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# Ghost number of group algebras

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A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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# Ghost number of group algebras

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by

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Department of Mathematics

A thesis submitted in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy

The School of Graduate and Postdoctoral Studies  
The University of Western Ontario  
London, Ontario, Canada

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# *Abstract*

The generating hypothesis for the stable module category of a finite group is the statement that if a map in the thick subcategory generated by the trivial representation induces the zero map in Tate cohomology, then it is stably trivial. It is known that the generating hypothesis fails for most groups. Generalizing work done for  $p$ -groups, we define the ghost number of a group algebra, which is a natural number that measures the degree to which the generating hypothesis fails. We describe a close relationship between ghost numbers and Auslander-Reiten triangles, with many results stated for a general projective class in a general triangulated category. We then compute ghost numbers and bounds on ghost numbers for many families of  $p$ -groups. For non- $p$ -groups, we introduce two other closely related invariants, the simple ghost number, which considers maps which are stably trivial when composed with any map from a simple module, and the strong ghost number, which considers maps which are ghosts after restriction to every subgroup of  $G$ . We produce the first computations of the ghost number for non- $p$ -groups. We prove that there are close relationships between the three invariants, and make computations of the new invariants for many families of groups. We also discuss how computational algebra can be applied to calculate the ghost number.

**Keywords:** Tate cohomology, stable module category, generating hypothesis, ghost map, GAP.

## *Co-Authorship*

Chapters 2 and 3 of this thesis are based on two consecutive papers [23, 24] that study the ghost numbers of  $p$ -groups and non- $p$ -groups, respectively. These are collaborated works with Dan Christensen. I have done most of the work to find the results and wrote down the first draft of both papers. Christensen has helped to improve the style of the papers, especially the introductions, to make them better for submission. The other chapters of the thesis are written solely by myself.

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# Chapter 1

## Introduction

In this thesis, we study the ghost number of group algebras for finite groups. Briefly speaking, the ghost number of a group algebra measures the failure of the *generating hypothesis* on the stable module category. This is motivated by the famous conjecture by Peter Freyd in stable homotopy theory, which states that if a map between two compact spectra is sent to zero by the stable homotopy group functor, then the map is null homotopic. The conjecture is referred to as the generating hypothesis and is still an open question. In [25], where Freyd made this conjecture, he also showed that it has many interesting consequences. For example, if the generating hypothesis holds for spectra, then the stable homotopy group functor is fully-faithful on compact spectra. The generating hypothesis can be generalised to a triangulated category, and has been studied in various cases, such as the derived category of a ring  $R$  and the stable module category of a group algebra  $kG$ . For the stable module category, it is known that the generating hypothesis fails for most groups (see Theorem 1.3.1). Hence we continue the study to see how badly the generating hypothesis can fail, and this is measured by the ghost number of the group algebra, which is the subject of the thesis.

This thesis is divided into five chapters. Chapter 1 is an introduction chapter where we introduce the background of the subject and summarize the results on ghost numbers of group algebras. Chapters 2 and 3 are based on two consecutive papers that study the ghost numbers of  $p$ -groups and non- $p$ -groups, respectively. They contain both theoretical and computational results. Chapter 4 focuses on applying computational algebra to study the group algebra. We present some improved code for GAP [26] to work on modular representations and provide examples of computations. The last chapter is

a conclusion chapter that briefly summarizes the results of the thesis and the relation between the chapters.

## 1.1 The generating hypothesis and its generalisation

In this section, we discuss the generalisation of the generating hypothesis to a triangulated category.

In homotopy theory, homotopy groups play a central role. They detect whether a nice space, such as a CW-complex, is contractible. Recall that the generating hypothesis for the stable homotopy category of spectra is the conjecture that if a map between two compact spectra is sent to zero by the stable homotopy group functor, then the map is null homotopic. Note that the stable homotopy category is a triangulated category. Hence we can apply the ideas from homotopy theory to other areas by generalisation from the stable homotopy category to a triangulated category.

**Definition 1.1.1.** A **triangulated category** is an additive category  $\mathbb{T}$  together with a translation functor  $\Sigma$  and a class  $\Delta$  of (distinguished) triangles  $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$  in  $\mathbb{T}$ , such that  $\Sigma$  is a self-equivalence of  $\mathbb{T}$  and  $\Delta$  satisfies the following axioms:

**TR1** For each  $X \in \mathbb{T}$ ,  $X \xrightarrow{id} X \rightarrow 0 \rightarrow \Sigma X$  is a triangle;

for each map  $f : X \rightarrow Y$  in  $\mathbb{T}$ , there exists a triangle  $X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X$ ; and the triangles are closed under isomorphisms.

**TR2** If  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  is a triangle, then  $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$  and  $\Sigma^{-1}Z \xrightarrow{-\Sigma^{-1}h} X \xrightarrow{f} Y \xrightarrow{g} Z$  are triangles too.

**TR3** Given a commutative square  $\beta \circ f = f' \circ \alpha$  in  $\mathbb{T}$ , complete the maps  $f$  and  $g$  into triangles. Then there exists a map  $\gamma$  making the following diagram into a map between triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \Sigma \alpha \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

**TR4** (The octahedral axiom) Let  $X \xrightarrow{f_1} Y \xrightarrow{f_2} W$  be maps in  $\mathbb{T}$ . Complete the maps  $f_1$  and  $f_2$  into triangles. Then there exists a commutative octahedron in  $\mathbb{T}$ :

$$\begin{array}{ccccc}
 & & \Sigma^{-1}V & & \\
 & & \swarrow & \searrow & \\
 & Y & \xrightarrow{\quad} & Z & \\
 f_1 \nearrow & & & & \searrow \\
 X & & & & \Sigma X \\
 & \downarrow f_2 & & \downarrow & \\
 & W & \xrightarrow{\quad} & U & \\
 & \searrow & \swarrow & \nearrow & \\
 & & V & & 
 \end{array} ,$$

such that  $X \rightarrow W \rightarrow U \rightarrow \Sigma X$  and  $\Sigma^{-1}V \rightarrow Z \rightarrow U \rightarrow V$  are also triangles.

If  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  is a triangle in  $\mathbb{T}$ , then  $Z$  is called the **cofibre** of  $f$  and  $X$  is called the **fibre** of  $g$ .

For example, the derived category  $D(R)$  for a ring  $R$  is a triangulated category. The stable module category  $\text{StMod}(kG)$  for a group algebra  $kG$  is also a triangulated category. We will give more details on  $\text{StMod}(kG)$  in Section 1.2.

In general, let  $\mathbb{T}$  be a triangulated category, and let  $\mathbb{S}$  be a set of distinguished objects in  $\mathbb{T}$ . We write  $[-, -]$  for hom-sets in  $\mathbb{T}$ . Then the set of functors  $[S, -]_*$  with  $S \in \mathbb{S}$  is analogous to the stable homotopy group functor in the sense that

$$\text{if } [S, M]_* = 0 \text{ for all } S \in \mathbb{S} \text{ and } M \in \text{Loc}\langle \mathbb{S} \rangle, \text{ then } M = 0.$$

Here  $\text{Loc}\langle \mathbb{S} \rangle$  is the localising subcategory of  $\mathbb{T}$  generated by  $\mathbb{S}$ . A full subcategory  $\mathbb{S}$  of  $\mathbb{T}$  is **localising** if it is closed under suspension, retracts, triangles, and arbitrary sums. The localising subcategory generated by  $\mathbb{S}$  is the smallest localising subcategory that contains  $\mathbb{S}$ , and is denoted by  $\text{Loc}\langle \mathbb{S} \rangle$ . If we do not require  $\mathbb{S}$  to be closed under arbitrary sums, then  $\mathbb{S}$  is said to be **thick**. The thick subcategory generated by  $\mathbb{S}$  is defined similarly and denoted by  $\text{Thick}\langle \mathbb{S} \rangle$ . We say that  $\mathbb{T}$  satisfies **the generating hypothesis with respect to  $\mathbb{S}$**  if the functors  $[S, -]_*$  are faithful on  $\text{Thick}\langle \mathbb{S} \rangle$  for all  $S \in \mathbb{S}$ .

Note that if  $\mathbb{S}$  consists of finitely many compact objects in  $\mathbb{T}$ , then

$$\text{Thick}\langle \mathbb{S} \rangle = \text{Loc}\langle \mathbb{S} \rangle \cap \text{compact objects in } \mathbb{T}.$$

In general, an object  $X$  in a triangulated category  $\mathbb{T}$  is **compact** if the canonical map  $\bigoplus [X, C_i] \rightarrow [X, \bigoplus C_i]$  is an isomorphism for all coproducts in  $\mathbb{T}$ . For example, finite

CW-complex are compact in spectra and perfect complex are compact in the derived category of a ring.

We will introduce the stable module category  $\text{StMod}(kG)$  of the group algebra  $kG$  in Section 1.2.2, and show that it is a triangulated category. Hence, taking  $\mathbb{S}$  to consist of the trivial representation  $k$ , we can state the generating hypothesis on  $\text{StMod}(kG)$  in this setting.

One can also consider the global generating hypothesis with respect to  $\mathbb{S}$  on  $\mathbb{T}$ , i.e., the statement that the functors  $[S, -]_*$  are faithful on  $\text{Loc}\langle\mathbb{S}\rangle$  for all  $S \in \mathbb{S}$ . This is studied by Hovey and Lockridge in the derived category of a ring spectrum  $E$ , and they show that the global generating hypothesis puts very strong constraints on  $E$ . The interested reader is referred to [27].

## 1.2 The generating hypothesis on the stable module category

In Section 1.2.1, we review some basic facts in representation theory and define the group cohomology and the Tate cohomology. Then, in Section 1.2.2, we describe the stable module category and state the generating hypothesis on it.

### 1.2.1 Background

We begin with the basic concepts in representation theory.

Let  $G$  be a finite group and  $k$  be a field. We define the group algebra  $kG$  to be the algebra over  $k$ , whose underlying space is the vector space generated by elements in  $G$ . Then a general element in  $kG$  is of the form  $\sum a_g \cdot g$  with  $a_g \in k$  and  $g \in G$ . We can multiply two basis elements  $g$  and  $h$  using the multiplication in  $G$  and extend this linearly to a general element. This defines the multiplication in  $kG$ , and if  $u$  is the unit in  $G$ , then  $1 \cdot u$  is the unit in  $kG$ . With an abuse of notation, we write 1 for the unit in  $kG$ . Then the  $kG$ -modules are exactly the representations of  $G$  over  $k$ , and the study of representation theory is the same as the study of modules over group algebras.

We write  $\text{Mod}(kG)$  for the category of  $kG$ -modules and  $\text{mod}(kG)$  for its full subcategory of finitely-generated modules. Then we make the following definitions for  $kG$ -modules.

**Definition 1.2.1.** Let  $G$  be a group and  $k$  be a field. A  $kG$ -module  $M$  is said to be **irreducible** or **simple** if it has no proper submodules. A  $kG$ -module  $M$  is **completely reducible** or **semisimple** if every submodule of  $M$  splits off as a summand. The group algebra  $kG$  is **semisimple** if every  $kG$ -module  $M$  is semisimple.

Maschke's Theorem provides a criterion for when the group algebra  $kG$  is semisimple:

**Theorem 1.2.2** (Maschke). *Let  $G$  a finite group and  $k$  be a field. The group algebra  $kG$  is semisimple if and only if the characteristic of  $k$  does not divide the order of the group  $G$ .*

If  $kG$  is semisimple and  $k$  is algebraically closed, then the representations of  $G$  over  $k$  can be described by the character table. It provides a complete list of all irreducible summands of the free module  $kG$ , or, more precisely, the character functions of these irreducible modules. Note that since  $kG$  is semisimple, these are all the irreducible modules, and every module in  $\text{mod}(kG)$  is a sum of the irreducible modules.

The following is an example of the character table of the group  $S_3$ :

	{1}	{3}	{2}
$S_3$	$id$	(12)	(123)
$\chi_{trivial}$	1	1	1
$\chi_{sign}$	1	-1	1
$\chi_2$	2	0	-1

When  $kG$  is not semisimple, we know that there are submodules that do not split off as summands. For example, consider the central element  $\sum_{g \in G} g$  in  $kG$ . It generates a two-sided ideal of  $kG$  of dimension one and, since  $\text{char}(k) \nmid |G|$  by Maschke's Theorem, the ideal is nilpotent. It follows that the ideal, considered as a submodule of  $kG$ , is not a direct summand. A  $kG$ -module  $M$  is said to be **indecomposable** if it has no proper summands. Note that if  $kG$  is semisimple, then indecomposable modules are the same as irreducible modules. Now we can state the Krull-Schmidt property of  $\text{mod}(kG)$ . It is a very important feature of the module category over a group algebra.

**Theorem 1.2.3** (Krull-Schmidt). *Let  $G$  be a group and  $k$  be a field. Let  $M$  be a finitely-generated  $kG$ -module. Suppose that  $M = M_1 \oplus \cdots \oplus M_k$  and  $M = N_1 \oplus \cdots \oplus N_l$  are two decompositions of  $M$  into indecomposable summands. Then  $k = l$ , and we can reorder the summands  $N_i$  so that  $M_i \cong N_i$  for each  $i$ .*

The non-semisimple case is much more complicated than the semisimple case because there are non-trivial extensions between simple modules. Hence we study the homological properties of the group algebra  $kG$  when it is not semisimple. We still need a lemma before we give the definition of Tate cohomology.

**Lemma 1.2.4.** *Projective modules and injective modules coincide in  $\text{mod}(kG)$ .*

*Proof.* We have a non-degenerate quadratic form

$$kG \times kG \longrightarrow k$$

that sends a pair  $(g, g') \in kG$  to  $\delta(g, g')$ . This gives us an isomorphism  $kG \cong kG^*$ . Hence  $kG$  is an injective module over itself and the lemma follows.  $\square$

Now let  $G$  be a finite group and  $k$  be a field whose characteristic divides the order of  $G$ . We define the group cohomology and Tate cohomology of a  $kG$ -module  $M$ .

**Definition 1.2.5.** Let  $G$  be a finite group and  $k$  be a field. Let

$$P_* : \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0$$

be a projective resolution of the trivial representation  $k$ . The  $n$ -th **group cohomology**  $H^n(G, M)$  of  $M$  is defined to be the  $n$ -th cohomology of the chain complex  $\text{Hom}(P_*, M)$  for  $n \geq 0$ .

If, instead of a projective resolution, we take a complete resolution

$$T_* : \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\partial_0} P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

of  $k$ , that is, a doubly infinite exact sequence of projective modules such that  $\text{im}(\partial_0) = k$ , then the  $n$ -th **Tate cohomology**  $\widehat{H}^n(G, M)$  of  $M$  is defined to be the  $n$ -th cohomology of the chain complex  $\text{Hom}(T_*, M)$  for  $n \in \mathbb{Z}$ . We can also replace the trivial module  $k$  by an arbitrary  $kG$ -module  $L$  and compute the resolutions  $P_*$  and  $T_*$  of  $L$ . The cohomology

of the chain complexes  $\text{Hom}(P_*, M)$  and  $\text{Hom}(T_*, M)$  of  $M$  are denoted by  $\text{Ext}^n(L, M)$  and  $\widehat{\text{Ext}}^n(L, M)$ , respectively.

### 1.2.2 The stable module category

We define the stable module category in this section and go over its basic properties. In particular, we show that the Tate cohomology functor is represented by the trivial  $kG$ -module  $k$  in  $\text{StMod}(kG)$ . Then we state the generating hypothesis and its variations on the stable module category.

Let  $G$  be a finite group and  $k$  be a field. Note that a projective  $kG$ -module has trivial reduced group cohomology and Tate cohomology. Hence, by Maschke's Theorem, there is no cohomology when  $\text{char}(k) \nmid |G|$ . Thus we assume that the characteristic of  $k$  divides the order of  $G$  and focus on non-projective modules. For  $M$  and  $N$  in  $\text{StMod}(kG)$ , we write  $\text{PHom}(M, N)$  for the subspace of  $\text{Hom}(M, N)$  that consists of maps between  $M$  and  $N$  that factor through a projective module. The stable module category  $\text{StMod}(kG)$  is a quotient category of the module category  $\text{Mod}(kG)$ , the hom-sets being the quotient

$$\underline{\text{Hom}}(M, N) = \text{Hom}(M, N) / \text{PHom}(M, N).$$

We write  $\text{stmod}(kG)$  for the full subcategory of finitely-generated modules in  $\text{StMod}(kG)$ . It will follow that two modules  $M$  and  $N$  are isomorphic in the stable category if and only if there exists  $P$  and  $Q$  projective, such that  $M \oplus P \cong N \oplus Q$ . In particular, projective modules are isomorphic to zero in the stable category. By the Krull-Schmidt property, two finitely-generated modules  $M$  and  $N$  are isomorphic in  $\text{stmod}(kG)$  if and only if they have the same projective-free summands.

The module category  $\text{Mod}(kG)$  is abelian, but there is no reason that  $\text{StMod}(kG)$  is abelian. However, we can show that  $\text{StMod}(kG)$  is a triangulated category. We define the desuspension and suspension functors first.

**Definition 1.2.6** (Desuspension and suspension). Let  $M$  be a  $kG$ -module. The **desuspension** of  $M$ , denoted by  $\Omega M$ , is the kernel in the short exact sequence

$$\Omega M \rightarrow P \xrightarrow{\epsilon} M,$$

where  $\epsilon$  is a surjection from a projective module  $P$  to  $M$ . Dually, the **suspension** of  $M$ , denoted by  $\Sigma M$ , or  $\Omega^{-1}M$ , is the cokernel in the short exact sequence

$$M \xrightarrow{\epsilon} P \rightarrow \Sigma M,$$

where  $\epsilon$  is an injection from  $M$  to an injective module  $P$ .

Desuspensions and suspensions are well-defined in the stable module category by Schanuel's Lemma. It is clear that they are inverses of each other.

**Lemma 1.2.7** (Schanuel's Lemma). *Let  $R$  be a ring, and let  $M$  be an  $R$ -module. Let  $P \xrightarrow{\epsilon} M$  and  $Q \xrightarrow{\theta} M$  be projective covers of  $M$  in  $\text{Mod}(R)$ . Then  $\ker \epsilon \oplus Q \cong \ker \theta \oplus P$ .*

*Remark 1.2.8.* For  $M$  and  $N$  in  $\text{Mod}(kG)$ , the tensor product  $M \otimes N$  can be viewed as a  $kG$ -module via the diagonal action. One can show that if  $P$  is a projective module, then  $P \otimes N$  is also projective. Since the functor  $- \otimes N$  preserves short exact sequences in  $\text{Mod}(kG)$ , it follows that we have a stable isomorphism  $\Omega k \otimes M \cong \Omega M$ . Here  $k$  is the trivial  $kG$ -module.

Now we can show that the cohomology groups are represented by hom-sets in  $\text{StMod}(kG)$ .

**Theorem 1.2.9.** *Let  $G$  be a finite group and  $k$  be a field whose characteristic divides the order of  $G$ . Then, for  $M$  and  $L$  in  $\text{StMod}(kG)$  and  $n \in \mathbb{Z}$ , there is a natural isomorphism*

$$\widehat{\text{Ext}}^n(L, M) \cong \underline{\text{Hom}}(L, \Sigma^n M) \cong \underline{\text{Hom}}(\Omega^n L, M).$$

*In particular, Tate cohomology is represented by the trivial representation  $k$ .  $\square$*

By usual homological algebra,  $\widehat{\text{Ext}}^1(L, M)$  is equivalent to the isomorphism classes of extensions between  $L$  and  $M$ . Then, by the theorem, given a short exact sequence  $M \rightarrow N \rightarrow L$ , there is a corresponding map in  $\underline{\text{Hom}}(L, \Sigma M)$ . More precisely, it is the



connecting map  $\delta : L \rightarrow \Sigma M$  in the following Snake-Lemma diagram

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & 0 & \longrightarrow & L \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & P & \longrightarrow & P & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \Sigma M & \longrightarrow & \Sigma N & \longrightarrow & 0
 \end{array} ,$$

where we choose  $P$  to be projective and the maps  $M \rightarrow P$  and  $N \rightarrow P$  to be injective. Then we use this to define a triangle

$$M \rightarrow N \rightarrow L \xrightarrow{\delta} \Sigma M$$

in  $\text{StMod}(kG)$ . Note that  $N \rightarrow L \oplus P \rightarrow \Sigma M$  is also a short exact sequence, and this corresponds the rotated triangle

$$N \rightarrow L \xrightarrow{\delta} \Sigma M \rightarrow \Sigma N.$$

Then, to compute the cofibre of a map  $f : M \rightarrow N$  in  $\text{StMod}(kG)$ , we replace  $f$  by an injection  $f'$  that is stably isomorphic to it, and then  $\text{coker}(f')$  is the cofibre of  $f$ .

Note that there is also a multiplication structure on the Tate cohomology  $\widehat{H}^n(G, k)$ . It is easy to describe the algebra structure of Tate cohomology using the natural isomorphism  $\widehat{H}^n(G, k) \cong \underline{\text{Hom}}(\Omega^n k, k)$ . For  $\zeta \in \underline{\text{Hom}}(\Omega^m(M), N)$  and  $\gamma \in \underline{\text{Hom}}(\Omega^n(L), M)$ , we define

$$\zeta\gamma := \zeta \circ \Omega^m(\gamma) \in \underline{\text{Hom}}(\Omega^{m+n}(L), N).$$

This makes  $\widehat{H}^*(G, k)$  into a graded commutative algebra. Similarly, we can define the multiplication on the group cohomology  $H^*(G, k)$ , and it is well-known [5, Section 4.2] that  $H^*(G, k)$  is finitely generated. On the other hand,  $\widehat{H}^*(G, k)$  is finitely generated if and only if the cohomology is periodic, i.e.,  $\Omega^n k \cong k$  for some  $n \geq 0$ .

We end this section with the generating hypothesis on the stable module category. Note that the Tate cohomology functor  $\widehat{H}^*(G, -)$  on  $\text{StMod}(kG)$  plays an analogous role to the stable homotopy functor, and it is represented by the trivial representation  $k$ . Hence we state the generating hypothesis on  $\text{StMod}(kG)$  as follows:

The generating hypothesis holds on  $\text{StMod}(kG)$  if the Tate cohomology functor  $\widehat{H}^*(G, -)$  is faithful on  $\text{Thick}\langle k \rangle$ .

We call a map in the kernel of the Tate cohomology functor a **ghost**. Then the generating hypothesis (with respect to the trivial representation  $k$ ) is the statement that every ghost in  $\text{Thick}\langle k \rangle$  is stably-trivial. It is important that we restrict to  $\text{Thick}\langle k \rangle$  here. In general,  $\text{Thick}\langle k \rangle$  can be a proper subcategory of  $\text{stmod}(kG)$ , the category of finitely-generated  $kG$ -modules. And whenever this is the case, there exists a non-projective  $kG$ -module  $M$ , whose Tate cohomology is trivial. Thus the identity map on  $M$  is a stably non-trivial ghost in  $\text{stmod}(kG)$ , but it is not in  $\text{Thick}\langle k \rangle$ . Restricting to  $\text{Thick}\langle k \rangle$  prevents this from happening. On the other hand, we can consider the generating hypothesis with respect to all simple modules. Since the simple modules generate the stable module category, there is no need to restrict to  $\text{Thick}\langle k \rangle$ . A map  $M \rightarrow N$  is called a **simple ghost**, if the composite of maps  $X \rightarrow M \rightarrow N$  is stably-trivial for any  $X$  that is a suspension of some simple module, and the **simple generating hypothesis** is the statement that every simple ghost in  $\text{stmod}(kG)$  is stably-trivial. If  $G$  is a  $p$ -group, then  $k$  is the only simple module and the simple generating hypothesis is equivalent to the generating hypothesis. One can also consider the **strong ghosts** in  $\text{StMod}(kG)$ , which are the maps whose restrictions to any subgroup are still ghosts. Simple ghosts and strong ghosts are studied in Chapter 3.

## 1.3 Background and literature review

In this section, we review the previous work in the study of the generating hypothesis and ghost numbers for group algebras. We begin in Section 1.3.1 with the results on the generating hypothesis on  $\text{StMod}(kG)$  and review some techniques used in the proof. Then we introduce projective classes in Section 1.3.2, and using this idea, we define ghost numbers in Section 1.3.3. Finally, we introduce strong ghosts and the strong generating hypothesis in Section 1.3.4.

### 1.3.1 The generating hypothesis

In a series of papers [9, 16, 18, 20], it is proved that the generating hypothesis holds in  $\text{StMod}(kG)$  if and only if the Sylow  $p$ -subgroup  $P$  of  $G$  is  $C_2$  or  $C_3$ .

**Theorem 1.3.1** (Benson, Carlson, Chebolu, Christensen and Mináč [9, 16, 18, 20]). *Let  $G$  be a finite group and  $k$  be a field whose characteristic  $p$  divides the order of  $G$ . The generating hypothesis holds for  $\text{StMod}(kG)$  if and only if the Sylow  $p$ -subgroup of  $G$  is  $C_2$  or  $C_3$ .  $\square$*

We review the techniques used to prove the theorem. Note that they are quite different for  $p$ -groups and non- $p$ -groups.

### 1.3.1.1 The induction technique

Let  $H$  be a subgroup of  $G$ , and let  $k$  be a field. It is well known that the induction functor is both left and right adjoint to the restriction functor:

$$\uparrow_H^G : \text{stmod}(kH) \rightleftarrows \text{stmod}(kG) : \downarrow_H^G.$$

Since the restriction of the trivial module is also trivial, then, by the adjunction

$$\underline{\text{Hom}}(\Omega^n k, f \uparrow_H^G) \cong \underline{\text{Hom}}(\Omega^n (k \downarrow_H^G), f),$$

we see that if  $f$  is a ghost in  $\text{stmod}(kH)$ , then  $f \uparrow_H^G$  is a ghost in  $\text{stmod}(kG)$ .

For a  $p$ -group  $G$ , since  $\text{Thick}\langle k \rangle = \text{stmod}(kG)$ , the induction technique becomes very useful. In particular, since the induction functor is faithful, it follows that if  $G$  is  $p$ -group and the generating hypothesis fails for a subgroup  $H$ , then the generating hypothesis fails for  $G$ . Hence, the work to disprove the generating hypothesis for a  $p$ -group can be reduced to the study of small groups.

However, the induction technique does not apply in general if  $G$  is not a  $p$ -group, since the image of a map induced up might not be in  $\text{Thick}\langle k \rangle$ .

### 1.3.1.2 Auslander-Reiten triangles.

Auslander-Reiten triangles are used to disprove the generating hypothesis for a general finite group. In general, a triangle  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$  in a triangulated category is called an **Auslander-Reiten triangle**, if

1.  $\gamma \neq 0$ ,

2. any map  $X \rightarrow Y'$  that is not split monic factors through  $\alpha$ ,
3. any map  $Y' \rightarrow Z$  that is not split epic factors through  $\beta$ .

Krause constructed Auslander-Reiten triangles in triangulated categories via Brown representability, and pointed out that this could be a counterexample to the generating hypothesis if the beginning-term  $X$  in the triangle is compact [30]. In the stable module category, Auslander-Reiten triangles have the form

$$\Omega^2 M \rightarrow H \rightarrow M \xrightarrow{\gamma} \Omega M,$$

with  $M$  being indecomposable and finitely-generated. One can show that if  $M \in \text{Thick}\langle k \rangle$  is indecomposable and not isomorphic to  $\Sigma^n k$  for any  $n \in \mathbb{Z}$ , then  $\gamma$  is a stably-non-trivial ghost in  $\text{Thick}\langle k \rangle$ . In [16], such a module  $M$  is proved to exist if the Sylow  $p$ -subgroup of  $G$  is not  $C_2$  or  $C_3$ , hence the generating hypothesis fails in  $\text{StMod}(kG)$  in this case.

### 1.3.2 Projective classes and the universal ghost

We introduce the idea of a projective class in this section. It is used throughout our study of the generating hypothesis and ghost numbers.

**Definition 1.3.2.** Let  $\mathbb{T}$  be a triangulated category. A **projective class** in  $\mathbb{T}$  consists of a class  $\mathcal{P}$  of objects of  $\mathbb{T}$  and a class  $\mathcal{I}$  of morphisms of  $\mathbb{T}$  such that:

- (i)  $\mathcal{P}$  consists of exactly the objects  $P$  such that every composite  $P \rightarrow X \rightarrow Y$  is zero for each  $X \rightarrow Y$  in  $\mathcal{I}$ ,
- (ii)  $\mathcal{I}$  consists of exactly the maps  $X \rightarrow Y$  such that every composite  $P \rightarrow X \rightarrow Y$  is zero for each  $P$  in  $\mathcal{P}$ ,
- (iii) for each  $X$  in  $\mathbb{T}$ , there is a triangle  $P \rightarrow X \rightarrow Y \rightarrow \Sigma P$  with  $P$  in  $\mathcal{P}$  and  $X \rightarrow Y$  in  $\mathcal{I}$ .

Note that the class  $\mathcal{P}$  is closed under arbitrary sums and retracts in  $\mathbb{T}$  and that the class  $\mathcal{I}$  is an ideal in  $\mathbb{T}$ . Also note that the map  $X \rightarrow Y$  satisfying the third condition in the definition is a (weakly) universal map out of  $X$  in  $\mathcal{I}$ . It is zero if and only if  $X$  is a retract of  $P$ , and then  $X \in \mathcal{P}$  and every map out of  $X$  in  $\mathcal{I}$  is zero.

Given a projective class  $(\mathcal{P}, \mathcal{I})$ , there is a sequence of **derived projective classes**  $(\mathcal{P}_n, \mathcal{I}^n)$  [21]. The ideal  $\mathcal{I}^n$  consists of all  $n$ -fold composites of maps in  $\mathcal{I}$ , and  $X$  is in  $\mathcal{P}_n$  if and only if it is a retract of an object  $M$  such that  $M$  sits inside a triangle  $P \rightarrow M \rightarrow Q \rightarrow \Sigma P$  with  $P \in \mathcal{P}_1 = \mathcal{P}$  and  $Q \in \mathcal{P}_{n-1}$ . For  $n = 0$ , we set  $\mathcal{P}_0$  to consist of all zero objects and  $\mathcal{I}^0$  to consist of all maps in  $\mathbb{T}$ . The sequence  $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \dots$  provides a filtration of the localising subcategory generated by  $\mathcal{P}$ .

In  $\text{StMod}(kG)$ , the ghosts form an ideal of a projective class  $(\mathcal{F}, \mathcal{G})$ , called the ghost projective class. Here  $\mathcal{G}$  consists of all ghosts in  $\text{StMod}(kG)$  and  $\mathcal{F}$  is generated by the trivial representation  $k$  by sums, retracts and suspensions. By assembling together the maps  $\Omega^{n_i} k \rightarrow M$  that represent the generators of  $\widehat{H}^*(G, M)$ , we form a map  $\bigoplus \Omega^{n_i} k \rightarrow M$  that is surjective on Tate cohomology. Hence the cofibre of  $\bigoplus \Omega^{n_i} k \rightarrow M$  is a universal ghost and  $(\mathcal{F}, \mathcal{G})$  is a projective class in  $\text{StMod}(kG)$ . Note that if the cohomology is periodic, then  $\bigoplus \Omega^{n_i} k$  can be chosen to be a finite sum and the universal ghost can be constructed within  $\text{stmod}(kG)$ . Note that if  $\mathcal{G}$  is the zero ideal, then the generating hypothesis holds. In general, the smallest integer  $n$  such that  $\mathcal{G}^n$  becomes zero provides a measurement of the failure of the generating hypothesis. We make this precise in the next section.

### 1.3.3 Ghost numbers

Since the generating hypothesis fails for  $\text{StMod}(kG)$  for most groups, we define the ghost number to measure the degree of its failure. For  $M \in \text{Thick}\langle k \rangle$ , the **ghost length** of  $M$  is the smallest integer  $n$ , such that every composite  $M \rightarrow M_1 \rightarrow \dots \rightarrow M_n$  of  $n$  ghosts in  $\text{Thick}\langle k \rangle$  is stably-trivial, and the **ghost number** of  $\text{StMod}(kG)$  is the upper bound of the ghost lengths of modules in  $\text{Thick}\langle k \rangle$ . With this terminology, the generating hypothesis holds on  $\text{StMod}(kG)$  if and only if the ghost number is 1.

Another closely related invariant, the generating number, is introduced similarly. For  $M \in \text{Thick}\langle k \rangle$ , the **generating length** of  $M$  is the smallest integer  $n$ , such that every composite  $M \rightarrow M_1 \rightarrow \dots \rightarrow M_n$  of  $n$  ghosts in  $\text{StMod}(kG)$  is stably-trivial, and the **generating number** of  $\text{StMod}(kG)$  is the upper bound of the generating lengths of modules in  $\text{Thick}\langle k \rangle$ . Clearly, the generating length is greater than or equal to the ghost length, and the same holds for the generating number and the ghost number.

Note that the generating length has better formal properties than the ghost length. Indeed, since  $(\mathcal{F}_n, \mathcal{G}^n)$  is a projective class in  $\text{StMod}(kG)$ , the generating length of  $M$

equals the smallest integer  $n$  such that  $M \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$ . See Section 2.3.4 for discussion on ghost length and generating length.

Computations of and bounds on ghost numbers and generating numbers of cyclic  $p$ -groups, abelian  $p$ -groups, and the quaternion group  $Q_8$  are given in [19]. It is also proved that the ghost number of a  $p$ -group  $G$  is always finite. The idea is to use the radical sequence of a module, which allows us to build up the module from  $k$  within finitely many steps, and the number of steps is universally bounded by the radical length of  $kG$ . It follows that the generating number, hence the ghost number of a  $p$ -group, is finite.

### 1.3.4 Strong ghosts

Another variation of the generating hypothesis is to consider strong ghosts. A map in  $\text{StMod}(kG)$  is called a **strong ghost** if its restriction to any subgroup  $H$  of  $G$  is a ghost. It follows from the results in [17] that every strong ghost in  $\text{stmod}(kG)$  is stably-trivial if and only if the Sylow  $p$ -subgroup  $P$  of  $G$  is  $C_2$ ,  $C_3$ , or  $C_4$ .

## 1.4 Results of the thesis

In this section, we summarize the main results of the thesis by chapters.

### 1.4.1 Ghost numbers of $p$ -groups

We continue the study of the ghost number of a  $p$ -group in Chapter 2 (which is based on [23]), improving on the results in [19]. We provide general bounds on ghost numbers as well as computations of ghost numbers of various  $p$ -groups.

#### 1.4.1.1 General lower bounds on the ghost number of a $p$ -group

For an Auslander-Reiten triangle  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$  in  $\text{Thick}\langle k \rangle$ , we show that if  $Z$  has generating length  $n$ , then  $\gamma$  is an  $(n-1)$ -fold ghost, i.e., the map  $\gamma$  is the longest possible composite of ghosts out  $Z$  (Proposition 2.3.6). This suggests that we can factorize  $\gamma$  as a composite of ghosts in  $\text{Thick}\langle k \rangle$  to find a lower bound on the ghost number.

For a  $p$ -group  $P$ , we consider the module  $M = M_{[p/2]} \uparrow_{C_p}^P$ , where  $C_p$  is a cyclic subgroup of order  $p$ , and  $M_{[p/2]}$  is the indecomposable  $C_p$ -module of radical length  $[p/2]$ . We find that the ghost length of  $M$  is equal to its radical length. As a result, we have Corollary 2.4.17, which says that, for a  $p$ -group  $P$ ,

$$\text{the ghost number of } kP \geq 1/3 \text{ the radical length of } kP,$$

and Proposition 2.4.33, which says that if  $P$  has order  $p^r$ ,

$$\text{the ghost number of } kP \geq (r-1)(p-1) + 1.$$

The inequalities recover that, for a  $p$ -group  $P$ , the generating hypothesis holds only if  $P = C_2$  or  $C_3$ . The results also apply to the quaternion group  $Q_8$  (Proposition 2.4.13) and the dihedral groups  $D_{4q}$  of order  $4q$  (Corollary 2.4.18), and improve the lower bounds on their ghost numbers shown in [19].

#### 1.4.1.2 The ghost numbers of $D_{4q}$ and $C_3 \times C_3$

We give an upper bound for the ghost number of a dihedral 2-group  $D_{4q}$ , with the aid of the classification theorem [4, Section 4.11]. This upper bound coincides with the lower bound we get in Corollary 2.4.18, so we can compute that (Corollary 2.4.25):

$$\text{the ghost number of } kD_{4q} = q + 1.$$

The classification of representations of the Kronecker Quiver is also applied to the study of the ghost number of  $C_3 \times C_3$ , and we have Theorem 2.4.28:

$$\text{the ghost number of } k(C_3 \times C_3) = 3.$$

#### 1.4.2 Ghost numbers of non- $p$ -groups

In Chapter 3 (which is based on [24]), we generalise the study of ghost numbers to arbitrary finite groups. We show that the ghost number is finite if  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$ , and compute the ghost numbers of various examples in this case. We also study simple ghosts and strong ghosts in Chapter 3.

### 1.4.2.1 Finiteness and lower bound of the ghost number of a non- $p$ -group

When the group  $G$  is a non- $p$ -group, we prove finiteness under the assumption that  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$ . Here  $\text{stmod}(B_0)$  is the full subcategory of  $\text{stmod}(kG)$  consisting of modules in the principal block  $B_0$ . It follows from the assumption that all the simple modules in the principal block have finite ghost lengths. Hence, given  $M \in \text{stmod}(B_0)$ , the semisimple modules that appear in the radical sequence of  $M$  have finite ghost lengths, and, like the  $p$ -group case, there is a *universal* upper bound for the ghost lengths of  $M \in \text{stmod}(B_0)$ . Hence the ghost number of  $kG$  is finite in this case (Theorem 3.4.7).

Now let  $e_0$  be the principal block idempotent of  $kG$ . Left multiplication by  $e_0$  provides a natural projection of  $\text{stmod}(kG)$  onto  $\text{stmod}(B_0)$ . Assuming  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$  again, the image of the functor  $e_0(-)$  lands inside  $\text{Thick}\langle k \rangle$ . We prove that the composite of functors  $e_0(-\uparrow) : \text{stmod}(kP) \rightarrow \text{stmod}(B_0)$  is faithful, where  $P$  is a Sylow  $p$ -subgroup of  $G$ . Since the functor  $e_0(-\uparrow)$  preserves ghosts, we get a lower bound for the ghost number of  $kG$  in this case (Proposition 3.4.10):

If  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$ , then

$$\text{the ghost number of } kG \geq \text{the ghost number of } kP.$$

### 1.4.2.2 Examples of ghost numbers of non- $p$ -groups

Recall that a map in  $\text{StMod}(kG)$  is a simple ghost if it is stably-trivial on any map from the suspensions of the simple modules. By comparing ghosts with simple ghosts, we get information about the ghost number. In the following examples:

1.  $G = A \times B$  is a direct product, with Sylow  $p$ -subgroup  $A$  (Corollary 3.4.2),
2. the Sylow  $p$ -subgroup  $P$  of  $G$  is cyclic and normal (Theorem 3.5.5), and
3. the dihedral group  $D_{2ql}$  of order  $2ql$  with  $l$  odd (Corollary 3.4.14),

we show that the simple modules in the principal block are suspensions of  $k$ . It follows that  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$  and, by Proposition 3.4.10, the ghost number of  $kG$  is greater than or equal to the ghost number of  $kP$ . On the other hand, we show that ghosts and



simple ghosts coincide in the principal block, so the *simple ghost number* provides an upper bound for the ghost number. Hence in these examples,

$$\text{the ghost number of } kG = \text{the ghost number of } kP.$$

### 1.4.2.3 Simple ghosts and the simple generating hypothesis

We have already seen that we need to consider simple ghosts in certain examples.

