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Polynomial Identities on Algebras with Actions

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POLYNOMIAL IDENTITIES ON ALGEBRAS WITH ACTIONS

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by

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Abstract

When an algebra is endowed with the additional structure of an action or a grading, one can often make striking conclusions about the algebra based on the properties of the structure-induced subspaces. For example, if A is an associative G-graded algebra such that the homogeneous component A_1 satisfies an identity of degree d, then Bergen and Cohen showed that A is itself a PI-algebra. Bahturin, Giambruno and Riley later used combinatorial methods to show that the degree of the identity satisfied by A is bounded above by a function of d and $|G|$. Utilizing a similar approach, we prove an analogue of this result which applies to associative algebras whose induced Lie or Jordan algebras are G-graded.

Group-gradings and actions by a group of automorphisms are examples of Hopf algebras acting on H-algebras. If H is finite-dimensional, semisimple, commutative, and splits over its base field, then it is known that A is an H-algebra precisely when the H-action on A induces a certain group-grading of A. We extend this duality to incorporate other natural H-actions. To this end, we introduce the notion of an oriented H-algebra. For example, if A has an action by a group of both automorphisms and anti-automorphisms, then A is not an H-algebra, but A is an oriented H-algebra. The vector space gradings associated to oriented H-algebra actions are not generally group-gradings, or even set-gradings. However, when A is a Lie algebra, the grading is a quasigroup-grading, and, when A is an associative algebra, the grading is what we call a Lie-Jordan-grading.

Lastly, we call certain H-polynomials in the free associative H-algebra essential, and show that, if an (associative) H-algebra A satisfies an essential H-identity of degree d, then A satisfies an ordinary identity of bounded degree. Furthermore, in the case when H is m-dimensional. semisimple and commutative, we prove that, if A^H satisfies an ordinary identity of degree d, then A satisfies an essential H-identity of degree dm. From this we are able to recover several well-known results as special cases.

Keywords: Noncommutative Algebra, Polynomial Identity Algebras, Hopf Algebras, Graded Algebras, Anti-automorphisms.

Co-Authorship Statement

Some results appearing within also appear in the following papers:

(1) C. Plyley and D. Riley, Identities of algebras with Hopf actions, (submitted).

(2) C. Plyley and D. Riley, Extending the duality between actions and group-gradings, (submitted).

(3) C. Plyley and D. Riley, Identities on Lie or Jordan group-graded associative algebras, Journal of Algebra 287 (2013), 160-168.

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Contents

Chapter 1

Introduction

In order to study the properties of an algebra A, we sometimes endow A with some additional structure; in this thesis, this structure usually takes the form of a prescribed action or a grading. Once this is done, properties of A can sometimes be determined based on the properties of the structure-induced subspaces of A. This tactic has proven especially effective in polynomial identity theory. For example, if an associative algebra is graded by a finite group, then it will satisfy a polynomial identity whenever the homogeneous identity component of the grading does. Similarly, if an associative algebra has an action by automorphisms, then it will satisfy a polynomial identity whenever the subspace of fixed points does. Recently, combinatorial techniques have been utilized in order to find quantitative versions of these classic theorems. One of our major goals in this thesis is to use combinatorial methods to formulate several new quantitative theorems which encapsulate a wider range of both actions and gradings on algebras.

This type of additional structure can often be formalized as the action of a finite-dimensional Hopf algebra. In fact, a powerful duality between Hopf algebra actions and group-gradings has long acted as a bridge between the theory of graded algebras and the study of actions. This duality has been known to have deep implications in several areas of mathematics, including polynomial identity theory. Another major goal in this thesis is to extend this duality to incorporate other naturally occurring Hopf algebra actions, and as a consequence, various new types of gradings.

More specifically, an algebra A is said to satisfy a polynomial identity if there is a (nonzero) polynomial in noncommuting variables which evaluates to zero upon the substitution from any elements of A (in this case, we say that A is a PI-algebra). For example, any commutative algebra satisfies the polynomial identity $x_1x_2 - x_2x_1 = 0$, any nilpotent algebra satisfies the identity $x_1 \cdots x_n = 0$, for some $n > 0$, and one may also show that any finite-dimensional algebra is a PI-algebra. The interest in polynomial identity theory resides in the fact that, although demonstratively larger than the classes of commutative and finite-dimensional algebras, the class of PI-algebras still enjoys many nice structural properties; further, PI-algebras have been instrumental in the resolution of several important mathematical problems (see Section 2.1 for more details).

Questions in polynomial identity theory typically focus on either structure theory, or the study of the identities that an algebra satisfies. In this thesis, we are interested in the combinatorial study of identities. For example, consider the following result of Bergen and Cohen ([BC]): if A is an associative algebra which is graded by a finite group G, then A satisfies a polynomial identity whenever the identity component A_1 does. The proof of this result used a structure theory approach, and no general information on the degree of the identity was obtained. However, utilizing a combinatorial approach, Bahturin, Giambruno and Riley ([BGR]) were able to formulate a quantitative version of this result, showing that the degree of the identity satisfied by A is bounded by an explicit function depending only on $|G|$ and the degree of the identity satisfied by A_1. Our first major result utilizes a similar approach to show that an associative algebra A whose induced Lie or Jordan algebra is group-graded satisfies a polynomial identity of explicitly bounded degree whenever the identity component of this grading does.

The motivating example of an algebra whose induced Lie or Jordan algebra is group-graded (we call such a grading a Lie-group-grading or Jordan-group-grading, respectively) is an associative algebra with an action by an involution. Recall that an involution is an anti-automorphism of order 2; for instance, the transpose map the matrix algebra $M_n(K)$ is an involution. If A is any algebra with involution, then (as we will see) A is Lie-$\mathbb{Z}_2$-graded and Jordan-$\mathbb{Z}_2$-graded; moreover, the identity component of the Lie-group-grading is the subspace of skew-symmetric elements, and the identity component of the Jordan-group-grading is the subspace of symmetric elements. Therefore, as a corollary to our first major result, we obtain a quantitative version of the following classic result of Amitsur [Am1]: an associative algebra which admits an involution satisfies a polynomial identity whenever the symmetric or skew-symmetric elements satisfy an identity.

As it happens, there is an extensive relationship between actions and gradings on algebras. The precise nature of this relationship is rather deep and delicate, although certain aspects of it have been long known. For instance, in certain cases, actions by automorphisms on an algebra are equivalent to group-gradings. The origin of this well-known and powerful duality is difficult to pinpoint; it was noticed in [BI] and [Pa2], but was likely known earlier (appearing in Cartier duals, for instance). Similarly, a duality between actions by derivations and group-gradings, which is also well-known, appears in [BK] but can be deduced from earlier results. Both of these dualities may be formalized as examples of a (more general) duality between actions of finite-dimensional Hopf algebras and group-gradings, as was demonstrated by Bergen and Cohen in [BC].

There are actions of interest not included in this duality, however. For instance, as evidenced by the above mentioned theorem of Amitsur, an action by a group of automorphisms and anti-automorphisms is an interesting piece of additional structure in polynomial identity theory. Unfortunately, such an action cannot be recognized as an (ordinary) Hopf algebra action. For our second major result, we extend the duality appearing in [BC] to incorporate more general Hopf actions and gradings, including actions by anti-automorphisms and anti-derivations. To accomplish this, we first introduce the notion of an oriented Hopf algebra action. Subsequently, we demonstrate that every oriented Hopf action on A induces a vector space decomposition which is a quasigroup-grading when A is a Lie algebra and a Lie-Jordan-grading when A is an associative algebra.

Our last objective is to show how to take an all-encompassing Hopf algebra approach to polynomial identities. The so-called polynomial H-identities of an algebra with the action of a Hopf algebra have been previously considered (for instance, [BL]). We study a type of identity called an essential H-identity, and as a consequence, we are able to encapsulate and generalize several theorems from polynomial identity theory, including the aforementioned theorems of Bahturin, Giambruno and Riley and Amitsur.

The thesis is organized as follows. In Chapter 2, we provide background information about polynomial identity theory and graded algebras. We outline some of the historical, and some of the more recent, results from polynomial identity theory so as to place the subsequent matter appropriately within the literature. In Chapter 3, we prove our new result on Lie-group-gradings and Jordan-group-gradings of associative algebras. In Chapter 4, we review the existing duality of Hopf algebra actions and group-gradings, and subsequently show how to extend this duality. In Chapter 5, we describe a unified Hopf algebra approach to the study of polynomial identities. In Chapter 6, we explicitly describe our results as they apply to the two main predominant examples of (finite-dimensional) Hopf algebra actions: the group algebra and the restricted universal enveloping algebra of a restricted Lie algebra. Finally, in Chapter 7, we provide some applications and mention a few open problems.

This thesis will attempt to be (mostly) self-contained, however, we assume that the reader is familiar with basic concepts of graduate algebra; namely, the elementary properties of vector spaces, groups, modules, algebras, character theory, and tensor products. The necessary background information can be found in [Ja2] and [Ja4], or [Ro2] for instance.

Fixed Notation

The following notation will be fixed throughout the thesis, except if explicitly stated otherwise.

- K will always denote an arbitrary field of characteristic $p \geq 0$, and all vector spaces, tensor products, algebras, and Hopf algebras are assumed to be over K.

- A will always denote a (nonassociative) algebra.

- G will always denote a finite group.
- $\mathfrak{g}$ will always denote a finite-dimensional restricted Lie algebra.
- S_n will always denote the symmetric group on n elements.
- Actions on algebras will be frequently denoted with exponential notation; for instance, if a group G acts on an algebra A, then we write $a^g := g(a)$.
- We shall denote by $\mathrm{Aut}(A)$ the group of all automorphisms on an algebra A, contained in $\mathrm{Aut}^*(A)$, the group of all automorphisms and anti-automorphisms on A (note that we do not allow anti-automorphisms to be automorphisms).
- We shall denote by $\mathrm{Der}(A)$ the restricted Lie algebra of all derivations on an algebra A (over a field of positive characteristic), contained in $\mathrm{Der}^*(A)$, the restricted Lie algebra of all derivations and anti-derivations an algebra A (note that we do not allow anti-derivations to be derivations).

Chapter 2

Polynomial Identities and Graded Algebras

This chapter is meant to provide the reader with an introduction into polynomial identity theory, and to provide much of the necessary background information that will be needed in the sequel. In the first section, we provide a brief historical account of some of the classical results from polynomial identity theory. In the second section, we discuss group-graded algebras, and we define some other types of gradings that are perhaps more general than the reader is accustomed to. In the third section, we discuss two classic results of Amitsur which serve as an impetus for much of what follows in the thesis. We conclude this chapter with a brief introduction to the combinatorial approach in polynomial identity theory, and we describe some key results which will be addressed in later chapters.

We take a moment to acknowledge that the results presented here are by no means a complete account of the theory; many of the interesting and important aspects of polynomial identity theory which are not presently needed have been omitted. For more complete accounts, the interested reader should refer to [Ja3] and [Ro1] for a general reference on polynomial identity theory, or [DF] and [GZ] for a reference more specific to our current cause.

2.1 Polynomial Identity Theory

In this section, all algebras are assumed to be associative. The theory of polynomial identities began to take shape in 1948 with an influential paper of Kaplansky ([Ka1]). Previously, while attempting to classify certain projective geometries, M. Hall (who was following the earlier work of Dehn ([De]) and Wagner ([Wa])) showed that a division algebra D with the property that

$$(xy - yx)^2 z - z(xy - yx)^2 = 0, \text{ for all } x, y, z \in D,$$

is finite-dimensional over its center ([Ha]). Kaplansky recognized that the precise nature of the relation that D satisfied (he called such a relation a polynomial identity) was inconsequential, and in fact a division algebra satisfying *any* polynomial identity is finite-dimensional over its center. Consequently, it was evident that satisfying a polynomial identity was an important property for an algebra to have, and Kaplansky's theorem (which we state shortly) became the starting point for the branch of algebra now called polynomial identity theory.

Formally, if $X = \{x_1, x_2, \ldots\}$ is a countable set, then we let $\mathcal{A}\langle X\rangle$ denote the free associative K-algebra on the generators X. Recall that $\mathcal{A}\langle X\rangle$ has a basis consisting of all words $x_{i_1} \cdots x_{i_n}$, where $x_{i_j} \in X$, $n \in \mathbb{N}$, and multiplication is defined by juxtaposition:

$$(x_{i_1} \cdots x_{i_n})(x_{j_1} \cdots x_{j_m}) = x_{i_1} \cdots x_{i_n} x_{j_1} \cdots x_{j_m}.$$

The elements of $\mathcal{A}\langle X\rangle$ are called polynomials, and we often write a polynomial $f \in \mathcal{A}\langle X\rangle$ as $f(x_1, \ldots, x_n)$ to point out that $x_1, \ldots, x_n$ are the indeterminates appearing in f. The degree of a polynomial, written $\deg(f)$, is defined as the total degree of f; that is, if we define the degree of a monomial by $\deg(x_{i_1}^{m_1} \cdots x_{i_n}^{m_n}) = \sum_{i=1}^{n} m_i$, then $\deg(f)$ is the maximum degree of a monomial appearing in f.

The algebra $\mathcal{A}\langle X\rangle$ has the following universal property: if A is an associative algebra, then any set-theoretic map $\varphi : X \to A$ completes uniquely to an algebra homomorphism $\bar{\varphi} : \mathcal{A}\langle X\rangle \to A$. We define polynomial identities as follows.

Definition 2.1.1. *Let $X = \{x_1, x_2, \ldots\}$, let $f(x_1, \ldots, x_n) \in \mathcal{A}\langle X\rangle$, and let A be an associative algebra.*

1. *If $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$, then f is called a polynomial identity of A.*

2. *If A satisfies any nonzero polynomial identity, then A is called a polynomial identity algebra (or a PI-algebra).*

3. *The set of polynomial identities of A is denoted by $Id(A)$.*

Clearly, $\mathrm{Id}(A)$ is an ideal of $\mathcal{A}\langle X\rangle$, and furthermore, $\mathrm{Id}(A)$ is invariant under the endomorphisms of $\mathcal{A}\langle X\rangle$. If a polynomial has the form:

$$\sum_{\sigma\in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}, \quad \alpha_\sigma \in K,$$

then it is called a multilinear polynomial. It is well-known, and not too difficult to show, that an algebra over any field which satisfies a polynomial identity of degree n also satisfies a multilinear polynomial identity of degree n. Further, over fields of characteristic 0, all the polynomial identities of an algebra are generated by the multilinear polynomial identities. Therefore, in many situations, we may restrict ourselves to the study of multilinear polynomials. For a more complete description of the multilinearization process, the reader may consult the monograph [GZ].

To get a better sense of the class of PI-algebras, we consider several preliminary examples. First, observe that every commutative algebra is a PI-algebra since $[x_1, x_2] = x_1x_2 - x_2x_1$ is a polynomial identity. Less obvious, but not difficult to see, is that every finite-dimensional algebra A satisfies the identity

$$\sum_{\sigma\in S_n} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)},$$

whenever $n > \dim(A)$ (this particular identity is called the standard identity of degree n). Thus, the matrix algebra $M_n(K)$ satisfies the standard identity of degree $n^2 + 1$. Other examples of PI-algebras include nilpotent algebras (by definition), and the Grassman algebra.

Problems in polynomial identity are usually approached using either structure theory or certain combinatorial techniques. The aforementioned theorem of Kaplansky, which is still considered one of the most important results in polynomial identity theory, gives us insight into the structural properties of PI-algebras.

Theorem 2.1.2 (Kaplansky, 1948)**.** *Every primitive algebra satisfying a polynomial identity is finite-dimensional over its center.*

Other important structure theorems emerged in the ensuing years, notably, Posner's Theorem ([Po]) and Artin's Theorem ([Ar]). Without getting into detail, we mention that all of these major theorems lend towards a common theme: an algebra which satisfies a polynomial identity (and some other hypotheses) has a relatively large center. In this thesis, we refrain from the structural aspect of PI-algebras, so that we may focus on the (more recent) combinatorial approach.

This combinatorial approach in polynomial identity theory is rooted in the study of the identities that an algebra satisfies. Given a concrete algebra A, it is generally a very hard problem to determine a set of generators of $\mathrm{Id}(A)$. In fact, this question is unknown even for $M_n(K)$ with $n \geq 3$. The most famous related problem is the Specht problem ([Sp]), which asks: *If A is a PI-algebra, then does Id*(A) *always have a finite generating set?* We briefly mention that a positive solution for associative algebras over fields of characteristic 0 was obtained by Kemer in 1987 (see [Ke]), and infinitely-generated counterexamples have been found for fields of positive characteristic (for a full exposition of the topic, see [BRV]).

Our interest involves quantitative questions regarding the minimal degree of an identity which an algebra satisfies. For instance, we already know that the matrix algebra $M_n(K)$ satisfies an identity of degree $n^2 + 1$, but we wish to know if it satisfies any identities of a lower degree. The famous Amitsur-Levitzki theorem ([AL]) provides us the answer.

Theorem 2.1.3 (Amitsur-Levitzki, 1950)**.** *The standard identity of degree* $2n$ *given by*

$$\sum_{\sigma \in S_{2n}} sign(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2n)},$$

is a polynomial identity for $M_n(K)$ *of minimal degree.*

In some sense, PI-algebras which satisfy smaller degree polynomials can be thought of as being closer to commutative algebras, and so once we ascertain that an algebra (or a class of algebras) satisfies a polynomial identity, we then seek the lowest degree identity that A satisfies; failing this, we try to find an upper bound on this degree. Combinatorial techniques have proven especially useful in this regard, and we will discuss some results of this nature at the end of this chapter.

For now, we conclude our brief history by mentioning some of the predominant applications of polynomial identity theory. Aside from being extensively researched in their own right, PI-algebras have proven a critical tool in several major developments, perhaps most notably appearing in Zelmanov's solution of the Restricted Burnside Problem (see [Ze1], [Ze2]). Recall that the Restricted Burnside Problem asks: *Are there only finitely many groups (up to isomorphism) of exponent n generated by m elements?* After major reductions by P. Hall and Higman ([HH]), Zelmanov successfully solved this problem in the affirmative by transferring it to a question involving the identities of Lie algebras. For this, he was awarded the Fields Medal in 1994. Along a similar line, we recall the famous Kurosh ([Ku]) problem (which is a ring-theoretic analogue of the Burnside problem): *Let A be a finitely generated associative algebra such that every element of A is algebraic. Is A finite-dimensional? If A is nil, is it nilpotent?* In 1946, Levitzki ([Le]) solved the nil question under the additional assumption that A was a PI-algebra, and in 1950, Kaplansky ([Ka2]) solved the general question for PI-algebras. However, in 1960, Golod ([Go]) constructed an example of a finitely generated nil algebra that is infinite-dimensional and not nilpotent. More information on the Kurosh problem can be found in the book by Hernstein [He] or in [DF].

2.2 Graded Algebras

When an algebra is endowed with the additional structure of a grading by a finite group, then the identity component of the grading retains a surprising amount of information about the algebra. In particular, in several cases, we can deduce that an algebra is a PI-algebra as soon as the identity component satisfies an identity. Before we discuss this, we take a minute to introduce the basic concept of a graded algebra, and we offer various refinements of this notion that will be used in the sequel.

In the most general sense, an algebra with a vector space decomposition such that the product of homogeneous components is again homogeneous is called a graded algebra. Most commonly, the indexing set is taken to be a group, however, we shall consider algebras which are graded over more general group-like structures. The most traditional definition of a group-graded algebra is as follows.

Definition 2.2.1. *Let G be a group and let A be a nonassociative algebra with a vector space decomposition given by*

$$\Gamma : A = \bigoplus_{g \in G} A_g.$$

If $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$, then A is said to be a G-graded algebra and Γ is called a G-grading of A.

Clearly, the identity component, A_1, of a group-graded algebra is a subalgebra of A. An obvious example of an algebra that is naturally group-graded is a group algebra.

Example 2.2.2. *Let K be a field, let G be any group, and denote by KG the group algebra of G over K. Recall that KG is the set of formal linear combinations of G with coefficients in K, where multiplication is induced by the group operation. It is clear that KG has a vector space decomposition given by*

$$KG = \bigoplus_{g \in G} Kg,$$

and since $KgKh \subseteq Kgh$, for all $g, h \in G$, this decomposition is a G-grading of KG.

It has been determined precisely when a group algebra is a PI-algebras (see [Pa1]). To further illustrate the idea PI-algebras generally have large centers, we mention that (in characteristic 0) it is necessary and sufficient that the group have an abelian subgroup of finite index. In the future, we are interested in the manner in which group algebras act on algebras, hence we are only interested in group algebras over finite groups.

For any algebra A which is graded by a group G, if N is a normal subgroup of G then it is possible to grade A by the group G/N; in this case, the homogeneous components are sums of the G-graded components with index belonging to a given coset.

Example 2.2.3. *If $A = \bigoplus_{g\in G} A_g$ is a G-graded algebra, and $N \trianglelefteq G$ is a normal subgroup, then A can be endowed with the group-grading*

$$A = \bigoplus_{x\in G/N} A_x, \text{ where } A_x = \bigoplus_{g\in x} A_g,$$

for each coset $x \in G/N$.

The following theorem demonstrates how we may use the additional structure that a group-grading offers in polynomial identity theory. Bergen and Cohen deduced the following theorem in [BC] from a theorem of Montgomery and Smith ([MS]). The proof of Theorem 2.2.4 is based on structure theory, and no general bound on the degree of the identity was disclosed.

Theorem 2.2.4 (Bergen and Cohen, 1986)**.** *Let G be a finite group, and let $A = \bigoplus_{g\in G} A_g$ be a group-graded associative algebra. If the identity component A_1 satisfies a polynomial identity, then A is a PI-algebra.*

We remark that that if A is a Lie algebra, then the analogous result was proved by Bahturin and Zaicev in [BZ], also without a bound on the degree of the identity.

It seems surprising (at first) that a single homogeneous component could determine important properties of an entire algebra, but when an algebra is group-graded, the identity component is somehow special in this regard. As an illustration, suppose that G is a finite group and that $A = \bigoplus_{g\in G} A_g$ is an associative G-graded algebra such that $A_1 = 0$. Then, by using the following general lemma about groups (which was proved in [BGR]), we can show that A must be nilpotent of degree $|G|$.

