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Structural Response Analyses of Piezoelectric Composites using NURBS

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Graduate Program in Mechanical and Materials Engineering

A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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STRUCTURAL RESPONSE ANALYSES OF PIEZOELECTRIC COMPOSITES USING NURBS

by
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Graduate Program in Engineering Science
Department of Mechanical and Materials Engineering

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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Variational method deduced on the basis of the minimum potential energy is an efficient method to find solutions for complex engineering problems. In structural mechanics, the total energy comprises strain energy, kinetic energy and the work done by external and internal loads. To obtain these, the displacement fields are required as a priori. This research is concerned with the development of a numerical method based on variational principles to analyze piezoelectric composite plates and solids. A Non-Uniform Rational B-Spline (NURBS) function is used for describing both the geometry and electromechanical displacement fields. Two dimensional plate models are formulated according to the first order shear deformable plate theory for mechanical displacement. The electric potential varies non-linearly through the thickness, this variation is modelled by a discrete layer-wise linear variation.

The matrix equations of motion are reported for piezoelectric sensors, actuators, and power harvesters. Normal mode summation technique is applied to study the frequency response of displacement, voltage and the power output. A full three dimensional model is also developed to study the dynamics of piezoelectric sandwich structures. Simulations are provided for thick plates using plate theory and three dimensional models to verify the applicability of those theories in their regime. Newmark’s direct integration technique and a fourth order Runge-Kutta method were used to study the transient vibration. The variational method developed in this thesis can be applied to other structural mechanics problems.

Keywords: NURBS, variational method, piezoelectric plates, vibrations, power harvesting, natural frequency, transient vibration.
The research work presented in this thesis is ready to be submitted in part or full to journals and conferences with the co-authorship of Professor Anand V. Singh. All the algorithms, programs, numerical simulations, manuscripts were prepared by Vijairaj. The project was supervised and the manuscript was edited by Professor Anand V. Singh.
I dedicate this thesis to my parents Vijayalakshmi and Raj, and my dearest brother Sanjayraj. Without their love and support, this journey wouldn’t be possible.
ACKNOWLEDGMENTS

I would like to sincerely thank my supervisor, Professor Anand V. Singh, whose help, advice and supervision were invaluable. It was a blessing to gain knowledge from his ideas and experience. A special thanks to Professor O. Remus Tutunea-Fatan and Professor Samuel F. Asokanathan for being part of the advisory committee and sharing their valuable inputs during progress review meetings.
Real happiness is cheap enough, yet how dearly we pay for its counterfeit.

-Hosea Ballou
**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi, \eta, \zeta$</td>
<td>Natural coordinates in a mapped space.</td>
</tr>
<tr>
<td>${ \phi }$</td>
<td>Voltage vector.</td>
</tr>
<tr>
<td>$n + 1$</td>
<td>Number of control points.</td>
</tr>
<tr>
<td>NURBS</td>
<td>Non Uniform Rational B-Spline.</td>
</tr>
<tr>
<td>$k$</td>
<td>Order of the NURBS curve.</td>
</tr>
<tr>
<td>$a, b, c$</td>
<td>Plate dimensions.</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>Displacement components in $x, y, z$ directions respectively.</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>Cartesian coordinates of a point.</td>
</tr>
<tr>
<td>${ \sigma }$</td>
<td>Stress tensor.</td>
</tr>
<tr>
<td>$[C]$</td>
<td>Elasticity matrix, in constitutive equation.</td>
</tr>
<tr>
<td>${ \varepsilon }$</td>
<td>Strain tensor.</td>
</tr>
<tr>
<td>$[e]$</td>
<td>Piezoelectric coupling matrix.</td>
</tr>
<tr>
<td>${ D }$</td>
<td>Dielectric displacement tensor.</td>
</tr>
<tr>
<td>$[\varepsilon]$</td>
<td>Dielectric matrix.</td>
</tr>
<tr>
<td>${ E' }$</td>
<td>Electric field.</td>
</tr>
<tr>
<td>$U$</td>
<td>Strain energy.</td>
</tr>
<tr>
<td>$T$</td>
<td>Kinetic energy.</td>
</tr>
<tr>
<td>$W$</td>
<td>Work done by the externally applied electrical and mechanical.</td>
</tr>
</tbody>
</table>

[K_m] Mechanical stiffness matrix.

[K_{me}] Electromechanical coupling matrix.

[K_e] Dielectric matrix.

\{Q\} Electrical charge density.

\{\Gamma\} Displacement vector.

\{F\} Mechanical load.

\(t\) Time.

[C] Proportional damping matrix, in equation of motion.

\(N_{i,k}(\xi)\) \(i^{th}\) blending function of order \(k\) in \(\xi\) space.

\(p_i\) \(i^{th}\) control point.

\(\xi_i\) \(i^{th}\) component of the knot vector.
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CHAPTER 1

Introduction to the Thesis

1.1 General introduction

Piezoelectric structures play an important role in the field of sensors and actuators. The development of sensors and actuators mechanisms in itself is intrinsically prominent, as they are much imminent in quantifying a physical phenomenon. Piezoelectric materials, in particular found wide application as sensors in vibration control of structures. One of the reason being, the piezoelectric materials inherent ability to convert the small mechanical deformations in the scale of micrometer to a corresponding electrical output. In addition to that, piezoelectric materials have high hardness, a high linear relationship between input and output and their ability to be formed in to many different shapes.

Tourmaline crystals were used by the people in Srilanka and India for centuries before the first documented discovery in Europe. They discovered that, tourmaline crystals attracted surrounding particles when heated or thrown in to fire. Albeit, this effect is due to pyro-electricity rather than piezoelectricity. Tourmaline crystals were then brought to Europe in the mid seventeenth century by the merchants of Dutch East India Company. In 1756, Aepinus reported the first scientific study on pyro electricity, in which he found, a heated tourmaline crystal produced electric potential at two ends [1]. The piezoelectric effect, as in its present context was first discovered by the brothers Jacques Curie and Pierre Curie in 1880. They showed that some crystals when compressed in particular directions developed charges, both negative and positive on their surfaces. They observed that the charges were proportional to the applied pressure and reduced to zero after removing the pressure. When electric charges are produced in crystals of certain classes under
mechanical action, it is called the “the direct piezoelectric effect”. Conversely if mechanical strains are produced under an applied voltage, the effect is known as the “the converse effect”. Though the experimental studies quite effectively validated the piezoelectric effect in crystals, the field of piezoelectricity remained more of a scientific accomplishment than finding a real world application. This scenario changed abruptly after world war one. Piezoelectric materials were used to transmit signals under water in devising resonators, actuators and transducers. The rest is history, one can find the proliferated application of piezoelectric materials in almost every field. Piezoelectric materials are used in multitude of disciplines like power harvesting, remote sensing devices, medical implants and recently the piezoelectric effects are even exploited in nanoscale devices.

In recent times, piezoelectric materials have been widely used as: sensors in structural health monitoring, a power source in micro scale electronic devices and actuators in micro pumps. In the beginning of the 19th century, the applications were different, piezoelectric materials were mainly used as transducers to transmit signal under water, sensors in measuring primarily force and acceleration. The materials used were also mostly naturally occurring crystals like quartz and Rochelle salt. However, over a period of time, more and more synthetic ceramics and synthetic piezoelectric materials were developed. These materials had very high electro mechanical coupling and can be tailored for specific application. Piezoelectric devices are finding more and more application at micro level vibration energy harvesters and nanoscale sensors.

1.2 Modelling techniques

The grassroots of understanding the piezoelectric effect was mainly inclined towards experimental research in the earlier days. The electromechanical coupling coefficients and the fundamental frequencies of crystals were mainly determined through experiments [2]. One of the earliest works documented in using piezoelectric material as sensor is reported by Keys [3] in 1921. However, it was the lifetime research work of people like Cady, Tiersten and Mindlin, who developed the fundamental mechanics for piezoelectricity as we know it today [4-6]. Once the crystallography and mechanics behind
the phenomena were laid out, the mathematical modelling caught the interest of many researchers. The work done by W.G. Cady [7] still remains as a relevant source in the field of piezoelectric crystal vibrations. This book provides valuable insights to piezoelectric crystal vibrations. The vibrations of naturally occurring crystals like, Tourmaline, Rochelle salt and quartz were documented. Another researcher, who stands out in the same period is, W.P. Mason [8]. His work primarily concentrated on piezoelectric transducers, wave filters and oscillators. Analytical closed form solutions are always limited or confined to certain classes of boundary value problems. In such cases, the boundary and loading conditions dictate the solutions. Birman [9] presented an approach to model composite plates with piezoelectric materials as stiffeners by considering them embedded inside the sandwich structure rather than on the surface. He assumed that the stiffeners were parallel to the middle surface. In the 1990’s piezoelectric materials saw a huge application as sensors. Often times in such applications, the structure were modelled using plate theory even if the thickness varied from moderately thin to thick. Various theories were proposed by researchers to model this two dimensional problem. In many cases an equivalent single layer or discrete layer approximation is made for mechanical displacements and a layer-wise mechanics for electric potential. Variations were made on the basis of the order of approximation in the thickness direction. This led to the development of various theories like, first order shear deformation theory (FSDT), zig-zag theory, and third order shear deformation theory [10, 11]. Incidentally, all these theories were native to plate vibration problem and extended to include the electro mechanical coupling. Ray et al. [12] reported the exact solutions for piezoelectric plates in cylindrical bending. They used closed-form solutions for stresses and displacements from the work of Pagano [13] and continued their work to study the dynamic analysis of square composite plates [14]. In 1995, Helinger and Brooks [15] studied the free vibration of composite piezoelectric plates in cylindrical bending. In their work, they assumed a classical plate theory without considering the influence of piezoelectric coupling in the free vibration analysis. However, the electromechanical coupling tends to reduce the stiffness and thereby reducing the fundamental frequencies. Analytical solutions for static analysis of laminated piezoelectric plates were presented by Heyliger [16] for simply supported boundary condition. Fernandes and Pouget [17, 18] used a layer-wise mechanics for electric potential and first order shear
deformation theory for mechanical displacements to model laminated piezoelectric plates. In their research, they used Fourier series for displacement and electric field functions. Displacement and electric potential distributions were reported for bimorphs and sandwich configurations. Sladek et al. [19] reported a meshless technique to model laminated piezoelectric actuators and performed static and transient response analyses. They obtained the integral equations of motion using a Heaviside step function and solved the system of ordinary differential equations by the finite difference technique. Circular thin unimorph actuator plates were studied by Dong et al. [20]. They provided closed form analytical solutions for deformations under mechanical and electrical loads. From the expressions for displacements they also obtained equations for an optimum thickness ratio for the piezoelectric layer in a sandwich configuration. Askari et al. [21] provided a Levy type analytical solution for the free vibration of rectangular plates. In their work they produced results for the influence of piezoelectric material thickness in a sandwich construction on natural frequencies. Functionally graded piezoelectric materials caught the attention of some researchers, A three dimensional static analysis was performed on functionally graded piezoelectric materials by Zhong and Shang [22]. An exponential variation of the material properties was assumed along the thickness direction in their work. Jodaei [23] derived a three dimensional elasticity solution for functionally graded annular piezoelectric plates by a differential quadrature method. His technique was partially analytical, as he used a state-space method along the thickness and one-dimensional differential quadrature in the radial direction. Moleiro et al. [24] provided closed form solutions for multilayered piezoelectric composite plates. They compared their results with Heyliger [15] and provided more results for various aspect ratios and lamina orientations. From the above literature, one could conclude that most of the research was concerned on developing a two-dimensional plate theory to model sensors and actuators. The closed form solutions were available only for certain geometry and boundary conditions.
1.3 Finite element method

Finite element method (FEM) played a vital role in the modelling of piezoelectric materials. Approximate numerical solutions can be obtained for complex shapes and different boundary conditions through FEM. This numerical method made the analysis possible for configurations which were not conceivable through analytical techniques. One of the earliest works on developing finite element technique for piezoelectric material was done by Allik and Hughes [25] in 1970. They presented the basic equations and variational formulation to arrive at the stiffness, mass, and electromechanical coupling matrices. Also a four node tetrahedron element was developed with each node having three displacement degrees of freedom and one electrical degree of freedom. Allik et al. [26] presented a finite element method for the dynamic analysis of piezoelectric solids. In their work, they employed an eight node hexahedron element with quadratic and quartic basis functions for displacement and electrical degrees of freedom. Ghandi and Hagood [27] used a similar type of eight node hexahedron element using isoparametric shape functions to analyze phase transitions in piezoelectric materials. This study also considered the response of piezoelectric materials subjected to a non-uniform electrical field. Lerch [28] presented a finite element technique for modelling piezoelectric transducers immersed in a fluid. He coupled the feedback from the fluid by using acoustic finite elements. Ha et al. [29] presented a three dimensional finite element formulation to study the static and dynamic response of laminated composite plates and verified their findings through experiments. Researchers focused more on two dimensional models in the early 1990’s. Primarily they were targeted to model thin piezoelectric sensors and actuators and were quite efficient in modeling with relatively fewer degrees of freedom compared to solid models. Kim et al. [30, 31] formulated a transitional element from solid to plate to model cantilever plate with embedded piezoelectric devices. They continued their studies to perform the dynamic analysis on the above model and compared it with experimental results. The above technique is significant in structural health monitoring applications, as the piezoelectric devices are mostly thin and bonded to a host structure. Saravanos and Heyliger [32] used discrete layer-wise mechanics for both displacement and electric potential to model smart composite structures. They obtained equations of motion in matrix form through variational
formulation for a four node element considering linear shape functions. A 20 node thermo piezoelectric element was developed by Koko et al. [33] to model smart composite structures. They also developed control algorithms using a linear quadratic regulator to obtain the feedback gain matrix. The finite element method was also used to study vibration control in smart structures [34,35]. Dietrich and Manfred [36] used a three dimensional formulation selecting reduced integration in the thickness direction to model thin plates. Numerical models were developed to study cracked piezoelectric materials in fracture mechanics by Qin [37]. The review article by Chen and Hasebe [38] documented the advancements in modelling cracks in piezoelectric materials. A brief survey reveals that finite element method is a method of choice to model piezoelectric materials as sensors and actuators. This is also true for applications like, active vibration control, structural health monitoring and fracture mechanics.

1.4 Power harvesting

Piezoelectric materials were widely used as vibration energy harvesters. The mechanical vibration induces strain, which in turn is converted to electrical energy. The energy harvested from vibrations due to ambient sources is usually in the range of microwatts, which can only serve as a power source for micro scale sensors and wireless devices. The application of piezoelectric materials as power harvesters got wide traction since Umeda et al. [39] studied the energy produced in a piezoelectric ceramic chip on impact loading. Williams and Yates [40] provided a simplified one dimensional model for a micro-electric power generator. Though the above model provided an approximate equation for power generating devices as the electromechanical coupling pronounced in piezoelectric materials was neglected. Poulin et al. [41] reported a study on comparing electromagnetic and piezoelectric devices as vibration energy harvesters. The study was primarily based on developing equivalent circuits for both systems and subjecting it to similar harmonic loads. The electromagnetic system produced more power than the piezoelectric device, however the energy density was higher for the later. The efficiency of piezoelectric ceramics as a power generator was studied by Michael and Lowell [42] who also reported that the maximum power output was obtained at a frequency lower than the
structural resonance of the system. Sodano et al. [43] developed fundamental equations for a piezoelectric power harvesting beam and included proportional mechanical damping for the inherent structural damping and air resistance. The power generated from a piezoelectric device is defined by the electromechanical coefficient of the material. Two coupling modes denoted by 3-1 and 3-3 respectively, of piezoelectric materials were traditionally exploited in power harvesting. The coupling coefficient of the 3-3 mode is much higher than that of the 3-1 mode. However, the vibrational energy required to operate in 3-3 mode is much higher. Various configurations of power harvesters were developed to generate maximum power under given working conditions. Several techniques which included adding a proof mass, an L-shaped harvester to induce more strain, a near triangular cantilever plate to create uniform strain, were reported in the literature [44, 45]. Zeng et al. [46] studied the positioning of piezoelectric materials in the structure for optimal power harvesting. They proposed optimal design realization for a cantilever bimorph type power generator. The influence of external circuits in power harvesting was studied by Zhu et al. [47]. They analyzed the influence of load resistance on the generated power from a cantilevered piezoelectric plate. Inman with other researchers [45, 48-51] published a series of papers on piezoelectric power harvesting. These notable studies included developing a beam model for piezoelectric powered harvesting with the effects of externally connected resistance on the power output. A cantilevered power harvester plate model was developed by Dutoit et al. [52] who provided non-dimensional power equations for beam and plate models.

