August 2014

Risk models with dependence and perturbation

Zhong Li
The University of Western Ontario

Supervisor
Kristina Sendova
The University of Western Ontario

Graduate Program in Statistics and Actuarial Sciences

A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

RISK MODELS WITH DEPENDENCE AND PERTURBATION

by

Zhong Li

Department of Statistical and Actuarial Sciences

A thesis submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy in Statistics

The School of Graduate and Postdoctoral Studies

The University of Western Ontario

London, Ontario, Canada

Abstract

In ruin theory, the surplus process of an insurance company is usually modeled by the classical compound Poisson risk model and the Sparre-Andersen risk model. Under these models, the claim amounts and inter-claim times are assumed to be independently distributed, which is not always appropriate in practice. In recent years, risk models relaxing the independence assumption have drawn intensive research attention. However, previous research mostly considers the so call dependent Sparre-Andersen risk model under which the pairwise events containing inter-claim time and the next claim amount remain independent of each other. In this thesis, we aim to examine the opposite case. Namely, the distribution of time until next claim depends on the size of previous claim amount. Explicit solutions for the Gerber-Shiu function are provided for arbitrary claim sizes and various ruin-related quantities are obtained as special cases. Numerical examples are also presented. The dependent insurance risk is further generalized to a perturbed version to incorporate small fluctuations of the underlying surplus process. Explicit solutions for the Gerber-Shiu function are deduced along with applications and examples. Lastly, we introduce the perturbed dependence structure into the dual risk model and study the ruin time problem. Exact solutions for the Laplace transform and the first moment of the time to ruin with an arbitrary gain-size distribution are obtained. Applications with numerical examples are provided to illustrate the impact of the dependence structure and the perturbation.

Keywords: non Sparre-Andersen dependence, diffusion, risk model, dual model, ruin time, Gerber-Shiu function.

Acknowledgements

I would like to express my sincere appreciation to my supervisor Prof. Kristina Sendova for her continuous support and valuable suggestions during my Ph.D. study. This thesis could never be possible without her guidelines and helps. More importantly, I want to thank her for teaching me how to become a better researcher and a better person.

I would also like to thank Prof. Hristo Sendov and my fellow students Yaofei Xiong and Chen Yang from my home department for the helpful discussions during the writing of this thesis.

Special thanks to those who never lose faith in me.

Contents

List of Figures

Chapter 1

Introduction

Consider the continuous-time surplus process for an insurer

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \ t \geq 0, \tag{1.1}$$

with initial surplus level $u \geq 0$ and constant premium rate $c > 0$, where $\{X_i, i = 1, 2, \ldots\}$ are claim-size random variables and the claim counting process $N(t)$ is a renewal process with interclaim times $\{V_i, i = 1, 2, \ldots\}$. When both $\{X_i\}$ and $\{V_i\}$ are sequences of independent and identically distributed (i.i.d.) random variables and their distributions are independent, process (1.1) is the Sparre-Andersen risk model. In addition, when $\{V_i, i = 1, 2, \ldots\}$ follows an exponential distribution, then $N(t)$ is a Poisson process and model (1.1) represents the classical compound Poisson risk model. Also, the linear component ct in model (1.1) may be generalized to a function $h(t)$ as long as h^{-1} is well defined. Sparre-Andersen risk model and classical compound Poisson risk model are widely studied in risk theory and ruin theory, however the i.i.d. assumption among individual claim sizes and inter-claim times, and the independence assumption between claim size and inter-claim time are usually inappropriate in practice.

Risk models involving various dependence structures have been studied intensively in the literature. One popular topic is assuming dependency among individual claims, for example, Müller and Pflug (2001) considered asymptotic ruin probabilities for risk models with dependent claim increments, Denuit et al. (2002) examined the impact of dependence between claim occurrences, and Cossette et al. (2002) constructed models allowing dependence among claims using risk classifying and copulas techniques. Risk processes with Markovian arrivals are also widely studied to model dependence, see for example, in Badescu et al. (2005), Ahn et al. (2007) and Badescu et al. (2007). A rising research interest in recent years is to relax the independence assumption between claim sizes and inter-claim times, among which the most persuasive works are articles by Albrecher and Teugels (2006), Boudreault et al. (2006), Cossette et al. (2008), Meng et al. (2008), Ambagaspitiya (2009), Badescu et al. (2009) and so on. However, previous research mostly concentrated on the dependence structure where the pairs of events (V_j, X_j) remain independent of each other, which allows the risk model preserves the independent increment assumption of the Sparre-Andersen risk model and is referred to as the dependent Sparre-Andersen risk model. For instance, Boudreault et al. (2006) studied a risk model with claim sizes depending on elapse time motivated by a natural catastrophe context. Cheung et al. (2010) summarized some general properties of the dependent Sparre-Andersen risk model. In this thesis, we aim to examine the opposite case when the distribution of the time until the next claim depends on the amount of the previous claim and generalize the dependence risk model to a perturbed version. More precisely, V_j and X_j are independent, but the next event of V_{j+1} depends on the previous pair (V_j, X_j). Consequently, the surplus process is no longer a dependent Spaarre-Andersen model. This kind of causal dependence model was first introduced by Albrecher and Boxma (2004) where the ultimate-survival probabilities are considered. Under the dependence setting, the amount of an individual claim may be viewed as a risk classifier for the insured, and we make different assumptions for the distribution of next inter-event time when the insured is classified to different group. Take car insurance for example, Gschlößl and Czado (2007) show there are empirical evidence that claim frequency and claim severity are negatively correlated in auto insurance. Then it is reasonable to assume that when a claim size is large, we expect the waiting time until next claim to be large as well. In addition to the causal dependence structure, we further assume that the premium rate also varies depending on claim sizes where claim sizes follow an arbitrary distribution, and analyze the Gerber-Shiu discounted penalty function that was proposed first in the paper by Gerber and Shiu (1998).

The Gerber-Shiu expected discounted penalty function is defined as

$$m(u) = \mathbb{E}\left\{e^{-\delta\tau}\, w(U(\tau-), |U(\tau)|)\, \mathbb{I}_{\{\tau<\infty\}} \,\middle|\, U(0) = u\right\}, \quad u \geq 0, \tag{1.2}$$

where $\delta \geq 0$ is the discount factor, $w(x_1, x_2)$, $x_1 \geq 0$, $x_2 > 0$ is a penalty function, τ is the infinite time to ruin random variable and $\mathbb{I}_{\{\tau<\infty\}}$ is an indicator function taking value 1 when $\tau < \infty$ and 0 otherwise, given that the initial surplus level is at $u \geq 0$. The Gerber-Shiu function is widely studied since it recovers a number of quantities of special interest in ruin theory, such as the probability of ultimate ruin, the Laplace transform of time to ruin, the joint and marginal distributions and moments of the surplus and the deficit r.v. $U(\tau-), |U(\tau)|$ and so on. We illustrate this by the following examples.

Example 1.1 *Probability of ultimate ruin.* Let $\delta = 0$ and $w(x_1, x_2) = 1$ for all $x_1 \geq 0$, $x_2 > 0$, then the Gerber-Shiu function (1.2) reduces to

$$\psi(u) = \mathbb{E}\left\{\mathbb{I}_{\{\tau<\infty\}} \,\middle|\, U(0) = u\right\} = \mathbb{P}\left\{\tau < \infty \,\middle|\, U(0) = u\right\},$$

which is the probability of ultimate ruin. □

Example 1.2 *Laplace transform of time to ruin.* Let $w(x_1, x_2) = 1$ for all $x_1 \geq 0$, $x_2 > 0$, then the Gerber-Shiu function reduces to the Laplace transform of the time to ruin random variable, denoted by

$$\varphi(u) = \mathbb{E}\left\{e^{-\delta\tau}\mathbb{I}_{\{\tau<\infty\}} \,\middle|\, U(0) = u\right\} = \int_0^\infty e^{-\delta t}\, f_\tau\,(t|u)\,\mathrm{d}t.$$

From the Laplace transform, we may compute the moments of time to ruin. □

Example 1.3 *Defective joint and marginal moments of the surplus and the deficit.* Let $\delta = 0$ and $w(x_1, x_2) = x_1^k x_2^l$ for all $x_1 \geq 0$, $x_2 > 0$, where k and l are nonnegative integers, then the Gerber-Shiu function presents the joint moments of the surplus and the deficit. With $k = 0$ or $l = 0$, we obtain the marginal moments. □

Example 1.4 *Joint defective distribution of the surplus and the deficit.* Let $\delta = 0$ and $w(x_1, x_2) = \mathbb{I}_{\{x_1 \leq x\}} \mathbb{I}_{\{x_2 \leq y\}}$, for all $x_1 \geq 0$, $x_2 > 0$, then the Gerber-Shiu function presents the joint defective distribution of the surplus and the deficit. With either $x \to \infty$ or $y \to \infty$, we obtain the marginal distributions. □

Example 1.5 *Defective distribution of the claim causing ruin.* Let $\delta = 0$ and $w(x_1, x_2) = \mathbb{I}_{\{x_1 + x_2 \leq z\}}$, for all $x_1 \geq 0$, $x_2 > 0$, then the Gerber-Shiu function produces the distribution of the size of the claim causing ruin. □

Another direction for generalizing the classical ruin model is adding a diffusion process to account for the fluctuations of aggregate premiums and aggregate claims. In actuarial science, risk models perturbed by Brownian motion are widely used when the underlying process is assumed to be subject to small changes at any point in time. For instance, when the surplus process incorporates some risky investment. This idea was first introduced by Gerber (1970) and modeled by

$$U_D(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma W(t), \; t \geq 0,$$

where $\sigma > 0$ is a constant and $W(t)$ is a standard Brownian motion. The perturbed risk models have been studied extensively, see for example, Dufresne and Gerber (1991), Gerber and Landry (1998), Tsai and Willmot (2002), Li and Garrido (2005) and Sarkar and Sen (2005) for the perturbed classical compound Poisson model and Sparre-Andersen model. Besides, Wan (2007), Li et al. (2009) and Mitric et al. (2010) consider a perturbed Sparre-Andersen model with threshold dividend strategy. In the above papers, various quantities of interest are analyzed under the perturbed risk model where it is assumed that the claim-size distribution and the inter-claim time distribution are independent.

A generalized Gerber-Shiu function for perturbed risk models was introduced by Tsai and Willmot (2002) based on the idea of Gerber and Landry (1998), and is defined as

$$m_D(u) = w_0\,\phi_{\mathrm{d}}(u) + \phi_{\mathrm{w}}(u), \quad u \geq 0, \tag{1.3}$$

where w_0 is a constant representing the penalty at ruin due to oscillation and

$$\begin{aligned}
\phi_{\mathrm{d}}(u) &= \mathbb{E}\left\{e^{-\delta\tau}\,\mathbb{I}_{\{\tau<\infty, U(\tau)=0\}}\,\middle|\,U(0)=u\right\},\\
\phi_{\mathrm{w}}(u) &= \mathbb{E}\left\{e^{-\delta\tau}\,w(U(\tau-), |U(\tau)|)\,\mathbb{I}_{\{\tau<\infty, U(\tau)<0\}}\,\middle|\,U(0)=u\right\}.
\end{aligned}$$

The component $\phi_{\mathrm{d}}(u)$ represents the Laplace transform of the time of ruin random variable τ if ruin is due to oscillation, while the summand $\phi_{\mathrm{w}}(u)$ corresponds to the penalty at ruin if caused by a claim. Here $\delta \geq 0$ is the discount factor, $w(x_1, x_2)$, $x_1, x_2 \geq 0$, is a penalty function for ruin caused by a claim with $w(0,0) = w_0$. At zero initial surplus $u = 0$, it implies $\phi_{\mathrm{w}}(0) = 0$, $\phi_{\mathrm{d}}(0) = 1$.

In recent years, insurance risk models with dependence structure and perturbed by diffusion have drawn substantial attention. For instance, Lu and Tsai (2007) analyze a Markov-modulated perturbed risk process where the distributions of interclaim times and claims sizes depend on an environmental Markov process. Also, Li and Ren (2013) consider the maximum severity of ruin under a perturbed risk process with Markovian arrivals. In Chapter 3, we study a perturbed risk model with interclaim-times depending on claim sizes following the dependence structure proposed by Albrecher and Boxma (2004). This perturbed risk model was studied previously by Zhou and Cai (2009). However, the authors considered only the ultimate-survival probabilities and derive a recursive solution for exponential claim sizes. We substantially extend the analysis to an explicit solution of the widely studied Gerber-Shiu function along with examples and clarify an open question formulated in Remark 3.2 of Zhou and Cai (2009). The advantage of our approach is that there are no constraints on the claim-size distribution and that our results are in terms of functions that are explicitly known. Another relevant recent contribution on the subject is Cheung and Landriault (2009) where the risk model with dependence structure of Albrecher and Boxma (2004) is considered as a special case of the Markov additive process and optimal dividend problem under a barrier strategy is studied. Here we focus on the solution for the generalized Gerber-Shiu function and its applications.

The idea of perturbed dependence structure may also be applied to the dual risk model. A dual risk model is suitable to analyze a revenue process for a line of business with steady expenses and occasional gains. An annuity business, an invention or mining company are examples of such businesses. This type of a setup may be modeled by the dual risk process

$$R(t) = u - ct + \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{1.4}$$

where $u > 0$ is the initial surplus, $c > 0$ represents the expense rate of the company, $N(t)$ is an event-counting process with inter-event times $\{V_i, i = 1, 2, \ldots\}$, and $\{X_i, i = 1, 2, \ldots\}$ are the amounts of the occasional gains. When $N(t)$ is a compound Poisson process, model (1.4) is called the compound Poisson dual model. When $\{V_i, i = 1, 2, \ldots\}$ and $\{X_i, i = 1, 2, \ldots\}$ are two sets of i.i.d. random variables and independent of each other, model (1.4) is the Sparre-Andersen dual model. The name of dual risk model is related to a duality between (1.4) and (1.4). The ruin model (1.1) with an absorbing barrier $b > u$ where ruin occurs at $U(t) = b$ instead of $u(t) = 0$ is equivalent to the dual model with initial capital $b - u$. To illustrate this, see Figure 1.1 for a sample path of such $U(t)$. Research on dual risk models has drawn rising interest in recent years. Avanzi et al. (2007), Afonso et al. (2013) and Bayraktar et al. (2013) study optimal dividend problems under the compound Poisson dual model with a barrier strategy. Ng (2009) considers the compound Poisson dual model with a threshold dividend strategy. Other articles related to dividend problems include Albrecher et.al (2008) who study a dual model with tax payments, Avanzi et al. (2013) who consider a dual model with periodic observation times, Bayraktar et al.

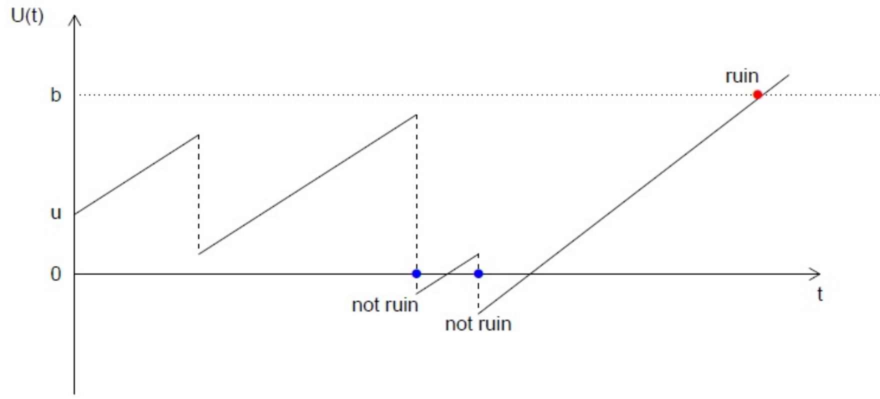

Figure 1.1: A sample path of U(t) equivalent to the dual model

(2014) who introduce a dual model with transaction cost, etc. Ruin-time problems under the Sparre-Andersen dual model with Erlang-n inter-gain times are studied in Landriault and Sendova (2011) and Rodríguez et al. (2013). Yang and Sendova (2014) further extend the analysis to the Sparre-Andersen dual model with generalized Erlang-n inter-gain times.

More recently, a dependence structure is implemented into a dual risk model in Albrecher et al. (2014) where the distribution of inter-gain times is assumed to depend on the size of the previous gain by comparing it to a fix threshold. We consider an extension of Albrecher et al. (2014) where the fixed threshold is generalized to random thresholds which is the dependence structure introduced by Albrecher and Boxma (2004) and we examine the ruin time. The dependence structure may describe a revenue process of a research company, when a certain research gain is large (or small), resources and talent will be drawn into (or out of) the company and that will affect the distribution of time until next gain. In addition, it is assumed that the expense rates also depend on the previous gain amount and the underlying surplus process is perturbed by a diffusion process.

Dual risk model with diffusion is given by

$$R_D(t) = u - ct + \sum_{i=1}^{N(t)} X_i + \sigma W(t), \quad t \geq 0, \sigma > 0, \tag{1.5}$$

where $W(t)$ is a standard Brownian motion, was first introduced by Avanzi and Gerber (2008). More recent works on dual risk models perturbed by diffusion include Avanzi et.al. (2011), Avanzi et al. (2014) and Liu and Chen (2014). As pointed out by Remark 2.2 in Avanzi and Gerber (2008), although model (1.4) is a limiting case of model (1.5) as $\sigma \to 0$, no formula under model (1.4) may be obtained as a limiting case of the respective formula under model (1.5) when dividend problems are concerned. Instead, under the perturbed dependent dual risk model that we consider, the convergence preserves very neatly for all ruin-related results (see Remark 4.2).

The rest of this thesis is structured as follows. Chapter 2 studies a ruin model with both inter-claim time and premium rating depending on claim sizes, where we derive the explicit solutions for the Gerber-Shiu function and its various applications. Chapter 3 considers a perturbed version of the ruin model with dependence between inter-claim time and claim sizes. In Chapter 4, a perturbed dual risk model with inter-gain distribution and expense rates depending on the size of previous gain is studied. Exact solutions for the Laplace transform of the ruin time with arbitrary gain-size distribution are obtained and the impacts of the dependence structure and perturbation are examined. Chapter 5 gives the conclusions and future research goals.

Chapter 2

An insurance risk model with dependence structure

In this chapter, we consider a continuous-time insurance risk process where both the interclaim-time distribution and premium rate both depend on the size of the previous claim. Explicit solution for the Gerber-Shiu discounted penalty function with arbitrary claim size distribution is derived utilizing the roots of a generalized Lundberg's equation. Lastly, applications with exponential thresholds are presented and a numerical example is provided.

2.1 Model description and notation

Suppose that the surplus process of an insurance company is modeled by

$$\begin{aligned} U(t) &= u + C(t) - \sum_{i=1}^{N(t)} X_i \\ &= u + c_1 \int_0^t \mathbb{I}_{\{J(s)=1\}}\, ds + c_2 \int_0^t \mathbb{I}_{\{J(s)=2\}}\, ds - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \end{aligned} \tag{2.1}$$

where the initial surplus is $U(0) = u \geq 0$, the premium received up to time t is $C(t)$, $N(t)$ is the claim-counting process and the claim amounts $\{X_i, i = 1, 2, \ldots\}$ are positive i.i.d random variables with cumulative distribution function $B(\cdot)$, density function $b(\cdot)$ and mean μ. Assume that the distribution of the waiting time until the next claim depends on the size of the previous claim by comparing it to random thresholds $\{Q_i, i = 1, 2, \ldots\}$. Suppose the thresholds $\{Q_i, i = 1, 2, \ldots\}$ are i.i.d. with c.d.f. $H(\cdot)$ and are independent of the claim sizes $\{X_i\}$. If the size of the claim X_j is larger than Q_j, then the time until next claim will follow an exponential distribution with mean $1/\lambda_1 > 0$; if X_j is smaller than Q_j, then the time until the next claim will follow another exponential distribution with mean $1/\lambda_2 > 0$ ($\lambda_1 \neq \lambda_2$). This causal dependence structure was first introduced by Albrecher and Boxma (2004). Under model (2.1), the thresholds $\{Q_i\}$ may be viewed as a risk indicator that governs the distribution of the waiting time until the next claim and may be deduced through, for example, the general population, a control group or past experience. In addition, the premium charged also varies depending on the same risk indicator, in response to the possible change in the distribution of interclaim time. If a claim X_j is larger than Q_j, we classify the insured as Class 1 and charge continuous premium at rate $c_1 > 0$; if X_j is smaller than Q_j, then we classify the insured as Class 2 and charge continuous premium at rate $c_2 > 0$. At any given time t, denote the class of the insured by $J(t)$. The premium collecting process $C(t)$ is a piecewise linear process. Notice that when $c_1 = c_2$, model (2.1) reduces to the semi-Markov dependent model in Albrecher and Boxma (2004). Assume that the positive-security-loading condition holds for model (2.1). Namely,

$$\frac{c_1}{\lambda_1}\mathbb{P}\{X > Q\} + \frac{c_2}{\lambda_2}\mathbb{P}\{X < Q\} > \mu, \tag{2.2}$$

which means in a probabilistic view, the insurance company charges a premium that is higher than the expected loss amount.

Given that the initial class of the insured is i, $i = 1, 2$, and the initial surplus is u, we analyze the Gerber-Shiu expected discounted penalty function

$$m_i(u) = \mathbb{E}\left\{e^{-\delta\tau_i}\, w(U(\tau_i-), |U(\tau_i)|)\, \mathbb{I}_{\{\tau_i<\infty\}} \,\middle|\, U(0) = u\right\}, \quad u \geq 0,\ i = 1, 2, \tag{2.3}$$

where $\delta \geq 0$ is the discount factor, $w(x_1, x_2), x_1, x_2 \geq 0$, is a penalty function, and τ_i, $i = 1, 2$, is the time to ruin random variable for Class i. Lastly, we introduce some notation and properties that are used throughout the chapter. Denote by

$$\zeta(u) = \int_u^\infty w(u, y - u)\, b(y)\, \mathrm{d}y. \tag{2.4}$$

Suppose that the Laplace transforms of $b(\cdot)$, $H(\cdot)$ and $\zeta(\cdot)$ exist for all $Re(s) \geq 0$. The Laplace transform of a real-valued function $f(\cdot)$ is denoted by

$$\widetilde{f}(s) = \int_0^\infty e^{-sy} f(y)\, \mathrm{d}y, \quad s \in \mathbb{C}.$$

Define the Translation operator T_s, $s \geq 0$, of a function $f(\cdot)$ as

$$T_s f(x) = \int_x^\infty e^{-s(y-x)} f(y)\, \mathrm{d}y, \quad x \geq 0\,, \tag{2.5}$$

which was first employed by Dickson and Hipp (2001), and has the following properties

$$\begin{aligned} T_s f(0) &= \widetilde{f}(s), \quad s \geq 0, \\ T_{s_1} T_{s_2} f(x) = T_{s_2} T_{s_1} f(x) &= \frac{T_{s_1} f(x) - T_{s_2} f(x)}{s_2 - s_1}, \quad s_1, s_2 \geq 0,\ s_1 \neq s_2, \\ T_{s_1} T_{s_2} f(0) = T_{s_2} T_{s_1} f(0) &= \frac{\tilde{f}(s_1) - \tilde{f}(s_2)}{s_2 - s_1}, \quad s_1, s_2 \geq 0,\ s_1 \neq s_2. \end{aligned} \tag{2.6}$$

See also Li and Garrido (2004) for the properties of the Translation operator T.

2.2 Generalized Lundberg's equation

First, we derive a system of integro-differential equations for the Gerber-Shiu function $m_i(u), i = 1, 2$, introduced by identity (2.3). Given the initial class is i, $i = 1, 2$, conditioning on the time and the amount of the first claim, we deduce

$$m_1(u) = \int_0^\infty e^{-\delta t} \lambda_1 e^{-\lambda_1 t} \left\{ \int_0^{u+c_1 t} \Big[\mathbb{P}\{y > Q_1\}\, m_1(u + c_1 t - y) + \mathbb{P}\{y < Q_1\}\, m_2(u + c_1 t - y)\Big] b(y)\, dy\right.$$

$$+\int_{u+c_1t}^{\infty} w(u+c_1t, y-u-c_1t)\, b(y)\, dy\Bigg\}\, dt, \tag{2.7}$$

$$m_2(u) = \int_0^{\infty} e^{-\delta t}\lambda_2 e^{-\lambda_2 t}\Bigg\{\int_0^{u+c_2t}\Big[\mathbb{P}\{y > Q_1\}\, m_1(u+c_2t-y) + \mathbb{P}\{y < Q_1\}\, m_2(u+c_2t-y)\Big]\, b(y)\, dy$$

$$+\int_{u+c_2t}^{\infty} w(u+c_2t, y-u-c_2t)\, b(y)\, dy\Bigg\}\, dt, \tag{2.8}$$

where

$$\mathbb{P}\{y > Q_1\} = H(y)\,,$$

$$\mathbb{P}\{y < Q_1\} = 1 - H(y) = \overline{H}(y)\,.$$

For simplicity, the following notation is introduced

$$\chi(y) := \overline{H}(y)\, b(y), \tag{2.9}$$

$$\xi(y) := H(y)\, b(y) = b(y) - \chi(y). \tag{2.10}$$

Changing the variable of integration t to $v = u + c_1 t$ in (2.7) and utilizing identities (2.4), (2.9) and (2.10), yields

$$m_1(u) = \int_u^{\infty} \frac{\lambda_1}{c_1}\, e^{-\left(\frac{\lambda_1+\delta}{c_1}\right)(v-u)} \left\{\int_0^{v}\Big[m_1(v-y)\xi(y) + m_2(v-y)\chi(y)\Big]\, dy + \zeta(v)\right\}\, dv. \tag{2.11}$$

Denote by

$$\gamma(t) := \int_0^{t}\Big[m_1(t-y)\xi(y) + m_2(t-y)\chi(y)\Big]\, dy + \zeta(t)\,,$$

then we may rewrite equation (2.11) by the definition of Translation Operator in (2.5) to

$$m_1(u) = \frac{\lambda_1}{c_1}\, T_{\frac{\lambda_1+\delta}{c_1}}\gamma(u)\,, \quad u \geq 0, \tag{2.12}$$

which implies

$$m_1(0) = \frac{\lambda_1}{c_1}\, T_{\frac{\lambda_1+\delta}{c_1}}\gamma(0) = \frac{\lambda_1}{c_1}\, \widetilde{\gamma}\left(\frac{\lambda_1+\delta}{c_1}\right).$$

Applying Laplace transforms to (2.12) and utilizing identity (2.6), we obtain

$$\widetilde{m}_1(s) = \frac{\lambda_1}{c_1}\, T_s T_{\frac{\lambda_1+\delta}{c_1}}\gamma(0)$$

$$= \frac{\lambda_1}{c_1} \cdot \frac{\widetilde{\gamma}\left(\frac{\lambda_1+\delta}{c_1}\right) - \widetilde{\gamma}(s)}{s - \dfrac{\lambda_1+\delta}{c_1}}$$
$$= \frac{\lambda_1}{c_1} \cdot \frac{\widetilde{\gamma}\left(\frac{\lambda_1+\delta}{c_1}\right) - \widetilde{\xi}(s)\,\widetilde{m}_1(s) - \widetilde{\chi}(s)\,\widetilde{m}_2(s) - \widetilde{\zeta}(s)}{s - \dfrac{\lambda_1+\delta}{c_1}}$$
$$= \frac{m_1(0) - \dfrac{\lambda_1}{c_1}\widetilde{\xi}(s)\,\widetilde{m}_1(s) - \dfrac{\lambda_1}{c_1}\widetilde{\chi}(s)\,\widetilde{m}_2(s) - \dfrac{\lambda_1}{c_1}\widetilde{\zeta}(s)}{s - \dfrac{\lambda_1+\delta}{c_1}}\,.$$

Further rearrangement of the terms produces

$$\left[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)\right]\widetilde{m}_1(s) + \frac{\lambda_1}{c_1}\widetilde{\chi}(s)\,\widetilde{m}_2(s) = m_1(0) - \frac{\lambda_1}{c_1}\widetilde{\zeta}(s)\,. \tag{2.13}$$

Similarly, we deduce from (2.8) that

$$m_2(u) = \frac{\lambda_2}{c_2}\, T_{\frac{\lambda_2+\delta}{c_2}}\gamma(u)\,, \qquad u \geq 0\,.$$

Applying Laplace transforms yields

$$\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]\widetilde{m}_2(s) + \frac{\lambda_2}{c_2}\widetilde{\xi}(s)\,\widetilde{m}_1(s) = m_2(0) - \frac{\lambda_2}{c_2}\widetilde{\zeta}(s)\,. \tag{2.14}$$

Together equations (2.13) and (2.14) provides a system of equations which $\widetilde{m}_1(s)$ and $\widetilde{m}_2(s)$ satisfy.

Multiply equation (2.13) by $\left[s - \dfrac{\lambda_2+\delta}{c_2} + \dfrac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]$ produces

$$\left[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)\right]\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]\widetilde{m}_1(s) + \frac{\lambda_1}{c_1}\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]\widetilde{\chi}(s)\,\widetilde{m}_2(s)$$
$$= \left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]m_1(0) - \frac{\lambda_1}{c_1}\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]\widetilde{\zeta}(s)\,,$$

in which replacing $\left[s - \dfrac{\lambda_2+\delta}{c_2} + \dfrac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]\widetilde{m}_2(s)$ by the expressions in (2.14) yields

$$\left[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)\right]\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]\widetilde{m}_1(s) + \frac{\lambda_1}{c_1}\widetilde{\chi}(s)\left[m_2(0) - \frac{\lambda_2}{c_2}\widetilde{\zeta}(s) - \frac{\lambda_2}{c_2}\widetilde{\xi}(s)\,\widetilde{m}_1(s)\right]$$

$$= \left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right] m_1(0) - \frac{\lambda_1}{c_1}\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right]\widetilde{\zeta}(s),$$

Then grouping all terms with $\widetilde{m}_1(s)$ to the left-hand side of the equation leads to

$$\left\{\left[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)\right]\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right] - \frac{\lambda_1\lambda_2}{c_1c_2}\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{m}_1(s)$$
$$= \left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right] m_1(0) - \frac{\lambda_1}{c_1}\widetilde{\chi}(s)\, m_2(0) - \frac{\lambda_1}{c_1}\left(s - \frac{\lambda_2+\delta}{c_2}\right)\widetilde{\zeta}(s). \tag{2.15}$$

Similarly, we multiply (2.14) by $\left[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)\right]$ and then replace $[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)]\widetilde{m}_1(s)$ by the expressions in (2.13), after rearranging it yields

$$\left\{\left[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)\right]\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right] - \frac{\lambda_1\lambda_2}{c_1c_2}\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{m}_2(s)$$
$$= \left[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)\right] m_2(0) - \frac{\lambda_2}{c_2}\widetilde{\xi}(s)\, m_1(0) - \frac{\lambda_2}{c_2}\left(s - \frac{\lambda_1+\delta}{c_1}\right)\widetilde{\zeta}(s). \tag{2.16}$$

The terms in front of $\widetilde{m}_i(s)$, $i = 1, 2$, in equations (2.15) and (2.16) are identical. Setting them to zero, produces the generalized Lundberg's equation for model (2.1),

$$\left[s - \frac{\lambda_1+\delta}{c_1} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s)\right]\left[s - \frac{\lambda_2+\delta}{c_2} + \frac{\lambda_2}{c_2}\widetilde{\chi}(s)\right] - \frac{\lambda_1\lambda_2}{c_1c_2}\widetilde{\xi}(s)\widetilde{\chi}(s) = 0,$$

or equivalently

$$\left(s - \frac{\lambda_1+\delta}{c_1}\right)\left(s - \frac{\lambda_2+\delta}{c_2}\right) - \frac{\lambda_1}{c_1}\left(\frac{\lambda_2+\delta}{c_2} - s\right)\widetilde{\xi}(s) - \frac{\lambda_2}{c_2}\left(\frac{\lambda_1+\delta}{c_1} - s\right)\widetilde{\chi}(s) = 0. \tag{2.17}$$

We analyze the roots of Lundberg's equation in Lemma 2.1 and Lemma 2.2.

Lemma 2.1 *When $\delta = 0$, the generalized Lundberg's equation* (2.17) *has exactly two roots with nonnegative real parts. These roots are distinct, real and one of them equals zero.*

Proof When $\delta = 0$, equation (2.17) reduces to

$$\left(s - \frac{\lambda_1}{c_1}\right)\left(s - \frac{\lambda_2}{c_2}\right) - \frac{\lambda_1}{c_1}\left(\frac{\lambda_2}{c_2} - s\right)\widetilde{\xi}(s) - \frac{\lambda_2}{c_2}\left(\frac{\lambda_1}{c_1} - s\right)\widetilde{\chi}(s) = 0. \tag{2.18}$$

One may verify easily that $s = 0$ is a root of (2.18) utilizing the relation $\widetilde{\xi}(0) + \widetilde{\chi}(0) = \widetilde{b}(0) = 1$ from the Laplace-transformed identity (2.10).

For $s \neq 0$, equation (2.18) may be rearranged to

$$s^2 - \frac{\lambda_1}{c_1}s - \frac{\lambda_2}{c_2}s + \frac{\lambda_1\lambda_2}{c_1c_2}\left[1 - \widetilde{\xi}(s) - \widetilde{\chi}(s)\right] + \frac{\lambda_1}{c_1}s\,\widetilde{\xi}(s) + \frac{\lambda_2}{c_2}s\,\widetilde{\chi}(s) = 0$$

$$s\left[s - \frac{\lambda_1}{c_1} - \frac{\lambda_2}{c_2} + \frac{\lambda_1}{c_1}\widetilde{\xi}(s) + \frac{\lambda_2}{c_2}\widetilde{\chi}(s) + \frac{\lambda_1\lambda_2}{c_1c_2}\cdot\frac{1-\widetilde{b}(s)}{s}\right] = 0\,.$$

We rewrite equation (2.18)as

$$s\left[g_1(s) - g_2(s)\right] = 0\,,$$

where

$$g_1(s) = s - \frac{\lambda_1}{c_1} - \frac{\lambda_2}{c_2}\,,$$

$$g_2(s) = -\frac{\lambda_1}{c_1}\widetilde{\xi}(s) - \frac{\lambda_2}{c_2}\widetilde{\chi}(s) - \frac{\lambda_1\lambda_2}{c_1c_2}\widetilde{\overline{B}}(s)\,.$$

The nonzero roots of equation (2.18) coincide with those of $g_1(s) - g_2(s) = 0$.

We analyze the roots of $g_1(s) - g_2(s) = 0$ by applying Rouché's theorem to a closed contour C, formed by the semi-circle $\{s : |s| = d,\ \mathrm{Re}(s) > 0\}$ in the right half plane and the imaginary axis, where d is a large enough constant. The functions $g_1(s)$ and $g_2(s)$ are analytic inside C and $g_1(s)$ has exactly one zero inside C. On the semi-circle part of the boundary of C, since $\mathrm{Re}(s) > 0$, we have

$$\left|\widetilde{\chi}(s)\right| < \widetilde{\chi}(0) \leq 1\,,$$

$$\left|\widetilde{\xi}(s)\right| < \widetilde{\xi}(0) \leq 1\,,$$

$$\left|\widetilde{\overline{B}}(s)\right| < \widetilde{\overline{B}}(0) = \mu\,.$$

Thus, comparing

$$|g_1(s)| \geq |s| - \left|\frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_2}\right| = d - \left(\frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_2}\right)$$

and

$$|g_2(s)| \leq \frac{\lambda_1}{c_1}\left|\widetilde{\xi}(s)\right| + \frac{\lambda_2}{c_2}\left|\widetilde{\chi}(s)\right| + \frac{\lambda_1\lambda_2}{c_1c_2}\left|\widetilde{\overline{B}}(s)\right| < \frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_2} + \frac{\lambda_1\lambda_2}{c_1c_2}\mu,$$

we obtain that $|g_2(s)| < |g_1(s)|$ on the semi-circle part of the boundary of C, for a sufficiently large d.