Lemma 2.2.5. *If G is a finite group, then any fixed word $w = g_1 g_2 \cdots g_{|G|d}$ in G contains a product of d consecutive subwords each with trivial evaluation.*

Indeed, in the above scenario, we can use Lemma 2.2.5 to deduce that any product of homogeneous elements in A of length $|G|$ contains a subword contained in $A_1 = 0$; hence $A^{|G|} = 0$. Along a similar line, if we suppose instead that the identity component satisfies $(A_1)^m = 0$, then we can infer that $A^{|G|m} = 0$.

If we consider group-gradings over infinite groups, we cannot make similar conclusions. For instance, the free associative algebra $\mathcal{A}\langle X\rangle$ has a $\mathbb{Z}$-grading given by

$$\mathcal{A}\langle X\rangle = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}\langle X\rangle_n,$$

where $\mathcal{A}\langle X\rangle_n = \{f \in \mathcal{A}\langle X\rangle \,|\, \deg(f) = n\}$, if $n \geq 0$, and $\mathcal{A}\langle X\rangle_n = 0$, if $n < 0$. It is clear that $\mathcal{A}\langle X\rangle_0 = K$ satisfies a polynomial identity, but $\mathcal{A}\langle X\rangle$ is not a PI-algebra. In particular, Theorem 2.2.4 does not extend to infinite groups.

We will also consider algebras which are graded over other group-like structures (the reader is referred to Appendix A for definitions of the various group-like structures). We now offer various refinements of Definition 2.2.1.

Definition 2.2.6. *Let S be a set and let A be an algebra with a vector space decomposition*

$$\Gamma : A = \bigoplus_{s\in S} A_s.$$

1. *Suppose that S is a magma (that is, a set with a binary operation). Then, whenever $A_{s_1}A_{s_2} \subseteq A_{s_1 s_2}$, for all $s_1, s_2 \in S$, Γ is called a magma-grading. If S happens to be a group (respectively, semigroup, quasigroup, etc.), then we also say that Γ is a group-grading (respectively, semigroup-grading, quasigroup-grading, etc.).*

2. *If S is just a set with the property that whenever, for every $s_1, s_2 \in S$, either $A_{s_1}A_{s_2} = 0$ or there exists a (unique) $s_3 \in S$ such that $A_{s_1}A_{s_2} \subseteq A_{s_3}$, then Γ is called a set-grading. In this case, we denote the partially defined binary operation on the support of Γ (that is, $\{s \in S \,|\, A_s \neq 0\}$) by $s_1 \cdot s_2 = s_3$. If G is a magma and there exists an embedding*

$(S,\cdot) \to G$, then the magma-grading $A = \bigoplus_{g\in G} A_g$, where $A_g = 0$, for all $g \notin S$, is called a realization of Γ as a magma-grading.

For example, we could have referred to the decomposition $\mathcal{A}\langle X\rangle = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}\langle X\rangle_n$ as a realization of the semigroup-grading $\bigoplus_{n\in\mathbb{N}} \mathcal{A}\langle X\rangle_n$ as a group-grading. A question of interest is whether or not all gradings on certain algebras can be realized as a certain type of grading. If A is an associative algebra, then the associativity of A assures that every set-grading can be realized as a semigroup-grading. If A is not associative, then the situation is less clear. In 1989, Patera and Zassenhaus claimed that all set-gradings of Lie algebras can be realized as semigroup-gradings ([PZ]). However, Elduque discovered a counterexample in [El1]. Consequently, gradings of Lie algebras which cannot be realized as semigroup-gradings have become a topic of interest (see [EK] for more details). In Chapter 7 we further discuss this topic. For now, we conclude this section with an example of a semigroup-grading that is not realizable as a group-grading.

Example 2.2.7. *Let $n \geq 2$, and consider the following vector space decomposition of $M_n(K)$:*

$$\Gamma : M_n(K) = \bigoplus_{1\leq i,j\leq n} E_{i,j},$$

where $E_{i,j}$ is the span of the matrix unit $e_{i,j}$, which has a 1 in the (i,j)-component and 0's elsewhere. Notice that Γ is a set-grading of $M_n(K)$ since $e_{i,j}e_{k,l} = e_{i,l}$, if $j = k$, and $e_{i,j}e_{k,l} = 0$, if $j \neq k$.

Let $S = \{e_{i,j} | 1 \leq i,j \leq n\}$ and denote by $(\cdot)$ the partially defined binary operation on S induced by Γ. We define $\bar{S} = S \cup \{0\}$, and we extend $(\cdot)$ onto $\bar{S}$ via:

- $e_{i,j} \cdot 0 = 0 \cdot e_{i,j} = 0$, *for all $e_{i,j} \in S$;*
- $0 \cdot 0 = 0$;
- $e_{i,j} \cdot e_{k,l} = 0$, *whenever $e_{i,j} \cdot e_{k,l}$ is not defined (in other words, $j \neq k$).*

Then $(\bar{S},\cdot)$ is a magma, and further, $(\cdot)$ is an associative operation on $\bar{S}$, and so $(\bar{S},\cdot)$ is *a semigroup. Thus,* Γ *is realized as a semigroup-grading via*

$$M_n(K) = \bigoplus_{s \in \bar{S}} A_s$$

where $A_s = E_{i,j}$*, if* $s = e_{i,j}$ *and* $A_s = 0$ *if* $s = 0$*. Further, observe that* Γ *cannot be realized as a group-grading. Indeed, since* $e_{i,i} \cdot e_{i,i} = e_{i,i}$*, for all i, it follows that any embedding will result in at least two idempotents, but a group may only have one idempotent.*

2.3 Amitsur's Theorems on Algebras with an Involution

We saw in the previous section how a group-graded structure can be utilized to determine certain properties of an algebra. If an associative algebra A comes endowed with another bit of additional structure in the form of an action by an involution, then two celebrated results of Amitsur allow us to make similar conclusions. Recall that an involution is a linear map $* : A \to A$ such that $(ab)^* = b^* a^*$ and $(a^*)^* = a$. Whenever an algebra admits an involution, we can define the following two subspaces:

- $\mathcal{S} = \{a \in A | \, a^* = a\}$, called the symmetric elements of A, and
- $\mathcal{K} = \{a \in A | \, a^* = -a\}$, called the skew-symmetric elements of A.

Amitsur proved ([Am1]) that if an associative algebra A admits an involution, then A is a PI-algebra whenever either $\mathcal{S}$ or $\mathcal{K}$ satisfy a polynomial identity. Note that $\mathcal{S}$ and $\mathcal{K}$ are not necessarily subalgebras of A.

Theorem 2.3.1 (Amitsur, 1968)**.** *Let A be an associative algebra with an involution. If the subspace of symmetric elements or skew-symmetric elements of A satisfies a polynomial identity, then A is a PI-algebra.*

There are many algebras that naturally admit involutions, for instance, the transpose map is an involution on $M_n(K)$. In this case, $\mathcal{S}$ and $\mathcal{K}$ are the usual symmetric and skew-symmetric matrices, respectively. Also, notice that for any Lie algebra L, there is an involution $*$ given by $x^* = -x$, for all $x \in L$. In this case, $\mathcal{S} = 0$ and $\mathcal{K} = L$, and thus Amitsur's theorem does not extend to nonassociative algebras. In general, determining exactly when an algebra admits an involution can be a difficult question, even when the algebra is known to admit an anti-automorphism of greater order (for instance, see [Le]).

Amitsur soon generalized Theorem 2.3.1 by considering a more general type of identity on A. If we set $X^* = \{x_1, x_1^*, x_2, x_2^*, \ldots\}$, then notice that $\mathcal{A}\langle X^*\rangle$ has a natural involution. The elements of $\mathcal{A}\langle X^*\rangle$ are called $*$-polynomials, and $*$-identities on an associative algebra A with involution are defined in a natural way: $f(x_1, x_1^*, \ldots, x_n, x_n^*) \in \mathcal{A}\langle X^*\rangle$ is a $*$-identity of A if $f(a_1, a_1^*, \ldots, a_n, a_n^*) = 0$, for any $a_1, \ldots, a_n \in A$. The following appears in [Am2].

Theorem 2.3.2 (Amitsur, 1969)**.** *Let A be an associative algebra with an involution. If A satisfies a $*$-identity, then A is a PI-algebra.*

This generalizes Theorem 2.3.1 since, if the symmetric elements satisfy the polynomial identity $f(x_1, \ldots, x_n)$, then A satisfies the $*$-identity $f(x_1 + x_1^*, \ldots, x_n + x_n^*)$; similarly, if the skew-symmetric elements satisfy the polynomial identity $f(x_1, \ldots, x_n)$, then A satisfies the $*$-identity $f(x_1 - x_1^*, \ldots, x_n - x_n^*)$. Amitsur used structure theory to prove his theorem, and similar to Theorem 2.2.4 no bound on the degree of the identity satisfied was given. In order to find bounded versions of these theorems, the incorporation of new combinatorial techniques was required, which we will review next.

2.4 A Combinatorial Approach to Polynomial Identity Theory

We conclude this background chapter by giving a brief discussion of the combinatorial approach that has been employed in recent years to attain quantitative versions of Theorem 2.2.4 and Theorem 2.3.2. In later chapters, we will utilize a similar approach to formulate some new results; consequently, in this section, we only give a brief preview, sparing the details for later.

The key ingredient used is the so-called codimension sequence of an algebra. This numerical sequence is used to prove that an algebra is a PI-algebra. We denote by

$$P_n = \mathrm{span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} | \sigma \in S_n\},$$

the subspace of $\mathcal{A}\langle X\rangle$ consisting of all multilinear polynomials in the variables $x_1, \ldots, x_n$. Clearly, $\dim P_n = n!$. The multilinear identities of a PI-algebra A belong to the spaces $P_n \cap \mathrm{Id}(A)$, and if A satisfies an identity of degree n, then $\dim(P_n \cap \mathrm{Id}(A)) < \dim(P_m \cap \mathrm{Id}(A))$, for all $n < m$. By analyzing the sequence of dimensions of $P_n \cap \mathrm{Id}(A)$, we can get a sense of the growth of the polynomial identities of A. For technical reasons, it is more convenient to consider the sequences of codimensions of A.

Definition 2.4.1. *Let A be an associative algebra, and let $\mathcal{A}\langle X\rangle$ be the free associative algebra on $X = \{x_1, x_2, \ldots\}$. The number*

$$c_n(A) = dim \frac{P_n}{P_n \cap Id(A)}$$

is called the n^{th}-codimension of A.

The following easy fact is important enough that we state it explicitly.

Remark 2.4.2. *An algebra A satisfies an identity of degree $n \geq 1$ if and only if $c_n(A) < n!$.*

The exact codimensions are explicitly known for very few algebras; here we only study the asymptotic behaviour of this sequence. It is worth noting that, for any PI-algebra A, the codimension sequence is invariant under extensions of the base field K, since we only consider multilinear polynomial identities.

The codimension sequence was first utilized by Regev in [Re1] when he proved that the tensor product of two PI-algebras is a PI-algebra (notably, Latyshev ([La]) later provided some modifications and simplified proofs). This was proved by demonstrating that if A satisfies an identity of degree d, then $c_n(A) \leq (d-1)^{2n}$, for any integers $2 \leq d \leq n$. In other words, the codimension sequence of a PI-algebra is exponentially bounded. The method to find this bound involved partitioning the monomials of P_n into two sets called 'd-good' and 'd-bad' monomials. In [Re1], Regev established an upper bound on the number of such monomials; in [Re2] this bound was sharpened to $\frac{(d-1)^{2n}}{(d-1)!}$. By showing that P_n is spanned by only the d-good monomials (modulo $\mathrm{Id}(A)$), the proof was completed. A similar approach was taken by Bahturin, Giambruno and Riley when they gave the following quantitative version of Theorem 2.2.4 in [BGR] .

Theorem 2.4.3 (The Bahturin-Giambruno-Riley Theorem)**.** *If G is a finite group, $A = \bigoplus_{g\in G} A_g$ is a group-graded associative algebra, and A_1 satisfies a polynomial identity of degree d, then A satisfies a polynomial identity of degree $\lceil e|G|(d|G|-1)^2\rceil$ (where e is the base of natural logarithms and $\lceil x\rceil$ is the least integer greater than or equal to x).*

In the ensuing chapter, we will offer both a Lie and Jordan analogue of this theorem.

Around the same time, Bahturin, Giambruno and Zaicev in [BGZ] gave the first quantitative version of Amitsur's Theorem 2.3.2. In the same way that Amitsur considered $*$-identities on an algebra A with involution, we can consider the following more general construction. Let G be a finite group which embeds into $\mathrm{Aut}^*(A)$, the group of all automorphisms and anti-automorphisms on A. Let $X^G = \{x_i^g | i \in \mathbb{Z}^+, g \in G\}$, and call $\mathcal{A}\langle X^G\rangle$ the free associative algebra with G-action. Notice that G acts on $\mathcal{A}\langle X^G\rangle$ in a natural way via $(x^{g_1})^{g_2} = x^{g_2 g_1}$, and this action extends to monomials by $(vw)^g = v^g w^g$, if $g \in \mathrm{Aut}(A)$, and $(vw)^g = w^g v^g$, if $g \in \mathrm{Aut}^*(A)/\,\mathrm{Aut}(A)$. The elements of $\mathcal{A}\langle X^G\rangle$ are called G-polynomials, and if a G-polynomial $f(x_{i_1}^{g_1}, \ldots, x_{i_n}^{g_n})$ has the property that $f(a_1^{g_1}, \ldots, a_n^{g_n}) = 0$, for all $a_1, \ldots, a_n \in A$, then f is called a G-identity. Bahturin, Giambruno, and Zaicev considered a special type of G-identity.

Definition 2.4.4. *A G-polynomial is called essential if it is of the form*

$$x_1^1 \cdots x_d^1 - \sum_{\substack{1\neq\sigma\in S_d,\\ g\in G^d}} \alpha_{\sigma,g} x_{\sigma(1)}^{g_1} \cdots x_{\sigma(d)}^{g_d}.$$

They proved the following theorem.

Theorem 2.4.5 (The Bahturin-Giambruno-Zaicev Theorem)**.** *Let A be an associative algebra, and suppose that G is a finite subgroup that embeds into* $\mathrm{Aut}^*(A)$*. If A satisfies an* *essential G-identity of degree d, then A satisfies an ordinary polynomial identity of degree bounded by a function of d and $|G|$.*

The precise (and complicated) nature of the function can be found in the original paper. We conclude this section by taking a moment to observe how the Bahturin-Giambruno-Zaicev Theorem improved Amitsur's Theorem 2.3.2 (using an argument found in [BGZ]). First, notice that a $*$-identity on an algebra with involution is just a G-identity with $G = \{1, *\}$. Now, if we suppose that A is an algebra with involution which satisfies a multilinear $*$-identity, then by multilinearizing, we can write this identity in the form

$$\sum_{g \in G^n} \alpha_g x_1^{g_1} \cdots x_n^{g_n} + \sum_{\substack{1 \neq \sigma \in S_n, \\ h \in G^n}} \beta_{\sigma,h} x_{\sigma(1)}^{h_1} \cdots x_{\sigma(n)}^{h_n}.$$

Further, we may assume that $\alpha_{(1,\ldots,1)} \neq 0$, since otherwise we may replace our identity with an identity of this form. Now, for each indeterminate $x_1, \ldots, x_n$, we substitute x_i with the indeterminate $y_{2i-1}y_{2i}$ to obtain a multilinear G-identity of degree $2n$. Since $\alpha_{(1,\ldots,1)} \neq 0$, this identity can be written as

$$y_1 \cdots y_{2d} + \sum_{\substack{1 \neq \sigma \in S_{2n}, \\ g \in G^{2n}}} \gamma_{\sigma,g} y_{\sigma(1)}^{g_1} \cdots y_{\sigma(2n)}^{g_{2n}},$$

which is an essential G-identity. Hence, Amitsur's Theorem 2.3.2 follows. It is worth noting that Bahturin, Sehgal and Zaicev offered similar results for certain nonassociative (including Lie) algebras in [BSZ].

Chapter 3

Lie-Group-Graded and Jordan-Group-Graded Associative Algebras

An algebra will sometimes admit a natural vector space decomposition that is not a group-grading (or even a set-grading), but can nevertheless be used to determine important properties of the algebra. In this chapter, our goal is demonstrate that any associative algebra whose induced Lie or Jordan algebra is group-graded satisfies a polynomial identity of bounded degree whenever the identity component of the grading satisfies a polynomial identity. For instance, any associative algebra with an involution can be naturally decomposed in such a way that the induced Lie and Jordan algebras are group-graded; hence, we will obtain, as a special case, Amitsur's Theorem 2.3.1. To this end, we assume throughout this chapter that A is an associative algebra and G is a finite group. Also,

- $A^{(-)}$ will henceforth denote the Lie algebra induced by A via the bracket operation $[a, b] = ab - ba$; and,

- $A^{(+)}$ will henceforth denote the Jordan algebra induced by A via the circle operation $a \circ b = ab + ba$.

For a reminder of the defining properties of Lie and Jordan algebras, the reader may consult Appendix B.

In the first section, we will define (what we call) Lie-group-gradings and Jordan-group-gradings, and we offer our motivating example of an algebra that is Lie-group-graded and Jordan-group-graded: an associative algebra with an involution. In the second section, we introduce graded identities and develop the necessary machinery in order to prove our main theorem. In the third section, we prove our main result using a method similar to the combinatorial approaches taken in [BGR] and [BGZ].

3.1 Lie-group-gradings and Jordan-group-gradings

Since the defining property of a group-grading is category dependent, and to simplify our notation, we make the following definitions.

Definition 3.1.1. *Let G be a finite group, and let A be an associative algebra with vector space decomposition $\Gamma : A = \bigoplus_{g\in G} A_g$. Then we say that*

1. *Γ is a Lie-G-grading of A if it is a G-grading of $A^{(-)}$;*
2. *Γ is a Jordan-G-grading of A if it is a G-grading of $A^{(+)}$; and,*
3. *Γ is an associative-G-grading of A whenever A is viewed fully as the underlying associative algebra.*

Notice that in cases (1) and (2), if $g, h \in G$ do not commute, then $[A_g, A_h] = 0$ and $A_g \circ A_h = 0$. Thus, unless G is abelian, an associative-G-grading of A may not necessarily be a Lie-G-grading or Jordan-G-grading. On the other hand, if it happens that $A = \bigoplus A_g$ is both a Lie-G-grading and Jordan-G-grading, then it is also an associative-G-grading, whenever the characteristic p is different from 2. Indeed, if $a \in A_g$ and $b \in A_h$ are homogeneous elements, then

$$2ab = [a, b] + a \circ b \in [A_g, A_h] + A_g \circ A_h \subseteq A_{gh}.$$

It is also clear that the identity component, A_1, of a Lie-G-graded algebra is a Lie subalgebra of $A^{(-)}$, and that the identity component of a Jordan-G-graded algebra is a Jordan subalgebra of $A^{(+)}$.

We demonstrate an important class of algebras with a natural vector space decomposition that can be viewed as either a Lie-group-grading or a Jordan-group-grading, but not as an associative-group-grading.

Example 3.1.2. *Let A be an associative algebra over field of characteristic not 2 that admits an involution $*$. Notice that if $\mathcal{S} = \{a \in A \,|\, a^* = a\}$ and $\mathcal{K} = \{a \in A \,|\, a^* = -a\}$, then*

$$\Gamma : A = \mathcal{S} \oplus \mathcal{K}$$

is a vector space decomposition of A. This follows since, for every $a \in A$, we have $a = s + k$ where

$$s = \frac{a + a^*}{2} \quad \text{and} \quad k = \frac{a - a^*}{2}.$$

Observe that Γ is not generally a set-grading of A, since, for instance, the product of a symmetric element and a skew-symmetric element is not necessarily homogeneous. However, since the following relations hold:

$$[\mathcal{S}, \mathcal{S}] \subseteq \mathcal{K},$$

$$[\mathcal{S}, \mathcal{K}] \subseteq \mathcal{S},$$

$$[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K},$$

it follows that $A = A_0 \oplus A_1$, where $A_0 = \mathcal{K}$ and $A_1 = \mathcal{S}$, is a Lie-$\mathbb{Z}_2$-grading of A. Similarly, since the following relations hold:

$$\mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S},$$

$$\mathcal{S} \circ \mathcal{K} \subseteq \mathcal{K},$$

$$\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{S},$$

it follows that $A = A_0 \oplus A_1$, where $A_0 = \mathcal{S}$ and $A_1 = \mathcal{K}$, is a Jordan-$\mathbb{Z}_2$-grading of A.

We are now ready to state our main theorem in this chapter. The function f mentioned in the theorem below is the same one used in the Bahturin-Giambruno-Zaicev (see [BGZ]); we will offer some additional details regarding f in Section 3.3, but the reader can refer to the original paper or [GZ] for a complete description.

Theorem 3.1.3. *Let G be a finite group, and let A be an associative algebra such that $A = \bigoplus_{g\in G} A_g$ is a Lie-G-grading or a Jordan-G-grading of A. If the subspace A_1 satisfies a polynomial identity, then the entire algebra A satisfies a polynomial identity. More pointedly, if A_1 satisfies a polynomial identity of degree d, then A satisfies a polynomial identity of degree bounded above by $f(d|G|, |G|)$.*

3.2 Graded Identities

An identity on a homogeneous component of a group-graded algebra (for instance, as in the Bahturin-Giambruno-Riley Theorem) can be viewed as an example of a more general type of identity called a graded identity. Whereas ordinary polynomial identities are evaluated upon the substitution of arbitrary elements of A, graded identities are evaluated only upon the substitution of elements from specified homogenous components of A. Traditionally, graded identities are defined only for group-graded algebras, but we will extend this notion so that it is applicable to more general types of gradings.

