefore be viewed as an optimization (maximization) problem with only non-negativity constraints, where we seek to determine the optimal values of  $m_a$ ,  $m_p$  and  $f_a$ . The optimal values are then used to set a value for the maximum reservation utility. The coordinated problem is described by model C below:

$$(C) \quad \max_{0 \leq f_a \leq f_L, m_a, m_p \geq 0} E[\Pi] = \mu \left( 1 - e^{-k(f_a + \frac{m_a + m_p}{c_h})} \right) (R_T + R_D) - c_a f_a - b_a m_a - b_p m_p. \quad (5.5)$$

Assuming the agent prefers applying minimal effort while the principal wishes to maximize likelihood of success, we have :

**Proposition 1** *The optimal solution of the Coordinated problem (C) is :*

$$f_a^C = \min \left\{ \max \left\{ 0, \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right) \right\}, f_L \right\} \quad (5.6)$$

$$m_a^C = 0 \quad (5.7)$$

$$m_p^C = \max \left\{ 0, c_h \left( \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{b_p c_h} \right) - f_L \right) \right\}. \quad (5.8)$$

Therefore an upper bound on the reservation utility is :

$$U_{max} = \mu \left( 1 - e^{-k \left( f_a^C + \frac{m_p^C}{c_h} \right)} \right) (R_T + R_D) - c_a f_a^C - b_p m_p^C. \quad (5.9)$$

All proofs are contained in the Appendix.

We recall that the agent's reservation utility refers to the potential payoff the agent is assumed to get if he chose to decline the principal's offer and pick another project instead. One implication of the result we just obtained (Proposition 1) is therefore that if the alternative project offers a potential reward greater than the maximum reservation utility as determined by Proposition 1 (5.1.1), then the agent will not even have to consider the principal's offer. This is because the most the agent can gain from accepting the principal's offer is still lower than the returns from the alternative choice the agent has.

#### **A lower bound on the agent's reservation utility, $U_{min}$**

The minimal utility that the agent should obtain to accept the co-development contract should be *at least* the maximum utility that he can get without contracting. Thus the maximum utility without contracting is equal to a lower bound for the agent's utility,  $U_{min}$ . We consider the agent-only model (Model A) where the agent tries to determine the optimal  $f_a$  and  $m_a$  to maximize his expected utility in the absence of the principal's participation. In this case we also do not consider the individual rationality and incentive compatibility constraints. We can therefore view the problem as an unconstrained optimization problem with non-negativity requirements for  $f_a$  and  $m_a$ . The agent-only problem is described by model A below:

$$(A) \quad \max_{0 \leq f_a \leq f_L, m_a \geq 0} E[\Pi] = \mu \left( 1 - e^{-k \left( f_a + \frac{m_a}{c_h} \right)} \right) R_T - c_a f_a - b_a m_a. \quad (5.10)$$

**Proposition 2** *The optimal solution of the Agent-only problem (A) is :*

$$f_a^A = \min \left\{ \max \left\{ 0, \frac{1}{k} \ln \left( \frac{k\mu R_T}{c_a} \right) \right\}, f_L \right\} \quad (5.11)$$

$$m_a^A = \max \left\{ 0, c_h \left( \frac{1}{k} \ln \left( \frac{k\mu R_T}{b_a c_h} \right) - f_L \right) \right\}. \quad (5.12)$$

*The agent's minimal reservation utility should therefore be :*

$$U_{min} = \mu \left( 1 - e^{-k \left( f_a^A + \frac{m_p^A}{c_h} \right)} \right) R_T - c_a f_a^A - b_a m_a^A. \quad (5.13)$$

## 5.2 Solutions to the Co-development Model

### 5.2.1 First Best Solution

To obtain the first best solution, we consider a relaxed form of model  $D$  where  $f_a$  is determined by the principal but the agent only accepts the co-development contract i.e. the Individual Rationality constraint must be satisfied but the Incentive Compatibility Constraint does not have to be satisfied. We label this model  $DF$ :

$$(DF) \quad \max_{0 \leq f_a \leq f_L, m_a, m_p, r \geq 0} E[\Pi] = \mu \left( 1 - e^{-k \left( f_a + \frac{m_a + m_p}{c_h} \right)} \right) ((1-r)R_T + R_D) - b_p m_p \quad (5.14)$$

s.t.

$$U_r \leq \mu \left( 1 - e^{-k \left( f_a + \frac{m_a + m_p}{c_h} \right)} \right) r R_T - c_a f_a - b_a m_a. \quad (5.15)$$

Since the principal intends to maximize her profit while satisfying the individual rationality constraint, it must be that in the first best scenario constraint (5.15) is binding, and hence is an equality. The equality can be rearranged to obtain  $r$ . Therefore the value of  $r$  set by the principal is

$$r = \frac{U_r + c_a f_a + b_a m_a}{\mu \left( 1 - e^{-k \left( f_a + \frac{m_a + m_p}{c_h} \right)} \right) R_T}. \quad (5.16)$$

Replacing  $r$  in the objective function with (5.16) and simplifying, we obtain an updated objective function :

$$\max_{0 \leq f_a \leq f_L, m_a, m_p, r \geq 0} E[\Pi] = \mu \left( 1 - e^{-k(f_a + \frac{m_a + m_p}{c_h})} \right) (R_T + R_D) - U_r - c_a f_a - b_a m_a - b_p m_p. \quad (5.17)$$

Comparing model  $C$ , (5.5), with the updated objective function of model  $DF$ , (5.17), we note that the only difference between the two is the  $-U_r$  term which is present in model  $DF$  but not in model  $C$ .  $U_r$  is not one of the decision variables but a constant within the objective function hence it does not affect the derivatives with respect to  $m_a$ ,  $m_p$  and  $f_a$  which are used to obtain the optimal solutions. It therefore follows that the optimal solution to model  $DF$  is identical to that of the coordinated problem, model  $C$  with the optimal value of  $r$  being as described in (5.16). The optimal solution to the problem is therefore the one described in Proposition 3 below.

**Proposition 3** *The optimal solution to model  $DF$  is :*

$$f_a^{DF} = \min \left\{ \max \left\{ 0, \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right) \right\}, f_L \right\} \quad (5.18)$$

$$m_p^{DF} = \max \left\{ 0, c_h \left( \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{b_p c_h} \right) - f_L \right) \right\} \quad (5.19)$$

$$m_a^{DF} = 0 \quad (5.20)$$

$$r^{DF} = \frac{U_r + c_a f_a^{DF} + b_a m_a^{DF}}{\mu \left( 1 - e^{-k \left( f_a^{DF} + \frac{m_a^{DF} + m_p^{DF}}{c_h} \right)} \right) R_T}. \quad (5.21)$$

There are three possible cases we can consider for the resulting net profit,  $E[\Pi]$ . If  $\frac{k\mu(R_T + R_D)}{c_a} \leq 1$ , then  $f_a^{DF} = 0$ . Since  $b_p c_h > c_a$ , then  $\frac{k\mu(R_T + R_D)}{c_a} \leq 1$  implies  $\frac{k\mu(R_T + R_D)}{b_p c_h} < 1$ , hence  $m_p^{DF} = 0$ . In this case  $E[\Pi] = -U_r$ .  $E[\Pi]$  is therefore only non-negative for  $U_r = 0$ .

If  $\frac{k\mu(R_T + R_D)}{c_a} > 1$  and  $\frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right) < f_L$  then  $f_a^{DF} = \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right) < f_L$  and  $m_p^{DF} = 0$ . Under this scenario the expected net profit is given by

$$E[\Pi] = \mu(R_T + R_D) - \frac{c_a}{k} - U_r - \frac{c_a}{k} \ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right).$$

The third case to consider is where  $\frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right) > f_L$ . In this case  $f_a^{DF} = f_L$  and  $m_p^{DF} = c_h \left( \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{b_p c_h} \right) - f_L \right)$ . The corresponding expected profit is

$$E[\Pi] = \mu(R_T + R_D) - \frac{b_p c_h}{k} - U_r - c_a f_L - b_p c_h \left( \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{b_p c_h} \right) - f_L \right).$$

### 5.2.2 Second Best Solution

The second best solution corresponds to an optimal co-development contract which ensures that the agent obtains at least his reservation utility if he participates and also induces the agent to put in an optimal level of effort into the the project. The co-development model, model  $D$ , represents the second best scenario. Solutions to  $(D)$  are partitioned according to the agent's optimal effort level,  $f_a$ , which is the effort level that maximizes the agent's utility. The optimal effort level,  $f_a^*$ , may fall into one of three possible regions which are :

$$(i) f_a^* = 0 \qquad (ii) 0 < f_a^* < f_L \qquad (iii) f_a^* = f_L.$$

The optimal solution to the co-development problem is therefore as described by Proposition 4.

**Proposition 4** *The optimal solution of the Co-development problem (D) is the one with the highest expected profit among the candidate solutions in Table 5.1.*

Table 5.1: Optimal Solutions to Co-Development Model (Second Best Solution)

$f_a$ range	Soln #	Required Conditions	$m_a$	$m_p$	$r$	$f_a^*$
$f_a^* = 0$	1a	$\frac{k\mu(R_T + R_D)}{b_p c_h} \leq 1$ ; $U_r = 0$	0	0	0	0
	1b	$\frac{kU_r}{c_a} + 1 \leq \frac{k\mu(R_T + R_D)}{b_p c_h}$ ; $\frac{k\mu(R_T + R_D)}{b_p c_h} > 1$	0	$\frac{c_a}{k} \ln \left( \frac{k\mu(R_T + R_D)}{b_p c_h} \right)$	$\frac{U_r}{\left(1 - \frac{b_p c_h}{k\mu(R_T + R_D)}\right) \mu R_T}$	0
$0 < f_a^* < f_L$	2a	$\sqrt{\frac{k\mu(R_T + R_D)}{c_a} - \ln \left( \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \right)}$ ; $1 < \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} < e^{kf_L}$	0	0	$\sqrt{\frac{c_a(R_T + R_D)}{k\mu R_T^2}}$	$\frac{1}{k} \ln \left( \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \right)$
	2b	$\frac{kU_r}{c_a} + kf_L + 1 < e^{kf_L}$	0	0	$-\frac{c_a}{k\mu R_T} W \left( -e^{-\frac{kU_r}{c_a}} \right)$ where $r > \frac{c_a}{k\mu R_T}$ and $W(\cdot)$ is the Lambert W function	$\frac{\mu R_T}{c_a} - \frac{U_r}{c_a} - \frac{1}{k}$
$f_a^* = f_L$	3a.i	$\frac{k\mu(R_T + R_D)}{b_p c_h} \leq e^{kf_L} \leq \frac{kU_r}{c_a} + kf_L + 1$	0	0	$\frac{U_r + c_a f_L}{(1 - e^{-kf_L}) \mu R_T}$	$f_L$
	3a.ii	$e^{kf_L} \leq \frac{kU_r}{c_a} + kf_L + 1$ $e^{kf_L} < \frac{k\mu(R_T + R_D)}{b_p c_h}$	0	$c_h \left( \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{b_p c_h} \right) - f_L \right)$	$\frac{U_r + c_a f_L}{\left(1 - \frac{b_p c_h}{k\mu(R_T + R_D)}\right) \mu R_T}$	$f_L$
		$e^{kf_L} \leq \sqrt{\frac{k\mu(R_T + R_D)}{c_a}}$ ; $\frac{kU_r}{c_a} + kf_L + 1 + \frac{b_p c_h}{b_a} \left( \frac{1}{k} \ln \left( \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \right) - f_L \right) \leq \sqrt{\frac{k\mu(R_T + R_D)}{c_a}}$	$c_h \left( \frac{1}{k} \ln \left( \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \right) - f_L \right)$	0	$\sqrt{\frac{c_a(R_T + R_D)}{k\mu R_T^2}}$	$f_L$
	3b	$\frac{kU_r}{c_a} + kf_L + 1 > e^{kf_L}$	0	$c_h \left( \frac{1}{k} \ln \left( \frac{kU_r}{c_a} + kf_L + 1 \right) - f_L \right)$	$\frac{c_a}{k\mu R_T} \left( \frac{kU_r}{c_a} + kf_L + 1 \right)$	$f_L$
	3c					

### 5.3 Conclusion

Following the work done in [35], we established a working model for our co-development problem and we stated propositions 1 and 2 which gave us reasonable working lower and upper bounds for the agent's reservation utility,  $U_r$ . Bounds on  $U_r$  essentially impose regions outside of which we may not expect feasible solutions to the problems. With these constraints in mind, we then proceeded to investigate the nature of the optimal contracts to the co-development problem, through solving the principal-agent model we have formulated.

We highlight that we were able to deduce analytical solutions for our problem mainly because we chose a conveniently structured function to describe the probability of success. This function we chose essentially made our problem convex in nature and hence more tractable. We also assumed fixed values of the borrowing costs,  $b_a$  and  $b_p$ . This may not necessarily reflect reality as borrowing costs may be non-constant functions of the amount borrowed.

We begin in the next section by solving the base co-development model numerically. We develop an algorithm for solving this problem and test its result against those obtained analytically. Once we verify that the numerical approach works well we can then adopt the method for other functions that describe the probability of success and the borrowing costs.

# Chapter 6

## Numerically Solving for Optimal Co-development Contracts

Development and testing of a numerical algorithm to the co-development model is the primary novel contribution of this thesis. While chapter 5 contained analytical solutions of the co-development problem first presented in [35], in this chapter we present a numerical method for solving the same problem. The numerical method for solving the base co-development model will allow us to go beyond the base model and solve modified versions of the model for which analytical results cannot be obtained. We begin with a discussion of the parameters we use for our numerical examples.

### 6.1 Setting Parameter Values

For our numerical examples we assume a unit of labour corresponds to an hour of work, with all work being carried out over the period of a year. We now describe the parameters we use for our base analysis. We set the unit cost of labour for the agent's internal workforce,  $c_a$ , at US\$40 per hour which is consistent with the 2012 average hourly wage rates of biological scientists and biochemists within the pharmaceutical and medicine manufacturing industry [40]. We assume the cost associated with the use of an external workforce is  $c_h = 1.25 \times c_a$ . According to a report by the Biotechnology Industry Organization [36], the average biotech company has fewer than 50 employees with 71% of them actually having fewer than 25 employees. Based



on this description, we suppose in our numerical example that the agent has an internal workforce of 25 individuals which can contribute a maximum effort level of  $f_L = 50,000$  units of labour over a year (40 hours a week over 50 weeks).

For the expected revenues from the test,  $R_T$ , and the drug sales,  $R_D$ , we adopted the estimated 2015-2019 annual sales figures of the melanoma drug vemurafenib (\$372 million) and its associated diagnostic test (\$10 million) [2]. Developed by Roche, vemurafenib is a targeted therapy for melanomas containing a particular protein mutation known as V600E BRAF. Upon noticing the effects the drug had on melanomas containing the mutation, Roche began a co-development partnership with diagnostics firm Plexxikon. The companion diagnostic to detect the V600E BRAF mutation was developed and approved in parallel with vemurafenib [9]. In our work, we make the assumption that the diagnostic test is drug-specific and will only be approved or cleared simultaneously with the companion drug as is recommended by the FDA [41]. We suppose that the need for a companion diagnostic is based on phase 2 results of the clinical trials [14] and the drug is about to enter phase 3, hence to represent uncertainty around the transition of the diagnostic from initial development to launch we set  $\mu$  equal to the product of the average probabilities of a drug entering phase 3 transitioning to launch (0.637) [42] and the successful development of companion a diagnostic(0.2) [2]. Therefore we set  $\mu = 0.1274$ . We will consider different values of  $k$ ,  $f_L$  and  $U_r$  to create different scenarios.

Table 6.1: A summary of our experimental parameters.

$f_L = 50,000$	$\mu = 0.1274$	$R_T = 10M$	$R_D = 372M$
$c_a = 40$	$c_h = 1.25 \cdot c_a$	$b_a = 1.07$	$b_p = 1.05$

The approach used to solve the problem numerically is parallel to that used earlier for the basic two actions, two output model in chapter 3. However, in this case we have three decision variables for the principal which are  $m_a$ ,  $m_p$  and  $r$ . All numerical calculations are carried out in C++ with the resulting data plotted using MATLAB. In calculating the optimal contracts based on the analytical results in Table 5.1, some of the results required use of the Lambert

W function. We made use of published C++ code available through the Computer Physics Communications Program Library [39]. We outline the key factors of the numerical approach in the following section.

### 6.1.1 The Numerical Approach

As in section 4.1.3 for the basic principal agent model, we rewrite the entire co-development model (the objective function and constraints) into an equivalent single expression to be optimized. We also need to specify some limited region in which we can search for the optimal solutions numerically. The reasonable space within which we can search for  $m_a^*$ ,  $m_p^*$  and the quantity  $r \times R_T$  is the interval  $[0, (R_T + R_D)]$ . This is equivalent to requiring that the largest amount of money to be invested be between 0 and the total anticipated revenue  $R_T + R_D$ . This also implies that the agent's total wage from the project,  $(r \times R_T)$  must be between 0 and  $R_T + R_D$ . These restrictions provide upper bounds on the decision variables and allow use of certain types of algorithms.

Unlike in the basic model of section 3.1 where all the constraints were inequalities, we note that in the co-development model, this is not necessarily the case. As we did while deriving the analytical solutions, we again note that the incentive compatibility constraint in the co-development problem can be considered as an equality that corresponds to the optimal value of  $f_a$ . There are three different scenarios we need to consider. For any given contract, the agent's optimal response in terms of  $f_a$  will satisfy at least one of the following three equalities :

$$i.) f_a^* = 0, \quad ii.) \frac{\partial E[U_A]}{\partial f_a} = 0, \quad iii.) f_a^* = f_L.$$

Each of these three equalities can be used in constructing objective functions that we can maximize to solve our problem. We have three such functions which we introduce below and refer to each of them as  $Y_i$ , for  $i = 1, 2, 3$ .