On the imaginary axis part of the boundary of C, we have $\mathrm{Re}(s) = 0$, which implies

$$\left|\widetilde{\chi}(s)\right| \le \widetilde{\chi}(0)\,,$$
$$\left|\widetilde{\xi}(s)\right| \le \widetilde{\xi}(0)\,,$$
$$\left|\widetilde{\overline{B}}(s)\right| \le \widetilde{\overline{B}}(0) = \mu\,.$$

On one hand, utilizing the positive-security-loading condition (2.2) with relations $\mathbb{P}\{X > Q\} = \widetilde{\xi}(0)$ and $\mathbb{P}\{X < Q\} = \widetilde{\chi}(0)$ which is

$$\mu < \frac{c_1}{\lambda_1}\widetilde{\xi}(0) + \frac{c_2}{\lambda_2}\widetilde{\chi}(0)\,,$$

we obtain

$$\begin{aligned}
|g_2(s)| &\le \frac{\lambda_1}{c_1}\left|\widetilde{\xi}(s)\right| + \frac{\lambda_2}{c_2}\left|\widetilde{\chi}(s)\right| + \frac{\lambda_1\lambda_2}{c_1c_2}\left|\widetilde{\overline{B}}(s)\right| \\
&\le \frac{\lambda_1}{c_1}\widetilde{\xi}(0) + \frac{\lambda_2}{c_2}\widetilde{\chi}(0) + \frac{\lambda_1\lambda_2}{c_1c_2}\mu \\
&< \frac{\lambda_1}{c_1}\widetilde{\xi}(0) + \frac{\lambda_2}{c_2}\widetilde{\chi}(0) + \frac{\lambda_1\lambda_2}{c_1c_2}\cdot\frac{c_1}{\lambda_1}\widetilde{\xi}(0) + \frac{\lambda_1\lambda_2}{c_1c_2}\cdot\frac{c_2}{\lambda_2}\widetilde{\chi}(0) \\
&= \frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_2}\,,
\end{aligned}$$

noting that $\widetilde{\xi}(0) + \widetilde{\chi}(0) = 1$. On the other hand, since $\mathrm{Re}(s) = 0$, we have

$$|g_1(s)| \ge \frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_2}\,,$$

thus $|g_2(s)| < |g_1(s)|$ holds on the imaginary axis part of the contour C as well. Applying Rouché's theorem to the contour C and letting the radius $d \to \infty$, we conclude that equation $g_1(s) - g_2(s) = 0$ has exactly one root in the positive half plane, which indicates that equation (2.18) has exactly one root in the positive half plane. Moreover, the root is real, since the complex roots of analytic functions that are presented in series form with only real coefficients come in conjugate pairs. Recall that zero is also a root of (2.18). Therefore, Lundberg's equation with $\delta = 0$ has exactly two roots with nonnegative real parts, where both are real roots and one of them is zero. ■

Lemma 2.2 *When $\delta > 0$, the generalized Lundberg's equation* (2.17) *has exactly two roots with nonnegative real parts. Moreover, they are distinct, positive and real.*

Proof We rewrite Lundberg's equation (2.17) as follows:

$$\widetilde{h}_1(s) - \widetilde{h}_2(s) = 0,$$

where

$$\widetilde{h}_1(s) = \left(s - \frac{\lambda_1 + \delta}{c_1}\right)\left(s - \frac{\lambda_2 + \delta}{c_2}\right), \tag{2.19}$$

$$\widetilde{h}_2(s) = \frac{\lambda_1}{c_1}\left(\frac{\lambda_2 + \delta}{c_2} - s\right)\widetilde{\xi}(s) + \frac{\lambda_2}{c_2}\left(\frac{\lambda_1 + \delta}{c_1} - s\right)\widetilde{\chi}(s). \tag{2.20}$$

We analyze the roots of Lundberg's equation $\widetilde{h}_1(s) - \widetilde{h}_2(s) = 0$ by applying Rouché's theorem to the same contour C as in the proof of Lemma 2.1. The equation $\widetilde{h}_1(s) = 0$ has exactly two roots inside the contour C, and $\widetilde{h}_1(s)$ and $\widetilde{h}_2(s)$ are analytic inside of C. On the semi-circle part of the boundary of C, we have

$$\begin{aligned}
\left|s - \frac{\lambda_1 + \delta}{c_1}\right| &\geq |s| - \left|\frac{\lambda_1 + \delta}{c_1}\right| = d - \left|\frac{\lambda_1 + \delta}{c_1}\right| > \frac{\lambda_1}{c_1}, \\
\left|s - \frac{\lambda_2 + \delta}{c_2}\right| &\geq |s| - \left|\frac{\lambda_2 + \delta}{c_2}\right| = d - \left|\frac{\lambda_2 + \delta}{c_2}\right| > \frac{\lambda_2}{c_2},
\end{aligned}$$

for a sufficiently large radius d. Hence, the above inequalities together with $\left|\widetilde{\xi}(s)\right| < \widetilde{\xi}(0)$ and $|\widetilde{\chi}(s)| < \widetilde{\chi}(0)$ yield

$$\begin{aligned}
\left|\widetilde{h}_2(s)\right| &\leq \frac{\lambda_1}{c_1}\left|s - \frac{\lambda_2 + \delta}{c_2}\right|\left|\widetilde{\xi}(s)\right| + \frac{\lambda_2}{c_2}\left|s - \frac{\lambda_1 + \delta}{c_1}\right|\left|\widetilde{\chi}(s)\right| \\
&< \left|s - \frac{\lambda_1 + \delta}{c_1}\right|\left|s - \frac{\lambda_2 + \delta}{c_2}\right|\widetilde{\xi}(0) + \left|s - \frac{\lambda_2 + \delta}{c_2}\right|\left|s - \frac{\lambda_1 + \delta}{c_1}\right|\widetilde{\chi}(0) \\
&= \left|s - \frac{\lambda_2 + \delta}{c_2}\right|\left|s - \frac{\lambda_1 + \delta}{c_1}\right| \\
&= \left|\widetilde{h}_1(s)\right|,
\end{aligned}$$

since $\widetilde{\xi}(0) + \widetilde{\chi}(0) = 1$. Consider now the part of the contour C on the imaginary axis. Since $\mathrm{Re}(s) = 0$ and $\delta > 0$, we have

$$\begin{aligned}
\left|\widetilde{h}_1(s)\right| &= \left|s - \frac{\lambda_2+\delta}{c_2}\right|\left|s - \frac{\lambda_1+\delta}{c_1}\right| \\
&= \left|s - \frac{\lambda_2+\delta}{c_2}\right|\left|s - \frac{\lambda_1+\delta}{c_1}\right|\widetilde{\xi}(0) + \left|s - \frac{\lambda_2+\delta}{c_2}\right|\left|s - \frac{\lambda_1+\delta}{c_1}\right|\widetilde{\chi}(0) \\
&\geq \left|s - \frac{\lambda_2+\delta}{c_2}\right|\left(\frac{\lambda_1+\delta}{c_1}\right)\widetilde{\xi}(0) + \left(\frac{\lambda_2+\delta}{c_2}\right)\left|s - \frac{\lambda_1+\delta}{c_1}\right|\widetilde{\chi}(0) \\
&> \frac{\lambda_1}{c_1}\left|s - \frac{\lambda_2+\delta}{c_2}\right|\widetilde{\xi}(0) + \frac{\lambda_2}{c_2}\left|s - \frac{\lambda_1+\delta}{c_1}\right|\widetilde{\chi}(0) \\
&\geq \frac{\lambda_1}{c_1}\left|s - \frac{\lambda_2+\delta}{c_2}\right|\left|\widetilde{\xi}(s)\right| + \frac{\lambda_2}{c_2}\left|s - \frac{\lambda_1+\delta}{c_1}\right|\left|\widetilde{\chi}(s)\right| \\
&\geq \left|\widetilde{h}_2(s)\right|.
\end{aligned}$$

Thus, $\left|\widetilde{h}_2(s)\right| < \left|\widetilde{h}_1(s)\right|$ holds on the boundary of the closed contour C and we may conclude that $\widetilde{h}_1(s) - \widetilde{h}_2(s) = 0$ has two roots inside of C, denoted by r and ρ. Letting $d \to \infty$ shows that r and ρ are the only two roots in the nonnegative half plane.

It remains to show that when $\delta > 0$, r and ρ are distinct and real. We know that as δ converges to 0, r and ρ converge to the roots of the simpler equation (2.18). Then, as δ converges to zero, one of r and ρ converges to zero and the other one converges to a strictly positive number, hence they are distinct. Moreover, we prove by contradiction that r and ρ are real numbers. Suppose r and ρ are complex roots of the analytic function $g_1(s) - g_2(s)$, then they must be a conjugated pair, i.e. $r = a + bi$ and $\rho = a - bi$ for some real numbers $a, b > 0$. When δ converges to zero, we know that one of the roots converges to zero, which indicates that a and b converge to 0 simultaneously. Then, the other root also converges to 0, which contradicts the fact that the other root converges to a strictly positive number. Thus, r and ρ are both real.

In addition, without lose of generality, we let $\rho < r$. Then, it follows that $\rho \in \left(0, \min\left\{\frac{\lambda_1+\delta}{c_1}, \frac{\lambda_2+\delta}{c_2}\right\}\right)$ and $r \in \left(\min\left\{\frac{\lambda_1+\delta}{c_1}, \frac{\lambda_2+\delta}{c_2}\right\}, \max\left\{\frac{\lambda_1+\delta}{c_1}, \frac{\lambda_2+\delta}{c_2}\right\}\right)$, since at $s = 0$, $\widetilde{h}_1(s) > \widetilde{h}_2(s)$; at $s = \min\left\{\frac{\lambda_1+\delta}{c_1}, \frac{\lambda_2+\delta}{c_2}\right\}$, $\widetilde{h}_1(s) < \widetilde{h}_2(s)$; and at $s = \max\left\{\frac{\lambda_1+\delta}{c_1}, \frac{\lambda_2+\delta}{c_2}\right\}$, $\widetilde{h}_1(s) > \widetilde{h}_2(s)$. ■

2.3 Gerber-Shiu expected discounted penalty function

In order to invert the Laplace transforms of (2.15) and (2.16) for $m_i(u)$, $i = 1, 2$, we need to solve for the values of $m_i(0)$, $i = 1, 2$, first. Lemmas 2.1 and 2.2 indicate that Lundberg's equation (2.17) has exactly two nonnegative roots for all $\delta \geq 0$. Denote these roots by r and ρ, for an arbitrary $\delta \geq 0$. When s takes the value of r or ρ, the right-hand sides of (2.15) and (2.16) also equal to zero. Moreover, when $s = r$ (or $s = \rho$), the right-hand sides of (2.15) and (2.16) are identical due to Lundberg's equation (2.17). Thus, only two equations are obtained

$$c_1\left[c_2 r - \lambda_2 - \delta + \lambda_2\widetilde{\chi}(r)\right] m_1(0) - \lambda_1 c_2 m_2(0)\widetilde{\chi}(r) - \lambda_1(c_2 r - \lambda_2 - \delta)\widetilde{\zeta}(r) = 0\,,$$

$$c_1\left[c_2 \rho - \lambda_2 - \delta + \lambda_2\widetilde{\chi}(\rho)\right] m_1(0) - \lambda_1 c_2 m_2(0)\widetilde{\chi}(\rho) - \lambda_1(c_2 \rho - \lambda_2 - \delta)\widetilde{\zeta}(\rho) = 0\,.$$

Solving this system of equations yields

$$m_1(0) = \frac{\lambda_1\widetilde{\chi}(\rho)(c_2 r - \lambda_2 - \delta)\widetilde{\zeta}(r) - \lambda_1\widetilde{\chi}(r)(c_2\rho - \lambda_2 - \delta)\widetilde{\zeta}(\rho)}{c_1\Big[(c_2 r - \lambda_2 - \delta)\widetilde{\chi}(\rho) - (c_2\rho - \lambda_2 - \delta)\widetilde{\chi}(r)\Big]}\,, \tag{2.21}$$

$$m_2(0) = \frac{\Big[c_2\rho - \lambda_2 - \delta + \lambda_2\widetilde{\chi}(\rho)\Big](c_2 r - \lambda_2 - \delta)\widetilde{\zeta}(r) - \Big[c_2 r - \lambda_2 - \delta + \lambda_2\widetilde{\chi}(r)\Big](c_2\rho - \lambda_2 - \delta)\widetilde{\zeta}(\rho)}{c_2\Big[(c_2 r - \lambda_2 - \delta)\widetilde{\chi}(\rho) - (c_2\rho - \lambda_2 - \delta)\widetilde{\chi}(r)\Big]}\,. \tag{2.22}$$

We rearrange (2.22) as

$$m_2(0) = \frac{(c_2\rho - \lambda_2 - \delta)(c_2 r - \lambda_2 - \delta)\Big[\widetilde{\zeta}(r) - \widetilde{\zeta}(\rho)\Big]}{c_2\Big[(c_2 r - \lambda_2 - \delta)\widetilde{\chi}(\rho) - (c_2\rho - \lambda_2 - \delta)\widetilde{\chi}(r)\Big]} + \frac{c_1\lambda_2}{c_2\lambda_1}\, m_1(0),$$

multiplying by λ_1/c_1 leads to a useful relation for some later results,

$$\frac{\lambda_2}{c_2} m_1(0) - \frac{\lambda_1}{c_1} m_2(0) \;=\; \frac{\lambda_1(c_2\rho - \lambda_2 - \delta)(c_2 r - \lambda_2 - \delta)\left[\widetilde{\zeta}(\rho) - \widetilde{\zeta}(r)\right]}{c_1 c_2\Big[(c_2 r - \lambda_2 - \delta)\widetilde{\chi}(\rho) - (c_2\rho - \lambda_2 - \delta)\widetilde{\chi}(r)\Big]}. \tag{2.23}$$

Substituting the solutions for $m_i(0)$, $i = 1, 2$, into (2.15) and (2.16), and inverting the Laplace transforms with respect to s results in the following theorem.

Theorem 2.3 *The Gerber-Shiu discounted penalty functions $m_1(u)$ and $m_2(u)$ defined in* (2.3) *satisfy the following system of defective-renewal equations,*

$$m_1(u) = \kappa_\delta \int_0^u m_1(u-y)\eta(y)\,dy + \sigma_1(u), \tag{2.24}$$

$$m_2(u) = \kappa_\delta \int_0^u m_2(u-y)\eta(y)\,dy + \sigma_2(u), \tag{2.25}$$

where

$$\begin{aligned}\kappa_\delta =& \frac{\lambda_1}{c_1}\cdot\frac{\lambda_2+\delta}{c_2}T_0T_rT_\rho\xi(0) + \frac{\lambda_1}{c_1}\cdot\frac{r}{r-\rho}T_0T_r\xi(0) - \frac{\lambda_1}{c_1}\cdot\frac{\rho}{r-\rho}T_0T_\rho\xi(0)\\ &+ \frac{\lambda_2}{c_2}\cdot\frac{\lambda_1+\delta}{c_1}T_0T_rT_\rho\chi(0) + \frac{\lambda_2}{c_2}\cdot\frac{r}{r-\rho}T_0T_r\chi(0) - \frac{\lambda_2}{c_2}\cdot\frac{\rho}{r-\rho}T_0T_\rho\chi(0),\end{aligned} \tag{2.26}$$

$$q_1 = \frac{\frac{\lambda_1}{c_1}\cdot\frac{\lambda_2+\delta}{c_2}T_0T_rT_\rho\xi(0)}{\kappa_\delta},\quad q_2 = \frac{\frac{\lambda_1}{c_1}\cdot\frac{r}{r-\rho}T_0T_r\xi(0)}{\kappa_\delta},\quad q_3 = \frac{-\frac{\lambda_1}{c_1}\cdot\frac{\rho}{r-\rho}T_0T_\rho\xi(0)}{\kappa_\delta},$$

$$q_4 = \frac{\frac{\lambda_2}{c_2}\cdot\frac{\lambda_1+\delta}{c_1}T_0T_rT_\rho\chi(0)}{\kappa_\delta},\quad q_5 = \frac{\frac{\lambda_2}{c_2}\cdot\frac{r}{r-\rho}T_0T_r\chi(0)}{\kappa_\delta},\quad q_6 = \frac{-\frac{\lambda_2}{c_2}\cdot\frac{\rho}{r-\rho}T_0T_\rho\chi(0)}{\kappa_\delta},$$

$$\eta(y) = q_1\frac{T_rT_\rho\xi(y)}{T_0T_rT_\rho\xi(0)} + q_2\frac{T_r\xi(y)}{T_0T_r\xi(0)} + q_3\frac{T_\rho\xi(y)}{T_0T_\rho\xi(0)} + q_4\frac{T_rT_\rho\chi(y)}{T_0T_rT_\rho\chi(0)} + q_5\frac{T_r\chi(y)}{T_0T_r\chi(0)} + q_6\frac{T_\rho\chi(y)}{T_0T_\rho\chi(0)}, \tag{2.27}$$

with r, ρ denoting the nonnegative roots of Lundberg's equation (2.17)*, and*

$$\sigma_1(u) = \frac{\lambda_1}{c_1}\left[\frac{\lambda_2+\delta}{c_2}T_rT_\rho\zeta(u) + \frac{rT_r\zeta(u)-\rho T_\rho\zeta(u)}{r-\rho}\right] + \left[\frac{\lambda_2}{c_2}m_1(0) - \frac{\lambda_1}{c_1}m_2(0)\right]T_rT_\rho\chi(u), \tag{2.28}$$

$$\sigma_2(u) = \frac{\lambda_2}{c_2}\left[\frac{\lambda_1+\delta}{c_1}T_rT_\rho\zeta(u) + \frac{rT_r\zeta(u)-\rho T_\rho\zeta(u)}{r-\rho}\right] - \left[\frac{\lambda_2}{c_2}m_1(0) - \frac{\lambda_1}{c_1}m_2(0)\right]T_rT_\rho\xi(u), \tag{2.29}$$

with $\left[\frac{\lambda_2}{c_2}m_1(0) - \frac{\lambda_1}{c_1}m_2(0)\right]$ *expressed in* (2.23). *Also, $\eta(y), y \geq 0$, is a probability density function and κ_δ is a constant satisfying* $0 < \kappa_\delta < 1$.

Proof For all $s \geq 0$ (except for r and ρ), we rearrange equations (2.15) and (2.16) as follows,

$$\widetilde{m}_1(s) = \frac{\widetilde{\alpha}_1(s) + \widetilde{\beta}_1(s)}{\widetilde{h}_1(s) - \widetilde{h}_2(s)}, \tag{2.30}$$

$$\widetilde{m}_2(s) = \frac{\widetilde{\alpha}_2(s) + \widetilde{\beta}_2(s)}{\widetilde{h}_1(s) - \widetilde{h}_2(s)}, \tag{2.31}$$

with $\widetilde{h}_1(s)$, $\widetilde{h}_2(s)$ defined in (2.19), (2.20), and

$$\widetilde{\alpha}_1(s) = \left(s - \frac{\lambda_2 + \delta}{c_2}\right) m_1(0), \tag{2.32}$$

$$\widetilde{\beta}_1(s) = \frac{\lambda_1}{c_1}\left(\frac{\lambda_2 + \delta}{c_2} - s\right)\widetilde{\zeta}(s) + \left[\frac{\lambda_2}{c_2} m_1(0) - \frac{\lambda_1}{c_1} m_2(0)\right]\widetilde{\chi}(s), \tag{2.33}$$

$$\widetilde{\alpha}_2(s) = \left(s - \frac{\lambda_1 + \delta}{c_1}\right) m_2(0),$$

$$\widetilde{\beta}_2(s) = \frac{\lambda_2}{c_2}\left(\frac{\lambda_1 + \delta}{c_1} - s\right)\widetilde{\zeta}(s) - \left[\frac{\lambda_2}{c_2} m_1(0) - \frac{\lambda_1}{c_1} m_2(0)\right]\widetilde{\xi}(s). \tag{2.34}$$

The transforms $\widetilde{\alpha}_1(s)$ and $\widetilde{h}_1(s)$ are polynomials in s of degree one and degree two, respectively. Implementing the Lagrange interpolation theorem, the following results are reached (a detailed derivation may be found in Boudreault et al., 2006, for example),

$$\widetilde{\alpha}_1(s) + \widetilde{\beta}_1(s) = (s - r)(s - \rho)\, T_s T_r T_\rho \beta_1(0), \tag{2.35}$$

$$\widetilde{h}_1(s) - \widetilde{h}_2(s) = (s - r)(s - \rho)\left[T_0 T_r T_\rho h_1(0) - T_s T_r T_\rho h_2(0)\right] = (s - r)(s - \rho)\left[1 - T_s T_r T_\rho h_2(0)\right], \tag{2.36}$$

where r and ρ are the two positive roots of Lundberg's equation. Inserting (2.35) and (2.36) into (2.30) yields

$$\widetilde{m}_1(s) = \frac{T_s T_r T_\rho \beta_1(0)}{1 - T_s T_r T_\rho h_2(0)}$$

or equivalently,

$$\widetilde{m}_1(s) = \widetilde{m}_1(s)\, T_s T_r T_\rho h_2(0) + T_s T_r T_\rho \beta_1(0). \tag{2.37}$$

Similarly for (2.31), we obtain

$$\widetilde{m}_2(s) = \widetilde{m}_2(s)\, T_s T_r T_\rho h_2(0) + T_s T_r T_\rho \beta_2(0). \tag{2.38}$$

In order to invert the Laplace transforms in (2.37) and (2.38), we need to derive the Laplace inversion for $T_s T_r T_\rho h_2(0)$, $T_s T_r T_\rho \beta_1(0)$ and $T_s T_r T_\rho \beta_2(0)$. Utilizing equation (2.6) repeatedly, we deduce from (2.20) that

$$
\begin{aligned}
T_sT_rT_\rho h_2(0) &= \frac{1}{r-\rho}\left[\frac{\widetilde{h}_2(\rho)-\widetilde{h}_2(s)}{s-\rho}-\frac{\widetilde{h}_2(r)-\widetilde{h}_2(s)}{s-r}\right]\\
&= \frac{\widetilde{h}_2(\rho)-\widetilde{h}_2(s)}{(s-\rho)(r-\rho)}-\frac{\widetilde{h}_2(r)-\widetilde{h}_2(s)}{(s-r)(r-\rho)}\\
&= \frac{\lambda_1}{c_1}\cdot\frac{\lambda_2+\delta}{c_2}\,T_sT_rT_\rho\xi(0)-\frac{\lambda_1}{c_1}\left[\frac{\rho\widetilde{\xi}(\rho)-s\widetilde{\xi}(s)}{(s-\rho)(r-\rho)}-\frac{r\widetilde{\xi}(r)-s\widetilde{\xi}(s)}{(s-r)(r-\rho)}\right]\\
&\quad+\frac{\lambda_2}{c_2}\cdot\frac{\lambda_1+\delta}{c_1}\,T_sT_rT_\rho\chi(0)-\frac{\lambda_2}{c_2}\left[\frac{\rho\widetilde{\chi}(\rho)-s\widetilde{\chi}(s)}{(s-\rho)(r-\rho)}-\frac{r\widetilde{\chi}(r)-s\widetilde{\chi}(s)}{(s-r)(r-\rho)}\right]\\
&= \frac{\lambda_1}{c_1}\cdot\frac{\lambda_2+\delta}{c_2}\,T_sT_rT_\rho\xi(0)-\frac{\lambda_1}{c_1}\cdot\frac{1}{r-\rho}\left[\frac{\rho\widetilde{\xi}(\rho)-s\widetilde{\xi}(s)}{s-\rho}-\frac{r\widetilde{\xi}(r)-s\widetilde{\xi}(s)}{s-r}\right]\\
&\quad+\frac{\lambda_2}{c_2}\cdot\frac{\lambda_1+\delta}{c_1}\,T_sT_rT_\rho\chi(0)-\frac{\lambda_2}{c_2}\cdot\frac{1}{r-\rho}\left[\frac{\rho\widetilde{\chi}(\rho)-s\widetilde{\chi}(s)}{s-\rho}-\frac{r\widetilde{\chi}(r)-s\widetilde{\chi}(s)}{s-r}\right]\\
&= \frac{\lambda_1}{c_1}\cdot\frac{\lambda_2+\delta}{c_2}\,T_sT_rT_\rho\xi(0)\\
&\quad-\frac{\lambda_1}{c_1}\cdot\frac{1}{r-\rho}\left[\frac{\rho\widetilde{\xi}(\rho)-\rho\widetilde{\xi}(s)+\rho\widetilde{\xi}(s)-s\widetilde{\xi}(s)}{s-\rho}-\frac{r\widetilde{\xi}(r)-r\widetilde{\xi}(s)+r\widetilde{\xi}(s)-s\widetilde{\xi}(s)}{s-r}\right]\\
&\quad+\frac{\lambda_2}{c_2}\cdot\frac{\lambda_1+\delta}{c_1}\,T_sT_rT_\rho\chi(0)\\
&\quad-\frac{\lambda_2}{c_2}\cdot\frac{1}{r-\rho}\left[\frac{\rho\widetilde{\chi}(\rho)-\rho\widetilde{\chi}(s)+\rho\widetilde{\chi}(s)-s\widetilde{\chi}(s)}{s-\rho}-\frac{r\widetilde{\chi}(r)-r\widetilde{\chi}(s)+r\widetilde{\chi}(s)-s\widetilde{\chi}(s)}{s-r}\right]\\
&= \frac{\lambda_1}{c_1}\cdot\frac{\lambda_2+\delta}{c_2}\,T_sT_rT_\rho\xi(0)-\frac{\lambda_1}{c_1}\left\{\frac{-\widetilde{\xi}(s)+\rho T_sT_\rho\xi(0)-\left[-\widetilde{\xi}(s)+rT_sT_r\xi(0)\right]}{r-\rho}\right\}\\
&\quad+\frac{\lambda_2}{c_2}\cdot\frac{\lambda_1+\delta}{c_1}\,T_sT_rT_\rho\chi(0)-\frac{\lambda_2}{c_2}\left\{\frac{-\widetilde{\chi}(s)+\rho T_sT_\rho\chi(0)-\left[-\widetilde{\chi}(s)+rT_sT_r\chi(0)\right]}{r-\rho}\right\}\\
&= \frac{\lambda_1}{c_1}\cdot\frac{\lambda_2+\delta}{c_2}\,T_sT_rT_\rho\xi(0)+\frac{\lambda_1}{c_1}\left[\frac{rT_sT_r\xi(0)-\rho T_sT_\rho\xi(0)}{r-\rho}\right]\\
&\quad+\frac{\lambda_2}{c_2}\cdot\frac{\lambda_1+\delta}{c_1}\,T_sT_rT_\rho\chi(0)+\frac{\lambda_2}{c_2}\left[\frac{rT_sT_r\chi(0)-\rho T_sT_\rho\chi(0)}{r-\rho}\right].
\end{aligned}
\tag{2.39}
$$

Therefore, the Laplace transform may be inverted to

$$
\begin{aligned}
T_rT_\rho h_2(u) = \frac{\lambda_1}{c_1}&\left[\frac{\lambda_2+\delta}{c_2}\,T_rT_\rho\xi(u)+\frac{rT_r\xi(u)-\rho T_\rho\xi(u)}{r-\rho}\right]\\
&+\frac{\lambda_2}{c_2}\left[\frac{\lambda_1+\delta}{c_1}\,T_rT_\rho\chi(u)+\frac{rT_r\chi(u)-\rho T_\rho\chi(u)}{r-\rho}\right].
\end{aligned}
\tag{2.40}
$$

Following a similar procedure, from (2.33) and (2.34) we obtain

$$\begin{aligned}T_sT_rT_\rho\beta_1(0) &= \frac{\lambda_1}{c_1}\left[\frac{\lambda_2+\delta}{c_2}\,T_sT_rT_\rho\zeta(0) + \frac{rT_sT_r\zeta(0)-\rho T_sT_\rho\zeta(0)}{r-\rho}\right]\\&\quad+\left[\frac{\lambda_2}{c_2}m_1(0)-\frac{\lambda_1}{c_1}m_2(0)\right]T_sT_rT_\rho\chi(0), \qquad (2.41)\\T_sT_rT_\rho\beta_2(0) &= \frac{\lambda_2}{c_2}\left[\frac{\lambda_1+\delta}{c_1}\,T_sT_rT_\rho\zeta(0) + \frac{rT_sT_r\zeta(0)-\rho T_sT_\rho\zeta(0)}{r-\rho}\right]\\&\quad-\left[\frac{\lambda_2}{c_2}m_1(0)-\frac{\lambda_1}{c_1}m_2(0)\right]T_sT_rT_\rho\xi(0), \qquad (2.42)\end{aligned}$$

and the inversion of these Laplace transforms yields

$$\begin{aligned}T_rT_\rho\beta_1(u) &= \frac{\lambda_1}{c_1}\left[\frac{\lambda_2+\delta}{c_2}\,T_rT_\rho\zeta(u) + \frac{rT_r\zeta(u)-\rho T_\rho\zeta(u)}{r-\rho}\right]\\&\quad+\left[\frac{\lambda_2}{c_2}m_1(0)-\frac{\lambda_1}{c_1}m_2(0)\right]T_rT_\rho\chi(u), \qquad (2.43)\\T_rT_\rho\beta_2(u) &= \frac{\lambda_2}{c_2}\left[\frac{\lambda_1+\delta}{c_1}\,T_rT_\rho\zeta(u) + \frac{rT_r\zeta(u)-\rho T_\rho\zeta(u)}{r-\rho}\right]\\&\quad-\left[\frac{\lambda_2}{c_2}m_1(0)-\frac{\lambda_1}{c_1}m_2(0)\right]T_rT_\rho\xi(u). \qquad (2.44)\end{aligned}$$

Utilizing (2.40), (2.43) and (2.44), we invert the Laplace transforms in (2.37) and (2.38) to

$$\begin{aligned}m_1(u) &= \int_0^u m_1(u-y)T_r\,T_\rho h_2(y)\,dy + T_rT_\rho\beta_1(u),\\m_2(u) &= \int_0^u m_2(u-y)T_r\,T_\rho h_2(y)\,dy + T_rT_\rho\beta_2(u).\end{aligned}$$

Employing the definitions of κ_δ, $\eta(y)$, $\sigma_1(u)$ and $\sigma_2(u)$ provided by equations (2.26) to (2.29) respectively, we obtain

$$\begin{aligned}m_1(u) &= \kappa_\delta\int_0^u m_1(u-y)\,\eta(y)\,\mathrm{d}y + \sigma_1(u),\\m_2(u) &= \kappa_\delta\int_0^u m_2(u-y)\,\eta(y)\,\mathrm{d}y + \sigma_2(u),\end{aligned}$$

which yields the system of renewal equations (2.24) and (2.25).

To demonstrate $\eta(y)$ is a proper p.d.f., we notice by comparing equalities (2.26) and (2.39) that

$$\kappa_\delta = T_0T_rT_\rho h_2(0)\,, \qquad (2.45)$$

hence

$$\eta(y) = \frac{T_r T_\rho h_2(y)}{\kappa_\delta} = \frac{T_r T_\rho h_2(y)}{T_0 T_r T_\rho h_2(0)} = \frac{T_r T_\rho h_2(y)}{\int_0^\infty T_r T_\rho h_2(y)\,\mathrm{d}y}. \tag{2.46}$$

Rearranging (2.40) yields

$$\begin{aligned}
T_r T_\rho h_2(u) &= \frac{\lambda_1}{c_1}\left[\frac{\lambda_2+\delta}{c_2}\,T_r T_\rho \xi(u) + \frac{rT_r\xi(u) - \rho T_r\xi(u) + \rho T_r\xi(u) - \rho T_\rho \xi(u)}{r-\rho}\right] \\
&\quad + \frac{\lambda_2}{c_2}\left[\frac{\lambda_1+\delta}{c_1}\,T_r T_\rho \chi(u) + \frac{rT_r\chi(u) - \rho T_r\chi(u) + \rho T_r\chi(u) - \rho T_\rho \chi(u)}{r-\rho}\right] \\
&= \frac{\lambda_1}{c_1}\left[\left(\frac{\lambda_2+\delta}{c_2} - \rho\right) T_r T_\rho \xi(u) + T_r\xi(u)\right] + \frac{\lambda_2}{c_2}\left[\left(\frac{\lambda_1+\delta}{c_1} - \rho\right) T_r T_\rho \chi(u) + T_r\chi(u)\right].
\end{aligned} \tag{2.47}$$

With loss of generality, let the two nonnegative roots of Lundberg's equation be $\rho < r$. From Lemmas 2.1 and 2.2, we have for arbitrary $\delta \geq 0$ that $\rho < \min\left\{\frac{\lambda_1+\delta}{c_1}, \frac{\lambda_2+\delta}{c_2}\right\}$. Then, all terms in (2.47) is positive for all $u \geq 0$, which implies $T_r T_\rho h_2(u) > 0$ for all $u \geq 0$. Recall (2.46) where

$$\eta(y) = \frac{T_r T_\rho h_2(y)}{\int_0^\infty T_r T_\rho h_2(y)\,\mathrm{d}y}.$$

Thus, $\eta(y)$ is positive for all $y \geq 0$. Moreover, $\int_0^\infty \eta(y)\,\mathrm{d}y = 1$, which confirms that $\eta(y)$ is a proper p.d.f.

To verify that (2.24) and (2.25) are defective renewal equations, it remains to show that $\kappa_\delta < 1$. We consider the cases $\delta > 0$ and $\delta = 0$ separately. When $\delta > 0$, recall (2.45) that $\kappa_\delta = T_0 T_r T_\rho h_2(0)$. Inserting $s = 0$ into (2.36) leads to

$$\kappa_\delta = T_0 T_r T_\rho h_2(0) = 1 - \frac{\tilde{h}_1(0) - \tilde{h}_2(0)}{r\rho}.$$

Utilizing (2.19), (2.20) and the Laplace-transformed relationship (2.10) at $s = 0$, we obtain

$$\kappa_\delta = 1 - \frac{\frac{\lambda_1+\delta}{c_1}\cdot\frac{\lambda_2+\delta}{c_2} - \frac{\lambda_1}{c_1}\cdot\frac{\lambda_2+\delta}{c_2}\,\tilde{\xi}(0) - \frac{\lambda_2}{c_2}\cdot\frac{\lambda_1+\delta}{c_1}\,\tilde{\chi}(0)}{r\rho}$$

$$
\begin{aligned}
&= 1 - \frac{\dfrac{\lambda_1\delta}{c_1c_2}\left[1-\tilde{\xi}(0)\right] + \dfrac{\lambda_2\delta}{c_1c_2}\left[1-\tilde{\chi}(0)\right] + \dfrac{\delta^2}{c_1c_2}}{r\rho} \\
&= 1 - \frac{\dfrac{\lambda_1\delta}{c_1c_2}\tilde{\chi}(0) + \dfrac{\lambda_2\delta}{c_1c_2}\tilde{\xi}(0) + \dfrac{\delta^2}{c_1c_2}}{r\rho} \\
&< 1\,,
\end{aligned}
$$

since $\lambda_1,\ \lambda_2,\ \delta,\ c_1,\ c_2,\ r,\ \rho > 0, \widetilde{\xi}(0) = \mathbb{P}\{X > Q\} \geq 0$ and $\widetilde{\chi}(0) = \mathbb{P}\{X < Q\} \geq 0$. Therefore, we conclude that $0 < \kappa_\delta < 1$ when $\delta > 0$.

When $\delta = 0$, without loss of generality, let the two nonnegative roots of Lundberg's equation be $r > 0$ and $\rho = 0$. Denote κ_δ as κ_0 to suggest that $\delta = 0$. Then, identity (2.26) reduces to

$$
\begin{aligned}
\kappa_0 &= \frac{\lambda_1\lambda_2}{c_1c_2}T_0T_0T_r\xi(0) + \frac{\lambda_1}{c_1}T_0T_r\xi(0) + \frac{\lambda_1\lambda_2}{c_1c_2}T_0T_0T_r\chi(0) + \frac{\lambda_2}{c_2}T_0T_r\chi(0) \\
&= \frac{\lambda_1\lambda_2}{c_1c_2}T_0T_0T_r\, b(0) + \frac{\lambda_1}{c_1}T_0T_r\xi(0) + \frac{\lambda_2}{c_2}T_0T_r\chi(0)\,,
\end{aligned} \tag{2.48}
$$

by equation (2.10). To prove that $\kappa_0 < 1$, utilizing Property 4 of Translation Operator T in Li and Garrido (2004) where

$$
T_0T_0T_r\, b(0) = \int_0^\infty u \cdot b(u)\, \mathrm{d}u = \mu\,,
$$

we rewrite (2.48) as follows

$$
\begin{aligned}
\kappa_0 &= \frac{\lambda_1\lambda_2}{c_1c_2}T_0T_0T_r\, b(0) + \frac{\lambda_1}{c_1}T_0T_r\xi(0) + \frac{\lambda_2}{c_2}T_0T_r\chi(0) \\
&= \frac{\lambda_1\lambda_2}{c_1c_2}\left[\frac{T_0T_0b(0) - T_0T_rb(0)}{r}\right] + \frac{\lambda_1}{c_1}\left[\frac{\tilde{\xi}(0)-\tilde{\xi}(r)}{r}\right] + \frac{\lambda_2}{c_2}\left[\frac{\tilde{\chi}(0)-\tilde{\chi}(r)}{r}\right] \\
&= \frac{\lambda_1\lambda_2}{c_1c_2r}\left[\mu - \frac{1-\tilde{b}(r)}{r}\right] + \frac{\lambda_1}{c_1r}\left[\tilde{\xi}(0)-\tilde{\xi}(r)\right] + \frac{\lambda_2}{c_2r}\left[\tilde{\chi}(0)-\tilde{\chi}(r)\right] \\
&= \frac{1}{r}\left[\frac{\lambda_1\lambda_2}{c_1c_2}\mu + \frac{\lambda_1}{c_1}\widetilde{\xi}(0) + \frac{\lambda_2}{c_1}\widetilde{\chi}(0)\right] + \frac{1}{r^2}\left[-\frac{\lambda_1\lambda_2}{c_1c_2} + \frac{\lambda_1\lambda_2}{c_1c_2}\tilde{b}(r) - \frac{\lambda_1}{c_1}r\tilde{\xi}(r) - \frac{\lambda_2}{c_2}r\tilde{\chi}(r)\right].
\end{aligned} \tag{2.49}
$$

Since r is a root of Lundberg's equation (2.18) when $\delta = 0$, we have

$$\left(\frac{\lambda_1}{c_1} - r\right)\left(\frac{\lambda_2}{c_2} - r\right) = \frac{\lambda_1\lambda_2}{c_1c_2}\tilde{b}(r) - \frac{\lambda_1}{c_1}r\tilde{\xi}(r) - \frac{\lambda_2}{c_2}r\tilde{\chi}(r).$$

Inserting this identity into (2.49) produces

$$\begin{aligned}
\kappa_0 &= \frac{1}{r}\left[\frac{\lambda_1\lambda_2}{c_1c_2}\mu + \frac{\lambda_1}{c_1}\widetilde{\xi}(0) + \frac{\lambda_2}{c_1}\widetilde{\chi}(0)\right] + \frac{1}{r^2}\left[-\frac{\lambda_1\lambda_2}{c_1c_2} + \left(\frac{\lambda_1}{c_1} - r\right)\left(\frac{\lambda_2}{c_2} - r\right)\right] \\
&= \frac{1}{r}\left[\frac{\lambda_1\lambda_2}{c_1c_2}\mu + \frac{\lambda_1}{c_1}\widetilde{\xi}(0) + \frac{\lambda_2}{c_1}\widetilde{\chi}(0)\right] + \frac{1}{r^2}\left[-r\left(\frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_2}\right)\right] + 1 \\
&= 1 + \frac{1}{r}\left\{\frac{\lambda_1\lambda_2}{c_1c_2}\mu + \frac{\lambda_1}{c_1}\left[\widetilde{\xi}(0) - 1\right] + \frac{\lambda_2}{c_1}\left[\widetilde{\chi}(0) - 1\right]\right\} \\
&= 1 + \frac{1}{r}\left[\frac{\lambda_1\lambda_2}{c_1c_2}\mu - \frac{\lambda_1}{c_1}\widetilde{\chi}(0) - \frac{\lambda_2}{c_2}\widetilde{\xi}(0)\right].
\end{aligned}$$

Utilizing the positive-security-loading condition (2.2) that

$$\frac{\lambda_1\lambda_2}{c_1c_2}\mu < \frac{\lambda_2}{c_2}\widetilde{\xi}(0) + \frac{\lambda_1}{c_1}\widetilde{\chi}(0),$$

which indicates $\left[\frac{\lambda_1\lambda_2}{c_1c_2}\mu - \frac{\lambda_1}{c_1}\widetilde{\chi}(0) - \frac{\lambda_2}{c_2}\widetilde{\xi}(0)\right] < 0$, hence $\kappa_0 < 1$. As a result, the proof that $0 < \kappa_\delta < 1$ is completed for both $\delta > 0$ and $\delta = 0$. ■

Remark 2.1 When $c_1 = c_2$, model (2.1) reduces to the model considered by Albrecher and Boxma (2004). Expressions (2.21) and (2.22) for $m_i(0)$, $i = 1, 2$, complement the system of equations (8) and (11) in Albrecher and Boxma (2004) where $\delta = 0$ and $w(x_1, x_2) = 1$ for all x_1, $x_2 \geq 0$. Moreover, (2.24) and (2.25) provide the explicit solutions for the Gerber-Shiu function $m_i(u)$, $i = 1, 2$. □

Remark 2.2 If premium rates c_1 and c_2 are set so that $\frac{\lambda_1}{c_1} = \frac{\lambda_2}{c_2}$, then we deduce from (2.21), (2.22), (2.24) and (2.25) that $m_1(0) = m_2(0)$ and $m_1(u) = m_2(u)$, which means that the effect of the dependence structure between interclaim times and claim sizes is offset and the model reduces to the classical compound Poisson model. □

2.4 Applications with exponential thresholds

The thresholds may be viewed as a criterion for classifying claims as large or small. Thus, it is natural to assume that the distribution of the thresholds is exponential. In this section, we assume that the random thresholds $\{Q_i,\ i = 1, 2, \ldots\}$ follow an exponential distribution with c.d.f. $H(y) = 1 - e^{-\nu y}, y \geq 0$, and derive the explicit expressions for the Gerber-Shiu function under consider some special cases. A numerical example is provided in section 2.4.3

2.4.1 Gerber-Shiu function with K_n-family claim sizes

Assume that the claim amounts $\{X_i,\ i = 1, 2, \ldots\}$ follow a distribution from the K_n family, i.e., the Laplace transform of the density function $b(\cdot)$ has the following form

$$\widetilde{b}(s) = \frac{p^{\star}_{k-1}(s)}{p_k(s)}, \quad k \in \mathbb{N}^{+}$$

where $p_k(s)$ is a polynomial in s of degree k with only negative zeros, $p^{\star}_{k-1}(s)$ is a polynomial in s of degree $k-1$ or less, both with leading constant 1 and $p_k(0) = p^{\star}_{k-1}(0)$. The K_n family is a general family of distributions that contains Erlang, Coxian, some phase-type distributions and their mixtures, which are common choices for modeling the claim-size random variables. The K_n family is also widely considered in applied probability areas (see Cohen, 1982, and Tijms, 1994).