In Section 3.3.1, we show that if the Sylow  $p$ -subgroup  $P$  of  $G$  is normal, then the simple ghost number of  $kP$  is equal to the ghost number of  $kG$  (Theorem 3.3.2).

In Section 3.5.2, we prove that the simple generating hypothesis holds for the group  $SL(2, p)$  at any prime  $p$  (Theorem 3.5.9). This is an interesting result because the generating hypothesis fails for its Sylow  $p$ -subgroup, the cyclic group of order  $p$ , when  $p \geq 5$ .

### 1.4.2.4 Strong ghost numbers

Recall that a map  $M \rightarrow N$  is a strong ghost in  $\text{StMod}(kG)$  if its restriction to any subgroup  $H$  of  $G$  is a ghost. Observe that the map  $M \rightarrow N$  is a strong ghost if and only if the composite of maps  $X \rightarrow M \rightarrow N$  is stably-trivial for any  $X$  that is a suspension of the module  $k\uparrow_H^G$  for some subgroup  $H$  of  $G$ . Such test objects generate  $\text{StMod}(kG)$ . Hence, for  $M \in \text{stmod}(kG)$ , we define its **strong ghost length** to be the smallest integer  $n$ , such that every composite of  $n$  strong ghosts in  $\text{stmod}(kG)$  out of  $M$  is stably-trivial. The **strong ghost number** of  $kG$  can be defined similarly as its ghost number and simple ghost number. Unlike ghosts or simple ghosts, both the restriction and the induction functors preserve strong ghosts, so the strong ghost number of  $kG$  equals that of  $kP$  (Proposition 3.6.4), where  $P$  is the Sylow  $p$ -subgroup of  $G$ .

For cyclic  $p$ -groups other than  $C_2$ ,  $C_3$ , and  $C_4$ , whose strong ghost numbers are 1, we compute in Theorem 3.6.6 that

$$\text{strong ghost number of } kG = \lceil \frac{p+1}{2} \rceil, \text{ when } |G| \neq 2, 3, \text{ or } 4.$$

Combining this with our earlier results about dihedral groups, we get an upper bound for the strong ghost number of dihedral groups in Theorem 3.6.7:

$$\text{strong ghost number of } kD_{4q} \leq 3.$$

The computation of strong ghost numbers suggests that the concept of a strong ghost is much stronger than a ghost.

### 1.4.3 Computations with GAP

In Chapter 4, we show how to apply GAP to the study of the group algebra. GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory [26]. The GAP package ‘reps’ has been developed to handle group representations in positive characteristic. The overall structure of the reps package was designed and most of it is written by Peter Webb, who is also the maintainer. Contributions were made by Dan Christensen, Roland Loetscher, Robert Hank, Bryan Simpkins, Brad Froehle and others.

We have improved the code used in GAP to compute the universal ghost and ghost length. Recall that the universal ghost is the cofibre of a map that is surjective on Tate cohomology, and to compute the cofibre of a map, we need to replace it by an injection. The new `ReplaceWithInj` function is faster than the previous version and uses less memory. We explain the idea of the function and how to implement it in Section 4.4.1, and we show that the code has the resulting cofibre as small as possible. We also discuss the `Simple` function in the same section, which is used in the `ReplaceWithInj` function. Given an indecomposable projective module  $P$ , it computes the corresponding simple module of  $P$ . The functions are presented in pseudo-code.

The new function makes computations of cofibres and universal ghosts more efficient. This allows us to apply the idea of a universal ghost to find an upper bound of the ghost length. More precisely, given  $M \in \text{stmod}(kG)$ , we can compute the  $n$ -fold universal ghost out  $M$ , and if the map is stably-trivial, then the ghost number of  $M$  is at most  $n$ . We implement this in Section 4.4.2. Note that the Tate cohomology is in general not finitely-generated, so we can only compute *unstable universal ghosts* within a certain range, i.e., maps that are stably-trivial on maps from  $\Sigma^i k$ , with  $l \leq i \leq m$  for some integers  $l$  and  $m$ . Nevertheless, if the  $n$ -fold unstable universal ghost out of  $M$  is stably-trivial, then  $n$  is still an upper bound of the ghost length of  $M$ . And we can enlarge the

range  $[l, m]$ , and get a decreasing sequence of upper bounds, whose limit is exactly the ghost length (Proposition 4.3.2). In the case when  $G$  has periodic cohomology, then the Tate cohomology is finitely-generated and the computation of the ghost length becomes a finite process.

Then we apply the functions to compute various examples. We make some computations with the group  $S_3 \times C_3$ , the first example where  $\text{Thick}\langle k \rangle \neq \text{stmod}(B_0)$ . We also present some computations with the group  $Q_8$  and  $C_9$  in Section 4.5.

## Chapter 2

# Ghost numbers of group algebras

## 2.1 Introduction

In modular representation theory, the Tate cohomology functor plays a central role, analogous to the role that the homotopy groups play in homotopy theory. Thus it is natural to study the *kernel* of Tate cohomology, that is, the collection of maps which induce the zero map in Tate cohomology. These maps are called **ghosts**, and are the topic of the present paper.

Let  $G$  be a finite group, and let  $k$  be a field whose characteristic  $p$  divides the order of  $G$ . We write  $\text{StMod}(kG)$  for the stable module category of  $kG$ , the triangulated category formed from the module category by killing the projectives,  $\text{stmod}(kG)$  for the full subcategory of finitely generated modules, and  $\text{Thick}\langle k \rangle$  for the thick subcategory generated by the trivial representation, a full subcategory of  $\text{stmod}(kG)$ . (See Section 2.2 for complete definitions and further background.)

The generating hypothesis (GH) for the stable module category is the statement that if a map in  $\text{Thick}\langle k \rangle$  induces the zero map in Tate cohomology, then it is stably trivial. Using the terminology of the first paragraph, this is equivalent to saying that all ghosts in  $\text{Thick}\langle k \rangle$  are trivial. This problem is motivated by Freyd's famous conjecture in homotopy theory [25], which is still open.

By work of Benson, Carlson, Chebolu, Christensen and Mináč (Theorem 2.2.1 below), it is known that the generating hypothesis fails for most groups. The extent to which it fails is measured by the **ghost number** of  $kG$ , which is the smallest number  $n$  such that every composite of  $n$  ghosts in  $\text{Thick}\langle k \rangle$  is stably trivial. With this terminology, the generating hypothesis is the statement that the ghost number is one. The ghost number was studied for  $p$ -groups in [19], but even for  $p$ -groups it was found to be difficult to calculate, and in most cases only crude bounds are known. It is a long-term goal to understand whether this invariant has a simple description in terms of other invariants of  $kG$ .

In this chapter we develop new techniques for the study of ghost numbers and use them to make new computations in many cases. For example, we make the first computations of the ghost numbers of group algebras of wild representation type at an odd prime ( $k(C_3 \times C_3)$  and others mentioned in the detailed summary below) as well as the first computations of the ghost numbers of non-abelian group algebras (the dihedral 2-groups). We also give many new bounds on ghost numbers, including lower bounds,

which are generally difficult to come by. As one example, we show that the ghost number is always at least one-third of the radical length, the first general lower bound we are aware of. Our work includes results which are quite general, in some cases applying to any projective class in any triangulated category.

Chapter 3 builds on the work here in order to compute the ghost numbers of non- $p$ -groups. For example, using the results on dihedral 2-groups, we are able to compute the ghost number of an arbitrary dihedral group at the prime 2.

We now give a summary of the contents of the paper. We begin in Section 2.2.1 by reviewing the stable module category. In Section 2.2.2 we recall the statement of the generating hypothesis in this situation and state the result of Benson, Carlson, Chebolu, Christensen and Mináč that says that the GH fails unless the Sylow  $p$ -subgroup of  $G$  is  $C_2$  or  $C_3$ . The ghost number, which measures the degree to which the GH fails, is best studied using the idea of a projective class, so we introduce projective classes and their associated invariants in Section 2.2.3. Briefly, a projective class consists of a collection  $\mathcal{P}$  of objects (thought of as “projective” building blocks) and an ideal  $\mathcal{I}$  of morphisms (the maps invisible to the objects in  $\mathcal{P}$ ) satisfying some axioms.

In Section 2.3 we present a variety of new results, many of which hold for arbitrary projective classes in arbitrary triangulated categories. For example, in Section 2.3.1, we give new bounds on the length of an object in a triangle in terms of the lengths of the other two objects and the filtration of the connecting homomorphism in the powers of the ideal. Then, in Section 2.3.2, we show that the connecting map  $\gamma : Z \rightarrow \Sigma X$  in an Auslander-Reiten triangle, which we call the **almost zero map**, has a remarkable property: if  $(\mathcal{P}, \mathcal{I})$  is any projective class such that there is a nonzero map from  $Z$  in  $\mathcal{I}^k$ , then  $\gamma$  is in  $\mathcal{I}^k$ . So the almost zero map is in some sense a universal example of a non-zero map from  $Z$ . We specialize to the case of the stable module category in Section 2.3.3, where we show that the **heart** of an indecomposable module  $M$  (the fibre of the almost zero map) has length which differs by at most one from  $M$ , with respect to any projective class. We also show that this is true for any summand of the heart, by showing that the lengths of the domain and codomain of any irreducible map differ by at most one. We finish Section 2.3 with Section 2.3.4, which describes the extent to which our results hold for the ghost length, the invariant used in defining the ghost number.

Section 2.4 contains detailed computational results on the ghost numbers of  $p$ -groups. We begin by recalling some background results in Section 2.4.1, such as the fact that the ghost number of  $kG$  is less than the nilpotency index of the Jacobson radical, as well as the fact that multiplication by  $x - 1$ , where  $x$  is a central element of  $G$ , is always a ghost. In Section 2.4.2 we show that the generating length invariant is in a precise sense a stabilized version of the socle length, and show that if these are equal for a module  $M$ , the same is true for  $\text{rad}(M)$  and  $M/\text{soc}(M)$ . This follows from a general result involving nested unstable projective classes in a triangulated category. We begin our computations in Section 2.4.3, where we study the ghost numbers of abelian  $p$ -groups. The main result here is an improved lower bound on the ghost number. This follows from a result giving a lower bound on the ghost length of induced modules for general  $p$ -groups. We also compute the exact ghost length of many modules over abelian  $p$ -groups. In Section 2.4.4 we show that the ghost number for the quaternion group  $Q_8$  is 3 or 4, improving the existing lower bound by 1. In Section 2.4.5, we compute the ghost length and generating length of certain modules induced up from a cyclic normal subgroup of a  $p$ -group, generalizing the technique used for  $Q_8$ . This is used in the same section to show that the ghost number and the radical length are within a factor of three of each other for any  $p$ -group. More precisely, we show that  $(\text{rad len } kG)/3 \leq \text{ghost num } kG < \text{rad len } kG$  for  $p$  odd, the first general lower bound we are aware of. For  $p = 2$ , the factor of 3 is replaced with a factor of 2. We also use the induction result in Section 2.4.6, where we show that the ghost number of the dihedral 2-group  $D_{4q}$  of order  $4q$  is exactly  $q + 1$ . This is the longest section of the paper. That the ghost length is at least  $q + 1$  follows immediately from the induction result of the previous section, but that it is no more than  $q + 1$  requires using the classification of  $kD_{4q}$ -modules. In Section 2.4.7 we show that the ghost number of  $k(C_3 \times C_3)$  is exactly 3. While  $k(C_3 \times C_3)$ -modules are not classifiable, we make use of the fact that certain quotients can be classified. Our argument also shows that the ghost number of the group algebra  $k(C_{p^r} \times C_{p^s})$ , for  $p^r, p^s > 2$ , is at most  $p^r + p^s - 3$ . It follows that the ghost number of  $k(C_3 \times C_{3^s})$  is  $3^s$  and that the ghost number of  $k(C_4 \times C_{2^s})$  is  $2^s + 1$ . We end the paper with Section 2.4.8, in which we give complete lists of the group algebras of  $p$ -groups with ghost numbers 1, 2 or 3, with the possible exception of  $kQ_8$ . We also prove that for each prime  $p$  there are gaps in the possible ghost numbers that can occur, and state a conjecture related to this.

## 2.2 The generating hypothesis and the ghost projective class

In this section, we recall background material which provides context to our results and which we use in our proofs.

### 2.2.1 The stable module category

Here we recall the basics of the stable module category. A good reference is [14].

Let  $G$  be a finite group, and let  $k$  be a field whose characteristic  $p$  divides the order of  $G$ . The **stable module category**  $\text{StMod}(kG)$  is a quotient category of the category  $\text{Mod}(kG)$  of left  $kG$ -modules by the ideal of maps that factor through a projective. Thus the objects of  $\text{StMod}(kG)$  are left  $kG$ -modules and the hom-sets are  $\underline{\text{Hom}}(M, N) = [M, N] := \text{Hom}(M, N)/\text{PHom}(M, N)$ , where  $\text{PHom}(M, N)$  denotes the stably trivial maps, i.e., those that factor through a projective module. Two modules  $M$  and  $N$  are isomorphic in the stable module category if and only if they have the same projective-free summands. In particular, projective modules are isomorphic to zero in the stable module category. We write  $\text{stmod}(kG)$  for the full subcategory of finitely generated  $kG$ -modules in  $\text{StMod}(kG)$ . (More precisely, we include all modules which are *stably isomorphic* to finitely generated  $kG$ -modules.)

The stable module category is a triangulated category. The desuspension  $\Omega M$  of a module  $M$  is the kernel of any surjection  $P \rightarrow M$  with  $P$  projective. This is well-defined in the stable module category by Schanuel's Lemma [14, Prop. 4.2], and we write  $\tilde{\Omega}M$  for the projective-free summand of  $\Omega M$ .

The group algebra  $kG$  is injective as a module over itself. In particular, this implies that projective modules and injective modules coincide in  $\text{mod}(kG)$ . The suspension  $\Sigma N$  of a module  $N$  is defined to be the cokernel of any injection  $N \rightarrow P$  with  $P$  injective. We will often write  $\Omega^{-1}N$  for  $\Sigma N$  since  $\Omega$  and  $\Sigma$  are inverse functors up to natural isomorphism.

Write  $k$  for the trivial representation and  $\text{Thick}\langle k \rangle$  for the thick subcategory generated by  $k$ , the smallest full triangulated subcategory of  $\text{StMod}(kG)$  that is closed under retracts and contains  $k$ . This is in fact a full subcategory of  $\text{stmod}(kG)$ , and plays a



central role in our formulation of the generating hypothesis. The localizing category generated by  $k$ , denoted  $\text{Loc}\langle k \rangle$ , is the smallest full triangulated subcategory of  $\text{StMod}(kG)$  that is closed under arbitrary coproducts and retracts and contains  $k$ .

### 2.2.2 The generating hypothesis

An important feature of the stable module category is that the Tate cohomology of a  $kG$ -module  $M$  is representable, i.e., we have a canonical isomorphism  $\hat{H}^n(G, M) \cong [\Omega^n k, M]$ .

We say that the **generating hypothesis (GH)** holds for the stable module category  $\text{StMod}(kG)$  if and only if the Tate cohomology functor  $\hat{H}^*(G, -)$  restricted to  $\text{Thick}\langle k \rangle$  is faithful. It has been shown that the GH fails for most group algebras [9, 16, 18, 20].

**Theorem 2.2.1** (Benson, Carlson, Chebolu, Christensen and Mináč). *Let  $G$  be a finite group, and let  $k$  be a field whose characteristic  $p$  divides the order of  $G$ . Then the GH holds for  $\text{StMod}(kG)$  if and only if the Sylow  $p$ -subgroup  $P$  of  $G$  is either  $C_2$  or  $C_3$ .*

It is worth pointing out here why we restrict to  $\text{Thick}\langle k \rangle$ . It is known that whenever the thick subcategory is not all of  $\text{stmod}(kG)$ , there are non-projective modules whose Tate cohomology is zero. The identity map on such a module is sent to zero by  $\hat{H}^*(G, -)$ , so the GH would be trivially false if we included such modules. Restricting to  $\text{Thick}\langle k \rangle$  prevents this from happening. In general, the stable module category is generated by the simple modules as a triangulated category. For a  $p$ -group  $G$ , the trivial representation  $k$  is the only simple module, so we have that  $\text{Thick}\langle k \rangle = \text{stmod}(kG)$  in this case.

We call a map in  $\text{StMod}(kG)$  that is in the kernel of the Tate cohomology functor a **ghost**. Thus the GH is the statement that all ghosts in  $\text{Thick}\langle k \rangle$  are stably trivial. When the GH fails, the vanishing of composites of ghosts gives a measure of the failure and leads to invariants of modules and of  $kG$ . This is formalized in the idea of a projective class.

### 2.2.3 The ghost projective class

**Definition 2.2.2.** Let  $\mathbb{T}$  be a triangulated category. A **projective class** in  $\mathbb{T}$  consists of a class  $\mathcal{P}$  of objects of  $\mathbb{T}$  and a class  $\mathcal{I}$  of morphisms of  $\mathbb{T}$  such that:

- (i)  $\mathcal{P}$  consists of exactly the objects  $P$  such that every composite  $P \rightarrow X \rightarrow Y$  is zero for each  $X \rightarrow Y$  in  $\mathcal{I}$ ,
- (ii)  $\mathcal{I}$  consists of exactly the maps  $X \rightarrow Y$  such that every composite  $P \rightarrow X \rightarrow Y$  is zero for each  $P$  in  $\mathcal{P}$ .
- (iii) for each  $X$  in  $\mathbb{T}$ , there is a cofibre sequence  $P \rightarrow X \rightarrow Y$  with  $P$  in  $\mathcal{P}$  and  $X \rightarrow Y$  in  $\mathcal{I}$ .

In this paper, we make the additional assumption that the projective class is **stable**, that is, that  $\mathcal{P}$  (or equivalently  $\mathcal{I}$ ) is closed under suspension and desuspension. With slight alterations, most of our results remain true without this assumption, but the extra bookkeeping complicates the arguments. The one exception is that in Section 2.4.2 we make use of an unstable projective class.

*Remark 2.2.3.* It follows from the definition that  $\mathcal{P}$  is closed under arbitrary coproducts and retracts, and  $\mathcal{I}$  is an ideal.

We write  $\mathcal{G}$  for the ideal of ghosts in the stable module category, and  $\mathcal{F}$  for all retracts of direct sums of suspensions of  $k$  in  $\text{StMod}(kG)$ . For a module  $M \in \text{StMod}(kG)$ , since  $\hat{H}^n(G, M) \cong [\Omega^n k, M]$ , we can form a map  $\oplus \Omega^i k \rightarrow M$  that is surjective on Tate cohomology by assembling sufficiently many homogeneous elements in  $\hat{H}^*(G, M)$ . Completing this map into a triangle in  $\text{StMod}(kG)$

$$\Omega U_M \rightarrow \oplus \Omega^i k \rightarrow M \xrightarrow{\phi_M} U_M, \quad (2.2.1)$$

we get a ghost  $\phi_M : M \rightarrow U_M$ . The map  $\phi_M$  is a (weakly) universal ghost in the sense that every ghost out of  $M$  factors through it, but the factorization is not necessarily unique. It follows easily that  $(\mathcal{F}, \mathcal{G})$  forms a projective class in  $\text{StMod}(kG)$ . This is called the **ghost projective class**.

While the ghost projective class is the focus of this paper, some of our results apply to any projective class, so we mention two other examples at this point: The **simple ghost projective class** is the projective class whose projectives are generated by all simple objects, and it was proposed for study in [12] as a way to avoid focusing on  $\text{Thick}\langle k \rangle$ . And the **strong ghost projective class** is the projective class whose ideal consists of the maps which are ghosts under restriction to every subgroup. (See [17] for more on this topic.)

For any projective class  $(\mathcal{P}, \mathcal{I})$ , there is a sequence of **derived projective classes**  $(\mathcal{P}_n, \mathcal{I}^n)$  [21]. The ideal  $\mathcal{I}^n$  consists of all  $n$ -fold composites of maps in  $\mathcal{I}$ , and  $X$  is in  $\mathcal{P}_n$  if and only if it is a retract of an object  $M$  that sits inside a cofibre sequence  $P \rightarrow M \rightarrow Q$  with  $P \in \mathcal{P}_1 = \mathcal{P}$  and  $Q \in \mathcal{P}_{n-1}$ . For  $n = 0$ , we let  $\mathcal{P}_0$  consist of all zero objects and  $\mathcal{I}^0$  consist of all maps in  $\mathbb{T}$ . The **length**  $\text{len}_{\mathcal{P}}(X)$  of an object  $X$  of  $\mathbb{T}$  with respect to  $(\mathcal{P}, \mathcal{I})$  is the smallest  $n$  such that  $X$  is in  $\mathcal{P}_n$ , if this exists. The fact that each pair  $(\mathcal{P}_n, \mathcal{I}^n)$  is a projective class implies that the length of  $X$  is equal to the smallest  $n$  such that every map in  $\mathcal{I}^n$  with domain  $X$  is trivial.

The length of a module  $M$  with respect to the ghost projective class is called the **generating length** of  $M$ , and this exists when  $M$  is in  $\text{Thick}\langle k \rangle$ . But since we are interested in the collection  $\mathcal{G}_t$  of ghosts in  $\text{Thick}\langle k \rangle$ , we also get another invariant. We describe both invariants, and the associated invariants of  $kG$ , in the following definition, generalizing the definition given in [19] for  $p$ -groups.

**Definition 2.2.4.**

- The **generating length**  $\text{gel}(M)$  of  $M \in \text{Thick}\langle k \rangle$  is the smallest  $n$  such that  $M \in \mathcal{F}_n$ . That is,  $\text{gel}(M) = \text{len}_{\mathcal{F}}(M)$ .
- The **ghost length**  $\text{gl}(M)$  of  $M \in \text{Thick}\langle k \rangle$  is the smallest integer  $n$  such that every map in  $(\mathcal{G}_t)^n$  with domain  $M$  is trivial.
- The **generating number** of  $kG$  is the least upper bound of the generating lengths of modules in  $\text{Thick}\langle k \rangle$ .
- The **ghost number** of  $kG$  is the least upper bound of the ghost lengths of modules in  $\text{Thick}\langle k \rangle$ .

With this terminology, the generating hypothesis is the statement that the ghost number of  $kG$  is 1.

Let  $M$  be in  $\text{Thick}\langle k \rangle$ . Since each  $(\mathcal{F}_n, \mathcal{G}^n)$  is a projective class and  $(\mathcal{G}_t)^n \subseteq \mathcal{G}^n$ , it follows that

$$\text{gl}(M) \leq \text{gel}(M)$$

and therefore that

$$\text{ghost number of } kG \leq \text{generating number of } kG.$$

When  $G$  has periodic Tate cohomology, the coproduct in (2.2.1) can be taken to be finite, and it follows that the ghost projective class restricts to a projective class in  $\text{Thick}\langle k \rangle$  [19]. This implies that equality holds in this case. We don't know whether equality holds in general, except for the trivial observation that  $M \cong 0$  if and only if  $\text{gl}(M) = 0$  if and only if  $\text{gl}(M) = 0$  and the less trivial fact that  $\text{gl}(M) = 1$  if and only if  $\text{gl}(M) = 1$  (see Corollary 2.3.7 or [16]). Thus the GH is equivalent to the generating number of  $kG$  being 1. See Remark 2.3.13 for further discussion of whether ghost length equals generating length.

## 2.3 Auslander-Reiten triangles and generating lengths

In this section, we explain how Auslander-Reiten triangles (in short, A-R triangles) provide examples of ghosts, and, more generally, of non-trivial maps in  $\mathcal{T}^n$  for  $n$  as large as possible, for any projective class  $(\mathcal{P}, \mathcal{I})$ . This extends the work of [16], where these triangles are called “almost split sequences.” Because we have in mind applications to other projective classes, in this section we state many of our results for a general projective class in a general triangulated category.

In Section 2.3.1, we give results about the relationship between the lengths of the objects in a triangle when one of the maps is in a power  $\mathcal{I}^m$  of the ideal. In Section 2.3.2, we recall A-R triangles and prove that the third map in an A-R triangle is the longest possible non-trivial composite of maps in  $\mathcal{I}$  with the given domain. In Section 2.3.3, we apply these results to the study of lengths in the stable module category, and also show a close relationship between lengths and irreducible maps. Finally, in Section 2.3.4 we explain the extent to which our results on generating length are true for ghost length.

### 2.3.1 Relations between the lengths of objects in a triangle

Consider a projective class  $(\mathcal{P}, \mathcal{I})$  in a triangulated category  $\mathbb{T}$ . Let

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

be a triangle in  $\mathbb{T}$ , where  $X$ ,  $Y$  and  $Z$  have finite lengths  $k$ ,  $n$  and  $l$ , respectively. We know that  $n \leq k + l$  [21]. Rotating the triangle, we also get  $l \leq n + k$  and  $k \leq n + l$ .

Here we show that when  $\gamma$  is in  $\mathcal{I}^m$ , one can refine these inequalities by subtracting  $m$  from  $l$ . Our methods also show that  $n \geq m$ . Note that  $\mathcal{I}^0$  consists of all maps in  $\mathbb{T}$ .

**Lemma 2.3.1.** *Let  $(\mathcal{P}, \mathcal{I})$  be a projective class in a triangulated category  $\mathbb{T}$ , and let*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

*be a triangle in  $\mathbb{T}$ , where  $X$ ,  $Y$  and  $Z$  have finite lengths  $k$ ,  $n$  and  $l$ , respectively, and  $\gamma \in \mathcal{I}^m$  with  $m \leq l$ . Then*

$$\text{len}_{\mathcal{P}}(Y) = n \leq \max(k - m + l, l).$$

Note that if  $m \geq l$ , then  $\gamma$  must be zero, and so the restriction to  $m \leq l$  is natural. When  $m = l$ , the triangle splits, and the lemma says that  $n \leq \max(k, l)$ .

*Proof.* Let  $n' = \max(k, m)$ , and let  $\phi : Y \rightarrow W$  be in  $\mathcal{I}^{n'}$ . Then  $\phi \circ \alpha$  is zero (since  $n' \geq k$ ), so  $\phi$  factors through a map  $\tilde{\phi} : Z \rightarrow W$ . We claim that  $\tilde{\phi}$  is in  $\mathcal{I}^m$ . Consider the diagram

$$\begin{array}{ccccc} & & & V & \\ & & & \downarrow \psi & \\ X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ & & \downarrow \phi & & \downarrow \tilde{\phi} & & \\ & & W & & & & \end{array}$$

with  $\psi : V \rightarrow Z$  being any map from an object  $V \in \mathcal{P}_m$ . Now  $\gamma \in \mathcal{I}^m$ , so  $\gamma \circ \psi$  is zero, and  $\psi$  factors through some map  $\tilde{\psi} : V \rightarrow Y$ . Hence  $\tilde{\phi} \circ \psi = \phi \circ \tilde{\psi}$  is zero (since  $n' \geq m$ ), and the claim follows. If  $g : W \rightarrow W'$  is in  $\mathcal{I}^{l-m}$ , then  $g \circ \tilde{\phi}$  is zero because  $Z$  has length  $l$ . Then  $g \circ \phi$  is zero, meaning that the length of  $Y$  is at most  $n' + l - m$ .  $\square$

**Lemma 2.3.2.** *Let  $(\mathcal{P}, \mathcal{I})$  be a projective class in a triangulated category  $\mathbb{T}$ , and let*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

*be a triangle in  $\mathbb{T}$ , where  $X$ ,  $Y$  and  $Z$  have finite lengths  $k$ ,  $n$  and  $l$ , respectively, and  $\gamma \in \mathcal{I}^m$  with  $m \leq l$ . Then*

$$\text{len}_{\mathcal{P}}(Y) = n \geq \max(k - l + m, m).$$

When  $m = l$ , this says that  $n \geq \max(k, l)$ , so the two lemmas together recover the fact that when the triangle splits,  $n = \max(k, l)$ .

*Proof.* We prove that the length of  $Y$  is at least  $k - l + m$ . The other inequality can be proved similarly.

Consider a map  $\phi : X \rightarrow W$  in  $\mathcal{T}^{l-m}$ . Since  $\phi \circ \Sigma^{-1}\gamma$  is in  $\mathcal{T}^l$  and has domain  $\Sigma^{-1}Z$  of length  $l$ , it is zero and  $\phi$  factors through a map  $\tilde{\phi} : Y \rightarrow W$ :

$$\begin{array}{ccccccc} \Sigma^{-1}Z & \xrightarrow{\Sigma^{-1}\gamma} & X & \longrightarrow & Y & \longrightarrow & Z \\ & & \downarrow \phi & \swarrow \tilde{\phi} & & & \\ & & W & & & & \end{array} .$$

Let  $g : W \rightarrow W'$  be in  $\mathcal{T}^n$ . Then  $g \circ \tilde{\phi}$  is zero because  $Y$  has length  $n$ , hence any map in  $\mathcal{T}^{n+l-m}$  with domain  $X$  is zero. This implies that  $k \leq n + l - m$ , i.e., that  $n \geq k - l + m$ . □

### 2.3.2 Auslander-Reiten triangles give composites of ghosts

We begin by recalling the definition.

**Definition 2.3.3.** Let  $\mathbb{T}$  be a triangulated category. A triangle  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$  is called an **Auslander-Reiten triangle**, if

- (a)  $\gamma \neq 0$ ,
- (b) any map  $X \rightarrow Y'$  that is not split monic factors through  $\alpha$ ,
- (c) any map  $Y' \rightarrow Z$  that is not split epic factors through  $\beta$ .

A map  $\alpha$  that is not split monic and satisfies (b) is said to be **left almost split**. Dually, a map  $\beta$  that is not split epic and satisfies (c) is said to be **right almost split**.

We know that Auslander-Reiten triangles exist in great generality.

**Theorem 2.3.4** (Krause, [30]). *Let  $\mathbb{T}$  be a triangulated category with all small coproducts, and suppose that all cohomological functors are representable. Let  $Z$  be a compact object in  $\mathbb{T}$  with local endomorphism ring. Then there exists an Auslander-Reiten triangle*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X.$$

*The triangle is unique up to a non-canonical isomorphism.* □

*Remark 2.3.5.* Let  $\beta$  be the second map in the A-R triangle above. One can show that, for any endomorphism  $g$  of  $Y$  with  $\beta g = \beta$ , the map  $g$  is an isomorphism (see [30]). We say that the map  $\beta$  is **right minimal** in this case. Dually, the first map  $\alpha$  in an A-R triangle is **left minimal**. A map  $\beta$  that is right almost split sits inside an Auslander-Reiten triangle if and only if it is right minimal [30].

For convenience, we call the map  $\gamma$  here the **almost zero map** with domain  $Z$ . It is unique up to an automorphism of  $\Sigma X$ . The following proposition follows from the definitions and the earlier lemmas.

**Proposition 2.3.6.** *Suppose that  $(\mathcal{P}, \mathcal{I})$  is a projective class on a triangulated category  $\mathbb{T}$ , and that*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

*is a distinguished triangle with  $\beta$  right almost split. If  $Z$  has finite length  $l$  and  $X$  has finite length  $k$  with respect to  $(\mathcal{P}, \mathcal{I})$ , then the third map  $\gamma$  is in  $\mathcal{I}^{l-1}$ , and*

$$\begin{aligned} k - 1 &\leq \text{len}_{\mathcal{P}}(Y) \leq k + 1, && \text{if } k \geq l; \\ l - 1 &\leq \text{len}_{\mathcal{P}}(Y) \leq l, && \text{if } k \leq l - 1. \end{aligned}$$

*For any summand  $S$  of  $Y$ ,  $\text{len}_{\mathcal{P}}(S) \leq \max(k + 1, l)$ .*

*Proof.* We test  $\gamma$  on all objects  $W$  in  $\mathcal{P}_{l-1}$ . Because  $Z$  has larger length than  $W$ , a map  $\phi : W \rightarrow Z$  cannot be split epic, so it factors through  $\beta$ . Hence  $\gamma \circ \phi$  is zero, which implies that  $\gamma \in \mathcal{I}^{l-1}$ .

The inequalities follow from Lemmas 2.3.1 and Lemma 2.3.2, with  $m = l - 1$ . The statement about the summand  $S$  follows immediately. □

Note in particular that for any A-R triangle, the almost zero map  $\gamma$  is an example of a non-zero map in the largest possible power of the ideal, for *any* projective class.

In the case when  $\mathbb{T}$  is  $\text{StMod}(kG)$  with  $G$  being a  $p$ -group, we know that ghosts and *dual ghosts* coincide [19]. Hence  $\gamma$  non-zero implies that  $k \geq l$ , and so we are in the first case of Proposition 2.3.6.

In the next section, we develop these ideas further.

### 2.3.3 Auslander-Reiten triangles, irreducible maps and lengths

The category  $\mathbf{StMod}(kG)$  satisfies the hypotheses on  $\mathbb{T}$  in Theorem 2.3.4, and its compact objects are precisely those in  $\mathbf{stmod}(kG)$ . For projective-free  $M \in \mathbf{stmod}(kG)$ , the stable endomorphism ring  $\underline{\mathbf{End}}(M)$  being local is equivalent to  $M$  being indecomposable. In this case, the Auslander-Reiten triangle has the form [4, 4.12.8]

$$\Omega^2 M \xrightarrow{\alpha} H(M) \xrightarrow{\beta} M \xrightarrow{\gamma} \Omega M.$$

The module  $H(M)$  is called the **heart** of  $M$ , and the triangle shows that it is also in  $\mathbf{stmod}(kG)$ .

The general theory we have set up in the last two sections applies to an A-R triangle for any projective class  $(\mathcal{P}, \mathcal{I})$  on  $\mathbf{StMod}(kG)$ . As a special case of Proposition 2.3.6, using that  $k = l$  in this case, we get

**Corollary 2.3.7.** *Let  $G$  be a finite group, let  $k$  be a field whose characteristic divides the order of  $G$ , and let  $(\mathcal{P}, \mathcal{I})$  be a projective class on  $\mathbf{StMod}(kG)$ . Consider the Auslander-Reiten triangle  $\Omega^2 M \xrightarrow{\alpha} H(M) \xrightarrow{\beta} M \xrightarrow{\gamma} \Omega M$  for some indecomposable non-projective module  $M$  in  $\mathbf{stmod}(kG)$  with finite length  $l$  with respect to  $(\mathcal{P}, \mathcal{I})$ . Then*

$$\text{len}_{\mathcal{P}}(M) - 1 \leq \text{len}_{\mathcal{P}}(H(M)) \leq \text{len}_{\mathcal{P}}(M) + 1,$$

and  $\gamma$  is a non-trivial map in  $\mathcal{I}^{l-1}$ . □

As above, we emphasize again that the same map  $\gamma : M \rightarrow \Omega M$  provides a map in  $\mathcal{I}^n$  with  $n$  maximal for *any* projective class  $(\mathcal{P}, \mathcal{I})$ . Put another way,  $\gamma$  is in the intersection of all projective class ideals that contain a non-trivial map from  $M$ .

*Remark 2.3.8.* One might hope that the heart  $H(M)$  always has larger generating length than  $M$  when  $\text{gel}(M)$  is less than the generating number of  $kG$ , but unfortunately this is not true in general. For example, take  $G = C_5 \times C_5$  and  $M = k\uparrow_{C_5}^G$ . One can compute that  $\text{gel}(M) = \text{gel}(H(M)) = 5$ , while the generating number of  $kG$  is at least 6 (Theorem 2.4.9).

Let  $S$  be an indecomposable non-projective summand of  $H(M)$ . Then, clearly,  $\text{len}_{\mathcal{P}}(S) \leq \text{len}_{\mathcal{P}}(H(M)) \leq \text{len}_{\mathcal{P}}(M) + 1$ . We will show below that  $\text{len}_{\mathcal{P}}(M) - 1 \leq \text{len}_{\mathcal{P}}(S)$  because of the right minimality of the map  $\beta$ .

We first need the notion of irreducible map.



**Definition 2.3.9.** Let  $G$  be a finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . A map  $\lambda : M \rightarrow N$  in  $\mathbf{StMod}(kG)$  is said to be **irreducible** if it is not split monic or split epic, and for any factorization  $\lambda = \nu \circ \mu$ , either  $\mu$  is split monic or  $\nu$  is split epic.

Irreducible maps are closely related to Auslander-Reiten triangles:

**Proposition 2.3.10** (Auslander and Reiten [2]). *Let  $M$  and  $N$  be indecomposable non-projective modules in  $\mathbf{stmod}(kG)$ . Then a map  $f : M \rightarrow N$  is irreducible if and only if the following equivalent conditions are satisfied:*

- (a)  $M$  is a summand of  $H(N)$  and  $f$  is the composite  $M \rightarrow H(N) \xrightarrow{\beta} N$ .
- (b)  $N$  is a summand of  $\Omega^{-2}H(M)$  and  $f$  is the composite  $M \xrightarrow{\Omega^{-2}\alpha} \Omega^{-2}H(M) \rightarrow N$ .  $\square$

Combining Corollary 2.3.7 and Proposition 2.3.10, one can prove

**Corollary 2.3.11.** *Let  $f : M \rightarrow N$  be an irreducible map with  $M$  and  $N$  non-projective indecomposables in  $\mathbf{stmod}(kG)$ , and let  $(\mathcal{P}, \mathcal{I})$  be a projective class on  $\mathbf{StMod}(kG)$ . If  $M$  and  $N$  have finite lengths with respect to  $(\mathcal{P}, \mathcal{I})$ , then*

$$\text{len}_{\mathcal{P}}(M) - 1 \leq \text{len}_{\mathcal{P}}(N) \leq \text{len}_{\mathcal{P}}(M) + 1.$$

*In particular, for  $M$  indecomposable and  $S$  any summand of  $H(M)$ , we have*

$$\text{len}_{\mathcal{P}}(M) - 1 \leq \text{len}_{\mathcal{P}}(S) \leq \text{len}_{\mathcal{P}}(M) + 1. \quad \square$$

### 2.3.4 Ghost lengths

The results of Sections 2.3.1 to 2.3.3 apply to the generating length of a module in  $\mathbf{StMod}(kG)$ , since generating length is the length with respect to the ghost projective class. When  $kG$  has periodic cohomology, there is a projective class on  $\mathbf{Thick}(k)$  whose ideal is  $\mathcal{G}_t$ , and ghost length is the length with respect to this projective class. In general, we don't know whether ghost length is a length with respect to a projective class, but we can still prove the analogue of half of Corollary 2.3.7:

**Proposition 2.3.12.** *Let  $G$  be a finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . Consider the Auslander-Reiten triangle  $\Omega^2 M \rightarrow H(M) \rightarrow M \rightarrow$*

$\Omega M$  for some indecomposable module  $M$  in  $\text{Thick}\langle k \rangle$ . Then the following holds:

$$\text{gl}(M) - 1 \leq \text{gl}(H(M))$$

*Proof.* We mimic the proof of Lemma 2.3.2. Suppose that  $\text{gl}(H(M)) = l - 1$ . We must prove that  $\text{gl}(M) \leq l$ . Since  $\text{gl}(M) = \text{gl}(\Omega^2 M)$ , it suffices to show that any map  $\phi : \Omega^2 M \rightarrow N$  in  $(\mathcal{G}_t)^l$  is stably trivial, where  $\mathcal{G}_t$  consists of ghosts between objects in  $\text{Thick}\langle k \rangle$ . Write  $\phi$  as  $\phi_2 \phi_1$ , where  $\phi_1$  is in  $\mathcal{G}_t$  and  $\phi_2$  is in  $(\mathcal{G}_t)^{l-1}$ . Then, by Proposition 2.3.6, the composite  $\phi_1 \Omega \gamma$  is stably trivial, so  $\phi_1$  factors through  $H(M)$ :

$$\begin{array}{ccccccc} \Omega M & \xrightarrow{\Omega \gamma} & \Omega^2 M & \longrightarrow & H(M) & \longrightarrow & M \xrightarrow{\gamma} \Omega M \\ & & \downarrow \phi_1 & \swarrow \psi & & & \\ & & W & & & & \\ & & \downarrow \phi_2 & & & & \\ & & N & & & & \end{array}$$

Now since  $\text{gl}(H(M)) = l - 1$ , the composite  $\phi_2 \psi$  is stably trivial and so  $\phi$  is stably trivial as well. □

The analogue of the other half of Corollary 2.3.7 would say that  $\text{gl}(H(M)) \leq \text{gl}(M) + 1$ , and we don't know whether this is true.