As we did for the basic PA problem in chapter 3, transformation of the original problem with multiple constraints into an unconstrained problem requires rewriting each of the constraints into different but related forms that are then embedded into  $Y_i$ . Each inequality of the

form  $g_k(x) \geq b_k$  is incorporated into  $Y_i$  as  $-\Lambda \cdot \frac{1}{g_k(x) - b_k}$  while an inequality of the form  $h_k(x) \leq d_k$  is transformed into  $-\Lambda \cdot \frac{1}{d_k - h_k(x)}$  where  $\Lambda$  is some positive number. Directly applying these transformation rules would imply that a non-negativity constraint of the form  $x_i \geq 0$  would be transformed into  $\frac{1}{x_i}$ . We note that in this form, our method will only produce results for the decision variables that are strictly positive. In order to allow for the decision variables to take the value 0, we transform the non-negativity constraint  $x_i \geq 0$  into  $\Lambda \cdot \frac{1}{e^{x_i}}$ . Equality constraints of the form  $q_k(x) = r_k$  are transformed into  $-\frac{[r_k - q_k(x)]^2}{\sqrt{\Lambda}}$  as suggested in [34]. With these rules in mind, we describe the three cases of  $Y_i$  that we will be optimizing.

**Case I,  $f_a^* = 0$**

When the agent's optimal action is  $f_a^* = 0$  we recall that this implies a scenario where

$$\frac{1}{k} \ln \left( \frac{k\mu R_T r}{c_a} \right) - \frac{m_a + m_p}{c_h} \leq 0.$$

Taking the equality constraint  $f_a^* = 0$  into account the corresponding objective function which we seek to optimize is:

$$\begin{aligned} Y_1 = & E[\Pi] - \Lambda \cdot \left( \frac{1}{\frac{m_a + m_p}{c_h} - \frac{1}{k} \ln \left( \frac{k\mu R_T r}{c_a} \right)} \right) \\ & - \Lambda \cdot \left( \frac{1}{-U_r + E[U_A]} \right) \\ & - \Lambda \cdot \left( \frac{1}{R_T + R_D - m_a} + \frac{1}{R_T + R_D - m_p} + \frac{1}{R_T + R_D - rR_T} + \frac{1}{e^{m_a}} + \frac{1}{e^{m_p}} \right). \end{aligned} \quad (6.1)$$

**Case II,  $0 \leq f_a^* \leq f_L$**

When the agent's optimal action is  $0 \leq f_a^* \leq f_L$ , this implies that the agent's optimal effort level coincides with the equality  $\frac{\partial E[U_A]}{\partial f_a} = 0$ . The resulting objective function we seek to

optimize in this case is

$$\begin{aligned}
Y_2 = E[\Pi] - \Lambda^{-0.5} & \left( \frac{\partial E[U_A]}{\partial f_a} \right)^2 \\
& - \Lambda \cdot \left( \frac{1}{-U_r + E[U_A]} \right) \\
& - \Lambda \cdot \left( \frac{1}{R_T + R_D - m_a} + \frac{1}{R_T + R_D - m_p} + \frac{1}{R_T + R_D - rR_T} + \frac{1}{e^{m_a}} + \frac{1}{e^{m_p}} \right). \quad (6.2)
\end{aligned}$$

**Case III,**  $f_a^* = f_L$

When the agent's optimal response to a proposed contract is  $f_L$ , then it must be that

$$\frac{1}{k} \ln \left( \frac{k\mu R_T r}{c_a} \right) - \frac{m_a + m_p}{c_h} \geq f_L.$$

The function we seek to maximize is therefore represented as

$$\begin{aligned}
Y_3 = E[\Pi] - \Lambda \cdot & \left( \frac{1}{-\frac{m_a + m_p}{c_h} + \ln \left( \frac{k\mu R_T r}{c_a} \right)} \right) \\
& - \Lambda \cdot \left( \frac{1}{-U_r + E[U_A]} \right) \\
& - \Lambda \cdot \left( \frac{1}{R_T + R_D - m_a} + \frac{1}{R_T + R_D - m_p} + \frac{1}{R_T + R_D - rR_T} + \frac{1}{e^{m_a}} + \frac{1}{e^{m_p}} \right). \quad (6.3)
\end{aligned}$$

### Solution Procedure

We now solve our problem using the methods analogous to those used for solving the basic two-actions, two-output model in section 4.1.3. We note that the above proposed objective functions (6.1, 6.2, 6.3), can be viewed as implying that the principal is considering three different 'types' of effort levels that the agent can implement. The principal then has to find the optimal values of  $m_a$ ,  $m_p$  and  $r$  that maximizes her profit given the type of effort that the agent implements.

The proposed objective functions already have the agent's reservation utility,  $U_r$ , embedded in them. We have included a term in the each of the  $Y_i$ 's that should ensure that the agent's

expected utility is at least his reservation utility. Any of the optimal solutions found from maximizing the objective functions above is therefore expected to satisfy the individual rationality constraint.

Once having determined the optimal contract values for each of the three cases of effort level, the principal then needs to check which among these three contracts satisfy the incentive compatibility constraint. To check whether the incentive compatibility constraint is satisfied, the principal needs to verify that given the contract values  $m_a$ ,  $m_p$  and  $r$ , the agent has no incentive to deviate from the effort level the principal is trying to induce. In other words, the values of  $m_a$ ,  $m_p$  and  $r$  should be such that the agent can not obtain greater utility from implementing a different effort level from the one the principal wishes the agent to implement. The contract the principal chooses to implement will therefore be the one that maximizes her expected profit among the ones that satisfy the incentive compatibility constraint.

### **Selecting a contract that satisfies incentive compatibility**

Suppose the principal prefers that the agent implement effort level  $f_a = \tilde{f}_a$ . If the incentive compatibility constraint is satisfied, then it means that given the contract values of  $m_a$ ,  $m_p$  and  $r$ , the agent should not be able to obtain higher expected utility from implementing an effort level different from  $\tilde{f}_a$ . We check to see whether a given contract satisfies these conditions using a random process which we outline below. This process is implemented for each of the three solutions obtained for the cases  $f_a^* = 0$ ,  $0 \leq f_a^* \leq f_L$  and  $f_a^* = f_L$ . Each of these three solutions consists of the four numerically obtained values of  $m_a$ ,  $m_p$ ,  $r$  and  $\tilde{f}_a$ .

#### Checking for incentive compatibility

1. Calculate,  $\tilde{U} = E[U_A(\tilde{f}_a)]$ , the agent's expected utility from implementing  $\tilde{f}_a$  given  $m_a$ ,  $m_p$  and  $r$ .
2. Randomly select  $N$  values of  $f_a$  from an interval of desired width,  $W$ . The interval should contain  $\tilde{f}_a$  and can, at its widest, be the interval  $[0, f_L]$ . In our work we use  $N = 30$  and an interval of width  $f_L/10$ . In section 6.2.1 we discuss why a much narrower interval than  $[0, f_L]$  may be better.

3. For each of the  $N$  random  $f_a$  values, calculate the corresponding expected utility for the agent given the values of  $m_a$ ,  $m_p$  and  $r$ .
4. Count the number of times,  $n$ , that  $\tilde{U} + tol$  is exceeded by the expected utilities obtained from any of the random  $f_a$  values.
5. If  $n$  exceeds a specific threshold value then conclude that the current solution violates the incentive compatibility constraint since the agent is able to achieve greater utility from implementing a different effort level than  $\tilde{f}_a$ . We use  $n = N/3$  in our work.

### Summary of Procedure

The first part of the method is to transform the co-development model into three unconstrained optimization problems each corresponding to the three possible forms of optimal effort:  $f_a^* = 0$ ,  $0 \leq f_a^* \leq f_L$  and  $f_a^* = f_L$ . The transformation involves subtracting a penalty function representing the constraints from the original objective function. The penalty function is multiplied by some positive number,  $\Lambda$ , to represent the original constraints. We use a sequence of successively smaller values of  $\Lambda$  and for each of these values of  $\Lambda$  we perform the following steps:

1. Transform the multiple constraint co-development model into an unconstrained maximization problem.
2. Identify a region within which all feasible candidate solutions,  $(m_a, m_p, r)$ , are contained. Feasible candidate solutions should satisfy IR and produce a positive profit for the principal.
3. Identify a feasible initial trial solution.
4. Determine the direction of steepest ascent from the current trial solution.
5. Move as far as feasible in the direction of steepest ascent (a feasible move is one that yields a trial solution that satisfies the IR constraint and produces positive profit for the principal).
6. Once at the next trial solution, return to step 4 and repeat until a desired number of iterations is reached or until the level of improvement in the objective value is sufficiently small.

As  $\Lambda \rightarrow 0$ , the obtained solutions should converge to some point which we then take as the optimal solution for the assumed form of  $f_a^*$ . Test the three solutions to determine the ones that satisfy the incentive compatibility constraint. Of these, the optimal contract is selected as the one that yields the highest expected profit for the principal.

## 6.2 Numerical Results

Tables 6.2 to 6.8 below contain a sample of the solutions to the co-development problem obtained using our numerical approach. The first row of each table shows the analytical solutions derived from Table 5.1 for a given set of parameters. The second row, shows the solution obtained under the assumption that  $f_a^* = 0$ , the second row is for  $0 \leq f_a^* \leq f_L$  and the last row is for  $f_a^* = f_L$ . Our base parameters are given in Table 6.1. We change two or three of these parameters in each experiment in order to see how our numerical method compares to the analytical solution. Of the three solutions obtained numerically, we then choose the one that yields the highest expected profit for the principal amongst the ones that satisfy the incentive compatibility constraint. We indicate the chosen contract with the double asterisk \*\*.

Table 6.2: Numerical Results :  $f_L = 5250$ ;  $U_r = 23,250,000$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$	IC Satisfied?
Analytical	5250	0	1,781,747	18.73	23,250,000	22,520,093	Yes
Numerical, $f_a = 0$	0	443,086	1,520,830	19	23,250,004	22,387,753	No
Numerical, $\frac{\partial E[U_A]}{\partial f_a} = 0$	5250	1,568,544	184,668	20.09	23,250,007	22,470,761**	Yes
Numerical, $f_a = f_L$	5250	582,851	1,481,309	19.09	23,250,013	22,563,972	No

Table 6.3: Numerical Results :  $f_L = 40,000$ ;  $U_r = 1,750,000$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$	IC Satisfied?
Analytical	24,006	0	0	3.463	3,051,853	40,242,579	Yes
Numerical, $f_a = 0$	0	3,035,363	2,910,152	4.134	1,750,012	38,123,397	Yes
Numerical, $\frac{\partial E[U_A]}{\partial f_a} = 0$	20,340	463,108	4	3.026	2,145,513	40,161,952**	Yes
Numerical, $f_a = f_L$	40,000	2,862,235	111,514	5.054	1,750,001	41,935,600	No



Table 6.4: Numerical Results :  $f_L = 10, 250$ ;  $U_r = 23, 250, 000$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$	IC Satisfied?
Analytical	10,250	0	1,535,921	18.89	23,250,000	22,584,993	Yes
Numerical, $f_a = 0$	0	1,195,999	1,285,979	19.39	23,250,022	22,446,833	Yes
Numerical, $\frac{\partial E[U_A]}{\partial f_a} = 0$	0	384,876	1,702,526	18.86	23,250,007	22,468,915**	Yes
Numerical, $f_a = f_L$	10,250	244,819	1,315,505	19.08	23,250,001	22,593,013	No

Table 6.5: Numerical Results :  $f_L = 40, 000$ ;  $U_r = 6, 000, 000$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$	IC Satisfied?
Analytical	29,413	0	0	5.947	6,000,000	38,920,919	Yes
Numerical, $f_a = 0$	0	2,727,110	3,543,918	7.318	6,000,010	33,911,762	Yes
Numerical, $\frac{\partial E[U_A]}{\partial f_a} = 0$	28,226	396,374	4	6.439	6,249,854	38,490,662**	Yes
Numerical, $f_a = f_L$	40,000	1,615,533	41,764	7.382	6,000,365	38,904,759	No

Table 6.6: Numerical Results :  $f_L = 30, 000$ ;  $U_r = 4, 750, 000$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$	IC Satisfied?
Analytical	0	0	2,264,677	3.769	4,750,000	41,013,889	Yes
Numerical, $f_a = 0$	0	2,259,851	80	5.71	4,777,389	40,941,279**	Yes
Numerical, $\frac{\partial E[U_A]}{\partial f_a} = 0$	0	2,420,198	13	7.314	6,655,302	39,037,223	No
Numerical, $f_a = f_L$	30,000	3,345,867	4	7.487	4,757,149	39,126,562	No

Table 6.7: Numerical Results :  $f_L = 20,000$ ;  $U_r = 2800$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[II]$	IC Satisfied?
Analytical	0	0	2,264,677	0.00222	2800	45,761,0897	Yes
Numerical, $f_a = 0$	0	906,803	1,419,703	0.7712	2807	45,739,092**	Yes
Numerical, $\frac{\partial E[U_A]}{\partial f_a} = 0$	0	3,299,495	80	3.929	1,468,368	43,601,627	No
Numerical, $f_a = f_L$	20,000	1,405,373	0	1.832	11,239	45,955,577	No

Table 6.8: Numerical Results :  $f_L = 22,000$ ;  $U_r = 33,000,000$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[II]$	IC Satisfied?
Analytical	22,000	0	1,125,426	26.91	33,000,000	13,037,228	Yes
Numerical, $f_a = 0$	0	1,119,944	1,070,532	27.18	33,000,010	12,735,402	No
Numerical, $\frac{\partial E[U_A]}{\partial f_a} = 0$	22,000	977,221	162,599	27.73	33,000,010	13,018,675	Yes
Numerical, $f_a = f_L$	22,000	313,565	797,994	27.18	33,000,001	13,029,546**	Yes

### 6.2.1 Analysis of the numerical results

We observe that in all cases explored so far and particularly those shown in the tables above, the numerically obtained optimal profits for the principal and the corresponding utility values for the agent are identical to those obtained analytically. What is interesting to note, however, is that the two methods regularly produce significantly different optimal values of  $m_a$ ,  $m_p$ ,  $f_a$  and  $r$  that ultimately yield identical expected payoffs for both the principal and agent.

#### Do the results make sense?

We can check graphically whether the solutions we obtained are reasonable. In order to make sense of the graphs, we need to keep in mind that the algorithm we have implemented works by finding the values of  $m_a$ ,  $m_p$  and  $r$  that maximize the principal's expected profit when  $f_a^*$  is fixed at 0, at  $f_L$  and at a point between 0 and  $f_L$  where  $\frac{\partial E[U_A]}{\partial f_a} = 0$ . Now the effect of  $m_a$ ,  $m_p$  and  $r$  is to determine the shape of the function that describes the agent's expected utility. The principal's goal is to offer the agent a contract with the values of  $m_a$ ,  $m_p$  and  $r$  such that the agent obtains his highest expected utility at the same effort level that the principal would like to induce. The principal wishes to induce an effort level that maximizes her expected profits. In order for this contract to be acceptable to the agent, the agent's expected utility from the contract should also exceed his reservation utility.

To completely demonstrate the graphical analysis of a solution, we consider here the fourth of our tabulated examples. This is the case whose results are carried in Table 6.5 on page 60. Now, for each of the proposed values of  $m_a$ ,  $m_p$  and  $r$ , we plot the agent's expected utility as a function of his effort level on the interval  $[0, f_L]$ . This will give us an indication of how 'good' a contract is from the principal's perspective.

**The analytical solution:**  $f_a^* \approx 29,413$  ( $m_a = 0$ ,  $m_p = 0$ ,  $r \approx 5.947$ )

This corresponds to the optimal solution obtained using the analytical results in Table 5.1 on page 49. The graphs in Figure 6.1 show the principal's expected profit and the agent's expected utility as functions of  $f_a$ . What we notice is that while the principal's profit function is entirely

increasing with effort, this is not the case with the agent's utility. The agent's utility function is increasing up to a point and decreasing afterward. His expected utility reaches its maximum at point  $f_a \approx 29,413$ .

It is interesting to note that the agent's utility in a wide interval about  $f_a = 29,413$  is virtually flat. In a way, this implies that the agent may be less sensitive to deviating from  $f_a = 29,413$  since his utility is somewhat constant around this region,  $f_a \in [26,000, 34,000]$ . In this sense it becomes clear then that an even better contract from the principal's perspective would be one that yields the same expected profit but with a much narrower region within which the agent can deviate without incurring a significant loss in utility. This is the reason why in comparing different numerically determined contracts, we settle on picking the one with the narrowest range of  $f_a$  within which the agent can deviate. While, this is not really an issue for the analytical as we can determine the exact optimal  $f_a$  value, this matters more in the numerical solutions when all the contract values are approximations.

**First numerical solution:**  $f_a^* = 0$  ( $m_a \approx 2,727,110, m_p \approx 3,543,918, r \approx 7.318$ )

We see in Figure 6.2 that our method does find values of  $m_a, m_p$  and  $r$  that maximize the agent's utility at  $f_a = 0$ . This contract also guarantees the agent his reservation utility for implementing the effort level  $f_a = 0$ . In this case the individual rationality constraint is seen to be binding. We observe too that for all effort levels near  $f_a = 0$ , the agent's expected utility is lower than that at  $f_a = 0$ . Hence we can conclude that the incentive compatibility constraint is satisfied. If it were that none of the other two proposed contracts offered better returns for the principal, then this would be the optimal contract.