1.5 Introduction to NURBS

The Non-Uniform Rational Basis Splines (NURBS) is the most common method of representing curves, surfaces and solids parametrically in geometric modelling packages [53]. NURBS have a wide application in the expanse of computer-aided design and animation as the most general form of representing a geometric model. Its applicability is quiet enhanced by its ability to represent shapes like circle, sphere, cylinder and free form shapes parametrically with few numbers of control points. In the finite element analysis both the geometry and unknown field functions are approximated by interpolating
polynomials, especially the Lagrange and Hermite polynomials. One of the main trade-offs while pursuing the conventional finite element method is that, the more complex the geometry is, the more complex becomes the meshing and the computational time increases. Analytical representation of curves, lines, surfaces and conics have different methods of representing each of them. This form of representation is easier for a designer to understand and get a physical context about the formulas and coordinates. However, it becomes counterintuitive as the geometry becomes complex. The intersection of surfaces with different forms of representation is often difficult to comprehend. Furthermore analytical methods are not suited to represent freeform curves and surfaces. Parametric curves like cubic splines, Bezier curves, B-splines and NURBS possess a common mathematical form for all type of geometrical entities. The data associated with parametric representation can be easily stored in matrix form and conversion between various formats is easier, i.e. the parametric curves can be manipulated and local changes made by simple steps. The first phase includes developing algorithms and programs for NURBS curves, surfaces and volumes, testing the numerical stability of programs, derivatives and numerical integration. The present work aims at proposing a computational method to study the dynamics of piezoelectric sensors, actuators, power harvesters using a numerical technique based on NURBS.

1.5.1 NURBS curves

A NURBS curve is given by Eq. (1.1) for a prescribed number of \( n+1 \) control points and order \( k \).

\[
C(\xi) = \frac{\sum_{i=0}^{n} N_{i,k}(\xi) w_i p_i}{\sum_{i=0}^{n} N_{i,k}(\xi) w_i}
\]

(1.1)

where the vector \( p_i \) is an array of the control points, \( w_i \) is the weights associated with the control points and \( N_{i,k} \) are the B-Spline basis functions (blending functions or shape functions), and \( \xi \) is the parametric variable. \( N_{i,k} \) can be evaluated by applying recursive
relation as given in the book by Cox/de Boor [53]. In one-dimensional case, it starts with piecewise constants and from there it forms a pyramidal hierarchy.

\[
N_{i,p}(\xi) = \begin{cases} 
1 & \text{for } \xi_i \leq \xi < \xi_{i+1} \\
0 & \text{else}
\end{cases}
\]  

\[
N_{i,1}(\xi) = \frac{\xi - \xi_i}{\xi_{i+1} - \xi_i} N_{i,1-1}(\xi) + \frac{\xi_{i+1} - \xi}{\xi_{i+1} - \xi_i} N_{i+1,1}(\xi)
\]  

The knot vector \(\xi_i\) is a set of parametric coordinates, which defines the limits of the knot spans and is an array of non-decreasing distinct numbers, the only condition that it needs to satisfy. The first and last knot values are repeated \(k\) times to ensure that the curve passes through the first and last control points. The knot vectors are parameterized in \(-1\) to \(1\) to take advantage of the Gauss integration and symmetric geometric conditions. The continuity of a curve depends on the order of the NURBS curve used. If the internal knot vectors are not repeated, the basis function is \(C^{p-1}\) continuous, where \(p\) being the degree of the NURBS curve. The relation between curve’s degree and number of control points enunciates that a NURBS curve becomes an equivalent Bezier curve, when the control points is equal to the order of the curve. Fig. 1.1(a) shows a cubic NURBS curve blended through six control points. Similarly, Fig. 1.1(b) is a fifth degree Bezier curve. Moving the fifth control point creates a global modification in the Bezier curve. This is because the blending functions in Bezier curve has global support and a change affects the entire curve. On the other hand, the cubic NURBS curve blended through six control points remained unaffected in Fig. 1.1(a) between first and third control point when the fifth control point is moved. The local modification of a NURBS curve is possible because the blending functions are associated with each control point. Moving the control point affects the curve to an extent where the blending function is non-zero. Also in NURBS curve the order is four and in the case of Bezier it is six. Curve shape control in NURBS can be achieved in three ways, viz. by modifying the control points itself, by varying the order of the curve, and by changing the weights. The order of the curve has a significant impact on its shape. As the order reduces the curve moves towards the control net and for \(k=1\), the curve reduces to control points.
Figure 1.2(a) shows the shape (or blending) functions of a cubic curve corresponding to six control points with uniform weights. The curve and the open control polygon made by joining the control points are shown in Fig. 1.2(b). It can be inferred that the blending functions are always positive, $N_{i,k} \geq 0$ for any parametric value. Fig. 1.2(c) shows the effect of increased weight, where the curve is pulled more towards the third control point. Knots are repeated thrice in Fig. 1.2(d) and it introduces $C^0$ continuity at the fourth control point. The knot insertion technique [54] is employed to elevate the degree, this is particularly important while using NURBS functions as basis function in analysis. The knots are inserted at certain parametric points, in order to make the curve remain parametrically and geometrically the same, the control points are modified. As a matter of fact, a NURBS curve can be constructed in many different ways. Various combination of control points, weights and curve order can result in a similar curve. The main aim of using the knot insertion technique is to increase the effective curve degree, which in turn can enrich the solution in the analysis. Knot removal is the reverse of the above knot insertion technique,
Fig. 1.2 NURBS curve and shape functions (a) Cubic NURBS basis functions for six control points (b) Parametric cubic curve for six control points, with uniform weights. (c) Parametric cubic curve for six control points, with non-uniform weights $w = \{113111\}$ (d) Knot repetition, $C^0$ continuity at fourth control coefficient [-1 -1 -1 -1 -0.4 -0.4 -0.4 0.6 1 1 1 1].

1.5.2 Differentiation

The first derivative of a NURBS curve at a parametric value $\xi$ can be obtained by normal differentiation [54] as given by the following equation.

$$C'(\xi) = \frac{\sum_{i=0}^{n} N'_{i,k}(\xi) w_i p_i}{\sum_{i=0}^{n} N'_{i,k}(\xi) w_i}$$

(1.3)
The first and second derivatives of the B-spline basis functions are obtained as,

\[
N'_{i,k}(\xi) = \frac{N_{i,k-1}(\xi) + (\xi - \xi_i) N'_{i,k-1}(\xi)}{\xi_{i+k+1} - \xi_i} + \frac{(\xi_{i+k} - \xi) N'_{i+1,k-1}(\xi)}{\xi_{i+k} - \xi_{i+1}} \tag{1.4}
\]

\[
N''_{i,k}(\xi) = \frac{2N'_{i,k-1}(\xi) + (\xi - \xi_i) N''_{i,k-1}(\xi)}{\xi_{i+k+1} - \xi_i} + \frac{(\xi_{i+k} - \xi) N''_{i+1,k-1}(\xi)}{\xi_{i+k} - \xi_{i+1}} \tag{1.5}
\]

The above Eq. (1.4) and (1.5) pose a recursion relation with blending functions and its derivative of the k-1\textsuperscript{th} order. For instance, the Fig. 1.3 below shows the first and second derivatives of the blending functions of a cubic curve with six control points. The first derivative of the blending functions forms piecewise parabolic Fig. 1.3(a) and the second derivative Fig. 1.3(b) results in linear functions.

![Fig. 1.3 Plot of the derivatives of blending functions: (a) First derivative (b) Second derivative](image)

Once the derivatives of the blending functions are obtained, the knot vectors are them modified to take account of the reduced curve order and the new NURBS curve can be obtained.

### 1.5.3 Numerical Integration

Gauss quadrature is used to perform the numerical integration of the NURBS curve. Unlike the conventional Legendre polynomials, the NURBS is a composite curve. So the
number of quadrature points \(n_{\text{int}}\) has to be modified. In this thesis, the composite integral of the curve is treated at each knot span level and a Gauss quadrature rule of \(2n-1\) is used, where \(n\) is the number of control points in each knot span. According to the work done by Stein [56], the number of integration points depends on the knot span and degree, providing an empirical relation \(n_{\text{int}} = 2 \times \text{degree} \times \text{knot span}.\) For a cubic basis function with five-knot spans, would result in 30 integration points. This seems to be a much higher number. In our convergent studies, the solution seems to converge even at half the number of the proposed integration points. Echter et al [57], studied the locking and unlocking characteristics of NURBS elements. In their work they used \(3 \times 3\) Gauss points for quadratic, \(3 \times 4\) Gauss points for cubic and a \(3 \times 5\) scheme for a cubic and quartic NURBS basis functions. The number of integration points used in their work is quite acquiescent with the results obtained in our analysis. Hughes et al proposed a slightly modified form of Gaussian quadrature for NURBS basis functions [58]. They used a separate algorithm to modify the standard Gauss integration points and weights, considering higher order continuity \(C^{p-1}\) and location of knot spans and their results are quite congruent with our results.

1.5.4 NURBS Surfaces and Solids

A NURBS curve is the fundamental entity in a model. Once a NURBS curve is constructed from a set of control points with predefined order and weights, a surface can be developed through more than one technique like sweeping, extruding and revolving. In the present context of the analysis, the quadrilateral surface of a plate is developed through number of surface patches. Let \(C_1(\xi)\) is a \(k\)th order parametric curve in \(\xi\) direction and \(C_2(\eta)\) is \(k\)th order curve in \(\eta\) direction. A surface patch \(S(\xi, \eta)\) is developed from the tensor product of two parametric curves as given in Eq. (1.6).

\[
S(\xi, \eta) = \sum_{i=0}^{n} \sum_{j=0}^{m} R_{i,j}(\xi, \eta) \ p_{i,j} 
\]

In which,
\[
R_{i,j}(\xi, \eta) = \frac{N_{i,j}(\xi)N_{j,k}(\eta) w_{i,j}}{\sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,j}(\xi) N_{j,k} w_{i,j}}
\]
Fig. 1.4 A NURBS surface and solid (a) Two dimensional elliptic surface (b) three dimensional trapezoidal solid

The curve can be of different order in both direction. Fig. 1.4(a) represents an elliptic plane surface, developed using a single patch, containing nine control points in both $\xi$ and $\eta$ direction. The elliptic surface lives inside the control net. Though the control net defines the surface generated, in reality only the control points on the boundary play a significant role in defining the shape. The intermediate control points are generated through a general interpolation technique. A complex model can have number of the sub-patches and in the present analysis a $C^0$ continuity is maintained at the intersection of sub-patches and the continuity inside a patch depends on the order of the corresponding parametric curves. A cubic NURBS curve is used throughout the analysis. The solution is enriched only by increasing either the number of patches or the number of control points in each patch. Developing a three dimensional solid from a surface is achieved by extruding the surface in the third dimension $\zeta$. Fig. 1.4(b) represents a three dimensional trapezoidal solid. A two-dimensional planar trapezoidal shape is developed first and then extruded along the thickness direction. Again, the shape of the solid is determined only by the control points on the edge and the intermediate points are obtained by interpolating between them. NURBS curves, surfaces and volumes possesses convex hull property, as the geometry is always enclosed within the control polygon or control net. The integration, differentiation, knot insertion/removal techniques, etc. developed for unidirectional NURBS curves can be extended to the surfaces and volumes.
1.6 NURBS - Analysis tool

One of the earliest application of splines and synthetic curves as basis functions for finite element analysis can be traced back to the 1980’s, when synthetic curves like Bezier and B-splines were used as shape functions in Ritz method. FEA formulations using spline functions as the basis functions appeared in 1984, when Moore et al. [59] modelled a geometrically nonlinear quadrilateral shell element with linearized rational B-spline functions. Fan and Luah [60] used B-spline functions to model both geometry and displacements. Repeated knots were used to enforce discontinuity in beams and with that they performed free vibration analysis having multiple open cracks. In 1991. Singh [61] proposed a computationally efficient numerical technique for the vibration of shells using Bezier polynomials. In his work, fifth order Bezier functions were used to analyze rotationally symmetric shell structures. Gontier and Vollmer [62] in 1994, reported static analysis of beam undergoing large rotations using Bezier functions. The notion of using NURBS as a basis functions in finite element analysis was investigated by Hughes and co-workers. In 2005, they reported a comprehensive study on NURBS based isogeometric analysis [58, 63]. They continued to work in isogeometric analysis and provided results on refinement and continuity between NURBS patches [64]. In this work they also reported various mesh refinement techniques for NURBS patches. Cottrell et al. [65] used NURBS functions to analyze structural vibration problem. Structural sizing and shape optimization of curved beam was done by Nagy et al. [66]. In their work, an isogeometric approach using NURBS was applied to obtain an optimized design by varying both weights and control point locations. Kim et al. [67] used a NURBS representation of trimmed surface and performed 2D elastic analysis essentially considering it as a lamina. They exported the CAD geometry and used that directly for analysis. Bazilevs et al. [68] studied the stability and errors in various mesh refinement methods on NURBS based finite element patches. They kept the CAD representation unchanged and modified the knots and degree of the curve for refinement. In this thesis, the NURBS functions are used as basis functions in the variational method. Using a variational technique enables us to satisfy only the Dirichlet
boundary condition. By solving the variational form of the differential equation, numerical solutions that satisfy the enforced boundary condition are obtained and discussed.

1.7 Research Objectives

The present work encompasses on developing a numerical method based on variational technique using non-uniform rational B-splines (NURBS). The primary objectives includes,

1. To assess and understand various modelling techniques and applications of piezoelectric materials through a literature survey.

2. First part of the research is involved in understanding the fundamentals of computer aided design and NURBS. The second part is involved in developing algorithms and programs in MATLAB and C++ environment for curves and surfaces using NURBS.

3. Once the rudimentary programs to draw a NURBS curve and surface were functional, programs were developed to study the dynamics of a beam and two-dimensional plate.

4. To study the bending and vibration characteristics of a sandwiched trapezoidal plate. The programs developed for two-dimensional plates are extended for a piezoelectric material.

5. To develop a generic computational model for vibrational energy harvesters using NURBS and find and analyse the frequency response of power and displacements. The plates are considered to have arbitrary or curved edges.

6. To study the statics and vibration characteristics of piezoelectric structures using a three dimensional solid model. The two-dimensional plate model developed in the previous section is extended to a full three dimensional model. Numerical studies are made to compare the efficiency of the three dimensional model to plate theory.
1.8 Thesis Organization

This thesis is organized in integrated article form. Presented in Chapter 1 are: a comprehensive literature review and general introduction to the thesis discussing various aspects of the present study. The study documents the previous research work available on piezoelectric materials, modelling techniques, finite element methods, power harvesting, non-conventional basis functions, and a general introduction to NURBS. Every effort is to present the review in a comprehensive form, yet it is difficult to review all the literature available on piezoelectricity.

Chapter two deals with studying the vibration characteristics of a trapezoidal sandwich plate. A variational method is presented on the statics and vibrations of composite first order shear deformable piezoelectric plates with NURBS functions. The results obtained using the computer codes developed in-house are validated by comparing them for the case of the cylindrical bending of piezoelectric plates. Results are also validated for the free vibration of isotropic trapezoidal plates. Maximum free end displacements are obtained for trapezoidal plates for various aspect ratios and skew angles. Forced vibration analysis is performed for various loading conditions.

Chapter three discusses about extending the variational method developed in previous chapter to plates with curved edges. A general curve fitting technique is also discussed in this section. The developed model is then validated by comparing the results of a circular sandwich plates with the data available in literature. In addition to actuator and sensor equations, equations are developed for a two dimensional power harvester, considering first order shear deformable plate theory and a layer-wise linear variation for electric potential. Frequency response curves were discussed for the power generated and displacements.

Chapter four is concerned about developing a full three dimensional variational method for piezoelectric structures using NURBS. The results obtained by three dimensional theory is compared with the two-dimensional plate theory and ANSYS. A normal mode summation technique is presented to reduce the multiple degree of freedom system to decoupled set of
equations. Forced vibration analysis is performed on the system of equations represented in state-space form using a Runge-Kutta fourth order technique.

Finally a general concluding remarks, contribution to the field of study, and future work is presented in chapter 5 encompassing all the research work.

References


CHAPTER 2

Bending and Vibration Analyses of Skewed Trapezoidal Laminated Piezoelectric Plates Using NURBS

2.1 Introduction

Piezoelectric materials have received attention due to their potential use as sensors and actuators in many sensing and vibration control systems [1]. The fundamental theory of piezoelectricity has been developed and reviewed by many authors [2-5]. Analytical models for piezoelectric actuators and sensors have been developed by several researchers for beams and plates under certain boundary conditions [6, 7]. Even though analytical solutions to piezoelectric problems are feasible in some cases, the coupled piezoelectric equations are generally cumbersome to solve analytically for arbitrary geometry and boundary conditions. One of the earliest works on crystal plate vibrations was done by Mindlin [8], who deduced equations from the three dimensional equations of linear piezoelectricity and provided solutions for the vibrations of quartz plates. Tiersten [7] published a book on piezoelectric plate vibration giving theories and relevant solution techniques. Yang [9] studied piezoelectric plates by assuming a linear variation of displacements and a cubic variation of electric potential in thickness direction. Owing to huge potential applications for sensors and actuators, piezoelectric materials received significant amount of attention in sensing and vibration control as reported in the survey paper by Rao and Sunar [10]. A similar review highlighting various types of simplifications through the thickness and characterizing the performance and quality of different laminate
theories was published by Saravanos and Heyliger [11]. Erturk and Inman presented the modelling and applications of piezoelectric plate as an energy harvester. Their work involved in developing various plate and beam models, optimizing the shape and experimental verifications for piezoelectric energy harvesters [12].

Eer Nisse [13] developed a method to examine the short circuit resonant properties of piezoelectric vibrators. It was shown that results had the agreement of better than three percent with the published experimental data for the lowest eight resonant frequencies of fully electroded thick barium titanate discs. Holland and Eer Nisse [14] explored the variational method further by treating some large classes of problems. Allik and Hughes [15] developed a finite element formulation for piezoelectricity and reported a tetrahedral element for three dimensional piezoelectric models.

