8-7-2014 12:00 AM

Valuation and Risk Measurement of Guaranteed Annuity Options under Stochastic Environment

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Statistics and Actuarial Sciences

VALUATION AND RISK MEASUREMENT OF GUARANTEED ANNUITY OPTIONS UNDER STOCHASTIC ENVIRONMENT

(Thesis format: Integrated Article)

by

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Graduate Program in Statistics and Actuarial Science

A thesis submitted in partial fulfillment
of the requirements for the degree of
Ph.D in Statistics

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Abstract

In recent years, there has been a rapid emergence of complex insurance products such as contract with option-embedded features. This thesis develops stochastic modelling frameworks for the accurate pricing and risk management of these products. We propose stochastic models for the evolution of the two main risk factors, the interest rate and mortality rate, which could also have a correlation structure. In particular, we focus on the analysis of a guaranteed annuity option (GAO). For the valuation problem, a general framework is put forward where correlated interest and mortality rates are modelled as affine-diffusion processes. A new concept of endowment-risk-adjusted measure is introduced to facilitate the calculation of the GAO value. The performance and computational efficiency of the proposed approach are examined through numerical experiments. As a natural offshoot of addressing GAO valuation, we derive the convex-order upper and lower bounds of GAO values by employing the comonotonicity theory. The results via Monte Carlo methodology are used as benchmarks in assessing the accuracy of the comonotonic-based approximated pricing results.

As an alternative to affine structure, we construct a more flexible modelling framework that incorporate regime-switching dynamics of interest and mortality rates in which the switching is governed by a continuous-time Markov chain. Three ways to embed the regime-switching approach to mortality modelling are considered. The corresponding endowment-risk-adjusted measures are constructed and employed to obtain more efficient GAO pricing formulae. An extension of the previous modelling set-up is further developed by integrating the affine structure and regime-switching feature. Both interest and mortality risk factors follow a correlated affine structure whilst their volatilities are modulated by a Markov chain process. The change of probability measure technique is again utilised to generate pricing expressions capable of significantly cutting down simulation and computing times.

Finally, the risk management aspect of GAO is investigated by evaluating various risk measurement metrics. The bootstrap technique is used to quantify standard error for the estimates of risk measures under a stochastic modelling framework in which death is the only decrement. The moment-based density approximation methods are applied to obtain analytic approximation of the distributions of the loss random variable that provide immediate solutions to the risk measure determination. We also find the relation between the desired accuracy level of risk measures and the required sample size through regression methodology. The sensitivity analyses of risk measures with respect to key parameters are studied demonstrating the utmost importance of reliable estimation and calibration of model parameters.

Keywords: change of probability measure, endowment-risk-adjusted measure, interest rate risk, mortality risk, annuity-contingent derivative, risk measures, regime-switching, Markov chain

Co-Authorship Statement

I hereby declare that this thesis incorporates materials that are direct results of several related research collaborations. The first two research outputs with my supervisors, Dr. Rogemar Mamon and Dr. Xiaoming Liu, have already been published. The remaining research works are manuscripts under review in various peer-reviewed journals or about to be submitted for review.

The content of chapter 2 was used as basis of a full paper (co-authored with Dr. Liu and Dr. Mamon), which was published in *Stochastics: (An International Journal of Probability and Stochastic Processes).* See reference [34] of chapter 4.

Chapter 3 appeared in the *Journal of Computational and Applied Mathematics* as a published paper (co-authored with Dr. Liu and Dr. Mamon); see reference [23] in chapter 2.

A manuscript (co-authored with Dr. Mamon, Dr. Liu and Anton Tenyakov) based on the content of chapter 4 was submitted in December 2013 to the journal *Insurance: Mathematics and Economics* (IME). Referee's report was received on 03 June 2014, and the manuscript is currently being revised for resubmission and potential publication in the IME's Longevity 9 Special Issue in November 2014.

Chapter 5 was already converted to a manuscript (co-authored with Dr. Mamon and Dr. Liu) for submission to the *European Actuarial Journal*.

The materials in chapter 6 are based on a manuscript (co-authored with Dr. Rogemar Mamon and Dr. Xiaoming Liu) that is about to be submitted to the *Journal of Risk and Insurance*.

Please note that this thesis employed an integrated-article style following Western's thesis guidelines. This means that each chapter can be read independently as it does not rely on other chapters. Every chapter is therefore deemed to be self-contained and can stand on its own.

With the exception of the guidance on formulating modelling frameworks and occasional help in dealing with numerical experiments from my supervisors, I certify that this document is a product of my own work and efforts.

This work is dedicated to my parents for their greatest love.

Acknowledgements

The completion of this work would not have been possible without the support of various individuals who firmly believe in my potential and capability. They kindly offered a lending hand whenever I need help. I, therefore, would like to express my sincere gratitude to my supervisors, instructors, family, and friends who were behind me all the way to reach this level of achievement.

First and foremost, my profound gratefulness goes to my discerning supervisor, Dr. Rogemar Mamon, for his continual guidance, unflinching support and unfaltering encouragement throughout my entire graduate education. His professionalism, scientific insights and meticulous supervision both ensured the accomplishment of this research work and enhanced many transferrable skills, which are priceless for my future career and endeavours. Likewise, I am thankful to my co-supervisor, Dr. Xiaoming Liu, from whom I learnt my first enlightening experience about research. I appreciate her many constructive suggestions over the course of finishing our co-authored papers and this thesis. I am also grateful to the faculty and staff of the Department of Statistical and Actuarial Sciences for their assistance and advice.

Many thanks to my friends who never hesitated to provide their moral support and consolation whenever I was in difficulties. The joyful moments spent with them will definitely be part of a lasting happy memory, and their friendship will be cherished forever.

Last but not least, I want to convey my great thanks to my parents for their purest love. Without their unconditional support and unselfish encouragement, I could not have weathered the hard times, and pursued my dreams, passion and aspirations with utmost persistence. They set themselves as the best example for me to emulate the values of courage, generosity and responsibility. My greatest thanks go to my mother, who is my lifelong mentor. She is full of wisdom and a woman of marked tenacity. My achievements are attributed to her for being the source of my strength and power.

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Chapter 1

Introduction

1.1 Background of stochastic modelling

Stochastic processes are sequences of random variables indexed with time. They are mathematical tools to model and describe time-related random events and phenomena. Stochastic models are utilised widely in the natural sciences, engineering, business and economics; see for instance, Ross [34] and Solberg [35], amongst others. Finance and actuarial science, in particular, have highly benefitted from stochastic methods especially in the valuation of derivatives and insurance products, risk management, asset allocation and credit risk analysis.

Insurance valuation and risk management techniques have been traditionally deterministic. But with the recent changes in the investment environments and insurance markets, the important uncertainty element needs to be adequately modelled. For example, many insurance companies have introduced new products with embedded options. These products have similar characteristics to derivatives traded in the financial market. Thus, the pricing of these products requires modern option pricing theory, which inevitably involves the use of stochastic calculus. Due to today's product sophistication, their value would depend on at least two risk factors, the most important of which are the interest and mortality rates. The risk factors are deemed correlated. Dealing with correlation represents a mathematical as well as a computational hurdle. This is one reason why in previous works, financial and mortality risks are assumed independent.

Correlation between the two risk factors are nonetheless supported by economic observations. On the one hand, a mortality decline or equivalently an increase in life expectancy can put a huge stress on the country's social programs and impacts its labour market. This in turn affects local and global economy. An improvement in life expectancy impacts national savings and demand on investments, which is directly linked to rate of returns. On the other hand, it is known that interest rate levels directly affect the economy. A high interest rate, for example, implies higher interest payments for debts, which could weaken the capacity of ordinary individuals, especially those saddled with mortgages and borrowings, to afford the much needed medical care and access to advancements in medicine thereby clearly impacting their longevity. It is therefore valid to incorporate the correlation of mortality and interest rates when designing a valuation model. This thesis proposes a stochastic framework to value and manage risks involved in guaranteed annuity options (GAOs) by allowing a dependence structure between the two most relevant risk factors.

1.2 Research objectives

To rectify the deficiency of traditional methods in valuation and setting of capital reserve for option-embedded insurance products, we build various stochastic models under which results of theoretical and numerical investigations constitute the contributions of this thesis. The main research objectives are detailed as follows.

1.2.1 An affine pricing framework for GAO addressing correlated risks

Based on the work of Jalen and Mamon [24], we propose a general risk-neutral framework to price a GAO. In this framework, the two risk factors follow affine diffusion dynamics, and their dependence is explicitly modelled. A challenge that arises from this pricing framework is the *direct calculation* of the price of GAO. Since the two risks are dependent, there is no closed-form solution for the GAO value. The calculation can become complicated and time-consuming. We address this problem via the change of measure technique and introduce the concept of endowment-risk-adjusted measure under which an endowment contract acts as a numéraire. Using both the forward measure, associated with the bond price as numéraire, and the endowment-risk-adjusted measure, an implementable expression for the GAO price is derived. The development of this model setting, change of measure approach and numerical results are presented in chapter 2. More details about the thesis structure is given in section 1.4.

1.2.2 A comonotonicity-based valuation method for GAO

Following the theme and set-up of the work described in subsection 1.2.1, we offer an alternative method to Monte Carlo simulation in determining the GAO price. As this work involves comonotonicity theory, we deal with the sums of lognormal random variables. Methodologies and approaches in this area were previously developed to find the distribution of sums of lognormal random variables to varying degrees of depth and treatment depending on certain theoretical or practical considerations. Such advances are highlighted in Dufresne [14], Leipnik [27] and Wu et al. [37]. Our approach is motivated by the works of Dhaene et al. [12, 13], and Liu et al. [28] that proposed the use of comonotonic bounds in approximating the sums of lognormal random variables. Based on the above-mentioned papers, we derive the upper and lower bounds of the price of GAO in convex order together with their quantile functions. Moreover, we investigate the accuracy of our comonotonic bounds by benchmarking them to simulated results generated from the previous work in subsection 1.2.1.

1.2.3 The pricing of GAO under a Markov regime-switching framework

In addition to affine model structures, regime-switching models are another type of models that are gaining popularity in finance and actuarial science. Developments in the applications of Markov regime-switching models to finance and economics can be found in Mamon and Elliott [30], and more recently in Mamon and Elliott [31]. Milidonis et al. [32] adopted a regime-switching approach in mortality modelling and made an extension to the original Lee-Carter model. In their paper, the error term of mortality index is switching between regimes; this set-up relaxes the normality assumption of the mortality index. So far, no research work employs a regime-switching model to price annuity-linked options. We aim to construct a new framework for the pricing of GAO in which both interest and mortality rates are functions of a continuous- or discrete-time Markov chain. In this work, we continue to apply the concept of endowment-risk-adjusted measure and derive a new transition probability matrix under the new measure. This emphasises the interplay of Girsanov and Bayes theorems.

1.2.4 Valuation of GAO under affine framework with regime-switching volatilities

The term structure models of interest rates, based on diffusion processes with constant parameters, may reasonably support the pricing of financial derivatives. However, when we value annuity-linked insurance products which often have long maturities, the constant volatility assumption may not be adequate to capture changes in the economy. The same holds true for mortality rates. Therefore, we relax the use of constant volatilities in the stochastic modelling of interest and mortality rates in the affine framework described in section 1.2.1. We assume volatilities of the risk factors evolve according to the dynamics of a continuous-time Markov chain. A modelling framework mixing the affine and correlated structure with the regime-switching feature will be constructed. The valuation process will still be carried out via the change of measure technique but further new results are obtained given the extended framework.

1.2.5 Risk measurement of GAO under one-decrement actuarial model framework

To complement the pricing of insurance products, it is also important for insurers to consider appropriate financial instruments in hedging risks to meet solvency requirements and maintain capital adequacy. Approaches in risk management and hedging of products with option-embedded features were studied by several authors (e.g., Hardy [22]). But in these previous works, the insurance risks are not all modelled stochastically. We aim to contribute to the research ideas under the paradigm of stochastic and dependent risk factors. The distribution of liabilities can be obtained by applying the moment-based density approximation approach. Different kinds of risk measurement such as the quantile measure and conditional tail expectation risk measure can be calculated from both the Monte Carlo methodology and the analytic approximation of the loss distribution.

1.3 Review of modelling financial and mortality risks

The purpose of this section is to survey briefly the stochastic modelling of both the interest and mortality rates with a view of employing them to price insurance contracts. We only mention papers that are pertinent to the goals of this thesis as outlined above.

1.3.1 Stochastic modelling of interest rates

The first account of using stochastic processes to model the movement of financial variables can be traced back to the work of Bachelier [1] who used Brownian motion to study the stock and option markets. Since then, stochastic methods were applied to various financial modelling endeavors and areas of finance. The field of interest rate theory hugely benefited from the advances made in option pricing theory. Various approaches have been proposed for the modelling of the term structure of interest rates in discrete and continuous time inspired by the pioneering work of Black and Scholes [5] in stock option valuation. The first interest rate model that has considerable impact and continues to permeate financial modelling is the Ornstein-Uhlenbeck-based model for the short rate r_t that was put forward by Vasiček [36]. Under any short-rate model, the goal is to calculate the zero-coupon bond price based on the stochastic differential equation that specifies the r_t process. The calculation is performed either by direct evaluation of the conditional expectation under a risk-neutral measure of a discounted pay-off or by solving a partial differential equation satisfied by the bond price. Cox et al. [11] also proposed an interest rate model based on a Bessel process that ensures positive rates.

As an alternative to the short-rate approach, Heath, Jarrow and Morton (HJM) [23] proposed to model the entire yield curve directly by specifying the dynamics of the forward-rate process. Based on the no-arbitrage conditions of the HJM approach, the implementation only requires the specification of the volatility function.

Since then various diffusion-based models for r_t as well as for the forward rate process were developed; see Brigo and Mercurio [7], James and Webber [25], amongst others. As reviewed below, starting in the 1990s, regime-switching term structure models began appearing and have then enriched the short-rate and HJM-type models by the inclusion of hidden Markov chains driving the evolution of parameters.

Elliott & Mamon [16] explored a Markov interest rate model giving a complete characterisation of the entire term structure under the assumption that the short rate is a function of a continuous-time Markov chain. In their paper, they proved the well-known Unbiased Expectation Hypothesis in economics holds but demonstrated that such relation between the short rate and forward rate holds, provided the expectation must be taken under a forward measure. Their result was employed to obtain an explicit stochastic dynamics for the forward rate. The analytical form of the bond price under the HJM pricing approach was presented as well. Pioneering works to model economic variables as a regime-switching process can be found in Hamilton [21]. Contributions from many authors to short-rate modelling then ensued. In these contributions, the parameters are ensured to change over time and controlled by a Markov state variable. Bekaert et al. [4], Evans and Lewis [19], Garcia and Perron [20] conducted empirical studies to test the validity of regime switches in interest rates.

Elliott and Mamon [15] provided an extended model combining diffusion- and Markov-based models. A two-factor Vasiček model was developed where the mean-reversion level changes according to the evolution of a continuous-time finite-state Markov chain. In their work, the analytical expression for the zero-coupon bond price was presented in terms of a fundamental matrix solution of a linear matrix differential equation. The validity of their bond pricing result was verified by checking consistency among the short-rate, forward-rate and yield rate processes. Some extensions of regime-switching interest rate modelling include the papers of Elliott and his research collaborators [17, 18].

1.3.2 Stochastic modelling of mortality rates

Pricing calculations in life insurance contracts and pension plans make use of a mortality assumption, commonly described as the annual probabilities of death rate or the force of mortality. In a traditional framework, these quantities are obtained using observed data. Given the future uncertainty on mortality levels due to medical breakthroughs and discoveries in pharmacology as well as the creation of insurance products with derivative features, researchers began giving attention to stochastic mortality models that are also compatible and consistent with stochastic models used in the pricing of financial products.

The beginning of stochastic modelling of mortality can be attributed to Lee and Carter (LC) [26] who proposed to model central death rates representing age-specific mortality. An essential ingredient of their method is a univariate mortality index that describes the variation of mortality patterns over time. Due to the absence of observable variable in their model, making traditional regression method invalid, singular-value decomposition was employed in their parameter estimation. A one-parameter life table was constructed and fitted to US mortality patterns. Forecasts of rates and life expectancy were obtained under the assumption that future trends would continue in the same way. The key feature of the LC methodology is that it gives allowance to uncertainty in mortality rate by modelling it as a stochastic process. Subsequent research works on stochastic mortality were built upon LC's methodology. Brouhns et al. [8] made some improvements to the LC model by using a Poisson random variation for the number of deaths instead of the additive error term in the original model. This is deemed more reasonable since the mortality rate is much more variable at older ages than at younger ages.

Another popular stochastic model describing mortality evolution was proposed by Cairns, Blake and Dowd [9], which exploited the relative simplicity of mortality curve at higher ages; the model is not designed, however, for lower ages. It has attracted much attention in pension plan valuation in the UK given its good performance in capturing mortality dynamics of older age groups. In contrast with the LC model, it incorporates two factors in describing mortality evolution. The first factor affects mortality rate dynamics at all ages in the same manner whilst the other one affects mortality rate dynamics at higher ages much more than at lower ages. Empirical evidence supports that both factors are necessary to achieve a satisfactory fit over the entire mortality term structure. One important advantage of their method is that it allows different improvements at different ages and at different times which can not be achieved in LC model.

In 2005, Luciano and Vigna [29] adopted affine processes to describe the evolution of mortality rate and provided detailed calibration using UK data. In their paper, they suggested that a non-mean-reverting process is more suitable to model mortality rate than a mean-reverting one. On the other hand, the addition of negative jumps into the diffusion process performed better to fit the real mortality data and forecast the mortality trend.

Research advances in modern finance have stimulated research developments in the field of insurance. Milidonis et al. [32] brought the concept of regime-switching approach into mortality rate modelling. In their paper, the advantages of applying regime-switching models into mortality rate modelling were highlighted. Through the investigation of the US population mortality index, they illustrated that there were structural changes in the underlying death probability for all age cohorts from all death causes, which provided evidence in the adoption of regime-switching models. Moreover, they applied the concept of regime-switching to model the error term of mortality index in the LC model. This captures the disturbances introduced by extreme observations over time and makes the error term non-normal.

1.3.3 Stochastic modelling in actuarial valuation

The booming market of sophisticated insurance products with benefits linked to financial variables along with various guarantees has provided impetus to the active use of stochastic modelling of both interest and mortality rates in the valuation of annuity-related products. Boyle and Hardy [6] discussed three major risks involved in GAOs. In their paper, interest rate risk and equity risk followed correlated affine processes whilst mortality rate remained deterministic and independent with the other two risks. They investigated to price GAOs via the change of measure technique assuming a swaption hedge. Ballotta and Haberman [3] examined the valuation of annuity-contingent options and extended the research results in Ballotta and Haberman [2], which assumed unsystematic mortality risk; they introduced an integrated framework to value GAO using option pricing methodology of modern finance. In their frameworks, a stochastic model for the evolution of mortality rate was considered whilst the term structure of interest rate evolves according to a single-factor HJM model. A fair value for GAO was derived through the change of measure technique. To make the estimation of the value of GAO implementable, Monte-Carlo simulation technique was applied. Moreover, the sensitivity of GAO prices with respect to key parameters was investigated. However, whilst the two types of risks are stochastic in their valuation, they are still assumed independent.

In Chu and Kwok [10], three analytical approximation methods were proposed for GAO pricing, namely, the stochastic duration approach, Edgeworth expansion and multi-factor affine interest rate model setting. The stochastic duration approach is based on the minimisation of the price error whilst the Edgeworth expansion method approximates the probability distribution of the annuity value at maturity of the contract. For the affine approximation, the concave exercise boundary is approximated by a hyperplane in order to obtain the exercise probability of the annuity option. The three analytic approximation methods were compared in terms of both numerical accuracy and computational efficiency and a sensitivity of GAO prices was performed.

Jalen and Mamon [24] proposed an integrated framework of stochastic mortality and interest rates to price insurance claims. They relaxed the independence assumption of the two risk factors. In their framework, the mortality rate was modelled as an affine-type diffusion process just like the short rate process. Through the change of measure technique, analytical expressions in mortality-linked insurance products were presented. Their approach provided new perspectives and methodology in the valuation of other insurance products under a more reasonable assumption that risk factors are dependent.

The paper of Liu et al. [28] illustrated the evaluation of the annuity rate defined as the conditional expected present value random variable of the annuity's future payments. The two risk factors were modelled as stochastic processes; the mortality rate followed the LC model and the short rate followed the Vasiček model. They applied the concept and properties of comonotonicity in the derivation of the convex-order lower and upper order of the annuity rate. The accuracy of the bounds was validated through numerical analysis. This approach has the advantage of mathematical tractability in computing the distribution function for the sum of comonotonic random variables, which could be adopted in the further calculation of other annuity-linked products.

The valuation of a related product, called guaranteed lifelong withdrawal benefit options with variable annuity, was described by Piscopo and Haberman [33]. In their paper, the equity risk followed the geometric Brownian motion whilst the mortality rate was based on the standard mortality tables with allowance for the possible perturbations having a regime-switching feature. They provided the fair value through Monte Carlo simulations under different scenarios and conducted sensitivity analysis to show the relation between the variation of parameters and the value of the product.

1.4 Structure of the thesis

This thesis consists of 7 chapters. The succeeding chapters are the compilation of related research papers (2 published, 1 under review and 2 for submission) on the valuation and risk measurement of GAOs with the stochastic modelling of risk factors.

In chapter 2 we propose a generalised pricing framework in which the dependence between the two risks can be explicitly modelled. We also utilise the change of measure technique to simplify the valuation expressions. We illustrate our methodology in the valuation of a GAO. Using both forward measure associated with the bond price as numéraire and the newly introduced concept of endowment-risk-adjusted measure, we derive a simplified formula for the GAO price under the generalised framework. Numerical results show that the methodology proposed in this work is highly efficient and accurate.

Chapter 3 presents an alternative way to value GAO under the model framework in chapter 2. Comonotonicity theory is applied to derive upper and lower bounds for the annuity rate in the convex order sense. These bounds provide accurate approximations for the value of GAOs. Numerical illustrations are included to show the accuracy and practical applicability of our comonotonic approximations for the GAO values benchmarked by simulated results in chapter 2.

In chapter 4, we consider three ways of developing a regime-switching approach in modelling the evolution of mortality rates for the purpose of pricing a GAO. This involves the extension of the Gompertz and non-mean reverting models as well as the adoption of a pure Markov model for the force of mortality. A continuous-time finite-state Markov chain is employed to describe the evolution of mortality model parameters which are then estimated using the filtered-based and least-squares methods. The adequacy of the regime-switching Gompertz model for the US mortality data is demonstrated via the goodness-of-fit metrics and likelihood-based selection criteria. A GAO is valued assuming the interest and mortality risk factors are switching regimes in accordance with the dynamics of two independent Markov chains. To obtain closed-form valuation formulae, we employ the change of measure technique with the pure endowment price as the numéraire. Numerical implementations are included to compare the results of the proposed approaches and those from the Monte Carlo simulations.

An extended modelling framework building from that in chapter 2 is proposed in chapter 5. The volatilities of the interest rate and mortality rate are regime-switching driven by a finite-state continuous time Markov chain. We derive the explicit solution to the endowment price which involves solving the linear system of ordinary differential equations by employing the forward measure. Utilising the endowment-risk-adjusted measure with endowment as the numéraire, we provide an efficient formula for GAO price as supported by numerical experiments producing results that have smaller errors and with less computing time.

Chapter 6 addresses the problem of setting capital reserves for a GAO. A modelling framework for the gross loss random variable is developed. We consider a one-decrement actuarial model for the gross loss in which death is the only decrement, and the financial and mortality risk factors follow correlated affine structures. Risk measures are determined using the moment-based density method and benchmarked with the Monte-Carlo simulation method. A bootstrap technique is utilised to assess the variability of risk measure estimates. We establish the relation between the level of desired risk measure accuracy and required sample size under the constraints of computing time and memory. A test of GAO sensitivity to model parameters demonstrates the need for accurate model estimation and calibration. Our numerical investigations should prove useful to insurers in adhering to certain regulatory requirements.

Lastly, we summarise the main findings and contributions of the thesis in chapter 7. The modelling setups we introduce will provide researchers and practitioners alike more flexible model choices in their quest of capturing the dynamics of real data relevant to actuarial valuation and risk management. Moreover, possible works motivated by this research in the areas of calibration, hedging of a GAO, and further improvement in the valuation process through Monte Carlo methodology are briefly mentioned in the last chapter.

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Chapter 2

A generalised pricing framework addressing correlated mortality and interest risks

2.1 Introduction

It is a well-accepted fact that annuity products are notably influenced by both interest and mortality risks. However, the methodology for dealing with these two risks is fundamentally oversimplified under the traditional actuarial approach. Mortality risk is usually regarded as secondary in importance compared to the volatile nature of interest risk. In addition, mortality risk is deemed diversifiable if the insurer holds a sufficiently large portfolio of similar contracts. As a result, mortality is traditionally modelled deterministically, whilst interest risk is modelled stochastically. Modern finance theory is then adopted for pricing and risk analysing annuity-related insurance products; see for example, Ballotta and Haberman [1], Boyle and Hardy [7], Lin and Tan [20], amongst others. Apparently, the deterministic modelling of mortality rates has the advantage that it makes the valuation problem more manageable since this implies mortality risk is independent from interest rate risk. Nonetheless, such framework assuming independence between the primary risk factors is too simplistic.

The perspective of deterministic mortality has been challenged in the last few years. It has been observed that recent mortality trends show some unprecedented improvement along with a great deal of uncertainty. The insurance industry as well as pension fund companies are thus exposed to substantial systematic mortality risk. Insurers underestimated the significance of mortality risk which led to emerging insolvency issues for many insurance companies that sold guaranteed annuity options (GAOs) between the late 70s and 80s. It caused the closure to new business in 2000 of Equitable Life, one of oldest life insurance companies in the UK. This insurance mishap has stimulated discussions on mortality/longevity risk and has since called for stochastic approach for mortality modelling; see Pitacco [30, 31] and the references therein. Biffis [3] explored the parallelism between interest and mortality rates and proposed affine-type stochastic models for mortality dynamics. In Luciano and Vigna [25], an empirical study found that non-mean reverting OU process fits the historical data better than the mean-reverting process.

This work contributes further to the methodology of affine mortality modelling. We put forward a generalised risk-neutral framework in which both mortality and interest risks follow affine dynamics and dependence between two risks is explicitly modelled. This chapter extends the paper of Jalen and Mamon [16] where only a specific form of dependence between mortality and interest is considered. Arguably, mortality risk can affect the economy which in turn affect interest rate movement. Therefore, it is desirable to have a mathematical framework that allows a dependence structure between these two risks. We apply this modelling framework to price a GAO, which is one of the most common and important life insurance products that have been trading since the 70s. Its significance stems from its ubiquity as part of the suite of products offered by insurance companies, pension funds and financial institutions to their clients. Many modern insurance products now have option-embedded features such as equity-linked annuities, variable annuities, equity-indexed annuities in addition to GAO. Thus, the pricing framework for these instruments is always of considerable interest to both researchers and practitioners; see for example, Cairns et al. [8], Cox et al. [10], Dahl and Moller [12], Kogure and Kurachi [17], Lin and Cox [19], Lin et al. [21], Wills and Sherris [33], and Yang et al. [35], amongst others.

Nonetheless, the majority of the aforementioned papers concerning this problem do not properly treat the correlation between interest and mortality rates; in particular, only one factor is considered stochastic and the other remaining factor is assumed deterministic. This work presents a generalised set-up and approach under which the pricing solution is obtained with great ease despite dependence between two stochastic factors. We demonstrate how to use the change of measure technique in the pricing of an annuity-linked option. More specifically, a new measure associated with the pure endowment as numéraire is constructed to solve the GAO pricing problem. We note that Dahl et al. [11] also utilised the change of measure technique to facilitate the pricing of survivor swaps. Nevertheless, whilst the likelihood process is constructed, the associated numéraire with the measure change is not categorically identified. In this work, we lay down the groundwork to get a simplified expression for the conditional expectation under the risk-neutral measure. By popularising this technique, which is not commonly used in actuarial science and insurance, it is hoped that its power and utility can be fully explored for other contingent claim valuation problems.

The formulation of the pricing framework is presented in section 2.2; in particular, this section outlines the assumptions for the models of interest and mortality rate processes and their dependence. An integrated set-up is also developed. In section 2.3, we describe the change of probability measure approach to aid the evaluation of conditional expectation necessary to determine the value of a GAO. The forward measure is revisited and the pure endowment-risk-adjusted measure is defined. Section 2.4 presents a numerical example illustrating the applicability and benefits of our proposed approach. Finally, in section 2.5, we provide some concluding remarks and further directions.

2.2 Valuation framework

The pricing of insurance and annuity products entails the inclusion of at least two types of uncertainty, which are the financial factors linked to interest rate and the random residual lifetime of insureds or annuitants linked to mortality or survival rates. To provide adequate and sound support to insurance and pension business, a coherent and integrated modelling framework is necessary. We present in this section a brief theoretical background and considerations on risk-neutral modelling of risk factors. A comprehensive discussion can be found in Biffis [3] and Cairns et al. [8]. We give general descriptions for each interest rate process and mortality rate process, and then form a combined modelling set-up. An important aspect of any valuation model or modelling approach is the balance between complexity and computational tractability of both pricing and parameter estimation. In the last subsection, we assume that both interest rate and mortality rate dynamics are modulated by affine processes. This implies that we are able to exploit the analytical tractability of these processes within the context of reflecting both factors into the valuation approach.

2.2.1 Interest rate model

The modern approach to valuation of bonds and interest rate derivatives employs martingale theory to obtain no-arbitrage price and hedging strategies. To value a contingent claim, we need the risk-free cash account B_t which is governed by the differential equation

$$dB_t = r_t B_t dt, \quad \text{or equivalently} \quad B_t = B_0 e^{\int_0^t r_u du}.$$

The process r_t is called the continuously compounded rate of interest for a riskless investment. We assume that a risk-neutral measure or the so-called martingale measure, Q, exists. Under Q, the discounted price of a risky asset is a martingale using B_t^{-1} as the discount factor or B_t as the numéraire. Thus, the bond price at time t for a zero-coupon bond paying \$1 at maturity $T > t$ is given by

$$B(t,T) = \mathbf{E}^Q \left[e^{-\int_t^T r_u du} \middle| \mathcal{R}_t \right],$$

where $\mathcal{R}_t$ is the filtration generated by the r_t process.

2.2.2 Mortality model

Let $\tau(t, x)$ be the future lifetime random variable attained at time t for an individual aged x at the initial time 0. Assume $\mu(t, x + t)$ is the force of mortality of the same individual at time t

with age $x + t$. Write

$$S(t, x) := e^{-\int_0^t \mu(s,\, x+s)\, ds}. \tag{2.1}$$

Under the assumption of deterministic mortality, $S(t, x)$ is the survival probability for a person currently aged x surviving for the next t years. Under the stochastic approach, however, this survival probability $S(t, x)$ itself becomes a random variable, and its value can only be observed at time t rather than at time 0. For the purpose of pricing, we need to calculate the expected value of the random variable $S(t, x)$. Thus,

$$\begin{aligned} P(0, t, x) : &= \mathbf{E}[\mathbf{I}_{\{\tau \geq t\}} | \mathcal{M}_0] = \mathbf{E}\left[S(t, x) \middle| \mathcal{M}_0 \right] \\ &= \mathbf{E}\left[e^{-\int_0^t \mu(s,\, x+s) ds} \middle| \mathcal{M}_0 \right], \end{aligned} \tag{2.2}$$

where $\mathbf{I}$ is an indicator function. Furthermore,

$$\begin{aligned} \mathbf{P}(t, T, x) : &= \mathbf{E}[\mathbf{I}_{\{\tau \geq T\}} | \mathcal{M}_t] = \mathbf{I}_{\{\tau \geq t\}}\, \mathbf{E}\left[\frac{S(T, x)}{S(t, x)} \middle| \mathcal{M}_t \right] \\ &= \mathbf{I}_{\{\tau \geq t\}}\, \mathbf{E}\left[e^{-\int_t^T \mu(s,\, x+s) ds} \middle| \mathcal{M}_t \right] \qquad (2.3) \\ &= \mathbf{I}_{\{\tau \geq t\}}\, P(t, T, x). \qquad (2.4) \end{aligned}$$

The distinction between $P(0, t, x)$ and $\mathbf{P}(t, T, x)$ given in equations (2.2) and (2.4) must be noted. Throughout the entire chapter, we employ the bold font to refer to the function that is conditional on survival up to time t, otherwise the regular font is used. We call $P(t, T, x) := \mathbf{E}\left[e^{-\int_t^T \mu(s,\, x+s) ds} \middle| \mathcal{M}_t \right]$ the *survival function* under the associated measure where the expectation is taken. When it is calculated under the real measure, $P(t, T, x)$ can be interpreted as the *central predicted survival function*. When it is calculated under the risk-neutral measure, $P(t, T, x)$ can be interpreted as *risk-adjusted survival function* to account for the adverse selection or to reflect the risk-premium adjustment on behalf of the insurance companies. In the succeeding discussion, we shall omit the reference to age x in the survival function and simply write it as $P(t, T)$.

2.2.3 Integrated model framework

We define the filtration $\mathcal{F}_t$ as $\mathcal{F}_t := \mathcal{R}_t \vee \mathcal{M}_t = \sigma(\mathcal{R}_t \cup \mathcal{M}_t)$, which refers to the joint filtration generated by both the r_t and μ_t processes. Under this generalised framework, we can value the survival benefit using no-arbitrage theory, with both interest and mortality rate being stochastic. For instance, let $M(t,T;x) = M(t,T)$ be the fair value of a survival benefit of \$1 payable at time T for a life aged x at time $t < T$. From the risk-neutral pricing principle, we have the survival benefit value given by

$$\mathbf{M}(t,T) = \mathbf{E}^Q\left[e^{-\int_t^T r_u du} \cdot \mathbf{I}_{\{\tau \geq T\}} \middle| \mathcal{F}_t\right] = \mathbf{I}_{\{\tau \geq t\}} \cdot \mathbf{E}^Q\left[e^{-\int_t^T r_u du} e^{-\int_t^T \mu_v dv} \middle| \mathcal{F}_t\right]. \tag{2.5}$$

For a general payoff function C_T conditional on the survival at time T, the value of C_t can be obtained as follows:

$$\mathbf{C}_t = \mathbf{E}^Q\left[e^{-\int_t^T r_u du} \cdot \mathbf{I}_{\{\tau \geq T\}} \cdot C_T \middle| \mathcal{F}_t\right] = \mathbf{I}_{\{\tau \geq t\}} \cdot \mathbf{E}^Q\left[e^{-\int_t^T r_u du} e^{-\int_t^T \mu_v dv} \cdot C_T \middle| \mathcal{F}_t\right]. \tag{2.6}$$

Remark: *The use of bold M and C in equations* (2.5) *and* (2.6)*, respectively, emphasises the conditioning on survival through the indicator function. In particular,*

$$\mathbf{M}(t,T) = I_{\{\tau \geq t\}} M(t,T), \text{ and } \mathbf{C}_t = I_{\{\tau \geq t\}} C_t .$$

2.2.4 Affine dynamics for mortality and interest risks

We assume that under a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$, where Q is a risk-neutral measure, the respective dynamics of the interest rate process r_t and force of mortality μ_t for an insured aged x at time 0 are given by

$$dr_t = a(b - r_t)dt + \sigma dW_t^1 \tag{2.7}$$

and

$$d\mu_t = c\mu_t dt + \xi dZ_t, \tag{2.8}$$

where $a,\ b,\ c,\ \sigma$ and ξ are positive constants, and $Z_t = \rho W_t^1 + \sqrt{1-\rho^2}W_t^2$. Here, W_t^1 and W_t^2 are independent standard Brownian motions. This means that Z_t in equation (2.8) is also a Brownian motion correlated with W_t^1. Both the initial values r_0 and μ_0 are assumed to be known at time 0.

The models specified in equations (2.7) and (2.8) indicate that the interest rate follows the well-known Vasiček model whilst the mortality rate process has the non-mean reverting Ornstein-Uhlenbeck (OU) specification proposed in Luciano and Vigna [25]. Such models have the drawback that theoretically it is possible to generate negative interest or mortality rates. The issue of negative interest rates has been widely discussed in the literature; this problem can be mitigated by appropriately choosing model parameter values or using the extended Vasiček model, i.e. the Hull and White model [cf. page 45 of Pelsser [29]]. The use of mortality model in (2.8) was justified by Luciano and Vigna [26] [cf. page 8], showing that the probability of negative mortality rates is negligible with the calibrated parameters. These particular interest and mortality rate models are employed in this work due to their tractability. They clearly facilitate the application of the change-of-measure approach in the evaluation of the risk-neutral conditional expectation for purpose of valuation, similar to the canonical example of Black-Scholes model in option pricing. Analytic expressions for the dynamics of the two underlying risk factors under the new measure can be derived under this modelling set-up, leading to a more implementable formula for the valuation of guaranteed annuity options (GAOs), as demonstrated in section 2.3.

The price $B(t,T)$ of a $T-$maturity zero-coupon bond at time $t < T$ is known to be given by

$$B(t,T) = \mathbf{E}^{Q}\left[e^{-\int_t^T r_u du} \middle| \mathcal{F}_t \right] = e^{-A(t,T)r_t + D(t,T)}, \tag{2.9}$$

where

$$A(t,T) = \frac{1 - e^{-a(T-t)}}{a} \quad \text{and} \tag{2.10}$$

$$D(t,T) = \left(b - \frac{\sigma^2}{2a^2} \right) [A(t,T) - (T-t)] - \frac{\sigma^2 A(t,T)^2}{4a}. \tag{2.11}$$

See Björk [5] or Mamon [27] for details of the result in (2.9).

2.3 The price calculation

2.3.1 The forward measure

The survival benefit in equation (2.5) can be expressed as the product of two expectations although one of the expectations is not necessarily under the same measure. In subsection 2.2.1, it was indicated that the risk-neutral measure Q is associated with the cash or money market account B_t as the numéraire. Now, we could also choose the bond price $B(t,T)$ as a numéraire. Associated with $B(t,T)$, we define the forward measure $\widetilde{Q}$ equivalent to the risk-neutral measure Q via the Radon-Nikodŷm derivative Λ_T by setting

$$\left.\frac{d\widetilde{Q}}{dQ}\right|_{\mathcal{F}_T} = \Lambda_T := \frac{e^{-\int_0^T r_u du} B(T,T)}{B(0,T)}. \tag{2.12}$$

Note that $B(T,T) = 1$ in equation (2.12). Under measure Q, Λ_T is a martingale, and for $t \le T$,

$$\Lambda_t = \mathbf{E}^Q[\Lambda_T|\mathcal{F}_t] = \frac{e^{-\int_0^t r_u du} B(t,T)}{B(0,T)}.$$

From Bayes' rule, we know that for any $\mathcal{F}_t$−measurable random variable H,

$$\mathbf{E}^{\widetilde{Q}}[H|\mathcal{F}_t] = \frac{\mathbf{E}^Q[\Lambda_T H|\mathcal{F}_t]}{\mathbf{E}^Q[\Lambda_T|\mathcal{F}_t]}, \tag{2.13}$$

which implies that

$$\mathbf{E}^{\widetilde{Q}}[H|\mathcal{F}_t] = \frac{\mathbf{E}^Q\left[e^{-\int_t^T r_u du} H \middle| \mathcal{F}_t\right]}{B(t,T)}.$$

Or equivalently,

$$\mathbf{E}^Q\left[e^{-\int_t^T r_u du} H \middle| \mathcal{F}_t\right] = B(t,T)\mathbf{E}^{\widetilde{Q}}[H|\mathcal{F}_t]. \tag{2.14}$$

Thus, equation (2.5) can be expressed as

$$\begin{aligned} \mathbf{M}(t,T) &= B(t,T)\mathbf{E}^{\widetilde{Q}}\left[\mathbf{I}_{\{\tau \ge T\}} \middle| \mathcal{F}_t\right] && (2.15)\\ &= \mathbf{I}_{\{\tau \ge t\}} B(t,T)\mathbf{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_v dv} \middle| \mathcal{F}_t\right]. && (2.16) \end{aligned}$$

We note that the term $\mathbf{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_v dv} \middle| \mathcal{F}_t\right] := P(t,T)$ in equation (2.16) is the survival function under $\widetilde{Q}$. Therefore, if we have the dynamics of μ_t under $\widetilde{Q}$ then the explicit solution for $P(t,T)$ follows.

Following the result given and established in Appendix of Mamon [27], we have

$$d\widetilde{W}_t^1 = dW_t^1 + A(t,T)\sigma dt \text{ and } d\widetilde{W}_t^2 = dW_t^2,$$

where $\widetilde{W}_t^1$ and $\widetilde{W}_t^2$ are standard Brownian motions under $\widetilde{Q}$, and the function $A(t,T)$ is specified in equation (2.10). Hence, the respective dynamics under $\widetilde{Q}$ of r_t and μ_t are given by the stochastic differential equations (SDEs)

$$dr_t = \left[ab - \sigma^2 A(t,T) - ar_t\right] dt + \sigma d\widetilde{W}_t^1 \tag{2.17}$$

and

$$\begin{aligned} d\mu_t &= c\mu_t dt + \rho\xi dW_t^1 + \sqrt{1-\rho^2}\,\xi dW_t^2 \\ &= (-\rho\sigma\xi A(t,T) + c\mu_t)dt + \rho\xi d\widetilde{W}_t^1 + \sqrt{1-\rho^2}\,\xi d\widetilde{W}_t^2 \\ &= (-\rho\sigma\xi A(t,T) + c\mu_t)dt + \xi d\widetilde{Z}_t, \end{aligned} \tag{2.18}$$

where $\widetilde{Z}_t = \rho\widetilde{W}_t^1 + \sqrt{1-\rho^2}\,\widetilde{W}_t^2$. From equation (2.18), we see that μ_t has an affine form with time-dependent drift. Note that, if there is no correlation between the processes r_t and μ_t, i.e. $\rho = 0$, the dynamics of μ_t does not change under the forward measure $\widetilde{Q}$. Formula (2.16) then reduces to the case when r_t and μ_t are independent.

Write $\alpha(t) := -\rho\sigma\xi A(t,T)$ and $b(t) := \int_0^t (-c)du = -ct$. Then

$$\mu_t = e^{-b(t)}\left(\mu_0 + \int_0^t e^{b(v)}\alpha(v)dv + e^{b(v)}\xi d\widetilde{Z}_v\right).$$

By letting

$$\gamma(t) = \int_t^T e^{-b(v)}dv = \frac{e^{cT} - e^{ct}}{c} = \frac{e^{ct}}{c}(e^{c(T-t)} - 1)$$

and employing the result in pp. 267-268 of Elliott and Kopp [15], we have

$$\begin{aligned} P(t,T) &= \mathbf{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_v dv}\middle|\mathcal{F}_t\right] \\ &= \mathbf{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_v dv}\middle|\mu_t\right] \quad \text{by the Markov property} \\ &= e^{-\mu_t\widetilde{G}(t,T)+\widetilde{H}(t,T)}, \end{aligned} \tag{2.19}$$

where

$$\widetilde{G}(t,T) = e^{b(t)}\int_t^T e^{-b(u)du} = e^{b(t)}\gamma(t) = \frac{(e^{c(T-t)} - 1)}{c} \tag{2.20}$$

and

$$\begin{aligned} \widetilde{H}(t,T) &= -\int_t^T \left(e^{b(u)}\alpha(u)\gamma(u) - \frac{1}{2}e^{2b(u)}\xi^2(u)\gamma^2(u)\right)du \\ &= \left(\frac{\rho\sigma\xi}{ac} - \frac{\xi^2}{2c^2}\right)[\widetilde{G}(t,T) - (T-t)] + \frac{\rho\sigma\xi}{ac}[A(t,T) - \phi(t,T)] + \frac{\xi^2}{4c}\widetilde{G}(t,T)^2 \end{aligned} \tag{2.21}$$

with $\phi(t,T) = \frac{1-e^{-(a-c)(T-t)}}{a-c}$.

Combining equations (2.9)–(2.11), (2.16), and (2.19)–(2.21), we have

$$\begin{aligned} M(t,T) &= e^{-(A(t,T)r_t+\widetilde{G}(t,T)\mu_t)+D(t,T)+\widetilde{H}(t,T)} \\ &= \beta(t,T)e^{-V(t,T)}, \end{aligned} \tag{2.22}$$

where $\beta(t,T) = e^{D(t,T)+\widetilde{H}(t,T)}$ and $V(t,T) = A(t,T)r_t + \widetilde{G}(t,T)\mu_t$.

2.3.2 The GAO and its valuation

We now consider the GAO valuation problem. A GAO can be viewed as a contract that gives the policyholder the right to convert the survival benefit into an annuity at a pre-specified conversion rate. This type of option first gained popularity in UK pension policies during the late 70s and 80s. Since then it became a common feature of policies sold in many countries.

The guaranteed conversion rate, g, can be quoted as an annuity/cash value ratio. According to Bolton et al. [6], the most commonly used guaranteed rate for males, aged 65 in UK in the 80s, was $g = \frac{1}{9}$, meaning that a £1000 cash value can be turned into an annuity of £111 per annum. If the guaranteed conversion rate is higher than the prevailing conversion rate, the GAO is of positive value; otherwise, the GAO is valueless since the policyholder could use the cash to obtain higher value of annuity from the primary market. Therefore, the moneyness of the GAO at maturity depends on the price of annuity available from the primary market at that time, which are determined by the prevailing interest and mortality rates.