By (2.9) with $H(y) = 1 - e^{\nu y}, y \geq 0$, we may write

$$\widetilde{\chi}(s) = \widetilde{b}(s + \nu) = \frac{q^{\star}_{k-1}(s)}{q_k(s)},$$

where $q_k(s) = p_k(s + \nu)$ is a polynomial in s of degree k with only negative zeros, $q^{\star}_{k-1}(s) = p^{\star}_{k-1}(s + \nu)$ is a polynomial in s of degree $k - 1$ or less, both with leading constant 1, since $\nu > 0$ is a constant. We rewrite the left-hand side of Lundberg's equation (2.17) utilizing identity (2.10) to

$$\widetilde{h}_1(s) - \widetilde{h}_2(s) = \left(s - \frac{\lambda_1 + \delta}{c_1}\right)\left(s - \frac{\lambda_2 + \delta}{c_2}\right) + \frac{\lambda_1}{c_1}\left(s - \frac{\lambda_2 + \delta}{c_2}\right)\widetilde{b}(s) + \left[\left(\frac{\lambda_2}{c_2} - \frac{\lambda_1}{c_1}\right)s + \frac{\lambda_1\delta - \lambda_2\delta}{c_1c_2}\right]\widetilde{\chi}(s).$$

Then Lundberg's equation becomes

$$\left(s - \frac{\lambda_1 + \delta}{c_1}\right)\left(s - \frac{\lambda_2 + \delta}{c_2}\right) + \frac{\lambda_1}{c_1}\left(s - \frac{\lambda_2 + \delta}{c_2}\right)\frac{p^\star_{k-1}(s)}{p_k(s)} + \left[\left(\frac{\lambda_2}{c_2} - \frac{\lambda_1}{c_1}\right)s + \frac{\lambda_1\delta - \lambda_2\delta}{c_1c_2}\right]\frac{q^\star_{k-1}(s)}{q_k(s)} = 0,$$

which may be rearranged as

$$\begin{aligned}\left(s - \frac{\lambda_1 + \delta}{c_1}\right)\left(s - \frac{\lambda_2 + \delta}{c_2}\right)p_k(s)\,q_k(s) + \frac{\lambda_1}{c_1}\left(s - \frac{\lambda_2 + \delta}{c_2}\right)p^\star_{k-1}(s)\,q_k(s)\\ + \left[\left(\frac{\lambda_2}{c_2} - \frac{\lambda_1}{c_1}\right)s + \frac{\lambda_1\delta - \lambda_2\delta}{c_1c_2}\right]q^\star_{k-1}(s)\,p_k(s) = 0,\end{aligned} \tag{2.50}$$

without changing the roots of the equation. The left-hand side of equation (2.50) is a polynomial in s of degree $2k + 2$ with leading coefficient 1. Hence, it has $2k + 2$ roots in total. Among these roots, exactly two are nonnegative by Lemmas 2.1 and 2.2, denoted as r and ρ. Therefore, the other $2k$ roots are in the left-hand complex plane, denoted as $R_1, \ldots, R_{2k}$. From now on, we assume that these roots are distinct. When this is not the case, the calculations may still be carried through but are more complex. In addition, the left-hand side of (2.50) equals $p_k(s)\,q_k(s)\left[\widetilde{h}_1(s) - \widetilde{h}_2(s)\right]$ and has leading coefficient 1, which in turn implies that

$$p_k(s)\,q_k(s)\left[\widetilde{h}_1(s) - \widetilde{h}_2(s)\right] = (s - r)(s - \rho)\prod_{l=1}^{2k}(s - R_l). \tag{2.51}$$

Notice that the polynomials $p_k(s)$ and $q_k(s)$ might share common terms, the negative roots will be reduced by number of shared terms. Suppose $p_k(s)$ and $q_k(s)$ have x terms in common, then the number of negative roots of Lundberg's equation reduces to $2k - x$, in which case we may simply replace the terms $2k$ by $2k - x$ in equation (2.51) and the following derivations.

Implementing (2.35) and (2.51) in (2.30), we have for all $s \geq 0$ (except for r and ρ),

$$\begin{aligned}
\widetilde{m}_1(s) &= \frac{\widetilde{\alpha}_1(s) + \widetilde{\beta}_1(s)}{\widetilde{h}_1(s) - \widetilde{h}_2(s)} \\
&= \frac{p_k(s)\, q_k(s) \left[\widetilde{\alpha}_1(s) + \widetilde{\beta}_1(s)\right]}{p_k(s)\, q_k(s) \left[\widetilde{h}_1(s) - \widetilde{h}_2(s)\right]} \\
&= \frac{p_k(s)\, q_k(s)(s-r)(s-\rho) T_s T_r T_\rho \beta_1(0)}{(s-r)(s-\rho) \prod_{l=1}^{2k}(s - R_l)} \\
&= \frac{p_k(s)\, q_k(s)}{\prod_{l=1}^{2k}(s - R_l)} T_s T_r T_\rho \beta_1(0).
\end{aligned} \tag{2.52}$$

Denote by

$$z(s) := p_k(s)\, q_k(s) - \prod_{l=1}^{2k}(s - R_l). \tag{2.53}$$

Since both $p_k(s)\, q_k(s)$ and $\prod_{l=1}^{2k}(s - R_l)$ are polynomials in s of degree $2k$ with leading coefficient 1, $z(s)$ is a polynomial of s of degree $2k - 1$ or less. Further, denote by

$$\mathcal{D}(s) := \prod_{l=1}^{2k}(s - R_l).$$

Then (2.52) may be rewritten as

$$\widetilde{m}_1(s) = \left[1 + \frac{z(s)}{\mathcal{D}(s)}\right] T_s T_r T_\rho \beta_1(0). \tag{2.54}$$

Observe that $\dfrac{z(s)}{\mathcal{D}(s)}$ is a rational function in s, which implies that it is the Laplace transform of some function $\ell(\cdot)$ with respect to s, i.e., $\widetilde{\ell}(s) = \dfrac{z(s)}{\mathcal{D}(s)}$. Applying the Heaviside expansion theorem, $\widetilde{\ell}(s)$ may be inverted to

$$\ell(u) = \sum_{j=1}^{2k} \frac{z(R_j)}{\mathcal{D}'(R_j)}\, e^{R_j u},$$

where by definition (2.53)

$$z(R_j) = p_k(R_j)\, q_k(R_j) = p_k(R_j)\, p_k(R_j + \nu).$$

Differentiating $\mathcal{D}(s)$ yields $\mathcal{D}'(R_j) = \prod_{l=1,\, l\neq j}^{2k} (R_j - R_l)$ for $j = 1, \ldots, 2k$. Thus,

$$\ell(u) = \sum_{j=1}^{2k} \frac{p_k(R_j)\, p_k(R_j + \nu)}{\prod_{l=1,\, l\neq j}^{2k}(R_j - R_l)}\, e^{R_j u}. \tag{2.55}$$

Then, we may invert the Laplace transform in (2.54) to

$$m_1(u) = T_r T_\rho \beta_1(u) + \ell(u) * T_r T_\rho \beta_1(u),$$

where $*$ denotes the convolution and $T_r T_\rho \beta_1(u)$ and $\ell(u)$ are expressed in (2.43) and (2.55). Similarly, from (2.31) we deduce that

$$m_2(u) = T_r T_\rho \beta_2(u) + \ell(u) * T_r T_\rho \beta_2(u),$$

with $T_r T_\rho \beta_2(u)$ and $\ell(u)$ expressed in (2.44) and (2.55).

2.4.2 Laplace transform of the time to ruin under exponential claim sizes

The Laplace transform of the time to ruin is one of the quantities of particular interest for insurance risk analysis. As shown in Example 1.2, let the penalty function $w(x_1, x_2) = 1$ for all $x_1, x_2 \geq 0$, then the Gerber-Shiu function (2.3) reduces to

$$\varphi_i(u) = \mathbb{E}\left[e^{-\delta\tau_i}\, \mathbb{I}_{\{\tau_i<\infty\}} \,|\, U(0) = u\right], \quad u \geq 0, i = 1, 2,$$

which is the Laplace transform of the time to ruin with respect to δ, given that the initial class of the insured is $i = 1, 2$ and the initial surplus is u. The transforms $\varphi_i(u)$, $i = 1, 2$, are useful for computing the moments of the time-to-ruin random variables τ_i, $i = 1, 2$. Moreover, by letting $\delta = 0$, $\varphi_i(u)$, $i = 1, 2$, yield the ultimate-ruin probabilities $\psi_i(u)$, $i = 1, 2$.

Assume the claim sizes $\{X_i\}$ are exponentially distributed with p.d.f. $b(y) = \epsilon e^{-\epsilon y}, y \geq 0$, which has Laplace transform $\widetilde{b}(s) = \dfrac{\epsilon}{s+\epsilon}$ and $\mu = \mathbb{E}\{X_1\} = \frac{1}{\epsilon}$. Employing (2.4) yields $\zeta(u) = \int_u^\infty b(y)dy = e^{-\epsilon u}$. Then for all $s \geq 0$,

$$T_s\zeta(0) = \widetilde{\zeta}(s) = \frac{1}{s+\epsilon}. \tag{2.56}$$

Utilizing (2.9) and (2.10), we obtain

$$\widetilde{\chi}(s) = \frac{\epsilon}{s+\nu+\epsilon}, \tag{2.57}$$

$$\widetilde{\xi}(s) = \widetilde{b}(s) - \widetilde{\chi}(s) = \frac{\epsilon\nu}{(s+\epsilon)(s+\nu+\epsilon)}, \tag{2.58}$$

and we may further derive utilizing (2.6) that

$$T_sT_\rho T_r\chi(0) = \frac{\epsilon}{(s+\nu+\epsilon)(\rho+\nu+\epsilon)(r+\nu+\epsilon)}, \tag{2.59}$$

$$T_sT_\rho T_r\xi(0) = \frac{\epsilon}{(s+\epsilon)(\rho+\epsilon)(r+\epsilon)} - \frac{\epsilon}{(s+\nu+\epsilon)(\rho+\nu+\epsilon)(r+\nu+\epsilon)}, \tag{2.60}$$

$$T_sT_\rho T_r\zeta(0) = \frac{1}{(s+\epsilon)(\rho+\epsilon)(r+\epsilon)}, \tag{2.61}$$

$$\frac{rT_sT_r\chi(0) - \rho T_sT_\rho\chi(0)}{r-\rho} = \frac{\epsilon(\nu+\epsilon)}{(s+\nu+\epsilon)(\rho+\nu+\epsilon)(r+\nu+\epsilon)}. \tag{2.62}$$

Implementing (2.57) and (2.58), Lundberg's equation (2.17) reduces to

$$\left(s - \frac{\lambda_1+\delta}{c_1}\right)\left(s - \frac{\lambda_2+\delta}{c_2}\right) + \frac{\lambda_1}{c_1}\left(s - \frac{\lambda_2+\delta}{c_2}\right)\frac{\epsilon\nu}{(s+\epsilon)(s+\nu+\epsilon)} + \frac{\lambda_2}{c_2}\left(s - \frac{\lambda_1+\delta}{c_1}\right)\frac{\epsilon}{s+\nu+\epsilon} = 0,$$

which may be rearranged to the following equation without change in the roots,

$$\left(s - \frac{\lambda_1+\delta}{c_1}\right)\left(s - \frac{\lambda_2+\delta}{c_2}\right)(s+\epsilon)(s+\nu+\epsilon) + \epsilon\nu\frac{\lambda_1}{c_1}\left(s - \frac{\lambda_2+\delta}{c_2}\right) + \epsilon\frac{\lambda_2}{c_2}\left(s - \frac{\lambda_1+\delta}{c_1}\right)(s+\epsilon) = 0. \tag{2.63}$$

The roots do not change because neither $s = -\epsilon$ nor $s = -\epsilon - \nu$ solves (2.63). Equation (2.63) is a fourth-order polynomial equation in s, which has four roots in the complex plane, among which exactly two are nonnegative by Lemmas 2.1 and 2.2, denoted by r and ρ as before. Then (2.63) has exactly two other roots with negative real part, denoted by R_1, R_2. The leading coefficient of the left-hand side of (2.63) is 1, which implies

$$(s+\epsilon)(s+\nu+\epsilon)\left[\widetilde{h}_1(s)-\widetilde{h}_2(s)\right]=(s-r)(s-\rho)(s-R_1)(s-R_2). \tag{2.64}$$

Inserting (2.35) and (2.64) into (2.30), we obtain that for all $s \geq 0$ (except for r and ρ),

$$\begin{aligned}
\widetilde{\varphi}_1(s) &= \frac{\widetilde{\alpha}_1(s)+\widetilde{\beta}_1(s)}{\widetilde{h}_1(s)-\widetilde{h}_2(s)} \\
&= \frac{[\widetilde{\alpha}(s)+\widetilde{\beta}_1(s)](s+\epsilon)(s+\nu+\epsilon)}{[\widetilde{h}_1(s)-\widetilde{h}_2(s)](s+\epsilon)(s+\nu+\epsilon)} \\
&= \frac{(s+\epsilon)(s+\nu+\epsilon)(s-\rho)(s-r)T_sT_\rho T_r\beta_1(0)}{(s-\rho)(s-r)(s-R_1)(s-R_2)} \\
&= \frac{(s+\epsilon)(s+\nu+\epsilon)T_sT_rT_\rho\beta_1(0)}{(s-R_1)(s-R_2)}.
\end{aligned}$$

Denote the numerator by

$$G(s) := (s+\epsilon)(s+\nu+\epsilon)\,T_sT_rT_\rho\beta_1(0)\,,$$

then

$$\widetilde{\varphi}_1(s) = \frac{G(s)}{(s-R_1)(s-R_2)}. \tag{2.65}$$

Employing (2.41) with auxiliary results (2.59) to (2.62) , we simplify $G(s)$ to

$$\begin{aligned}
G(s) &= (s+\epsilon)(s+\nu+\epsilon)\,T_sT_rT_\rho\beta_1(0) \\
&= (s+\epsilon)(s+\nu+\epsilon)\Bigg\{\frac{\lambda_1}{c_1}\left(\frac{\lambda_2+\delta}{c_2}\right)T_sT_\rho T_r\zeta(0) \\
&\qquad + \left(\frac{\lambda_1}{c_1}\right)\frac{rT_sT_r\zeta(0)-\rho T_sT_\rho\zeta(0)}{r-\rho} + \left[\frac{\lambda_2}{c_2}\varphi_1(0)-\frac{\lambda_1}{c_1}\varphi_2(0)\right]T_sT_\rho T_r\chi(0)\Bigg\} \\
&= (s+\epsilon)(s+\nu+\epsilon)\Bigg\{\frac{\frac{\lambda_1}{c_1}\left(\frac{\lambda_2+\delta}{c_2}\right)}{(r+\epsilon)(\rho+\epsilon)(s+\epsilon)} \\
&\qquad + \frac{\left(\frac{\lambda_1}{c_1}\right)\epsilon}{(r+\epsilon)(\rho+\epsilon)(s+\epsilon)} + \frac{\left[\frac{\lambda_2}{c_2}\varphi_1(0)-\frac{\lambda_1}{c_1}\varphi_2(0)\right]\epsilon}{(r+\epsilon+\nu)(\rho+\epsilon+\nu)(s+\epsilon+\nu)}\Bigg\}
\end{aligned}$$

$$= \frac{\frac{\lambda_1}{c_1}\left(\frac{\lambda_2+\delta}{c_2}+\epsilon\right)}{(\rho+\epsilon)(r+\epsilon)}\,(s+\epsilon+\nu)+\frac{\left[\frac{\lambda_2}{c_2}\varphi_1(0)-\frac{\lambda_1}{c_1}\varphi_2(0)\right]\epsilon}{(\rho+\epsilon+\nu)(r+\epsilon+\nu)}\,(s+\epsilon)\,, \tag{2.66}$$

which is a polynomial of degree 1 in s. Moreover, inserting (2.56) and (2.57) into (2.23) produces

$$\frac{\lambda_2}{c_2}\varphi_1(0)-\frac{\lambda_1}{c_1}\varphi_2(0)=\frac{\lambda_1(r+\epsilon+\nu)(\rho+\epsilon+\nu)(c_2\rho-\lambda_2-\delta)(c_2r-\lambda_2-\delta)}{c_1c_2\epsilon(r+\epsilon)(\rho+\epsilon)(c_2r+c_2\rho+c_2\epsilon+c_2\nu-\lambda_2-\delta)}. \tag{2.67}$$

Since $G(s)$ is a polynomial of degree 1 and R_1, R_2 are in the left-half of the complex plane, applying the Heaviside expansion theorem to (2.65), the inversion of the Laplace transforms yields

$$\varphi_1(u)=\frac{G(R_1)}{R_1-R_2}\,e^{R_1u}+\frac{G(R_2)}{R_2-R_1}\,e^{R_2u},\quad u\geq 0, \tag{2.68}$$

where $G(\cdot)$ is expressed in (2.66), and R_1, R_2 are the only two roots of Lundberg's equation (2.63) with negative real parts.

Similarly for $\widetilde{\varphi}_2(s)$, denote by

$$J(s):=(s+\epsilon)(s+\nu+\epsilon)T_sT_rT_\rho\beta_2(0)\,,$$

it follows from (2.31) that

$$\widetilde{\varphi}_2(s)=\frac{J(s)}{(s-R_1)(s-R_2)}\,.$$

We deduce from (2.42), utilizing results (2.59) to (2.62), that $J(s)$ is also a polynomial of degree one in s, where

$$\begin{aligned}J(s)&=(s+\epsilon)\,(s+\nu+\epsilon)\,T_sT_rT_\rho\beta_2(0)\\&=(s+\epsilon)\,(s+\nu+\epsilon)\left\{\frac{\lambda_2}{c_2}\left[\frac{\lambda_1+\delta}{c_1}\,T_sT_rT_\rho\zeta(0)+\frac{rT_sT_r\zeta(0)-\rho T_sT_\rho\zeta(0)}{r-\rho}\right]\right.\\&\qquad\left.-\left[\frac{\lambda_2}{c_2}m_1(0)-\frac{\lambda_1}{c_1}m_2(0)\right]T_sT_rT_\rho\xi(0)\right\}\end{aligned}$$

$$= \frac{\frac{\lambda_2}{c_2}\left(\frac{\lambda_1+\delta}{c_1}+\epsilon\right) - \left[\frac{\lambda_2}{c_2}\varphi_1(0) - \frac{\lambda_1}{c_1}\varphi_2(0)\right]\epsilon}{(\rho+\epsilon)(r+\epsilon)}\,(s+\epsilon+\nu) \;+\; \frac{\left[\frac{\lambda_2}{c_2}\varphi_1(0) - \frac{\lambda_1}{c_1}\varphi_2(0)\right]\epsilon}{(\rho+\epsilon+\nu)(r+\epsilon+\nu)}\,(s+\epsilon)\,, \tag{2.69}$$

with $\left[\frac{\lambda_2}{c_2}\varphi_1(0) - \frac{\lambda_1}{c_1}\varphi_2(0)\right]$ expressed in (2.67). Applying the Heaviside expansion theorem again yields

$$\varphi_2(u) = \frac{J(R_1)}{R_1 - R_2}\,e^{R_1 u} + \frac{J(R_2)}{R_2 - R_1}\,e^{R_2 u}, \quad u \geq 0, \tag{2.70}$$

where $J(\cdot)$ is defined in (2.69), and R_1, R_2 are the two roots of Lundberg's equation (2.63) with negative real parts.

Together (2.68) and (2.70) give us the explicit expressions for the Laplace transform of the time to ruin $\varphi_i(u)$, $i = 1, 2$, under the exponential setting. Furthermore, if we insert $\delta = 0$ (which implies $\rho = 0$) into the expressions for $\varphi_i(u)$, $i = 1, 2$, the ultimate ruin probabilities $\psi_i(u), i = 1, 2$ are obtained.

In addition, we derive the first moment of the time to ruin τ_i for $i = 1, 2$. In order to differentiate $\varphi_i(u), i = 1, 2$ with respect to δ, we introduce the following notation. Denote the four roots r, ρ, R_1 and R_2 of the Lundberg's equation (2.63) as $\mathcal{A}_1(\delta), \mathcal{A}_2(\delta), \mathcal{A}_3(\delta)$ and $\mathcal{A}_4(\delta)$ respectively, where $\mathcal{A}_1(\delta)$ and $\mathcal{A}_2(\delta)$ are interchangeable, as well as $\mathcal{A}_3(\delta)$ and $\mathcal{A}_4(\delta)$. Let

$$\begin{aligned}
\Omega_1(\delta) &:= \frac{1}{[\mathcal{A}_1(\delta)+\epsilon][\mathcal{A}_2(\delta)+\epsilon]}\,,\\
\Omega_2(\delta) &:= \frac{1}{[\mathcal{A}_1(\delta)+\epsilon+\nu][\mathcal{A}_2(\delta)+\epsilon+\nu]}\,,\\
\nabla(\delta) &:= \frac{\lambda_1[c_2\mathcal{A}_1(\delta)-\delta-\lambda_2][c_2\mathcal{A}_2(\delta)-\delta-\lambda_2]}{c_1c_2\epsilon[c_2\mathcal{A}_1(\delta)+c_2\mathcal{A}_2(\delta)-\delta-\lambda_2+c_2\epsilon+c_2\nu]}\,,\\
V(\delta) &:= \left[\frac{\lambda_2}{c_2}\varphi_1(0) - \frac{\lambda_1}{c_1}\varphi_2(0)\right] \;=\; \frac{\Omega_1(\delta)}{\Omega_2(\delta)}\nabla(\delta)\,,
\end{aligned}$$

by equation (2.67). Then (2.66) and (2.69) may be expressed as

$$G(\mathcal{A}_j(\delta)) = \frac{\lambda_1}{c_1c_2}(\delta+\lambda_2+c_2\epsilon)[\mathcal{A}_j(\delta)+\epsilon+\nu]\,\Omega_1(\delta) + \epsilon\,[\mathcal{A}_j(\delta)+\epsilon]\nabla(\delta)\,\Omega_1(\delta)\,, \qquad j = 3, 4,$$

$$J(\mathcal{A}_j(\delta)) = \left[\frac{\lambda_2}{c_1c_2}(\delta + \lambda_1 + c_1\epsilon) - \epsilon V(\delta)\right][\mathcal{A}_j(\delta) + \epsilon + \nu]\,\Omega_1(\delta) - \epsilon\,[\mathcal{A}_j(\delta) + \epsilon]\nabla(\delta)\,\Omega_1(\delta)\,, \quad j = 3, 4.$$

Differentiation of Lundberg's equation (2.63) with respect to δ yields

$$\mathcal{A}_i'(\delta) = \frac{\left[\Lambda_{1,i} + \Lambda_{2,i}\right](\mathcal{A}_i + \epsilon)(\mathcal{A}_i + \epsilon + \nu) + \lambda_2\epsilon(\mathcal{A}_i + \epsilon) + \lambda_1\epsilon\nu}{\left[c_2\Lambda_{1,i} + c_1\Lambda_{2,i}\right](\mathcal{A}_i + \epsilon)(\mathcal{A}_i + \epsilon + \nu) + \Lambda_{1,i}\Lambda_{2,i}(2\mathcal{A}_i + 2\epsilon + \nu) + \lambda_2\epsilon(\Lambda_{1,i} + c_1\mathcal{A}_i + c_1\epsilon) + c_2\lambda_1\epsilon\nu}\,,$$

for $i = 1, 2, 3, 4$, where $\mathcal{A}_i$ stands for $\mathcal{A}_i(\delta)$ and

$$\Lambda_{1,i} = c_1\mathcal{A}_i(\delta) - \lambda_1 - \delta\,, \qquad \Lambda_{2,i} = c_2\mathcal{A}_i(\delta) - \lambda_2 - \delta\,.$$

With $\mathcal{A}_i'(\delta)$ known, we are able to derive the following first-order derivatives, $\Omega_1'(\delta)$, $\Omega_2'(\delta)$, $\nabla'(\delta)$, $V'(\delta)$, $\frac{\partial}{\partial\delta}G(\mathcal{A}_i(\delta))$ and $\frac{\partial}{\partial\delta}J(\mathcal{A}_i(\delta))$, assuming that these derivatives exist when δ is close to 0. Then, differentiating (2.68) and (2.70) with respect to δ produces

$$\begin{aligned}
\frac{\partial}{\partial\delta}\varphi_1(u) &= \left[\frac{\mathcal{A}_3'(\delta)G(\mathcal{A}_3(\delta))}{\mathcal{A}_3(\delta) - \mathcal{A}_4(\delta)}u + \frac{\frac{\partial}{\partial\delta}G(\mathcal{A}_3(\delta))}{\mathcal{A}_3(\delta) - \mathcal{A}_4(\delta)} - \frac{G(\mathcal{A}_3(\delta))(\mathcal{A}_3'(\delta) - \mathcal{A}_4'(\delta))}{\mathcal{A}_3(\delta) - \mathcal{A}_4(\delta)}\right]e^{\mathcal{A}_3(\delta)u} \\
&\quad + \left[\frac{\mathcal{A}_4'(\delta)G(\mathcal{A}_4(\delta))}{\mathcal{A}_4(\delta) - \mathcal{A}_3(\delta)}u + \frac{\frac{\partial}{\partial\delta}G(\mathcal{A}_4(\delta))}{\mathcal{A}_4(\delta) - \mathcal{A}_3(\delta)} - \frac{G(\mathcal{A}_4(\delta))(\mathcal{A}_4'(\delta) - \mathcal{A}_3'(\delta))}{\mathcal{A}_4(\delta) - \mathcal{A}_3(\delta)}\right]e^{\mathcal{A}_4(\delta)u}, \\
\frac{\partial}{\partial\delta}\varphi_2(u) &= \left[\frac{\mathcal{A}_3'(\delta)J(\mathcal{A}_3(\delta))}{\mathcal{A}_3(\delta) - \mathcal{A}_4(\delta)}u + \frac{\frac{\partial}{\partial\delta}J(\mathcal{A}_3(\delta))}{\mathcal{A}_3(\delta) - \mathcal{A}_4(\delta)} - \frac{J(\mathcal{A}_3(\delta))(\mathcal{A}_3'(\delta) - \mathcal{A}_4'(\delta))}{\mathcal{A}_3(\delta) - \mathcal{A}_4(\delta)}\right]e^{\mathcal{A}_3(\delta)u} \\
&\quad + \left[\frac{\mathcal{A}_4'(\delta)J(\mathcal{A}_4(\delta))}{\mathcal{A}_4(\delta) - \mathcal{A}_3(\delta)}u + \frac{\frac{\partial}{\partial\delta}J(\mathcal{A}_4(\delta))}{\mathcal{A}_4(\delta) - \mathcal{A}_3(\delta)} - \frac{J(\mathcal{A}_4(\delta))(\mathcal{A}_4'(\delta) - \mathcal{A}_3'(\delta))}{\mathcal{A}_4(\delta) - \mathcal{A}_3(\delta)}\right]e^{\mathcal{A}_4(\delta)u}.
\end{aligned}$$

Hence, the first moment of the time to ruin τ_i, $i = 1, 2$, when the initial class is i and the initial surplus is u, may be obtained as

$$\mathbb{E}\left\{\tau_i \,\middle|\, \tau_i < \infty, U(0) = u\right\} = \frac{\mathbb{E}\{\tau_i\,\mathbb{I}_{\{T_i<\infty\}}|U(0) = u\}}{\mathbb{P}\{\tau_i < \infty|U(0) = u\}} = \frac{-\frac{\partial}{\partial\delta}\varphi_i(u)\big|_{\delta=0}}{\psi_i(u)}, \quad i = 1, 2. \qquad (2.71)$$

2.4.3 Numerical Example

Assume that thresholds $Q_i \sim \text{Exp}(2)$, claim sizes $X_i \sim \text{Exp}(1)$, $c_1 = c_2 = 2$, $\lambda_1 = 3$, $\lambda_2 = 1$ and $\delta = 0$. Then, Lundberg's equation is

$$4s^4 + 8s^3 - 15s^2 - s = 0,$$

which has four roots, yielding $r = 1.22575$, $\rho = 0$, $R_1 = -0.06452$ and $R_2 = -3.16124$. Since $\delta = 0$, the ultimate ruin probabilities may be calculated from (2.68) and (2.70) as

$$\psi_1(u) = 0.9384\, e^{-0.0645u} + 0.0068\, e^{-3.1612u},$$
$$\psi_2(u) = 0.8669\, e^{-0.0645u} + 0.0029\, e^{-3.1612u}.$$

To compute the first moment of finite ruin time τ_i, $i = 1, 2$, we first derive some constants

$$\begin{aligned}
\mathcal{A}_1'(0) &= 0.602416 & \mathcal{A}_2'(0) &= 5 \\
\mathcal{A}_3'(0) &= -4.614023 & \mathcal{A}_4'(0) &= 0.01141974 \\
\Omega_1'(0) &= -2.368031 & \Omega_2'(0) &= -0.1427141 \\
\nabla'(0) &= 1.494321 & V'(0) &= 11.39155 \\
\frac{\partial}{\partial\delta} G(\mathcal{A}_3(\delta))\Big|_{\delta=0} &= -18.06103 & \frac{\partial}{\partial\delta} G(\mathcal{A}_4(\delta))\Big|_{\delta=0} &= -1.383135 \\
\frac{\partial}{\partial\delta} J(\mathcal{A}_3(\delta))\Big|_{\delta=0} &= -32.22943 & \frac{\partial}{\partial\delta} J(\mathcal{A}_4(\delta))\Big|_{\delta=0} &= -0.586625
\end{aligned}$$

Then, plugging the above into (2.71), we obtain

$$\mathbb{E}\{\tau_1\, \mathbb{I}_{\{\tau_1<\infty\}} | U(0) = u\} = (4.43061 + 4.32993u)\, e^{-0.0645u} - (0.45684 + 0.00008u)\, e^{-3.1612u},$$
$$\mathbb{E}\{\tau_2\, \mathbb{I}_{\{\tau_2<\infty\}} | U(0) = u\} = (9.11269 + 4.00003u)\, e^{-0.0645u} - (0.19376 + 0.00003u)\, e^{-3.1612u},$$

and consequently, the expected time of ruin given that ruin occurs in finite time is

$$\mathbb{E}\left\{\tau_1 \,\middle|\, \tau_1 < \infty, U(0) = u\right\} = \frac{(4.43061 + 4.32993u)\, e^{-0.0645u} - (0.45684 + 0.00008u)\, e^{-3.1612u}}{0.9384\, e^{-0.0645u} + 0.0068\, e^{-3.1612u}},$$
$$\mathbb{E}\left\{\tau_2 \,\middle|\, \tau_2 < \infty, U(0) = u\right\} = \frac{(9.11269 + 4.00003u)\, e^{-0.0645u} - (0.19376 + 0.00003u)\, e^{-3.1612u}}{0.8669\, e^{-0.0645u} + 0.0029\, e^{-3.1612u}}.$$

Chapter 3

An insurance risk model with dependence and diffusion

In this chapter, we consider a perturbed version of an insurance risk model with interclaim-time distribution depends on the size of the previous claim. We assume that the surplus process of the insurer is perturbed by a Brownian motion to account for small fluctuations. Explicit solutions for the Gerber-Shiu discounted penalty function are derived for arbitrary claim sizes. Special cases of the Gerber-Shiu function when claim sizes come from the K_n-family are deduced. A numerical example is provided to illustrative the impact of the perturbation.

3.1 Model description and preliminary results

Suppose that the surplus process of an insurance company is modeled by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma W(t), \quad t \geq 0, \tag{3.1}$$

with initial surplus $u \geq 0$ and constant premium rate c. Claims occur with a dependence structure described in Albrecher and Boxma (2004). Namely, claim sizes $\{X_1, X_2, \dots\}$

are i.i.d random variables with cumulative distribution function $B(\cdot)$, probability density function $b(\cdot)$ and mean μ. If a claim X_i is larger than some threshold Q_i, then the process is classified to class 1 and the time until next claim follows an exponential distribution with rate λ_1; if X_i is smaller than Q_i, then the process is classified to class 2 and the time until next claim follows an exponential distribution with rate λ_2. Suppose that thresholds Q_i are i.i.d random variable with distribution function $H(\cdot)$ and are independent from X_i. In addition, $\sigma > 0$ is a parameter and $W(t)$ is a standard Brownian motion with $W(0) = 0$ and $W(t) \sim \mathcal{N}(0, t)$ for any fixed $t > 0$. The diffusion process may also represent the insurer's investment, where the parameter σ indicates how the risky investments affect the underlying surplus process. Assume that the positive-security-loading condition

$$\mu < \frac{c}{\lambda_1}\mathbb{P}\{X > Q\} + \frac{c}{\lambda_2}\mathbb{P}\{X \leq Q\} \tag{3.2}$$

holds for the model.

Given the initial claim occurs at rate of λ_i, the generalized expected discounted penalty function introduced by (1.3) is denoted as

$$m_{D,i}(u) = w_0\, \phi_{\mathrm{d},i}(u) + \phi_{\mathrm{w},i}(u), \quad u > 0,\ i = 1, 2, \tag{3.3}$$

where

$$\phi_{\mathrm{d},i}(u) = \mathbb{E}\left\{e^{-\delta\tau_i}\, \mathbb{I}_{\{\tau_i<\infty, U(\tau)=0\}} \,\middle|\, U(0) = u\right\}, \quad i = 1, 2\,,$$
$$\phi_{\mathrm{w},i}(u) = \mathbb{E}\left\{e^{-\delta\tau_i}\, w(U(\tau-), |U(\tau)|)\, \mathbb{I}_{\{\tau<\infty, U(\tau)<0\}} \,\middle|\, U(0) = u\right\}, \quad i = 1, 2\,.$$

The summand $\phi_{\mathrm{w}}(u)$ corresponds to the penalty at ruin if caused by a claim, while the component $\phi_{\mathrm{d},i}(u)$ represents the Laplace transform of the time of ruin random variable τ_i due to oscillation. At zero initial surplus $u = 0$, by definition

$$\phi_{\mathrm{d},i}(0) = 1, \quad \phi_{\mathrm{w},i}(0) = 1.$$

Recall the notation introduced in Chapter 2. The Laplace transform of a function $f(\cdot)$ is denoted by

$$\widetilde{f}(s) = \int_0^\infty e^{-sy} f(y)\, \mathrm{d}y, \quad s \in \mathbb{C}.$$

The Translation operator $T_s,\ s \geq 0$, of a real-valued function $f(\cdot)$ is defined by

$$T_s f(x) = \int_x^\infty e^{-s(y-x)} f(y)\, \mathrm{d}y,$$

and has the following properties

$$T_s f(0) = \tilde{f}(s), \quad s \geq 0,$$

$$T_{s_1} T_{s_2} f(x) = T_{s_2} T_{s_1} f(x) = \frac{T_{s_1} f(x) - T_{s_2} f(x)}{s_2 - s_1}, \quad s_1, s_2 \geq 0,\ s_1 \neq s_2.$$

$$T_{s_1} T_{s_2} f(0) = T_{s_2} T_{s_1} f(0) = \frac{\tilde{f}(s_1) - \tilde{f}(s_2)}{s_2 - s_1}, \quad s_1, s_2 \geq 0,\ s_1 \neq s_2.$$

3.2 Integro-differential equations and Lundberg's equation

In this section, we will derive a system of integro-differential equations for $\phi_{\mathrm{w},i}(u)$, $i = 1, 2$ and $\phi_{\mathrm{d},i}(u)$, $i = 1, 2$ respectively and analyze the generalized Lundberg's equation under model (3.1).