*Remark 2.3.13.* A related question is whether the generating length and ghost length always agree. We know of no counterexamples. However, Corollary 2.3.7 implies that the longest composite of ghosts starting from a given module  $M$  in  $\text{Thick}\langle k \rangle$  can always be attained by a map in  $(\mathcal{G}^m)_t$ , the intersection of  $\mathcal{G}^m$  and  $\text{Thick}\langle k \rangle$ . Thus if  $(\mathcal{G}_t)^m = (\mathcal{G}^m)_t$ , then the ghost length and generating length agree. Note that a related statement for the objects of  $\mathcal{P}$ , i.e., that  $(\mathcal{P}^c)_n = (\mathcal{P}_n)^c$ , where the superscript  $c$  means to take the intersection with the compact objects, is known to be true [13, 2.2.4].

## 2.4 Ghost numbers of $p$ -groups

In this section we study finite  $p$ -groups, using the fact that  $\text{Thick}\langle k \rangle = \text{stmod}(kG)$ . We begin in Section 2.4.1 by recalling several results that we will use. In Section 2.4.2 we show that the generating length invariant is a stabilized version of the socle length, and give a result that shows that if these are equal for a module  $M$ , the same is true for

$\text{rad}(M)$  and  $M/\text{soc}(M)$ . Then we give new computations of bounds on ghost numbers for various  $p$ -groups: abelian  $p$ -groups in Section 2.4.3, the quaternion group  $Q_8$  in Section 2.4.4, dihedral 2-groups in Section 2.4.6, and the groups  $C_{p^r} \times C_{p^s}$  in Section 2.4.7. In several cases we determine the ghost number completely, such as for  $D_{4q}$ ,  $C_3 \times C_{3^s}$  and  $C_4 \times C_{2^s}$ . In Section 2.4.5, we compute the ghost length and generating length of certain modules induced up from a cyclic normal subgroup. This is used in the same section to show that the ghost number and the radical length are within a factor of three of each other for any  $p$ -group. It is also used in Section 2.4.6 in the computation of the ghost number of  $kD_{4q}$  and in Section 2.4.8, where we classify group algebras with small ghost number and put constraints on which ghost numbers can occur.

When we write “ $p$ -group”, we always mean “finite  $p$ -group”.

### 2.4.1 Background

We recall the following theorem, and then explain the terminology and give an idea of the proof.

**Theorem 2.4.1** (Chebolu, Christensen and Mináč [19]). *Let  $G$  be a  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Then the generating length of a  $kG$ -module  $M$  is at most its radical length, and the following inequalities hold:*

$$\text{ghost number of } kG \leq \text{generating number of } kG < \text{nilpotency index of } J(kG) \leq |G|.$$

*In particular, the ghost number of  $kG$  is finite in this case.* □

Let  $G$  be any finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . Let  $J = J(kG)$  be the Jacobson radical of  $kG$ , i.e., the largest nilpotent ideal of  $kG$ . The nilpotency index of  $J(kG)$  is the smallest integer  $m$  such that  $J^m = 0$ , and for any module  $M$ , we have a radical series

$$M = \text{rad}^0(M) \supseteq \text{rad}^1(M) \supseteq \text{rad}^2(M) \supseteq \cdots \supseteq 0,$$

with  $\text{rad}^n(M) = J^n M$ , and a socle series

$$0 = \text{soc}^0(M) \subseteq \text{soc}^1(M) \subseteq \text{soc}^2(M) \subseteq \cdots \subseteq M,$$

with  $\text{soc}^n(M)$  consisting of the elements of  $M$  annihilated by  $J^n$ . The radical length of  $M$  is the smallest integer  $n$  such that  $\text{rad}^n(M) = 0$ . This is equal to the socle length of  $M$ , the smallest integer  $m$  such that  $\text{soc}^m(M) = M$ . The successive quotients in the sequences are direct sums of simple modules.

If  $G$  is a  $p$ -group, then each quotient is a direct sum of  $k$ 's, so the generating length of a module  $M$  is less than or equal to its radical length. Note that the nilpotency index of  $J(kG)$  is exactly the radical length of  $kG$ , and if  $M$  is a projective-free  $kG$ -module, it always has smaller radical length than  $kG$ . The theorem then follows.

The following lemma is proved by studying Tate cohomology in degrees 0 and  $-1$ . We write  $\text{rad}(M)$  for  $\text{rad}^1(M)$  and  $\text{soc}(M)$  for  $\text{soc}^1(M)$ .

**Lemma 2.4.2** (Chebolu, Christensen and Mináč [19]). *Let  $G$  be a  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Let  $f : M \rightarrow N$  be a map in  $\text{Mod}(kG)$  between projective-free modules  $M$  and  $N$ . Then:*

$$(a) \text{soc}(M) \subseteq \ker(f) \text{ iff } [k, f] = 0.$$

$$(b) \text{im}(f) \subseteq \text{rad}(N) \text{ iff } [\Omega^{-1}k, f] = 0.$$

*In particular, if  $f$  represents a ghost in the stable category, then both inclusions hold.  $\square$*

As a corollary, we get

**Corollary 2.4.3** (Chebolu, Christensen and Mináč [19]). *Let  $G$  be a  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Let  $f : M \rightarrow N$  be a map in  $\text{Mod}(kG)$  between projective-free modules  $M$  and  $N$ . If  $f$  is an  $l$ -fold ghost, then:*

$$(a) \text{soc}^l(M) \subseteq \ker(f).$$

$$(b) \text{im}(f) \subseteq \text{rad}^l(N). \quad \square$$

The next lemma provides ghosts with a particular form.

**Lemma 2.4.4** (Benson, Chebolu, Christensen and Mináč [9]). *Let  $G$  be a  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Let  $x \in G$  be a central element. Then left multiplication by  $x - 1$  on a  $kG$ -module  $M$  is a ghost.  $\square$*

Note that in general there are ghosts not of this form. Nevertheless these ghosts work well for abelian groups in providing lower bounds for ghost numbers (see Section 2.4.3). It is not hard to check that if  $G$  is a cyclic  $p$ -group with generator  $g$ , then  $g - 1$  is a universal ghost.

## 2.4.2 Generating and socle lengths

We now show that the generating length is a stabilized version of the socle length. In this section we allow our projective classes to be unstable, that is, we don't assume that the projectives are closed under suspension and desuspension.

Let  $G$  be a  $p$ -group, let  $k$  be a field of characteristic  $p$ , and let  $M$  be a  $kG$ -module. Note that  $\text{soc}(M)$  contains exactly the image of maps from  $k$ . So, when we build up  $M$  in a socle sequence in Theorem 2.4.1, we are only using maps from  $k$ , not all suspensions of  $k$ . This suggests that we consider the unstable projective class generated by  $k$  in  $\text{StMod}(kG)$ . We will show that the length with respect to this projective class is exactly the socle length for projective-free modules in  $\text{stmod}(kG)$ .

Note that the regular representation  $kG$  is the only indecomposable projective  $kG$ -module, and  $\text{soc}(kG) \cong k$  is its unique minimal left submodule. Thus any map  $kG \rightarrow M$  in  $\text{Mod}(kG)$  with  $M$  projective-free has  $\text{soc}(kG)$  in its kernel, since the map cannot be injective. It follows that a map  $\oplus k \rightarrow M$  in  $\text{Mod}(kG)$  with  $M$  projective-free is stably trivial if and only if it is the zero map. For finitely generated modules, a similar argument shows that the same is true for a map  $M \rightarrow \oplus k$  in  $\text{mod}(kG)$  with  $M$  projective-free.

**Proposition 2.4.5.** *Let  $G$  be a  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Let  $(\mathcal{P}, \mathcal{I})$  be the unstable projective class in  $\text{StMod}(kG)$  generated by  $k$ . Then a map  $f : M \rightarrow N$  between projective-free objects  $M$  and  $N$  is in  $\mathcal{I}$  if and only if it is represented by a map  $f$  such that  $\text{soc}(M) \subseteq \ker(f)$ . Hence, if  $M$  is finitely-generated and projective-free, the length of  $M$  with respect to  $(\mathcal{P}, \mathcal{I})$  is exactly its socle length.*

*Proof.* That  $f \in \mathcal{I}$  is equivalent to  $\text{soc}(M) \subseteq \ker(f)$  is Lemma 2.4.2 (a).

Now let  $M$  be projective-free. Then  $M \rightarrow M/\text{soc}(M)$  is a universal map in  $\mathcal{I}$ . It follows that  $M \rightarrow M/\text{soc}^k(M)$  is universal in  $\mathcal{I}^k$ . If  $M$  has socle length  $n$ , then  $M \in \mathcal{P}^n$  and  $M \rightarrow M/\text{soc}^{n-1}(M)$  is non-zero. If further  $M$  is finitely-generated, then the universal map  $M \rightarrow M/\text{soc}^{n-1}(M) \cong \oplus k$  is stably non-trivial, by the remarks preceding this proposition. Thus  $M$  has length  $n$  with respect to  $(\mathcal{P}, \mathcal{I})$ .  $\square$

Note that the stable projective class generated by  $k$  in  $\text{StMod}(kG)$  is exactly the ghost projective class. Thus the generating length is indeed the socle length stabilized and is generally less than or equal to the socle length. We have also recovered Theorem 2.4.1 from this observation. In Section 2.4.5, we are going to prove that the generating number of  $kG$  is within a factor of 3 of the socle length of  $kG$ .

Here we show that if the generating length of a module  $M \in \text{StMod}(kG)$  happens to equal its socle length (see, for example, Proposition 2.4.10 and Theorem 2.4.15), then the same holds for  $\text{rad}(M)$  and  $M/\text{soc}(M)$ , a result that we will use in Section 2.4.6 when studying dihedral groups.

**Proposition 2.4.6.** *Let  $k$  be a field of characteristic  $p$ , and let  $G$  be a  $p$ -group. Assume that  $M \in \text{StMod}(kG)$  has generating length equal to its radical length. Then  $\text{gel}(M/\text{soc}(M)) = \text{gel}(M) - 1$ , and similarly  $\text{gel}(\text{rad}(M)) = \text{gel}(M) - 1$ .*

*Proof.* Since the generating length of  $M$  is strictly less than the nilpotency index of  $J(kG)$ ,  $M$  is projective-free. The proposition is then a special case of the following more general lemma. □

**Lemma 2.4.7.** *Let  $\mathbb{T}$  be a triangulated category, and let  $(\mathcal{P}, \mathcal{I})$  and  $(\mathcal{P}', \mathcal{I}')$  be (possibly unstable) projective classes on  $\mathbb{T}$  such that  $\mathcal{P}' \subseteq \mathcal{P}$ . Suppose that  $M \in \mathbb{T}$  has  $\text{len}_{\mathcal{P}'}(M) = \text{len}_{\mathcal{P}}(M) = m$  and that there exist  $L \in \mathcal{P}'_{m-n}$  and  $N \in \mathcal{P}'_n$  with a triangle*

$$L \rightarrow M \rightarrow N.$$

*Then*

$$\text{len}_{\mathcal{P}'}(L) = \text{len}_{\mathcal{P}}(L) = m - n, \text{ and } \text{len}_{\mathcal{P}'}(N) = \text{len}_{\mathcal{P}}(N) = n.$$

*Proof.* We have that  $\text{len}_{\mathcal{P}'}(L) \leq m - n$  and  $\text{len}_{\mathcal{P}'}(N) \leq n$ . But  $\text{len}_{\mathcal{P}'}(L) + \text{len}_{\mathcal{P}'}(N) \geq m = (m - n) + n$ , so the equalities follow for  $(\mathcal{P}', \mathcal{I}')$ . Since  $\mathcal{P}' \subseteq \mathcal{P}$ , the same results hold for  $(\mathcal{P}, \mathcal{I})$  too. □

Intuitively, this easy fact says that when  $\text{len}_{\mathcal{P}'}(M) = \text{len}_{\mathcal{P}}(M)$ , the related object  $L$  can be built from  $\mathcal{P}'$  as efficiently as it can be built from  $\mathcal{P}$ . It applies to generating lengths and socle lengths.

We now provide examples of computations of ghost numbers of certain groups, improving on results in [19].

### 2.4.3 Ghost numbers of abelian $p$ -groups

We first prove a general proposition. It generalizes [9, Lemma 2.3] and [19, Prop. 5.10].

**Proposition 2.4.8.** *Let  $k$  be a field of characteristic  $p$ , and let  $H$  be a non-trivial subgroup of a  $p$ -group  $G$ . Assume that there exists a central element  $x$  in  $G$ . Let  $l$  be the smallest positive integer such that  $x^l \in H$ . Suppose that  $M \in \text{StMod}(kH)$  has generating length  $m \geq 1$ . Then  $\text{gel}(M \uparrow^G) \geq \text{gel}(M) + (l - 1)$ , and*

$$\text{generating number of } kG \geq \text{generating number of } kH + (l - 1).$$

*Suppose that  $M \in \text{stmod}(kH)$  has ghost length  $n \geq 1$ . Then  $\text{gl}(M \uparrow^G) \geq \text{gl}(M) + (l - 1)$ , and*

$$\text{ghost number of } kG \geq \text{ghost number of } kH + (l - 1).$$

*Proof.* For brevity, we write  $\downarrow$  for  $\downarrow_H^G$  and  $\uparrow$  for  $\uparrow_H^G$ . Let  $f : M \rightarrow N$  be a non-trivial  $(m - 1)$ -fold ghost in  $\text{StMod}(kH)$ . We will show that  $(x - 1)^{l-1} \circ f \uparrow$  is stably non-trivial. Since ghosts induce up to ghosts and  $x - 1$  is a ghost, it follows that there exists a non-trivial composite of  $(m - 1) + (l - 1)$  ghosts in  $\text{StMod}(kG)$ .

Consider the map  $M \xrightarrow{i} M \uparrow \downarrow \xrightarrow{f \uparrow \downarrow} N \uparrow \downarrow \xrightarrow{(x-1)^{l-1} \downarrow} N \uparrow \downarrow \xrightarrow{r} N$ , where  $i$  and  $r$  are the natural maps. To be more explicit,  $M \uparrow_H^G = kG \otimes_H M$ ,  $i(\alpha) = 1 \otimes \alpha$  and  $r(g \otimes \alpha) = g\alpha$  if  $g \in H$  and is zero otherwise. By naturality of the inclusion, the composite equals  $M \xrightarrow{f} N \xrightarrow{i} N \uparrow \downarrow \xrightarrow{(x-1)^{l-1} \downarrow} N \uparrow \downarrow \xrightarrow{r} N$ . Since  $x^i \notin H$  for  $i \leq l - 1$ , the map  $N \xrightarrow{i} N \uparrow \downarrow \xrightarrow{(x-1)^{l-1} \downarrow} N \uparrow \downarrow \xrightarrow{r} N$  is simply multiplication by  $(-1)^{l-1}$ , an isomorphism. Since  $N$  is stably non-zero, it follows that  $(x - 1)^{l-1} \downarrow \circ f \uparrow \downarrow$  and therefore  $(x - 1)^{l-1} \circ f \uparrow$  are stably non-trivial.

The result on ghost length and ghost number can be proved similarly by replacing  $\text{StMod}(kG)$  with  $\text{stmod}(kG)$ .  $\square$

We can apply this proposition to abelian groups.

**Theorem 2.4.9.** *Let  $k$  be a field of characteristic  $p$ , and let  $A = C_{p^r} \times C_{p^{r_1}} \times \cdots \times C_{p^{r_l}}$  be an abelian  $p$ -group. Then*

$$m - p^r + \left\lceil \frac{p^r - 1}{2} \right\rceil \leq \text{ghost number of } kA \leq \text{generating number of } kA \leq m - 1,$$

where  $m$  is the nilpotency index of  $J(kA)$ , and  $p^r$  is the order of the smallest cyclic summand.

When the prime  $p$  is greater than 2, the result here improves on that in [19], where the lower bound for the ghost number of  $kA$  is given by  $m - p^r + p^{r-1} = m - p^r + \lceil (p^r - 1)/p \rceil$ .

Note that since

$$m = 1 + (p^r - 1) + (p^{r_1} - 1) + \cdots + (p^{r_l} - 1),$$

our lower bound can also be written as

$$\left\lceil \frac{p^r - 1}{2} \right\rceil + (p^{r_1} - 1) + \cdots + (p^{r_l} - 1).$$

Also note that when  $A$  is cyclic, we have  $m = p^r$ , and the lower bound  $d = \lceil \frac{p^r - 1}{2} \rceil$  here is exactly the ghost number of  $A$  [19, Thm. 5.4].

*Proof.* Let  $g$  be a generator of  $C_{p^r}$ , and let  $g_i$  be a generator of  $C_{p^{r_i}}$ ,  $i = 1, 2, \dots, l$ . Write  $d = \lceil \frac{p^r - 1}{2} \rceil$ . By the proof of [19, Prop. 5.3],  $kC_{p^r}$  has ghost number  $d$ . We can now apply Proposition 2.4.8 by successively including the summands  $C_{p^{r_i}}$  to obtain

$$\text{ghost number of } kA \geq d + (p^{r_1} - 1) + \cdots + (p^{r_l} - 1).$$

The other inequalities are from Theorem 2.4.1. □

Proposition 2.4.8 allows us to make this explicit. Let  $M = N \uparrow_{C_{p^r}}^A$ , with  $N = kC_{p^r}/(g - 1)^d$ . Note that  $(g - 1)^{d-1}$  is a stably non-trivial  $(d - 1)$ -fold ghost on  $N$  in  $\text{stmod}(kC_{p^r})$  and, since  $A$  is abelian, the self map  $(g - 1) \uparrow_{C_{p^r}}^A$  on  $M$  is simply left multiplication by  $g - 1$ . Hence we have a particular form for the non-trivial  $(m - p^r + d - 1)$ -fold ghost on  $M$ :

$$\theta = (g - 1)^{d-1} (g_1 - 1)^{p^{r_1} - 1} \cdots (g_l - 1)^{p^{r_l} - 1}.$$

More generally, we have the following result.

**Proposition 2.4.10.** *Let  $k$  be a field of characteristic  $p$ , let  $A = C_{p^{r_1}} \times C_{p^{r_2}} \times \cdots \times C_{p^{r_l}}$  be an abelian  $p$ -group, and let  $M_i$  be an indecomposable  $C_{p^{r_i}}$ -module of dimension  $n_i$  for each  $i$ . Then the  $A$ -module  $M = M_1 \otimes \cdots \otimes M_l$  has radical length  $1 + (n_1 - 1) + \cdots + (n_l - 1)$ . If  $n_i \leq \frac{p^{r_i}}{2}$  for some  $i$ , then the generating length of  $M$  equals its radical length.*



Before proving the proposition, we state the following lemma.

**Lemma 2.4.11** ([29, Theorem 1.2]). *Let  $G$  be a  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Then the elements  $h - 1$  with  $h \neq 1$  form a basis for  $\text{rad}(kG)$ . It follows that the products  $(h_1 - 1) \cdots (h_n - 1)$  with  $h_i \neq 1$  span  $\text{rad}^n(kG)$ .  $\square$*

Note that it suffices to consider generators of the group  $G$  when we generate  $\text{rad}^n kG$  as a sub-module. We can now compute the radical length of the module  $M$  and prove the proposition.

*Proof of Proposition.* Let  $g_i$  be a generator of  $C_{p^{r_i}}$ . Then the various  $g_i - 1$  with  $1 \leq i \leq l$  generate  $\text{rad}(kG)$ . We regard  $M_i$  as the quotient  $kC_{p^{r_i}} / (g_i - 1)^{n_i}$ , so the elements  $(g_i - 1)^j$  with  $0 \leq j \leq n_i - 1$  form a basis of  $M_i$ . Now let  $m = (n_1 - 1) + \cdots + (n_l - 1)$ . Since any  $(m + 1)$ -fold product of the elements  $g_i - 1$  has to be zero in  $M$ ,  $\text{rad}^{m+1}(M) = 0$ . On the other hand, the element  $(g_1 - 1)^{n_1-1} \otimes \cdots \otimes (g_l - 1)^{n_l-1} \in M$  is non-zero and spans  $\text{rad}^m(M)$ . It follows that the radical length of  $M$  is  $m + 1$ .

To prove the last statement, without loss of generality we can assume that  $n_1 \leq \frac{p^{r_1}-1}{2}$ . We then consider the restriction of  $M$  to  $H = C_{p^{r_1}}$ . Note that we have a vector space isomorphism

$$M \downarrow_H \cong \bigoplus_{i_2=0}^{n_2-1} \cdots \bigoplus_{i_l=0}^{n_l-1} M_1.$$

Since  $G$  acts componentwise, this is actually an isomorphism of  $kH$ -modules, and we have  $kH$ -maps  $i : M_1 \rightarrow M \downarrow_H$  sending  $\alpha$  to  $\alpha \otimes 1 \otimes \cdots \otimes 1$  and  $r : M \downarrow_H \rightarrow M_1$  sending  $\alpha \otimes (g_2 - 1)^{i_2} \otimes \cdots \otimes (g_l - 1)^{i_l}$  to  $(-1)^{i_2 + \cdots + i_l} \alpha$  for  $0 \leq i_k \leq n_k - 1$ .

We can form the  $m$ -fold ghost  $f = (g_1 - 1)^{n_1-1} \cdots (g_l - 1)^{n_l-1}$  on  $M$ . And one can check that  $r \circ f \downarrow_H \circ i$  is  $\pm(g_1 - 1)^{n_1-1}$  on  $M_1$ , which is stably non-trivial. Hence  $f$  is stably non-trivial and the ghost length of  $M$  is at least  $m + 1$ . Since this is also the radical length of  $M$ , we have  $\text{gl}(M) = \text{gel}(M) = m + 1$ .  $\square$

*Remark 2.4.12.* We don't know which of the lower bound and upper bound better approximates the ghost number in general, but we suspect that the lower bound is better. We show in Section 2.4.7 that the upper bound can be refined by 1 for rank 2 abelian  $p$ -groups  $C_{p^r} \times C_{p^s}$ , with  $p^r, p^s \geq 3$ . In particular, the lower bound we have here is the exact ghost number for the group  $C_3 \times C_3$ .

#### 2.4.4 Ghost number of the quaternion group $Q_8$

In this section, we study the quaternion group  $Q_8 = \langle \epsilon, i, j \mid \epsilon^2 = 1, i^2 = j^2 = (ij)^2 = \epsilon \rangle$  over a field  $k$  of characteristic 2. It has been shown in [19] that the ghost number of  $kQ_8$  is 2, 3, or 4.

**Proposition 2.4.13.** *Let  $k$  be a field of characteristic 2. Then there exists a stably non-trivial double ghost in  $\text{stmod}(kQ_8)$ . Hence*

$$3 \leq \text{ghost number of } kQ_8 \leq \text{generating number of } kQ_8 \leq 4.$$

*Proof.* We have a quotient map from  $Q_8$  to the Klein four group  $V$  that identifies  $\epsilon$  with 1. We also write  $i$  and  $j$  for the generators of  $V$ . The rank one free  $kV$ -module can be viewed as a  $kQ_8$ -module, and we write  $kV$  for it. It has radical length 3, and we will show that it admits a stably non-trivial double ghost, hence  $\text{gl}(kV) = \text{gel}(kV) = 3$ .

Right multiplication  $R_{i+1}$  on  $kV$  by  $i + 1$  is a left  $kQ_8$ -map, and we claim that it is a ghost. To see this, consider the short exact sequence

$$0 \rightarrow kV \xrightarrow{i} kQ_8 \rightarrow kV \rightarrow 0$$

of left  $kQ_8$ -modules, where the kernel  $kV$  is generated by  $\epsilon + 1$  in  $kQ_8$ . It follows from this sequence that  $\Omega kV = kV$  and that  $\Omega R_{i+1} = R_{i+1}$ .

Thus to show that  $R_{i+1}$  is a ghost, we just need to check that it is stably trivial on maps from  $k$ . Multiplication by  $i + 1$  kills the socle of  $kV$ , which is generated by  $1 + i + j + ij$ , so this follows from Lemma 2.4.2(a).

Next we show that there is a non-trivial double ghost. For any map  $f : kQ_8 \rightarrow kV$ , the composite  $fi$  is zero, since  $\epsilon + 1$  acts trivially on  $kV$ . Thus a  $kQ_8$ -map  $kV \rightarrow kV$  is stably trivial if and only if it is zero. As a result, multiplication by  $(i + 1)(j + 1)$  on  $kV$  is stably non-trivial, and we get the desired double ghost.

It follows that the ghost number of  $kQ_8$  is at least 3. The nilpotency index of  $J(kQ_8)$  is 5, so the generating number of  $kQ_8$  is at most 4.  $\square$

*Remark 2.4.14.* The map  $R_{(i+1)(j+1)} = R_{1+i+j+ij} : kV \rightarrow kV$  constructed in the proof is in fact the almost zero map with domain  $kV$  in  $\text{stmod}(kQ_8)$ . To see this, we consider the inclusion  $\text{rad}(kV) \rightarrow kV$ . Since this map is not split-epi, its composition with the almost

zero map  $\gamma : kV \rightarrow kV$  factors through a projective module  $P$ . But  $P$  is also injective, thus we can change  $\gamma$  by a map factoring through  $P$  to ensure that  $\text{rad}(kV) \subseteq \ker(\gamma)$ . Since  $kV/\text{rad}(kV) \cong \text{soc}(kV) \cong k$  and  $\text{soc}(kV)$  is generated by the element  $1 + i + j + ij$ , it must be that  $R_{1+i+j+ij}$  is the almost zero map (up to a scalar factor). This gives another proof that this map is stably non-trivial.

In the next section, we generalize the technique used here.

### 2.4.5 $p$ -groups with cyclic normal subgroups

In Section 2.4.3, we produced ghosts using left multiplication by  $x - 1$  for abelian groups. More generally, in Lemma 2.4.4, we saw that left multiplication by  $x - 1$  for  $x$  a central element produces a ghost. For a non-central element, in order to produce a left module map, one must consider *right* multiplication, when this makes sense, and indeed we used this technique in Section 2.4.4 to produce ghosts for  $Q_8$ . However, it is not always true that right multiplication by  $x - 1$  produces ghosts. Generalizing the known examples, we show that if  $M$  is induced up from a cyclic normal subgroup, then right multiplication by  $x - 1$  on  $M$  is well-defined and is a ghost.

**Theorem 2.4.15.** *Let  $C_{p^r}$  be a cyclic normal subgroup of a  $p$ -group  $G$ , and let  $k$  be a field of characteristic  $p$ . Let  $M_n$  be an indecomposable  $kC_{p^r}$ -module of dimension  $n$ , and write  $M = M_n \uparrow^G$ . Then, for each  $x \in G$ , one can define the right multiplication map  $R_{x-1}$  on  $M$  and it is a ghost. Moreover, if  $n \leq \lceil \frac{p^r-1}{2} \rceil$ , then  $\text{gl}(M) = \text{gel}(M) = \text{rad} \text{len } M$ .*

Note that for  $n = 1$ , we have  $M \cong kH \downarrow_G$ , where  $H = G/C_{p^r}$  and the restriction is taken along the quotient map. Thus the ghosts in the previous section are examples of this construction.

*Proof.* Let  $g$  be a generator of  $C_{p^r}$ . We can identify  $M_n$  with the left submodule of  $kC_{p^r}$  generated by  $(g - 1)^{p^r-n}$ , and so we have a short exact sequence of  $kC_{p^r}$ -modules:

$$0 \rightarrow M_n \rightarrow kC_{p^r} \rightarrow M_{p^r-n} \rightarrow 0,$$

where  $M_{p^r-n}$  is an indecomposable  $kC_{p^r}$ -module of dimension  $p^r - n$ . Inducing up, we get

$$0 \rightarrow M_n \uparrow^G \xrightarrow{i} kG \xrightarrow{p} M_{p^r-n} \uparrow^G \rightarrow 0. \quad (2.4.1)$$

The inclusion  $i$  identifies  $M = M_n \uparrow^G$  with the left submodule of  $kG$  generated by  $(g-1)^{p^r-n}$ . Since  $C_{p^r} \leq G$  is normal, this submodule is actually a sub-bimodule. Thus the right multiplication map  $R_{x-1} : M \rightarrow M$  is well-defined and is a left  $kG$ -module map, for each  $x \in G$ . We must show that it is a ghost.

Since (2.4.1) is in fact a short exact sequence of bimodules,  $R_{x-1}$  is two-periodic as a left  $kG$ -map, so it suffices to check that  $R_{x-1}$  is left stably-trivial on maps from  $k$  and  $\Omega^{-1}k$ . By Lemma 2.4.2, this is equivalent to  $\text{soc}_L(M) \subseteq \ker(R_{x-1})$  and  $\text{im}(R_{x-1}) \subseteq \text{rad}_L(M)$ , where we use subscripts to indicate left and right socles and radicals. Clearly,  $\text{soc}_R(M) \subseteq \ker(R_{x-1})$  and  $\text{im}(R_{x-1}) \subseteq \text{rad}_R(M)$ . Now  $\text{soc}_L(kG) = \text{soc}_R(kG) \cong k$ , so  $\text{soc}_L(M) = \text{soc}_R(M) \cong k$ , which gives the first inclusion. And one can also show that  $\text{rad}_L(M) = \text{rad}_R(M)$ , which gives the second inclusion.

To prove the last claim, let  $n \leq \lceil \frac{p^r-1}{2} \rceil$  and assume that  $\text{rad len } M = l$ . We want to construct an  $(l-1)$ -fold ghost. Note that  $\text{soc}_L(M) = \text{soc}_R(M) = \text{rad}_R^{l-1}(M) = M(g_1-1) \cdots (g_{l-1}-1)$  for some  $g_1, \dots, g_{l-1}$  in  $G$ , so the  $(l-1)$ -fold ghost  $f := R_{g_{l-1}-1} \circ \cdots \circ R_{g_1-1}$  takes  $M$  onto its socle. For any map  $h : kG \rightarrow M$ , the composite  $hi$  is zero, since the image of  $i$  is generated by  $(g-1)^{p^r-n}$  which acts trivially on  $M$  since  $n \leq p^r - n$ . Thus a map  $M \rightarrow M$  is stably trivial if and only if it is zero, and so our  $(l-1)$ -fold ghost  $f$  is stably non-trivial. Thus  $l \leq \text{gl}(M) \leq \text{gel}(M) \leq \text{rad len } (M) = l$ , and we are done.  $\square$

*Remark 2.4.16.* As in Remark 2.4.14, we can also see that  $f$  is non-trivial using the theory of Auslander-Reiten triangles. There is a canonical inclusion  $j$  of  $M$  into  $M_{p^r-n} \uparrow^G = \Omega M$  induced from the  $kC_{p^r}$ -map  $M_n \rightarrow M_{p^r-n}$ , and one can show that the composite  $jf$  is exactly the almost zero map out of  $M$ .

Note that any  $p$ -group  $G$  has a non-trivial center, hence a cyclic normal subgroup  $C_p$ . Applying the theorem to the short exact sequence of groups  $C_p \rightarrow G \rightarrow H$ , we get

**Corollary 2.4.17.** *Let  $G$  be a  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Then*

$$\frac{1}{2} \text{rad len } kG \leq \text{ghost num } kG \leq \text{gen num } kG < \text{rad len } kG,$$

when  $p$  is even, and

$$\frac{1}{3} \text{rad len } kG \leq \text{ghost num } kG \leq \text{gen num } kG < \text{rad len } kG,$$

when  $p$  is odd.

*Proof.* Choose a cyclic normal subgroup  $C_p$  of  $G$ , and let  $M = M_n \uparrow^G$ , where  $M_n$  is an indecomposable  $kC_p$ -module of dimension  $n = \lceil \frac{p-1}{2} \rceil$ . Since  $\text{rad len } M = \text{gl}(M) \leq \text{ghost num } kG$ , we only need to show that  $2(\text{rad len } M) \geq \text{rad len } kG$  for  $p$  even and  $3(\text{rad len } M) \geq \text{rad len } kG$  for  $p$  odd. By (2.4.1), we know that

$$\text{rad len } M + \text{rad len } M_{p-n} \uparrow^G \geq \text{rad len } kG.$$

For  $p$  even,  $p - n = n$ , and so the result follows.

For  $p$  odd,  $p - n = n + 1$ . We will show that  $2(\text{rad len } M) \geq \text{rad len } M_{n+1} \uparrow^G$ , and the corollary will follow. There is a short exact sequence

$$0 \rightarrow M \rightarrow M_{n+1} \uparrow^G \rightarrow M_1 \uparrow^G \rightarrow 0,$$

induced up from  $C_p$ -maps, and one sees that  $M_1 \uparrow^G$  is a submodule of  $M$  again by inducing up the  $C_p$ -map  $k \rightarrow M_n$ . It follows that

$$2(\text{rad len } M) \geq \text{rad len } M + \text{rad len } M_1 \uparrow^G \geq \text{rad len } M_{n+1} \uparrow^G,$$

and we are done. □

We expect that for odd primes, the lower bound can be improved to an expression that is generically close to  $(\text{rad len } kG)/2$ .

### 2.4.6 Ghost numbers of dihedral 2-groups

Our next goal is to study the dihedral 2-groups. We will show that the ghost number and generating number of  $kD_{4q}$  are both  $q + 1$ . Here we write  $D_{4q}$  for the dihedral 2-group of order  $4q$ , with  $q$  a power of 2:

$$D_{4q} = \langle x, y \mid x^2 = y^2 = 1, (xy)^q = (yx)^q \rangle.$$

It has a normal cyclic subgroup  $C_{2q}$ , generated by  $g = xy$ .

Since  $kC_{2q}$  has ghost number  $q$ , which is realized by the ghost length of  $M = kC_{2q}/(g-1)^q$  [19, Prop. 5.3], the ghost length of  $N = M \uparrow_{C_{2q}}^{D_{4q}}$  is at least  $q$  in  $\text{stmod}(kD_{4q})$ . By Theorem 2.4.15, we actually have  $\text{gl}(N) = \text{gel}(N) = \text{rad len } N$ . Note that  $(xy)^q \in$

$D_{4q}$  is central of order 2 and that  $M \cong k\uparrow_{C_2}^{C_{2q}}$ , hence  $N = M\uparrow_{C_{2q}}^{D_{4q}} \cong k\uparrow_{C_2}^{D_{4q}} \cong kD_{2q}\downarrow_{D_{4q}}^{D_{2q}}$ , where the restriction is along the quotient map in the short exact sequence  $C_2 \rightarrow D_{4q} \rightarrow D_{2q}$ . It is not hard to see that the radical length of  $kD_{2q}$  is  $q + 1$  (see Remark 2.4.20) and that its  $q$ -th radical is generated by  $((y - 1)(x - 1))^{\frac{q}{2}} = ((x - 1)(y - 1))^{\frac{q}{2}}$  (which makes sense for  $q = 1$  since we have identified  $x = y$  in that case). Thus we have proved the following consequence of Theorem 2.4.15:

**Corollary 2.4.18.** *Let  $k$  be a field of characteristic 2. Then the ghost number of  $kD_{4q}$  is at least  $q + 1$ . In fact,  $\text{gl}(N) = \text{gel}(N) = q + 1$ , where  $N = k\uparrow_{C_2}^{D_{4q}}$ .  $\square$*

The proof of Theorem 2.4.15 shows that an explicit  $q$ -fold ghost  $N \rightarrow N$  is given by  $R_{((x-1)(y-1))^{\frac{q}{2}}}$ .

To get upper bounds for the generating numbers of dihedral 2-groups, we need classification theorems [4].

Let  $\Lambda = k\langle X, Y \rangle / (X^2, Y^2)$  be the quotient of the free algebra on two non-commuting variables. In  $kD_{4q}$ , writing  $X = x - 1$  and  $Y = y - 1$ , one can show that  $(XY)^r - (YX)^r = (xy)^r - (yx)^r$  for  $r$  a power of 2, and so  $kD_{4q} \cong \Lambda / ((XY)^q - (YX)^q)$  [4, Lemma 4.11.1].

In the isomorphism  $kD_{4q} \cong \Lambda / ((XY)^q - (YX)^q)$ , we have implicitly assumed that the characteristic of  $k$  is 2. However, for the classification we describe below,  $k$  can have any characteristic, and we apply it in this generality in the next section.

$\Lambda$ -modules are classifiable. Let  $\mathcal{W}$  be the set of words in the **direct letters**  $a$  and  $b$  and the **inverse letters**  $a^{-1}$  and  $b^{-1}$ , such that  $a$  and  $a^{-1}$  are always followed by  $b$  or  $b^{-1}$  and vice versa, together with the “**zero length word**” 1.

Given  $C = l_1 \cdots l_n \in \mathcal{W}$ , where each  $l_i$  is a direct or inverse letter, let  $M(C)$  be the vector space over  $k$  with basis  $z_0, \dots, z_n$  on which  $\Lambda$  acts according to the schema

$$kz_0 \xleftarrow{l_1} kz_1 \xleftarrow{l_2} kz_2 \cdots kz_{n-1} \xleftarrow{l_n} kz_n,$$

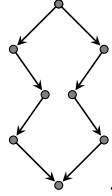
with  $X$  acting via  $a$  and  $Y$  acting via  $b$ . For example, if  $C = ab^{-1}a^{-1}$ , then the schema is

$$kz_0 \xleftarrow{a} kz_1 \xrightarrow{b} kz_2 \xrightarrow{a} kz_3$$



Thus, when  $k$  has characteristic 2, a complete list of indecomposable  $kD_{4q}$ -modules, up to isomorphism, consists of the  $\Lambda$ -modules satisfying one of these three conditions.  $\square$

*Remark 2.4.20.* The identification  $kD_{4q} \cong \Lambda/((XY)^q - (YX)^q)$  yields that  $kD_{4q} = M((ab)^q(ba)^{-q}, id)$ . It is not hard to see from the schema of  $M((ab)^q(ba)^{-q}, id)$  that it has radical length  $2q + 1$ . Here is an illustration for  $q = 2$ :



The module  $N = k \uparrow_{C_2}^{D_{4q}} = kD_{4q} \otimes_{kC_2} k$  is the quotient of  $kD_{4q}$  where we identify  $(xy)^q$  with 1, in other words,  $(xy)^{\frac{q}{2}} = (yx)^{\frac{q}{2}}$ , for  $q > 1$ . This is equivalent to  $(XY)^{\frac{q}{2}} = (YX)^{\frac{q}{2}}$ . Hence  $N = M((ab)^{\frac{q}{2}}(ba)^{-\frac{q}{2}}, id)$  and it follows that  $N$  has radical length  $q + 1$ .

We want to prove that the generating number of  $kD_{4q}$  does not exceed  $q + 1$ . Note that when  $q = 1$ , the dihedral group  $D_4$  is just  $C_2 \times C_2$ , and the claim follows from Theorem 2.4.9, so we assume that  $q \geq 2$  from now on unless otherwise stated.

Now let  $M$  be an indecomposable  $kD_{4q}$ -module. By Theorem 2.4.19, it corresponds to a word satisfying one of the conditions (a), (b) or (c). Then  $\text{soc}(M)$  contains the submodule spanned by the vector spaces at positions of the form  $b^{-1}a$  or  $a^{-1}b$  (interpreted cyclically if  $M$  is of the second kind). Such a position exists if  $M$  is of the second kind since the condition that the word is not a power of a smaller word forces the word to contain both direct and inverse letters. However, such positions are removed in  $M/\text{soc}(M)$ , so the indecomposable summands of  $M/\text{soc}(M)$  are of the first kind and correspond to words not containing  $b^{-1}a$  or  $a^{-1}b$ .