Hwang and Park [16] presented a finite element formulation using classical laminated plate theory in the Hamilton’s energy functional. Lee and Moon [17] developed a set of piezo-polymer devices by using polyvinylidene difluoride (PVDF) with different ply layouts. An equivalent single layer theory with third degree interpolation through the thickness for mechanical displacement components was developed by Mitchell and Reddy [18] for piezoelectric composites. They applied discrete linear interpolation for the potential function and solved equations by applying Navier’s method for the static deflections and natural frequencies of symmetric simply supported plates embedded with piezoelectric materials. Kim et al. [19] used flat shell elements for the plates structure and three dimensional elements for MEMS scaled piezoelectric devices.

Fernandes and Pouget [20, 21] considered equivalent single layer approach for mechanical stiffness and linear distribution of electric potential along the thickness. In doing so, they used Fourier series to define the displacement functions and produced results for bimorph and sandwich plates in cylindrical bending. Lage et al. [22] presented static and free vibration analyses on piezoelectric laminated plate structures, where they used mixed layerwise finite element model. Similar analyses on rectangular piezoelectric bimorph were performed by Wang [23], who developed a two dimensional isoparametric finite element method based on the first order shear deformable plate theory and layerwise
linear interpolation through the thickness for the electric potential function. Kapuria and Kulkarni [24] developed an element with four physical nodes and one electrical node based on the zigzag theory. They also studied the transient response analysis of skewed sandwich plate under step and pulse type electromechanical excitations. The performance of this method was assessed by comparing results from the analysis with 3D finite element using ABAQUS. Loja et al. [25] studied the static and free vibration analyses of sandwich plates composed of functionally graded core and piezoelectric skins by B-spline finite strip method. This selective literature review, pertaining to the mechanics of plates, piezoelectricity and composites reveals that in most cases the studies have been limited to rectangular shapes. Also most of the work provided solution techniques for a certain class of problems. It is also realized that apart from finite element technique a general methodology to study the piezoelectric plate structures as an actuators and sensors are few and far between.

A NURBS based variational method is proposed in the present work for composite plates with piezoelectric constituents in a sandwich construction. A two dimensional formulation is presented for static and vibration analyses of the first order shear deformable plates. The geometry is defined by four straight edges on the mid-plane and the prescribed coordinates of the four corner points in the Cartesian system. The domain is then mapped into a square using natural coordinates and NURBS functions. The choice of a quadrangular domain renders high versatility needed for modeling different shaped plates such as skewed, trapezoidal, near triangular and others. The mechanical displacement and rotation components as well as the electrical field function in the plane of the plate are expressed by NURBS in the same manner as done for the geometry. Variation of the mechanical displacement components through the thickness is taken to be linear. However, the electric potential is nonlinear through the thickness. To model this nonlinear variation, the piezoelectric layers are divided into sub-layers and the electric potential is assumed to vary linearly through each sub-layer [20]. Results from the present analyses are validated for piezoelectric plates under the cylindrical bending [21]. Similarly, the free vibration analysis is performed and the eigenvalues are successfully compared and verified for an isotropic trapezoidal plate [26]. Additional results for the static deflections and natural frequencies
of skewed trapezoidal cantilevered sandwich plates with piezoelectric skins are also presented and discussed. The transient vibration analysis is performed by the Newmark’s direct integration method for sandwich plates under different loadings such as impulsive, short duration step and half sine wave. The fast Fourier transforms (FFTs) of the transient responses are studied to determine the contributing natural frequencies.

2.2 Theoretical development

2.2.1 NURBS Representation of a quadrilateral domain.

A quadrilateral domain as shown in Fig. 2.1 is chosen aiming to propose a formulation that can be easily applied to study plate bending and vibration problems of many different shapes. The mid-plane geometry is defined parametrically using $\zeta$ and $\eta$ as coordinates along the edges. Since NURBS offers a comprehensive mathematical representation for all types of conics and arbitrary shapes, the mathematical modelling can be dealt in a unified way for various plate shapes.

Fig. 2.1 Quadrilateral plate.
The mathematical description of a NURBS curve contains a set of basis functions, degree and control points/coefficients [27]. The NURBS curve can be defined parametrically in \( \zeta \) direction by \( n \) control points and \( k^{th} \) degree as given in Eq (2.1).

\[
C(\zeta) = \sum_{i=0}^{n} R_{i,k}(\zeta) p_i \quad \text{and} \quad R_{i,k}(\zeta) = \frac{N_{i,k}(\zeta) w_i}{\sum_{i=0}^{n} N_{i,k}(\zeta) w_i}.
\]

(2.1)

Here, \( R_{i,k}(\zeta) \) is the rational basis functions on \( \zeta \in [-1, +1] \), \( p_i \) is a vector of \( n+1 \) control points, \( w_i \) contains the weights and \( N_{i,k}(\zeta) \) is the \( k^{th} \) degree B-spline basis functions, which can be evaluated by applying recursive relation given by Cox and de Boor [27].

\[
N_{i,0}(\zeta) = \begin{cases} 1 & \text{for } \xi_i \leq \zeta < \xi_{i+1} \\ 0 & \text{else} \end{cases}
\]

(2.2)

\[
N_{i,k}(\zeta) = \frac{\xi - \xi_i}{\xi_{i+k} - \xi_i} N_{i,k-1}(\zeta) + \frac{\xi_{i+k} - \xi}{\xi_{i+k} - \xi_{i+1}} N_{i+1,k-1}(\zeta)
\]

(2.3)

Knot vector \( \xi_i \) is a set of parametric coordinates that defines the limits of the knot spans. Depending on the degree, each shape function is non-zero only over a certain span. The desired order of continuity is attainable by repeating the knots at a particular nodal point over the knot span. An open uniform knot vector has the following form,

\[
\xi_i = \begin{cases} 0, & 0 \leq i < k + 1 \\ i - k + 1, & k + 1 \leq i \leq n \\ n - k + 2, & n < i \leq n + k + 1 \end{cases}
\]

(2.4)

\[
\xi_i = \{ 0, \ldots, 0, \underbrace{\xi_{k+1}, \ldots, \xi_n, 1, \ldots, 1}_k \}
\]

(2.5)

The first and last knot values in Eq.(2.5) are repeated \((k+1)\) times to ensure that the curve interpolates first and last control points. Essentially, the knot vectors are not required to be parameterized from zero to one. Rather, it can be an array of non-decreasing numbers and is parameterized to be \( \xi_i = \{-1, \ldots, 1\} \) in the present work for expedience in
computation. Similarly, the NURBS curve is constructed in the $\eta$ direction with different number of control points. Thus, using $(n+1)$ control points in $\zeta$ direction and $(m+1)$ in $\eta$ direction, a control net of $p = (n+1) \times (m+1)$ grid points is defined below in Eq. (2.6).

$$S(\xi, \eta) = \sum_{i=0}^{n} \sum_{j=0}^{m} R_{i,j}(\xi, \eta) \ p_{i,j}$$

(2.6)

In the above, $w_{i,j}$ are the weights associated with $p_{i,j}$ and $R_{i,j}(\xi, \eta)$ is given as

$$R_{i,j}(\xi, \eta) = \frac{N_{i,j}(\xi)N_{j,k}(\eta)w_{i,j}}{\sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,j}(\xi)N_{j,k}w_{i,j}}$$

(2.7)

The blending function $R_{i,j}(\xi, \eta)$ can be used in Eq. (2.6) along with the Cartesian coordinates of the grid points to define the mid-plane surface of the plate.

### 2.2.2 First Order Shear Deformable Piezoelectric Plates.

In this section, the governing equations are developed beginning with the very basic equations for the first order shear deformable plates. The plate is assumed to be made from the fiber reinforced composites with piezoelectric material as insert and the rotary inertia and transverse shear are included in the formulation. Piezoelectric materials are inherently anisotropic and possess coupled mechanical and electrical properties described by the following relationships [15].

$$\{\sigma\}' = [C]\{\varepsilon\}' - [e]^T\{E\}'$$

$$\{D\}' = [e]\{\varepsilon\}' + [\varepsilon]\{E\}'$$

(2.8)

In the above, $\{\sigma\}'$, $[C]$, and $\{\varepsilon\}'$ are the stress tensor, elasticity evaluated at a constant electric field, and the strain tensor. Similarly, $[e],\{D\}',[\varepsilon]$, and $\{E\}'$ correspond to the piezoelectric coupling constant matrix, electric displacement vector, dielectric constant matrix at constant strain and the electric field vector respectively. The electric field vector
\{E'\} is related to the electric potential \(\phi\) by the following Maxwell’s equation of electrostatics.

\[
\{E'\} = -\{\nabla\} \phi'
\]  

(2.9)

It is assumed that the transverse deformation of the plate is small compared to the plate thickness \((h)\) and also that the material is initially free of stress and electric displacements. The mechanical displacement components at an arbitrary point in the plate are defined by \(u' = u + z \beta_1, \ v' = v + z \beta_2\) and \(w' = w\). Where \(u, v\) and \(w\) are the mid-plane displacement components along \(x, y\) and \(z\) coordinates. Symbols \(\beta_1\) and \(\beta_2\) are the components of rotation of the normal to the plate. Readers should refer to the work of Tanveer and Singh [28] for other details of notations and steps involved in the formulation.

### 2.2.3 NURBS Representation of displacement and electrical potential fields

Similar to the geometry, the displacement and electric field functions are represented by NURBS Each control point has five mechanical degrees of freedom \(U, V, W, B_1\) and \(B_2\) as expressed below.

\[
\begin{align*}
    u(\xi, \eta) &= \sum_{i=0}^{r} \sum_{j=0}^{s} R_{i,j}(\xi, \eta) U_{i,j} \\
    v(\xi, \eta) &= \sum_{i=0}^{r} \sum_{j=0}^{s} R_{i,j}(\xi, \eta) V_{i,j} \\
    w(\xi, \eta) &= \sum_{i=0}^{r} \sum_{j=0}^{s} R_{i,j}(\xi, \eta) W_{i,j} \\
    \beta_1(\xi, \eta) &= \sum_{i=0}^{r} \sum_{j=0}^{s} R_{i,j}(\xi, \eta) B_{1i,j} \\
    \beta_2(\xi, \eta) &= \sum_{i=0}^{r} \sum_{j=0}^{s} R_{i,j}(\xi, \eta) B_{2i,j}
\end{align*}
\]  

(2.10)

The number of unknowns for each of \(\{u \ v \ w \ \beta_1 \ \beta_2\}\) in Eq. (2.11) is \(q = (r+1) \times (s+1)\), where \(r\) and \(s\) are the number of unknown control points of the displacement functions in \(\xi\) and \(\eta\) directions respectively. Equation (2.11) can be expressed conveniently as.

\[
[\Delta] = [\overline{R}(\xi, \eta)] [\Gamma]
\]  

(2.11)
where, \( \mathbf{R}(\xi, \eta) \) is the blending function matrix and

\[
\{ \mathbf{\Gamma} \}' = \{ U_1 \ V_1 \ W_1 \ B_1 \_1 \ B_2 \_1 \ - \ - \ - U_2 \ V_2 \ W_2 \ B_1 \_2 \ B_2 \_2 \} 
\]

(2.12)

In the present work, the in-plane displacement components \( u' \) and \( v' \) vary linearly through the thickness, while the transverse displacement component \( w' \) is constant. It is not so for the electric potential function \( \phi \), as it varies nonlinearly along the thickness. Therefore, the piezoelectric layer is divided into a number of sub-layers and \( \phi \) is characterized as linear in a given sub-layer. For example: if \( L \) is the number of sub-layers, the electric potential for the \( k \)th sub-layer can be expressed using the separation of variables as \( \phi^{(k)} = \{ g_k \} \{ \phi_k \ phi_{k+1} \}^T \),

where \( g_k = (z_{k+1} - z_k)^{-1} \left[ (z_{k+1} - z) (z - z_k) \right] \) is a linear function in \( z \). The electric potentials for the \( k \)th sub-layer at the bottom and top interfaces are given by \( \phi_k \) and \( \phi_{k+1} \) respectively and are independent of the \( z \) coordinate. Therefore, the electric potential function at the \( k \)th interface, i.e. bottom plane of the \( k \)th sub-layer, can be defined similar to the displacement field functions as

\[
\phi^{(k)}(\xi, \eta) = \sum_{i=0}^{c} \sum_{j=0}^{s} R_{i,j}(\xi, \eta) \Phi_{i,j}^{(k)}
\]

(2.13)

In the above Eq. (2.13), \( \Phi_{i,j}^{(k)} \) corresponds to the degrees of freedom for the potential function on the \( k \)th interface and is analogous to a displacement or rotation component in Eq. (2.11).

### 2.3 Variational method

So far, the basic piezoelectric plate relations are developed and defined in terms of NURBS functions. The displacement and electric field functions are next applied to the kinetic energy \( (T) \), the strain energy \( (U) \) and the work done \( (W) \) by externally applied
mechanical and electrical forces. These energy terms are subsequently substituted into the Hamilton’s energy functional $\Pi = T - U - W$ as follows.

$$\int_\eta^\tau \delta \Pi \, dt = \delta \int_\eta^\tau (T - U - W) \, dt = 0$$ \hspace{1cm} (2.14)

The strain and kinetic energies require integration over the volume, while the integration on the work done may involve either surface or volume integrations. First, the integration is performed analytically over the thickness ($h$) and then Gauss quadrature is used over the area of the mid-plane. The number of integration points needed for highly accurate results depends on the knot span and order of the NURBS function. Finally, the following coupled equations of motion can be arrived at [15].

$$[M][\ddot{\Gamma}] + [K_m]\{\Gamma\} + [K_{me}]\{\varphi\} = \{F(t)\}$$ \hspace{1cm} (2.15)

$$[K_{em}]\{\Gamma\} + [K_e]\{\varphi\} = \{Q(t)\}$$

Equation (2.15) presents simultaneous coupled differential equations. These two equations can be combined and the final equation is motion in terms of mechanical displacement can be written as,

$$[M][\ddot{\Gamma}] + ([K_m] - [K_{me}][K_e]^{-1}[K_{em}])\{\Gamma\} = \{F(t) - [K_{me}][K_e]^{-1}\{Q(t)\}\}$$ \hspace{1cm} (2.16)

Equation (2.16) represents the dynamic equation of a piezoelectric plate in terms of mechanical displacements. The equation is valid to perform the forced vibration analysis of the plate. Also the equation for free vibration analysis can be obtained if the forcing function on the right hand side is dropped. Similarly, the equations for the static analysis are deduced by removing time dependency from Eq. (2.15). For numerical modeling, the displacement parameters are normalized with respect to thickness of the plate and the dielectric constants are modified by a factor of $10^{+10}$ to prevent any ill conditioning of matrices.
2.4 Numerical results and discussion

A general skewed quadrilateral configuration as shown in Fig. 2.1 is chosen for the plate geometry so that the applicability of the present study can be significantly broadened. The parametric indices are introduced as follows, \( \alpha \) is the skew angle and \( a \) being the distance between the mid-points of parallel edges 2 and 4. Similarly, \( b \) and \( c \) represent the lengths of edges 4 and 3.

2.4.1 Rectangular piezoelectric plate in cylindrical bending.

To validate the accuracy of the present method, numerical results are generated under cylindrical bending condition for both sensor and actuator modes and then compared with data available in the literature [20] for a PZT–4 piezoelectric rectangular plate. The material properties of PZT-4 are taken from the work of Polit and Bruant [29]. Assumptions made under this condition are that the stress, strain and electric potential do not change in the \( y \) direction. This is easily implemented by setting \( v \) and \( \beta_2 \) to zero everywhere in the plate which is supported on edges 2 and 4 with boundary conditions \( u=w=0 \) at \( x=0 \) and \( w=0 \) at \( x=a \). Dimensions are taken as \( a=25 \text{mm}, b=c=12.5 \text{mm}, \alpha=0^\circ \) and thickness \( h=1 \text{mm} \).

Table 2.1 Central displacement and electric potential, in sensor mode for the applied load of 1 KN/m\(^2\)

<table>
<thead>
<tr>
<th>a/h</th>
<th>Sub-layers</th>
<th>Central displacement (w) in ( \mu \text{m} )</th>
<th>Electric potential ( \phi ) in volt</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
<td>0.04358</td>
<td>0.04298</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.04326</td>
<td>0.04270</td>
</tr>
<tr>
<td>50</td>
<td>2</td>
<td>5.3110</td>
<td>5.3270</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>5.3110</td>
<td>5.3270</td>
</tr>
</tbody>
</table>
Table 2.2 Maximum lateral displacement, as an actuator for applied load of 50V

<table>
<thead>
<tr>
<th>a/h</th>
<th>Sub-layers</th>
<th>Lateral displacement (w) in μm</th>
<th>Present method</th>
<th>Fernandes and Pouget[20]</th>
<th>Percentage difference with reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
<td>0.1643</td>
<td></td>
<td>0.1637</td>
<td>0.365</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.1643</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>2</td>
<td>0.8216</td>
<td></td>
<td>0.8203</td>
<td>0.158</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.8217</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the sensor mode, a closed circuit condition is created by grounding the top and bottom layers and applying a uniformly distributed load of 1KN/m² on the top surface. Similarly, electrical potential of 50V is applied on the top and bottom faces across the thickness for the actuator mode. Since, the applied voltage at the top and bottom faces are equal and opposite, the plate bends the in direction of the axis of polarization. Analysis is performed using different aspect ratios and number of sub-layers on a model of 3×3 patches with each patch having 25 control points. The maximum central deflection (w) and the electric potential are calculated and presented in Tables 2.1 and 2.2 along with the results, from the work of Fernandes and Pouget [20]. An excellent agreement between the results is seen in these tables by considering only six layers.