**Second numerical solution:**  $0 \leq f_a^* \leq f_L$  ( $f_a^* \approx 28,226, m_a \approx 396,374, m_p \approx 4, r \approx 6.439$ )

First, we note that the obtained value of  $m_p \approx 4$  in this case is not too different from the analytically obtained value of  $m_p = 0$ . We recall that  $m_a$  and  $m_p$  represent the amounts that are borrowed to finance external effort at unit cost  $c_h$ . Now in this case, the parameters we have are such that  $c_h = 5 \cdot c_a = 200$ . Hence  $m_a = 396,374$  translates to  $396,374/200 \approx 1982$  additional

units of external effort. The total amount of effort expended in this contract is thus not too different from the one derived analytically. This solution can therefore be seen to resemble the one obtained analytically.

In terms of the individual rationality constraint, the agent is guaranteed his reservation utility by accepting this contract. We note however, that unlike the analytical solution, in this case the agent will obtain more than his reservation utility by implementing  $f_a^*$ .  $f_a^*$  also maximizes the agent's utility hence this solution satisfies the incentive compatibility constraint.

**Third numerical solution:**  $f_a^* = f_L$  ( $f_a^* = 40,000, m_a \approx 1,697,499, m_p \approx 4, r \approx 7.4496$ )

What we note first about this solution is that of the three numerical contracts, this one offers the principal the highest expected profit. We then only need to check whether this contract provides the agent with an amount that is at least his reservation utility. This contract seeks to induce an effort level of  $f_a = f_L = 40,000$ . We note that for all feasible effort levels that are upwards of about  $f_a = 12,500$ , the agent will indeed obtain an amount that is at least his reservation utility.

While the principal prefers that the agent implement  $f_a = f_L$ , we see from the graph of his expected utility that the agent can actually improve his expected utility by implementing an effort level significantly different from  $f_L$ . In this case, given  $m_a, m_p$  and  $r$  as described above, the agent's optimal response is not to work at level  $f_a = f_L$  but instead to implement  $f_a \approx 23,000$ . This therefore implies that this solution does not satisfy the incentive compatibility constraint.

### Picking the best contract

Between the two contracts above that satisfied both the individual rationality and incentive compatibility constraints, we now pick the one that provides the principal with higher expected profit. Comparing the expected profit of \$33,911,762 from inducing  $f_a^* = 0$ , versus selecting  $f_a^*$  between 0 and  $f_L$  with corresponding expected profit of \$38,490,662, the latter contract is the optimal one. In this case it results in higher payoffs for both the principal and the agent.

### 6.3 Optimal Contracts with $m_a = 0$

In chapter 5, we observed that the optimal contracts presented in Table 5.1 had the property  $m_a^* \times m_p^* = 0$ , with  $m_a^* = 0$  in all but one of the cases. In our numerical results, we observe that while a significant number of solutions had  $m_p^* \approx 0$ , this was not the case for  $m_a^*$ . It is therefore interesting to investigate the kind of solutions we would obtain numerically with  $m_a$  fixed at zero. Fixing the value of  $m_a$  also effectively reduces the size of the search space. Implementing the change only requires minimal adjustment to our numerical method. In particular we need to exclude  $m_a$  from all calculations of the optimal step size when iterating from one candidate solution to the next. Tables 6.9-6.11 contain some results obtained by implementing our numerical method with  $m_a = 0$ .

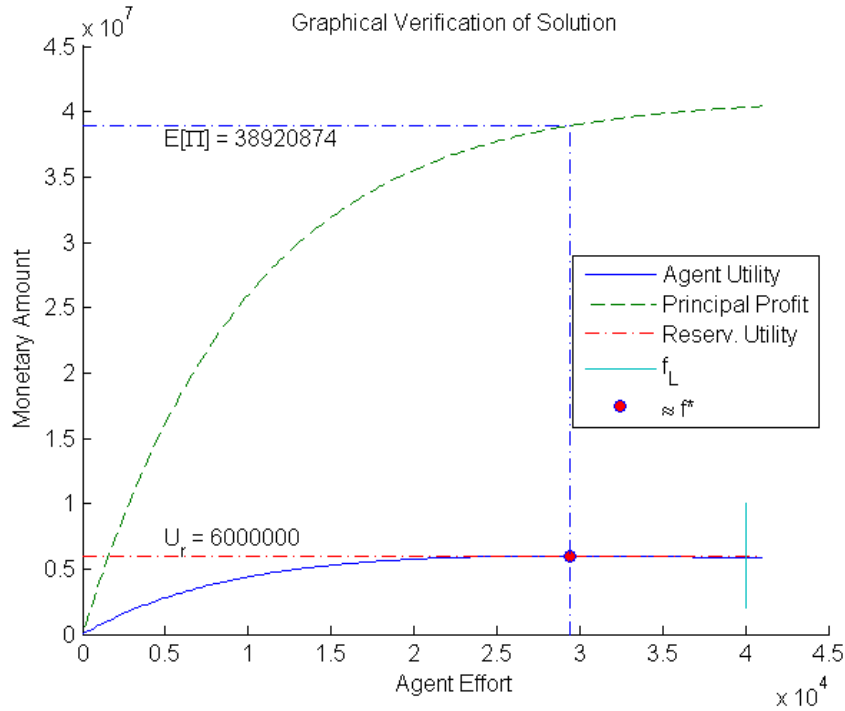


Figure 6.1: Plots of the agent’s reservation utility, the principal’s expected profit and agent’s expected utility as functions of agent’s effort,  $f_a$ , given the values of  $m_a$ ,  $m_p$  and  $r$  obtained analytically.

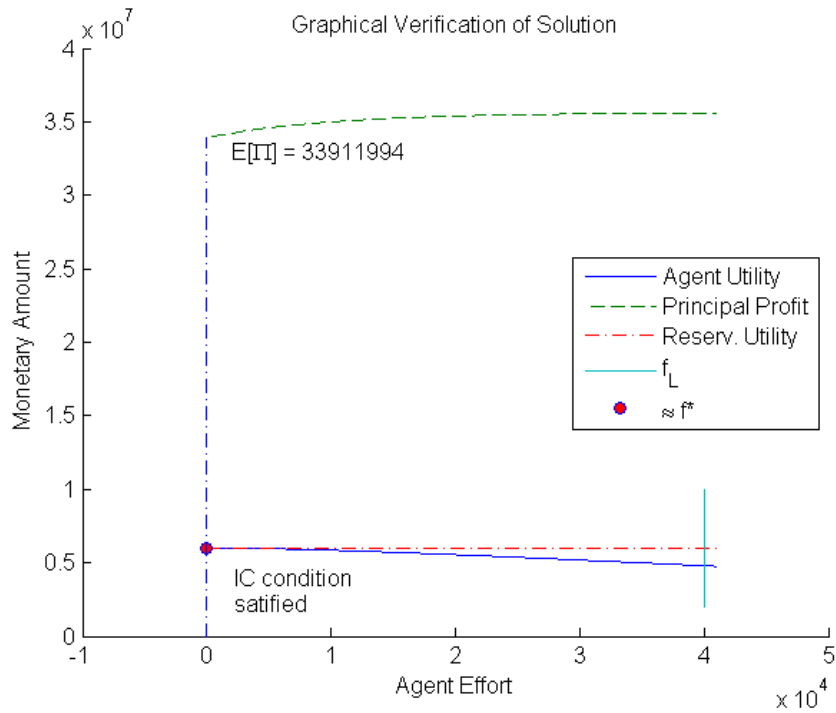


Figure 6.2: Plots of the agent’s reservation utility; the principal’s expected profit and agent’s expected utility as functions of agent’s effort,  $f_a$ , given the values of  $m_a$ ,  $m_p$  and  $r$  obtained numerically with  $f_a$  fixed at 0.

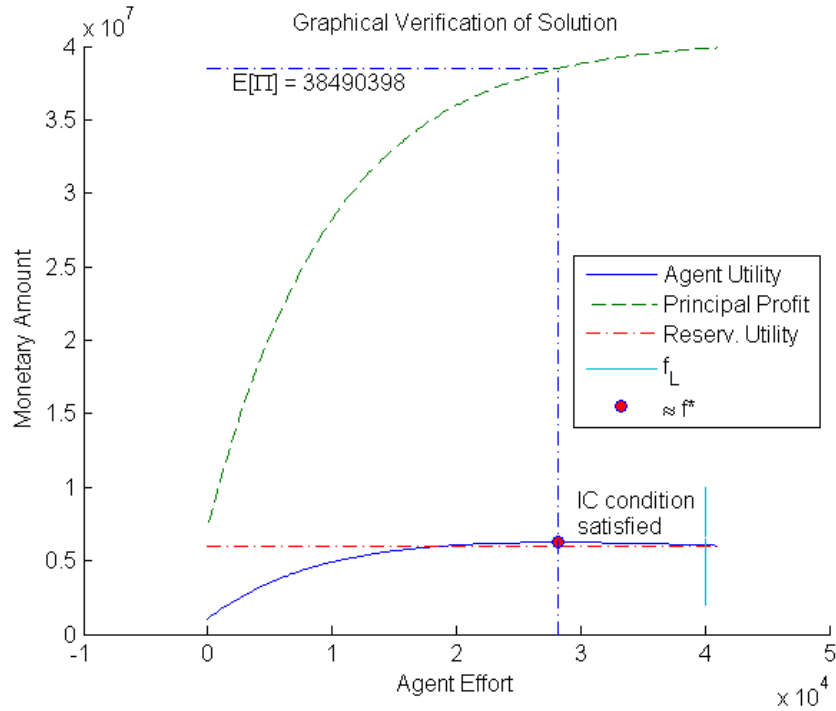


Figure 6.3: Plots of the agent’s reservation utility; the principal’s expected profit and agent’s expected utility as functions of agent’s effort,  $f_a$ , given the values of  $m_a$ ,  $m_p$  and  $r$  obtained numerically with  $\frac{\partial E[U_A]}{\partial f_a}$  fixed at 0.

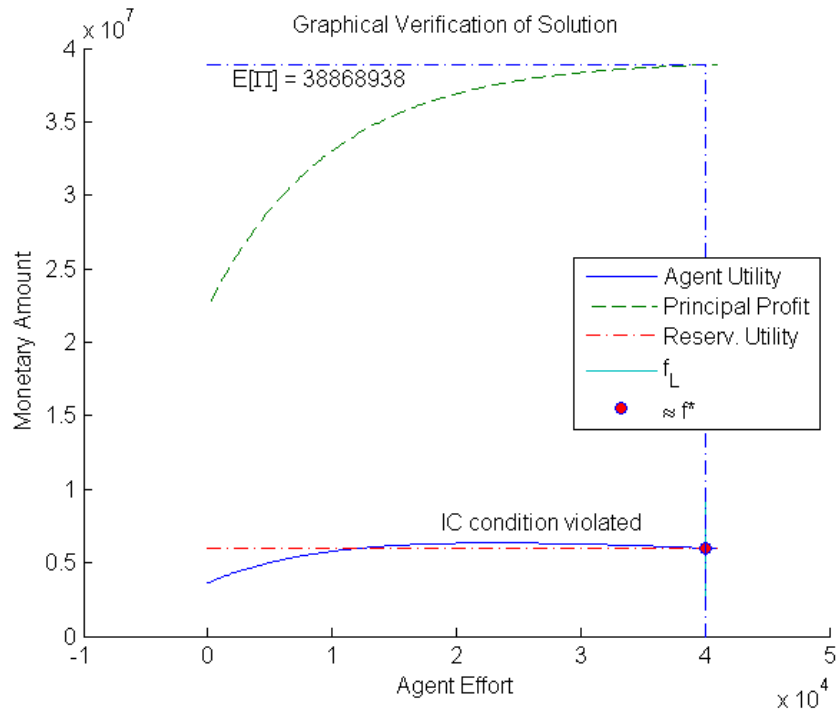


Figure 6.4: Plots of the agent’s reservation utility; the principal’s expected profit and agent’s expected utility as functions of agent’s effort,  $f_a$ , given the values of  $m_a$ ,  $m_p$  and  $r$  obtained numerically with  $f_a$  fixed at  $f_L$ .



Table 6.9: Numerical Results :  $f_L = 10,250$ ;  $U_r = 23,250,000$ ;  $m_a = 0$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$	IC Satisfied?
Analytical	10,250	0	1,535,921	18.89	23,250,000	22,584,993	Yes
Numerical, $f_a = 0$	0	0	2,265,927	18.45	23,250,009	22,513,878	Yes
Numerical, $\frac{\partial E[U_a]}{\partial f_a} = 0$	10,250	0	1,527,107	18.89	23,250,017	22,579,841**	Yes
Numerical, $f_a = f_L$	10,250	0	1,724,967	18.79	23,250,000	22,641,222	No

Table 6.10: Numerical Results :  $f_L = 40,000$ ;  $U_r = 1,750,000$ ;  $m_a = 0$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$	IC Satisfied?
Analytical	24,006	0	0	3.463	3,051,853	40,242,579	Yes
Numerical, $f_a = 0$	0	0	6,290,548	1.435	1,750,010	38,216,360	Yes
Numerical, $\frac{\partial E[U_a]}{\partial f_a} = 0$	24,180	0	5	3.524	3,122,294	40,241,240**	Yes
Numerical, $f_a = f_L$	40,000	0	21,854	2.678	1,750,000	44,412,176	No

Table 6.11: Numerical Results :  $f_L = 30,000$ ;  $U_r = 4,750,000$ ;  $m_a = 0$ 

	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$	IC Satisfied?
Analytical	0	0	2,264,677	3.769	4,750,000	41,013,889	Yes
Numerical, $f_a = 0$	0	0	2,255,559	3.77	4,750,011	41,013,790**	Yes
Numerical, $\frac{\partial E[U_a]}{\partial f_a} = 0$	0	0	2,393,992	3.76	4,750,004	40,997,747	No
Numerical, $f_a = f_L$	30,000	0	808,052	4.717	4,750,001	41,386,969	No

Setting  $m_a = 0$  significantly increases the proximity of the numerically calculated solutions to those obtained analytically. In contrast to the initial set of numerical solutions obtained without fixing  $m_a$ , when  $m_a$  is fixed at zero, we note that the numerical solutions are much more consistent with the exact optimal solutions. This is not just in terms of the expected payoffs,  $E[U_A]$  and  $E[\Pi]$ , but also in terms of the other decision variables,  $f_a$ ,  $r$  and  $m_p$ .

## 6.4 Conclusion

In this chapter we were able to develop and test a numerical approach to solving a principal-agent problem presented in the form of a co-development model. In terms of the expected payoffs for both principal and agent, when compared to the exact solutions obtained analytically, we found that our method was able to yield results that were similar to the analytical ones. However, in most cases the numerically obtained values of  $m_a$ ,  $m_p$  and  $r$  are found to be significantly different from the analytical values. The corresponding optimal response for the agent,  $f_a$ , is thus usually significantly different from the analytical value.

Our observations indicate that near optimality, the co-development model is not very sensitive to the values  $m_a$ ,  $m_p$  and  $r$ . This suggests that in order to obtain the same objective (expected net profit), the pharmaceutical company may have considerable leeway in how it structures the contract it offers the biotech firm. For example, considering the results in Table 6.6, we see that using the optimal contracts determined both numerically and analytically, the drug manufacturer and the biotech firm stand to get expected returns of \$41 million and \$4.75 million respectively. In both cases, we see that the biotech firm is expected to expend no effort (beyond the intellectual property it brings) and it is preferable for all work to be done by an external workforce. However, the solutions differ in that the numerically determined contract requires that all funding for the project be sourced by the agent while the analytically determined contract requires that the principal source the funds instead.

However, we highlight that while the principal's expected profit is not very sensitive to the values of  $m_a$ ,  $m_p$  and  $r$ , satisfying the incentive compatibility (IC) constraint becomes the key

factor that determines the eventual values of the contract decision variables. Results in the last two rows of Table 6.4 are a good example of how sensitive the IC constraint can be to the decision variables. The last two candidate solutions contained in the table are not too far apart in terms of  $m_a^*$ ,  $m_p^*$  and  $r^*$ . The resulting expected payoffs for the principal are also identical with one yielding  $E[\Pi] = 22,468,915$  and the other yielding  $E[\Pi] = 22,593,013$ . However, we see that only the first of the two contracts satisfies the incentive compatibility constraint. The other contract therefore fails to satisfy all model constraints even though the values of  $m_a$ ,  $m_p$  and  $r$  are quite similar to those of a contract that satisfies all required constraints.

Based on the performance of our numerical approach for the base co-development model, we know that we now have a viable method for solving more complicated models. For those models, we can then rely on our numerical approach to derive optimal contracts. We consider in the next chapter, a co-development model where the borrowing cost for the agent is variable in the amount borrowed. We then look at how we could solve the co-development model when other functions are used to describe the probability of success.

# Chapter 7

## Co-development Model Extensions

### 7.1 Extensions to the base model : The agent's borrowing cost, $b_a$ , considered as a function of the amount borrowed,

$m_a$

Propositions 1 to 4 in chapter 5 represent solutions obtained by and under the assumptions specified for the base model proposed by the authors of [35]. We now present an extension to the work done by the authors of [35] by investigating the optimal solutions to the co-development problem under an additional set of conditions that enhance the model's approximation to actual practice. Our ability to extend beyond the base co-development model and derive solutions for the modified problem is the primary highlight of this thesis. Specifically, we adjust assumptions about the nature of the borrowing cost,  $b_a$ . We now assume that the agent's borrowing cost is not constant but rather a variable linear function of the amount borrowed by the agent. This variability in the cost of borrowing may be reflective of associated costs like administration fees and one time up-front fees. An example of such a borrowing facility for smaller enterprises is the Canada Small Business Financing (CSBF) Loan [37].