Similarly, the indecomposable summands of  $\text{rad}(M)$  are of the first kind and correspond to words not containing  $ba^{-1}$  or  $ab^{-1}$ . It follows that the indecomposable summands of  $\text{rad}(M/\text{soc}(M))$  are of the first kind and correspond to words not containing  $b^{-1}a$ ,  $a^{-1}b$ ,  $ba^{-1}$  or  $ab^{-1}$ . Thus the words must consist entirely of direct or inverse letters. But since  $M(C) \cong M(C^{-1})$ , we can assume that the words only contain direct letters. By (a), the possible words are  $(ab)^{q-1}a$ ,  $(ba)^{q-1}b$ , or subwords of these. And we can prove



**Lemma 2.4.21.** *Let  $M$  be a  $kD_{4q}$ -module of the first kind, with  $q \geq 2$ . If  $M$  corresponds to a word that only contains direct letters, then its generating length is less than or equal to  $q$ .*

*Proof.* We are going to show that

$$\text{gel}(M((ab)^r a)) \leq q \quad \text{and} \quad \text{gel}(M((ab)^r)) \leq q$$

for  $0 \leq r \leq q-1$ , the case of words starting with  $b$  being similar.

Since  $D_{4q}$  is a 2-group, the generating length of a module is always no more than its radical length, hence its dimension. So, for any word  $C$ ,  $\text{gel}(M(C)) \leq \dim M(C) = |C| + 1$ , where  $|C|$  denotes the number of letters in  $C$ . Thus we are done if  $r \leq q/2 - 1$ .

To handle  $r \geq q/2$ , we temporarily introduce the following notation for modules with symmetry under reflection when exchanging  $X$  with  $Y$ . For a word  $u$ , write  $u'$  for the inverse word with all  $a$ s and  $b$ s exchanged, so for example  $(ab^{-1}ab) = a^{-1}b^{-1}ab^{-1}$ . Write  $M'(u)$  for  $M(uu')$  and  $M'(u, \phi)$  for  $M(uu', \phi)$ . Then  $kD_{4q} = M'((ab)^q, id)$ , and one can see that  $\tilde{\Omega}k = M'((b^{-1}a^{-1})^{q-1}b^{-1})$  and  $\tilde{\Omega}^{-2}k = M'((ab)^{q-1}ab^{-1})$ . It follows that we have short exact sequences

$$0 \rightarrow k \rightarrow \tilde{\Omega}^{-2}k \rightarrow M((ab)^{q-1}a) \oplus M((ba)^{q-1}b) \rightarrow 0$$

and

$$0 \rightarrow k \rightarrow \tilde{\Omega}k \rightarrow M((ab)^{q-1}) \oplus M((ba)^{q-1}) \rightarrow 0.$$

Since  $q \geq 2$ , one sees that  $\text{gel}(M((ab)^{q-1}a)) = \text{gel}(M((ab)^{q-1})) \leq 2$ , which handles the case  $r = q-1$ .

Now for  $r \leq q-2$ ,  $M((ab)^r a)$  and  $M((ab)^r)$  embed in  $M((ab)^{q-1})$ . Thus their ghost lengths are no more than the codimension plus two, and one can check that this is no more than  $q$  when  $r \geq q/2$ .  $\square$

In general, for a  $p$ -group  $G$  and a  $kG$ -module  $M$ , we know that  $M/\text{rad}(M)$  and  $\text{soc}(M)$  are sums of trivial modules. Thus  $\text{rad}(M)$  is the fibre of a map  $M \rightarrow \oplus k$  and  $M/\text{soc}(M)$  is the cofibre of a map  $\oplus k \rightarrow M$ . So

$$\text{gel}(M) \leq \text{gel}(\text{rad}(M)) + 1 \quad \text{and} \quad \text{gel}(M) \leq \text{gel}(M/\text{soc}(M)) + 1.$$

Hence

$$\text{gel}(M) \leq \text{gel}(\text{rad}(M/\text{soc}(M))) + 2,$$

and so by Lemma 2.4.21 and the discussion preceding it, the generating number of  $kD_{4q}$  does not exceed  $q + 2$ . This is one more than the correct answer. We will show in Proposition 2.4.26 that the module  $M((ab)^{\frac{q}{2}-1}a)$  has length  $q$ , so we can't improve this bound by improving Lemma 2.4.21.

We will have to be a bit more clever in the construction to get the exact generating number. The above process takes two steps to produce a module  $\text{rad}(M/\text{soc}(M))$  whose summands involve only direct letters, by removing “top” and “bottom” elements. We next show that we can add top elements instead of removing them, with the same effect, and as a result we will be able to do both steps at the same time.

**Lemma 2.4.22.** *Let  $M$  be a non-projective indecomposable module, with corresponding word  $C$ . There exists a short exact sequence*

$$0 \rightarrow M \rightarrow M' \rightarrow \oplus k \rightarrow 0,$$

where the indecomposable summands of  $M'$  are of the first kind and correspond to words that contain no  $ab^{-1}$  or  $ba^{-1}$ .

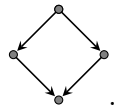
*Proof.* First suppose that  $M$  is of the first kind. If  $C$  contains no  $ab^{-1}$  or  $ba^{-1}$ , we simply set  $M'$  to be  $M$ . Otherwise, assume for example that  $C$  contains  $ab^{-1}$  and factor the word  $C$  as  $L_1L_2$ , with  $L_1$  ending with  $a$  and  $L_2$  starting with  $b^{-1}$ . Write  $z$  for the basis element of  $M(C)$  corresponding to the vertex connecting  $L_1$  with  $L_2$ , and write  $z_i$  for the corresponding basis element in  $M(L_i)$ ,  $i = 1, 2$ . Then we have a short exact sequence  $M \rightarrow M(L_1) \oplus M(L_2) \rightarrow k$ , where the first map takes  $z$  to  $z_1 - z_2$  and does the natural thing on the other basis elements, and the second map takes  $z_1$  and  $z_2$  to 1 in  $k$  and the other basis elements to 0. More generally, we can write  $C = L_1L_2 \cdots L_n$ , broken at the spots  $a^{-1}b$  and  $b^{-1}a$ , and set  $M' = \oplus M(L_i)$ .

Now suppose that  $M = M(C, \phi)$  is of the second kind, where  $\phi : V \rightarrow V$  is an indecomposable automorphism. We can assume that  $C = a^{-1}Lb$  up to inverse and cyclic permutation. Fix a basis  $v_1, \dots, v_n$  of  $V$ , where  $n = \dim(V)$ . Let  $M'' = \oplus_{i=1}^n M_i$ , with each  $M_i = M(C)$ . We write  $w_i$  and  $z_i$  for the basis elements in  $M_i$  corresponding to the beginning and end of the word  $C$ . Then we have a short exact sequence  $M(C, \phi) \rightarrow M'' \rightarrow V$ , where the first map sends  $v_i$  to  $\phi(w_i) - z_i$  for the first vertex and does the

natural thing on the other vertices, and the second map sends  $w_i$  to  $v_i$ ,  $z_i$  to  $\phi(v_i)$  and the other basis elements to 0. Here we regard  $V$  as a module with trivial action. Repeating the process for a module of the first kind, we get a short exact sequence  $M'' \rightarrow M' \rightarrow \oplus k$ . It is not hard to see that the cokernel of the composite  $M \rightarrow M'' \rightarrow M'$  also has a trivial action, and we are done.  $\square$

Note that the short exact sequence is represented by a map  $\oplus \Omega k \rightarrow M$ , and this makes it possible to combine it with a map  $\oplus k \rightarrow M$ .

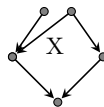
**Example 2.4.23.** We illustrate an example for  $q = 2$ . Write  $kV$  for the module  $M(a^{-1}b^{-1}ab, id_k)$ :



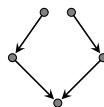
We begin by defining a cofibre sequence

$$\Omega k \rightarrow kV \rightarrow M(a^{-1}b^{-1}ab) \rightarrow k.$$

To see what the maps are, first consider the module



which has  $kV$  as a codimension 1 submodule. We can choose a basis so that this becomes  $M' = M(a^{-1}b^{-1}ab)$



and the map  $M' \rightarrow k$  takes both top points to  $k$  and has kernel  $kV$ . Then  $M'$  corresponds to a word that does not contain  $ba^{-1}$  or  $ab^{-1}$ , and the summands of  $M'/\text{soc}(M') \cong M(a) \oplus M(b)$  correspond to words that only contain direct letters. Note that the map from  $k$  to  $\text{soc}(M')$  factors through  $kV \rightarrow M'$ , so we can combine the two steps to get a cofibre sequence

$$\Omega k \oplus k \rightarrow kV \rightarrow M(a) \oplus M(b) \rightarrow k \oplus \Sigma k.$$

By Lemma 2.4.21, the generating length of the third term is at most  $q$ , which is 2 in our case.

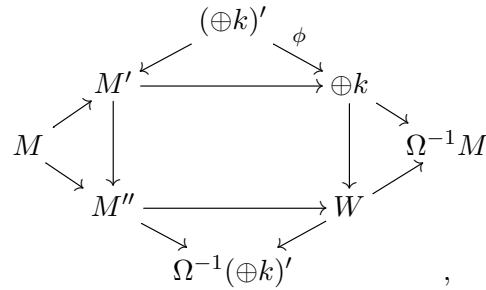
Now we are ready to prove

**Theorem 2.4.24.** *Let  $k$  be a field of characteristic 2. Then the generating number of  $kD_{4q}$  is at most  $q + 1$ , for all  $q \geq 1$ .*

*Proof.* The case when  $q = 1$  is dealt with in Theorem 2.4.9, so we prove the theorem for  $q \geq 2$ .

Let  $M$  be a non-projective indecomposable module, with corresponding word  $C$ . In the short exact sequence  $M \rightarrow M' \rightarrow \oplus k$  from Lemma 2.4.22, the indecomposable summands of  $M'$  correspond to words that contain no  $ab^{-1}$  or  $ba^{-1}$ . Hence the indecomposable summands of  $M'' = M'/\text{soc}(M')$  correspond to words of direct letters, and  $\text{gel}(M'') \leq q$ .

We can form the octahedron



where  $(\oplus k)'$  is  $\text{soc}(M')$ .

The proof will be finished once we show that  $\text{gel}(W) = 1$ . Here  $W$  is the cofibre of a map  $\phi$  between direct sums of trivial modules. Such a map is the sum of an identity map and a zero map. Hence  $W$  is a direct sum of trivial modules  $k$  and the modules  $\Omega^{-1}k$ , so  $\text{gel}(W) = 1$ . □

**Corollary 2.4.25.** *Let  $k$  be a field of characteristic 2. Then the ghost number and generating number of  $kD_{4q}$  are  $q + 1$  for all  $q \geq 1$ .* □

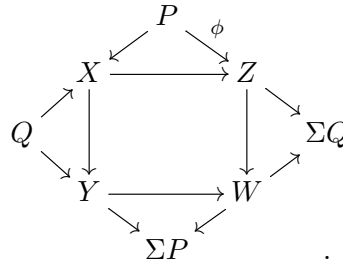
We now summarize and generalize the idea in the proof of the Theorem. Suppose that we start building an object  $Q$  from  $P$ ,  $Y$  and  $Z$  by first using a triangle

$$P \rightarrow X \rightarrow Y \rightarrow \Sigma P$$

and then using a triangle

$$Q \rightarrow X \rightarrow Z \rightarrow \Sigma Q.$$

Then we can form the octahedron



Assume that  $P$  has length  $m$ ,  $Y$  has length  $n$ , and  $Z$  has length  $l$ . Then the length of  $Q$  does not exceed  $m + n + l$ . Indeed,  $n + \text{len}(W)$  bounds the length of  $Q$ . For example, if  $\phi$  is in  $\mathcal{I}^s$  for some positive integer, we have  $\text{len}(W) \leq m + l - s$  by Lemma 2.3.1. Or, if  $\phi = 0$ , then  $W \cong Z \oplus \Sigma P$  and the two steps can be combined. This is analogous to the fact in topology that when a second cell is attached to a  $CW$ -complex without touching a first cell, then they can be attached to the complex at the same time.

We finish this section by computing the generating lengths of the modules  $M(ab)^r$  and  $M((ab)^r a)$ , with  $r \leq q/2 - 1$ . Note that there is a category automorphism on  $\text{StMod}(kD_{4q})$  induced by the group automorphism on  $D_{4q}$  that exchanges  $x$  and  $y$ . It exchanges the  $a$ 's and  $b$ 's in the word which an indecomposable module corresponds to and preserves the ghost projective class. As a result,

$$\text{gel}(M((ab)^r)) = \text{gel}(M((ba)^r)) \text{ and } \text{gel}(M((ab)^r a)) = \text{gel}(M((ba)^r b))$$

for  $D_{4q}$ -modules with  $0 \leq r \leq q - 1$ .

Recall from Corollary 2.4.18 that the module  $M = kD_{2q}$  in  $\text{StMod}(kD_{4q})$  has its generating length equal to its radical length  $q+1$ . By Proposition 2.4.6,  $\text{gel}(\text{rad}(M/\text{soc}(M))) = \text{gel}(M) - 2 = q - 1$ . Note that  $M = M((ab)^{l+1}(a^{-1}b^{-1})^{l+1}, id)$ , where  $l = q/2 - 1$ , so  $\text{rad}(M/\text{soc}(M)) \cong M((ab)^l) \oplus M((ba)^l)$ . Then, since exchanging  $a$ 's and  $b$ 's preserves

the generating length,

$$\text{gel}(M((ab)^l)) = \text{gel}(M((ba)^l)) = q - 1.$$

It follows that

$$\text{gel}(M((ab)^r)) = 2r + 1 \text{ if } r \leq l, \text{ and}$$

$$\text{gel}(M((ab)^r a)) = 2(r + 1) \text{ if } r \leq l - 1.$$

We need to be a bit trickier to handle the module  $M((ab)^l a)$ .

**Proposition 2.4.26.** *The  $kD_{4q}$ -module  $M((ab)^l a)$  has generating length  $q$ , where  $l = q/2 - 1$ .*

*Proof.* We have a triangle

$$\Sigma k \oplus k \rightarrow M \rightarrow M((ab)^l a) \oplus M((ba)^l b),$$

where the map  $\Sigma k \rightarrow M$  is a surjection.

Hence  $\text{gel}(M((ab)^l a) \oplus M((ba)^l b)) \geq q$ . Since its radical length is  $q$ , this must be an equality. Then, using the symmetry again,

$$\text{gel}(M((ab)^l a)) = \text{gel}(M((ba)^l b)) = q. \quad \square$$

### 2.4.7 Ghost number of $C_{p^r} \times C_{p^s}$

Let  $G = C_{p^r} \times C_{p^s}$ . In this section we show that

$$\text{the ghost number of } kG \leq \text{the generating number of } kG \leq p^r + p^s - 3$$

and give the exact result when  $p^r$  is 3 or 4. Note that a general upper bound for the generating number for a  $p$ -group is given by the radical length of  $kG$  minus 1 (Theorem 2.4.1). This gives  $p^r + p^s - 2$  for the group  $C_{p^r} \times C_{p^s}$ , and our result refines this upper bound by 1. To keep the indices simple, we give a detailed proof for the group  $C_3 \times C_3$  at the prime 3, and we indicate how to modify the proof to cover the general case. We are going to show that the composite of any three ghosts is stably trivial for the group  $C_3 \times C_3$ , using Theorem 2.4.19.

Here is an overview of our strategy. Given a finitely generated projective-free module  $N$  with radical length  $n$  and an  $l$ -fold ghost  $g : N \rightarrow N_1$  in  $\text{Mod}(kG)$ , where  $N_1$  is an arbitrary projective-free module, we can form the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{g} & N_1 \\ \downarrow p_1 & & \uparrow h \\ N/\text{rad}^{n-l}(N) & \xrightarrow{p_2} & N/\text{soc}^l(N). \end{array}$$

The  $l$ -fold ghost  $g$  factors through  $N/\text{soc}^l(N)$  by Corollary 2.4.3, and the canonical projection  $N \rightarrow N/\text{soc}^l(N)$  factors through  $N/\text{rad}^{n-l}(N)$  because  $\text{rad}^{n-l}(N) \subseteq \text{soc}^l(N)$ . If we have a good control over the modules  $N/\text{rad}^{n-l}(N)$  or  $N/\text{soc}^l(N)$ , we can factorize a long composite of ghosts as an  $l$ -fold ghost  $g : N \rightarrow N_1$  followed by another composite of ghosts  $f : N_1 \rightarrow N_2$ , and check whether  $f$  is stably trivial on  $N/\text{rad}^{n-l}(N)$  or  $N/\text{soc}^l(N)$ . For example, we can take  $l$  to be  $n - 1$ , so that  $N/\text{rad}(N)$  is a sum of trivial modules. Hence, if the map  $f$  is a ghost, the composite  $f \circ g$  is stably trivial, and so we have reproved that the generating length of  $N$  is at most its radical length  $n$  (Theorem 2.4.1). If we want to improve the bound, we need to choose  $l$  smaller. We will take  $l = n - 2$ .

The relevance of Theorem 2.4.19 is that there is an isomorphism  $k(C_{p^r} \times C_{p^s}) \cong k[X, Y]/(X^{p^r}, Y^{p^s})$ , where  $X = x - 1$  and  $Y = y - 1$ , and  $x$  and  $y$  are the generators of the cyclic summands. Under this isomorphism,  $\text{rad}(k(C_{p^r} \times C_{p^s})) \cong (X, Y)$  and  $\text{rad}^2(k(C_{p^r} \times C_{p^s})) \cong (X^2, XY, Y^2)$ . Therefore  $k(C_{p^r} \times C_{p^s})/\text{rad}^2(k(C_{p^r} \times C_{p^s})) \cong \Lambda'$ , where  $\Lambda' = \Lambda/(XY, YX) \cong k[X, Y]/(X^2, Y^2, XY)$  and  $\Lambda = k\langle X, Y \rangle/(X^2, Y^2)$  is the ring from Section 2.4.6. Thus when  $M$  is a  $k(C_{p^r} \times C_{p^s})$ -module,  $M/\text{rad}^2(M)$  will be a  $\Lambda'$ -module. Up to isomorphism, the indecomposable  $\Lambda'$ -modules biject with the  $\Lambda$ -modules of Theorem 2.4.19 satisfying conditions (a) or (b) for  $q = 1$ . Condition (c) is excluded by the requirement that  $XY$  be in the kernel.

Our proof will use this classification, so we will make it more explicit. A module satisfying condition (a) is of the first kind. If it has odd dimension, it is either the trivial module  $k$ ; the module  $M((b^{-1}a)^n)$  for some positive integer  $n$ , which we say has shape “ $W$ ”; or the module  $M((ab^{-1})^n)$  for some positive integer  $n$ , which we say has shape “ $M$ ”. For example, the “ $M$ ” module  $M((ab^{-1})^3)$  looks like



A module of the first kind with even dimension is one of the above with one end removed.

One can check that a module satisfying condition (b) of Theorem 2.4.19 corresponds to the word  $b^{-1}a$ , up to inverse and cyclic permutation. Recall that the additional data one needs to specify are a vector space  $V$  with an indecomposable automorphism  $\phi$ . Since  $\phi$  is indecomposable, one can choose a basis  $\{v_1, v_2, \dots, v_m\}$  for  $V$  such that  $\phi(v_i) = v_{i+1}$  for  $i < m$ . Thus we can view such a module as a quotient of an “ $M$ ” module, with a relation that identifies the right bottom basis element with a linear combination of the other bottom basis elements, as specified by  $\phi(v_m)$ .

We point out that this is very similar to the classification of  $kV$ -modules given in [4, Theorem 4.3.3], where  $k$  has characteristic 2.

Recall that the radical length of  $k(C_{p^r} \times C_{p^s})$  is  $p^r + p^s - 1$ . If  $N$  is projective-free, then its radical length  $n$  is at most  $p^r + p^s - 2$ , so we pick  $l = p^r + p^s - 4$ . Note that  $N/\text{rad}^2(N)$  and  $N/\text{soc}^l(N)$  are naturally  $\Lambda'$ -modules. And we have the following lemma, which helps describe summands of  $N/\text{soc}^l(N)$ .

**Lemma 2.4.27.** *Let  $G = C_{p^r} \times C_{p^s}$  be an abelian  $p$ -group of rank 2 with generators  $x$  and  $y$ , respectively, and let  $k$  be a field of characteristic  $p$ . Write  $X = x - 1$  and  $Y = y - 1$  in  $kG$ , and let  $l = p^r + p^s - 4$ . Suppose  $M$  is a  $kG$ -module containing elements  $z_0, z_2$ , and  $z_4$  such that  $Yz_0 - Xz_2$  and  $Yz_2 - Xz_4$  are in  $\text{soc}^l(M)$ . If  $p^s \geq 3$ , then  $Xz_0$  and  $Xz_2$  are in  $\text{soc}^l(M)$ . Similarly, if  $p^r \geq 3$ , then  $Yz_2$  and  $Yz_4$  are in  $\text{soc}^l(M)$ .*

Intuitively, this is saying that we cannot have a “W”-shape in the module  $M/\text{soc}^l(M)$ . In particular, only  $k$ ,  $M(ab^{-1})$  and  $M((ab^{-1})^2)$  can appear as indecomposable summands of  $M/\text{soc}^l(M)$  if  $M$  is projective-free and  $p^r, p^s \geq 3$ . Note that to exclude a module like  $M(a)$ , one takes  $z_2 = z_4 = 0$ , so the “W” isn’t visible in this case.

*Proof.* Assume that  $p^s \geq 3$ . To show that  $Xz_0 \in \text{soc}^l(M)$ , we need to show that it is killed by  $\text{rad}^l(kG)$ , which is generated by  $X^{p^r-1}Y^{p^s-3}$ ,  $X^{p^r-2}Y^{p^r-2}$  and  $X^{p^r-3}Y^{p^s-1}$  (where the last one is omitted if  $p^r = 2$ ). We compute

$$\begin{aligned} X^{p^r-1}Y^{p^s-3}Xz_0 &= X^{p^r}Y^{p^s-3}z_0 = 0, \\ X^{p^r-2}Y^{p^r-2}Xz_0 &= X^{p^r-1}Y^{p^s-3}Yz_0 = X^{p^r}Y^{p^s-3}z_2 = 0, \end{aligned}$$



and

$$X^{p^r-3}Y^{p^s-1}Xz_0 = X^{p^r-2}Y^{p^s-2}Yz_0 = X^{p^r-1}Y^{p^s-3}Yz_2 = X^{p^r}Y^{p^s-3}z_4 = 0,$$

where we have made use of the fact that  $Yz_0 - Xz_2$  and  $Yz_2 - Xz_4$  are killed by the generators. Hence  $Xz_0 \in \text{soc}^l(M)$ . Similarly,

$$\begin{aligned} X^{p^r-1}Y^{p^s-3}Xz_2 &= 0, \\ X^{p^r-2}Y^{p^s-2}Xz_2 &= X^{p^r-1}Y^{p^s-3}Yz_2 = X^{p^r}Y^{p^s-3}z_4 = 0, \end{aligned}$$

and

$$X^{p^r-3}Y^{p^s-1}Xz_2 = X^{p^r-3}Y^{p^s-1}Yz_0 = 0.$$

Hence  $Xz_2 \in \text{soc}^l(M)$ . The other case is symmetrical. □

We are now ready to prove the main theorem.

**Theorem 2.4.28.** *Let  $G = C_3 \times C_3$  with generators  $x$  and  $y$ , respectively, and let  $k$  be a field of characteristic 3. Then the ghost number of  $kG$  is 3.*

*Proof.* Theorem 2.4.9 gives a lower bound of 3, so it suffices to show that the composite of any three ghosts in  $\text{Mod}(kG)$  out of a finitely-generated module is stably trivial. As we have explained, we consider the diagram

$$\begin{array}{ccccccc} N & \xrightarrow{g_1} & N_1 & \xrightarrow{g_2} & N_2 & \xrightarrow{g_3} & N_3 \\ & & \downarrow p_1 & & \uparrow h & & \\ N/\text{rad}^2(N) & \xrightarrow{p_2} & & & & & N/\text{soc}^2(N), \end{array}$$

where  $g_1, g_2,$  and  $g_3$  are ghosts in  $\text{Mod}(kG)$  and  $N, N_1, N_2,$  and  $N_3$  are projective-free. Note that this diagram commutes in the module category. We will show that the composite  $g_3 \circ h \circ p_2$  is stably trivial, by restricting to each indecomposable summand  $M$  of  $N/\text{rad}^2(N)$ . We divide the summands  $M$  into four cases, and write  $j$  for the inclusion map  $M \rightarrow N/\text{rad}^2(N)$ .

**Case 1:**  $M$  is not of the form  $k, M(ab^{-1})$  or  $M((ab^{-1})^2)$ .

We claim that  $\text{soc}(M) \subseteq \ker(p_2 \circ j)$ , hence  $p_2 \circ j$  factors through a sum of trivial modules. Therefore, since  $g_3$  is a ghost, the composite  $g_3 \circ h \circ p_2 \circ j$  is stably trivial. We actually show that  $p_1^{-1}j(\text{soc}(M)) \subseteq \text{soc}^2(N)$ , which suffices, since  $p_2$  kills  $\text{soc}^2(N)$ . Observe using the classification that since  $M$  is not  $k$ ,  $M(ab^{-1})$  or  $M((ab^{-1})^2)$ , the elements  $X(z_0)$ ,  $X(z_2)$ ,  $Y(z_2)$  and  $Y(z_4)$  span  $\text{soc}(M)$  as  $z_0, z_2$ , and  $z_4$  vary over elements satisfying  $Y(z_0) = X(z_2)$  and  $Y(z_2) = X(z_4)$ . Suppose that we have  $s \in p_1^{-1}j(\text{soc}(M))$ , say  $p_1(s) = j(X(z_0))$  for some  $z_0 \in M$  satisfying the above relations. Since  $p_1$  is surjective, we have  $\tilde{z}_0, \tilde{z}_2$ , and  $\tilde{z}_4 \in N$  that project to  $j(z_0), j(z_2)$ , and  $j(z_4)$ , respectively. Then  $p_1(Y(\tilde{z}_0)) = p_1(X(\tilde{z}_2))$  and  $p_1(Y(\tilde{z}_2)) = p_1(X(\tilde{z}_4))$ . Since  $N$  is projective-free, its radical length is at most 4, hence  $\text{rad}^2(N) \subseteq \text{soc}^2(N)$ . Now we can apply Lemma 2.4.27 and see that  $X(\tilde{z}_0) \in \text{soc}^2(N)$ . It follows that  $s \in \text{soc}^2(N)$  because  $p_1(s) = p_1(X(\tilde{z}_0))$ . The other cases when  $p_1(s) = j(Xz_2), j(Yz_2)$ , or  $j(Yz_4)$  are similar.

**Case 2:**  $M = M(ab^{-1})$ .

The map  $p_1$  is surjective, so  $g_3hp_2$  has its image in  $\text{rad}^3(N_3)$ , using Corollary 2.4.3 and the fact that the diagram commutes in  $\text{Mod}(kG)$ .  $M$  has a basis  $\{z, Xz, Yz\}$  for some  $z$  and the map  $g_3hp_2$  sends  $z$  to an element of the form  $X^2Yw_1 + XY^2w_2$ . After restriction to  $M$ ,  $g_3hp_2$  factors through the injective module which is free on two generators  $v_1$  and  $v_2$  via the maps sending  $z$  to  $X^2Yv_1 + XY^2v_2$ ,  $v_1$  to  $w_1$  and  $v_2$  to  $w_2$ . Thus  $g_3hp_2$  is stably trivial on  $M$ .

**Case 3:**  $M = M((ab^{-1})^2)$ .

The module  $M((ab^{-1})^2)$  has schema  $kz_0 \xleftarrow{X} kz_1 \xrightarrow{Y} kz_2 \xleftarrow{X} kz_3 \xrightarrow{Y} kz_4$ . By considering the injective hull of  $M((ab^{-1})^2)$ , which is free on three generators, we see that a map out of it is stably trivial if it sends  $z_1$  to  $XY^2w_1 + X^2Yw_2$  and  $z_3$  to  $XY^2w_2 + X^2Yw_3$  for some elements  $w_1, w_2$ , and  $w_3$ . This is equivalent to  $z_1$  being sent to  $X\alpha$  and  $z_3$  being sent to  $Y\alpha$  for some  $\alpha$  in the  $2^{nd}$  radical.

To prove that this is the case, we form the following diagram:

$$\begin{array}{ccccccc}
 \tilde{\Omega}^{-2}k & \dashrightarrow & N & \xrightarrow{g_1} & N_1 & \xrightarrow{g_2} & N_2 & \xrightarrow{g_3} & N_3 \\
 & & \downarrow p_1 & & \nearrow hp_2 & & & & \\
 M((ab^{-1})^2) & \xrightarrow{j} & N/\text{rad}^2(N) & & & & & & 
 \end{array}$$

Writing  $g = g_3 \circ g_2 \circ g_1$ , we will show below that we can choose  $\tilde{z}_1$  and  $\tilde{z}_3$  in  $N$  with

$$g(\tilde{z}_1) = g_3 h p_2 j(z_1), \quad g(\tilde{z}_3) = g_3 h p_2 j(z_3), \quad \text{and} \quad Y \tilde{z}_1 = X \tilde{z}_3.$$

Since  $\tilde{\Omega}^{-2}k$  is the free module on two generators  $u_1$  and  $u_2$  subject to the relation  $Y u_1 = X u_2$ , the last displayed equality allows us to construct the dotted map  $f$ , by sending the generators to  $\tilde{z}_1$  and  $\tilde{z}_3$ , respectively. We will now show that

$$g(\tilde{z}_1) = X\alpha \quad \text{and} \quad g(\tilde{z}_3) = Y\alpha$$

for some  $\alpha \in \text{rad}^2(N_3)$ . Since  $g_1$  is a ghost, the composite  $g_1 f$  is stably trivial. It follows that, modulo  $\text{soc}^2(N_1)$ ,  $g_1(\tilde{z}_1) = X\alpha'$  and  $g_1(\tilde{z}_3) = Y\alpha'$  for some  $\alpha' \in N_1$ . Since  $g_3 g_2$  is a double ghost, it kills  $\text{soc}^2(N_1)$  and takes  $\alpha'$  into  $\text{rad}^2(N_3)$ . Hence we can set  $\alpha = g_3 g_2(\alpha')$ .

We still need to pick the  $\tilde{z}_1$  and  $\tilde{z}_3$ . First choose  $\tilde{z}_1'$  and  $\tilde{z}_3'$  in  $N$  that project to  $j(z_1)$  and  $j(z_3)$  in  $M((ab^{-1})^2)$ , respectively. The difference  $Y \tilde{z}_1' - X \tilde{z}_3'$  is in  $\text{rad}^2(N)$ , say  $Y \tilde{z}_1' - X \tilde{z}_3' = Y\beta - X\gamma$  for some  $\beta$  and  $\gamma \in \text{rad}(N)$ . We set  $\tilde{z}_1 = \tilde{z}_1' - \beta$  and  $\tilde{z}_3 = \tilde{z}_3' - \gamma$  so that  $Y \tilde{z}_1 = X \tilde{z}_3$ . By Corollary 2.4.3,  $g(\beta) = g(\gamma) = 0$ , hence

$$g(\tilde{z}_1) = g(\tilde{z}_1') = g_3 h p_2 j(z_1) \quad \text{and} \quad g(\tilde{z}_3) = g(\tilde{z}_3') = g_3 h p_2 j(z_3).$$

**Case 4:**  $M = k$  is trivial.

Then clearly  $g_3 \circ h \circ p_2$  is stably trivial when restricted to  $M$ , since  $g_3$  is a ghost.  $\square$

Since we don't require the modules  $N_1$ ,  $N_2$ , and  $N_3$  to be finitely-generated in the proof, we have actually proved a stronger result, a bound for the generating number, giving:

**Corollary 2.4.29.** *Let  $k$  be a field of characteristic 3. Then the generating number of  $k(C_3 \times C_3)$  is 3.  $\square$*

*Remark 2.4.30.* The arguments in this section go through for the group  $G = C_{p^r} \times C_{p^s}$  with  $2 < p^r \leq p^s$ , and we get that the generating number of  $kG$  is less than or equal to  $p^r + p^s - 3$ . Theorem 2.4.9 gives a lower bound of  $\left\lceil \frac{p^r - 1}{2} \right\rceil + p^s - 1$ . In particular, if  $p^r = 3$ , the ghost number of  $kG$  is  $p^s$ , and if  $p^r = 4$ , the ghost number of  $kG$  is  $p^s + 1$ .

We now indicate the modifications needed in the proof of the general case. Instead of  $g_2$  being a ghost, we take it to be a  $(p^r + p^s - 5)$ -fold ghost. Then the map  $h$  has domain  $N/\text{soc}^{p^r+p^s-4}(N)$ . In Case 1, one checks that  $p_1^{-1}j(\text{soc}(M)) \subseteq \text{soc}^{p^r+p^s-4}(N)$ . In Case 2, the map  $g_3hp_2$  sends  $z \in M(ab^{-1})$  to an element of the form  $X^{p^r-1}Y^{p^s-2}w_1 + X^{p^r-2}Y^{p^s-1}w_2$ . In Case 3, a map out of  $M((ab^{-1})^2)$  is stably trivial if it sends  $z_1$  to  $X\alpha$  and  $z_3$  to  $Y\alpha$  for some  $\alpha$  in the  $(p^r + p^s - 4)^{\text{th}}$  radical. Case 4 is unchanged.

### 2.4.8 Possible ghost numbers for group algebras

In this Section, we classify group algebras with certain small ghost numbers, and also put constraints on which ghost numbers can occur. Whenever we write  $kG$ ,  $k$  can be any field whose characteristic divides the order of  $G$ .

In [19] it is shown that the abelian groups  $G$  such that the ghost number of  $kG$  is 2 are  $C_4$ ,  $C_2 \times C_2$  and  $C_5$ . The results of the previous section and Theorem 2.4.9 give a complete list of abelian  $p$ -groups of ghost number 3:

**Proposition 2.4.31.** *Let  $G$  be an abelian  $p$ -group. Then the ghost number of  $kG$  is 3 if and only if  $G$  is  $C_7$ ,  $C_3 \times C_3$ , or  $C_2 \times C_2 \times C_2$  if and only if the generating number of  $kG$  is 3.  $\square$*

Below we will extend this to non-abelian  $p$ -groups, with one ambiguous group. We first recall a consequence of Jennings' formula which will also be useful in studying the gaps in the possible ghost numbers.

**Lemma 2.4.32** ([4, Thm. 3.14.6]). *Let  $k$  be a field of characteristic  $p$ . If  $G$  is a group of order  $p^r$ , then*

$$\begin{aligned} \text{nilpotency index of } J(k(C_p^r)) &\leq \text{nilpotency index of } J(kG) \\ &\leq \text{nilpotency index of } J(k(C_{p^r})). \end{aligned}$$

Note that the nilpotency index of  $J(k(C_p^r))$  is  $r(p - 1) + 1$ .

**Proposition 2.4.33.** *Let  $k$  be a field of characteristic  $p$ . If  $G$  is a group of order  $p^r$ , then the ghost number of  $kG$  is at least  $(r - 1)(p - 1) + 1$ .*

*Proof.* The group  $G$  has a quotient  $H$  of order  $p^{r-1}$ . By Theorem 2.4.15,  $\text{rad len}(kH)$  is a lower bound for the ghost number of  $kG$ . Now by the previous lemma,  $\text{rad len}(kH) \geq (r - 1)(p - 1) + 1$ , so we are done.  $\square$

**Theorem 2.4.34.** *The following is a complete list of the  $p$ -groups  $G$  such that  $kG$  has the specified ghost number:*

- 1: the abelian groups  $C_2$  and  $C_3$ ;
- 2: the abelian groups  $C_4$ ,  $C_2 \times C_2$  and  $C_5$ ;
- 3: the abelian groups  $C_7$ ,  $C_3 \times C_3$  and  $C_2 \times C_2 \times C_2$ , the dihedral group  $D_8$  of order 8, and possibly the quaternion group  $Q_8$ , which has ghost number 3 or 4.

*In each case, except possibly for  $Q_8$ , the generating number equals the ghost number.*

*Proof.* The case of ghost number 1 is the main result of [9].

A non-abelian  $p$ -group must have order  $p^r$  for  $r \geq 3$ , so by Proposition 2.4.33 it must have ghost number at least 3. Thus a  $p$ -group of ghost number 2 must be abelian, and this case is proved in [19].

The only ways for  $(r-1)(p-1)+1$  to equal 3 are  $p^r = 8$  or 9. The non-abelian groups of order 8 are  $D_8$  and  $Q_8$ , which are discussed in Corollary 2.4.25, Theorem 2.4.28 and Corollary 2.4.29, and there are no non-abelian groups of order 9. The abelian case is Proposition 2.4.31.  $\square$

Next we observe that, for a fixed prime  $p$ , not all positive integers can be the ghost number of some  $p$ -group. For example, since the generating hypothesis fails for  $p > 3$ , the number 1 cannot be the ghost number of a  $p$ -group with  $p > 3$ . On the other hand, the elementary abelian 2-group of rank  $l$  has ghost number  $l-1$ , so every positive integer can be a ghost number at the prime 2. Here is a result giving gaps in the possible ghost numbers at odd primes.

**Theorem 2.4.35.** *Let  $p$  be an odd prime, and let  $k$  be a field of characteristic  $p$ . Write  $(l_1, l_2, l_3, \dots)$  for the increasing sequence of integers that are ghost numbers of the group algebras  $kG$ , with  $G$  being a  $p$ -group. Then  $l_1 = \frac{p-1}{2}$ ,*

$$\frac{3(p-1)}{2} \leq l_2 = \text{ghost number of } C_p \times C_p \leq 2p-3,$$

*and  $\min(\frac{p^2-1}{2}, 2p-1) \leq l_3$ .*

*Proof.* We know that the ghost number of  $C_p$  is  $\frac{p-1}{2}$  and that of  $C_{p^2}$  is  $\frac{p^2-1}{2}$  [19, Thm. 5.4]. And the ghost number of  $C_p \times C_p$  is constrained by Theorems 2.4.9 and Remark 2.4.30:

$$\frac{3(p-1)}{2} \leq \text{ghost number of } C_p \times C_p \leq 2p-3.$$

By Proposition 2.4.33, the groups of order  $p^r$  with  $r \geq 3$  have ghost numbers at least  $2p-1$ . Comparing these numbers, we get

$$\frac{p-1}{2} \leq 2p-3 \leq \min\left(\frac{p^2-1}{2}, 2p-1\right),$$

and the theorem follows. □

Thus one sees that for large primes there are large gaps in the sequence of possible ghost numbers.

Observe that when  $p \geq 5$ ,

$$\text{the ghost number of } k(C_p^3) \leq 3p-3 \leq \frac{p^2-1}{2} = \text{the ghost number of } kC_{p^2},$$

where the first inequality uses Theorem 2.4.1. And by Theorem 2.4.9 and Proposition 2.4.33, the ghost number of  $k(C_p^r)$  is no more than the ghost number of any  $p$ -group with larger size. We conjecture that this is also true for groups of the same size, which would imply that  $l_3$  is the ghost number of  $k(C_p^3)$  when  $p \geq 5$ . The following conjecture should be viewed as the stabilized version of Lemma 2.4.32.