Proposition 3.1 *Functions $\phi_{\mathrm{w},i}(u)$, $i = 1, 2$ in* (3.3) *satisfy the following systems of integro-differential equations*

$$(\lambda_1 + \delta)\, \phi_{\mathrm{w},1}(u) = c\phi'_{\mathrm{w},1}(u) + D\phi''_{\mathrm{w},1}(u) + \lambda_1 \int_0^u \left[\phi_{\mathrm{w},1}(u-y)\xi(y) + \phi_{\mathrm{w},2}(u-y)\chi(y)\right] \mathrm{d}y + \lambda_1 \zeta(u)\,, \tag{3.4}$$

$$(\lambda_2 + \delta)\, \phi_{\mathrm{w},2}(u) = c\phi'_{\mathrm{w},2}(u) + D\phi''_{\mathrm{w},2}(u) + \lambda_2 \int_0^u \left[\phi_{\mathrm{w},1}(u-y)\xi(y) + \phi_{\mathrm{w},2}(u-y)\chi(y)\right] \mathrm{d}y + \lambda_2 \zeta(u)\,; \tag{3.5}$$

and $\phi_{\mathrm{d},i}(u)$*,* $i = 1, 2$ *satisfy the following systems of integro-differential equations*

$$(\lambda_1 + \delta)\,\phi_{\mathrm{d},1}(u) = c\phi'_{\mathrm{d},1}(u) + D\phi''_{\mathrm{d},1}(u) + \lambda_1 \int_0^u [\xi(y)\phi_{\mathrm{d},1}(u-y) + \chi(y)\phi_{\mathrm{d},2}(u-y)]\mathrm{d}y, \quad (3.6)$$

$$(\lambda_2 + \delta)\,\phi_{\mathrm{d},2}(u) = c\phi'_{\mathrm{d},1}(u) + D\phi''_{\mathrm{d},2}(u) + \lambda_2 \int_0^u [\xi(y)\phi_{\mathrm{d},1}(u-y) + \chi(y)\phi_{\mathrm{d},2}(u-y)]\mathrm{d}y, \quad (3.7)$$

where

$$\chi(y) = \overline{H}(y)b(y), \quad (3.8)$$

$$\xi(y) = H(y)b(y) = b(y) - \chi(y), \quad (3.9)$$

$$\zeta(u) = \int_u^\infty w(u, y-u)b(y)\,\mathrm{d}y, \quad u > 0. \quad (3.10)$$

and $D = \frac{1}{2}\sigma^2$*.*

Proof For $\phi_{\mathrm{w},i}(u)$, $i = 1, 2$, considering a small time interval of length dt and conditioning on the amount of the first claim that might have occurred in that interval, we obtain

$$\begin{aligned}\phi_{\mathrm{w},1}(u) = {} & (1-\lambda_1 dt)e^{-\delta dt}\,\mathbb{E}\left\{\phi_{\mathrm{w},1}(u + cdt + \sigma W(dt))\right\} \\ & + \lambda_1 dt e^{-\delta dt}\,\mathbb{E}\Bigg\{\int_0^{u+cdt+\sigma W(dt)} \Big[\mathbb{P}\{y > Q_1\}\phi_{\mathrm{w},1}(u + cdt + \sigma W(dt) - y) \\ & \qquad + \mathbb{P}\{y < Q_1\}\phi_{\mathrm{w},2}(u + cdt + \sigma W(dt) - y)\Big] b(y)\,\mathrm{d}y \\ & + \int_{u+cdt+\sigma W(dt)}^\infty w(u + cdt + \sigma W(dt), y - u - cdt - \sigma W(dt))b(y)\,\mathrm{d}y\Bigg\} + o(dt),\end{aligned} \quad (3.11)$$

$$\begin{aligned}\phi_{\mathrm{w},2}(u) = {} & (1-\lambda_2 dt)e^{-\delta dt}\,\mathbb{E}\left\{\phi_{\mathrm{w},2}(u + cdt + \sigma W(dt))\right\} \\ & + \lambda_2 dt e^{-\delta dt}\,\mathbb{E}\Bigg\{\int_0^{u+cdt+\sigma W(dt)} \Big[\mathbb{P}\{y > Q_1\}\phi_{\mathrm{w},1}(u + cdt + \sigma W(dt) - y) \\ & \qquad + \mathbb{P}\{y < Q_1\}\phi_{\mathrm{w},2}(u + cdt + \sigma W(dt) - y)\Big] b(y)\,\mathrm{d}y \\ & + \int_{u+cdt+\sigma W(dt)}^\infty w(u + cdt + \sigma W(dt), y - u - cdt - \sigma W(dt))b(y)\,\mathrm{d}y\Bigg\} + o(dt),\end{aligned} \quad (3.12)$$

where $\mathbb{P}\{y > Q_1\} = H(y)$ and $\mathbb{P}\{y < Q_1\} = 1 - H(y) = \overline{H}(y)$. Applying Taylor expansion to $\phi_{\mathrm{w},i}(u+cdt+\sigma W(dt))$, $i = 1, 2$, and utilizing the facts that $\mathbb{E}\{W(dt)\} = 0$ and $\mathbb{E}\{W^2(dt)\} = dt$ results in (see Tsai and Willmot, 2002)

$$\mathbb{E}\left\{\phi_{\mathrm{w},1}(u + cdt + \sigma W(dt))\right\} = \phi_{\mathrm{w},1}(u) + c\phi'_{\mathrm{w},1}(u)dt + \frac{1}{2}\sigma^2\phi''_{\mathrm{w},1}(u)dt + o(dt).$$

Hence, identity (3.11) is simplified to

$$\begin{aligned}\phi_{\mathrm{w},1}(u) =&(1 - \lambda_1 dt)e^{-\delta dt}\left[\phi_{\mathrm{w},1}(u) + c\phi'_{\mathrm{w},1}(u)dt + \frac{1}{2}\sigma^2\phi''_{\mathrm{w},1}(u)dt\right] \\ &+ \lambda_1 dte^{-\delta dt}\left\{\int_0^{u+cdt}\left[H(y)\phi_{\mathrm{w},1}(u + cdt - y) + \overline{H}(y)\phi_{\mathrm{w},2}(u + cdt - y)\right]b(y)\,\mathrm{d}y\right. \\ &\left.+ \int_{u+cdt}^{\infty} w(u + cdt, y - u - cdt)b(y)\,\mathrm{d}y\right\} + o(dt).\end{aligned}$$

Dividing both sides by dt, denoting $D = \frac{1}{2}\sigma^2$ and letting $dt \to 0$ yields

$$(\lambda_1 + \delta)\phi_{\mathrm{w},1}(u) = c\phi'_{\mathrm{w},1}(u) + D\phi''_{\mathrm{w},1}(u) + \lambda_1\int_0^u\left[H(y)\phi_{\mathrm{w},1}(u - y) + \overline{H}(y)\phi_{\mathrm{w},2}(u - y)\right]b(y)\,\mathrm{d}y + \lambda_1\zeta(u).$$

Introducing the notation (3.8), (3.9) and (3.10), we obtain

$$(\lambda_1 + \delta)\phi_{\mathrm{w},1}(u) = c\phi'_{\mathrm{w},1}(u) + D\phi''_{\mathrm{w},1}(u) + \lambda_1\int_0^u\left[\phi_{\mathrm{w},1}(u - y)\xi(y) + \phi_{\mathrm{w},2}(u - y)\chi(y)\right]\,\mathrm{d}y + \lambda_1\zeta(u),$$

which is equation (3.4). Similarly, we deduce from (3.12) that

$$(\lambda_2 + \delta)\phi_{\mathrm{w},2}(u) = c\phi'_{\mathrm{w},2}(u) + D\phi''_{\mathrm{w},2}(u) + \lambda_2\int_0^u\left[\phi_{\mathrm{w},1}(u - y)\xi(y) + \phi_{\mathrm{w},2}(u - y)\chi(y)\right]\,\mathrm{d}y + \lambda_2\zeta(u),$$

which is equation (3.5). Together (3.4) with (3.5) provide a system of integro-differential equations that $\phi_{\mathrm{w},1}(u)$ and $\phi_{\mathrm{w},2}(u)$ satisfy.

To find a system of integro-differential equations for $\phi_{\mathrm{d},1}(u)$ and $\phi_{\mathrm{d},2}(u)$, we follow similar arguments as for the derivation of equation (6) in Gerber and Landry (1998). Namely, we deduce

$$\begin{aligned}(\lambda_1 + \delta)\phi_{\mathrm{d},1}(u) &= c\phi'_{\mathrm{d},1}(u) + D\phi''_{\mathrm{d},1}(u) + \lambda_1\int_0^u\left[H(y)\phi_{\mathrm{d},1}(u - y) + \overline{H}(y)\phi_{\mathrm{d},2}(u - y)\right]b(y)\,\mathrm{d}y, \\ (\lambda_2 + \delta)\phi_{\mathrm{d},2}(u) &= c\phi'_{\mathrm{d},1}(u) + D\phi''_{\mathrm{d},2}(u) + \lambda_2\int_0^u\left[H(y)\phi_{\mathrm{d},1}(u - y) + \overline{H}(y)\phi_{\mathrm{d},2}(u - y)\right]b(y)\,\mathrm{d}y,\end{aligned}$$

which are equations (3.6) and (3.7). ■

Applying Laplace transforms to equations (3.4), (3.5), (3.6) and (3.7), we reach the following proposition.

Proposition 3.2 *The Laplace transforms* $\widetilde{\phi}_{\mathrm{w},i}(s)$, $i = 1, 2$ *satisfy the following equations*

$$\begin{aligned}&\left\{\left[s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D} + \frac{\lambda_1}{D}\widetilde{\xi}(s)\right]\left[s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D} + \frac{\lambda_2}{D}\widetilde{\chi}(s)\right] - \frac{\lambda_1\lambda_2}{D^2}\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{\phi}_{\mathrm{w},1}(s)\\&= \left[s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D} + \frac{\lambda_2}{D}\widetilde{\chi}(s)\right]\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\widetilde{\chi}(s)\phi'_{\mathrm{w},2}(0) - \frac{\lambda_1}{D}\left[s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right]\widetilde{\zeta}(s),\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\left\{\left[s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D} + \frac{\lambda_1}{D}\widetilde{\xi}(s)\right]\left[s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D} + \frac{\lambda_2}{D}\widetilde{\chi}(s)\right] - \frac{\lambda_1\lambda_2}{D^2}\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{\phi}_{\mathrm{w},2}(s)\\&= \left[s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D} + \frac{\lambda_1}{D}\widetilde{\xi}(s)\right]\phi'_{\mathrm{w},2}(0) - \frac{\lambda_2}{D}\widetilde{\xi}(s)\phi'_{\mathrm{w},1}(0) - \frac{\lambda_2}{D}\left[s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right]\widetilde{\zeta}(s).\end{aligned} \tag{3.14}$$

and Laplace transforms $\widetilde{\phi}_{\mathrm{d},i}(s)$, $i = 1, 2$ *satisfy the following equations*

$$\begin{aligned}&\left\{\left[s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D} + \frac{\lambda_1}{D}\widetilde{\xi}(s)\right]\left[s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D} + \frac{\lambda_2}{D}\widetilde{\chi}(s)\right] - \frac{\lambda_1\lambda_2}{D^2}\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{\phi}_{\mathrm{d},1}(s)\\&= \left(s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right)\left[s + \frac{c}{D} + \phi'_{\mathrm{d},1}(0)\right]\\&\quad + \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{c}{D}\cdot\frac{\lambda_2-\lambda_1}{D}\right]\widetilde{\chi}(s) + \frac{\lambda_2-\lambda_1}{D}s\widetilde{\chi}(s),\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\left\{\left[s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D} + \frac{\lambda_1}{D}\widetilde{\xi}(s)\right]\left[s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D} + \frac{\lambda_2}{D}\widetilde{\chi}(s)\right] - \frac{\lambda_1\lambda_2}{D^2}\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{\phi}_{\mathrm{d},2}(s)\\&= \left(s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right)\left[s + \frac{c}{D} + \phi'_{\mathrm{d},2}(0)\right]\\&\quad - \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{c}{D}\cdot\frac{\lambda_2-\lambda_1}{D}\right]\widetilde{\xi}(s) - \frac{\lambda_2-\lambda_1}{D}s\widetilde{\xi}(s).\end{aligned} \tag{3.16}$$

Proof Applying Laplace transforms to (3.4) and (3.5), assuming that $\lim_{u\to\infty} e^{-su}\phi_{\mathrm{w},i}(u) = 0$ and $\lim_{u\to\infty} e^{-su}\phi'_{\mathrm{w},i}(u) = 0$ hold for $i = 1, 2$, yields

$$\begin{aligned}(\lambda_1+\delta)\int_0^\infty e^{-su}\phi_{\mathrm{w},1}(u)\mathrm{d}u &= (\lambda_1+\delta)\,\widetilde{\phi}_{\mathrm{w},1}(s)\\&= c\left[s\widetilde{\phi}_{\mathrm{w},1}(s) - \phi_{\mathrm{w},1}(0)\right] + D\left[s^2\widetilde{\phi}_{\mathrm{w},1}(s) - s\phi_{\mathrm{w},1}(0) - \phi'_{\mathrm{w},1}(0)\right]\\&\quad + \lambda_1\widetilde{\xi}(s)\widetilde{\phi}_{\mathrm{w},1}(s) + \lambda_1\widetilde{\chi}(s)\widetilde{\phi}_{\mathrm{w},2}(s) + \lambda_1\widetilde{\zeta}(s),\end{aligned}$$

$$
\begin{aligned}
(\lambda_1+\delta)\int_0^\infty e^{-su}\phi_{\mathrm{w},2}(u)\mathrm{d}u &= (\lambda_2+\delta)\ \widetilde{\phi}_{\mathrm{w},2}(s)\\
&= c\left[s\widetilde{\phi}_{\mathrm{w},2}(s)-\phi_{\mathrm{w},2}(0)\right]+D\left[s^2\widetilde{\phi}_{\mathrm{w},2}(s)-s\phi_{\mathrm{w},2}(0)-\phi'_{\mathrm{w},2}(0)\right]\\
&\quad+\lambda_2\widetilde{\xi}(s)\widetilde{\phi}_{\mathrm{w},1}(s)+\lambda_2\widetilde{\chi}(s)\widetilde{\phi}_{\mathrm{w},2}(s)+\lambda_2\widetilde{\zeta}(s).
\end{aligned}
$$

Implementing $\phi_{\mathrm{w},1}(0)=\phi_{\mathrm{w},2}(0)=0$ produces

$$
\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\widetilde{\phi}_{\mathrm{w},1}(s)=D\phi'_{\mathrm{w},1}(0)-\lambda_1\widetilde{\chi}(s)\widetilde{\phi}_{\mathrm{w},2}(s)-\lambda_1\widetilde{\zeta}(s), \tag{3.17}
$$

$$
\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]\widetilde{\phi}_{\mathrm{w},2}(s)=D\phi'_{\mathrm{w},2}(0)-\lambda_2\widetilde{\xi}(s)\widetilde{\phi}_{\mathrm{w},1}(s)-\lambda_2\widetilde{\zeta}(s). \tag{3.18}
$$

We multiply (3.17) by $\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]$ and substitute $\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]\widetilde{\phi}_{\mathrm{w},2}(s)$ by the right-hand side of (3.18), and grouping the terms with $\widetilde{\phi}_{\mathrm{w},1}(s)$ leads to

$$
\begin{aligned}
&\left\{\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]-\lambda_1\lambda_2\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{\phi}_{\mathrm{w},1}(s)\\
&=D\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]\phi'_{\mathrm{w},1}(0)-D\lambda_1\widetilde{\chi}(s)\phi'_{\mathrm{w},2}(0)-\lambda_1\left[Ds^2+cs-\lambda_2-\delta\right]\widetilde{\zeta}(s).
\end{aligned}
$$

Similarly, multiplying (3.18) by $\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]$ and substituting $\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\widetilde{\phi}_{\mathrm{w},1}(s)$ by the right-hand side of (3.17) produces

$$
\begin{aligned}
&\left\{\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]-\lambda_1\lambda_2\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{\phi}_{\mathrm{w},2}(s)\\
&=D\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\phi'_{\mathrm{w},2}(0)-D\lambda_2\widetilde{\xi}(s)\phi'_{\mathrm{w},1}(0)-\lambda_2\left[Ds^2+cs-\lambda_1-\delta\right]\widetilde{\zeta}(s).
\end{aligned}
$$

Dividing the above equations by D^2 yields equations (3.13) and (3.14) representing the Laplace transforms of $\phi_{\mathrm{w},1}(u)$ and $\phi_{\mathrm{w},2}(u)$ respectively.

For equations (3.6) and (3.7), assume that $\lim_{u\to\infty}e^{-su}\phi_{\mathrm{d},i}(u)=0$ and $\lim_{u\to\infty}e^{-su}\phi'_{\mathrm{d},i}(u)=0$ hold for $i=1,2$, applying Laplace transforms produces

$$
\begin{aligned}
&(\lambda_1+\delta)\ \widetilde{\phi}_{\mathrm{d},1}(s)\\
&\quad=c\left[s\widetilde{\phi}_{\mathrm{d},1}(s)-\phi_{\mathrm{d},1}(0)\right]+D\left[s^2\widetilde{\phi}_{\mathrm{d},1}(s)-s\phi_{\mathrm{d},1}(0)-\phi'_{\mathrm{d},1}(0)\right]+\lambda_1\widetilde{\xi}(s)\widetilde{\phi}_{\mathrm{d},1}(s)+\lambda_1\widetilde{\chi}(s)\widetilde{\phi}_{\mathrm{d},2}(s),
\end{aligned}
$$

$$(\lambda_2+\delta)\,\widetilde{\phi}_{\mathrm{d},2}(s)$$
$$= c\left[s\widetilde{\phi}_{\mathrm{d},2}(s)-\phi_{\mathrm{d},2}(0)\right]+D\left[s^2\widetilde{\phi}_{\mathrm{d},2}(s)-s\phi_{\mathrm{d},2}(0)-\phi'_{\mathrm{d},2}(0)\right]+\lambda_2\widetilde{\xi}(s)\widetilde{\phi}_{\mathrm{d},1}(s)+\lambda_2\widetilde{\chi}(s)\widetilde{\phi}_{\mathrm{d},2}(s).$$

It follows from the definition of $\phi_{d,i}(u)$, $i=1,2$ that $\phi_{\mathrm{d},1}(0)=\phi_{\mathrm{d},2}(0)=1$. Thus, the above equations may be simplified to

$$\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\widetilde{\phi}_{\mathrm{d},1}(s)=D\phi'_{\mathrm{d},1}(0)+c+Ds-\lambda_1\widetilde{\chi}(s)\widetilde{\phi}_{\mathrm{d},2}(s), \tag{3.19}$$
$$\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]\widetilde{\phi}_{\mathrm{d},2}(s)=D\phi'_{\mathrm{d},2}(0)+c+Ds-\lambda_2\widetilde{\xi}(s)\widetilde{\phi}_{\mathrm{d},1}(s).$$

Similarly to the rearranging procedure of (3.17) and (3.18), we obtain

$$\begin{aligned}
&\left\{\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]-\lambda_1\lambda_2\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{\phi}_{\mathrm{d},1}(s)\\
&=D\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]\phi'_{\mathrm{d},1}(0)-D\lambda_1\widetilde{\chi}(s)\phi'_{\mathrm{d},2}(0)+c\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]\\
&\quad+Ds\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]-c\lambda_1\widetilde{\chi}(s)-Ds\lambda_1\widetilde{\chi}(s),\\
&\left\{\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\left[Ds^2+cs-\lambda_2-\delta+\lambda_2\widetilde{\chi}(s)\right]-\lambda_1\lambda_2\widetilde{\xi}(s)\widetilde{\chi}(s)\right\}\widetilde{\phi}_{\mathrm{d},2}(s)\\
&=D\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\phi'_{\mathrm{d},2}(0)-D\lambda_2\widetilde{\xi}(s)\phi'_{\mathrm{d},1}(0)+c\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]\\
&\quad+Ds\left[Ds^2+cs-\lambda_1-\delta+\lambda_1\widetilde{\xi}(s)\right]-c\lambda_2\widetilde{\xi}(s)-Ds\lambda_2\widetilde{\xi}(s).
\end{aligned}$$

Dividing these by D^2 and further rearranging yields equations (3.15) and (3.16). ■

Observe that the terms in front of $\widetilde{\phi}_{\mathrm{w},i}(s)$ and $\widetilde{\phi}_{\mathrm{d},i}(s)$ in equations (3.13), (3.14), (3.15) and (3.16) are identical. Setting them to be equal to zero, provides a generalized Lundberg's equation

$$\begin{aligned}
&\left(s^2+\frac{c}{D}s-\frac{\lambda_1+\delta}{D}\right)\left(s^2+\frac{c}{D}s-\frac{\lambda_2+\delta}{D}\right)\\
&\qquad+\frac{\lambda_1}{D}\left(s^2+\frac{c}{D}s-\frac{\lambda_2+\delta}{D}\right)\widetilde{\xi}(s)+\frac{\lambda_2}{D}\left(s^2+\frac{c}{D}s-\frac{\lambda_1+\delta}{D}\right)\widetilde{\chi}(s)=0.
\end{aligned} \tag{3.20}$$

The roots of Lundberg's equation play an important role in deducing the solutions for the Gerber-Shiu functions. In the following Lemmas, we will show that equation (3.20) has exactly two nonnegative roots. Moreover, these roots are distinct and real.

Lemma 3.3 *For $\delta = 0$, Lundberg's equation* (3.20) *has exactly two roots with non-negative real parts, which are distinct, real and one of them equals zero.*

Proof When $\delta = 0$, Lundberg's equation (3.20) reduces to

$$\begin{aligned}&\left(s^2 + \frac{c}{D}s - \frac{\lambda_1}{D}\right)\left(s^2 + \frac{c}{D}s - \frac{\lambda_2}{D}\right)\\&\qquad + \frac{\lambda_1}{D}\left(s^2 + \frac{c}{D}s - \frac{\lambda_2}{D}\right)\widetilde{\xi}(s) + \frac{\lambda_2}{D}\left(s^2 + \frac{c}{D}s - \frac{\lambda_1}{D}\right)\widetilde{\chi}(s) = 0\,. \qquad (3.21)\end{aligned}$$

One may easily verify that $s = 0$ is a root of equation (3.21) utilizing the relation $\widetilde{\xi}(0) + \widetilde{\chi}(0) = \widetilde{b}(0) = 1$. To analyze the nonzero roots, we rearrange equation (3.21) to

$$\left(s^2 + \frac{c}{D}s\right)\left[g_{D,1}(s) + g_{D,2}(s)\right] = 0\,,$$

where

$$\begin{aligned}g_{D,1}(s) &= s^2 + \frac{c}{D}s - \frac{\lambda_1 + \lambda_2}{D}\,,\\ g_{D,2}(s) &= \frac{\lambda_1}{D}\widetilde{\xi}(s) + \frac{\lambda_2}{D}\widetilde{\chi}(s) + \frac{\frac{\lambda_1\lambda_2}{D^2}\widetilde{\overline{B}}(s)}{s + \frac{c}{D}}\,,\end{aligned}$$

since

$$\widetilde{\overline{B}}(s) = \frac{1 - \widetilde{b}(s)}{s}\,, \quad s \neq 0\,.$$

Observe that the roots of equation (3.21) in the positive half plane coincide with those of $g_{D,1}(s) + g_{D,2}(s) = 0$. To analyze the zeros of $g_{D,1}(s) + g_{D,2}(s)$, we consider a closed contour C formed by the imaginary axis and the semi-circle $\{s : |s| = d, \operatorname{Re}(s) \geq 0\}$, where d is a sufficiently large constant. The functions $g_{D,1}(s)$ and $g_{D,2}(s)$ are analytic inside and on C. It is straightforward that quadratic function $g_{D,1}(s)$ has exactly one zero inside C. We will show that $\left|g_{D,2}(s)\right| < \left|g_{D,1}(s)\right|$ on the boundary of C and apply Rouché's theorem.

On the semi-circle part of the boundary of C, utilizing $\operatorname{Re}(s) \geq 0$ and triangle inequality yield that

$$|s| < \left|s + \frac{c}{D}\right|\,,$$

$$\left|\widetilde{\xi}(s)\right| \le \widetilde{\xi}(0) \le 1\,,$$

$$\left|\widetilde{\chi}(s)\right| \le \widetilde{\chi}(0) \le 1\,,$$

$$\left|\widetilde{\overline{B}}(s)\right| \le \widetilde{\overline{B}}(0) = \mu\,.$$

Employing these inequalities, we compare

$$\begin{aligned}
\left|g_{D,1}(s)\right| &= \left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\lambda_2}{D}\right| \\
&\ge \left|s^2 + \frac{c}{D}s\right| - \left|\frac{\lambda_1+\lambda_2}{D}\right| \\
&> |s||s| - \left(\frac{\lambda_1+\lambda_2}{D}\right) \\
&= d^2 - \left(\frac{\lambda_1+\lambda_2}{D}\right)
\end{aligned}$$

to

$$\begin{aligned}
\left|g_{D,2}(s)\right| &\le \frac{\lambda_1}{D}\left|\widetilde{\xi}(s)\right| + \frac{\lambda_2}{D}\left|\widetilde{\chi}(s)\right| + \frac{\lambda_1\lambda_2}{D^2}\frac{\left|\widetilde{\overline{B}}(s)\right|}{\left|s+\frac{c}{D}\right|} \\
&\le \frac{\lambda_1}{D} + \frac{\lambda_2}{D} + \frac{\lambda_1\lambda_2}{D^2}\cdot\frac{\mu}{d} \\
&< \frac{\lambda_1}{D} + \frac{\lambda_2}{D} + \frac{\lambda_1\lambda_2}{D^2}\mu\,,
\end{aligned}$$

which yields $\left|g_{D,2}(s)\right| < \left|g_{D,1}(s)\right|$ on the semi-circle part of the contour C, since d is a large constant.

On the imaginary axis part of the contour C, $\mathrm{Re}(s) = 0$ implies $\left|s+\frac{c}{D}\right| \ge \frac{c}{D}$, $\left|\widetilde{\xi}(s)\right| \le \widetilde{\xi}(0)$, $\left|\widetilde{\chi}(s)\right| \le \widetilde{\chi}(0)$ and $\left|\widetilde{\overline{B}}(s)\right| \le \widetilde{\overline{B}}(0) = \mu$, which leads to

$$\begin{aligned}
\left|g_{D,2}(s)\right| &\le \frac{\lambda_1}{D}\left|\widetilde{\xi}(s)\right| + \frac{\lambda_2}{D}\left|\widetilde{\chi}(s)\right| + \frac{\lambda_1\lambda_2}{D^2}\frac{\left|\widetilde{\overline{B}}(s)\right|}{\left|s+\frac{c}{D}\right|} \\
&\le \frac{\lambda_1}{D}\widetilde{\xi}(0) + \frac{\lambda_2}{D}\widetilde{\chi}(0) + \frac{\lambda_1\lambda_2}{Dc}\mu.
\end{aligned}$$

Utilizing the relations $\widetilde{\xi}(0) = \mathbb{P}\{X > Q\}$ and $\widetilde{\chi}(0) = \mathbb{P}\{X \le Q\}$, the positive-security-loading condition (3.2) may be rewritten as

$$\mu < \frac{c}{\lambda_1}\widetilde{\xi}(0) + \frac{c}{\lambda_2}\widetilde{\chi}(0)\,, \tag{3.22}$$

and thus

$$\begin{aligned}\left|g_{D,2}(s)\right| &< \frac{\lambda_1}{D}\widetilde{\xi}(0) + \frac{\lambda_2}{D}\widetilde{\chi}(0) + \frac{\lambda_1\lambda_2}{Dc}\cdot\frac{c}{\lambda_1}\widetilde{\xi}(0) + \frac{\lambda_1\lambda_2}{Dc}\cdot\frac{c}{\lambda_2}\widetilde{\chi}(0)\\ &= \frac{\lambda_1}{D} + \frac{\lambda_2}{D}.\end{aligned}$$

Meanwhile, since $\mathrm{Re}(s) = 0$, $\mathrm{Im}(s^2) = 0$ and $-s^2 \geq 0$, we have

$$\begin{aligned}\left|g_{D,1}(s)\right| &= \left|\frac{c}{D}s - \left(\frac{\lambda_1 + \lambda_2}{D} - s^2\right)\right|\\ &\geq \left|\mathrm{Re}\left[\frac{c}{D}s - \left(\frac{\lambda_1 + \lambda_2}{D} - s^2\right)\right]\right|\\ &= \frac{\lambda_1 + \lambda_2}{D} - s^2\\ &\geq \frac{\lambda_1 + \lambda_2}{D}.\end{aligned}$$

Comparing $\left|g_{D,1}(s)\right|$ with $\left|g_{D,2}(s)\right|$ shows that $\left|g_{D,1}(s)\right| > \left|g_{D,2}(s)\right|$ also holds on the imaginary axis part of the contour C.

Applying Rouché's theorem on the closed contour C and letting $d \to \infty$, we may conclude that $g_{D,1}(s) + g_{D,2}(s) = 0$ has exactly one root in the positive half plane, which implies that equation (3.21) also has exactly one root in the positive half plane. Moreover, the root is real, since the complex roots of analytic functions that are presented in series form with only real coefficients come in conjugate pairs. Recall that zero is also a root of (3.21). Therefore, Lundberg's equation with $\delta = 0$ has exactly two roots with nonnegative real parts, where both are real roots and one of them is zero. ■

Lemma 3.4 *For $\delta > 0$, Lundberg's equation* (3.20) *has exactly two roots with nonnegative real parts, which are distinct, positive and real.*

Proof We rewrite equation (3.20) as

$$f_1(s) + f_2(s) = 0,$$

where

$$
\begin{aligned}
f_1(s) &= \left(s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right)\left(s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right),\\
f_2(s) &= \frac{\lambda_1}{D}\left(s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right)\widetilde{\xi}(s) + \frac{\lambda_2}{D}\left(s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right)\widetilde{\chi}(s).
\end{aligned}
$$

Rouché's theorem states that if functions $f_1(s)$ and $f_2(s)$ are analytic inside and on some closed contour C and $|f_2(s)| < |f_1(s)|$ on the boundary of C, then $f_1(s)$ and $f_1(s) + f_2(s)$ have the same number of zeros inside C. Consider such a closed contour C in the complex plane, formed by the semi-circle $\{s : |s| = d, \mathrm{Re}(s) \geq 0\}$ and the imaginary axis, where d is a sufficiently large constant. The functions $f_1(s)$ and $f_2(s)$ are analytic inside and on C, and $f_1(s)$ has two zeros inside C. We will show that $|f_2(s)| < |f_1(s)|$ on the boundary of C.

On the semi-circle part of the boundary of C, it follows from $\mathrm{Re}(s) \geq 0$ that $\left|\widetilde{\xi}(s)\right| \leq \widetilde{\xi}(0)$ and $\left|\widetilde{\chi}(s)\right| \leq \widetilde{\chi}(0)$. By the triangle inequality, we obtain that

$$
\begin{aligned}
\left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right| &\geq \left|s^2 + \frac{c}{D}s\right| - \left|\frac{\lambda_1+\delta}{D}\right| > |s||s| - \left(\frac{\lambda_1+\delta}{D}\right) = d^2 - \left(\frac{\lambda_1+\delta}{D}\right) > \frac{\lambda_1}{D},\\
\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right| &\geq \left|s^2 + \frac{c}{D}s\right| - \left|\frac{\lambda_2+\delta}{D}\right| > |s||s| - \left(\frac{\lambda_2+\delta}{D}\right) = d^2 - \left(\frac{\lambda_2+\delta}{D}\right) > \frac{\lambda_2}{D}.
\end{aligned}
$$

The above inequalities together with the fact that $\widetilde{\xi}(0) + \widetilde{\chi}(0) = 1$ yield

$$
\begin{aligned}
|f_2(s)| &\leq \frac{\lambda_1}{D}\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\left|\widetilde{\xi}(s)\right| + \frac{\lambda_2}{D}\left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\left|\widetilde{\chi}(s)\right|\\
&< \left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\widetilde{\xi}(0) + \left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\widetilde{\chi}(0)\\
&= \left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\\
&= \left|f_1(s)\right|.
\end{aligned}
$$

When s is on the imaginary axis part of the boundary of contour C, we have $\mathrm{Re}(s) = 0$, $\mathrm{Im}(s^2) = 0$ and $-s^2 \geq 0$, then

$$
\begin{aligned}
\left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right| &= \left|\frac{c}{D}s - \left(-s^2 + \frac{\lambda_1+\delta}{D}\right)\right| \geq -s^2 + \frac{\lambda_1+\delta}{D} \geq \frac{\lambda_1+\delta}{D},\\
\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right| &= \left|\frac{c}{D}s - \left(-s^2 + \frac{\lambda_2+\delta}{D}\right)\right| \geq -s^2 + \frac{\lambda_2+\delta}{D} \geq \frac{\lambda_2+\delta}{D}.
\end{aligned}
$$

Hence,

$$\begin{aligned}
|f_1(s)| &= \left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right| \\
&= \left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\widetilde{\xi}(0) + \left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\widetilde{\chi}(0) \\
&\geq \frac{\lambda_1+\delta}{D}\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\widetilde{\xi}(0) + \frac{\lambda_2+\delta}{D}\left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\widetilde{\chi}(0) \\
&> \frac{\lambda_1}{D}\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\widetilde{\xi}(0) + \frac{\lambda_2}{D}\left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right|\widetilde{\chi}(0) \\
&\geq \frac{\lambda_1}{D}\left|s^2 + \frac{c}{D}s - \frac{\lambda_2+\delta}{D}\right|\left|\widetilde{\xi}(s)\right| + \frac{\lambda_2}{D}\left|s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right||\widetilde{\chi}(s)| \\
&\geq |f_2(s)|\,.
\end{aligned}$$

Applying Rouché's theorem on the closed contour C, we may conclude that $f_1(s)+f_2(s)$ has two zeros insides the contour C. Denote these roots by ϱ and $\mathfrak{r}$. Letting $d \to \infty$ indicates that ϱ and $\mathfrak{r}$ are the only roots of $f_1(s) + f_2(s) = 0$ in the right half plane. It remains to show that they are real and distinct. As δ converges to 0, $\mathfrak{r}$ and ϱ converge to the roots of the simpler equation (3.21), which means one of $\mathfrak{r}$ and ϱ converges to zero and the other one converges to a strictly positive number, hence they are distinct. Moreover, we prove by contradiction that $\mathfrak{r}$ and ϱ are real numbers. Suppose $\mathfrak{r}$ and ϱ are complex roots of the analytic function $f_1(s) - f_2(s)$, then they must be a conjugated pair, i.e. $\mathfrak{r} = a + bi$ and $\varrho = a - bi$ for some real numbers $a, b > 0$. When δ converges to zero, we know that one of the roots converges to zero, which indicates that a and b converge to 0 simultaneously. Then, the other root also converges to 0, which contradicts the fact that the other root converges to a strictly positive number. Thus, $\mathfrak{r}$ and ϱ are both real. ■

From Lemmas 3.3 and 3.4, we conclude that for any $\delta \geq 0$ Lundberg's equation has exactly two distinct nonnegative roots and one of them converges to zero as $\delta \to 0$. For the rest of the paper, these roots are denoted by ϱ and $\mathfrak{r}$.