We first present the notion of a free associative-group-graded algebra. For a finite group G of order k, let $X^{(G)}$ be the disjoint union of k-many countably infinite sets $X^{(G)} = \bigcup_{g\in G} X_{(g)}$, where $X_{(g)} = \{x_1^{(g)}, x_2^{(g)}, \ldots\}$. Let $\mathcal{A}\langle X^{(G)}\rangle$ be the free associative algebra on $X^{(G)}$. A monomial $x_{i_1}^{(g_1)} \cdots x_{i_t}^{(g_t)} \in \mathcal{A}\langle X^{(G)}\rangle$ is said to have homogeneous degree $g_1 \cdots g_t$ (as opposed to its total degree, which is t). Then

$$\mathcal{A}\langle X^{(G)}\rangle = \bigoplus_{g\in G} \mathcal{A}\langle X^{(G)}\rangle_{(g)},$$

where $\mathcal{A}\langle X^{(G)}\rangle_{(g)}$ is the subspace of $\mathcal{A}\langle X^{(G)}\rangle$ spanned by all monomials having homogeneous degree g, and further, this vector space decomposition is an associative-G-grading.

We will denote by $\mathcal{A}\langle X\rangle^{\mathrm{gr}}$ the algebra $\mathcal{A}\langle X^{(G)}\rangle$ when it is endowed with this grading, and we call its elements graded polynomials.

We now make the following general definitions: it is important to note that we do not assume that the vector space decomposition $A = \bigoplus_{g\in G} A_g$ satisfies any implicit algebraic properties, such as being an associative-G-grading.

Definition 3.2.1. *Let G be a finite group, and let A be an associative algebra admitting a vector space decomposition $A = \bigoplus_{g\in G} A_g$.*

1. *A graded identity of A is a graded polynomial $f(x_1^{(g_1)}, \dots, x_n^{(g_n)})$ with the property that $f(a_1, \dots, a_n) = 0$, for all $a_1 \in A_{g_1}, \dots, a_n \in A_{g_n}$.*

2. *$Id^{gr}(A)$ denotes the set of all graded identities of A.*

3. *For each monomial $w = x_{i_1}^{(g_1)} \cdots x_{i_n}^{(g_n)} \in \mathcal{A}\langle X\rangle^{gr}$ and permutation $\sigma \in S_n$, we denote $w_\sigma = x_{i_{\sigma(1)}}^{(g_{\sigma(1)})} \cdots x_{i_{\sigma(n)}}^{(g_{\sigma(n)})}$.*

4. *We denote*

$$P_n^{gr} = span\{w_\sigma | \, w = x_1^{(g_1)} \cdots x_n^{(g_n)}, g_1, \dots, g_n \in G, \sigma \in S_n\};$$

the integer

$$c_n^{gr}(A) = dim \frac{P_n^{gr}}{P_n^{gr} \cap Id^{gr}(A)}$$

is called the n^{th} G-graded codimension of A.

We have already seen an example of an algebra which satisfies a graded identity but not an ordinary polynomial identity: the free associative algebra, when decomposed as $\mathcal{A}\langle X\rangle = \bigoplus \mathcal{A}\langle X\rangle_n$, satisfies the graded identity $x_1^{(0)} x_2^{(0)} - x_2^{(0)} x_1^{(0)}$.

We identify ordinary polynomials with graded polynomials by setting $z_i = \sum_{g\in G} x_i^{(g)}$, for each $i \geq 1$. Under the identification $x_i \to z_i$, we may regard $P_n \subseteq P_n^{\mathrm{gr}}$. Further, observe that $f(x_1, \dots, x_m) \in \mathcal{A}\langle X\rangle$ is a polynomial identity of A if and only if $f(z_1, \dots, z_m) \in \mathcal{A}\langle X\rangle^{gr}$ is a graded identity of A. Hence, we may also suppose that $P_n \cap \mathrm{Id}^{\mathrm{gr}}(A) = P_n \cap \mathrm{Id}(A)$. Using these facts, we can deduce the following well-known result (which holds even when A is not necessarily associative-G-graded); a proof can be found in [BGR] for example.

Lemma 3.2.2. *Let G be a finite group, and let A be an associative algebra admitting a vector space decomposition $A = \bigoplus_{g\in G} A_g$. Then the following inequality holds:*

$$c_n(A) \leq c_n^{gr}(A).$$

Thus, an algebra $A = \bigoplus_{g\in G} A_g$ satisfies an ordinary polynomial identity of degree n whenever $c_n^{gr}(A) < n!$.

Note that $\mathrm{Id}^{\mathrm{gr}}(A)$ is always an associative ideal. The algebra $\mathcal{A}\langle X\rangle^{gr}$ has the following universal property: if $A = \bigoplus A_g$ is an associative-G-graded algebra, then any set-theoretic map $\varphi : X \to A$ such that $\varphi(X_g) \subseteq A_g$, for all $g \in G$, extends uniquely to a G-graded homomorphism $\bar{\varphi} : \mathcal{A}\langle X\rangle^{gr} \to A$. Therefore, if $\bar{\Phi}$ is the set of all such homomorphisms, then $\mathrm{Id}^{\mathrm{gr}}(A) = \bigcap_{\bar{\varphi}\in\bar{\Phi}} \ker\bar{\varphi}$ is a graded ideal that is invariant under all G-graded endomorphisms of $\mathcal{A}\langle X\rangle^{\mathrm{gr}}$. However, if A is only Lie-G-graded or Jordan-G-graded, then the homomorphisms $\bar{\varphi}$ are not necessarily G-graded homomorphisms. Furthermore, in these situations, the ideal $\mathrm{Id}^{\mathrm{gr}}(A)$ is not necessarily invariant under the G-graded endomorphisms of $\mathcal{A}\langle X\rangle^{\mathrm{gr}}$. Yet, we can make some more limited observations. First we require a definition.

Definition 3.2.3. *Let G be a finite group, and let A be an associative algebra admitting a vector space decomposition $A = \bigoplus_{g\in G} A_g$.*

1. *$L\langle X^{(G)}\rangle$ will denote the free Lie subalgebra of $\mathcal{A}\langle X^{(G)}\rangle^{(-)}$ generated by the set $X^{(G)}$. For each $g \in G$, we denote*

$$L\langle X^{(G)}\rangle_{(g)} = L\langle X^{(G)}\rangle \cap \mathcal{A}\langle X^{(G)}\rangle_{(g)}.$$

2. *$J\langle X^{(G)}\rangle$ will denote the free Jordan subalgebra of $\mathcal{A}\langle X^{(G)}\rangle^{(+)}$ generated by the set $X^{(G)}$. For each $g \in G$, we denote*

$$J\langle X^{(G)}\rangle_{(g)} = J\langle X^{(G)}\rangle \cap \mathcal{A}\langle X^{(G)}\rangle_{(g)}.$$

We end this section by offering some technical results which we utilize in our proof of Theorem 3.1.3. The first lemma says that, if A is Lie or Jordan-G-graded, then, although $\mathrm{Id}^{gr}(A)$ is not invariant under G-graded endomorphisms of $\mathcal{A}\langle X\rangle^{gr}$, it is invariant under graded homomorphisms $\mathcal{A}\langle X^{(G)}\rangle \to L\langle X^{(G)}\rangle$. This is obvious and the proof is omitted.

Lemma 3.2.4. *Let G be a finite group, let A be an associative algebra admitting a vector space decomposition $A = \bigoplus_{g\in G} A_g$, and let $f(x_1^{(g_1)}, \dots, x_n^{(g_n)})$ be a graded identity of A. Then the following statements hold.*

1. *If $A = \bigoplus_{g\in G} A_g$ is Lie-G-graded and $z_i \in L\langle X^{(G)}\rangle_{(g_i)}$, for each $i = 1, \dots, n$, then $f(z_1, \dots, z_n)$ is also a graded identity of A.*

2. *If $A = \bigoplus_{g\in G} A_g$ is Jordan-G-graded and $z_i \in J\langle X^{(G)}\rangle_{(g_i)}$, for each $i = 1, \dots, n$, then $f(z_1, \dots, z_n)$ is also a graded identity of A.*

Lemma 3.2.5. *Let G be any finite group. Then the following statements hold.*

1. *Suppose that $g_1, \dots, g_m$ in G pairwise commute, and put $g = g_1 \cdots g_m$. Then every Lie (respectively, Jordan) product of $x_{i_1}^{(g_1)}, \dots, x_{i_m}^{(g_m)}$ lies in $L\langle X^{(G)}\rangle_{(g)}$ (respectively, $J\langle X^{(G)}\rangle_{(g)}$).*

2. *$L\langle X^{(G)}\rangle = \bigoplus_{g\in G} L\langle X^{(G)}\rangle_{(g)}$ if and only if G is abelian, in which case $L\langle X^{(G)}\rangle$ is a G-graded Lie algebra.*

3. *$J\langle X^{(G)}\rangle = \bigoplus_{g\in G} J\langle X^{(G)}\rangle_{(g)}$ if and only if G is abelian, in which case $J\langle X^{(G)}\rangle$ is a G-graded Jordan algebra.*

Proof. Statement (1) is obvious. The proof of (3) is analogous to (2), so we prove only (2). Suppose that $L\langle X\rangle = \bigoplus_{g\in G} L\langle X^{(G)}\rangle_{(g)}$ is G-graded and $g, h \in G$ are such that $gh \neq hg$. Then $[x_1^{(g)}, x_2^{(h)}] \in L\langle X^{(G)}\rangle_{(gh)} \cap L\langle X^{(G)}\rangle_{(hg)} = 0$, which is impossible. Conversely, if G is abelian, then the fact that $L\langle X^{(G)}\rangle = \bigoplus_{g\in G} L\langle X^{(G)}\rangle_{(g)}$ follows from (1). When G is abelian, the associative-G-grading on $\mathcal{A}\langle X\rangle^{\mathrm{gr}}$ induces an natural Lie-G-grading on $\mathcal{A}\langle X^{(G)}\rangle$, and hence on $L\langle X^{(G)}\rangle$. □

We adopt the convention that Lie monomials $[a_1, \ldots, a_m]$ and Jordan monomials $a_1 \circ \cdots \circ a_m$ are left-normed. We require one more technical result before we proceed to the proof of our main theorem.

Lemma 3.2.6. *Let G be any finite group, let $A = \bigoplus_{g\in G} A_g$ be a Lie-G-graded (respectively, Jordan-G-graded) algebra, and let $m \geq 2$ be an integer. Then, either the Lie (respectively, Jordan) monomial $[x_1^{(g_1)}, \ldots, x_m^{(g_m)}]$ (respectively, $x_1^{(g_1)} \circ \cdots \circ x_m^{(g_m)}$) is homogeneous of degree $g_1 \cdots g_m$ or it is a graded identity of A.*

Proof. We prove only the case where A is Lie-G-graded because the Jordan-G-graded case is similar. If $g_1g_2 \neq g_2g_1$, then $[a_1^{(g_1)}, a_2^{(g_2)}] \in A_{g_1g_2} \cap A_{g_2g_1} = 0$, for all $a_1 \in A_{g_1}$, $a_2 \in A_{g_2}$, and so $[x_1^{(g_1)}, x_2^{(g_2)}]$ is a graded identity of A. Otherwise, $[x_1^{(g_1)}, x_2^{(g_2)}]$ is homogeneous of degree g_1g_2 by part (1) of Lemma 3.2.5. In general, if $[x_1^{(g_1)}, \ldots, x_m^{(g_m)}]$ is a graded identity of A, then so is $[x_1^{(g_1)}, \ldots, x_{m+1}^{(g_{m+1})}]$ since $\mathrm{Id}^{\mathrm{gr}}(A)$ is an ideal of $\mathcal{A}\langle X\rangle^{gr}$. Otherwise, by the induction hypothesis, we can assume that $z = [x_1^{(g_1)}, \ldots, x_m^{(g_m)}]$ is homogeneous of degree $g = g_1 \cdots g_m$, in which case the claim follows from the $m = 2$ case applied to $[z, x_{m+1}^{(g_{m+1})}]$. □

3.3 Lie and Jordan Analogues of the Bahturin-Giambruno-Riley Theorem

We are now in a position to prove Theorem 3.1.3. To this end, we assume in this section that $A = \bigoplus_{g\in G} A_g$ is Lie-G-graded (or Jordan-G-graded) and that A_1 satisfies a polynomial identity of degree d.

We fix $t \geq d|G|$ and $\bar{g} = (g_1, \ldots, g_t) \in G^t$. We denote the subspace of all multilinear polynomials in the variables $x_1^{(g_1)}, \ldots, x_t^{(g_t)}$ by

$$P_t^{\bar{g}} = \mathrm{span}\{x_{\sigma(1)}^{(g_{\sigma(1)})} \cdots x_{\sigma(t)}^{(g_{\sigma(t)})} | \, \sigma \in S_t\}.$$

Observe that $\dim P_t^{\bar{g}} = \mathrm{t!}$.

Our present goal is to demonstrate that every monomial $w = x_{i_1}^{(g_1)} \cdots x_{i_t}^{(g_t)} \in P_t^{\bar{g}}$ lies in $\mathrm{span}\{w_\sigma | \, 1 \neq \sigma \in S_t\} + \mathrm{Id}^{\mathrm{gr}}(A)$.

Proposition 3.3.1. *Let G be any finite group, and suppose that A is either a Lie-G-graded or Jordan-G-graded algebra such that A_1 satisfies a nontrivial polynomial identity of degree d. Then $w \in span\{w_\sigma | 1 \neq \sigma \in S_t\} + Id^{gr}(A)$, for all multilinear monomials $w = x_{i_1}^{(g_1)} \cdots x_{i_t}^{(g_t)}$ with $t \geq d|G|$.*

Proof. We consider only the case when A is Lie-G-graded: a similar proof works substituting $x \circ y$ for $[x, y]$.

Let w be given, and put $W = \text{span}\{w_\sigma | 1 \neq \sigma \in S_t\}$. By Lemma 2.2.5, we can factor $w = aw_1 \cdots w_d b$ into submonomials where $w_1, \ldots, w_d$ have homogeneous degree 1. Thus, we can write $w_1 = y_1^{(g_1)} \cdots y_m^{(g_m)}$ for some $y_1^{(g_1)}, \ldots, y_m^{(g_m)} \in X^{(G)}$ with the property that $g_1 \cdots g_m = 1$. We claim that

$$w \equiv a\bar{w}_1 w_2 \cdots w_d b \pmod{W},$$

where $\bar{w}_1$ is the Lie monomial $[y_1^{(g_1)}, \ldots, y_m^{(g_m)}]$. To see why, first observe that

$$w \equiv a[y_1^{(g_1)}, y_2^{(g_2)}] y_3^{(g_3)} \cdots y_m^{(g_m)} w_2 \cdots w_d b \pmod{W}$$

since $a y_2^{(g_2)} y_1^{(g_1)} y_3^{(g_3)} \cdots y_m^{(g_m)} w_2 \cdots w_d b \in W$. Repeated application of this same sort of argument proves the claim. Applying the same procedure to each w_i yields

$$w \equiv a\bar{w}_1 \cdots \bar{w}_d b \pmod{W}.$$

Next, notice that Lemma 3.2.6 implies that either $\bar{w}_i$ is homogeneous or $\bar{w}_i$ is a graded identity of A. Consequently, without loss of generality, we may assume that every $\bar{w}_i$ is homogeneous for otherwise $w \in W + \text{Id}^{\text{gr}}(A)$, as required. Thus, since each w_i is homogeneous of degree 1, each $\bar{w}_i$ lies in $L\langle X^{(G)} \rangle_{(1)}$.

By the usual multilinearization process, we can assume the nontrivial polynomial identity satisfied by A_1 is of the form

$$f(x_1^{(1)}, \ldots, x_d^{(1)}) = x_1^{(1)} \cdots x_d^{(1)} - \sum_{1 \neq \tau \in S_t} \alpha_\tau x_{\tau(1)}^{(1)} \cdots x_{\tau(d)}^{(1)},$$

for some scalars α_τ. Since A is Lie-G-graded and each $\bar{w}_i \in L\langle X^{(G)} \rangle_{(1)}$, it follows from part (1) of Lemma 3.2.4 that $f(\bar{w}_1, \ldots, \bar{w}_d) \in \text{Id}^{\text{gr}}(A)$. Therefore,

$$w \equiv \sum_{\tau \in S_t, \tau \neq 1} \alpha_\tau a\bar{w}_{\tau(1)} \cdots \bar{w}_{\tau(d)} b \pmod{W + \text{Id}^{\text{gr}}(A)}.$$

Because each nontrivial permutation τ contributes only nontrivial permutations of the indeterminates in w, it follows that $w \in W + \mathrm{Id}^{\mathrm{gr}}(A)$ in every case. $\square$

Next, we impose a total order on the multilinear monomials $x_{i_1} \cdots x_{i_n} \in P_n$ according to their subscripts, by requiring that $x_i < x_j$, whenever $i < j$, and extending lexicographically from left to right. This extends to a partial order of the graded monomials by disregarding the homogeneous degrees. We define the following special type of monomial.

Definition 3.3.2. *Let $w \in P_n$ be a multilinear monomial. Then w is called m-decomposable if it can be represented in the form:*

$$w = aw_1w_2 \cdots w_m b,$$

where $w_1, \ldots, w_m$ are nonempty monomials such that:

a) *The first indeterminate in w_i is greater than the first indeterminate in w_j whenever $i < j$;*

b) *The first indeterminate in w_i is greater than any other indeterminate in w_i.*

If w has no m-decompositions, then it is called m-indecomposable. Also, the total number of m-indecomposable monomials in P_n shall be denoted by $a_m(n)$.

Observe that if $w = aw_1w_2 \cdots w_m b$ is an m-decomposition, then any nontrivial permutation of the submonomials $w_1w_2 \cdots w_m$ results in a monomial which is smaller in the lexicographic order.

Our next objective is to demonstrate that, for every $n \geq t \geq d|G|$, the space $P_n^{\bar{g}}$ is spanned by the t-indecomposable monomials (modulo $P_n^{\bar{g}} \cap \mathrm{Id}^{gr}(A)$). From this it will follow that

$$\dim \frac{P_n^{\bar{g}}}{P_n^{\bar{g}} \cap Id^{gr}(A)} < a_t(n),$$

and consequently that $c_n^{gr}(A) < a_t(n)|G|^n$.

Proposition 3.3.3. *Let G be a finite group, and suppose that A is either a Lie-G-graded or Jordan-G-graded algebra such that A_1 satisfies a non-trivial polynomial identity of degree d. Fix $n \geq t \geq d|G|$ and $\bar{g} = (g_1, \ldots, g_n) \in G^n$, and put $w = x_1^{(g_1)} \cdots x_n^{(g_n)}$ and $P_n^{\bar{g}} =$*

span$\{w_\sigma | \sigma \in S_n\}$. *Then* $P_n^{\bar{g}}$ *is spanned by the set* $\{w_\sigma | w_\sigma$ *is t-indecomposable*$\}$ *modulo* $P_n^{gr} \cap Id^{gr}(A)$.

Proof. As usual, we prove only the Lie-G-graded case because the Jordan-G-graded case is so similar. It suffices to show that, whenever w_σ is t-decomposable, $w_\sigma \in W + \mathrm{Id}^{\mathrm{gr}}(A)$, where W is the span of all the monomials w_τ ($\tau \in S_t$) smaller than w_σ in the lexicographic ordering.

By assumption, we can factor $w_\sigma = aw_1w_2\cdots w_tb$ according to its t-decomposition. For each $w_j = x_{\sigma(i_j)}^{(g_{\sigma(i_j)})} \cdots x_{\sigma(i_j+k_j)}^{(g_{\sigma(i_j+k_j)})}$, we denote

$$\bar{w}_j = [x_{\sigma(i_j)}^{(g_{\sigma(i_j)})}, \ldots, x_{\sigma(i_j+k_j)}^{(g_{\sigma(i_j+k_j)})}].$$

Recall that the first indeterminate of each w_j is the greatest indeterminate appearing in w_j. Thus, as in the proof of Proposition 3.3.1, it follows that

$$w_\sigma \equiv a\bar{w}_1 \cdots \bar{w}_t b \pmod{W}.$$

Furthermore, by Lemma 3.2.6, either a given $\bar{w}_j$ is a graded identity of A or $\bar{w}_j$ is homogeneous of degree $h_j \in G$, say. Consequently, we can assume that every w_j is homogeneous since otherwise

$$w_\sigma \equiv a\bar{w}_1 \cdots \bar{w}_t b \equiv 0 \pmod{W + \mathrm{Id}^{\mathrm{gr}}(A)},$$

as required. Now, by Proposition 3.3.1, given indeterminates $y_1^{(h_1)}, \ldots, y_t^{(h_t)} \in X$, there exist scalars $\alpha_\tau \in K$, such that

$$y_1^{(h_1)} \cdots y_t^{(h_t)} \equiv \sum_{1 \neq \tau \in S_t} \alpha_\tau y_{\tau(1)}^{(h_{\tau(1)})} \cdots y_{\tau(t)}^{(h_{\tau(t)})} \pmod{\mathrm{Id}^{\mathrm{gr}}(A)}.$$

Therefore, by Lemma 3.2.4,

$$a\bar{w}_1 \cdots \bar{w}_t b \equiv \sum_{1 \neq \tau \in S_t} \alpha_\tau a\bar{w}_{\tau(1)} \cdots \bar{w}_{\tau(t)} b \pmod{\mathrm{Id}^{\mathrm{gr}}(A)}.$$

Since $w_\sigma = aw_1w_2\cdots w_tb$ is a t-decomposition, expanding all the Lie monomials on the right produces only associative monomials smaller than w_σ in the lexicographic ordering. This yields

$$w_\sigma \equiv a\bar{w}_1 \cdots \bar{w}_t b \equiv \sum_{\tau \in S_t, \tau \neq 1} \alpha_\tau a\bar{w}_{\tau(1)} \cdots \bar{w}_{\tau(t)} b \equiv 0 \pmod{W + \mathrm{Id}^{\mathrm{gr}}(A)},$$

as required. □

An upper bound of $a_m(n)$ was demonstrated in [BGZ]. We refer the reader to the original paper or [GZ] for complete details; we will only briefly outline the key constructions here.

Fix m and n and any positive integer c. Put $t = m + \lfloor \log_2 c \rfloor$, $N = 2^{t2^{t+1}}$, $p_2 = 2^{t2^t}$, and define p_j, for each $j > 2$, to be the integer for which

$$\underbrace{\log_N \ldots \log_N}_{j-2} p_j = p_2.$$

Finally, set $f(m, c) = \log_2 p_m$. The following lemma was proved in [BGZ].