We consider a function of the form  $b_a(m_a) = b_0 + \gamma m_a$ , where  $b_0 \geq 1$  and  $\gamma > 0$  is some constant. We assume that  $m_a$  is non-negative and in general  $b'_a(m_a)$  cannot be negative (i.e. the cost of borrowing is not expected to decline as the amount borrowed is increased). We con-

sider first the case where  $b_a(0) \geq b_p$ . This is a reasonable assumption since from the lender's perspective lending to the small biotech firm would likely be deemed riskier than lending to the larger pharmaceutical company. With  $b_a(m_a) \geq b_p$ , obtaining some analytical results for some of the models we previously solved is straightforward. Specifically, we are still able to solve models *A* and *C* which we used previously to determine the upper bound and lower bound of  $U_r$ , respectively. Once we relax the requirement that  $b_a(m_a)$  be constant or  $b_a(m_a) \geq b_p$  and replace that requirement with a less restrictive  $b_a(m_a) = b_0 + \gamma m_a$  where  $b_0 \geq 1$  and not necessarily greater than  $b_p$ , then we have to rely on numerically solving the problem. Solving the full co-development problem will also require the numerical approach. We highlight that while not too likely, the case with  $b_0 \leq b_p$  reflects a scenario where the biotech firm is actually able to secure funding at a lower rate than the pharmaceutical firm. This could perhaps be through a government provided facility for small businesses such as the US SBA loans [38].

In the two sections that follow, we examine analytical approaches to models **A** and **C** given the variable borrowing costs. We then transition to solving the full co-development problem numerically under variable borrowing functions and different functions to describe the probability of success.

### 7.1.1 Revised Model C

By obtaining the optimal solution to model *C* earlier, we were able to establish an upper bound on  $U_r$ . We now investigate the implications of a variable borrowing cost on the optimal solution to model *C*. We refer to the model with this modified assumption as model  $\check{C}$ .

When we consider the derivatives for model  $\check{C}$  with respect to  $f_a$ ,  $m_a$ ,  $m_p$  respectively we note that the only difference between the derivatives of model *C* and of model  $\check{C}$  is with differentiation with respect to  $m_a$ . In this case we have :

$$\frac{\partial E[\Pi]}{\partial m_a} = \frac{k\mu}{c_h}(R_T + R_D)e^{-k\left(f_a + \frac{m_a + m_p}{c_h}\right)} - b_a(m_a) - m_a b'_a(m_a) \quad (7.1)$$

where  $b'_a(m_a)$  is the derivative of  $b_a$  with respect to  $m_a$ .

The optimal value of  $m_a$  in Proposition 1 was determined on the basis that  $\frac{\partial E[\Pi]}{\partial m_a} \leq \frac{\partial E[\Pi]}{\partial m_p}$ .

Since we assume that  $b'_a(m_a) \geq 0$ , we therefore observe in model  $\check{C}$  that same relationship between the derivative with respect to  $m_a$  and the derivative with respect to  $m_p$ . The optimal value of  $m_a$  in model  $\check{C}$  is therefore identical to that in model  $C$ . Hence the overall optimal solution of model  $\check{C}$  is the same as that of model  $C$ , (i.e.  $m_a^{\check{C}} = 0$ ). This implies that the upper bound for the agent's reservation utility is independent of whether or not the agent's borrowing costs are fixed or variable.

### 7.1.2 Revised Model A

We used model  $A$  to determine the lowest rational value of the agent's reservation utility in the co-development project. In this case, we are reconsidering model  $A$  where  $m_a$  is dependent on  $b_a$ . We refer to this model as model  $\check{A}$ . In deriving the optimal solution for model  $\check{A}$ , we adopt the same approach used for solving model  $A$ , however we note the difference in the derivative with respect to  $m_a$ . In this case

$$\frac{\partial E[\Pi]}{\partial m_a} = \frac{k\mu R_T}{c_h} e^{-k(f_a + \frac{m_a}{c_h})} - b_a(m_a) - m_a b'_a(m_a). \quad (7.2)$$

To investigate the optimal values of  $m_a$  under the three different cases where the optimal  $f_a$  values are  $f_a = 0$ ,  $0 < f_a < f_L$  and  $f_a = f_L$ .

$f_a^{\check{A}} = 0$   
 $f_a^{\check{A}} = 0$  is only optimal if  $\frac{\partial E[\Pi]}{\partial f_a} \leq 0$ . By the same arguments given in the proof of Proposition 2, we know that  $\frac{\partial E[\Pi]}{\partial m_a} < \frac{\partial E[\Pi]}{\partial f_a} = 0$ . Hence it must be that the optimal value of  $m_a$  is at 0. Therefore, when  $f_a^{\check{A}} = 0$ ;  $m_a^{\check{A}} = 0$ .

$0 < f_a^{\check{A}} < f_L$   
 If the optimal effort level,  $f_a^{\check{A}}$ , for the agent is between 0 and the upper limit  $f_L$ , then it must be that  $\frac{\partial E[\Pi]}{\partial f_a} \Big|_{f_a=f_a^{\check{A}}} = 0$ . Solving  $\frac{\partial E[\Pi]}{\partial f_a} = 0$  for  $f_a$ , we obtain  $f_a^{\check{A}} = \frac{1}{k} \ln\left(\frac{k\mu R_T}{c_a}\right)$ . Now,  $\frac{\partial E[\Pi]}{\partial f_a} = 0$  again implies that  $\frac{\partial E[\Pi]}{\partial m_a} < 0$ , hence the optimal value of  $m_a$  must be  $m_a^{\check{A}} = 0$ . Therefore, when  $f_a^{\check{A}} = \frac{1}{k} \ln\left(\frac{k\mu R_T}{c_a}\right)$ ;  $m_a^{\check{A}} = 0$ .

$\frac{f_a^{\check{A}}}{f_a^{\check{A}}} = f_L$   
 $f_a^{\check{A}} = f_L$  implies that  $\left. \frac{\partial E[\Pi]}{\partial f_a} \right|_{f_a=f_a^{\check{A}}} \geq 0$ . This implies that for a corresponding optimal value of  $m_a$ , we can consider the value of  $m_a$  such that  $\frac{\partial E[\Pi]}{\partial m_a} = 0$ . We can therefore determine the optimal  $m_a$  by solving :

$$\frac{\partial E[\Pi]}{\partial m_a} = \frac{k\mu R_T}{c_h} e^{-k\left(f_a + \frac{m_a}{c_h}\right)} - b_a(m_a) - m_a \cdot b'_a(m_a) = 0 \quad (7.3)$$

$$\Rightarrow m_a = \frac{1}{b'_a(m_a)} \left( \frac{k\mu R_T}{c_h} e^{-k\left(f_L + \frac{m_a}{c_h}\right)} - b_a(m_a) \right). \quad (7.4)$$

We can solve (7.3) for the optimal  $m_a$  using a simple iterative approach such as the bisection method where we seek a root that lies within the interval  $[0, m_L]$ , where  $m_L$  is an upper bound on the value of  $m_a$ . We make use of some properties of  $b_a(m_a)$  to determine  $m_L$ .

Since  $b_a(m_a) \geq 1$  and  $b'_a(m_a) > 0$ , then (7.4) implies that if  $m_a \geq 0$  then it must be that

$$\frac{k\mu R_T}{c_h} e^{-k\left(f_L + \frac{m_a}{c_h}\right)} - b_a(m_a) \geq 0 \quad (7.5)$$

$$\Rightarrow \frac{k\mu R_T}{c_h} e^{-k\left(f_L + \frac{m_a}{c_h}\right)} \geq 1 \quad (7.6)$$

$$\Rightarrow m_a \leq c_h \left[ \frac{1}{k} \ln \left( \frac{k\mu R_T}{c_h} \right) - f_L \right] = m_L. \quad (7.7)$$

We therefore consider (7.7) as a suitable value of  $m_L$ , an upper bound of the optimal  $m_a$ .

### A closed form solution for optimal $m_a$ in model $\check{A}$ , when $f_a = f_L$

We observe that if we assume that the function  $b_a(m_a)$  is linear in  $m_a$  we are able to deduce a closed form solution to the problem of determining the optimal amount to borrow. This closed form solution makes use of the Lambert W function. Since  $b_a(m_a) = b_p + \gamma m_a$  and  $b_p = b_a(0)$ ,

(7.3) can be written as

$$\frac{k\mu R_T}{c_h} e^{-kf_a} e^{-\frac{k}{c_h} m_a} - b_p - 2\gamma m_a = 0$$

$$\text{now let } H = \frac{k\mu R_T}{c_h} e^{-kf_L} \quad (7.8)$$

$$\Rightarrow H e^{-\frac{k}{c_h} m_a} - b_p - 2\gamma m_a = 0$$

$$H = (2\gamma m_a + b_p) e^{\frac{k}{c_h} m_a}$$

$$\frac{Hk}{2\gamma c_h} e^{\frac{b_p k}{2\gamma c_h}} = \left( \frac{k}{c_h} m_a + \frac{b_p k}{2\gamma c_h} \right) e^{\left( \frac{k}{c_h} m_a + \frac{b_p k}{2\gamma c_h} \right)}$$

$$\Rightarrow m_a = \frac{c_h}{k} \left[ W \left( \frac{kH}{2\gamma c_h} e^{\left( \frac{b_p k}{2\gamma c_h} \right)} \right) - \frac{b_p k}{2\gamma c_h} \right] \quad (7.9)$$

where  $W(\cdot)$  represents the Lambert W function.

This therefore implies that the optimal solution to model  $\check{A}$  is

$$f_a^{\check{A}} = \min \left\{ \max \left\{ 0, \frac{1}{k} \ln \left( \frac{k\mu R_T}{c_a} \right) \right\}, f_L \right\} \quad (7.10)$$

$$m_a^{\check{A}} = \max \left\{ 0, \frac{c_h}{k} \left[ W \left( \frac{kH}{2\gamma c_h} e^{\left( \frac{b_p k}{2\gamma c_h} \right)} \right) - \frac{b_p k}{2\gamma c_h} \right] \right\} \quad (7.11)$$

with  $H$  as described in 7.8.

*The agent's minimal reservation utility should therefore be :*

$$U_{min} = \mu \left( 1 - e^{-k \left( f_a^{\check{A}} + \frac{m_a^{\check{A}}}{c_h} \right)} \right) R_T - c_a f_a^{\check{A}} - b_a m_a^{\check{A}}. \quad (7.12)$$

## 7.2 Solutions for contracts where borrowing costs are not fixed

In the previous chapter, we assumed that the agent faced a constant borrowing cost. Analytical and numerical results for seven different parameter sets were provided in tables 6.2 to 6.8. For these same parameters, we now determine the optimal contracts under the assumption that  $b_a = b_0 + \gamma m_a$ . As we indicated earlier, if the agent's lowest possible borrowing cost is greater than the principal's borrowing costs, that is  $b_0 > b_p$ , then we generally do not expect our analytical results to change. However, if we allow  $b_0$  to sometimes be lower than  $b_p$  then some of



the rules by which we derived our optimal contracts to the co-development problem no longer hold and we have to switch to numerical computations.

Implementing this change will only require the slight adjustment of substituting  $b_a$  with  $b_0 + \gamma m_a$  in the code we used in the previous chapter. It will be of interest to see how sensitive the expected returns are to the assumed structure of the borrowing costs. We also noted in earlier results that in most cases, the optimal contracts resulted in the agent getting expected utility that was very close to his reservation utility. We will also check how far from the agent's reservation utility the agent earns in this case.

We present in the following tables results for a sample of cases where the agent's borrowing costs are variable. The first row of each table shows the solution to each problem as previously obtained with a fixed  $b_a$ . The rest of the rows contain results for variable cost functions.

Table 7.1: Variable  $b_a$ :  $f_L = 5250$ ;  $U_r = 23,250,000$ ;  $b_p = 1.05$

$b_a(m_a)$	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$
$b_a = 1.07$	5250	1,568,544	184,668	20.09	23,250,007	22,470,761
$b_a = 1.01 + 10^{-7} \times m_a$	5250	246,092	1,456,295	18.99	23,250,004	22,466,866
$b_a = 1.01 + 3 \times 10^{-7} \times m_a$	0	563,427	1,676,478	18.99	23,250,015	22,440,521
$b_a = 1.03 + 10^{-6} \times m_a$	0	321,688	1,768,087	18.88	23,250,011	22,380,611
$b_a = b_p + 10^{-7} \times m_a$	5250	238,092	1,465,142	18.99	23,250,005	22,458,140

Table 7.2: Variable  $b_a$ :  $f_L = 40,000$ ;  $U_r = 1,750,000$ ;  $c_h = 200$ ;  $b_p = 1.05$ 

$b_a(m_a)$	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$
$b_a = 1.07$	20,340	463,108	4	3.026	2,145,513	40,161,952
$b_a = 1.01 + 10^{-7} \times m_a$	19,675	1,130,334	4	3.952	2,578,596	40,165,512
$b_a = 1.01 + 3 \times 10^{-7} \times m_a$	18,301	1,236,149	5	3.632	1,787,903	40,232,606
$b_a = 1.03 + 10^{-6} \times m_a$	22,529	105,587	5	3.149	2,591,395	40,202,768
$b_a = b_p + 10^{-7} \times m_a$	22,267	58,486	4	2.997	2,465,412	40,150,078

Table 7.3: Variable  $b_a$ :  $f_L = 40,000$ ;  $U_r = 6,000,000$ ;  $c_h = 200$ ;  $b_p = 1.05$ 

$b_a(m_a)$	$f_a^*$	$m_a^*$	$m_p^*$	$r^*$	$E[U_A]$	$E[\Pi]$
$b_a = 1.07$	28,226	396,374	4	6.439	6,249,854	38,490,662
$b_a = 1.01 + 10^{-7} \times m_a$	29,801	109	80	6.183	6,284,389	38,718,700
$b_a = 1.01 + 3 \times 10^{-7} \times m_a$	29,946	264,400	4	7.159	7,234,668	37,811,880
$b_a = 1.03 + 10^{-6} \times m_a$	29,455	935	4	5.975	6,032,598	38,897,576
$b_a = b_p + 10^{-7} \times m_a$	27,485	536,735	4	6.414	6,079,378	38,513,259

What is most apparent from these experiments is that the principal's expected profits are not very sensitive to the agent's borrowing cost even when the agent's borrowing cost is allowed to take values lower than the principal's. What is interesting is that in maintaining the levels of expected returns for the principal and agent, the optimal values of  $m_p$  and  $r$  also stay

relatively unchanged. In nearly all instances, we observe that changing  $b_a$  only seems to affect the eventual values of  $f_a$  and  $m_a$  with the optimal  $f_a$  increasing as the optimal  $m_a$  decreases and vice-versa.

These observations lead to a significant insight into the co-development problem. The results suggest that the pharmaceutical company may not have to concern itself much with the biotech firm's associated borrowing costs. If borrowing costs are too high, the project will rely more on the biotech firm's internal workforce and less on an external workforce. This is because of the inverse relationship between the amount of internal labour effort,  $f_a$ , and the amount of external labour effort at cost  $m_a$ . As borrowing costs increase, it becomes more costly to obtain a required amount of total effort through external labour at cost  $m_a$  than to obtain the required effort using the internal workforce at cost  $c_a \times f_a$ . We notice then that the agent's borrowing costs only push  $m_a$  and  $f_a$  in opposite directions in a manner that preserves the principal's expected profits. The principal therefore is largely unaffected by the agent's borrowing costs.

### 7.3 Numerical Results for different functions describing probability of success

In the original co-development model, we represented the probability of success with the expression  $p(f_a, m_a, m_p) = 1 - e^{-k(f_a + (m_a + m_p)/c_h)}$ . The key properties of this function that we require of any function that we could use to describe this probability are that  $p(f_a, m_a, m_p)$  is increasing in its arguments but with diminishing marginal returns. Once we deviate from the original form of  $p(f_a, m_a, m_p)$ , we may have to rely on numerical computations. We cannot directly apply the same code we used in past cases and have to make some significant changes. Previously we had three different objective functions based on the optimal value of  $f_a$  as a function of  $m_a$ ,  $m_p$  and  $r$ . We were able to do this because we had a conveniently chosen  $p(f_a, m_a, m_p)$  which permitted us to easily deduce the structure of the agent's optimal responses.

We introduce a change in our numerical approach which will allow us to obtain solutions to the co-development problem without explicitly solving for the optimal effort level from  $\frac{\partial E[U_A]}{\partial f_a} = 0$ . In transforming the original co-development problem into an unconstrained problem, instead of using a transformation of  $\frac{\partial E[U_A]}{\partial f_a} = 0$  or an expression for  $f_a$  derived from solving  $\frac{\partial E[U_A]}{\partial f_a} = 0$ , we substitute either of the two with the forward finite difference,  $\frac{p(f_a + \Delta, m_a, m_p) - p(f_a, m_a, m_p)}{\Delta}$  at  $f_a$  where  $\Delta \ll 1$ .

We carry out a gradient search procedure from a randomly generated point,  $(f_a, m_a, m_p, r)$ , which satisfies the individual rationality constraint and produces a non-negative expected profit for the principal. As was the case in the previous chapter, we begin and carry out the search for optimal  $m_a$ ,  $m_p$  and  $r \times R_T$  in the interval  $[0, (R_T + R_D)]$ . All trial values of  $f_a$  are confined within the interval  $[0, f_L]$ . For every initial trial solution, if the final point  $(f_a, m_a, m_p, r)$  that is obtained yields a higher expected net profit than the current ‘best’ solution, it is tested for incentive compatibility. If it passes both these tests, then it becomes the current best solution.