2.4.2 Simply supported symmetric isotropic trapezoidal plates.

Chopra and Durvasula [26] presented a detailed study of both the natural frequencies and nodal patterns of simply supported trapezoidal plates. Their method is based on the Fourier sine series in transformed non-orthogonal coordinates for the displacement field in the Galerkin method. A 3×3 patches with each patch having 25 control points and a fourth order NURBS curve are used in the present analysis. Non-dimensional angular frequency $\lambda$ with a/h=50 and c/b = 0.20 are calculated by the present method and
recorded in Table 2.3 for $a/b = 0.5$, skew angle $\alpha = 0$ and $a/b=1.0$. Here $\rho$ and $D$ are the mass density and flexural rigidity respectively.

Table 2.3 Non-dimensional frequency $\lambda = \omega \left( \frac{b^2}{\pi^2} \right) \sqrt{\frac{\rho h}{D}}$ for the simply supported symmetric isotropic trapezoidal plate with $a/h=50$ and $c/b=0.20$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$a/b = 0.5$</th>
<th></th>
<th>$a/b = 1$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ref [26]</td>
<td>Present</td>
<td>Ref [26]</td>
<td>Present</td>
</tr>
<tr>
<td>1</td>
<td>1.9376</td>
<td>1.9159</td>
<td>3.8242</td>
<td>3.8124</td>
</tr>
<tr>
<td>2</td>
<td>4.0555</td>
<td>4.0265</td>
<td>8.0707</td>
<td>8.0312</td>
</tr>
<tr>
<td>3</td>
<td>5.0874</td>
<td>5.0172</td>
<td>9.7898</td>
<td>9.7542</td>
</tr>
<tr>
<td>4</td>
<td>7.0252</td>
<td>6.9464</td>
<td>13.6640</td>
<td>13.5748</td>
</tr>
<tr>
<td>5</td>
<td>8.0199</td>
<td>7.9184</td>
<td>16.6810</td>
<td>16.5920</td>
</tr>
<tr>
<td>6</td>
<td>9.9950</td>
<td>9.8275</td>
<td>18.2350</td>
<td>18.1318</td>
</tr>
<tr>
<td>7</td>
<td>10.6940</td>
<td>10.5528</td>
<td>20.9340</td>
<td>20.7746</td>
</tr>
<tr>
<td>8</td>
<td>12.2630</td>
<td>11.9084</td>
<td>24.5670</td>
<td>24.3987</td>
</tr>
</tbody>
</table>

Frequencies from the present method are found consistently lower than their results [26], but match very well with the largest discrepancy of 3% at the eighth mode. Lower values of the frequencies can be attributed to the flexible behavior of NURBS functions which are composite curves with high order continuity.
2.4.3 Cantilevered skewed sandwich trapezoidal plates.

After successfully verifying results from the present method with those from the literature, a skewed sandwich quadrilateral plate is examined. Let us consider a basic sandwich configuration consisting of a main supporting structure made of silicon between two thin ZnO layers polarized in the z direction. The layer thicknesses from the bottom to the top are $0.2h$, $0.6h$, and $0.2h$ where $h$ represents the overall thickness of the sandwich plate. Other dimensions considered in this case are: $a = 25\text{mm}$, $b = 12.5\text{mm}$, $0 \leq c \leq 12.5\text{mm}$ and $50 \leq (a/h) \leq 100$. Each piezoelectric layer is divided into six sub-layers. The material properties are obtained from the work of Polit and Bruant [29]. The static analysis is performed for both applied mechanical load and electrical load.

Fig. 2.2 Maximum free end displacement of a cantilevered sandwich plate, for an applied voltage of 100V and $\alpha= 0^\circ, 15^\circ, 30^\circ$ and $45^\circ$. 
Fig. 2.2 shows the plot of the maximum deflection against $c/b$ at the free end under an applied electric load of 100V. The skew angles considered are in the range of $0^\circ \leq \alpha \leq 45^\circ$. The base to tip ratio, $c/b$ is varied in the range of $0.0 \leq c/b \leq 1.0$, which covers plate geometry from a triangle to parallelogram. The maximum deflection at $c/b=0$ decreases with increasing skew angle $\alpha$, but is just the opposite at $c/a=1$. However, the deflection increases consistently with increasing $c/b$ due to the fact that the surface area of the plate increases as long as $a$ and $b$ are kept the same. Also as the $a/h$ ratio increases the plate becomes thinner and the displacements are higher, as inferred from Fig.2.2

![Graph showing maximum free end displacement vs. c/b](image)

Fig. 2.3 Maximum free end transverse displacement of a sandwich plate, for a uniformly distributed load of 50kN/m² and $\alpha= 0^\circ, 15^\circ, 30^\circ$ and $45^\circ$.

A uniformly distributed transverse load of 50 $kN/m^2$ is applied on the top surface of the plate. The top and bottom piezoelectric layers are grounded. The maximum free end
displacements are provided in Fig. 2.3 and it shows a similar trend as seen in Fig. 2.2 for the electrical load.

The free vibration analysis is performed for the cantilevered sandwich piezoelectric plate for various aspect ratios and skew angles. The electrical boundary conditions are set to zero on the top and bottom surface the of the plate. A 3×3 NURBS patch containing 36 control points in each patch is used in the analysis. The number of piezoelectric sub layers is kept at six. For the frequency analysis the forcing functions in the right hand side of the Eq. (2.16) is dropped resulting in a standard eigenvalue problem.

\[
[M][\ddot{\mathbf{\Gamma}}] + [K_{eq}][\mathbf{\Gamma}] = 0 \quad (2.17)
\]

In which, \( [K_{eq}] = [K_n] - [K_{me}][K_m]^{-1}[K_{me}] \)

As it can be inferred the plate stiffness is reduced because of the piezoelectric effect and thereby reducing the fundamental frequencies of the plate. The eigenvalues are obtained from Eq. (2.17), using matrix inverse iteration technique. Fig. 2.4(a–f) shows the changing trend of the first five modes of the non-dimensional natural frequency \( \lambda = h\sqrt{\rho/C_{ij}} \) against \((c/b)\). To investigate the influence of the taper ratio, the first five fundamental frequencies are analyzed in the range of \( 0 \leq (c/b) \leq 1 \) covering triangular to parallelogram plates. The other plate dimensions remained same as that of static analysis.
Fig. 2.4 First five non-dimensional natural frequencies of a cantilever sandwich plate with various skew angles and aspect ratios. In (a) and (b) the variation of natural frequencies are shown for $a/h = 50$ and in (c) and (d) the natural frequencies are given for $a/h = 100$.

Fig. 2.4a and 2.4b shows the variation of the first five non-dimensional natural frequency for $a/h=50$ and skew angle $0 \leq \alpha \leq 45^\circ$ at $15^\circ$ interval. The taper ratio $(c/b)$, is varied in the range of $0 \leq (c/b) \leq 1$. Similarly, Fig. 4c and 4d contain results for $a/h = 100$. Frequencies are seen to increase with the skew angle $\alpha$ and decrease with $c/b$ for all the aspect ratios. Also seen particularly at higher modes is the tendency of the modal curves flipping from symmetric to asymmetric.

\subsection*{2.4.4 Cantilevered skewed sandwich trapezoidal plates: transient vibration}

Transient response analysis of trapezoidal plate is carried out by Newmark’s direct integration method under. Both mechanical and electrical loads are considered in this study. The integration parameters are, $\alpha = 0.391$ and $\delta = 0.75$. The time step $\delta t$ is assumed to be less than two percent of the fundamental frequency of the plate under consideration. Assuming the time step in such order resulted in a stable solution without inducing artificial damping. Numerical experiments were performed for cantilevered sandwich plate with ZnO/Si/ZnO configuration. The plate dimensions are, $a = 25$ mm, $b/a = 0.5$, $a/h = 50$, $c/b = 0.48$ and skew angle is zero. The displacement time history are reported in Fig 2.5(a-j) and discussed here.
(c) Non-dimensional displacement vs. non-dimensional time.

(d) Non-dimensional amplitude vs. non-dimensional frequency.
Fig. 2.5 Displacement history and FFT plots of cantilevered trapezoidal plate under various mechanical and electrical loads. (a) and (b) Half sine unit point load applied at the midpoint of the free end. (c) and (d) Unit point step load applied at the midpoint of the free end. (e) and (f) Impulse point load applied at the midpoint of the free end. (g) and (h) For an electrical step load of 50V. (i) and (j) Impulsive electrical load of 50V.

A mechanical point load applied at the midpoint of the free end of the cantilevered sandwich plate. Parametric studies are performed for various loading functions like sinusoidal, step, and impulsive. The displacement history is recorded at a node closer to the fixed end of the cantilever plate. Fig 2.5(a) shows plot of the non-dimensional transverse displacement under the sinusoidal load versus the non-dimensional time $\hat{t} = t/(h\sqrt{\rho/C_{11}})$. The load is applied for $3e^{+04}$ units. The maximum amplitude appears at the time of $1.5e^{+04}$ units and once the load is removed the plate vibrates about its mean position. The displacement history comprises of more than one fundamental frequency. In order to identify the contributing fundamental frequencies, the fast Fourier transform is obtained for the displacement history as shown in Fig. 2.5(b). The peaks are seen to occur at frequencies corresponding to the fundamental frequencies. Figs. 2.5(c)-2.5(f) show results for the case of step and impulsive point mechanical loads. Similarly, the transient response results under a step and impulsive electrical load of 50V is presented in Figs. 2.5(g)-2.5(j).
Table 2.4 Non-dimensionalized frequency $\bar{\Omega} = \Omega \times 10^{-6}$ for the cantilevered sandwich trapezoidal plate with $a = 25$ mm, $b/a = 0.5$, $a/h = 50$, $c/b = 0.48$, $\alpha = 0$ and

$$\Omega = \frac{1}{2}(\omega h / \pi)\sqrt{\rho / C_n}.$$ 

<table>
<thead>
<tr>
<th>Mode</th>
<th>Free Vibration</th>
<th>Applied point load</th>
<th>Applied voltage</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Impulse</td>
<td>Step</td>
</tr>
<tr>
<td>1</td>
<td>81.72</td>
<td>-</td>
<td>81.0</td>
</tr>
<tr>
<td>2</td>
<td>440.81</td>
<td>440.6</td>
<td>441.6</td>
</tr>
<tr>
<td>3</td>
<td>544.01</td>
<td>539.8</td>
<td>540.7</td>
</tr>
<tr>
<td>4</td>
<td>1261.45</td>
<td>1265.0</td>
<td>1263.0</td>
</tr>
<tr>
<td>5</td>
<td>1390.89</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>1583.37</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>2296.34</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>2658.32</td>
<td>2611.0</td>
<td>-</td>
</tr>
<tr>
<td>9</td>
<td>3451.91</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>3772.62</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

The contribution of multiple frequencies is particularly more pronounced when the load is applied impulsively. The results from the fast Fourier transforms and free vibration analysis are summarized in Table 2.4. Since the impulse and step loads are of the suddenly applied type with zero build up time, they are seen to excite more natural modes than the
sinusoidal load which begins with zero and builds up to its peak at 1.5e+04 time units and returns to zero. The direct integration method is seen to endorse the free vibration results.

2.5 Concluding remarks.

A variational method is proposed based on the NURBS for the static and vibration analyses of composite first order shear deformable piezoelectric plates. A general representation for both geometry and displacement fields were implemented using NURBS functions. The present formulation using a quadrangular domain provided versatility to study the skewed, trapezoidal, near-triangular and other shaped plates without much modifications. The method and computer codes developed in-house are validated by comparing results for the cylindrical static bending of piezoelectric plates and the free vibration of isotropic trapezoidal plates. Additional results for the static deflections and natural frequencies of cantilevered skewed trapezoidal sandwich plates with piezoelectric skins are also reported and discussed in this work. The present study concludes with the transient vibration analysis by the Newmark’s direct integration method of sandwich plates subjected to impulsive, step with short duration and half sine wave loads. The computational method presented in this study is expected to be applied to model piezoelectric actuating and sensing devices of skewed and trapezoidal shapes. Since NURBS functions are used to define the geometry, it has the potential application of modelling complex geometries with fewer number of control points. The method presented in this work is seen to be accurate and efficient and expected to be used in the design and analysis in the field of sensor, actuators and vibration energy harvesting.
References


CHAPTER 3

A Plate Model for Piezoelectric Vibrational Energy Harvesters Using NURBS

3.1 Introduction

Piezoelectric materials are widely used as energy source in low-powered sensors. The ability of piezoelectric materials to generate energy from vibration under ambient conditions is exploited and used in industrial automation systems, medical implants and wireless devices. Unlike other actuator and energy harvesting systems, piezoelectric devices possesses high sensitivity, high mechanical strength, very few to no hard-wired connections and are devoid of any hysteresis. Piezoelectric materials are also often distributed or embedded in smart structures to enable self-monitoring capabilities. These applications employ piezoelectric devices of varied shape, thus requiring a comprehensive modeling for various shapes and configurations. The piezoelectric effect was reported in the 1880 by Curie brothers and ever since great interest by researchers and scientists has been shown owing to its vast rage of application as sensors and actuators. Mindlin studied the forced thickness shear and flexural vibrations of piezoelectric crystal plates, continued working on this topic and published a series of papers in the next three of decades [1, 2]. The monograph published in 1964 by W.G. Cady [3] serves as one of the finest reference, in piezoelectricity. Tiersten published a monograph on linear piezoelectric plate vibrations [4] with detailed development of the mechanics of piezoelectric plate vibrations. Many analytical techniques were proposed to analyze the piezoelectric sandwich structures, under
certain boundary conditions. Crawley developed an analytical model for embedded piezoelectric material under shear load for a beam and also experimentally verified the results for surface bonded piezoelectric actuators [5]. C.K. Lee developed an analytical model for distributed sensors, in which he used a generalized lower order laminate theory using classical laminate plate theory [6]. Osama and Ahmed obtained analytical solutions for laminated piezoelectric sandwich plates using a Lévy type solution technique along with state space approach for rectangular plates [7]. Analytical solutions for circular piezoelectric plates have been presented by some researchers. Takano et al. [8] performed non-axisymmetric analysis for the vibration and applied the results to an ultrasonic motor. Free vibration of laminated piezoelectric circular plates and discs was considered by Heyliger and Ramirez [9]. They presented results for thin plates and thick discs. Wang et al. [10] studied vibrations of circular plates bonded with piezoelectric layers to the top and bottom. Results from their analytical method are compared with those by finite element analysis. Dong et al. [11] produced simple formulas for engineering design and optimization of a circular unimorph piezoelectric actuator. While there are considerable amount of work that has been done on analytical and semi analytical techniques, usually such methods are limited by the geometric shape and boundary conditions of the problems. Theories of laminated plates with integrated sensors and actuators were published [6, 12] and several review studies pertaining to the analytical and numerical solutions for piezoelectric composite material structures were reported [13–16]. While there are considerable amount of work has been done on analytical and semi analytical techniques, usually such methods were limited by the geometric shape and boundary conditions of the problem being considered. Variational formulation of piezoelectricity was developed by Holland and Eer Nisse [17]. They obtained solutions for several classes of problems, which were unsolvable by the techniques of the time. In the late 1960s and 1970s, the finite element methods in structural mechanics saw a rapid growth. Allik and Hughes [18] extended this variational approach and developed a general method for the electro-elastic analysis using tetrahedral finite elements for the piezoelectric three dimensional problems. Varadan et al. studied the static analysis of cantilever plate as sensor and actuator using finite elements, in their work they provided results for thin plate with circular piezoelectric element and modeled cantilever plate using a combination of brick, transition, and plate
elements [19]. It is difficult to cover the enormous amount of literature available on finite element modelling of piezoelectric plates, nevertheless one can refer to the works of Benjeddou [20], for a comprehensive review on finite element modelling of piezoelectric structures until 2000. In this review, he documented various formulations based on plate, beam and solid models.