Lemma 3.3.4. *If $n \geq f(m, c)$, then $a_m(n) < n!(\frac{1}{c})^n$.*

We are now ready to finish the proof of Theorem 3.1.3. It follows immediately from the following corollary and Proposition 3.3.3.

Corollary 3.3.5. *Let G be a finite group, and suppose that A is either a Lie-G-graded or Jordan-G-graded such that A_1 satisfies a non-trivial polynomial identity of degree d. Then, for each integer $n \geq f(d|G|, |G|)$, we have*

$$c_n(A) \leq c_n^{gr}(A) \leq |G|^n a_{d|G|}(n) < n!$$

As an illustration of Theorem 3.1.3, observe the following corollary, which follows from Example 3.1.2 (and also from the Bahturin-Giambruno-Zaicev Theorem, obtained by other means). This is a quantitative version of Amitsur's Theorem 2.3.1.

Corollary 3.3.6. *Let A be an algebra over a field of characteristic not 2 which admits an involution $*$. If either $\mathcal{S}$ or $\mathcal{K}$ satisfy a polynomial identity of degree d, then the entire algebra A satisfies a polynomial identity of degree bounded above by $f(2d, 2)$.*

In Chapter 4 and Chapter 6, we will demonstrate that (in certain circumstances) if A is an associative algebra such that $G \leq \mathrm{Aut}^*(A)$, then A can be endowed with a Lie-group-grading and a Jordan-group-grading (in the form of what we will come to call a Lie-Jordan-group-grading). In this case, the respective identity components of the gradings are the subspace of invariants, $A^G = \{a \in A | \, a^g = a\}$, and the subspace of skew-invariants, ${}^G A = \{a \in A | \, a^g = -a\}$. Consequently, we will be able to deduce that if either of these subspaces satisfies a nontrivial polynomial identity of degree d, then A satisfies a polynomial identity of degree bounded above by $f(d|G|, |G|)$. In order that we may construct these gradings, we need to further examine the relationship that exists between actions on algebras and gradings in detail; we begin our investigation into this topic in Chapter 4.

Chapter 4

Hopf Algebra Actions and Dualities

In this chapter, we discuss the relationship between actions and gradings on algebras. Sometimes, an action or a grading can be formalized as the action of a Hopf algebra, and as we will see, we can sometimes utilize the dual nature of Hopf algebras to produce powerful dualities between these two concepts.

For example, when a Hopf algebra H is finite-dimensional, commutative, semisimple, and split, certain actions of H are equivalent to group-gradings; special cases of this duality include the well-known duality of actions by automorphisms and group-gradings, and the duality of actions by derivations and group-gradings. Our primary goal in this chapter is to extend this relationship to include more general Hopf algebra actions (which we call oriented Hopf actions); in particular, we wish to include actions by anti-automorphisms and anti-derivations.

In the first section, we describe the notion of a Hopf algebra. We will only recall the basic properties that are needed, additional details can be found in [Sw] or [Mo]. Following this, we introduce Hopf algebra actions and we present the Bergen and Cohen Duality between certain finite-dimensional Hopf actions and group-gradings. Note that the monograph [Mo] contains a detailed exposition of Hopf algebra actions on algebras. In the second section, we describe a new type of Hopf action that we call an oriented Hopf action. In the final section of this chapter, we extend the Bergen and Cohen Duality to incorporate our oriented Hopf algebra actions, and resultantly, we incorporate gradings which are more general than group-gradings. We remind the reader that in this section, except when explicitly stated otherwise, we do not assume that A is associative.

4.1 Hopf Algebras and Duality of Hopf Actions

4.1.1 Hopf Algebras

Our goal in this section is to define what a Hopf algebra is, and also to recall some basic concepts from linear algebra that will be needed. Hopf algebras are surprisingly pervasive structures in mathematics which are prompted by the notion of the dual of a unital associative algebra. As we will see, a Hopf algebra is a structure that is simultaneously a unital associative algebra and a counital coassociative coalgebra which satisfies several compatibility conditions and possesses a particular anti-automorphism (called the antipode). In this thesis, we only consider Hopf algebras that are finite-dimensional.

Before we begin, we recall some basic definitions. If H is a vector space with basis $\{e_1, \ldots, e_n\}$, then the linear dual $\mathrm{Hom}(H, K)$ is a vector space with (dual) basis $\{p_{e_i} | 1 \leq i \leq n\}$, where $p_{e_i}(e_j) = \delta_{i,j}$ (here, δ is Kronecker's delta). We recall the transpose of a linear map.

Definition 4.1.1. *Let V and W be vector spaces, and let $\varphi : V \to W$ be a linear map. The transpose of φ is the map $\varphi^* : Hom(W, K) \to Hom(V, K)$, defined by $\varphi^*(f)(v) = f(\varphi(v))$, for all $v \in V, f \in Hom(W, K)$.*

Algebras and Coalgebras

We begin by defining the algebraic structures that a Hopf algebra encapsulates. We choose to define associative unital algebras using diagrams, so that this definition may be more easily dualized. An associative unital algebra is a triple (A, m, u) consisting of: a vector space A, a multiplication map $m : A \otimes A \to A$, and a unit map $u : K \to A$, such that the following diagrams commute.

Associativity:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
& A \otimes A \arrow[dd, "m"] & \\
K \otimes A \arrow[ur, "u\otimes id"] \arrow[dr, "\cong"'] & & A \otimes K \arrow[ul, "id\otimes u"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

Dualizing this definition (that is, turning all the arrows around) yields a vector space with a product expansion map and a map into the base field. Formally, we define a coassociative counital coalgebra as a triple (C, Δ, ε) consisting of: a vector space C, a map $\Delta : C \to C \otimes C$ called *comultiplication* or the *coproduct*, and a map $\varepsilon : C \to K$ called the *counit*, for which the following diagrams commute:

Coassociativity:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes id"] \\
C \otimes C \arrow[r, "id\otimes\Delta"] & C \otimes C \otimes C
\end{tikzcd}$$

Counit:

$$\begin{tikzcd}
& C \otimes C \arrow[dl, "\epsilon\otimes id"'] \arrow[dr, "id\otimes\epsilon"] & \\
K \otimes C & & C \otimes K \\
& C \arrow[uu, "\Delta"] \arrow[ul, "1\otimes id"] \arrow[ur, "id\otimes 1"'] &
\end{tikzcd}$$

If (A, m, u) is a finite-dimensional algebra, then $(\mathrm{Hom}(A, K), m^*, u^*)$ is a counital coassociative coalgebra; similarly, whenever (C, Δ, ε) is a finite-dimensional coalgebra, $(\mathrm{Hom}(C, K), \Delta^*, \varepsilon^*)$ is an associative unital algebra. This is not necessarily true in the infinite-dimensional case.

Sweedler Notation and Cocommutativity in Coalgebras

If C is a countial coassociative coalgebra, then coproducts can be written as finite sums

$$\Delta(c) = \sum_{i=1}^{n} a_i \otimes b_i, \qquad \text{where } a_i, b_i \in C.$$

In order to simplify notation, we will henceforth adopt the conventional *Sweedler notation* (as introduced in the monograph [Sw]) by writing

$$\Delta(c) = \sum_{c} c_{(1)} \otimes c_{(2)}, \qquad \text{for all } c \in C.$$

It is understood that $c_{(1)}$ and $c_{(2)}$ do not represent any specific elements in C, and that the summands in this expression are not necessarily related. To illustrate why such a notation is useful, notice that when we apply the coproduct map twice, then the coassociativity condition implies that

$$\Delta^2(c) = \sum_{c} \Delta(c_{(1)}) \otimes c_{(2)} = \sum_{c} c_{(1)} \otimes \Delta(c_{(2)}).$$

Using Sweedler notation, we may write both of these expressions as

$$\Delta^2(c) = \sum_{c} c_{(1)} \otimes c_{(2)} \otimes c_{(3)},$$

so that the components $c_{(i)}$ are each assumed to lie in the i^{th} tensor factor.

If an element c in a coalgebra C satisfies

$$\Delta(c) = \sum_{c} c_{(1)} \otimes c_{(2)} = \sum_{c} c_{(2)} \otimes c_{(1)},$$

then c is called *cocommutative*. If C is spanned by cocommutative elements, then C is called a cocommutative coalgebra.

Bialgebras and Hopf Algebras

When a vector space is simultaneously endowed with the structure of an algebra and a coalgebra, we can impose certain compatibility conditions to ensure that the various operations behave well together; further, we would like that our structure may be dualized accordingly. To that end, if (H, m, u) is a unital associative algebra and that (H, Δ, ε) is a counital coassociative coalgebra, we will consider when the following condition holds:

- Δ and ε are algebra homomorphisms.

One may demonstrate this is equivalent to saying that m and u are coalgebra homomorphisms. If this holds, then $(H, m, u, \Delta, \varepsilon)$ is called a *bialgebra*.

Finally, if $(H, m, u, \Delta, \epsilon)$ is a bialgebra, then a K-linear map $S : H \to H$ such that

$$\sum_h S(h_{(1)})h_{(2)} = \sum_h h_{(1)}S(h_{(2)}) = \varepsilon(h)1,$$

for all $h \in H$, is called an *antipode*. A bialgebra with an antipode $(H, m, u, \Delta, \varepsilon, S)$ is called a *Hopf algebra*. Whenever the context is clear, we refer to H as a Hopf algebra omitting the collection of maps.

We take a moment to consider the reasoning behind these compatibility conditions to illustrate that the complicated definition of a Hopf algebra is not futile; the reader should consult Bergman's excellent article [Bg] which affably describes the motivation behind Hopf algebras (from our perspective at least, as many ulterior motivations exist).

Looking ahead, our applications for Hopf algebras will involve when an algebra A has the additional structure of an H-module action via a mapping $h \cdot a \in A$. The conditions we have imposed on H allow us to study such actions in a reasonable way. For example, actions which expand on products in a manner conducive with the coproduct Δ (in the sense that $h \cdot (ab) = \sum_h (h_{(1)} \cdot a)(h_{(2)} \cdot b)$) are important. By imposing coassociativity on H, we ensure that $h \cdot ((ab)c) = h \cdot (a(bc))$. Similarly, the associativity of H ensures that, for $x, y, z \in H$, the two expressions $x \cdot ((yz) \cdot a)$ and $(xy) \cdot (z \cdot a)$ are equal. We would also like to be able to expand $xy \cdot (ab)$ in the two obvious ways; and it turns out that this condition is expressed by saying that Δ is an algebra homomorphism.

Dual Hopf Algebras and Other Examples

As expected, if $(H, m, u, \Delta, \varepsilon, S)$ is a finite-dimensional Hopf algebra, then it is easy to see that $(\mathrm{Hom}(H, K), \Delta^*, \varepsilon^*, m^*, u^*, S^*)$ is also a finite-dimensional Hopf algebra. In keeping with convention, we denote this Hopf algebra by H^*, and when the context is clear we also denote the underlying algebra $(\mathrm{Hom}(H, K), \Delta^*, \varepsilon^*)$ by H^*. Observe that multiplication in

H^* is the convolution operation (abbreviated $\Delta^* = *$),

$$(\alpha * \beta)(h) = \sum_h \alpha(h_{(1)})\beta(h_{(2)}), \text{ for all } \alpha, \beta \in \mathrm{Hom}(H, K),$$

and the unit element is $\varepsilon^* = \varepsilon$.

We introduce an important multiplicative subgroup existing in any Hopf algebra. For any H, elements $h \in H$ satisfying $\Delta(h) = h \otimes h$ and $\varepsilon(h) = 1$ are called group-like elements. The set of all group-like elements of H is denoted by $G(H)$. We will observe that $G(H)$ is a group under the multiplication of H. Indeed, $G(H)$ is not empty since $1 \in G(H)$, and furthermore $G(H)$ is closed under multiplication (since comultiplication and counit are algebra maps). Further, by the compatibility condition of the antipode, for every $g \in G(H)$, we have $gS(g) = S(g)g = 1$, and since $S(g) \in G(H)$, we have that $G(H)$ is a group.

When we consider a finite-dimensional Hopf algebra H and its dual, it is a routine observation that $G(H^*) = \mathrm{Alg}(H, K)$, the set of algebra maps from H to K. Further, $G(H^*)$ is a (finite) group of linearly independent units in H^* (linear independence is shown in [Sw], page 55). This group is important in the sequel, and so we formally state this definition. Additional properties of the group $G(H^*)$ can also be found in [Mo].

Definition 4.1.2. *For a finite-dimensional Hopf algebra H, we define $G(H^*) = Alg(H, K)$ to be the set of all algebra maps from H to K. Note that $(G(H^*), *)$ is a group of linearly independent units contained in H^*.*

We conclude by presenting examples of the Hopf algebras which will be of most interest to us.

Example 4.1.3. *Let G be group a and let K be a field. The group algebra KG is made into a Hopf algebra by setting the following operations.*

- *Coproduct: $\Delta(g) = g \otimes g$, for all $g \in G$;*
- *Counit: $\varepsilon(g) = 1$, for all $g \in G$, and;*
- *Antipode: $S(g) = g^{-1}$.*

We note that KG is commutative if and only if G is abelian, finite-dimensional if and only if G is finite, and always cocommutative.

Example 4.1.4. *Let $\mathfrak{g}$ be a restricted Lie algebra over a field K of characteristic $p > 0$. The restricted universal enveloping algebra of $\mathfrak{g}$, denoted by $u(\mathfrak{g})$ (see Appendix C for the definition of $u(\mathfrak{g})$), is made into a Hopf algebra by setting the following operations.*

- *Coproduct:* $\Delta(x) = x \otimes 1 + 1 \otimes x$*, for all* $x \in \mathfrak{g}$*;*
- *Counit:* $\varepsilon(x) = 0$*, for all* $x \in \mathfrak{g}$*, and;*
- *Antipode:* $S(x) = 0$*, for all* $x \in \mathfrak{g}$.

We note that $u(\mathfrak{g})$ is not commutative unless $\mathfrak{g}$ is abelian, finite-dimensional if and only if $\mathfrak{g}$ is finite-dimensional, and always cocommutative.

Due to its prevalence in the ensuing section, we take a moment to describe the important Hopf algebra which arises as the dual of a group algebra.

Example 4.1.5. *Let G be a finite (multiplicative) group, and let KG be the group Hopf algebra. Recall that $(KG)^*$ has a basis $\{p_g | \, g \in G\}$, given by $p_g(x) = \delta_{g,x}$, for all $g, x \in G$. Recall multiplication is the convolution operation, $(p_g * p_h)(x) = \delta_{g,h,x}$. A routine check verifies the other Hopf operations in $(KG)^*$ are as follows.*

- *Coproduct:* $\Delta(p_g) = \sum_{xy=g} p_x \otimes p_y$.
- *Counit:* $\varepsilon(p_g) = p_{\delta_{g,1}}$.
- *Antipode:* $S(p_g) = p_{g^{-1}}$.

Notice that the dual basis consists of orthogonal idempotents which sum to 1. The Hopf algebra $(KG)^*$ is finite-dimensional, always commutative, and cocommutative whenever G is abelian.

4.1.2 Duality Between Hopf Actions and Gradings

In this section we consider when a Hopf algebra H acts on an algebra A. By using the dual nature of Hopf algebras, we are able to build some interesting relationships between various types of Hopf actions. We reiterate that, for the remainder of this thesis, all Hopf algebras are assumed to be finite-dimensional. If A has the structure of an H-module via some mapping $h \cdot a \in A$, then we make the following definition.

Definition 4.1.6. *If A is a (left) H-module via the mapping $h \cdot a \in A$, and*

$$h \cdot (ab) = \sum_h (h_{(1)} \cdot a)(h_{(2)} \cdot b)$$

is satisfied for all $a, b \in A$, $h \in H$, then A is called a (left) H-algebra. If A happens to be associative and unital, then we further require that $h \cdot 1_A = \epsilon(h)1_A$, for all $h \in H$.

We begin with a worked example. Suppose that G is a finite group and $A = \bigoplus_{g\in G} A_g$ is a group-grading. There is a natural action on A by the set of projection maps, $\{p_g | \, g \in G\}$ (that is, $p_g \cdot (\sum_{h\in G} a_h) = a_g$, where each $a_h \in A_h$). Under function composition, this set spans a subalgebra of the linear transformations of A, and moreover, it consists of orthogonal idempotents which sum to 1. It follows that this subalgebra is isomorphic to the underlying algebra of $(KG)^*$. Further, notice that if $a \in A_g$ and $b \in A_h$ are homogeneous, then, for each $x \in G$,

$$p_x \cdot (ab) = \begin{cases} ab, & \text{if } x = gh; \\ 0, & \text{if } x \neq gh. \end{cases}$$

Consequently, $p_x \cdot (ab) = \sum_{yz=x}(p_y \cdot a)(p_z \cdot b)$, and so any G-grading of A can be recognized as a $(KG)^*$-algebra structure on A.

On the other hand, suppose that an algebra A has a structure of a $(KG)^*$-algebra via a mapping $p_g \cdot a \in A$. In this case, we define $p_g \cdot A = A_g$, for all $g \in G$. Recalling from Example 4.1.5 that $\{p_g | \, g \in G\}$ are orthogonal idempotents with sum 1, it is not hard to see that $A = \bigoplus_{g\in G} A_g$ is a vector space decomposition of A. Further, if $a \in A_{g_1}$ and $b \in A_{g_2}$,

$$p_{g_1 g_2} \cdot (ab) = \sum_{xy=g_1 g_2} (p_x \cdot a)(p_y \cdot b) = ab.$$

Hence $ab \in A_{g_1 g_2}$ and $A = \bigoplus_{g\in G} A_g$ is a group-grading of A. Thus we deduce that an algebra A is G-graded if and only if A is a $(KG)^*$-algebra.

This leads us to the key result on Hopf actions, which appears in [BC]. Suppose that H is any finite-dimensional commutative Hopf algebra that is semisimple as an algebra and splits over K. In this case, H has a basis of orthogonal idempotents whose sum is 1. Since the dual basis in H^* is precisely $\mathrm{Alg}(H, K) = G(H^*)$, and since $\dim H = \dim H^*$, it follows that $H^* = KG(H^*)$ and consequently $H = (KG(H^*))^*$.

Similarly to the above example, it follows that every H-module A admits a vector space decomposition

$$A = \bigoplus_{\varphi\in G(H^*)} A_\varphi, \text{ where } A_\varphi = \{a \in A |\, h \cdot a = \varphi(h)a, \text{ for all } h \in H\},$$

and conversely, that every graded vector space $A = \bigoplus_{\varphi\in G(H^*)} A_\varphi$ has a natural H-module action induced by $h \cdot a_\varphi = \varphi(h)a_\varphi$, for every $a_\varphi \in A_\varphi$ and $h \in H$. Bergen and Cohen proved that every associative H-algebra is graded by the group $(G(H^*), *)$ in [BC]. In fact, this assertion holds equally true for all H-algebras, and the converse also holds. We summarize these facts with the following duality theorem.

Theorem 4.1.7 (The Bergen and Cohen Duality)**.** *Let H be a Hopf algebra that is a finite-dimensional, commutative, semisimple, and splits over K. Then an algebra A is an H-algebra if and only if $A = \bigoplus_{\varphi\in(G(H^*),*)} A_\varphi$ is a group-grading.*

We remark that special cases of this duality appeared earlier in the literature, for instance, see [Wt]. We now present examples of finite-dimensional H-algebra actions. The first example involves group algebras; more on this duality can be found in Chapter 6.

Example 4.1.8. *Let G be a finite group and let $H = KG$ be the group (Hopf) algebra. If A is a KG-algebra, then it follows that*

$$g \cdot (ab) = \sum_g (g_{(1)} \cdot a)(g_{(2)} \cdot b) = (g \cdot a)(g \cdot b), \textit{ for all } a, b \in A.$$

Since $(\cdot)$ *is a module map, by defining* $a^g := g \cdot a$*, for all* $g \in G$*, it follows that* G *acts as automorphisms on* A*. Conversely, if* $G \leq \mathrm{Aut}(A)$*, then* A *is a* KG*-algebra via the action* $g \cdot a := a^g$*. So* G *embeds into* $\mathrm{Aut}(A)$ *if and only if* A *is a* KG*-algebra.*

If we suppose further that KG *satisfies the hypotheses of the Bergen and Cohen Duality, then* A *is a* KG*-algebra if and only if* A *is graded by the group* $(G(H^*), *)$*. By recognizing that* $(G(H^*), *) \cong \hat{G}$ *(see Chapter 6), we observe that Theorem 4.1.7 implies the well-known duality:* G *embeds into* $\mathrm{Aut}(A)$ *if and only if*

$$A = \bigoplus_{\chi \in \hat{G}} A_\chi, \text{ where } A_\chi = \{a \in A \mid a^g = \chi(g)a, \text{ for all } g \in G\},$$

is a $\hat{G}$*-grading of* A*.*

Another pertinent example of an H-action involves the restricted universal enveloping algebra of a restricted Lie algebra; again, additional details may be found in Chapter 6.

Example 4.1.9. *Let* K *be a field of characteristic* $p > 0$*, let* $\mathfrak{g}$ *be a restricted Lie algebra, and let* $H = u(\mathfrak{g})$ *be the restricted universal enveloping (Hopf) algebra. If* A *is a* $u(\mathfrak{g})$*-algebra, then it follows that*

$$x \cdot (ab) = \sum_x (x_{(1)} \cdot a)(x_{(2)} \cdot b) = (x \cdot a)b + a(x \cdot b), \text{ for all } a, b \in A.$$

Since $(\cdot)$ *is a module map, and by defining* $a^x = x \cdot a$*, for all* $x \in \mathfrak{g}$*, it follows that* $\mathfrak{g}$ *acts as derivations on* A*. Conversely, if* $\mathfrak{g} \leq \mathrm{Der}(A)$*, then* A *is a* $u(\mathfrak{g})$*-algebra via the action* $x \cdot a := a^x$*. Hence, a restricted Lie algebra* $\mathfrak{g}$ *embeds into* $\mathrm{Der}(A)$ *if and only if* A *is a* $u(\mathfrak{g})$*-algebra.*

If we suppose further that $u(\mathfrak{g})$ *satisfies the hypotheses of the Bergen and Cohen Duality, then* A *is a* $u(\mathfrak{g})$*-algebra if and only if* A *is graded by the group* $(G(H^*), *)$*. By recognizing that* $(G(H^*), *) \cong \mathbb{Z}_p^n$ *(see Chapter 6), we observe that Theorem 4.1.7 implies the following duality:* $\mathfrak{g}$ *embeds into* $\mathrm{Der}(A)$ *if and only if*

$$A = \bigoplus_{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_p^n} A_{(\lambda_1, \ldots, \lambda_n)}, \text{ where } A_{(\lambda_1, \ldots, \lambda_n)} = \bigcap_{i=1}^{n} \{a \in A \mid a^{\delta_i} = \lambda_i a\},$$

is a $\mathbb{Z}_p^n$*-grading of* A*.*

4.2 Oriented Hopf Algebras and Their Actions

Our goal in this section is to define and describe oriented Hopf actions on algebras. In naturally occurring examples, Hopf algebras act on algebras in manners that are not captured by Definition 4.1.6 (for instance, a group which acts on A as both automorphisms and anti-automorphisms, or a restricted Lie algebra acting on A as both derivations and anti-derivations). Hence, we offer the following generalization.