Let $a_x(T)$ be the prevailing annuity rate in the primary market. Since the annuity payments can be considered as a sequence of survival benefit \$1 at the beginning of each year, we can use equation (2.22) to get

$$\begin{aligned} a_x(T) &= \sum_{n=0}^{\infty} \mathbf{E}^Q \left[e^{-\int_T^{T+n} r_u du} e^{-\int_T^{T+n} \mu_v dv} \middle| \mathcal{F}_T \right] \\ &= \sum_{n=0}^{\infty} M(T,T+n) = \sum_{n=0}^{\infty} \beta(T,T+n)e^{-V(T,T+n)}, \end{aligned} \tag{2.23}$$

where $\beta(T, T+n) = e^{D(T,T+n)+\widetilde{H}(T,T+n)}$ and $V(T, T+n) = A(T, T+n)r_T + \widetilde{G}(T, T+n)\mu_T$.

Then the payoff function of the GAO at time T, based on each one dollar cash amount, is

$$\mathbf{C}_T = \mathbf{I}_{\{\tau \geq T\}}[g \cdot a_x(T) - 1]^+ = g\, \mathbf{I}_{\{\tau \geq T\}} \left[a_x(T) - \frac{1}{g} \right]^+ .$$

Our valuation problem is to determine the price of GAO at time 0, which is

$$\begin{aligned} P_{GAO} &= \mathbf{E}^Q \left[e^{-\int_0^T r_u du} \mathbf{C}_T \middle| \mathcal{F}_0 \right] \\ &= g\, \mathbf{E}^Q \left[e^{-\int_0^T r_u du} e^{-\int_0^T \mu_v dv} (a_x(T) - K)^+ \middle| \mathcal{F}_0 \right], \end{aligned} \tag{2.24}$$

where $a_x(T)$ is defined in equation (2.23) and K is $1/g$.

In the next subsection, we employ a change of numéraire technique to evaluate equation (2.24) straightforwardly despite the dependence between r_t and μ_t, and the complicated form of $a_x(T)$. We invoke the idea of change of probability measures in order to explicitly price contingent claims; see Dahl et al. [11] and Jalen and Mamon [16]. We then show how to derive the dynamics μ_t and r_t under this new measure that will facilitate the GAO price calculation.

2.3.3 Endowment-risk-adjusted measure for GAOs

We introduce a new measure associated with the pure endowment $M(t, T)$ as the numéraire. The new measure is then called the *endowment-risk-adjusted measure* $\widehat{Q}$. The measure is defined via the Radon-Nikodŷm derivative

$$\frac{d\widehat{Q}}{dQ} := \eta_T = \frac{e^{-\int_0^T r_u du} M(T, T)}{M(0, T)}. \tag{2.25}$$

Note that $M(T, T) = \mathbf{I}_{\{\tau \geq T\}}$ in equation (2.25). Since η_T is a martingale, then for $t \leq T$,

$$\eta_t = \mathbf{E}^Q[\eta_T | \mathcal{F}_t] = \frac{e^{-\int_0^t r_u du} M(t, T)}{M(0, T)}. \tag{2.26}$$

Utilising $\widehat{Q}$, equation (2.24) can be re-written as

$$P_{GAO} = g\, \mathbf{E}^Q \left[e^{-\int_0^T r_u du} e^{-\int_0^T \mu_v dv} \middle| \mathcal{F}_0 \right] \mathbf{E}^{\widehat{Q}} \left[(a_x(T) - K)^+ \middle| \mathcal{F}_0 \right]$$

based on the similar Bayes' rule argument following equation (2.13).

From equations (2.5), (2.23) and (2.24), we have

$$\begin{aligned} P_{GAO} &= g\, M(0,T)\, \mathbf{E}^{\widehat{Q}}\left[(a_x(T) - K)^+ \middle| \mathcal{F}_0 \right] \\ &= g\, M(0,T)\, \mathbf{E}^{\widehat{Q}}\left[\left(\sum_{n=0}^{\infty} \beta(T, T+n) e^{-V(T,T+n)} - K \right)^+ \middle| \mathcal{F}_0 \right]. \end{aligned} \tag{2.27}$$

To evaluate equation (2.27), we need the dynamics under $\widehat{Q}$ of the mortality and interest rate processes. We consider the dynamics of $e^{-\int_0^t r_u du} M(t,T) = e^{-\int_0^t r_u du} B(t,T) P(t,T) := X_t$ in (2.26). Suppose $X_t^1 := e^{-\int_0^t r_u du} B(t,T)$ and $X_t^2 := P(t,T)$. We are interested to find dX_t where $X_t = X_t^1 X_t^2$.
From equations (2.9) and (2.7), one may verify that

$$dX_t^1 = -\sigma A(t,T) X_t^1 dW_t^1. \tag{2.28}$$

Let $Y(t) := -\widetilde{G}(t,T)\mu_t + \widetilde{H}(t,T)$ in equation (2.19), i.e., $X_t^2 := P(t,T) = e^{Y(t)}$. Then we have

$$dY(t) = \left(\frac{\partial \widetilde{H}(t,T)}{\partial t} - \frac{\partial \widetilde{G}(t,T)}{\partial t} \mu_t - c\mu_t \widetilde{G}(t,T) \right) dt - \xi \widetilde{G}(t,T) dZ_t$$

and therefore

$$\begin{aligned} dX_t^2 &= \frac{1}{2} e^{Y(t)} \left(\xi \widetilde{G}(t,T) \right)^2 dt + e^{Y(t)} dY(t) \\ &= e^{Y(t)} \left[\frac{1}{2} \left(\xi \widetilde{G}(t,T) \right)^2 dt + \left\{ \frac{\partial \widetilde{H}(t,T)}{\partial t} - \left(\frac{\partial \widetilde{G}(t,T)}{\partial t} + c\widetilde{G}(t,T) \right) \mu_t \right\} dt - \xi \widetilde{G}(t,T) dZ_t \right] \\ &= X_t^2 \left[\left\{ \frac{\partial \widetilde{H}(t,T)}{\partial t} + \frac{1}{2} (\xi \widetilde{G}(t,T))^2 - \left(\frac{\partial \widetilde{G}(t,T)}{\partial t} + c\widetilde{G}(t,T) \right) \mu_t \right\} dt - \xi \widetilde{G}(t,T) dZ_t \right], \end{aligned} \tag{2.29}$$

where $dZ_t = \rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2$.

Since $X_t = X_t^1 X_t^2$, we have

$$\begin{aligned}
dX_t &= X_t^1 dX_t^2 + X_t^2 dX_t^1 + \left(-X_t^1 \sigma A(t,T)\right)\left(-X_t^2 \xi \widetilde{G}(t,T)\right)\rho dt \\
&= X_t^1 X_t^2 \Bigg(\frac{\partial \widetilde{H}(t,T)}{\partial t} + \frac{1}{2}(\xi \widetilde{G}(t,T))^2 - \left(\frac{\partial \widetilde{G}(t,T)}{\partial t} + c\widetilde{G}(t,T) \right)\mu_t dt - \xi \widetilde{G}(t,T) dZ_t \\
&\quad -\sigma A(t,T) dW_t^1 + \rho\sigma\xi A(t,T)\widetilde{G}(t,T)dt \Bigg) \\
&= X_t \left[\left(\frac{\partial \widetilde{H}(t,T)}{\partial t} + \rho\sigma\xi A(t,T)\widetilde{G}(t,T) + \frac{1}{2}(\xi \widetilde{G}(t,T))^2 - \left(\frac{\partial \widetilde{G}(t,T)}{\partial t} + c\widetilde{G}(t,T) \right)\mu_t \right) dt \right] \\
&\quad + X_t \left[-\sigma A(t,T) dW_t^1 - \xi \widetilde{G}(t,T)(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2) \right]. \qquad (2.30)
\end{aligned}$$

Note that the dt term of equation (2.30) must be identically zero since $X(t) = e^{-\int_0^t r_u du} M(t,T)$ is a martingale process (being a discounted process) under Q. That is,

$$\frac{dX_t}{X_t} = -\left(\sigma A(t,T) dW_t^1 + \xi \widetilde{G}(t,T)(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2) \right). \qquad (2.31)$$

Utilising equation (2.31), we find that

$$\begin{aligned}
d(\ln X_t) &= \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{(X_t)^2}(dX_t)^2 \\
&= \left[-\frac{1}{2}\left(\sigma A(t,T) + \rho\xi \widetilde{G}(t,T) \right)^2 - \frac{1}{2}(1-\rho^2)(\xi \widetilde{G}(t,T))^2 \right] dt \\
&\quad - \left[\sigma A(t,T) + \rho\xi \widetilde{G}(t,T) \right] dW_t^1 - \sqrt{1-\rho^2}\, \xi \widetilde{G}(t,T) dW_t^2. \qquad (2.32)
\end{aligned}$$

The dynamics specified in equation (2.32) allows us to identify the relations of the $\widehat{Q}$−standard Brownian motions $\widehat{W}_t^1$ and $\widehat{W}_t^2$ to W_t^1 and W_t^2, respectively. We find that to change measure from Q to $\widehat{Q}$, the corresponding Brownian motions are given by

$$d\widehat{W}_t^1 = dW_t^1 + \left(\sigma A(t,T) + \rho\xi \widetilde{G}(t,T) \right) dt, \qquad (2.33)$$

$$d\widehat{W}_t^2 = dW_t^2 + \sqrt{1-\rho^2}\, \xi \widetilde{G}(t,T) dt. \qquad (2.34)$$

Finally, under $\widehat{Q}$, the stochastic dynamics for r_t and μ_t are easily obtained as

$$dr_t = \left(ab - \sigma(\sigma A(t,T) + \rho\xi \widetilde{G}(t,T)) - ar_t \right) dt + \sigma d\widehat{W}_t^1, \qquad (2.35)$$

$$d\mu_t = \left(c\mu_t - \rho\sigma\xi A(t,T) - \xi^2 \widetilde{G}(t,T) \right) dt + \xi d\widehat{Z}_t, \qquad (2.36)$$

where $\widehat{Z}_t = \rho \widehat{W}_t^1 + \sqrt{1-\rho^2}\, \widehat{W}_t^2$. From the above SDEs, both r_t and μ_t processes have the form of an extended Vasiček model and their distributions are immediate (see the result in pp. 267-268 of Elliott and Kopp [15]). More precisely, under measure $\widehat{Q}$, (r_t, μ_t) is a bivariate normal random variable, with the following parameters:

$$\begin{aligned}
\mathbf{E}^{\widehat{Q}}[r_t] &= e^{-at}r_0 + b(1-e^{-at}) - \frac{\sigma^2}{2a^2}\Big[(1-e^{-at})(2-e^{-aT}(e^{at}+1))\Big] \\
&\quad -\frac{\rho\sigma\xi}{c}\Big[\frac{e^{cT}(e^{-ct}-e^{-at})}{a-c} - \frac{1-e^{-at}}{a}\Big], &(2.37)\\
\mathrm{Var}^{\widehat{Q}}[r_t] &= \frac{\sigma^2}{2a}\Big[1-e^{-2at}\Big], &(2.38)\\
\mathbf{E}^{\widehat{Q}}[\mu_t] &= e^{ct}\mu_0 - \frac{\xi^2}{c^2}\Big[\frac{e^{cT}(e^{ct}-e^{-ct})}{2} - e^{ct}+1\Big] + \frac{\rho\sigma\xi}{a}\Big[\frac{e^{-aT}(e^{at}-e^{ct})}{a-c} - \frac{e^{ct}-1}{c}\Big], &(2.39)\\
\mathrm{Var}^{\widehat{Q}}[\mu_t] &= \frac{\xi^2}{2c}\Big[e^{2ct}-1\Big], &(2.40)
\end{aligned}$$

and

$$\mathrm{Cov}^{\widehat{Q}}[r_t,\mu_t] = \frac{\rho\sigma\xi}{a-c}\Big[1-e^{-(a-c)t}\Big]. \qquad (2.41)$$

2.4 Numerical illustration

In this section, we provide a numerical experiment in calculating the price of GAO based on both formulae (2.27) and (2.24). Direct implementation of formula (2.24) is a brute-force method of coming up with a GAO price. On the other hand, the use of equation (2.27) is a more efficient and accurate approach of getting a GAO value. We use Monte Carlo simulation method to obtain the value of the GAO price in both formulae. In equation (2.27), the function $M(0,T)$ is given explicitly in (2.22) assuming that r_0 and μ_0 are known. To calculate the expectation component of equation (2.27), we note that $V(T,T+n) = A(T,T+n)r_T + \widetilde{G}(T,T+n)\mu_T$. Therefore the summation term depends only on the value of r_t and μ_t at maturity time T. This means that the simulated pair (r_T,μ_T) are all we need in the calculation of the GAO price using formula (2.27). The bivariate normal distribution of (r_t,μ_t) under measure $\widehat{Q}$ is specified by the parameters given from equations (2.37) to (2.41).

To compute the price of the GAO based on equation (2.24), however, we would need to generate the sample path under Q for each process r_t and μ_t given in equations (2.7) and (2.8), respectively. We subdivide the time period $[0, T]$ into m equal subintervals with fixed length $\Delta t = \dfrac{T}{m}$ and define $t_i = i\Delta t, i = 0, 1, \ldots, m$. At each time step, we generate the sample paths of r_t and μ_t as follows:

$$r_{t_i} = r_{t_{i-1}} + (ab - ar_{t_{i-1}})\Delta t + \sigma\sqrt{\Delta t}\epsilon^1_{t_i} \tag{2.42}$$

and

$$\begin{aligned} \mu_{t_i} &= \mu_{t_{i-1}} + c\mu_{t_{i-1}}\Delta t + \xi\sqrt{\Delta t}Z_{t_i} \\ &= \mu_{t_{i-1}} + c\mu_{t_{i-1}}\Delta t + \xi\sqrt{\Delta t}(\rho\epsilon^1_{t_i} + \sqrt{(1-\rho^2)}\epsilon^2_{t_i}), \end{aligned} \tag{2.43}$$

where $\{\epsilon^1_{t_i}\}_{i=1,\ldots,m}$ and $\{\epsilon^2_{t_i}\}_{i=1,\ldots,m}$ are two independent sequences of standard normal random variables.

The integrals in (2.24) are then approximated using the trapezoidal rule, i.e.,

$$\int_0^t r_u\,du \approx \frac{\Delta t}{2}\left[r_0 + r_m + 2\sum_{k=1}^{m-1} r_k\right], \tag{2.44}$$

and

$$\int_0^t \mu_v\,dv \approx \frac{\Delta t}{2}\left[\mu_0 + \mu_m + 2\sum_{k=1}^{m-1} \mu_k\right]. \tag{2.45}$$

Consequently, we obtained numerical values for $e^{-\int_0^t r_u\,du}$ and $e^{-\int_0^t \mu_v\,dv}$. The r_m and μ_m values at the end of each path are used to calculate $a_x(T)$ in equation (2.24).

Our numerical results are obtained by generating 50,000 sample paths. The parameters employed for the interest rate model (2.7) and mortality model (2.8) are given in Table 2.1. The mortality model parameters are based on the values provided in Luciano and Vigna [25]. In Table 2.2, we display the price of the GAO based on a cohort born in 1935 assumed to hold GAO contracts maturing at age 65. The GAO is evaluated at age 50, i.e. 15 years before maturity. In our calculation, we also assumed that the maximum age is 100 so that there are at most 35 annuity payments.

In the first column of Table 2.2, we present the correlation coefficient between the interest and

Table 2.1: Parameter values used in the numerical experiment in chapter 2.

Parameter set for numerical analysis
Contract specification
$g = 11.1\%$, $T = 15$, $n = 35$;
Interest rate model
$a = 0.15$, $b = 0.045$, $\sigma = 0.03$, $r_0 = b$;
Mortality model
$c = 0.1$, $\xi = 0.0003$, $\mu_0 = 0.006$.

mortality rates. The price calculated under the endowment-risk-adjusted measure approach and under the risk-neutral measure direct approach are given in the second and third columns, respectively. It is apparent that as the correlation between interest and mortality rates varies from negative to positive, GAO prices increase. This is consistent with the fact that when interest and mortality rates are negatively correlated, the two risk factors collectively act as a "natural hedge" against the overall uncertainty of the GAOs, and consequently the price of the GAO is reduced. Conversely, this "natural hedge" disappears as the two risk factors become positively correlated leading to an increasing trend in the GAO values. The numbers enclosed in parentheses are the standard errors of the price estimates, which indicate that the results based on formula (2.27) are more accurate than the results based on formula (2.24). Moreover, as previously mentioned above, our method that utilises both the forward measure and endowment-risk-adjusted measure is efficient. This is supported by the highly significant difference between times of completion in the price calculation, as exhibited in the last row of Table 2.2, for the two methods. Clearly, there is so much to be gained in employing the proposed approach.

2.5 Conclusions

We showed in this work how to price a GAO under a generalised framework of stochastic mortality and interest risk factors where the dependence between two risks is explicitly modelled.

Table 2.2: Actuarial prices for GAO under two different methods in chapter 2.
Numbers in parentheses are standard errors.

	GAO value	
ρ	Using equation (2.27)	Using equation (2.24)
-1.0	0.0904026 (0.0003836)	0.0903010 (0.0005953)
-0.9	0.0920624 (0.0003915)	0.0920976 (0.0006043)
-0.8	0.0943914 (0.0004018)	0.0944001 (0.0006373)
-0.7	0.0962496 (0.0004116)	0.0963117 (0.0006410)
-0.6	0.0986634 (0.0004217)	0.0986722 (0.0006737)
-0.5	0.1003584 (0.0004320)	0.1006036 (0.0006908)
-0.4	0.1023678 (0.0004412)	0.1024359 (0.0007232)
-0.3	0.1042501 (0.0004485)	0.1041982 (0.0007421)
-0.2	0.1067286 (0.0004587)	0.1066445 (0.0007556)
-0.1	0.1088653 (0.0004687)	0.1087849 (0.0007788)
0	0.1110679 (0.0004790)	0.1110713 (0.0007985)
0.1	0.1131000 (0.0004896)	0.1131335 (0.0008286)
0.2	0.1153378 (0.0005001)	0.1152761 (0.0008528)
0.3	0.1174438 (0.0005131)	0.1172690 (0.0008922)
0.4	0.1197348 (0.0005241)	0.1197303 (0.0009240)
0.5	0.1218968 (0.0005350)	0.1219112 (0.0009375)
0.6	0.1246585 (0.0005464)	0.1246414 (0.0009520)
0.7	0.1263725 (0.0005509)	0.1264038 (0.0009845)
0.8	0.1290466 (0.0005682)	0.1289118 (0.0010281)
0.9	0.1317430 (0.0005813)	0.1316604 (0.0010497)
1.0	0.1338156 (0.0005901)	0.1338989 (0.0010943)
time (in seconds)	3.25	955.37

Both the forward measure associated with the bond price as numéraire and the newly introduced concept of endowment-risk-adjusted measure are used to derive a simplified expression for the GAO price, despite dependence between two stochastic factors. The advantages of our approach are underscored by the following novelty highlights. Firstly, without a change of numéraire, the direct calculation of the GAO value would be challenging due to the complicated nature of the C_t process as well as the dependence between interest rate and mortality rate processes. Secondly, simulating the valuation formula directly under the risk-neutral measure would be time-consuming because one needs to deal with correlated random variables driving the r_t and μ_t dynamics. The discretisation involved in the simulation entails longer computation time and magnifies simulation errors. In comparison, our proposed approach is both efficient and accurate in calculating the numerical value for the price. Finally, the techniques used in this work can certainly be applied and naturally extended to contingent claims with option-embedded features and whose risk factors are assumed to follow any affine-type mortality and interest rate processes.

It has to be noted that from the valuation formula for GAO in equation (2.27), we have to deal with a sum of lognormal random variables. Whilst we have obtained explicitly the dynamics for the interest and mortality risk factors in section 2.3 and their distributions under the endowment-risk-adjusted measure can also be easily extracted from the SDEs, the evaluation of the conditional expectation remains a hurdle. The problem of determining the distribution of sums of lognormal random variables was tackled by several authors to varying degrees of depth and treatment. Developments in this area are highlighted in the research results of Dufresne [14], Leipnik [18] and Wu et al. [34]. Alternatively, one can use a comonotonicity-based approach to approximate the sum of lognormal random variables when the random variables in the summation are highly correlated [cf. Liu et al. [22] and Liu et al. [23]]. Comonotonic upper and lower bounds can then be used to obtain accurate approximations for the value of GAOs.

The interest and mortality rate models used in this work are adopted mainly due to their combined mathematical tractability. The negativity of interest rate in the Vasiček model can be fixed, for example, by the Cox-Ingersoll-Ross model. Mortality dynamics can be made more realistic by using a continuous-time version of the Lee-Carter model [cf. Biffis et al. [4]], which is well-accepted for its nice property in fitting empirical data. Needless to say, one will have to circumvent new computational challenges and deal with an entirely different set of calculations associated with the endowment-risk-adjusted measure considered in section 2.3.

The issue of hedging mortality derivatives is an important but challenging problem. A major obstacle in constructing an effective hedging strategy for GAOs is the lack of a trading market for mortality risk. In addition, the options written by insurance companies often have very long maturities usually from 10 to 30 years, which makes the modelling of underlying risks difficult. As a result, early research in hedging GAOs (e.g. Pelsser [29]) proposed the use of static option replication as a partial solution for insurance companies to hedge their exposure to embedded options in their portfolios. Recently, Luciano et al. [24] considered the delta-gamma hedging of mortality and interest rate risks under the independence assumption of these risks. However, as noted in Dhaene et al. [13], an independence relation that is observed in the real world is not necessarily preserved in the pricing world. The EU's Solvency II Directive strongly recommends the testing of capital adequacy requirements based on the explicit assumption of mutual dependence between financial markets and life/health insurance markets including the dependence between interest and mortality risks; refer to QSI5 [32]. Therefore, our generalised framework provides a plausible starting point for further research investigations towards this direction.

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Chapter 3

A comonotonicity-based valuation method for guaranteed annuity options

3.1 Introduction

There are many recent financial innovations featuring mortality-related guarantees in traditional life insurance and annuity products as well as mortality-linked securities in the capital market. Examples of the former include guaranteed annuity option (GAO), guaranteed minimum death benefit (GMDB), guaranteed minimum income benefit (GMIB) whilst examples of the latter include European Investment Bank longevity bonds, the Swiss Re mortality bond and survivor swaps. The modelling and pricing issues of guarantees embedded in insurance products, including GAO, GMIB, GMDB, etc., are discussed in Hardy [28]. Whilst the financial risks are the main concern in the valuation of GMIB and GMDB as shown in Kijima and Wong [30], Lin and Tan [32], Lin et al. [33] and Milevsky and Salisbury [38], and the interplay between financial risks and mortality risk plays an important role in pricing GAOs as pointed out by Boyle and Hardy [12]. For an overview on the development of mortality risk-embedded products, see Blake & Burrows [8], Blake et al. [9, 10], Wills & Sherris [47], and the references therein. For recent progress on using capital market solutions to counterpoising longevity risk, see the annual updates published by Pensions Institute and the references therein; for example, the latest annual update by Black et al. [6].

These innovations reflect the contemporaneous recognition of the stochastic nature of mortality risk. Empirical studies of mortality changes and mortality predictions can be found in Tuljapurkar and Boe [44], a series of working papers by the CMI (UK) (e.g., [15], [16], [17]), and the review paper by Pitacco [42]. Consequently, stochastic mortality requires adequate modelling and valuation methods. Pioneering research works in this area exploited the structural similarity between interest and mortality rates, and proposed to use diffusion models for mortality; see Ballotta and Haberman [2], Biffis [3], Blackburn and Sherris [7], Cairns et al. [13], Dahl [18], Dahl et al. [19], Milevsky and Promislow [37] and the references therein. It is worth pointing out that there are some fundamental differences between the two rates; see Norberg [41], therefore specification of the model for mortality needs to be addressed carefully. Utilising modern finance theory, this approach can provide a sound and rich modelling framework, capable of incorporating financial risk factors, for the fair valuation of mortality-linked contracts of various kinds.

In the aforementioned works, mortality risks are often assumed independent from financial risks, particularly interest risk, for convenience. As noted in Dhaene et al. [25], the real-world independence between the two risk factors is not equivalent to the pricing-world independence: it has been shown that an independence relation that is observed in the real-world often fails to be maintained in the pricing-world. Therefore, it is more reasonable to have a pricing framework that allows dependence between mortality and interest rates. On the other hand, the manifestation of dependence between these two risk factors in the real-world has been studied, for instance, in the work of Nicolini [40], which demonstrated that the decline of adult mortality at the end of the 17th century can be one of the causes driving the decline of interest rate in pre-industrial England. In the past few years, we have witnessed how the decline of mortality or equivalently increase in life expectancy puts a considerable stress on the social programmes of various countries. It is viewed that fiscal crisis, which is the supposed inability of the state to raise more tax revenues to fund its programmes, could ensue. With declining mortality patterns, the state has to deal with an ageing population; this implies that social security and health expenditure per capita must rise considerably to maintain the same level of service. The associated price for this predicament is sourced out from levies on the economically active who are in a declining proportion of the population. A serious fiscal situation, where government's budget deficit is higher than expected, creates uncertainty and instability in both the local and global economy. This in turn negatively affects the financial markets including returns on investments. In a resolute response, EU's Solvency II Directive created new insurance risk management practices for capital adequacy requirements based on the explicit assumption of mutual dependence between financial markets and life/health insurance markets including the dependence between interest and mortality risks; cf. Quantitative Impact Study 5: Technical Specifications [43]. The consequential impact of such dependence is also underscored by Christiansen and Steffensen [14] in the context of stress testing.

This work contributes along this research direction of introducing a dependence structure between the dynamics of mortality and interest rates, following the approach of Jalen and Mamon [29]. More precisely, under the generalised risk-neutral pricing set-up, both mortality and interest risks are modelled by affine processes and are correlated. We illustrate that the well-developed techniques in modern finance theory are applicable and allow for explicit expression for the value of mortality-linked contracts, including fundamental pure endowments as well as the more complicated GAOs. We employ the change of numéraire technique twice to derive an efficiently implementable GAO valuation formula. We then use the comonotonicity theory to provide analytic upper and lower bounds for the value of the GAOs.

We present the building blocks of the modelling framework in section 3.2 along with the necessary changes of probability measures designed to tackle both stochastic and correlated risk factors encountered in contingent claim valuation. Section 3.3 deals with the essential concepts related to comonotonicity vital for pricing applications and presents the derivation of comonotonicity-based upper and lower bounds. In section 3.4, we present a numerical implementation illustrating the applicability and accuracy of our proposed methodology. We provide conclusions in section 3.5.

3.2 The generalised valuation framework

Actuarial valuation of mortality-linked contracts involves at least two types of uncertainty: the uncertainty related to interest risk and the uncertainty related to mortality risk. In this section, we introduce a generalized modelling framework based on the short rate process r_t and the force of mortality rate process μ_t on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$, where Q is a risk-neutral measure and $\mathcal{F}_t$ is the joint filtration generated by r_t and μ_t. The framework adopted here follows those of Biffis [3], Cairns et al. [13], Dahl [18] and Milevsky and Promislow [37]. However, instead of assuming the independence of mortality evolution and financial market, we explicitly introduce a dependence structure between the dynamics of mortality and interest rates. As we have seen in the past, demographic factor interacts with economy and has financial impact, whether it is caused by longevity risk and its associated financial burden or catastrophic events such as those massive earthquakes in Kobe, 1995; China, 2008; Japan, 2011, etc.

3.2.1 The modelling assumptions

Under a risk-neutral measure Q, we assume that r_t follows the well-known Vasiček model, i.e., r_t has the dynamics given by

$$dr_t = a(b - r_t)dt + \sigma dW_t^1, \tag{3.1}$$

where $a,\ b$ and σ are positive constants and W_t^1 is a standard Brownian motion.

Moreover, the force of mortality μ_t for an insured aged x at time 0 evolves according to

$$d\mu_t = c\mu_t dt + \xi dZ_t, \tag{3.2}$$

where c and ξ are positive constants and Z_t is a standard Brownian motion correlated with W_t^1 so that

$$dW_t^1 dZ_t = \rho dt.$$

In other words, $Z_t = \rho W_t^1 + \sqrt{1-\rho^2}W_t^2$, where W_t^2 is a standard Brownian motion independent of W_t^1 . Under this model setting, the joint filtration $\mathcal{F}_t (= \mathcal{F}_t^{r_t} \vee \mathcal{F}_t^{\mu_t})$ is also generated by $\mathcal{F}_t^{W_1} \vee \mathcal{F}_t^{W_2}$, and both r_t and μ_t are processes adapted to $\mathcal{F}_t$.

The interest rate process in equation (3.1) follows a Vasiček model and such model is used for valuation tractability. On the other hand, the mortality rate model is a non-mean reverting affine process, a feature that distinguishes it from interest rates. An empirical study by Luciano and Vigna [35] confirmed and supported this general well-known observation, which found that non-mean reverting OU process fits historical data better than the mean-reverting process. This is the reason why we do not deem Cox-Ingersoll-Ross model, for example, to be appropriate even though it could produce positive mortality rates. The issue on the possibility of negative mortality rates can be circumvented by choosing values for c and ξ properly, similar to the use of Vasiček model for interest rates. The use of dynamics in equation (3.2) for the force of mortality is definitely far from being perfect let alone ideal. Mortality dynamics can be made more realistic by introducing, for example, regime-switching characteristics into the model and we shall explore such extension in future works.

It is well known that the price at time t of a zero-coupon bond paying \$1 at maturity $T > t$ is given by

$$B(t,T) = \mathbf{E}^Q \left[e^{-\int_t^T r_u du} \middle| \mathcal{F}_t \right] = e^{-A(t,T)r_t + D(t,T)}, \tag{3.3}$$

where

$$\begin{aligned} A(t,T) &= \frac{1-e^{-a(T-t)}}{a} \quad \text{and} & (3.4)\\ D(t,T) &= \left(b - \frac{\sigma^2}{2a^2}\right)[A(t,T)-(T-t)] - \frac{\sigma^2 A(t,T)^2}{4a}. & (3.5) \end{aligned}$$

For a life aged x, let $M(t,T;x) = M(t,T)$ be the fair value at time t of a unit pure endowment payment at time T ($T > t$), provided that (x) is alive at time t. From the risk-neutral pricing principle, we have

$$\mathbf{M}(t,T) = \mathbf{E}^Q \left[e^{-\int_t^T r_u du} \cdot \mathbf{I}_{\{\tau \ge T\}} \middle| \mathcal{F}_t \right] = \mathbf{I}_{\{\tau \ge t\}} \cdot \mathbf{E}^Q \left[e^{-\int_t^T r_u du} e^{-\int_t^T \mu_v dv} \middle| \mathcal{F}_t \right], \tag{3.6}$$

where τ is the future lifetime random variable for (x). In the second equality of formula (3.6), the indicator function $\mathbf{I}_{\{\tau \ge t\}}$ is used to emphasize that the fair value of a pure endowment is defined only conditionally, i.e. contingent on the survival of the insured. The pure endowment has a positive value at time t only if the insured survives to time t, a feature that is not present in the pricing of default-free bonds, see Norberg [41]. In equation (3.6), we use bold $\mathbf{M}$ to represent such conditional requirement on survival as reflected by the indicator function, and plain M to just represent the value of the pure endowment given that (x) is alive at time t. That is

$$M(t,T) = \mathbf{E}^Q\left[e^{-\int_t^T r_u du} e^{-\int_t^T \mu_v dv} \middle| \mathcal{F}_t\right] \text{and } \mathbf{M}(t,T) = \mathbf{I}_{\{\tau \geq t\}} M(t,T). \tag{3.7}$$

Similarly, let $a_x(T)$ be the risk neutral value, evaluated at time T, of a life annuity contract paying \$1 to the insured (x) annually conditional on his/her survival at the moment of payments, provided that (x) is alive at the time of valuation. Since a life annuity can be viewed as combinations of pure endowments, we can write

$$a_x(T) = \sum_{n=0}^{\infty} \mathbf{E}^Q\left[e^{-\int_T^{T+n} r_u du} e^{-\int_T^{T+n} \mu_v dv} \middle| \mathcal{F}_T\right]. \tag{3.8}$$

Following the same line of reasoning, the time t risk neutral value of a call option to acquire a life annuity for (x) at time T $(T > t)$ with strike price K can be calculated from

$$\begin{aligned} \mathbf{c}(t,T) &= \mathbf{E}^Q\left[e^{-\int_t^T r_u du} \cdot \mathbf{I}_{\{\tau \geq T\}}(a_x(T) - K)^+ \middle| \mathcal{F}_t\right] && (3.9) \\ &= \mathbf{I}_{\{\tau \geq t\}} \mathbf{E}^Q\left[e^{-\int_t^T r_u du} e^{-\int_t^T \mu_v dv} (a_x(T) - K)^+ \middle| \mathcal{F}_t\right]. && (3.10) \end{aligned}$$

As in (3.7), we can use $c(t,T)$ only for the value of this call option:

$$c(t,T) = \mathbf{E}^Q\left[e^{-\int_t^T r_u du} e^{-\int_t^T \mu_v dv} (a_x(T) - K)^+ \middle| \mathcal{F}_t\right]. \tag{3.11}$$

Here we would like to remark that $c(t,T)$ can be used for the valuation of a particular type of GAO. A GAO is a contract that gives the policyholder the right to convert his/her survival benefit into an annuity at a pre-specified conversion rate, a feature first becoming popular in UK pension policies during the late 70s and 80s. According to Bolton et al. [11], the most commonly used guaranteed conversion rate g for males, aged 65 in UK in the 80s, was that a £1000 cash value can be turned into an annuity of £111 per annum, i.e. $g = \frac{1}{9}$. If an annuity is sold at a rate higher than $1/g$ in the primary market, the GAO is of positive value; otherwise, the GAO is valueless. Therefore, letting $K = 1/g$ in equation (3.11), the price for the aforementioned GAO at time 0 can be expressed as follows.

$$\begin{aligned} P_{GAO} &= \mathbf{E}^Q\left[e^{-\int_0^T r_u du} e^{-\int_0^T \mu_v dv}(g a_x(T) - 1)^+ \middle| \mathcal{F}_0\right] \\ &= g\mathbf{E}^Q\left[e^{-\int_0^T r_u du} e^{-\int_0^T \mu_v dv}(a_x(T) - \frac{1}{g})^+ \middle| \mathcal{F}_0\right] \\ &= gc(0,T). \end{aligned} \tag{3.12}$$

For general description of GAOs, see Bolton et al. [11] and Wilkie et al. [46].
We remark that, although one can write evaluation formulae (3.6)–(3.12), it is not easy to get a closed-form solution for them as in the case of (3.3) because the dynamics of r_t and μ_t are correlated. Earlier works that valuate life insurance and annuity products under stochastic mortality modelling frameworks [2, 4, 13, 37] usually assume independence condition between mortality and interest rates. Under independence condition, $M(t,T)$ is equal to the product of the bond price and survival function; i.e.,

$$M(t,T) = \mathbf{E}^Q\left[e^{-\int_t^T r_u du}\middle|\mathcal{F}_t\right]\mathbf{E}^Q\left[e^{-\int_t^T \mu_v dv}\middle|\mathcal{F}_t\right].$$

The separation of interest and mortality risk factors in the valuation of pure endowment as well as life annuity greatly simplifies the calculation. However, this specific advantage does not extend to equation (3.11) or (3.12) because the annuity term in equations (3.11) and (3.12) is correlated with both μ_t and r_t.

It is worth noting that we have chosen to use the risk-neutral approach for the fair valuation of insurance products. This approach may be debatable because currently there is no transparent market for the GAOs and their underlying variables. The rationale behind this approach is that we can view a risk-neutral price as a natural benchmark in setting the initial price of a derivative product that is still not trading and just being introduced in the market. This approach is consistent with the methodology employed in the financial pricing of exotic and structured products as well as over-the-counter derivatives where there is no liquid market for such instruments. In this study, one may view that the risk-neutral value obtained for the GAO serves as a benchmark that can be adjusted by individual insurance companies in setting their desired gross premiums.

3.2.2 Valuation formulae obtained via change of numéraire technique

Now we use the well-developed change of numéraire technique to derive more tractable valuation formulae for mortality-linked contracts mentioned in the previous subsection. Consider the bond price $B(t,T)$ as a numéraire. Associated with $B(t,T)$ is the forward measure $\widetilde{Q}$ equivalent to Q. Given that the theoretical underpinnings of forward measure can be found in standard quantitative finance textbooks Björk [5], Musiela and Rutkowski [39], the pure endowment price involving $\widetilde{Q}$ is given, without derivation, as follows:

$$\mathbf{M}(t,T) = B(t,T)\mathbf{E}^{\widetilde{Q}}\left[\mathbf{I}_{\{\tau\geq T\}}\middle|\mathcal{F}_t\right] = \mathbf{I}_{\{\tau\geq t\}}B(t,T)\mathbf{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_v dv}\middle|\mathcal{F}_t\right]. \tag{3.13}$$

The term $\mathbf{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_v dv}\middle|\mathcal{F}_t\right]$ in equation (3.13) is calculated under $\widetilde{Q}$. The advantage of changing to measure $\widetilde{Q}$ is that two correlated risk factors in (3.6) could be dealt with separately. Therefore, if we have the dynamics of μ_t then its explicit solution under $\widetilde{Q}$ follows.

Following the result given and established in Appendix of Mamon [36], we derive the respective dynamics for r_t and μ_t under $\widetilde{Q}$ as follows:

$$dr_t = \left[ab - \sigma^2 A(t,T) - ar_t\right]dt + \sigma d\widetilde{W}_t^1 \tag{3.14}$$

and

$$\begin{aligned} d\mu_t &= (-\rho\sigma\xi A(t,T) + c\mu_t)dt + \rho\xi d\widetilde{W}_t^1 + \sqrt{1-\rho^2}\,\xi d\widetilde{W}_t^2 \\ &= (-\rho\sigma\xi A(t,T) + c\mu_t)dt + \xi d\widetilde{Z}_t, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} d\widetilde{W}_t^1 &= dW_t^1 + A(t,T)\sigma dt, \\ d\widetilde{W}_t^2 &= dW_t^2 \end{aligned}$$

and

$$\widetilde{Z}_t = \rho\widetilde{W}_t^1 + \sqrt{1-\rho^2}\,\widetilde{W}_t^2.$$

Note that $\widetilde{W}_t^1$, $\widetilde{W}_t^2$ and $\widetilde{Z}_t$ are standard Brownian motions under $\widetilde{Q}$, and $A(t,T)$ is specified in equation (3.4). From equation (3.15), we see that μ_t has an affine form with time-dependent drift. Since $\rho \neq 0$, changing to the forward measure $\widetilde{Q}$ alters the dynamics of μ_t under $\widetilde{Q}$ as expected.

Write $\alpha(t) := -\rho\sigma\xi A(t,T)$ and $b(t) := \displaystyle\int_0^t (-c)du = -ct$. Then

$$\mu_t = e^{-b(t)}\left(\mu_0 + \int_0^t e^{b(v)}\alpha(v)dv + e^{b(v)}\xi d\widetilde{Z}_v\right).$$

By letting

$$\gamma(t) = \int_t^T e^{-b(v)}dv = \frac{e^{cT} - e^{ct}}{c} = \frac{e^{ct}}{c}(e^{c(T-t)} - 1)$$

and employing the result in pp. 267-268 of Elliott and Kopp [27], we have

$$\mathbf{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_v dv}\middle|\mathcal{F}_t\right] = e^{-\mu_t\widetilde{G}(t,T)+\widetilde{H}(t,T)}, \tag{3.16}$$

where

$$\widetilde{G}(t,T) = e^{b(t)}\int_t^T e^{-b(u)du} = e^{b(t)}\gamma(t) = \frac{(e^{c(T-t)} - 1)}{c} \tag{3.17}$$

and

$$\begin{aligned}\widetilde{H}(t,T) &= -\int_t^T \left(e^{b(u)}\alpha(u)\gamma(u) - \frac{1}{2}e^{2b(u)}\xi^2(u)\gamma^2(u)\right)du \\ &= \left(\frac{\rho\sigma\xi}{ac} - \frac{\xi^2}{2c^2}\right)[\widetilde{G}(t,T) - (T-t)] + \frac{\rho\sigma\xi}{ac}[A(t,T) - \phi(t,T)] + \frac{\xi^2}{4c}\widetilde{G}(t,T)^2\end{aligned} \tag{3.18}$$

with $\phi(t,T) = \frac{1-e^{-(a-c)(T-t)}}{a-c}$. Combining equations (3.3)–(3.5), (3.7), (3.13), and (3.16)–(3.18), we have

$$M(t,T) = e^{-(A(t,T)r_t+\widetilde{G}(t,T)\mu_t)+D(t,T)+\widetilde{H}(t,T)} = \beta(t,T)e^{-V(t,T)}, \tag{3.19}$$

where $\beta(t,T) = e^{D(t,T)+\widetilde{H}(t,T)}$ and $V(t,T) = A(t,T)r_t + \widetilde{G}(t,T)\mu_t$. Combining (3.7), (3.8) and (3.19), an explicit expression for $a_x(T)$ is given by

$$a_x(T) = \sum_{n=0}^{\infty}\beta(T,T+n)e^{-V(T,T+n)}, \tag{3.20}$$

with $\beta(T,T+n) = e^{D(T,T+n)+\widetilde{H}(T,T+n)}$ and $V(T,T+n) = A(T,T+n)r_T + \widetilde{G}(T,T+n)\mu_T$.

We shall now perform a second change of measure to facilitate the evaluation of equation (3.11). This new measure is associated with the pure endowment $M(t,T)$ as the numéraire and defined via the Radon-Nikodŷm derivative

$$\frac{d\widehat{Q}}{dQ} := \eta_T = \frac{e^{-\int_0^T r_u du} M(T,T)}{M(0,T)}. \tag{3.21}$$

Using Itô calculus, it can be shown that the stochastic dynamics for r_t and μ_t under $\widehat{Q}$ are given by

$$dr_t = \left(ab - \sigma(\sigma A(t,T) + \rho\xi\widetilde{G}(t,T)) - ar_t\right)dt + \sigma d\widehat{W}_t^1, \tag{3.22}$$

$$d\mu_t = \left(c\mu_t - \rho\sigma\xi A(t,T) - \xi^2\widetilde{G}(t,T)\right)dt + \xi d\widehat{Z}_t, \tag{3.23}$$

where

$$\begin{aligned} d\widehat{W}_t^1 &= dW_t^1 + \left(\sigma A(t,T) + \rho\xi\widetilde{G}(t,T)\right)dt, \\ d\widehat{W}_t^2 &= dW_t^2 + \sqrt{1-\rho^2}\,\xi\widetilde{G}(t,T)dt, \end{aligned}$$

and

$$\widehat{Z}_t = \rho\widehat{W}_t^1 + \sqrt{1-\rho^2}\,\widehat{W}_t^2.$$

Note that $\widehat{W}_t^1$, $\widehat{W}_t^2$ and $\widehat{Z}_t$ are standard Brownian motions under $\widehat{Q}$. These tell us that both r_t and μ_t processes have the form of an extended Vasiček model and their distributions are immediate. More precisely, under measure $\widehat{Q}$, (r_t, μ_t) is a bivariate normal random variable with the following parameters:

$$\begin{aligned} \mathbf{E}^{\widehat{Q}}[r_t] &= e^{-at}r_0 + b(1-e^{-at}) - \frac{\sigma^2}{2a^2}\left[(1-e^{-at})(2-e^{-aT}(e^{at}+1))\right] \\ &\quad -\frac{\rho\sigma\xi}{c}\left[\frac{e^{cT}(e^{-ct}-e^{-at})}{a-c} - \frac{1-e^{-at}}{a}\right], \\ \mathrm{Var}^{\widehat{Q}}[r_t] &= \frac{\sigma^2}{2a}\left[1-e^{-2at}\right], \\ \mathbf{E}^{\widehat{Q}}[\mu_t] &= e^{ct}\mu_0 - \frac{\xi^2}{c^2}\left[\frac{e^{cT}(e^{ct}-e^{-ct})}{2} - e^{ct}+1\right] + \frac{\rho\sigma\xi}{a}\left[\frac{e^{-aT}(e^{at}-e^{ct})}{a-c} - \frac{e^{ct}-1}{c}\right], \\ \mathrm{Var}^{\widehat{Q}}[\mu_t] &= \frac{\xi^2}{2c}\left[e^{2ct}-1\right], \end{aligned}$$

and

$$\mathrm{Cov}^{\widehat{Q}}[r_t,\mu_t] = \frac{\rho\sigma\xi}{a-c}\left[1-e^{-(a-c)t}\right].$$

It is worth mentioning that although the means of r_t and μ_t are different under different measures, their variance and covariance were preserved under the change of measures, as must be expected from the gist of Girsanov theorem in which under measure changes, only the process's drift changes but the volatility does not.