While the original form of  $p(f_a, m_a, m_p)$  was strictly convex, we now solve the co-development model using other functions that are not necessarily convex. Figure 7.3 shows the general shapes of the functions we use to describe the probability of success. We present some of the results obtained using the different functions in tables 7.3-7.7.

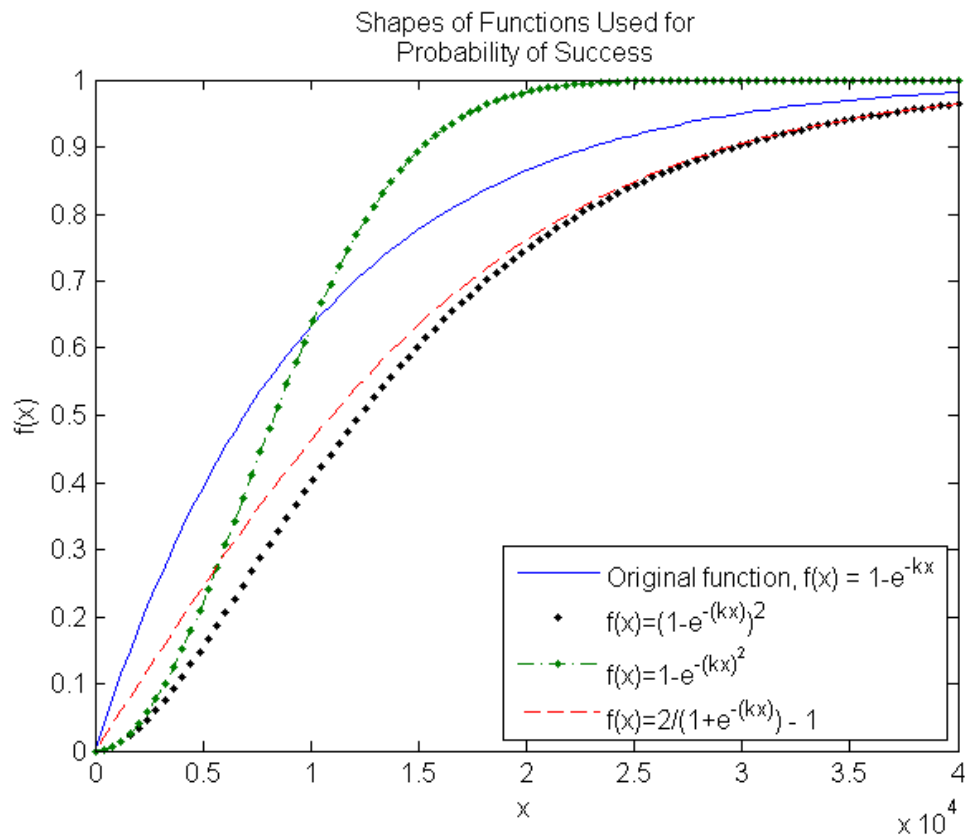


Figure 7.1: The general shapes of the different functions describing probability of success.

Table 7.4: Different  $p(f_a, m_a, m_p)$ :  $f_L = 5250$ ;  $U_r = 23,250,000$ ;  $b_p = 1.05$

$p(f_a, m_a, m_p)$	$f_a^*$	$m_a^*$	$m_p^*$	Total Effort	$r^*$	$E[U_A]$	$E[III]$
$1 - e^{-k\left(\frac{m_a+m_p}{c_h}\right)}$	5250	0	1,781,747	40,885	18.73	23,250,000	22,520,093
$\left(1 - e^{-k\left(\frac{m_a+m_p}{c_h}\right)}\right)^2$	362	3,439,131	0	69,144	21.54	23,699,463	21,176,365
$1 - e^{-k^2\left(\frac{m_a+m_p}{c_h}\right)^2}$	164	1,480,575	0	29,776	20.09	23,996,684	23,072,469
$\frac{2}{1 + e^{-k\left(\frac{m_a+m_p}{c_h}\right)}} - 1$	5250	482,967	1,636,330	14,909	19.14	23,250,061	22,148,117

Table 7.5: Different  $p(f_a, m_a, m_p)$ :  $f_L = 40,000$ ;  $U_r = 1,750,000$ ;  $c_h = 200$ ;  $b_p = 1.05$

$p(f_a, m_a, m_p)$	$f_a^*$	$m_a^*$	$m_p^*$	Total Effort	$r^*$	$E[U_A]$	$E[III]$
$1 - e^{-k\left(\frac{m_a+m_p}{c_h}\right)}$	24,006	0	0	24,006	3.463	3,051,853	40,242,579
$\left(1 - e^{-k\left(\frac{m_a+m_p}{c_h}\right)}\right)^2$	4115	10,981,908	0	59,025	11.73	2,952,094	33,533,843
$1 - e^{-k^2\left(\frac{m_a+m_p}{c_h}\right)^2}$	26,767	3,404,157	0	43,788	6.005	2,937,741	41,015,924
$\frac{2}{1 + e^{-k\left(\frac{m_a+m_p}{c_h}\right)}} - 1$	40,000	696,930	5017	43,510	10.80	11,068,797	34,007,953

Table 7.6: Different  $p(f_a, m_a, m_p)$ :  $f_L = 40,000$ ;  $U_r = 6,000,000$ ;  $c_h = 200$ ;  $b_p = 1.05$ 

$p(f_a, m_a, m_p)$	$f_a^*$	$m_a^*$	$m_p^*$	Total Effort	$r^*$	$E[U_A]$	$E[III]$
$1 - e^{-k\left(f_a + \frac{m_a + m_p}{c_h}\right)}$	29,413	0	0	29,413	5.947	6,000,000	38,920,919
$\left(1 - e^{-k\left(f_a + \frac{m_a + m_p}{c_h}\right)}\right)^2$	40,000	722,251	5017	43,611	10.80	11,043,231	34,014,126
$1 - e^{-k^2\left(f_a + \frac{m_a + m_p}{c_h}\right)}$	30,147	1,970,214	0	39,998	9.231	8,445,775	36,906,995
$\frac{2}{1 + e^{-k\left(f_a + \frac{m_a + m_p}{c_h}\right)}} - 1$	40,000	678,400	0	43,392	11.08	11,423,321	33,664,038

Table 7.7: Different  $p(f_a, m_a, m_p)$ :  $f_L = 30,000$ ;  $U_r = 4,750,000$ 

$p(f_a, m_a, m_p)$	$f_a^*$	$m_a^*$	$m_p^*$	Total Effort	$r^*$	$E[U_A]$	$E[III]$
$1 - e^{-k\left(f_a + \frac{m_a + m_p}{c_h}\right)}$	0	0	2,264,677	45,294	3.769	4,750,000	41,013,889
$\left(1 - e^{-k\left(f_a + \frac{m_a + m_p}{c_h}\right)}\right)^2$	1835	4,709,873	0	96,032	10.05	7,692,197	35,855,082
$1 - e^{-k^2\left(f_a + \frac{m_a + m_p}{c_h}\right)}$	1312	2,016,846	0	41,649	6.674	6,291,883	40,164,410
$\frac{2}{1 + e^{-k\left(f_a + \frac{m_a + m_p}{c_h}\right)}} - 1$	1835	4,709,873	0	94,197	10.05	7,692,208	35,855,082

The functions describing the probability of success appear to have the most impact on the optimal ‘total amount of effort’,  $\left(f_a + \frac{m_a + m_p}{c_h}\right)$ , which we present as Total Effort in tables 7.3-7.7. If we consider the results in table 7.3 as an example, we observe that the values of  $E[\Pi]$  remain largely unaffected as we change the form of  $p(f_a, m_a, m_p)$ . However, the total effort required to maintain the payoff levels changes drastically as  $p(f_a, m_a, m_p)$  is varied. Compared to the principal’s expected profit, the agent’s expected utility is significantly more sensitive to the changes in  $p(f_a, m_a, m_p)$ . We observe that there are cases where the agent’s expected payoff nearly doubles while the principal’s payoff changes by a far smaller margin as  $p(f_a, m_a, m_p)$  is changed.

However, since we note that  $p(f_a, m_a, m_p)$  has the potential to significantly affect the expected profits for the principal, it implies that the principal may need to exercise due diligence in understanding how to model the translation of the agent’s effort into results. Perhaps before finding the optimal structure, determining  $p(f_a, m_a, m_p)$  may be the most important aspect of principal’s problem; more than the determination of the agent’s utility function,  $\mu$  or the projections for  $R_T$  and  $R_D$ . Since the function  $p(f_a, m_a, m_p)$  translates the biotech firm’s intellectual property and expertise into the final project outcome, the pharmaceutical firm therefore has to put considerable effort or resources to gain some understanding of the biotech firm’s knowledge in order to better formulate  $p(f_a, m_a, m_p)$ .

## 7.4 Conclusion

In this chapter we carried out two different sets of experiments. We began by modifying our solution concept so as to investigate the consequences of the agent having variable borrowing cost. We found that having borrowing cost that was variable had little impact on the overall expected payoffs for both the principal and agent. The key implications of these findings are that in designing a contract to offer the diagnostics firm, the pharmaceutical company may not have to direct much focus on the diagnostic firm’s associated borrowing costs.



In the second set of experiments, we reconsidered the co-development problem with different functions describing the probability of success,  $p(f_a, m_a, m_p)$ . Changing the functional form of  $p(f_a, m_a, m_p)$  had most apparent impact on total effort input as described by the quantity  $\left(f_a + \frac{m_a + m_p}{c_h}\right)$ . We noted that while the optimal amount of total effort changed drastically in response to a changing  $p(f_a, m_a, m_p)$ , it was possible for the expected returns for the principal and agent to stay relatively unchanged. Overall, we determined that the formulation of  $p(f_a, m_a, m_p)$  would likely have a significant impact on the pharmaceutical firm's expected profits hence there was need to be as accurate as possible in modeling the translation of effort into success.

# Chapter 8

## Conclusion

### 8.1 Discussion

The principal-agent problem is a framework commonly used to model contractual arrangements between an employer and employee. In this thesis we adopted the framework to model a companion diagnostic co-development project between a larger pharmaceutical firm as the principal and a smaller biotech company as the agent. The pharmaceutical firm sought to design a profit maximizing contract that would be acceptable to the biotech firm and also induce the biotech firm to work in a manner that maximized the pharmaceutical firm's expected profits.

For the base co-development model, we were able to produce a set of results contained in Table 5.1 of which one represented the optimal contract if it existed for the given set of parameters. Since we were able to deduce exact analytical results, we therefore had a benchmark that we could use for assessing the performance of any numerical algorithms we developed. With this in mind, we then presented a numerical approach that could be used for solving the base co-development model. Presentation and testing of the numerical method for solving the co-development problem and its extensions in chapters 6 and 7 respectively is the primary highlight of this thesis. This same numerical approach can then be used as a basis for solving other forms of principal-agent problems with moral hazard.

We observed that in terms of expected returns for both principal and agent, the numerical

method produced results that were consistent with those obtained analytically. What was interesting to note was that these expected payoffs would mostly be achieved at different effort and investment levels from those obtained analytically. The analytical approach mostly produced contracts that required that the companion diagnostics firm do as much as possible first before any extra investment was sourced for any additional workforce. The additional funding almost always had to be accessed by the pharmaceutical firm. In contrast to this, the numerically obtained contracts typically required a mix of the biotech company's own effort and investment amounts provided mostly by the biotech firm. The additional investments were required even while the biotech firm was not utilizing the full capabilities of its internal workforce.

We further adjusted our numerical method so that we could have more leeway in how we modeled the borrowing costs and probability of success. We found that changing the functions that describe the biotech firm's borrowing costs had limited impact on the expected payoffs of both the biotech and pharmaceutical firms. On the other hand, changing the function representing the probability of success had significant impact on the payoffs. To a greater extent the pharmaceutical firm had the more sensitive expected payoff of the two.

### **8.1.1 Future Work**

We have been able to develop a numerical approach for solving two instances of the principal-agent problem - a basic two actions, two output problem and our co-development model. For future work it will be interesting to see how well this method performs for more general principal-agent problems especially those where the contracting parties are not assumed to be risk neutral. In our co-development model, we implicitly assumed the companion diagnostic test had perfect sensitivity and specificity and hence no chance of false positives and false negatives. It would be interesting to incorporate test performance into the co-development model to improve the model's approximation to actual practice.

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# **Appendix A**

## **Variable and Symbol Definitions**

Table A.1: Definitions of Variables and Symbols

Symbol	Description	Additional Notes
$a$	agent's action	an element of some non-empty set $A$ containing all possible agent actions
$x$	observed output	a random variable, dependent on agent's actions/effort
$w(x)$	agent's wage	a function of observed output, $x$
$p(x a)$	density function of output given takes action $a$	nondecreasing in $a$
$P(x a)$	distribution function of output	
$U_p(\cdot)$	principal's utility function	increasing in principal's payoff
$U_A(\cdot)$	agent's utility function	increasing in agent's payoff
$U_r$	agent's reservation utility	$U_r \geq 0$
$\pi_h$	high output	$\pi_l < \pi_h$
$\pi_l$	low output	$\pi_l < \pi_h$
$p_h$	probability of high output given high effort	$p_l < p_h$
$p_l$	probability of high output given low effort	$p_l < p_h$
$w_h$	agent's wage given high output	$w_h \geq U_r$
$w_l$	agent's wage given low output	$w_l \geq U_r$
$C_h$	cost of implementing high effort for agent	$C_h > C_l$
$C_l$	cost of implementing low effort for agent	$C_h > C_l$
$\omega$	$\omega = (C_h - C_l)/(p_h - p_l)$	
$\Pi$	Principal's net profit from co-development	
$R_D$	revenue from sale of companion drug	
$R_T$	revenue from sale of companion diagnostic test	

Table A.1 continued. Definitions of Variables and Symbols

Symbol	Description	Additional Notes
$r$	agent's share of revenue from diagnostic test	$r \geq 0$
$f_a$	agent's effort level	$0 < f_a < f_L$
$f_L$	maximum effort level the agent can expend	$f_L > 0$
$c_a$	cost of using agent's internal workforce per unit of effort	$c_a < c_h$
$c_h$	cost of hiring additional labour per unit of effort	$c_a < c_h$
$m_p$	amount borrowed by principal to use for external workforce	$m_p \geq 0$
$m_a$	amount borrowed by agent to use for external workforce	$m_a \geq 0$
$b_p$	cost of borrowing amount $m_p$ for the principal	$b_p \geq 1$
$b_a$	cost of borrowing amount $m_a$ for the agent	$b_a \geq b_p$
$k$	constant for translating effort into probability of success	$k > 0$
$\theta$	Bernoulli random variable to represent the random state of nature	$E[\theta] = \mu$
$\mu$	the probability of success for the Bernoulli random variable $\theta$	

# Appendix B

## Proof of Proposition 1

We obtain the partial derivatives of  $E[\Pi]$  with respect to  $f_a$ ,  $m_a$  and  $m_p$ :

$$\frac{\partial E[\Pi]}{\partial f_a} = k\mu(R_T + R_D)e^{-k(f_a + \frac{m_a + m_p}{c_h})} - c_a \quad (\text{B.1})$$

$$\frac{\partial E[\Pi]}{\partial m_a} = \frac{k\mu}{c_h}(R_T + R_D)e^{-k(f_a + \frac{m_a + m_p}{c_h})} - b_a \quad (\text{B.2})$$

$$\frac{\partial E[\Pi]}{\partial m_p} = \frac{k\mu}{c_h}(R_T + R_D)e^{-k(f_a + \frac{m_a + m_p}{c_h})} - b_p \quad (\text{B.3})$$

We note that for positive values of the parameters  $k$ ,  $c_h$ ,  $\mu$ ,  $R_T$  and  $R_D$ , the second derivatives are less than or equal to zero ( i.e.  $\frac{\partial^2 E[\Pi]}{\partial f_a^2} \leq 0$  and  $\frac{\partial^2 E[\Pi]}{\partial m_a^2} = \frac{\partial^2 E[\Pi]}{\partial m_p^2} \leq 0$ ). This implies that each of the partial derivatives above, (B.1, B.2, B.3), are strictly decreasing for positive parameter values.

### *Determining $m_a^C$*

Since  $b_a \geq b_p$ , we know from (B.2) and (B.3) that  $\frac{\partial E[\Pi]}{\partial m_a} \leq \frac{\partial E[\Pi]}{\partial m_p}$ . Therefore at any point it is always more preferable to obtain an additional unit of  $m_p$  than of  $m_a$  which implies  $m_a^C = 0$ . It is preferable to fund any level of production at cost  $b_p$  than at  $b_a$ . There is no upper bound on both  $m_a$  and  $m_p$ , hence in the coordinated problem, any required amount of funding is obtained at lower cost  $b_p$ , which means  $m_a^C = 0$ .

### *Determining $f_a^C$ and $m_p^C$*

We note that for  $f_a$  there are three critical points to consider:

- a)  $f_a = 0$  (Lower limit of  $f_a$ )
- b)  $0 < f_a < f_L$  where  $\frac{\partial E[\Pi]}{\partial f_a} \Big|_{f_a=f_a^C} = 0$  (Stationary point of  $E[\Pi]$ )
- c)  $f_a = f_L$  (Upper limit of  $f_a$ )

We evaluate the optimal values of  $m_p$  based on the optimal effort levels for the coordinated problem,  $f_a^C$ .