Definition 4.2.1. *Let H be a Hopf algebra, and let A be an algebra.*

1. *Suppose that H has a vector space decomposition given by*

$$\Gamma : H = H_+ \oplus H_-.$$

   If Γ is a $\mathbb{Z}_2$-grading of H as an algebra (but not necessarily as a coalgebra), then Γ will be called an orientation of H, and the pair (H, Γ) will be referred to as an oriented Hopf algebra. We shall say that Γ is trivial when $H = H_+$.

2. *Suppose that $H = H_+ \oplus H_-$ is an oriented Hopf algebra. If an algebra A is a (left) H-module such that*

$$h_+ \cdot (ab) = \sum_{h_+} (h_{(1)} \cdot a)(h_{(2)} \cdot b), \text{ for all } h_+ \in H_+, \text{ and}$$
$$h_- \cdot (ab) = \sum_{h_-} (h_{(2)} \cdot b)(h_{(1)} \cdot a), \text{ for all } h_- \in H_-,$$

   then A is an oriented H-algebra. If A is associative and unital, then we further require that $h \cdot 1_A = \varepsilon(h)1_A$, for all $h \in H$.

Our choice of terminology is motivated by the notion of an oriented group G and the $\mathbb{Z}_2$-grading it induces on the group algebra KG (see the following example). Moreover, the terms ‘graded Hopf algebra’ and ‘super Hopf algebra’ are used ambiguously in the literature. Notice that every H-algebra is an oriented H-algebra via the trivial orientation. We also remark that an oriented H-algebra action is a special case of a generalized H-action, as defined by Berele in [Be].

Example 4.2.2. *Recall that an oriented group is defined to be a group G together with a group homomorphism $\varphi : G \to \{-1, 1\}$. If G is any such group, then we may orient the Hopf algebra KG by setting*

$$\Gamma : KG = KG_+ \oplus KG_-,$$

where $KG_+ = span\{g|\,\varphi(g) = 1\}$, and $KG_- = span\{g|\,\varphi(g) = -1\}$. Since φ is a homomorphism, Γ is a $\mathbb{Z}_2$-grading of KG.

Next, we introduce a kind of conjugation operation. In order for this to make proper sense, we shall assume, for the remainder of this chapter, that our characteristic p is different from 2.

Definition 4.2.3. *Let $H = H_+ \oplus H_-$ be an oriented Hopf algebra, let $h = h_+ + h_- \in H$, where $h_+ \in H_+$ and $h_- \in H_-$, and let $\alpha, \beta \in Hom(H, K)$.*

1. *The conjugate of h is $\bar{h} = h_+ - h_-$.*

2. *The conjugate of α is the map $\bar{\alpha} \in Hom(H, K)$ given by $\bar{\alpha}(h) = \alpha(\bar{h})$, for every $h \in H$. Set*

$$D_+ = \{\alpha|\,\bar{\alpha} = \alpha\} \text{ and } D_- = \{\alpha|\,\bar{\alpha} = -\alpha\}.$$

It follows that

$$Hom(H, K) = D_+ \oplus D_-$$

since every $\alpha = \alpha_+ + \alpha_-$, where

$$\alpha_+ = \frac{1}{2}(\alpha + \bar{\alpha}) \in D_+ \text{ and } \alpha_- = \frac{1}{2}(\alpha - \bar{\alpha}) \in D_-.$$

3. *We shall write $H^\star = Hom(H, K)$ to indicate the non-associative algebra with multiplication given by oriented convolution*

$$\alpha \star \beta = \overline{\alpha * \beta}.$$

4. *Let A be an oriented H-algebra. Then*

$$A^H = A_\varepsilon = \{a \in A|\,a^h = \varepsilon(h)a, \text{ for all } h \in H\}$$

is the set of all H-invariants of A, while

$$^{H}A = A_{\bar{\varepsilon}} = \{a \in A |\, a^h = \bar{\varepsilon}(h)a, \text{ for all } h \in H\}$$

is the set of all skew H-invariants of A.

We summarize a few straightforward properties of conjugation as follows.

Lemma 4.2.4. *Let $H = H_+ \oplus H_-$ be an oriented Hopf algebra. Then the following statements hold for every $\alpha \in Hom(H, K)$.*

1. $\bar{\alpha} = \alpha_+ - \alpha_-$ *and* $\bar{\bar{\alpha}} = \alpha$.
2. $\alpha \in D_+$ *if and only if* $\alpha(H_-) = 0$*, while* $\alpha \in D_-$ *if and only if* $\alpha(H_+) = 0$.

For the applications we have in mind, we will be interested to know exactly when $H^\star$ is associative. Given the abstract nature of oriented convolution, this is sometimes difficult to ascertain. Therefore, we offer the following characterizations.

Theorem 4.2.5. *Let $H = H_+ \oplus H_-$ be an oriented Hopf algebra. Then the following conditions are equivalent.*

1. *The algebra $H^\star$ is associative.*
2. *The algebra $H^\star$ has a unital element.*
3. *For every $\alpha \in Hom(H, K)$, the following property holds:*

$$\alpha * \bar{\varepsilon} = \bar{\alpha} = \bar{\varepsilon} * \alpha.$$

   In particular, $\bar{\varepsilon} * \bar{\varepsilon} = \varepsilon$.

4. *The decomposition $H^\star = D_+ \oplus D_-$ is the sum of ideals.*
5. *$H^\star$ is an associative algebra with multiplication given by $\alpha \star \beta = \alpha * \beta * \bar{\varepsilon}$ and unity $\bar{\varepsilon}$; moreover, $H^\star = D_+ \oplus D_-$ is a sum of ideals.*

Furthermore, in the case when (1)-(5) hold, conjugation $H^* \to H^\star : \alpha \mapsto \bar{\alpha}$ is an algebra isomorphism.

Proof. Let $\alpha, \beta, \gamma \in \mathrm{Hom}(H,K)$, and suppose that $\star$ is an associative operation. Since $(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma)$, we have $\overline{(\alpha * \beta)} * \gamma = \alpha * \overline{(\beta * \gamma)}$. If we choose $\beta = \gamma = \varepsilon$, we obtain $\bar{\alpha} = \alpha * \bar{\varepsilon}$. Likewise, if we take $\alpha = \beta = \varepsilon$, we obtain $\bar{\varepsilon} * \gamma = \bar{\gamma}$. So, (1) implies (3).

Next suppose that (2) holds, and observe that $H^\star$ has unital element τ precisely when $\alpha * \tau = \bar{\alpha} = \tau * \alpha$, for every α. So, by the associativity of $*$,

$$(\alpha \star \beta) \star \gamma = \overline{\overline{(\alpha * \beta)} * \gamma} = ((\tau * (\alpha * \beta)) * \gamma) * \tau = \overline{\alpha * \overline{(\beta * \gamma)}} = \alpha \star (\beta \star \gamma).$$

Thus, (2) implies (1).

Observe that (3) is equivalent to saying $\bar{\varepsilon}$ in the unital element of $H^\star$; hence, (3) implies (2).

It remains then to show that (3) and (4) are equivalent. Suppose then that (3) holds, and let $\alpha \in D_+$ and $\beta \in \mathrm{Hom}(H,K)$. Then

$$\beta \star \alpha = \overline{\beta * \alpha} = (\beta * \alpha) * \bar{\varepsilon} = \beta * \bar{\alpha} = \beta * \alpha = \overline{\beta \star \alpha}.$$

Hence, $\beta \star \alpha \in D_+$ and D_+ is a left ideal of $H^\star$. Likewise, D_+ is a right ideal of $H^\star$. The proof that D_- is an ideal is similar. On the other hand, if (4) holds, then, for every $\alpha \in D_+$,

$$\alpha \star \varepsilon_+ = \alpha \star \varepsilon_+ + \alpha \star \varepsilon_- = \alpha \star \varepsilon = \overline{\alpha * \varepsilon} = \bar{\alpha} = \alpha.$$

By symmetry, $\varepsilon_+ \star \alpha = \alpha$ as well, and so ε_+ is unity in D_+. Similarly, $-(\varepsilon_-)$ is unity in D_-. Therefore, for every $\alpha \in \mathrm{Hom}(H,K)$,

$$\alpha \star \bar{\varepsilon} = (\alpha_+ + \alpha_-) \star (\varepsilon_+ - \varepsilon_-) = \alpha_+ + \alpha_- = \alpha = \bar{\varepsilon} \star \alpha,$$

as required.

Finally, suppose that the equivalent statements (1)-(5) hold. Then clearly conjugation is a linear map preserving unity; moreover, for every $\alpha, \beta \in H^*$, we have

$$\overline{\alpha * \beta} = \overline{\alpha * (\bar{\varepsilon} * \bar{\varepsilon}) * \beta} = \overline{\bar{\alpha} * \bar{\beta}} = \bar{\alpha} \star \bar{\beta}.$$

□

We remark that the subspaces D_+ and D_- are ideals in H^* if and only if they are ideals in $H^\star$.

Corollary 4.2.6. *Let $H = H_+ \oplus H_-$ be a nontrivial oriented Hopf algebra (that is, $H \neq H_+$). Then either $\bar{\varepsilon} \neq \varepsilon$ or $H^\star$ is not associative.*

Proof. Suppose, to the contrary, that $H^\star$ is associative and $\bar{\varepsilon} = \varepsilon$. Then part (3) of Theorem 4.2.5 implies that $\alpha = \bar{\alpha}$, for every $\alpha \in \mathrm{Hom}(H, K)$. However, this is impossible. For instance, let $x \in H_-$ be nonzero, and let $\alpha \in \mathrm{Hom}(H, K)$ be the map induced by the projection of H onto Kx. Then $\alpha(x) = 1$, but $\bar{\alpha}(x) = -1$. □

Lemma 4.2.7. *Let $H = H_+ \oplus H_-$ be an oriented Hopf algebra. Then the following statements hold.*

1. *If $\varphi \in G(H^*)$, then $\bar{\varphi} \in G(H^*)$.*
2. *The set $G(H^*)$ is closed under $\star$.*
3. *The magma $(G(H^*), \star)$ is a quasigroup.*

Proof. We leave the proof of (1) to the reader. Since $G(H^*)$ is closed under $*$, (2) follows easily from (1). It remains to demonstrate that, for every $\varphi, \psi \in G(H^*)$, there exist unique elements $x, y \in G(H^*)$ such that $\varphi \star x = \psi$ and $y \star \varphi = \psi$. Since $\bar{\psi} \in G(H^*)$ and $(G(H^*), *)$ is a group, there exists a unique $x \in G(H^*)$ such that $\varphi * x = \bar{\psi}$, which is equivalent to saying $\varphi \star x = \overline{\varphi * x} = \bar{\bar{\psi}} = \psi$. The other case follows by symmetry. □

If $H = H_+ \oplus H_-$ is an oriented Hopf algebra, then to avoid ambiguity, we shall henceforth set the following notation:

- G_H^* will denote the group $(G(H^*), *)$;
- $G_H^\star$ will denote the quasigroup $(G(H^*), \star)$; and,
- G_H will denote the set $G(H^*)$.

Lemma 4.2.7 and the proof of Theorem 4.2.5 essentially form a proof of the following result.

Theorem 4.2.8. *Let $H = H_+ \oplus H_-$ be an oriented Hopf algebra. Then $G_H^\star$ is a quasigroup such that the following properties are equivalent.*

1. *$G_H^\star$ is a semigroup.*

2. *$G_H^\star$ is a loop.*

3. *For every $\varphi \in G$, $\varphi * \bar{\varepsilon} = \bar{\varphi} = \bar{\varepsilon} * \varphi$. In particular, $\bar{\varepsilon} * \bar{\varepsilon} = \varepsilon$.*

4. *$G_H^\star$ is a group with multiplication given by $\varphi \star \psi = \varphi * \psi * \bar{\varepsilon}$ and identity element $\bar{\varepsilon}$.*

Furthermore, if (1)-(4) hold, conjugation $G^* \to G^\star : \varphi \mapsto \bar{\varphi}$ is an isomorphism of groups.

Corollary 4.2.9. *Let $H = H_+ \oplus H_-$ be a nontrivial oriented Hopf algebra such that H is finite-dimensional, commutative, semisimple, and splits over K. Then either $\bar{\varepsilon} \neq \varepsilon$ or $G_H^\star$ is not associative. Consequently, if $\dim(H)$ is odd, then $G_H^\star$ is not associative.*

Proof. Suppose, to the contrary, that $G_H^\star$ is associative and $\bar{\varepsilon} = \varepsilon$. Then, by part (3) of Theorem 4.2.8, $\varphi * \varepsilon = \bar{\varphi} = \bar{\varepsilon} * \varphi$, for every $\varphi \in G_H$. Hence, by part (2) of Lemma 4.2.4 forces $\alpha(H_-) = 0$, for every $\alpha \in G_H$. However, this is impossible since G_H forms a basis of $\mathrm{Hom}(H, K)$ under the hypotheses on H. Finally, suppose that $G_H^\star$ is associative and $\dim(H) = |G_H|$ is odd. Then, as above, $\bar{\varepsilon} * \bar{\varepsilon} = \varepsilon$, and so $\bar{\varepsilon} = \varepsilon$, a contradiction. □

We close this section with the construction of an orientation that will be useful later.

Lemma 4.2.10. *Let H be a Hopf algebra that is a finite-dimensional, commutative, semisimple, and splits over K. Suppose that H has even dimension, and let $\{e_1, \ldots, e_{2n}\}$ be its basis of orthogonal idempotents. Then $H = H_+ \oplus H_-$ is an orientation of H, where*

$$H_+ = \mathit{span}\{e_1 + e_2, e_3 + e_4, \ldots, e_{2n-1} + e_{2n}\} \text{ and}$$

$$H_- = \mathit{span}\{e_1 - e_2, e_3 - e_4, \ldots, e_{2n-1} - e_{2n}\}.$$

4.3 A Duality Between Oriented Hopf Actions and Gradings

Our goal in this section is to extend the Bergen and Cohen Duality to include oriented Hopf actions. In particular, we show that oriented H-algebra actions on a Lie algebra are dual to certain quasigroup-gradings; whereas, in the case of associative algebras, we show that oriented Hopf actions are dual to, what we call, Lie-Jordan-gradings.

Throughout this section, we assume that $H = H_+ \oplus H_-$ is an oriented Hopf algebra. In addition, we assume that H satisfies the hypotheses of Theorem 4.1.7. Furthermore, because the gradings that we will introduce below only make proper sense when G_H^* is abelian, we shall also assume that H is cocommutative. We thus set the following notation.

Definition 4.3.1. *We say that a Hopf algebra H satisfies the duality hypothesis if H is a finite-dimensional, commutative, semisimple, cocommutative Hopf algebra which splits over K.*

Recall now that every H-module A has a vector space decomposition

$$A = \bigoplus_{\varphi \in G_H} A_\varphi, \text{ where } A_\varphi = \{a \in A \,|\, h \cdot a = \varphi(h)a \text{ for all } h \in H\},$$

and, conversely, that every graded vector space $A = \bigoplus_{\varphi \in G_H} A_\varphi$ has a natural H-module action induced by $h \cdot a_\varphi = \varphi(h)a_\varphi$, for every $a_\varphi \in A_\varphi$ and $h \in H$. Consequently, A is an oriented H-algebra if and only if, for all homogeneous $a \in A_\varphi$, $b \in A_\psi$, $h_+ \in H_+$, and $h_- \in H_-$, we have

$$h_+ \cdot (ab) = (\varphi * \psi)(h_+)ab, \text{ and}$$
$$h_- \cdot (ab) = (\varphi * \psi)(h_-)ba.$$

Using these facts, we will obtain dualities between actions and gradings in three special cases: when A is commutative, anti-commutative, or associative. In order to handle the associative case, we make use of the induced Lie algebra, $A^{(-)}$, and the induced Jordan algebra, $A^{(+)}$. Recall that since our characteristic p is assumed not to be 2, associative multiplication decomposes via the identity

$$2ab = a \circ b + [a, b]$$

into commutative and anti-commutative operations. We will make use of the following simple observation:

Lemma 4.3.2. *Let A be an oriented H-algebra. If A is associative and H is cocommutative, then $A^{(-)}$ and $A^{(+)}$ are oriented H-algebras under the induced H-action.*

Commutative Algebras

If A happens to be commutative, then A is an oriented H-algebra if and only if A is an H-algebra. Thus, A is an oriented H-algebra precisely when $A = \bigoplus_{\varphi \in G_H^*} A_\varphi$ is a group-grading.

Anti-commutative Algebras

Let A be an anti-commutative algebra (for instance, a Lie algebra). If A is an oriented H-algebra, then

$$(\varphi * \psi)(h_+) = (\varphi \star \psi)(h_+), \text{ and}$$
$$(\varphi * \psi)(h_-) = -(\varphi \star \psi)(h_-),$$

and it follows that the vector space decomposition $A = \bigoplus_{\varphi \in G_H^\star} A_\varphi$ described above is a quasigroup-grading.

Conversely, if $A = \bigoplus_{\varphi \in G_H^\star} A_\varphi$ is a quasigroup-grading, then $h \cdot (ab) = (\varphi \star \psi)(h)(ab)$, for all $h \in H$ and homogeneous $a \in A_\varphi$ and $b \in A_\psi$. Similarly to the last paragraph, but in reverse, one can verify that this property is equivalent to A being an oriented H-algebra.

Now suppose that $G_H^\star$ is either a semigroup or a loop. Then, by Theorem 4.2.8, $G_H^\star$ is a group such that, for every $\varphi, \psi \in G_H$, we have $\bar{\varphi} \star \bar{\psi} = \overline{\varphi * \psi}$. Thus we have the following:

Theorem 4.3.3. *Let $H = H_+ \oplus H_-$ be an oriented Hopf algebra satisfying the duality hypothesis, and let A be an anti-commutative algebra. Then the following statements hold.*

1. *A is an oriented H-algebra if and only if the vector space decomposition $A = \bigoplus_{\varphi \in G_H^\star} A_\varphi$ is a quasigroup-grading of A.*

2. *Suppose that the quasigroup $G_H^\star$ is a either a semigroup or a loop, and define $B_\varphi = A_{\bar{\varphi}}$, for each $\varphi \in G_H$. Then A is an oriented H-algebra if and only if the vector space decomposition $A = \bigoplus_{\varphi \in G_H^*} B_\varphi$ is a group-grading.*

We recall that determining when a Lie algebra admits a quasigroup-grading may be of some independent interest, due to the ongoing search for set-gradings of Lie algebras which cannot be realized as semigroup-gradings. Recall that Zassenhaus and Patera once asserted that such things did not exist ([PZ]). In Chapter 7 we offer an example of how to use the above duality to construct a counterexample to this claim.

Associative Algebras

Let A be an associative algebra, and suppose that A is an oriented H-algebra. Then, since H is cocommutative, $A^{(+)}$ and $A^{(-)}$ are also oriented H-algebras by Lemma 1.3. Hence, from the discussion above, $A^{(+)} = \bigoplus_{\varphi \in G^*} A_\varphi$ is a group-grading and $A^{(-)} = \bigoplus_{\varphi \in G^\star} A_\varphi$ is a quasigroup-grading. These facts prompt the following definition.

Definition 4.3.4. *Let $H = H_+ \oplus H_-$ be an oriented Hopf algebra, and let A be an associative algebra with an H-module action. Then a vector space decomposition $A = \bigoplus_{\varphi \in G_H} A_\varphi$ will be called a Lie-Jordan-grading of A whenever $A^{(+)} = \bigoplus_{\varphi \in G_H^*} A_\varphi$ is a group-grading and $A^{(-)} = \bigoplus_{\varphi \in G_H^\star} A_\varphi$ is a quasigroup-grading. If the operations $*$ and $\star$ coincide, we shall say that a Lie-Jordan-group-grading is trivial.*

Notice that a Lie-Jordan-group-grading is trivial precisely when it is a group-grading over G_H^*. It follows from above that every oriented H-algebra action on an associative algebra A gives rise to a Lie-Jordan-grading of A. On the other hand, if $A = \bigoplus_{\varphi \in G_H} A_\varphi$

is a Lie-Jordan-grading of A, then, for every $a \in A_\varphi$, $b \in A_\psi$, the induced H-action on A satisfies

$$h \cdot (ab) = \frac{1}{2}(h \cdot (a \circ b) + h \cdot [a,b]) = \frac{1}{2}((\varphi * \psi)(h)(a \circ b) + (\varphi \star \psi)(h)[a,b]).$$

Consequently, for $h_+ \in H_+$ and $h_- \in H_-$, we see that

$$h_+ \cdot (ab) = (\varphi * \psi)(h_+)(ab), \text{ and}$$
$$h_- \cdot (ab) = (\varphi * \psi)(h_-)(ba);$$

that is, A is an oriented H-algebra. We summarize as follows.

Theorem 4.3.5. *Let $H = H_+ \oplus H_-$ be an oriented Hopf algebra satisfying the duality hypothesis, and let A be an associative algebra. Then A is an oriented H-algebra if and only if the vector space decomposition $A = \bigoplus_{\varphi \in G_H} A_\varphi$ is a Lie-Jordan-grading of A.*

We remark that, in the case where $H = KG$, any Lie-Jordan-group-grading of an associative algebra A is, in particular, a Jordan-group-grading of A. Moreover, according to the above construction, the identity component of this grading is $A_\varepsilon = A^G = \{a \in A |\, a^g = a, \text{ for all } g \in G\}$. Hence, in the case that A is an oriented KG-algebra, we can derive from Theorem 3.1.3 that A is a PI-algebra of bounded degree whenever A^G is.