**Conjecture 2.4.36.** *Let  $k$  be a field of characteristic  $p$ . If  $G$  is a  $p$ -group of order  $p^r$ , then*

$$\text{ghost number of } k(C_p^r) \leq \text{ghost number of } kG \leq \text{ghost number of } k(C_{p^r}).$$

## Chapter 3

# Ghost numbers of group algebras

## II

### 3.1 Introduction

In this paper, we study several closely related invariants of a group algebra  $kG$ , where  $G$  is a finite group, and  $k$  is a field whose characteristic  $p$  divides the order of  $G$ . To describe these invariants, we work in the stable module category  $\text{StMod}(kG)$ , which is the triangulated category formed from the category of  $kG$ -modules by killing the maps that factor through a projective. A map  $f$  in  $\text{StMod}(kG)$  is called a **ghost** if it induces the zero map in Tate cohomology, or equivalently, if  $\underline{\text{Hom}}(\Omega^i k, f) = 0$  for each  $i \in \mathbb{Z}$ . Our most basic invariant is the **ghost number** of  $kG$ , which is the smallest  $n$  such that every composite of  $n$  ghosts in  $\text{Thick}\langle k \rangle$  is zero. Here  $\text{Thick}\langle k \rangle$  denotes the thick subcategory generated by the trivial module. When there are no non-trivial ghosts in  $\text{Thick}\langle k \rangle$  (so  $n = 1$ ), we say that the **generating hypothesis** holds for  $kG$ . This is motivated the Freyd's generating hypothesis in stable homotopy theory [25], which is still an open question. In a series of papers [9, 16, 18, 20] (with a minor correction made below), it has been shown that the generating hypothesis holds for  $kG$  if and only if the Sylow  $p$ -subgroup of  $G$  is  $C_2$  or  $C_3$ . However, computing the ghost number in cases where it is larger than one has proven to be difficult. Some preliminary work was done in [19], where the ghost numbers of cyclic  $p$ -groups were computed, and various upper and lower bounds were obtained in other cases. Substantial progress was made in our previous paper [23], where we computed the ghost numbers of  $k(C_3 \times C_3)$  and other algebras of wild representation type, as well as the ghost numbers of dihedral 2-groups, the first non-abelian computations.

In this chapter, we extend the past work in two different ways. Our initial motivation was to produce the first computations of ghost numbers for non- $p$ -groups. For a  $p$ -group,  $\text{Thick}\langle k \rangle$  coincides with  $\text{stmod}(kG)$ , the full subcategory of finitely generated modules, which allows one to use induction from a subgroup to produce ghosts in  $\text{Thick}\langle k \rangle$ . But for a general  $p$ -group,  $\text{Thick}\langle k \rangle$  is usually a proper subcategory of  $\text{stmod}(kG)$ , which makes things more delicate. Nevertheless, we obtain a variety of exact computations of ghost numbers in this setting, e.g., for all dihedral groups at all primes, as well as new bounds. One of our new techniques is to produce ghosts for  $kG$  by inducing up a ghost from a subgroup and then projecting onto the principal block. We show that this composite is faithful, and so when  $\text{Thick}\langle k \rangle$  coincides with the principal block of  $\text{stmod}(kG)$ , we are able to use this technique to study the ghost number of  $kG$ . As an example, we prove that the ghost number is finite in this situation. Our main results on ghost numbers are described in the detailed summary below.



Our work on non- $p$ -groups led us to realize the importance of another invariant in this setting, which is the **simple ghost number**, a concept suggested in [12]. A **simple ghost** is a map  $f$  such that  $\underline{\text{Hom}}(\Omega^i S, f) = 0$  for each simple module  $S$  and each  $i \in \mathbb{Z}$ , and the **simple ghost number** of  $kG$  is the smallest  $n$  such that every composite of  $n$  simple ghosts in  $\text{stmod}(kG)$  is trivial. The point here is that  $\text{stmod}(kG)$  is the thick subcategory generated by the simple modules, so this is exactly analogous to the ghost number, with the trivial module  $k$  replaced by the set of all simple modules. Moreover, for a  $p$ -group,  $k$  is the only simple module, so the two notions coincide. It turns out that there is a close relationship between the simple ghost number of  $kG$  and the ghost number of  $kP$ , where  $P$  is a Sylow  $p$ -subgroup of  $G$ , and by studying both invariants at once we can make many more computations. Again, these are described in the detailed summary below.

One of the most important techniques in our work is the use of induction and restriction, which brings us to the third and final invariant that we study in this paper. A **strong ghost** is a map  $f$  whose restriction to every subgroup is a ghost, or equivalently, such that  $\underline{\text{Hom}}(\Omega^i k \uparrow_H^G, f) = 0$  for each subgroup  $H$  of  $G$  and each  $i \in \mathbb{Z}$ . The **strong ghost number** of  $kG$  is the smallest  $n$  such that every composite of  $n$  strong ghosts in  $\text{stmod}(kG)$  is trivial. This follows the same pattern as above, since  $\text{stmod}(kG)$  is the thick subcategory generated by the test objects  $k \uparrow_H^G$ . Unlike the other invariants, one can show that the strong ghost number of  $kG$  equals the strong ghost number of  $kP$ , and so it suffices to study  $p$ -groups. Below we summarize our computations of and bounds on strong ghost numbers.

The overall organization of the paper is as follows. In Section 3.2, we introduce general concepts that will be of use in the rest of the paper and recall some background material on modular representation theory. Sections 3.3, 3.4 and 3.5 study both the ghost number and the simple ghost number, and are distinguished by the assumptions placed on the group: In Section 3.3, we assume that the Sylow  $p$ -subgroup of  $G$  is normal. In Section 3.4, we assume that  $\text{Thick}\langle k \rangle$  coincides with the principal block. And in Section 3.5, we assume that the Sylow  $p$ -subgroup is cyclic. Finally, in Section 3.6, we study the strong ghost number.

Note that there is some overlap in the assumptions made in Sections 3.3, 3.4 and 3.5. For example, in Section 3.4.1 we study groups whose Sylow  $p$ -subgroup is a direct factor, and these groups satisfy the assumptions of Sections 3.3 and 3.4. This includes the case of  $p$ -groups. And in Section 3.5.1, we study groups with a cyclic normal Sylow  $p$ -subgroup,

and these satisfy the assumptions of all three sections. In general, the assumptions made are independent, except that Sunil Chebolu and Jan Mináč have an unpublished proof that when the Sylow  $p$ -subgroup is cyclic,  $\text{Thick}\langle k \rangle$  coincides with the principal block. (This may be one of those results that is “known to the experts”.)

We now summarize the main results of each section in more detail. In Section 3.2.1, working in a general triangulated category, we define the Freyd length and Freyd number with respect to a set  $\mathbb{P}$  of test objects. The Freyd number generalizes the ghost number, simple ghost number and strong ghost number defined above. We also recall the closely related concept of length with respect to a projective class, and we prove general results about both of these invariants. In Section 3.2.2, we recall the basics of the stable module category, and in Section 3.2.3 we formally introduce ghosts and simple ghosts, specializing the Freyd length and Freyd number to these two situations.

In Section 3.3 we assume that our group  $G$  has a normal Sylow  $p$ -subgroup  $P$ . Under this assumption, in Section 3.3.1 we show that a map in  $\text{StMod}(kG)$  is a simple ghost if and only if its restriction to  $P$  is a ghost, and show that the simple ghost number of  $kG$  is equal to the ghost number of  $kP$ . It follows that when  $P$  is normal, the simple generating hypothesis holds if and only if  $P$  is  $C_2$  or  $C_3$ . (We don’t have a characterization of when the simple generating hypothesis holds in general, but we do know that it does not depend only on the Sylow  $p$ -subgroup. See Section 3.5.2.) In Section 3.3.2, we apply this result to the group  $A_4$  at the prime 2, deducing that the simple ghost number is 2 and that the ghost number is between 2 and 4. We also give an example of a ghost for  $A_4$  whose restriction to the Sylow  $p$ -subgroup is not a ghost.

In Section 3.4, we focus on groups whose principal block is generated by  $k$  in the sense that  $\text{stmod}(B_0) = \text{Thick}\langle k \rangle$  (or, equivalently,  $\text{StMod}(B_0) = \text{Loc}\langle k \rangle$ ). We show that this holds when the Sylow  $p$ -subgroup  $P$  is a direct factor, in Section 3.4.1, using a result that shows that there is an equivalence between  $\text{stmod}(kP)$  and  $\text{Thick}_G\langle k \rangle$ . This last result corrects an error in [20]; see the comments after Theorem 3.4.1. In Section 3.4.2, we show that if  $\text{stmod}(B_0) = \text{Thick}\langle k \rangle$ , then the ghost number of  $kG$  is finite. We prove this by using a comparison to the simple ghost number, which is finite for any  $G$ . We conjecture that the ghost number is finite for general  $G$ . This is related to a question proposed in [8]. (See Remark 3.4.8.) Still assuming that the principal block is generated by  $k$ , we show that the ghost number of  $kG$  is greater than or equal to the ghost number of  $kP$ , by first showing that the composite of inducing up from  $P$  to  $G$  followed by projection onto the principal block is faithful. In Section 3.4.3, working at

the prime 2, we show that for a dihedral group  $D_{2ql}$  of order  $2ql$ , with  $q$  a power of 2 and  $l$  odd, the principal block is generated by  $k$  and the ghost number of  $D_{2ql}$  is equal to the ghost number of the Sylow 2-subgroup  $D_{2q}$ , which was shown to be  $\lfloor \frac{q}{2} + 1 \rfloor$  in [23]. By computing the simple ghost lengths of modules in non-principal blocks, we are also able to show that the simple ghost number of  $D_{2ql}$  is again  $\lfloor \frac{q}{2} + 1 \rfloor$ .

Section 3.5 studies the case when the Sylow  $p$ -subgroup  $P$  is cyclic. In Section 3.5.1, we assume that  $P$  is cyclic and normal, and show that every simple module in the principal block is a suspension of the trivial module. It follows that  $\text{stmod}(B_0) = \text{Thick}\langle k \rangle$  and that a map in  $\text{Thick}\langle k \rangle$  is a ghost if and only if it is a simple ghost. Thus the simple ghost number of  $kG$ , the ghost number of  $kG$  and the ghost number of  $kP$  are all equal. Since  $P$  is a cyclic  $p$ -group, its ghost number is known [19]. In particular, this allows us to compute the ghost numbers of the dihedral groups at an odd prime. Combined with the results above, this completes the computation of the ghost numbers of the dihedral groups, at any prime. The group  $SL(2, p)$  has a cyclic Sylow  $p$ -subgroup  $P$ , but it is not normal. By studying the normalizer  $L$  of  $P$  and applying the results of Section 3.5.1 to  $L$ , we show in Section 3.5.2 that the simple generating hypothesis holds for  $SL(2, p)$  over a field  $k$  of characteristic  $p$ . Along the way, we find that there is an equivalence  $\text{stmod}(kG) \rightarrow \text{stmod}(kL)$ , but that the simple generating hypothesis does *not* hold for  $kL$ .

In Section 3.6 we study strong ghosts. We begin in Section 3.6.1 by showing that the strong ghost number of a group algebra  $kG$  equals the strong ghost number of  $kP$ , where  $P$  is a Sylow  $p$ -subgroup of  $G$ . Then we compute the strong ghost numbers of cyclic  $p$ -groups in Section 3.6.2. Finally, in Section 3.6.3, we show that the strong ghost number of a dihedral 2-group  $D_{4q}$  is between 2 and 3, with the upper bound being the non-trivial result.

## 3.2 Background

In this section, we provide background material that will be used throughout the paper. In Section 3.2.1, we define invariants of a triangulated category  $\mathbb{T}$  which depend on a set  $\mathbb{P}$  of test objects, and prove general results about these invariants. In Section 3.2.2, we recall some background results about the stable module category of a finite group. In Section 3.2.3, we apply the general theory to two sets of test objects in the stable

module category of a group, giving rise to invariants called the ghost number and the simple ghost number.

### 3.2.1 The generating hypothesis and related invariants

We begin this section by stating the generating hypothesis with respect to a set of objects in a triangulated category and defining invariants, the Freyd length and the length, which measure the degree to which the generating hypothesis fails. Motivated by this, we recall the definition of a projective class. Then, working in a general triangulated category, we study the relationship between the lengths (and Freyd lengths) of an object with respect to different projective classes. We also compare lengths in different categories by using the pullback projective class.

Let  $\mathcal{T}$  be a triangulated category, and let  $\mathbb{P}$  be a set of objects in  $\mathcal{T}$ . The thick subcategory generated by  $\mathbb{P}$ , denoted  $\text{Thick}\langle\mathbb{P}\rangle$ , is the smallest full triangulated subcategory of  $\mathcal{T}$  that is closed under retracts and contains  $\mathbb{P}$ . It is easy to see that  $\mathbb{P}$  detects zero objects in  $\text{Thick}\langle\mathbb{P}\rangle$ , i.e., if  $M \in \text{Thick}\langle\mathbb{P}\rangle$  and  $[\Sigma^i\mathbb{P}, M] = 0$  for all  $P \in \mathbb{P}$  and  $i \in \mathbb{Z}$ , then  $M \cong 0$ . Here we write  $[-, -]$  for the hom-sets in  $\mathcal{T}$ .

The **generating hypothesis** for the set of test objects  $\mathbb{P}$  is the statement that  $\mathbb{P}$  detects trivial *maps* in  $\text{Thick}\langle\mathbb{P}\rangle$ , i.e., if  $f$  is a map in  $\text{Thick}\langle\mathbb{P}\rangle$  and  $[\Sigma^i P, f] = 0$  for all  $P \in \mathbb{P}$  and  $i \in \mathbb{Z}$ , then  $f$  is the zero map [12].

When the generating hypothesis for  $\mathbb{P}$  fails, there is a natural invariant which measures the degree to which it fails. Let  $\mathcal{I}$  denote the class of maps such that  $[\Sigma^i P, f] = 0$  for all  $P \in \mathbb{P}$  and  $i \in \mathbb{Z}$ , and write  $\mathcal{I}_t$  for such maps in  $\text{Thick}\langle\mathbb{P}\rangle$ . The **Freyd length**  $\text{len}_{\mathbb{P}}^{\text{F}}(X)$  of an object  $X$  in  $\text{Thick}\langle\mathbb{P}\rangle$  with respect to  $\mathbb{P}$  is the smallest number  $n$  such that every composite  $X \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$  of  $n$  maps in  $\mathcal{I}_t$  is zero. The **Freyd number** of  $\mathcal{T}$  with respect to  $\mathbb{P}$  is the least upper bound of the Freyd lengths of the objects in  $\text{Thick}\langle\mathbb{P}\rangle$ . With this terminology, the generating hypothesis holds for  $\mathbb{P}$  if and only if the Freyd number of  $\mathcal{T}$  with respect to  $\mathbb{P}$  is 1.

It turns out to be fruitful to consider a related invariant, where none of the objects are required to lie in  $\text{Thick}\langle\mathbb{P}\rangle$ . The **length**  $\text{len}_{\mathbb{P}}(X)$  of an object  $X$  in  $\mathcal{T}$  with respect to  $\mathbb{P}$  is the smallest number  $n$  such that every composite  $X \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$  of  $n$  maps in  $\mathcal{I}$  is zero, if this exists (which is the case when  $X \in \text{Thick}\langle\mathbb{P}\rangle$ ). This is clearly at least as big as the Freyd length, but has better formal properties which make it easier to work

with. These properties are best expressed in terms of the *projective class* generated by  $\mathbb{P}$ . To motivate the definition, note that  $\langle \mathbb{P} \rangle$  detects the same maps in  $\mathbb{T}$  as  $\mathbb{P}$  does, where  $\langle \mathbb{P} \rangle$  denotes the closure of  $\mathbb{P}$  under retracts, sums, suspensions and desuspensions. Moreover, it is easy to show ([21]) that  $\mathcal{P} := \langle \mathbb{P} \rangle$  and  $\mathcal{I}$  determine each other in the sense of the following definition:

**Definition 3.2.1.** Let  $\mathbb{T}$  be a triangulated category. A **projective class** in  $\mathbb{T}$  consists of a class  $\mathcal{P}$  of objects of  $\mathbb{T}$  and a class  $\mathcal{I}$  of morphisms of  $\mathbb{T}$  such that:

- (i)  $\mathcal{P}$  consists of exactly the objects  $P$  such that every composite  $P \rightarrow X \rightarrow Y$  is zero for each  $X \rightarrow Y$  in  $\mathcal{I}$ ,
- (ii)  $\mathcal{I}$  consists of exactly the maps  $X \rightarrow Y$  such that every composite  $P \rightarrow X \rightarrow Y$  is zero for each  $P$  in  $\mathcal{P}$ .
- (iii) for each  $X$  in  $\mathbb{T}$ , there is a triangle  $P \rightarrow X \rightarrow Y \rightarrow \Sigma P$  with  $P$  in  $\mathcal{P}$  and  $X \rightarrow Y$  in  $\mathcal{I}$ .

Our main examples will be projective classes of the form  $(\langle \mathbb{P} \rangle, \mathcal{I})$ , which we call the **(stable) projective class generated by  $\mathbb{P}$** .

Given a projective class  $(\mathcal{P}, \mathcal{I})$ , there is a sequence of **derived projective classes**  $(\mathcal{P}_n, \mathcal{I}^n)$  [21]. The ideal  $\mathcal{I}^n$  consists of all  $n$ -fold composites of maps in  $\mathcal{I}$ , and  $X$  is in  $\mathcal{P}_n$  if and only if it is a retract of an object  $M$  that sits inside a triangle  $P \rightarrow M \rightarrow Q \rightarrow \Sigma P$  with  $P \in \mathcal{P}_1 = \mathcal{P}$  and  $Q \in \mathcal{P}_{n-1}$ . For  $n = 0$ , we let  $\mathcal{P}_0$  consist of all zero objects and  $\mathcal{I}^0$  consist of all maps in  $\mathbb{T}$ .

Extending the definition above to any projective class, we define the **length**  $\text{len}_{\mathcal{P}}(X)$  of an object  $X$  in  $\mathbb{T}$  with respect to  $(\mathcal{P}, \mathcal{I})$  to be the smallest number  $n$  such that every map in  $\mathcal{I}^n$  with domain  $X$  is trivial. The fact that each pair  $(\mathcal{P}_n, \mathcal{I}^n)$  is a projective class implies that the length of  $X$  is equal to the smallest  $n$  such that  $X \in \mathcal{P}_n$ . When  $\mathcal{P} = \langle \mathbb{P} \rangle$ , we write  $\text{len}_{\mathbb{P}}(X)$  as above.

We note that different sets of objects can generate the same projective class but different thick subcategories, so the Freyd length depends on the choice of generating set  $\mathbb{P}$ , not just on the projective class  $\langle \mathbb{P} \rangle$  it generates.

The following lemma is a direct consequence of the definition of a projective class. This idea is used in comparing the ghost length and the simple ghost length of a module.

**Lemma 3.2.2.** *Let  $\mathsf{T}$  be a triangulated category, and let  $(\mathcal{P}, \mathcal{I})$  and  $(\mathcal{Q}, \mathcal{J})$  be projective classes on  $\mathsf{T}$ . Then we have the following relationships:*

- *If  $M$  has finite length with respect to  $(\mathcal{P}, \mathcal{I})$ , then*

$$\mathrm{len}_{\mathcal{P}_n}(M) = \left\lceil \frac{\mathrm{len}_{\mathcal{P}}(M)}{n} \right\rceil.$$

- *If  $\mathcal{Q} \subseteq \mathcal{P}$ , then*

$$\mathrm{len}_{\mathcal{P}}(M) \leq \mathrm{len}_{\mathcal{Q}}(M).$$

- *If  $\mathcal{Q} \subseteq \mathcal{P}_n$ , then*

$$\mathrm{len}_{\mathcal{P}}(M) \leq n \mathrm{len}_{\mathcal{P}_n}(M) \leq n \mathrm{len}_{\mathcal{Q}}(M).$$

*Proof.* To show  $\mathrm{len}_{\mathcal{P}_n}(M) = \left\lceil \frac{\mathrm{len}_{\mathcal{P}}(M)}{n} \right\rceil$ , we actually need to prove two inequalities:

$$\mathrm{len}_{\mathcal{P}_n}(M) \leq \left\lceil \frac{\mathrm{len}_{\mathcal{P}}(M)}{n} \right\rceil \quad \text{and} \quad \mathrm{len}_{\mathcal{P}}(M) \leq n \mathrm{len}_{\mathcal{P}_n}(M). \quad (3.2.1)$$

For the second inequality, let  $\mathrm{len}_{\mathcal{P}_n}(M) = m$ . Then  $M \in (\mathcal{P}_n)_m \subseteq \mathcal{P}_{mn}$ , which means that  $\mathrm{len}_{\mathcal{P}}(M) \leq mn$ . Equivalently, we can prove the inequality using the inclusion  $\mathcal{I}^{mn} \subseteq (\mathcal{I}^n)^m$ , i.e., if every  $m$ -fold composite of  $n$ -fold composites of maps in  $\mathcal{I}$  out of  $M$  is trivial, then every  $mn$ -fold composite of maps in  $\mathcal{I}$  out of  $M$  is trivial.

Using the inclusions the other way, i.e.,  $(\mathcal{P}_n)_m \supseteq \mathcal{P}_{mn}$  and  $\mathcal{I}^{mn} \subseteq (\mathcal{I}^n)^m$ , one can prove that  $\mathrm{len}_{\mathcal{P}_n}(M) \leq \lceil \mathrm{len}_{\mathcal{P}}(M)/n \rceil$ .

The other inequalities in the lemma follow with similar proofs.  $\square$

The analog of Lemma 3.2.2 for Freyd lengths is a bit more subtle because of the need to take into account the appropriate thick subcategories. For example, if  $(\langle \mathbb{P} \rangle, \mathcal{I})$  and  $(\langle \mathbb{Q} \rangle, \mathcal{J})$  are projective classes and  $\mathbb{Q} \subseteq \mathbb{P}$ , then clearly  $\mathcal{I} \subseteq \mathcal{J}$ . But the inclusion  $\mathrm{Thick}\langle \mathbb{Q} \rangle \subseteq \mathrm{Thick}\langle \mathbb{P} \rangle$  goes in the other direction, so in general there is no inclusion between  $\mathcal{I}_t$  and  $\mathcal{J}_t$ .

Nevertheless, if we include assumptions which control the thick subcategories, then most of the results go through. We simply work with  $(\mathcal{I}_t)^n$  instead of  $\mathcal{I}^n$ . However, one difference is that we only have an inclusion  $(\mathcal{I}_t)^{mn} \subseteq ((\mathcal{I}^n)_t)^m$ , rather than an equality,

and as a result, we lose the first inequality from equation (3.2.1). In the next lemma, we give a result which we will use later.

**Lemma 3.2.3.** *Let  $\mathsf{T}$  be a triangulated category, and let  $(\langle \mathbb{P} \rangle, \mathcal{I})$  and  $(\langle \mathbb{Q} \rangle, \mathcal{J})$  be projective classes on  $\mathsf{T}$  generated by sets  $\mathbb{P}$  and  $\mathbb{Q}$ . If  $\mathbb{P} \subseteq \text{Thick}\langle \mathbb{Q} \rangle$ ,  $\mathbb{Q} \subseteq \langle \mathbb{P} \rangle_n$  and  $M \in \text{Thick}\langle \mathbb{P} \rangle$ , then*

$$\text{len}_{\mathbb{P}}^{\mathbb{F}}(M) \leq n \text{len}_{\mathbb{Q}}^{\mathbb{F}}(M).$$

*Proof.* Let  $m = \text{len}_{\mathbb{Q}}^{\mathbb{F}}(M)$ . We must show that any composite  $M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{mn}$  of maps in  $\mathcal{I}$  with the  $M_i$  in  $\text{Thick}\langle \mathbb{P} \rangle$  is zero. The inclusion  $\mathbb{P} \subseteq \text{Thick}\langle \mathbb{Q} \rangle$  tells us that  $\text{Thick}\langle \mathbb{P} \rangle \subseteq \text{Thick}\langle \mathbb{Q} \rangle$ , so these maps are in  $\text{Thick}\langle \mathbb{Q} \rangle$ . The inclusion  $\mathbb{Q} \subseteq \langle \mathbb{P} \rangle_n$  tells us that  $\mathcal{I}^n \subseteq \mathcal{J}$ . Thus the above composite is an  $m$ -fold composite of maps in  $\mathcal{J} \cap \text{Thick}\langle \mathbb{Q} \rangle$ , and so is zero by the definition of  $m$ .  $\square$

Consider a triangle

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow \Sigma M'$$

in  $\mathsf{T}$ . We know that  $\text{len}(M) \leq \text{len}(M') + \text{len}(M'')$  by [21, Note 3.6]. We will prove the analog for Freyd lengths.

**Lemma 3.2.4.** *Let  $\mathsf{T}$  be a triangulated category with a set of test objects  $\mathbb{P}$  and  $\mathcal{I}_t$  be the class of maps in  $\text{Thick}\langle \mathbb{P} \rangle$  that are trivial on  $\mathbb{P}$ . Let  $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow \Sigma M'$  be a triangle in  $\mathsf{T}$ . If  $M'$  and  $M''$  have finite Freyd lengths, then*

$$\text{len}_{\mathbb{P}}^{\mathbb{F}}(M) \leq \text{len}_{\mathbb{P}}^{\mathbb{F}}(M') + \text{len}_{\mathbb{P}}^{\mathbb{F}}(M'').$$

*Proof.* Let  $n = \text{len}_{\mathbb{P}}^{\mathbb{F}}(M')$  and  $l = \text{len}_{\mathbb{P}}^{\mathbb{F}}(M'')$ . We want to show that any map  $\phi : M \rightarrow N$  in  $(\mathcal{I}_t)^{n+l}$  is trivial. Write  $\phi$  as  $\phi_2\phi_1$ , where  $\phi_1$  is in  $(\mathcal{I}_t)^n$  and  $\phi_2$  is in  $(\mathcal{I}_t)^l$ . Then, since  $\text{len}_{\mathbb{P}}^{\mathbb{F}}(M') = n$ , the composite  $\phi_1\alpha$  is stably trivial and  $\phi_1$  factors through  $M''$ :

$$\begin{array}{ccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\ & & \downarrow \phi_1 & \swarrow \psi & \\ & & W & & \\ & & \downarrow \phi_2 & & \\ & & N & & \end{array}$$

Now since  $\text{len}_{\mathbb{P}}^{\mathbb{F}}(M'') = l$ , the composite  $\phi_2\psi$  is trivial and so  $\phi$  is trivial as well.  $\square$

Now we explain how to compare lengths in different categories, using the pullback projective class.

**Definition 3.2.5.** Let  $U : \mathbb{T} \rightarrow \mathbb{S}$  be a triangulated functor between triangulated categories, together with a left adjoint  $F : \mathbb{S} \rightarrow \mathbb{T}$  that is also triangulated, and let  $(\mathcal{P}, \mathcal{I})$  be a projective class on  $\mathbb{S}$ . We define

$$\mathcal{I}' := \{M \rightarrow N \text{ in } \mathbb{T} \text{ such that } UM \rightarrow UN \text{ is in } \mathcal{I}\} = U^{-1}(\mathcal{I}).$$

Then  $\mathcal{I}'$  forms the ideal of a projective class on  $\mathbb{T}$  with relative projectives

$$\mathcal{P}' = \{\text{retracts of } FP \text{ for } P \text{ in } \mathcal{P}\} = \langle F(\mathcal{P}) \rangle.$$

The projective class  $(\mathcal{P}', \mathcal{I}')$  on  $\mathbb{T}$  is called the **pullback** of  $(\mathcal{P}, \mathcal{I})$  along the right adjoint  $U$  [22]. It is the projective class on  $\mathbb{T}$  generated by the class of objects  $F(\mathcal{P})$ .

One readily sees that the following relationships hold, since  $F$  sends  $\mathcal{P}$  into  $\mathcal{P}'$  and  $U$  sends  $\mathcal{I}'$  into  $\mathcal{I}$ .

**Lemma 3.2.6.** *Suppose we are in the above situation and that  $M \in \mathbb{S}$  and  $N \in \mathbb{T}$ . Then*

$$\text{len}_{\mathcal{P}}(M) \geq \text{len}_{\mathcal{P}'}(FM),$$

*and, if the functor  $U$  is faithful,*

$$\text{len}_{\mathcal{P}'}(N) \leq \text{len}_{\mathcal{P}}(UN). \quad \square$$

### 3.2.2 The stable module category

Let  $G$  be a finite group, and let  $k$  be a field whose characteristic  $p$  divides the order of  $G$ . The stable module category  $\text{StMod}(kG)$  is a quotient category of the module category  $\text{Mod}(kG)$ . For  $kG$ -modules  $M$  and  $N$ , the hom-set  $\underline{\text{Hom}}(M, N)$  in  $\text{StMod}(kG)$  is the quotient  $\text{Hom}(M, N)/\text{PHom}(M, N)$ , where  $\text{PHom}(M, N)$  consists of the maps that factor through a projective module. Then  $\text{StMod}(kG)$  is a triangulated category with triangles coming from short exact sequences in  $\text{Mod}(kG)$ . Two modules  $M$  and  $N$  are said to be **stably isomorphic** if they are isomorphic in the stable module category, and this holds if and only if their projective-free summands are isomorphic as  $kG$ -modules.



We use the symbol  $\cong$  for isomorphism as  $kG$ -modules, unless otherwise stated. The desuspension  $\Omega M$  of a module  $M$  is defined to be the kernel in any short exact sequence

$$0 \longrightarrow \Omega M \longrightarrow Q \longrightarrow M \longrightarrow 0,$$

where  $Q$  is a projective  $kG$ -module. Note that  $\Omega M$  is well-defined in the stable module category, and we denote by  $\tilde{\Omega}M$  the projective-free summand of  $\Omega M$ . We write  $\text{stmod}(kG)$  for the full subcategory of finitely generated modules in  $\text{StMod}(kG)$ . (More precisely, we include all modules which are stably isomorphic to finitely generated  $kG$ -modules.) We refer to [14] for more background on  $\text{StMod}(kG)$ .

Now let  $P$  be a Sylow  $p$ -subgroup of  $G$ . We consider the adjunction

$$\uparrow^G : \text{StMod}(kP) \rightleftarrows \text{StMod}(kG) : \text{res} = \downarrow_P,$$

with  $\uparrow^G$  as a left adjoint. We quote the following important facts in modular representation theory for further use:

**Lemma 3.2.7** ([4]). *Let  $G$  be a finite group, let  $k$  be a field whose characteristic  $p$  divides the order of  $G$ , and let  $P$  be a Sylow subgroup of  $G$ . Then the following hold:*

(i) *The restriction functor  $\downarrow_P : \text{StMod}(kG) \rightarrow \text{StMod}(kP)$  is faithful.*

(ii) *Each  $kG$ -module  $M$  is a summand of the module  $M \downarrow_P \uparrow^G$ .*

(iii) *A  $kG$ -module  $Q$  is projective if and only if its restriction  $Q \downarrow_P$  is projective.* □

**Theorem 3.2.8** (Mackey's Theorem [4]). *Let  $L$  and  $H$  be subgroups of  $G$ , and let  $V$  be a  $kH$ -module. Then*

$$(V \uparrow_H^G) \downarrow_L \cong \bigoplus_{s \in L \backslash G / H} (sV) \downarrow_{L \cap sHs^{-1}} \uparrow^L.$$

*Here  $sV = s \otimes V$  is the corresponding  $k(sHs^{-1})$ -module for  $s \in G$ , and the sum is taken over the double coset representatives.* □

### 3.2.3 Ghost lengths and simple ghost lengths in $\text{StMod}(kG)$

By the **generating hypothesis** on  $\text{StMod}(kG)$ , we mean the generating hypothesis with respect to the set  $\{k\}$  containing only the trivial module. Since Tate cohomology is represented by  $k$ , the associated ideal  $\mathcal{G}$  consists of **ghosts** in  $\text{StMod}(kG)$ , i.e., maps

which induce the zero map in Tate cohomology. Thus the generating hypothesis is the statement that there are no non-trivial maps in  $\text{Thick}\langle k \rangle$  which induce the zero map in Tate cohomology. The projective class  $(\mathcal{F}, \mathcal{G})$  generated by  $k$  has  $\mathcal{F} = \langle k \rangle$ , summands of direct sums of suspensions and desuspensions of  $k$ . We call  $(\mathcal{F}, \mathcal{G})$  the **ghost projective class**. When we need to indicate the dependence on the group, we write  $(\mathcal{F}^G, \mathcal{G}_G)$ .

For a module  $M \in \text{Thick}\langle k \rangle$ , its **ghost length**  $\text{gl}(M)$  is defined to be its Freyd length with respect to  $\{k\}$ , and the **ghost number** of  $kG$  is the Freyd number of  $\text{StMod}(kG)$  with respect to  $\{k\}$ . With this terminology, the generating hypothesis is the statement that the ghost number of  $kG$  is 1.

Since the restriction functor preserves the trivial module, we can induce up a (non-trivial) ghost from a subgroup of  $G$  to get a (non-trivial) ghost of  $G$ . This provides a very convenient tool when we study  $p$ -groups. However, the inducing up technique has limited use for a general finite group, since the ghosts, when induced up, do not always land in  $\text{Thick}\langle k \rangle$ , which is often smaller than  $\text{stmod}(kG)$ .

In general, the stable module category is generated by the set  $\mathbb{S}$  of simple modules. This suggests that we examine the projective class  $(\mathcal{S}, \text{s}\mathcal{G})$  generated by  $\mathbb{S}$ , which we call the **simple ghost projective class**, and compare it to the ghost projective class. Here  $\mathcal{S} = \langle \mathbb{S} \rangle$ , and the maps in  $\text{s}\mathcal{G}$  are called **simple ghosts**. The **simple generating hypothesis** for  $kG$  is the generating hypothesis with respect to  $\mathbb{S}$ . The Freyd length (respectively number) with respect to  $\mathbb{S}$  will be called the **simple ghost length** (respectively **number**). For  $M \in \text{stmod}(kG)$ , the simple ghost length is denoted by  $\text{sgl}(M) = \text{len}_{\mathbb{S}}^{\text{F}}(M)$ . Note that while the ghost length is only defined for  $M \in \text{Thick}\langle k \rangle$ , the simple ghost length is defined for all  $M \in \text{stmod}(kG)$  since  $\text{stmod}(kG) = \text{Thick}\langle \mathbb{S} \rangle$ . If  $G$  is a  $p$ -group, then  $\mathbb{S} = \{k\}$ , so the simple ghost projective class and the ghost projective class coincide.

*Remark 3.2.9.* The radical series of a  $kG$ -module  $M$  gives a construction of  $M$  using simple modules, showing that  $\text{len}_{\mathbb{S}}(M)$  is at most the radical length of  $M$ , since the pair  $(\mathcal{S}_n, \text{s}\mathcal{G}^n)$  is a projective class, as described in Section 3.2.1. Therefore,

$$\text{sgl}(M) = \text{len}_{\mathbb{S}}^{\text{F}}(M) \leq \text{len}_{\mathbb{S}}(M) \leq \text{rad len}(M) \leq \text{rad len}(kG).$$

This shows that the simple ghost number of  $kG$  is finite. In particular, for  $P$  a  $p$ -group, the ghost number of  $kP$  is finite. In Conjecture 3.4.9 we assert that the ghost number of  $kG$  is always finite, but this is an open question.

In the last section of the paper, we will study another projective class on  $\text{StMod}(kG)$ , which is called the strong ghost projective class.

### 3.3 Groups with normal Sylow $p$ -subgroups

In this section, we assume that our group  $G$  has a normal Sylow  $p$ -subgroup  $P$ . Under this assumption, in Section 3.3.1 we show that the simple ghost number of  $kG$  is equal to the ghost number of  $kP$ . In Section 3.3.2, we apply this result to the group  $A_4$  at the prime 2, deducing that the simple ghost number is 2 and that the ghost number is between 2 and 4.

#### 3.3.1 The simple projective class as a pullback

In this section, we show that the simple ghost projective class on  $\text{StMod}(kG)$  is the pullback of the ghost projective class on  $\text{StMod}(kP)$ , under the assumption that the Sylow  $p$ -subgroup  $P$  is normal in  $G$ . Then we show that simple ghost lengths in  $\text{StMod}(kG)$  are the same as ghost lengths in  $\text{StMod}(kP)$ . The main result of this section should be viewed as the stabilised version of the next lemma:

**Lemma 3.3.1** ([1, Lemma 5.8]). *Let  $k$  be a field of characteristic  $p$ , and let  $G$  be a finite group with a normal Sylow  $p$ -subgroup  $P$ . Let  $M$  be a  $kG$ -module. Then  $\text{rad}(M)\downarrow_P = \text{rad}(M\downarrow_P)$ . It follows that the radical sequence of  $M$  coincides with that of  $M\downarrow_P$ . In particular,  $M$  is semisimple if and only if  $M\downarrow_P$  is.  $\square$*

We write  $(\langle \mathcal{F}^P \uparrow^G \rangle, \text{res}^{-1}(\mathcal{G}_P))$  for the pullback of  $(\mathcal{F}^P, \mathcal{G}_P)$  along the restriction functor. Then, by Lemma 3.2.7(ii), we have  $\mathcal{F}^G \subseteq \langle \mathcal{F}^P \uparrow^G \rangle$ . Equivalently,  $\text{res}^{-1}(\mathcal{G}_P) \subseteq \mathcal{G}_G$ , i.e., if a map in  $\text{StMod}(kG)$  restricts to a ghost in  $\text{StMod}(kP)$ , then it is a ghost. (Note that we write  $\downarrow_P$  for the restriction functor except when considering preimages, in which case we write  $\text{res}^{-1}$ .) We can describe  $\text{res}^{-1}(\mathcal{G}_P)$  more precisely when  $P$  is normal in  $G$ .

**Theorem 3.3.2.** *Let  $k$  be a field of characteristic  $p$ , and let  $G$  be a finite group with a normal Sylow  $p$ -subgroup  $P$ . Then the projective classes  $(\mathcal{S}, \text{s}\mathcal{G})$  and  $(\langle \mathcal{F}^P \uparrow^G \rangle, \text{res}^{-1}(\mathcal{G}_P))$  on  $\text{StMod}(kG)$  coincide, and for  $M \in \text{stmod}(kG)$  and  $L \in \text{stmod}(kP)$ , we have*

$$\text{sgl}(M) = \text{gl}(M\downarrow_P) \quad \text{and} \quad \text{gl}(L) = \text{sgl}(L\uparrow^G).$$

Hence

$$\text{simple ghost number of } kG = \text{ghost number of } kP.$$

In particular, the simple generating hypothesis holds for  $kG$  if and only if  $P \cong C_2$  or  $P \cong C_3$ .

The first claim of the theorem is saying that a map in  $\text{StMod}(kG)$  is a simple ghost if and only if its restriction to  $P$  is a ghost.

*Proof.* We first show that both functors  $\uparrow^G$  and  $\text{res} = \downarrow_P$  preserve the test objects. The containment  $\text{res}(\mathcal{S}) \subseteq \mathcal{F}^P$  follows directly from Lemma 3.3.1. To see that  $\langle \mathcal{F}^P \uparrow^G \rangle \subseteq \mathcal{S}$ , by Lemma 3.3.1 it suffices to check that  $k \uparrow^G \downarrow_P \cong \oplus k$ , and this is true by Mackey's theorem (Theorem 3.2.8). Finally, by Lemma 3.2.7(ii), we have inclusions  $\mathcal{S} \subseteq \langle \text{res}(\mathcal{S}) \uparrow^G \rangle \subseteq \langle \mathcal{F}^P \uparrow^G \rangle$ , hence  $\mathcal{S} = \langle \mathcal{F}^P \uparrow^G \rangle$ . It follows immediately that  $s\mathcal{G} = \text{res}^{-1}(\mathcal{G}_P)$ , and so  $s\mathcal{G} \downarrow_P \subseteq \mathcal{G}_P$ . Note that we also have that  $\mathcal{G}_P \uparrow^G \subseteq s\mathcal{G}$ , using that  $\text{res}(\mathcal{S}) \subseteq \mathcal{F}^P$  and that  $\uparrow^G$  is right adjoint to restriction.