A similar change of numéraire argument based on equation (3.21) yields

$$c(t,T) = M(t,T)\mathbf{E}^{\widehat{Q}}\left[(a_x(T) - K)^+ \middle| \mathcal{F}_t\right],$$

where $M(t,T)$ and $a_x(T)$ are given in (3.19) and (3.20), respectively. In particular, the GAO price in (3.12) can now be written as

$$P_{GAO} = gM(0,T)\mathbf{E}^{\widehat{Q}}\left[(a_x(T) - \frac{1}{g})^+ \middle| \mathcal{F}_0\right]. \tag{3.24}$$

Remark: *In the subsequent numerical calculation, the upper limit of the sum in the annuity term $a_x(T)$ (see equation (3.20)) is set to 35, which means the annuity payments start from age 65 and cease at age 100 or the time of death whichever occurs earlier.*

3.3 Comonotonicity bounds for GAO values

From equation (3.24), the GAO is a European-style call option whose payoff is dependent on the sum of correlated lognormal random variables. In general, an analytical expression for the distribution of sums of lognormal random variables is not available. In this work, we offer an alternative to Monte Carlo simulation in obtaining numerical results for the GAO price. Sums of lognormal random variables frequently occur in many areas of the mathematical and engineering sciences such as finance, actuarial science, signal theory and telecommunications, economics, reliability, biology, ecology, atmospheric sciences and geology. Certain methodologies and approaches were developed to determine the distribution of sums of lognormal random variables to varying degrees of depth and treatment depending on particular theoretical interest or practical considerations. Advances in this field are highlighted in the research results of Dufresne [26], Leipnik [31] and Wu et al. [48]. Nonetheless, many of these methods require the assumption of independence amongst the lognormal random variables, otherwise the accuracy of the approximation is compromised as the number of lognormal random variables becomes very large. Our approach is motivated by the work of Dhaene et al. [23, 24] and Liu et al. [34] that proposed the use of comonotonicity-based convex order bounds to approximate the sums of lognormal random variables. An overview of comonotonicity and its applications in finance and insurance can be found in Deelstra et al. [20].

3.3.1 Definition of comonotonicity and quantile additivity property

The random variables $X_1, X_2, ..., X_n$ are said to be comonotonic if and only if there exist a random variable Z and non-decreasing (or non-increasing) functions $h_1, h_2, ..., h_n$ such that $(X_1, X_2, ..., X_n)$ is distributed as $(h_1(Z), h_2(Z), ..., h_n(Z))$. Comonotonicity means all random variables move in the same direction so that any two of possible outcomes $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ can be ordered componentwise.

For our purpose, the relevant mathematical property concerning the sum of comonotonic random variables is the quantile additivity for $S = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n$, $\alpha_i \geq 0$. This property states that

$$F^{-1}_{\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n}(p) = \sum_{i=1}^{n} \alpha_i F^{-1}_{X_i}(p),$$

where $F^{-1}_X(p)$ is the inverse function of X and $0 < p < 1$.

Let $\mathbf{E}[(X - d)^+]$ represent the stop-loss premium of a random variable X with retention d. It can be shown that if $X_1, X_2, ..., X_n$ are comonotonic then

$$\mathbf{E}[(S - d)^+] = \sum_{i=1}^{n} \mathbf{E}[(\alpha_i X_i - d_i)^+], \quad \text{for } F^{-1}_S(0) < d < F^{-1}_S(1), \tag{3.25}$$

where d_i is determined by

$$d_i = F^{-1}_{\alpha_i X_i}(F_S(d)), \; i = 1, 2, \ldots, n.$$

Note that here we assume the X_i's have strictly increasing distribution functions. In the general situation, one may adopt the α-inverse function to define the d_i's in (3.25), as described in Dhaene et al. [23]. A proof of the additivity properties of quantiles and stop-loss premiums can be found in Dhaene et al. [23] as well.

The calculation of stop-loss premium is similar in nature to that required for the GAO price given in equation (3.24). Therefore, we can make use of the additivity property of the stop-loss premium to decompose the expectation in equation (3.24) into a sum of expectations based on each random variable X_i assuming $X_1, X_2, ..., X_n$ are comonotonic. When each X_i is lognormally distributed, i.e. $\ln(X_i) \sim N(\mu_i, \sigma_i^2)$, equation (3.25) becomes

$$\mathbf{E}^{\widehat{Q}}[(S - K)^+] = \sum_{i=1}^{\infty} \alpha_i e^{\mu_i + \frac{1}{2}\sigma_i^2} \Phi\Big(\sigma_i - \Phi^{-1}(F_S(d))\Big) - K(1 - F_S(d)). \tag{3.26}$$

Equation (3.26) is an established result from Dhaene et al. [23] that provides a Black-Scholes-type formula for the price of a call option written on the sum S, which only depends on the marginal distribution of X_i's and S. Since F_{X_i}'s are strictly increasing and continuous, $F_S(d)$ can be uniquely determined from solving

$$\sum_{i=1}^{n} \alpha_i e^{-\mu_i + \sigma_i \Phi^{-1}(F_S(d))} = d.$$

Consider the truncated sum S in equation (3.20) defined by

$$S := \sum_{n=0}^{34} \beta(T, T+n) e^{-V(T,T+n)}. \tag{3.27}$$

Equation (3.27) implies that S is a sum of highly correlated random variables although it is not a comonotonic sum. Therefore, we need to find comonotonic approximations for S in order to make use of the additive property.

3.3.2 Comonotonic upper and lower bounds for the GAO Price

We construct approximations under the notion of convex order relation. This type of relation orders two random variables with the same mean in terms of their variability. More precisely, X is said to be smaller than Y in convex order, symbolised by $X \preceq_{cx} Y$, if and only if

$$\mathbf{E}[X] = \mathbf{E}[Y], \ \mathbf{E}[(X - d)^+] \leq \mathbf{E}[(Y - d)^+], \quad \text{for any } d \in (-\infty, \infty). \tag{3.28}$$

To simplify the notation, write

$$\alpha : = (\alpha_1, \alpha_2, \ldots) = (\beta(T,T), \beta(T,T+1), \ldots), \tag{3.29}$$

$$Y : = (Y_1, Y_2, \ldots) = (V(T,T), V(T,T+1), \ldots), \tag{3.30}$$

$$X : = (X_1, X_2, \ldots) = (e^{-Y_1}, e^{-Y_2}, \ldots). \tag{3.31}$$

Then $S = \sum_{n=0}^{34} \beta(T,T+n)e^{-V(T,T+n)} = \alpha X^\top = \alpha_1 X_1 + \alpha_2 X_2 + \ldots$, where $\alpha_i \geq 0$ and $\top$ is the transpose of a vector. Each Y_i is normally distributed with parameters

$$\begin{aligned} \mu_i &:= \mathbf{E}^{\widehat{Q}}[Y_i] = \mathbf{E}^{\widehat{Q}}[V(T,T+i-1)] \\ &= \mathbf{E}^{\widehat{Q}}\left[A(T,T+i-1)r_T + \widetilde{G}(T,T+i-1)\mu_T\right] \\ &= A(T,T+i-1)\mathbf{E}^{\widehat{Q}}[r_T] + \widetilde{G}(T,T+i-1)\mathbf{E}^{\widehat{Q}}[\mu_T], \end{aligned} \tag{3.32}$$

$$\begin{aligned} \sigma_i^2 &:= \mathrm{Var}^{\widehat{Q}}[Y_i] = \mathrm{Var}^{\widehat{Q}}[V(T,T+i-1)] \\ &= A^2(T,T+i-1)\mathrm{Var}^{\widehat{Q}}[r_T] + \widetilde{G}^2(T,T+i-1)\mathrm{Var}^{\widehat{Q}}[\mu_T] \\ &\quad +2A(T,T+i-1)\widetilde{G}(T,T+i-1)\mathrm{Cov}^{\widehat{Q}}[r_T,\mu_T], \end{aligned} \tag{3.33}$$

$$\begin{aligned} \sigma_{ij} &:= \mathrm{Cov}^{\widehat{Q}}[Y_i, Y_j] = \mathrm{Cov}^{\widehat{Q}}[V(T,T+i-1), V(T,T+j-1)] \\ &= A(T,T+i-1)A(T,T+j-1)\mathrm{Var}^{\widehat{Q}}[r_T] + \widetilde{G}(T,T+i-1)\widetilde{G}(T,T+j-1)\mathrm{Var}^{\widehat{Q}}[\mu_T] \\ &\quad +\Big(A(T,T+i-1)\widetilde{G}(T,T+j-1) + A(T,T+j-1)\widetilde{G}(T,T+i-1)\Big)\mathrm{Cov}^{\widehat{Q}}[r_T,\mu_T], \end{aligned} \tag{3.34}$$

for $i, j = 1, 2, \ldots$, and functions $A(T,T+i)$ and $\widetilde{G}(T,T+i)$ are specified in equations (3.4) and (3.17).

Therefore, the marginal distribution of X_i is lognormal with respective mean and variance

$$\begin{aligned} \mathbf{E}^{\widehat{Q}}[X_i] &= e^{-\mu_i + \frac{1}{2}\sigma_i^2}, \\ \mathrm{Var}^{\widehat{Q}}[X_i] &= e^{-2\mu_i + \sigma_i^2}(e^{\sigma_i^2} - 1). \end{aligned}$$

Now we look for the convex order upper and lower bounds for S, denoted by S^c and S^l, respectively. Following Dhaene et al. [24], we define S^c as

$$S^c = \sum_{i=1}^{35} \alpha_i e^{-\mu_i + \sigma_i \Phi^{-1}(U)}, \tag{3.35}$$

where α_i, μ_i, and σ_i are given in equations (3.29), (3.32) and (3.33), respectively. In equation (3.35), $U \sim$ Uniform(0,1) and $\Phi^{-1}(\cdot)$ is the inverse function of the $N(0,1)$ distribution. This upper bound is the convex-largest sum for S that one can obtain from any random vector $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$ with the same marginal distributions as the components in equation (3.27). In other words, for any random vector $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$ with the same marginal distributions as $(X_1, X_2, \ldots, X_n)$, it is always the case that

$$\alpha_1 \tilde{X}_1 + \alpha_2 \tilde{X}_2 + \cdots + \alpha_n \tilde{X}_n \preceq_{cx} S^c.$$

To construct the lower comonotonic bound for S, we first choose a random variable Λ (which must be "close" to S) and then use the conditional expectation $\mathbf{E}[S|\Lambda]$ as the comonotonic lower bound. This construction is in accordance with Theorem 10 in Dhaene et al. [23]. We choose Λ to be the first-order Taylor approximation of S. That is, $\Lambda = \sum_{i=1}^{35} \alpha_i Y_i$ where Y_i is defined in equation (3.30) and Λ is normally distributed with parameters

$$\begin{aligned} \mu_\Lambda := \mathbf{E}^{\widehat{Q}}[\Lambda] &= \sum_{i=1}^{35} \alpha_i \mu_i, \\ \sigma_\Lambda^2 := \mathrm{Var}^{\widehat{Q}}[\Lambda] &= \sum_{i=1}^{35} \sum_{j=1}^{35} \alpha_i \alpha_j \sigma_{ij}. \end{aligned}$$

Moreover, (Y_i, Λ) is a bivariate normal random variable and

$$\mathrm{Cov}^{\widehat{Q}}[Y_i, \Lambda] = \sum_{j=1}^{35} \alpha_j \sigma_{ij}.$$

We define the correlation coefficient between Y_i and Λ as

$$r_i := \frac{\mathrm{Cov}^{\widehat{Q}}[Y_i, \Lambda]}{\sigma_i \cdot \sigma_\Lambda}.$$

Since $Y_i|\Lambda \sim N\left(\mu_i + r_i\frac{\sigma_i}{\sigma_\Lambda}(\Lambda - \mu_\Lambda), (1-r_i)^2\sigma_i^2\right)$, we have

$$\mathbf{E}[X_i|\Lambda] = e^{-\mu_i - r_i\frac{\sigma_i}{\sigma_\Lambda}(\Lambda-\mu_\Lambda)+\frac{1}{2}(1-r_i)^2\sigma_i^2}. \tag{3.36}$$

Note that $\mathbf{E}[X_i|\Lambda]$ is a lognormal random variable being a function of Λ. We replace $\frac{\Lambda-\mu_\Lambda}{\sigma_\Lambda}$ by $\Phi^{-1}(V)$, where $\Phi^{-1}(\cdot)$ is the inverse function of a standard normal random variable and $V \sim$ Uniform(0, 1). Hence, equation (3.36) can be rewritten as

$$\mathbf{E}[X_i|\Lambda] = e^{-\mu_i - r_i\sigma_i\Phi^{-1}(V)+\frac{1}{2}(1-r_i)^2\sigma_i^2}$$

so that the convex lower bound of S is

$$S^l = \mathbf{E}[S|\Lambda] = \sum_{i=1}^{35}\alpha_i\mathbf{E}[X_i|\Lambda] = \sum_{i=1}^{35}\alpha_i e^{-\mu_i - r_i\sigma_i\Phi^{-1}(V)+\frac{1}{2}(1-r_i)^2\sigma_i^2}. \tag{3.37}$$

To sum up, S^c and S^l defined in equations (3.35) and (3.37) are the convex order upper and lower bounds for S, i.e., $S^l \preceq_{cx} S \preceq_{cx} S^c$. Since both S^c and S^l are comonotonic sum of lognormal random variables, we can now apply formula (3.26) to them to derive upper and lower bounds for the GAO price. We get

$$\mathbf{E}^{\widehat{Q}}[(S^c - K)^+] = \sum_{i=1}^{35}\alpha_i e^{-\mu_i+\frac{1}{2}\sigma_i^2}\Phi(\sigma_i - \Phi^{-1}(F_{S^c}(K))) - K(1 - F_{S^c}(K)), \tag{3.38}$$

where $F_{S^c}(K)$ satisfies

$$\sum_{i=1}^{35}\alpha_i e^{-\mu_i+\sigma_i\Phi^{-1}(F_{S^c}(K))} = K.$$

Also,

$$\mathbf{E}^{\widehat{Q}}[(S^l - K)^+] = \sum_{i=1}^{35}\alpha_i e^{-\mu_i+\frac{1}{2}(1-r_i)^2\sigma_i^2+\frac{1}{2}r_i^2\sigma_i^2}\Phi(r_i\sigma_i - \Phi^{-1}(F_{S^l}(K))) - K(1 - F_{S^l}(K)), \tag{3.39}$$

where $F_{S^l}(K)$ satisfies

$$\sum_{i=1}^{35}\alpha_i e^{-\mu_i+\frac{1}{2}(1-r_i)^2\sigma_i^2 - r_i\sigma_i\Phi^{-1}(F_{S^l}(K))} = K.$$

The quantile functions for the upper and lower bounds, S^c and S^l, are then given by

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{35}\alpha_i e^{-\mu_i+\sigma_i\Phi^{-1}(p)} \tag{3.40}$$

and

$$F_{S^l}^{-1}(p) = \sum_{i=1}^{35}\alpha_i e^{-\nu_i+r_i\sigma_i\Phi^{-1}(p)+\frac{1}{2}(1-r_i)^2\sigma_i^2}, \tag{3.41}$$

respectively. For more details on how to derive formulae (3.38)-(3.41), readers are referred to section 3.4.2 of Dhaene et al. [24].

3.4 Numerical illustration

We examine the reasonableness of our approximations to the sum of lognormal random variables. We shall also provide approximate bounds for the GAO price based on equations (3.38) and (3.39) and compare them to the simulated "true" values based on formula (3.24). The values of the model parameters used in the numerical implementation are given in Table 3.1.

Table 3.1: Parameter values used in the numerical experiment in chapter 3.

Parameter set for numerical analysis
Contract specification
$g = 11.1\%$, $T = 15$, $n = 35$;
Interest rate model
$a = 0.15$, $b = 0.045$, $\sigma = 0.03$, $r_0 = b$;
Mortality model
$c = 0.1$, $\xi = 0.0003$, $\mu_0 = 0.006$.

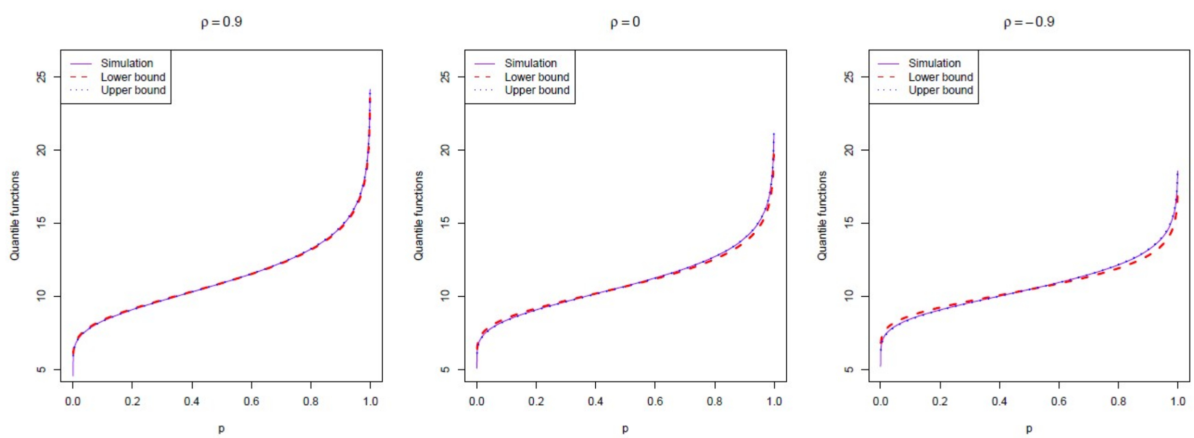

Figure 3.1: Quantile functions of S, S^c and S^l with $\rho = 0.9$, $\rho = 0$, and $\rho = -0.9$.

Figure 3.1 depicts the plots of quantile functions for S^c and S^l obtained using equations (3.40) and (3.41). The quantile plots are accompanied by a middle plot for S generated using 50,000 simulations with $\rho = 0.9$, $\rho = 0$, and $\rho = -0.9$. With $\rho = 0.9$, we see that both the upper

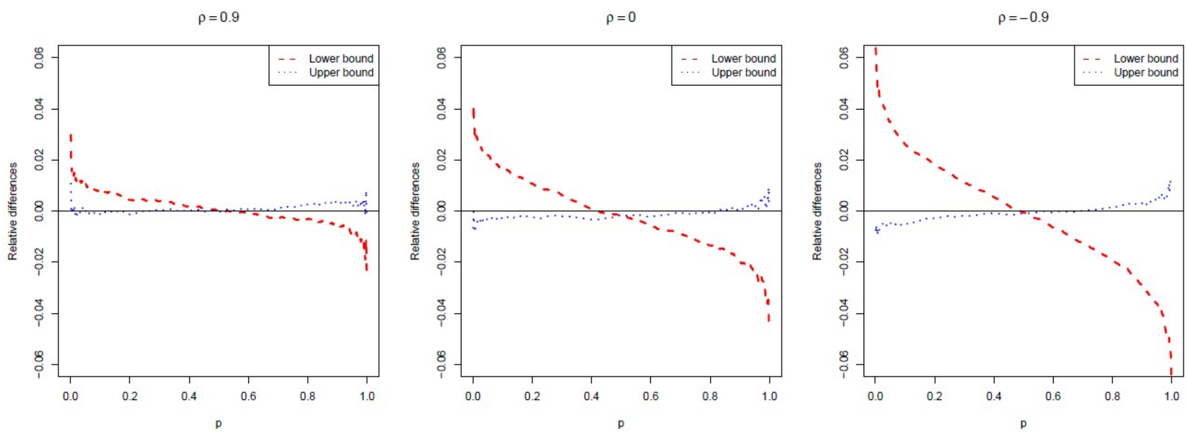

Figure 3.2: Relative differences between the approximated quantiles of S^c (or S^l) and the simulated quantiles, with $\rho = 0.9$, $\rho = 0$, and $\rho = -0.9$.

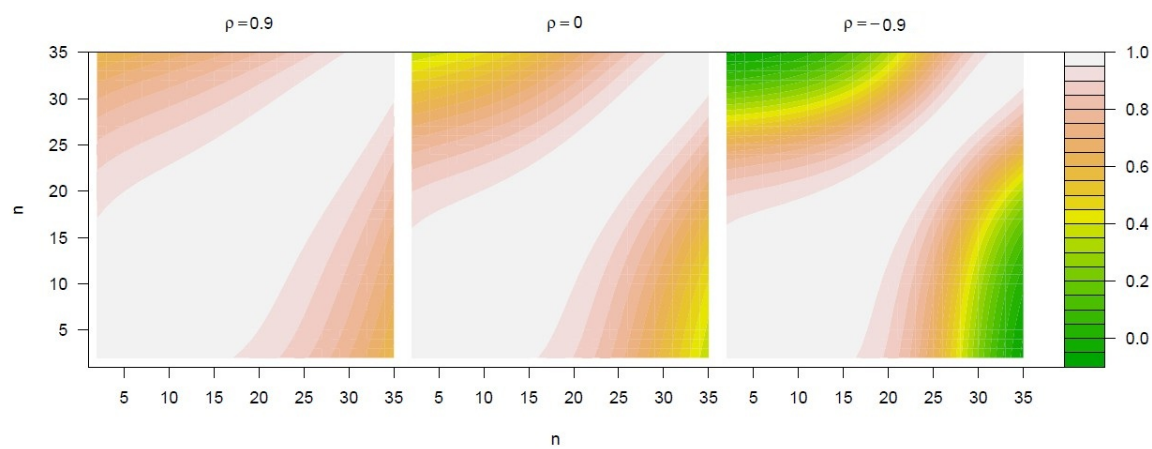

Figure 3.3: Contour maps with $\rho = 0.9$, $\rho = 0$, and $\rho = -0.9$.

and lower bounds are very accurate, and the differences from the "true" values are negligible. When ρ changes from positive to negative, the upper bound remains extremely accurate, whilst the lower bound becomes less accurate although it is still a reasonably good approximation. This is validated by Figure 3.2 where the relative differences between the approximated quantile functions and the simulated quantiles are displayed.

In previous research works Denuit [21, 22], lower bounds generally give more accurate approximations. However, in our case, upper bounds are better than lower bounds when benchmarked to simulated values. To find plausible rationale for this observation, we draw the contour map based on the correlation coefficients amongst the individual random variables X_i's contained in the sum S in equation (3.27). The contour maps with different ρ's are displayed in Figure 3.3.

From Figure 3.3, it appears that the good performance of the upper bounds could be attributed to the fact that the X_i's are highly positive correlated, particularly, when $\rho > 0$. When $\rho = -0.9$, that is, when interest rate r_t and mortality rate μ_t are negatively correlated, the correlation between X_i and X_j goes to around 0 when i and j are far apart. This kind of dependency structure may be addressed by selecting another Λ that could lead to improved lower bounds. Interested readers are referred to Vanduffel et al. [45] for more details regarding optimal approximation.

The values of the GAO price based on the upper and lower comonotonic approximations S^c and S^l can then be calculated analytically. In Table 3.2, we provide these calculated values together with the GAO price utilising formula (3.24) with 50,000 simulations; the numbers enclosed in parentheses are standard errors. Figure 3.4 is a plot of GAO prices versus the parameter ρ. Indeed, the upper bounds based on S^c are very close to the "true" values for any ρ, $-1 \leq \rho \leq +1$. The accuracy of the lower bounds are more pronounced when ρ is positive. Overall, our numerical experiments provide support for the highly adequate performance of our proposed comonotonic approximations.

3.5 Conclusions

We presented a particular important application of comonotonicity theory in the valuation of GAOs. This contributes to the available methods of pricing annuity-linked contracts by specifically circumventing the issue of dealing with the sums of a large number of lognormal random variables. Using Monte Carlo simulated "true" values as our benchmark, we demonstrated the accuracy of our lower and upper bound approximations to a notable degree. We designed a valuation framework that incorporates the stochastic nature of both mortality and interest rates as well as their correlation structure. Our approach made use of the change of probabil-

Table 3.2: Actuarial prices for GAO under different methods in chapter 3.
Numbers in parentheses are standard errors.

	GAO value			
ρ	Simulated values		Lower bound	Upper bound
-1.0	0.0904026	(0.0003836)	0.0878802	0.0905889
-0.9	0.0920624	(0.0003916)	0.0898968	0.0925702
-0.8	0.0943914	(0.0004018)	0.0919473	0.0945658
-0.7	0.0962496	(0.0004116)	0.0940276	0.0965768
-0.6	0.0986634	(0.0004217)	0.0961350	0.0986044
-0.5	0.1003584	(0.0004320)	0.0982680	0.1006494
-0.4	0.1023678	(0.0004412)	0.1004255	0.1027128
-0.3	0.1042501	(0.0004485)	0.1026068	0.1047953
-0.2	0.1067286	(0.0004587)	0.1048118	0.1068977
-0.1	0.1088653	(0.0004687)	0.1070402	0.1090209
0	0.1110679	(0.0004790)	0.1092922	0.1111654
0.1	0.1131000	(0.0004896)	0.1115679	0.1133321
0.2	0.1153378	(0.0005001)	0.1138677	0.1155216
0.3	0.1174438	(0.0005131)	0.1161919	0.1177346
0.4	0.1197348	(0.0005241)	0.118541	0.1199718
0.5	0.1218968	(0.0005350)	0.1209154	0.1222339
0.6	0.1246585	(0.0005464)	0.1233156	0.1245216
0.7	0.1263725	(0.0005509)	0.1257422	0.1268356
0.8	0.1290466	(0.0005682)	0.1281958	0.1291765
0.9	0.1317430	(0.0005813)	0.1306770	0.1315453
1.0	0.1338156	(0.0005901)	0.1339424	0.1331865

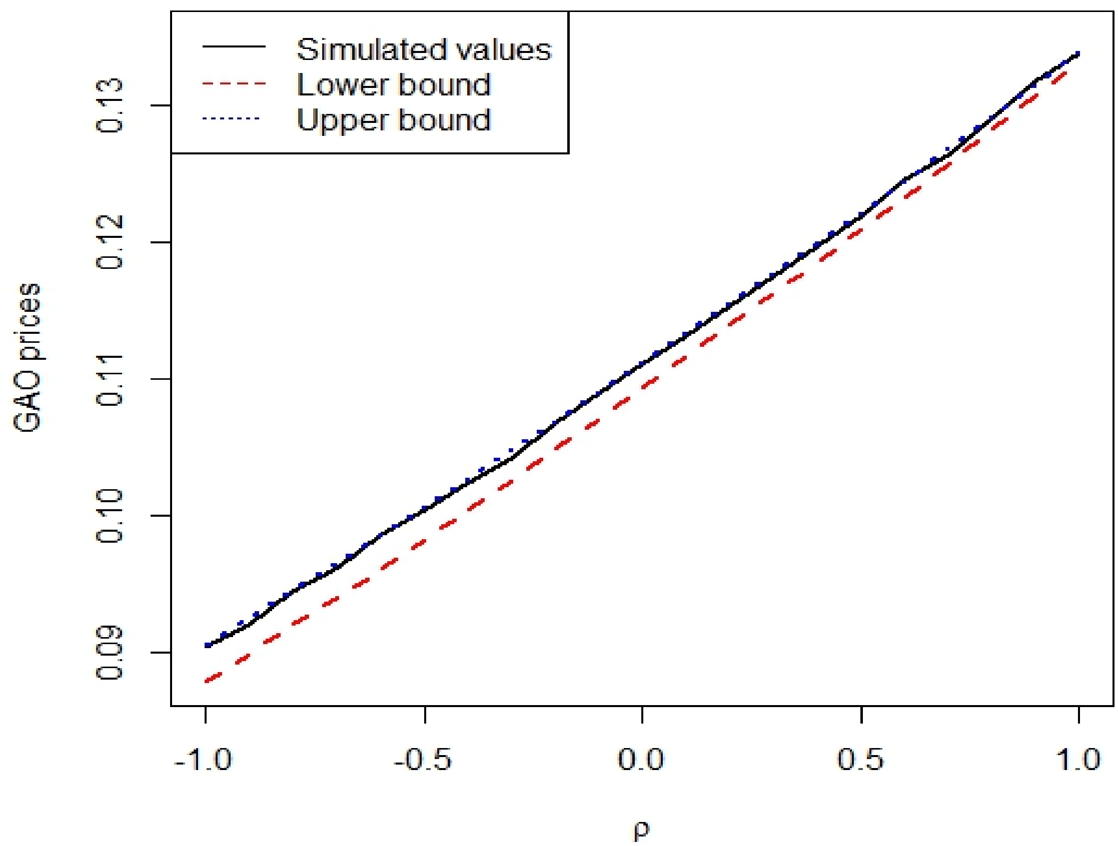

Figure 3.4: The values of the GAO price with its upper and lower approximations.

ity measures technique with the appropriate choice of numéraire to considerably facilitate the evaluation of conditional expectation for valuation. The proposed methodology here may be adopted to value other insurance products with option-embedded features such as equity-linked annuities, equity indexed annuities and variable annuities.

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Chapter 4

Mortality modelling with regime-switching for the valuation of GAO

4.1 Introduction

Regime-switching (RS) models have gained popularity in modelling financial time series due to their ability to capture dynamic changes exhibited by the stochastic behaviour of economic and financial variables observed over time. Research works on RS models can be divided into two kinds: threshold models and Markov RS models. The difference between these two lies on the trigger of regime shifts, say, an observed variable for the former and a Markov chain for the latter, see Meyers [39] for details. Goldfeld and Quandt [24] proposed the switching regression model for housing markets data and this was viewed as one of the earliest applications of Markov RS models in economics. Markov-switching models in discrete time setting were extensively studied by Hamilton [26, 27] with special emphasis on economic and financial modelling. Since then, various interest rate models were developed that incorporate regime-switching characteristics either in the rate level itself or in the parameter dynamics. See Ang and Bekaert [1], Elliott and Mamon [16, 17], Elliott and Siu [18], Garcia and Perron [21], Lewis [32], Mamon [36], amongst others. Moreover, RS models have been widely used in asset price modelling and pricing equity options, see for instance, Elliott et al. [19], Hardy [28] and Yuen and Yang [52].

The increasing utility of regime-switching models in finance has influenced research on their applications in actuarial science. Milidonis et al. [40] introduced a regime-switching approach to model mortality dynamics and highlighted favourable features of regime-switching models in mortality modelling. The distinguishing features of RS models include the capacity to identify structural changes and the flexibility to make parameter estimates change as time evolves depending on the dictates of the data. In their paper, they adopted a general RS model that switches between two geometric Brownian motions to model the annual US mortality index. In addition, they applied RS approach in modelling the time-varying mortality index under the Lee-Carter mortality framework.

Prior to 2000, techniques of mortality modelling in practice have been traditionally deterministic. Since Gompertz [25] first proposed the law of mortality asserting that the person's probability of dying increases at a constant exponential rate as age increases, many research works were developed based upon it, see for instance Finch and Pike [20], Shklovskii [44] and Trachtenber [46]. It is well-accepted that the Gompertz's law holds between the ages of 30 and 90 over a wide time range of mortality data; see Spiegelman [43] and Wetterstrand [49]. Tenenbein and Vanderhoof [45] provided biophysical implications of the foundation of such a law. In Wetterstrand [49], the ultimate mortality experience from life insurance for 1948-1977 was examined and changes of parameters in the law were described. At higher ages, say 90, the fluctuation of mortality is not easily dealt with by the Gompertz mortality law due to small sample sizes (cf. Bell and Miller [5]) and data aggregation problems (cf. Gavrilova and Gavrilov [22]). But, as mortality keeps on improving so that more people survive longer than 90 years, the small-sample size problem is mitigated, and it is reasonable to expect that the exponential aging law for humans is sustained at these ages. Some works on the modelling of mortality at higher ages include Beddington et al. [4] and Bongaarts [8].

In this chapter, the rationality of the Gompertz law is examined as our starting point using US data from 1933-2009. The yearly mortality rates were found to follow an exponential increasing trend or the logarithm of the mortality rates has a linear form, which is in agreement with the Gompertz model; this result was determined using the regression method. Our analysis of the mortality patterns demonstrate dynamic changes in the parameters of the Gompertz model as the years go by. We, therefore, enrich the Gompertz model by putting forward a regime-switching model based on Gompertz law. We shall coin the term RS Gompertz model (RSGM) for the first new model proposed in this work. Specifically, we employ a continuous-time Markov chain to modulate the parameters of the Gompertz model. The Markov chain captures the switching in the level of the rate governing the parameters' movement.

To provide data-based evidence of the capability of the proposed mortality model, we include an empirical study demonstrating the goodness-of-fit and likelihood-based measures supporting the superiority of the Markov-switching model compared to its one-regime counterpart. In particular, the hidden Markov model filtering approach described in Mamon et al. [37] is employed and applied to US mortality data spanning a period of nearly eight decades to provide dynamic estimates of model parameters.

The primary goal in developing a mortality model with good adequacy is to support the pricing of insurance and annuity products. Many product innovations in recent times were introduced, and their pricing and reserving present new challenges. It is now common for products to have investment guarantees in them and therefore, they involve embedded options (cf. Bolton et al. [7], and Boyle and Hardy [9]). The stochastic modelling of two key factors, namely, the interest and mortality rates (see Ballota and Haberman [3], and Milevsky and Promislow [38]) is the most important consideration in valuation. Although there have been substantial research achievements in insurance pricing under a stochastic environment as shown in Dahl [13] and Hardy [29], progress is little to nil in the pricing of products especially those with long-dated horizons under a consistent framework in which both interest and mortality rates are regime-switching. This work constructs a new framework whereby the interest rate process follows a pure Markov RS model whilst the mortality rates are deemed to follow a regime-switching mortality model. The pricing, hedging and reserving problem for guaranteed annuity options (GAOs) were studied in Liu et al. [33] and Wilkie et al. [50], amongst others. In this work, we further propose two alternative RS mortality models motivated by the limitation of the RSGM, which does not provide analytical solution to the survival probability. We then consider the

GAO pricing along with a numerical implementation and investigation of price sensitivity to parameters of a combined regime-switching models under the two alternatives. Aiming for an analytic pricing solution for GAO as our target product, we develop a Gompertz model with RS regression parameters as well as extend the methodology in Elliott and Mamon [17]. We then derive the endowment price under the assumption that the two factors are driven by independent Markov chains under each alternative modelling set-up. Consequently, we use the so-called endowment-risk-adjusted measure, which was first introduced in Liu et al. [33] and show the significant benefits of the change of measure technique over Monte-Carlo-based implementation. In addition, we show the flexibility of RS models in accommodating a range of underlying states in our model framework by changing the states of each random factor.

The essence of this chapter is to offer three ways of formulating a regime-switching mortality framework, where the first framework is well-supported by the data, and the other two are less sophisticated than the first one but they are designed to produce GAO analytic pricing solutions. To attain our objectives, this chapter is organised as follows. Section 4.2 presents the proposed three regime-switching mortality models. An analysis of the mortality trend using the US data for the period 1933-2009 validates the utility of the RSGM. Two alternative RS models are provided along with their analytical solutions to the survival index. The pure Markov interest rate model is described including the derivation of bond price in section 4.3. In section 4.4, we formulate the RS framework to value a GAO by integrating a Markov interest rate model and the alternative mortality models put forward in this work. Interest and mortality processes are governed by two independent Markov chains for tractability. As will be illustrated, the measure-change method facilitates efficiently the pricing implementation. Numerical results are given in section 4.5. Some remarks and an indication of future works conclude this work in section 4.6.

4.2 Mortality models

4.2.1 Model 1 (M1): Gompertz model with BM- and Markov-switching parameters

4.2.1.1 Mortality analysis

The stochasticity of the mortality rate is documented in Pitacco [41] and Tuljapurkar and Boe [47]. A proposed model aimed to capture salient features of a mortality process must be supported by empirical evidence. So, before embarking on the full development of a mortality model, we first examine the US mortality data from 1933 to 2009 available and downloadable from the Human Mortality Database (URL: www.mortality.org). Since we are aiming at pricing pension products, we focus on older ages, in particular, from 50 to 100.

To understand better the evolution of mortality rates, we plot in 3 dimensions the mortality rates as a function of age and year as shown in Figures 4.1. Moreover, we display the log mortality rates through age for certain specific years in Figure 4.2, and the graphs show an approximately linear trend each year. This, consistent with the traditional Gompertz law, is based on the model that for a life aged x at time t, the force of mortality model is given by

$$\log \mu(x,t) = a(t) + b(t)\, x. \tag{4.1}$$

However, we find that for each year the log of mortality has different intercept and slope. This leads us to find the patterns of these two regression parameters shown in Figure 4.3. We see that in the long run, the intercept in the Gompertz model has a decreasing behaviour whilst the slope has an increasing trend. This is in agreement with the consensus that the time series of slopes and intercepts of mortality curves are negatively correlated, see Yashin et al. [51]. The graphs also show that the movements of these parameters vary over time. The rate of increase or decrease of model parameters in (4.1) is high for certain time periods and low in other time periods. The ANOVA tests and the analysis of residuals illustrated by the contour map in Figure 4.4 show that the linear model fits the log of mortality rate well.

The study of US mortality data above signifies that mortality rates can be described by an exponential increasing functional form. We note that this phenomenon varies with time. This serves as our inspiration in putting forward the RS approach for mortality modelling, which is detailed in the next section.

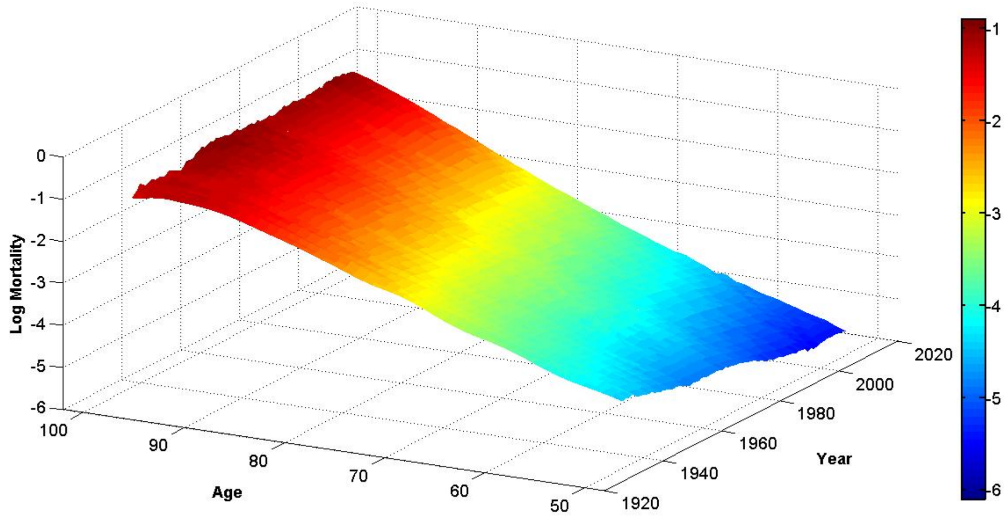


Figure 4.1: 3-dimensional plot of mortality rates versus years and ages.

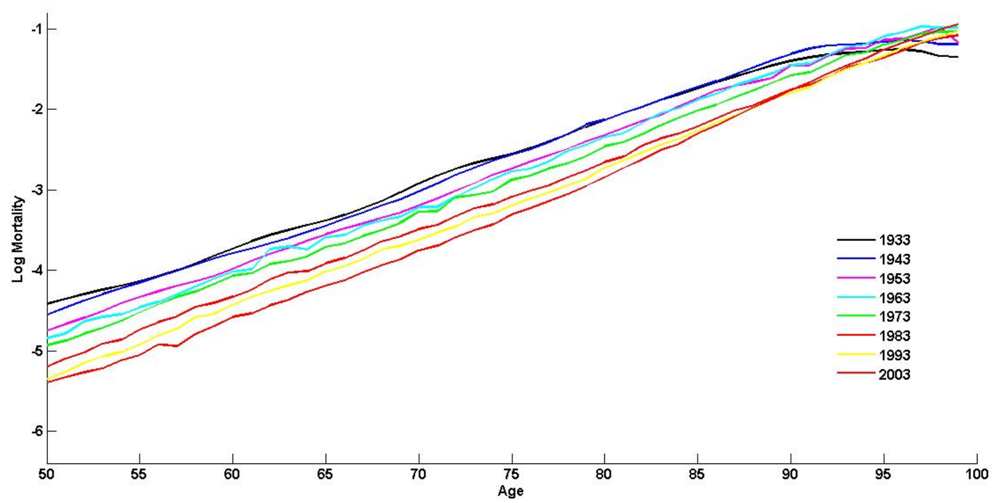
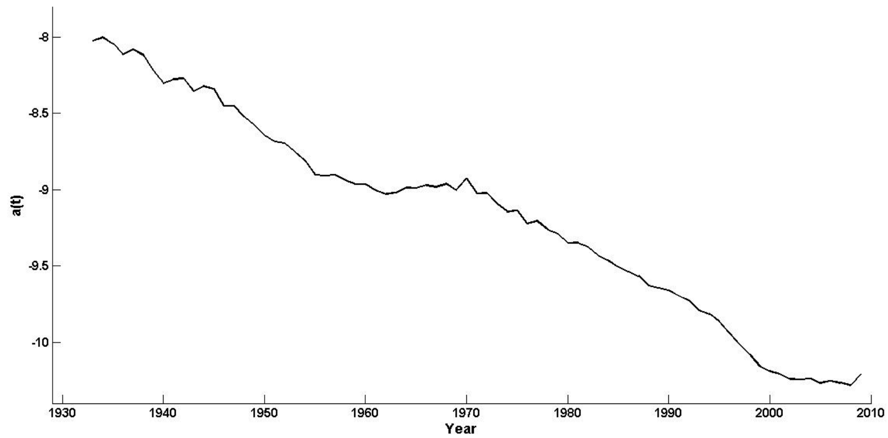
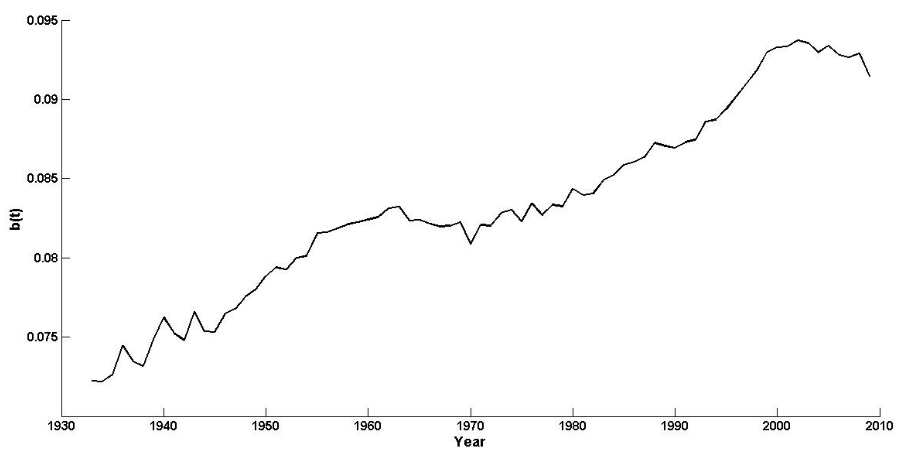


Figure 4.2: US mortality rate in different years for the period 1933–2009.



(a) $a(t)$



(b) $b(t)$

Figure 4.3: Evolution of parameters in the Gompertz model based on 1933–2009 US data.

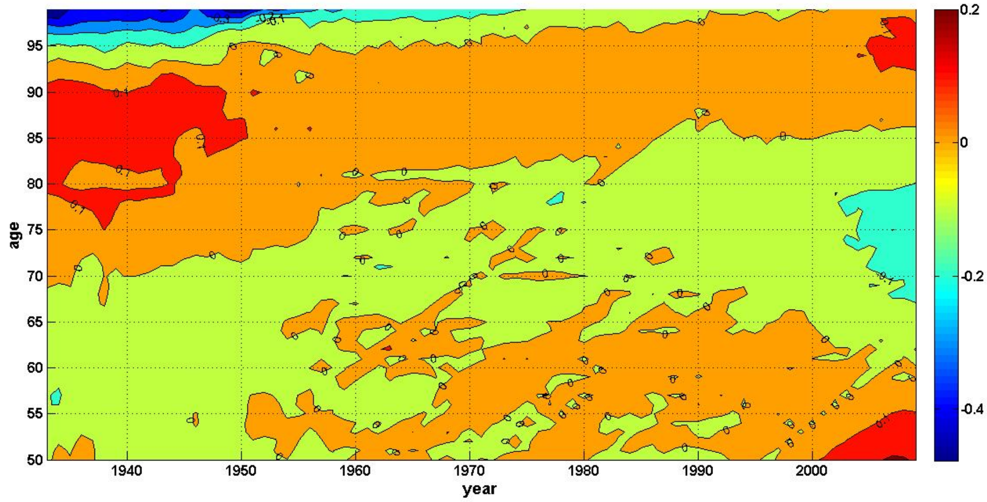


Figure 4.4: Contour map of residuals.

4.2.1.2 Model description

From the mortality analysis above, it is observed that the regression parameters of the mortality model in equation (4.1) exhibit varying levels with respect to time. Thus, for a life aged x at time t, we propose that the force of mortality follows

$$\log \mu(x, t, \boldsymbol{y}_t) = a(\boldsymbol{y}_t) + b(\boldsymbol{y}_t)x, \tag{4.2}$$

where $\boldsymbol{y}_t,\ t \geq 0$ is a finite-state Markov chain with state space $S_{\boldsymbol{y}} = \left\{\mathbf{s}^1_{\boldsymbol{y}}, \mathbf{s}^2_{\boldsymbol{y}}, \ldots, \mathbf{s}^m_{\boldsymbol{y}}\right\}$. The points in $S_{\boldsymbol{y}}$ are also associated with the unit vectors $\left\{\mathbf{e}^1_{\boldsymbol{y}}, \mathbf{e}^2_{\boldsymbol{y}}, \ldots, \mathbf{e}^m_{\boldsymbol{y}}\right\}$ where $\mathbf{e}^i_{\boldsymbol{y}} = (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^m$. The unconditional distribution of $\boldsymbol{y}_t$ is

$$\boldsymbol{q}_t = \mathbf{E}[\boldsymbol{y}_t] = (q^1_t, q^2_t, \ldots, q^m_t),$$

where

$$q^i_t = P\left[\boldsymbol{y}_t = \boldsymbol{e}^i_{\boldsymbol{y}}\right] = \mathbf{E}\left[\langle \boldsymbol{e}^i_{\boldsymbol{y}}, \boldsymbol{y}_t\rangle\right],$$

with $\langle\cdot,\cdot\rangle$ denotes the usual scalar product of two vectors in $\mathbb{R}^m$ and P is the probability measure under which we observe the data. Here, $\boldsymbol{y}_t$ is a semimartingale with representation

$$\boldsymbol{y}_t = \boldsymbol{y}_0 + \int_0^t \boldsymbol{\Lambda}\boldsymbol{y}_u\, du + \boldsymbol{n}_t,$$

where $\boldsymbol{\Lambda}$ is the intensity matrix of $\boldsymbol{y}_t$ and $\boldsymbol{n}_t$ is a $(\mathcal{J}_t, P)$ martingale. We assume that this measure P is equivalent to a pricing measure $\widetilde{P}$ in GAO valuation. As noted in Mamon [36], any diffusion can be approximated by a Markov chain so that even without the white-noise driven error term, the model in equation (4.2) is sufficient to capture the randomness in mortality-rate behaviour.