In fact, using the above dualities, several similar remarks to this one can be made. In Chapter 6, we elaborate further on these special cases of the duality theorems, but first, we provide a unified approach to polynomial identities in Chapter 5 which will organize these results in an all-encompassing manner.

Chapter 5

Essential H-identities

In this chapter, we return to the study of polynomial identities, and offer a unified Hopf algebra approach which will encompass many of the aforementioned results as special cases. Throughout this chapter, H denotes an m-dimensional Hopf algebra over a field K of characteristic $p \geq 0$ with a basis $\mathfrak{B} = \{e_1, \ldots, e_m\}$.

In the first section, we define H-identities, which generalize the previous notions of G-identities (as used in Amitsur's theorems on involutions and the Bahturin-Giambruno-Zaicev Theorem) and graded identities (as used in the Bahturin-Giambruno-Riley Theorem and Theorem 3.1.3). Subsequently, we define and discuss essential H-identities, and use this notion to prove our main results. Finally, we further elaborate on the difference between essential H-identities and ordinary polynomial identities on the subalgebra of invariants, $A^H = \{a \in A | \, a^h = \varepsilon(h)a\}$.

5.1 H-identities

If an algebra A is an H-module, then we can consider the H-identities on A. We denote the free associative K-algebra on the set $\{x_i^{e_j} | \, e_j \in \mathfrak{B}, i \in \mathbb{Z}^+\}$ by $\mathcal{A}\langle X|H\rangle$. Elements in $\mathcal{A}\langle X|H\rangle$ are called H-polynomials. We identify $x_i^h = \sum_{j=1}^m \alpha_j x_i^{e_j}$, for all linear combinations $h = \sum_{j=1}^m \alpha_j e_j \in H$, and $x_i = x_i^1$, for $1 = 1_H$. There is a natural H-action on $\mathcal{A}\langle X|H\rangle$ given by

$$h \cdot (x_{i_1}^{h_1} \cdots x_{i_n}^{h_n}) = \sum_h x_{i_1}^{h_{(1)}h_1} \cdots x_{i_n}^{h_{(n)}h_n},$$

where $\Delta^{n-1}(h) = \sum_h h_{(1)} \otimes \cdots \otimes h_{(n)}$. In this way, $\mathcal{A}\langle X|H\rangle$ is the free associative H-algebra on X in the sense that it satisfies the following universal property: given any set map $\varphi : X \to A$ to an associative H-algebra A, there is a unique algebra homomorphism extension $\bar{\varphi} : \mathcal{A}\langle X|H\rangle \to A$ of φ such that $\bar{\varphi}(f^h) = h \cdot (\varphi(f))$, for every $f \in \mathcal{A}\langle X|H\rangle$ and $h \in H$.

Now let A be any associative algebra, and suppose that A is an H-module (but not necessarily an H-algebra). An H-polynomial $f(x_1, \ldots, x_n) \in \mathcal{A}\langle X|H\rangle$ is an H-identity of A if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$, where the H-action on A is used in the evaluation of $f(x_1, \ldots, x_n)$. Notice, however, that if we denote by $\mathrm{Id}^H(A)$ the ideal in $\mathcal{A}\langle X|H\rangle$ of all H-identities of A, that $\mathrm{Id}^H(A)$ is not closed under endomorphisms unless A is an H-algebra.

The following definition generalizes the notion of an essential G-identity (where G is a group acting by automorphisms and anti-automorphisms) given in Chapter 2.

Definition 5.1.1. *A multilinear H-polynomial is called essential if it is of the form*

$$x_1 \cdots x_d - \sum_{1 \neq \sigma \in S_d, b \in \mathfrak{B}^d} \alpha_{\sigma,b} x_{\sigma(1)}^{b_1} \cdots x_{\sigma(d)}^{b_d},$$

where $b = (b_1, \ldots, b_d)$.

5.2 Essential H-identities on H-algebras

We are now ready to state our first result of this chapter.

Theorem 5.2.1. *Let H be an m-dimensional Hopf algebra, and let A be an associative H-algebra. If A satisfies an essential H-identity of degree d, then A satisfies a polynomial identity of degree at most $\lceil em(d-1)^2 \rceil$.*

Here e denotes the base of natural logarithms and $\lceil x \rceil$ is the least integer greater than or equal to a real number x. We remark that the Bahturin-Giambruno-Riley Theorem can be formulated as follows: if H is a semisimple commutative m-dimensional Hopf algebra, and A is an H-algebra such that

$$A^H = \{a \in A |\, h \cdot a = \varepsilon(h)a, \text{ for every } h \in H\}$$

satisfies a polynomial identity of degree d, then A satisfies an identity of degree at most $\lceil em(dm-1)^2 \rceil$. In Section 5.3, we will show how to induce an essential H-identity in this case. We also remark that, if H is a finite-dimensional Hopf algebra that is not semisimple, then there exists an associative H-algebra A such that A^H satisfies a polynomial identity but A does not, as shown by Bahturin and Linchenko in [BL]. Thus, we have:

Corollary 5.2.2. *Let H be a finite-dimensional Hopf algebra that is not semisimple. Then there exists an associative H-algebra A such that A^H satisfies a polynomial identity but A does not satisfy an essential H-identity.*

Before we begin the proof of Theorem 5.2.1, we make a few more definitions. Let A be an associative algebra with a given H-module action. We do not assume that A is an H-algebra. For each positive integer n, set

$$P_n^H = \operatorname{span}_K\{x_{\sigma(1)}^{b_1} \cdots x_{\sigma(n)}^{b_n} |\, \sigma \in S_n, b_i \in \mathfrak{B}\}.$$

That is, P_n^H is the space of all multilinear H-polynomials in the indeterminates $x_1, \ldots, x_n$. The n^{th} H-codimension of A is given by

$$c_n^H(A) = \dim \frac{P_n^H}{P_n^H \cap \operatorname{Id}^H(A)},$$

respectively. Key to our proof is the observation that, since $c_n(A) \leq c_n^H(A)$, for every $n \geq 1$, A satisfies an ordinary polynomial identity of degree n whenever $c_n^H < n!$. See Lemma 5 in [Gd], for example. We impose a left lexicographic partial order on the monomials in P_n^H by setting $x_i^{b_1} < x_j^{b_2}$, whenever $i < j$, and extending this ordering left lexicographically. Let $\sigma \in S_n$. We recall Regev's ([Re1]) definition of d-good (and d-bad) monomials.

Definition 5.2.3. *The monomial $x_{\sigma(1)} \cdots x_{\sigma(n)}$ is called d-bad if there are integers $1 \leq i_1 < \cdots < i_d \leq n$ such that $\sigma(i_1) > \cdots > \sigma(i_d)$; otherwise, $x_{\sigma(1)} \cdots x_{\sigma(n)}$ is called d-good.*

Now, in order to prove Theorem 5.2.1, suppose that an associative H-algebra A satisfies an essential H-identity

$$f = x_1 \cdots x_d - \sum_{1\neq\sigma\in S_d, b\in\mathfrak{B}^d} \alpha_{\sigma,b} x_{\sigma(1)}^{b_1} \cdots x_{\sigma(d)}^{b_d}.$$

We claim that, for all integers $n \geq d$, P_n^H is spanned by the set of all d-good monomials modulo $\mathrm{Id}^H(A)$. Indeed, suppose that this were false. Then there exists a monomial $w = x_{i_1}^{b_1} \cdots x_{i_n}^{b_n}$ minimal in the lexicographical order that is d-bad and not a linear combination of d-good monomials modulo $\mathrm{Id}^H(A)$. As such, we may write $w = uw_d \cdots w_1$ with the property that, whenever $j > k$, the left variable in w_j is greater than the left variable in w_k. Then, because A is an H-algebra,

$$uf(w_d, \dots, w_1) = w - u\Big(\sum_{1\neq\sigma\in S_d, b\in\mathfrak{B}^d} \alpha_{\sigma,b} w_{\sigma(d)}^{b_1} \cdots w_{\sigma(1)}^{b_d}\Big)$$

is a well-defined H-identity on A. Therefore, since each monomial in the sum on the right is smaller than w in the lexicographical order, it is a linear combination of d-good monomials. In other words, w is a linear combination of d-good monomials modulo $\mathrm{Id}^H(A)$, a contradiction.

Since it is known that the number of d-good permutations in S_n is no greater than $\frac{(d-1)^{2n}}{(d-1)!}$ (see [Re2]), it follows that

$$c_n^H(A) \leq m^n \frac{(d-1)^{2n}}{(d-1)!}.$$

Thus, substituting $n = \lceil em(d-1)^2 \rceil$ into the inequality $(\frac{n}{e})^n < n!$ (see [FR]), allows us to conclude

$$c_n(A) \leq c_n^H(A) \leq m^n(d-1)^{2n} < n!,$$

so that A satisfies a polynomial identity of degree $\lceil em(d-1)^2 \rceil$. This completes the proof of Theorem 5.2.1 and proves the following fact interesting in its own right.

Theorem 5.2.4. *Let H be an m-dimensional Hopf algebra, and let A be an associative H-algebra. If A satisfies an essential H-identity of degree d, then*

$$c_n^H(A) \leq (m(d-1)^2)^n,$$

for all n, so that the H-codimensions of A grow at most exponentially in n.

5.3 Essential H-identities on Lie and Jordan H-algebras

In this section, A will always denote an associative algebra, although A will also be viewed as the Lie algebra $A^{(-)}$, and as the Jordan algebra $A^{(+)}$. Below, we investigate H-module actions on A with the property that either $A^{(+)}$ or $A^{(-)}$ is an H-algebra for which we propose the following terminology.

Definition 5.3.1. *Let A be an associative algebra, and let H be a Hopf algebra.*

1. *We shall say A is a Lie H-algebra whenever $A^{(-)}$ is an H-algebra.*

2. *Similarly, A is a Jordan H-algebra whenever $A^{(+)}$ is an H-algebra.*

Since Lie algebras and Jordan algebras are anti-commutative and commutative, respectively, it is natural to assume that H is cocommutative. Thus, in this section, H will always denote a cocommutative Hopf algebra. Under this assumption, if A is an associative H-algebra, then A is both a Lie H-algebra and a Jordan H-algebra (via the induced H-action). In particular, the free associative H-algebra, $\mathcal{A}\langle X|H\rangle$, is both a Lie H-algebra and a Jordan H-algebra. Analogously to the notation introduced in Chapter 3, we shall write $L\langle X|H\rangle$ (respectively, $J\langle X|H\rangle$) for the Lie H-subalgebra (respectively, Jordan H-subalgebra) generated by X in $\mathcal{A}\langle X|H\rangle^{(-)}$ (respectively, $\mathcal{A}\langle X|H\rangle^{(+)}$). It follows that $L\langle X|H\rangle$ (respectively, $J\langle X|H\rangle$) is the free Lie H-algebra (respectively, free Jordan H-algebra) on X, in the sense discussed for associative H-algebras.

The function $f(d,m)$ in our next result is the same used to prove the main theorem in [BGZ] and Theorem 3.1.3.

Theorem 5.3.2. *Let H be an m-dimensional cocommutative Hopf algebra, and let A be an associative algebra that is either a Lie or Jordan H-algebra. If A satisfies an essential H-identity of degree d, then A satisfies a polynomial identity of degree bounded by the function $f(d,m)$.*

The proof of Theorem 5.3.2 proceeds along a line similar to the proof of Theorem 5.2.1

but uses instead the notion of d-indecomposable monomials in place of d-good monomials (recall d-decomposable monomials were defined in Definition 3.3.4).

We prove Theorem 5.3.2 only in the case when A is a Lie H-algebra as the case when A is a Jordan H-algebra is similar. Thus, suppose that A is a Lie H-algebra that satisfies an essential H-identity

$$f(x_1, \ldots, x_d) = x_1 \cdots x_d - \sum_{1 \neq \sigma \in S_d, b \in \mathfrak{B}^d} \alpha_{\sigma,b} x_{\sigma(1)}^{b_1} \cdots x_{\sigma(d)}^{b_d}.$$

We claim that, for all integers $n \geq d$, P_n^H is spanned by the set of d-indecomposable monomials modulo $\mathrm{Id}^H(A)$. Suppose then, to the contrary, that this were false. Then there exists a monomial $w = x_{i_1}^{b_1} \cdots x_{i_n}^{b_n}$ in P_n^H minimal in lexicographical ordering that is d-decomposable and is not a linear combination of d-indecomposable monomials. Let $w = uw_d \cdots w_1 v$ be its d-decomposition. For each $1 \leq j \leq d$, write $w_j = y_1^{b_{j_1}} \cdots y_l^{b_{j_l}}$ (we use y's in place of x's for simplicity) and set $\bar{w}_j = [y_1^{b_{j_1}}, \ldots, y_l^{b_{j_l}}]$. It follows that

$$w \equiv u\bar{w}_d \bar{w}_{d-1} \cdots \bar{w}_1 v \pmod{W},$$

where W is the span of all the monomials $x_{i_{\sigma(1)}}^{b_{\sigma(1)}} \cdots x_{i_{\sigma(n)}}^{b_{\sigma(n)}}$ which are smaller than w in the lexicographical ordering. Indeed, writing $w_d = y_1^{b_{j_1}} \cdots y_l^{b_{j_l}}$, we have

$$w \equiv u[y_1^{b_{j_1}}, y_2^{b_{j_2}}] y_3^{b_{j_3}} \cdots y_l^{b_{j_l}} w_{d-1} \cdots w_1 v \pmod{W}$$

since $uy_2^{b_{j_2}} y_1^{b_{j_1}} y_3^{b_{j_3}} \cdots y_l^{b_{j_l}} w_{d-1} \cdots w_1 v \in W$. Repeated application of this same sort of argument applied to each w_i yields the assertion. Now, since each $\bar{w}_{\sigma(i)}$ lies in $L\langle X|H\rangle$, the free Lie H-algebra on X, and $A^{(-)}$ is an H-algebra, the evaluation of $\bar{w}_{\sigma(i)}^{b_{d-i+1}}$ in A is unambiguous. Consequently,

$$uf(\bar{w}_d, \ldots, \bar{w}_1)v = u\bar{w}_d \cdots \bar{w}_1 v - u\Big(\sum_{1 \neq \sigma \in S_d, b \in \mathfrak{B}^d} \alpha_{\sigma,b} \bar{w}_{\sigma(d)}^{b_1} \cdots \bar{w}_{\sigma(1)}^{b_d}\Big)v$$

is a well-defined H-identity on A. Expanding all the Lie monomials in the sum on the right produces only associative monomials in W; thus, $w \in \mathrm{Id}^H(A) + W$, contrary to assumption. This proves the claim.

In order to complete the proof of Theorem 5.3.2, we recall that $a_d(n)$ denotes the number of d-indecomposable words in P_n. We have just seen that $c_n^H(A) \leq a_d(n)m^n$. Consequently, using Lemma 3.3.4 we have

$$c_n(A) \leq c_n^H(A) \leq a_d(n)m^n < n!,$$

for all $n \geq f(d,m)$. In particular, A satisfies a polynomial identity of degree $f(d,m)$, proving Theorem 5.3.2. Thus, by Lemma 4.7 in [GR],

$$c_n(A) \leq (f(d,m)-1)^{2n},$$

for all $n \geq f(d,m)$, yielding the following result.

Corollary 5.3.3. *Let H be an m-dimensional co-commutative Hopf algebra, and let A be an associative algebra that is either a Lie or Jordan H-algebra. If A satisfies an essential H-identity of degree d, then, for all $n \geq f(d,m)$,*

$$c_n^H(A) \leq (m(f(d,m)-1)^2)^n,$$

so that the H-codimensions of A grow at most exponentially in n.

5.4 Essential H-Identities Versus Ordinary Identities on A^H

Next we further explore the relationship between essential H-identities and ordinary identities on A^H. Recall from Corollary 5.2.2 that, if H is not semisimple, then there exists an associative H-algebra A such that A^H satisfies a polynomial identity, but A does not satisfy an essential H-identity.

Theorem 5.4.1. *Let H be an m-dimensional semisimple commutative Hopf algebra, and let A be an associative algebra with an H-module action.*

1. *If A satisfies an essential H-identity of degree d, then A^H satisfies a polynomial identity of degree d.*

2. *Conversely, if A^H satisfies a polynomial identity of degree d, and one of the following conditions holds:*

   (a) *A is an H-algebra, or*

   (b) *H is co-commutative and A is a Lie or Jordan H-algebra,*

   then A satisfies an essential H-identity of degree dm.

Proof. It suffices to assume that the base field is algebraically closed; consequently, H is isomorphic to $(KG(H^*))^*$ as Hopf algebras (see [BC]). Hence, we set $G = G(H^*)$ and we may assume that $H = (KG)^*$. Let $\mathfrak{B} = \{e_1, \dots, e_m\} = \{\rho_g \,|\, g \in G\}$ be the standard dual basis of $(KG)^*$. Then $\mathfrak{B}$ is a complete set of orthogonal idempotents of H such that $e_1 + \cdots + e_m = 1$. Furthermore, $e_1 := \rho_1$ is the augmentation map on KG, so that $A^H = e_1 \cdot A$. Thus, if $f(x_1, \dots, x_d)$ is an essential identity of A, then $g = f(x_1^{e_1}, \dots, x_d^{e_1})$ is an H-identity on A which may be viewed as an ordinary identity on A^H. This proves (1).

To prove (2), we first remark that a nonassociative algebra A is an H-algebra precisely when A is G-graded with $A_g = \rho_g \cdot A$, for each $g \in G$. This was proved for associative algebras in [BC], but the same proof works for all nonassociative algebras as noted in [PR2]. Now notice that, if A^H satisfies a polynomial identity

$$f(x_1, \dots, x_d) = x_1 \cdots x_d - \sum_{1 \neq \sigma \in S_d} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}$$

of degree d, then obviously it satisfies some multilinear degree dm consequence, which we shall denote by g. It follows that $g(x_1^{e_1}, \dots, x_{dm}^{e_1})$ is an H-identity of A. Using the fact that $e_1 = 1 - e_2 - \cdots - e_m$ and expanding linearly, we obtain

$$\begin{aligned} g(x_1^{e_1}, \dots, x_{dm}^{e_1}) = x_1^1 \cdots x_{dm}^1 &- \sum_{1 \neq b \in \mathfrak{B}^{dm}} \gamma_b x_1^{b_1} \cdots x_{dm}^{b_{dm}} \\ &- \sum_{1 \neq \tau \in S_{dm}, b \in \mathfrak{B}^{dm}} \delta_{\tau,b} x_{\tau(1)}^{b_1} \cdots x_{\tau(dm)}^{b_{dm}}. \end{aligned} \tag{5.1}$$

Let W be the span of monomials of the type in the second sum of (5.1). Then A satisfies an essential H-identity as soon as

$$g(x_1^{e_1}, \dots, x_{dm}^{e_1}) \equiv x_1^1 \cdots x_{dm}^1 \pmod{\mathrm{Id}^H(A) + W}.$$

According to Lemma 2.2.5, for any word of length dm in G, there exists a string of d consecutive subwords each with trivial evaluation. Hence, we may write each of the monomials in the first sum of (5.1) as $w = x_1^{b_1} \cdots x_{dm}^{b_{dm}} = uw_1 \cdots w_d v$, where each submonomial w_i is of the form $w_i = x_{i_1}^{\rho_{g_1}} \cdots x_{i_l}^{\rho_{g_l}}$ with $g_1 \cdots g_l = 1$.

Consider now the case when A is an H-algebra. As remarked above, this means A is G-graded. Thus, each w_i evaluates into $A_1 = A^H$, and so

$$uf(w_1, \ldots, w_d)v = w - u(\sum_{1 \neq \sigma \in S_d} \alpha_\sigma w_{\sigma(1)} \cdots w_{\sigma(d)})v \in \mathrm{Id}^H(A).$$

It follows that $w \in W + \mathrm{Id}^H(A)$, and so A satisfies an essential H-identity.

Now suppose that H is co-commutative, as in condition (b). This is equivalent to supposing that the group G is abelian. We shall only prove the case when A is a Lie H-algebra: the case when A is a Jordan H-algebra being analogous. Let w be one of the monomials appearing in the first sum of (5.1) and write $w = uw_1 \cdots w_d v$, where each $w_i = x_{i_1}^{\rho_{g_1}} \cdots x_{i_l}^{\rho_{g_l}}$ with $g_1 \cdots g_l = 1$, as before. As in the proof of Theorem 5.3.2, we have

$$w \equiv u\bar{w}_1 \cdots \bar{w}_d v \pmod{W},$$

where each $\bar{w}_i = [x_{i_1}^{\rho_{g_1}}, \cdots, x_{i_l}^{\rho_{g_l}}]$. Since A is a Lie H-algebra, $A^{(-)}$ is G-graded. Hence, each $\bar{w}_i$ evaluates into $A_1 = A^H$, and so

$$uf(\bar{w}_1, \cdots, \bar{w}_d)v = u\bar{w}_1 \cdots \bar{w}_d v - u(\sum_{1 \neq \sigma \in S_d} \alpha_\sigma \bar{w}_{\sigma(1)} \cdots \bar{w}_{\sigma(d)})v \in \mathrm{Id}^H(A).$$

Therefore, because

$$u(\sum_{1 \neq \sigma \in S_d} \alpha_\sigma \bar{w}_{\sigma(1)} \cdots \bar{w}_{\sigma(d)})v \in W,$$

each $w \in W + \mathrm{Id}^H(A)$ and A satisfies an essential H-identity, as before. □

Combining Theorems 5.2.1, 5.3.2 and 5.4.1 yields the following result.