3.3 Explicit solution for the Gerber-Shiu function

To find an explicit solution for the Gerber-Shiu function, we need to invert the Laplace transforms in (3.13), (3.14), (3.15) and (3.16). First, we determine the unknowns constants $\phi'_{\mathrm{w},1}(0)$, $\phi'_{\mathrm{w},2}(0)$, $\phi'_{\mathrm{d},1}(0)$ and $\phi'_{\mathrm{d},2}(0)$ in these equations, utilizing the nonnegative roots of Lundberg's equation ϱ and $\mathfrak{r}$. When s takes the value ϱ or $\mathfrak{r}$, the right-hands sides of (3.13) and (3.14) also equal zero. Moreover, when $s = \varrho$ (or $s = \mathfrak{r}$), the right-hand sides of (3.13) and (3.14) are identical. Thus, we may solve a system of linear equations for $\phi'_{\mathrm{w},1}(0)$ and $\phi'_{\mathrm{w},2}(0)$ which is

$$\left[\varrho^2 + \frac{c}{D}\varrho - \frac{\lambda_2+\delta}{D} + \frac{\lambda_2}{D}\widetilde{\chi}(\varrho)\right]\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\widetilde{\chi}(\varrho)\,\phi'_{\mathrm{w},2}(0) = \frac{\lambda_1}{D}\left[\varrho^2 + \frac{c}{D}\varrho - \frac{\lambda_2+\delta}{D}\right]\widetilde{\zeta}(\varrho),$$
$$\left[\mathfrak{r}^2 + \frac{c}{D}\mathfrak{r} - \frac{\lambda_2+\delta}{D} + \frac{\lambda_2}{D}\widetilde{\chi}(\mathfrak{r})\right]\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\widetilde{\chi}(\mathfrak{r})\,\phi'_{\mathrm{w},2}(0) = \frac{\lambda_1}{D}\left[\mathfrak{r}^2 + \frac{c}{D}\mathfrak{r} - \frac{\lambda_2+\delta}{D}\right]\widetilde{\zeta}(\mathfrak{r}).$$

The solution yields

$$\phi'_{\mathrm{w},1}(0) = \frac{\lambda_1\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\widetilde{\zeta}(\varrho)\widetilde{\chi}(\mathfrak{r}) - \lambda_1\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\widetilde{\zeta}(\mathfrak{r})\widetilde{\chi}(\varrho)}{D\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\widetilde{\chi}(\mathfrak{r}) - D\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\widetilde{\chi}(\varrho)}, \tag{3.23}$$

$$\begin{aligned}\phi'_{\mathrm{w},2}(0) = {} & \frac{\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\widetilde{\zeta}(\varrho)\left[D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta + \lambda_2\widetilde{\chi}(\mathfrak{r})\right]}{D\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\widetilde{\chi}(\mathfrak{r}) - D\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\widetilde{\chi}(\varrho)} \\ & - \frac{\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\widetilde{\zeta}(\mathfrak{r})\left[D\varrho^2 + c\varrho - \lambda_2 - \delta + \lambda_2\widetilde{\chi}(\varrho)\right]}{D\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\widetilde{\chi}(\mathfrak{r}) - D\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\widetilde{\chi}(\varrho)}.\end{aligned} \tag{3.24}$$

Rearrange (3.24) to

$$\phi'_{\mathrm{w},2}(0) = \frac{\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\left[\widetilde{\zeta}(\varrho) - \widetilde{\zeta}(\mathfrak{r})\right]}{D\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\widetilde{\chi}(\mathfrak{r}) - D\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\widetilde{\chi}(\varrho)} + \frac{\lambda_2}{\lambda_1}\phi'_{\mathrm{w},1}(0),$$

which leads to a useful quantity

$$\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right] = \frac{-\lambda_1\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\left[\widetilde{\zeta}(\varrho) - \widetilde{\zeta}(\mathfrak{r})\right]}{D^2\left(D\varrho^2 + c\varrho - \lambda_2 - \delta\right)\widetilde{\chi}(\mathfrak{r}) - D^2\left(D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta\right)\widetilde{\chi}(\varrho)}. \tag{3.25}$$

Similarly, from (3.15) and (3.16) we derive a system of linear equations and solve for $\phi'_{\mathrm{d},1}(0)$ and $\phi'_{\mathrm{d},2}(0)$ yielding

$$\phi'_{\mathrm{d},1}(0) = \frac{-\left(\frac{c}{D} + \varrho\right)\left[D\varrho^2 + c\varrho - \lambda_2 - \delta + (\lambda_2 - \lambda_1)\widetilde{\chi}(\varrho)\right]\lambda_1\widetilde{\chi}(\mathfrak{r})}{\left[D\varrho^2 + c\varrho - \lambda_2 - \delta + \lambda_2\widetilde{\chi}(\varrho)\right]\lambda_1\widetilde{\chi}(\mathfrak{r}) - \left[D\mathfrak{r}^2 + c\mathfrak{r} - \lambda_2 - \delta + \lambda_2\widetilde{\chi}(\mathfrak{r})\right]\lambda_1\widetilde{\chi}(\varrho)}$$

$$+\frac{\left(\frac{c}{D}+\mathfrak{r}\right)\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+(\lambda_2-\lambda_1)\widetilde{\chi}(\mathfrak{r})\right]\lambda_1\widetilde{\chi}(\varrho)}{\left[D\varrho^2+c\varrho-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\varrho)\right]\lambda_1\widetilde{\chi}(\mathfrak{r})-\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\mathfrak{r})\right]\lambda_1\widetilde{\chi}(\varrho)}, \tag{3.26}$$

$$\begin{aligned}\phi'_{\mathrm{d},2}(0)=&\frac{-\left(\frac{c}{D}+\varrho\right)\left[D\varrho^2+c\varrho-\lambda_2-\delta+(\lambda_2-\lambda_1)\widetilde{\chi}(\varrho)\right]\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\mathfrak{r})\right]}{\left[D\varrho^2+c\varrho-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\varrho)\right]\lambda_1\widetilde{\chi}(\mathfrak{r})-\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\mathfrak{r})\right]\lambda_1\widetilde{\chi}(\varrho)}\\&+\frac{\left(\frac{c}{D}+\mathfrak{r}\right)\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+(\lambda_2-\lambda_1)\widetilde{\chi}(\mathfrak{r})\right]\left[D\varrho^2+c\varrho-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\varrho)\right]}{\left[D\varrho^2+c\varrho-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\varrho)\right]\lambda_1\widetilde{\chi}(\mathfrak{r})-\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\mathfrak{r})\right]\lambda_1\widetilde{\chi}(\varrho)},\end{aligned} \tag{3.27}$$

$$\begin{aligned}\left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0)\right]=&\frac{\left(\frac{c}{D}+\varrho\right)\left[D\varrho^2+c\varrho-\lambda_2-\delta+(\lambda_2-\lambda_1)\widetilde{\chi}(\varrho)\right]\left(D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta\right)}{D\left[D\varrho^2+c\varrho-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\varrho)\right]\widetilde{\chi}(\mathfrak{r})-D\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\mathfrak{r})\right]\widetilde{\chi}(\varrho)}\\&-\frac{\left(\frac{c}{D}+\mathfrak{r}\right)\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+(\lambda_2-\lambda_1)\widetilde{\chi}(\mathfrak{r})\right]\left(D\varrho^2+c\varrho-\lambda_2-\delta\right)}{D\left[D\varrho^2+c\varrho-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\varrho)\right]\widetilde{\chi}(\mathfrak{r})-D\left[D\mathfrak{r}^2+c\mathfrak{r}-\lambda_2-\delta+\lambda_2\widetilde{\chi}(\mathfrak{r})\right]\widetilde{\chi}(\varrho)}.\end{aligned} \tag{3.28}$$

Denote the two real roots of equation $s^2+\frac{c}{D}s-\frac{\lambda_i+\delta}{D}=0$ by A_i and $-a_i$ such that $A_i, a_i>0$ for $i=1,2$, i.e.,

$$s^2+\frac{c}{D}s-\frac{\lambda_1+\delta}{D}=(s-A_1)(s+a_1), \tag{3.29}$$

$$s^2+\frac{c}{D}s-\frac{\lambda_2+\delta}{D}=(s-A_2)(s+a_2). \tag{3.30}$$

Notice that

$$a_i=\frac{c}{D}+A_i, \qquad \text{for } i=1,2.$$

Dividing equations (3.13) and (3.14) by $(s+a_1)(s+a_2)$ produces

$$\begin{aligned}&\left[(s-A_1)(s-A_2)+\frac{\lambda_1}{D}\cdot\frac{s-A_2}{s+a_1}\widetilde{\xi}(s)+\frac{\lambda_2}{D}\cdot\frac{s-A_1}{s+a_2}\widetilde{\chi}(s)\right]\widetilde{\phi}_{\mathrm{w},1}(s)\\&\qquad=\frac{s-A_2}{s+a_1}\phi'_{\mathrm{w},1}(0)+\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\frac{\widetilde{\chi}(s)}{(s+a_1)(s+a_2)}-\frac{\lambda_1}{D}\cdot\frac{s-A_2}{s+a_1}\widetilde{\zeta}(s),\\&\left[(s-A_1)(s-A_2)+\frac{\lambda_1}{D}\cdot\frac{s-A_2}{s+a_1}\widetilde{\xi}(s)+\frac{\lambda_2}{D}\cdot\frac{s-A_1}{s+a_2}\widetilde{\chi}(s)\right]\widetilde{\phi}_{\mathrm{w},2}(s)\\&\qquad=\frac{s-A_1}{s+a_2}\phi'_{\mathrm{w},2}(0)-\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\frac{\widetilde{\xi}(s)}{(s+a_1)(s+a_2)}-\frac{\lambda_2}{D}\cdot\frac{s-A_1}{s+a_2}\widetilde{\zeta}(s).\end{aligned}$$

For all $s \geq 0$ except for $s = \varrho$ or $s = \mathfrak{r}$, the above equations may be rewritten as

$$\widetilde{\phi}_{\mathrm{w},1}(s) = \frac{\phi'_{\mathrm{w},1}(0) + \widetilde{\vartheta}_1(s)}{\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s)}, \tag{3.31}$$

$$\widetilde{\phi}_{\mathrm{w},2}(s) = \frac{\phi'_{\mathrm{w},2}(0) + \widetilde{\vartheta}_2(s)}{\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s)}, \tag{3.32}$$

where

$$\widetilde{h}_{D,1}(s) = (s - A_1)(s - A_2), \tag{3.33}$$

$$\widetilde{h}_{D,2}(s) = \frac{\lambda_1}{D} \cdot \frac{A_2 + a_1}{s + a_1}\widetilde{\xi}(s) - \frac{\lambda_1}{D}\widetilde{\xi}(s) + \frac{\lambda_2}{D} \cdot \frac{A_1 + a_2}{s + a_2}\widetilde{\chi}(s) - \frac{\lambda_2}{D}\widetilde{\chi}(s), \tag{3.34}$$

$$\begin{aligned}\widetilde{\vartheta}_1(s) = &-\frac{A_2 + a_1}{s + a_1}\phi'_{\mathrm{w},1}(0) + \left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\frac{\widetilde{\chi}(s)}{(s + a_1)(s + a_2)} \\ &- \frac{\lambda_1}{D}\widetilde{\zeta}(s) + \frac{\lambda_1}{D} \cdot \frac{A_2 + a_1}{s + a_1}\widetilde{\zeta}(s),\end{aligned} \tag{3.35}$$

$$\begin{aligned}\widetilde{\vartheta}_2(s) = &-\frac{A_1 + a_2}{s + a_2}\phi'_{\mathrm{w},2}(0) - \left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\frac{\widetilde{\xi}(s)}{(s + a_1)(s + a_2)} \\ &- \frac{\lambda_2}{D}\widetilde{\zeta}(s) + \frac{\lambda_2}{D} \cdot \frac{A_1 + a_2}{s + a_2}\widetilde{\zeta}(s).\end{aligned} \tag{3.36}$$

From equations (3.31) and (3.32), we derive the following proposition.

Proposition 3.5 *The Laplace transforms $\widetilde{\phi}_{\mathrm{w},i}(s)$, $i = 1, 2$, satisfy*

$$\widetilde{\phi}_{\mathrm{w},1}(s) = \frac{T_s T_\varrho T_\mathfrak{r} \vartheta_1(0)}{1 - T_s T_\varrho T_\mathfrak{r} h_{D,2}(0)}, \tag{3.37}$$

$$\widetilde{\phi}_{\mathrm{w},2}(s) = \frac{T_s T_\varrho T_\mathfrak{r} \vartheta_2(0)}{1 - T_s T_\varrho T_\mathfrak{r} h_{D,2}(0)}, \tag{3.38}$$

where

$$h_{D,2}(u) = \frac{\lambda_1(A_2 + a_1)}{D}\int_0^u e^{-a_1(u-y)}\xi(y)\,dy + \frac{\lambda_2(A_1 + a_2)}{D}\int_0^u e^{-a_2(u-y)}\chi(y)\,dy - \frac{\lambda_1}{D}\xi(u) - \frac{\lambda_2}{D}\chi(u), \tag{3.39}$$

$$\begin{aligned}\vartheta_1(u) = &-(A_2 + a_1)\,\phi'_{\mathrm{w},1}(0)\,e^{-a_1 u} + \left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\int_0^u \frac{e^{-a_1(u-y)} - e^{-a_2(u-y)}}{a_2 - a_1}\,\chi(y)\,dy \\ &- \frac{\lambda_1}{D}\zeta(u) + \frac{\lambda_1(A_2 + a_1)}{D}\int_0^u e^{-a_1(u-y)}\zeta(y)\,dy,\end{aligned} \tag{3.40}$$

$$\vartheta_2(u) = -(A_1 + a_2)\,\phi'_{w,2}(0)\,e^{-a_2 u} - \left[\frac{\lambda_2}{D}\phi'_{w,1}(0) - \frac{\lambda_1}{D}\phi'_{w,2}(0)\right]\int_0^u \frac{e^{-a_1(u-y)} - e^{-a_2(u-y)}}{a_2 - a_1}\,\xi(y)\,dy$$
$$- \frac{\lambda_2}{D}\zeta(u) + \frac{\lambda_2\,(A_1 + a_2)}{D}\int_0^u e^{-a_2(u-y)}\zeta(y)\,dy. \tag{3.41}$$

Proof The two nonnegative roots of Lundberg's equation are denoted by ϱ and $\mathfrak{r}$, which implies $\widetilde{h}_{D,1}(\varrho) = \widetilde{h}_{D,2}(\varrho)$ and $\widetilde{h}_{D,1}(\mathfrak{r}) = \widetilde{h}_{D,2}(\mathfrak{r})$. Since $\widetilde{h}_{D,1}(s)$ is a second-order polynomial of s, an application of the Lagrange's interpolation theorem with the properties of the Translation operator yields that (see Boudreault et al., 2006, page 274-275, for more detailed derivations),

$$\begin{aligned}\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s) &= (s-\varrho)(s-\mathfrak{r})\left[T_0 T_\varrho T_\mathfrak{r} h_{D,1}(0) - T_s T_\varrho T_\mathfrak{r} h_{D,2}(0)\right]\\ &= (s-\varrho)(s-\mathfrak{r})\left[1 - T_s T_\varrho T_\mathfrak{r} h_{D,2}(0)\right].\end{aligned} \tag{3.42}$$

When $s = \varrho$ or $s = \mathfrak{r}$, the numerator of (3.31) also equals zero, which indicates that

$$\widetilde{\vartheta}_1(\varrho) = \widetilde{\vartheta}_1(\mathfrak{r}) = -\phi'_{w,1}(0)\,,$$

and thus

$$T_\mathfrak{r} T_\varrho \vartheta_1(0) = 0\,.$$

Hence, the numerator of (3.31) may be rewritten to

$$\begin{aligned}\phi'_{w,1}(0) + \widetilde{\vartheta}_1(s) &= (\varrho - s)T_s T_\varrho \vartheta_1(0)\\ &= (\varrho - s)(\mathfrak{r} - s)\frac{T_s T_\varrho \vartheta_1(0) - T_r T_\varrho \vartheta_1(0)}{(\mathfrak{r} - s)}\\ &= (s-\varrho)(s-\mathfrak{r})T_s T_\varrho T_\mathfrak{r} \vartheta_1(0).\end{aligned} \tag{3.43}$$

Inserting equalities (3.42) and (3.43) into (3.31) produces the desired equation (3.37). Utilizing a similar procedure, we obtain equation (3.38) from (3.32). In addition, applying Laplace inversion to identities (3.34), (3.35) and (3.36) yields the functions $h_{D,2}(u)$, $\vartheta_1(u)$ and $\vartheta_2(u)$ defined in (3.39), (3.40) and (3.41) respectively. ■

For the transforms $\widetilde{\phi}_{d,1}(s)$ and $\widetilde{\phi}_{d,2}(s)$, dividing both equations (3.15) and (3.16) by $(s+a_1)(s+a_2)$ produces that

$$\begin{aligned}&\left[(s-A_1)(s-A_2)+\frac{\lambda_1}{D}\cdot\frac{s-A_2}{s+a_1}\widetilde{\xi}(s)+\frac{\lambda_2}{D}\cdot\frac{s-A_1}{s+a_2}\widetilde{\chi}(s)\right]\widetilde{\phi}_{d,1}(s)\\&\quad=\frac{s-A_2}{s+a_1}\left[s+\frac{c}{D}+\phi'_{d,1}(0)\right]+\left[\frac{\lambda_2}{D}\phi'_{d,1}(0)-\frac{\lambda_1}{D}\phi'_{d,2}(0)+\frac{\lambda_2-\lambda_1}{D}\left(\frac{c}{D}+s\right)\right]\frac{\widetilde{\chi}(s)}{(s+a_1)(s+a_2)},\end{aligned}\tag{3.44}$$

$$\begin{aligned}&\left[(s-A_1)(s-A_2)+\frac{\lambda_1}{D}\cdot\frac{s-A_2}{s+a_1}\widetilde{\xi}(s)+\frac{\lambda_2}{D}\cdot\frac{s-A_1}{s+a_2}\widetilde{\chi}(s)\right]\widetilde{\phi}_{d,2}(s)\\&\quad=\frac{s-A_1}{s+a_2}\left[s+\frac{c}{D}+\phi'_{d,2}(0)\right]-\left[\frac{\lambda_2}{D}\phi'_{d,1}(0)-\frac{\lambda_1}{D}\phi'_{d,2}(0)+\frac{\lambda_2-\lambda_1}{D}\left(\frac{c}{D}+s\right)\right]\frac{\widetilde{\xi}(s)}{(s+a_1)(s+a_2)}.\end{aligned}\tag{3.45}$$

Substituting the terms

$$\begin{aligned}\frac{s-A_2}{s+a_1}\left[s+\frac{c}{D}+\phi'_{d,1}(0)\right]&=\frac{s-A_2}{s+A_1+\frac{c}{D}}\left[s+\frac{c}{D}+A_1-A_1+\phi'_{d,1}(0)\right]\\&=s-A_2+\frac{s-A_2}{s+a_1}\left[\phi'_{d,1}(0)-A_1\right]\\&=s-A_1-A_2+\phi'_{d,1}(0)+\frac{A_2+a_1}{s+a_1}\left[A_1-\phi'_{d,1}(0)\right]\end{aligned}$$

and

$$\frac{s-A_1}{s+a_2}\left[s+\frac{c}{D}+\phi'_{d,2}(0)\right]=s-A_1-A_2+\phi'_{d,2}(0)+\frac{A_1+a_2}{s+a_2}\left[A_2-\phi'_{d,2}(0)\right]$$

into equations (3.44) and (3.45), we obtain for all $s\geq 0$ (except for $s=\varrho$ or $s=\mathfrak{r}$),

$$\widetilde{\phi}_{d,1}(s)=\frac{\widetilde{\varsigma}_1(s)+\widetilde{\varpi}_1(s)}{\widetilde{h}_{D,1}(s)-\widetilde{h}_{D,2}(s)}\,,\tag{3.46}$$

$$\widetilde{\phi}_{d,2}(s)=\frac{\widetilde{\varsigma}_2(s)+\widetilde{\varpi}_2(s)}{\widetilde{h}_{D,1}(s)-\widetilde{h}_{D,2}(s)}\,,\tag{3.47}$$

where $\widetilde{h}_{D,1}(s)$, $\widetilde{h}_{D,2}(s)$ are defined in (3.33), (3.34) and

$$\begin{aligned}\widetilde{\varsigma}_1(s)&=s-A_1-A_2+\phi'_{d,1}(0)\,,\\\widetilde{\varpi}_1(s)&=\frac{A_2+a_1}{s+a_1}\left[A_1-\phi'_{d,1}(0)\right]+\left[\frac{\lambda_2}{D}\phi'_{d,1}(0)-\frac{\lambda_1}{D}\phi'_{d,2}(0)+\frac{\lambda_2-\lambda_1}{D}\left(\frac{c}{D}+s\right)\right]\frac{\widetilde{\chi}(s)}{(s+a_1)(s+a_2)}\,,\end{aligned}\tag{3.48}$$

$$\widetilde{\varsigma}_2(s) = s - A_1 - A_2 + \phi'_{\mathrm{d},2}(0)\,,$$
$$\widetilde{\varpi}_2(s) = \frac{A_1 + a_2}{s + a_2}\left[A_2 - \phi'_{\mathrm{d},2}(0)\right] - \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{\lambda_2 - \lambda_1}{D}\left(\frac{c}{D} + s\right)\right]\frac{\widetilde{\xi}(s)}{(s + a_1)(s + a_2)}\,. \tag{3.49}$$

Rearranging (3.46) and (3.47) leads to the proposition below.

Proposition 3.6 *When ruin is caused by oscillation, the Laplace transforms $\widetilde{\phi}_{\mathrm{d},i}(s)$, $i = 1, 2$, satisfy*

$$\widetilde{\phi}_{\mathrm{d},1}(s) = \frac{T_s T_\varrho T_{\mathfrak{r}} \varpi_1(0)}{1 - T_s T_\varrho T_{\mathfrak{r}} h_{D,2}(0)}, \tag{3.50}$$
$$\widetilde{\phi}_{\mathrm{d},2}(s) = \frac{T_s T_\varrho T_{\mathfrak{r}} \varpi_2(0)}{1 - T_s T_\varrho T_{\mathfrak{r}} h_{D,2}(0)}, \tag{3.51}$$

where

$$\begin{aligned}
\varpi_1(u) = {} & (A_2 + a_1)\left[A_1 - \phi'_{\mathrm{d},1}(0)\right]e^{-a_1 u} + \frac{\lambda_2 - \lambda_1}{D}\int_0^u \frac{a_2 e^{-a_2(u-y)} - a_1 e^{-a_1(u-y)}}{a_2 - a_1}\chi(y)\,dy, \\
& + \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{c\,(\lambda_2 - \lambda_1)}{D^2}\right]\int_0^u \frac{e^{-a_1(u-y)} - e^{-a_2(u-y)}}{a_2 - a_1}\chi(y)\,dy,
\end{aligned} \tag{3.52}$$
$$\begin{aligned}
\varpi_2(u) = {} & (A_1 + a_2)\left[A_2 - \phi'_{\mathrm{d},2}(0)\right]e^{-a_2 u} - \frac{\lambda_2 - \lambda_1}{D}\int_0^u \frac{a_2 e^{-a_2(u-y)} - a_1 e^{-a_1(u-y)}}{a_2 - a_1}\xi(y)\,dy \\
& - \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{c\,(\lambda_2 - \lambda_1)}{D^2}\right]\int_0^u \frac{e^{-a_1(u-y)} - e^{-a_2(u-y)}}{a_2 - a_1}\xi(y)\,dy,
\end{aligned} \tag{3.53}$$

and $h_{D,2}(u)$ is defined in (3.39).

Proof For $i = 1, 2$, $\widetilde{\varsigma}_i(s)$ is a linear function of s with $\widetilde{\varsigma}_i(\varrho) = -\widetilde{\varpi}_i(\varrho)$ and $\widetilde{\varsigma}_i(\mathfrak{r}) = -\widetilde{\varpi}_i(\mathfrak{r})$. Utilizing the properties of the Translation operator that are listed in Li and Garrido (2004) and applying Lagrange's interpolation theorem (see Boudreault et al., 2006, page 274) leads to

$$\widetilde{\varsigma}_i(s) + \widetilde{\varpi}_i(s) = (s - \varrho)(s - \mathfrak{r})\,T_s T_\varrho T_{\mathfrak{r}} \varpi_i(0), \quad i = 1, 2. \tag{3.54}$$

Together with identity (3.42), we obtain from (3.46) and (3.47)

$$\widetilde{\phi}_{\mathrm{d},1}(s) = \frac{T_s T_\varrho T_{\mathfrak{r}} \varpi_1(0)}{1 - T_s T_\varrho T_{\mathfrak{r}} h_{D,2}(0)},$$

$$\widetilde{\phi}_{\mathrm{d},2}(s) = \frac{T_s T_\varrho T_\mathfrak{r} \varpi_2(0)}{1 - T_s T_\varrho T_\mathfrak{r} h_{D,2}(0)}.$$

Moreover, inverting the Laplace transforms in (3.48) and (3.49) yields $\varpi_1(u)$ and $\varpi_2(u)$. ■

Utilizing Propositions 3.5 and 3.6, we reach the following theorem, which is the main result of this chapter.

Theorem 3.7 *The Gerber-Shiu function $m_{D,i}(u)$, $i = 1,2$, defined in* (3.3) *satisfies the following defective renewal equations*

$$m_{D,1}(u) = \kappa_{D,\delta} \int_0^u m_{D,1}(u-y)\eta_D(y)\,dy + \sigma_{D,1}(u), \tag{3.55}$$

$$m_{D,2}(u) = \kappa_{D,\delta} \int_0^u m_{D,2}(u-y)\eta_D(y)\,dy + \sigma_{D,2}(u), \tag{3.56}$$

where

$$\kappa_{D,\delta} = \int_0^\infty T_\varrho T_\mathfrak{r} h_{D,2}(y) dy, \tag{3.57}$$

$$\eta_D(y) = \frac{T_\varrho T_\mathfrak{r} h_{D,2}(y)}{T_0 T_\varrho T_\mathfrak{r} h_{D,2}(0)}, \tag{3.58}$$

$$\sigma_{D,1}(u) = T_\varrho T_\mathfrak{r} \vartheta_1(u) + w_0 T_\varrho T_\mathfrak{r} \varpi_1(u), \tag{3.59}$$

$$\sigma_{D,2}(u) = T_\varrho T_\mathfrak{r} \vartheta_2(u) + w_0 T_\varrho T_\mathfrak{r} \varpi_2(u), \tag{3.60}$$

with ϱ, $\mathfrak{r}$ denoting the nonnegative roots of Lundberg's equation (3.20), *and $h_{D,2}(u)$, $\vartheta_1(u)$, $\vartheta_2(u)$, $\varpi_1(u)$ and $\varpi_2(u)$ defined in* (3.39), (3.40), (3.41), (3.52) *and* (3.53). *Moreover, $\eta_D(y)$, $y \geq 0$ is a probability density function and $\kappa_{D,\delta}$ is a constant satisfying* $0 < \kappa_{D,\delta} < 1$.

Proof Combining (3.37), (3.38), (3.50), (3.51) and (3.3), we obtain

$$\widetilde{m}_{D,i}(s) = \frac{T_s T_\varrho T_\mathfrak{r} \vartheta_i(0) + w_0 T_s T_\varrho T_\mathfrak{r} \varpi_i(0)}{1 - T_s T_\varrho T_\mathfrak{r} h_{D,2}(0)}, \qquad i = 1,2.$$

Applying Laplace inversion yields

$$m_{D,1}(u) = T_0 T_\varrho T_\mathfrak{r} h_{D,2}(0) \int_0^u m_{D,1}(u-y) \frac{T_\varrho T_\mathfrak{r} h_{D,2}(y)}{T_0 T_\varrho T_\mathfrak{r} h_{D,2}(0)}\,\mathrm{d}y + T_\varrho T_\mathfrak{r} \vartheta_1(u) + w_0 T_\varrho T_\mathfrak{r} \varpi_1(u),$$

$$m_{D,2}(u) = T_0 T_\varrho T_\mathfrak{r} h_{D,2}(0) \int_0^u m_{D,2}(u-y) \frac{T_\varrho T_\mathfrak{r} h_{D,2}(y)}{T_0 T_\varrho T_\mathfrak{r} h_{D,2}(0)} \,\mathrm{d}y + T_\varrho T_\mathfrak{r} \vartheta_2(u) + w_0 T_\varrho T_\mathfrak{r} \varpi_2(u).$$

Introducing notation (3.57), (3.58), (3.59) and (3.60), we reach the desired equations (3.55) and (3.56). To verify equations (3.55) and (3.56) are of defective-renewal type, we still need to show that $\eta_D(y)$ is a proper probability density function and $0 < \kappa_{D,\delta} < 1$.

Let

$$\widetilde{\iota}_1(s) = \frac{\widetilde{\xi}(s)}{s + a_1}, \tag{3.61}$$

$$\widetilde{\iota}_2(s) = \frac{\widetilde{\chi}(s)}{s + a_2}, \tag{3.62}$$

then

$$\iota_1(u) = \int_0^u e^{-a_1(u-y)} \xi(y)\mathrm{d}y = e^{-a_1 u} \int_0^u e^{a_1 y} \xi(y)\mathrm{d}y,$$

$$\iota_2(u) = \int_0^u e^{-a_2(u-y)} \chi(y)\mathrm{d}y = e^{-a_2 u} \int_0^u e^{a_2 y} \chi(y)\mathrm{d}y.$$

Recall (3.33) and rewrite (3.34) as

$$\widetilde{h}_{D,1}(s) = (s - A_1)(s - A_2),$$

$$\widetilde{h}_{D,2}(s) = \frac{\lambda_1}{D}(A_2 - s)\widetilde{\iota}_1(s) + \frac{\lambda_2}{D}(A_1 - s)\widetilde{\iota}_2(s).$$

The nonnegative roots of $\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s) = 0$ coincide with the nonnegative roots of Lundberg's equation which are ϱ and $\mathfrak{r}$ in Lemma 3.3 and Lemma 3.4. Notice that $\widetilde{h}_{D,1}(s)$ and $\widetilde{h}_{D,2}(s)$ have similar forms with equations (2.19) and (2.20) in Chapter 2 which are

$$\widetilde{h}_1(s) = \left(s - \frac{\lambda_1 + \delta}{c_1}\right)\left(s - \frac{\lambda_2 + \delta}{c_2}\right),$$

$$\widetilde{h}_2(s) = \frac{\lambda_1}{c_1}\left(\frac{\lambda_2 + \delta}{c_2} - s\right)\widetilde{\xi}(s) + \frac{\lambda_2}{c_2}\left(\frac{\lambda_1 + \delta}{c_1} - s\right)\widetilde{\chi}(s).$$

By a similar argument to the one leading to equation (2.47), taking into account the admissible rages of ϱ and $\mathfrak{r}$, we may conclude that $T_\varrho T_\mathfrak{r} h_{D,2}(u) > 0$ for all $u > 0$. Hence, $\eta_D(y) = \frac{T_\varrho T_\mathfrak{r} h_{D,2}(y)}{T_0 T_\varrho T_\mathfrak{r} h_{D,2}(0)}$ is a proper probability density function, and it follows that $\kappa_{D,\delta} > 0$.

It remains to show that $\kappa_\delta < 1$, which we prove separately in the cases $\delta > 0$ and $\delta = 0$. When $\delta > 0$, it follows from Lemma 3.4 that $\varrho > 0$ and $\mathfrak{r} > 0$. By definition of $\kappa_{D,\delta}$ in (3.57) and letting $s = 0$ in (3.42) gives

$$\kappa_{D,\delta} = T_0 T_\varrho T_\mathfrak{r} h_{D,2}(0) = 1 - \frac{\widetilde{h}_{D1}(0) - \widetilde{h}_{D,2}(0)}{\varrho \mathfrak{r}},$$

then inserting identities (3.33) and (3.34) with $s = 0$ yields

$$\kappa_{D,\delta} = 1 - \frac{A_1 A_2 - \frac{\lambda_1 A_2}{D a_1}\widetilde{\xi}(0) - \frac{\lambda_2 A_1}{D a_2}\widetilde{\chi}(0)}{\varrho \mathfrak{r}}.$$

Utilizing the relations $A_1 a_1 = \frac{\lambda_1+\delta}{D}$, $A_2 a_2 = \frac{\lambda_2+\delta}{D}$ from (3.29), (3.30), and $\widetilde{\xi}(0) + \widetilde{\chi}(0) = 1$, we obtain

$$\begin{aligned}
\kappa_{D,\delta} &= 1 - \frac{A_1 A_2 a_1 a_2 - \frac{\lambda_1}{D} A_2 a_2 \widetilde{\xi}(0) - \frac{\lambda_2}{D} A_1 a_1 \widetilde{\chi}(0)}{a_1 a_2 \varrho \mathfrak{r}} \\
&= 1 - \frac{\frac{\lambda_1+\delta}{D}\frac{\lambda_2+\delta}{D}\left[\widetilde{\xi}(0) + \widetilde{\chi}(0)\right] - \frac{\lambda_1}{D}\cdot\frac{\lambda_2+\delta}{D}\widetilde{\xi}(0) - \frac{\lambda_2}{D}\cdot\frac{\lambda_1+\delta}{D}\widetilde{\chi}(0)}{a_1 a_2 \varrho \mathfrak{r}} \\
&= 1 - \frac{\delta}{D}\cdot\frac{\frac{\lambda_2+\delta}{D}\widetilde{\xi}(0) + \frac{\lambda_1+\delta}{D}\widetilde{\chi}(0)}{a_1 a_2 \varrho \mathfrak{r}} \qquad (3.63) \\
&< 1,
\end{aligned}$$

since $D, \delta, \lambda_1, \lambda_2, a_1, a_2, \varrho, \mathfrak{r} > 0$, $\widetilde{\xi}(0) = \mathbb{P}\{X > Q\} > 0$ and $\widetilde{\chi}(0) = \mathbb{P}\{X < Q\} > 0$.

When $\delta = 0$, we let the unique positive root of Lundberg's equation be $\mathfrak{r} > 0$ and let $\varrho = 0$. Observe that $\mathfrak{r}$ and 0 are also zeros of $\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s)$. Denote $\kappa_{D,\delta}$ as $\kappa_{D,0}$ to suggest that $\delta = 0$. Then,

$$\kappa_{D,0} = T_0 T_0 T_\mathfrak{r} h_{D,2}(0) = \frac{T_0 T_0 h_{D,2}(0) - T_0 T_\mathfrak{r} h_{D,2}(0)}{\mathfrak{r}}. \qquad (3.64)$$

From Property 4 of Li and Garrido (2004), we have for any $s \in \mathbb{C}$,

$$\begin{aligned}
T_s T_s h_{D,2}(0) &= -\frac{\mathrm{d}}{\mathrm{d}s} T_s h_{D,2}(0) \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\lambda_1}{D}\cdot\frac{A_2 - s}{s + a_1}\int_0^\infty e^{-sy}\xi(y)\mathrm{d}y + \frac{\lambda_2}{D}\cdot\frac{A_1 - s}{s + a_2}\int_0^\infty e^{-sy}\chi(y)\mathrm{d}y\right]
\end{aligned}$$

$$= \frac{\lambda_1}{D}\left[\frac{a_1 + A_2}{(s+a_1)^2}\widetilde{\xi}(s) + \frac{A_2 - s}{s + a_1}\int_0^\infty e^{-sy} y\,\xi(y)\mathrm{d}y\right]$$
$$+ \frac{\lambda_2}{D}\left[\frac{a_2 + A_1}{(s+a_2)^2}\widetilde{\chi}(s) + \frac{A_1 - s}{s + a_2}\int_0^\infty e^{-sy} y\,\chi(y)\mathrm{d}y\right].$$

Inserting $s = 0$ and utilizing $\frac{\lambda_1}{D} = A_1 a_1$, $\frac{\lambda_2}{D} = A_2 a_2$ and $\xi(y) + \chi(y) = b(y)$ produces

$$\begin{aligned} T_0 T_0 h_{D,2}(0) &= \frac{\lambda_1}{D}\left[\frac{a_1 + A_2}{a_1^2}\widetilde{\xi}(0) + \frac{A_2}{a_1}\int_0^\infty y\,\xi(y)\mathrm{d}y\right] + \frac{\lambda_2}{D}\left[\frac{a_2 + A_1}{a_2^2}\widetilde{\chi}(0) + \frac{A_1}{a_2}\int_0^\infty y\,\chi(y)\mathrm{d}y\right] \\ &= \frac{A_1 a_1 + A_1 A_2}{a_1}\widetilde{\xi}(0) + \frac{A_2 a_2 + A_1 A_2}{a_2}\widetilde{\chi}(0) + A_1 A_2\,\mu. \end{aligned}$$

Substituting the above into (3.64) yields

$$\begin{aligned} \kappa_{D,0} &= \frac{1}{\mathfrak{r}}\left[\frac{A_1 a_1 + A_1 A_2}{a_1}\widetilde{\xi}(0) + \frac{A_2 a_2 + A_1 A_2}{a_2}\widetilde{\chi}(0) + A_1 A_2\,\mu - \frac{\widetilde{h}_{D,2}(0) - \widetilde{h}_{D,2}(\mathfrak{r})}{\mathfrak{r}}\right] \\ &= \frac{1}{\mathfrak{r}}\left[\frac{A_1 a_1 + A_1 A_2}{a_1}\widetilde{\xi}(0) + \frac{A_2 a_2 + A_1 A_2}{a_2}\widetilde{\chi}(0) + A_1 A_2\,\mu - \frac{\widetilde{h}_{D1}(0) - \widetilde{h}_{D1}(\mathfrak{r})}{\mathfrak{r}}\right] \\ &= \frac{1}{\mathfrak{r}}\left[\frac{A_1 a_1 + A_1 A_2}{a_1}\widetilde{\xi}(0) + \frac{A_2 a_2 + A_1 A_2}{a_2}\widetilde{\chi}(0) + A_1 A_2\,\mu - \frac{-\mathfrak{r}^2 + (A_1 + A_2)\,\mathfrak{r}}{\mathfrak{r}}\right] \\ &= 1 - \frac{1}{\mathfrak{r}}\left\{(A_1 + A_2) - \left[\frac{A_1 a_1 + A_1 A_2}{a_1}\widetilde{\xi}(0) + \frac{A_2 a_2 + A_1 A_2}{a_2}\widetilde{\chi}(0) + A_1 A_2\,\mu\right]\right\}. \end{aligned} \tag{3.65}$$

From the positive-security-loading condition (3.2), we have

$$\begin{aligned} \mu &< \frac{c}{\lambda_1}\widetilde{\xi}(0) + \frac{c}{\lambda_2}\widetilde{\chi}(0), \\ \frac{\lambda_1}{D}\cdot\frac{\lambda_2}{D}\mu &< \frac{c}{D}\cdot\frac{\lambda_2}{D}\widetilde{\xi}(0) + \frac{c}{D}\cdot\frac{\lambda_1}{D}\widetilde{\chi}(0), \\ A_1 a_1 A_2 a_2\,\mu &< \frac{c}{D}\cdot A_2 a_2 \widetilde{\xi}(0) + \frac{c}{D} A_1 a_1 \widetilde{\chi}(0) \\ A_1 A_2\,\mu &< \frac{\frac{c}{D}\cdot A_2}{a_1}\widetilde{\xi}(0) + \frac{\frac{c}{D}\cdot A_1}{a_2}\widetilde{\chi}(0). \end{aligned}$$

Then, utilizing the relations $a_1 = A_1 + \frac{c}{D}$ and $a_2 = A_2 + \frac{c}{D}$, we deduce that

$$\begin{aligned} &\frac{A_1 a_1 + A_1 A_2}{a_1}\widetilde{\xi}(0) + \frac{A_2 a_2 + A_1 A_2}{a_2}\widetilde{\chi}(0) + A_1 A_2\,\mu \\ &< \frac{A_1 a_1 + A_1 A_2}{a_1}\widetilde{\xi}(0) + \frac{A_2 a_2 + A_1 A_2}{a_2}\widetilde{\chi}(0) + \frac{\frac{c}{D}\cdot A_2}{a_1}\widetilde{\xi}(0) + \frac{\frac{c}{D}\cdot A_1}{a_2}\widetilde{\chi}(0) \\ &= \frac{A_1 a_1 + A_2\left(A_1 + \frac{c}{D}\right)}{a_1}\widetilde{\xi}(0) + \frac{A_2 a_2 + A_1\left(A_2 + \frac{c}{D}\right)}{a_2}\widetilde{\chi}(0) \end{aligned}$$

$$= \frac{A_1a_1 + A_2a_1}{a_1}\widetilde{\xi}(0) + \frac{A_2a_2 + A_1a_2}{a_2}\widetilde{\chi}(0)$$
$$= A_1 + A_2\,.$$

Therefore, from (3.65) we may conclude that $\kappa_{D,0} < 1$. ■

3.4 Applications

3.4.1 K_n family claim sizes

In this section, we derive the explicit expressions for the Gerber-Shiu function under model (3.1) when the Laplace transforms $\widetilde{\xi}(s)$ and $\widetilde{\chi}(s)$ belong to the K_n family. One typical example is when the thresholds are exponentially distributed and the claim amounts follow a distribution from the K_n family. Assume that the random thresholds $\{Q_i,\ i = 1, 2, \ldots\}$ follow an exponential distribution with c.d.f. $H(y) = 1 - e^{-\nu y}, y \geq 0$, and the claim amounts $\{X_i,\ i = 1, 2, \ldots\}$ follow a distribution from the K_n family, i.e., the Laplace transform of the density function $b(\cdot)$ has the following form

$$\widetilde{b}(s) = \frac{p^{\star}_{k-1}(s)}{p_k(s)}, \quad k \in \mathbb{N}^+,$$

where $p_k(s)$ is a polynomial in s of degree k with only negative zeros, $p^{\star}_{k-1}(s)$ is a polynomial in s of degree $k-1$ or less, both with leading coefficient 1 and $p_k(0) = p^{\star}_{k-1}(0)$. Then, by (2.9) with $H(y) = 1 - e^{\nu y}, y \geq 0$, we may write

$$\widetilde{\chi}(s) = \widetilde{b}(s+\nu) = \frac{q^{\star}_{k-1}(s)}{q_k(s)}, \tag{3.66}$$

where $q_k(s)$ is another polynomial in s of degree k with only negative zeros, $q^{\star}_{k-1}(s)$ is another polynomial in s of degree $k-1$ or less, and both with leading coefficient 1 since $\nu > 0$ is a constant.