Although the piezoelectric phenomenon has been studied since the 1960’s, the notion of piezoelectric energy harvesting received more attention in the last decade and a half. In this period, piezoelectric energy harvesters have been recognized as a source of power for electronic instruments at the micro scale. Also, such devices are used to charge batteries in difficult-to-reach MEMS and provide self-sustenance. The electricity generated due to vibrations under ambient conditions can be recovered to charge batteries in electronic devices. The fundamental governing equations obtained for sensors and actuators can be extended to model piezoelectric power harvesters. William and Yates [21] analyzed electromagnetic generator for micro scaled systems using a single degree of freedom (SDOF) lumped system subject to an ambient base excitation. Lu et al. [22] modelled a piezoelectric micro generator operating in transverse mode and presented the case study of power efficiency for PZN-8%PT crystal and PZT-PIC 255. They drew some design regulations based on their numerical simulation. Inman and Erturk [23] published a book on piezoelectric power harvesting, documenting their works in the field of energy harvesting using piezoelectric, electromechanical, electromagnetic and thermoelectric techniques. They also discussed various modelling techniques for piezoelectric power harvesters, addressed some issues in the modelling techniques used by researchers of different subject areas, stipulated modifications on most commonly used beam models, and presented a modified version considering piezoelectric coupling and influence of the load resistance [24]. Sodano et al. [25] provided a simple theoretical beam model for a piezoelectric power harvesting device and verified their models with experimental results. Dutoit et al. [26] presented design considerations in their study of micro-electromechanical scale piezoelectric energy harvesters and validated models by comparing with published results. Sohn et al. [27] adopted finite element method to evaluate electrical power generating capabilities of commercially available piezofilms and conducted experimental
Guyomar et al. [28] proposed a nonlinear processing technique for output voltage from a piezoelectric transducer, thereby enhancing the power conversion. They also reported an increase in power efficiency of up to 200 percent. Shu and Lien [29] used equations of a one dimensional beam model to analyze the performance of a cantilevered piezoelectric energy harvester. The influence of electromechanical coupling factor and resistance on the optimal power output was discussed. Anton and Sodano [30] reviewed various topics on power harvesting piezoelectric devices including the design of geometries for enhanced efficiency. These efficiency enhancements were discussed under different sections as piezoelectric configurations, circuitry and power storage methods, implantable and wearable power supplies, ambient fluid flows, MEMS, self-powered sensors and performance of piezoelectric materials. A general equivalent circuit model of piezoelectric cantilever beam energy harvesters containing was proposed by Elvin and Elvin [31], who described that their work could be applied to different generator geometries and included higher vibrational modes in their analysis. Xiong and Oyadiji [32] optimized a two layered power harvester using modal approach, they varied the location and mass of the spacer that was used to connect the piezoelectric and substrate layer. A vibration model of beams with non-uniform width was investigated by Dietel and Garcia [33] and focused on the concentration of strain in sections in order to harvest the optimum energy from piezoelectric bimorph. They studied parametrically the dependence of energy transferred on the ratio of tip-mass-to-beam-mass. Companion papers on the analytical and experimental studies respectively have been published by Kim et al. [34, 35]. They developed models for clamped circular unimorph piezoelectric plate to analyze the influence of geometric design parameters and electrode configuration on the amount of energy harvested by applying the transverse pressure. In their work, analytical results are supported experimentally with the help of three piezoelectric energy generators. Performance of circular piezoelectric bimorph plate with centrally attached mass used for energy harvesting was studied by Jiang and Hu [36]. Their analytical solution for the flexural motion shows output power density to increase initially with load impedance, reach a maximum and finally decrease monotonically. Zheng et al. [37] studied the sensitivities of strain energy while optimizing the location and placement of piezoelectric material. Works cited above reveals that essentially beam models have been studied by researchers for the electric power generation.
As the shape and location of the piezoelectric material plays an important role in the amount of power generated, a much more comprehensive model is needed for the better understanding. This present study is concerned with a general variational method based on non-uniform rational B-spline (NURBS) function for modeling arbitrarily shaped piezoelectric sandwich plates. Equations of motion for a piezoelectric plate structure are formulated on the Reissner-Mindlin’s first order shear deformable plate theory. A layer wise linear scheme is considered in order to include the nonlinear variation of electric potential through the thickness. The non-uniform rational B-spline (NURBS) functions are used to define the geometry and unknown field functions. The coupled matrix equations of motion are obtained for piezoelectric plates as sensor and actuator. The present method is validated by performing static and dynamic characteristics of circular piezoelectric plates for which the closed formed solutions are available in the literature [10, 11]. Finally, the equations are extended further with the help of the normal mode summation method to investigate piezoelectric energy harvesters. The present study is expected to add a suitable modeling technique for piezoelectric plates of arbitrary shapes.

### 3.2 Geometrical representation in NURBS.

The mathematical description of a NURBS curve contains a set of basis functions, order and control points/coefficients. The NURBS curve defined by \( n+1 \) control points is can be written as [38],

\[
C(\xi) = \sum_{i=0}^{n} R_{i,k}(\xi) \ p_i \quad \text{and} \quad R_{i,k}(\xi) = \frac{N_{i,k}(\xi) w_i}{\sum_{i=0}^{n} N_{i,k}(\xi) w_i} .
\]  

(3.1)

Here in the Eq. (3.1), \( p_i \) are the control points, \( w_i \) comprises the weights, \( k \) is the order and \( N_{i,k} \) contains the components of B-Spline basis function, \( \xi \) is the parametric variable. The basis function \( N_{i,k} \) can be evaluated by applying recursive relation given by Cox/de Boor [38]. For a one-dimensional case, the blending functions starts with piecewise constants, following which it forms a pyramid of hierarchy described by Eq. (3.2)
\[ N_{i,0}(\xi) = \begin{cases} 1 & \text{for } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{else} \end{cases} \]  
(3.2)

\[ N_{i,k}(\xi) = \frac{\xi - \xi_i}{\xi_{i+k-1} - \xi_i} N_{i,k-1}(\xi) + \frac{\xi_{i+k} - \xi}{\xi_{i+k} - \xi_{i+1}} N_{i+1,k-1}(\xi) \]

Fig. 3.1 Quadrilateral middle surface of an arbitrary shaped plate

Knot vector \( \xi_i \) is a set of non-decreasing parametric coordinates that defines the limits of the knot span. An open uniform knot vector has the form of

\[ \xi_i = \begin{cases} 0, & 0 \leq i < k + 1 \\ i - k + 1, & k + 1 \leq i \leq n + 1 \\ n - k + 2, & n + 1 < i \leq n + k + 1 \end{cases} \]  
(3.3)

\[ \xi_i = \left\{ 0, \ldots, 0, \xi_{k+1}, \ldots, \xi_n, 1, \ldots, 1 \right\} \]
Once the knot vectors are defined for a predefined set of control points and order, the NURBS curve can be constructed using Eq. (3.1). Similarly the first derivative $C(\xi)$ of a NURBS curve can be evaluated at any given parametric value $\xi$ from

$$
C'(\xi) = \frac{\sum_{i=0}^{n} N'_{i,k}(\xi) w_i p_i}{\sum_{i=0}^{n} N'_{i,k}(\xi) w_i}
$$

(3.4)

In which, the first derivative of the B-spline basis functions are obtained as

$$
N'_{i,k}(\xi) = \frac{N_{i,k-1}(\xi) + (\xi - \xi_i)N'_{i,k-1}(\xi)}{\xi_{i+k} - \xi_i} \left( \frac{(\xi_{i+k} - \xi)N'_{i+1,k-1}(\xi) - N_{i+1,k-1}(\xi)}{\xi_{i+k} - \xi_{i+1}} \right)
$$

(3.5)

Similarly, Eq. (3.3) can be used to obtain the NURBS curve for the $\eta$ direction. Finally, the NURBS surface patch is a tensor product of univariate NURBS curves in $\xi$ and $\eta$ directions as shown in Eq. (3.6).

$$
S(\xi, \eta) = \sum_{i=0}^{n} \sum_{j=0}^{m} R_{i,j}(\xi, \eta) \ p_{i,j},
$$

(3.6)

Where,

$$
R_{i,j}(\xi, \eta) = \frac{N_{i,j}(\xi)N_{j,k}(\eta) w_{i,j}}{\sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,j}(\xi) N_{j,k} w_{i,j}}
$$

In Eq. (3.6) $p_{i,j}$ represents points defining the control net as shown in Fig. 3.1. Similarly, $w_{i,j}$ are the weights associated with those control points. The order and number of control points in $\eta$ direction is defined by $l$ and $m+1$ respectively. The number and location of control points depend on the shape of the plate. As shown in Fig. 3.1, the quadrilateral planar surface is defined by four edges. Determining the location of control points along the edges of plate poses two possible scenarios. On one hand, there can be geometry for which the control points are known or the geometry is of a regular conic section like rectangle or circle and on the other hand the geometry is known by its prescribed coordinates. In the second case, it is required to obtain the control points which represent the geometry appropriately. To avoid a non-linear situation, a NURBS curve of predefined
order and weights is selected and fitted through a set of known curve points [38]. For instance, let \( n \) be the number of curve points defined along the curve, then the parametric value corresponding to the curve point is obtained using the chord length between the points. Using these parametric values, the blending function matrix is obtained for \( n \) control points. This can be shown as in Eq. (3.7),

\[
\begin{align*}
C_1(\xi_1) &= N_1(\xi_1)P_1 + N_1(\xi_1)P_2 + \cdots + N_n(\xi_1)P_n \\
C_2(\xi_2) &= N_1(\xi_2)P_1 + N_2(\xi_2)P_2 + \cdots + N_n(\xi_2)P_n \\
C_3(\xi_3) &= N_1(\xi_3)P_1 + N_2(\xi_3)P_2 + \cdots + N_n(\xi_3)P_n \\
&\vdots \\
C_p(\xi_p) &= N_1(\xi_p)P_1 + N_p(\xi_p)P_2 + \cdots + N_n(\xi_p)P_n
\end{align*}
\]  

(3.7)

In the above equation \( C \) represent curve points and \( P \) control points. If there are equal number of control points and curve points \( (n=p) \), then the blending function matrix will be square. In this case the control points are obtained directly by the inverting the blending function matrix \( (M) \), and the resulting curve passes through all the curve points.

\[ \{P\} = [M]^{-1}\{C\} \]  

(3.8)

If we choose to fit a curve with less number of control points than the number of curve points, the NURBS curve is then an approximate fit through the curve points and the order of the blending function matrix will be then ‘\( p \times n \)’. The system of equation can be solved as shown in Eq. (3.9).

\[ \{P\} = [[M]^T[M]]^{-1}[M]^T\{C\} \]  

(3.9)

Once the control points of the four outer edges are obtained the plate control net is developed by interpolating the intermediate points in the same way as that of outer edges. Having defined the control net, the surface is developed by using the Eq. (3.6). Each of the NURBS patch possesses \( C^{k-2} \) continuity, where \( k \) being the order of the curve. However, a \( C^0 \) continuity is maintained between patches. Three types of refinements can be accomplished using the present method, the patches are formulated in such a way that both \( p \)-type and \( h \)-type analysis can be performed without modifications in the program. In
addition to that, the NURBS patches allows us to increase the order \((k)\) thereby attaining a higher continuity inside each patch. Remarkably this type of refinement does not increase the total degrees of freedom of the system and computational effort.

### 3.3 Displacement Fields and Equation of Motion.

In this section, the response field functions for a first order shear deformable plate is introduced. The plate under considerations has uniform thickness ‘\(h\)’ and is assumed to undergo deformation at an arbitrary point \(u'=u+z\beta_1, \ v'=v+z\beta_2\) and \(w'=w\), where \(u, v\) and \(w\) represent the mid-plane displacements along the \(x, y\) and \(z\) coordinates and \(\beta_1, \beta_2\) are the components of rotation of the normal to mid-plane. Equation (3.6) is applied to the displacement and rotation components with unknown control coefficients. For example, \(u\) is represented by

\[
\begin{align*}
  u(\xi, \eta) &= \sum_{i=0}^r \sum_{j=0}^s R_{i,j}(\xi, \eta) U_{i,j} \\
  \text{The number of unknowns for a patch depends on the control coefficients chosen along } \xi \text{ and } \eta \text{ directions. For example, } U_{i,j} \text{ can be written to be } U_q, \text{ where } q = (r+1)(s+1). \text{ The remaining four displacement and rotation components can be expressed similarly. Therefore, the mechanical displacement component vector } \{\Delta\}^T = \{u \ v \ w \ \beta_1 \ \beta_2\} \text{can be written in terms of blending functions as,}
\end{align*}
\]

\[
\{\Delta\} = [\overline{R}(\xi, \eta)] \{\Gamma\} \tag{3.11}
\]

In which, \([\overline{R}(\xi, \eta)]_{5 \times 5} = \begin{bmatrix} R_1(\xi, \eta) & R_2(\xi, \eta) & R_3(\xi, \eta) & - & R_q(\xi, \eta) \end{bmatrix}\) and

\[
\{\Gamma\}^T = \begin{bmatrix} U_1 & V_1 & W_1 & B_{11} & B_{21} & - & - & U_q & V_q & W_q & B_{1q} & B_{2q} \end{bmatrix}
\]

Having defined the displacement field functions in terms of blending functions \(R_i(\xi, \eta)\), it is applied in the energy expression Eq. (3.14). Variation of the potential function \(\phi\) is not linear along the thickness. The above non-linearity is modeled by a layer-wise linear approach. This is achieved by discretizing the material along the thickness direction in to
sub-layers and approximating the electric field to vary linearly in each sub-layer. Convergence of electric potential distribution could be obtained by increasing the number of layers. Let L be the number of sub-layers, the electric potential for the \( k \)-th sub-layer can be expressed as a linear variation of potential function over the thickness.

\[
\phi^{(k)} = \{g_k \} \{\phi_{k} \ \phi_{k+1} \}^T, \quad \text{where,} \quad g_k = (z_{k+1} - z_k)^{-1} [(z_{k+1} - z) (z - z_k)]
\]

Similar to the displacement functions the electric potential can be written as \( \phi^{(i)}(\xi, \eta) = [\bar{R}(\xi, \eta)]\{\phi\}' \) for each parallel plane to the plate. To understand it, the displacement fields for each component of \( \{\Delta\} \) are defined only the mid-plane of the plate, whereas the potential fields are defined for \( l+1 \) layers, \( l \) being the number layers. Now that both displacement functions and electrical field functions are defined using NURBS functions, they are applied in the strain displacement and electric field equations. The linearized piezoelectric constitutive equations are given in Eq. (3.12), considering an infinitesimal deformation and small electric fields [18].

\[
\{\sigma\} = [C] \{\epsilon'\} - [e]^T \{E\} \tag{3.12}
\]

\[
\{D\} = [e]\{\epsilon'\} + [\varepsilon]\{E\}
\]

In the above equation, the parameters \( \{\sigma\}, \ [C], \ \{\epsilon'\}, \ [e] \) and \( \{E\} \) represents the stress, elasticity, strain, piezoelectric strain constant and electric field matrices. \( \{D\} \) and \( \{\varepsilon\} \) are the electric displacement and dielectric permittivity matrices. The electric potential \( \phi \) is obtained from electric field by the following relationship,

\[
E_x = -\frac{\partial \phi}{\partial x}, \quad E_y = -\frac{\partial \phi}{\partial y} \quad \text{and} \quad E_z = -\frac{\partial \phi}{\partial z}
\]

Since, the electrical field is applied only in z direction, it is convenient to assume zero electric fields in x and y directions. Finally the electric field is related to the voltage and piezoelectric plate thickness \( (t_i) \) as, \( E_z = V / t_i \). The dynamics of piezoelectric plate can be analyzed as a boundary value problem by solving the coupled equation Eq. (3.12) and
Maxwell’s equation Eq. (3.13) To derive the equation of motion for the laminated piezoelectric plate, we used Hamilton’s variational principle [10], accordingly the variation of total potential energy $\Pi$ is,

$$
\int_{t_1}^{t_2} \delta \left[ \Pi \right] dt = \int_{t_1}^{t_2} (T - U - W) dt = 0
$$

(3.14)

In the above equation $T$ represents kinetic energy, $U$ is the strain energy and $W$ is the work done by applied electrical and mechanical loads. The method of obtaining the governing equations in matrix form is documented in numerous times in literature [20]. Without going into the detailed derivation, the following two coupled equations of motion can be obtained.

$$
[M]\{\ddot{\Gamma}(t)\} + [C]\{\dot{\Gamma}(t)\} + [K_m]\{\Gamma(t)\} + [K_{me}]\{\varphi(t)\} = \{F(t)\}
$$

(3.15)

$$
[K_{me}]\{\Gamma(t)\} + [K_e]\{\varphi(t)\} = \{Q(t)\}
$$

(3.16)

In the above equation, $[K_m]$, $[K_{me}]$, $[C]$, $[M]$ and $[K_e]$ represents the mechanical stiffness, electro-mechanical coupling, Rayleigh’s proportional damping, mass and the dielectric matrices respectively. Similarly vectors $\{\dot{\Gamma}(t)\}$, $\{\varphi(t)\}$, $\{F(t)\}$ and $\{Q(t)\}$ correspond to the mechanical degrees of freedom, electrical degrees of freedom, mechanical consistent load and electrical load respectively. The above Eq. (3.15) and (3.16) can be combined together and the equation of motion of the system in terms of mechanical degrees of freedom can be written as,

$$
[M]\{\ddot{\Gamma}(t)\} + [C]\{\dot{\Gamma}(t)\} + ([K_m] - [K_{me}] [K_e]^{-1} [K_m])\{\Gamma(t)\} = \\
\{F(t)\} - [K_{me}] [K_e]^{-1} \{Q(t)\}
$$

(3.17)

The right hand side contains both mechanical and electrical excitations and can be set to zero for the free vibration analysis to evaluate the natural frequencies and associated mode shapes. Also, this equation has been used to examine the vibrational behaviors of sensors and actuators.
3.4 Numerical Results.

The present method is validated by comparing the results with the data available in literature. Two numerical investigations are performed in this section. Firstly the static displacements of a circular unimorph actuator is studied. It is followed by the free vibration analysis of laminated circular and elliptic sandwich plates. The present work is also extended to derive equations for piezoelectric power harvester for which the case studies are made with straight and curved edges, demonstrating the versatility of the present method.