Case 1 :  $f_a^C = 0$

Since  $\frac{\partial^2 E[\Pi]}{\partial f_a^2} \leq 0$ ,  $f_a = 0$  is optimal only if  $\frac{\partial E[\Pi]}{\partial f_a} \Big|_{f_a=0} \leq 0$ . Now, from (B.3), we have :

$$c_h \frac{\partial E[\Pi]}{\partial m_p} = k\mu(R_T + R_D)e^{-k(f_a + \frac{m_a + m_p}{c_h})} - b_p c_h \quad (\text{B.4})$$

Since  $c_h > c_a$  and  $b_p \geq 1$ , we have  $b_p c_h \geq c_h > c_a$ , which implies that  $c_h \frac{\partial E[\Pi]}{\partial m_p} < \frac{\partial E[\Pi]}{\partial f_a}$ .

Therefore  $\frac{\partial E[\Pi]}{\partial f_a} \leq 0$  implies  $\frac{\partial E[\Pi]}{\partial m_p} < 0$ . Since  $m_p \geq 0$  (non-negative), when  $f_a^C = 0$ , we have  $\frac{\partial E[\Pi]}{\partial m_p} < 0$  which means the optimal value of  $m_p$  has to be  $m_p^C = 0$ . Therefore when  $f_a^C = 0$ ;  $m_a^C = 0$  and  $m_p^C = 0$ . In this case the resulting expected profit is  $E[\Pi] = 0$ .

Case 2 :  $0 < f_a^C < f_L$

If the optimal value  $f_a^C$  is strictly between 0 and  $f_L$ , then it must be that  $\frac{\partial E[\Pi]}{\partial f_a} \Big|_{f_a=f_a^C} = 0$ .

Solving for  $f_a$  we obtain

$$f_a^C = \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right)$$

where  $\ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right) < k f_L$  by definition of the case. We determined in case 1 above that

$\frac{\partial E[\Pi]}{\partial m_p} < \frac{\partial E[\Pi]}{\partial f_a}$  hence  $\frac{\partial E[\Pi]}{\partial f_a} = 0$  implies  $\frac{\partial E[\Pi]}{\partial m_p} < 0$ . This implies again that the optimal

value of  $m_p$  cannot be larger than zero hence  $m_p^C = 0$ . Therefore when  $f_a^C = \frac{1}{k} \ln \left( \frac{k\mu(R_T + R_D)}{c_a} \right)$ ;  $m_a^C =$

0,  $m_p^C = 0$ . The corresponding optimal profit is

$$E[\Pi] = \mu(R_T + R_D) - \frac{c_a}{k} - \frac{c_a}{k} \ln\left(\frac{k\mu(R_T + R_D)}{c_a}\right)$$

Case 3 :  $f_a^C = f_L$

If the optimal value of  $f_a$  is such that  $f_a^C \neq 0$  and  $f_a^C \neq \frac{1}{k} \ln\left(\frac{k\mu(R_T + R_D)}{c_a}\right)$ , then it must be that  $\left.\frac{\partial E[\Pi]}{\partial f_a}\right|_{f_a=f_a^C} > 0$ . This also means  $\frac{\partial E[\Pi]}{\partial m_p} = 0$  cannot be ruled out. In this case, since we only have a non-negativity constraint for  $m_p$ , the optimal value of  $m_p^C$  occurs when  $\frac{\partial E[\Pi]}{\partial m_p} = 0$ . Solving for  $m_p$  we obtain  $m_p^C = c_h \left(\frac{1}{k} \ln\left(\frac{k\mu(R_T + R_D)}{b_p c_h}\right) - f_L\right)$ . Therefore when  $f_a^C = f_L$ ;  $m_a^C = 0$ ,  $m_p^C = c_h \left(\frac{1}{k} \ln\left(\frac{k\mu(R_T + R_D)}{b_p c_h}\right) - f_L\right)$ .  $\square$

In this case, expected profit

$$E[\Pi] = \mu(R_T + R_D) - \frac{b_p c_h}{k} - c_a f_L - b_p c_h \left(\frac{1}{k} \ln\left(\frac{k\mu(R_T + R_D)}{b_p c_h}\right) - f_L\right)$$

# Appendix C

## Proof of Proposition 2

To derive the solution for model A, we begin by obtaining the partial derivatives :

$$\frac{\partial E[\Pi]}{\partial f_a} = k\mu R_T e^{-k\left(f_a + \frac{m_a}{c_h}\right)} - c_a \quad (\text{C.1})$$

$$\frac{\partial E[\Pi]}{\partial m_a} = \frac{k\mu R_T}{c_h} e^{-k\left(f_a + \frac{m_a}{c_h}\right)} - b_a \quad (\text{C.2})$$

### *Determining $f_a^A$ and $m_a^A$*

We consider again the three critical points of  $E[\Pi]$  which are  $f_a = 0$ ,  $0 < f_a < f_L$  and  $f_a = f_L$ .

#### Case 1 : $f_a^A = 0$

For a maximization problem with non-negativity constraints,  $f_a = 0$  is only optimal if  $\frac{\partial E[\Pi]}{\partial f_a} \leq 0$ . As in the proof of Proposition 1, we have  $c_h \frac{\partial E[\Pi]}{\partial m_a} < \frac{\partial E[\Pi]}{\partial f_a}$ , hence  $\frac{\partial E[\Pi]}{\partial m_a} < 0$ . Since  $m_a$  is non-negative, the fact that  $\frac{\partial E[\Pi]}{\partial m_a} < 0$  implies the optimal value of  $m_a$  occurs at 0. Therefore when  $f_a^A = 0$ ;  $m_a^A = 0$ .

#### Case 2 : $0 < f_a^A < f_L$

An optimal value satisfying  $0 < f_a^C < f_L$  must occur when  $\left. \frac{\partial E[\Pi]}{\partial f_a} \right|_{f_a=f_a^A} = 0$ . Solving for  $f_a$ , we obtain  $f_a = \frac{1}{k} \ln\left(\frac{k\mu R_T}{c_a}\right)$ .  $\frac{\partial E[\Pi]}{\partial f_a} = 0$  implies that  $\frac{\partial E[\Pi]}{\partial m_a} < 0$ . The optimal value of  $m_a$  must therefore be  $m_a^A = 0$ . Therefore when  $f_a^A = \frac{1}{k} \ln\left(\frac{k\mu R_T}{c_a}\right)$ ;  $m_a^A = 0$ .

Case 3 :  $f_a^A = f_L$

$f_a^A = f_L$  must imply that  $\left. \frac{\partial E[\Pi]}{\partial f_a} \right|_{f_a=f_a^A} > 0$ . Since  $m_a$  has no upper limit, it must be that the optimal value of  $m_a$  occurs at  $\frac{\partial E[\Pi]}{\partial m_a} = 0$ . Solving for  $m_a$ , we obtain  $m_a^A = c_h \left( \frac{1}{k} \ln\left(\frac{k\mu R_T}{b_a c_h}\right) - f_L \right)$   $\square$



# Appendix D

## Proof of Proposition 4

In the second best solution, the agent and the principal's actions are not coordinated. Therefore the agent considers optimality from the perspective of his utility function while the principal considers optimality from the perspective of her expected profit. As previously described, the solutions will be derived by cases based on the agent's optimal effort level,  $f_a^*$ . The agent chooses his optimal effort level in response to the values of  $m_a$ ,  $m_p$  and  $r$ , proposed by the principal in the co-development contract. Each of the proposed values of  $m_a$ ,  $m_p$  and  $r$  can either be equal to zero or greater than zero. Each solution should satisfy the incentive compatibility constraint. While every solution will satisfy the individual rationality constraint, at optimality the constraint can be either binding or not. We will investigate the structure of each of the possible optimal contracts and the required conditions necessary for a solution to be optimal. We begin our investigation of the optimal solutions by looking first at the agent's royalty percentage,  $r$ .

We note that since the agent's expected utility is

$$E[U_A] = \mu \left( 1 - e^{-k(f_a + \frac{m_a + m_p}{c_h})} \right) r R_T - c_a f_a - b_a m_a, \quad (\text{D.1})$$

and all other variables and parameters are non-negative, then the agent can only have a non-negative expected utility for  $r = 0$  when both  $f_a$  and  $m_a$  are zero, otherwise if either  $m_a > 0$  or  $f_a > 0$ , then he can only have a non-negative expected utility when  $r > 0$ .

If a contract is optimal and the individual rationality constraint (participation constraint) is

binding, then at optimality the agent's expected utility,  $E[U_A]$  will be equal to his reservation utility,  $U_r$ . Therefore in this case we can rearrange the individual rationality constraint to obtain  $r$ :

$$\begin{aligned} U_r &= \mu \left( 1 - e^{-k(f_a^* + \frac{m_a^* + m_p^*}{c_h})} \right) r^* R_T - c_a f_a^* - b_a m_a^* \\ \Rightarrow r^* &= \frac{U_r + c_a f_a^* + b_a m_a^*}{\mu \left( 1 - e^{-k(f_a^* + \frac{m_a^* + m_p^*}{c_h})} \right) R_T} \end{aligned} \quad (D.2)$$

### Determining the optimal values of $m_a$ , $m_p$ , $r$ and $f_a$

We begin by obtaining the derivative of  $E[U_A]$  with respect to the agent's effort level, and the derivatives of  $E[\Pi]$  with respect to  $m_a$  and  $m_p$ .

$$\frac{\partial E[U_A]}{\partial f_a} = k\mu R_T r e^{-k(f_a + \frac{m_a + m_p}{c_h})} - c_a \quad (D.3)$$

In the cases where the individual rationality constraint is binding,  $r$  is as described by (D.2), hence we can substitute  $r$  in (D.1) to obtain:

$$\frac{\partial E[U_A]}{\partial f_a} = k\mu \left( \frac{U_r + c_a f_a + b_a m_a}{\mu \left( 1 - e^{-k(f_a + \frac{m_a + m_p}{c_h})} \right) R_T} \right) R_T e^{-k(f_a + \frac{m_a + m_p}{c_h})} - c_a \quad (D.4)$$

$$\frac{\partial E[\Pi]}{\partial m_a} = \frac{k\mu}{c_h} (R_T + R_D) e^{-k(f_a + \frac{m_a + m_p}{c_h})} - b_a \quad (D.5)$$

$$\frac{\partial E[\Pi]}{\partial m_p} = \frac{k\mu}{c_h} (R_T + R_D) e^{-k(f_a + \frac{m_a + m_p}{c_h})} - b_p \quad (D.6)$$

Case 1a :  $f_a^* = 0$ ,  $m_a^* = 0$ ,  $m_p^* = 0$ ,  $r^* \geq 0$

Since  $b_a \geq b_p \geq 1$  and  $c_h > 0$ , then from the perspective of the principal we know  $\frac{\partial E[\Pi]}{\partial m_a} \leq \frac{\partial E[\Pi]}{\partial m_p}$  hence we only need to have  $\frac{\partial E[\Pi]}{\partial m_p} \leq 0$  as a condition for  $m_a = m_p = 0$  to be optimal.

When  $f_a^* = m_a^* = m_p^* = r^* = 0$  we have:

$$\begin{aligned} \frac{\partial E[\Pi]}{\partial m_p} &= \frac{k\mu}{c_h}(R_T + R_D)e^{-k(f_a^* + \frac{m_a^* + m_p^*}{c_h})} - b_p \leq 0 \\ &\Rightarrow \frac{k\mu}{c_h}(R_T + R_D) - b_p \leq 0 \\ &\quad \frac{k\mu(R_T + R_D)}{b_p c_h} \leq 1 \end{aligned} \quad (D.7)$$

Now when  $f_a^* = m_a^* = m_p^* = 0$  the corresponding expected profit, assuming the agent's reservation utility is not violated, is  $E[\Pi] = -U_r$ . Therefore the only way the principal can obtain a non-negative expected profit is if the agent's reservation utility,  $U_r$ , is equal to zero which would imply that  $E[\Pi] = 0$ . As explained in the previous subsection, when  $f_a = m_a = m_p = 0$ , the corresponding value of  $r$  can only be  $r = 0$ .

Case 1b :  $f_a^* = 0, m_a^* = 0, m_p^* > 0$

Suppose the principal proposes a contract with  $m_a = 0$  and  $m_p$  and  $r$  greater than zero. If we assume that the individual rationality constraint is not binding, then the principal's expected profit is described by

$$E[\Pi] = \mu(1 - e^{-k\frac{m_p}{c_h}})(R_T - rR_T + R_D) - b_p m_p \quad (D.8)$$

$$\Rightarrow \frac{\partial E[\Pi]}{\partial r} = \mu R_T e^{-k\frac{m_p}{c_h}} \quad (D.9)$$

From (D.9), the principal can not infer an optimal value of  $r$ , hence no contract with an optimal  $r$  with respect to  $E[\Pi]$  can be set when individual rationality constraint is not binding, therefore an appropriate value of  $r$  can be determined by using the individual rationality constraint as an equality.

We consider a scenario where the individual rationality is binding. If the agent sets his effort level at  $f_a = 0$  then satisfying the incentive compatibility constraint implies that the agent's optimal expected utility coincides with  $f_a = 0$ . This can only be the case if  $\frac{\partial E[U_A]}{\partial f_a} \leq 0$  and  $\frac{\partial^2 E[U_A]}{\partial f_a^2} \leq 0$  where  $\frac{\partial E[U_A]}{\partial f_a}$  is given by (D.3). Since  $k\mu R_T r \geq 0$ , then for any non-negative values of  $f_a, m_a$  and  $m_p$ ,  $\frac{\partial^2 E[U_A]}{\partial f_a^2} \leq 0$ . Now, since  $f_a = m_a = 0$  and using the (D.4),

$\frac{\partial E[U_A]}{\partial f_a} \leq 0$  implies that

$$\frac{kU_r}{c_a} + 1 \leq \frac{k\mu(R_T + R_D)}{b_p c_h} \quad (\text{D.10})$$

which we consider as a required condition for an optimal solution that has  $f_a^* = 0$  when  $m_a^* = 0$ ,  $m_p^* > 0$

There is no upper bound for the possible values of  $m_p$  and thus the optimal value of  $m_p$  occurs when  $\frac{\partial E[\Pi]}{\partial m_p} = 0$ . If  $f_a^* = 0$  and  $m_a^* = 0$  then from (D.6), we can obtain  $m_p^*$  when :

$$\begin{aligned} \frac{\partial E[\Pi]}{\partial m_p} &= 0 \\ \Rightarrow m_p^* &= \frac{c_h}{k} \ln \left( \frac{k\mu(R_T + R_D)}{b_p c_h} \right) \end{aligned} \quad (\text{D.11})$$

A required condition for  $m_p^* > 0$  is therefore that  $\frac{k\mu(R_T + R_D)}{b_p c_h} > 1$ .

Since  $m_p^* = \frac{c_h}{k} \ln \left( \frac{k\mu(R_T + R_D)}{b_p c_h} \right)$  and  $f_a^* = m_a^* = 0$ , from (D.2) we obtain the optimal value of  $r$  as

$$r^* = \frac{U_r}{\left(1 - \frac{b_p c_h}{k\mu(R_T + R_D)}\right) \mu R_T} \quad (\text{D.12})$$

Case 2a :  $0 < f_a^* < f_L$ ,  $m_a^* = 0$ ,  $m_p^* = 0$

Suppose the principal proposes a contract with  $m_a = m_p = 0$  and the individual rationality constraint is satisfied but not necessarily binding. If the agent's corresponding optimal effort level is between 0 and  $f_L$  then it must be that at  $f_a^*$ ,  $\frac{\partial E[U_A]}{\partial f_a} = 0$ . We solve for the optimal effort level using (D.3) with  $m_a$  and  $m_p$  set to zero :

$$\begin{aligned} \frac{\partial E[U_A]}{\partial f_a} &= kr\mu R_T e^{-kf_a} - c_a = 0 \\ \Rightarrow f_a^* &= \frac{1}{k} \ln \left( \frac{k\mu r R_T}{c_a} \right) \end{aligned} \quad (\text{D.13})$$

Now for the principal, setting  $m_a = m_p = 0$  and with  $f_a$  as described by (D.13), then an optimal value of  $r$  occurs when  $\frac{\partial E[\Pi]}{\partial r} = 0$ . Now in this case we have :

$$\begin{aligned} E[\Pi] &= \mu(1 - e^{-kf_a^*})((1 - r)R_T + R_D) \\ &= \mu \left(1 - \frac{c_a}{kr\mu R_T}\right) (R_T - rR_T + R_D) \\ \text{hence } \frac{\partial E[\Pi]}{\partial r} &= -\mu R_T + \frac{c_a(R_T + R_D)}{kr^2 R_T} \end{aligned}$$

Solving  $\frac{\partial E[\Pi]}{\partial r} = 0$  for  $r^*$ , we get

$$r^* = \sqrt{\frac{c_a(R_T + R_D)}{k\mu R_T^2}} \quad (\text{D.14})$$

We now substitute this value of  $r^*$  back into the  $f_a^*$  described by (D.13) to get

$$f_a^* = \frac{1}{k} \ln \left( \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \right) \quad (\text{D.15})$$

Since  $0 < f_a^* < f_L$  then

$$1 < \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} < e^{kf_L} \quad (\text{D.16})$$

Since the individual rationality constraint must be satisfied, a required condition for this solution is that  $E[U_A] \geq U_r$ . Using the values of  $r$  and  $f_a$  as specified by (D.14) and (D.15) respectively with  $m_a = m_p = 0$ ,  $E[U_A] \geq U_r$  implies that :

$$\frac{kU_r}{c_a} + 1 \leq \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} - \ln \left( \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \right) \quad (\text{D.17})$$

Case 2b :  $0 < f_a^* < f_L$ ,  $m_a^* = 0$ ,  $m_p^* = 0$

Suppose the individual rationality constraint is binding and the agent's optimal effort level is  $0 < f_a^* < f_L$ . Then this implies that

$$\frac{1}{k} \ln \left( \frac{k\mu r R_T}{c_a} \right) < f_L. \quad (\text{D.18})$$

A non-negative value of  $f_a^*$  implies that a feasible value of  $r$  must satisfy  $r > \frac{c_a}{k\mu R_T}$ . If  $m_a = m_p = 0$  and  $r$  is given by (D.2) we find that (D.18) is equivalent to the condition that

$$\frac{kU_r}{c_a} + kf_L + 1 < e^{kf_L} \quad (\text{D.19})$$

At optimal the value of  $r$  should also coincide with the value of  $f_a$  such that  $\frac{\partial E[U_A]}{\partial f_a} = 0$ . We previously found the value of optimal  $r$  to be of the form given in (5.21), and plugging this  $r$  into the binding individual rationality constraint with  $m_a = m_p = 0$ , we find that the optimal  $r$  should also satisfy

$$\frac{kU_r}{c_a} + 1 = \frac{k\mu r R_T}{c_a} - \ln\left(\frac{k\mu r R_T}{c_a}\right) \quad (\text{D.20})$$

We can then solve for  $r$  using the Lambert W function to obtain

$$r^* = -\frac{c_a}{k\mu R_T} W\left(-e^{-\left(\frac{kU_r}{c_a} + 1\right)}\right) \quad (\text{D.21})$$

where  $W(\cdot)$  denotes the Lambert W function. Since  $k$ ,  $U_r$  and  $c_a$  are non-negative, we can easily verify that the argument of  $W(\cdot)$  in our case satisfies  $-\frac{1}{e} \leq -e^{-(kU_r/c_a + 1)} \leq 1$ . For  $x$  between  $-\frac{1}{e}$  and 0,  $W(x)$  is double-valued. This therefore implies that there are two values of  $r$  that satisfy (D.20). The principal's expected profit can be computed with both values of  $r$  to determine the preferable value between the two.