We now prove that  $\text{sgl}(L \uparrow^G) = \text{gl}(L)$ , with the other equality following similarly. Since the induction functor takes a non-trivial ghost in  $\text{stmod}(kP)$  into a non-trivial simple ghost in  $\text{stmod}(kG)$ , we get  $\text{sgl}(L \uparrow^G) \geq \text{gl}(L)$  for  $L \in \text{stmod}(kP)$ .

To show that  $\text{sgl}(L \uparrow^G) \leq \text{gl}(L)$ , we claim that the natural isomorphism  $\alpha : \underline{\text{Hom}}_G(L \uparrow^G, M) \rightarrow \underline{\text{Hom}}_P(L, M \downarrow_P)$  takes simple ghosts to ghosts. Indeed, if  $g : L \uparrow^G \rightarrow M$  is a simple ghost, then the morphism  $\alpha(g)$  is the composite  $L \xrightarrow{\eta} L \uparrow^G \downarrow_P \xrightarrow{g \downarrow_P} M \downarrow_P$ , and is a ghost. It follows that  $\text{sgl}(L \uparrow^G) \leq \text{gl}(L)$ .  $\square$

*Remark 3.3.3.* One can also consider the *unstable* projective classes generated by the simple modules on  $\text{StMod}(kG)$  and  $\text{StMod}(kP)$ . We write  $(\mathcal{S}_u, s\mathcal{G}_u)$  for the unstable projective class generated by the simple modules on  $\text{StMod}(kG)$  and  $(\mathcal{F}_u, \mathcal{G}_u)$  for the unstable projective class generated by the trivial module on  $\text{StMod}(kP)$ . Here  $\mathcal{S}_u$  consists of retracts of direct sums of simple modules in  $\text{StMod}(kG)$  and  $\mathcal{F}_u$  consists of direct sums of the trivial module in  $\text{StMod}(kP)$ .

For a projective-free  $kP$ -module  $L$ , the radical length of  $L$  is exactly the length with respect to the projective class  $(\mathcal{F}_u, \mathcal{G}_u)$ . And Lemma 3.3.1 says that if  $M$  is a projective-free  $kG$ -module and  $L$  is a projective-free  $kP$ -module, then

$$\text{len}_{\mathcal{S}_u}(M) = \text{rad len}(M \downarrow_P) \quad \text{and} \quad \text{rad len}(L) = \text{len}_{\mathcal{S}_u}(L \uparrow^G).$$

Moreover, the projective classes  $(\mathcal{S}_u, s\mathcal{G}_u)$  and  $(\langle \mathcal{F}_u \uparrow^G \rangle, \text{res}^{-1}(\mathcal{G}_u))$  are the same on  $\text{StMod}(kG)$ . Hence we see that Theorem 3.3.2 and Lemma 3.3.1 are stable and unstable versions of each other.

*Remark 3.3.4.* When the Sylow  $p$ -subgroup is not normal, there is no obvious relationship between the simple ghost number of  $G$  and the ghost number of its Sylow  $p$ -subgroup, or between their radical lengths. See Section 3.5.2 for more discussion.

### 3.3.2 The group $A_4$ at the prime 2

In this section, we show that in general the restriction functor from a finite group  $G$  to a Sylow  $p$ -subgroup  $P$  does not preserve ghosts. We also compute the simple ghost number of  $kA_4$  at the prime 2 and give bounds on its ghost number.

Let  $G$  be  $A_4$ , the alternating group on 4 letters, and set  $p = 2$ , so  $P = V$ , the Klein four group, is normal in  $A_4$ . It is known that  $\text{Thick}_{A_4}\langle k \rangle = \text{stmod}(kA_4)$  [20]. For convenience, we assume that  $k$  contains a third root of unity  $\zeta$ , i.e.,  $\mathbb{F}_4 \subseteq k$ . Then  $k \uparrow_V^G \cong k \oplus k_\zeta \oplus k_{\bar{\zeta}}$ . Here  $k_\zeta$  is the one-dimensional module with the cyclic permutation (123) acting as  $\zeta$  and elements of even order acting as the identity, and similarly for  $k_{\bar{\zeta}}$ . Note that by Lemmas 3.2.7(ii) and 3.3.1, these are all the simple  $kA_4$ -modules, i.e.,  $\mathbb{S} = \{k, k_\zeta, k_{\bar{\zeta}}\}$ . By Theorem 3.3.2, a map restricts to a ghost in  $\text{stmod}(kV)$  if and only if it is a simple ghost in  $\text{stmod}(kA_4)$ . Since  $k_\zeta \not\cong \tilde{\Omega}^i k$  for all  $i \in \mathbb{Z}$ , the class of  $kA_4$ -modules  $\mathcal{F} = \langle k \rangle$  is strictly contained in  $\mathcal{S} = \langle \mathbb{S} \rangle$ , or equivalently, simple ghosts are strictly contained in ghosts. Therefore, there exists a ghost in  $\text{stmod}(kA_4)$  which does not restrict to a ghost in  $\text{stmod}(kP)$ .

For a specific example, we consider the connecting map  $\gamma : k_\zeta \rightarrow \Omega k_\zeta$  in the Auslander-Reiten triangle [4, Section 4.12]

$$\Omega^2 k_\zeta \longrightarrow E \longrightarrow k_\zeta \xrightarrow{\gamma} \Omega k_\zeta$$

associated to the simple module  $k_\zeta$ . Since  $\gamma$  is stably non-trivial, it is not a simple ghost. But since  $k_\zeta \notin \mathcal{F}$ , the map is a ghost, by [16, Theorem 2.1].

We now compute the simple ghost number of  $kA_4$  and give bounds on the ghost number. We are able to get an upper bound for the ghost number of  $A_4$ , since the simple modules have bounded ghost lengths.

**Proposition 3.3.5.** *Let  $k$  be a field of characteristic 2. Assume that  $k$  contains a third root of unity  $\zeta$ . Then*

$$\text{simple ghost number of } kA_4 = \text{ghost number of } kV = 2,$$

and

$$2 \leq \text{ghost number of } kA_4 \leq 4.$$

*Proof.* By Theorem 3.3.2, the simple ghost number of  $kA_4$  is equal to the ghost number of  $kV$ , which is known to be 2 (see [19]).

Since  $\text{stmod}(kA_4) = \text{Thick}\langle k \rangle$  and every simple ghost is a ghost, the ghost number of  $kA_4$  is at least 2. On the other hand, there is a short exact sequence

$$\tilde{\Omega}^2 k \rightarrow \tilde{\Omega} k_\zeta \oplus \tilde{\Omega} k_{\bar{\zeta}} \rightarrow k$$

in  $\text{mod}(kA_4)$  (see [4, Section 4.17]). It follows that  $\mathbb{S} \subseteq \mathcal{F}_2$ . Thus, by Lemma 3.2.3, the ghost number of  $kA_4$  is at most twice the simple ghost number.  $\square$

Note that  $\text{stmod}(kA_4) = \text{Thick}\langle k \rangle$ . In the next section, we prove finiteness under a weaker hypothesis.

### 3.4 Groups whose principal block is generated by $k$

In this section, we further our study of the ghost number of a group algebra  $kG$  by making use of the fact that the thick subcategory  $\text{Thick}\langle k \rangle$  generated by  $k$  is contained in  $\text{stmod}(B_0)$ , where  $B_0$  is the principal block of  $kG$ , and  $\text{stmod}(B_0)$  consists of modules in  $\text{stmod}(kG)$  whose projective-free summands are in the principal block  $B_0$ . The reader is referred to [1] and [4] for background on block theory.

We focus on the case in which  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$ . In Section 3.4.1, we show that this holds when the Sylow  $p$ -subgroup  $A$  is a direct factor. In this situation, we prove that  $\text{stmod}(kA)$  is equivalent to  $\text{Thick}_G\langle k \rangle$ , and use these results to show that the ghost numbers of  $kA$  and  $kG$  agree.

In Section 3.4.2, we show that when  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$ , the ghost number of  $kG$  is finite. The finiteness of the ghost number remains an open question without this

hypothesis. We also show that in general the composite of functors

$$e_0(-\uparrow^G) : \text{StMod}(kP) \rightarrow \text{StMod}(B_0)$$

is faithful, where  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $e_0$  is the principal idempotent, which allows us to prove that the ghost number of  $kG$  is at least as large as the ghost number of  $kP$  when  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$ . We quote Theorem 3.4.11 which provides conditions equivalent to  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$ .

Finally, in Section 3.4.3, we use this material to compute the ghost numbers of the dihedral groups at the prime 2. In addition, we give a block decomposition of each dihedral group and compute its simple ghost number.

### 3.4.1 Direct products

In this section, we study the ghost number of certain direct products, making a slight correction to a result in [20]. Let  $k$  be a field of characteristic  $p$ , and  $G = A \times B$  with  $A$  being a  $p$ -group, and the order of  $B$  being coprime to  $p$ . (That is,  $A$  is the Sylow  $p$ -subgroup of  $G$ .) Write  $i : A \rightarrow A \times B$  for the inclusion of  $A$  into  $G$  and  $\pi : A \times B \rightarrow A$  for the projection onto  $A$ . Then  $\pi i = \text{id}_A$ .

We will prove the following result. Recall that for a class  $\mathbb{P}$  of objects,  $\text{Loc}\langle \mathbb{P} \rangle$  denotes the localizing category generated by  $\mathbb{P}$ , i.e., the smallest full triangulated subcategory that is closed under arbitrary coproducts and retracts and contains  $\mathbb{P}$ .

**Theorem 3.4.1.** *Let  $k$  be a field of characteristic  $p$ , and let  $G = A \times B$  with  $A$  a  $p$ -group and the order of  $B$  coprime to  $p$ . Then the projection  $\pi : G \rightarrow A$  induces a triangulated functor  $\pi^* : \text{StMod}(kA) \rightarrow \text{StMod}(kG)$  that preserves the trivial representation  $k$ , and it restricts to triangulated equivalences*

$$\pi^* : \text{StMod}(kA) \rightarrow \text{Loc}_G\langle k \rangle$$

and

$$\pi^* : \text{stmod}(kA) \rightarrow \text{Thick}_G\langle k \rangle.$$

*The inverse functors are the restriction functors. Moreover, the image of  $\pi^*$  consists of the  $kG$ -modules whose projective-free summands have trivial  $B$ -actions.*

This theorem corrects the statement of Lemma 4.2 in [20], which has  $\text{StMod}(kG)$  in place of  $\text{Loc}_G\langle k \rangle$  and  $\text{stmod}(kG)$  in place of  $\text{Thick}_G\langle k \rangle$ . That statement is false whenever  $B$  is non-trivial. The problem is that the restriction functor  $\text{StMod}(kG) \rightarrow \text{StMod}(kA)$  is not full. For example, writing  $kB$  for the  $kG$ -module on which  $A$  acts trivially, note that  $kB \cong k \uparrow_A^G$ . Then one can see that the dimension of  $\underline{\text{Hom}}_G(kB, kB)$  is  $|B|$ , while the dimension of  $\underline{\text{Hom}}_A(kB \downarrow_A, kB \downarrow_A)$  is  $|B|^2$ . The correction is simply to restrict attention to  $\text{Loc}_G\langle k \rangle$ . The uses of Lemma 4.2 in [20] can be replaced with the above theorem and the fact that  $\text{Thick}_G\langle k \rangle = \text{stmod}(B_0)$  (Corollary 3.4.4 below), so all of the main results of [20] are correct.

*Proof of Theorem.* We first note that the functor  $\pi^* : \text{Mod}(kA) \rightarrow \text{Mod}(kG)$  induced by  $\pi : G \rightarrow A$  passes down to the stable module categories. To prove this, it suffices to show that if  $P$  is a projective  $kA$ -module, then  $\pi^*P$  is projective. Since  $\pi i = \text{id}$ , the restriction of  $\pi^*P$  to  $A$  is  $P$ , and since  $A$  is the Sylow  $p$ -subgroup of  $G$ , it follows from Lemma 3.2.7(iii) that  $\pi^*P$  is projective. It is easy to see that the functor  $\pi^* : \text{StMod}(kA) \rightarrow \text{StMod}(kG)$  is triangulated and preserves coproducts and the trivial representation.

Let  $\text{im}(\pi^*)$  be the essential image of  $\pi^*$  in  $\text{StMod}(kG)$ . The modules in  $\text{im}(\pi^*)$  are exactly those whose projective-free summands have trivial  $B$ -actions. It follows that  $\pi^*$  is full and that  $\text{im}(\pi^*)$  is closed under coproducts. Since  $i^*\pi^* = \text{id}$ , the functor  $\pi^*$  is also faithful. Thus  $\pi^*$  induces a triangulated equivalence between  $\text{StMod}(kA)$  and  $\text{im}(\pi^*)$ . Because  $\text{StMod}(kA) = \text{Loc}_A\langle k \rangle$  and  $\pi^*$  is triangulated, we get that  $\text{im}(\pi^*)$  is contained in  $\text{Loc}_G\langle k \rangle$  and that  $\text{im}(\pi^*)$  is triangulated. Hence  $\text{im}(\pi^*) = \text{Loc}_G\langle k \rangle$ , and we get the triangulated equivalence  $\pi^* : \text{StMod}(kA) \rightarrow \text{Loc}_G\langle k \rangle$ . Clearly, the restriction functor  $i^* : \text{Loc}_G\langle k \rangle \rightarrow \text{StMod}(kA)$  on the localizing subcategory generated by  $k$  is inverse to  $\pi^*$ . Restricting to compact objects, we get the equivalence  $\pi^* : \text{stmod}(kA) \rightarrow \text{Thick}_G\langle k \rangle$ , since  $\text{Thick}\langle k \rangle$  consists of exactly the compact objects in  $\text{Loc}\langle k \rangle$  by [32, Lemma 2.2].  $\square$

As a corollary, we can compute the ghost number of  $kG$ .

**Corollary 3.4.2.** *In the same set-up as above, the following holds:*

$$\text{ghost number of } kG = \text{ghost number of } kA,$$

*In particular, the generating hypothesis holds for  $kG$  if and only if it holds for  $kA$  if and only if  $A$  is  $C_2$  or  $C_3$ .*  $\square$



To show that  $\text{Thick}\langle k \rangle = \text{stmod}(B_0)$ , we compute the principal block idempotent using the next formula:

**Theorem 3.4.3** ([31, Theorem 1]). *Let  $k$  be a field of characteristic  $p$ , let  $G$  be a finite group, and let  $e_0 = \sum \epsilon_g g$  be the principal block idempotent in  $kG$  with each  $\epsilon_g$  in  $k$ . Then*

$$\epsilon_g = |\{(u, s) \in G_p \times G_{p'} \mid us = g\}| |G_{p'}|^{-1}$$

for a  $p$ -regular element  $g \in G$ , and  $\epsilon_g = 0$  if  $g$  is not  $p$ -regular.  $\square$

We say that  $g$  is  **$p$ -regular** if its order is not divisible by  $p$ ; otherwise it is said to be  **$p$ -singular**; an exception is that the identity element  $1$  is both  $p$ -regular and  $p$ -singular. We write  $G_p$  for the set of  $p$ -singular elements and  $G_{p'}$  for the set of  $p$ -regular elements.

**Corollary 3.4.4.** *Let  $k$  be a field of characteristic  $p$ . Let  $G = A \times B$ , with  $A$  being the Sylow  $p$ -subgroup of  $G$ . Then  $\text{Thick}_G\langle k \rangle = \text{stmod}(B_0)$  and  $\text{Loc}_G\langle k \rangle = \text{StMod}(B_0)$ .*

The conditions  $\text{Thick}_G\langle k \rangle = \text{stmod}(B_0)$  and  $\text{Loc}_G\langle k \rangle = \text{StMod}(B_0)$  are equivalent by Theorem 3.4.11.

*Proof.* We compute that the principal idempotent  $e_0$  is  $\frac{1}{|B|}(\sum_{b \in B} b)$ , using Theorem 3.4.3. Since  $be_0 = e_0$  for each  $b \in B$ , the projective-free modules in  $\text{stmod}(B_0)$  and  $\text{StMod}(B_0)$  all have trivial  $B$  actions. Thus, by Theorem 3.4.1, the claim follows.  $\square$

One can also prove the corollary using Theorem 3.4.11.

Note that the only simple module in  $\text{stmod}(B_0)$  is the trivial module  $k$  in this case. Indeed, since  $A \triangleleft G$  is normal, a simple module  $S$  has trivial  $A$ -action (Lemma 3.3.1); and if  $S$  is in  $\text{stmod}(B_0)$ , then it has trivial  $B$ -action too, by Theorem 3.4.1. Hence  $S$  is the trivial module  $k$ .

*Remark 3.4.5.* One can check that the algebra map  $kA \rightarrow k(A \times B) \xrightarrow{e_0} e_0(k(A \times B))$  is an isomorphism. It induces the equivalence  $\text{stmod}(B_0) \rightarrow \text{stmod}(kA)$  with inverse  $\pi^*$ . This also explains why we need to shrink the domain of the functor  $i^*$  to get an equivalence.

We combine the discussion in Section 3.3.1 and the results of this section in the next proposition.

**Proposition 3.4.6.** *Let  $k$  be a field of characteristic  $p$ . Let  $G = A \times B$ , with  $A$  being the Sylow  $p$ -subgroup of  $G$ . Then, for  $M \in \mathbf{stmod}(B_0)$ ,*

$$\mathrm{gl}(M) = \mathrm{sgl}(M) = \mathrm{gl}(M \downarrow_A),$$

and for  $N \in \mathbf{stmod}(kA)$ ,

$$\mathrm{sgl}(N \uparrow) = \mathrm{gl}(N) = \mathrm{gl}(e_0(N \uparrow)) = \mathrm{sgl}(e_0(N \uparrow)).$$

*Proof.* Since the trivial module  $k$  is the only simple module in  $\mathbf{stmod}(B_0)$ ,  $\mathrm{gl}(M) = \mathrm{sgl}(M)$  for  $M \in \mathbf{stmod}(B_0)$ . The equalities  $\mathrm{sgl}(M) = \mathrm{gl}(M \downarrow_A)$  and  $\mathrm{sgl}(N \uparrow) = \mathrm{gl}(N)$  are from Theorem 3.3.2. That  $\mathrm{gl}(N) = \mathrm{gl}(e_0(N \uparrow))$  for  $N \in \mathbf{stmod}(kA)$  is a result of Theorem 3.4.1, as one checks that the functor  $e_0(-\uparrow)$  is isomorphic to the equivalence  $\pi^* : \mathbf{stmod}(kA) \rightarrow \mathbf{Thick}_G\langle k \rangle$ . The last equality is a special case of the first.  $\square$

Note that one can't expect  $\mathrm{gl}(N) = \mathrm{gl}(e_0(N \uparrow))$  for groups that aren't direct products, even when  $\mathbf{Thick}\langle k \rangle = \mathbf{stmod}(B_0)$ . For example, this fails for  $A_4$ , using the discussion in Section 3.3.2 and the fact that  $e_0 = 1$  in this case.

### 3.4.2 Finiteness of the ghost number and a lower bound

Let  $G$  be a finite group, let  $k$  be a field whose characteristic  $p$  divides the order of  $G$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . In this section, assuming that  $\mathbf{Thick}\langle k \rangle = \mathbf{stmod}(B_0)$ , we prove that the ghost number of  $kG$  is finite (Theorem 3.4.7) and is greater than or equal to the ghost number of  $kP$  (Proposition 3.4.10).

**Theorem 3.4.7.** *Let  $k$  be a field of characteristic  $p$ , and let  $G$  be a finite group with Sylow  $p$ -subgroup  $P$ . Suppose that  $\mathbf{Thick}_G\langle k \rangle = \mathbf{stmod}(B_0)$ . Then the ghost number of  $kG$  is finite.*

In particular, the theorem holds for any  $p$ -group  $G$ , where  $\mathbf{Thick}_G\langle k \rangle = \mathbf{stmod}(B_0) = \mathbf{stmod}(kG)$ , recovering [19, Theorem 4.7]. Our proof follows the approach used in Proposition 3.3.5 for the alternating group  $A_4$ .

*Proof.* Recall that the simple ghost number of  $kG$  is finite (Remark 3.2.9). It then follows that, since there are no non-zero maps between different blocks, the Freyd number of

$kG$  with respect to  $\mathbb{Q} := \mathbb{S} \cap B_0$ , the set of simple modules in the principle block, is finite. On the other hand, since  $\text{Thick}_G\langle k \rangle = \text{stmod}(B_0)$  and  $\mathbb{Q}$  is a finite set, we have  $\mathbb{Q} \subseteq \mathcal{F}_n = \langle \mathbb{P} \rangle_n$  for some  $n$ , where  $\mathbb{P} = \{k\}$ . It then follows from Lemma 3.2.3 that the ghost number of  $kG$  is bounded above by  $n$  times the Freyd number of  $kG$  with respect to  $\mathbb{Q}$ , and thus is finite.  $\square$

We call the Freyd number of  $kG$  with respect to  $\mathbb{Q}$  the **simple ghost number of  $B_0$** .

*Remark 3.4.8.* Note that each  $M \in \text{Thick}\langle k \rangle$  has finite ghost length. But we need to find an universal upper bound to prove finiteness of the ghost number. One idea is to look at the radical sequence as was done for  $p$ -groups in [19]. When  $\text{Thick}_G\langle k \rangle = \text{stmod}(B_0)$ , the simple modules that can appear in the radical sequence for  $M \in \text{Thick}_G\langle k \rangle$  all have finite ghost lengths. However, whether the ghost number is finite when  $\text{Thick}_G\langle k \rangle \neq \text{stmod}(B_0)$  remains open, since we cannot answer the following question proposed in [8]: does there exist a simple module in the principal block with vanishing Tate cohomology? Indeed, if there exists a simple module in  $\text{stmod}(B_0)$  but not in  $\text{Thick}_G\langle k \rangle$ , and its Tate cohomology does not vanish, then it can appear in the radical sequence of a module  $M \in \text{Thick}_G\langle k \rangle$ . Hence the proof here does not apply to the case where  $\text{Thick}_G\langle k \rangle \neq \text{stmod}(B_0)$ .

We state the question in the general case as a conjecture:

**Conjecture 3.4.9.** *Let  $G$  be a finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . Then the ghost number of  $kG$  is finite.*

Now we determine a lower bound for the ghost number of  $kG$ . Note that for a group  $G$  with subgroup  $H$ , the induction functor sends ghosts to ghosts and is faithful. However, induction does not preserve  $\text{Thick}\langle k \rangle$  in general, so this technique is of limited use in computing the ghost number of  $G$ . To try to remedy this, we can consider the composite  $e_0(-\uparrow^G)$  of induction with projection onto the principal block. This will provide us with a ghost in  $\text{Thick}_G\langle k \rangle$  if we assume that  $\text{Thick}_G\langle k \rangle = \text{stmod}(B_0)$ .

Note that we have adjunctions

$$\uparrow^G : \text{stmod}(kH) \rightleftarrows \text{stmod}(kG) : \downarrow_H \quad \text{and} \quad e_0(-) : \text{stmod}(kG) \rightleftarrows \text{stmod}(B_0) : j,$$

where  $j$  denotes the inclusion. We show that the composite  $e_0(-\uparrow^G)$  is faithful in the case where  $H$  is a Sylow  $p$ -subgroup.

**Proposition 3.4.10.** *Let  $k$  be a field of characteristic  $p$ , and let  $G$  be a finite group with Sylow  $p$ -subgroup  $P$ . Then the functor*

$$e_0(-\uparrow^G) : \text{stmod}(kP) \rightarrow \text{stmod}(B_0)$$

*is faithful. In particular, if  $\text{Thick}_G(k) = \text{stmod}(B_0)$ , then*

$$\text{ghost number of } kG \geq \text{ghost number of } kP.$$

We don't know of a counterexample to the last inequality.

*Proof.* It suffices to show that the unit map

$$\begin{aligned} M &\longrightarrow j(e_0(M\uparrow^G))\downarrow_P \\ m &\longmapsto e_0 \otimes m \end{aligned}$$

of the composite adjunction is split monic.

It is well known that  $\uparrow^G$  is both left and right adjoint to  $\downarrow_P$ , with unit map  $\eta : M \rightarrow M\uparrow^G\downarrow_P$  sending  $m$  to  $1 \otimes m$ , and counit map  $\epsilon : M\uparrow^G\downarrow_P \rightarrow M$  sending  $g \otimes m$  to  $gm$  if  $g \in P$  and to 0 if  $g \notin P$ .

The unit map for the adjunction  $e_0(-) : \text{stmod}(kG) \rightleftarrows \text{stmod}(B_0) : j$  is the natural projection  $N \rightarrow j(e_0N)$  by left multiplication by  $e_0$ . Since the stable module category  $\text{stmod}(kG)$  decomposes into blocks, it is easy to check that  $e_0(-)$  is also right adjoint to  $j$ , with counit the natural inclusion  $j(e_0N) \rightarrow N$ .

The composite

$$M \rightarrow j(e_0(M\uparrow^G))\downarrow_P \rightarrow M\uparrow^G\downarrow_P \xrightarrow{\epsilon} M$$

sends  $m$  to  $\epsilon(e_0 \otimes m)$ . We show that it is an isomorphism. Since  $P$  is a  $p$ -subgroup of  $G$  and the only possible non-zero coefficient  $\epsilon_h$  for  $h \in P$  is  $\epsilon_1 = |G_{p'}|^{-1}$  by Theorem 3.4.3, one sees that  $\epsilon(e_0 \otimes m) = \epsilon_1 m$ . But  $\epsilon_1$  is invertible in  $k$ , so the composite is an isomorphism. It follows that  $M \rightarrow (e_0(M\uparrow^G))\downarrow_P$  is split monic and the functor  $e_0(-\uparrow^G)$  is faithful.

It is clear that the composite  $e_0(-\uparrow^G)$  preserves ghosts. Hence  $\text{gl}(e_0(L\uparrow^G)) \geq \text{gl}(L)$  for  $L \in \text{stmod}(kP)$ , and the ghost number of  $kG$  is greater than or equal to the ghost number of  $kP$ .  $\square$

We quote the next theorem to end this section. It provides conditions for checking whether  $\text{Thick}_G(k) = \text{stmod}(B_0)$ . Recall that a finite group is said to be  $p$ -**nilpotent** if  $G_{p'}$ , the set of  $p$ -regular elements of  $G$ , forms a subgroup.

**Theorem 3.4.11** ([6, Theorem 1.4]). *Let  $G$  be a finite group, and let  $k$  be a field of characteristic  $p$ . Then  $\text{Thick}(k) = \text{stmod}(B_0)$  if and only if  $\text{Loc}(k) = \text{StMod}(B_0)$  if and only if the centralizer of every element of order  $p$  is  $p$ -nilpotent.  $\square$*

### 3.4.3 Dihedral groups at the prime 2

Let  $G = D_{2ql}$  be a dihedral group of order  $2ql$ , where  $q$  is a power of 2 and  $l$  is odd, with presentation  $D_{2ql} = \langle x, y \mid x^{ql} = y^2 = (xy)^2 = 1 \rangle$ . Let  $k$  be a field of characteristic 2. In this section, we will determine the ghost number and simple ghost number of  $kD_{2ql}$  by analyzing the blocks. (See Theorem 3.5.7 for the ghost number of  $kD_{2ql}$  at an odd prime.)

We can compute the principal block idempotent of  $kD_{2ql}$  using Theorem 3.4.3 and the fact that  $l = 1$  in  $k$ .

**Lemma 3.4.12.** *The 2-regular elements of  $D_{2ql}$  are exactly those in the subgroup  $C_l = \langle x^q \rangle$ . The principal idempotent is  $e_0 = 1 + x^q + x^{2q} + \dots + x^{(l-1)q}$ .  $\square$*

We regard  $D_{2q}$  as the subgroup of  $D_{2ql}$  generated by  $x^l$  and  $y$ , and so we have a natural unital algebra map  $\alpha : kD_{2q} \rightarrow kD_{2ql} \rightarrow e_0kD_{2ql}$ . Note that  $D_{2q}$  is a Sylow 2-subgroup of  $D_{2ql}$ .

**Lemma 3.4.13.** *The algebra map  $\alpha : kD_{2q} \rightarrow e_0kD_{2ql}$  is an isomorphism.*

*Proof.* As an algebra,  $e_0kD_{2ql}$  is generated by  $e_0x$  and  $e_0y$ . Clearly,  $e_0y = \alpha(y)$  is in the image of  $\alpha$ . And since  $l$  is odd and  $e_0x^q = e_0$ , we see that  $e_0x = e_0x^{kl}$  for some integer  $k$ . Hence the map  $\alpha$  is surjective. Since  $e_0kD_{2ql}$  is projective as a  $kD_{2ql}$ -module, its dimension is at least  $2q$ , which equals the dimension of  $kD_{2q}$ , so  $\alpha$  has to be an isomorphism.  $\square$

As a corollary, we can compute the ghost number of  $kD_{2ql}$ .

**Corollary 3.4.14.** *The thick subcategory generated by  $k$  is the same as the principal block,*

$$\text{Thick}_{D_{2ql}}\langle k \rangle = \text{stmod}(e_0 k D_{2ql}),$$

and the ghost number of  $k D_{2ql}$  is  $\lfloor \frac{q}{2} + 1 \rfloor$ .

*Proof.* Since  $\alpha$  is an isomorphism, it induces an equivalence

$$\text{stmod}(k D_{2q}) \rightarrow \text{stmod}(e_0 k D_{2ql})$$

that sends  $M$  to  $e_0(M \uparrow_{D_{2q}}^{D_{2ql}})$ . The first statement follows from the facts that this equivalence sends  $k$  to  $k$  and that  $\text{Thick}_{D_{2q}}\langle k \rangle = \text{stmod}(k D_{2q})$ . It also follows that

$$\text{the ghost number of } k D_{2ql} = \text{the ghost number of } k D_{2q}.$$

The second statement then follows from [23, Corollary 4.25], which shows that the ghost number of  $k D_{2q}$  is  $\lfloor \frac{q}{2} + 1 \rfloor$ .  $\square$

So, in this case, the lower bound given by Proposition 3.4.10 is an equality.

We next consider the simple ghost number of  $k D_{2ql}$ .

*Remark 3.4.15.* Note that the only simple module in the principal block is  $k$ , by Lemma 3.4.13. Also, the inverse to the equivalence  $\text{stmod}(k D_{2q}) \rightarrow \text{stmod}(e_0 k D_{2ql})$  is given by restriction. It follows that, for  $M \in \text{stmod}(e_0 k D_{2ql})$ , we have

$$\text{sgl}(M) = \text{gl}(M) = \text{gl}(M \downarrow_{D_{2q}}).$$

To compute the simple ghost number of  $k D_{2ql}$ , it remains to consider the non-principal blocks. From now on, we assume that  $k$  contains an  $l$ -th primitive root of unity  $\zeta$ . Let  $C_{ql}$  be the cyclic subgroup of  $D_{2ql}$  generated by  $x$ . We will show that inducing up is fully-faithful on each non-principal block, using the following lemmas.

It is not hard to compute the idempotent decomposition of 1 in  $k C_{ql}$ .

**Lemma 3.4.16.** *The identity  $1 \in k C_{ql}$  has an decomposition into orthogonal primitive idempotents:*

$$1 = \sum_{i=0}^{l-1} e_i, \text{ with } e_i = \sum_{j=0}^{l-1} (\zeta^i x^q)^j.$$

The block corresponding to  $e_i$  has exactly one simple module  $k_i$ , the one-dimensional module on which  $x^q$  acts as  $\zeta^{l-i}$ .

*Proof.* It is easy to check that the  $e_i$ 's are orthogonal and idempotent, and that  $e_i k_i = k_i$ . It is well known that the  $k_i$ 's are a complete list of simple  $kC_{ql}$ -modules, so it follows that the idempotents are primitive.  $\square$

Since conjugation by  $y$  in  $D_{2ql}$  takes  $e_0$  to  $e_0$  and  $e_i$  to  $e_{l-i}$  for  $i > 0$ , we can deduce the idempotent decomposition for  $kD_{2ql}$ .

**Lemma 3.4.17.** *The identity  $1 \in kD_{2ql}$  has a decomposition into orthogonal primitive central idempotents:*

$$1 = e_0 + \sum_{i=1}^{\frac{l-1}{2}} e'_i, \text{ with } e'_i = e_i + e_{l-i} = \sum_{j=0}^{l-1} (\zeta^i x^q)^j + \sum_{j=0}^{l-1} (\zeta^{l-i} x^q)^j.$$

Moreover, the block corresponding to  $e'_i$  has exactly one simple module, namely  $S_i := k_i \uparrow_{C_{ql}}^{D_{2ql}}$ . It follows that  $\text{stmod}(e'_i kD_{2ql}) = \text{Thick}_{D_{2ql}} \langle S_i \rangle$ .

*Proof.* Clearly, the  $e'_i$ 's are orthogonal central idempotents. They are primitive since there are exactly  $(l+1)/2$  simple  $kD_{2ql}$ -modules [1, Theorem 3.2]. It follows that there is exactly one simple module in each block.

Define  $S_i$  to be  $k_i \uparrow_{C_{ql}}^{D_{2ql}} = kD_{2ql} \otimes_{C_{ql}} k_i$ , where  $k_i$  is the simple  $kC_{ql}$ -module defined in Lemma 3.4.16. With respect to the basis  $\{1 \otimes 1, y \otimes 1\}$  of  $kD_{2ql} \otimes_{C_{ql}} k_i$ , it is easy to check that  $S_i$  is represented using the following matrices:

$$x^l \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x^q \mapsto \begin{bmatrix} \zeta^{l-i} & 0 \\ 0 & \zeta^i \end{bmatrix}, \quad \text{and} \quad y \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And from this representation, one sees quickly that  $S_i \downarrow_{C_{ql}} = k_i \oplus k_{l-i}$ . The action of  $y$  on  $S_i$  exchanges  $k_i$  and  $k_{l-i}$ , hence, as  $kD_{2ql}$ -modules, both  $k_i$  and  $k_{l-i}$  generate the whole module  $S_i$ . Thus  $S_i$  is a simple module. It is also clear that the module  $S_i$  is in the block  $e'_i kD_{2ql}$ , and so  $\text{stmod}(e'_i kD_{2ql}) = \text{Thick}_{D_{2ql}} \langle S_i \rangle$ .  $\square$

We next provide a list of all the indecomposable  $kC_{ql}$ -modules. The result can be found in [1, p. 14, 34]. Recall that for each  $1 \leq n \leq q$  there is a unique indecomposable

$kC_q$ -module  $M_n$  of radical length  $n$ , and that these are all of the indecomposable  $kC_q$ -modules.

**Lemma 3.4.18** ([1]). *The modules  $e_i(M_n \uparrow^{C_{ql}})$ , for  $1 \leq n \leq q$  and  $0 \leq i < l$ , are a complete list of the indecomposable  $kC_{ql}$ -modules.  $\square$*

Now we can show that the induction functor induces an equivalence between the non-principal blocks of  $kC_{ql}$  and  $kD_{2ql}$ .

**Proposition 3.4.19.** *For  $i \neq 0$ , let  $B_i = e_i kC_{ql}$  be a non-principal block of  $kC_{ql}$ . Then the composite of functors*

$$\text{stmod}(B_i) \longrightarrow \text{stmod}(kC_{ql}) \xrightarrow{\uparrow_{C_{ql}}^{D_{2ql}}} \text{stmod}(kD_{2ql})$$

*is fully-faithful, hence induces an equivalence  $\text{stmod}(B_i) \rightarrow \text{stmod}(e'_i kD_{2ql})$ .*

*Proof.* We begin by showing that  $\uparrow_{C_{ql}}^{D_{2ql}}$  is fully-faithful when restricted to  $\text{stmod}(B_i)$ . Let  $M := e_i(M_n \uparrow^{C_{ql}})$  be one of the indecomposable  $kC_{ql}$ -modules described in Lemma 3.4.18, and write  $N := M_n \uparrow^{C_{ql}}$ . Using Mackey's Theorem, we have  $M \uparrow^{D_{2ql}} \downarrow_{C_{ql}} \cong e_i(N) \oplus y(e_i(N)) = e_i(N) \oplus e_{l-i}(N)$ , and the natural map  $M \xrightarrow{\eta} M \uparrow \downarrow \cong e_i(N) \oplus e_{l-i}(N)$  is an isomorphism onto  $e_i(N)$ .

Because  $\uparrow$  is left adjoint to  $\downarrow$ , the following diagram commutes

$$\begin{array}{ccc} \underline{\text{Hom}}_{C_{ql}}(M, M) & \xrightarrow{\uparrow} & \underline{\text{Hom}}_{D_{2ql}}(M \uparrow, M \uparrow) \\ & \searrow \eta_* & \downarrow \cong \\ & & \underline{\text{Hom}}_{C_{ql}}(M, M \uparrow \downarrow). \end{array}$$

By the discussion in the previous paragraph,  $\eta_*$  is an isomorphism, and so  $\uparrow$  is as well. Since this is true for every indecomposable in  $\text{stmod}(B_i)$ , it follows that the induction functor is fully-faithful when restricted to  $\text{stmod}(B_i)$ , and induces a triangulated equivalence between  $\text{stmod}(B_i)$  and its essential image. Since  $\text{stmod}(B_i) = \text{Thick}_{C_{ql}} \langle k_i \rangle$  (Lemma 3.4.16) and  $k_i \uparrow = S_i$ , the essential image of  $\text{stmod}(B_i)$  is  $\text{stmod}(e'_i kD_{2ql}) = \text{Thick}_{D_{2ql}} \langle S_i \rangle$  (Lemma 3.4.17), and the claim follows.  $\square$

*Remark 3.4.20.* Note that the inverse of the equivalence is given by the composite of restriction and then projection onto the block  $e_i kC_{ql}$ .



We can now compute the simple ghost number of  $kD_{2ql}$ .

**Theorem 3.4.21.** *For  $M \in \text{stmod}(e'_i kD_{2ql})$  with  $i \neq 0$ , we have*

$$\text{sgl}(M) = \text{sgl}(M \downarrow_{C_{ql}}) = \text{gl}(M \downarrow_{C_q}).$$

For  $M \in \text{stmod}(e_0 kD_{2ql})$ , we have

$$\text{sgl}(M) = \text{gl}(M \downarrow_{D_{2q}}).$$

Hence the simple ghost number of  $kD_{2ql}$  = the ghost number of  $kD_{2q}$  =  $\lfloor \frac{q}{2} + 1 \rfloor$ .

*Proof.* We have equivalences

$$\text{stmod}(B_i) \rightarrow \text{stmod}(e'_i kD_{2ql}) \text{ and } \text{stmod}(kD_{2q}) \rightarrow \text{stmod}(e_0 kD_{2ql}).$$

The equivalences preserve simple modules, hence radical lengths and simple ghost lengths. Then, for  $M \in \text{stmod}(e'_i kD_{2ql})$ , we have  $\text{sgl}(M) = \text{sgl}(e_i(M \downarrow_{C_{ql}})) = \text{sgl}(e_{l-i}(M \downarrow_{C_{ql}}))$  by Proposition 3.4.19 and Remark 3.4.20. Since  $M \downarrow_{C_{ql}} = e_i(M \downarrow_{C_{ql}}) \oplus e_{l-i}(M \downarrow_{C_{ql}})$ , it follows that

$$\text{sgl}(M) = \text{sgl}(M \downarrow_{C_{ql}}).$$

And by Theorem 3.3.2,  $\text{sgl}(M \downarrow_{C_{ql}}) = \text{gl}(M \downarrow_{C_q})$ .