The respective dynamics of the $a(t)$ and $b(t)$ processes are assumed to follow

$$da(t) = \alpha(\boldsymbol{y}_t)dt + \beta(\boldsymbol{y}_t)dW_t^1 \tag{4.3}$$

and

$$db(t) = \gamma(\boldsymbol{y}_t)dt + \zeta(\boldsymbol{y}_t)dW_t^2, \tag{4.4}$$

where $\boldsymbol{y}_t$ and W_t^1 as well as W_t^2 are independent processes defined under P. The dynamics in (4.3) and (4.4) were discretised and have the respective representation

$$\xi_k := a_k - a_{k-1} = \alpha(\boldsymbol{y}_k) + \beta(\boldsymbol{y}_k)\epsilon_k^1 = \langle \boldsymbol{\alpha}, \boldsymbol{y}_k \rangle + \langle \boldsymbol{\beta}, \boldsymbol{y}_k \rangle \epsilon_k^1, \tag{4.5}$$

and

$$\eta_k := b_k - b_{k-1} = \gamma(\boldsymbol{y}_k) + \zeta(\boldsymbol{y}_k)\epsilon_k^2 = \langle \boldsymbol{\gamma}, \boldsymbol{y}_k \rangle + \langle \boldsymbol{\zeta}, \boldsymbol{y}_k \rangle \epsilon_k^2, \quad k = 1, 2, \ldots, \tag{4.6}$$

where $\epsilon_k^i \sim N(0, 1)$, for $i = 1,\ 2$.

The plot of the ξ_k process is depicted in Figure 4.5 which clearly shows regime switches through time in the mean and volatility levels. Although we see that $\{a_k\}$ and $\{b_k\}$ are negatively correlated from Figure 4.3, we assume that ϵ_k^1 and ϵ_k^2 are independent for the purpose of filtering implementation. However, some dependence structure between the two series is captured since both are functions of the same Markov chain $\boldsymbol{y}_t$.

Recursive filters derived using the change of probability measure technique described in Mamon et al. [37] are then implemented to the ξ_k-data series. We obtain the maximum likelihood estimates for the parameters, and their dynamic evolutions under the two-state setting are exhibited in Figures 4.6 - 4.8 from 1933-2009.

When the entire data set is used without processing a window of data, i.e., static estimation is considered, we get the parameter values for the 1-state, 2-state, and 3-state settings shown in

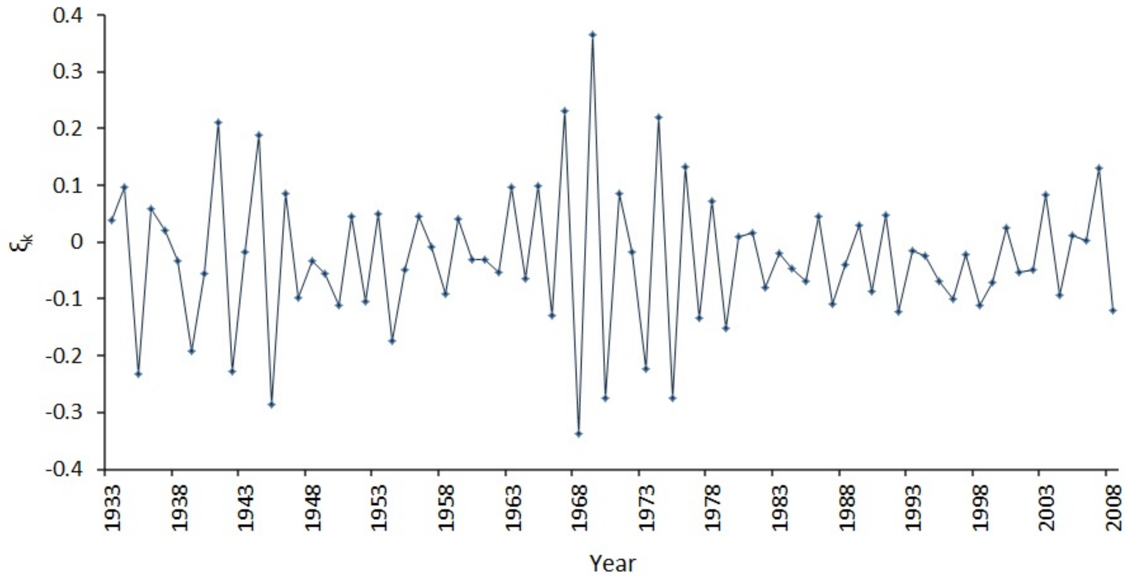


Figure 4.5: ξ_k process.

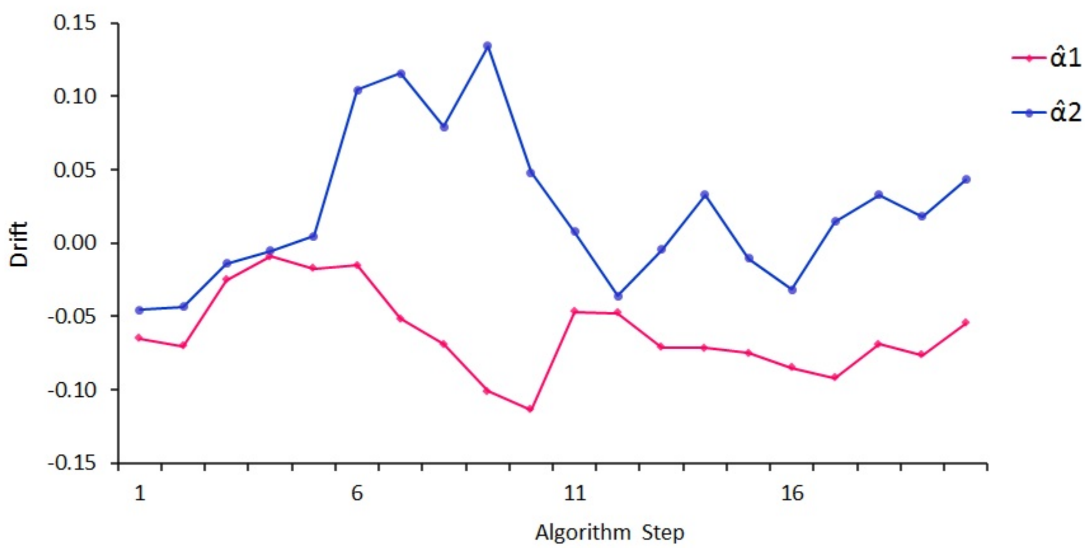


Figure 4.6: Evolution of $\hat{\alpha}$ under the 2-state setting.

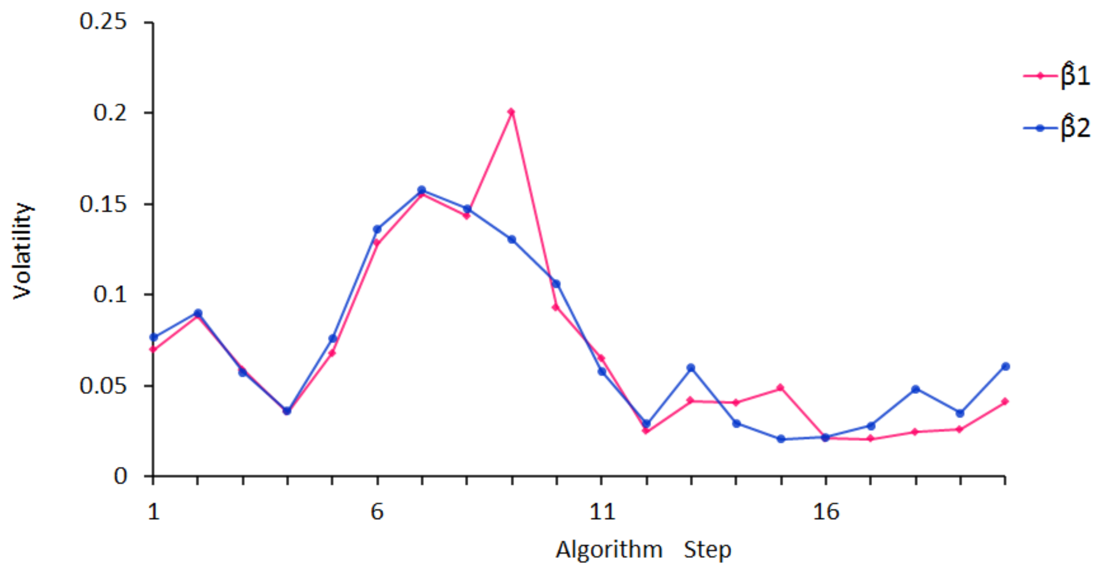


Figure 4.7: Evolution of $\hat{\beta}$ under the 2-state setting.

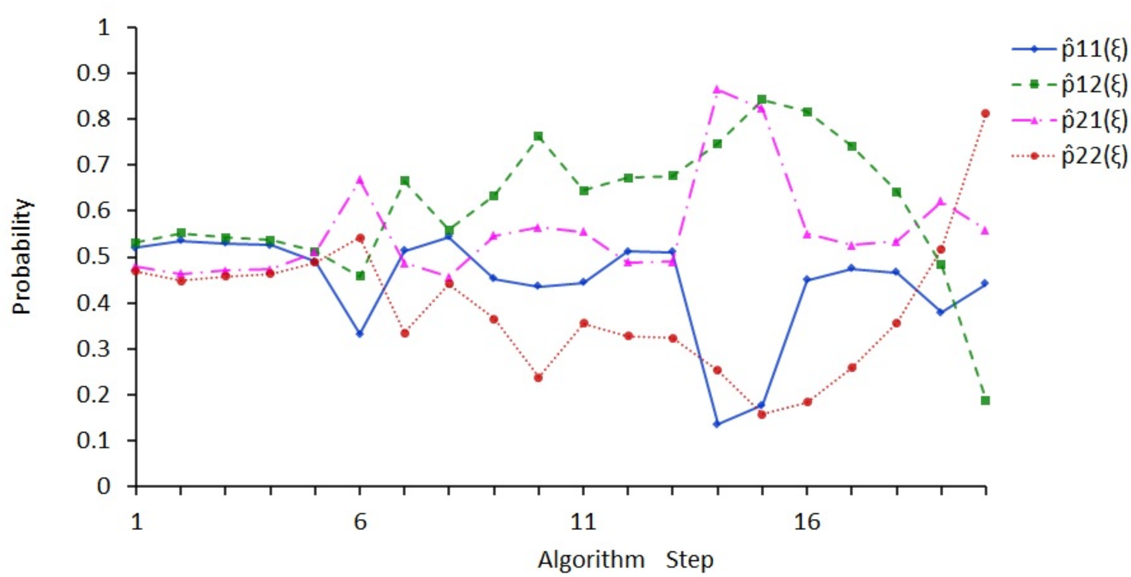


Figure 4.8: Evolution of the transition probabilities for ξ_k under the 2-state setting.

Table 4.1: Estimated parameters of ξ_k for different state settings under static calibration.

Parameters	1-state	2-state	3-state
α_1	-0.02893	-0.02371	-0.02703
α_2	- -	-0.03496	0.03201
α_3	- -	- -	-0.06987
β_1	0.12353	0.17798	0.18488
β_2	- -	0.06071	0.04165
β_3	- -	- -	0.04029
$p_{12}(\xi)$	- -	0.06084	0.06038
$p_{13}(\xi)$	- -	- -	0.00100
$p_{21}(\xi)$	- -	0.04164	0.00900
$p_{23}(\xi)$	- -	- -	0.99000
$p_{31}(\xi)$	- -	- -	0.05450
$p_{32}(\xi)$	- -	- -	0.50948

Table 4.1.

From Table 4.2, we see that the 2-state Markov switching model provides the best fit for the data in accordance with both the Akaike information criterion (AIC) and Bayes information criterion (BIC). The second column also gives the calculated likelihood (LL) values.

Similarly, we show the plot of the η_k process depicted in Figure 4.9, which again exhibit regime switches in means and volatility through time. The maximum likelihood estimates for η_k's model parameters, and their dynamic evolutions under the two-state setting are exhibited in Figures 4.10 - 4.12.

With the entire data set as input (i.e., static estimation), we get the parameter values under the 1-state, 2-state, and 3-state settings shown in Table 4.3.

Table 4.2: Goodness-of-fit test for different models of ξ_k.

State setting	LL	BIC	AIC
1-state	64.0696	59.7389	62.0696
2-state	79.3603	66.36801	73.3603
3-state	81.0457	55.0613	69.0457

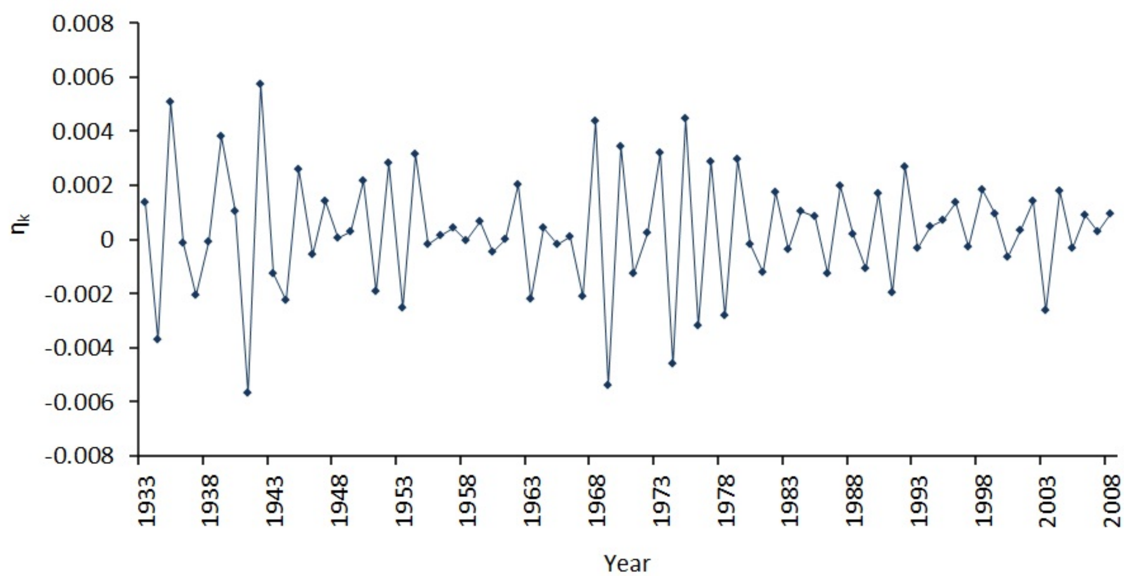


Figure 4.9: η_k process.

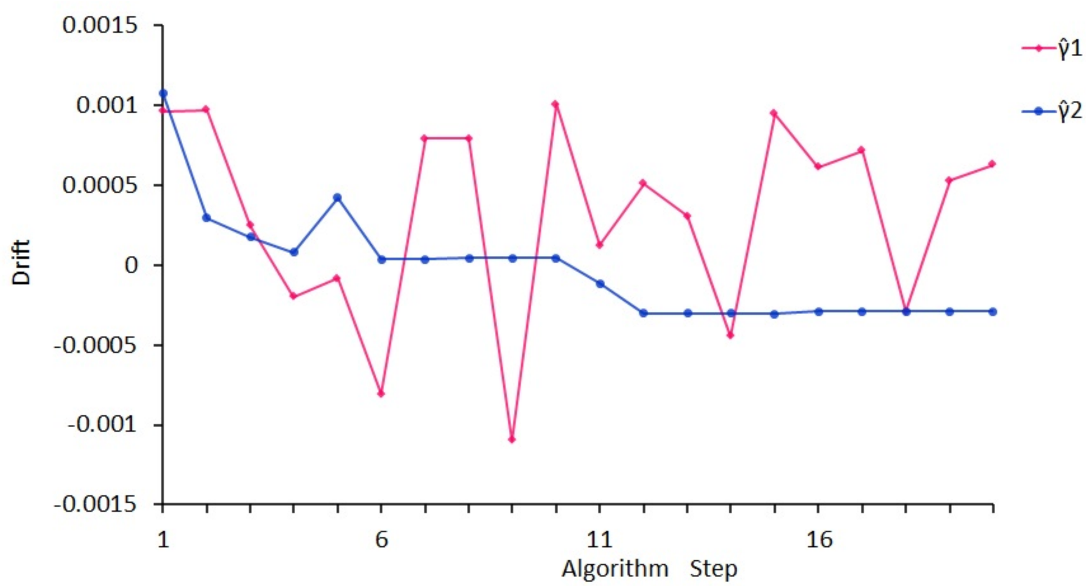


Figure 4.10: Evolution of $\hat{\gamma}$ under the 2-state setting.

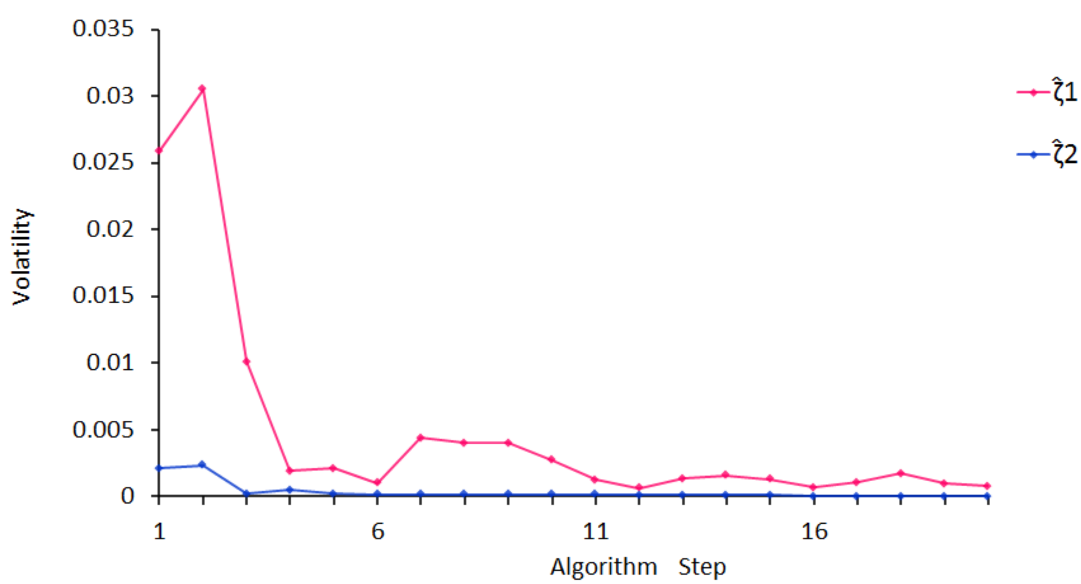


Figure 4.11: Evolution of $\hat{\zeta}$ under the 2-state setting.

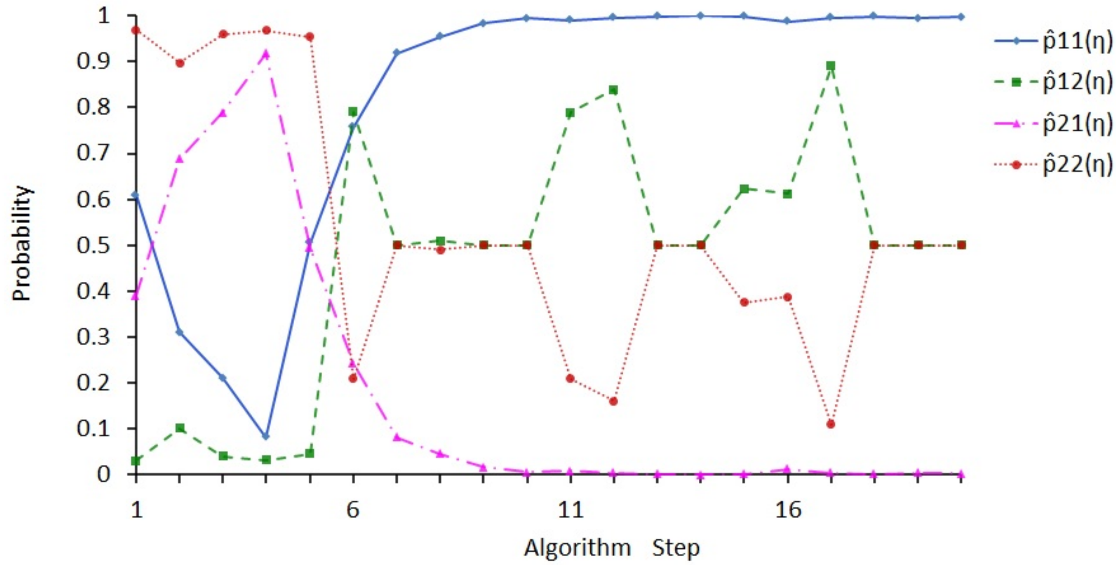


Figure 4.12: Evolution of the transition probabilities for η_k under the 2-state setting.

From Table 4.4, we see that the 2-state Markov switching model provides the best fit for the data in accordance with both the Akaike information criterion (AIC) and Bayes information criterion (BIC). The estimated LL values are also displayed in the second column.

4.2.2 Model 2 (M2): Gompertz with pure Markov-switching parameters

The RSGM termed as M1 does not have an analytical solution to survival probability, and therefore the GAO price does not also have an analytic representation. Analytic pricing solutions are not only elegant, but useful and relevant in practice. In particular, they enable the sensitivity analysis of model parameters and possible calibration of the model using market data (if available and reliable). To keep the focus of this work on GAO pricing development and the justification of the regime-switching framework, sensitivity analysis and calibration are relegated as future directions of this research.

To strike a balance between the empirical analyses in the previous subsection and the objective of getting a closed-form solution for the survival index (and hence, of the GAO price),

Table 4.3: Estimated parameters of η_k for different state settings under static calibration.

Parameters	1-state	2-state	3-state
γ_1	0.00031	0.00030	-0.00373
γ_2	- -	0.00026	0.00232
γ_3	- -	- -	-0.000396
ζ_1	0.00022	0.00130	0.00131
ζ_2	- -	0.00308	0.00148
ζ_3	- -	- -	0.00010
$p_{12}(\eta)$	- -	0.05595	0.99000
$p_{13}(\eta)$	- -	- -	0.00100
$p_{21}(\eta)$	- -	0.07528	0.12639
$p_{23}(\eta)$	- -	- -	0.87261
$p_{31}(\eta)$	- -	- -	0.02955
$p_{32}(\eta)$	- -	- -	0.58343

Table 4.4: Goodness-of-fit test for different models of η_k.

State setting	LL	BIC	AIC
1-state	343.0608	338.7301	341.0608
2-state	354.8772	341.8850	348.8772
3-state	355.3551	329.3707	343.3551

we modify the specification of the model parameters as

$$a(\boldsymbol{y}_t) = \alpha(\boldsymbol{y}_t)t + a(0) \tag{4.7}$$

$$b(\boldsymbol{y}_t) = \gamma(\boldsymbol{y}_t)t + b(0) \tag{4.8}$$

in the force of mortality model

$$\log \mu(x, t, \boldsymbol{y}_t) = a(\boldsymbol{y}_t) + b(\boldsymbol{y}_t)x.$$

The above formulation reflects the fact that empirical data support linear trends of the two parameters in the long run, and both the time and age effects are still being modelled. Now, define the random variable $\tau(x, t)$ as the future lifetime attained at time t for a life with initial age x at time 0. Under the assumption of the force of mortality above, the survival function giving the probability of surviving in the next t years for (x) is $s(x, t) = e^{-\int_0^t \mu(x+u,u,\boldsymbol{y}_u)\,du}$. For the purpose of pricing, the survival index $S(t, T)$ is then the expected value, under a risk-neutral measure $\widetilde{P}$, of the survival probability to time T for an x-aged individual alive at time t. Therefore, with $\chi(\cdot)$ being the indicator function,

$$\begin{aligned} S(t,T) &= \mathbf{E}^{\widetilde{P}}\left[\chi_{\{\tau \geq T\}} \middle| \mathcal{J}_t\right] = \mathbf{E}^{\widetilde{P}}\left[\frac{s(x,T)}{s(x,t)} \middle| \mathcal{J}_t\right] \\ &= \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu(x+u,u,\boldsymbol{y}_u)\,du} \middle| \mathcal{J}_t\right]. \end{aligned} \tag{4.9}$$

Following the method in Elliott and Mamon [17], we have

$$\begin{aligned} S(t,T) &= \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu(x+u,u,\boldsymbol{y}_{t,u})\,du} \middle| \mathcal{J}_t\right] = \left\langle e^{(\boldsymbol{\Lambda}(T-t)-\mathbf{D}(T)+\mathbf{D}(t))}\boldsymbol{y}_t,\ \mathbf{1}\right\rangle \\ &= \left\langle \boldsymbol{y}_t,\ e^{\left(\boldsymbol{\Lambda}^\top(T-t)-\mathbf{D}(T)+\mathbf{D}(t)\right)}\mathbf{1}\right\rangle, \end{aligned} \tag{4.10}$$

where $\mathbf{D}(t)$ is a diagonal matrix with representation

$$\mathbf{D}(u) = \begin{bmatrix} \sum_0^\infty \frac{\delta_1\gamma_1^n}{(2n+1)n!}(u+\nu_1)^{(2n+1)} & & & \\ & \sum_0^\infty \frac{\delta_2\gamma_2^n}{(2n+1)n!}(u+\nu_2)^{(2n+1)} & & \\ & & \ddots & \\ & & & \sum_0^\infty \frac{\delta_m\gamma_m^n}{(2n+1)n!}(u+\nu_m)^{(2n+1)} \end{bmatrix}.$$

See details of derivation in Appendix A.1.

4.2.3 Model 3 (M3): Regime-switching Luciano-Vigna mortality model

Non-mean-reverting affine processes were justified to model mortality rate by Luciano and Vigna [35]. The mortality for a life aged x is expressed as

$$d\mu_t = c\mu_t dt + \sigma d\widetilde{W}_t, \tag{4.11}$$

with $c > 0$ and $\sigma > 0$. Under this model, which was employed as well in Liu et al. [34], the survival index $S(t,T)$ can be easily obtained as

$$S(t,T) = \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu_s\,ds} | \mathcal{F}_t\right] = e^{-H(t,T)\mu_t + G(t,T)}, \tag{4.12}$$

where

$$\begin{aligned} H(t,T) &= \frac{e^{c(T-t)-1}}{c} \\ G(t,T) &= \frac{\sigma^2}{2c}\left[H(t,T) - (T-t)\right] + \frac{\sigma^2 H(t,T)^2}{4c}. \end{aligned}$$

To capture the changes of mortality rate as time goes by, we come up with a regime-switching non-mean reverting mortality model, which under a risk-neutral measure has the representation

$$d\mu_t = c\mu_t dt + \sigma_t d\widetilde{W}_t. \tag{4.13}$$

The volatility component σ_t is a stochastic process driven by an m-state pure Markov chain $\boldsymbol{y}_t$. The semi-martingale representation of $\boldsymbol{y}_t$ is

$$d\boldsymbol{y}_t = \boldsymbol{\Lambda}\boldsymbol{y}_t dt + d\boldsymbol{n}_t.$$

Given the state-space association of $\boldsymbol{y}_t$ with the canonical basis of $\mathbb{R}^m$, the volatility can be expressed as $\sigma_t = \langle \boldsymbol{\sigma}, \boldsymbol{y}_t \rangle$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m)$. Applying first the results for the Hull-White model with deterministic volatility, the survival index is then given by

$$S(t,T,\boldsymbol{y}_t) = e^{-H(t,T)\mu_t + G(t,T,\boldsymbol{y}_t)}, \tag{4.14}$$

where

$$G(t,T,\boldsymbol{y}_t) = \frac{1}{2}\int_t^T H(t,u)^2 \sigma_u^2\,du = \frac{1}{2}\int_t^T \langle \boldsymbol{\phi}_u, \boldsymbol{y}_u \rangle\,du, \tag{4.15}$$

with the ith component of the vector $\boldsymbol{\phi}_u$ being $\left[\frac{1}{c}\left(e^{c(T-u)}-1\right)\sigma_i\right]^2$. Following Elliott and Mamon [16] when the volatility is Markov-modulated, we obtain the survival index

$$\begin{aligned} S(t,T) &= e^{-H(t,T)\mu_t}\langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t,\ \mathbf{1}\rangle \\ &= \left\langle \boldsymbol{y}_t,\ e^{-H(t,T)\mu_t}\boldsymbol{\Pi}_{t,T}^{\top}\mathbf{1}\right\rangle, \end{aligned} \tag{4.16}$$

where $\boldsymbol{\Pi}_{t,T}$ is the fundamental solution to a linear system ordinary differential equations (ODEs) involving the $G(t,T,\boldsymbol{y}_t)$ function; see details in Appendix A.2.

4.2.4 Summary of mortality models

To compare present mortality models with our proposed model, we look at several features highlighting certain advantages and limitations in Table 4.5. Clearly, our regime-switching extensions provide more flexibility than the models in the current literature.

Table 4.5: Comparison of proposed regime-switching mortality models versus current models.

	Features					
Model	RS	nontrivial correlation[1]	two factors[2]	tractability[3]	dynamic estimation[4]	No. of Yes's
Gompertz	No	No	No	Yes	No	1
Lee-Carter	No	No	Yes	No	No	1
Cairn-Blake-Dowd	No	Yes	Yes	No	No	2
Luciano-Vigna	No	No	No	Yes	Yes	2
RS-GBM	Yes	No	No	No	Yes	2
RSGM (M1)	Yes	Yes	Yes	No	Yes	4
M2	Yes	Yes	Yes	Yes	Yes	5
M3	Yes	No	No	Yes	Yes	3

1: Allowance for different improvements at different ages and different times

2: Capacity to incorporate both age and time effects

3: Closed-form solution to the survival index (similar to bond price)

4: Dynamic parameter estimation

4.3 Interest rate model

As previously mentioned, the two risk factors affecting the value of GAO are the interest and mortality risks. In order to value a GAO, a dependable interest rate model is needed as well. Within the last decade or so, there are several papers highlighting the applications of RS models to interest rate modelling. However, the current practice in mortality modelling as far as RS models go still remains primordial. The Makov-modulated Gompertz model should contribute to the enlargement of the collection of available models in the literature. In our framework, we combine together the RS models for both risk factors. In other words, both interest and mortality rates have regime-switching mechanisms via two independent continuous-time Markov chains that are both defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \widetilde{P})$, where $\widetilde{P}$ is risk-neutral. The filtration $\mathcal{F}_t$ is defined as the joint filtration generated by $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$, which are the respective Markov chains driving the interest and mortality rates, i.e., $\mathcal{F}_t := \mathcal{H}_t \vee \mathcal{J}_t$, where $\mathcal{H}_t := \sigma(\boldsymbol{x}_t)$ and $\mathcal{J}_t := \sigma(\boldsymbol{y}_t)$.

Note that in the previous section, we provided an empirical examination of mortality data to back up the model choice and the functional form for the logarithm of the force of mortality. For both the interest and mortality rates, risk-neutral estimates of parameters, which include transition probabilities or intensity rates and states of the Markov chains are necessary to proceed with pricing. The issue of parameter estimation under risk-neutral measure, i.e., using available market data is an inverse problem that is outside the scope of this work. We simply assume that a calibration method (either formal or ad-hoc), such as a least-squares method, is available that gives parameters consistent with the risk-neutral valuation principle. This enables us to discuss pricing in the sequel under the proposed joint interest and mortality rate models.

Furthermore, whilst the independence between the two risk factors limits the general applicability of the proposed framework, this assumption renders mathematical tractability of pricing results. It should also be viewed as a preliminary development where improvements to incorporate correlation structure can be made further in the future. However, the use of regime-switching for both factors that yield a consistent pricing framework merits its discussion. The modelling pursuit in this work parallels the initial attempts of previous authors (e.g., Biffis [6]) when the stochastic nature of both interest and mortality rates was factored into insurance product valuation. But such attempts also started with the assumption of independence between the two rate processes. Only very recently that such a deficiency was addressed albeit at a gradual pace using some advanced mathematical techniques and only for cases of models with no-regime switching feature yet; see for example, Liu et al. [34].

4.3.1 Pure Markov interest rate model

The theory of term structure models of interest rates is well-developed and there are many stochastic interest rate models to choose from depending on one's objectives, sophistication and preferences. An encyclopedic account of these models can be found in Brigo and Mercurio [10], and James and Webber [31]. In this work, we choose an interest rate model with a simple structure yet rich enough to include the important regime-switching feature. We follow the interest rate model in Mamon [36] whereby the short-term interest rate process r_t evolves as a function of a continuous-time Markov chain. Define $\boldsymbol{x}_t$, $t \geq 0$, as a finite-state Markov chain with state space $S_{\boldsymbol{X}} = \left\{\mathbf{s}^1_{\boldsymbol{X}}, \mathbf{s}^2_{\boldsymbol{X}}, \ldots, \mathbf{s}^n_{\boldsymbol{X}}\right\}$. The points $\mathbf{s}^i_{\boldsymbol{X}}$ can be points in $\mathbb{R}^n$. For ease of calculation, the points in $S_{\boldsymbol{X}}$ are associated with the unit vectors $\left\{\mathbf{e}^1_{\boldsymbol{X}}, \mathbf{e}^2_{\boldsymbol{X}}, \ldots, \mathbf{e}^n_{\boldsymbol{X}}\right\}$. In particular, $\mathbf{e}^i_{\boldsymbol{X}} = (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^n$, and $\top$ denotes the transpose of a vector or matrix. If r_t is assumed to be a function of $\boldsymbol{x}_t$ and given its state space as the canonical basis of $\mathbb{R}^n$, then r_t can be represented as

$$r_t = r(\boldsymbol{x}_t) = \langle \boldsymbol{r}, \boldsymbol{x}_t \rangle, \tag{4.17}$$

where $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$ is a vector in $\mathbb{R}^n$.

The unconditional distribution of $\boldsymbol{x}_t$ is

$$\boldsymbol{p}_t = \mathbf{E}^{\widetilde{P}}[\boldsymbol{x}_t] = (p^1_t, p^2_t, \ldots, p^n_t),$$

where

$$p^i_t = \widetilde{P}\left[\boldsymbol{x}_t = \boldsymbol{e}^i_{\boldsymbol{X}}\right] = \mathbf{E}^{\widetilde{P}}\left[\langle \boldsymbol{e}^i_{\boldsymbol{X}}, \boldsymbol{x}_t \rangle\right].$$

Let $p_{ij}(t) = \widetilde{P}\left[\boldsymbol{x}_{t+s} = \mathbf{e}^{j}_{\boldsymbol{x}} \middle| \boldsymbol{x}_s = \mathbf{e}^{i}_{\boldsymbol{x}}\right]$ be the transition probability from state i to j over time period $[s, s+t]$ and $\boldsymbol{P}(t)$ denote the matrix whose i, jth entry is $p_{ij}(t)$. From the Kolmogorov's forward equation, a homogeneous Markov chain satisfies

$$\frac{d\boldsymbol{P}(t)}{dt} = \boldsymbol{\Theta}\boldsymbol{P}(t),\ \boldsymbol{P}(0) = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix and $\boldsymbol{\Theta} = (\theta_{ij})$ is the intensity matrix. That is, $\theta_{ij} \geq 0$ if $i \neq j$ with $\sum_{j=1}^{n} \theta_{ij} = 0$ for $1 \leq i, j \leq n$ and

$$\theta_{ij} = \lim_{t \to 0} \frac{p_{ij}(t)}{t} = p'_{ij}(0).$$

This implies that the transition probability matrix is given by

$$\boldsymbol{P}(t) = e^{t\boldsymbol{\Theta}}.$$

As stated in Mamon [36], $\boldsymbol{x}_t$ is a semi-martingale with representation

$$\boldsymbol{x}_t = \boldsymbol{x}_0 + \int_0^t \boldsymbol{\Theta}\boldsymbol{x}_u\, du + \boldsymbol{m}_t,$$

where $\{\boldsymbol{m}_t\}$ is a sequence of martingale increments.

The Markov-switching interest rate model considered here gives rise to an analytic solution for the zero-coupon bond price $B(t,T)$ (see details in [36]) expressed as

$$\begin{aligned} B(t,T) &= \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T r(\boldsymbol{x}_u)du} | \mathcal{F}_t\right] = \left\langle e^{(\boldsymbol{\Theta}-\mathbf{R})(T-t)}\boldsymbol{x}_t,\ \mathbf{1}\right\rangle \\ &= \left\langle \boldsymbol{x}_t,\ e^{(\boldsymbol{\Theta}-\mathbf{R})^\top (T-t)}\mathbf{1}\right\rangle, \end{aligned} \tag{4.18}$$

where $\mathbf{R}$ is a diagonal matrix, i.e., $\mathbf{R} = \text{diag}(r_1, r_2, \ldots, r_n)$ and $\mathbf{1} = (1, 1, \ldots, 1)^\top$.

4.4 Valuation of GAO

Guaranteed annuity options are options embedded in certain pension policies that afford the policyholders the right to convert the proceeds into an annuity at a guaranteed rate. These were first issued in 1839 recorded in the report of the Insurance Institute of London [30] and gained much popularity in the 1970's through the 1980's in the UK. Recent research works on this area focus on developing efficient and accurate pricing and hedging approaches; see Ballotta and Haberman [2, 3], Chu and Kwok [12], Liu et al. [33, 34], amongst others.

For a person aged x, the payoff function of a GAO at time T based on one dollar cash amount is

$$C_T = \chi_{\{\tau\geq T\}}[g\, a_x(T) - 1]^+ = g\,\chi_{\{\tau\geq T\}}\left[a_x(T) - \frac{1}{g}\right]^+ ,$$

where $a_x(T)$ denotes the annuity rate. Therefore, the value of a GAO for a life aged x is given by

$$\begin{aligned} V_{GAO} &= \mathbf{E}^{\widetilde{P}}\left[e^{-\int_0^T r(\boldsymbol{x}_u)du} C_T \middle| \mathcal{F}_0\right] \\ &= g\,\mathbf{E}^{\widetilde{P}}\left[e^{-\int_0^T r(\boldsymbol{x}_u)du} e^{-\int_0^T \mu(x+v,v,\boldsymbol{y}_v)dv}(a_x(T) - K)^+ \middle| \mathcal{F}_0\right], \end{aligned} \tag{4.19}$$

where $K = \frac{1}{g}$.