To find the corresponding optimal value of  $f_a$ , we note that (D.20) above is equivalent to

$$\frac{kU_r}{c_a} + 1 = \frac{k\mu r R_T}{c_a} - kf_a^*$$

where  $f_a^*$  is as given in (D.13). Solving for the optimal effort level, we find that  $f_a^*$  is as shown below

$$f_a^* = \frac{\mu r R_T}{c_a} - \frac{U_r}{c_a} - \frac{1}{k} \quad (\text{D.22})$$

Case 3a.i :  $f_a^* = f_L$ ,  $m_a^* = 0$ ,  $m_p^* = 0$

Suppose the principal proposes a contract with  $m_a = m_p = 0$  and the individual rationality

constraint is binding. If the agent's optimal effort level,  $f_a^* = f_L$ , then from (D.2) we know that the corresponding optimal value of  $r$  under this case would be

$$r^* = \frac{U_r + c_a f_L}{(1 - e^{-k f_L}) \mu R_T} \quad (\text{D.23})$$

If the agent's optimal effort level  $f_a^* = f_L > 0$  then it must be that  $\frac{\partial E[U_A]}{\partial f_a} \geq 0$  at  $f_a = f_L$ . Substituting  $r$  in  $\frac{\partial E[U_A]}{\partial f_a} = k \mu r R_T e^{-k f_L} - c_a \geq 0$  with (D.23) where  $m_a = m_p = 0$  we can obtain the equivalent expression:

$$\frac{k U_r}{c_a} + k f_L + 1 \geq e^{k f_L}$$

Optimal  $m_a$  and  $m_p$  values of zero imply that  $\frac{\partial E[\Pi]}{\partial m_p} \leq 0$  hence :

$$\frac{k \mu}{c_h} (R_T + R_D) e^{-k f_L} - b_p \leq 0 \quad \Rightarrow \quad \frac{k \mu (R_T + R_D)}{b_p c_h} \leq e^{k f_L}$$

Hence a required condition for the solution  $f_a = f_L$ ,  $m_a = m_p = 0$  to be optimal is that

$$\frac{k \mu (R_T + R_D)}{b_p c_h} \leq e^{k f_L} \leq \frac{k U_r}{c_a} + k f_L + 1 \quad (\text{D.24})$$

*Case 3a.ii :  $f_a^* = f_L$ ,  $m_a^* = 0$ ,  $m_p^* > 0$*

An optimal  $m_p > 0$  occurs where  $\frac{\partial E[\Pi]}{\partial m_p} = 0$ . We can solve  $\frac{\partial E[\Pi]}{\partial m_p} = 0$  for the optimal  $m_p$  with  $f_a^* = f_L$  and  $m_a^* = 0$  to obtain

$$m_p^* = c_h \left( \frac{1}{k} \ln \left( \frac{k \mu (R_T + R_D)}{b_p c_h} \right) - f_L \right) \quad (\text{D.25})$$

A required condition for  $m_p^* > 0$  is therefore that  $\frac{1}{k} \ln \left( \frac{k \mu (R_T + R_D)}{b_p c_h} \right) > f_L$  which means

$$\frac{k \mu (R_T + R_D)}{b_p c_h} > e^{k f_L} \quad (\text{D.26})$$

Also if  $f_L > 0$  and  $f_a^* = f_L$  then it must be that with  $m_p$  as described above in (D.25) and  $m_a = 0$  and  $r$  given by (D.2), then  $\frac{\partial E[U_A]}{\partial f_a} \geq 0$  which implies that

$$e^{k f_L} \leq \frac{k U_r}{c_a} + k f_L + 1 \quad (\text{D.27})$$

With  $m_a = 0$ ,  $f_a = f_L$  and  $m_p$  as described above in (D.25) we obtain the optimal value of  $r$  as

$$r = \frac{U_r + c_a f_L}{\left(1 - \frac{b_p c_h}{k\mu(R_T + R_D)}\right)\mu R_T} \quad (\text{D.28})$$

Case 3b :  $f_a^* = f_L$ ,  $m_a^* > 0$ ,  $m_p^* = 0$

Previously we determined that  $\frac{\partial E[U_A]}{\partial f_a}$  reaches zero at a point where  $f_a = \frac{1}{k} \ln \left( \frac{k\mu R_T r}{c_a} \right) - \frac{m_a + m_p}{c_h}$  with  $m_p$  fixed at 0 in this case. Now if the agent exerts an effort level of  $f_a = f_L$ , then it may be that  $\frac{\partial E[U_A]}{\partial f_a} \geq 0$ , as  $f_L$  is the upper bound of the agent's actual capabilities.  $f_L$  may in fact be lower than or equal to the value of  $f_a$  at which  $E[U_A]$  is maximized. Therefore we consider a case where

$$f_L \leq \frac{1}{k} \ln \left( \frac{k\mu R_T r}{c_a} \right) - \frac{m_a}{c_h} \quad (\text{D.29})$$

We determined earlier that the optimal value of  $r$  where  $\frac{\partial E[\Pi]}{\partial r} = 0$  is as given in (D.14). Substituting this value of  $r$  into (D.29) we obtain

$$e^{kf_L} \leq \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \quad (\text{D.30})$$

If  $m_a^*$  is optimal, then solving  $\frac{\partial E[\Pi]}{\partial m_p} = 0$  for  $m_a$  with  $m_p = 0$  and  $f_a = f_L$  we obtain an optimal  $m_a$  value of

$$m_a^* = c_h \left( \frac{1}{k} \ln \left( \frac{k\mu R_T r}{c_a} \right) - f_L \right) \quad (\text{D.31})$$

Substituting  $r^*$  into (D.31) we obtain the optimal  $m_a$  as

$$m_a = c_h \left( \frac{1}{k} \ln \left( \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \right) - f_L \right) \quad (\text{D.32})$$

These values of  $m_a$ ,  $r$ ,  $f_a$  and  $m_p$  should satisfy the individual rationality constraint,  $E[U_A] \geq U_r$  which means

$$\frac{kU_r}{c_a} + kf_L + 1 + \frac{b_a c_h k}{c_a} \left( \frac{1}{k} \ln \left( \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \right) - f_L \right) \leq \sqrt{\frac{k\mu(R_T + R_D)}{c_a}} \quad (\text{D.33})$$



Case 3c :  $f_a^* = f_L, m_a^* = 0, m_p^* > 0$

Suppose that  $\frac{\partial E[U_A]}{\partial f_a} = 0$  when  $f_a = f_L$ . This implies that

$$\begin{aligned} k\mu R_T r e^{-k\left(f_L + \frac{m_p}{c_h}\right)} - c_a &= 0 \\ m_p &= c_h \left[ \frac{1}{k} \ln \left( \frac{k\mu R_T r}{c_a} \right) - f_L \right] \end{aligned} \quad (\text{D.34})$$

Now the optimal value of  $r$  must also satisfy (D.2) when  $m_a = 0, f_a = f_L$  and with  $m_p$  as described above in (D.34). This gives an  $r$  value of

$$r = \frac{c_a}{k\mu R_T} \left( \frac{kU_r}{c_a} + kf_L + 1 \right) \quad (\text{D.35})$$

Substituting back (D.35) for  $r$  in (D.34) we obtain

$$m_p = c_h \left[ \frac{1}{k} \ln \left( \frac{kU_r}{c_a} + kf_L + 1 \right) - f_L \right] \quad (\text{D.36})$$

An optimal  $m_p > 0$  implies that

$$\begin{aligned} \frac{1}{k} \ln \left( \frac{kU_r}{c_a} + kf_L + 1 \right) &> f_L \\ \Rightarrow \frac{kU_r}{c_a} + kf_L + 1 &> e^{kf_L} \quad \square \end{aligned}$$

# Appendix E

## C++ Code for solving Co-Development Model

The code and header files required for calculations involving the Lambert W function can be found at [39].

### E.1 Header Files

#### E.1.1 MyFxns.h

**Declaration of variables and functions used**

```
#ifndef MYFXNS_H
#define MYFXNS_H
#include<vector>
using namespace std;

// Declare variables and Vectors
extern double h , pen , pen2, EPS, RT, RD, c_a, c_h, b_p, b_a, Mu, k, ur, f_L;
extern int ParaSet;
extern vector<double> Parameters;
extern vector<double> result1, result2, result3;
```

```
void ParameterSet(const int ParaSet);
int Contour(const int ParaSet);

vector<double> Numerical(const int ParaSet, const int type);
vector<double> search(const double k, const double ur, const double f_L);
vector<double> search(const int ParaSet);

// fxn for Principle's profit
double principal( const double m_a, const double m_p, const double r, const double Age);

// fxn for Agent's utility
double agent(
const double m_a, const double m_p, const double r, const double Agent_f);

// NUMERICAL FUNCTIONS

// Optimal effort  $f_a = \max(0, \cdot)$  where  $\text{del}E[U] / \text{delf} = 0$ 
double f_a(const int type, const double m_a, const double m_p, const double rRT0);

// Principal's net profit
double E_Pi(const int type, const double m_a, const double m_p, const double rRT0);

// Agent's expected utility
double U(const int type, const double m_a, const double m_p, const double rRT0);

// SUMT objective fxn (transformed problem)
double Z(const int type, const double m_a, const double m_p, const double rRT0);

// Contract Finder (Checks IC constraint)
```

```
int ContractFinder(const vector<double> SolnContainer);

#endif
```

### **E.1.2 MatlabFxn.h**

**in-line declaration of functions that have the same functionality as specific MATLAB functions**

```
#ifndef MATLABFXNS_H
#define MATLABFXNS_H
#include<vector>

// mimics MATLAB linspace
std::vector<double> linspace(const double a, const double b, const int n);

// mimics MATLAB rand
double urand();

// mimics MATLAB min
double min(const double a, const double b);

// mimics MATLAB max
double max(const double a, const double b);

#endif
```

## **E.2 CPP Files**

### **E.2.1 Main.cpp**

**file containing main() method**

```
#include "MyFxn.h"
#include <iostream>
#include <ctime>
#include <iomanip>
#include <fstream>
using namespace std;

int ParaSet = 0;
int main(){
//overall timer
clock_t t0;
t0 = clock();

int prsn = 8; // output precision
// for the Parameter set ...
for (int ParaSet = 1; ParaSet <= 7; ParaSet++){
//Print out results for each parameter set
cout << "Parameter Set :" << ParaSet << endl << endl;
vector<double> Analytical(7);
Analytical = search(ParaSet);
cout << "ANALYTICAL: " << endl;
for (int iii = 1; iii < 7; iii++){
cout << setprecision(prsn) << Analytical[iii] << " "; }
cout << endl << endl;

time_t tic = time(0);
vector<double> result1, result2, result3, Null;
result1 = Numerical(ParaSet, 1); //f_a = 0
result2 = Numerical(ParaSet, 2); // f_a>0
result3 = Numerical(ParaSet, 3); // f_a =f_L
```

```

Null = { 0 };
if (result1 == Null || result2 == Null || result3 == Null){
cin.get();
return 0; }
cout << "NUMERICALS: " << endl;
for (int iii = 0; iii <= 5; iii++){
cout << setprecision(prsn) << result1[iii] << " "; }
cout << endl;
for (int iii = 0; iii <= 5; iii++){
cout << setprecision(prsn) << result2[iii] << " "; }
cout << endl;
for (int iii = 0; iii <= 5; iii++){
cout << setprecision(prsn) << result3[iii] << " "; }
cout << endl;

vector<double> SolnContainer(18);
for (int i = 0; i < 18; i++){
if (i<6)
SolnContainer[i] = result1[i];
else if (i>5 && i < 12)
SolnContainer[i] = result2[i - 6];
else
SolnContainer[i] = result3[i - 12]; }
int bestSolnType = ContractFinder(SolnContainer);
cout <<endl
<< "-----" << endl; }

cout << "Total Time : " << setprecision(prsn) <<
double((clock() - t0)) / CLOCKS_PER_SEC
<< " seconds....." << endl << endl;

```

```
cout << "press ENTER to exit";  
cin.get();  
return 0;}
```

## E.2.2 MatlabFxnns.cpp

```
#include "MatlabFxnns.h"  
  
using namespace std;  
// mimics MATLAB rand  
double urand(){ return double(rand()) / RAND_MAX; }  
  
// mimics MATLAB min  
double min(const double a, const double b){  
if (a < b) return a;  
else return b;  
}  
  
// mimics MATLAB max  
double max(const double a, const double b){  
if (a < b) return b;  
else return a;  
}  
  
// mimics MATLAB linspace fxn; returns a vector type of size n  
vector<double> linspace(const double a, const double b, const int n){  
  
vector<double> lspace(n); // the (n) refers to number of elements  
double intvl = (b - a) / (n - 1);
```





```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% * /
```

```
switch (ParaSet){
```

```
case 1:
```

```
f_L = 10250; ur = 23.25e6; break;
```

```
case 2:
```

```
    f_L = 5250; ur = 23.25e6; break;
```

```
case 3:
```

```
c_h = 5 * c_a; f_L = 40e3; ur = 1.75e6; break;
```

```
case 4:
```

```
c_h = 5 * c_a; f_L = 40e3; ur = 6e6; break;
```

```
case 5:
```

```
c_h = 1.25*c_a; f_L = 30e3; ur = 4.75e6; break;
```

```
case 6:
```

```
c_h = 1.25*c_a; f_L = 20e3; ur = 2.8e3; break;
```

```
case 7:
```

```
c_h = 1.25*c_a; f_L = 22e3; ur = 33e6; break;
```

```
case 8:
```

```
RT = 10E6; RD = 372E6; c_a = 40; c_h = 1.25*c_a;
```

```
b_p = 1.05; b_a = 1.07; k = 2e-4; Mu = 0.1274;
```

```
f_L = 10250; ur = 23.25e6; break;
```

```
case 9:
```

```

RT = 10E6;      RD = 372E6;      c_a = 40;      c_h = 1.25*c_a;
b_p = 1.05;     b_a = 1.07;     k = 5e-4;     Mu = 0.1274;
f_L = 10250;   ur = 23.25e6;   break;

case 10:
c_h = 5 * c_a; f_L = 50e3;   ur = 1000;   break;
}
}

// fxn for Principle's profit
double principal(
const double m_a, const double m_p, const double r, const double Agent_f){
return Mu*(1 - exp(-k*(Agent_f + (m_a + m_p) / c_h)))
*((1 - r)*RT + RD) - b_p*m_p;}

// fxn for Agent's utility
double agent(
const double m_a, const double m_p, const double r, const double Agent_f){
return Mu*(1 - exp(-k*(Agent_f + (m_a + m_p) / c_h)))*r*RT
- c_a*Agent_f - b_a*m_a;}

////////////////////////////////////
// NUMERICAL FUNCTIONS

// Optimal effort f_a = max(0, .) where delU / delf = 0
double f_a(const int type,
const double m_a, const double m_p, const double rRT0){

switch (type){
default:

```

```

return min(f_L, max(0, 1 / k * log(k*Mu*rRT0 / c_a)
- (m_a + m_p) / c_h)); break;
case 1 :
return 0; break;
case 3: return f_L; break;
} }

// Principal's net profit
double E_Pi(const int type,
const double m_a, const double m_p, const double rRT0){

return Mu* (1 - exp(-k * (f_a(type, m_a, m_p, rRT0) + (m_a + m_p) / c_h)))
*(RT - rRT0 + RD) - b_p*m_p;
}

// Agent's expected utility
double U(const int type,
const double m_a, const double m_p, const double rRT0){

return Mu*(1 - exp(-k * (f_a(type, m_a, m_p, rRT0) + (m_a + m_p) / c_h)))
*rRT0 - b_a*m_a - c_a*f_a(type, m_a, m_p, rRT0);
}

// SUMT objective fxns for each case
double Z(const int type,
const double m_a, const double m_p, const double rRT0){

switch (type){
default:
return