3.4.1 Static analysis on circular unimorph actuator:

A circular unimorph plate essentially contains a piezoelectric plate bonded to a host structure and is subject to bending. In the present analysis a thin circular unimorph plate is analyzed in actuator mode, subjecting it to both electric field and a point load. Dong et al. [11] presented analytical closed form solution for the transverse displacement of circular unimorph actuator. The piezoelectric material is chosen as lead zirconate titanate (PZT-4) and the host structure is steel with Young’s modulus 210Gpa and Poisson’s ratio 0.313. The plate dimensions are \( t_p = 0.4\text{mm} \) and \( t_h = 0.2\text{mm} \). Where, \( t_p \) and \( t_h \) are the thickness of piezoelectric and host material respectively. The numerical tests are performed for simply supported plate with different patch configurations and loading conditions. In all of the analyses the number of sub-layers for a piezoelectric layer is kept at eight and a cubic NURBS function is used. For displacement functions, there are 36 control points on each NURBS patch. Similarly, a patch contains 16 control points to define the geometry. As a case study, downward point load of 1N is applied at the center. Also, an electric field of 250V/mm is applied simultaneously to the top and bottom of the unimorph. The electrical load is very high in magnitude compared to the point load and hence the plate bends in the positive z direction. The central deflection is presented in Table 3.1. It is observed that results from the present method agree well with the analytical solutions for thin circular plates. Apparently, as the plate gets thicker the percentage difference increases, e.g. the difference is 5.52 percent for, \( a/h = 10 \). The analytical solution tends to overestimate the
deformation for thicker plates. Nevertheless the central deflections agree very well when the plates are thin. This discrepancy may be due to their assumptions like linear variation of the electric potential along the thickness assumed in the analytical solution [11] which is based on the thin plate theory.

Table 3.1. Central displacement of a simply supported circular unimorph plate under an applied electric potential of 150V and point load of 1N at its center.

<table>
<thead>
<tr>
<th>Patch</th>
<th>Central displacement (μm)</th>
<th>a/h = 100</th>
<th>a/h = 75</th>
<th>a/h = 50</th>
<th>a/h = 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1×1</td>
<td></td>
<td>4.3430</td>
<td>2.7168</td>
<td>1.0858</td>
<td>0.0317</td>
</tr>
<tr>
<td>2×2</td>
<td></td>
<td>5.5871</td>
<td>3.1315</td>
<td>1.3568</td>
<td>0.0403</td>
</tr>
<tr>
<td>3×3</td>
<td></td>
<td>5.6205</td>
<td>3.1498</td>
<td>1.3807</td>
<td>0.0527</td>
</tr>
<tr>
<td>4×4</td>
<td></td>
<td>5.6214</td>
<td>3.1509</td>
<td>1.3810</td>
<td>0.0530</td>
</tr>
<tr>
<td>5×5</td>
<td></td>
<td>5.6220</td>
<td>3.1511</td>
<td>1.3811</td>
<td>0.0531</td>
</tr>
<tr>
<td>Ref [11]</td>
<td></td>
<td>5.6244</td>
<td>3.1637</td>
<td>1.4061</td>
<td>0.0562</td>
</tr>
<tr>
<td>Percentage difference</td>
<td></td>
<td>0.0426</td>
<td>0.4000</td>
<td>1.7800</td>
<td>5.5200</td>
</tr>
</tbody>
</table>
3.4.2 Free vibration of laminated circular and elliptic sandwich plates:

To test the present formulation further, the free vibration analysis is performed for elliptical plates with various a/b ratios. The results are validated for circular plate (a/b = 1.0) under clamped and simply supported boundary conditions. The geometric shape of the plate is varied from being circle to an ellipse by changing a/b ratio. The cross section of the plate is shown in fig 3.2, the thickness of top and bottom piezoelectric layers is \( h_1 \) and \( h_2 \). The piezoelectric layer is of PZT4, the host structure is steel and the radius is 600mm [10]. A 3×3 NURBS patch containing 36 control points as mentioned above for the static analysis is used in the analysis. The first six natural frequencies are obtained for clamped circular plate and compared with Wang et al. [10] as presented in Table 3.2. The frequencies tend to agree well at lower modes and the maximum percentage difference observed is 0.75 for the sixth mode. The first six fundamental frequencies are also presented as new results in Table 2 for three elliptic plates having a/b = 1.5, 2 and 3. Numerical calculations are also repeated for the case of the simply supported circular (a/b = 1) and elliptic (a/b = 2) plates and the results are presented in Table 3.3 for four thickness ratios \((h_1 + h_3)/h_2\) of values 1/12, 1/10, 1/8, and 1/5 respectively. For this case as well, the present results are in close agreement with those of Wang et al. [29].

![Cross section of piezoelectric sandwich plate.](image)
Table 3.2 Natural frequencies of clamped piezoelectric circular and elliptic plates.

<table>
<thead>
<tr>
<th>Mode number</th>
<th>a/b = 1</th>
<th>a/b=1.5</th>
<th>a/b=2</th>
<th>a/b=3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
<td>Reference [29]</td>
<td>Difference (%)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>902.51</td>
<td>902.47</td>
<td>0.004</td>
<td>680.06</td>
</tr>
<tr>
<td>2</td>
<td>1872.38</td>
<td>1878.17</td>
<td>0.308</td>
<td>1121.53</td>
</tr>
<tr>
<td>3</td>
<td>1873.55</td>
<td>-</td>
<td>-</td>
<td>1645.34</td>
</tr>
<tr>
<td>4</td>
<td>3063.13</td>
<td>3081.08</td>
<td>0.582</td>
<td>1747.05</td>
</tr>
<tr>
<td>5</td>
<td>3066.32</td>
<td>-</td>
<td>-</td>
<td>2238.11</td>
</tr>
<tr>
<td>6</td>
<td>3487.05</td>
<td>3513.43</td>
<td>0.750</td>
<td>2550.14</td>
</tr>
</tbody>
</table>
Table 3.3 Natural frequencies of piezoelectric elliptic plate with simply supported edges

<table>
<thead>
<tr>
<th>Mode</th>
<th>Thickness ratio ((h_1+h_3)/h_2) and (a/b = 1)</th>
<th>1/12</th>
<th>1/10</th>
<th>1/8</th>
<th>1/5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
<td>Ref [29]</td>
<td>Present</td>
<td>Ref [29]</td>
<td>Present</td>
</tr>
<tr>
<td>1</td>
<td>432.53</td>
<td>432.19</td>
<td>434.74</td>
<td>435.63</td>
<td>438.20</td>
</tr>
<tr>
<td>2</td>
<td>1197.81</td>
<td>-</td>
<td>1203.40</td>
<td>-</td>
<td>1212.22</td>
</tr>
<tr>
<td>3</td>
<td>1202.35</td>
<td>1217.69</td>
<td>1207.94</td>
<td>1227.49</td>
<td>1216.78</td>
</tr>
<tr>
<td>4</td>
<td>2203.66</td>
<td>-</td>
<td>2113.49</td>
<td>-</td>
<td>2228.73</td>
</tr>
<tr>
<td>5</td>
<td>2213.55</td>
<td>2244.33</td>
<td>2223.62</td>
<td>2262.44</td>
<td>2239.56</td>
</tr>
<tr>
<td>6</td>
<td>2552.98</td>
<td>2604.22</td>
<td>2564.47</td>
<td>2625.23</td>
<td>2582.67</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode</th>
<th>Thickness ratio ((h_1+h_3)/h_2) and (a/b = 2)</th>
<th>1/12</th>
<th>1/10</th>
<th>1/8</th>
<th>1/5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
<td>Ref [29]</td>
<td>Present</td>
<td>Ref [29]</td>
<td>Present</td>
</tr>
<tr>
<td>1</td>
<td>288.63</td>
<td>290.02</td>
<td>292.20</td>
<td>299.47</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>509.47</td>
<td>511.86</td>
<td>515.63</td>
<td>528.22</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>823.60</td>
<td>827.41</td>
<td>833.44</td>
<td>853.57</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1001.70</td>
<td>1006.27</td>
<td>1013.50</td>
<td>1037.67</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1234.60</td>
<td>1240.26</td>
<td>1249.21</td>
<td>1279.13</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1347.22</td>
<td>1353.29</td>
<td>1362.90</td>
<td>1395.08</td>
<td></td>
</tr>
</tbody>
</table>
3.5 Vibration energy harvesting

Equations (3.15) and (3.16) are extended in this section from the perspective of their application for the vibration energy harvesters (VEH) fed by scalar transverse base excitation $a_s(t)$. The mechanical force vector in Eq. (3.15) takes the form $\{F(t)\} = [M]\{\Lambda\}a_s(t)$, where vector $\{\Lambda\}$ is used to obtain the mass contribution in the $z$ direction and $a_s(t)$ is the base acceleration. Performing the response analysis on the matrix equations of motion by a direct integration method can be a cumbersome process. Therefore, the normal mode summation technique is used to reduce the computational effort. Vibration energy harvesters during its operation depend on the strain developed in the material and the strain produced therein is proportional to the electrical charge developed. For the maximum power output, the charges produced should be of same polarity necessitating that the deformed shape of the plate should not have a negative curvature. The first mode shape of a cantilever plate satisfies this requirement. Now, when a cantilevered plate is operated closer to its first natural frequency, the contributions from the higher modes are negligible. Thus instead of performing a full time integration simulation for the electro mechanical equations, the modal approach seems to be more convenient. The displacement vector can be written as $\{\Gamma(x, y, t)\} = [Z]\{q(t)\}$ in which the modal matrix $[Z]$ is composed of the eigenvectors in the columns normalized by the mass matrix and its order depends on the number of natural modes considered in the analysis. By substituting this displacement form into Eq. (3.15) and (3.16) and pre multiplying with $[Z]^T$ the following equations are obtained after the application of the orthogonal properties of the eigenvectors.

$$\begin{align*}
[\dot{q}(t)] + [2\xi_\omega][\dot{q}(t)] + [\omega^2] [q(t)] + [Z]^T[K_{me}][\varphi(t)] &= [Z]^T[F(t)] \\
[K_{em}][Z][q(t)] + [K_e][\varphi(t)] &= [Q(t)]
\end{align*}$$

(3.18) (3.19)
By differentiating Eq. (3.19) with respect to time, writing \( \{ \dot{Q}(t) \} = [R]^{-1} \{ \varphi(t) \} \) wherein \([R]\) is a diagonal matrix with the equivalent load resistance \( R_L \) as diagonals and taking the Laplace transform, the following is obtained.

\[
\{ \varphi(s) \} = -s[\overline{K}_e]^{-1}[K_{em}][Z]\{Q(s)\}
\]  \hspace{1cm} (3.20)

Taking Laplace transform of Eq. (3.18) and using Eq. (3.20),

\[
(s^2[I] + s[K_D] + [\omega]^2)\{Q(s)\} = [Z]^T\{F(s)\}
\]  \hspace{1cm} (3.21)

where \([K_D] = [2\xi\omega] - [Z]^T[K_{me}][\overline{K}_e]^{-1}[K_{em}][Z]\) and \([\overline{K}_e] = s[K_e] - [R]^{-1}\). Now, the solution of Eq. (3.21) provides the displacements and then Eq. (3.20) is used to get the voltage. Finally, the power is obtained from \( P(t) = V^2(t)/R \).

### 3.5.1 Sandwich plate power harvesters

Two examples of sandwich cantilever plates are presented in this section based on their geometries. The first example is a regular rectangular piezoelectric sandwiched plate, while the second a piezoelectric plate having curved edges. In both cases, the plates are subjected to a base acceleration of 3 m/s\(^2\) purely in the z direction. The geometrical configuration of the plate consists of two layers of PZT-5H bonded to a silicon substrate. All three layers are assumed to have the same thickness. The piezoelectric layers are polarized in the positive z direction resulting in a parallel configuration. This makes the power harvester to operate in 3-1 mode and the power obtained is dictated by the electromechanical coefficient \( d_{31} \). The fundamental mode corresponds to the bending mode in a cantilevered plate. Dimensions for the rectangular plate are: \( l = 100mm, b = 20mm \) and total thickness \( h = 0.75mm \). A 3×3 cubic NURBS patch each containing 36 control points is used for the displacement field functions. The geometry is also defined using the same 3×3 patch configuration. However, only four control points per patch is necessary to define the geometry of the rectangular plate. Geometry for the sandwich plate with curved edges as shown in Fig 3 is sketched in a commercial CAD environment package (Rhino) from
which the control points are directly exported. The control points for the edge 1 is given in appendix A.2. Edge 3 is the mirror image of edge 1. Edges 2 and 4 are straight lines. Other dimensions for this case are: \( l=100\text{mm}, h = 0.75\text{mm}, b = 20\text{mm} \) and \( c = 10\text{mm} \).

![diagram](image)

**Fig. 3.3** sandwiched piezoelectric plate with curved edges.

The displacement at the free end of the plate against the dimensionless frequency \( \Omega \) are presented in Fig. 3.4(a) and 3.4(b) respectively for the rectangular and curved edge cantilever sandwich plates. It is evident from Eq. (3.21) that the equivalent impedance \( K_D \) connected to the piezoelectric plate causes a damping effect. It should be noted here that \( K_D \) contains Rayleigh’s damping and the damping effect of the piezoelectric current due to externally connected resistance and load. This plays an important role in the operation and design of the harvester. The optimum power output not only depends on the shape and configuration of the harvester, but also on the externally connected circuitry and load resistance. As the load resistance increases, the fundamental frequency is increased, e.g. the fundamental frequency of the rectangular plate is 35.19 Hz without any load resistance and 37.45 Hz with 42000 ohm. This increase in the fundamental frequency is due to the
coupled piezoelectric effect produced by the load resistance. This effect essentially causes the plate’s stiffness to increase and the free end displacement to reduce.

Fig. 3.4 Maximum displacement for an applied base acceleration of 3m/s², (a) Straight edge (b) curved edge
Fig. 3.5 Electrical output voltage as a function of non-dimensional frequency. (a) Straight edge (b) curved edge
Fig. 3.6 Electrical power output versus non-dimensional frequency (a) Rectangular plate (b) Curved edge
Fig. 3.7 Variation of power versus load resistance.

Fig. 3.8 Power versus proof mass
Fig. 3.5(a) and 3.5(b) depicts the frequency response of output voltage as the resistance is increased. In both examples, voltage seems to be increasing as the load resistance is increased. The difference in maximum power between the two examples are significant because of the difference in their fundamental frequencies. Similarly, the frequency response of the output power for the two cases are given in Fig. 3.6(a) and 3.6(b) where it is seen that the power output increases with the resistance as well. Further investigation on the effect of load resistance on the output power is carried out and the results are shown in Fig. 3.7. Initially the power output increases up to an optimum level with the load resistance and then slowly decreases. The maximum power is attained at 20100 and 12520 ohm for the rectangular and curved edge plates. The frequency is fixed at 35.19Hz for rectangular plate and 39.22Hz for the curved edge in the above analysis.

The piezoelectric power harvesters are often required to harvest energy from vibrations under ambient conditions at which the frequencies at are very low in most cases. On the other hand, the piezoelectric materials are mostly ceramics and the dimensions are very small making them to have very high fundamental frequencies. This makes the piezoelectric harvesting devices to produce very low power. The problem can be substantially reduced by adding a proof mass at the tip. The attached proof mass will be usually of a material with high density. Since the added mass to the system reduces the fundamental frequency, a feasible condition to operate at near resonance can be conveniently created. Fig. 3.8 shows the variation of power with regards to the proof mass for the rectangular plate. Simulation is performed, following the lumped mass approach for various mass percentages from 10 to 30 percent. An equivalent load resistance of 20100 ohm is assumed and the mass and rotary inertia about z direction are lumped at the appropriate control points. From the above numerical simulations, few conclusions could be drawn. The efficiency of the power harvester depends on the mass, shape, magnitude and frequency of the base acceleration, material property, and externally connected circuitry. Obtaining an optimized power harvester involves designing one involving all these parameters. The general formulation developed in the present work can be applied to a generalized plate models.
3.6 Conclusion

A NURBS based numerical model is proposed in this study for the analysis of laminated piezoelectric plates of arbitrary shape. The main advantage of the present technique is that it enables to model complex geometric shapes accurately with fewer degrees of freedom. The formulation is based on first order shear deformable plate theory in which the in-plane displacement components vary linearly over the thickness. The same cannot be assumed for the variation of the electrical potential along the thickness. Therefore, a layer-wise linear variation of electric potential is considered in order to retain the overall nonlinear variation in the thickness direction. The numerical simulation results are successfully corroborated by comparing with the static analysis of a uniform circular actuator [10]. Also, the free vibration analysis was performed for circular and elliptic laminated piezoelectric plates and the results are compared. Finally, numerical studies are carried out for rectangular and curved edged piezoelectric power harvesters. The influence of load resistance in the output power and free end displacements are analyzed and presented. The present approach is efficient and expected to help in the design of piezoelectric power harvesting devices.
References


CHAPTER 4

A three dimensional formulation and analysis of piezoelectric composites using NURBS