Corollary 5.4.2. *Let H be an m-dimensional semisimple commutative Hopf algebra, and let A be an associative algebra with an H-module action with the property that A^H satisfies a polynomial identity of degree d.*

1. *If A is an H-algebra, then A satisfies a polynomial identity of degree bounded by $\lceil em(dm-1)^2 \rceil$.*

2. *If H is co-commutative and A is either a Lie or Jordan H-algebra, then A satisfies a polynomial identity of degree bounded by $f(dm, m)$.*

Part (1) is equivalent to the Bahturin-Giambruno-Riley Theorem. Part (2) is equivalent to Theorem 3.1.3: if G is a finite abelian group, A is an associative algebra such that either $A^{(-)}$ or $A^{(+)}$ is G-graded, and A_1 satisfies an identity of degree d, then A satisfies an identity of degree $f(d|G|, |G|)$.

Chapter 6

Worked Examples

In this chapter, we offer a detailed account of the two predominant examples of finite-dimensional Hopf algebra actions on algebras. First, we investigate when the Hopf algebra in question is $H = KG$, the group Hopf algebra of a finite group, and subsequently, we investigate when $H = u(\mathfrak{g})$, the restricted universal enveloping Hopf algebra of a finite-dimensional restricted Lie algebra. In each case, we begin by describing sufficient conditions for H to be semisimple, finite-dimensional, commutative, cocommutative, and split. Following this, we describe well-known dualities between actions and group-gradings, which are actually examples of the Bergen-Cohen Duality involving these particular Hopf algebras. Subsequently, we apply the theory presented in Chapter 4 to extend these dualities. Finally, we discuss the applications to polynomial identity theory.

6.1 The Case of Group Algebras

In this section, our goal is to go into greater detail on how the theory presented in the previous two chapters relates to the most well-known case of Hopf algebra actions on algebras; hence, in this section, we assume that $H = KG$.

The Duality Hypothesis on $H = KG$

To guarantee that the duality hypothesis (Definition 4.3.1) holds on $H = KG$, we clearly need G to be a finite abelian group. Further, by Maschke's Theorem, KG is semisimple if and only if the characteristic p is 0 or does not divide $|G|$. Lastly, by a theorem of Brauer (see [Ja2], Theorem 5.25), if G has exponent m, then any field containing a primitive m^{th} root of unity is a splitting field of G. Hence, for example, the duality hypothesis holds on G whenever G is a finite abelian group and K is an algebraically closed field of characteristic $p = 0$ or $p > 0$ and G has no p-torsion. Note that these hypotheses imply that $G \cong \hat{G} = \{\chi_1, \ldots, \chi_k\}$, and each character $\chi_i \in \hat{G}$ is a homomorphism.

It is well-known that when the duality hypothesis holds on $H = KG$, then the unique basis of orthogonal idempotents summing to 1 is given by $\mathfrak{B} = \{f_\chi | \chi \in \hat{G}\}$, where

$$f_\chi = \frac{1}{|G|}\sum_{g \in G} \chi(g^{-1})g, \text{ for all } \chi \in \hat{G}.$$

See [GZ] for details. It is clear that the dual basis to $\mathfrak{B}$ is in fact $\hat{G}$.

The Standard Duality of Actions by Automorphisms and $\hat{G}$-gradings

The best known duality between actions and gradings is the following proposition, mentioned in [BI] and [Pa2], but known earlier to specialists like Cartier. It follows from the Bergen and Cohen Duality but is frequently used in its own right; hence we take a moment to view its explicit construction.

Proposition 6.1.1 (The Duality Between Actions by Automorphisms and Group-gradings)**.** *Let A be an algebra and let G be a group such that the duality hypotheses holds on KG. Then G embeds into* $\mathrm{Aut}(A)$ *if and only if*

$$A = \bigoplus_{\chi \in \hat{G}} A_\chi, \ \textit{where}\ A_\chi = \{a \in A | \, a^g = \chi(g)a, \textit{ for all } g \in G\},$$

is a group-grading of A.

Proof. Given an embedding of G into $\mathrm{Aut}(A)$, we can decompose A into a direct sum of eigenspaces with respect to the characters of G:

$$A = \bigoplus_{\chi \in \hat{G}} A_\chi, \text{ where } A_\chi = \{a \in A \mid a^g = \chi(g)a, \text{ for all } g \in G\}. \tag{6.1}$$

Furthermore, for every $a \in A_\chi$, $b \in A_\psi$, and $g \in G$,

$$(ab)^g = a^g b^g = (\chi\psi)(g)(ab).$$

Hence, $ab \in A_{\chi\psi}$ and (6.1) is a group-grading of A. Conversely, if $A = \bigoplus_{\chi \in \hat{G}} A_\chi$ is a group-grading, we can define a G-action on A by setting

$$a_\chi^g = \chi(g)a_\chi,$$

for each $g \in G$ and $a_\chi \in A_\chi$, and extending linearly. If $a \in A_\chi$ and $b \in A_\psi$, then the grading forces $ab \in A_{\chi\psi}$. Thus,

$$(ab)^g = (\chi\psi)(g)ab = a^g b^g,$$

and so G embeds into $\mathrm{Aut}(A)$. □

Duality Theorems for Actions by Anti-automorphisms

The original motivation behind the duality theorems in Section 4.3 was a desire to extend Proposition 6.1.1 to include anti-automorphisms. Observe that any group G which embeds into $\mathrm{Aut}^*(A)$ gives rise to an orientation of the Hopf algebra KG. Indeed, by defining $\sigma : G \to \{-1, 1\}$ as the group homomorphism given by $\sigma(g) = 1$, if $g \in \mathrm{Aut}(A)$, and $\sigma(g) = -1$, if $g \in \mathrm{Aut}^*(A) \backslash \mathrm{Aut}(A)$, it follows that G is an oriented group. Hence by setting

$$G_+ = \{g \in G \mid \sigma(g) = 1\}, \text{ and}$$
$$G_- = \{g \in G \mid \sigma(g) = -1\},$$

it follows (as in Example 4.2.2) that $H = KG_+ \oplus KG_-$ is an orientation of H.

We proceed to consider the quasigroup $G_H^\star$. First, observe that $\bar{\varepsilon}$ is the linear extension of σ. We are interested in whether or not $G_H^\star$ is a group; as such, we offer the following easy lemma.

Lemma 6.1.2. *Let G be any oriented group, and let $H = KG_+ \oplus KG_-$ be the induced oriented Hopf algebra. Then $H^\star$ is an associative algebra and $G_H^\star$ is a group with multiplication $\varphi \star \psi = \varphi * \psi * \bar{\varepsilon}$ and identity $\bar{\varepsilon}$.*

Proof. Let $\varphi \in \mathrm{Hom}(H, K)$, and let g be a homogeneous element in G. Notice that $\bar{g} = \sigma(g)g = \bar{\varepsilon}(g)g$. Therefore,

$$(\varphi * \bar{\varepsilon})(g) = \varphi(g)\bar{\varepsilon}(g) = \varphi(\bar{g}) = \bar{\varphi}(g),$$

and so, by Theorems 4.2.5 and 4.2.8, we are done. □

We will now demonstrate that, when the duality hypothesis holds on KG, embeddings of G into $\mathrm{Aut}^*(A)$ are oriented KG-algebras, similar to how embeddings of G into $\mathrm{Aut}(A)$ are KG-algebras. We also remark on how to induce a group-grading from the subgroup of automorphisms, given a subgroup of $\mathrm{Aut}^*(A)$.

Lemma 6.1.3. *Let G be a group such that the duality hypothesis holds on KG, and let A be an algebra. Then the following statements hold.*

1. *If G embeds into $\mathrm{Aut}^*(A)$, then, via the induced orientation, KG is an oriented Hopf algebra and A is an oriented KG-algebra.*

2. *If G is an oriented group, $KG = KG_+ \oplus KG_-$ is the induced orientation, and A is an oriented KG-algebra, then G embeds into $\mathrm{Aut}^*(A)$ in such a way that $G_+ \subseteq \mathrm{Aut}(A)$ and $G_- \subseteq \mathrm{Aut}^*(A)$.*

3. *Suppose $G \leq \mathrm{Aut}^*(A)$, and let $G_+ = G \cap \mathrm{Aut}(A)$. Then $A_{\chi^{G_+}} = A_\chi + A_{\overline{\chi}}$, for all $\chi \in \hat{G}$, where χ^{G_+} is the restriction of the character χ to G_+. The sum of all $A_{\chi^{G_+}}$ is a group-grading of A by $\hat{G}_+$.*

Proof. Suppose that G embeds into $\mathrm{Aut}^*(A)$. Then G is an oriented group and KG is an oriented Hopf algebra as seen above. Write $g \cdot a = a^g$, for all $a \in A$ and $g \in G$. It follows that

$$g \cdot (ab) = (ab)^g = \begin{cases} a^g b^g = (g \cdot a)(g \cdot b), & \text{if } g \in G_+, \\ b^g a^g = (g \cdot b)(g \cdot a), & \text{if } g \in G_-. \end{cases}$$

This proves (1). To prove (2), write $a^g = g \cdot a$ and reverse the proof of (1).

To prove (3), notice that G_+ is the kernel of σ, and thus, without loss of generality, $|G : G_+| = 2$. Since $\bar{\chi}^{G_+} = \chi^{G_+}$, for every $\chi \in \hat{G}$, we can write $\hat{G} = \{\chi_1, \ldots, \chi_n, \bar{\chi}_1, \ldots, \bar{\chi}_n\}$, say. Because $|G : G_+| = 2$, it follows that $|\hat{G_+}| = |G_+|$, and so KG_+ also satisfies the duality hypothesis. Clearly, $A_\chi + A_{\bar{\chi}} \subseteq A_{\chi^{G_+}}$, for every $\chi \in \hat{G}$. Conversely, let $a \in A_{\chi_i^{G_+}}$ and write $a = \sum a_{\chi_j} + a_{\bar{\chi}_j}$. Then, for every $g \in G_+$,

$$\begin{aligned} a^g &= \chi_i(g)\left(\sum_{j=1}^n a_{\chi_j} + a_{\bar{\chi}_j}\right) \\ &= \sum_{j=1}^n \chi_j(g)\left(a_{\chi_j} + a_{\bar{\chi}_j}\right). \end{aligned}$$

Hence, for each $1 \le j \le n$, either $a_{\bar{\chi}_j} + a_{\chi_j} = 0$ or $\chi_j(g) = \chi_i(g)$, for all $g \in G_+$. So, if there exists $j \neq i$ such that $a_{\bar{\chi}_j} + a_{\chi_j} \neq 0$, then $\chi_i(g) = \bar{\chi}_i(g) = \chi_j(g) = \bar{\chi}_j(g)$, for every $g \in G_+$. However, this is impossible because $|G : G_+| = 2$. □

As consequence of Lemmas 5.2 and 5.3 and Theorems 4.3.3 and 4.3.5, we have the following corollary. It generalizes Proposition 6.1.1 in the case when A is a Lie or associative algebra.

Corollary 6.1.4. *Let G be a group such that $H = KG$ satisfies the duality hypothesis. Then the following statements hold.*

1. *Suppose that A is a Lie algebra and set $B_\varphi = A_{\bar{\varphi}}$, for each $\varphi \in G_H$. Then the following statements are equivalent:*

   (a) *G embeds into* $\mathrm{Aut}^*(A)$.

   (b) *$A = \bigoplus_{\varphi \in G_H^\star} A_\varphi$ is a group-grading.*

   (c) *$A = \bigoplus_{\varphi \in G_H^*} B_\varphi$ is a group-grading.*

2. *Suppose that A is an associative algebra. Then G embeds into* $\mathrm{Aut}^*(A)$ *if and only if $A = \bigoplus_{\varphi \in G_H} A_\varphi$ is a Lie-Jordan-group-grading.*

Polynomial Identity Theorem for $H = KG$

The following corollary is the interpretation of the results outlined in Chapter 5 for the case where $H = KG$. In particular, we obtain the Baturin-Giambruno-Zaicev Theorem, from which Amitsur's Theorem 2.3.2 is deduced.

Corollary 6.1.5. *Let G be a finite group of order m, let A be an associative algebra, and let $H = KG$.*

1. *If $G \leq \mathrm{Aut}(A)$ and A satisfies an essential H-identity of degree d, then A satisfies a polynomial identity of degree $\lceil em(d-1)^2 \rceil$.*

2. *If $G \leq \mathrm{Aut}^*(A)$ (or, more generally, if $G \leq \mathrm{Aut}(A^{(+)})$) and A satisfies an essential H-identity of degree d, then A satisfies a polynomial identity of degree bounded by the function $f(d, m)$.*

Proof. To prove (1), notice that A is an H-algebra. Thus, we may apply Theorem 5.2.1. To prove (2), observe that KG is co-commutative and $A^{(+)}$ is a KG-algebra. In this case, we may apply Theorem 5.3.2. □

6.2 The Case of Restricted Universal Enveloping Algebras

In this section, we provide details on how the theory presented in the previous chapters relates to another well-known case of Hopf algebra actions; namely, we assume that $H = u(\mathfrak{g})$.

The Duality Hypothesis on $u(\mathfrak{g})$

Suppose that $\mathfrak{g}$ is a restricted Lie algebra over a field of characteristic $p > 0$. Determining when the duality hypotheses holds on $H = u(\mathfrak{g})$ is less clear than the group algebra case; as such, we make the following observations.

If the duality hypothesis holds on $u(\mathfrak{g})$, then $\mathfrak{g}$ is finite-dimensional; in this case, $u(\mathfrak{g})$ is semisimple if and only if $\mathfrak{g}$ is abelian and its p-map is injective (see [Ho]). To guarantee that $u(\mathfrak{g})$ splits over K, we could assume that K is algebraically closed, which would further allow us to assume that $\mathfrak{g}$ is an (abelian) n-torus; namely, $\mathfrak{g}$ has a basis $\{\delta_1, \ldots, \delta_n\}$ consisting of toral elements (see [SF], Theorems 2.36 and 2.37). Recall that $\delta \in \mathfrak{g}$ is called toral if $\delta^p = \delta$. On the other hand, assuming $\mathfrak{g}$ is an n-torus is sufficient to satisfy the duality hypothesis, as we shall see below.

Lemma 6.2.1. *Let $\mathfrak{g}$ be an n-torus with toral basis $\{\delta_1, \ldots, \delta_n\}$ over a field K of characteristic $p > 0$, and put $H = u(\mathfrak{g})$. Then the following statements hold.*

1. *The set of all ordered monomials $\{e_1^{(\lambda_1)} \cdots e_n^{(\lambda_n)} | \, 0 \leq \lambda_i \leq p-1\}$, where, for each $1 \leq i \leq n$,*

$$e_i^{(0)} = 1 - \delta_i^{p-1} \text{ and}$$
$$e_i^{(\lambda_i)} = - \sum_{1 \leq j \leq p-1} \lambda_i^{-j} \delta_i^j \quad (\lambda_i \neq 0),$$

is a basis of orthogonal idempotents of $u(\mathfrak{g})$ which sum to 1.

2. *The duality hypothesis holds on $u(\mathfrak{g})$.*

3. *Let $p_{(\lambda_1, \ldots, \lambda_n)}$ be the element in the dual basis of $Hom(H, K)$ corresponding to the idempotent $e_1^{(\lambda_1)} \cdots e_n^{(\lambda_n)}$. Then $p_{(\lambda_1, \ldots, \lambda_n)}(\delta_i) = \lambda_i$, for every i. Hence, $G_H^* \cong (\mathbb{Z}_p^n, +)$.*

Proof. First notice that $\delta_i e_i^{(\lambda_i)} = \lambda_i e_i^{(\lambda_i)}$, for every i. It follows that each $e_i^{(\lambda_i)}$ is an idempotent. Now suppose $\lambda_i \neq \mu_i$, for some i. Then $\lambda_i^{-1} \mu_i \neq 1$, and so

$$\begin{aligned}
e_i^{(\lambda_i)} e_i^{(\mu_i)} &= - \left(\sum_{1 \leq j \leq p-1} \lambda_i^{-j} \delta_i^j \right) e_i^{(\mu_i)} \\
&= - \left(\sum_{1 \leq j \leq p-1} (\lambda^{-1} \mu_i)^j \right) e_i^{(\mu_i)} \\
&= -(0) e_i^{(\mu_i)}.
\end{aligned}$$

Thus, $e_1^{(\lambda_1)} \cdots e_n^{(\lambda_n)}$ and $e_1^{(\mu_1)} \cdots e_n^{(\mu_n)}$ are orthogonal idempotents, and

$$\{e_1^{(\lambda_1)} \cdots e_n^{(\lambda_n)} | \, 0 \leq \lambda_i \leq p-1\}$$

is a linearly independent set of p^n elements. By Jacobson's version of the PBW Theorem for restricted universal enveloping algebras (see [SF], for example), $\dim(H) = p^n$. Statements (1) and (2) now follow.

To prove (3), observe that, since $\mathfrak{g}$ is an n-torus, defining $q_{(\lambda_1,\dots,\lambda_n)}(\delta_i) = \lambda_i$, for each i, induces a well-defined algebra map $q_{(\lambda_1,\dots,\lambda_n)}$ from H to K. Because $q_{(\lambda_1,\dots,\lambda_n)}(e_k^{(\mu_k)}) = \delta_{(\lambda_i,\mu_k)}$, it follows that $q_{(\lambda_1,\dots,\lambda_n)} = p_{(\lambda_1,\dots,\lambda_n)}$. Hence, for every i,

$$\begin{aligned}\left(p_{(\lambda_1,\dots,\lambda_n)} * p_{(\mu_1,\dots,\mu_n)}\right)(\delta_i) &= \left(p_{(\lambda_1,\dots,\lambda_n)} + p_{(\mu_1,\dots,\mu_n)}\right)(\delta_i)\\ &= \lambda_i + \mu_i\\ &= p_{(\lambda_1+\mu_1,\dots,\lambda_n+\mu_n)}(\delta_i),\end{aligned}$$

and so $G_H^* \cong \mathbb{Z}_p^n$. □

The Standard Duality Between Actions by Derivations and $\mathbb{Z}_p^n$-gradings

We will explicitly describe another well-known duality, which follows from the Bergen and Cohen Duality.

Proposition 6.2.2 (Duality Between Actions by Derivations and Group-Gradings)**.** *Let K be a field of characteristic $p > 0$, let $\mathfrak{g}$ be the n-torus, and let A be a K-algebra. Then $\mathfrak{g}$ embeds into* $\mathrm{Der}(A)$ *if and only if*

$$A = \bigoplus_{(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}_p^n} A_{(\lambda_1,\dots,\lambda_n)}, \textit{ where } A_{(\lambda_1,\dots,\lambda_n)} = \bigcap_{i=1}^{n}\{a \in A|\, a^{\delta_i} = \lambda_i a\},$$

is a group-grading.

To demonstrate how this duality works, we suppose first that $\mathfrak{g}$ embeds into $\mathrm{End}(A)$. Then, since each basis element δ_i of $\mathfrak{g}$ satisfies $\delta_i^p - \delta_i = 0$, we can decompose A into the eigenspaces of δ_i corresponding to its eigenvalues $\lambda_i = 0, 1, \dots, p-1$. From these eigenvalues, we form the additive group $G = \mathbb{Z}_p^n$, whose elements we denote by $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$. It follows that

$$A = \bigoplus_{\bar{\lambda}\in G} A_{\bar{\lambda}}, \text{ where } A_{\bar{\lambda}} = \bigcap_{i=1}^{n}\{a \in A|\, a^{\delta_i} = \lambda_i a\},$$

is a vector space decomposition of A. Furthermore, if $\mathfrak{g} \leq \mathrm{Der}(A)$, for every $a \in A_{\bar{\lambda}}$ and $b \in A_{\bar{\mu}}$, then

$$(ab)^{\delta_i} = a^{\delta_i}b + ab^{\delta_i} = (\lambda_i + \mu_i)ab,$$

for each i, so that this is indeed a G-grading of A. Conversely, suppose that $G = \mathbb{Z}_p^n$ and that $A = \bigoplus_{\bar{\lambda}\in G} A_{\bar{\lambda}}$ is any vector space decomposition. Then, for each $i = 1, \ldots, n$, we may define $\delta_i \in \mathrm{End}(A)$ by $a^{\delta_i} = \lambda_i a$, for every $a \in A_{\bar{\lambda}}$ and $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in G$. Furthermore, if this vector space decomposition is a G-grading of A, then, for every homogeneous $a \in A_{\bar{\lambda}}$ and $b \in A_{\bar{\mu}}$, we have $ab \in A_{\bar{\lambda}+\bar{\mu}}$, so that

$$(ab)^{\delta_i} = (\lambda_i + \mu_i)ab = a^{\delta_i}b + ab^{\delta_i}.$$

Notice, as well, that each $\delta_i = \delta_i^p$ since each $\lambda_i = \lambda_i^p$. Consequently, each δ_i is a toral derivation and $\{\delta_1, \ldots, \delta_n\}$ spans an n-torus which embeds into $\mathrm{Der}(A)$.

Duality Theorems for Actions by Anti-derivations

We now describe how to extend the above duality to include actions by anti-derivations on A. Recall that a K-linear map δ is called an anti-derivation of A if $(ab)^\delta = b^\delta a + ba^\delta$, for all $a, b \in A$. First, we observe that $\mathrm{Der}^*(A)$ admits a natural $\mathbb{Z}_2$-grading: $\mathrm{Der}^*(A) = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_+ = \mathrm{Der}(A)$ and $\mathfrak{g}_-$ is the subspace of all anti-derivations on A. Now let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a $\mathbb{Z}_2$-graded restricted Lie algebra. We shall call $\mathfrak{g}$ a $\mathbb{Z}_2$-graded n-torus if $\mathfrak{g}$ is an n-torus with a homogeneous basis $\Delta = \{\delta_1, \ldots, \delta_n\}$ consisting of toral elements.

Let

$$B = \{\delta_{i_1}^{s_1} \cdots \delta_{i_n}^{s_n} | \, 0 \leq s_i \leq p - 1\}$$

be the corresponding basis of PBW monomials of $u(\mathfrak{g})$. Define $\sigma : \Delta \to \{-1, 1\}$ by $\sigma(\delta) = 1$, if $\delta \in \mathfrak{g}_+$, and $\sigma(\delta) = -1$, if $\delta \in \mathfrak{g}_-$, extend σ to $u(\mathfrak{g})$ via $\sigma(\delta_{i_1}^{s_1} \cdots \delta_{i_n}^{s_n}) = \sigma(\delta_{i_1})^{s_1} \cdots \sigma(\delta_{i_n})^{s_n}$, for each $\delta_{i_1}^{s_1} \cdots \delta_{i_n}^{s_n}$ in B, and put

$$u(\mathfrak{g})_+ = \mathrm{span}\{b \in B | \, \sigma(b) = 1\} \text{ and } u(\mathfrak{g})_- = \mathrm{span}\{b \in B | \, \sigma(b) = -1\},$$

so that $u(\mathfrak{g}) = u(\mathfrak{g})_+ \oplus u(\mathfrak{g})_-$.