For  $M \in \text{stmod}(e_0 kD_{2ql})$ , we have seen in Remark 3.4.15 that

$$\text{sgl}(M) = \text{gl}(M) = \text{gl}(M \downarrow_{D_{2q}}).$$

Since the ghost number of  $C_q$  is  $\lfloor q/2 \rfloor$  (Lemma 3.6.5), and the ghost number of  $D_{2ql}$  is  $\lfloor \frac{q}{2} + 1 \rfloor$  [23, Corollary 4.25], it follows that the simple ghost length is maximized by  $\text{sgl}(M)$  for some  $M \in \text{stmod}(e_0 kD_{2ql})$ , and that the simple ghost number of  $kD_{2ql}$  equals its ghost number.  $\square$

### 3.5 Groups with cyclic Sylow $p$ -subgroups

We consider a group  $G$  with a cyclic Sylow  $p$ -subgroup  $P$  in this section. When the Sylow  $p$ -subgroup is normal, we know from Section 3.3.1 that simple ghost lengths can be computed by restricting to  $P$ . We show in Section 3.5.1 that, when  $P$  is also

cyclic, the simple ghost length of a module in the principal block is equal to its ghost length and that the finitely generated modules in the principal block are exactly those in  $\text{Thick}\langle k \rangle$ . We use this to compute the ghost numbers of dihedral groups at odd primes. In Section 3.5.2, we study the group  $SL(2, p)$  at the prime  $p$ , which has a cyclic Sylow  $p$ -subgroup which is not normal. Nevertheless, by restricting to the normalizer  $L$  of  $P$ , we are able to show that the simple generating hypothesis holds for  $SL(2, p)$  for any  $p$ , even though it fails for  $L$  and  $P$  when  $p > 3$ .

### 3.5.1 The case of a cyclic normal Sylow $p$ -subgroup

Let  $k$  be a field of characteristic  $p$ , and let  $G$  be a finite group with cyclic Sylow  $p$ -subgroup  $C_{p^r}$ . We assume that  $k$  is algebraically closed and that  $C_{p^r}$  is normal in  $G$ .

Since  $P \leq G$  is normal, Theorem 3.3.2 says that

$$\text{sgl}(M) = \text{gl}(M \downarrow_P)$$

for  $M \in \text{stmod}(kG)$ . In this section, using that  $P$  is in addition cyclic, we are going to show that

$$\text{sgl}(M) = \text{gl}(M)$$

for  $M \in \text{stmod}(B_0)$ , as we found for direct products in Proposition 3.4.6.

Our approach is as follows. We will show that all simple modules in the principal block  $\text{StMod}(B_0)$  are suspensions of the trivial module  $k$ . Hence the simple ghost projective class and the ghost projective class coincide when both are pulled back to  $\text{StMod}(B_0)$ . It then follows that  $\text{Thick}\langle k \rangle$  equals  $\text{stmod}(B_0)$  and that for  $M$  in  $\text{stmod}(B_0)$ , its ghost length is the same as its simple ghost length.

We say that a  $kG$ -module  $M$  is **uniserial** if the successive quotients in the radical sequence associated to  $M$  are simple. Note that this is equivalent to the successive quotients in the socle sequence associated to  $M$  being simple.

An important fact about the representations of  $G$  when its Sylow  $p$ -subgroup is normal and cyclic is that the indecomposable modules are uniserial:

**Theorem 3.5.1** ([1, pp. 42–43]). *Let  $G$  be a finite group, and let  $k$  be a field of characteristic  $p$ . Assume that the Sylow  $p$ -subgroup  $P$  of  $G$  is normal and cyclic. Then there are*

*finitely many indecomposable  $kG$ -modules. Every indecomposable module  $M$  is uniserial and is characterised by its radical length and the simple module  $M/\text{rad}(M)$ .  $\square$*

Recall that in general there is a bijection between indecomposable projective  $kG$ -modules and simple  $kG$ -modules given by the assignment that sends a projective module  $Q$  to its radical quotient  $Q/\text{rad}(Q)$  [1, Theorem 5.3]. The inverse sends a simple module to its projective cover, i.e. the unique indecomposable projective module that surjects onto it. Also note that for a projective  $kG$ -module  $Q$ , we have an isomorphism  $Q/\text{rad}(Q) \cong \text{soc}(Q)$  [1, Theorem 6.6].

When  $P = C_{p^r} \leq G$  is cyclic and normal, we can say more.

**Lemma 3.5.2.** *Let  $G$  be a finite group with cyclic normal Sylow  $p$ -subgroup  $P = C_{p^r}$ , let  $k$  be a field of characteristic  $p$ , and let  $Q$  be the projective cover of the trivial module  $k$ . If  $S$  is a simple module, then  $Q \otimes S$  is its projective cover.*

*Proof.* First note that  $Q \otimes M$  is projective for any  $kG$ -module  $M$  [1, Lemma 7.4], so  $Q \otimes S$  is projective.

To see that  $Q \otimes S$  is indecomposable, first note that  $S \cong k \otimes S \subseteq \text{soc}(Q \otimes S)$ . Since  $Q \otimes S$  is projective, so is its restriction to  $P$ , by Lemma 3.2.7(iii). Since projective  $kP$ -modules are free, this restriction must have rank  $\dim S$  and socle  $k^{\oplus \dim S}$ . Then, by Lemma 3.3.1, the dimension of  $\text{soc}(Q \otimes S)$  must also be  $\dim S$ , and so we actually have  $S \cong \text{soc}(Q \otimes S)$ . Thus  $Q \otimes S$  is indecomposable.

We have seen that  $Q \otimes S$  is an indecomposable projective, and it comes with a surjection onto  $S$ , so it must be the projective cover of  $S$ .  $\square$

We continue to write  $Q$  for the projective cover of the trivial module  $k$ . The proof above shows that  $Q \downarrow_{kC_{p^r}} = kC_{p^r}$ . It follows that the radical layers of  $Q$  are all 1-dimensional. Now, let  $W$  be the 1-dimensional simple module  $\text{rad}(Q)/\text{rad}^2(Q)$ . We show that  $W \cong \tilde{\Omega}^2 k$ .

**Lemma 3.5.3.** *The module  $W$  is isomorphic to the double desuspension of the trivial module  $k$ , i.e.,  $W \cong \tilde{\Omega}^2 k$ . Moreover, each composition factor of  $Q$  is a tensor power  $W^{\otimes n}$  of the module  $W$ .*

*Proof.* The map  $Q \otimes W \rightarrow W$  lifts through the quotient map  $\pi : \text{rad}(Q) \rightarrow W$  and gives a map  $f : Q \otimes W \rightarrow \text{rad}(Q)$ . Since  $\ker(\pi) = \text{rad}^2(Q)$  is the unique maximal submodule

of  $\text{rad}(Q)$ , the map  $f$  is surjective. As we saw for  $Q$ , the radical layers of  $Q \otimes W$  are also all 1-dimensional, so  $Q$  and  $Q \otimes W$  have the same radical length. Since the radical length of  $\text{rad}(Q)$  is one less than that of  $Q \otimes W$ , the composite

$$W \rightarrow Q \otimes W \xrightarrow{f} \text{rad}(Q)$$

is zero, where the map  $W \rightarrow Q \otimes W$  is the inclusion of the last radical (which equals the socle) of  $Q \otimes W$ . By comparing dimensions, one sees that this is a short exact sequence. And since  $\text{rad}(Q) \cong \tilde{\Omega}k$ , we have  $W \cong \tilde{\Omega}^2k$ .

To see that the composition factors of  $Q$  are  $W^{\otimes n}$ , first note that

$$\text{rad}^n(Q)/\text{rad}^{n+1}(Q) \cong \text{rad}^{n-1}(Q \otimes W)/\text{rad}^n(Q \otimes W)$$

for  $1 \leq n \leq p^r - 1$ . We get these isomorphisms by comparing the radical layers along the surjective map  $f : Q \otimes W \rightarrow \text{rad}(Q)$ , using that both  $Q$  and  $Q \otimes W$  have 1-dimensional layers. On the other hand,  $\text{rad}^n(Q \otimes W) \cong \text{rad}^n(Q) \otimes W$ , since tensoring with  $W$  preserves the radical layers. Thus

$$\text{rad}^{n-1}(Q \otimes W)/\text{rad}^n(Q \otimes W) \cong (\text{rad}^{n-1}(Q)/\text{rad}^n(Q)) \otimes W.$$

Combining the two displayed isomorphisms and using that  $W = \text{rad}(Q)/\text{rad}^2(Q)$ , it follows inductively that

$$\text{rad}^n(Q)/\text{rad}^{n+1}(Q) \cong W^{\otimes n}. \quad \square$$

Note that  $M \otimes W \cong \tilde{\Omega}^2M$  for any module  $M$ . In particular,  $W^{\otimes n} \cong \tilde{\Omega}^{2n}k$ . Also note that, more generally, the indecomposable projective module  $Q \otimes S$  is uniserial with composition factors  $W^{\otimes n} \otimes S$ . Thus the following lemma, together with Lemma 3.5.3, implies that the modules  $W^{\otimes n}$  are all the simple modules in  $\text{StMod}(B_0)$ . (See also [1, Exercise 13.3].) Thus the simple modules in  $\text{StMod}(B_0)$  are all in  $\mathcal{F}$ , and so simple ghosts and ghosts agree in the principal block.

**Lemma 3.5.4** ([1, Proposition 13.3]). *Let  $k$  be a field of characteristic  $p$ , and let  $G$  be a finite group with a cyclic normal Sylow  $p$ -subgroup  $C_{p^r}$ . Then two simple modules  $S$  and  $T$  are in the same block if and only if there exists a sequence of simple modules*

$$S = S_1, S_2, \dots, S_m = T$$

such that  $S_i$  and  $S_{i+1}$  are composition factors of an indecomposable projective  $kG$ -module, for  $1 \leq i < m$ . □

We can use the above observations to compute the ghost number of  $kG$ .

**Theorem 3.5.5.** *Let  $k$  be a field of characteristic  $p$ , and let  $G$  be a finite group with a cyclic normal Sylow  $p$ -subgroup  $C_{p^r}$ . Then  $\text{Thick}_G\langle k \rangle = \text{stmod}(B_0)$ , and a map in  $\text{Thick}_G\langle k \rangle$  is a ghost if and only if its restriction to  $\text{stmod}(kC_{p^r})$  is a ghost. As a result,*

$$\text{ghost number of } kG = \text{ghost number of } kC_{p^r} = \lfloor p^r/2 \rfloor.$$

Moreover, let  $M$  be a uniserial  $kG$ -module of radical length  $l$  in  $\text{Thick}_G\langle k \rangle$ . Then

$$\text{gl}(M) = \text{sgl}(M) = \min(l, p^r - l).$$

In particular, using the natural terminology, the ghost number of  $kG$  is equal to the simple ghost number of  $B_0$ .

*Proof.* Since the simple modules in the principal block are contained in  $\mathcal{F}$ , the pullback of the simple ghost projective class to  $\text{StMod}(B_0)$  coincides with the pullback of the ghost projective class to  $\text{StMod}(B_0)$ . It follows that  $\text{Thick}_G\langle k \rangle = \text{stmod}(B_0)$ , and  $\text{gl}(M) = \text{sgl}(M)$  for a module  $M$  in  $\text{stmod}(B_0)$ . Since  $P \leq G$  is normal,  $\text{sgl}(M) = \text{gl}(M \downarrow_P)$  for  $M \in \text{stmod}(kG)$ , by Theorem 3.3.2. Hence

$$\text{gl}(M) = \text{sgl}(M) = \text{gl}(M \downarrow_P),$$

and we can compute the ghost lengths in  $kG$  by restricting to  $kC_{p^r}$ . The ghost lengths in  $kC_{p^r}$  are computed in [19] (summarized in Lemma 3.6.5 below). □

*Remark 3.5.6.* We give a concrete description of the module  $W$  [1, Exercise 5.3]. Let  $x$  be a generator of the cyclic group  $C_{p^r}$ . Then the one-dimensional module  $W$  is given by the group homomorphism that sends  $g \in G$  to  $\overline{\alpha(g)} \in k^\times$ , where  $\alpha(g)$  is the integer such that  $gxg^{-1} = x^{\alpha(g)}$  and  $\overline{\alpha(g)}$  is its image under the canonical map  $\mathbb{Z} \rightarrow k$ . If we further compose this map with the self map on  $k^\times$  that takes  $\alpha$  to  $\alpha^n$ , we get the module  $W^{\otimes n}$ . Since  $\overline{\alpha(g)}$  lands in  $\mathbb{F}_p \subseteq k$ , we always have  $W^{\otimes(p-1)} = k$ .

Let  $M$  be a non-projective uniserial module with radical length  $l \geq 2$ . We give an explicit construction of a (weakly) universal simple ghost out of  $M$ . Let  $W^*$  be the dual

of  $W$ , so  $W \otimes W^* \cong k$ . We have

$$M/\text{rad}(M) \cong (\text{rad}(M)/\text{rad}^2(M)) \otimes W^* \cong \text{rad}(M \otimes W^*)/\text{rad}^2(M \otimes W^*). \quad (3.5.1)$$

To see that the first isomorphism holds, note that it holds for the module  $Q$ , hence for the modules  $Q \otimes S$  with  $S$  simple. Since  $M$  is a quotient of one of the uniserial modules  $Q \otimes S$ , the isomorphism holds for  $M$  too. Recall by Theorem 3.3.2 that a map  $f$  is simple ghost if and only if its restriction to a Sylow  $p$ -subgroup is a ghost. And for a  $p$ -group  $P$ , we know that a ghost  $g : M \rightarrow N$  has  $\text{im}(g) \subseteq \text{rad}(N)$  and  $\text{soc}(M) \subseteq \ker(g)$  by [19, Corollary 2.6]. Hence we consider the short exact sequences

$$0 \rightarrow \text{soc}(M) \rightarrow M \xrightarrow{\pi} M/\text{soc}(M) \rightarrow 0$$

and

$$0 \rightarrow \text{rad}(M \otimes W^*) \xrightarrow{i} M \otimes W^* \rightarrow M \otimes W^*/\text{rad}(M \otimes W^*).$$

Equation (3.5.1) implies that  $M/\text{soc}(M) \cong \text{rad}(M \otimes W^*)$ . Now let  $g$  be the composite  $M \xrightarrow{\pi} M/\text{soc}(M) \cong \text{rad}(M \otimes W^*) \xrightarrow{i} M \otimes W^*$ . Then  $\text{im}(g) \subseteq \text{rad}(N)$  and  $\text{soc}(M) \subseteq \ker(g)$ . By Lemma 3.3.1, the inclusions still hold when restricted to the normal Sylow  $p$ -subgroup  $C_{p^r}$ . Since  $\Omega^2 k \cong k$  in  $\text{stmod}(kC_{p^r})$ , the proof of [19, Proposition 2.1] shows that  $g \downarrow_{C_{p^r}}$  is a ghost. So by Theorem 3.3.2, the map  $g$  is a simple ghost. One can check that the fibre of  $g$  is  $\text{soc}(M) \oplus \Omega(M \otimes W^*/\text{rad}(M \otimes W^*))$ . Thus  $g$  is a weakly universal simple ghost. This process can be iterated, producing composites  $M \rightarrow M \otimes (W^*)^n$  of  $n$  simple ghosts which are nonzero for  $n < \text{sgl}(M)$ . If  $M$  is in the principal block, then these simple ghosts are ghosts, and so we have exhibited the ghosts predicted by Theorem 3.5.5.

**Theorem 3.5.7.** *Let  $D_{2ql}$  be a dihedral group, with  $q$  a power of 2 and  $l$  odd. Let  $k$  be a field of characteristic  $p$  which divides  $2ql$ . If  $p$  is odd, then the ghost number of  $kD_{2ql}$  is  $\lfloor p^r/2 \rfloor$ , where  $p^r$  is the  $p$ -primary part of  $l$ . If  $p$  is even, then the ghost number of  $kD_{2ql}$  is  $\lfloor q/2 + 1 \rfloor$ .*

*Proof.* If  $p$  is odd, then its Sylow  $p$ -group is cyclic and normal, so its ghost number is given by Theorem 3.5.5. If  $p$  is even, then its ghost number was computed in Corollary 3.4.14.  $\square$

### 3.5.2 The simple generating hypothesis for the group $SL(2, p)$

In this section, we show that the simple generating hypothesis holds for  $kG$ , where  $G$  is the group  $SL(2, p)$  of order  $p(p-1)(p+1)$  and  $k$  is a field of characteristic  $p$ . Background on representations of  $SL(2, p)$  can be found in [1, p. 14, 75]. We will also need to know about representations of the normalizer  $N(P)$  of  $P$  in  $SL(2, p)$ , which illustrates the results of Section 3.5.1.

We let  $P \leq G$  consist of all elements of the form  $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ .  $P$  has order  $p$  and is a Sylow  $p$ -subgroup of  $G$ . Let  $L = N(P)$  be the normalizer of  $P$  in  $G$ . It consists of the elements of the form  $\begin{pmatrix} a & 0 \\ c' & 1/a \end{pmatrix}$ .

For  $i \in \mathbb{Z}$ , consider the one-dimensional simple module  $S_i$  of  $L$  given by the group map  $L \rightarrow k^\times$  that sends  $\begin{pmatrix} a & 0 \\ c' & 1/a \end{pmatrix}$  to  $a^i$ . Note that  $S_0 = k$  is the trivial representation. Clearly,  $S_i \cong S_j$  if and only if  $i \equiv j \pmod{p-1}$  and  $S_i \otimes S_j \cong S_{i+j}$ . These are all of the simple  $kL$ -modules, since there can be at most  $p-1$  non-isomorphic indecomposable projective  $kL$ -modules.

Applying the discussion in Section 3.5.1 to the group  $L$ , one obtains a  $kL$ -module  $W \cong \tilde{\Omega}^2 k$ . By Remark 3.5.6, one can check that  $W \cong S_{-2}$ . It follows that  $kL$  has two blocks, with the module  $S_i$  in the principal block if and only if  $i$  is even. Moreover,  $S_{-2i} \cong W^{\otimes i} \cong \tilde{\Omega}^{2i} k$ , using Lemma 3.5.3, so all of the simple modules in the principal block are suspensions of the trivial module  $k$ . By Theorems 3.3.2 and 3.5.5, the simple ghost number of  $kL$ , the ghost number of  $kL$  and the ghost number of  $kP$  are all equal to  $\lfloor p/2 \rfloor$ .

We will show below that the simple ghost number of  $kG$  is actually 1, which is surprising since the simple generating hypothesis fails for its subgroups  $P$  and  $L$  when  $p > 3$ . It is even more surprising in view of the next result, which shows that  $\text{stmod}(kG)$  and  $\text{stmod}(kL)$  are equivalent.

The Sylow  $p$ -subgroup  $P$  is cyclic of order  $p$ . Thus it is a trivial intersection subgroup of  $G$  ( $gPg^{-1} \cap P$  is either  $P$  or trivial), and we have an equivalence between  $\text{stmod}(kG)$  and  $\text{stmod}(kL)$  by restriction and inducing up:

**Theorem 3.5.8** ([1, Theorems 10.1, 10.3]). *Let  $G$  be a finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . Let  $P$  be a Sylow  $p$ -subgroup of  $G$  and let*

$L = N(P)$  be the normalizer of  $P$  in  $G$ . Assume that  $P$  is a trivial intersection subgroup of  $G$ . Then the restriction functor

$$\text{stmod}(kG) \rightarrow \text{stmod}(kL)$$

is an equivalence, with inverse given by the inducing up functor. □

Note that the equivalence preserves the trivial representation  $k$  both ways, so the ghost number of  $kG$  equals that of  $kL$ , which is  $\lfloor p/2 \rfloor$ , by Theorem 3.5.5. But the equivalence does not preserve simple modules.

By Theorem 3.5.8, to study the simple ghost number of  $\text{StMod}(kG)$ , it is equivalent to study the pullback projective class of  $(\mathcal{S}, \text{s}\mathcal{G})$  on  $\text{StMod}(kL)$ , i.e. the projective class on  $\text{StMod}(kL)$  generated by the modules  $S\downarrow_L$ , for  $S$  a simple  $kG$ -module. We are going to show that this projective class contains all finitely-generated modules. It will then follow that the simple generating hypothesis holds for  $kG$ .

**Theorem 3.5.9.** *Let  $G = SL(2, p)$ . Every module in  $\text{stmod}(kG)$  is a direct sum of suspensions of simple modules. In particular, the simple generating hypothesis holds for  $kG$ .*

Note that despite the equivalence of Theorem 3.5.8, we already observed that the simple generating hypothesis does not hold for  $kL$  unless  $p \leq 3$ .

*Proof.* By the remarks immediately preceding the theorem, it suffices to show that the modules  $S\downarrow_L$ , with  $S$  a simple module in  $\text{stmod}(kG)$ , generate everything in  $\text{stmod}(kL)$  under direct sums, suspensions and retracts.

By Theorem 3.5.1, the indecomposable  $kL$ -modules are  $M_{i,j}$ , for  $1 \leq i \leq p$  and  $0 \leq j \leq p - 2$ , where  $M_{i,j}$  has radical length  $i$  and radical quotient  $M/\text{rad}(M) \cong S_j$ . It thus suffices to show that each module  $M_{i,j}$  is a suspension of some  $S\downarrow_L$ . For convenience, in the following we will interpret the subscript  $j$  modulo  $p - 1$ .

There are  $p$  simple  $kG$ -modules [1, p. 14], and we write  $V_1, \dots, V_p$  for their restrictions to  $L$ . The  $kL$ -module  $V_i$  is uniserial of radical length  $i$ , with radical quotient  $V_i/\text{rad}(V_i) \cong S_{i-1}$  [1, p. 76], so  $V_i = M_{i,i-1}$ . Note that the module  $V_1$  is trivial and the module  $V_p$  is projective. The case  $p = 2$  follows immediately, since  $L = C_2$ , and  $M_{1,0} = V_1 \cong k$



and  $M_{2,0} = M_{2,1} = V_2 \cong kC_2$  are the only two indecomposable  $kL$ -modules. Thus we assume that  $p$  is odd.

Recall that  $W \cong \tilde{\Omega}^2(k)$ , hence  $- \otimes W$  is isomorphic to the functor  $\Omega^2(-)$  on  $\text{stmod}(kL)$ . Since  $- \otimes W$  preserves radical lengths and shifts the simple module  $S_j$  to  $S_{j-2}$ , we have a stable isomorphism  $\Omega^{2k}V_i \cong M_{i,i-1-2k}$  for  $k \in \mathbb{Z}$ . This gives all modules  $M_{i,j}$  where  $i + j$  is odd.

To get the modules  $M_{i,j}$  with  $i + j$  even and  $1 \leq i < p$ , note that  $V_p \otimes S_{p-i-1}$  is the projective cover of  $V_{p-i}$ . It follows that  $\tilde{\Omega}V_{p-i}$  has radical length  $i$  and radical quotient  $S_{i-2}$ , i.e.,  $\tilde{\Omega}V_{p-i} \cong M_{i,i-2}$ . Then we can apply  $\Omega^{2k}$  again to obtain the modules  $M_{i,j}$  where  $i + j$  is even.  $\square$

In general, for which groups the simple generating hypothesis holds remains open.

## 3.6 Strong ghosts

In Section 3.6.1, we motivate and define strong ghosts and show that the strong ghost number of a group algebra  $kG$  equals the strong ghost number of  $kP$ , where  $P$  is a Sylow  $p$ -subgroup of  $G$ . In Section 3.6.2, we compute the strong ghost numbers of cyclic  $p$ -groups. In Section 3.6.3, we show that the strong ghost number of a dihedral 2-group  $D_{4q}$  is between 2 and 3.

### 3.6.1 The strong ghost projective class

If  $H$  is a subgroup of a finite group  $G$ , then it is rare for the restriction functor from  $G$  to  $H$  to preserve ghosts. For example, we saw in Section 3.3.2 that restriction from the group  $A_4$  to its Sylow  $p$ -subgroup  $P$  does not preserve ghosts. As another example, if  $G$  is a  $p$ -group and  $N \leq G$  is any normal subgroup, then the restriction from  $G$  to  $N$  does not preserve ghosts, since  $k\uparrow_N^G$  is indecomposable [1, Theorem 8.8] and is not a suspension of  $k$ . Strong ghosts, which were introduced in [17], will by definition restrict to ghosts.

**Definition 3.6.1.** Let  $G$  be a finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . A map in  $\text{StMod}(kG)$  is called a **strong ghost** if its restriction to  $\text{StMod}(kH)$  is a ghost for every subgroup  $H$  of  $G$ .

It follows immediately that the restriction of a strong ghost to any subgroup is again a strong ghost.

In [17], Carlson, Chebolu and Mináč study strong ghosts in  $\text{Thick}\langle k \rangle$ , but their results imply the following theorem, which says that most groups admit strong ghosts in  $\text{stmod}(kG)$ :

**Theorem 3.6.2** (Carlson, Chebolu and Mináč [17]). *Let  $G$  be a finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . Then every strong ghost in  $\text{stmod}(kG)$  is stably trivial if and only if the Sylow  $p$ -subgroup of  $G$  is  $C_2$ ,  $C_3$ , or  $C_4$ .*

Note that in passing from ghosts to strong ghosts, we only get one more  $p$ -group, namely  $C_4$ , where all strong ghosts are stably trivial.

We next observe that strong ghosts form an ideal of a projective class and use this in further study of strong ghosts.

Let  $H$  be a subgroup of  $G$ . We know that the restriction functor

$$\downarrow_H : \text{StMod}(kG) \rightarrow \text{StMod}(kH)$$

is both left and right adjoint to the induction functor

$$\uparrow^G : \text{StMod}(kH) \rightarrow \text{StMod}(kG).$$

The pullback (see Definition 3.2.5) of the ghost projective class along the restriction functor consists of maps in  $\text{StMod}(kG)$  which restrict to ghosts in  $\text{StMod}(kH)$ . The intersection of such ideals when  $H$  ranges over all subgroups of  $G$  consists of exactly the strong ghosts and again forms an ideal of a projective class: the relative projectives are obtained from modules of the form  $k\uparrow_H^G$  by closing under suspensions, desuspensions, direct sums and retracts. This is the **strong ghost projective class** on  $\text{StMod}(kG)$  and is denoted by  $(\text{st}\mathcal{F}, \text{st}\mathcal{G})$ . (In the terminology of [21], it is the *meet* of the pullbacks.)

Note that we can set  $\mathbb{P} = \{k\uparrow_H^G \mid H \text{ is a subgroup of } G\}$  in  $\text{StMod}(kG)$ , and this generates exactly the strong ghost projective class. Since every  $kG$ -module  $M$  is a summand of  $M\downarrow_P\uparrow_P^G$ , where  $P$  is a Sylow  $p$ -subgroup of  $G$ , and induction is a triangulated functor, we have that  $\text{Thick}_G\langle \mathbb{P} \rangle = \text{stmod}(kG)$ . Hence, using the terminology in Section 3.2.1, Theorem 3.6.2 is the statement that the generating hypothesis with respect to  $\mathbb{P}$  holds in  $\text{StMod}(kG)$  if and only if the Sylow  $p$ -subgroup of  $G$  is  $C_2$ ,  $C_3$ , or  $C_4$ .

For  $M \in \mathbf{stmod}(kG)$ , we define the **strong ghost length** of  $M$ , denoted by  $\text{stgl}(M)$ , to be the Freyd length with respect to  $\mathbb{P}$ , i.e.,  $\text{stgl}(M) = \text{len}_{\mathbb{P}}^{\mathbb{F}}(M)$ . The **strong ghost number** of  $kG$  is defined to be the Freyd number of  $\mathbf{StMod}(kG)$  with respect to  $\mathbb{P}$ .

One can show that strong ghosts induce up to strong ghosts by proving the dual statement, i.e., that relative projectives restrict to relative projectives. This follows from Mackey's Theorem (Theorem 3.2.8) and the observation that  $s(\Omega_H^n k) \cong \Omega_{sHs^{-1}}^n k$  [17]. Since the induction functor is always faithful, one obtains the following result:

**Proposition 3.6.3** (Carlson, Chebolu and Mináč [17]). *Let  $G$  be a finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . Let  $H$  be a subgroup of  $G$ . If  $g$  is a stably non-trivial strong ghost in  $\mathbf{StMod}(kH)$ , then  $g^{\uparrow G}$  is a stably non-trivial strong ghost in  $\mathbf{StMod}(kG)$ .* □

Next, we prove that the induction functor preserves strong ghost lengths.

**Proposition 3.6.4.** *Let  $G$  be a finite group, and let  $k$  be a field whose characteristic divides the order of  $G$ . Let  $H$  be a subgroup of  $G$ . Then for any  $M$  in  $\mathbf{stmod}(kH)$ ,  $\text{stgl}(M^{\uparrow G}) = \text{stgl}(M)$ , and so the strong ghost number of  $kG$  is at least as big as the strong ghost number of  $kH$ . Moreover, if  $P$  is a Sylow  $p$ -subgroup of  $G$ , then*

$$\text{strong ghost number of } kP = \text{strong ghost number of } kG.$$

*Proof.* The proof is essentially the same as the proof of Theorem 3.3.2. By Proposition 3.6.3, we have  $\text{stgl}(M^{\uparrow G}) \geq \text{stgl}(M)$ . Conversely, since the natural isomorphism  $\alpha : \underline{\text{Hom}}_G(M^{\uparrow G}, L) \rightarrow \underline{\text{Hom}}_H(M, L \downarrow_H)$  preserves strong ghosts,  $\text{stgl}(M^{\uparrow G}) \leq \text{stgl}(M)$ .

When  $P$  is a Sylow  $p$ -subgroup of  $G$ , the restriction functor is faithful by Lemma 3.2.7(i). The last equality follows. □

### 3.6.2 Strong ghost numbers of cyclic $p$ -groups

We study the strong ghost numbers of cyclic  $p$ -groups in this section. Our result suggests that the notion of a strong ghost is much stronger than that of a ghost.

We first review ghost lengths in  $\mathbf{stmod}(kC_{p^r})$ , following [19, Section 5.1].

**Lemma 3.6.5.** *Let  $G = C_{p^r}$  be a cyclic group of order  $p^r$  with generator  $g$ , let  $k$  be a field of characteristic  $p$ , and let  $M_n$  be the indecomposable  $kC_{p^r}$ -module of radical length*

$n$ . Then the self map  $g-1$  on  $M_n$  is a weakly universal ghost, i.e., any ghost with domain  $M_n$  factors through  $g-1$ . Moreover  $\text{gl}(M_n) = \min(n, p^r - n)$  and the ghost number of  $kG$  is  $\lfloor p^r/2 \rfloor$ .

*Proof.* That the map  $g-1$  is a ghost is proved in [9, Lemma 2.2]. It is weakly universal, since it fits into a triangle

$$k \oplus \Sigma k \longrightarrow M_n \xrightarrow{g-1} M_n \longrightarrow k \oplus \Sigma k.$$

The  $l$ -fold composite  $(g-1)^l$  on  $M_n$  is stably trivial if and only if  $l \geq \min(n, p^r - n)$  (see [19, Propositions 5.2, 5.3]). Hence  $\text{gl}(M_n) = \min(n, p^r - n)$ . Since all indecomposables are of this form, the ghost number of  $kG$  is  $\lfloor p^r/2 \rfloor$ .  $\square$

**Theorem 3.6.6.** *Let  $G = C_{p^r}$  be a cyclic group of order  $p^r$ , let  $k$  be a field of characteristic  $p$ , and let  $M_n$  be the indecomposable  $kC_{p^r}$ -module of radical length  $n < p^r$ . Writing  $N = \min(n, p^r - n) = \text{gl}(M_n)$ , we have the following:*

(i) *If  $N \leq p^{r-1}$ , then*

$$\text{stgl}(M_n) = \begin{cases} 1, & \text{if } N \mid p^r, \\ 2, & \text{otherwise.} \end{cases}$$

(ii) *If  $N > p^{r-1}$ , then*

$$\text{stgl}(M_n) = \left\lceil \frac{N}{p^{r-1}} \right\rceil = \left\lceil \frac{\text{gl}(M_n)}{p^{r-1}} \right\rceil.$$

*It follows that*

$$\text{strong ghost number of } kG = \begin{cases} \left\lceil \frac{p+1}{2} \right\rceil, & \text{if } p = 2 \text{ and } r \geq 3, \text{ or } p \text{ is odd and } r \geq 2, \\ \left\lceil \frac{p-1}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

*Proof.* We divide the proof into three cases:

**Case 1:** We first determine the indecomposable modules in  $\text{st}\mathcal{F}$ , i.e., those of strong ghost length 1. The set  $\text{st}\mathcal{F}$  is generated by  $\mathbb{P} = \{k \uparrow_{C_{p^j}}^{C_{p^r}} = M_{p^{r-j}} \mid 1 \leq j \leq r\}$ , and so an indecomposable module  $M_n$  is in  $\mathbb{P}$  if and only if  $n \mid p^r$ . Since  $\text{st}\mathcal{F}$  also contains the suspensions of modules in  $\mathbb{P}$  and  $\Sigma M_n \cong M_{p^r-n}$ , it follows that  $\text{stgl}(M_n) = 1$  if and only if  $n \mid p^r$  or  $(p^r - n) \mid p^r$ , i.e.,  $N \mid p^r$ .

This implies that  $\mathbb{P} \subseteq \mathcal{F}_{p^{r-1}}$ , or equivalently, that  $\mathcal{G}^{p^{r-1}} \subseteq \text{st}\mathcal{G}$ , which will be useful below.

**Case 2:** For  $N < p^{r-1}$ , we show that  $M_n$  is contained in  $\text{st}\mathcal{F}_2$ . Indeed, for such  $n$  we have a triangle

$$M_n \oplus \Sigma M_n \longrightarrow M_{p^{r-1}} \xrightarrow{(g-1)^N} M_{p^{r-1}} \longrightarrow M_n \oplus \Sigma M_n,$$

where  $g$  is a generator of  $C_{p^r}$ . Hence  $M_n \in \text{st}\mathcal{F}_2$  and  $\text{stgl}(M_n) \leq 2$ , completing the proof of (i).

**Case 3:** We compute the strong ghost length of  $M_n$  for  $N > p^{r-1}$ . By the previous observation, the self map  $(g-1)^{p^{r-1}}$  on  $M_n$  is a strong ghost. This map fits into the triangle

$$M_{p^{r-1}} \oplus \Sigma M_{p^{r-1}} \longrightarrow M_n \xrightarrow{(g-1)^{p^{r-1}}} M_n \longrightarrow M_{p^{r-1}} \oplus \Sigma M_{p^{r-1}},$$

with fibre in  $\text{st}\mathcal{F}$ , so it is a weakly universal strong ghost. By Lemma 3.6.5, its  $j$ th power is stably trivial if and only if  $jp^{r-1} \geq N = \text{gl}(M_n)$ . The equality in (ii) then follows.

The calculation of the strong ghost number follows from these results:

When  $p = 2$ , the ghost number of  $C_{2^r}$  is  $2^{r-1}$ , hence all  $C_{2^r}$ -modules are dealt with in (i), and the strong ghost number of  $C_{2^r}$  is 2 provided  $r \geq 3$ , and 1 otherwise.

When  $p$  is odd, the modules in (ii) dominate. The strong ghost length is maximized when  $N = (p^r - 1)/2$  (the ghost number of  $C_{p^r}$ ) and is

$$\left\lceil \frac{p^r - 1}{2p^{r-1}} \right\rceil = \left\lceil \frac{p - \frac{1}{p^{r-1}}}{2} \right\rceil,$$

which simplifies to the desired expressions. □

### 3.6.3 Strong ghost numbers of dihedral 2-groups

In this section we find an upper bound for the strong ghost number of a dihedral 2-group, using the result from the previous section on the strong ghost numbers of cyclic  $p$ -groups.

We write  $D_{4q}$  for the dihedral 2-group of order  $4q$ , with  $q$  a power of 2:

$$D_{4q} = \langle x, y \mid x^2 = y^2 = 1, (xy)^q = (yx)^q \rangle.$$

It has a normal cyclic subgroup  $C_{2q}$ , generated by  $g = xy$ . We prove the following theorem on the strong ghost number of  $D_{4q}$ :

**Theorem 3.6.7.** *Let  $D_{4q}$  be the dihedral 2-group of order  $4q$ , with  $q = 2^r$  and  $r \geq 1$ . Then*

$$2 \leq \text{the strong ghost number of } kD_{4q} \leq 3.$$

Recall that the strong generating hypothesis fails for  $kD_{4q}$  by Theorem 3.6.2, so the strong ghost number of  $D_{4q}$  is at least 2 for  $r \geq 0$ . When  $r = 0$ , so  $D_{4q} \cong C_2 \times C_2$ , the strong ghost number is 2. Since, for a  $p$ -group, the strong ghost length is bounded above by the ghost length, and by [19, Corollary 5.13], the ghost number of  $C_2 \times C_2$  is also 2.

Our goal will be to prove the upper bound. We will make use of the notation from [4] (see also [23, Section 4.6]), where the indecomposable  $kD_{4q}$ -modules are written using words in the letters  $a$  and  $b$ . By the proof of [23, Theorem 4.24], every non-projective indecomposable  $kD_{4q}$ -module  $M$  sits in a triangle  $\Omega W \rightarrow M \rightarrow M'' \rightarrow W$ , where  $M''$  is a sum of modules of the form  $M((ab)^s)$  and  $M((ab)^s a)$ , for  $0 \leq s < q$  (and the same, with  $a$  and  $b$  reversed), and  $W$  is a sum of suspensions and desuspensions of the trivial module. Thus, by Lemma 3.2.4, it will suffice to show that the modules  $M((ab)^s)$  and  $M((ab)^s a)$  have strong ghost length at most 2.

*Proof of Theorem.* By the discussion above, it suffices to show that

$$\text{stgl}(M((ab)^s)) \leq 2 \quad \text{and} \quad \text{stgl}(M((ab)^s a)) \leq 2$$

for  $0 \leq s < q$ .

It will be convenient to make the following notational convention: when we write  $(ab)^{\frac{m}{2}}$ , we mean  $aba \cdots$  with  $m$  letters in total. For example,  $(ab)^{\frac{5}{2}} = ababa$ . In addition,  $(ba)^{-\frac{m}{2}}$  denotes  $((ba)^{\frac{m}{2}})^{-1}$ , so  $(ba)^{-\frac{5}{2}} = b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$ . Let  $M = M((ab)^{\frac{q}{2}}(ba)^{-\frac{q}{2}}, \text{id}) \cong k \uparrow_{C_2}^{D_{4q}} \cong M_q \uparrow_{C_{2q}}^{D_{4q}}$ , which has strong ghost number 1. Similarly, for  $0 \leq m \leq q - 1$ , we write  $M'_m$  for the module  $M((ab)^{\frac{m}{2}}(ba)^{-\frac{m}{2}}, \text{id})$ , which has  $2m$  letters in total. Then  $M'_m \cong M_m \uparrow_{C_{2q}}^{D_{4q}}$ , as one can check that  $(1 - xy)^m (yxyx \cdots) = XYXY \cdots - YXYX \cdots$

( $m$  factors in each expression) by induction. Thus, by Proposition 3.6.4,  $M'_m$  has strong ghost length at most 2, since  $M_m$  does. Inducing up the triangle  $M_q \xrightarrow{(1-g)^m} M_q \rightarrow M_m \oplus M_{2q-m} \rightarrow M_q$  in  $\text{stmod}(kC_{2q})$ , we get the triangle

$$M \xrightarrow{(1-xy)^m} M \xrightarrow{\alpha} M'_m \oplus M'_{2q-m} \xrightarrow{\beta} M. \tag{3.6.1}$$

Let  $j : M'_m \oplus M'_{2q-m} \rightarrow k$  be zero on  $M'_m$  and non-zero on  $M'_{2q-m}$ . Then the composite  $j\alpha$  is stably trivial. One can check this fact by looking at the adjoint of  $j\alpha$ . Similarly, let  $i : k \rightarrow M'_m \oplus M'_{2q-m}$  be zero on  $M'_m$  and non-zero on  $M'_{2q-m}$ . The composite  $\beta i$  is stably trivial as well.