4.4.1 Pure endowment price

With the modelling set-ups for the two key risk factors in the previous sections, we are now ready to derive the value of a survival benefit of a unit amount payable at time T for a life aged x at time $t < T$. By the risk-neutral pricing principle, assuming that the life is alive at time t, we have the survival benefit value given by

$$M(t,T) = \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T r(\boldsymbol{x}_{t,u})\,du}\chi_{\{\tau\geq T\}}\middle|\mathcal{F}_t\right] = \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T r(\boldsymbol{x}_{t,u})\,du} e^{-\int_t^T \mu(x+v,v,\boldsymbol{y}_{t,v})\,dv}\middle|\mathcal{F}_t\right]. \tag{4.20}$$

Under the assumption that the two Markov chains are independent, the price of the pure endowment is then

$$M(t,T) = \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T r(\boldsymbol{x}_{t,u})\,du}\middle|\mathcal{F}_t\right]\mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu(x+v,v,\boldsymbol{y}_{t,v})\,dv}\middle|\mathcal{F}_t\right]. \tag{4.21}$$

4.4.1.1 Pure endowment price under M2

We obtained the bond price (4.18) in section 4.3 and the survival index (4.10) under M2 in section 4.2.2. Therefore, the pure endowment price can be expressed as

$$\begin{aligned} M(t,T) &= B(t,T)S(t,T) \\ &= \left\langle e^{(\mathbf{\Theta}-\mathbf{R})(T-t)}\boldsymbol{x}_t,\ \mathbf{1}\right\rangle \left\langle e^{(\mathbf{\Lambda}(T-t)-\mathbf{D}(T)+\mathbf{D}(t))}\boldsymbol{y}_t,\ \mathbf{1}\right\rangle \\ &= \left\langle \boldsymbol{x}_t,\ e^{(\mathbf{\Theta}-\mathbf{R})^\top(T-t)}\mathbf{1}\right\rangle \left\langle \boldsymbol{y}_t,\ e^{\left(\mathbf{\Lambda}^\top(T-t)-\mathbf{D}(T)+\mathbf{D}(t)\right)}\mathbf{1}\right\rangle. \end{aligned} \tag{4.22}$$

Since annuity rate can be viewed as the sum of pure endowments with different maturities, we obtain the expression for $a_x(T)$ as

$$\begin{aligned} a_x(T) &= \sum_{n=0}^{\infty} \mathbf{E}^{\widetilde{P}}\left[e^{-\int_T^{T+n} r(\boldsymbol{x}_u)du} e^{-\int_T^{T+n} \mu\left(x+v,v,\boldsymbol{y}_v\right)dv} \middle| \mathcal{F}_T \right] \\ &= \sum_{n=0}^{\infty} M(T,T+n) \\ &= \sum_{n=1}^{\infty} \left\langle e^{n(\mathbf{\Theta}-\mathbf{R})}\boldsymbol{x}_T,\ \mathbf{1}\right\rangle \left\langle e^{(n\mathbf{\Lambda}-\mathbf{D}(T+n)+\mathbf{D}(T))}\boldsymbol{y}_T,\ \mathbf{1}\right\rangle \\ &= \sum_{n=1}^{\infty} \left\langle \boldsymbol{x}_T,\ e^{n(\mathbf{\Theta}-\mathbf{R})^\top}\mathbf{1}\right\rangle \left\langle \boldsymbol{y}_T,\ e^{\left(n\mathbf{\Lambda}^\top-\mathbf{D}(T+n)+\mathbf{D}(T)\right)}\mathbf{1}\right\rangle. \end{aligned} \tag{4.23}$$

4.4.1.2 Pure endowment price under M3

Similarly, given the respective analytical solutions of the bond price (4.18) and survival index (4.16), we can get the price of the pure endowment as

$$M(t,T) = \left\langle \boldsymbol{x}_t, e^{(\mathbf{\Theta}-\mathbf{R})^\top(T-t)}\mathbf{1}\right\rangle e^{-H(t,T)\mu_t} \left\langle \mathbf{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1}\right\rangle. \tag{4.24}$$

From equation (4.24), $a_x(T)$ can be expressed as

$$\begin{aligned} a_x(T) &= \sum_{n=0}^{\infty} \mathbf{E}^{\widetilde{P}}\left[e^{-\int_T^{T+n} r(\boldsymbol{x}_u)du} e^{-\int_T^{T+n} \mu(\boldsymbol{y}_v)dv} \middle| \mathcal{F}_T \right] \\ &= \sum_{n=0}^{\infty} M(T,T+n) \\ &= \sum_{n=0}^{\infty} \left\langle \boldsymbol{x}_T, e^{n(\mathbf{\Theta}-\mathbf{R})^\top}\mathbf{1}\right\rangle e^{-H(t,T)\mu_t} \left\langle \mathbf{\Pi}_{T,T+n}\boldsymbol{y}_T, \mathbf{1}\right\rangle. \end{aligned} \tag{4.25}$$

4.4.2 Endowment-risk-adjusted measure

The closed-form solutions of the endowment price in equations (4.22) and (4.24) will aid in the simplification of the GAO value under M2 and M3, respectively. This can be accomplished by using the change of measure technique pioneered in Geman et al. [23], and also utilised in the work of Liu et al. [33]. In our case, we employ the concept of endowment-risk-adjusted measure by choosing the pure endowment price as the numéraire. The Radon-Nikodŷm derivative of the endowment-risk-adjusted measure $\widehat{P}$ with respect to the risk-neutral measure $\widetilde{P}$ is defined as

$$\left.\frac{d\widehat{P}}{d\widetilde{P}}\right|_{\mathcal{F}_T} := \Upsilon_{0,T} = \frac{e^{-\int_0^T r(\boldsymbol{x}_{0,u})\,du} M(T,T)}{M(0,T)}.$$

From the Bayes' rule, if ω is a contingent claim, we get

$$\mathbf{E}^{\widehat{P}}\left[\omega|\mathcal{F}_t\right] = \frac{\mathbf{E}^{\widetilde{P}}\left[\Upsilon_{0,T}\omega\middle|\mathcal{F}_t\right]}{\mathbf{E}^{\widetilde{P}}\left[\Upsilon_{0,T}\middle|\mathcal{F}_t\right]}.$$

Consequently,

$$\mathbf{E}^{\widetilde{P}}\left[(e^{-\int_t^T r(\boldsymbol{x}_{t,u})\,du}e^{-\int_t^T \mu\left(x+v,v,\boldsymbol{y}_{t,v}\right)dv})\omega\middle|\mathcal{F}_t\right] = M(t,T)\mathbf{E}^{\widehat{P}}\left[\omega|\mathcal{F}_t\right]. \tag{4.26}$$

So, after changing measure and with $\omega = (a_x(T) - K)^+$, the price of GAO in equation (4.19) becomes

$$V_{GAO} = gM(0,T)\,\mathbf{E}^{\widehat{P}}\left[(a_x(T) - K)^+\middle|\mathcal{F}_T\right]. \tag{4.27}$$

4.4.2.1 GAO price under M2

Recall that under M2, the annuity rate is given as

$$a_x(T) = \sum_{n=1}^{\infty}\left\langle \boldsymbol{x}_T,\ e^{n(\boldsymbol{\Theta}-\mathbf{R})^\top}\mathbf{1}\right\rangle\left\langle \boldsymbol{y}_T,\ e^{\left(n\boldsymbol{\Lambda}^\top-\mathbf{D}(T+n)+\mathbf{D}(T)\right)}\mathbf{1}\right\rangle$$

This implies that $a_x(T)$ is a discrete random variable with $n \times m$ possible outcomes. Its probability mass function is

$$\begin{aligned}\pi_{ij} &= \widehat{P}(a_x(T) = v_{ij})\\ &= \widehat{p}_T^{\,i}\widehat{q}_T^{\,j},\end{aligned}$$

where

$$v_{ij} = \sum_{n=1}^{\infty} \left\langle e^{i}_{\boldsymbol{x}},\ e^{n(\boldsymbol{\Theta}-\mathbf{R})^{\top}}\mathbf{1}\right\rangle \left\langle e^{j}_{\boldsymbol{y}},\ e^{\left(n\boldsymbol{\Lambda}^{\top}-\mathbf{D}(T+n)+\mathbf{D}(T)\right)}\mathbf{1}\right\rangle, \tag{4.28}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Therefore, using equations (4.22) and (4.28), equation (4.27) becomes

$$\begin{aligned} V^{M2}_{GAO} &= gM(0,T)\,\mathbf{E}^{\widehat{\mathbf{P}}}\left[\left(\sum_{n=1}^{\infty} M(T, T+n) - K\right)^{+}\middle|\mathcal{F}_T\right] \\ &= g\left\langle \boldsymbol{x}_0,\ e^{T(\boldsymbol{\Theta}-\mathbf{R})^{\top}}\mathbf{1}\right\rangle\left\langle \boldsymbol{y}_0,\ e^{\left(T\boldsymbol{\Lambda}^{\top}-\mathbf{D}(T)+\mathbf{D}(0)\right)}\mathbf{1}\right\rangle \sum_{i}\sum_{j} \pi_{ij}\left(v_{ij} - K\right)^{+}. \end{aligned} \tag{4.29}$$

In order to calculate the GAO value using the formula in equation (4.29), we require the transition probability matrix (or intensity matrix) of $\boldsymbol{x}_T$ and $\boldsymbol{y}_T$ under the endowment-risk-adjusted measure. Given (4.26) which is obtained from the Bayes' rule and the independence assumption of $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$, we have

$$\begin{aligned} \mathbf{E}^{\widehat{P}}\left[\boldsymbol{x}_T|\mathcal{F}_t\right] &= \frac{\mathbf{E}^{\widetilde{P}}\left[\Upsilon_{0,T}\boldsymbol{x}_T\middle|\mathcal{F}_t\right]}{\mathbf{E}^{\widetilde{P}}\left[\Upsilon_{0,T}\middle|\mathcal{F}_t\right]} \\ &= \frac{\mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T r(\boldsymbol{x}_{t,u})\,du}e^{-\int_t^T \mu(x+v,v,\boldsymbol{y}_{t,v})\,dv}\boldsymbol{x}_T\middle|\mathcal{F}_t\right]}{M(t,T)} \\ &= \frac{\mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T r(\boldsymbol{x}_{t,u})\,du}\boldsymbol{x}_T\middle|\mathcal{F}_t\right]S(t,T)}{M(t,T)} \\ &= \frac{e^{(\boldsymbol{\Theta}-\mathbf{R})(T-t)}\boldsymbol{x}_t}{\langle e^{(\boldsymbol{\Theta}-\mathbf{R})(T-t)}\boldsymbol{x}_t, \mathbf{1}\rangle}. \end{aligned} \tag{4.30}$$

Similarly,

$$\begin{aligned} \mathbf{E}^{\widehat{P}}\left[\boldsymbol{y}_T\middle|\mathcal{F}_t\right] &= \frac{\mathbf{E}^{\widetilde{P}}\left[\Upsilon_{0,T}\boldsymbol{y}_T\middle|\mathcal{F}_t\right]}{\mathbf{E}^{\widetilde{P}}\left[\Upsilon_{0,T}\middle|\mathcal{F}_t\right]} \\ &= \frac{B(t,T)\mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu(x+v,v,\boldsymbol{y}_{t,v})\,dv}\boldsymbol{y}_T\middle|\mathcal{F}_t\right]}{M(t,T)} \\ &= \frac{e^{(\boldsymbol{\Lambda}(T-t)-\mathbf{D}(T)+\mathbf{D}(t))}\boldsymbol{y}_t}{\langle e^{(\boldsymbol{\Lambda}(T-t)-\mathbf{D}(T)+\mathbf{D}(t))}\boldsymbol{y}_t, \mathbf{1}\rangle}. \end{aligned} \tag{4.31}$$

Thus, the unconditional distribution of $\boldsymbol{x}_t$ under the endowment-risk-adjusted measure is

$$\mathbf{E}^{\widehat{P}}\left[\boldsymbol{x}_t\right] = \widehat{\boldsymbol{p}}_t = \left(\hat{p}_t^1, \hat{p}_t^2, \ldots, \hat{p}_t^n\right), \tag{4.32}$$

where

$$\begin{aligned}
\hat{p}_t^i &= \widehat{P}\left(\boldsymbol{x}_t = \boldsymbol{e}_{\boldsymbol{x}}^i\right) \\
&= \mathbf{E}^{\widehat{P}}\left[\langle \boldsymbol{e}_{\boldsymbol{x}}^i, \boldsymbol{x}_t\rangle\right] \\
&= \langle \mathbf{E}^{\widehat{P}}\left[\boldsymbol{x}_t\right], \boldsymbol{e}_{\boldsymbol{x}}^i\rangle \\
&= \left\langle \frac{e^{t(\boldsymbol{\Theta}-\mathbf{R})}\boldsymbol{x}_0}{\langle e^{t(\boldsymbol{\Theta}-\mathbf{R})}\boldsymbol{x}_0,\ \mathbf{1}\rangle}, \mathbf{e}_{\boldsymbol{x}}^i \right\rangle.
\end{aligned} \tag{4.33}$$

On the other hand, the unconditional distribution of $\boldsymbol{y}_t$ under the endowment-risk-adjusted measure is

$$\mathbf{E}^{\widehat{P}}\left[\boldsymbol{y}_t\right] = \widehat{\boldsymbol{q}}_t = \left(\hat{q}_t^1, \hat{q}_t^2, \ldots, \hat{q}_t^m\right), \tag{4.34}$$

where

$$\begin{aligned}
\hat{q}_t^i &= \widehat{P}\left(\boldsymbol{y}_t = \boldsymbol{e}_{\boldsymbol{y}}^i\right) \\
&= \mathbf{E}^{\widehat{P}}\left[\langle \boldsymbol{e}_{\boldsymbol{y}}^i, \boldsymbol{y}_t\rangle\right] \\
&= \left\langle \frac{e^{(t\boldsymbol{\Lambda}-\mathbf{D}(t)+\mathbf{D}(0))}\boldsymbol{y}_0}{\langle e^{(t\boldsymbol{\Lambda}-\mathbf{D}(t)+\mathbf{D}(0))}\boldsymbol{y}_0,\ \mathbf{1}\rangle},\ \boldsymbol{e}_{\boldsymbol{y}}^i \right\rangle.
\end{aligned} \tag{4.35}$$

Under the new measure, $\boldsymbol{y}_t$ is a nonhomogeneous Markov chain. However, both the Chapman-Kolmogorov equation and Kolmogorov forward equation still hold; see details in Ross [42].

4.4.2.2 GAO price under M3

Given the analytical solution to annuity rate by equation (4.25), the price of GAO becomes

$$\begin{aligned}
V_{GAO}^{M3} &= gM(0,T)\,\mathbf{E}^{\widehat{\mathbf{P}}}\left[\left(\sum_{n=1}^{\infty} M(T,T+n) - K\right)^{+}\middle|\mathcal{F}_T\right] \\
&= g\left\langle \boldsymbol{x}_0,\ e^{T(\boldsymbol{\Theta}-\mathbf{R})^{\top}}\right\rangle e^{-H(0,T)\mu_0}\left\langle \boldsymbol{\Pi}_{0,T}\boldsymbol{y}_0, \mathbf{1}\right\rangle \mathbf{E}^{\widehat{P}}\left[\left(\sum_{n=0}^{\infty}\left\langle \boldsymbol{x}_T,\ e^{n(\boldsymbol{\Theta}-\mathbf{R})^{\top}}\right\rangle e^{-H(T,T+n)\mu_T}\left\langle \boldsymbol{\Pi}_{T,T+n}\boldsymbol{y}_T, \mathbf{1}\right\rangle - K\right)^{+}\middle|\mathcal{F}_T\right].
\end{aligned} \tag{4.36}$$

To obtain the GAO price using (4.36), we need to obtain the transition probability matrices of $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ and the dynamics of μ_t under the new measure $\widehat{P}$. The distribution of $\boldsymbol{x}_t$ remains the same as equation (4.32). Note that

$$\mathbf{E}^{\widehat{P}}\left[\boldsymbol{y}_T|\mathcal{F}_t\right] = \frac{\mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu_s\,ds}\boldsymbol{y}_T|\mathcal{F}_t\right]}{S(t,T)}. \tag{4.37}$$

Also, we have

$$\begin{aligned} S(t,T) &= \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu_s\,ds}|\mathcal{F}_t\right] \\ &= \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu_s\,ds}\langle \boldsymbol{y}_T, \mathbf{1}\rangle|\mathcal{F}_t\right] \\ &= \left\langle \mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu_s\,ds}\boldsymbol{y}_T|\mathcal{F}_t\right], \mathbf{1}\right\rangle. \end{aligned} \tag{4.38}$$

Therefore, we obtain

$$\mathbf{E}^{\widetilde{P}}\left[e^{-\int_t^T \mu_s\,ds}\boldsymbol{y}_T|\mathcal{F}_t\right] = e^{-H(t,T)\mu_t}\mathbf{\Pi}_{t,T}\boldsymbol{y}_t\ . \tag{4.39}$$

Consequently,

$$\mathbf{E}^{\widehat{P}}\left[\boldsymbol{y}_T|\mathcal{F}_t\right] = \frac{e^{-H(t,T)\mu_t}\mathbf{\Pi}(t,T)\boldsymbol{y}_t}{e^{-H(t,T)\mu_t}\langle \mathbf{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1}\rangle} = \frac{\mathbf{\Pi}_{t,T}\boldsymbol{y}_t}{\langle \mathbf{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1}\rangle}. \tag{4.40}$$

The unconditional distribution of $\boldsymbol{y}_t$ under the new measure $\widehat{P}$ is then given as

$$\mathbf{E}^{\widehat{P}}\left[\boldsymbol{y}_t\right] = \widehat{\boldsymbol{q}}_t = \left(\hat{q}_t^1, \hat{q}_t^2, \ldots, \hat{q}_t^m\right), \tag{4.41}$$

where

$$\begin{aligned} \hat{q}_t^i &= \widehat{P}\left(\boldsymbol{y}_t = \boldsymbol{e}_{\boldsymbol{y}}^i\right) \\ &= \mathbf{E}^{\widehat{P}}\left[\langle \boldsymbol{e}_{\boldsymbol{y}}^i, \boldsymbol{y}_t\rangle\right] \\ &= \left\langle \frac{\mathbf{\Pi}_{0,t}\boldsymbol{y}_0}{\langle \mathbf{\Pi}_{0,t}\boldsymbol{y}_0, \mathbf{1}\rangle}, \boldsymbol{e}_{\boldsymbol{y}}^i\right\rangle. \end{aligned} \tag{4.42}$$

As indicated above and in connection with equation (4.36), besides the two Markov chains $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$, the (affine) dynamics of μ_t under the new measure $\widehat{P}$ are needed to price a GAO. Since $e^{-\int_0^t r_s\,ds}M(t,T)$ is an $(\mathcal{F}_t, \widetilde{P})$-martingale, we have $\dfrac{dM(t,T)}{M(t,T)} - r_t dt = \sigma_M(t)d\widetilde{W}_t$ and $\widehat{W}_t = \widetilde{W}_t - \displaystyle\int_0^t \sigma_M(u)\,du$ by Girsanov theorem. Following the method in Mamon [36], we have $\sigma_M(u) = H(t,T)\sigma_u$. Therefore, the dynamics of μ_t under the new measure $\widehat{P}$ is given as

$$d\mu_t = \left(c\mu_t - H(t,T)\sigma_u^2\right)dt + \sigma_u d\widehat{W}_u. \tag{4.43}$$

Itô's lemma shows that

$$\mu_t = e^{ct}\left[\mu_0 + \int_0^t e^{-cu}(-H(u,T)\sigma_u^2)\,du + \int_0^t e^{-cu}\sigma_u\,d\widehat{W}_u\right] \tag{4.44}$$

is the solution to (4.43).

4.5 Numerical illustration

This section contains a numerical experiment comparing the efficiency between formula (4.29), (4.36) and formula (4.19) in pricing a GAO. Under direct calculation of GAO price using (4.19), we discretise the time period to calculate the two integrals in the equation. We divide each year period into 252 equal subintervals with fixed length $\Delta t = \frac{1}{252}$, i.e., the total number of subintervals over the period $(0, T)$ is $N = 252T$ and generate sample paths of the two Markov chains $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ given their transition probability matrices. By the trapezoidal rule, the two integrals in equation (4.19) are then respectively approximated as

$$\int_0^T r(\boldsymbol{x}_u)\,du \approx \frac{\Delta t}{2}\left[r(\boldsymbol{x}_0) + r(\boldsymbol{x}_T) + 2\sum_{k=1}^{N-1} r(\boldsymbol{x}_{k\Delta t})\right], \tag{4.45}$$

and

$$\int_0^T \mu(x+v, v, \boldsymbol{y}_v)\,dv \approx \frac{\Delta t}{2}\left[\mu(x, 0, \boldsymbol{y}_0) + \mu(x+T, T, \boldsymbol{y}_T) + 2\sum_{k=1}^{N-1} \mu(x + k\Delta t, k\Delta t, \boldsymbol{y}_{k\Delta t})\right]. \tag{4.46}$$

Under M2, apart from the two integrals that need to be approximated, we have to calculate the annuity rate $a_x(T)$. By equation (4.23), the simulated pair $(\boldsymbol{x}_T, \boldsymbol{y}_T)$ are all we need in the calculation of the GAO price, which can be obtained as the final values of the sample path of the two Markov chains. Using the measure-change technique proposed in this work, we can obtain the *exact* value of GAO from equation (4.29). This is because $M(0, T)$ can be calculated if we know the initial states of the two Markov chains $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ from equation (4.22) whilst the component with the expectation term can be computed directly after obtaining the distributions of $\boldsymbol{x}_T$ and $\boldsymbol{y}_T$ under the new measure. Whilst under M3, the GAO price can be calculated through (4.36) by generating $\boldsymbol{x}_T$, $\boldsymbol{y}_T$ and μ_T via equations (4.33), (4.42) and (4.44), respectively.

All numerical results under direct calculation using the Monte-Carlo simulation method rely on 10,000 simulated sample paths. We assume the same rates of transition between the states for both the Markov chains $\boldsymbol{x}_t$ in the interest model and $\boldsymbol{y}_t$ in the mortality model. To be more specific, $\theta_{ij} = \lambda_{ij} = 1,\ i \neq j$. The parameters of the interest rate model, and mortality models M1, M2 and M3 are given in Table 4.6. Systematic trial-and-error approach coupled with sensitivity analyses were employed to set the parameters in our numerical experiments. More specifically, we start with the two-state model by setting each parameter correspond to one small and one large value. In practice, this step is guided by the standard deviation of certain values (e.g., mean or volatility) calculated from the entire data set. The small and large values in the 2-state model together with their midpoint provide a framework for the three-state model. Likewise, for the four-state model we consider the small and large values as endpoints of an interval. This interval is partitioned into three equally spaced sub-intervals to generate four points of the partition that will support the four-state setting. We then evaluate the efficiency of our approach and examine the sensitivity of prices to various regime pairs. This kind of price sensitivity is straightforward and must be distinguished from the usual price sensitivity exercise mentioned in subsection 4.2.2, which is more exhaustive as it entails varying of individual parameters for the interest and mortality models.

In Tables 4.7 to 4.9, we display the GAO prices together with their standard errors based on a cohort data aged 50 whose GAO contracts mature at age 65 under different combinations of regime number pairs for the two risk factors. In our calculation, the maximum age is assumed to be 100 which means that there are no more than 35 annual payments.

Each pair of values in the first column refers to the number of regimes we choose for the interest rate and mortality rate models. Under each regime pair, we present the prices under the three mortality models along with their standard and relative errors. We see that M1 is computationally intensive; that is, it takes about 30 hours to obtain prices even with 56 parallel processors. This is because under M1, there is no analytical solution to the survival probability function and therefore we cannot employ the change of measure technique to obtain GAO prices and resort instead to direct Monte Carlo method. Moreover, under each sample path, we have to estimate the annuity rate using Monte Carlo simulation as well, which is an embedded simulation resulting to an enormous computing endeavour. From Tables 4.8 to 4.9, we see that the prices under both methods (i.e., our measure change-based method and direct Monte Carlo method) are close to each other under M2 and M3. We notice as well that the results under multi-regime set-ups are bounded by the results under the one-regime set-ups. The greater the number of regimes (i.e., comparison of cases with 2 or more regimes), the smaller the RSE (relative standard error) and SEM (standard error of the mean). Such observation is consistent with the fact that the randomness can be captured better with more regimes. Also, as the number of regimes rises, we find that the differences between prices from a multi-regime model and price from the middle-point scenario under a one-regime model become smaller. This proves the improvement in accuracy when we increase the number of regimes. Moreover, the prices are highly sensitive to interest rates driven by a pure Markov model. Thus, careful and accurate setting of parameters is necessary given the substantial impact of the interest rate model in the GAO value.

Since we cannot apply the measure-change technique to M1 to get the GAO price, we can only adopt the Monte Carlo methodology. Using parallel computing, it takes 28 hours and 30 hours to accomplish the task of valuation under the single- and multi-regime settings, respectively. For M2, however, the only computing hurdle involved is the calculation of the exponential matrix in formula (4.29). This is handled very easily by the rooted function package of the statistical software R. It takes less than 1 second to obtain pricing results using measure change compared to 160 seconds using Monte Carlo method. In Table 4.9, the (1,1) setting under M2 is a deterministic case and does not actually need either simulation or measure-change technique at all. This is the reason why the prices under both methods are exactly the same, and there are no RSEs and SEMs for them. The (1,1) setting prices are exhibited for comparison with those under the (2,2), (3,3) and (4,4) settings. Apart from the calculation of the exponential matrix, solving the system of ODEs in M3 is another challenge to obtain the GAO price. But, once the model parameters are selected, the total computing times to obtain a numerical value using the formulae derived from the measure-change technique, namely, (4.29) and (4.36) are greatly reduced in contrast to the times of implementing formula (4.19).

4.6 Conclusions

In this chapter, we put forward three regime-switching mortality models. The empirical evidence from the US mortality data supports the use of RSGM for mortality rates. That is, the

Table 4.6: Parameter set for the numerical experiment in regime-switching model framework.

Contract specification

$g = 11.1\%$, $T = 15$, $n = 35$;

Interest rate model: $r_t = r(\boldsymbol{x}_t) = \langle \mathbf{r}, \boldsymbol{x}_t \rangle$. See equation (4.17).

2-state RS model: $(r_1,\ r_2) = (0.02, 0.08)$

3-state RS model: $(r_1,\ r_2,\ r_3) = (0.02, 0.05, 0.08)$

4-state RS model: $(r_1,\ r_2,\ r_3,\ r_4) = (0.02, 0.04, 0.06, 0.08)$

$\theta_{ij} = 1$ for $i \neq j$

RSGM: $\log \mu(x, t, \boldsymbol{y}_t) = a(\boldsymbol{y}_t) + b(\boldsymbol{y}_t)x$. See equations (4.2), (4.3) and (4.4).

$a(0) = -8$, $b(0) = 0.07$;

2-state RS model: $\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \zeta_2 \end{bmatrix} = \begin{bmatrix} -0.04 & 0.0008 & 0.008 & 0.0008 \\ -0.01 & 0.0002 & 0.004 & 0.0004 \end{bmatrix}$

3-state RS model: $\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \zeta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \zeta_3 \end{bmatrix} = \begin{bmatrix} -0.04 & 0.0008 & 0.008 & 0.0008 \\ -0.025 & 0.005 & 0.006 & 0.0006 \\ -0.01 & 0.002 & 0.004 & 0.0004 \end{bmatrix}$

4-state RS model: $\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \zeta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \zeta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \zeta_4 \end{bmatrix} = \begin{bmatrix} -0.04 & 0.008 & 0.008 & 0.0008 \\ -0.03 & 0.006 & 0.0066 & 0.00066 \\ -0.02 & 0.004 & 0.0055 & 0.00055 \\ -0.01 & 0.002 & 0.004 & 0.0004 \end{bmatrix}$

$\lambda_{ij} = 1$ for $i \neq j$

RSPMM: $\log \mu(x, t, \boldsymbol{y}_t) = a(\boldsymbol{y}_t) + b(\boldsymbol{y}_t)x$. See equations (4.2), (4.7) and (4.8).

$a(0) = -8$, $b(0) = 0.07$;

2-state RS model: $\begin{bmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{bmatrix} = \begin{bmatrix} -0.07 & 0.0005 \\ -0.03 & 0.0001 \end{bmatrix}$

3-state RS model: $\begin{bmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \\ \alpha_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} -0.07 & 0.0005 \\ -0.05 & 0.0003 \\ -0.03 & 0.0001 \end{bmatrix}$

4-state RS model: $\begin{bmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \\ \alpha_3 & \gamma_3 \\ \alpha_4 & \gamma_4 \end{bmatrix} = \begin{bmatrix} -0.07 & 0.0005 \\ -0.055 & 0.00035 \\ -0.04 & 0.0002 \\ -0.03 & 0.0001 \end{bmatrix}$

RSNMRM: $d\mu_t = c\mu_t dt + \sigma_t d\widetilde{W}_t$. See equation (4.13).

$c = 0.07$, $\mu 0 = 0.006$;

2-state RS model: $(\sigma_1,\ \sigma_2) = (0.0008, 0.0002)$

3-state RS model: $(\sigma_1,\ \sigma_2,\ \sigma_3) = (0.0008, 0.0005, 0.0002)$

4-state RS model: $(\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4) = (0.0008, 0.0006, 0.0004, 0.0002)$

Table 4.7: Actuarial prices for GAO under RSGM (M1).

Regime pair	Parameter set			Price (mean)	RSE	SEM	Time (h)[1]
	r	(α, β)	(γ, η)				
(1,1)	0.02	(-0.01, 0.002)	(0.002,0.0002)	0.6680910	0.00389%	0.0000260	28
(1,1)	0.02	(-0.04, 0.008)	(0.008,0.0008)	0.1088857	0.13882%	0.0001512	28
(2,2)	(0.02, 0.08)	(-0.04, 0.008) (-0.01, 0.002)	(0.008,0.0008) (0.002,0.0002)	0.0906555	0.17318%	0.0001570	30
(3,3)	(0.02,0.05,0.08)	(-0.04, 0.008) (-0.025, 0.005) (-0.01, 0.002)	(0.008,0.0008) (0.005,0.0005) (0.002,0.0002)	0.0891200	0.10862%	0.0000968	30
(1,1)	0.05	(-0.04, 0.008)	(0.008,0.0008)	0.0157539	0.16789%	0.0000264	28
(1,1)	0.05	(-0.025, 0.005)	(0.005,0.0005)	0.0877859	0.02490%	0.0000219	28
(1,1)	0.05	(-0.01, 0.002)	(0.002,0.0002)	0.1937441	0.00397%	0.0000077	28
(4,4)	(0.02,0.04,0.06,0.08)	(-0.04, 0.008) (-0.03,0.006) (-0.02, 0.004) (-0.01, 0.002)	(0.008,0.0008) (0.06,0.0006) (0.004,0.0004) (0.002,0.0002)	0.0866900	0.08638%	0.0000749	30
(1,1)	0.08	(-0.01, 0.002)	(0.002,0.0002)	0.0261796	0.00435%	0.0000011	28
(1,1)	0.08	(-0.04, 0.008)	(0.008,0.0008)	0.0000000			28

1: Parallel computing using 56 CPUs

Table 4.8: Actuarial prices for GAO under two different methods for M2.

Regime pair	Parameter set			Monte Carlo simulation (using eqn (4.19))				Proposed approach (using eqn (4.29))	
	r	a	γ	Price (mean)	RSE	SEM	Time (s)	Price	Time (s)
(1,1)	0.02	-0.07	0.0005	0.4591484	–	–	< 1	0.4591484	< 1
(1,1)	0.02	-0.03	0.0001	0.3603371	–	–	< 1	0.3603371	< 1
(2,2)	(0.02, 0.08)	-0.07 -0.03	0.0005 0.0001	0.0880022	0.13938%	0.0001227	160	0.0878127	< 1
(3,3)	(0.02,0.05,0.08)	-0.07 -0.05 -0.03	0.0005 0.0003 0.0001	0.0863534	0.08940%	0.0000772	160	0.0864020	< 1
(1,1)	0.05	-0.07	0.0005	0.1020477	–	–	< 1	0.1020477	< 1
(1,1)	0.05	-0.05	0.0003	0.0849910	–	–	< 1	0.0849910	< 1
(1,1)	0.05	-0.03	0.0001	0.0675727	–	–	< 1	0.0675727	< 1
(4,4)	(0.02,0.04,0.06,0.08)	-0.07 -0.055 -0.04 -0.03	0.0005 0.00035 0.0002 0.0001	0.0846000	0.06806%	0.0000576	160	0.0847391	< 1
(1,1)	0.08	-0.07	0.0005	0.0000000	–	–	< 1	0.0000000	< 1
(1,1)	0.08	-0.03	0.0001	0.0000000	–	–	< 1	0.0000000	< 1

Table 4.9: Actuarial prices for GAO under two different methods for M3.

Regime	Parameter set		Monte Carlo simulation (using eqn (4.19))				Proposed approach (using eqn (4.36))			
pair	r	σ	Price (mean)	RSE	SEM	Time (s)	Price (mean)	RSE	SEM	Time (s)
(1,1)	0.02	0.0008	0.4293627	0.39965%	0.0017159	70	0.4305314	0.35127%	0.0015123	9
(1,1)	0.02	0.0005	0.4104248	0.24128%	0.0009903	70	0.4112082	0.21358%	0.0008782	9
(1,1)	0.02	0.0002	0.4009524	0.09497%	0.0003808	70	0.4012513	0.08440%	0.0003387	9
(1,1)	0.05	0.0008	0.0901351	0.64169%	0.0005784	70	0.0905728	0.58687%	0.0005315	9
(1,1)	0.05	0.0005	0.0840867	0.40542%	0.0003409	70	0.0843770	0.35145%	0.0002965	9
(1,1)	0.05	0.0002	0.0811261	0.16283%	0.0001321	70	0.0812362	0.15162%	0.0001232	9
(2,2)	(0.02,0.08)	(0.0008, 0.0002)	0.0885169	0.47427%	0.0004198	1200	0.0889869	0.45300%	0.0004031	700
(3,3)	(0.02,0.05,0.08)	(0.0008,0.0005,0.0002)	0.0858787	0.43995%	0.0003778	1200	0.0858274	0.43648%	0.0003746	700
(4,4)	(0.02,0.04,0.06,0.08)	(0.0008,0.0004,0.0006,0.0002)	0.0855836	0.42361%	0.0003625	1200	0.0855506	0.41451%	0.0003546	700
(1,1)	0.08	0.0008	0.0027966	3.13173%	0.0000876	70	0.0027544	2.85321%	0.0000786	9
(1,1)	0.08	0.0005	0.0005446	4.95820%	0.0000270	70	0.0005185	4.39670%	0.0000228	9
(1,1)	0.08	0.0002	0.0000005	86.07199%	0.0000004	70	0.0000005	57.99959%	0.0000003	9

mortality rates follow a Gompertz model having parameters being driven by a Markov chain. Such support for the proposed model based on US mortality data was established via the successful application of HMM filtering technique. Two of our RS mortality models give closed-form solutions to the survival index. Under these two RS mortality models, we developed a regime-switching framework combining mortality and interest risk factors with the independence assumption and priced a GAO accordingly. The analytical solution of a pure endowment price was derived. This was employed to introduce the concept of an endowment-risk-adjusted measure, which facilitated the derivation of the transition probability matrices under a measure change. We provided numerical results under different combinations of regimes for both Markov chains driving the two risk factors. Our measure-change methodology offers a viable approach to determine the price of a GAO. In particular, its efficiency is beyond question when compared to that of the Monte-Carlo simulation method.

We acknowledge a few limitations of this work, but also recognised that such limitations are opportunities to explore as part of future research directions. It should be noted that some empirical evidence was provided to justify the appropriateness of M1 but none for M2 and M3. This is because the empirical justification for both M2 and M3 would also require implementation of pertinent HMM-based filtering equations. Such equations entail a separate and comprehensive development, which is better left as another future research endeavour.

The performance of the proposed approach could be further assessed by looking at the adequacy of capital requirements via the calculation of quantile risk measures for GAO. This would entail calibration methods such as maximum likelihood method, hidden Markov filtering techniques, etc to estimate the model parameters. In addition, analysis of the out-of-sample forecasting capabilities could be another way to gauge the approach's effectiveness. As mentioned above, it is necessary to obtain risk-neutral parameters compatible with risk-neutral valuation. This poses several challenges including availability of data, market liquidity, and mathematical tools that can provide market-consistent parameters. Development of approaches that can circumvent these challenges would be valuable for researchers and practitioners alike.

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Chapter 5

Pricing a guaranteed annuity option under correlated and regime-switching risk factors

5.1 Introduction

New developments and innovations in the insurance market give rise to insurance products with embedded options. These products have similar characteristics to financial derivatives but whose values depend on at least two important risk factors, the most important of which are the interest and mortality rates. The recent works by Liu et al. [16, 17] argue that correlation between these two risk factors has significant pricing effects, and hence, it must be incorporated in a valuation modelling framework.

In [17], a generalised framework was constructed, whereby two correlated diffusion processes were employed to model the dependence of interest and mortality rates but whose volatilities are constant. However, due to the long-term maturities of insurance contracts, reliable mathematical models are needed for the long-term stochastic behaviour of interest and mortality rates. Over a long time period, changes in macro-economic and social conditions may cause economic structural changes. Thus, the one-factor term-structure models for interest and mortality rates with constant volatility are no longer adequate to capture random changes over time. For interest rate modelling, multi-factor models emerged to describe the evolution of interest rate's term structure dependent on several factors; see Chen [4], Duffie and Kan [6], Longstaff and Schwartz [18], amongst others. Alternatively, Elliott et al. (cf. [8], [10] and [11]) proposed a series of Markov-modulated affine models for interest rate and presented explicit solution to the bond price. In this work, we follow the modelling set-up in Elliott and Siu [11] but keep the interest rate's mean-reverting level constant for simplicity and tractability.

It has to be noted as well that over the past 50 years, many countries have experienced significant mortality improvement. So, longevity risk is now a well-acknowledged issue affecting annuity and insurance products. Life expectancies have been improving at an accelerating and faster pace than anticipated. Historical data, however, show that such improvement in mortality still exhibits random patterns, and therefore flexible stochastic models must be built to respond to this development. On the one hand continued advancements and discoveries in medical care and healthy lifestyle awareness will greatly contribute to the population's longevity but on the other hand, there is also evidence showing that some other factors may hinder mortality improvement. For example, there are various situations in the present society and current environments that cause epidemic obesity and coronary heart diseases. There are also pandemics happening from time to time, and viruses and bacteria developing resistance to antibiotics and other treatments. Severe air pollution and consequences from acts of terrorism are extreme instances that further curb mortality improvements. Therefore, a stochastic volatility model of mortality evolution is necessary to capture this stylised fact in the long run.

Our contribution in this work is the extension of model setting in Liu et al. [17] by allowing for the interest and mortality rates' volatilities to be regime-switching according to the dictates of a continuous-time finite-state Markov chain. The forward measure is introduced with the bond price as the numéraire, which is determined following the idea in Elliott and Mamon [8]. We derive the survival probability under the forward measure and in turn, the explicit solution to the pure endowment price is obtained. The valuation of GAO, an option-embedded insurance product with an emerging popularity, is examined. We provide an efficient pricing method under the endowment-risk-adjusted measure associated with the price of pure endowment. We conduct numerical experiments to show that our approach is more efficient than the Monte-Carlo method.

This chapter is organised as follows. In section 5.2, we formulate the framework to price the GAO under which the interest and mortality rates follow correlated affine structures and their volatilities are regime-switching driven by a Markov chain. Section 5.3 presents the explicit solutions to the zero-coupon bond price, survival index and the pure endowment price by utilising the forward measure. We find the GAO price by employing the new measure called endowment-risk-adjusted measure. Numerical illustrations in section 5.4 demonstrate the efficiency in pricing GAO via the technique of changing reference probability measures. In the last section, we give some concluding remarks.

5.2 Modelling set-up

Assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ supporting all stochastic processes considered in the valuation of GAO. The probability measure Q is a risk-neutral measure, and the interest and mortality rates follow affine structures. We modify the Vasiček model for the interest rate r_t through the stochastic differential equation (SDE)

$$dr_t = a(b - r_t)dt + \sigma_t\, dW_t, \tag{5.1}$$

where W_t is a standard Brownian motion under Q whilst a and b are constants. Calibration for the purpose of valuation implementation of the Vasiček model using bond prices from the market is discussed in Rodrigo and Mamon [22].

The Ornstein-Uhlenbeck (OU) process, in which the Vasiček model is a special case, was also justified for modelling the mortality rate μ_t; see details in Milevsky and Promislow [21]. For the μ_t process, our model is given by

$$d\mu_t = c\mu_t\, dt + \xi_t\, dZ_t. \tag{5.2}$$

Here, c is a constant whilst Z_t is another standard Brownian motion correlated with W_t, and satisfies $dZ_t = \rho\, dW_t + \sqrt{1-\rho^2} dW'_t$, where W'_t is a standard Brownian motion independent from W_t.

Under this framework, we suppose that the respective volatilities σ_t and ξ_t in the models (5.1) and (5.2) are driven by the same finite-state Markov chain $\boldsymbol{y}_t$ in continuous time. For the r_t process, the volatility dynamics is

$$\sigma_t = \sigma(\boldsymbol{y}_t) = \langle \boldsymbol{\sigma}, \boldsymbol{y}_t \rangle .$$

Similarly, the volatility for the mortality rate follows

$$\xi_t = \xi(\boldsymbol{y}_t) = \langle \boldsymbol{\xi}, \boldsymbol{y}_t \rangle , \tag{5.3}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_n)$. This representation describes the different levels of volatilities that the processes could attain at time t. We assume that $\boldsymbol{y}_t$ is a homogeneous Markov chain and the state space of $\boldsymbol{y}_t$ takes one of the unit vectors in the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$ belonging to $\mathbb{R}^n$, where $\boldsymbol{e}_i = (0, \dots, 1, \dots, 0)^\top$, i.e., the ith element is 1 and 0 elsewhere. The unconditional distribution of $\boldsymbol{y}_t$ is expressed as $\boldsymbol{p}_t = \mathbf{E}^Q[\boldsymbol{y}_t] = (\boldsymbol{p}_t^1, \boldsymbol{p}_t^2, \dots, \boldsymbol{p}_t^n)$ and $\boldsymbol{p}_t^i = \mathbf{E}^Q[\langle \boldsymbol{e}_i, \boldsymbol{y}_t \rangle]$. Moreover, $\boldsymbol{y}_t$ is a semimartingale process satisfying

$$d\boldsymbol{y}_t = \boldsymbol{\Gamma}_t \boldsymbol{y}_t dt + d\boldsymbol{n}_t, \tag{5.4}$$

where $\boldsymbol{n}_t$ is a martingale increment and $\boldsymbol{\Gamma}_t = \boldsymbol{\Gamma}$ is the intensity matrix. By the Kolmogorov forward equation,

$$d\boldsymbol{p}_t = \boldsymbol{\Gamma}\, \boldsymbol{p}_t\, dt \tag{5.5}$$

with initial value $\boldsymbol{p}_0$.

5.3 Derivation of the endowment price

A pure endowment is contract that promises to pay the holder a stated sum if he survives a specified period but nothing in case of prior death. Given the models for interest and mortality rates, we can price the pure endowment as

$$M(t,T) = \mathbf{E}^Q \left[e^{-\int_t^T r_u\, du} e^{-\int_t^T \mu_v\, dv} | \mathcal{F}_t \right]. \tag{5.6}$$

Due to the dependence relation between these rates, we can not separate the expectation into a product of two expectations. To facilitate the valuation of pure endowment, we employ the forward measure associated with bond price as the numéraire.

5.3.1 Bond price

From standard interest-rate theory, the price of a zero-coupon bond with maturity T at time t ($T > t$) is

$$B(t,T) = \mathbf{E}^{Q}\left[e^{-\int_t^T r_u(\boldsymbol{y}_t)\,du}|\mathcal{F}_t\right].$$

If we know the trajectory of $\boldsymbol{y}_t$ then from Elliott and Kopp [9]

$$B(t,T,\boldsymbol{y}_t) = e^{-A(t,T)r_t + D(t,T,\boldsymbol{y}_t)}, \tag{5.7}$$

where $A(t,T)$ and $D(t,T,\boldsymbol{y}_t)$ are deterministic functions independent of r_t. When $\boldsymbol{y}_t$ is random, we take another condition expectation with respect to $\{\mathcal{F}_t\}$ and obtain

$$A(t,T) = \frac{1-e^{-a(T-t)}}{a}. \tag{5.8}$$

and

$$\begin{aligned} D(t,T,\boldsymbol{y}_t) &= \int_t^T (-abA(u,T) + \frac{1}{2}A^2(u,T)\sigma_u^2)\,du \\ &= -b(T-t) + bA(t,T) + \int_t^T \langle \boldsymbol{\phi}_u, \boldsymbol{y}_u \rangle\,du. \end{aligned} \tag{5.9}$$

Here, $\boldsymbol{\phi}_u = \frac{1}{2}A^2(u,T)\boldsymbol{\sigma}^2 = \left(\frac{1}{2}A^2(u,T)\sigma_1^2, \ldots, \frac{1}{2}A^2(u,T)\sigma_n^2\right)$. Therefore, the bond price representation can be written as

$$B(t,T,\boldsymbol{y}_t) = e^{-A(t,T)r_t - b(T-t) + bA(t,T)}\, e^{\int_t^T \langle \boldsymbol{\phi}_u, \boldsymbol{y}_u \rangle\,du}.$$

To further evaluate the bond price, it remains to evaluate the expectation of $e^{\int_t^T \langle \boldsymbol{\phi}_u, \boldsymbol{y}_u \rangle\,du}$. Following Elliott and Mamon [8], we get

$$\begin{aligned} B(t,T) &= e^{-A(t,T)r_t - b(T-t) + bA(t,T)} \langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1} \rangle \\ &= \left\langle e^{-A(t,T)r_t - b(T-t) + bA(t,T)} \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1} \right\rangle, \end{aligned} \tag{5.10}$$

where $\mathbf{\Pi}_{t,T}$ is the fundamental matrix solution to the linear matrix differential equation

$$d\mathbf{\Pi}_{t,s} = \boldsymbol{H}(t,s)\mathbf{\Pi}_{t,s}\,ds$$

with initial value $\mathbf{\Pi}_{t,t} = \boldsymbol{I}$, the identity matrix. The matrix $\boldsymbol{H}(t,s)$ is an $n \times n$. It has the form $\boldsymbol{H}(t,s) = \boldsymbol{J}(t,s) + \mathbf{\Gamma}^{\top}$, where $\boldsymbol{J}(t,s)$ is a time-varying diagonal matrix with the elements $\boldsymbol{\phi}_s$ in the diagonal, i.e.,

$$\boldsymbol{J}(t,s) = \begin{bmatrix} \frac{1}{2}A^2(t,s)\sigma_1^2 & & & \\ & \frac{1}{2}A^2(t,s)\sigma_2^2 & & \\ & & \ddots & \\ & & & \frac{1}{2}A^2(t,s)\sigma_n^2 \end{bmatrix}.$$

5.3.2 Survival index

With the analytic solution of the bond price (5.10), we can use it as the numéraire associated with the forward measure to calculate the pure endowment price. Define the Radon-Nikodŷm derivative as

$$\left.\frac{d\widetilde{Q}}{dQ}\right|_{\mathcal{F}_T} = \Lambda_T := \frac{e^{-\int_0^T r_u du}\,B(T,T)}{B(0,T)}. \tag{5.11}$$

Under measure Q, Λ_T is a martingale and for $t \le T$,

$$\Lambda_t = \mathbf{E}^Q[\Lambda_T|\mathcal{F}_t] = \frac{e^{-\int_0^t r_u du}B(t,T)}{B(0,T)}.$$

From the Bayes' rule for conditional expectation, we know that for any $\mathcal{F}_t$−measurable random variable ζ,

$$\begin{aligned} \mathbf{E}^{\widetilde{Q}}[\zeta|\mathcal{F}_t] &= \frac{\mathbf{E}^Q[\Lambda_T\zeta|\mathcal{F}_t]}{\mathbf{E}^Q[\Lambda_T|\mathcal{F}_t]} \\ &= \frac{\mathbf{E}^Q\left[e^{-\int_t^T r_u\,du}\zeta|\mathcal{F}_t\right]}{B(t,T)}. \end{aligned} \tag{5.12}$$

This implies that

$$\begin{aligned} M(t,T) &= \mathbf{E}^{Q}\left[e^{-\int_t^T r_u du}e^{-\int_t^T \mu_v\, dv}\middle|\mathcal{F}_t\right] \\ &= B(t,T)\mathbf{E}^{\widetilde{Q}}[e^{-\int_t^T \mu_v\, dv}|\mathcal{F}_t] \\ &= B(t,T)\widetilde{S}(t,T), \end{aligned} \tag{5.13}$$

where $\widetilde{S}(t,T) := \mathbf{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_v dv}\middle|\mathcal{F}_t\right]$ is the survival function under $\widetilde{Q}$. To find its explicit solution, we require the dynamics of μ_t and $\boldsymbol{y}_t$ under the forward measure $\widetilde{Q}$.

Following the procedure in the Appendix of Mamon [20], the corresponding Brownian motion $\widetilde{W}_t$ under $\widetilde{Q}$ is given by

$$d\widetilde{W}_t = dW_t + A(t,T)\sigma_t dt \text{ and } d\widetilde{W}'_t = dW'_t,$$

where $\widetilde{W}_t$ and $\widetilde{W}'_t$ are independent standard Brownian motions under $\widetilde{Q}$. Hence, the dynamics of r_t and μ_t under $\widetilde{Q}$ are given by the respective SDEs

$$dr_t = \left[ab - \sigma_t^2 A(t,T) - ar_t\right]dt + \sigma_t d\widetilde{W}_t \tag{5.14}$$

and

$$\begin{aligned} d\mu_t &= (-\rho\sigma_t\xi_t A(t,T) + c\mu_t)dt + \rho\xi_t d\widetilde{W}_t + \sqrt{1-\rho^2}\,\xi_t d\widetilde{W}'_t \\ &= (-\rho\sigma_t\xi_t A(t,T) + c\mu_t)dt + \xi_t d\widetilde{Z}_t, \end{aligned} \tag{5.15}$$

where $\widetilde{Z}_t = \rho\widetilde{W}_t + \sqrt{1-\rho^2}\,\widetilde{W}'_t$.