4.1 Introduction

Piezoelectric materials have received attention due to their potential use as sensors and actuators in many sensing and vibration control systems [1]. The fundamental constitutive equations for piezoelectric devices has been developed and reviewed by many authors [2-5]. Researchers developed various theories based on, beam, plate, shell and three dimensional models. Analytical models for piezoelectric actuators and sensors have been developed by several researchers [6-8]. The electro-mechanical coupling in piezoelectric materials is often difficult to model, though researchers were able to model the phenomena using analytical techniques for certain boundary and loading conditions. The coupled electro mechanical equations are reasonably complex to obtain closed form solution in most cases. Many researchers have used the finite element method as a viable modelling technique. Allik and Hughes [9] implemented a finite element formulation for electro elasticity and developed tetrahedral and hexahedron element for vibration analysis. Hwang and Park [10] analyzed vibration control of laminated plates with integrated piezoelectric sensors using finite element method, they used a quadrilateral plate element with 12 degrees of freedom. Piezoelectric shell theory was developed by many researchers, including; Tzou and Garde [11] analyzed electro-elastic plates analytically using Kirchhoff-Love assumptions, and although their results are acceptable for thin plates, they diverge for
moderately thick sandwich plates. Two-dimensional plate theory involving a layer-wise approximation of electric potential produced a viable modelling technique for moderately thick plates. Saravanos [12] developed the mechanics of mixed laminate theory for composite shell structures, representing a layer-wise distribution of electric potential and he reported numerical results for cylindrical laminated piezoelectric panels. Heyliger et al [13] used finite element method to develop a discrete-layer shell theory, which is applicable to general shells of revolution. A number of plate and shell theories have been formulated based on order of expansion and variation of electric potential along the thickness. As inferred from the literature, piezoelectric materials were generally used as sensors, actuators and as power harvesters. In majority of these applications the thickness of the piezoelectric devices were relatively small compared to other dimensions. Accordingly, researchers used a two-dimensional plate or shell model with layer-wise mechanics and higher order polynomial approximation for electric potential in thickness direction [13-15], these models are quite adequate for the purpose they serve. Having said that, as the plate gets thicker, the two dimensional models are not quite efficient. Also in all of the above models the electric potential variation is subjected to some degree of approximation. These limitations in two dimensional modelling warrants the development of a three dimensional modelling technique. Ghandi and Hagood [16] developed a 8 node element, having three mechanical degrees of freedom and one electrical degree of freedom, to model phase transition of electro-mechanical material. Koko et al. [17] used a 20 node thermo-piezoelectric element for modelling smart composite structures. They reported a comprehensive analysis on controlling the vibrations of a composite structure using piezoelectric materials. Varadan et al [18] performed transient response analysis on MEMS scale piezoelectric sensors, they used a combination of 20 node solid element, 13 node transition element and a 9 node shell element to model a cantilever plate with embedded piezoelectric sensor. Even though, three dimensional modelling of piezoelectric materials proves to be an efficient method, it has some limitations; the associated difficulty includes locking effects and increase in degrees of freedom. The shear locking effect is encountered while modelling relatively thin plates. Braess and Kaltenbacher [19] used selective integration technique in thickness direction to alleviate locking problem, while using a three
dimensional solid element. It is evident from the above brief literature survey that a much needed attention has to be paid towards modelling piezoelectric crystals and thick plates.

The present study aims to propose an efficient and reliable numerical technique based on the non-uniform rational B-splines are capable of modelling complex geometrical shapes using fewer control points. The NURBS patch also has higher order continuity within each patch, compared to traditional finite elements. This translates into fewer number of degrees of freedom while pursuing a full three dimensional solid patch formulation. Another advantage of using a three dimensional model is its inherent ability to model the electromechanical coupling effectively. This flexibility in curve generation is exploited further to define the geometry as well as displacement fields in the variational method for three dimensional electro-elastic problems. The present method is successfully tested against the commercially available finite element code ANSYS. Static analysis performed on cantilevered trapezoidal sandwich piezoelectric plate reveals discrepancies of using a first order shear deformable theory for considerably thick sandwich plates. Finally a transient response analysis is done on a cantilevered sandwich piezoelectric thick sandwich plate by decoupling the equation of motion in space and time. Runge-Kutta fourth order technique is used to solve the state-space equations.

4.2 Constitutive equation

A piezoelectric material is described by arrays of material coefficients which relate stress to strain and their electrical counterparts. The electromechanical constitutive equations for a piezoelectric material [9], involving mechanical stresses and electric fields can be written as,

$$\{\sigma'\} = [C]\{\varepsilon'\} - [e]^T\{E'\} \tag{4.1}$$

$$\{D'\} = [e]\{\varepsilon'\} + [\varepsilon]\{E'\} \tag{4.2}$$

Where \{\sigma'\} is the stress tensor, \([C]\) is the elasticity matrix, \{\varepsilon'\} is the strain tensor. \{e\}, \{D\}, \{\varepsilon\}, and \{E'\} represents piezoelectric coupling matrix, dielectric displacement tensor,
dielectric matrix and electric field. The electric field \( E' \) is related to electric potential \( \phi \) through Maxwell’s equation \( E' = -\{\nabla\}\phi' \).

### 4.3 NURBS Definitions of Geometry and Displacement – Electrical Fields

The mathematical description of a NURBS curve involves a set of basis functions, order and control points/coefficients. For example, a curve \( C(\zeta) \) in parametric space \( \zeta \) can be defined by \( n+1 \) control points \( P_i \) as given below [20].

\[
C(\zeta) = \sum_{i=0}^{n} R_{i,k} (\zeta) \ p_i
\]  

(4.3)

The function \( R_{i,k} (\zeta) = \frac{N_{i,k} (\zeta) w_i}{\sum_{i=0}^{n} N_{i,k} (\zeta) w_i} \) and \( N_{i,k} (\zeta) \) blends the curve under consideration through the set of given control points \( P_i \). Also required is a set of non-decreasing parametric coordinates \( \zeta_i \) is the knot vector which defines the limits of the NURBS curve. An open uniform knot vector is constructed through the following equation.

\[
\xi_i = \begin{cases} 
0, & 0 \leq i < k + 1 \\
i - k + 1, & k + 1 \leq i \leq n + 1 \\
n - k + 2, & n + 1 < i \leq n + k + 1 
\end{cases}
\]  

(4.4)

Depending upon the values of \( n \) and \( k \), the open uniform knot vector appears as follows.

\[
\xi_i = \left\{ \underbrace{0, \ldots, 0}_{k+1}, \underbrace{\xi_{k+1}, \ldots, \xi_n}_{k+1}, \underbrace{1, \ldots, 1}_{k+1} \right\}
\]  

(4.5)

A parametric NURBS curve can be differentiated \( k-1 \) times continuously, provided the knot vector is without multiplicity. The derivatives of a NURBS curve, therefore, can be explicitly obtained by formal differentiation.
Where, \( N_{i,k}'(\xi) = \frac{N_{i,k-1}(\xi) + (\xi - \xi_i)N_{i,k-1}'(\xi)}{\xi_{i+k-1} - \xi_i} + \frac{(\xi_{i+k} - \xi)N_{i+1,k-1}'(\xi) - N_{i+1,k-1}(\xi)}{\xi_{i+k} - \xi_{i+1}} \)

In the above equation, derivatives of a \( k \)th order blending function depend recursively on the \((k-1)\)th order blending functions. Once the unidirectional NURBS curves are defined in \( x, y, z \) directions, a NURBS solid can be obtained by the tensor product of the parametric curves in three directions. The order of the curve and control points in each direction can be different. The number of control points and order can be changed through knot insertion/removal techniques [20].

\[
G(\xi, \eta, \zeta) = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{q=0}^{l} R_{i,j,q}(\xi, \eta, \zeta) \, p_{i,j,q}
\]

Where, \( R_{i,j,q}(\xi, \eta, \zeta) = \frac{N_{i,j,\xi}(\xi)N_{j,k,\eta}(\eta)N_{q,k,\zeta}(\zeta)w_{i,j,q}}{\sum_{r=0}^{n} \sum_{j=0}^{m} \sum_{q=0}^{l} N_{r,j,\xi}(\xi)N_{j,k,\eta}(\eta)N_{q,k,\zeta}(\zeta)w_{r,j,q}} \)

\( P_{i,j,q} \) represents the control net comprising the control points of the NURBS solid and \( w_{i,j,q} \) are the weights associated with the control points, each of which is defined by its \( x, y, z \) coordinates. The geometry is first defined and then one has to deal with the structural response of the piezoelectric continuum. For this three displacement components and one charge are taken as the degrees of freedom at a point in the continuum. Therefore, there are four degrees of freedom at a control point, namely mechanical displacements \( u, v, w \) in \( x, y, z \) direction respectively and electric potential \( \phi \). These degrees of freedom are also represented by NURBS in terms of blending functions and associated control coefficients as,
\[
\begin{align*}
    u(\xi, \eta, \zeta) &= \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{q=0}^{l} R_{i,j,q}(\xi, \eta, \zeta) U_{i,j,q} \\
    v(\xi, \eta, \zeta) &= \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{q=0}^{l} R_{i,j,q}(\xi, \eta, \zeta) V_{i,j,q} \\
    w(\xi, \eta, \zeta) &= \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{q=0}^{l} R_{i,j,q}(\xi, \eta, \zeta) W_{i,j,q} \\
    \phi(\xi, \eta, \zeta) &= \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{q=0}^{l} R_{i,j,q}(\xi, \eta, \zeta) \phi_{i,j,q}
\end{align*}
\]

(4.8)

In this case, the number and location of control points in the continuum are different from those taken to define the geometry. Once the field functions are defined they are used in the variational formulation through strain displacement relations.

### 4.4 Equations of motion

The coupled equations of motion for piezoelectric material is obtained by minimizing the following Hamilton’s energy functional \( II \).

\[
\int_{t_1}^{t_2} \delta \Pi \, dt = \delta \int_{t_1}^{t_2} (T - U - W) \, dt = 0
\]

(4.9)

The strain energy \( (U) \), kinetic energy \( (T) \) and the work done by the externally applied electrical and mechanical loads \( (W) \) are expressed in terms of the above described displacement and electrical field functions. The procedure to obtain the equation of motion from the energy functional is well documented in the literature [9] and hence it is only described below.

The total kinetic energy, potential energy and the work done, including the effects of electromechanical coupling for the entire volume is in Eq. (4.10) are given below,

\[
T = \frac{1}{2} \int \rho \left[ \left( \frac{\partial u'}{\partial t} \right)^2 + \left( \frac{\partial v'}{\partial t} \right)^2 + \left( \frac{\partial w'}{\partial t} \right)^2 \right] \, dV
\]

\[
U = \frac{1}{2} \int \left( \{\varepsilon'\} \{\sigma\} - \{E\}' \{D\} \right) \, dV
\]

(4.10)
\[
W = \int_{A} \left( \{p\}^T \{\Delta\} - q(\varphi) \right) dx dy
\]

The strain displacement relations are used to write the energy functions in terms of displacement and electrical field functions. Gauss quadrature is used to perform the volume integration. As the NURBS functions are synthetic composite curves, the integral is calculated at each knot span. The number of control points in each solid module is kept arbitrary, so is the number of modules in the formulation. The stiffness and mass matrices are computed at the knot span level. When the number of control points are different in each direction of the patch, additional control points are obtained using knot insertion technique [21] and then assembled to form the global system matrices. The consistent load vector is obtained by work equivalence method. The coupled matrix equations of motion for a piezoelectric continuum is written below.

\[
[M]\{\ddot{\Gamma}(t)\} + [K_m]\{\Gamma(t)\} + [K_{me}]\{\phi(t)\} = \{F(t)\} \tag{4.11}
\]

\[
[K_{me}]\{\Gamma(t)\} + [K_e]\{\phi(t)\} = \{Q(t)\} \tag{4.12}
\]

In the above, \([M]\) is the consistent mass matrix, \([K_m]\) the mechanical stiffness matrix, \([K_{me}]\) the electromechanical coupling matrix, \([K_e]\) the dielectric matrix, \([Q]\) the electrical charge density, \([\Gamma]\) the displacement vector, \([F]\) the mechanical load and \(\phi\) the voltage vector. The present formulation allows for three types of refinement, viz. the number of solid modules, number of control points in a module and order of NURBS respectively. The two equations are combined to make one in terms of the mechanical degrees of freedom for the free vibration analysis.

\[
[M]\{\dddot{\Gamma}\} + ([K_m] - [K_{me}][K_e]^T[K_{me}])\{\Gamma\} = 0 \tag{4.13}
\]

The Rayleigh proportional damping matrix \([C]\) can be introduced in the matrix equation of motion to study the forced vibration of the system. The proportional damping matrix \([C]\) is obtained from the linear combination of stiffness and mass matrices.
Finally, the governing equation in terms of mechanical displacements for the forced vibration analysis of a piezoelectric structure is

$$[M][\dot{\mathbf{\Gamma}}] + [C][\mathbf{\Gamma}] + ([K_{\text{me}}] - [K_{\text{ac}}][K_{\text{ac}}]^{-1}[K_{\text{me}}])[\mathbf{\Gamma}] = \{F(t) - [K_{\text{ac}}][K_{\text{ac}}]^{-1}[Q(t)]\}$$  \hspace{0.5cm} (4.14)

The formulation concerning the piezoelectric three dimensional medium is evaluated and applied for the following numerical case studies for static and vibration analyses.

### 4.5 Numerical Case Studies

The three dimensional formulation developed in the previous sections is numerically investigated in this section with regards to its suitability, applicability and accuracy along with the correctness of the in-house developed computer codes. The basic NURBS module (or block) in this regard consists of a hexahedral solid contained by six plane quadrilateral faces, is defined by the x, y, z coordinates of the eight corner points. The block is mapped by the natural coordinates ($\xi, \eta, \zeta$) in three dimensions. To create the geometry, each edge is treated as a curve with ends as the control points. Blending functions of degree one, i.e. $k=1$, are used to express $(x, y, z)$ in $(\xi, \eta, \zeta)$ for an edge. Cubic blending NURBS curve with five control knots in each of the natural coordinates is used for displacement and potential field $u, v, w, \phi$. The interior control knots are generated through linear interpolation giving the block 5×5×5 control points. Each solid module contains 375 mechanical degrees of freedom and 125 electrical degrees of freedom. Computer codes are developed for static, free vibration and transient response analyses on various configurations of piezoelectric composites. Numerical results along with convergence studies are presented and discussed in the following considering various module configurations.
4.5.1 Static analysis of a trapezoidal cantilevered sandwich plate

A cantilevered tapered sandwich plate as shown in Fig. 4.1 is considered first for the static analysis. The plate thickness is assumed to be uniform with each layer of equal thickness. The other dimensions are: \(a = 15\text{cm}, b = 10\text{cm}, c = 5\text{cm}\). The sandwich configuration includes a top and bottom layers made of ZNO and the core material is silicon. The piezoelectric effect depends on the poling direction and the direction of the applied load. In this case the piezoelectric material is polarized in positive \(z\) direction. The properties of ZNO is given in the appendix A.1. A uniformly distributed load of 1kN/m\(^2\) is applied on the top surface. Numerical simulations are performed using both two dimensional and three dimensional NURBS models using patches and blocks for various aspect ratios and mesh configurations. The results are compared with two dimensional plate theory considering FSDT and a layer-wise linear variation approximation for electric potential. In both cases, the order of the NURBS functions are kept at four.

![Fig. 4.1 Trapezoidal plate configuration.](image)

The maximum free end displacements are provided in Table 4.1, for three mesh configurations and various aspect ratios. For plate model, a 3\times3 patch having 36 control points in each patch and six sub layers has 1280 mechanical degrees of freedom and 1792 electrical degrees of freedom. On the other hand for a three dimensional solid model with 3\times3\times3 configuration and 125 control points per module, there are 6591 mechanical and 2197 electrical degrees of freedom. Generally the displacements obtained using three dimensional formulation are found higher as compared to the plate theory. The differences
are quite apparent as the ratio $a/h$ decreases. For large $a/h$ ratios, both plate theory and full three dimensional model produced close results having the difference of less than 0.2 percent. As the plate thickness goes over $a/h=5$, the difference is close to eight percent.

Table 4.1 Maximum free end displacement of cantilevered trapezoidal sandwich plate for a uniformly distributed load of 1kN/m2

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>Plate (displacement in $\mu$m)</th>
<th>Solid (displacement in $\mu$m)</th>
<th>Difference in percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1x1</td>
<td>2x2</td>
<td>3x3</td>
</tr>
<tr>
<td>100</td>
<td>921.4</td>
<td>1127.2</td>
<td>1135.01</td>
</tr>
<tr>
<td>75</td>
<td>407.11</td>
<td>472.34</td>
<td>472.48</td>
</tr>
<tr>
<td>50</td>
<td>139.17</td>
<td>144.56</td>
<td>144.25</td>
</tr>
<tr>
<td>25</td>
<td>17.09</td>
<td>17.82</td>
<td>17.8</td>
</tr>
<tr>
<td>10</td>
<td>1.14</td>
<td>1.16</td>
<td>1.16</td>
</tr>
<tr>
<td>5</td>
<td>0.13</td>
<td>0.14</td>
<td>0.15</td>
</tr>
</tbody>
</table>

As mentioned earlier, the non-linear distribution of electric potential across the thickness is modelled by a layer-wise linear approximation in the first order shear deformable plates. This distribution is plotted along the piezoelectric layer as shown in Fig. 4.2 for very thick plate having $a/h=5$. Also added to this figure is the distribution of the electric potential from the 3D model. It is inferred from Fig 4.2 that the three dimensional solid model estimates higher electric potential across the thickness up to 10.3 percent. There is no approximation
involved in modelling the electric potential variation along the thickness in three dimensional model and it sustains the electro mechanical coupling effectively.