Implementing identities (3.29), (3.30), (3.9) and (3.66), the Lundberg's equation (3.20) becomes

$$\begin{aligned}&(s-A_1)\,(s-A_2)\,(s+a_1)\,(s+a_2)\\&\qquad+\frac{\lambda_1}{D}\,(s-A_2)\,(s+a_2)\left[\frac{p^{\star}_{k-1}(s)}{p_k(s)}-\frac{q^{\star}_{k-1}(s)}{q_k(s)}\right]+\frac{\lambda_2}{D}\,(s-A_1)\,(s+a_1)\,\frac{q^{\star}_{k-1}(s)}{q_k(s)}=0\,,\end{aligned}$$

which may be rearranged by multiplying by $p_k(s)q_k(s)$ to

$$\begin{aligned}&(s-A_1)\,(s-A_2)\,(s+a_1)\,(s+a_2)\,p_k(s)q_k(s)\\&\quad+\frac{\lambda_1}{D}\,(s-A_2)\,(s+a_2)\,[q_k(s)p^{\star}_{k-1}(s)-p_k(s)q^{\star}_{k-1}(s)]+\frac{\lambda_2}{D}\,(s-A_1)\,(s+a_1)\,q^{\star}_{k-1}(s)=0\,,\end{aligned} \tag{3.67}$$

without changing the positive roots of the equation. The left-hand side of equation (3.67) is a polynomial in s of degree $2k+4$ with leading coefficient 1, which indicates that it has $2k+4$ zeros in total. Among these roots, exactly two are nonnegative by Lemmas 3.3 and 3.4, denoted as ϱ and $\mathfrak{r}$. Therefore, the other $2k+2$ roots have negative real parts, denoted as $R_1,\ldots,R_{2k+2}$. From now on, we assume that these roots are distinct. By comparing to (3.33) and (3.34), we see that the left-hand side of (3.67) equals $(s+a_1)\,(s+a_2)\,p_k(s)q_k(s)\left[\widetilde{h}_{D,1}(s)-\widetilde{h}_{D,2}(s)\right]$, which implies

$$(s+a_1)\,(s+a_2)\,p_k(s)\,q_k(s)\left[\widetilde{h}_{D,1}(s)-\widetilde{h}_{D,2}(s)\right]=(s-\varrho)(s-\mathfrak{r})\prod_{l=1}^{2k+2}(s-R_l). \tag{3.68}$$

Inserting (3.33) and (3.34) into (3.13) yields

$$\widetilde{\phi}_{\mathrm{w},1}(s)=\frac{(s-A_2)(s+a_2)\phi'_{\mathrm{w},1}(0)+\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\widetilde{\chi}(s)-\frac{\lambda_1}{D}(s-A_2)(s+a_2)\widetilde{\zeta}(s)}{(s+a_1)\,(s+a_2)\left[\widetilde{h}_{D,1}(s)-\widetilde{h}_{D,2}(s)\right]}, \tag{3.69}$$

and denote the latter part of the numerator by

$$\begin{aligned}\widetilde{\vartheta}_{K,1}(s)&=\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\widetilde{\chi}(s)-\frac{\lambda_1}{D}(s-A_2)(s+a_2)\widetilde{\zeta}(s)\\&=\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\widetilde{\chi}(s)-\frac{\lambda_1}{D}\left(s^2+\frac{c}{D}s-\frac{\lambda_2+\delta}{D}\right)\widetilde{\zeta}(s).\end{aligned} \tag{3.70}$$

Since $(s - A_2)(s + a_2)\,\phi'_{\mathrm{w},1}(0)$ is a quadratic function in s and ϱ, $\mathfrak{r}$ are simple zeros of the numerator of $\widetilde{\phi}_{\mathrm{w},1}(s)$, we deduce that the numerator of (3.69) satisfies

$$(s - A_2)(s + a_2)\,\phi'_{\mathrm{w},1}(0) + \widetilde{\vartheta}_{K,1}(s) = (s-\varrho)(s-\mathfrak{r})\left[\phi'_{\mathrm{w},1}(0) + T_s T_\varrho T_{\mathfrak{r}} \vartheta_{K,1}(0)\right]. \tag{3.71}$$

Employing relations (3.68) and (3.71) in (3.69) yields that

$$\begin{aligned}
\widetilde{\phi}_{\mathrm{w},1}(s) &= \frac{(s-\varrho)(s-\mathfrak{r})\left[\phi'_{\mathrm{w},1}(0) + T_s T_\varrho T_{\mathfrak{r}} \vartheta_{K,1}(0)\right]}{(s+a_1)\,(s+a_2)\left[\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s)\right]} \\
&= \frac{p_k(s)\,q_k(s)(s-\varrho)(s-\mathfrak{r})\left[\phi'_{\mathrm{w},1}(0) + T_s T_\varrho T_{\mathfrak{r}} \vartheta_{K,1}(0)\right]}{p_k(s)\,q_k(s)\,(s+a_1)\,(s+a_2)\left[\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s)\right]} \\
&= \frac{p_k(s)\,q_k(s)(s-\varrho)(s-\mathfrak{r})\left[\phi'_{\mathrm{w},1}(0) + T_s T_\varrho T_{\mathfrak{r}} \vartheta_{K,1}(0)\right]}{(s-\varrho)(s-\mathfrak{r})\prod_{l=1}^{2k+2}(s-R_l)} \\
&= \frac{p_k(s)\,q_k(s)}{\prod_{l=1}^{2k+2}(s-R_l)}\left[\phi'_{\mathrm{w},1}(0) + T_s T_\varrho T_{\mathfrak{r}} \vartheta_{K,1}(0)\right].
\end{aligned} \tag{3.72}$$

To obtain an explicit expression for $T_s T_\varrho T_{\mathfrak{r}} \vartheta_{K,1}(0)$ from (3.70), we deduce some general results first. Let

$$\begin{aligned}
\widetilde{\mathfrak{z}}_1(s) &= s\,\widetilde{\zeta}(s)\,, \\
\widetilde{\mathfrak{z}}_2(s) &= s^2\,\widetilde{\zeta}(s)\,,
\end{aligned}$$

then utilizing relation (2.6), we obtain

$$\begin{aligned}
T_s T_\varrho T_{\mathfrak{r}}\,\mathfrak{z}_1(0) &= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\widetilde{\mathfrak{z}}_1(\varrho) - \widetilde{\mathfrak{z}}_1(s)}{s-\varrho} - \frac{\widetilde{\mathfrak{z}}_1(\mathfrak{r}) - \widetilde{\mathfrak{z}}_1(s)}{s-\mathfrak{r}}\right] \\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\varrho\widetilde{\zeta}(\varrho) - s\widetilde{\zeta}(s)}{s-\varrho} - \frac{\mathfrak{r}\widetilde{\zeta}(\mathfrak{r}) - s\widetilde{\zeta}(s)}{s-\mathfrak{r}}\right] \\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\varrho\widetilde{\zeta}(\varrho) - \varrho\widetilde{\zeta}(s) + \varrho\widetilde{\zeta}(s) - s\widetilde{\zeta}(s)}{s-\varrho} - \frac{\mathfrak{r}\widetilde{\zeta}(\mathfrak{r}) - \mathfrak{r}\widetilde{\zeta}(s) + \mathfrak{r}\widetilde{\zeta}(s) - s\widetilde{\zeta}(s)}{s-\mathfrak{r}}\right] \\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\varrho T_s T_\varrho \zeta(0) - \widetilde{\zeta}(s) - \mathfrak{r} T_s T_{\mathfrak{r}} \zeta(0) + \widetilde{\zeta}(s)\right] \\
&= \frac{\varrho T_s T_\varrho \zeta(0) - \mathfrak{r} T_s T_{\mathfrak{r}} \zeta(0)}{\mathfrak{r}-\varrho}\,,
\end{aligned}$$

$$
\begin{aligned}
T_s T_\varrho T_\mathfrak{r}\, \mathfrak{z}_2(0) &= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\widetilde{\mathfrak{z}_2}(\varrho)-\widetilde{\mathfrak{z}_2}(s)}{s-\varrho} - \frac{\widetilde{\mathfrak{z}_2}(\mathfrak{r})-\widetilde{\mathfrak{z}_2}(s)}{s-\mathfrak{r}}\right] \\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\varrho^2\widetilde{\zeta}(\varrho)-s^2\widetilde{\zeta}(s)}{s-\varrho} - \frac{\mathfrak{r}^2\widetilde{\zeta}(\mathfrak{r})-s^2\widetilde{\zeta}(s)}{s-\mathfrak{r}}\right] \\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\varrho^2\widetilde{\zeta}(\varrho)-\varrho^2\widetilde{\zeta}(s)+\varrho^2\widetilde{\zeta}(s)-s^2\widetilde{\zeta}(s)}{s-\varrho} - \frac{\mathfrak{r}^2\widetilde{\zeta}(\mathfrak{r})-\mathfrak{r}^2\widetilde{\zeta}(s)+\mathfrak{r}^2\widetilde{\zeta}(s)-s^2\widetilde{\zeta}(s)}{s-\mathfrak{r}}\right] \\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\varrho^2 T_s T_\varrho \zeta(0) - (\varrho+s)\,\widetilde{\zeta}(s) - \mathfrak{r}^2 T_s T_\mathfrak{r}\zeta(0) + (\mathfrak{r}+s)\,\widetilde{\zeta}(s)\right] \\
&= \frac{\varrho^2 T_s T_\varrho\zeta(0) - \mathfrak{r}^2 T_s T_\mathfrak{r}\zeta(0)}{\mathfrak{r}-\varrho} + \widetilde{\zeta}(s)\,.
\end{aligned}
$$

Utilizing the above results, we obtain from (3.70) that

$$
\begin{aligned}
T_s T_\varrho T_\mathfrak{r}\vartheta_{K,1}(0) = &\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right] T_s T_\mathfrak{r} T_\varrho \chi(0) + \frac{\lambda_1}{D}\left[\frac{\mathfrak{r}^2 T_s T_\mathfrak{r}\zeta(0) - \varrho^2 T_s T_\varrho\zeta(0)}{\mathfrak{r}-\varrho} - \widetilde{\zeta}(s)\right] \\
&+ \frac{\lambda_1}{D}\cdot\frac{c}{D}\left[\frac{\mathfrak{r} T_s T_\mathfrak{r}\zeta(0) - \varrho T_s T_\varrho\zeta(0)}{\mathfrak{r}-\varrho}\right] + \frac{\lambda_1}{D}\cdot\frac{\lambda_2+\delta}{D} T_s T_\mathfrak{r} T_\varrho \zeta(0)\,. \qquad (3.73)
\end{aligned}
$$

Denote by

$$
\widetilde{\ell_D}(s) := \frac{p_k(s)\, q_k(s)}{\prod_{l=1}^{2k+2}(s-R_l)},
$$

applying the Heaviside expansion theorem, $\widetilde{\ell_D}(s)$ may be inverted to

$$
\begin{aligned}
\ell_D(u) &= \sum_{j=1}^{2k+2} \frac{p_k(R_j)\, q_k(R_j)}{\frac{\partial}{\partial s}\left[\prod_{l=1}^{2k+2}(s-R_l)\right]_{s=R_j}} e^{R_j u} \\
&= \sum_{j=1}^{2k+2} \frac{p_k(R_j)\, p_k(R_j+\epsilon)}{\prod_{l=1,\, l\neq j}^{2k+2}(R_j-R_l)} e^{R_j u}\,. \qquad (3.74)
\end{aligned}
$$

Inverting the Laplace transform in (3.72), we obtain

$$
\phi_{\mathrm{w},1}(u) = \phi'_{\mathrm{w},1}(0)\,\ell_D(u) + \ell_D(u) * T_\varrho T_\mathfrak{r}\vartheta_{K,1}(u), \qquad (3.75)
$$

where $*$ denotes the convolution of functions and inverting (3.73) yields

$$
\begin{aligned}
T_\varrho T_\mathfrak{r}\vartheta_{K,1}(u) = &\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right] T_\mathfrak{r} T_\varrho \chi(u) + \frac{\lambda_1}{D}\left[\frac{\mathfrak{r}^2 T_\mathfrak{r}\zeta(u) - \varrho^2 T_\varrho\zeta(u)}{\mathfrak{r}-\varrho} - \zeta(u)\right] \\
&+ \frac{\lambda_1}{D}\cdot\frac{c}{D}\left[\frac{\mathfrak{r} T_\mathfrak{r}\zeta(u) - \varrho T_\varrho\zeta(u)}{\mathfrak{r}-\varrho}\right] + \frac{\lambda_1}{D}\cdot\frac{\lambda_2+\delta}{D} T_\mathfrak{r} T_\varrho \zeta(u).
\end{aligned}
$$

Similarly for $\widetilde{\phi}_{\mathrm{w},2}(s)$, rewrite (3.14) to

$$\widetilde{\phi}_{\mathrm{w},2}(s) = \frac{(s-A_1)(s+a_1)\,\phi'_{\mathrm{w},2}(0) + \widetilde{\vartheta}_{K,2}(s)}{(s+a_1)\,(s+a_2)\left[\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s)\right]}, \tag{3.76}$$

where

$$\begin{aligned}\widetilde{\vartheta}_{K,2}(s) &= -\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\widetilde{\xi}(s) - \frac{\lambda_2}{D}(s-A_1)(s+a_1)\widetilde{\zeta}(s)\\ &= -\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\widetilde{\xi}(s) - \frac{\lambda_2}{D}\left(s^2 + \frac{c}{D}s - \frac{\lambda_1+\delta}{D}\right)\widetilde{\zeta}(s).\end{aligned}$$

Rearranging (3.76) and applying Laplace inversion leads to

$$\phi_{\mathrm{w},2}(u) = \phi'_{\mathrm{w},2}(0)\,\ell_D(u) + \ell_D(u) * T_\varrho T_{\mathfrak{r}}\vartheta_{K,2}(u), \tag{3.77}$$

where

$$\begin{aligned}T_\varrho T_{\mathfrak{r}}\vartheta_{K,2}(u) = &-\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]T_{\mathfrak{r}}T_\varrho\xi(u) + \frac{\lambda_2}{D}\left[\frac{\mathfrak{r}^2T_{\mathfrak{r}}\zeta(u) - \varrho^2T_\varrho\zeta(u)}{\mathfrak{r}-\varrho} - \zeta(u)\right]\\ &+ \frac{\lambda_2}{D}\cdot\frac{c}{D}\left[\frac{\mathfrak{r}T_{\mathfrak{r}}\zeta(u) - \varrho T_\varrho\zeta(u)}{\mathfrak{r}-\varrho}\right] + \frac{\lambda_2}{D}\cdot\frac{\lambda_1+\delta}{D}T_{\mathfrak{r}}T_\varrho\zeta(u).\end{aligned}$$

and $\ell_D(u)$ is defined in (3.74).

For ruin caused by oscillation, substituting (3.33) and (3.34) into (3.15) yields

$$\widetilde{\phi}_{\mathrm{d},1}(s) = \frac{(s-A_2)(s+a_2)\phi'_{\mathrm{d},1}(0) + (s-A_2)(s+a_2)\left(s+\frac{c}{D}\right) + \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{\lambda_2-\lambda_1}{D}\left(\frac{c}{D}+s\right)\right]\widetilde{\chi}(s)}{(s+a_1)\,(s+a_2)\left[\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s)\right]}. \tag{3.78}$$

Comparing (3.19) with (3.17), we introduce the following notation

$$\widetilde{\zeta}_d(s) := \left(s + \frac{c}{D}\right). \tag{3.79}$$

We rewrite the numerator of (3.78) into

$$(s-A_2)(s+a_2)\,\phi'_{\mathrm{d},1}(0) + \widetilde{\varpi}_{K,1}(s)\,,$$

where

$$\begin{aligned}\widetilde{\varpi}_{K,1}(s) &= (s-A_2)(s+a_2)\left(s+\frac{c}{D}\right)+\left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0)+\frac{\lambda_2-\lambda_1}{D}\left(\frac{c}{D}+s\right)\right]\widetilde{\chi}(s)\\ &= \left(s^2+\frac{c}{D}s-\frac{\lambda_2+\delta}{D}\right)\widetilde{\zeta}_d(s)\\ &\quad +\left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0)+\frac{\lambda_2-\lambda_1}{D}\cdot\frac{c}{D}\right]\widetilde{\chi}(s)+\frac{\lambda_2-\lambda_1}{D}s\widetilde{\chi}(s)\,. \end{aligned}\tag{3.80}$$

Since $(s-A_2)(s+a_2)\,\phi'_{\mathrm{d},1}(0)$ is a polynomial in s of degree 2 and ϱ, $\mathfrak{r}$ are zeros of the numerator of (3.78), we obtain

$$(s-A_2)(s+a_2)\,\phi'_{\mathrm{d},1}(0)+\widetilde{\varpi}_{K,1}(s)=(s-\varrho)(s-\mathfrak{r})\left[\phi'_{\mathrm{d},1}(0)+T_sT_\varrho T_{\mathfrak{r}}\varpi_{K,1}(0)\right].\tag{3.81}$$

Inserting relations (3.68) and (3.81) into the denominator and the numerator of identity (3.78) respectively yields

$$\begin{aligned}\widetilde{\phi}_{\mathrm{d},1}(s) &= \frac{(s-A_2)(s+a_2)\,\phi'_{\mathrm{d},1}(0)+\widetilde{\varpi}_{K,1}(s)}{(s+a_1)\,(s+a_2)\left[\widetilde{h}_{D,1}(s)-\widetilde{h}_{D,2}(s)\right]}\\ &= \frac{p_k(s)\,q_k(s)\left[(s-A_2)(s+a_2)\,\phi'_{\mathrm{d},1}(0)+\widetilde{\varpi}_{K,1}(s)\right]}{p_k(s)\,q_k(s)\,(s+a_1)\,(s+a_2)\left[\widetilde{h}_{D,1}(s)-\widetilde{h}_{D,2}(s)\right]}\\ &= \frac{p_k(s)\,q_k(s)(s-\varrho)(s-\mathfrak{r})\left[\phi'_{\mathrm{d},1}(0)+T_sT_\varrho T_{\mathfrak{r}}\varpi_{K,1}(0)\right]}{(s-\varrho)(s-\mathfrak{r})\prod_{l=1}^{2k+2}(s-R_l)}\\ &= \frac{p_k(s)\,q_k(s)\left[\phi'_{\mathrm{d},1}(0)+T_sT_\varrho T_{\mathfrak{r}}\varpi_{K,1}(0)\right]}{\prod_{l=1}^{2k+2}(s-R_l)}\,. \end{aligned}\tag{3.82}$$

To obtain an explicit expression for $T_sT_\varrho T_{\mathfrak{r}}\varpi_{K,1}(0)$, we first derive some auxiliary results. Utilizing relation (2.6), we deduce from identity (3.79) that

$$\begin{aligned}T_sT_{\mathfrak{r}}\zeta_d(0) &= \frac{\widetilde{\zeta}_d(s)-\widetilde{\zeta}_d(\mathfrak{r})}{\mathfrak{r}-s}=\frac{\mathfrak{r}+\frac{c}{D}-s-\frac{c}{D}}{\mathfrak{r}-s}=-1\,,\\ T_sT_\varrho\zeta_d(0) &= \frac{\widetilde{\zeta}_d(s)-\widetilde{\zeta}_d(\varrho)}{\varrho-s}=\frac{\varrho+\frac{c}{D}-s-\frac{c}{D}}{\varrho-s}=-1\,,\\ T_sT_\varrho T_{\mathfrak{r}}\zeta_d(0) &= \frac{T_sT_\varrho\zeta_d(0)-T_sT_{\mathfrak{r}}\zeta_d(0)}{\mathfrak{r}-\varrho}=0\,.\end{aligned}$$

Let

$$
\begin{aligned}
\widetilde{\mathcal{P}}_1(s) &= s\,\widetilde{\zeta}_d(s)\,,\\
\widetilde{\mathcal{P}}_2(s) &= s^2\,\widetilde{\zeta}_d(s)\,,\\
\widetilde{Q}(s) &= s\,\widetilde{\chi}(s)
\end{aligned}
$$

then

$$
\begin{aligned}
T_sT_\varrho T_\mathfrak{r}\mathcal{P}_1(0) &= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\widetilde{\mathcal{P}}_1(\varrho)-\widetilde{\mathcal{P}}_1(s)}{s-\varrho}-\frac{\widetilde{\mathcal{P}}_1(\mathfrak{r})-\widetilde{\mathcal{P}}_1(s)}{s-\mathfrak{r}}\right]\\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\varrho\left(\varrho+\frac{c}{D}\right)-s\left(s+\frac{c}{D}\right)}{s-\varrho}-\frac{\mathfrak{r}\left(\mathfrak{r}+\frac{c}{D}\right)-s\left(s+\frac{c}{D}\right)}{s-\mathfrak{r}}\right]\\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\left(-\varrho-s-\frac{c}{D}\right)-\left(-\mathfrak{r}-s-\frac{c}{D}\right)\right]\\
&= 1\,,\\
T_sT_\varrho T_\mathfrak{r}\mathcal{P}_2(0) &= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\widetilde{\mathcal{P}}_2(\varrho)-\widetilde{\mathcal{P}}_2(s)}{s-\varrho}-\frac{\widetilde{\mathcal{P}}_2(\mathfrak{r})-\widetilde{\mathcal{P}}_2(s)}{s-\mathfrak{r}}\right]\\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\varrho^2\left(\varrho+\frac{c}{D}\right)-s^2\left(s+\frac{c}{D}\right)}{s-\varrho}-\frac{\mathfrak{r}^2\left(\mathfrak{r}+\frac{c}{D}\right)-s^2\left(s+\frac{c}{D}\right)}{s-\mathfrak{r}}\right]\\
&= \frac{1}{\mathfrak{r}-\varrho}\left[-\left(\varrho^2+\varrho s+s^2\right)-(\varrho+s)\frac{c}{D}+\left(\mathfrak{r}^2+\mathfrak{r}s+s^2\right)+(\mathfrak{r}+s)\frac{c}{D}\right]\\
&= s+\varrho+\mathfrak{r}+\frac{c}{D}\,,\\
T_sT_\varrho T_\mathfrak{r}Q(0) &= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\widetilde{Q}(\varrho)-\widetilde{Q}(s)}{s-\varrho}-\frac{\widetilde{Q}(\mathfrak{r})-\widetilde{Q}(s)}{s-\mathfrak{r}}\right]\\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\varrho\widetilde{\chi}(\varrho)-s\widetilde{\chi}(s)}{s-\varrho}-\frac{\mathfrak{r}\widetilde{\chi}(\mathfrak{r})-s\widetilde{\chi}(s)}{s-\mathfrak{r}}\right]\\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\frac{\varrho\widetilde{\chi}(\varrho)-\varrho\widetilde{\chi}(s)+\varrho\widetilde{\chi}(s)-s\widetilde{\chi}(s)}{s-\varrho}-\frac{\mathfrak{r}\widetilde{\chi}(\mathfrak{r})-\mathfrak{r}\widetilde{\chi}(s)+\mathfrak{r}\widetilde{\chi}(s)-s\widetilde{\chi}(s)}{s-\mathfrak{r}}\right]\\
&= \frac{1}{\mathfrak{r}-\varrho}\left[\varrho T_sT_\varrho\chi(0)-\widetilde{\chi}(s)-\varrho T_sT_\mathfrak{r}\chi(0)+\widetilde{\chi}(s)\right]\\
&= \frac{\varrho T_sT_\varrho\chi(0)-\mathfrak{r}T_sT_\mathfrak{r}\chi(0)}{\mathfrak{r}-\varrho}\,.
\end{aligned}
$$

Employing the above results, we derive $T_s T_\varrho T_\mathfrak{r} \varpi_{K,1}(0)$ from identity (3.80) that

$$\begin{aligned} T_s T_\varrho T_\mathfrak{r} \varpi_{K,1}(0) &= \frac{\widetilde{\varpi}_{K,1}(\varrho) - \widetilde{\varpi}_{K,1}(s)}{(s-\varrho)(\mathfrak{r}-\varrho)} - \frac{\widetilde{\varpi}_{K,1}(\mathfrak{r}) - \widetilde{\varpi}_{K,1}(s)}{(s-\mathfrak{r})(\mathfrak{r}-\varrho)} \\ &= s + \varrho + \mathfrak{r} + \frac{c}{D} + \frac{c}{D} + \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{\lambda_2 - \lambda_1}{D} \cdot \frac{c}{D}\right] T_s T_\varrho T_\mathfrak{r} \chi(0) \\ &\quad + \frac{\lambda_2 - \lambda_1}{D}\left[\frac{\varrho T_s T_\varrho \chi(0) - \mathfrak{r} T_s T_\mathfrak{r} \chi(0)}{\mathfrak{r} - \varrho}\right]. \end{aligned}$$

Plugging the above into (3.82) gives

$$\begin{aligned} \widetilde{\phi}_{\mathrm{d},1}(s) = & \frac{p_k(s)\, q_k(s)\left[s + \varrho + \mathfrak{r} + \frac{c}{D} + \frac{c}{D} + \phi'_{\mathrm{d},1}(0)\right]}{\prod_{l=1}^{2k+2}(s - R_l)} \\ & + \frac{p_k(s)\, q_k(s)}{\prod_{l=1}^{2k+2}(s - R_l)} \left\{ \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{\lambda_2 - \lambda_1}{D} \cdot \frac{c}{D}\right] T_s T_\varrho T_\mathfrak{r} \chi(0) \right. \\ & \qquad \left. + \frac{\lambda_2 - \lambda_1}{D}\left[\frac{\varrho T_s T_\varrho \chi(0) - \mathfrak{r} T_s T_\mathfrak{r} \chi(0)}{\mathfrak{r} - \varrho}\right] \right\}. \end{aligned}$$

Applying Heavside expansion theorem to invert the Laplace transform yields

$$\begin{aligned} \phi_{\mathrm{d},1}(u) = & \sum_{j=1}^{2k+2} \frac{p_k(R_j)\, p_k(R_j + \epsilon)\left[R_j + \varrho + \mathfrak{r} + \frac{c}{D} + \frac{c}{D} + \phi'_{\mathrm{d},1}(0)\right]}{\prod_{l=1,\, l\neq j}^{2k+2}(R_j - R_l)} e^{R_j u} \\ & + \ell_D(u) * \left\{ \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{\lambda_2 - \lambda_1}{D} \cdot \frac{c}{D}\right] T_\varrho T_\mathfrak{r} \chi(u) + \frac{\lambda_2 - \lambda_1}{D}\left[\frac{\varrho T_\varrho \chi(u) - \mathfrak{r} T_\mathfrak{r} \chi(u)}{\mathfrak{r} - \varrho}\right] \right\}, \end{aligned} \tag{3.83}$$

with $\ell_D(u)$ defined in (3.74).

Similarly for $\widetilde{\phi}_{\mathrm{d},2}(s)$, inserting (3.33), (3.34) and (3.79) into (3.16) yields

$$\begin{aligned} \widetilde{\phi}_{\mathrm{d},2}(s) &= \frac{(s - A_1)(s + a_1)\phi'_{\mathrm{d},2}(0) + \widetilde{\varpi}_{K,2}(s)}{(s + a_1)\,(s + a_2)\left[\widetilde{h}_{D,1}(s) - \widetilde{h}_{D,2}(s)\right]} \\ &= \frac{p_k(s)\, q_k(s)\left[\phi'_{\mathrm{d},2}(0) + T_s T_\varrho T_\mathfrak{r} \varpi_{K,2}(0)\right]}{\prod_{l=1}^{2k+2}(s - R_l)}, \end{aligned}$$

in which

$$\begin{aligned} \widetilde{\varpi}_{K,2}(s) = & \left(s^2 + \frac{c}{D}s - \frac{\lambda_1 + \delta}{D}\right)\widetilde{\zeta}_d(s) \\ & - \left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0) + \frac{\lambda_2 - \lambda_1}{D} \cdot \frac{c}{D}\right]\widetilde{\xi}(s) + \frac{\lambda_2 - \lambda_1}{D} s\widetilde{\xi}(s), \end{aligned}$$

$$T_sT_\varrho T_\mathfrak{r}\varpi_{K,2}(0) = s+\varrho+\mathfrak{r}+\frac{c}{D}+\frac{c}{D}-\left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0)+\frac{\lambda_2-\lambda_1}{D}\cdot\frac{c}{D}\right]T_sT_\varrho T_\mathfrak{r}\xi(0) + \frac{\lambda_2-\lambda_1}{D}\left[\frac{\varrho T_sT_\varrho\xi(0)-\mathfrak{r}T_sT_\mathfrak{r}\xi(0)}{\mathfrak{r}-\varrho}\right],$$

and inverting the Laplace transform $\widetilde{\phi}_{\mathrm{d},2}(s)$ yields

$$\begin{aligned}\phi_{\mathrm{d},2}(u) = &\sum_{j=1}^{2k+2}\frac{p_k(R_j)\,p_k(R_j+\epsilon)\left[R_j+\varrho+\mathfrak{r}+\frac{c}{D}+\frac{c}{D}+\phi'_{\mathrm{d},2}(0)\right]}{\prod_{l=1,\,l\neq j}^{2k+2}(R_j-R_l)}e^{R_ju}\\ &+\ell_D(u)*\left\{-\left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0)+\frac{\lambda_2-\lambda_1}{D}\cdot\frac{c}{D}\right]T_\varrho T_\mathfrak{r}\xi(u)+\frac{\lambda_2-\lambda_1}{D}\left[\frac{\varrho T_\varrho\xi(u)-\mathfrak{r}T_\mathfrak{r}\xi(u)}{\mathfrak{r}-\varrho}\right]\right\},\end{aligned} \tag{3.84}$$

where $\ell_D(u)$ is defined in (3.74). The solutions for Gerber-Shiu functions $\phi_{\mathrm{w},i}(u)$ and $\phi_{\mathrm{d},i}(u)$ for $i=1,2$ are complete by (3.75), (3.77), (3.83) and (3.84).

3.4.2 Exponential claim sizes and ruin time

The Laplace transform of the time to ruin is one of the quantities of particular interest for insurance risk analysis. This is why we examine it separately. Let the penalty function $w(x_1,x_2)=1$ for all $x_1,x_2\geq 0$, which implies $w_0=1$, then the Gerber-Shiu function (1.3) reduces to

$$\varphi(u)=\phi_\mathrm{d}(u)+\varphi_\mathrm{w}(u),\quad u\geq 0,\quad u\geq 0,$$

where

$$\varphi_\mathrm{w}(u)=\mathbb{E}\left\{e^{-\delta\tau}\,\mathbb{I}_{\{\tau<\infty,U(\tau)<0\}}\,\middle|\,U(0)=u\right\}.$$

Given an initial state i, $i=1,2,$, we denote the Laplace transform of the time to ruin by

$$\varphi_i(u)=\phi_{\mathrm{d},i}(u)+\varphi_{\mathrm{w},i}(u),\quad u\geq 0,\ i=1,2.$$

Assume that the random thresholds $\{Q_i,\ i=1,2,\ldots\}$ follow an exponential distribution with c.d.f. $H(y)=1-e^{-\nu y}, y\geq 0$, and the claim sizes $\{X_i,\ i=1,2,\ldots\}$ are exponentially distributed with p.d.f. $b(y) = \epsilon e^{-\epsilon y}, y \geq 0$, which has Laplace transform

$$\widetilde{b}(s) = \frac{\epsilon}{s+\epsilon},$$

and $\mu = \mathbb{E}\{X_1\} = \frac{1}{\epsilon}$. Employing (3.10) and (3.8) yields

$$\widetilde{\zeta}(s) = \frac{1}{s+\epsilon}, \quad s \geq 0, \tag{3.85}$$

$$\widetilde{\chi}(s) = \frac{\epsilon}{s+\nu+\epsilon}, \quad s \geq 0, \tag{3.86}$$

$$\widetilde{\xi}(s) = \widetilde{b}(s) - \widetilde{\chi}(s) = \frac{\epsilon}{s+\epsilon} - \frac{\epsilon}{s+\nu+\epsilon}, \quad s \geq 0.$$

We derive the following auxiliary relations

$$\begin{aligned}
T_sT_\varrho T_\mathfrak{r} b(0) &= \frac{\epsilon}{(s+\epsilon)(\varrho+\epsilon)(\mathfrak{r}+\epsilon)},\\
T_sT_\varrho T_\mathfrak{r} \chi(0) &= \frac{\epsilon}{(s+\nu+\epsilon)(\varrho+\nu+\epsilon)(\mathfrak{r}+\nu+\epsilon)},\\
T_sT_\varrho T_\mathfrak{r} \xi(0) &= \frac{\epsilon}{(s+\epsilon)(\varrho+\epsilon)(\mathfrak{r}+\epsilon)} - \frac{\epsilon}{(s+\nu+\epsilon)(\varrho+\nu+\epsilon)(\mathfrak{r}+\nu+\epsilon)},\\
T_sT_\varrho T_\mathfrak{r} \zeta(0) &= \frac{1}{(s+\epsilon)(\varrho+\epsilon)(\mathfrak{r}+\epsilon)},\\
\frac{\mathfrak{r}T_sT_\mathfrak{r}\chi(0) - \varrho T_sT_\varrho\chi(0)}{\mathfrak{r}-\varrho} &= \frac{\epsilon(\nu+\epsilon)}{(s+\nu+\epsilon)(\varrho+\nu+\epsilon)(\mathfrak{r}+\nu+\epsilon)},\\
\frac{\mathfrak{r}T_sT_\mathfrak{r}\xi(0) - \varrho T_sT_\varrho\xi(0)}{\mathfrak{r}-\varrho} &= \frac{\epsilon^2}{(s+\epsilon)(\varrho+\epsilon)(\mathfrak{r}+\epsilon)} - \frac{\epsilon(\nu+\epsilon)}{(s+\nu+\epsilon)(\varrho+\nu+\epsilon)(\mathfrak{r}+\nu+\epsilon)},\\
\frac{\mathfrak{r}T_sT_\mathfrak{r}\zeta(0) - \varrho T_sT_\varrho\zeta(0)}{\mathfrak{r}-\varrho} &= \frac{\epsilon}{(s+\epsilon)(\varrho+\epsilon)(\mathfrak{r}+\epsilon)},\\
\frac{\mathfrak{r}^2T_sT_\mathfrak{r}\zeta(0) - \varrho^2 T_sT_\varrho\zeta(0)}{\mathfrak{r}-\varrho} - \widetilde{\zeta}(s) &= \frac{-\epsilon^2}{(s+\epsilon)(\varrho+\epsilon)(\mathfrak{r}+\epsilon)},
\end{aligned}$$

utilizing the property of T in (2.6) repeatedly.

The exponential distribution is a member of the K_n family of distributions with $k = 1$ where

$$p_k(s) = s + \epsilon,$$

$$q_k(s) = s + \nu + \epsilon.$$

Inserting identities (3.85), (3.86) and (3.73) into (3.72) with the above quantities yields

$$
\begin{aligned}
&\widetilde{\varphi}_{\mathrm{w},1}(s)\\
&= \frac{(s+\epsilon)(s+\nu+\epsilon)\left\{\phi'_{\mathrm{w},1}(0) + \frac{\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\epsilon}{(s+\nu+\epsilon)(\varrho+\nu+\epsilon)(\mathrm{r}+\nu+\epsilon)} + \frac{\lambda_1}{D}\cdot\frac{-\epsilon^2+\frac{c}{D}\epsilon+\frac{\lambda_2+\delta}{D}}{(s+\epsilon)(\varrho+\epsilon)(\mathrm{r}+\epsilon)}\right\}}{\prod_{l=1}^{4}(s-R_l)}\\
&= \frac{(s+\epsilon)(s+\nu+\epsilon)\phi'_{\mathrm{w},1}(0) + (s+\epsilon)\frac{\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\epsilon}{(\varrho+\nu+\epsilon)(\mathrm{r}+\nu+\epsilon)} + (s+\nu+\epsilon)\frac{\lambda_1}{D}\cdot\frac{-\epsilon^2+\frac{c}{D}\epsilon+\frac{\lambda_2+\delta}{D}}{(\varrho+\epsilon)(\mathrm{r}+\epsilon)}}{\prod_{l=1}^{4}(s-R_l)},
\end{aligned}
$$

where $R_1,\ldots,R_4$ are the negative roots of the Lundberg's equation. Observe that the numerator of $\widetilde{\varphi}_{\mathrm{w},1}(s)$ is a quadratic function in s and we denote it by $P_{\mathrm{w},1}(s)$. Therefore,

$$
\widetilde{\varphi}_{\mathrm{w},1}(s) = \frac{P_{\mathrm{w},1}(s)}{\prod_{l=1}^{4}(s-R_l)}.
$$

Similarly, we deduce

$$
\widetilde{\varphi}_{\mathrm{w},2}(s) = \frac{P_{\mathrm{w},2}(s)}{\prod_{l=1}^{4}(s-R_l)},
$$

where

$$
\begin{aligned}
P_{\mathrm{w},2}(s) &= (s+\epsilon)(s+\nu+\epsilon)\phi'_{\mathrm{w},2}(0) - (s+\nu+\epsilon)\frac{\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\epsilon}{(\varrho+\epsilon)(\mathrm{r}+\epsilon)}\\
&\quad + (s+\epsilon)\frac{\left[\frac{\lambda_2}{D}\phi'_{\mathrm{w},1}(0) - \frac{\lambda_1}{D}\phi'_{\mathrm{w},2}(0)\right]\epsilon}{(\varrho+\nu+\epsilon)(\mathrm{r}+\nu+\epsilon)} + (s+\nu+\epsilon)\frac{\lambda_2}{D}\cdot\frac{-\epsilon^2+\frac{c}{D}\epsilon+\frac{\lambda_1+\delta}{D}}{(\varrho+\epsilon)(\mathrm{r}+\epsilon)}.
\end{aligned}
$$

Applying the Heaviside expansion theorem to invert the Laplace transforms yields

$$
\begin{aligned}
\varphi_{\mathrm{w},1}(u) &= \frac{P_{\mathrm{w},1}(R_1)}{(R_1-R_2)(R_1-R_3)(R_1-R_4)}e^{R_1 u} + \frac{P_{\mathrm{w},1}(R_2)}{(R_2-R_1)(R_2-R_3)(R_2-R_4)}e^{R_2 u}\\
&\quad + \frac{P_{\mathrm{w},1}(R_3)}{(R_3-R_1)(R_3-R_2)(R_3-R_4)}e^{R_3 u} + \frac{P_{\mathrm{w},1}(R_4)}{(R_4-R_1)(R_4-R_2)(R_4-R_3)}e^{R_4 u},\\
\varphi_{\mathrm{w},2}(u) &= \frac{P_{\mathrm{w},2}(R_1)}{(R_1-R_2)(R_1-R_3)(R_1-R_4)}e^{R_1 u} + \frac{P_{\mathrm{w},2}(R_2)}{(R_2-R_1)(R_2-R_3)(R_2-R_4)}e^{R_2 u}\\
&\quad + \frac{P_{\mathrm{w},2}(R_3)}{(R_3-R_1)(R_3-R_2)(R_3-R_4)}e^{R_3 u} + \frac{P_{\mathrm{w},2}(R_4)}{(R_4-R_1)(R_4-R_2)(R_4-R_3)}e^{R_4 u}.
\end{aligned}
$$

Similar computation may be carried out for the case when is ruin due to oscillation, which leads to

$$\widetilde{\varphi}_{\mathrm{d},1}(s) = \frac{P_{d,1}(s)}{\prod_{l=1}^{4}(s - R_l)},$$
$$\widetilde{\varphi}_{\mathrm{d},2}(s) = \frac{P_{d,2}(s)}{\prod_{l=1}^{4}(s - R_l)},$$

where $P_{d,1}(s)$ and $P_{d,2}(s)$ are polynomials with the following forms

$$\begin{aligned} P_{d,1}(s) &= (s+\epsilon)(s+\nu+\epsilon)\left[s+\varrho+\mathfrak{r}+\frac{c}{D}+\frac{c}{D}+\phi'_{\mathrm{d},1}(0)\right] \\ &\quad + (s+\epsilon)\,\frac{\left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0)+\frac{\lambda_2-\lambda_1}{D}\cdot\frac{c}{D}\right]\epsilon-\frac{\lambda_2-\lambda_1}{D}\cdot\epsilon\,(\nu+\epsilon)}{(\varrho+\nu+\epsilon)\,(\mathfrak{r}+\nu+\epsilon)}, \\ P_{d,2}(s) &= (s+\epsilon)(s+\nu+\epsilon)\left[s+\varrho+\mathfrak{r}+\frac{c}{D}+\frac{c}{D}+\phi'_{\mathrm{d},2}(0)\right] \\ &\quad - (s+\nu+\epsilon)\,\frac{\left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0)+\frac{\lambda_2-\lambda_1}{D}\cdot\frac{c}{D}\right]\epsilon}{(\varrho+\epsilon)\,(\mathfrak{r}+\epsilon)} \\ &\quad + (s+\epsilon)\,\frac{\left[\frac{\lambda_2}{D}\phi'_{\mathrm{d},1}(0)-\frac{\lambda_1}{D}\phi'_{\mathrm{d},2}(0)+\frac{\lambda_2-\lambda_1}{D}\cdot\frac{c}{D}\right]\epsilon}{(\varrho+\nu+\epsilon)\,(\mathfrak{r}+\nu+\epsilon)} \\ &\quad + (s+\nu+\epsilon)\,\frac{\frac{\lambda_2-\lambda_1}{D}\cdot\epsilon^2}{(\varrho+\epsilon)\,(\mathfrak{r}+\epsilon)} - (s+\epsilon)\,\frac{\frac{\lambda_2-\lambda_1}{D}\cdot\epsilon\,(\nu+\epsilon)}{(\varrho+\nu+\epsilon)\,(\mathfrak{r}+\nu+\epsilon)}. \end{aligned}$$

Inverting the Laplace transforms $\widetilde{\varphi}_{\mathrm{d},1}(s)$ and $\widetilde{\varphi}_{\mathrm{d},2}(s)$ produces

$$\begin{aligned} \varphi_{\mathrm{d},1}(u) &= \frac{P_{\mathrm{d},1}(R_1)}{(R_1-R_2)(R_1-R_3)(R_1-R_4)}e^{R_1u} + \frac{P_{\mathrm{d},1}(R_2)}{(R_2-R_1)(R_2-R_3)(R_2-R_4)}e^{R_2u} \\ &\quad + \frac{P_{\mathrm{d},1}(R_3)}{(R_3-R_1)(R_3-R_2)(R_3-R_4)}e^{R_3u} + \frac{P_{\mathrm{d},1}(R_4)}{(R_4-R_1)(R_4-R_2)(R_4-R_3)}e^{R_4u}, \\ \varphi_{\mathrm{d},2}(u) &= \frac{P_{\mathrm{d},2}(R_1)}{(R_1-R_2)(R_1-R_3)(R_1-R_4)}e^{R_1u} + \frac{P_{\mathrm{d},2}(R_2)}{(R_2-R_1)(R_2-R_3)(R_2-R_4)}e^{R_2u} \\ &\quad + \frac{P_{\mathrm{d},2}(R_3)}{(R_3-R_1)(R_3-R_2)(R_3-R_4)}e^{R_3u} + \frac{P_{\mathrm{d},2}(R_4)}{(R_4-R_1)(R_4-R_2)(R_4-R_3)}e^{R_4u}. \end{aligned}$$

The moments of the time to ruin random variable may be computed from its Laplace transform. Furthermore, if we let $\delta = 0$ which implies $\varrho = 0$, the Laplace transform of the time to ruin $\varphi_{\mathrm{w},i}(u), \varphi_{\mathrm{d},i}(u)$, $i = 1, 2$, reduce to the ultimate ruin probabilities, which we illustrate by the following numerical example.