The distribution of the process $\boldsymbol{y}_t$ changes as well under $\widetilde{Q}$. To obtain the transition probability matrix under $\widetilde{Q}$, we set $\zeta = \boldsymbol{y}_T$ in equation (5.12) so that

$$\mathbf{E}^{\widetilde{Q}}[\boldsymbol{y}_T|\mathcal{F}_t] = \frac{\mathbf{E}^{Q}\left[e^{\int_t^T r_u\, du}\boldsymbol{y}_T|\mathcal{F}_t\right]}{B(t,T)}. \tag{5.16}$$

Note that

$$\begin{aligned} B(t,T) &= \mathbf{E}^{Q}\left[e^{-\int_t^T r_u\, du}|\mathcal{F}_t\right] \\ &= \mathbf{E}^{Q}\left[e^{-\int_t^T r_u\, du}\langle\boldsymbol{y}_T,\mathbf{1}\rangle|\mathcal{F}_t\right] \\ &= \left\langle\mathbf{E}^{Q}\left[e^{-\int_t^T r_u\, du}\boldsymbol{y}_T|\mathcal{F}_t\right],\mathbf{1}\right\rangle. \end{aligned} \tag{5.17}$$

Comparing equations (5.10) and (5.17), we obtain

$$\mathbf{E}^{Q}\left[e^{-\int_t^T r_u\,du}\boldsymbol{y}_T|\mathcal{F}_t\right] = e^{-A(t,T)r_t-b(T-t)+bA(t,T)}\mathbf{\Pi}_{t,T}\boldsymbol{y}_t. \tag{5.18}$$

Substituting equations (5.18) and (5.10) into equation (5.16), we have

$$\mathbf{E}^{\widetilde{Q}}\left[\boldsymbol{y}_T|\mathcal{F}_t\right] = \frac{\mathbf{\Pi}_{t,T}\boldsymbol{y}_t}{\langle\mathbf{\Pi}_{t,T}\boldsymbol{y}_t,\mathbf{1}\rangle}. \tag{5.19}$$

Equation (5.15) shows that μ_t still has an affine form with a regime-switching volatility under $\widetilde{Q}$. Therefore, we can reapply the results in Elliott and Mamon [8] to calculate the survival function $\widetilde{S}(t,T)$. The only difficulty in this process is solving for the intensity matrix of the Markov chain $\widetilde{\boldsymbol{y}}_t$ under $\widetilde{Q}$. Equation (5.19) implies that under the forward measure $\widetilde{Q}$ the Markov chain $\widetilde{\boldsymbol{y}}_t$ is no longer homogeneous. However, the Kolmogorov forward differential equation still holds, that is,

$$\frac{\partial}{\partial t}\widetilde{\boldsymbol{p}}_{v,t} = \widetilde{\boldsymbol{p}}_{v,t}\widetilde{\mathbf{\Gamma}}_t\ , \tag{5.20}$$

subject to $\widetilde{\boldsymbol{p}}_{v,v} = \mathbf{I}$. Here, $\widetilde{\mathbf{\Gamma}}_t$ for $t \geq 0$ is a one-parameter family of matrices whose off-diagonal entries are the transition rates and $\widetilde{\boldsymbol{p}}_{v,t}$ is the vector of conditional probabilities given the starting point v under $\widetilde{Q}$, which can be obtained from (5.19) as $\widetilde{\boldsymbol{p}}_{v,t} = \mathbf{E}^{\widetilde{Q}}[\widetilde{\boldsymbol{y}}_t|\mathcal{F}_v]$. This means

$$\widetilde{\boldsymbol{p}}^{\,i}_{v,t} = \mathbf{E}^{\widetilde{Q}}\left[\langle\boldsymbol{e}_i,\boldsymbol{y}_t\rangle|\mathcal{F}_v\right] = \left\langle\frac{\mathbf{\Pi}_{v,t}\boldsymbol{y}_v}{\langle\mathbf{\Pi}_{v,t}\boldsymbol{y}_v,\mathbf{1}\rangle}, e_i\right\rangle. \tag{5.21}$$

The survival function is then given by

$$\begin{aligned}\widetilde{S}(t,T) &= e^{-\widetilde{G}(t,T)\mu_t}\langle\widetilde{\mathbf{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t,\mathbf{1}\rangle \\ &= \langle e^{-\widetilde{G}(t,T)\mu_t}\widetilde{\mathbf{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t,\mathbf{1}\rangle\end{aligned} \tag{5.22}$$

with $\widetilde{G}(t,T) = \frac{e^{c(T-t)}-1}{c}$. In equation (5.22), $\widetilde{\mathbf{\Pi}}_{t,T}$ is the fundamental matrix solution to the linear differential system

$$d\widetilde{\mathbf{\Pi}}_{t,s} = \widetilde{\boldsymbol{H}}(t,s)\widetilde{\mathbf{\Pi}}_{t,s}\,ds$$

at time $s = T$, where $\widetilde{H}(t,s) = \widetilde{\boldsymbol{J}}(t,s) + \widetilde{\mathbf{\Gamma}}_s^{\top}$ and $\widetilde{\boldsymbol{J}}(t,s)$ is the diagonal matrix

$$\widetilde{\boldsymbol{J}}(t,s) = \begin{bmatrix} \rho A(s,T)\widetilde{G}(s,T)\sigma_1\xi_1 + \frac{1}{2}\widetilde{G}(s,T)^2\xi_1^2 & & & \\ & \rho A(s,T)\widetilde{G}(s,T)\sigma_2\xi_2 + \frac{1}{2}\widetilde{G}(s,T)^2\xi_2^2 & & \\ & & \ddots & \\ & & & \rho A(s,T)\widetilde{G}(s,T)\sigma_n\xi_n + \frac{1}{2}\widetilde{G}(s,T)^2\xi_n^2 \end{bmatrix}.$$

Therefore, combining equations (5.10) and (5.22), the pure endowment price can be expressed as

$$\begin{aligned} M(t,T) &= \mathbf{E}^{Q}\left[e^{-\int_t^T r_u\,du}e^{-\int_t^T \mu_v\,dv}|\mathcal{F}_t\right] \\ &= B(t,T)\widetilde{S}(t,T) \\ &= e^{-A(t,T)r_t-b(T-t)+bA(t,T)-\widetilde{G}(t,T)\mu_t}\langle \mathbf{\Pi}_{t,T}\boldsymbol{y}_t,\mathbf{1}\rangle\langle \widetilde{\mathbf{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t,\mathbf{1}\rangle \\ &= e^{-A(t,T)r_t-b(T-t)+bA(t,T)-\widetilde{G}(t,T)\mu_t}\langle \mathbf{\Pi}_{t,T}Y_t+\widetilde{\mathbf{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t,\mathbf{1}\rangle\,. \end{aligned} \tag{5.23}$$

5.4 GAO valuation

As previously indicated, GAO provides the insured the right to choose between a fund value and a life annuity with guaranteed annual payments. Although its first issue dates back to 1839, it only became popular starting in the United Kingdom in the 1970-80s. Its option-embedded feature makes it attractive giving rise to higher demand. Nonetheless, the unpredictable nature of interest and mortality rates, heavily affecting the GAO value, caused solvency problems to some companies offering this product. Research on GAO valuation and its risk management continues to grow; see Ballotta et al. [1, 2] and Wilkie et al. [23]. This is largely motivated by the advent of insurance products with similar features and are currently being traded in the market such as the guaranteed minimum income benefit (GMIB) type products.

For simplicity, we do not consider choices in investing the fund but assume that a cash amount for the fund is attained at the maturity date of the contract. Applying the risk-neutral pricing theory, the GAO price is defined as the expected value of its discounted payoff. In particular, the payoff function is $C_T = (ga_x(T) - 1)^+$, where $a_x(T)$ denotes the whole life annuity with unit annual payment for a life aged x at time 0 and g is the guaranteed rate determined by the insurance company. The price of GAO can then be expressed as

$$\begin{aligned} P_{GAO} &= \mathbf{E}^{Q}\left[e^{-\int_0^T r_u\,du}e^{-\int_0^T \mu_v\,dv}C_T\right] \\ &= \mathbf{E}^{Q}\left[e^{-\int_0^T r_u\,du}e^{-\int_0^T \mu_v\,dv}\left(g\,a_x(T)-1\right)^+\right] \\ &= g\mathbf{E}^{Q}\left[e^{-\int_0^T r_u\,du}e^{-\int_0^T \mu_v\,dv}\left(a_x(T)-K\right)^+\right], \end{aligned} \tag{5.24}$$

with $K = \frac{1}{g}$.

To facilitate the pricing of a GAO, we employ a different measure called endowment-risk-adjusted measure $\widehat{Q}$, which first appeared in Liu et al. [16]. The introduction of a regime-switching framework is new in this work. Choosing the pure endowment price as the numéraire, we define the Radon-Nikodŷm derivative $\widehat{Q}$ with respect to Q by setting

$$\frac{d\widehat{Q}}{dQ} := \Theta_T = \frac{e^{-\int_0^T r_u du}M(T,T)}{M(0,T)}. \tag{5.25}$$

Applying the Bayes' rule for conditional expectation similar to what was done in the construction of the forward measure $\widetilde{Q}$, the price of GAO is given by

$$P_{GAO} = gM(0,T)\mathbf{E}^{\widehat{Q}}\left[\left(a_x(T)-K\right)^+\right]. \tag{5.26}$$

Recall by definition that the annuity $a_x(T)$ is the summation of prices for pure endowments with different maturities. Therefore, given the pure endowment price in equation (5.23), the annuity $a_x(T)$ is represented by

$$\begin{aligned} a_x(T) &= \sum_{n=1}^{\infty}\mathbf{E}^{Q}\left[e^{-\int_T^{T+n} r_u\,du}e^{-\int_T^{T+n} \mu_v\,dv}|\mathcal{F}_T\right] \\ &= \sum_{n=1}^{\infty} M(T,T+n) \\ &= \sum_{n=1}^{\infty} e^{-A(T,T+n)r_T-bn+bA(T,T+n)-\widetilde{G}(T,T+n)\mu_T}\left\langle \mathbf{\Pi}_{T,T+n}\boldsymbol{y}_T+\widetilde{\mathbf{\Pi}}_{T,T+n}\widetilde{\boldsymbol{y}}_T,\mathbf{1}\right\rangle. \end{aligned} \tag{5.27}$$

From equations (5.26) and (5.27), we see that in order to value GAO, the dynamics of r_t, μ_t and $\boldsymbol{y}_t$ under the new measure $\widehat{Q}$ must be calculated. These can be obtained by applying the martingale property.

Write

$$X_t := e^{-\int_0^t r_u\,du} M(t,T) = e^{-\int_0^t r_u\,du} B(t,T)\widetilde{S}(t,T) = X_t^1\,X_t^2,$$

where $X_t^1 = e^{-\int_0^t r_u\,du} B(t,T)$ and $X_t^2 = \widetilde{S}(t,T)$. By the martingale property, we know that

$$dX_t^1 = X_t^1\left[-A(t,T)\sigma_t\,dW_t + \frac{\langle \boldsymbol{\Pi}_{t,T} d\boldsymbol{n}_t, \mathbf{1}\rangle}{\langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1}\rangle}\right]. \tag{5.28}$$

From Itô's lemma,

$$\begin{aligned}
dX_t^2 &= X_t^2\Bigg[\left(\frac{1}{2}\widetilde{G}(t,T)^2\xi_t^2 - \frac{\partial \widetilde{G}(t,T)}{\partial t}\mu_t - c\widetilde{G}(t,T)\mu_t + \frac{\langle \widetilde{\boldsymbol{H}}(t,T)\widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}\right)dt \\
&\quad + \frac{\langle \widetilde{\boldsymbol{\Pi}}_{t,T} d\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle} - \widetilde{G}(t,T)\xi_t\,dZ_t\Bigg] \\
&= X_t^2\Bigg[\left(\frac{1}{2}\widetilde{G}(t,T)^2\xi_t^2 - \frac{\partial \widetilde{G}(t,T)}{\partial t}\mu_t - c\widetilde{G}(t,T)\mu_t + \frac{\langle \widetilde{\boldsymbol{H}}(t,T)\widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}\right. \\
&\quad \left. + \frac{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{\Gamma}}_t\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}\right)dt + \frac{\langle \widetilde{\boldsymbol{\Pi}}_{t,T} d\widetilde{\boldsymbol{n}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle} - \widetilde{G}(t,T)\xi_t\,dZ_t\Bigg].
\end{aligned} \tag{5.29}$$

Combining equations (5.28) and (5.29), we have

$$\begin{aligned}
dX_t &= X_t^1\,dX_t^2 + X_t^2\,dX_t^1 \\
&= X_t^1 X_t^2\Bigg[\left(\frac{1}{2}\widetilde{G}(t,T)^2\xi_t^2 - \frac{\partial \widetilde{G}(t,T)}{\partial t}\mu_t - c\widetilde{G}(t,T)\mu_t + \frac{\langle \widetilde{\boldsymbol{H}}(t,T)\widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}\right. \\
&\quad \left. + \frac{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{\Gamma}}_t\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}\right)dt + \frac{\langle \boldsymbol{\Pi}_{t,T} d\boldsymbol{n}_t, \mathbf{1}\rangle}{\langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1}\rangle} + \frac{\langle \widetilde{\boldsymbol{\Pi}}_{t,T} d\widetilde{\boldsymbol{n}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle} - A(t,T)\sigma_t\,dW_t - \widetilde{G}(t,T)\xi_t\,dZ_t\Bigg] \\
&= -X_t\left[A(t,T)\sigma_t\,dW_t + \widetilde{G}(t,T)\xi_t\,dZ_t - \frac{\langle \boldsymbol{\Pi}_{t,T} d\boldsymbol{n}_t, \mathbf{1}\rangle}{\langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1}\rangle} - \frac{\langle \widetilde{\boldsymbol{\Pi}}_{t,T} d\widetilde{\boldsymbol{n}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}\right] \qquad (5.30) \\
&= -X_t\Bigg[(A(t,T)\sigma_t + \rho\widetilde{G}(t,T)\xi_t)\,dW_t + \sqrt{1-\rho^2}\widetilde{G}(t,T)\xi_t\,dW_t' \\
&\quad - \frac{\langle \boldsymbol{\Pi}_{t,T} d\boldsymbol{n}_t, \mathbf{1}\rangle}{\langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1}\rangle} - \frac{\langle \widetilde{\boldsymbol{\Pi}}_{t,T} d\widetilde{\boldsymbol{n}}_t, \mathbf{1}\rangle}{\langle \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\rangle}\Bigg]. \qquad (5.31)
\end{aligned}$$

Equation (5.30) is justified by the martingale property. So by Girsanov theorem, the dynamics of W_t and W_t' are

$$d\widehat{W}_t = (A(t,T)\sigma_t + \rho\widetilde{G}(t,T)\xi_t)dt + dW_t \tag{5.32}$$

$$d\widehat{W}_t' = \sqrt{1-\rho^2}\widetilde{G}(t,T)\xi_t\,dt + dW_t'. \tag{5.33}$$

Consequently, the dynamics of r_t and μ_t under $\widehat{Q}$ become

$$\begin{aligned} dr_t &= a(b-r_t)dt + \sigma_t\left[d\widehat{W}_t - (A(t,T)\sigma_t + \rho\widetilde{G}(t,T)\xi_t)dt\right] \\ &= (ab - ar_t - A(t,T)\sigma_t^2 - \rho\widetilde{G}(t,T)\sigma_t\xi_t)dt + \sigma_t d\widehat{W}_t \qquad (5.34) \\ d\mu_t &= c\mu_t dt + \xi_t\left[\rho(d\widehat{W}_t - (A(t,T)\sigma_t + \rho\widetilde{G}(t,T)\xi_t)dt) + \sqrt{1-\rho^2}(d\widehat{W}'_t - \sqrt{1-\rho^2}\xi_t\, dt)\right] \\ &= \left(c\mu_t - \rho A(t,T)\sigma_t\xi_t - \widetilde{G}(t,T)^2\xi_t^2\right)dt + \xi_t\, d\widehat{Z}_t. \qquad (5.35) \end{aligned}$$

The change of measure brings about a corresponding transformation to the distribution of the Markov chain $\boldsymbol{y}_t$. Such transformed distribution is necessary to calculate the GAO price using equation (5.26) along with r_t and μ_t. To keep it distinct from $\boldsymbol{y}_t$ under the original measure Q, we denote the Markov chain under $\widehat{Q}$ as $\widehat{\boldsymbol{y}}_t$. Following the similar idea to that in getting the distribution of $\boldsymbol{y}_t$ under the forward measure $\widetilde{Q}$, we have

$$\mathbf{E}^{\widehat{Q}}\left[\widehat{\boldsymbol{y}}_T|\mathcal{F}_t\right] = \frac{\boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t + \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t}{\left\langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t + \widetilde{\boldsymbol{\Pi}}_{t,T}\widetilde{\boldsymbol{y}}_t, \mathbf{1}\right\rangle}. \qquad (5.36)$$

Equation (5.36) tells us that $\widehat{\boldsymbol{y}}_t$ is also a non-homogeneous Markov chain. Therefore, the vector of conditional probabilities of $\widehat{\boldsymbol{y}}_t$ given a starting point v can be obtained in a way similar to that of getting $\widetilde{\boldsymbol{y}}_t$. So,

$$\widehat{\boldsymbol{p}}_{v,t} = \mathbf{E}^{\widehat{Q}}\left[\widehat{\boldsymbol{y}}_t|\mathcal{F}_v\right] = (\widehat{\boldsymbol{p}}^1_{v,t}, \widehat{\boldsymbol{p}}^2_{v,t}, \ldots, \widehat{\boldsymbol{p}}^n_{v,t})$$

with $\widehat{\boldsymbol{p}}^i_{v,t} = \left\langle e_i, \frac{\boldsymbol{\Pi}_{v,t}\boldsymbol{y}_v + \widetilde{\boldsymbol{\Pi}}_{v,t}\widetilde{\boldsymbol{y}}_v}{\left\langle \boldsymbol{\Pi}_{v,t}\boldsymbol{y}_v + \widetilde{\boldsymbol{\Pi}}_{v,t}\widetilde{\boldsymbol{y}}_v, \mathbf{1}\right\rangle}\right\rangle$ for $i = 1, 2, \ldots, n$. Given the dynamics of the interest and mortality rates under different probability measures, we can price GAO using our proposed change of measure technique described in equation (5.26), and the result can be compared with the usual Monte-Carlo method using equation (5.24).

5.5 Numerical illustrations

In this section, we present the results of a numerical experiment to assess the efficiency of formula (5.26). We show that it is superior to (5.24) in computing times. Under direct calculation of GAO price using (5.24), we need to generate sample paths of the Markov chain $\boldsymbol{y}_t$ through its transition probability matrix and the diffusion processes for r_t and μ_t given by equations (5.1) and (5.2). To do this, we discretise each year time period into $N = 252$ subintervals, each of which has the fixed length $\Delta t = \frac{1}{N}$. We apply the Euler discretisation scheme to approximate the evolutions of r_t and μ_t over the time period $[0, T]$. The discretisations are

$$r_{(i+1)\Delta t} = r_{i\Delta t} + (ab - ar_{i\Delta t})\Delta t + \sigma_{i\Delta t}\sqrt{\Delta t}\epsilon_{i\Delta t} \tag{5.37}$$

$$\mu_{(i+1)\Delta t} = \mu_{i\Delta t} + c\mu_{i\Delta t}\Delta t + \xi_{i\Delta t}\sqrt{\Delta t}(\rho\epsilon_{i\Delta t} + \sqrt{(1-\rho^2)}\epsilon'_{i\Delta t}), \tag{5.38}$$

where $\{\epsilon_{i\Delta t}\}_{i=1,\ldots,NT}$ and $\{\epsilon'_{i\Delta t}\}_{i=1,\ldots,NT}$ are two independent sequences of standard normal random variables and $\sigma_{i\Delta t} = \langle \boldsymbol{\sigma}, \boldsymbol{y}_{i\Delta t}\rangle$, $\xi_{i\Delta t} = \langle \boldsymbol{\xi}, \boldsymbol{y}_{i\Delta t}\rangle$ with $\{\boldsymbol{y}_{i\Delta t}\}_{i=1,2,\ldots,NT}$ forming a path of the Markov chain $\boldsymbol{y}_t$. The integrals in equation (5.24) can be approximated using the Trapezoidal Rule expressed as

$$\int_0^T r_u\, du \approx \frac{\Delta t}{2}\left[r_0 + r_T + 2\sum_{k=1}^{NT-1} r_k\right], \tag{5.39}$$

and

$$\int_0^T \mu_v\, dv \approx \frac{\Delta t}{2}\left[\mu_0 + \mu_T + 2\sum_{k=1}^{NT-1} \mu_k\right]. \tag{5.40}$$

Numerical values of $e^{-\int_0^T r_u\, du}$ and $e^{-\int_0^T \mu_v\, dv}$ can then be obtained. The terminal values r_T, μ_T and $\boldsymbol{y}_T$ of each path are used to calculate $a_x(T)$ in equation (5.27). Additionally, sample paths of $\widetilde{\boldsymbol{y}}_t$ are also needed for the annuity valuation in equation (5.27).

With our proposed approach in calculating the GAO price using equation (5.26), we do not need to obtain the entire evolution of r_t and μ_t during the time period $[0, T]$. The values r_T and μ_T are determined by the equations

$$r_T = e^{-aT}\left[r_0 + \int_0^T e^{au}\left(ab - A(u,T)\sigma_u^2 - \rho\widetilde{G}(u,T)\sigma_u\xi_u\right)du + \int_0^T e^{au}\sigma_u\, d\widehat{W}_u\right] \tag{5.41}$$

$$\mu_T = e^{cT}\left[\mu_0 - \int_0^T e^{-cu}\left(\rho A(u,T)\sigma_u\xi_u + \widetilde{G}(u,T)^2\xi_u^2\right)du + \int_0^T e^{-cu}\xi_u\, d\widehat{W}_u\right]. \tag{5.42}$$

This greatly reduces the calculation time and improve the accuracy of the pricing results.

Our numerical results are obtained from 50,000 samples generated through Monte-Carlo methodology. The parameters we use are based on Jalen and Mamon [14] and Luciano and Vigna [19] which are given in Table 5.1. Tables 5.2 - 5.6 exhibit the prices of GAO based on a cohort born in 1935 and assumed to hold GAO contracts with maturity at age 65. The maximum survival age is assumed to be 100. This implies that the annuity pays out at most 35 annual payments. The first two columns provide the number of regimes and the corresponding volatility levels in interest rate and mortality rate models. The prices calculated from equations (5.24) and (5.26) under each regime pair along with the standard errors of the mean (SEM) and computing time are shown in the succeeding columns.

From Tables 5.2 to 5.6, the prices under both methods are close to each other. In addition, the prices under multiple regimes are bounded by those under one regime. The SEMs go down as more regimes are chosen supporting the fact that randomness can be captured better and thus results become more accurate by increasing the number of regimes. Moreover, we also find that as ρ varies from negative to positive, the prices of GAO increases; this is consistent with the fact that when two risk factors are negatively correlated there is an offsetting of uncertainties, a mechanism that serves like a natural hedge. A particular hurdle in the implementation of the numerical experiments is the approximation of the transition probability matrices and the intensity matrices of non-homogeneous Markov chain under the forward measure and endowment-risk-adjusted measure. This step is time-consuming because when generating the non-homogeneous Markov chains we have to discretise the time period and obtain the transition probability matrices during each time subinterval by solving quantities of linear ordinary differential equations.

5.6 Conclusions

We proposed a modelling framework, where the interest and mortality rates are correlated and the dynamics of each risk factor have regime-switching affine structures, to support the GAO valuation. The correlation introduced through the diffusion components of the risk factors and the underlying Markov chain driving the switching of regimes adequately describes the rates' relation and dynamics. The change of measure technique was employed to obtain the explicit solution to the pure endowment price. In particular, we utilised the forward measure associ-

Table 5.1: Parameter set for the numerical experiment in chapter 5.

Contract specification

$g = 11.1\%$, $T = 15$, $n = 35$;

Interest rate model: $dr_t = a(b - r_t)dt + \sigma_t\, dW_t$. See equation (5.1).

$a = 0.09$, $b = 0.045$, $r_0 = b$;

Mortality model: $d\mu_t = c\mu_t dt + \xi_t\, dZ_t$. See equation (5.2).

$c = 0.07$, $\mu_0 = 0.006$;

Regime-switching volatilities (σ, ξ):

2-state RS model: $\begin{bmatrix} \sigma_1 & \xi_1 \\ \sigma_2 & \xi_2 \end{bmatrix} = \begin{bmatrix} 0.008 & 0.0008 \\ 0.002 & 0.0002 \end{bmatrix}$

3-state RS model: $\begin{bmatrix} \sigma_1 & \xi_1 \\ \sigma_2 & \xi_2 \\ \sigma_3 & \xi_3 \end{bmatrix} = \begin{bmatrix} 0.008 & 0.0008 \\ 0.005 & 0.0005 \\ 0.002 & 0.0002 \end{bmatrix}$

4-state RS model: $\begin{bmatrix} \sigma_1 & \xi_1 \\ \sigma_2 & \xi_2 \\ \sigma_3 & \xi_3 \\ \sigma_4 & \xi_4 \end{bmatrix} = \begin{bmatrix} 0.008 & 0.0008 \\ 0.006 & 0.0006 \\ 0.004 & 0.0004 \\ 0.002 & 0.0002 \end{bmatrix}$

$\gamma_{ij} = 1$ for $i \neq j$.

Table 5.2: Actuarial prices for GAO under two different methods given $\rho = 0.9$.

Regime pair	Parameter set σ	Parameter set ξ	Monte Carlo simulation (using eqn (5.24)) Price (mean)	SEM	Time (m)	Proposed approach (using eqn (5.26)) Price (mean)	SEM	Time (m)
(1,1)	0.008	0.0008	0.1684685	0.00082	15	0.1687068	0.00059	0.25
(2,2)	(0.008,0.002)	(0.0008, 0.0002)	0.1415551	0.00051	50	0.1411626	0.00035	26
(3,3)	(0.008,0.005,0.002)	(0.0008,0.0005,0.0002)	0.1386103	0.00048	50	0.1386492	0.00034	26
(4,4)	(0.008,0.006,0.004,0.002)	(0.0008,0.0006,0.0004,0.0002)	0.1365523	0.00046	50	0.1367995	0.00033	26
(1,1)	0.005	0.0005	0.1319101	0.00061	15	0.1320392	0.00032	0.25
(1,1)	0.002	0.0002	0.1152363	0.00022	15	0.1152269	0.00012	0.25

Table 5.3: Actuarial prices for GAO under two different methods given $\rho = 0.5$.

Regime pair	Parameter set σ	Parameter set ξ	Monte Carlo simulation (using eqn (5.24)) Price (mean)	SEM	Time (m)	Proposed approach (using eqn (5.26)) Price (mean)	SEM	Time (m)
(1,1)	0.008	0.0008	0.1564501	0.00078	15	0.1568094	0.00051	0.25
(2,2)	(0.008,0.002)	(0.0008, 0.0002)	0.1356113	0.00044	50	0.1353042	0.00023	26
(3,3)	(0.008,0.005,0.002)	(0.0008,0.0005,0.0002)	0.1323058	0.00041	50	0.1327754	0.00021	26
(4,4)	(0.008,0.006,0.004,0.002)	(0.0008,0.0006,0.0004,0.0002)	0.1315971	0.00039	50	0.1318359	0.00020	26
(1,1)	0.005	0.0005	0.1280335	0.00049	15	0.1286647	0.00028	0.25
(1,1)	0.002	0.0002	0.1147828	0.00021	15	0.1145978	0.00011	0.25

Table 5.4: Actuarial prices for GAO under two different methods given $\rho = 0$.

Regime pair	Parameter set σ	Parameter set ξ	Monte Carlo simulation (using eqn (5.24)) Price (mean)	SEM	Time (m)	Proposed approach (using eqn (5.26)) Price (mean)	SEM	Time (m)
(1,1)	0.008	0.0008	0.1427846	0.00066	15	0.1430349	0.00041	0.25
(2,2)	(0.008,0.002)	(0.0008, 0.0002)	0.1284802	0.00035	50	0.1282701	0.00018	26
(3,3)	(0.008,0.005,0.002)	(0.0008,0.0005,0.0002)	0.1269821	0.00033	50	0.1265669	0.00017	26
(4,4)	(0.008,0.006,0.004,0.002)	(0.0008,0.0006,0.0004,0.0002)	0.1257165	0.00032	50	0.1258862	0.00016	26
(1,1)	0.005	0.0005	0.1235888	0.00039	15	0.1233202	0.00023	0.25
(1,1)	0.002	0.0002	0.1140863	0.00017	15	0.1140126	0.00009	0.25

Table 5.5: Actuarial prices for GAO under two different methods given $\rho = -0.5$.

Regime pair	Parameter set σ	ξ	Monte Carlo simulation (using eqn (5.24)) Price (mean)	SEM	Time (m)	Proposed approach (using eqn (5.26)) Price (mean)	SEM	Time (m)
(1,1)	0.008	0.0008	0.1298425	0.00053	15	0.1295196	0.00030	0.25
(2,2)	(0.008,0.002)	(0.0008, 0.0002)	0.1217010	0.00025	50	0.1215940	0.00014	26
(3,3)	(0.008,0.005,0.002)	(0.0008,0.0005,0.0002)	0.1209566	0.00023	50	0.1207913	0.00013	26
(4,4)	(0.008,0.006,0.004,0.002)	(0.0008,0.0006,0.0004,0.0002)	0.1201273	0.00022	50	0.1201987	0.00012	26
(1,1)	0.005	0.0005	0.1187349	0.00029	15	0.1189467	0.00018	0.25
(1,1)	0.002	0.0002	0.1133012	0.00016	15	0.1133344	0.00007	0.25

Table 5.6: Actuarial prices for GAO under two different methods given $\rho = -0.9$.

Regime pair	Parameter set σ	ξ	Monte Carlo simulation (using eqn (5.24)) Price (mean)	SEM	Time (m)	Proposed approach (using eqn (5.26)) Price (mean)	SEM	Time (m)
(1,1)	0.008	0.0008	0.1200346	0.00038	15	0.1197627	0.00020	0.25
(2,2)	(0.008,0.002)	(0.0008, 0.0002)	0.1165428	0.00015	50	0.1164483	0.00010	26
(3,3)	(0.008,0.005,0.002)	(0.0008,0.0005,0.0002)	0.1162165	0.00014	50	0.1160103	0.00009	26
(4,4)	(0.008,0.006,0.004,0.002)	(0.0008,0.0006,0.0004,0.0002)	0.1158628	0.00013	50	0.1158934	0.00009	26
(1,1)	0.005	0.0005	0.1151399	0.00022	15	0.1152547	0.00012	0.25
(1,1)	0.002	0.0002	0.1128203	0.00009	15	0.1127606	0.00005	0.25

ated with the bond price as a numéraire and solved two linear systems of ordinary differential equations. The Kolmogorov forward equation was applied to get the intensity matrices of non-homogeneous Markov chains, which are rarely considered in the implementation based on our literature search. The employment of the newly constructed measure called endowment-risk-adjusted measure by choosing the endowment price as the numéraire enables efficient valuation of GAO. This was demonstrated by the numerical experiments under different correlation structures and regime-switching settings.

This work concentrated on the aspect of pricing an insurance product with option-embedded characteristics. In practice, calibration of model parameters is necessary to price and manage risks of long-term contracts. These could be achieved by least-square and maximum likelihood methods. For dynamic calibration, a natural choice for our set-up is the hidden Markov model (HMM) filtering technique, but must be tailored to correlated OU processes. This is a future research direction that could be explored further.

An extension of our modelling framework to provide greater flexibility may be accomplished by imposing that every parameter in the mortality and interest rates' SDEs is regime switching. The correlation parameter could also depend on the Markov chain; having a regime-switching correlation matrix from the point of view of model calibration and HMM filtering methodology requires extensive work.

Within our proposed model setting, there are other types of option-embedded benefits that could be priced and analysed as they gain popularity and their demands increase. Two examples are the guaranteed minimum death benefit and guaranteed minimum withdrawal benefit products. Finally, we note that we adopted the OU process to model the cohort mortality, which ignores the age pattern. This deficiency can be rectified by incorporating the age factor into the mortality model. The most popular models that contain both age and time effects are the Lee-Carter [15] and Cairns-Blake-Dowd [3] models, but such models must also be modified to incorporate a regime-switching volatility feature for accurate pricing of annuity-linked insurance products.

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Chapter 6

Risk measurement of a guaranteed annuity option under a stochastic modelling framework

6.1 Introduction

Risk measurement is an important component of business insurance and examines the insurer's capability in fulfilling its future obligation once a product is sold. Various kinds of risk measures have emerged in the last few decades, each with certain desirable features; see Balbas et al. [5], Sereda et al. [40], and Wirch and Hardy [47], amongst others. Value at risk (VaR), first introduced by Markowitz [32] and Roy [39], stands out amongst many competing risk measures due to its simple implementation in practice. But, its inability to preserve the subadditivity property is its major drawback. The concept of a "coherent" risk measure was then proposed by Artzner et al. [4] to rectify the deficiency of VaR. A representative of coherent risk measures, conditional tail expectation (CTE) has been commonly used in recent years as the alternative to VaR. In Canada's life insurance regulatory framework, the Office of the Superintendent of Financial Institutions requires insurers to use CTE over one year for the supervisory target level (cf. [33]).

Denneberg [12] and Wang et al. [46] developed a class of risk measures called distortion risk measures which demonstrate the theoretical results of coherent risk measures. These measures are defined by distortion functions such as the proportional hazard (PH) function [44], lookback function (LB) [21] and Wang Transform (WT) function [43]. Wirch and Hardy [47] proved that the distortion risk measures are coherent if and only if the distortion function is concave for positive losses; then, came the introduction of the Beta function as a distortion function. The theory of spectral risk measure (SRM) was developed by Acerbi [1, 2] in which this type of risk measure is linked to the user's risk aversion.

Although many risk measures were put forward, there is no consensus which one is the best for risk management. Sortino and Satchell [41] concluded that there is no single risk measure that is universally acceptable because any proposed risk measure would have its own limitations. Rachev et al. [37] argued that an ideal measure does not exist but it is reasonable to search for risk measures ideal for the specific problem under investigation.

The insurance industry standard keeps evolving in response to new economic conditions and in an effort to set high levels of safety and effectiveness. Given growing uncertainties nowadays, regulatory authorities and entities with oversight functions require higher levels of safety via capital requirements to address companies' insolvency issues. The emergence of contracts like unit-linked life insurance contingencies with guaranteed minimum payoffs entails cautious risk assessment given the interaction of several risk factors.

When option-embedded insurance products began appearing in the market, various papers immediately dealt with its valuation; for example, Ballotta and Haberman [6], Boyle and Hardy [9], Liu et al. [29], amongst others. The focus of this chapter is the estimation of extreme losses that may cause solvency problem for companies. Equitable Life taught us a valuable lesson on the importance of assessing solvency capital adequately. For a long time Equitable Life held small reserves to cover against adverse events due to high interest rates. However, when interest rates fell along with the unanticipated mortality improvement Equitable had to put itself up for sale and close new businesses in 2000. Many insurance companies still use classical methods to evaluate their risks, which cannot offer provisions against capital solvency.

In this work, we aim to evaluate the capital requirement of a guaranteed annuity option (GAO) through the above-mentioned risk measures and construct a relationship among these risk measures. To attain these objectives, we develop a framework to model the loss (profit is viewed as negative loss) of GAO in which the risk factors are stochastic and correlated with each other. Since it is not easy to identify the underlying distribution of the loss exactly, we adopt the Monte-Carlo simulation to get the approximate empirical distribution and obtain the estimates of the risk measures. However, it is known that the sample variability limits the applications of Monte-Carlo method. That is, different values are obtained under different sample paths. To address the accuracy and credibility of the estimated risk measures, we employ the bootstrap method to estimate the variation. Through regression, we also determine in advance the number of sample replicates needed to achieve the target sampling error and vice versa.

As an alternative approach, we employ the methodology of density approximation to estimate the distribution of the loss random variable. We address the problem of finding the underlying unknown probability distribution function (PDF) and the corresponding cumulative density function (CDF) of the population given samples drawn from a population. Various techniques can be utilised for density approximation; see for instance, Baron and Sheu [7], Devroye and Györfy [14], amongst others. Traditionally, there are two principal approaches for density estimation: parametric, which makes stringent assumptions about the density; and nonparametric, which is essentially a distribution-free approach. Kernel density estimation is one of the most widely-used non-parametric techniques to model densities because it provides a flexible framework to represent multi-modal densities. However, the requirements of high memory and computational complexity limit its applicability in practice. Moment-based density approximation, on the other hand, is an easier way to approximate the density when the moments of a given distribution are available; see Provost [36]. In this work, we utilise the moment-based method to approximate the distribution of the GAO losses given the samples generated from the expression of the loss random variable. We compare this approach with the commonly used non-parametric kernel density estimation, which is executed using the function *density* provided in the software **R** with a Gaussian kernel and the bandwidth is the standard deviation of the smoothing kernel.

This chapter is structured as follows. We construct the modelling framework of the correlated risk factors and evaluate the loss of a GAO in section 6.2 under the assumption of no charges. Section 6.3 presents some well-known risk measures with an elaboration of their attractive properties and limitations. In section 6.4, we illustrate numerical results covering three related aspects. Firstly, we provide numerical risk measures from empirical CDF and from approximated distributions through moment-based density approximation method. Secondly, the accuracy of the risk measures is examined through the bootstrap method. The byproduct of this process is that we could obtain the number of replicates for a given desired standard error according to a relation derived using a regression method. Finally, a sensitivity analysis provides an examination of the impact of each parameter on various risk measures, which could be useful for parameter control and calibration. Finally, section 6.5 concludes.

6.2 The loss random variable associated with a GAO

6.2.1 Description of a GAO contract

GAOs were designed to make pension contracts more attractive. For policyholders who plan to receive annuities upon retirement, they offer protection against poor market performance during the accumulation phase and adverse interest rate experience at annuitisation. The single or regular premiums are invested into a mutual or separate fund managed by the insurance company. For simplicity, we assume an insured purchases the GAO contract at a single premium $F(0) = \$1$ paid at time 0.

At retirement time T, the policyholder aged x has the option to choose from the greater of the fund value denoted as $F(T)$ and a life annuity converted from maturity proceeds of the fund with annual payment $g\,F(T)$ where g is referred to as the guaranteed rate determined by the insurance company. Thus, the life annuity has the value of $g\,F(T)\,a_x(T)$, where $a_x(T)$ is the whole life annuity-immediate defined as the expected present value of the future unit annual payments. The payoff function of GAO is then

$$C_T = [g\,F(T)\,a_x(T) - F(T)]^+ = g\,F(T)\left[a_x(T) - \frac{1}{g}\right]^+. \tag{6.1}$$

With the assumptions and framework above, we define the gross loss of GAO as the present value of future payments that the insurance company needs to make without charges. Charges are amounts that a policyholder must pay when investing in a fund. In practice, these may include surrender charge, mortality and expense risk charge, administrative fees, and other expenses that reduce the value of the account and consequently the return of the original investment. From equation (6.1), we see that at maturity time T, the better the performance of the fund, the more payments the insurance company will have to incur. Therefore, it is reasonable to invest the solvency capital in the same asset as the fund, as described in Hardy [19]. We write ${}_tp_x$ as the probability of life aged (x) surviving after t years. Consequently, the loss L per $F(0)$ premium can be obtained by discounting the payoff function C_T using the financial discounting factor $\dfrac{F(T)}{F(0)}$ and actuarial discounting factor ${}_Tp_x$. So,

$$L = \frac{F(0)}{F(T)}\,{}_Tp_x\,C_T = g\,{}_Tp_x\left[a_x(T) - \frac{1}{g}\right]^+. \tag{6.2}$$

Intuitively, equation (6.2) says that for a pool of GAO contracts with large enough size, approximately ${}_Tp_x$ proportion contracts are still in force at time T. We may average the remaining losses at time T out of all original contracts and treat them as if every contract reduces to the portion ${}_Tp_x$ at time T of its original size by the strong law of large numbers.

6.2.2 Modelling framework

From equation (6.2), it is apparent that there are two key risk factors in a GAO contract: the interest rate r_t and mortality rate μ_t. We do not model the asset $F(t)$ that the fund is invested in but rather assume simply that the fund will accumulate to $F(T)$ at time T ready to be annuitise if the GAO is exercised at time T. We follow the framework in [28] in modelling the two risk factors whose dynamics under a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ are given by

$$dr_t = a(b - r_t)dt + \sigma dX_t \tag{6.3}$$

and

$$d\mu_t = c\mu_t dt + \xi dY_t, \tag{6.4}$$

where $a,\ b,\ c,\ \sigma$ and ξ are positive constants. Here $Y_t = \theta X_t + \sqrt{1-\theta^2}X_t^1$ and X_t, X_t^1 are independent standard Brownian motions. So, θ is the correlation between X_t and Y_t. All the initial values r_0 and μ_0 are assumed to be known at time 0.

Remarks: *The probability measure P denotes the objective measure since we are interested in the capital allocation under the real world. As stated in Hardy [19], the projection of true distributions of outcomes for equity-linked product or portfolio should be under the real-world measure, whilst the risk-neutral measure equivalent to real-world measure, is just a device to simplify the price of an option as an expected value and only relevant to pricing and replication. Moreover, when modelling guaranteed maturity benefits, current market statistics, which are used to back out risk-neutral measure, may not provide sufficient market information since the guaranteed maturity benefits often have longer maturities than the traded options. They vary with term to maturity so that it is hard to assert that current market conditions can provide an appropriate assumption when analysing future cash flows.*

Following Liu et al. [28], the annuity $a_x(T)$ is defined as the sum of pure endowments with different maturities and can be expressed as

$$\begin{aligned} a_x(T) &= \sum_{n=0}^{\infty} \mathbf{E}^P \left[e^{-\int_T^{T+n} r_u\,du} e^{-\int_T^{T+n} \mu_v\,dv} \middle| \mathcal{F}_T \right] \\ &= \sum_{n=0}^{\infty} M(T, T+n) \\ &= \sum_{n=0}^{\infty} \beta(T, T+n) e^{-V(T,T+n)}, \end{aligned} \tag{6.5}$$

with $\beta(T, T+n) = e^{D(T,T+n)+\widetilde{H}(T,T+n)}$ and $V(T, T+n) = A(T, T+n)r_T + \widetilde{G}(T, T+n)\mu_T$.
Here,

$$\begin{aligned}
A(T, T+n) &= \frac{1-e^{-an}}{a} \\
D(T, T+n) &= \left(b - \frac{\sigma^2}{2a^2}\right)[A(T, T+n) - n] - \frac{\sigma^2 A(T, T+n)^2}{4a} \\
\widetilde{G}(T, T+n) &= \frac{(e^{cn} - 1)}{c} \\
\widetilde{H}(T, T+n) &= \left(\frac{\rho\sigma\xi}{ac} - \frac{\xi^2}{2c^2}\right)[\widetilde{G}(T, T+n) - n] + \frac{\rho\sigma\xi}{ac}[A(T, T+n) - \phi(T, T+n)] + \frac{\xi^2}{4c}\widetilde{G}(T, T+n)^2
\end{aligned}$$

with $\phi(T, T+n) = \frac{1-e^{-(a-c)n}}{a-c}$.

Substituting equation (6.5) into equation (6.2), the gross loss of GAO becomes

$$L = g\, e^{-\int_0^T \mu_v\, dv} \left[\sum_{n=0}^{\infty} \beta(T, T+n) e^{-V(T,T+n)} - K\right]^+ \tag{6.6}$$

with $K = \frac{1}{g}$.

6.3 Description of risk measures

A risk measure is defined as a functional mapping from a loss random variable to the set of real numbers. In particular, $\rho : \mathscr{X} \to \mathbb{R}$ is called a risk measure if it satisfies the following conditions:

$\mathcal{P}_1$. Translation invariance: for any $a \in \mathbb{R}$ and a fixed $X \in \mathscr{X}$, $\rho(X + a) = \rho(X) + a$;

$\mathcal{P}_2$. Monotonicity: for $X \neq Y$, X, $Y \in \mathscr{X}$, $\rho(X) \neq \rho(Y)$.

6.3.1 Quantile-based risk measures

VaR gained greater acceptance to measure risk in financial and actuarial fields due to its ease of implementation. It is a quantile-based risk measure defined as the loss in market value that can only be exceeded with a probability of at most $1 - \alpha$, i.e., VaR is the 100α percentile of the loss distribution. Specifically, for a risk X over a given period [0, T] and $0 < \alpha < 1$, the $100\alpha\%$ VaR, denoted as $\mathrm{VaR}_\alpha(X)$, is

$$\mathrm{VaR}_\alpha(X) = \inf\{x : P(X \leq x) \geq \alpha\}. \tag{6.7}$$

However, VaR fails to satisfy the sub-additivity property and ignores the potential loss beyond the confidence level, which limits its use in reporting to regulators or clients. Artzner et al. [4] introduce the concept of coherent risk measures. In addition to properties $\mathcal{P}_1$ and $\mathcal{P}_2$, a coherent risk measure satisfies:

$\mathcal{P}_3$. Sub-additivity: for any $X,\ Y \in \mathscr{X}$, $\rho(X+Y) \leq \rho(X) + \rho(Y)$;

$\mathcal{P}_4$. Positive homogeneity: for any $X \in \mathscr{X},\ \lambda \geq 0, \rho(\lambda X) = \lambda\rho(X)$.

As an alternative to VaR, the conditional tail expectation (CTE), which is employed in practice, is a coherent risk measure and rectifies the shortcoming of VaR. Moreover, it takes into account what the loss will be when the worst event occurs with probability of $1-\alpha$. The CTE is defined as the expected loss given that the loss values fall into the worst $(1-\alpha)$ part of the loss distribution. In other words,

$$CTE_\alpha = \mathbf{E}[X|X > VaR_\alpha]. \tag{6.8}$$

Compared with VaR, CTE considers the entire tail of the loss distribution but ignores usable information at the opposite side of the distribution. Wang [43] saw opportunity in the drawback of the CTE and proposed a new coherent measure, which adjusts accordingly extreme low frequency and high severity loss. This is further described in the next subsection.

6.3.2 Distortion risk measures

There are two ways of introducing distortion measures, viz. by axiomatic definition and via definition motivated by the economic theory of choice under uncertainty. The distortion risk measures under the second approach are mostly based on dual utility theory. In the second approach, a distortion measure is defined by a distortion function; the risk is valued under the distortion probability measure and not under the original probability measure; see Yaari [49]. Through the axiomatic definition approach, Wang et al. [46] developed axioms on law invariance, monotonicity, comonotonic additivity, and continuity to investigate the price of insurance risks. It is shown that risk measures meet those such axiomatic properties if and only if the risk measures have the Choquet integral representation under the distortion probability measure.