![Graph](image)

**Fig. 4.2** Variation of electric potential across the thickness

### 4.5.2 Free Vibration Analysis of Rectangular Cantilevered Sandwich Plates.

The piezoelectric devices are often used as vibration energy harvesters which operate close to their fundamental frequencies for optimum power generation. Therefore, the free vibration analysis is important to understand the system’s characteristics. In such cases, a vibration problem cannot be treated purely as mechanical, since the electromechanical coupling tends to lower the stiffness of the structure, thereby reducing the frequency.

A rectangular cantilevered sandwich plate having configuration of ZnO/Si/ZnO and dimensions of $a = 15 \text{cm}$ and $b = c = 10 \text{cm}$ is considered for the free vibration analysis. The
eigenvalues are obtained from the homogeneous equation of motion using direct matrix iterative technique. Table 2 contains the values of the first five natural frequencies for three $a/h$ ratios 100, 50, and 5 respectively. There are three sets of results corresponding to analyses based on first order shear deformable plate, 3D model from the present method and the 3D analysis using ANSYS. The three dimensional model is built in ANSYS using solid 226 coupled field element for the piezoelectric material and solid 185 for the host. The assembled model in ANSYS has 129546 degrees of freedom. On the contrary, the total mechanical and electrical degrees of freedom in the NURBS based solid model is 8788. The results in Table 4.2 shows a resounding agreement between the frequencies and modes shapes from the present NURBS solid and ANSYS models. ANSYS models as expected has 14.7 times the degrees of freedom than the present method. This clearly shows a huge advantage of the present method over the conventional finite element method. While comparing the frequencies between the plate and solid models, it is found that there is a very good agreement in the fundamental frequencies from the plate and solid NURBS models. The odd modes show reasonably good agreement, while it is not so for the even modes. As the plate gets thicker, the difference in fundamental frequencies seems to be increasing. Fig 4.3 shows the mode shapes for the free vibration of the plate with $a/h=5$. The first four mode shapes from both analyses agree. It is noted that the fifth mode in the solid’s case is purely an in-plane mode.
### Table 4.2 Natural frequencies of a cantilevered rectangular sandwich piezoelectric plate

<table>
<thead>
<tr>
<th>a/h</th>
<th>Mode number</th>
<th>Plate Natural frequency in rad/s</th>
<th>Solid Natural frequency in rad/s</th>
<th>ANSYS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3x3</td>
<td>1x1x3</td>
<td>2x2x3</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>389.41</td>
<td>401.10</td>
<td>385.89</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1432.25</td>
<td>1298.75</td>
<td>1222.75</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2370.85</td>
<td>2610.73</td>
<td>2445.10</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4746.66</td>
<td>4354.15</td>
<td>4215.69</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>5998.89</td>
<td>6235.87</td>
<td>6000.84</td>
</tr>
<tr>
<td>50</td>
<td>1</td>
<td>802.53</td>
<td>832.25</td>
<td>799.16</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2846.34</td>
<td>2801.22</td>
<td>2433.82</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4729.73</td>
<td>5167.47</td>
<td>4910.08</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>9419.43</td>
<td>8565.32</td>
<td>8381.81</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>11935.58</td>
<td>13147.34</td>
<td>12971.67</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>7650.73</td>
<td>7742.75</td>
<td>7610.94</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>19921.32</td>
<td>21003.68</td>
<td>19971.42</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>23101.42</td>
<td>22651.35</td>
<td>20015.25</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>39532.85</td>
<td>43461.33</td>
<td>41021.36</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>56512.66</td>
<td>63014.18</td>
<td>59519.74</td>
</tr>
</tbody>
</table>
Fig. 4.3 First five mode shapes of cantilevered rectangular sandwich piezoelectric plate with $a/h = 5$
4.5.3 Free Vibration Analysis of Cantilevered Piezoelectric Prismatic Bar

A cantilevered prismatic PZT4 piezoelectric bar is studied as a benchmark problem. The length is \( a = 15\text{cm} \) and the cross sectional dimensions are \( b = c = 5\text{cm} \). A 3\times3\times3 grid of solid modules with 125 control points in each is used in the analysis. The model contains 6591 mechanical degrees of freedom and 2197 electrical degrees of freedom to a sum of 8788. The same bar is modeled in ANSYS environment using solid 226 coupled field elements with a total of 65088 degrees of freedom. The properties of PZT4 are given in the appendix A.1. Table 4.3 shows the natural frequencies of the cantilevered prismatic bar in Hz from the present method and ANSYS. The results show extremely close agreement.

The mode shapes for modes 1–6 from the present method are shown in Fig. 4.4, though not reported in this study, the mode shapes from ANSYS matches closely with the ones in Fig. 4.4

Table 4.3 First eight natural frequencies in hertz.

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D solid</td>
<td>1125.63</td>
<td>1141.52</td>
<td>2980.05</td>
<td>4947.47</td>
<td>5124.10</td>
<td>5684.19</td>
<td>8936.09</td>
<td>10681.60</td>
</tr>
<tr>
<td>ANSYS</td>
<td>1121.80</td>
<td>1151.41</td>
<td>2981.73</td>
<td>4939.74</td>
<td>5134.86</td>
<td>5554.89</td>
<td>8910.20</td>
<td>10740.93</td>
</tr>
</tbody>
</table>
Fig. 4.4 First six mode shapes of cantilevered prismatic piezoelectric bar
4.5.4 Transient vibration of cantilevered plate.

After investing the free vibration of the cantilevered piezoelectric sandwich rectangular plate with $a/h = 5$, the transient response analysis is carried on it using 3D NURBS block. In a $3 \times 3 \times 3$ configuration, there are 6591 mechanical and 2197 electrical degrees of freedom in the model. It is a computationally intensive process to study the transient response of this problem using direct numerical integration techniques. It is possible to use Newmark’s direct integration and state space methods on the entire set of equations, but the size can be quite forbidding. Moreover it is well documented that only first few fundamental frequencies contribute in transient vibration. In order to overcome such limitations, normal mode summation technique is used. In this method, the differential equation of motion is decoupled in space and time and a proportional Rayleigh damping is also conveniently added to the equation.

Let, $\Gamma = Z(x) q(t)$, where $Z(x)$ is the modal matrix containing eigenvectors in columns and $x$ represents the space coordinate. The size of the modal matrix depends on the number of modes of vibration being considered. The eigenvectors are normalized with mass matrix. By substituting it into the decoupled equation of motion, pre-multiplying by $[Z]^T$ and using the orthogonal properties of the eigenvectors, the equation of motion is reduced considerably to a set of $n$ second order ordinary differential equations in time. Where $n$ being the number of modes considered, therefore Eq. (4.14) reduces to,

$$\{\ddot{\textbf{q}}\} + [2\xi \omega_i]\{\dot{\textbf{q}}\} + [\omega_i^2]\{\textbf{q}\} = \{P\}_i f(t)$$ (4.15)

Here, $\omega_i$ is the $i^{th}$ natural frequency, $P_i$ is the modal participation factor for the $i^{th}$ mode. Damping and frequency matrices $[2\xi \omega_i]$ and $[\omega_i^2]$ are diagonal. Once the mechanical displacements are found, the voltages are obtained from the coupled equation of motion.

Each decoupled equations in time domain are then represented in state-space form and solved using fourth order Runge-Kutta method. A time step of $\Delta t < 1/\omega$ is chosen, where $\omega$ being the first fundamental frequency. The plate is subjected to a point load of 1 kN applied at the mid-point of the free edge, opposite to the clamped boundary. Three types of loading conditions are considered, a step, half-sine and impulse. Both step and half sine
loads are applied for only 0.02 seconds and then removed. The displacement-time histories in m, at location A (a/3, 0, 0) for both damped and undamped conditions are obtained and reported in the following.

**Fig. 4.5 Step response of cantilevered rectangular plate**
Fig. 4.6 Sinusoidal response of cantilevered rectangular plate

Fig. 4.7 Impulse response of cantilevered rectangular plate
A damping ratio of 1 percent is used in the analysis. And first five modes are considered in the normal mode summation technique. As inferred from Fig 4.5, the plate oscillates about a mean position for the duration of the load and then the response drops and continues about the zero-mean reference once the load is removed. The damped oscillation follows the undamped case, but the amplitude reduces with the progress of time. Fig 4.6 shows the displacement-time history for the half sinusoidal input load. The displacement follows the applied load very closely in this case as well by riding a pseudo half-sine response curve and then taking a simple form of zero mean sinusoidal wave after 0.02 seconds. There is a very little oscillation in the response contributed by higher order frequencies. Fig 4.7 presents the displacement history for an impulsive point load. Compared to step and sinusoidal loads, the impulse load tends to trigger more natural frequencies than the fundamental. It is obvious from the Fig 4.8 that the displacement
history is composed of more than one wave form. Fig. 4.8 shows the fast Fourier transform of the displacement-time history for the third load case. Three peaks in the FFT plot are found at frequencies 1197.34 Hz, 3066.19 Hz, and 6370.53 Hz respectively. These frequencies matches closely with the first, third and fourth natural frequencies obtained from the free vibration analysis of the plate. Since the problem is symmetric about the horizontal axis of the plate, the asymmetric mode two is clearly not participating in the response.

4.6 Closing Remarks

A numerical study is made on the static and dynamic analysis of piezoelectric structures by a three dimensional variational method in which NURBS are used to represent both the geometric coordinates and displacement and charge field functions. A full three dimensional variational method is developed using constitutive equations of electroelasticity without simplifications and with full electromechanical coupling. Static analysis is performed on cantilevered trapezoidal sandwich piezoelectric plate using both three and two dimensional theories. This study reveals the discrepancies of using a first order shear deformable theory for considerably thick sandwich plates. Free vibration analysis is performed on a prismatic bar to validate the efficiency of the present NURBS based method. Finally a transient response analysis is done on a cantilevered sandwich piezoelectric solid. The matrix dynamic equations are reduced to decoupled set of second order differential equations using modal summation technique and solved using a fourth order Runge-Kutta technique. The displacement time histories are obtained for step, sinusoidal and impulse loading conditions.

- The fundamental frequencies from the plate and solid models of the piezoelectric sandwich plates are close. Hence, the use of the first order shear deformable plate model is quite adequate in sensing, actuating and power harvesting devices, wherein only the fundamental mode of vibration is triggered to a great extent.

- The present NURBS based approach and ANSYS yield very close results. However, there is a huge difference in the numbers of the degrees of freedom in the two models. This can be important particularly if the transient response analyses are required.
• FFT of the displacement time histories reveal that only the fundamental mode appears to contribute the most.

References


CHAPTER 5

Conclusion and Future Works

5.1 Introduction

Rayleigh-Ritz method for the vibrations of simple structures has a history of more than a century. Different versions of it were introduced by others, but the applications were limited to very simple structures for approximate solutions. One of the most cited paper on the vibration of rectangular plate is from the work of Young [1]. His paper was published in 1950 and he computed natural frequencies and associated mode shapes using beam’s characteristic functions. The digital computer and energy method came in the second half of 1950’s and then a large number of researchers in structural mechanics concentrated on the development of numerical methods around the digital computers. A discrete form of energy method, named the finite element method was introduced in 1956 and became the research area in engineering science for the next three decades [2]. Differential equations were replaced by matrix equations, which along with solution methods were tailored to the capabilities of the computers.

A significant effort has been made in this study to develop a general energy based numerical method utilizing NURBS are the characteristic function to analyze the static and dynamic characteristics of piezoelectric materials. The displacement based method developed here can be used as a viable modelling technique for wide range of structural mechanics problems. Limitations in conventional finite element technique like mesh regeneration and mesh approximation can be quite easily alleviated by the present method. Chronology of the work done including the tools developed are presented as follows.

In this thesis, the theories of plates and shells were thoroughly reviewed along with the structural response behaviours of piezoelectric materials when embedded with different
elastic materials. Method to apply the electromechanical properties of piezoelectricity was understood from the point of view of implementing into the smart material structures.

Computer codes were developed in C++ and MATLAB environment for the analysis of isotropic first order shear deformable plates. The codes were then expanded in the direction of dealing with electro-elastic problems. NURBS were used to define the geometry and the mechanical and potential field functions.

Chapter two contains a NURBS based variational technique for the static and vibration analyses of trapezoidal plates. The method amalgamates approximations of the Reissner-Mindlin plate theory and piecewise linear approximation of electric potential along the thickness. The NURBS basis functions are introduced as geometric and displacement field functions for the analysis of piezoelectric plates of uniform thickness. The displacements and electric potential are compared with data available in the open literature for the cylindrical bending in sensor and actuator modes. Results are also provided for the fundamental frequencies of various taper ratios and skew angles. Transient response and associated FFT plots are provided for a trapezoidal plate.

In chapter three, a NURBS curve fitting technique is presented to model piezoelectric plates with curved edges which can either be cad generated or defined by functions in the plate. Results produced for circular plates are validated with the available data in the literature. New results are documented for elliptical plates. The equations developed for sensors and actuators are then extended for vibration energy harvesters (VEH) subjected to base acceleration. Parametric studies are made for a VEH with a rectangular and curved edge profiles. Normal mode summation technique has been implemented to analyse a multi degree freedom system. Frequency response analyses performed for displacement, voltage and power output for both straight and curved edged profiles are presented and discussed.

The method is extended further to develop a three dimensional solid model in chapter four. The solid brick modules are formulated to have arbitrary number of control points and curve order. Each control point has four degrees of freedom, e.g. three mechanical and one electrical. Sandwich plate is modelled in both environments. Two and
three dimensional analyses yield comparable results for thin plates. The plate model has many times less number of degrees of freedom than that for the solid model. Numerical studies are performed to compare the efficiency and computational effort involved in pursuing two and three dimensional analyses. Results obtained for free vibration analysis of considerably a thick piezoelectric sandwich plate from the plate, solid and finite element models are successfully validated. Commercial FEA package ANSYS is used for the finite element results. A normal mode summation technique is also provided to study the transient response of piezoelectric sandwich plate. Distributed and point loads consistent with the formulation are calculated using work-energy principle. The number of degrees of freedom in all analysis were considerably less than finite element method and the results fare well with analytical techniques in some benchmark studies.

NURBS are very well behaved functions and easy to blend between sections, when the structure is divided into sub-patches. Cubic blending function provides $C^2$ continuity, reliability and stability in the solution by eliminating numerical fluctuations in the convergence of the results.

5.2 Major Contributions to the Field of Study and Experience Gained.

The research has been carried in the field of computational solid mechanics dealing with piezoelectric composite plates and solids. During the course of this study, the Ritz – Type solution is generalized and the principles of the variational formulations are thoroughly reviewed, understood and modified as needed.

- Algorithms for NURBS curves, surfaces and volumes are developed and converted into computer programs for numerical differentiation, integration and knot removal/insertion.
- A consistent variational method to study vibrational characteristics of piezoelectric composite first order shear deformable plates has been presented. One set mechanical displacement and rotation components are used for layers of piezoelectric and host materials.
• In-plane displacement components vary linearly in the thickness direction, but electric potential function is modelled as piecewise linear only in the piezoelectric thick.
• Matrix equations of motion are developed for the application of the plate as sensors and actuators. Numerical studies are made on trapezoidal cantilevered plates.
• Circular and elliptic sandwich plate vibrations are studied in details.
• The coupled matrix equations of motion are modified so that the plate model can be studied as vibration energy harvesters. The normal mode approach is applied to separate equations in space and time. The transient solutions were obtained by fourth order Runge-Kutta method.
• It is shown that only $C^0$ continuity at the interfaces of two patches or solid modules yield accurate results.
• Two dimensional plate model is modified for application in a full three dimensional solid module for the static and dynamic analyses.
• Comparative studies on sandwich plates are made by three methods. The first is based on the Reissner-Mindlin’s first order shear deformation plate theory. The second is the three dimensional one based upon NURBS and finally using a commercial FEA package ANSYS.

5.3 Future works

The present method is developed to take advantage of the CAD representation of the geometry. The geometric definition of all the models that has been considered in this research has been represented in NURBS. Having said that the profiles considered in the present work are regular, planar and are easily developable. The algorithms developed are robust enough to accommodate these profiles. This warrants a sufficient study on developing and extending the present work to accommodate more complex geometries. A uniform weight distribution is used throughout the analysis, except for models for few cases involving curved edges. The availability of weights gives an additional flexibility in geometric modelling, which could be exploited in future case studies. There are number of assumptions made throughout this thesis and the developed model is only good to the extent
the assumptions are made. One of the reasons for relatively lower number of degrees of freedom in the three dimensional formulation compared to ANSYS is the implementation of higher order solid elements. Since a prismatic bar is studied in the present analysis, modelling such geometry did not pose any restriction on element formulation. All the solid blocks used for the analysis are the same. Once the geometry becomes complex, assembling them becomes complex as well. In chapter four, a desired result is obtained using three dimensional solid blocks over the plate elements using FSDT. Even though the number of degrees of freedom are considerable reduced by normal mode summation technique, static analysis still required more computational effort. One of the possible recommendation would be to implement the plate theory using higher order approximations or discrete layer theories to match the accuracy of three dimensional model.

Finally, the vibration energy harvester model developed in the thesis is concerned about a two dimensional cantilever plate subjected to a uniform base acceleration. In reality, this situation is more desired than to happen. Normally such devices will be subject to the ambient excitations that are intermittent and chaotic in nature. A random vibration analysis has to be carried out in order to estimate any realistic amount of power that could be obtained from the harvester.

References


### Appendix

#### A.1 Properties of Piezoelectric Materials

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## CURRICULUM VITAE

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