3.4.3 Numerical Example

Assume that the thresholds $Q_i \sim \text{Exp}(2)$, claim sizes $X_i \sim \text{Exp}(1)$, $c = 2$, $\lambda_1 = 3$, $\lambda_2 = 1$, $\delta = 0$ and $\sigma = 1$. Then, we find the Lundberg's equation as

$$s\left(s^5 + 12s^4 + 43s^2 + 26s^2 - 70s - 4\right) = 0.$$

Lundberg's equation has six roots in total that are $\varrho = 0$, $\mathfrak{r} = 0.947295$, $R_1 = -0.056081$, $R_2 = -3.198969$, $R_3 = -4.846122 - 0.22772i$ and $R_4 = -4.846122 + 0.22772i$. Utilizing the results from the previous section, we obtain

$$\begin{aligned}
\psi_{\mathrm{w},1}(u) &= 0.77420\, e^{-0.05608u} - 0.09450\, e^{-3.19897u} \\
&\qquad - [0.67971 \cos(0.22772u) - 2.56055 \sin(0.22772u)]\, e^{-4.84612u}, \\
\psi_{\mathrm{w},2}(u) &= 0.72281\, e^{-0.05608u} - 0.05911\, e^{-3.19897u} \\
&\qquad - [0.66369 \cos(0.22772u) + 0.53646 \sin(0.22772u)]\, e^{-4.84612u},
\end{aligned}$$

and

$$\begin{aligned}
\psi_{\mathrm{d},1}(u) &= 0.18431\, e^{-0.05608u} + 0.13906\, e^{-3.19897u} \\
&\qquad + [0.67663 \cos(0.22772u) - 2.82182 \sin(0.22772u)]\, e^{-4.84612u}, \\
\psi_{\mathrm{d},2}(u) &= 0.17208\, e^{-0.05608u} + 0.08699\, e^{-3.19897u} \\
&\qquad + [0.74093 \cos(0.22772u) + 0.56990 \sin(0.22772u)]\, e^{-4.84612u}.
\end{aligned}$$

Combining the above gives us the ultimate-ruin probabilities

$$\begin{aligned}
\psi_1(u) &= 0.95851\, e^{-0.05608u} + 0.04456\, e^{-3.19897u} \\
&\qquad - [0.00308 \cos(0.22772u) + 0.26127 \sin(0.22772u)]\, e^{-4.84612u}, \\
\psi_2(u) &= 0.89488\, e^{-0.05608u} + 0.02788\, e^{-3.19897u} \\
&\qquad + [0.07724 \cos(0.22772u) + 0.03344 \sin(0.22772u)]\, e^{-4.84612u}.
\end{aligned}$$

We may change the value of σ to examine the impact of perturbation under the dependent insurance risk model. Notice that when $\sigma = 0$, this example reduces to the one in Section 2.4.3. Figure 3.1 shows the ultimate-ruin probabilities comparing to the non-perturbed case as $\sigma \to 0$. The different initial conditions under $\sigma = 0$ and $\sigma > 0$ is the reason we treat the perturbed and unperturbed version of the dependence model separately.

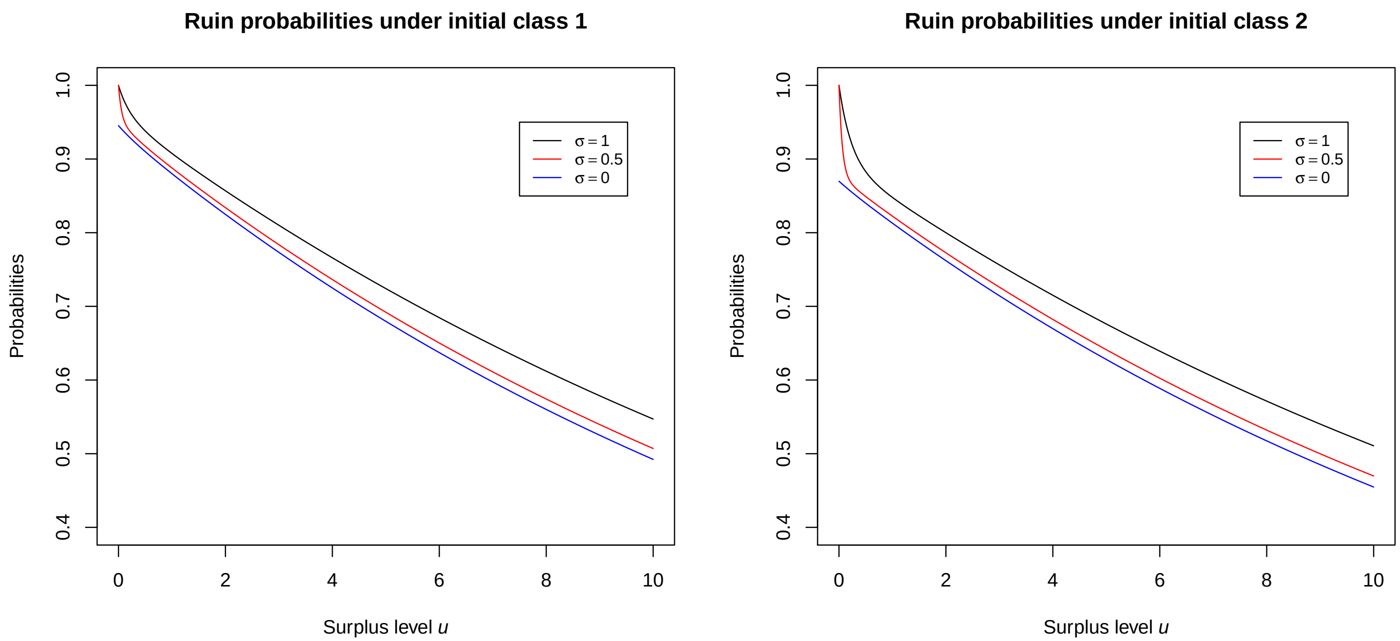

Figure 3.1: Ruin probabilities under the dependent insurance risk model with diffusion

Chapter 4

A perturbed dual risk model with dependence

Dual risk models may be used to model the revenue process of a company with constant expense rate and occasional gains. In this chapter, we consider a dual risk model with both inter-gain distribution and expense rate depending on the size of the previous gain. In addition, we assume that the surplus process is perturbed by a Brownian motion. Exact solutions for the Laplace transform and the first moment of the time to ruin with an arbitrary gain-size distribution are obtained. Applications with numerical illustrations are provided to examine the impact of the dependence structure and the perturbation.

4.1 Model description and notation

Consider the dual risk model

$$R(t) = u - c_1 \int_0^t \mathbb{I}_{\{J(s)=1\}}\, ds - c_2 \int_0^t \mathbb{I}_{\{J(s)=2\}}\, ds + \sum_{i=1}^{M(t)} X_i + \sigma W(t), \quad t \geq 0, \sigma > 0, \qquad (4.1)$$

where $u > 0$ is the initial surplus, $W(t)$ is a standard Brownian motion, and the gain-size random variables $\{X_i,\ i = 1, 2, \dots\}$ are i.i.d. with density function $p(y)$, $y > 0$, and c.d.f.

$P(y)$, $y > 0$. The gain counting process $M(t)$ is a renewal process with intergain times $\{Z_i,\ i = 1, 2, \ldots\}$. Assume that the distribution of the time until the next gain depends on the previous gain amount by comparing it to a random threshold, similar to Albrecher and Boxma (2004). Suppose the thresholds $\{Q_i,\ i = 1, 2, \ldots\}$ are i.i.d. with c.d.f. $H(y)$, $y > 0$, and are independent of the gain sizes. The thresholds $\{Q_i\}_{i=1}^{\infty}$ play the role of a classifier. If the size of a gain X_j is smaller than Q_j, then the revenue process is in Class 1 where the time until next gain follows an exponential distribution with mean $1/\lambda_1 > 0$ and expense rate is c_1 until the arrival of the next gain. If X_j is larger than Q_j, then the revenue process is in Class 2 where the time until next gain follows another exponential distribution with mean $1/\lambda_2 > 0$ ($\lambda_1 \neq \lambda_2$) and the expense rate is c_2. At any given time t, $J(t)$ represents which class the process falls in. The model may describe the revenue process of a research company, where the thresholds represent the competition or the industrial average. When a certain research gain is larger (or smaller) than the average for the industry, resources and talent will be drawn into (or out of) the company. This action will influence the waiting-time distribution of the next gain and the company should adjust the expense rate accordingly.

For comparison, a non-perturbed version of the dual model (4.1) is given by

$$R(t) = u - c_1 \int_0^t \mathbb{I}_{\{J(s)=1\}}\, ds - c_2 \int_0^t \mathbb{I}_{\{J(s)=2\}}\, ds + \sum_{i=1}^{M(t)} X_i, \quad t \geq 0, \tag{4.2}$$

which is a limiting case of model (4.1) as $\sigma \to 0$.

Next, we introduce some notation that will be used in the rest of this chapter. Given that the initial class of the revenue process is i, $i = 1, 2$, denote the Laplace transform of the random time of ruin $\tau_i = \inf\{t : R(t) = 0\}$ as

$$\Phi_i(u) = \mathbb{E}\left[e^{-\delta\tau_i}\, \mathbb{I}(\tau_i < \infty)\middle| R(0) = u\right], \qquad u > 0,\ \delta \geq 0,\ i = 1, 2, \tag{4.3}$$

which implies

$$\lim_{u \to 0} \Phi_i(u) = 1\,, \qquad i = 1, 2.$$

Assume further that the net-profit condition

$$\frac{c_1}{\lambda_1}\int_0^\infty p(y)\overline{H}(y)\mathrm{d}y + \frac{c_2}{\lambda_2}\int_0^\infty p(y)H(y)\mathrm{d}y < \mathbb{E}\{X_1\} \tag{4.4}$$

holds. For simplicity, we denote by

$$\chi(y) := p(y)\overline{H}(y), \tag{4.5}$$

$$\xi(y) := p(y)H(y), \tag{4.6}$$

$$D := \frac{1}{2}\sigma^2, \tag{4.7}$$

and by

$$\widetilde{f}(s) = \int_0^\infty e^{-sy} f(y)\,\mathrm{d}y, \quad s \in \mathbb{C},$$

the Laplace transform of a real-valued function f.

4.2 Laplace transform of the time to ruin

This section is dedicated to the explicit solutions to second order integro-differential equations satisfied by the Laplace transform of the time to ruin. First, we derive these equations in Proposition 4.1. Second, we provide their solutions in Theorem 4.2.

Proposition 4.1 *The Laplace transforms of the time to ruin* $\Phi_1(u)$ *and* $\Phi_2(u)$ *defined in* (4.3) *satisfy the system of integro-differential equations*

$$-\frac{D}{c_1}\Phi_1''(u) + \Phi_1'(u) + \frac{\lambda_1+\delta}{c_1}\Phi_1(u) = \frac{\lambda_1}{c_1}\int_0^\infty \left[\Phi_1(u+y)\chi(y) + \Phi_2(u+y)\xi(y)\right]dy, \tag{4.8}$$

$$-\frac{D}{c_2}\Phi_2''(u) + \Phi_2'(u) + \frac{\lambda_2+\delta}{c_2}\Phi_2(u) = \frac{\lambda_2}{c_2}\int_0^\infty \left[\Phi_1(u+y)\chi(y) + \Phi_2(u+y)\xi(y)\right]dy, \tag{4.9}$$

with boundary condition $\lim_{u\to 0}\Phi_1(u) = \lim_{u\to 0}\Phi_2(u) = 1$.

Proof First we consider when the revenue process is initially in class 1, that is the fist gain arrives at rate of λ_1.

Consider a sufficiently small time interval $[0, dt)$ and condition on whether a gain occurs in this time interval, we then obtain

$$\begin{aligned}\Phi_1(u) &= (1-\lambda_1 dt)\,\Phi_1(u|\text{there is no gain in [0,dt)}) + \lambda_1 dt\,\Phi_1(u|\text{there is one gain in [0,dt)}) + o(dt)\\ &= (1-\lambda_1 dt)\,\mathbb{E}\left[e^{-\delta dt}\Phi_1\left(u - c_1 dt + \sigma W(dt)\right)\right]\\ &\quad + \lambda_1 dt e^{-\delta dt}\mathbb{E}\left[\int_0^\infty \Phi_1\left(u - c_1 dt + \sigma W(dt) + y\right) p(y)\mathbb{P}\{Q > y\}\,\mathrm{d}y\right.\\ &\qquad\qquad \left. + \int_0^\infty \Phi_2\left(u - c_1 dt + \sigma W(dt) + y\right) p(y)\mathbb{P}\{Q < y\}\,\mathrm{d}y\right] + o(dt).\end{aligned} \tag{4.10}$$

Applying Taylor expansion yields

$$\mathbb{E}\left[\Phi_1(u - c_1 dt + \sigma W(dt))\right] = \Phi_1(u) - c_1\Phi_1'(u)dt + \frac{1}{2}\sigma^2\Phi_1''(u)dt + o(dt)\,,$$

since $\mathbb{E}[W(dt)] = \mathbb{E}[W^3(dt)] = 0$ and $\mathbb{E}[W^2(dt)] = \frac{1}{2}\sigma^2$. Implementing the above result in equation (4.10) and dividing it by dt produces

$$\begin{aligned}\left(\lambda_1 e^{-\delta dt} + \frac{1-e^{-\delta dt}}{dt}\right)\Phi_1(u) &= (1-\lambda_1 dt)\,e^{-\delta dt}\left[-c_1\Phi_1'(u) + \frac{1}{2}\sigma^2\Phi_1''(u)\right]\\ &\quad + \lambda_1 e^{-\delta dt}\,\mathbb{E}\left\{\int_0^\infty \Phi_1\left(u - c_1 dt + \sigma W(dt) + y\right) p(y)\overline{H}(y)\mathrm{d}y\right.\\ &\qquad \left. + \int_0^\infty \Phi_2\left(u - c_1 dt + \sigma W(dt) + y\right) p(y)H(y)\mathrm{d}y\right\} + \frac{o(dt)}{dt}.\end{aligned}$$

Letting dt converge to zero gives

$$(\lambda_1 + \delta)\,\Phi_1(u) = \frac{1}{2}\sigma^2\Phi_1''(u) - c_1\Phi_1'(u) + \lambda_1\left[\int_0^\infty \Phi_1\left(u+y\right) p(y)\overline{H}(y)\mathrm{d}y + \int_0^\infty \Phi_2\left(u+y\right) p(y)H(y)\mathrm{d}y\right].$$

Rearranging the above yields

$$-\frac{D}{c_1}\Phi_1''(u) + \Phi_1'(u) + \frac{\lambda_1+\delta}{c_1}\Phi_1(u) = \frac{\lambda_1}{c_1}\int_0^\infty \left[\Phi_1\left(u+y\right)\chi(y) + \Phi_2\left(u+y\right)\xi(y)\right]\mathrm{d}y\,,$$

where $\chi(y)$, $\xi(y)$ and D are defined in (4.5), (4.6) and (4.7) respectively. Similarly, given the revenue process is initially in Class 2, we may deduce the equation

$$-\frac{D}{c_2}\Phi_2''(u) + \Phi_2'(u) + \frac{\lambda_2+\delta}{c_2}\Phi_2(u) = \frac{\lambda_2}{c_2}\int_0^\infty \left[\Phi_1\left(u+y\right)\chi(y) + \Phi_2\left(u+y\right)\xi(y)\right]\mathrm{d}y,$$

which completes the proof. ■

As seen in the following theorem, the solutions to equations (4.8) and (4.9) have a fairly simple form. Moreover, it is straightforward to determine all relevant constants.

Theorem 4.2 *A solution to the system of equations* (4.8) *and* (4.9) *is*

$$\Phi_1^*(u) = \ell_1 e^{-\rho_1 u} + \vartheta_1 e^{-\rho_2 u}, \tag{4.11}$$

$$\Phi_2^*(u) = \ell_2 e^{-\rho_1 u} + \vartheta_2 e^{-\rho_2 u}, \tag{4.12}$$

where $\ell_1, \ell_2, \vartheta_1, \vartheta_2, \rho_1$ and ρ_2 are nonzero constants. Here ρ_1 and ρ_2 are the only two roots with positive real parts to Lundberg's fundamental equation

$$\frac{\frac{\lambda_1}{c_1}\widetilde{\chi}(s)}{\frac{\lambda_1+\delta}{c_1} - s - \frac{D}{c_1}s^2} + \frac{\frac{\lambda_2}{c_2}\widetilde{\xi}(s)}{\frac{\lambda_2+\delta}{c_2} - s - \frac{D}{c_2}s^2} = 1, \tag{4.13}$$

The constants $\ell_1, \ell_2, \vartheta_1$ and ϑ_2 are determined by the following system of linear equations

$$\ell_1 + \vartheta_1 = 1 \tag{4.14}$$

$$\ell_2 + \vartheta_2 = 1 \tag{4.15}$$

$$\frac{\lambda_2}{c_2} \cdot \frac{\frac{\lambda_1+\delta}{c_1} - \rho_1 - \frac{D}{c_1}\rho_1^2}{\frac{\lambda_2+\delta}{c_2} - \rho_1 - \frac{D}{c_2}\rho_1^2}\, \ell_1 = \frac{\lambda_1}{c_1}\, \ell_2 \tag{4.16}$$

$$\frac{\lambda_2}{c_2} \cdot \frac{\frac{\lambda_1+\delta}{c_1} - \rho_2 - \frac{D}{c_1}\rho_2^2}{\frac{\lambda_2+\delta}{c_2} - \rho_2 - \frac{D}{c_2}\rho_2^2}\, \vartheta_1 = \frac{\lambda_1}{c_1}\, \vartheta_2. \tag{4.17}$$

Proof We first show that $\Phi_1^*(u)$ and $\Phi_2^*(u)$ solve equation (4.8). Inserting $\Phi_1^*(u)$ as defined in identity (4.11) into the left-hand side of equation (4.8) yields

$$\begin{aligned}
&-\frac{D}{c_1} \cdot \frac{\partial^2}{\partial^2 u}\Phi_1^*(u) + \frac{\partial}{\partial u}\Phi_1^*(u) + \frac{\lambda_1+\delta}{c_1}\Phi_1^*(u) \\
&= -\frac{D}{c_1}\left(\rho_1^2 \ell_1\, e^{-\rho_1 u} + \rho_2^2 \vartheta_1\, e^{-\rho_2 u}\right) + \left(-\rho_1 \ell_1\, e^{-\rho_1 u} - \rho_2 \vartheta_1\, e^{-\rho_2 u}\right) + \frac{\lambda_1+\delta}{c_1}\left(\ell_1\, e^{-\rho_1 u} + \vartheta_1\, e^{-\rho_2 u}\right) \\
&= \left(\frac{\lambda_1+\delta}{c_1} - \rho_1 - \frac{D}{c_1}\rho_1^2\right)\ell_1 e^{-\rho_1 u} + \left(\frac{\lambda_1+\delta}{c_1} - \rho_2 - \frac{D}{c_1}\rho_2^2\right)\vartheta_1 e^{-\rho_2 u}.
\end{aligned} \tag{4.18}$$

Then inserting $\Phi_1^*(u)$ and $\Phi_2^*(u)$ into the right-hand side of equation (4.8) which is

$$\frac{\lambda_1}{c_1}\int_0^\infty \left[\Phi_1^*(u+y)\,\chi(y) + \Phi_2^*(u+y)\,\xi(y)\right] \mathrm{d}y$$

$$= \frac{\lambda_1}{c_1}\int_0^\infty \left[\ell_1\, e^{-\rho_1(u+y)}\chi(y) + \vartheta_1\, e^{-\rho_2(u+y)}\chi(y) + \ell_2\, e^{-\rho_1(u+y)}\xi(y) + \vartheta_2\, e^{-\rho_2(u+y)}\xi(y)\right] \mathrm{d}y\,,$$

by the definition of Laplace transform we have

$$\begin{aligned}
&\frac{\lambda_1}{c_1}\int_0^\infty \left[\Phi_1^*(u+y)\chi(y) + \Phi_2^*(u+y)\,\xi(y)\right]\mathrm{d}y \\
&\quad = \frac{\lambda_1\ell_1}{c_1}\, e^{-\rho_1 u}\widetilde{\chi}(\rho_1) + \frac{\lambda_1\vartheta_1}{c_1}\, e^{-\rho_2 u}\widetilde{\chi}(\rho_2) + \frac{\lambda_1\ell_2}{c_1}\, e^{-\rho_1 u}\widetilde{\xi}(\rho_1) + \frac{\lambda_1\vartheta_2}{c_1}\, e^{-\rho_2 u}\widetilde{\xi}(\rho_2) \\
&\quad = \left[\frac{\lambda_1\ell_1}{c_1}\widetilde{\chi}(\rho_1) + \frac{\lambda_1\ell_2}{c_1}\widetilde{\xi}(\rho_1)\right]e^{-\rho_1 u} + \left[\frac{\lambda_1\vartheta_1}{c_1}\widetilde{\chi}(\rho_2) + \frac{\lambda_1\vartheta_2}{c_1}\widetilde{\xi}(\rho_2)\right]e^{-\rho_2 u} \\
&\quad = \left[\frac{\lambda_1}{c_1}\widetilde{\chi}(\rho_1) + \frac{\lambda_2}{c_2}\cdot\frac{\frac{\lambda_1+\delta}{c_1} - \rho_1 - \frac{D}{c_1}\rho_1^2}{\frac{\lambda_2+\delta}{c_2} - \rho_1 - \frac{D}{c_2}\rho_1^2}\widetilde{\xi}(\rho_1)\right]\ell_1 e^{-\rho_1 u} \\
&\qquad + \left[\frac{\lambda_1}{c_1}\widetilde{\chi}(\rho_2) + \frac{\lambda_2}{c_2}\cdot\frac{\frac{\lambda_1+\delta}{c_1} - \rho_2 - \frac{D}{c_1}\rho_2^2}{\frac{\lambda_2+\delta}{c_2} - \rho_2 - \frac{D}{c_2}\rho_2^2}\widetilde{\xi}(\rho_2)\right]\vartheta_1 e^{-\rho_2 u} \\
&\quad = \left[\frac{\frac{\lambda_1}{c_1}\widetilde{\chi}(\rho_1)}{\frac{\lambda_1+\delta}{c_1} - \rho_1 - \frac{D}{c_1}\rho_1^2} + \frac{\frac{\lambda_2}{c_2}\widetilde{\xi}(\rho_1)}{\frac{\lambda_2+\delta}{c_2} - \rho_1 - \frac{D}{c_2}\rho_1^2}\right]\left(\frac{\lambda_1+\delta}{c_1} - \rho_1 - \frac{D}{c_1}\rho_1^2\right)\ell_1 e^{-\rho_1 u} \\
&\qquad + \left[\frac{\frac{\lambda_1}{c_1}\widetilde{\chi}(\rho_2)}{\frac{\lambda_2+\delta}{c_2} - \rho_2 - \frac{D}{c_2}\rho_2^2} + \frac{\frac{\lambda_2}{c_2}\widetilde{\xi}(\rho_2)}{\frac{\lambda_2+\delta}{c_2} - \rho_2 - \frac{D}{c_2}\rho_2^2}\right]\left(\frac{\lambda_2+\delta}{c_2} - \rho_2 - \frac{D}{c_2}\rho_2^2\right)\vartheta_1 e^{-\rho_2 u}, \qquad (4.19)
\end{aligned}$$

utilizing equalities (4.16) and (4.17). The left-hand side expression (4.18) matches the right-hand side expression (4.19), since both ρ_1 and ρ_2 satisfy Lundberg's equation (4.13), that is,

$$\left[\frac{\frac{\lambda_1}{c_1}\widetilde{\chi}(\rho_1)}{\frac{\lambda_1+\delta}{c_1} - \rho_1 - \frac{D}{c_1}\rho_1^2} + \frac{\frac{\lambda_2}{c_2}\widetilde{\xi}(\rho_1)}{\frac{\lambda_2+\delta}{c_2} - \rho_1 - \frac{D}{c_2}\rho_1^2}\right] = \left[\frac{\frac{\lambda_1}{c_1}\widetilde{\chi}(\rho_2)}{\frac{\lambda_2+\delta}{c_2} - \rho_2 - \frac{D}{c_2}\rho_2^2} + \frac{\frac{\lambda_2}{c_2}\widetilde{\xi}(\rho_2)}{\frac{\lambda_2+\delta}{c_2} - \rho_2 - \frac{D}{c_2}\rho_2^2}\right] = 1\,.$$

Thus, $\Phi_1^*(u)$ and $\Phi_2^*(u)$ solve equation (4.8).

Similarly, $\Phi_1^*(u)$ and $\Phi_2^*(u)$ also solve equation (4.9), which indicates together $\Phi_1^*(u)$ with $\Phi_2^*(u)$ is a solution for the system of equations (4.8) and (4.9). Moreover, equations (4.14) and (4.15) are obtained from the initial conditions $\Phi_1^*(0) = \Phi_2^*(0) = 1$. The four linearly independent equations (4.14) to (4.17) uniquely determine constants $\ell_1, \ell_2, \vartheta_1$ and ϑ_2 in $\Phi_1^*(u)$ and $\Phi_2^*(u)$. ■

We now need to demonstrate that the statement in Theorem 4.2 that there are only two roots with positive real roots to the Lundberg's equation.

Lemma 4.3 *For all $\delta \geq 0$, the generalized Lundberg's equation* (4.13) *under model* (4.1) *has exactly two roots with positive real parts.*

Proof We separate the proof into two cases: $\delta > 0$ and $\delta = 0$. For $\delta > 0$, the proof is same as for Lemma 3.4 in Chapter 3.

For $\delta = 0$, we rewrite the Lundberg's equation (4.13) in the form

$$l_1(s) - l_2(s) = 0\,,$$

where

$$\begin{aligned} l_1(s) &= \left(s^2 + \frac{c_1}{D}s - \frac{\lambda_1}{D}\right)\left(s^2 + \frac{c_2}{D}s - \frac{\lambda_2}{D}\right), \\ l_2(s) &= \frac{\lambda_1}{D}\left(s^2 + \frac{c_2}{D}s - \frac{\lambda_2}{D}\right)\widetilde{\chi}(s) + \frac{\lambda_2}{D}\left(s^2 + \frac{c_1}{D}s - \frac{\lambda_1}{D}\right)\widetilde{\xi}(s)\,. \end{aligned}$$

For $i = 1, 2$, denote the real roots to $s^2 + \frac{c_i}{D}s - \frac{\lambda_i}{D} = 0$ by α_i and $-\beta_i$ with $\alpha_i, \beta_i > 0$. Hence,

$$\begin{aligned} s^2 + \frac{c_1}{D}s - \frac{\lambda_1}{D} &= (s - \alpha_1)\,(s + \beta_1)\,, \\ s^2 + \frac{c_2}{D}s - \frac{\lambda_2}{D} &= (s - \alpha_2)\,(s + \beta_2)\,, \\ \alpha_1\beta_1 &= \frac{\lambda_1}{D}\,, \\ \alpha_2\beta_2 &= \frac{\lambda_2}{D}\,. \end{aligned}$$

Let $z = \frac{\kappa - s}{\kappa}$, where $\kappa > 0$ is a sufficiently large number, and define C as the circle $\{s : |z| = 1\}$. On the boundary of C except for the point $s = 0$, we have

$$\begin{aligned} |l_1(s)| &= |s - \alpha_1|\,|s + \beta_1|\,|s - \alpha_2|\,|s + \beta_2| \\ &= |s - \alpha_1|\,|s + \beta_1|\,|s - \alpha_2|\,|s + \beta_2|\widetilde{\chi}(0) + |s - \alpha_1|\,|s + \beta_1|\,|s - \alpha_2|\,|s + \beta_2|\widetilde{\xi}(0) \\ &> (-\alpha_1\beta_1)\,|s - \alpha_2|\,|s + \beta_2|\left|\widetilde{\chi}(s)\right| + (-\alpha_2\beta_2)\,|s - \alpha_1|\,|s + \beta_1|\left|\widetilde{\xi}(s)\right| \\ &\geq |l_2(s)|\,, \end{aligned}$$

since $|s| > 0$ and $Re(s) > 0$. Also, $|l_1(s)| = |l_2(s)|$ at $s = 0$.

To apply Corollary 2 in Klimenok (2001), we still need to consider the first derivatives of l_1 and l_2 with respect to z, where

$$\begin{aligned}
\left(\frac{\partial l_1}{\partial z} + \frac{\partial l_2}{\partial z}\right)\bigg|_{z=1} &= \left(\frac{\partial l_1}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial l_2}{\partial s}\cdot\frac{\partial s}{\partial z}\right)\bigg|_{s=0} \\
&= \left[\frac{\partial l_1}{\partial s}\cdot(-\kappa) + \frac{\partial l_2}{\partial s}\cdot(-\kappa)\right]\bigg|_{s=0} \\
&= (-\kappa)\left\{-\frac{\lambda_1 c_2}{D^2} - \frac{\lambda_2 c_1}{D^2} + \frac{\lambda_1 c_2}{D^2}\widetilde{\chi}(0) + \frac{\lambda_2 c_1}{D^2}\widetilde{\xi}(0) - \frac{\lambda_1\lambda_2}{D^2}\cdot\left[\frac{\partial}{\partial s}\widetilde{p}(s)\bigg|_{s=0}\right]\right\} \\
&= (-\kappa)\left\{\frac{\lambda_1 c_2}{D^2}\left[\widetilde{\chi}(0)-1\right] + \frac{\lambda_2 c_1}{D^2}\left[\widetilde{\xi}(0)-1\right] + \frac{\lambda_1\lambda_2}{D^2}\,\mathbb{E}\{X_1\}\right\} \\
&= \kappa\left[\frac{\lambda_1 c_2}{D^2}\widetilde{\xi}(0) + \frac{\lambda_2 c_1}{D^2}\widetilde{\chi}(0) - \frac{\lambda_1\lambda_2}{D^2}\,\mathbb{E}\{X_1\}\right] \\
&= \kappa\cdot\frac{\lambda_1\lambda_2}{D^2}\left[\frac{c_2}{\lambda_2}\widetilde{\xi}(0) + \frac{c_1}{\lambda_1}\widetilde{\chi}(0) - \mathbb{E}\{X_1\}\right]. \qquad (4.20)
\end{aligned}$$

Since κ, λ_1, λ_2, $D > 0$ and the net-profit condition (4.4) states

$$\begin{aligned}
\frac{c_1}{\lambda_1}\int_0^\infty p(y)\overline{H}(y)\mathrm{d}y + \frac{c_2}{\lambda_2}\int_0^\infty p(y)H(y)\mathrm{d}y &< \mathbb{E}\{X_1\} \\
\frac{c_1}{\lambda_1}\widetilde{\chi}(0) + \frac{c_2}{\lambda_2}\widetilde{\xi}(0) &< \mathbb{E}\{X_1\},
\end{aligned}$$

we have in (4.20) that

$$\left(\frac{\partial l_1}{\partial z} + \frac{\partial l_2}{\partial z}\right)\bigg|_{z=1} < 0\,.$$

In addition, $l_1(0) > 0$. Applying Corollary 2 in Klimenok (2001) yields that $l_1(s) - l_2(s)$ has the same number of zeros as $l_1(s)$ does in the interior of the circle C. Inside C, function $l_1(s)$ has exactly two zeros, and so does $l_1(s) - l_2(s)$. Letting $\kappa \to \infty$, we conclude that $l_1(s) - l_2(s)$ has exactly two zeros in the positive half plane. ■

Utilizing the Laplace transform of the time to ruin, we may deduce the first moment of the ruin time, given that ruin occurs in finite time, to be

$$\mathbb{E}\left[\tau_i|\tau_i < \infty\right] = \frac{-\frac{\partial}{\partial\delta}\,\Phi_i^*(u)|_{\delta=0}}{\Phi_i^*(u)|_{\delta=0}}, \qquad i = 1, 2. \qquad (4.21)$$

Denote by

$$\Lambda_i(x) := \frac{\lambda_i + \delta}{c_i} - x - \frac{D}{c_i}x^2, \qquad i = 1, 2,$$
$$\Upsilon(x) := \frac{\Lambda_1(x)}{\Lambda_2(x)}.$$

Taking derivatives of $\Lambda_1(\rho_j)$, $\Lambda_2(\rho_j)$ and $\Upsilon(\rho_j)$ for $j = 1, 2$ with respect to δ yields

$$\frac{\partial}{\partial\delta}\Lambda_i(\rho_j) = \frac{1}{c_i} - \left(1 + \frac{2D}{c_i}\rho_j\right)\frac{\partial\rho_j}{\partial\delta}, \qquad i = 1, 2, \quad j = 1, 2,$$
$$\frac{\partial}{\partial\delta}\Upsilon(\rho_j) = \frac{\frac{\partial}{\partial\delta}\Lambda_1(\rho_j) - \left[\frac{\partial}{\partial\delta}\Lambda_2(\rho_j)\right]\Upsilon(\rho_j)}{\Lambda_2(\rho_j)}, \qquad j = 1, 2. \tag{4.22}$$

The unknown functions $\frac{\partial\rho_j}{\partial\delta}$, $j = 1, 2$, may be derived from Lundberg's equation (4.13), where

$$\frac{\lambda_1}{c_1} \cdot \frac{\left[\frac{\partial\widetilde{\chi}(\rho_j)}{\partial\rho_j} \cdot \frac{\partial\rho_j}{\partial\delta}\right]\Lambda_1(\rho_j) - \widetilde{\chi}(\rho_j) \cdot \frac{\partial}{\partial\delta}\Lambda_1(\rho_j)}{\Lambda_1(\rho_j)^2} + \frac{\lambda_2}{c_2} \cdot \frac{\left[\frac{\partial\widetilde{\xi}(\rho_j)}{\partial\rho_j} \cdot \frac{\partial\rho_j}{\partial\delta}\right]\Lambda_2(\rho_j) - \widetilde{\xi}(\rho_j) \cdot \frac{\partial}{\partial\delta}\Lambda_2(\rho_j)}{\Lambda_2(\rho_j)^2} = 0,$$

or equivalently,

$$\left[\frac{\lambda_1}{c_1} \cdot \frac{1}{\Lambda_1(\rho_j)} \cdot \frac{\partial\widetilde{\chi}(\rho_j)}{\partial\rho_j}\right]\frac{\partial\rho_j}{\partial\delta} - \frac{\lambda_1}{c_1} \cdot \frac{\widetilde{\chi}(\rho_j)}{\Lambda_1(\rho_j)^2}\left[\frac{1}{c_1} - \left(1 + \frac{2D}{c_1}\rho_j\right)\frac{\partial\rho_j}{\partial\delta}\right]$$
$$+ \left[\frac{\lambda_2}{c_2} \cdot \frac{1}{\Lambda_2(\rho_j)} \cdot \frac{\partial\widetilde{\xi}(\rho_j)}{\partial\rho_j}\right]\frac{\partial\rho_j}{\partial\delta} - \frac{\lambda_2}{c_2} \cdot \frac{\widetilde{\xi}(\rho_j)}{\Lambda_2(\rho_j)^2}\left[\frac{1}{c_2} - \left(1 + \frac{2D}{c_2}\rho_j\right)\frac{\partial\rho_j}{\partial\delta}\right] = 0,$$

which may also be restated as

$$\left[\frac{\lambda_1}{c_1} \cdot \frac{1}{\Lambda_1(\rho_j)} \cdot \frac{\partial\widetilde{\chi}(\rho_j)}{\partial\rho_j} + \frac{\lambda_1}{c_1} \cdot \frac{\widetilde{\chi}(\rho_j)}{\Lambda_1(\rho_j)^2}\left(1 + \frac{2D}{c_1}\rho_j\right) + \frac{\lambda_2}{c_2} \cdot \frac{1}{\Lambda_2(\rho_j)} \cdot \frac{\partial\widetilde{\xi}(\rho_j)}{\partial\rho_j} + \frac{\lambda_2}{c_2} \cdot \frac{\widetilde{\xi}(\rho_j)}{\Lambda_2(\rho_j)^2}\left(1 + \frac{2D}{c_2}\rho_j\right)\right]\frac{\partial\rho_j}{\partial\delta}$$
$$= \frac{\lambda_1}{c_1^2} \cdot \frac{\widetilde{\chi}(\rho_j)}{\Lambda_1(\rho_j)^2} + \frac{\lambda_2}{c_2^2} \cdot \frac{\widetilde{\xi}(\rho_j)}{\Lambda_2(\rho_j)^2},$$

or as

$$\frac{\partial\rho_j}{\partial\delta} = \frac{\frac{\lambda_1}{c_1^2} \cdot \frac{\widetilde{\chi}(\rho_j)}{\Lambda_1(\rho_j)^2} + \frac{\lambda_2}{c_2^2} \cdot \frac{\widetilde{\xi}(\rho_j)}{\Lambda_2(\rho_j)^2}}{\frac{\lambda_1}{c_1} \cdot \frac{1}{\Lambda_1(\rho_j)} \cdot \frac{\partial\widetilde{\chi}(\rho_j)}{\partial\rho_j} + \frac{\lambda_1}{c_1} \cdot \frac{\widetilde{\chi}(\rho_j)}{\Lambda_1(\rho_j)^2}\left(1 + \frac{2D}{c_1}\rho_j\right) + \frac{\lambda_2}{c_2} \cdot \frac{1}{\Lambda_2(\rho_j)} \cdot \frac{\partial\widetilde{\xi}(\rho_j)}{\partial\rho_j} + \frac{\lambda_2}{c_2} \cdot \frac{\widetilde{\xi}(\rho_j)}{\Lambda_2(\rho_j)^2}\left(1 + \frac{2D}{c_2}\rho_j\right)}. \tag{4.23}$$