A distortion risk measure is defined as the distorted expectation of any loss random variable X related to a distortion function $\chi : [0,1] \to [0,1]$, which is a non-decreasing function with $\chi(0) = 0$ and $\chi(1) = 1$. If $S_X(x)$ denotes the decumulative function of the loss random variable, $\chi(S_X(x))$ can be viewed as a distorted decumulative distribution function. When the loss random variable X can take any real number, the distortion risk measure can be expressed as the Choquet integral

$$\rho_X(x) = -\int_{-\infty}^{0} [1 - \chi(S_X(x))]\, dx + \int_{0}^{\infty} \chi(S_X(x))\, dx. \tag{6.9}$$

In addition to the basic properties of risk measures, the distortion risk measure also satisfies the following:

$\mathcal{P}_5$. Conditional state independence: If X and Y have the same distribution, $\rho(X) = \rho(Y)$;

$\mathcal{P}_6$. Continuity:

$$\lim_{d\to 0} \rho((X-d)^+) = \rho(X^+);\ \lim_{d\to\infty} \rho(\min\{X;d\}) = \rho(X);\ \lim_{d\to-\infty} \rho(\max\{X;d\}) = \rho(X). \tag{6.10}$$

Note that the VaR and CTE fall into the class of distortion risk measures with χ functions defined below. The distortion function defining the quantile risk measure is

$$\chi(S_X(x)) = \begin{cases} 0 & \text{if } 0 \le S_X(x) < 1-\alpha, \\ 1 & \text{if } 1-\alpha \le S_X(x) \le 1. \end{cases}$$

On the other hand, the distortion function for CTE risk measure is

$$\chi(S_X(x)) = \begin{cases} \frac{S_X(x)}{1-\alpha} & \text{if } 0 \le S_X(x) < 1-\alpha, \\ 1 & \text{if } 1-\alpha \le S_X(x) \le 1. \end{cases}$$

There are other well-known distortion risk measures. The proportional hazard (PH) transform proposed by Wang [42] is a special subclass of coherent distortion functions which preserves a larger set of properties and useful in calculating insurance premiums. The distortion function χ for PH transform is given by

$$\chi(S_X(x)) = (S_X(x))^{\gamma} \;\; for\, \gamma \in (0, 1], \tag{6.11}$$

where γ is a risk aversion parameter and lower γ corresponds to a higher security level.

Another well-known distortion risk measure is the Wang transform (WT) [43]. The distortion transform is expressed as

$$\chi(S_X(x)) = \Phi(\Phi^{-1}(S_X(x)) + \Phi^{-1}(\iota)) \tag{6.12}$$

where the parameter $\iota \in [0, 1]$.

An alternative transform to PH is the lookback (LB) transform introduced by Hürlimann [21] with the associated distortion function

$$\chi(S_X(x)) = S_X(x)^{\eta}(1 - \eta \, \log(S_X(x))) \tag{6.13}$$

where $\eta \in (0, 1]$.

As discussed in Hürlimann [22], a coherent distortion risk measure follows the usual stochastic order and the usual stop-loss order, which is a desirable property since more risks should be penalised with a higher risk measure value. However, counterexamples in Hürlimann [22] demonstrated that some distortion risk measures, such as CTE or WT, failed to gain capital relief when reducing the risks due to the absence of degree-two tail-preserving feature. It was further pointed out in [22] that a sound coherent distortion risk measure must preserve some higher degree stop-loss orders, and conditions on the distortion functions of PH and LB were derived in order to guarantee that such risk measures are degree-two tail-preserving coherent.

Convexity is another desired property as it captures the idea that diversification should not increase the risk. Formally, this means

$\mathcal{P}_7$. Convexity: given $\beta \in [0, 1]$, $\rho(\lambda X + (1 - \lambda)Y) \neq \lambda\rho(X) + (1 - \lambda)\rho(Y)$.

It was shown in Wirch and Hardy [47] that for positive losses, a distortion risk measure is coherent if and only if the distortion function is concave.

6.3.3 Spectral risk measures

Spectral risk measures (SRMs) are closely related to coherent risk measures. The SRM preserve the subadditivity property as well as the coherence property, and it is a weighted average of the quantiles of a loss distribution. The weights reflect the users risk aversion. As Acerbi [2] suggested, SRM can be used to evaluate capital requirement whilst Overbeck [34] demonstrated how it can be applied for capital allocation. The difficulty of using SRM comes from the choice of the risk aversion function, i.e., the weighting function. The commonly used risk aversion functions are exponential and power functions, the respective risk measures are called the exponential spectral risk measure (ESRM) and power spectral risk measure (PSRM); see Dowd et, al. [16].

An SRM is defined as

$$\rho_w = \int_0^1 w(p)q(p)\,dp \tag{6.14}$$

where $w(p)$ is a weighing function representing the attitude to risk and q_p is the p-quantile.

In this work, we adopt the ESRM and PSRM with the corresponding weighting functions

$$w_{ES}(p) = \frac{\kappa e^{-\kappa(1-p)}}{(1 - e^{-\kappa})} \tag{6.15}$$

and

$$w_{PS}(p) = \delta p^{\delta-1}. \tag{6.16}$$

SRMs benefit from the free choice of weighting functions based on the risk tolerance of the user.

Related to the previous subsection, Wirch and Hardy [47] proved that if the distortion function χ is concave, the resulting distortion risk measure is spectral.

6.4 Numerical illustrations

6.4.1 Valuation of risk measures

6.4.1.1 Monte Carlo method

Since it is not possible to identify the exact distribution of the loss of GAO defined in section 6.2, we first use the Monte Carlo methodology to evaluate the risk measures. We generate N replicates of the loss random variable L given by equation (6.6). The replicates are then arranged in ascending order, i.e., $L_{(0)} \leq L_{(1)} \leq \ldots \leq L_{(N)}$. Three candidates can be chosen as an estimate of the VaR_α: $L_{(N\alpha)}$, $L_{(N\alpha+1)}$ and the interpolated value between $L_{(N\alpha)}$ and $L_{(N\alpha+1)}$; this assumes that we use the simulated sample as the empirical distribution of L. There is no guarantee that one of the three estimates is better than the other two. Each is liable to be biased though the bias tends to be very small for a large sample size. Hardy [20] indicated that the latter two estimates provide lower bias for the right tail of the loss distribution.

Here, we use $L_{(N\alpha+1)}$ as the estimate of the VaR_α. That is,

$$\widehat{V}aR_\alpha(L) = L_{(N\alpha+1)}. \tag{6.17}$$

If $N\alpha$ is not an integer, we round it off the usual way.

As defined above, the estimate of CTE is the mean of the worst losses in the $1-\alpha$ of the loss distribution. Thus, the estimate of the CTE, assuming $N\alpha$ is an integer, is given by

$$\widehat{C}TE_\alpha(L) = \frac{\sum_{N\alpha+1}^{N} L_{(j)}}{N(1-\alpha)}. \tag{6.18}$$

When estimating the distortion risk measures, we apply the Choquet integral in equation (6.9). Given the ordered samples $L_{(1)}, L_{(2)}, \ldots, L_{(N)}$ of L, we know that the decumulative function is

$$S_L(L_{(i)}) = 1 - \frac{i}{N}, \quad i = 1,\ 2,\ \ldots,\ N \tag{6.19}$$

as the probability that each element occurs is $\frac{1}{N}$. Hence, the decumulative function of risk measures under the distortion probability measure $\chi \circ \mathbb{P}$ can be obtained as $\chi(S_L(L_{(i)}))$.

We obtain the approximation of the distortion risk measure by approximating the Choquet integral using the left Riemann sum, i.e.,

$$\begin{aligned}\rho_\chi(L) &= -\int_{-\infty}^{0}[1-\chi(S_L(l))]\,dl + \int_0^\infty \chi(S_L(l))\,dl \\ &\approx \sum_{0}^{N-1}\chi(S_L(L_{(i)}))(L_{(i+1)} - L_{(i)}),\end{aligned} \tag{6.20}$$

assuming $L_{(0)} = 0$. Applying the distortion functions χ given in section 6.3 in conjunction with equation (6.20), we get the approximate value of the distortion risk measure. The SRM can be approximated by getting the quantiles through the empirical cumulative density function (ECDF) of L.

6.4.1.2 Moment-based density approximation

As described above, the risk measures depend on the cumulative density function (or decumulative probability function) of the loss random variable. A natural alternative to evaluate risk measures is to provide an analytical approximation of the distribution of the loss random variable. We adopt the moment-based density approximation method proposed by Provost [36]. This method requires the moments of the loss random variable. Although these are not available, we can use the sample moments derived from a relatively large sample size of replicates for implementation.

The underlying theory of moment-based density approximation states that given the moments of a random variable, the density of such a random variable is approximated as a product of (i) a polynomial of degree n and (ii) a base density function whose tail behaviour is congruent to that of the density to be approximated. The parameters of the base distribution are obtained by equating the moments of the random variable to those of the base distribution. The polynomial coefficients are determined by equating the first n moments of the random variable to those of the density that needs to be approximated. This method is detailed in Provost [36].

The histogram of the generated samples shows that there is an apparent difficulty in estimating the distribution of L due to the truncation occurring at 0. From equation (6.6), we have

$$L = \begin{cases} L_p = g\, e^{-\int_0^T \mu_v\, dv} \left[\sum_{n=0}^{\infty} \beta(T, T+n) e^{-V(T,T+n)} - K\right] & \text{if } L_p > 0, \\ 0 & \text{if } L_p \leq 0. \end{cases}$$

Knowing the CDF of L_p, denoted as F_{L_p}, the respective CDF and PDF of L are given by the expressions

$$F_L(l) = \begin{cases} F_{L_p}(0) & \text{if } l \leq 0, \\ Pr[L_p \leq l | L_p > 0] = F_{L_p}(l) & \text{if } l > 0, \end{cases} \tag{6.21}$$

and

$$f_L(l) = \begin{cases} F_{L_p}(0) & \text{if } l \leq 0, \\ f_{L_p}(l) & \text{if } l > 0. \end{cases} \tag{6.22}$$

From the preliminary examination of the characteristics of the distribution of L_P and with the aid of calculated moments, we choose the t and normal distributions as base densities and assume a polynomial degree of $n = 10$. The choice of the polynomial degree is justified by: (i) the availability of maximum moments constraint by the base distribution itself and (ii) the influence (or the lack thereof) of the higher degree terms of the polynomial. From Table 6.1, we see that as the degree n increases, the magnitude of the parameters decreases. As n goes up to 10, they become very close to 0. This tells us that degrees beyond 10 is not necessary as the higher-order terms have very little effect on the distribution.

Under the assumption of a t-base density, we make the transformation $X_t := \dfrac{L_p - u}{q}$. We calculate from the samples the first 10 moments of L_P denoted by $\mu_{L_p}(j),\ j = 1,\ ,2, \ldots,\ 10$ and of X_t denoted by $\mu_{X_t}(j),\ j = 1,\ ,2, \ldots,\ 10$. Here, the transformation is employed to facilitate the establishment of the density approximants identical to those obtained in terms of certain orthogonal polynomials. The random variable X_t follows a t distribution whose density is denoted by $\Psi_{X_t}(x)$ and its first $2n = 20$ theoretical moments denoted by $m_{X_t}(j),\ j = 1,\ 2, \ldots,\ 20$

can be obtained. The parameters u, q and d (degrees of freedom of a t distribution) are determined by setting $\mu_{X_t}(i) = m_{X_t}(i)$ (in terms of the $\mu_{L_p}(i)$ expressions) for $i = 1, 2, 3, 4$. That is,

$$\begin{aligned}
u &= \mu_{L_p}(1) \\
d &= 4 + \frac{2\left(3\left(\frac{\mu_{L_p}(3)-u^3}{3u}\right)^2 - 1\right)}{\left(\mu_{L_p}(4) - u^4 - \frac{6u^2\left(\mu_{L_p}(3)-u^3\right)}{3u}\right)} \\
q &= \sqrt{\frac{\left(\mu_{L_p}(3) - u^3\right)(d-2)}{3ud}}.
\end{aligned}$$

The approximated density of X_t is then

$$f_{X_t}(x) = \Psi_{X_t}(x) \sum_{m=0}^{10} h_t(m)\, x^m,$$

where $h_t(i),\ i = 0, 1, \ldots, 10$ are the polynomial coefficients determined by $(h_t(0),\ h_t(1), \ldots, h_t(n))^\top = M_t^{-1} (\mu_{X_t}(0), \ldots, \mu_{X_t}(n))^\top$ and M_t is an $(n+1)\times(n+1)$ matrix whose $(i+1)th$ row has the entries $(m_{X_t}(i),\ m_{X_t}(i+1), \ldots, m_{X_t}(i+n))$. Therefore, the density of L_p using the t distribution as the base density is approximated as

$$f_{L_p}(l) = \frac{\Gamma(\frac{d+1}{2})}{q\,\Gamma(\frac{d}{2})\sqrt{\pi d}} \left(1 + \frac{1}{d}\left(\frac{l-u}{q}\right)^2\right)^{-\frac{d+1}{2}} \sum_{m=0}^{n} \frac{h_t(m)}{q} \left(\frac{l-u}{q}\right)^m . \tag{6.23}$$

When we choose the normal distribution as the base density, the transformation is $X_n := \dfrac{L_p - \theta}{\nu}$. The density of the standard normal random variable is given as $\Phi_{X_n}(x)$ and its first 20 theoretical moments denoted by $m_{X_n}(i),\ i = 1, 2, \ldots, 10$ can be obtained. The parameters θ and ν are

$$\begin{aligned}
\theta &= \mu_{L_p}(1) \\
\nu &= \sqrt{\mu_{L_p}(2) - \mu_{L_p}(1)^2}
\end{aligned}$$

The approximated density of L_p can be similarly derived as

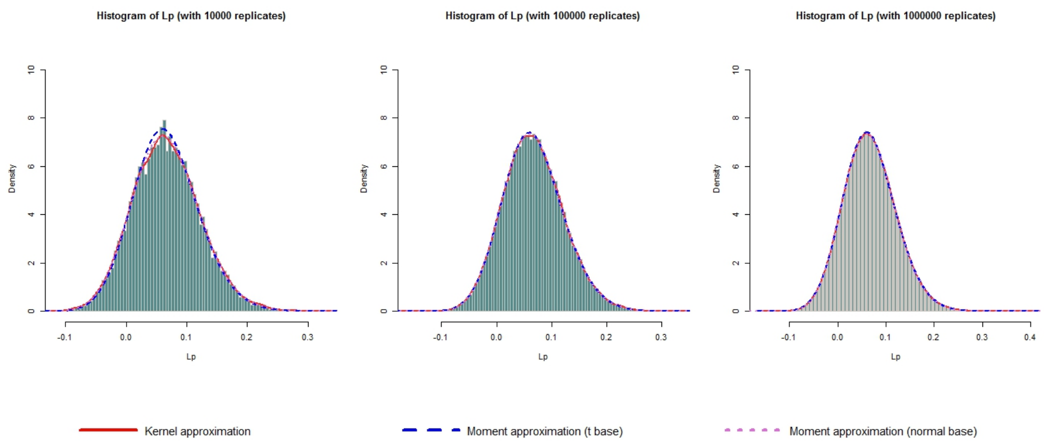
$$f_{L_p}(l) = \frac{1}{\sqrt{2\pi}\nu} e^{-\frac{(l-\theta)^2}{2\nu^2}} \sum_{m=0}^{n} \frac{h_n(m)}{\nu} \left(\frac{l-u}{\nu}\right)^m, \tag{6.24}$$

where $h_n(i),\ i = 0,\ 1,\ \ldots,\ 10$ are polynomial coefficients obtained from $(h_n(0),\ h_n(1), \ldots,\ h_n(n))^\top = M_n^{-1}\ (\mu_{X_n}(0),\ \mu_{X_n}(1),\ \ldots,\ \mu_{X_n}(n))^\top$ and the $(i+1)th$ row of the $(n+1) \times (n+1)$ matrix M_n is $(m_{X_n}(i),\ m_{X_n}(i+1),\ \ldots,\ m_{X_n}(i+n))$. The calculated coefficients of the polynomials in the approximated distributions under both the t and normal base densities are exhibited in Table 6.1 for different N number of replicates.

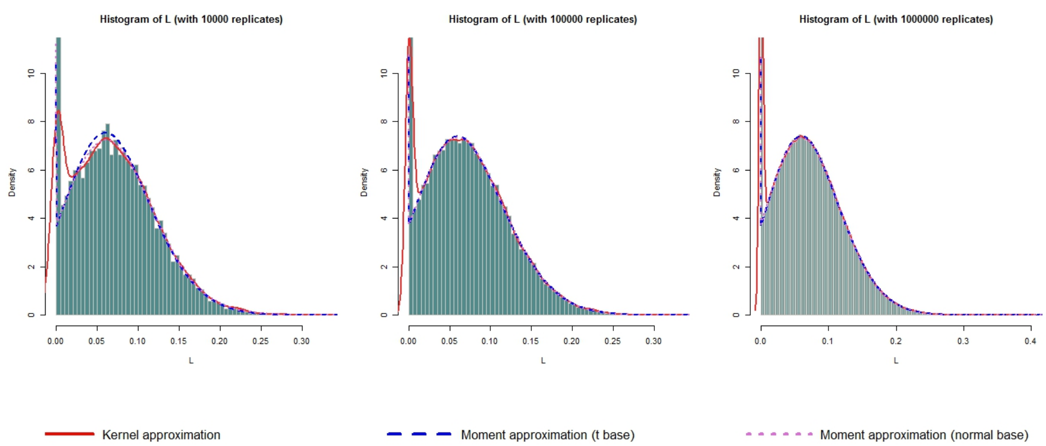
Table 6.1: Estimated polynomial coefficients in the density approximants under t and normal base densities.

Degree	*t* base			normal base		
	N = 10000	**N = 100000**	**N = 1000000**	**N = 10000**	**N = 100000**	**N = 1000000**
0	1.046344e+00	1.014913e+00	1.011061e+00	1.006584e+00	1.005781e+00	9.993913e-01
1	-1.373447e-01	-1.436735e-01	-1.359162e-01	-1.053166e-01	-1.241525e-01	-1.166146e-01
2	-2.947464e-02	6.543860e-03	1.447174e-02	-3.582583e-04	-6.825984e-03	1.911297e-02
3	5.868048e-02	5.916338e-02	5.535582e-02	3.160496e-02	3.652886e-02	3.104721e-02
4	-5.023485e-03	-8.354558e-03	-1.029980e-02	-6.549157e-03	3.521606e-05	-1.477202e-02
5	-3.731151e-03	-3.410847e-03	-3.122710e-03	5.008525e-04	2.140068e-03	3.732859e-03
6	3.263180e-04	4.784531e-04	6.115308e-04	9.469542e-04	-2.736820e-04	2.497891e-03
7	7.202648e-05	5.965667e-05	5.571442e-05	7.176375e-06	-2.328274e-04	-4.676521e-04
8	-5.439679e-06	-8.358958e-06	-1.084437e-05	-9.062617e-06	5.991452e-05	-1.466368e-04
9	-3.681676e-07	-2.792040e-07	-2.741076e-07	2.365550e-06	7.314094e-06	1.754877e-05
10	2.161631e-08	3.738611e-08	4.873466e-08	1.791662e-07	-1.319306e-06	3.957992e-06

With the determination of the approximated density of L_p in equations (6.23) and (6.24), the calculation of the CDF of L_p denoted by F_{L_p} is immediate. Consequently, the approximated distribution of L can be derived via equations (6.21) and (6.22). The results are depicted in Figures 6.1 - 6.2.



(a) Histogram of Lp



(b) Histogram of L

Figure 6.1: Histogram approximation.

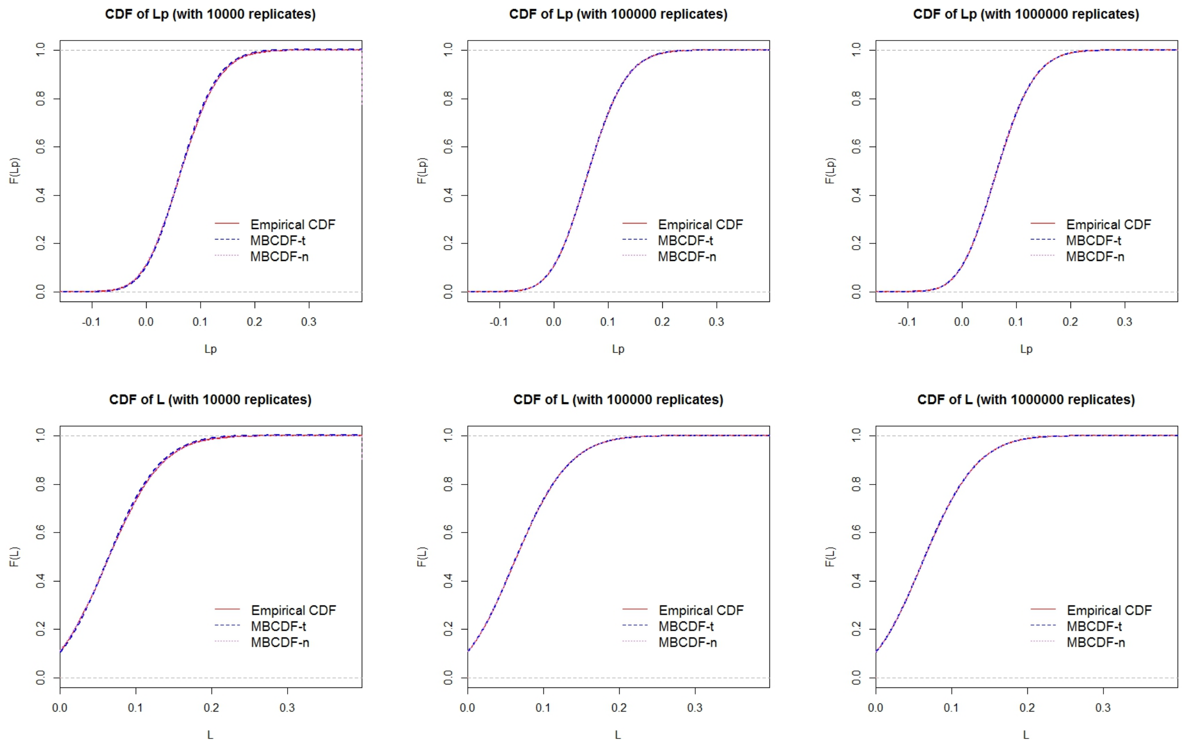


Figure 6.2: CDF approximation.

The Kolmogorov-Smirnov test is used to assess the goodness of fit of the approximated densities. In order to illustrate that the sample we used to estimate the distributions is not a special case and prove that other samples also fit very well to the approximated distributions, we generate 3 other sets of samples (labelled samples 2, 3 and 4), and perform the Kolmogorov-Smirnov test. The test is conducted under three sample sizes: $N = 10,000$, $N = 100,000$ and $N = 1,000,000$. In each sample size, we investigate the goodness of fit under both the t and normal density bases, and provide the p-values together with the Kolmogorov-Smirnov distance statistic presented in brackets.

Table 6.2 exhibits that the approximated distributions under the t base and normal base densities both fit well the other sets of data. However, the normal base density does a better job than the t base density. Moreover, as we increase the sample size, the approximants has better fit. This result is consistent with the Glivenko-Cantelli theorem in which the distance statistic converges to 0 almost surely whenever the sample size goes to infinity and the sample observations comes from the distribution.

Table 6.2: Goodness-of-fit (Kolmogorov-Smirnov) test.

Sample	N = 10,000		N = 100,000		N = 1,000,000	
Number	*t*-base	normal-base	*t*-base	normal-base	*t*-base	normal-base
1.	0.4676 (0.0082)	0.9986 (0.0064)	0.9489 (0.0023)	0.9923 (0.0019)	0.9022 (0.0015)	0.9983 (0.0005)
2.	0.9349 (0.0076)	0.9283 (0.0077)	0.5433 (0.0036)	0.7002 (0.0032)	0.3001 (0.0016)	0.3677 (0.0013)
3.	0.5806 (0.0110)	0.9139 (0.0079)	0.8114 (0.0028)	0.9738 (0.0022)	0.5608 (0.0011)	0.7644 (0.0009)
4.	0.7346 (0.0097)	0.8232 (0.0089)	0.9524 (0.0023)	0.9854 (0.0019)	0.5888 (0.0014)	0.8180 (0.0010)

Note: Sample 1 was used to approximate the distribution of the GAO's loss random variable.

Parameter values used in the computation of the risk measures are displayed in Table 6.3. The numerical values for the risk measures under different numbers of replicates through ECDF, moment-driven CDF with t density base function (MCDF-t) and moment-driven CDF with normal base density function (MCDF-n) are given in Table 6.4. On the one hand, in this table we present the risk measures we have introduced above under different parameters. On the other hand, the difference between the simulated results and those from moment-based approach decreases as the number of sample paths increases, showing that moment-based density approximation method provides a better fit for a large sample size.

6.4.2 Analysis of accuracy

Our results are based on the samples generated through Monte Carlo simulation without knowing the exact distribution of the loss random variable. Monte Carlo method is widely used in statistical and other sciences when the underlying processes are too complicated to yield analytic solutions for certain statistics. The method offers the good features of simplicity, independence from the dimension of random variables, and easy parallelisation. The main drawback is that its implementation is time-consuming. Other limitations are the effect of sampling variability and difficulty in error calculation. That is, different simulations yield different results.

Note that the risk measures above are based on a particular simulation. Hence, results will

Table 6.3: Parameter values used in the numerical experiment in chapter 6.

Parameter set for numerical analysis
Contract specification
$g = 11.1\%$, $T = 15$, $n = 35$;
Interest rate model
$a = 0.15$, $b = 0.045$, $\sigma = 0.01$, $r_0 = b$;
Mortality model
$c = 0.1$, $\xi = 0.0005$, $\mu_0 = 0.006$;

Table 6.4: Risk measures of gross loss for GAO under different sample sizes. Numbers refer to percentage of the premium.

Risk	**N = 10,000**			**N = 100,000**			**N = 1,000,000**		
measures	**Empirical CDF**	**MCDF-t**	**MCDF-n**	**Empirical CDF**	**MCDF-t**	**MCDF-n**	**Empirical CDF**	**MCDF-t**	**MCDF-n**
VaR(α = 0.90)	13.8286	13.6450	13.8750	13.8646	13.8750	13.8550	13.8199	13.8350	13.8150
VaR(α = 0.95)	16.1527	16.8250	16.1550	16.1000	16.1450	16.1350	16.0568	16.0750	16.0550
VaR(α = 0.99)	20.9684	20.8150	20.7450	20.5446	20.5450	20.5550	20.4372	20.4150	20.4550
CTE(α = 0.90)	16.9309	16.4152	16.9370	16.8693	16.8727	16.8665	16.7901	16.8001	16.7913
CTE(α = 0.95)	18.9558	18.1954	18.9662	18.8494	18.8421	18.8453	18.7486	18.7485	18.7546
CTE(α = 0.99)	22.7461	22.3502	22.7482	22.8205	22.7814	22.7959	22.7263	22.6820	22.7125
WT(ι = 0.10)	14.1938	14.6033	14.2045	14.1073	14.1062	14.1060	14.0781	14.0790	14.0781
WT(ι = 0.05)	16.4716	16.5204	16.4915	16.3337	16.3313	16.3300	16.3093	16.3132	16.3093
WT(ι = 0.01)	20.8677	20.7574	20.9565	20.5992	20.5918	20.5814	20.6739	20.7040	20.6733
PH(γ = 0.90)	7.4506	7.3539	7.4512	7.4461	7.4472	7.4462	7.4456	7.4472	7.4454
PH(γ = 0.50)	10.9552	10.5228	10.9546	10.9015	10.9012	10.8997	10.8997	10.9037	10.8994
PH(γ = 0.10)	23.5293	23.0479	23.5345	23.3422	23.3612	23.3413	25.2823	25.4283	25.2912
LB(η = 0.90)	12.6122	12.2383	12.6193	12.5585	12.5581	12.5581	12.5327	12.5329	12.5327
LB(η = 0.50)	17.7672	17.3185	17.8507	17.5889	17.5844	17.5879	17.6509	17.6729	17.6498
LB(η = 0.10)	30.4555	30.8563	30.5028	30.3099	30.3715	30.3505	34.7219	34.9579	34.7775
ESRM(κ = 1)	8.3689	8.2509	8.3704	8.3592	8.3596	8.3595	8.3556	8.3572	8.3555
ESRM(κ = 20)	17.6068	17.9456	17.6215	17.5224	17.5182	17.5195	17.4470	17.4483	17.4453
ESRM(κ = 100)	21.9289	21.4146	22.0203	21.7378	21.7126	21.7283	21.6692	21.6266	21.6486
PSRM(δ = 1)	6.9204	6.8505	6.9215	6.9207	6.9217	6.9210	6.9228	6.9250	6.9225
PSRM(δ = 20)	17.6921	17.2066	17.7067	17.6073	17.6031	17.6044	17.5307	17.5322	17.5300
PSRM(δ = 100)	21.9429	21.4259	22.0341	21.7501	21.7256	21.7415	21.6823	21.6370	21.6617

change due to sample variation inherent in simulation methodology. Figure 6.3 shows variation of risk measures with respect to several simulations and sample sizes. As the sample size is increased to $N = 100,000$, the values of risk measures fluctuate but converge albeit at a low speed. This motivates us to analyse the sampling errors which determines the accuracy and credibility of the calculated risk measures.

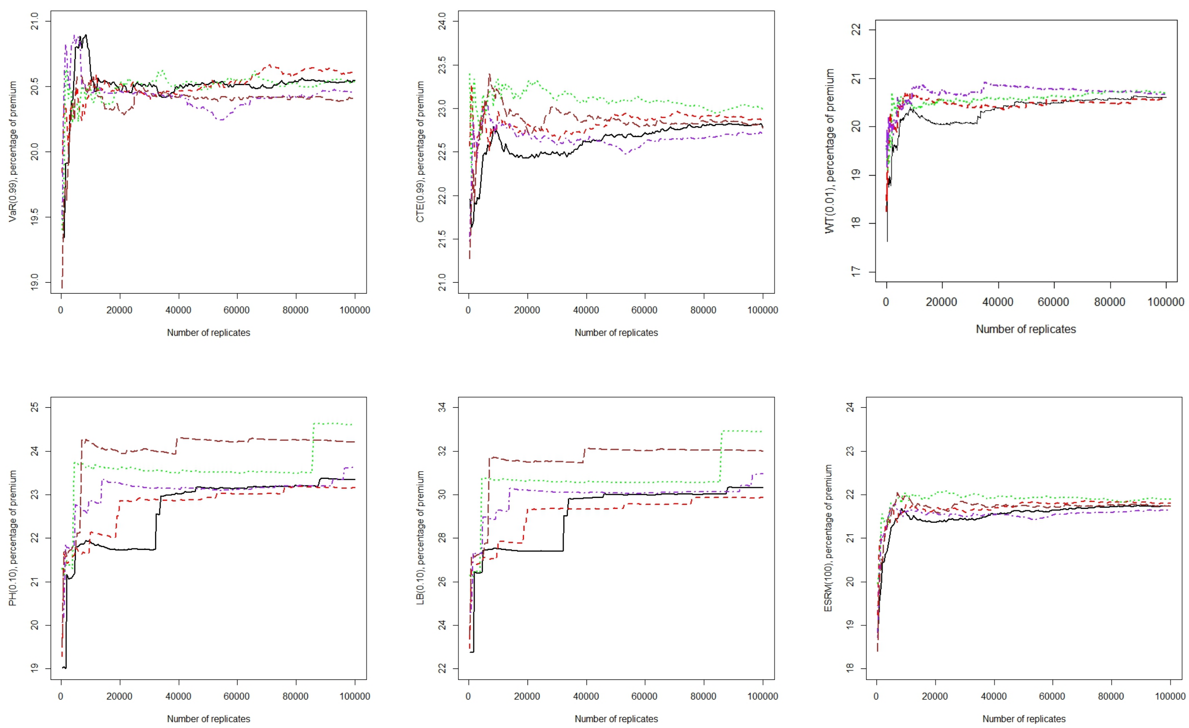


Figure 6.3: Risk measures under different samples.

We employ the bootstrap approach to quantify the standard errors. Bootstrap is a simulation-based statistical analysis too that provides measures of accuracy to sample estimates. The law of bootstrap comes from the plug-in principle, that is, the statistic calculated from the sample can be used to estimate the parameter in the population. For common estimators and under general distribution assumptions, the bootstrap distribution can be useful in describing the behaviour of quantities being estimated, such as standard error, skewness, bias and quantities for confidence interval construction; see Chihara and Hestergerg [11].

More sophisticated applications of the bootstrap method to risk measurement can also be found in Kim [24], Kim and Hardy [25, 26], and Manistre and Hancock [31]. For our application, we choose $CTE(0.99)$ as the subject of our study. We generate $M = 10,000$ samples, each with $N = 10,000$ replicates and calculate the sampling distribution of $CTE(0.99)$. The bootstrap distribution is obtained by resampling $B = 10,000$ times from the sample paths chosen randomly. The comparison of the sampling distribution with the bootstrap distribution is shown in Figure 6.4 from which we observe that the bootstrap distribution closely resembles with the sampling distribution.

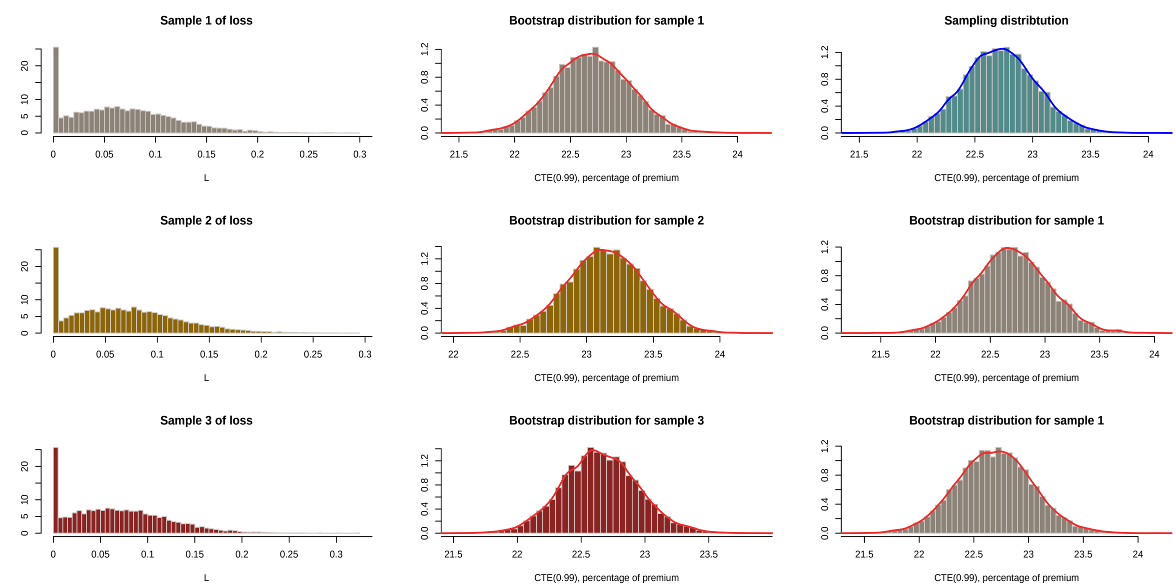


Figure 6.4: Sampling distribution versus bootstrap distribution.

The indispensable advantage of using bootstrap methodology to evaluate the accuracy of risk measures over the classical Monte Carlo method lies in the time savings and computer memory. For example, $M = 10,000$ times of sampling each with a sample size of $N = 10,000$ consumes around 1000 hours whilst $B = 10,000$ times of bootstrap from a sample size of $N = 10,000$ only needs around 30 seconds. The bootstrap standard errors of the risk measures given in Table 6.4 are calculated under $B = 100$, $B = 1000$ and $B = 10,000$ which are shown in Table 6.5. We see that there is not much difference in the results under these three numbers of bootstrap replicates. Hence, $B = 100$ should be enough for the assessment of risk measures accuracy considering time and computer storage constraints (cf. Chernick and LaBudde, [10]).

Table 6.5: Bootstrap standard error of loss for GAO under different sample sizes (percentage).

Bootstrap	N = 10000			N = 100000			N = 1000000		
standard errors	B = 100	B = 1000	B = 10000	B = 100	B = 1000	B = 10000	B = 100	B = 1000	B = 10000
VaR(α = 0.90)	0.091994	0.097163	0.093352	0.033562	0.032422	0.033472	0.010494	0.010025	0.010178
VaR(α = 0.95)	0.012773	0.012578	0.012192	0.042815	0.041874	0.041412	0.013021	0.012896	0.011960
VaR(α = 0.99)	0.345039	0.341920	0.332964	0.099950	0.091801	0.091007	0.024658	0.023541	0.023700
CTE(α = 0.90)	0.119333	0.124479	0.127527	0.040905	0.038329	0.039061	0.011961	0.012251	0.012355
CTE(α = 0.95)	0.167521	0.165146	0.165443	0.054229	0.051183	0.050375	0.016315	0.016028	0.016222
CTE(α = 0.99)	0.324589	0.309199	0.308211	0.103793	0.095824	0.094474	0.031056	0.030897	0.031059
WT(ϵ = 0.10)	0.106533	0.101247	0.104336	0.033638	0.031386	0.031162	0.010355	0.010379	0.010364
WT(ϵ = 0.05)	0.154435	0.146169	0.149173	0.046112	0.044414	0.044320	0.015446	0.015380	0.015439
WT(ϵ = 0.01)	0.323941	0.322814	0.319924	0.090342	0.089635	0.086748	0.042117	0.041973	0.041652
PH(γ = 0.90)	0.051485	0.052217	0.052723	0.016534	0.016594	0.016502	0.005524	0.005219	0.005213
PH(γ = 0.50)	0.081997	0.079770	0.080238	0.025079	0.023898	0.024040	0.008621	0.008651	0.008597
PH(γ = 0.10)	0.833469	0.818151	0.794971	0.374531	0.394621	0.383049	0.109264	0.107965	0.108267
LB(η = 0.90)	0.080972	0.079486	0.078125	0.026719	0.025326	0.024849	0.008102	0.008097	0.007992
LB(η = 0.50)	0.221471	0.219490	0.216308	0.059959	0.059593	0.058768	0.031038	0.030970	0.030610
LB(η = 0.10)	1.507544	1.477509	1.436696	0.884943	0.874943	0.854648	0.153169	0.151289	0.149468
ESRM(κ = 1)	0.056407	0.055677	0.054389	0.017582	0.017415	0.017753	0.005878	0.005571	0.005584
ESRM(κ = 20)	0.138879	0.137562	0.138985	0.046772	0.041886	0.042579	0.013510	0.013454	0.013500
ESRM(κ = 100)	0.284437	0.267147	0.267433	0.086535	0.079620	0.078859	0.026085	0.025978	0.026085
PSRM(δ = 1)	0.051145	0.050922	0.049919	0.015935	0.015124	0.015534	0.005107	0.005046	0.005063
PSRM(δ = 20)	0.140097	0.139868	0.140277	0.046734	0.042276	0.042977	0.013644	0.013575	0.013723
PSRM(δ = 100)	0.284677	0.267644	0.267915	0.086703	0.079765	0.078998	0.026131	0.026024	0.026132

Clearly from Table 6.5, increasing the number of replicates reduces the standard error within a certain range but this achieved at the expense of greater time and machine computation resource. In practice, we have to strike a balance between computational expense and result accuracy. We demonstrate how to obtain the needed number of replicates for a given target sampling error referred to as bootstrap standard errors. If a quantity φ can be expressed as $\varphi = \displaystyle\int \varphi(x) f_X(x)\, dx$, then given a sample of N replicates $X_1,\ X_2,\ \ldots,\ X_N$ generated from the density $f_X(x)$, the estimate of φ is given by $\widehat{\varphi} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \varphi(X_i)$. The variance of the estimate is proportional to the square root of the sample size N by the Central Limit Theorem.

For distortion risk measures we know that for non-negative loss, if χ is continuous and differentiable, by the chain rule of derivatives and from equation (6.9) we have

$$\begin{aligned} \rho_\chi(L) &= \int_0^\infty \chi(S_L(l))\, dl \\ &= l\chi(S_L(l))\Big|_0^\infty - \int_0^\infty l\chi'\, f_L(l)\, dl \\ &= \int_0^\infty h(l)\, f_L(l)\, dl, \end{aligned} \tag{6.25}$$

where χ' is the derivative of χ and $h(l) = -l\chi'$. We use the fact that $\chi(S_L(\infty)) = \chi(0) = 0$ to get equation (6.25). For non-continuous distortion risk measures such as CTE, we could construct a function $h(l)$ so that the CTE could be described as the integral of the product of $h(l)$ and the density function. By definition and in accordance with equation (6.8), the CTE could be written as

$$CTE_\alpha(l) = \mathbf{E}\left[L|L > VaR_\alpha\right] = \int h(l) f_L(l)\, dl,$$

where

$$h(l) = \begin{cases} l & \text{if } l \leq \mathrm{VaR}_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

In the succeeding experimental investigation, we choose $CTE(0.99)$ as our object of study. Given samples generated by the Monte Carlo method, the estimate of CTE_α is $\widehat{CTE}_\alpha = \frac{1}{N}\sum_{n=1}^{N} h(L_n)$. By the Strong Law of Large Numbers, $\widehat{CTE}_\alpha(L) \to CTE_\alpha$ as $N \to \infty$. The Central Limit Theorem also tells us that the standard error of $\widehat{CTE_\alpha}$ is proportional to $\sqrt{N}$ for a large enough sample size N. That is,

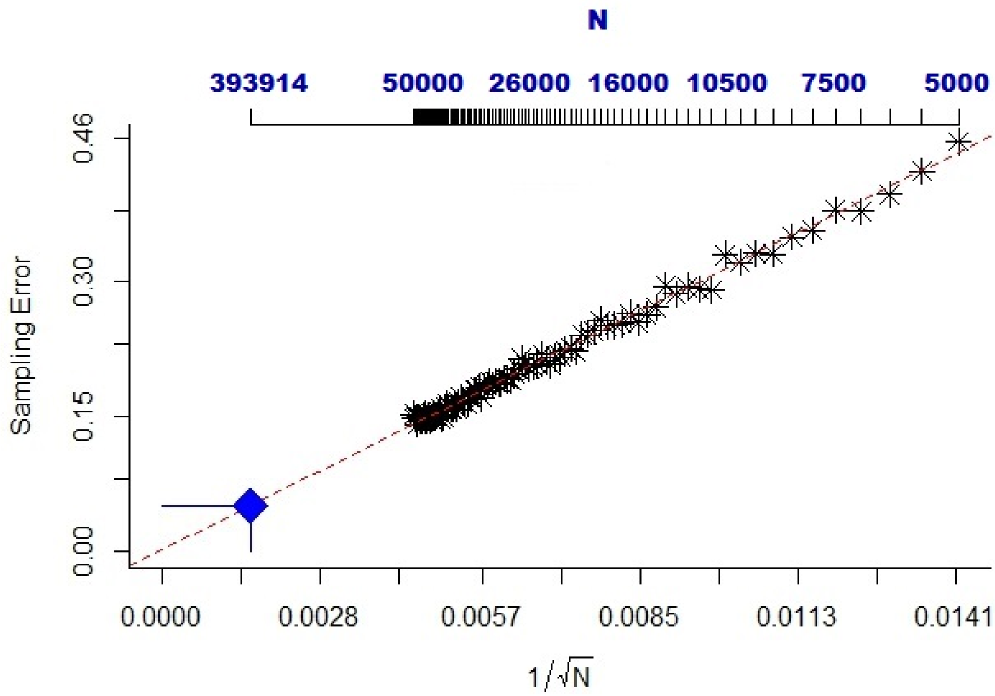
$$SE(CTE_\alpha) = \frac{\epsilon}{\sqrt{(N)}}. \tag{6.26}$$

Therefore, if we set a desired value for ϵ, we could find out how many replicates are needed to achieve the target sampling error. That is,

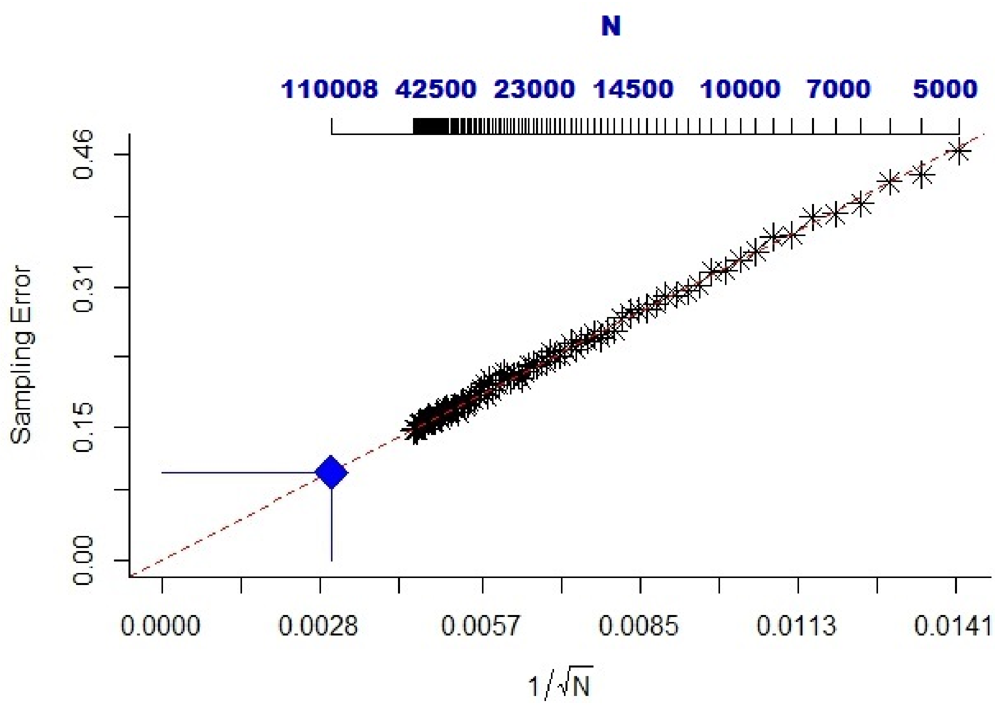
$$N = \left(\frac{\epsilon}{SE(CTE_\alpha)}\right)^2. \tag{6.27}$$

Having N replicates generated from L, we choose a sequence of m subsets, each with N_i elements, $N_i \leq N,\ i = 1,\ 2,\ \ldots,\ m$, and apply B times bootstrap sampling to each subset. The standard error of the B number of $CTE(0.99)$s can be used as an approximation of the sampling error. In total, we have m pairs of sampling errors SE_i with respect to the number of replicates N_i, $i = 1,\ 2,\ \ldots,\ m$. In our experiment, we examine the relation between the standard errors of $CTE(0.99)$ and the sample size through 1000 times of bootstrap under two different replicates $N = 50,000$ and $N = 100,000$. Since a small sample size has high variation affecting the reliability of the regression results, we choose two minimum replicates, say 1000 and 5000 for comparison. The results are shown in Figures 6.5 – 6.6. The estimated sample size given a sampling error and the estimated sampling error given the sample size are depicted as diamond points in these graphs. Moreover, when we remove the effect of fluctuation that a small sample size brings on the regression results, we get a better fit and consequently, better approximations are attained.