```

```

E_Pi(type, m_a, m_p, rRT0) - pen * (
1 / (-ur + U(type, m_a, m_p, rRT0)) +
pow(pen, -1.5) * (
-c_a + k*Mu*rRT0*
exp(-k*(f_a(type, m_a, m_p, rRT0) + (m_a + m_p) / c_h))) *
(
-c_a + k*Mu*rRT0*
exp(-k*(f_a(type, m_a, m_p, rRT0) + (m_a + m_p) / c_h)))
+
1 / (RT + RD - m_a) + 1 / (RT + RD - m_p) + 1 / (RT + RD - rRT0) +
1 / exp(m_a) + 1 / exp(m_p) + 1 / rRT0); break;
case 1:
return
E_Pi(type, m_a, m_p, rRT0) - pen*(
1 / (-ur + U(type, m_a, m_p, rRT0)) +
1 / (
(m_a + m_p) / c_h - 1 / k*log(k*Mu*rRT0 / c_a)) +
1 / (RT + RD - m_a) + 1 / (RT + RD - m_p) + 1 / (RT + RD - rRT0) +
1 / exp(m_a) + 1 / exp(m_p) + 1 / rRT0); break;
case 3:
return
E_Pi(type, m_a, m_p, rRT0) - pen*(
1 / (U(type, m_a, m_p, rRT0) - (ur + 1)) +
pen2 / (((1 / k * log(k*Mu*rRT0 / c_a) - (m_a + m_p) / c_h) - (f_L + 1)) +
pen2 / ((k*Mu*rRT0 * exp(-k*(f_L + (m_a + m_p) / c_h))) - (c_a + 1)) +
1 / (RT + RD - m_a) + 1 / (RT + RD - m_p) + 1 / (RT + RD - rRT0) +
1 / exp(m_a) + 1 / exp(m_p) + 1 / rRT0); break;
}}

```

## E.2.4 search.cpp

```
/* For a given set of parameters, this function determines the
the best contract type from the the analytical options */
```

```
#include <cmath>
#include <vector>
#include "MyFxn.h"
#include "LambertW.h"

using namespace std;

vector<double> search(
const double k, const double ur, const double f_L){

// declaring the values to store
const int vectorSize = 7; const int endo = vectorSize - 1;
vector<double> optimalSoln(vectorSize);
vector<double> currentSoln(vectorSize);

// INITIALIZE optimalSoln
optimalSoln[0] = 9; // if not soln keep it at 9
for (int i = 1; i < 7; i++) optimalSoln[i] = 0;
double m_a, m_p, r, Agent_f, E_U, E_Pi;
m_a = m_p = r = Agent_f = E_U = E_Pi = 0;

// Define  $k*m \Rightarrow k*\mu*(RT + RD)$ ;
double m =  $\mu*(RT + RD)$ ;
double solnNum;
```

```

// SEARCH FOR BEST SOLUTION STARTS HERE

// SOLUTION 1a;
if ((k*m / (b_p*c_h) <= 1) && (ur == 0)){
solnNum = 1;
currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
//currentSoln = { solnNum, m_a, m_p, r, Agent_f, E_U, E_Pi };
optimalSoln = currentSoln; // if condition satisfied, becomes optimal
}

// SOLUTION 1b;
if (((k*m / (b_p*c_h)) > 1) && ((k*m / (b_p*c_h)) >= (k*ur / c_a + 1))){
solnNum = 2;
m_a = 0;
m_p = c_h / k * log(k*m / (b_p*c_h));
r = ur / ((1 - (b_p*c_h) / (k*m))*Mu*RT);
Agent_f = 0;
E_Pi = principal(m_a, m_p, r, Agent_f); // (k, Mu, m_a, m_p, r, Agent_f);
E_U = agent(m_a, m_p, r, Agent_f);
currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
if (((E_Pi > optimalSoln[endo]) && ((E_U - ur) > -1))){ // % if current soln better,
optimalSoln = currentSoln;
} }

// SOLUTION 2a;
if (((1 < sqrt(k*m / c_a)) && (sqrt(k*m / c_a) < exp(k*f_L))
&& (sqrt(k*m / c_a) - log(sqrt(k*m / c_a)) >= k*ur / c_a + 1))){
solnNum = 3;
m_a = 0;
m_p = 0;

```

```

r = sqrt(c_a*(RT + RD) / (k*Mu* RT*RT));
Agent_f = 1 / k * log(sqrt((k*Mu*(RT + RD)) / c_a));
E_Pi = principal(m_a, m_p, r, Agent_f);    //(k, Mu, m_a, m_p, r, Agent_f);
E_U = agent(m_a, m_p, r, Agent_f);
currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
if (((E_Pi > optimalSoln[endo]) && ((E_U - ur) > -1))){// % if current soln better, re
optimalSoln = currentSoln;
} }

// SOLUTION 2b;
if ((exp(k*f_L) > k*ur / c_a + k*f_L + 1)){
solnNum = 4;
m_a = 0;
m_p = 0;

// Upper Branch LambertW
r = -c_a / (k*Mu*RT) * utl::LambertW(0, -exp(-k*ur / c_a - 1));
Agent_f = Mu*r*RT / c_a - ur / c_a - 1 / k;
E_Pi = principal(m_a, m_p, r, Agent_f);    //(k, Mu, m_a, m_p, r, Agent_f);
E_U = agent(m_a, m_p, r, Agent_f);
currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
if (((E_Pi > optimalSoln[endo]) && ((E_U - ur) > -1))){// % if current soln better, re
optimalSoln = currentSoln;
}

// Lower Branch LambertW
r = -c_a / (k*Mu*RT) * utl::LambertW(-1, -exp(-k*ur / c_a - 1));
Agent_f = Mu*r*RT / c_a - ur / c_a - 1 / k;
E_Pi = principal(m_a, m_p, r, Agent_f);    //(k, Mu, m_a, m_p, r, Agent_f);
E_U = agent(m_a, m_p, r, Agent_f);

```

```

currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
if (((E_Pi > optimalSoln[endo]) && ((E_U - ur) > -1))){// % if current soln better,
optimalSoln = currentSoln;
} }

```

```

// SOLUTION 3a.i;
if (((k*m / (b_p*c_h) <= exp(k*f_L)) &&
(exp(k*f_L) <= k*ur / c_a + k*f_L + 1) && (exp(k*f_L) > 1))) {
solnNum = 5;
m_a = 0;
m_p = 0;
r = (ur + c_a*f_L) / ((1 - exp(-k*f_L))*Mu*RT);
Agent_f = f_L;
E_Pi = principal(m_a, m_p, r, Agent_f); // (k, Mu, m_a, m_p, r, Agent_f);
E_U = agent(m_a, m_p, r, Agent_f);
currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
if (((E_Pi > optimalSoln[endo]) && ((E_U - ur) > -1))){// % if current soln better,
optimalSoln = currentSoln;
} }

```

```

// SOLUTION 3a.ii;
if (((exp(k*f_L) < (k*m / (b_p*c_h))) &&
((k*m / (b_p*c_h)) <= (k*ur / c_a + k*f_L + 1)))) {
solnNum = 6;
m_a = 0;
m_p = c_h*(1 / k*log(k*m / (b_p*c_h)) - f_L);
r = (ur + c_a*f_L) / ((1 - b_p*c_h / (k*m)) * Mu*RT);
Agent_f = f_L;
E_Pi = principal(m_a, m_p, r, Agent_f); // (k, Mu, m_a, m_p, r, Agent_f);

```



```

E_U = agent(m_a, m_p, r, Agent_f);
currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
if (((E_Pi > optimalSoln[endo]) && ((E_U - ur) > -1))){// % if current soln better, re
optimalSoln = currentSoln;
} }

// SOLUTION 3b;
if (((sqrt(k*m / c_a) >= exp(k*f_L)) &&
(sqrt(k*m / c_a) >= (k*ur / c_a + k*f_L + 1 + (k*b_a*c_h / c_a)*
(1 / k * log(sqrt(k*m / c_a)) - f_L))))){
solnNum = 7;
m_a = c_h*(1 / k*log(sqrt(k*m / c_a)) - f_L);
m_p = 0;
r = sqrt(c_a*(RT + RD) / (k*Mu*RT*RT));
Agent_f = f_L;
E_Pi = principal(m_a, m_p, r, Agent_f); // (k, Mu, m_a, m_p, r, Agent_f);
E_U = agent(m_a, m_p, r, Agent_f);
currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
if (((E_Pi > optimalSoln[endo]) && ((E_U - ur) > -1))){// % if current soln better, re
optimalSoln = currentSoln;
} }

// SOLUTION 3c;
if ((exp(k*f_L) <= (k*ur / c_a + k*f_L + 1))) {
solnNum = 8;
m_a = 0;
m_p = c_h*(1 / k * log(k*ur / c_a + k*f_L + 1) - f_L);
r = c_a / (k*Mu*RT)*(k*ur / c_a + k*f_L + 1);
Agent_f = f_L;
E_Pi = principal(m_a, m_p, r, Agent_f); // (k, Mu, m_a, m_p, r, Agent_f);

```

```

E_U = agent(m_a, m_p, r, Agent_f);
currentSoln = { solnNum, Agent_f, m_a, m_p, r, E_U, E_Pi };
if (((E_Pi > optimalSoln[endo]) && ((E_U - ur) > -1))){// % if current soln better,
optimalSoln = currentSoln;
} }

return optimalSoln;

}

vector<double> search(const int ParaSet){
ParameterSet(ParaSet);
return search(k, ur, f_L);
}

```

### E.2.5 Numerical.cpp

**primary code for numerical approach**

```

#include <iostream>
#include "MatlabFxn.h"
#include "MyFxn.h"
#include <vector>
#include<ctime>

using namespace std;

vector<double> Numerical(const int ParaSet, const int type){

// will keep track of best soln from diff initial pts

```

```

double CurrentSoln = 0, fCurrent = 0,
maCurrent = 0, mpCurrent = 0, rRTCurrent = 0, UCurrent = 0;

// load Parameters
ParameterSet(ParaSet);

double ma, mp, rRT, Echeck, Ucheck; int N = 10;
vector<double> ma_values(N), mp_values(N), rRT_values(N);

// generate nCurr random feasible points and iterate frm there
int nCurr = 100;
for (int Currents = 1; Currents <= nCurr; Currents++){

// Randomly Generate initital trial soln
// and Check Feasibility

//overall timer
clock_t loopTimeStart;
loopTimeStart = clock();
int loopcount = 0;
do{
loopcount++;
ma = urand() * (RT + RD);
mp = urand() * (RT + RD);
rRT = urand() *(RT + RD);
Echeck = E_Pi(type, ma, mp, rRT);
Ucheck = U(type, ma, mp, rRT);
if ((clock() - loopTimeStart)/CLOCKS_PER_SEC > 10){
cout << "taking too long generating feasible random initial solution"
<< endl << "Random Trial Number : " << Currents << endl

```

```

<<"Number of generation attempts : "<<loopcount<<endl
<< "Last attempt : "
<< "ma = " << ma << "; mp = " << mp << "; rRT = " << rRT << endl
<< "E_Pi = " << Echeck << "; E_U = " << Ucheck << endl
<< "Consider changing method of generation" << endl;
cout << "press ENTER twice to quit" << endl;
cin.get();
return{ 0 };
}

} while ((Echeck < 0) || (Ucheck < ur));

for (int trials = 1; trials <= 10; trials++){

int opt_indx = 0; // will keep track of optimal indx
//calc gradient
double mah, mph, rRTh, del_ma, del_mp, del_rRT;
//double h = 0.1, pen = 100, pen2 = 1, EPS = 1e-3;
mah = ma + h; mph = mp + h; rRTh = rRT + h;
del_ma = (Z(type, mah, mp, rRT) - Z(type, ma, mp, rRT)) / h;
del_mp = (Z(type, ma, mph, rRT) - Z(type, ma, mp, rRT)) / h;
del_rRT = (Z(type, ma, mp, rRTh) - Z(type, ma, mp, rRT)) / h;

// if gradient is "zero" we are done
if (((abs(del_ma) < EPS) && (abs(del_mp) < EPS)
&& (abs(del_rRT) < EPS))){
trials = 11;
}

// Finding upper bound on L (stepsize for gradient search)

```

```

double L_for_ma, L_for_mp, L_for_rRT, L_ryt, L_lft;
L_for_ma = max((RT + RD - ma) / del_ma, -ma / del_ma);
L_for_mp = max((RT + RD - mp) / del_mp, -mp / del_mp);
L_for_rRT = max((RT + RD - rRT) / del_rRT, -rRT / del_rRT);
L_ryt = min(L_for_ma, min(L_for_mp, L_for_rRT));
L_lft = 0;

// for (int L_search = 0; L_search <= 7; L_search++){
int L_search = 0; bool quitLsearch = 0;
do{
L_search++;
//*****
vector<double> L_values(N), Z_values(N);

L_values = linspace(L_lft, L_ryt, N);
for (int ii = 0; ii < N; ii++){
Z_values[ii] = 0;
ma_values[ii] = ma + L_values[ii] * del_ma;
mp_values[ii] = mp + L_values[ii] * del_mp;
rRT_values[ii] = rRT + L_values[ii] * del_rRT;
}

// Calculate the Z value for each L trial
double top_Z = 0;
for (int z = 0; z < N; z++){
Z_values[z] =
Z(type, ma_values[z], mp_values[z], rRT_values[z]);

// Check that each of the Z values dsnt violate IR
// if IR violated, make Zvalue = -Inf

```

```
double IRcheck;
IRcheck =
U(type, ma_values[z], mp_values[z], rRT_values[z]);
if (IRcheck < ur) Z_values[z] = -1e9;

// keep track of best case
if (Z_values[z] >= top_Z){
opt_indx = z;
top_Z = Z_values[z];
} }

L_lft = L_values[opt_indx];

if (opt_indx == N - 1) { quitLsearch = 1; }
else {
L_ryt = L_values[opt_indx + 1];
}

} while (L_search <= 7 && quitLsearch == 0); // end search for best L

ma = ma_values[opt_indx];
mp = mp_values[opt_indx];
rRT = rRT_values[opt_indx];

} // ends for trials

double NewSoln = E_Pi(type, ma, mp, rRT);

if (NewSoln > CurrentSoln){
fCurrent = f_a(type, ma, mp, rRT);
```

```

CurrentSoln = NewSoln;
maCurrent = ma;
mpCurrent = mp;
rRTCurrent = rRT;
UCurrent = U(type, ma, mp, rRT);
} }

double r = rRTCurrent / RT;

return // vector containing optimal solution
{ fCurrent, maCurrent, mpCurrent, r , UCurrent, CurrentSoln };
}

```

## E.2.6 ContractFinder

**Code for checking IC and determining best contract**

```

#include <iostream>
#include "MyFxns.h"
#include "MatlabFxns.h"

using namespace std;

// Contract Finder
int ContractFinder(const vector<double> SolnContainer){

if (!(SolnContainer.size() % 6 == 0)){
cerr << "The solution container must contain 6,12 or 18 elements";
return 1;
}
}

```

```

}

int Num2Compare = (SolnContainer.size())/6 - 1;

ParameterSet(ParaSet);

int Best = -1; // -1->No contract, 0->f_a = 0; 1 -> 0<f_a<f_L
// 2->f_a = f_LZ

double BestUP = -1; // Stores best current expected profit
bool a, b, c; // to store contract types that satisfy IC
a = b = c = 0;
int nn = 30;
double SearchInterval = f_L / 10;

double dist = 0.5 * SearchInterval;
double AcceptTol = nn / 3;

for (int counta = 0; counta <= Num2Compare; counta++){

int iter = -1;
double f_A = SolnContainer[++iter + (6 * counta)];
double m_A = SolnContainer[++iter + (6 * counta)];
double m_P = SolnContainer[++iter + (6 * counta)];
double rr = SolnContainer[++iter + (6 * counta)];
double U_A = SolnContainer[++iter + (6 * counta)];
double U_P = SolnContainer[++iter + (6 * counta)];

double Search_lft = min(max(0, (f_A - dist)), (f_L - 2 * dist));
double Search_ryt = min(f_L, max((f_A + dist), (2 * dist)));

```



```

// Randomly generate nn different f_a values
// for each of the nn rndm f values calculate E[U]
int NumBetter = 0;
for (int tests = 1; tests <= nn; tests++){

// Randomly generate f_a value
double random_f = Search_lft + ((Search_ryt - Search_lft)*urand());
// cout << "random f " << random_f << " : ";
// calculate E[U]
double AgentUtility =
agent(m_A, m_P, rr, random_f);

if (AgentUtility >= U_A){
NumBetter++;
} }

/*% if all solutions are either lower than for f_a or
      close enough to f_a then then change send to
      hold solution... then compare to current best*/
bool cond1 = ( NumBetter <=AcceptTol);

bool cond2 = (U_P >= BestUP);

if (cond1 && cond2){
BestUP = U_P;
Best = counta;

switch (counta){
case 0: a = true; break;
case 1: b = true; break;

```

```
case 2: c = true; break;
} } }

// determine the best of the 3 contracts
int BestIndx = Best; //

cout<<endl << "BEST NUMERICAL CONTRACT : " ;
if (BestIndx == -1){
cout << endl
<< "\t No optimal Contract from given Parameters!!" <<
endl; }
else{
cout << "IC satisfied : " << a << b << c << endl;
cout << "Type " << BestIndx + 1 << endl;
for (int iii = 0; iii <= 5; iii++){
cout << SolnContainer[iii + (6 * BestIndx)] << " "; }
cout << endl << endl; }
return BestIndx + 1;}
```

# Curriculum Vitae

**Name:** Rodney Tembo

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