The kernel of the non-zero map  $M'_{2q-m} \rightarrow k$  is  $M((ab)^{q-\frac{m+1}{2}}(ba)^{\frac{m+1}{2}-q})$ , which we denote  $K_{2q-m}$ . We then form the octahedron

$$\begin{array}{ccccc}
 & & \Omega k & & \\
 & & \swarrow & \searrow & \\
 & M \oplus \Omega k & \longrightarrow & M'_m \oplus K_{2q-m} & \\
 M \nearrow \gamma & \downarrow & & \downarrow \psi & \searrow \phi \\
 & M & \xrightarrow{\alpha} & M'_m \oplus M'_{2q-m} & \longrightarrow M \\
 (1-xy)^m \searrow & & & & \nearrow \beta \\
 & & 0 & \searrow j & \\
 & & & k & 
 \end{array} . \tag{3.6.2}$$

We can use  $K_{2q-m}$  to build the module  $M((ab)^{q-\frac{m}{2}-1})$ , using the triangle

$$k \xrightarrow{\theta} K_{2q-m} \longrightarrow M((ab)^{q-\frac{m}{2}-1}) \oplus M((ba)^{q-\frac{m}{2}-1}) \longrightarrow \Sigma k.$$

Now consider the map  $0 + \theta : k \rightarrow M'_m \oplus K_{2q-m}$ . Since  $\phi(0 + \theta) = \beta\psi(0 + \theta) = \beta i$  is stably trivial, we get another octahedron

$$\begin{array}{ccccccc}
 & & & & k & & \\
 & & & & \swarrow 0+\theta & \searrow 0 & \\
 & & & & M'_m \oplus K_{2q-m} & \xrightarrow{\phi} & M \\
 M \oplus \Omega k \nearrow & & & & \downarrow & & \searrow \Sigma\gamma \\
 & & & & M'_m \oplus M((ab)^{q-\frac{m}{2}-1}) \oplus M((ba)^{q-\frac{m}{2}-1}) & \longrightarrow & M \oplus \Sigma k \\
 & & & & & & \nearrow f \\
 & & & & & & \Sigma k \\
 & & & & & & \searrow
 \end{array} , \tag{3.6.3}$$

and the triangle

$$M \oplus \Omega k \longrightarrow M'_m \oplus M((ab)^{q-\frac{m}{2}-1}) \oplus M((ba)^{q-\frac{m}{2}-1}) \longrightarrow M \oplus \Sigma k \xrightarrow{f} M \oplus k$$

shows that  $\text{stgl}(M((ab)^s))$  and  $\text{stgl}(M((ab)^{s-1}a)) \leq 2$  for  $\frac{q-1}{2} \leq s \leq q-1$ .

Note that in order to get  $M((ab)^{q-1})$ , we set  $m = 0$ , hence the map  $(1 - xy)^m$  is the identity on  $M$  and  $M((ab)^{q-1})$  is a summand of the cofibre of the non-trivial map  $k \rightarrow \Omega k$ .

To construct the modules  $M((ab)^s)$  and  $M((ab)^s a)$  for  $s$  small, we first suspend the map  $\gamma$  to get a triangle

$$M \xrightarrow{\Sigma\gamma} M \oplus k \longrightarrow M'_{2q-m} \oplus M((ba)^{-\frac{m}{2}}(ab)^{\frac{m}{2}}) \xrightarrow{\Sigma\phi} M.$$

Then we have

$$\Omega k \xrightarrow{\theta'} M((ba)^{-\frac{m}{2}}(ab)^{\frac{m}{2}}) \longrightarrow M((ba)^{\frac{m}{2}}) \oplus M((ab)^{\frac{m}{2}}) \longrightarrow k,$$

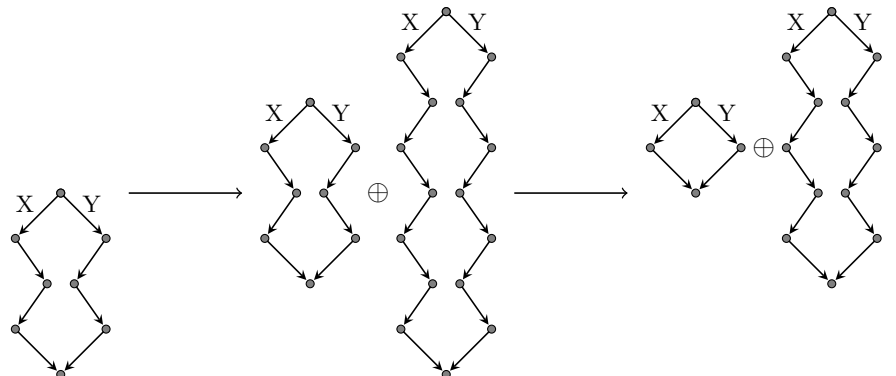
with  $\Sigma\phi(0 + \theta')$  stably trivial and we get a triangle similar to the one above:

$$M \oplus \Omega k \xrightarrow{g} M \oplus k \longrightarrow M((ab)^{\frac{m}{2}}) \oplus M((ba)^{\frac{m}{2}}) \oplus M'_{2q-m} \longrightarrow M \oplus k.$$

Thus  $\text{stgl}(M((ab)^s))$  and  $\text{stgl}(M((ab)^{s-1}a)) \leq 2$  for  $1 \leq s \leq \frac{q-1}{2}$ .

The two remaining cases are  $M((ab)^0) = k$  and  $M((ab)^{q-1}a) \cong k \uparrow_{(1,y)}^{D_{4q}}$ , both of which have strong ghost length 1, so we are done.  $\square$

We illustrate the triangles in the first octahedron as follows, taking  $q = 4$  and  $m = 2$ . The triangle (3.6.1) corresponds to a short exact sequence



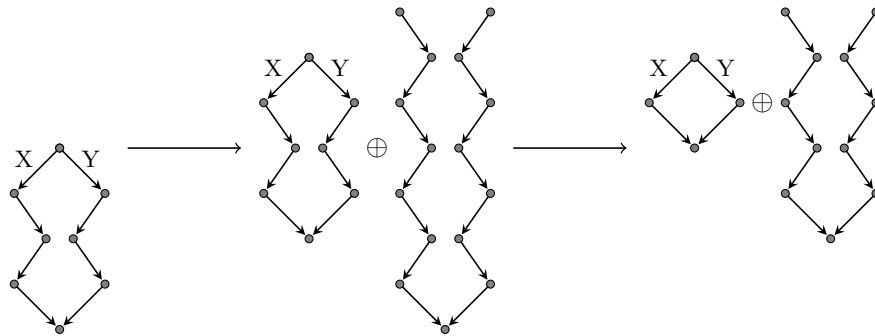


where a free module  $kD_{16}$  has been included. In these diagrams, the downward-left arrows indicate the action of  $X = x - 1$  and the downward-right arrows indicate the action of  $Y = y - 1$ .

And the triangle

$$M \xrightarrow{\gamma} M \oplus \Omega k \longrightarrow M'_2 \oplus M((ab)^2 ab^{-1} (ba)^{-2}) \xrightarrow{\phi} M$$

appearing in (3.6.2) and (3.6.3) has  $\Omega k$  in place of the free summand and corresponds to a short exact sequence



One can check that the map  $0 + \theta : k \rightarrow M'_2 \oplus M((ab)^2 ab^{-1} (ba)^{-2})$  factors through the middle term.

## Chapter 4

# Computations with GAP

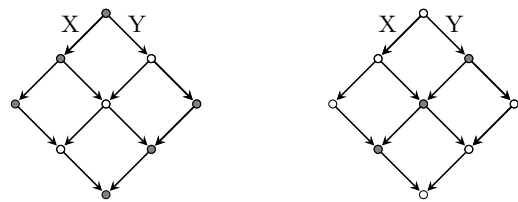
### 4.1 Introduction

In this chapter, we discuss how to apply GAP to compute ghost lengths for some examples. GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory [26]. The GAP package ‘reps’ handles group representations in positive characteristic. Its overall structure was designed and most of it written by Peter Webb, who is also the maintainer. Contributions were made by Dan Christensen, Roland Loetscher, Robert Hank, Bryan Simpkins, Brad Froehle and others.

### 4.2 The group $S_3 \times C_3$ at the prime 3

We begin with a motivating example. Let  $G = C_3 \times S_3$ , and let  $k$  be a field of characteristic 3. We write  $x$  for a generator of  $C_3$ ,  $y = (123)$  for an element of order 3 in  $S_3$  and  $z = (12)$  for an element of order 2 in  $S_3$ . Thus  $G$  is a group on three generators  $x$ ,  $y$ , and  $z$  subject to the relations  $x^3 = y^3 = z^2 = 1$ ,  $xy = yx$ ,  $xz = zx$ , and  $yz = zy^2$ .

There are two simple  $kG$ -modules  $k$  and  $\epsilon$ . Here  $k$  is the trivial representation and  $\epsilon$  is a 1-dimensional module with  $z$  acting as  $-1$ . They correspond to the indecomposable projective modules:



where we use a solid dot for  $k$  and a circle for  $\epsilon$ . The arrows down-left indicate the action of  $X = 1 - x$ , and the arrows down-right indicate the action of  $Y = y - y^2$ . Note that  $Xz = zX$ , while  $Yz = -zY$ . The correspondence is a result of the following lemma.

**Lemma 4.2.1.** [4, Theorem 1.6.3] *Let  $P$  be an indecomposable projective  $kG$ -module. Then  $P/\text{rad}(P)$  is simple and  $P/\text{rad}(P) \cong \text{soc}(P)$ .  $\square$*

With an abuse of notation, we write  $\epsilon$  for its restrictions to  $C_3 \times C_2$  and  $S_3$ . Since the principal idempotent of  $kG$  is 1 [31], both  $k$  and  $\epsilon$  are in the principal block. However,  $\epsilon$  is not in  $\text{Thick}_G\langle k \rangle$ . Indeed, by restricting to  $C_3 \times C_2$ , one sees easily that  $\epsilon$  is not in

the principal block of  $k(C_3 \times C_2)$ , hence cannot be in  $\text{Thick}_{C_3 \times C_2}\langle k \rangle$ . Since restriction is triangulated, it follows that  $\epsilon \notin \text{Thick}_G\langle k \rangle$ .

More generally, we know that there are only 6 indecomposable  $k(C_3 \times C_2)$ -modules:



It is clear that the first three modules are in  $\text{Thick}_{C_3 \times C_2}\langle k \rangle$ . We know that  $\epsilon$  is not in  $\text{Thick}_{C_3 \times C_2}\langle k \rangle$ , and the fifth module is isomorphic to  $\Omega\epsilon$  in  $\text{stmod}(k(C_3 \times C_2))$ , hence is not in  $\text{Thick}_{C_3 \times C_2}\langle k \rangle$  either. The last module is projective as a  $k(C_3 \times C_2)$ -module, hence is in  $\text{Thick}_{C_3 \times C_2}\langle k \rangle$ .

Since the restriction functor is triangulated, we deduce the following proposition.

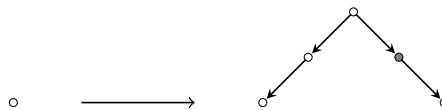
**Proposition 4.2.2.** *Let  $M$  be a  $kG$ -module. If  $M$  is in  $\text{Thick}\langle k \rangle$ , then the modules*



*cannot be summands of  $M \downarrow_{C_3 \times C_2}$ .*

Conversely, we can view the  $k(C_3 \times C_2)$ -modules as  $kG$ -modules with trivial  $y$ -action. Again, it is easy to see that the first three modules listed above are in  $\text{Thick}_G\langle k \rangle$ . One also sees that the three-dimensional modules are induced up from the subgroup  $S_3$ , as  $k \uparrow^G$  and  $\epsilon \uparrow^G$ . Since  $\Omega^2 k \cong \epsilon$  in  $\text{stmod}(kS_3)$ , the last module is a double suspension of the third one in  $\text{stmod}(kG)$ , hence is in  $\text{Thick}_G\langle k \rangle$ . It then follows that the other two modules are not in  $\text{Thick}_G\langle k \rangle$ , and we conjecture that the converse of the proposition is also true.

For example, we consider the cokernel  $M$  of a non-zero map  $f$



that sends  $\epsilon$  to the difference of the bottom elements. By Proposition 4.2.2, the domain and codomain of  $f$  are not in  $\text{Thick}_G\langle k \rangle$ . Nevertheless, we expect  $M$  to be in  $\text{Thick}_G\langle k \rangle$ . Note that this is equivalent to showing that  $M$  has finite generating length.

### 4.3 A computational method to calculate the generating length

Before we show how to apply computational methods to calculate the generating length, we need to introduce some notation. Let  $\mathsf{T}$  be a triangulated category and  $\mathbb{P}$  be a finite set of compact objects in  $\mathsf{T}$ . Recall that  $\langle \mathbb{P} \rangle$  denotes the closure of  $\mathbb{P}$  under retracts, direct sums, suspensions, and desuspensions, and this constitutes part of a projective class  $(\langle \mathbb{P} \rangle, \mathcal{I})$ . Then we can inductively define  $\langle \mathbb{P} \rangle_1 = \langle \mathbb{P} \rangle$  and  $\langle \mathbb{P} \rangle_n$  to consist of the objects  $X$  that are retracts of some object  $M$  such that  $M$  sits in a triangle  $P \rightarrow M \rightarrow Q$  with  $P \in \langle \mathbb{P} \rangle$  and  $Q \in \langle \mathbb{P} \rangle_{n-1}$ . Now we set  $\langle \mathbb{P} \rangle^c$  to be the closure of  $\mathbb{P}$  under retracts, finite direct sums, suspensions, and desuspensions, and define  $\langle \mathbb{P} \rangle_n^c$  in the same way as  $\langle \mathbb{P} \rangle_n$ , with  $\langle \mathbb{P} \rangle$  replaced by  $\langle \mathbb{P} \rangle^c$ . Writing  $\mathsf{T}^c$  for the collection of compact objects in  $\mathsf{T}$ , it is not hard to see that  $\langle \mathbb{P} \rangle^c = \langle \mathbb{P} \rangle \cap \mathsf{T}^c$ . More generally, we have

**Lemma 4.3.1.** *[13, Proposition 2.2.4] Let  $\mathsf{T}$  be a triangulated category and let  $\mathbb{P}$  be a set of compact objects in  $\mathsf{T}$ . With the notion described above,*

$$\langle \mathbb{P} \rangle_n^c = \langle \mathbb{P} \rangle_n \cap \mathsf{T}^c.$$

*In particular,  $\text{Thick}\langle \mathbb{P} \rangle = \text{Loc}\langle \mathbb{P} \rangle \cap \mathsf{T}^c$ .* □

We have chosen the notation to be consistent with that in Chapter 3. It is slightly different than that of [13]. Note that we have a filtration of  $\text{Thick}\langle \mathbb{P} \rangle$  by

$$\langle \mathbb{P} \rangle^c \subseteq \langle \mathbb{P} \rangle_2^c \subseteq \cdots \subseteq \langle \mathbb{P} \rangle_n^c \subseteq \cdots \subseteq \text{Thick}\langle \mathbb{P} \rangle.$$

Now consider  $\mathbb{P} = \{k\}$  in  $\text{StMod}(kG)$ . We write  $\mathbb{P}(-m, m)$  for the set  $\{\Sigma^i k \mid -m \leq i \leq m\}$  of finitely many suspensions of  $k$  contained in  $\langle \mathbb{P} \rangle$ . Recall that  $\langle \mathbb{P}(-m, m) \rangle$  is part of a projective class. Given  $M \in \text{Thick}\langle k \rangle$ , we write  $\text{gel}_m(M)$  for the length of  $M$  with respect to  $\mathbb{P}(-m, m)$ . Since  $\mathbb{P}(-m, m) \subseteq \mathbb{P}(-m-1, m+1) \subseteq \cdots \subseteq \langle \mathbb{P} \rangle$ , we get a decreasing sequence greater than or equal to  $\text{gel}(M)$ :

$$\text{gel}_m(M) \geq \text{gel}_{m+1}(M) \geq \cdots \geq \text{gel}(M).$$

Moreover, since  $M \in \langle \mathbb{P} \rangle_n^c$  for some positive integer  $n$ , there are only finitely many spheres  $\Sigma^{n_i} k$  needed to build up  $M$  in  $n$  steps. Hence there exists an integer  $m$ , such that  $M \in \langle \mathbb{P}(-m, m) \rangle_n^c$ , and, as a result of Lemma 4.3.1, we get

**Proposition 4.3.2.** *Let  $G$  be a finite group and  $k$  be a field whose characteristic divides the order of  $G$ . Let  $M$  be a module in  $\text{Thick}\langle k \rangle$ . Then  $\text{gel}(M) = \lim_{n \rightarrow \infty} \text{gel}_n(M)$ .  $\square$*

Since  $(\text{gel}_n(M))$  is a sequence of integers, we have  $\text{gel}_m(M) = \text{gel}(M)$  for  $m$  large. Using the formal property of a projective class, we are going to show that, for each integer  $n$ ,  $\text{gel}_n(M)$  can be computed by a finite process. In particular, if the cohomology of  $kG$  has periodicity  $n$ , then  $\text{gel}(M) = \text{gel}_n(M)$  and the computation of the generating length of  $M$  is a finite process. We recall the following lemma on the basic property of a projective class.

**Lemma 4.3.3.** *Let  $\mathbb{T}$  be a triangulated category, and  $(\mathcal{P}, \mathcal{I})$  be a (possibly unstable) projective class on  $\mathbb{T}$ . Let  $M$  be an object in  $\mathbb{T}$ . Then the following are equivalent:*

1.  $M$  is in  $\mathcal{P}_n$ .
2. Every  $n$ -fold composite of maps in  $\mathcal{I}$  out of  $M$  is zero.
3. The  $n$ -fold composite of universal maps in  $\mathcal{I}$  out of  $M$  is zero.

To implement this idea, we first compute the (unstable) universal ghost  $f : M \rightarrow N$  in the range  $[-m, m]$ . Since there are only finitely many suspensions of  $k$  needed, this is a finite computation. If  $f$  is stably trivial, then we know that  $M$  actually has generating length 1. Otherwise, we can make a recursive call to compute the universal ghost out of  $N$ , and test whether the composite of the universal ghosts is stably trivial. Finally, the first integer  $n$  such that  $n$ -fold composite of universal ghosts out of  $M$  is stably trivial is the generating length of  $M$  in the range  $[-m, m]$ . We present the method in pseudo-code:

```

GhostLengthHelper = function with inputs: a map f from M to N,
                                     an integer n
    g = universal ghost g from N to L
    if f composed with g is stably trivial then
        return f and n
    return GhostLengthHelper(f composed with g, n+1)

```

```

GhostLength of M = GhostLengthHelper(the identity map on M, 1)

```

**Example 4.3.4.** With the help of the `GhostLength` function, we can compute that the four dimensional module  $M$  we considered in Section 4.2 has  $\text{gel}_3(M) = 3$ , and so

$\text{gl}(M) \leq \text{gel}(M) \leq 3$ . Now we show that

$$\text{gl}(M) = \text{gel}(M) = 3.$$

To compute the lower bound, we consider left multiplication by the central element  $1 - x$  on  $M$ . Restricting to  $C_3 \times C_3$ , we know that  $1 - x$  is a ghost and  $(1 - x)^2$  is stably non-trivial. Then, by Theorem 3.3.2,  $1 - x$  is a simple ghost, hence a ghost, on  $M$ . Since the restriction functor to the Sylow  $p$ -subgroup is faithful by Lemma 3.2.7, and  $1 - x$  is a map in  $\text{Thick}_G\langle k \rangle$ , the ghost length of  $M$  is at least 3. Note that it follows directly from Theorem 3.3.2 that the simple ghost length of  $M$  is 3, but this does not guarantee that the simple ghosts are in  $\text{Thick}_G\langle k \rangle$ .

*Remark 4.3.5.* We remark here that there is not a universal choice of  $N$  such that  $\text{gel}_N(M) = \text{gel}(M)$  for all  $M \in \text{Thick}\langle k \rangle$ . Indeed, if the group cohomology is not periodic, then  $\text{gel}_N(\Omega^n k) = \text{gel}(\Omega^n k)$  if and only if  $N \geq |n|$ , and the number  $N$  can grow infinitely large. Note that the numbers  $\text{gel}_n(M)$  give upper bounds of the ghost length of  $M$ . Hence if a lower bound of the ghost length of  $M$  is known, we can hope to get the exact answer for the ghost length of  $M$ . It would also be interesting to know whether there is a way to compute lower bounds for the ghost length which converge to the correct answer.

## 4.4 The ReplaceWithInj function and related functions

We have improved the GAP code used in the reps package to compute the universal ghost and ghost length. We introduce the `ReplaceWithInj` function in this section, which is essential for computing the universal ghost. We also show the relation of `ReplaceWithInj` with other functions.

### 4.4.1 The ReplaceWithInj function and the Simple function

Recall that the universal ghost is the cofibre of a map that is surjective on Tate cohomology, and computing the cofibre depends on a function that replaces a map by an injection that is stably equivalent to it. For simplicity, we write  $f + g$  for the map  $M \rightarrow N \oplus P$ , where  $f : M \rightarrow N$  and  $g : M \rightarrow P$  are maps out of  $M$ . If  $P$  is projective, then the maps  $f$  and  $f + g$  are stably equivalent. Now let  $\{P_i\}$  be the set of non-isomorphic

indecomposable projective  $kG$ -modules, and let  $\mathcal{B}_i$  be a basis for  $\text{Hom}(M, P_i)$ . Observe that the natural map

$$\alpha : M \rightarrow \bigoplus_i (\bigoplus_{g \in \mathcal{B}_i} P_i)$$

is injective. Then for any map  $f : M \rightarrow N$ , the map  $f + \alpha$  is a replacement of  $f$  by an injection. But in this way, we will have added more maps than we need to  $f$ . For example, we don't need the maps  $g$  with  $\ker(f + g) = \ker(f)$ . And we can do better than this. We need a lemma before we state the condition that we will put on  $g$ .

**Lemma 4.4.1.** *Let  $f : M \rightarrow N$  be a map in  $\text{mod}(kG)$ . Then the map  $f$  is injective if and only if, for any simple module  $S$ , the map*

$$\text{Hom}(S, f) : \text{Hom}(S, M) \rightarrow \text{Hom}(S, N)$$

*is injective.*

*Proof.* Since  $\ker(\text{Hom}(S, f)) \cong \text{Hom}(S, \ker(f))$ , the map  $f$  being injective implies that  $\text{Hom}(S, f)$  is injective for any  $S \in \text{mod}(kG)$ . Conversely, if  $\text{Hom}(S, \ker(f)) = 0$  for all simple modules, then, since the simple modules generate the module category,  $\ker(f) = 0$  and  $f$  is injective.  $\square$

It follows from the lemma that we only need to add to  $f$  those maps  $g$  that shrink  $\ker(\text{Hom}(S, f))$ . More precisely, let  $P$  be an indecomposable projective module, and let  $g$  be a map from  $M$  to  $P$ . Then, to decide whether we want to replace  $f$  by  $f + g$ , it suffices to check the condition

$$\ker(\text{Hom}(S, f + g)) \subsetneq \ker(\text{Hom}(S, f)), \tag{4.4.1}$$

where  $S = P/\text{rad}(P)$  is the corresponding simple module of  $P$ . Indeed, if  $S' \not\cong S$  is another simple module, then  $\text{Hom}(S', P) = 0$ , and since  $\ker(\text{Hom}(S, f + g)) = \ker(\text{Hom}(S, f)) \cap \ker(\text{Hom}(S, g))$ , there is no need to test  $g$  on  $S'$ . It follows that we can work with one simple module at a time. Note that if we have replaced  $f$  by  $f' = f + g$ , then we can replace the condition in Equation 4.4.1 by  $\ker(\text{Hom}(S, f' + g)) \subsetneq \ker(\text{Hom}(S, f'))$ , and if  $\{g_1, g_2, \dots, g_l\}$  is a basis for  $\text{Hom}(M, P)$ , then

$$\ker(\text{Hom}(S, \sum_{i=1}^l (g_i))) = \ker(\text{Hom}(S, \alpha)) = 0,$$



where  $\alpha : M \rightarrow \bigoplus_i(\bigoplus_{g \in \mathcal{B}_i} P_i)$  is the injection we started with. Hence, for each indecomposable projective module  $P$ , we can use the following pseudo-code to produce a replacement  $f$  such that  $\ker(\text{Hom}(S, f)) = 0$  for  $S = P/\text{rad}(P)$ .

```

f = a given map from M to N
P = an indecomposable projective module
S = the corresponding simple module of P
for g in a basis for Hom(M, P)
  if ker(Hom(S, f+g)) is strictly contained in ker(Hom(S, f)) then
    replace f by f+g
  continue the loop over g until ker(Hom(S, f)) = 0
return f

```

Then, by Lemma 4.4.1, we can loop the preceding process over all indecomposable projective modules and produce a replacement by an injection. But we still need to describe how to determine whether  $\ker(\text{Hom}(S, f + g)) \subsetneq \ker(\text{Hom}(S, f))$ . This is done by a rank computation. We form the map  $\beta : \bigoplus S \rightarrow M$ , where the sum ranges over a basis for  $\text{Hom}(S, M)$ . Then we compare the dimensions of  $\text{im}((f + g) \circ \beta)$  and  $\text{im}(f \circ \beta)$  in the diagram

$$\begin{array}{ccccc}
 & & N \oplus P & & \\
 & & \nearrow & & \searrow \\
 \bigoplus S & \xrightarrow{\beta} & M & \xrightarrow{f} & N.
 \end{array}$$

It is clear that  $\text{rank}((f + g) \circ \beta) \geq \text{rank}(f \circ \beta)$ . Since  $\bigoplus S$  is semi-simple, the equality holds if and only if  $\ker(\text{Hom}(S, f + g)) = \ker(\text{Hom}(S, f))$ . In other words, the following conditions are equivalent:

1.  $\ker(\text{Hom}(S, f + g)) \subsetneq \ker(\text{Hom}(S, f))$ ,
2.  $\text{rank}((f + g) \circ \beta) > \text{rank}(f \circ \beta)$ .

Note that  $\text{rank}(f \circ \beta)$  is at most  $\text{rank}(\beta)$ , and this is equivalent to  $\ker(\text{Hom}(S, f)) = 0$ , so we can break out the loop over the basis for  $\text{Hom}(M, P)$  when  $\text{rank}(f \circ \beta) = \text{rank}(\beta)$ . We can also check at the same time whether  $f$  is injective or not and, if yes, we return  $f$  to avoid the extra loop over the other projective modules. To conclude the discussion, we display the function “ReplaceWithInj” in the following pseudo-code:

```

f = a given map from M to N
if Rank(f) == dimension of M then    % f is injective
    return N and f
L = list of non-isomorphic indecomposable projectives
for P in L
    S = the corresponding simple module of P
    b = map from a sum of S to M, ranging over a basis for Hom(S, M)
    r = Rank(f composed with b)
    rankb = Rank(b)
    if r != rankb then
        % r not maximal, so need to loop over a basis for Hom(M, P)
        for g in a basis for Hom(M, P)
            newf = f+g
            newr = Rank(newf composed with b)
            if newr > r then
                f = newf
                r = newr
                N = direct sum of N and P
            if r == rankb then          % r is maximal
                if Rank(f) == dimension of M then
                    return N and f
            break out of the loop over the basis for Hom(M, P)

```

*Remark 4.4.2.* We remark here that the code we just presented actually produces an optimal answer. That is, the replacement we produce is always minimal, unless the map  $f$  itself contains a stably-trivial summand, in which case we need to exclude the summand. To see that the process is optimal, observe first that  $\ker((f+g)\circ\beta) \subseteq \ker(f\circ\beta)$  is the kernel of the composite

$$\ker(f \circ \beta) \rightarrow \oplus S \xrightarrow{\beta} M \xrightarrow{g} P.$$

Since  $\ker(f \circ \beta)$  is a direct sum of copies of the simple module  $S$  and  $P$  is the corresponding projective module, the image of this composite is either zero or isomorphic to  $S$ . It follows that, when we replace  $f$  by  $f + g$ , we always have

$$\text{rank}((f + g) \circ \beta) = \text{rank}(f \circ \beta) + \dim(S).$$

Thus, to replace a map  $f : M \rightarrow N$  by an injection, we need to add exactly

$$\frac{\text{rank}(\beta) - \text{rank}(f \circ \beta)}{\dim(S)}$$

copies of the projective module  $P$  to  $N$ , as our code will do. Since this number is independent of the choice of a basis for  $\text{Hom}(M, P)$ , our code is optimal.

Note that the new code we introduced depends on a decomposition function to find all indecomposable projective modules and, for each indecomposable projective module, we need to find the corresponding simple module. We describe how to do these now.

It follows from Lemma 4.2.1 that there is a self map on  $P$

$$f : P \rightarrow P/\text{rad}(P) \cong \text{soc}(P) \rightarrow P,$$

with  $\text{im}(f) \cong S$ , and we can compute the image of *all* self maps on  $P$  to find  $S$  as the image whose dimension is the smallest. But this is not very efficient. So we consider  $M = \text{im}(f)$ , the image of an arbitrary self map  $f$  on  $P$ . Then  $M$  also satisfies the condition that  $M/\text{rad}(M) \cong \text{soc}(M) \cong S$ , being both a submodule and a quotient module of  $P$ . Hence, we can replace  $P$  by  $M$  to work with a smaller hom-set, and find  $S$  as the image of a self map on  $M$ . To implement this, we can loop over all self maps  $f$  on  $P$  and compute  $M = \text{im}(f)$ . Then, if  $M$  is a proper submodule of  $P$ , we replace  $P$  by  $M$  and make a recursive call and compute the images of self maps on  $M$ . The recursion will end with a module  $S$  that has no proper submodules. In other words,  $S$  is simple. Note that if  $\text{Hom}(M, M)$  has dimension 1, then the map  $M \rightarrow M/\text{rad}(M) \cong \text{soc}(M) \rightarrow M$  is an isomorphism, hence  $M$  is simple, and we can return  $M$  in this case. Then, assuming that  $P$  is an indecomposable projective module, we can find the corresponding simple module  $S$  using the following pseudo-code:

```
Simple = a function with one input P
  M = P
  hom = Hom(M, M)
  if hom has dimension 1 then
    return M
  for all maps f in hom
    if 0 < Rank(f) < dimension of M then
      M = im(f)
      return Simple(M)
  return M
```

*Remark 4.4.3.* Note that not every simple module  $S$  has  $\dim(\text{Hom}(M, M)) = 1$  when the field  $k$  is small, so, in general, we have to search over all self maps on  $M$ . Also note that, for an arbitrary module  $M$ ,  $\dim(\text{Hom}(M, M)) = 1$  does not imply that  $M$  is simple. For a counterexample, take  $G = S_3$ , the symmetric group on three letters and consider the two dimensional module  $M = \tilde{\Omega}k$ , where the condition  $M/\text{rad}(M) \cong \text{soc}(M)$  fails. But for the modules  $M$  that arises in the algorithm, the condition always holds.

#### 4.4.2 Other functions related to `ReplaceWithInj`

We show in this section how the `ReplaceWithInj` function can be used in other functions.

##### 1. `Cofibre` and `Suspension`.

With the `ReplaceWithInj` function, we can compute the cofibre of a map  $f$ . In particular, replacing the zero map out of  $M$ , we get an injection of  $M$  into a projective module, and its cofibre is the suspension of  $M$ . Since the `ReplaceWithInj` function provides an optimal answer, the suspension of  $M$  we get is projective-free. `Cofibre` is also essential in the `GhostLength` function, where we need to compute universal ghosts.

##### 2. `CreateRandomModule`.

We can create random modules in `Thick⟨k⟩` using cofibres. The function `CreateRandomModule` takes a random map  $f : P \rightarrow Q$  between random modules  $P$  and  $Q$  that are sums of suspensions of  $k$  and computes that cofibre  $R_1$ . Note that  $R_1$  has generating length at most 2. Iterating the process  $n$ -times, we can build up a module  $R_n$  of length at most  $n + 1$ . Note that the function depends on the number of summands that we allow in each step and the number of steps  $n$  that we take.

##### 3. `IsStablyTrivial`.

Let  $f : M \rightarrow P$  be an injection of  $M$  into a projective module. Then since  $P$  is also injective, every map from  $M$  to a projective module factors through  $f$ . Hence `ReplaceWithInj` provides an algorithm to detect whether a map  $g : M \rightarrow N$  is stably-trivial or not, by checking whether it factors through  $f$ .

##### 4. `ReplaceWithSurj`, `Fibre` and `Desuspension`.

Note that the pseudo-code we present in `ReplaceWithInj` is dualizable, so we can write the dual functions `ReplaceWithSurj`, `Fibre` and `Desuspension`.

## 4.5 More examples

We give more examples of computations in this section.

### 4.5.1 Comparing new code with old code

We begin with an easy computation of suspensions of the trivial representation for the alternating group  $A_4$  and the field  $GF(4)$  to compare the different versions of the `Suspension` function. We iterate `Suspension` to compute  $\Sigma^{14}(k)$  and test the time used.

$\Sigma^{14}(k)$	Dimension	Time
new function	29	0.7 s
old function without decomposing	37	13.2 s
old function with decomposing	29	45.4 s

It is clear from the table that our new function gives the optimal answer with less time. The old `ReplaceWithSurj` adds a free module to the codomain, so it may produce some projective summands in the cofibre. In the example, it raises the dimension of  $\Sigma^{14}k$  by 8. To get the optimal answer, we ask GAP to compute the projective-free summand, but this has taken much more time.

### 4.5.2 Computations in $C_9$ and $Q_8$

We test our code for the cyclic group  $C_9$  of order 9 with  $k = GF(3)$  and the quaternion group  $Q_8$  of order 8 with  $k = GF(2)$ . Note that the cohomology of  $C_9$  has periodicity 2 and that the cohomology of  $Q_8$  has periodicity 4. Also note that the generating number of  $kC_9$  is 4 and that the generating number of  $kQ_8$  is 3 or 4. In the examples, we create modules using the `CreateRandomModule` function, and keep the cofibres  $R_n$  with  $n \geq 3$ , so that  $R_n$  can have lengths greater than or equal to 4. Then we compute their generating lengths.

For the group  $C_9$ , we first take  $n = 4$  and record the dimensions and lengths of  $R_3$  and  $R_4$ . We performed 6 trials and get

n	3	4	3	4	3	4	3	4	3	4	3	4
Dimension	17	22	30	29	17	8	22	15	7	15	7	16
Length	1	2	2	3	1	1	2	2	3	4	2	2

The process seldom produces a module that achieves that generating number.

But if we take  $n = 17$ , then we have created some  $kC_9$ -modules of length 4:

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Dimension	22	14	20	19	11	11	11	12	11	19	18	18	8	16	9
Length	2	2	3	4	4	4	4	4	3	3	3	3	3	3	2

It is interesting to note that the lengths can decrease as we take more steps to build up the modules.

For the group  $Q_8$ , we are looking for some module that has length 4. It would then follow that the generating number of  $Q_8$  should be 4. We have tried to build up  $kQ_8$ -modules with  $n$  up to 100, but in all the examples, there are no  $kQ_8$ -modules of length 4, strongly suggesting that the generating number of  $kQ_8$  is 3.

**Conjecture 4.5.1.** *Let  $G = Q_8$  and  $k$  be a field of characteristic 2. Then*

$$\text{generating number of } kQ_8 = 3.$$

For evidence, here is the result when we built up  $kQ_8$ -modules with  $n = 9$ . We allowed up to 20 summands in each step to build up the modules.

n	3	4	5	6	7	8	9
Dimension	39	33	61	57	55	63	55
Length	3	3	2	3	3	3	3

## Chapter 5

# Conclusion

We provide a brief summary of the thesis in this chapter and describe the relation between the chapters.

The thesis focuses on the study of the stable module category  $\text{StMod}(kG)$ . Since  $\text{StMod}(kG)$  is a triangulated category, we can generalize the generating hypothesis from the stable homotopy category of spectra to  $\text{StMod}(kG)$ . In  $\text{StMod}(kG)$ , the generating hypothesis is the statement that the Tate cohomology functor  $\widehat{H}^n(G, -)$  is faithful on the subcategory  $\text{Thick}\langle k \rangle$ . In a series of papers [9, 16, 18, 20], Benson, Carlson, Chebolu, Christensen and Mináč proved that the generating hypothesis holds in  $\text{StMod}(kG)$  if and only if the Sylow  $p$ -subgroup  $P$  of  $G$  is  $C_2$  or  $C_3$ . Since the generating hypothesis fails in  $\text{StMod}(kG)$  in most cases, we study the ghost number of  $kG$ , which measures the failure of the generating hypothesis in  $\text{StMod}(kG)$ . It is the smallest integer  $n$  such that every  $n$ -fold composite of ghosts in  $\text{Thick}\langle k \rangle$  is stably trivial. This is first studied in [19] for a  $p$ -group, where it is shown that the ghost number of  $kG$  is always finite in this case. There are also some computations of and bounds on ghost numbers given in [19]. The ghost number is best described using the idea of a projective class [21], and this has been used throughout the thesis. The notation and background that we need here are introduced in Chapter 1. It also contains a literature review of the previous work in Section 1.3 and a summary of the main results of the thesis in Section 1.4.

In Chapter 2, which is based on [23], we continue the study of the ghost number of a group algebra. We have improved on the results in [19] and provided new computations for  $p$ -groups (See Section 2.4). And in general, we have proved that, for  $p$ -groups, the ghost number and the radical length of  $kG$  are within a constant factor of each other

(Corollary 2.4.17). More precisely, let  $G$  be a  $p$ -group, and let  $k$  be a field of characteristic  $p$ . Then

$$\frac{1}{3} \text{rad len } kG \leq \text{ghost num } kG \leq \text{gen num } kG < \text{rad len } kG.$$

Note that the trivial module  $k$  is the only simple module when  $G$  is a  $p$ -group, so the induction technique is very useful in our study of  $p$ -groups. This fails in the case of an arbitrary finite group. On the other hand, we have proved results on Auslander-Reiten triangles that apply to a general projective class in a triangulated category in Section 2.3.2. For example, we consider the simple ghosts and the strong ghosts in Chapter 3, where the results apply.

In Chapter 3, which is based on [24], we generalize the study of ghost numbers to arbitrary finite groups. In general, since  $\text{Thick}\langle k \rangle \neq \text{StMod}(kG)$ , a module induced up from a subgroup might not be in  $\text{Thick}\langle k \rangle$  and the induction technique fails. Hence we consider the projection onto the principal block  $B_0$  of  $kG$ . Under the assumption that  $\text{Thick}\langle k \rangle = \text{StMod}(B_0)$ , we show in Section 3.4.2 that

$$\text{ghost number of } kP \leq \text{ghost number of } kG,$$

with  $P$  being a Sylow  $p$ -subgroup of  $G$ , and that

$$\text{ghost number of } kG \text{ is finite.}$$

Examples of computations of ghost numbers are given in Sections 3.4 and 3.5. We have also introduced the simple ghost number in Section 3.2.3 and the strong ghost number in Section 3.6, and we show that they are closely related to the ghost number.

In Chapter 4, we apply computational algebra to the study of ghost numbers. We introduce a method to compute the generating number in Section 4.3, and then we describe the improved GAP code in the reps package to compute universal ghosts and ghost length in Section 4.4. We have made computations for the group  $S_3 \times C_3$  at the prime 3, the first example where  $\text{Thick}\langle k \rangle \neq \text{StMod}(B_0)$  (See Example 4.3.4). And for the quaternion group  $Q_8$  of order 8, we have experimental data that suggests Conjecture 4.5.1, which says that

$$\text{generating number of } kQ_8 = 3.$$



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