To derive $\frac{\partial}{\partial\delta}\,\Phi_i^*(u)|_{\delta=0}$ for $i = 1, 2$ in identity (4.21), we still need to express $\frac{\partial \ell_i}{\partial\delta}$ and $\frac{\partial \vartheta_i}{\partial\delta}$. Solving the system of linear equations (4.14) to (4.17) yields

$$\begin{aligned} \ell_1 &= \frac{\Upsilon(\rho_2) - \frac{c_2\lambda_1}{c_1\lambda_2}}{\Upsilon(\rho_2) - \Upsilon(\rho_1)} & \qquad \ell_2 &= \frac{c_1\lambda_2}{c_2\lambda_1}\Upsilon(\rho_1)\cdot \ell_1 \\ \vartheta_1 &= 1 - \ell_1 & \qquad \vartheta_2 &= 1 - \ell_2\,. \end{aligned} \tag{4.24}$$

Also, summing (4.16) and (4.17) leads to

$$\Upsilon(\rho_1)\cdot \ell_1 + \Upsilon(\rho_2)\,(1-\ell_1) = \frac{c_2\lambda_1}{c_1\lambda_2}\,.$$

After differentiation with respect to δ, we obtain

$$\frac{\partial \ell_1}{\partial\delta} = \frac{(1-\ell_1)\cdot\frac{\partial}{\partial\delta}\Upsilon(\rho_2) + \ell_1\cdot\frac{\partial}{\partial\delta}\Upsilon(\rho_1)}{\Upsilon(\rho_2)-\Upsilon(\rho_1)}\,. \tag{4.25}$$

Similarly, differentiating ℓ_2 in (4.24), we deduce

$$\frac{\partial \ell_2}{\partial\delta} = \frac{c_1\lambda_2}{c_2\lambda_1}\left[\frac{\partial \ell_1}{\partial\delta}\cdot\Upsilon(\rho_1) + \ell_1\cdot\frac{\partial}{\partial\delta}\Upsilon(\rho_1)\right], \tag{4.26}$$

where $\frac{\partial}{\partial\delta}\Upsilon(\rho_j)$ for $j = 1, 2$, are given in (4.22). Differentiating $\Phi_1^*(u)$ and $\Phi_2^*(u)$ with respect to δ yields

$$\frac{\partial}{\partial\delta}\,\Phi_i^*(u) = \left(\frac{\partial \ell_i}{\partial\delta} - \ell_i\frac{\partial\rho_1}{\partial\delta}\cdot u\right)e^{-\rho_1 u} - \left[\frac{\partial \ell_i}{\partial\delta} + (1-\ell_i)\,\frac{\partial\rho_2}{\partial\delta}\cdot u\right]e^{-\rho_2 u}, \qquad i = 1, 2,$$

where $\frac{\partial\rho_j}{\partial\delta}$ for $j = 1, 2$, and $\frac{\partial \ell_i}{\partial\delta}$ for $i = 1, 2$, may be found in (4.23), (4.25) and (4.26). Finally, the mean of the time of ruin, given that ruin occurs in finite time, is

$$\mathbb{E}\left[\tau_i|\tau_i < \infty\right] = \frac{-\left(\frac{\partial \ell_i}{\partial\delta} - \ell_i\frac{\partial\rho_1}{\partial\delta}\cdot u\right)e^{-\rho_1 u} + \left[\frac{\partial \ell_i}{\partial\delta} + (1-\ell_i)\,\frac{\partial\rho_2}{\partial\delta}\cdot u\right]e^{-\rho_2 u}\Big|_{\delta=0}}{\ell_i e^{-\rho_1 u} + (1-\ell_i)\,e^{-\rho_2 u}\Big|_{\delta=0}}, \qquad i = 1, 2.$$

Remark 4.1 When $\lambda_1 = \lambda_2$ and $c_1 = c_2$, the model (4.1) reduces to the compound Poisson dual model with diffusion in Avanzi and Gerber (2008). □

Remark 4.2 As $\sigma \to 0$, model (4.1) converges to model (4.2). Under the non-perturbed model (4.2), all results in Proposition 4.1, Theorem 4.2, Lemma 4.3 and the first moment of time to ruin still hold by letting $D = 0$. The reason is that in both cases of dual models — the perturbed one and the unperturbed one — the initial conditions coincide: $\lim_{u\to 0}\Phi_i(u) = 1$ and $\lim_{u\to\infty}\Phi_i(u) = 0,\ i = 1,2.$ $\square$

To demonstrate the results in Remark 4.2, we consider a non-perturbed version of the dependent dual risk model as defined in (4.2). Denote by $\phi_1(u)$, $i = 1,2$ for the Laplace transform of time to ruin under the non-perturbed model. Given that the first gain arrives at the rate of λ_1, conditioning on the time and amount of first gain that might occur, we have

$$\phi_1(u) = \int_0^{\frac{u}{c_1}} \lambda_1 e^{-\lambda_1 t} e^{-\delta t} \int_0^\infty \phi_1(u - c_1 t + y)p(y)\mathbb{P}\{Q > y\} + \phi_2(u - c_1 t + y)p(y)\mathbb{P}\{Q < y\}\,\mathrm{d}y\mathrm{d}t + e^{-\left(\frac{\lambda_1+\delta}{c_1}\right)u}.$$

Changing the variable of integration t to $v = u - c_1 t$ yields

$$\begin{aligned}
\phi_1(u) &= \frac{\lambda_1}{c_1}\int_0^u e^{-\left(\frac{\lambda_1+\delta}{c_1}\right)(u-v)} \int_0^\infty \phi_1(v + y)p(y)\overline{H}(y) + \phi_2(v + y)p(y)H(y)\,\mathrm{d}y\mathrm{d}v + e^{-\left(\frac{\lambda_1+\delta}{c_1}\right)u} \\
&= \frac{\lambda_1}{c_1}e^{-\left(\frac{\lambda_1+\delta}{c_1}\right)u}\int_0^u e^{\left(\frac{\lambda_1+\delta}{c_1}\right)v} \int_0^\infty \phi_1(v + y)\chi(y) + \phi_2(v + y)\xi(y)\,\mathrm{d}y\mathrm{d}v + e^{-\left(\frac{\lambda_1+\delta}{c_1}\right)u}.
\end{aligned}$$

Similarly, given that the first gain arrives at the rate of λ_2, we obtain

$$\phi_2(u) = \frac{\lambda_2}{c_2}e^{-\left(\frac{\lambda_2+\delta}{c_2}\right)u}\int_0^u e^{\left(\frac{\lambda_2+\delta}{c_2}\right)v} \int_0^\infty \phi_1(v + y)\chi(y) + \phi_2(v + y)\xi(y)\,\mathrm{d}y\mathrm{d}v + e^{-\left(\frac{\lambda_2+\delta}{c_2}\right)u}.$$

Differentiating the above equations with respect to u, after some rearranging produces

$$\begin{aligned}
\phi_1'(u) + \frac{\lambda_1+\delta}{c_1}\phi_1(u) &= \frac{\lambda_1}{c_1}\int_0^\infty \phi_1(u + y)\chi(y)\mathrm{d}y + \frac{\lambda_1}{c_1}\int_0^\infty \phi_2(u + y)\xi(y)\mathrm{d}y\,, \\
\phi_2'(u) + \frac{\lambda_2+\delta}{c_2}\phi_2(u) &= \frac{\lambda_2}{c_2}\int_0^\infty \phi_1(u + y)\chi(y)\mathrm{d}y + \frac{\lambda_2}{c_2}\int_0^\infty \phi_2(u + y)\xi(y)\mathrm{d}y\,,
\end{aligned}$$

which are equations (4.8) and (4.9) with $D = 0$. All the rest of the results follows straightforwardly by letting $D = 0$.

Remark 4.3 When $D = 0$, rewriting the system of integro-differential equations (4.8) and (4.9) by changing the variable of integration to $z = u + y$ produces

$$\phi_1'(u) + \frac{\lambda_1+\delta}{c_1}\phi_1(u) = \frac{\lambda_1}{c_1}\int_0^\infty k_1(u - z)\phi_1(z)\,\mathrm{d}z + \frac{\lambda_1}{c_1}\int_0^\infty k_2(u - z)\phi_2(z)\,\mathrm{d}z,$$

$$\phi_2'(u) + \frac{\lambda_2 + \delta}{c_2}\phi_2(u) = \frac{\lambda_2}{c_2}\int_0^\infty k_1(u-z)\phi_1(z)\,\mathrm{d}z + \frac{\lambda_2}{c_2}\int_0^\infty k_2(u-z)\phi_2(z)\,\mathrm{d}z.$$

where the kernels $k_1(x) = \chi(-x)\,\mathbb{I}(x \le 0)$ and $k_2(x) = \xi(-x)\,\mathbb{I}(x \le 0)$ are introduced. This kind of convolution-type system of integro-differential equations is also of interest in other areas such as applied physics, see for example equation (1.1) in Khachatryan and Khachatryan (2009). Theorem 4.2 provides a possible solution to some of its special cases.

□

4.3 Numerical illustrations

In this section, numerical examples are provided to apply the results in the previous section. Meanwhile, we examine the impact of the dependence structure and the perturbation to the underlying risk respectively.

In Example 4.1, we compare the dependent dual model to one with an independent setting, where the revenue process follows a perturbed compound Poisson dual model with parameter λ_1 with probability $\mathbb{P}\{X < Q\}$ and it follows another perturbed compound Poisson dual model with parameter λ_2 with probability $\mathbb{P}\{X > Q\}$. All other parameters remain the same. Let $\mathbb{P}\{X < Q\} = w$, the Laplace transform of the time to ruin is

$$\varphi(u) = w\,\varphi_1(u) + (1-w)\,\varphi_2(u),$$

where $\varphi_1(u)$ and $\varphi_2(u)$ correspond to the two perturbed compound Poisson dual models with net-profit conditions satisfied. For $i = 1, 2$, we have

$$\varphi_i(u) = e^{-r_i u},$$

where r_i is the unique positive solution to

$$-\frac{D}{c_i}s^2 - s + \frac{\lambda_i + \delta}{c_i} = \frac{\lambda_i}{c_i}\widetilde{p}(s).$$

The impact of the dependence structure is considered under different gain-size assumptions.

Example 4.2 is constructed to demonstrate how the change in the volatility of the diffusion process affects the dependent dual risk model and shows the ruin-related results converge to the unperturbed version of the model as $D \to 0$.

Example 4.1 Suppose that the gain sizes $\{X_i\}_{i=1}^{\infty}$ are Coxian(3) distributed with Laplace transform

$$\widetilde{p}(s) = \frac{3s+2}{(s+1)^2\,(s+2)}$$

and $\mathbb{E}\{X_1\} = 1$. Let $\lambda_1 = 2.5$, $\lambda_2 = 1.5$, $c_1 = c_2 = 1$, $D = 0.5$, $\delta = 0$ and the random thresholds follow an exponential distribution with $H(y) = 1 - e^{-1/3\,y}$, $y > 0$. By definition, we have $\widetilde{\chi}(s) = \widetilde{p}\left(s + \frac{1}{3}\right) = \frac{3s+3}{\left(s+\frac{4}{3}\right)^2\left(s+\frac{7}{3}\right)}$ and $\widetilde{\xi}(s) = \widetilde{p}(s) - \widetilde{\chi}(s)$. Inserting these into Lundberg's equation (4.13), we obtain the only two positive roots to be $\rho_1 = 0.726401$ and $\rho_2 = 1.199512$. Given the initial class of the process, the ultimate-ruin probabilities are

$$\Psi_1(u) = \Phi_1(u)\Big|_{\delta=0} = 0.702211\,e^{-0.726401u} + 0.2977893\,e^{-1.199512u},$$
$$\Psi_2(u) = \Phi_2(u)\Big|_{\delta=0} = 1.247830\,e^{-0.726401u} - 0.247830\,e^{-1.199512u}.$$

We may compare the ruin probabilities under the stationary state of the perturbed dependent dual model

$$\Psi_s(u) = w\,\Psi_1(u) + (1-w)\,\Psi_2(u) = 0.853230\,e^{-0.726401u} + 0.146770\,e^{-1.199512u},$$

to the independent case, where

$$\psi(u) = \varphi(u)\Big|_{\delta=0} = w\,\varphi_1(u) + (1-w)\,\varphi_2(u) = 0.723214\,e^{-0.965404u} + 0.276786\,e^{-0.399137u}.$$

The ruin probability results are shown in the left panel of Figure 4.1. These results indicate the presence of the dependence structure reduces the underlying risk and failing to realize the dependence will overestimate the risk.

Similar computations and comparisons may be carried out for other distributions, such as a heavy-tailed Pareto or an exponential distribution. Figure 4.1 shows the impact of the dependence structure under different gain-size distributions with the same means where all the other parameters in Example 4.1 remain unchanged. For comparison, a graph for the case $\sigma = 0$ is provided in Figure 4.2. The impact of dependence structure is greater for distributions with heavier tail. □

Example 4.2 Let $\lambda_1 = 2.5$, $\lambda_2 = 0.5$, $c_1 = c_2 = 1$ and $D = 0$. Assume that the gain sizes $\{X_i,\}_{i=1}^{\infty}$ are exponentially distributed with mean 1 and the random thresholds follow another exponential distribution with $H(y) = 1 - e^{-1/3\,y}$, $y > 0$. The net profit condition $\frac{1}{2.5}(\frac{3}{4}) + \frac{1}{0.5}(\frac{1}{4}) < 1$ is satisfied. For ruin probabilities, let $\delta = 0$ and Lundberg's equation is

$$\frac{5}{2} \cdot \frac{1}{\left(\frac{5}{2} - s\right)\left(s + \frac{4}{3}\right)} + \frac{1}{6} \cdot \frac{1}{\left(\frac{1}{2} - s\right)(s+1)\left(s + \frac{4}{3}\right)} = 1\,.$$

The only two positive roots are $\rho_1 = 0.166675$, $\rho_2 = 1.686141$. Ultimate-ruin probabilities are

$$\phi_1(u) = 0.739776\, e^{-0.166675u} + 0.260224\, e^{-1.686141u},$$

$$\phi_2(u) = 1.035710\, e^{-0.166675u} - 0.035710\, e^{-1.686141u}.$$

First moments of ruin time are

$$\mathbb{E}\,[\tau_1|\tau_1 < \infty] = \frac{(6.290285 + 3.418592\,u)\; e^{-0.166675u} + (-6.290285 + 0.350133\,u)\; e^{-1.686141u}}{0.739776\, e^{-0.166675u} + 0.260224\, e^{-1.686141u}}\,,$$

$$\mathbb{E}\,[\tau_2|\tau_2 < \infty] = \frac{(-0.837642 + 4.786135\,u)\; e^{-0.166675u} + (0.837642 - 0.048048\,u)\; e^{-1.686141u}}{1.035710\, e^{-0.166675u} - 0.035710\, e^{-1.686141u}}\,.$$

Similar computations may be carried out for D taking various values. Figure 4.3 illustrates how the perturbation increases the underlying risk by showing how the change in D affects the ruin probabilities and the expected time to ruin under this example. □

Remark 4.4 According to arguments in Albrecher et al. (2014), by interchanging the jumps and inter-event times, the event of ruin under dual model (4.2) when $c_1 = c_2 = 1$

coincides with the ruin under the dependent Sparre-Andersen risk model in Boudreault et al. (2006) with exponential claim sizes. More precisely, the Sparre-Andersen risk model may be described by

$$U(t) = u + \pi t - \sum_{j=1}^{N(t)} Z_j, \quad t \geq 0,$$

with i.i.d. inter-claim times $\{X_i\}_{i=1}^{\infty}$ and claim sizes $\{Z_1, Z_2, \dots\}$ depending on the claim elapse time.

In Example 4.2, the parameters are set to be equivalent to model A in Boudreault et al. (2006), Section 7. To obtain the ruin probabilities under model A utilizing the results from the proposed dual dependence model, we have

$$\begin{aligned}
\psi_A(u) &= \mathbb{E}\left[\phi(u + \pi X_1)\right] \\
&= \int_0^{\infty} \phi_1(u+t)\, e^{-t} e^{-1/3t} + \phi_2(u+t)\, e^{-t}\left[1 - e^{-1/3t}\right] \mathrm{d}t \\
&= 0.690457\, e^{-0.166675u} + 0.084714\, e^{-1.686141u},
\end{aligned}$$

which provides an alternative and simpler way to obtain the numerical results in Boudreault et al. (2006), page 282, which coincide to the extent of a small rounding error. □

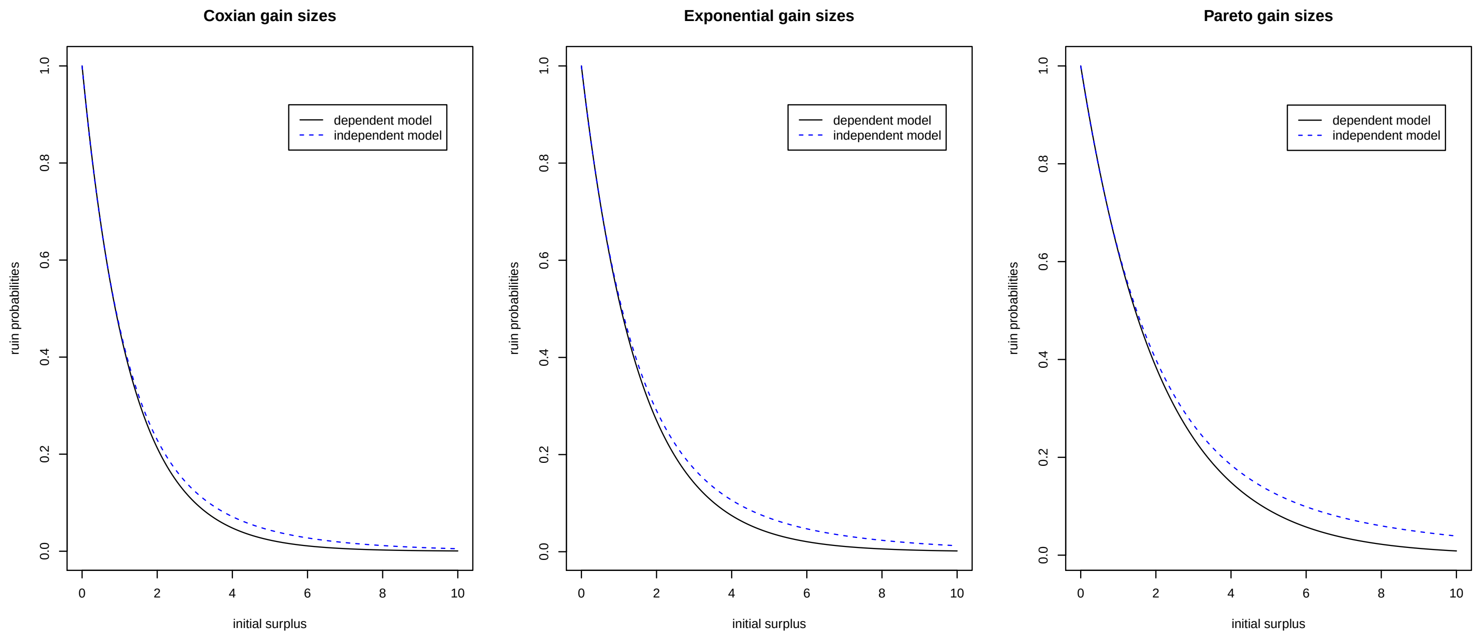

Figure 4.1: Impact of the dependence structure under different gain-size distributions ($\sigma = 1$)

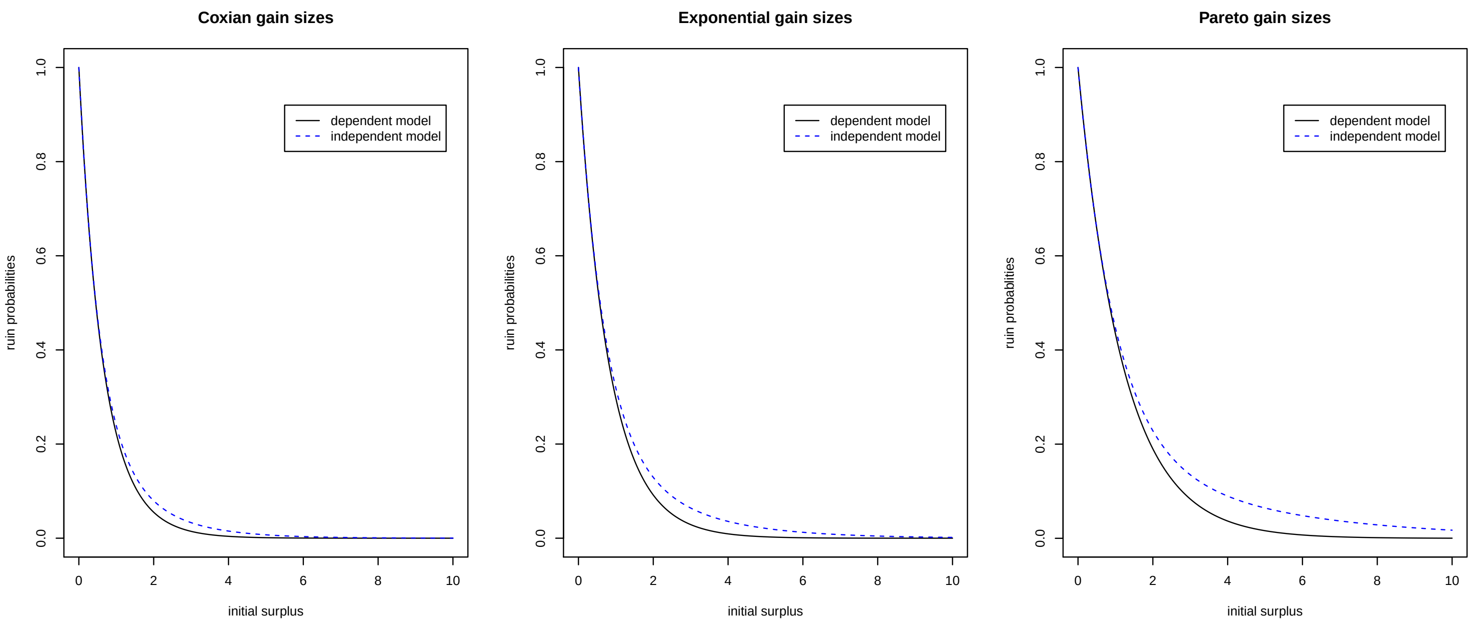

Figure 4.2: Impact of the dependence structure under different gain-size distributions ($\sigma = 0$)

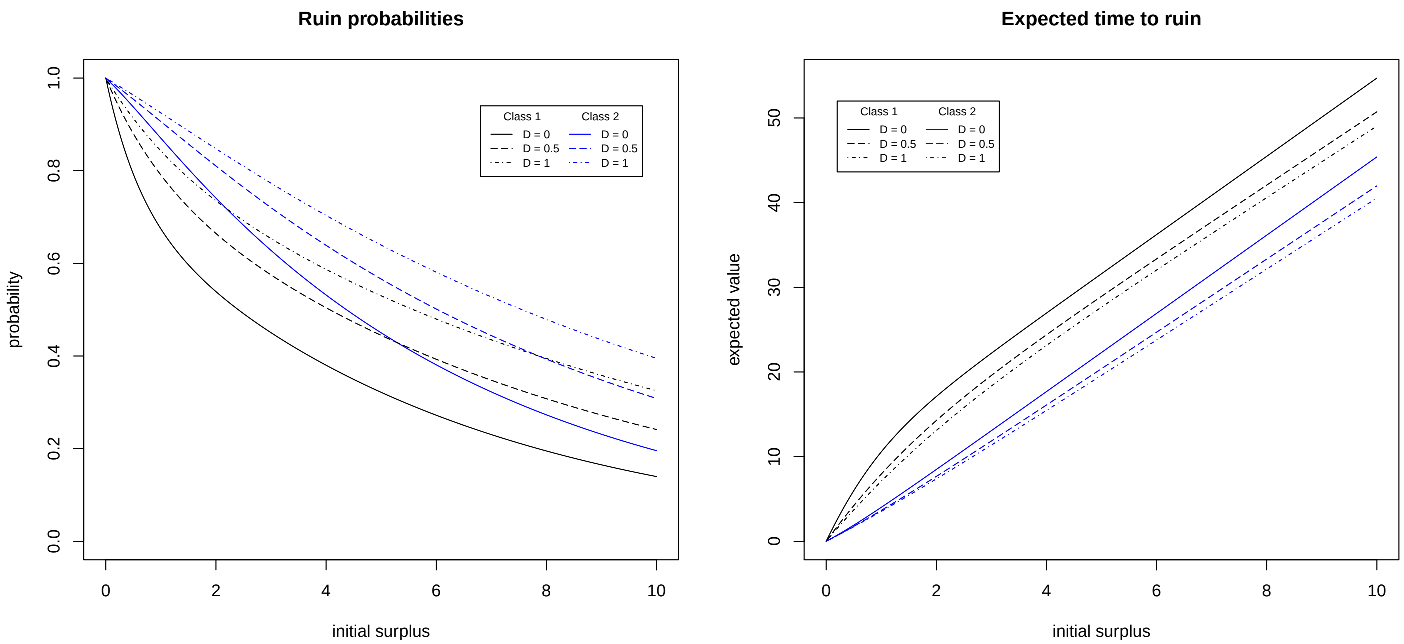

Figure 4.3: Impact of the perturbation to the dependent dual risk model ($D = \frac{\sigma^2}{2}$)

Chapter 5

Concluding remark and further research

Under the classical compound Poisson risk model and the Sparre-Andersen risk model, one crucial assumption is that the interclaim times and the claim sizes are independent. However, this assumption might be inappropriate in practice. In this thesis, we consider a dependent risk model where the assumption about independent increments, which is fundamental for the Sparre-Andersen risk model, no longer holds. In addition, we assume the underlying risk process is perturbed by a Brownian motion to account for small fluctuations, which is also a more realistic assumption. Lastly, the idea of dependence and diffusion combined is implemented to the dual risk model.

For the two insurance risk models, explicit solutions for the Gerber-Shiu function are obtained for arbitrary claim sizes along with applications. Various applications under special cases of the Gerber-Shiu function along with examples are provided. Exact solution for the Laplace transform and first moment of time to ruin are deduced under the dual risk model. Further research may include extending these risk models with dependence and diffusion to more generalized settings. For instance, increasing the risk classifier to higher dimensions or generalizing the underlying risk process to a more complex stochastic process, such as Markovian arrival process.

Bibliography

Afonso, L.B., Cardoso, R.M.R., Egídio dos Reis, A.D., 2013. Dividend problems in the dual model, *Insurance: Mathematics and Economics* 53(3), 906-918.

Ahn, S., Badescu, A.L., Ramaswami, V., 2007. Time dependent analysis of finite buffer fluid flows and risk models with a dividend barrier, *Queueing Systems* 55, 207-222.

Albrecher, H., Badescu, A., Landriault, D., 2008. On the dual risk model with tax payments. *Insurance: Mathematics and Economics* 42, 1086-1094.

Albrecher, H., Boxma, O.J., 2004. A ruin model with dependence between claim sizes and claim intervals, *Insurance: Mathematics and Economics* 35, 245-254.

Albrecher, H., Boxma, O.J., 2005. On the discounted penality in a Markov-dependent risk model, *Insurance: Mathematics and Economics* 37, 650-672.

Albrecher, H., Boxma, O.K., Ivanovs, J., 2014. On simple ruin expressions in dependent Sparre Andersen risk models, *Journal of Applied Probability* 51(1), 293-296.

Albrecher, H., Teugels, J.L., 2006. Exponential behaviour in the presence of dependece in risk theory, *Journal of Applied Probability* 43 (1), 257-273.

Ambagaspitiya, R.S., 2009. Ulimate ruin probability in the Sparre Andersen model with dependent claim sizes and claim occurrence times, *Insurance: Mathematics and Economics* 44, 464-472.

Avanzi, B., Cheung, E.C.K, Wong, B., Woo, J.-K., 2013. On a periodic dividend barrier strategy in the dual model with continuous monitioring of solvency, *Insurance: Mathematics and Economics* 52(1), 98-113.

Avanzi, B., Gerber, H.U, 2008. Optimal dividends in the dual model with diffusion, *Astin Bulletin* 38(2), 653-667.

Avanzi, B., Gerber, H.U., Shiu, E.S.W., 2007. Optimal dividends in the dual model, *Insurance: Mathematics and Economics* 41(1), 111-123.

Avanzi, B., Shen, J., Wong, B., 2011. Optimal dividends and capital injections in the dual model with diffusion, *Astin Bulletin* 41(2), 611-644.

Avanzi, B., Tu, V., Wong, B., 2014. On optimal periodic dividend strategies in the dual model with diffusion, *Insurance: Mathematics and Economics* 55, 210-224.

Bayraktar, E., Kyprianou, A.E., Yamazaki, K., 2013. On optimal dividends in the dual model, *Astin Bulletin* 43(3), 359-372.

Badescu, A.L., Breuer, L., Drekic, S., Latouche, G., Stanford, D.A., 2005. The surplus prior to ruin and the deficit at ruin for a correlated risk process, *Scandinavian Actuarial Journal* 2005, 433-445.

Badescu, A.L., Cheung, E.C.K., Landriault, D., 2009. Dependent risk models with bivariate phase-type distributions, *Journal of Applied Probability* 46, 113-131.

Badescu, A.L., Drekic, S., Landriault, D., 2007. Analysis of a threshold dividend strategy for a MAP risk model, *Scandinavian Actuarial Journal* 2007, 227-247.

Bayraktar, E., Kyprianou, A.E., Yamazaki, K., 2014. Optimal dividends in the dual model under transaction costs, *Insurance: Mathematics and Economics* 54(1), 133-143.

Boudreault, M., Cossette, E., Landriault, D., Marceau, E., 2006. On a risk model with dependence between interclaim arrivals and claim sizes, *Scandinavian Actuarial Journal* 2006(5), 265-285.

Cheung, E.C.K., Landriault, D., 2009. Perturbed MAP risk models with dividend barrier strategies, *Journal of Applied Probability* 46, 521-541.

Cheung, E.C.K., Landriault, D., Willmot, G.E., Woo, J.-K., 2010. Structural properties of Gerber-Shiu functions in dependent Sparre Andersen models, *Insurance: Mathematics and Economics* 46, 117-126.

Cohen, J.W., 1982. *The Single Server Queue* (rev. ed.). North-Holland Pub. Co., Amsterdam.

Cossette, H., Gaillardetz, P., Marceau, E., Rioux, J., 2002. On two dependent individual risk models, *Insurance: Mathematics and Economics* 30, 153-166.

Cossette, E., Marceau, E., Marri, F, 2009. On the compound Poisson risk model with dependence based on a generalized Farlie-Gumbel-Morgenstern coupla, *Insurance: Mathematics and Economics* 43(3), 444-455.

Denuit, M., Lefevre, C., Utev, S., 2002. Measuring the impact of dependence between claims occurrences, *Insurance: Mathematics and Economics* 30, 1-19.

Dickson, D.C.M., Hipp, C., 2001. On the time to ruin for Erlang(2) risk process, *Insurance: Mathematics and Economics* 29, 333-344.

Dufresne, F., Gerber, H.U., 1991. Risk theory fot the compound Poisson process that is perturbed by diffusion, *Insurance: Mathematics and Economics* 10, 51-59.

Gerber, H.U., 1970. An extension of the renewal equation and its application in the collective theory of risk. *Scandinavian Actuarial Journal* 1970, 205-210.

Gerber, H.U., Landry, B., 1998. On the discounted penalty at ruin in a jump-diffusion and the perpetual put option, *Insurance: Mathematics and Economics* 22, 263-276.

Gerber, H.U., Shiu, E.S.W., 1998. On the time value of ruin, *North American Actuarial Journal* 2, 48-78.

Gschlößl, S., Czado, C., 2007. Spatial modelling of claim frequency and claim size in non-life insurance, *Scandinavian Actuarial Journal* 2007(3), 202-225.

Khachatryan, A.Kh., Khachatryan, Kh.A., 2009. On some systems of convolution-type first-order integrodifferential equations on the semiaxis, *Ukrainian Mathematical Journal* 61(9), 1511-1528.

Klimenok, V., 2001. On the modification of Rouche's theorem for the queueing theory problems, *Queueing Systems* 38, 431-434.

Landriault, D., Sendova, K.P., 2011. A direct approach to a first-passage problem with applications in risk theory. *Stochastic Models* 27, 388-406.

Li, S., Garrido, J., 2004. On ruin for the Erlang(n) risk process, *Insurance: Mathematics and Economics* 34, 391C408.

Li, S., Garrido, J., 2005. The Gerber-Shiu function in a Sparre Andersen risk process perturbed by diffusion, *Scandinavian Actuarial Journal* 3, 161-186.

Li, S., Ren, J, 2013. The maximum severity of ruin in a perturbed risk process with Markovian arrivals, *Statistics and Probability Letters* 83, 993-998.

Li, Z., Sendova, K.P., in press. On a ruin model with both interclaim times and premiums depending on claim sizes, *Scandinavian Actuarial Journal*.

Li, B., Wu, R., Song, M., 2009. A renewal jump-diffusion process with threshold dividend strategy, *Journal of Computational and Applied Mathematics*, 228, 41-55.

Liu, X., Chen, Z., 2014. Dividende problems in the dual model with diffsion and expentially distributed observation *Statistics and Probability Letters* 87, 175-183.

Lu, Y., Tsai, C.C.-L., 2007. The expected discounted penalty at ruin for a Markov-modulated risk process perturbed by diffusion. *North American Actuarial Journal* 11(2), 136-152.

Meng, Q., Zhang, X., Guo, J., 2008. On a risk model with dependence between claim sizes and claim intervals, *Statistics and Probability Letters* 78, 1727-1734.

Mitric, I-R., Sendova, K.P., Tsai, C.C-L., 2010. On a multi-threshold compound Poisson process perturbed by diffusion, *Statistics and Probability Letters* 80, 366-375.

Müller, A., Pflug, G., 2001, Asymptotic ruin probabilities for risk processes with dependent increments, *Insurance: Mathematics and Economics* 28, 381-392.

Ng, A.C.Y., 2009. On a dual model with a dividend threshold, *Insurance: Mathematics and Economics* 44(2), 315-324.

Rodríguez, E., Cardoso, R.M.R., Egídio dos Reis, A.D., 2013. Some advances on the Erlang(n) dual risk model. *CEMAPRE preprint*.

Sarkar, K., Sen, A., 2005. Weak convergence approach to compound Poisson risk processes perturbed by diffusion, *Insurance: Mathematics and Economics* 36, 421-432.

Tijms, H.C., 1994. *Stochastic Models : An Algorithmic Approach.* John Wiley & Sons, Chichester.

Tsai, C.C.-L., Willmot, G.E., 2002. A generalized defective renewal equation for the surplus process perturbed by diffusion, *Insurance: Mathematics and Economics* 30, 51-66.

Wan, N., 2007. Dividend payments with a threshold strategy in the compound Poisson risk model perturbed by diffusion, *Insurance: Mathematics and Economics* 40, 509-523.

Yang, C., Sendova, K.P., 2014. The ruin time under the Sparre-Andersen dual model, *Insurance: Mathematics and Economics* 54, 28-40.

Zhou, M., Cai, J., 2009. A perturbed risk model with dependence between premium rates and claim sizes, *Insurance: Mathematics and Economics* 45, 382-392.

Curriculum Vitae

Name	Zhong Li
Education	Ph.D. in Statistics, Actuarial Science University of Western Ontario, London, Ontario 2011 - 2014
	M.Sc. in Statistics, Actuarial Science University of Western Ontario, London, Ontario 2010 - 2011
	B.Sc., Honors Specialization in Statistical Sciences University of Western Ontario, London, Ontario, Canada 2004 - 2010
Related Work Experience	Guest Lecturer Department of Statistical and Actuarial Sciences, UWO, London, ON 2014. 02
	Tutorial Instructor Department of Statistical and Actuarial Sciences, UWO, London, ON 2012. 09 - 2012. 12

Teaching Assistant

Department of Statistical and Actuarial Sciences, UWO, London, ON

2010. 09 - 2014. 04

Research Assistant

Department of Statistical and Actuarial Sciences, UWO, London, ON

2010. 09 - 2014. 08

Awards and Scholarships

The Faculty of Science Special Award, UWO

2011 - 2014

Western Graduate Research Scholarship, UWO

2010 - 2014

Graduate Teaching Assistantship, UWO

2010 - 2014

Publications

Li, Z. and Sendova, K.P., On a ruin model with both interclaim times and premiums depending on claim sizes. *Scandinavian Actuarial Journal*, in press. Doi10.1080/03461238.2013.811096

Li, Z. and Sendova, K.P., On a perturbed risk model with interclaim times depending on the claim sizes, submitted.

Li, Z., Sendova, K.P., Yang, C., On a perturbed dual risk model with dependence between inter-gain times and gain sizes, submitted.

Yang, C., Sendova, K.P., Li, Z., On the dual model with expense rate changing at the positive jumps, submitted.