Conversely, with computing time and memory as constraints in generating N replicates, we can get the approximated sampling error from equation (6.26). Figures 6.7 and 6.8 show the results of sampling errors given the number of replicates. Clearly, a better fit is obtained under the setting of 5000 minimum sample size.

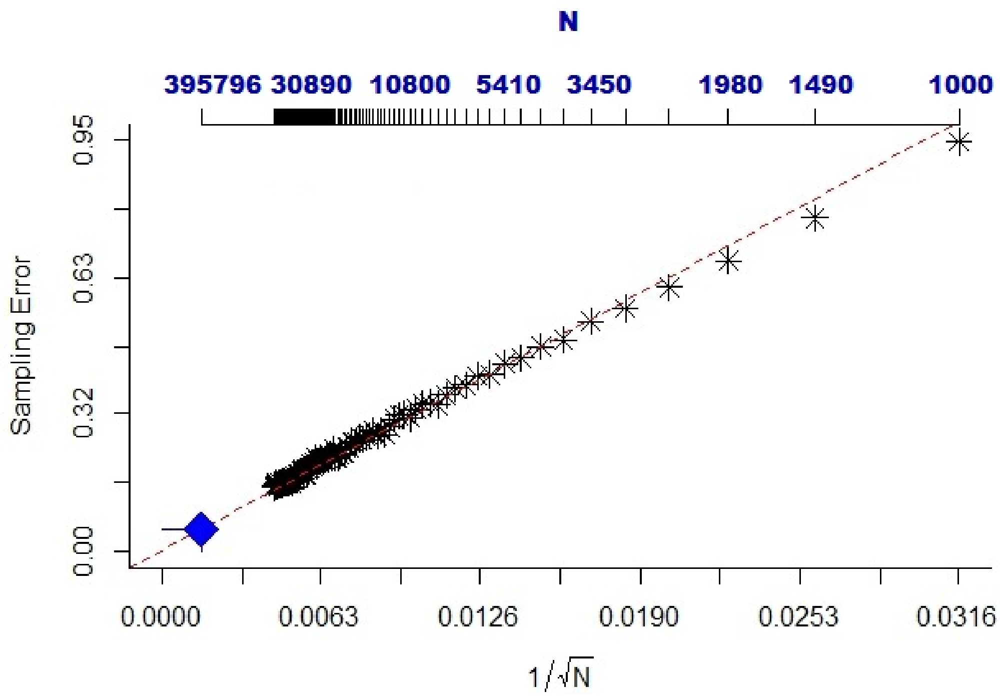


(a) $SE = 0.05$

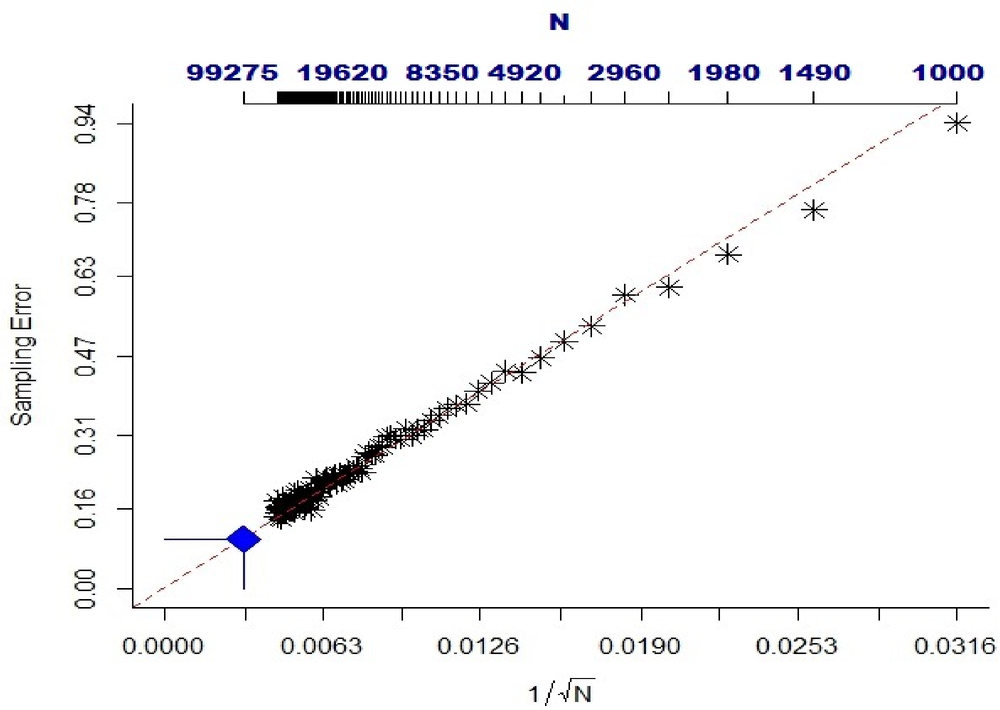


(b) $SE = 0.10$

Figure 6.5: Sample size estimate with 5000 as minimum sample size.

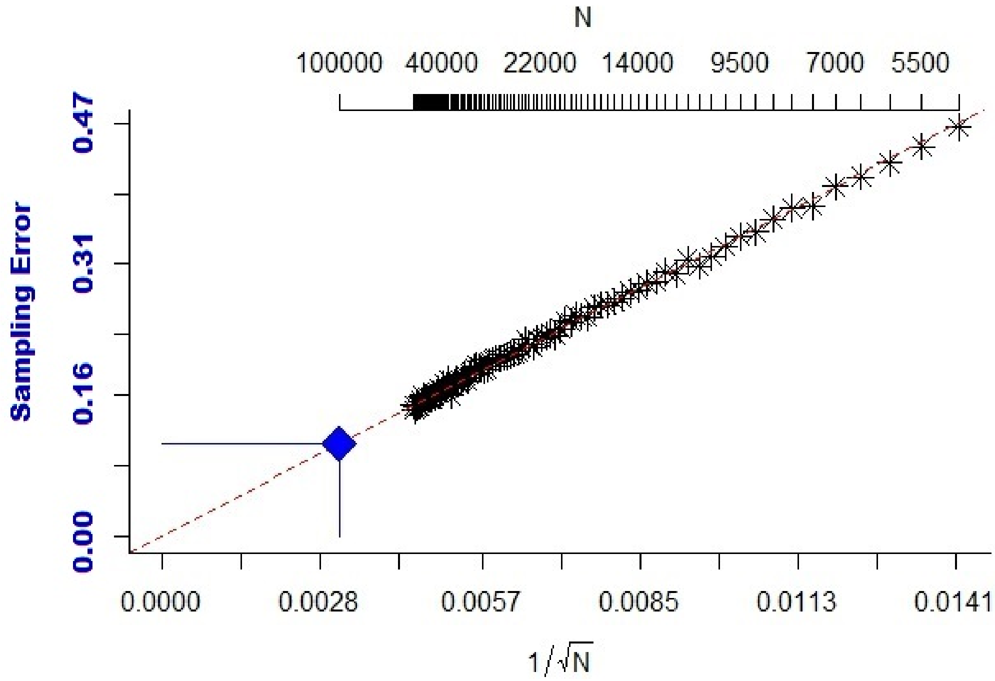


(a) $SE = 0.05$

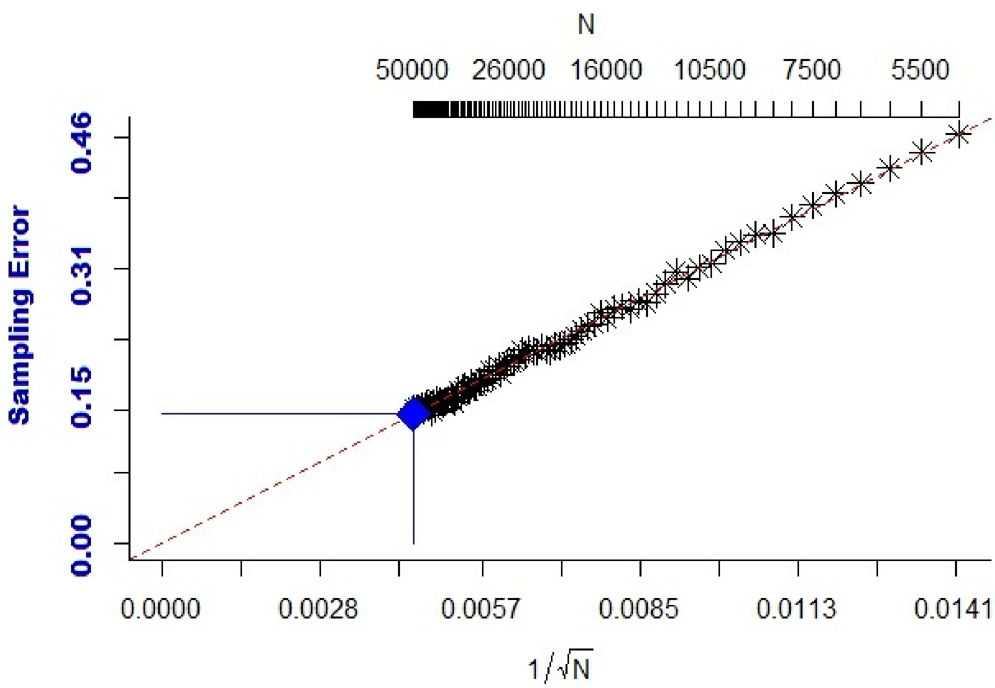


(b) $SE = 0.10$

Figure 6.6: Sample size estimate with 1000 as minimum sample size.

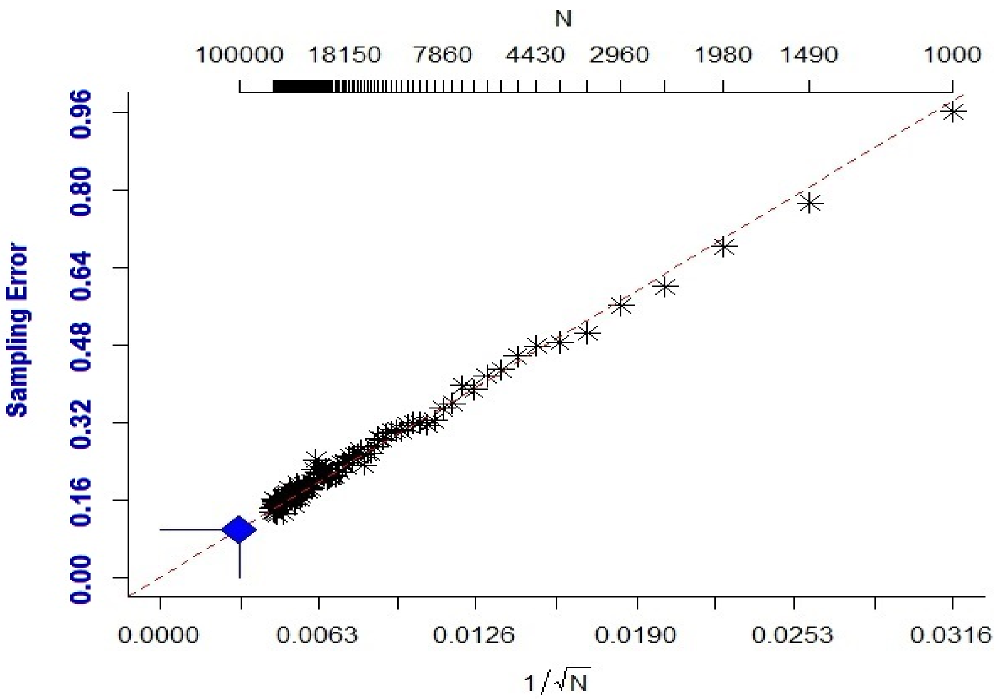


(a) N = 100000

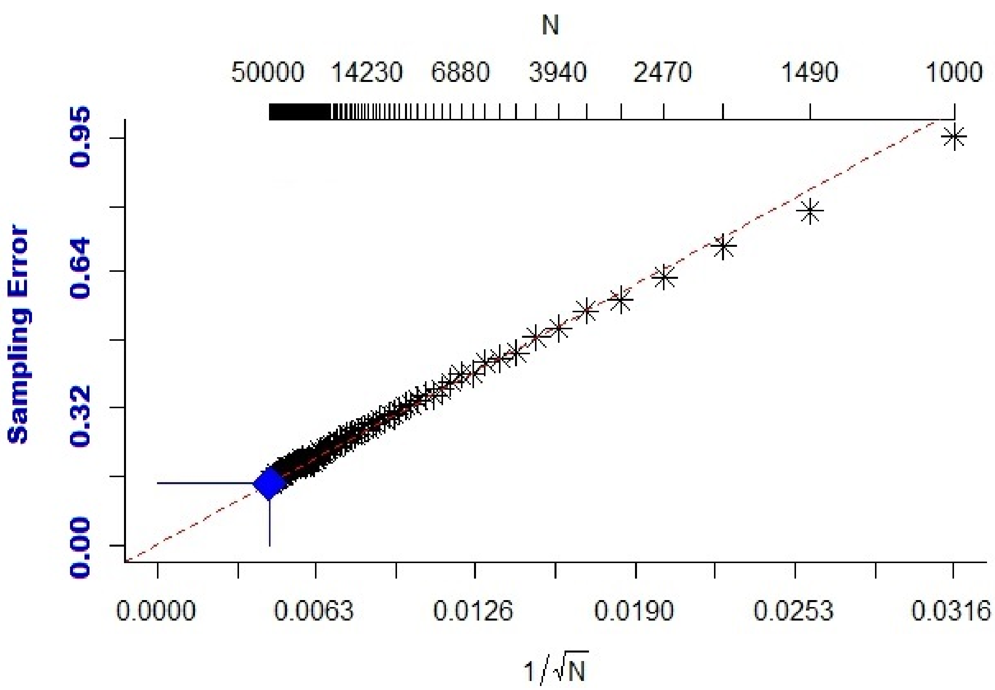


(b) N = 50000

Figure 6.7: Sampling error estimate with 5000 as minimum sample size.



(a) N = 100000



(b) N = 50000

Figure 6.8: Sampling error estimate with 1000 as minimum sample size.

6.4.3 Sensitivity analysis

From Table 6.4, we find substantial variation in risk measures given different parameters. We are therefore interested in finding how parameters in our modelling framework influence the risk measures. This leads to examining the sensitivity of risk measures with respect to various parameters with the values in Table 6.3 used as benchmarks.

6.4.3.1 Impact of interest rate assumptions

The Vasiček model is employed to describe the evolution of the interest rate process. The three parameters in this model are the mean-reverting rate a, mean level b, and volatility σ. Figure 6.9 shows that a has a negative influence on the risk measures, that is, the greater a is, the smaller the risk measures are. Also, as a increases, the rate at which the risk measures decline slows down. There is a workable range of a in the assessment risk measures; when it is large, the risk measures go close to 0. The mean level b has a similar effect to a in so far as the behaviour of risk measures goes. That is, if b is increased, the risk measures will decrease to 0 although at a faster speed. The relationship between b and the risk measures is approximately linear. A valid range of b in this experiment is roughly the interval $[0,\ 0.1]$. Intuitively, higher volatility will produce greater risk measure value and this is confirmed by the numerical results as we perturb σ. The impact of parameters in the interest rate model is graphically shown in Figures 6.9.

6.4.3.2 Impact of mortality rate assumptions

Figure 6.10 demonstrates the impact of parameters in the mortality rate model on the risk measures. From the left plot, the parameter c shows similar pattern of influence to the parameter a on risk measures. The valid range for c is about $(0, 0.14)$. The risk measures are more sensitive to the volatility ξ of the mortality rate model than the volatility σ of the interest rate model. This is due to the fact that mortality rate influence both the discounting and annuity factors in the gross loss whilst the interest rate only affects the annuity factor. When ξ is greater than 0.0075, the risk measures go beyond 100% of the loss and increase at an extremely faster rate.

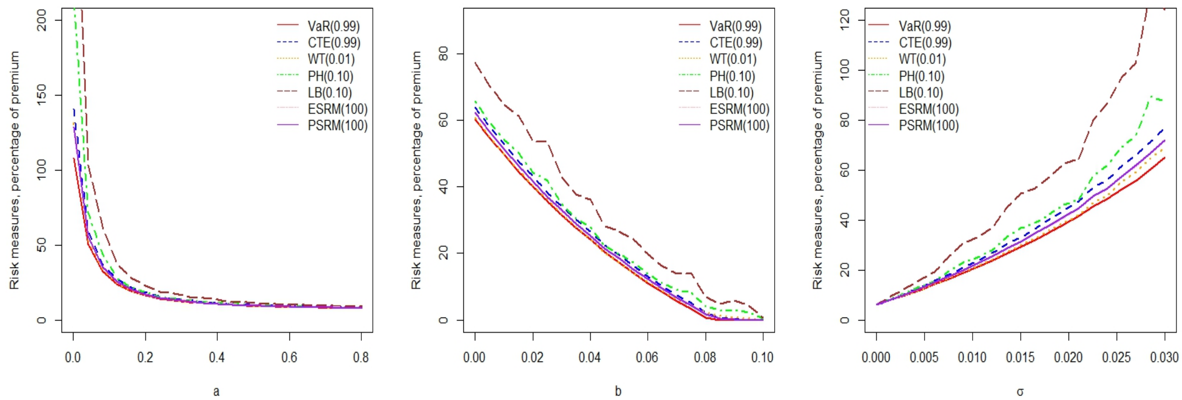


Figure 6.9: The impact of parameters in the interest rate model on risk measures.

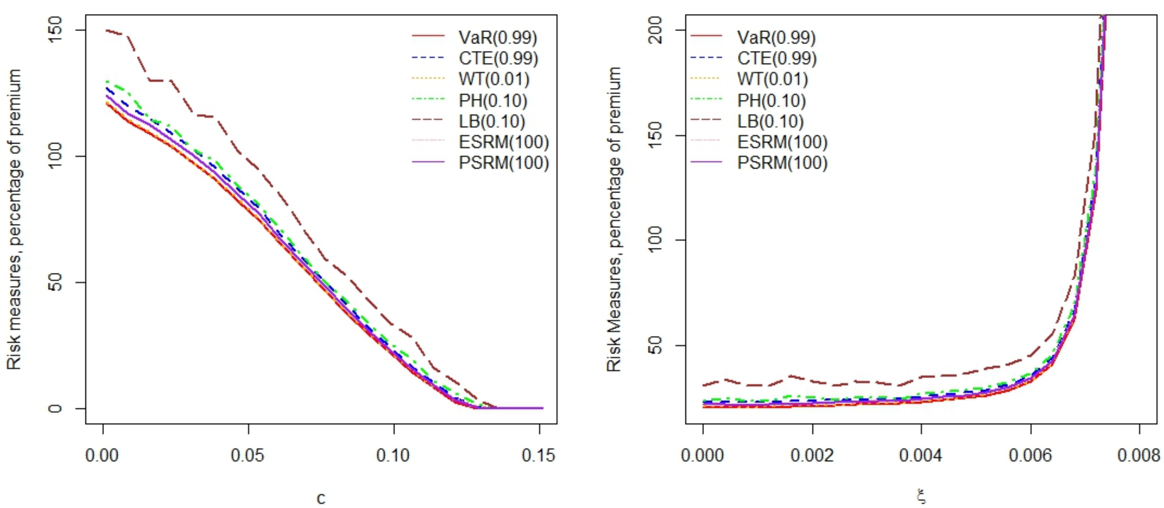


Figure 6.10: The impact of parameters in the mortality rate model on risk measures.

The above sensitivity analysis suggests that when evaluating risk measures, we need to calibrate accurately the parameters. Otherwise, risk assessment becomes problematic and could lead to a wrong determination of capital solvency.

6.4.3.3 Final remark

Previously, we note that the prices of GAO in chapter 4 are highly sensitive to interest rates driven by a pure Markov chain (cf. page 95). However, in this particular contribution (cf. subsection 6.4.3.1), risk measures are less sensitive to interest rates than to mortality rates. This is coming from the fact that when we were doing pricing, we assumed a dollar cash payment (i.e., $F(T) = 1$). In the risk-measurement setting, we take the stochastic nature of $F(T)$ which "acts" as the discounting factor. More specifically, we observe in equation (6.2) the double effect of μ_t through the loss function L.

6.5 Conclusions

In this work, we demonstrated the evaluation of risk measures on the gross loss of GAO under a stochastic modelling framework. The interest and mortality rates have correlated affine structures. We employed the moment-based density approximation method to estimate the loss distribution and calculated risk measures with Monte-Carlo results as benchmark. To address the accuracy of these estimates, we adopted bootstrap method to calculate their standard errors, the so-called sampling errors. By establishing the relation between sample size and standard error of risk measures, the required number of replicates is known for a desired standard error and vice versa. Furthermore, we conducted local sensitivity analyses (that is, we varied one parameter at a time by a small amount around a fixed value and gauged the effect of individual perturbations on risk measures) to study the impact of interest and mortality models' parameters on risk measures. Our analyses provided insights on how risk measures behave as parameters are changed, and affirmed the importance of having accurate parameter estimates for risk management implementation.

It has to be noted that our evaluation of risk measures are under the gross loss assumption and charges were not considered. In practice, there are various charges affecting the insurance business such as administrative fees and surrender charges. To compute risk measures with charges included, we need to make correct and realistic assumptions on fees. For example, the probability of withdrawal before maturity causing surrender charges is often assumed to be a constant. However, the likelihood of withdrawal depends on economic and social conditions, and it is typically correlated with interest and mortality rates. Such withdrawal probability clearly also requires mathematical modelling and any model needs to be calibrated to pertinent data. Therefore, we may extend our work to measure risks of GAO under a two-decrement actuarial model by incorporating stochastic withdrawal probability.

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Chapter 7

Concluding remarks

7.1 Summary and commentaries

In this thesis, we put forward various stochastic models for the evolution of financial and mortality risk factors in the context of pricing and risk management of GAOs. Compared to the current literature, we heavily emphasised the need to accommodate for a dependence structure between financial risk and mortality risk. Our framework developments contribute to the widening of available technology in dealing with option-embedded insurance products that are becoming more popular these days. Throughout this entire research work, we took advantage of the power of change of measure technique in GAO valuation leading to substantial reduction in computing time and standard errors. Our proposed methodologies demonstrated the benefits that can be gained, and such can further be applied to other products with both financial and insurance features. We introduced the applications of comonotonicity theory, and for risk measurement, the moment-based density approximation method is advanced as an alternative to the Monte Carlo simulation method. Both techniques outperform the Monte Carlo method with respect to both computing time and accuracy. Certainly, they have the potential for greater development and applications that could eventually solve efficiently some relatively challenging actuarial and financial valuation problems that are currently in existence.

In chapter 2, we built a framework based on interest rate and mortality models admitting a dependence assumption between two risk factors. The employment of the forward measure and our newly constructed endowment-risk-adjusted measure notably aided the procedure of pricing GAOs as shown by the comparison results with the usual Monte Carlo simulation with respect to computing time and errors.

Explicit pricing solutions are desired and preferred to simulation-based results because the former is exact and has important implications to implement hedging and sensitivity analyses. Nonetheless, analytic pricing solutions are not easily obtainable for complicated financial products under stochastic models. We applied comonotonicity concepts and generated explicit bounds to GAO price as an alternative to Monte Carlo method in chapter 3. The upper and lower bounds of the GAO prices under the framework in chapter 2 were obtained together with their distributions and quantile functions. The principles of the methodology proposed in this chapter may benefit the valuation of other option-embedded insurance products.

In chapter 4, a regime-switching approach was developed owing to its ability to capture structural changes in financial and insurance risks. The regime-switching feature was reflected in three ways, namely, (i) through a Gompertz model with BM- and Markov-switching parameters, (ii) via a Gompertz with pure Markov-switching parameters, and (iii) through a regime-switching Luciano-Vigna mortality model. Along with the pure Markov interest rate model, we provided comprehensive derivations of implementable GAO pricing expressions using again the concept of endowment-risk-adjusted measure. The numerical results corroborated the benefit of the change of measure technique under the three regime-switching frameworks.

The extension of the modelling framework in chapter 2 was considered in chapter 5. We maintained the dependent affine structures but relaxed the constant volatility assumptions by having a regime-switching volatility dictated by the movement of a Markov chain. The pricing of GAOs was facilitated again using the change of measure methodology.

In chapter 6, we turned our focus on the risk measurement of GAOs. We followed the framework in chapter 2 in modelling the gross loss random variable. For simplicity, charges and related fees were excluded. The moment-based density approximation approach was employed to find analytic approximation of the distribution of the GAO's loss random variable. Different kinds of risk measures were calculated through both Monte Carlo simulation and using the approximated distributions. A bootstrap technique was applied to get standard errors of the risk measure results. A particular contribution, obtained from regression method, that is important in efficiently carrying out numerical calculations is the establishment of the relation between the sample size and the required accuracy of risk measure values.

7.2 Future research directions

The utility of the change of measure technique under various stochastic modelling frameworks for the valuation of GAOs is ubiquitous in this thesis. Related works may be further explored in relation to this endeavour and technique for the purpose of dealing with insurance products having option-embedded characteristics. Such opportunities for future research investigations, as elaborated below, arise from various limitations of this thesis and certain aspects not extensively treated in the course of our modelling and analyses.

Mortality rate is more complicated than interest rate to model due to stylised facts concerning the former. In our framework, the evolution of mortality rate is modelled under the simple assumption that each cohort has the same pattern of evolution. But, a further improvement could be attained by incorporating the age effect, similar to some well-known mortality models proposed by Cairns et al. [1] and Lee & Carter [5] although these models lack regime-switching capacity. We may modify our modelling framework by adopting these mortality models and utilise the change of measure technique to price annuity-linked insurance products.

In this thesis, we also adopt parameter estimates from previous research for the purpose of ease of comparison and exploiting what already worked in the past. But assuming the availability of relevant real data from the market sooner or later, it is important to develop efficient and robust estimation algorithms for all stochastic models developed in this thesis. These could include the methods of maximum likelihood estimate (MLE), least squares (LS), and hidden Markov model (HMM) filtering; see Elliott et al. [4] for HMM. The associated statistical inference questions arising after estimation of these models are rich sources of problems in continuing this kind of research.

Hedging risks for insurance companies is more difficult than hedging financial risks due to insurance market being less mature and less liquid in terms of trading insurance products with derivative components. Traditionally, hedging mortality risks is achieved by coupling life insurance and annuity products. However, the advent of new type insurance contracts, especially finsurance products, requires new advanced hedging methodology. The newly introduced hedging methods proposed by Luciano et al. [6] and Melnikov and Romaniuk [7] may be aptly tweaked and applied into our modelling framework to hedge insurance products with option-embedded features.

Monte Carlo methodology is frequently used in our work to obtain numerical results. However, the sample variability and the cost of time and computer memory limit its applications when dealing with complex models. Instead of increasing the number of replicates to reduce result uncertainties, several variance reduction methods could be used, for example, antithetic variate, control variable, and importance sampling. We may employ one or a combination of several such methods to reduce the computing time and errors in our valuation process and especially for risk measurement given the apparent need for simulation.

The characterisation of the underlying distribution of the loss random variable is not straightforward when we incorporate dependence structure into the modelling framework of complex insurance products. An alternative way to Monte Carlo methodology in pricing such products is to approximate the distribution via density approximation methods such as moment-based density and kernel density approximation. We may adopt these methods to approximate the distribution of loss variable in the model framework of chapter 2 and chapter 5 to value GAOs and other guaranteed maturity contracts.

To culminate this research direction, we note that in our work on GAO pricing, we only consider two risks: interest rate and mortality rate, and we are assuming that the proceeds at maturity from the single premium paid end up to a dollar cash amount. However, as we know the attractiveness of option-embedded insurance products stems from not only their function as a pension plan but also as a financial investment vehicle. Therefore, the incorporation of investment risk, which must be correlated with interest rate, into our modelling framework would be more realistic. We may employ the change of measure technique and conditional comonotonicity detailed in Cheung [2, 3] to price GAOs or other insurance products with guarantees under the presence of investment risk.

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Appendix A

Derivation of survival probability in chapter 4

The following two appendices are devoted to the derivation of the survival probability $S(t, T)$ under the two regime-switching mortality models M2 and M3.

A.1 Derivation of survival probability under M2

Under the mortality model M2, the dynamics of $a(t)$ and $b(t)$ are given by

$$a(t) = \alpha(\boldsymbol{y}_t)t + a(0)$$
$$b(t) = \gamma(\boldsymbol{y}_t)t + b(0)$$

where

$$\alpha(\boldsymbol{y}_t) = \langle \boldsymbol{\alpha}, \boldsymbol{y}_t \rangle$$
$$\gamma(\boldsymbol{y}_t) = \langle \boldsymbol{\gamma}, \boldsymbol{y}_t \rangle$$

and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, $a(0)$ and $b(0)$ are the initial values of a and b.

Let $\Psi_{t,u} = e^{-\int_t^u \mu(x+v,v,\boldsymbol{y}_{t,v})\,dv}$. We then get the dynamics

$$d(\Psi_{t,u}\boldsymbol{y}_{t,u}) = \Psi_{t,u}\left[\boldsymbol{\Lambda}\boldsymbol{y}_{t,u}du + d\boldsymbol{n}_u\right] + \boldsymbol{y}_{t,u}\left[-\mu(x+u,u,\boldsymbol{y}_{t,u})\Psi_{t,u}du\right].$$

In integral form,

$$\Psi_{t,T}\boldsymbol{y}_{t,T} = \boldsymbol{y}_t + \int_t^T \Psi_{t,u}\boldsymbol{\Lambda}\boldsymbol{y}_{t,u}\,du + \int_t^T \Psi_{t,u}\,d\boldsymbol{n}_u - \int_t^T \mu(x+u,u,\boldsymbol{y}_{t,u})\Psi_{t,u}\boldsymbol{y}_{t,u}\,du.$$

This implies that

$$\mathbf{E}\left[\Psi_{t,T}\boldsymbol{y}_{t,T}\middle|\boldsymbol{y}_t\right] = \boldsymbol{y}_t + \int_t^T \boldsymbol{\Lambda}\mathbf{E}\left[\Psi_{t,u}\boldsymbol{y}_{t,u}\middle|\boldsymbol{y}_t\right]du - \int_t^T \mathbf{E}\left[\mu(x+u,u,\boldsymbol{y}_{t,u})\Psi_{t,u}\boldsymbol{y}_{t,u}\middle|\boldsymbol{y}_t\right]du. \tag{A.1}$$

Note that

$$\mu(x+u,u,\boldsymbol{y}_{t,u})\Psi_{t,u}\boldsymbol{y}_{t,u} = \mathbf{G}(u)\Psi_{t,u}\boldsymbol{y}_{t,u},$$

where

$$\mathbf{G}(u) = \begin{bmatrix} \delta_1 e^{\gamma_1(u+\nu_1)^2} & & & \\ & \delta_2 e^{\gamma_2(u+\nu_2)^2} & & \\ & & \ddots & \\ & & & \delta_m e^{\gamma_m(u+\nu_m)^2} \end{bmatrix}$$

is a diagonal matrix with

$$\begin{aligned} \delta_i &= e^{a(0)+b(0)x-\frac{(\alpha_i+\gamma_i x+b(0))^2}{4\gamma_i}} \\ \nu_i &= \frac{\alpha_i+\gamma_i x+b(0)}{2\gamma_i}, \end{aligned}$$

for $i = 1,\ 2,\dots,m$.

Define $\psi_{t,T} := \mathbf{E}\left[\Psi_{t,T}\boldsymbol{y}_{t,T}\middle|\boldsymbol{y}_t\right]$. Thus, equation (A.1) becomes

$$\begin{aligned} \psi_{t,T} &= \boldsymbol{y}_t + \int_t^T \boldsymbol{\Lambda}\psi_{t,u}\,du - \int_t^T \mathbf{G}(u)\psi_{t,u}\,du \\ &= \boldsymbol{y}_t + \int_t^T (\boldsymbol{\Lambda}-\mathbf{G}(u))\,\psi_{t,u}\,du. \end{aligned}$$

Differentiating the above equation with respect to T, we have

$$d\psi_{t,T} = \psi_{t,T}(\boldsymbol{\Lambda}-\mathbf{G}(T))dT. \tag{A.2}$$

When solving equation (A.2), we have to calculate the integral which has the form like $\int e^{x^2}\,dx$. Although e^{x^2} does not have an elementary antiderivative, we can provide a Taylor series representation for it. It is known that

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \ldots + \frac{x^{2n}}{n!} + \ldots$$

and so,

$$\int e^{x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \int \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}.$$

Therefore, the solution of the differential equation is

$$\psi_{t,T} = e^{(\mathbf{\Lambda}(T-t)-\mathbf{D}(T)+\mathbf{D}(t))}\boldsymbol{y}_t,$$

where

$$\mathbf{D}(u) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{\delta_1 \gamma_1^n}{(2n+1)n!}(u+\nu_1)^{(2n+1)} & & & \\ & \sum_{n=0}^{\infty} \frac{\delta_2 \gamma_2^n}{(2n+1)n!}(u+\nu_2)^{(2n+1)} & & \\ & & \ddots & \\ & & & \sum_{n=0}^{\infty} \frac{\delta_m \gamma_m^n}{(2n+1)n!}(u+\nu_m)^{(2n+1)} \end{bmatrix}.$$

Integrating the matrix $\mathbf{G}(u)$, we have

$$
\begin{aligned}
&\int_t^T \mathbf{G}(u)\,du \\
&= \begin{bmatrix} \int_t^T \delta_1 e^{\gamma_1(u+\nu_1)^2}\,du & & & \\ & \int_t^T \delta_2 e^{\gamma_2(u+\nu_2)^2}\,du & & \\ & & \ddots & \\ & & & \int_t^T \delta_m e^{\gamma_m(u+\nu_m)^2}\,du \end{bmatrix} \\
&= \begin{bmatrix} \int_{\sqrt{|\gamma_1|}(t+\nu_1)}^{\sqrt{|\gamma_1|}(T+\nu_1)} \frac{\delta_1}{\sqrt{|\gamma_1|}} e^{g_1 v^2}\,dv & & & \\ & \int_{\sqrt{|\gamma_2|}(t+\nu_2)}^{\sqrt{|\gamma_2|}(T+\nu_2)} \frac{\delta_2}{\sqrt{|\gamma_2|}} e^{g_2 v^2}\,dv & & \\ & & \ddots & \\ & & & \int_{\sqrt{|\gamma_m|}(t+\nu_m)}^{\sqrt{|\gamma_m|}(T+\nu_m)} \frac{\delta_m}{\sqrt{|\gamma_m|}} e^{g_m v^2}\,dv \end{bmatrix} \\
&= \begin{bmatrix} \int_{\sqrt{|\gamma_1|}(t+\nu_1)}^{\sqrt{|\gamma_1|}(T+\nu_1)} \frac{\delta_1}{\sqrt{|\gamma_1|}} \sum_{n=0}^{\infty} \frac{g_1^n v^{2n}}{n!}\,dv & & & \\ & \int_{\sqrt{|\gamma_2|}(t+\nu_2)}^{\sqrt{|\gamma_2|}(T+\nu_2)} \frac{\delta_2}{\sqrt{|\gamma_2|}} \sum_{n=0}^{\infty} \frac{g_2^n v^{2n}}{n!}\,dv & & \\ & & \ddots & \\ & & & \int_{\sqrt{|\gamma_m|}(t+\nu_m)}^{\sqrt{|\gamma_m|}(T+\nu_m)} \frac{\delta_m}{\sqrt{|\gamma_m|}} \sum_{n=0}^{\infty} \frac{g_m^n v^{2n}}{n!}\,dv \end{bmatrix} \\
&= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{\delta_1 \gamma_1^n \left((T+\nu_1)^{2n+1}-(t+\nu_1)^{2n+1}\right)}{(2n+1)n!} & & & \\ & \sum_{n=0}^{\infty} \frac{\delta_2 \gamma_2^n \left((T+\nu_2)^{2n+1}-(t+\nu_2)^{2n+1}\right)}{(2n+1)n!} & & \\ & & \ddots & \\ & & & \sum_{n=0}^{\infty} \frac{\delta_m \gamma_m^n \left((T+\nu_m)^{2n+1}-(t+\nu_m)^{2n+1}\right)}{(2n+1)n!} \end{bmatrix} \\
&= \mathbf{D}(T) - \mathbf{D}(t),
\end{aligned}
$$

where $g_i = sgn(\gamma_i)$ and sgn is the sign function.

Write

$$
S(t,T) := \mathbf{E}\left[e^{-\int_t^T \mu(x+u,u,\boldsymbol{y}_{t,u})\,du} \middle| \mathcal{J}_t \right].
$$

Hence,

$$\begin{aligned}
S(t,T) &= \mathbf{E}\left[e^{-\int_t^T \mu(x+u,u,\boldsymbol{y}_{t,u})\,du} \middle| \boldsymbol{y}_t \right] \\
&= \mathbf{E}\left[e^{-\int_t^T \mu(x+u,u,\boldsymbol{y}_{t,u})\,du} \langle \boldsymbol{y}_T, \mathbf{1}\rangle \middle| \boldsymbol{y}_t \right] \\
&= \left\langle \mathbf{E}\left[e^{-\int_t^T \mu(x+u,u,\boldsymbol{y}_{t,u})\,du} \boldsymbol{y}_T \middle| \boldsymbol{y}_t \right], \mathbf{1} \right\rangle \\
&= \left\langle e^{(\boldsymbol{\Lambda}(T-t)-\mathbf{D}(T)+\mathbf{D}(t))} \boldsymbol{y}_t,\ \mathbf{1} \right\rangle \\
&= \left\langle \boldsymbol{y}_t,\ e^{(\boldsymbol{\Lambda}^\top(T-t)-\mathbf{D}(T)+\mathbf{D}(t))} \mathbf{1} \right\rangle.
\end{aligned} \tag{A.3}$$

A.2 Derivation of survival probability under M3

In order to obtain the survival index under M3, we need to find $\mathbf{E}_{\boldsymbol{y}_t}[e^{G(t,T,\boldsymbol{y}_t)}]$, i.e., the expectation of $e^{G(t,T,\boldsymbol{y}_t)}$. Define $\boldsymbol{\kappa}_{t,s} = e^{\frac{1}{2}\int_t^s \langle \boldsymbol{\phi}_u, \boldsymbol{y}_u \rangle\,du}$, which has the differential form

$$d\boldsymbol{\kappa}_{t,s} = \frac{1}{2} \langle \boldsymbol{\phi}_s, \boldsymbol{y}_s \rangle \boldsymbol{\kappa}_{t,s}\, ds.$$

Therefore,

$$\begin{aligned}
d\boldsymbol{\kappa}_{t,s}\boldsymbol{y}_{t,s} &= \boldsymbol{\kappa}_{t,s} d\boldsymbol{y}_{t,s} + \boldsymbol{y}_{t,s} d\boldsymbol{\kappa}_{t,s} \\
&= \boldsymbol{\kappa}_{t,s}\left[\boldsymbol{\Lambda}\boldsymbol{y}_{t,s} ds + d\boldsymbol{n}_s\right] + \boldsymbol{y}_{t,s}\left[\frac{1}{2}\langle \boldsymbol{\phi}_s, \boldsymbol{y}_s \rangle \boldsymbol{\kappa}_{t,s}\right] ds \\
&= \left[\boldsymbol{\Lambda}\boldsymbol{\kappa}_{t,s}\boldsymbol{y}_{t,s} + \frac{1}{2}\langle \boldsymbol{\phi}_s, \boldsymbol{y}_s \rangle \boldsymbol{\kappa}_{t,s}\boldsymbol{y}_{t,s}\right] ds + \boldsymbol{\kappa}_{t,s} d\boldsymbol{n}_s.
\end{aligned} \tag{A.4}$$

The integral form of (A.4) is then

$$\boldsymbol{\kappa}_{t,s}\boldsymbol{y}_{t,s} = \boldsymbol{y}_t + \int_t^s \boldsymbol{\Lambda}\boldsymbol{\kappa}_{t,v}\boldsymbol{y}_{t,v}\, dv + \frac{1}{2}\int_t^s \langle \boldsymbol{\phi}_v, \boldsymbol{y}_v \rangle \boldsymbol{\kappa}_{t,v}\boldsymbol{y}_{t,v}\, dv + \int_t^s \boldsymbol{\kappa}_{t,v}\, d\boldsymbol{n}_v. \tag{A.5}$$

Taking expectations on both sides of (A.5), we obtain

$$\widetilde{\boldsymbol{\kappa}}_{t,s} := \mathbf{E}\left[\boldsymbol{\kappa}_{t,s}\boldsymbol{y}_{t,s}\right] = \boldsymbol{y}_t + \int_t^s \boldsymbol{\Lambda}\mathbf{E}\left[\boldsymbol{\kappa}_{t,v}\boldsymbol{y}_{t,v}\right] dv + \frac{1}{2}\int_t^s \mathbf{E}\left[\langle \boldsymbol{\phi}_v, \boldsymbol{y}_v \rangle \boldsymbol{\kappa}_{t,v}\boldsymbol{y}_{t,v}\right] dv. \tag{A.6}$$

Note that $\langle \boldsymbol{\phi}_v, \boldsymbol{y}_v \rangle \boldsymbol{\kappa}_{t,v}\boldsymbol{y}_{t,v} = \boldsymbol{K}(v)\boldsymbol{\kappa}_{t,v}\boldsymbol{y}_{t,v}$ where $\boldsymbol{K}(v)$ is a time-varying diagonal matrix expressed as

$$\boldsymbol{K}(v) = \begin{bmatrix} (\frac{e^{c(T-v)}-1}{c}\sigma_1)^2 & & & \\ & (\frac{e^{c(T-v)}-1}{c}\sigma_2)^2 & & \\ & & \ddots & \\ & & & (\frac{e^{c(T-v)}-1}{c}\sigma_m)^2 \end{bmatrix}.$$

Therefore, $\widetilde{\boldsymbol{\kappa}}_{t,s} = \boldsymbol{y}_t + \int_t^s \left[\boldsymbol{\Lambda}^\top + \frac{1}{2}\boldsymbol{K}(v)\right]\widetilde{\boldsymbol{\kappa}}_{t,v}\, dv$. This is equivalent to solving the differential equation

$$\frac{d}{ds}\boldsymbol{\kappa}_{t,s} = \widetilde{\boldsymbol{K}}(s)\boldsymbol{\kappa}_{t,s},$$

with

$$\begin{aligned} \widetilde{\boldsymbol{\kappa}}_{t,t} &= \boldsymbol{y}_t \\ \widetilde{\boldsymbol{K}}(s) &= \boldsymbol{\Lambda}^\top + \frac{1}{2}\boldsymbol{K}(s). \end{aligned}$$

Let $\boldsymbol{\Pi}_{t,s}$ be the fundamental matrix satisfying the first-order ODE $\frac{d}{ds}\boldsymbol{\Pi}_{t,s} = \widetilde{\boldsymbol{K}}(s)\boldsymbol{\Pi}_{t,s}$. Then $\boldsymbol{\kappa}_{t,s} = \boldsymbol{\Pi}_{t,s}\boldsymbol{y}_t$ so that

$$\begin{aligned} \mathbf{E}\left[e^{\frac{1}{2}\int_t^T \langle \boldsymbol{\phi}_v, \boldsymbol{y}_v\rangle\, dv} | \mathcal{F}_t\right] &= \mathbf{E}\left[e^{\frac{1}{2}\int_t^T \langle \boldsymbol{\phi}_v, \boldsymbol{y}_v\rangle\, dv} \langle \boldsymbol{y}_T, \mathbf{1}\rangle | \mathcal{F}_t\right] \\ &= \langle \widetilde{\boldsymbol{\kappa}}_{t,T}, \mathbf{1}\rangle \\ &= \langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t, \mathbf{1}\rangle. \end{aligned}$$

It follows that the survival index is

$$\begin{aligned} S(t,T) &= e^{-H(t,T)\mu_t}\langle \boldsymbol{\Pi}_{t,T}\boldsymbol{y}_t,\ \mathbf{1}\rangle \\ &= \left\langle \boldsymbol{y}_t,\ e^{-H(t,T)\mu_t}\boldsymbol{\Pi}_{t,T}^\top\mathbf{1}\right\rangle, \end{aligned}$$

which reconciles with equation (4.16).

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