Lemma 6.2.3. *Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a $\mathbb{Z}_2$-graded n-torus, and let $H = u(\mathfrak{g})$. Then the following statements hold.*

1. *$H = u(\mathfrak{g})_+ \oplus u(\mathfrak{g})_-$ is an orientation of H such that $\bar{\varepsilon} = \varepsilon$.*

2. *Multiplication in the quasigroup $G_H^\star$ is given by*

$$(p_{(\lambda_1,\ldots,\lambda_n)} \star p_{(\mu_1,\ldots,\mu_n)})(\delta_i) = \sigma(\delta_i)(\lambda_i + \mu_i), \text{ for each } i.$$

   Thus, if we define a new binary operation $\star$ on $\mathbb{Z}_p^n$ by

$$\lambda \star \mu = (\sigma(\delta_1)(\lambda_1 + \mu_1), \ldots, \sigma(\delta_n)(\lambda_n + \mu_n)),$$

   for each $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n)$ in $\mathbb{Z}_p^n$, then $G_H^\star$ and $(\mathbb{Z}_p^n, \star)$ are isomorphic as quasigroups.

3. *Suppose that $\mathfrak{g} \neq \mathfrak{g}_+$. Then the algebra $H^\star$ is neither associative nor unital; moreover, the quasigroup $G_H^\star$ is neither a semigroup nor a loop.*

Proof. Since $\mathfrak{g}$ is the n-torus, it is easy to see B is closed under multiplication. The fact that $H = u(\mathfrak{g})_+ \oplus u(\mathfrak{g})_-$ is an orientation is an easy consequence. Since $\varepsilon(x) = 0$, for all $x \in \mathfrak{g}$, and $1 \in u(\mathfrak{g})_+$, it follows that $\bar{\varepsilon} = \varepsilon$. This proves (1). Part (2) follows from part (3) of Lemma 6.2.1. Since $\bar{\varepsilon} = \varepsilon$, part (3) follows from Corollaries 4.2.6 and 4.2.9. □

The proof of the next lemma is analogous to that of Lemma 6.1.3.

Lemma 6.2.4. *Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a $\mathbb{Z}_2$-graded n-torus, let $H = u(\mathfrak{g}) = u(\mathfrak{g})_+ \oplus u(\mathfrak{g})_-$ be the induced oriented Hopf algebra, and let A be an algebra. Then the following statements hold.*

1. *If $\mathfrak{g}$ embeds into $\mathrm{Der}^*(A)$ as a $\mathbb{Z}_2$-graded restricted Lie algebra, then A is an oriented $u(\mathfrak{g})$-algebra.*

2. *If A is an oriented $u(\mathfrak{g})$-algebra, then $\mathfrak{g}$ embeds into $\mathrm{Der}^*(A)$ in such a way that $\mathfrak{g}_+ \subseteq \mathrm{Der}(A)$ and $\mathfrak{g}_- \subseteq \mathrm{Der}^*(A) \backslash \mathrm{Der}(A)$.*

The preceding results, together with Theorems 4.3.3 and 4.3.5, imply the following corollary.

Corollary 6.2.5. *Let* $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *be a* $\mathbb{Z}_2$*-graded n-torus and let* $H = u(\mathfrak{g}) = u(\mathfrak{g})_+ \oplus u(\mathfrak{g})_-$ *be the induced oriented Hopf algebra. Then the following statements hold.*

1. *Let A be a Lie algebra. Then* $\mathfrak{g}$ *embeds into* $\mathrm{Der}^*(A)$ *as a* $\mathbb{Z}_2$*-graded Lie algebra if and only if* $A = \bigoplus_{\varphi \in G_H^\star} A_\varphi$ *is a quasigroup-grading (which is not a group-grading unless* $\mathfrak{g} = \mathfrak{g}_+$*).*

2. *Let A be an associative algebra. Then* $\mathfrak{g}$ *embeds into* $\mathrm{Der}^*(A)$ *as a* $\mathbb{Z}_2$*-graded Lie algebra if and only if* $A = \bigoplus_{\varphi \in G_H} A_\varphi$ *is a Lie-Jordan-grading (which is not a Lie-Jordan-group-grading unless* $\mathfrak{g} = \mathfrak{g}_+$*).*

Polynomial Identity Applications for $H = u(\mathfrak{g})$

As before, we interpret the results from Chapter 5 for this particular choice of Hopf algebra. We omit the proof of this corollary which is similar to that of Corollary 6.1.5.

Corollary 6.2.6. *Let* $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *be a* $\mathbb{Z}_2$*-graded m-dimensional restricted Lie algebra, let A be an associative algebra, and let* $H = u(\mathfrak{g})$*, the restricted universal enveloping algebra of* $\mathfrak{g}$.

1. *If* $\mathfrak{g} \leq \mathrm{Der}(A)$ *and A satisfies an essential H-identity of degree d, then A satisfies a polynomial identity of degree* $\lceil em(d-1)^2 \rceil$.

2. *If* $\mathfrak{g}$ *embeds into* $\mathrm{Der}^*(A)$ *as a* $\mathbb{Z}_2$*-graded restricted Lie algebra (or, more generally, if* $\mathfrak{g} \leq \mathrm{Der}(A^{(+)})$*), and A satisfies an essential H-identity of degree d, then A satisfies a polynomial identity of degree bounded by the function* $f(d,m)$.

Chapter 7

Applications and Open Problems

In this chapter, we present some other applications and questions for future research. Our first application is a method for constructing non-semigroup-graded Lie algebras (which were once thought not to exist). After this, we briefly discuss how to our results relate to the open problem of determining when an algebra admits an involution. Lastly, we mention a problem to the ones addressed in this thesis that involves Lie algebras rather than associative algebras.

7.1 Non-semigroup-graded Lie Algebras

Our first application involves a method for finding gradings of Lie algebras which cannot be recognized as semigroup-gradings (see [EK] for a current survey of the topic). Patera and Zassenhaus asserted in [PZ] that all Lie algebra set-gradings can be realized as semigroup-gradings, but a counterexample was found by Elduque in [El1]. The structure of non-semigroup-gradings on Lie algebras has since become an area of investigation. We saw in the previous section that every modular Lie algebra that admits a toral anti-derivation has a grading over a quasigroup that is not a semigroup. We note, however, that such a set-grading could, simultaneously, be realized as a semigroup-grading.

Proposition 7.1.1. *Let A be a Lie algebra over a field of odd characteristic p that admits a* *toral anti-derivation* δ*. Let* $\mathfrak{g}$ *be the* $\mathbb{Z}_2$*-graded* 1*-torus given by* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = 0 \oplus span\{\delta\}$*, and let* Γ *be the induced set-grading*

$$\Gamma : A = \bigoplus_{\lambda \in \mathbb{Z}_p} A_\lambda, \text{ where } A_\lambda = \{a \in A | \, a^\delta = \lambda a\}.$$

Then the following statements hold.

1. *The set-grading* Γ *cannot be realized as a semigroup-grading if and only if there exist* $\lambda, \mu, \nu \in \mathbb{Z}_p$ *such that* $\lambda \neq \nu$ *and each of the subspaces*

$$[A_\lambda, A_\mu], \;\; [A_{-(\lambda+\mu)}, A_\nu], \;\; [A_\mu, A_\nu], \;\; [A_\lambda, A_{-(\mu+\nu)}]$$

   is nonzero.

2. *If* Γ *is a fine grading, then* Γ *can always be realized as a semigroup-grading.*

Proof. Let $\alpha \cdot \beta$ denote the partially defined binary operation on the support of Γ. From Lemma 6.2.3 and Corollary 6.2.5, if $\alpha \cdot \beta$ is defined, then $\alpha \cdot \beta = \alpha \star \beta = -(\alpha + \beta)$. Now suppose Γ cannot be realized as a semigroup-grading; in other words, that there exist $\lambda, \mu, \nu \in \mathbb{Z}_p$ such that $(\lambda \cdot \mu) \cdot \nu \neq \lambda \cdot (\mu \cdot \nu)$. In particular, since these products are defined, $[A_\lambda, A_\mu]$, $[A_{-(\lambda+\mu)}, A_\nu]$, $[A_\mu, A_\nu]$, and $[A_\lambda, A_{-(\mu+\nu)}]$ are each nonzero. Moreover,

$$\lambda + \mu - \nu = (\lambda \cdot \mu) \cdot \nu \neq \lambda \cdot (\mu \cdot \nu) = -\lambda + \mu + \nu$$

implies $\lambda \neq \nu$. The converse follows by reversing this argument, proving (1).

To prove (2), suppose that Γ is fine; namely, that $\dim(A_\lambda) \leq 1$, for each $\lambda \in \mathbb{Z}_p$. Suppose now, to the contrary, that Γ cannot be realized as a semigroup-grading. Then there exists λ, μ, ν with the properties described in part (1). Since Γ is fine, $[A_\lambda, A_\mu] \neq 0$ implies that $A_{-(\lambda+\nu)} = [A_\lambda, A_\mu]$, while $[A_{-(\lambda+\mu)}, A_\nu] \neq 0$ implies that $A_{\lambda+\mu-\nu} = [[A_\lambda, A_\mu], A_\nu] \neq 0$. It now follows from the Jacobi identity that $(\lambda \cdot \mu) \cdot \nu = \lambda + \mu - \nu$ coincides with $\mu + \nu - \lambda$ or $\nu + \lambda - \mu$. However, since $\lambda \neq \nu$, we must have that $\lambda + \mu - \nu = \nu + \lambda - \mu$, and therefore $\mu = \nu$. Thus, the symmetric argument applied to $\lambda \cdot (\mu \cdot \nu)$ yields $\lambda = \mu = \nu$, our desired contradiction. □

Example 7.1.2. *Let K be a field of odd characteristic p, and consider the metabelian Lie algebra $L = span\{a, u, v, w\}$, where multiplication is given by*

$$[a, u] = u, \ \ [a, w] = v, \ \ [a, v] = w,$$

the product of all other basis elements defined to be 0. Define a linear map δ on L via

$$a^\delta = u^\delta = 0, \ v^\delta = v, \ \textit{and}\ w^\delta = -w.$$

It is clear that $\delta^3 = \delta$, and so $\delta^p = \delta$ is a toral map. A straightforward check shows that δ is an anti-derivation. Thus, L admits the quasigroup-grading

$$L = \bigoplus_{\lambda \in (\mathbb{Z}_p, \star)} L_\lambda, \ \textit{where}\ L_\lambda = \{a \in L |\, a^\delta = \lambda a\},$$

which can be simplified to

$$L = L_{-1} \oplus L_0 \oplus L_1.$$

By Proposition 7.1.1, this grading cannot be realized as a semigroup-grading; for instance, take $\lambda = \mu = 0$ and $\nu = 1$.

Hence we obtain a counterexample to the assertion made in [PZ]. We remark that this example was first obtained by Elduque in [El2] by other means.

An open problem related to the above discussion is the following, which was posed by Elduque and Kochetov in [EK].

Open Problem 7.1.3. *Is every set-grading on a finite-dimensional simple Lie algebra over $\mathbb{C}$ a semigroup-grading?*

In [EK], it is shown that semigroup-grading can be replaced with group-grading.

7.2 Algebras with Involution

In this section, we revisit a familiar example and present a tool which may be of some independent interest. The following is an open question.

Open Problem 7.2.1. *When does a given algebra A admit an involution?*

Even in the case where A admits an anti-automorphism of higher order, this is unknown. Here, we describe how this condition is equivalent to A possessing a nontrivial Lie-Jordan-$\mathbb{Z}_2$-grading.

Let A be an associative algebra over a field of characteristic not 2, and suppose that A admits an involution T. Then the group $G = \{e, T\}$ (where e is the identity map on A) induces a nontrivial orientation

$$H = KG = KG_+ \oplus KG_- = Ke \oplus KT.$$

Clearly H satisfies the duality hypothesis. Indeed,

$$f_1 = \frac{1}{2}(e + T) \text{ and } f_2 = \frac{1}{2}(e - T)$$

are the basis of orthogonal idempotents. The corresponding dual basis $\{p_1, p_2\}$ is defined by

$$p_1(e) = p_1(T) = 1, \text{ and}$$

$$p_2(e) = 1, p_2(T) = -1;$$

in other words, $p_1 = \varepsilon$ and $p_2 = \bar{\varepsilon}$, so that $(G(H), *) = \{\varepsilon, \bar{\varepsilon}\} = \hat{G}$. By Corollary 6.1.4, we know that A admits the Lie-Jordan-group-grading

$$A = A_\varepsilon \oplus A_{\bar{\varepsilon}}.$$

In this case, $A_\varepsilon = \{a \in A|\, a^T = a\}$ is the subspace of symmetric elements, and $A_{\bar{\varepsilon}} = \{a \in A|\, a^T = -a\}$ is the subspace of skew-symmetric elements.

Conversely, let $G = \langle t \rangle$ be an oriented group of order 2 such that

$$H = KG = KG_+ \oplus KG_- = Ke \oplus Kt$$

is the induced orientation, and suppose that $A = A_\varepsilon \oplus A_{\bar{\varepsilon}}$ is a Lie-Jordan-group-grading. Then, t acts on A via $a^t = \varepsilon(t)a_\varepsilon + \bar{\varepsilon}(t)a_{\bar{\varepsilon}} = a_\varepsilon - a_{\bar{\varepsilon}}$, and it is easy to verify that $(ab)^t = b^t a^t$, for every $a, b \in A$, as expected. This allows us to make the following conclusion.

Proposition 7.2.2. *An associative algebra A over a field of characteristic not 2 admits an involution precisely when A admits a nontrivial Lie-Jordan-group-grading over a group of order 2.*

7.3 An Analogue for Lie Algebras

One may also consider polynomial identities of Lie algebras in an analogous way. Bahturin and Zaicev proved in [BZ] that when the identity component of a group-graded Lie algebra satisfies a Lie polynomial identity, then the algebra itself satisfies an identity. No bound on the degree of the identity was found, however. More general non-associative algebras were studied in [BSZ], and it was shown that if A is a Lie algebra which is G-graded and satisfies an essential Lie identity of degree d, then A satisfies an identity of degree bounded by a function depending on $|G|$ and d. In general, such an identity does not imply an identity on the homogeneous component A_1, and as far as the author is aware, it remains an open question as to whether or not a similar bound exists for this case.

Open Problem 7.3.1. *If $A = \bigoplus_{g \in G} A_g$ is a group-graded Lie algebra such that the identity component A_1 satisfies a polynomial identity of degree d, then is there an explicit function which bounds the degree of the identity satisfied by A?*

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Appendix A

Group Like Structures

For the convenience of the reader, we recall the definitions of various 'group-like' structures. A group-like structure consists of a set with a single binary operation which may or may not satisfy various additional properties. We will assume that the binary operation is closed, but in general we do not impose any other conditions.

Definition A.0.2. *A magma is a set G together with a binary operation $\cdot : G \times G \to G$.*

If G is a magma, then depending on the properties of the operation $(\cdot)$, we may refer to G by one of the following names:

- **Quasigroup** - If G has divisibility, that is, for each $g, h \in G$, there exist unique elements $x, y \in G$ such that $a \cdot x = b$ and $y \cdot a = b$, then G is called a quasigroup.

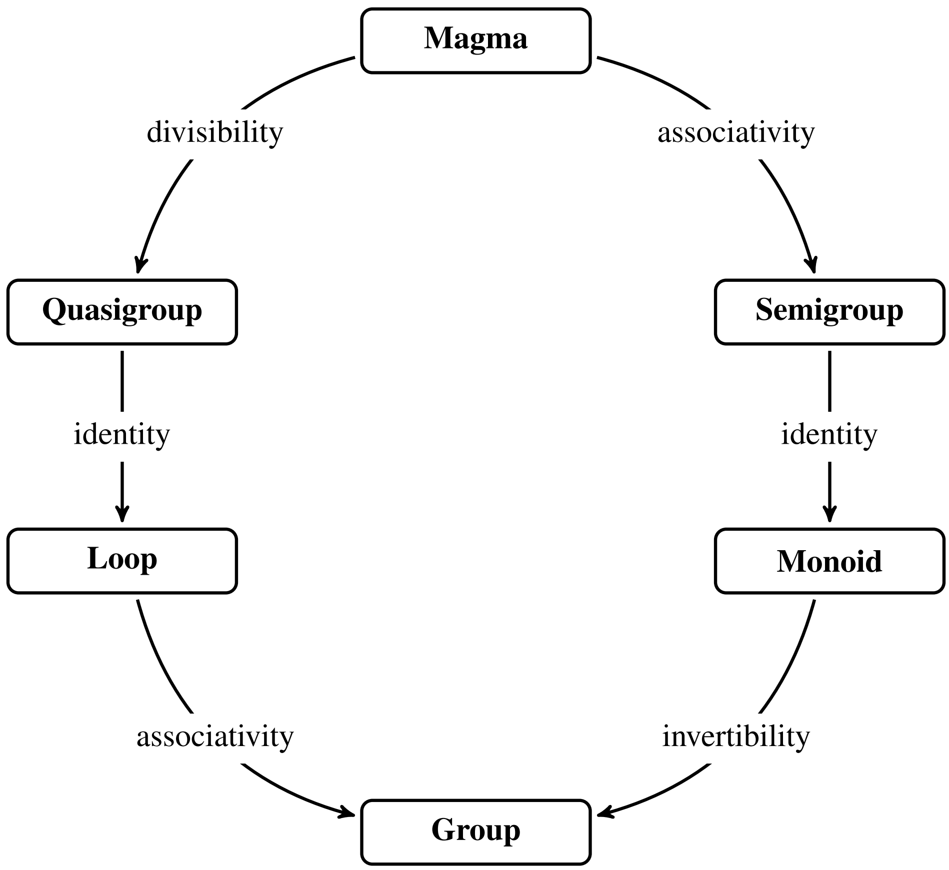
- **Loop** - If G is a quasigroup with identity, that is, there exists an element $e \in G$ such that $g \cdot e = e \cdot g = g$, for all $g \in G$, then G is called a loop.

- **Semigroup** - If G is an associative magma, that is, $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$, for all $g_1, g_2, g_3 \in G$, then G is called a semigroup.

- **Monoid** - If G is a semigroup with identity, then G is called a monoid.

- **Group** - If G is an associative loop, or equivalently, a monoid with invertibility, then G is called a group.

The following diagram may be helpful to remember the relationship between these structures.



Appendix B

Lie and Jordan Algebras

B.1 Lie Algebras

A Lie algebra L is an algebra over a field K, whose product we denote by $[,] : L \times L \to L$, which satisfies the following axioms, for all $x, y, z \in L$:

1. $[x, x] = 0$;

2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

The second property is referred to as the Jacobi identity. Note that if the characteristic of K is not 2, then a Lie algebra is anti-commutative: $[x, y] = -[y, x]$.

B.2 Jordan Algebras

A Jordan algebra A is an algebra over a field K such that the following axioms hold, for all $a, b \in A$:

1. $ab = ba$;

2. $(ab)(aa) = a(b(aa))$.

That is, the first property says that A is commutative, and the second property is called the Jordan identity.

Appendix C

Restricted Universal Enveloping Algebras

We briefly recall the restricted universal enveloping algebra of a restricted Lie algebra. Additional properties and details can be found in [Ja1] or [SF], for example. To begin, we define the universal enveloping algebra of a Lie algebra; this is an associative unital construction which maintains the important properties of the Lie algebra and contains it as a subalgebra.

Recall that if L is any Lie algebra, then given a unital associative algebra $U(L)$ and a Lie algebra homomorphism $i : L \to U(L)^{(-)}$, we say that $U(L)$ is the universal enveloping algebra of L if it satisfies the following universal property: for any unital associative algebra A and Lie algebra homomorphism $\varphi : L \to A^{(-)}$, there exists a unique unital algebra homomorphism $\bar{\varphi} : U(L) \to A$ such that $\varphi = i \circ \bar{\varphi}$.

The important Poincaire-Birkhoff-Witt theorem (known as the PBW theorem) gives a description of the universal enveloping algebra.

Theorem C.0.1. *Let L be a Lie algebra with a totally ordered basis* $\{x_i | i \in I\}$*. The monomials* $x_{i_1}^{s_1} \cdots x_{i_n}^{s_n}$*, where* $x_{i_1} < \cdots < x_{i_n}$ *and* $s_i \geq 0$*, form a basis for* $U(L)$*.*

Now let $\mathfrak{g}$ be a Lie algebra over a field of positive characteristic p. If $\mathfrak{g}$ is paired with an additional operation, denoted $x \to x^{[p]}$, such that the following conditions hold:

- $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^{[p]}$, for all $x \in \mathfrak{g}$;
- $(\alpha x)^{[p]} = \alpha^p x^{[p]}$, for all $\alpha \in K, x \in \mathfrak{g}$;
- $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i}$, where $s_i(x, y)$ is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(tx + y)^{p-1}(x)$,

then $\mathfrak{g}$ is called a restricted Lie algebra.

When $\mathfrak{g}$ is a restricted Lie algebra, we can consider the restricted universal enveloping algebra, denoted $u(\mathfrak{g})$. We construct the ordinary universal enveloping algebra $U(\mathfrak{g})$, and set $u(\mathfrak{g}) = U(\mathfrak{g})/I$, where I is the two-sided ideal generated by elements of the form $x^p - x^{[p]}$. The PBW theorem for restricted enveloping algebras is as follows.

Theorem C.0.2. *Let $\mathfrak{g}$ be a restricted Lie algebra with a totally ordered basis $\{x_i | \, i \in I\}$. The monomials $x_{i_1}^{s_1} \cdots x_{i_n}^{s_n}$, where $x_{i_1} < \cdots < x_{i_n}$ and $0 \leq s_i \leq p-1$, form a basis for $u(\mathfrak{g})$.*

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